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Spécialité : Mathématiques

par

Abdelkoddousse Ahdida

Ecole Doctorale : MATHÉMATIQUES ET SCIENCES ET TECHNOLOGIES DE
L'INFORMATION ET DE LA COMMUNICATION

***Processus matriciels : simulation et modélisation
de la dépendance en finance***

Thèse dirigée par Aurélien Alfonsi et Bernard Lapeyre préparée au CERMICS, Ecole
des Ponts ParisTech
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Résumé

Le première partie de cette thèse est consacrée à la simulation des équations différentielles stochastiques définies sur le cône des matrices symétriques positives. Nous présentons de nouveaux schémas de discrétisation d'ordre élevé pour ce type d'équations différentielles stochastiques, et étudions leur convergence faible. Nous nous intéressons tout particulièrement au processus de Wishart, souvent utilisé en modélisation financière. Pour ce processus nous proposons à la fois un schéma exact en loi et des discrétisations d'ordre élevé. A ce jour, cette méthode est la seule qui soit utilisable quels que soient les paramètres intervenant dans la définition de ces modèles. Nous montrons, par ailleurs, comment on peut réduire la complexité algorithmique de ces méthodes et nous vérifions les résultats théoriques sur des implémentations numériques.

Dans la deuxième partie, nous nous intéressons à des processus à valeurs dans l'espace des matrices de corrélation. Nous proposons une nouvelle classe d'équations différentielles stochastiques définies dans cet espace. Ce modèle peut être considéré comme une extension du modèle Wright-Fisher (ou processus Jacobi) à l'espace des matrices de corrélation. Nous étudions l'existence faible et forte des solutions. Puis, nous explicitons les liens avec les processus de Wishart et les processus de Wright-Fisher multi-allèles. Nous démontrons le caractère ergodique du modèle et donnons des représentations de Girsanov susceptibles d'être employées en finance. En vue d'une utilisation pratique, nous explicitons deux schémas de discrétisation d'ordre élevé. Cette partie se conclut par des résultats numériques illustrant le comportement de la convergence de ces schémas.

La dernière partie de cette thèse est consacrée à l'utilisation de ces processus pour des questions de modélisation multi-dimensionnelle en finance. Une question importante de modélisation, aujourd'hui encore difficile à traiter, est l'identification d'un type de modèle permettant de calibrer à la fois le marché des options sur un indice et sur ses composants. Nous proposons, ici, deux types de modèles : l'un à corrélation locale et l'autre à corrélation stochastique. Dans ces deux cas, nous expliquons quelle procédure on doit adopter pour obtenir une bonne calibration des données de marché.

Mots clé : EDS, processus de Wishart, processus Affines, processus de Wright-Fisher et Jacobi, modélisation en grande dimension, les options sur panier, la volatilité implicite de l'indice, skew , Calibration.

Abstract

After a short introduction (in French) to the multi dimensional modelling for index pricing problems , the first part of the thesis treats the simulation of stochastic differential equations defined on the cone of symmetric positive semi-definite matrices. Indeed, we present several second order discretization schemes associated to a general class of affine processes defined on $\mathcal{S}_d^+(\mathbb{R})$. We study also their weak convergence. We pay a special attention to Wishart processes, which are considered as a particular case of this class and have been frequently used in finance. In this case, we give an exact scheme and a third order discretization one. To the best of our knowledge, this is the first exact sampling of the Wishart distribution without any restrictions on its parameters. Some algorithm are proposed in order to enhance all scheme in term of computation of time. We show numerical illustrations of our convergence and compare it to the theoretical rate.

We then focus on other type of processes defined on the correlation matrix space. For this purposes, We propose a new stochastic differential equation defined on $\mathcal{C}\mathcal{R}_d(\mathbb{R})$. We prove the weak and the strong existence of such solutions. These processes are considered as the extension of Wright-Fisher processes (or Jacobi process) on correlation matrices. We shed light on a useful connection with Wishart processes and Wright-Fisher multi-alleles. Moreover, we explicitly present their moments, which enable us to describe the ergodic limit. Other results about Girsanov representations are also given. Finally, in order to use these processes in practice, we propose second order discretization schemes based on two different methods. Numerical experiments are presented to show the convergence.

The last part is devoted to multi dimension modelling in finance for baskets and indices pricing. After giving a mathematical analysis of models defined either by the correlation matrix or in the positive semi-definite semi positive one, we ask if we find the adequate structure of correlation models which is able to calibrate both the index options market and the single options market related to each component of this index. For this purpose, we propose two types of modelling, the first uses a local model correlation and the second derives from a pure stochastic correlation model. Moreover, we explain different routines that have been used for improved calibration.

Keywords: SDEs, Wishart process, Affine process, Wright-Fisher and Jacobi process, multi dimension modelling, Basket options, index implied volatility, skew , Calibration.

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Chapitre 1

Introduction

La thèse que je présente se décompose en deux grandes parties. La première partie est consacrée à l'étude mathématique des méthodes numériques pour la simulation des processus, dit affines, étant définis sur le cône des matrices symétriques positives. L'étude se focalisera en particulier sur les matrices de Wishart, qui présentent un intérêt très prononcé en finance. La deuxième partie portera sur l'analyse stochastique et l'étude des méthodes d'approximation pour des processus définis, cette fois ci, dans l'espace des matrices de corrélation. En effet, je présenterai une nouvelle équation différentielle stochastique à retour à la moyenne, et à valeur dans l'espace des matrices de corrélation. Enfin, ce résultat me permettra de proposer un modèle de corrélation stochastique pour l'évaluation des options sur panier.

Ce premier chapitre a pour objectif d'introduire les principaux résultats de cette thèse d'une manière générale sans rentrer dans les détails techniques. Nous commencerons en particulier par expliquer les enjeux et la motivation de ces études. De plus amples détails seront donnés par la suite dans les chapitres correspondants.

1.1 Littérature : de la volatilité, à la corrélation.

L'acheteur d'une option ou bien d'un produit dérivé quelconque est exposé à plusieurs risques. Dans une perspective de contrôler, voire même d'éliminer ces incertitudes, il est important sur le plan financier, de les identifier, mettre en place les outils de gestion pour les comprendre, et enfin élaborer les stratégies adéquates pour les quantifier.

Les recherches sur l'aptitude du marché à prévoir les cours boursier en identifiant le risque, à travers les options, remontent aussi loin qu'en 1900, lorsque, Louis Bachelier termine sa thèse de doctorat intitulée 'Théorie de la spéculation'. Il est considéré comme un des premiers à vouloir utiliser la théorie et les techniques mathématiques sophistiquées pour expliquer le comportement des marchés financiers. Il a supposé que le prix d'un actif pourrait être expliqué par une tendance linéaire en temps, et un bruit représentant les fluctuations du marché. L'amplitude de ces fluctuations est définie par un intervalle, et '*cet intervalle est proportionnel à la racine carré de temps*' [10]. Ce fameux bruit est considéré par la suite comme le mouvement brownien. Il est aussi le précurseur de l'identification de la marche de hasard [10] (ou de 'diable'), bien connu dans la littérature financière par la suite. En revanche, le modèle de Bachelier présente un défaut non négligeable, puisque le prix d'un actif pourrait prendre des valeurs négatives.

Après plus un demi siècle, Black, Scholes et Merton ont élaboré (en même temps, mais pas ensemble) le même schéma de raisonnement, mais cette fois ci, appliqué à la rentabilité d'un actif donné. Si on note par S_t le prix de l'actif, ils considèrent qu'entre un intervalle de temps infinitésimal $[t, t + dt]$, la rentabilité $\frac{S_{t+dt} - S_t}{S_t}$ est donnée par une tendance linéaire en temps, et un mouvement brownien (représentant le bruit de marché) ayant un écart type proportionnel à la racine de l'intervalle. Sur le plan mathématique, le sous-jacent est défini par un mouvement brownien géométrique et donné par l'EDS suivante, sous l'espace risque neutre noté \mathcal{Q} :

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

où $(B_t)_{t \geq 0}$ est un mouvement brownien, et σ représente la volatilité de la rentabilité de sous-jacent et r est le taux sans risque. Pour un strike $K > 0$, maturité $T > 0$ et volatilité fixée $\sigma > 0$, si on note par $Call_{BS}(T, K, \sigma)$ le prix d'un call, sous le modèle de Black & Scholes, alors le prix s'écrit sous la forme $C(T, K, \sigma) = \mathbb{E}^{\mathcal{Q}} [e^{-rT}(S_T - K)^+]$. On définit ainsi la volatilité implicite, notée par $(\Sigma_{T,K})_{T>0, K>0}$, la solution unique de l'équation suivante :

$$\forall T > 0, \forall K > 0, Call_{BS}(T, K, \Sigma_{T,K}) = Prix^M(T, K)$$

où $Prix^M(T, K)$ correspond au prix sur le marché du call à maturité T et strike K . Dans le marché, on observe que cette volatilité implicite n'est pas constante et dépend étroitement à la fois de sa maturité et de son strike. Comme le modèle de Black & Scholes considère que la volatilité est constante, cette hypothèse mène à des difficultés pour l'évaluation et la calibration des prix des options. Cependant, l'avantage indéniable du modèle (ou bien plutôt, de la formule de Black & Scholes) reste la traduction très simple et rapide des prix des options européennes en terme de volatilité pour un sous-jacent donné. Les recherches ont penché ainsi vers des modèles plus riches, ayant des considérations plus générales par rapport à la structure de la volatilité.

Dans la littérature on peut identifier trois grandes classes de modélisation. La première est la volatilité locale introduite par Dupire [33], où ce dernier a considéré que la volatilité est une fonction déterministe du niveau de prix du sous-jacent. La nouvelle EDS de l'actif, sous l'espace de probabilité risque neutre, s'écrit sous la forme suivante :

$$\frac{dS_t}{S_t} = r dt + \sigma(t, S_t) dW_t,$$

où la volatilité $\sigma(\cdot, \cdot)$ est une fonction suffisamment régulière. Dans un souci de reproduire les mêmes prix de marché des options vanilles, Dupire a établi une équation différentielle permettant d'identifier la structure de σ à partir d'un continuum des prix des calls :

$$\frac{K^2}{2} \sigma^2(T, K) = \frac{\partial_T C(T, K) + rK \partial_K C(T, K)}{\partial_K^2 C(T, K)}$$

La deuxième classe de modélisation considère que la volatilité dispose de sa propre dynamique stochastique, ayant ainsi un risque non forcément incorporé dans les prix des sous-jacent. Le modèle de Heston [51] s'inscrit dans ce cadre de raisonnement, et considère que l'actif $(S_t)_{t \geq 0}$ et sa volatilité instantanée $(V_t)_{t \geq 0}$ sont une solution de l'EDS suivante :

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{V_t} dW_t \\ dV_t &= \kappa(\bar{V} - V_t) dt + \sigma \sqrt{V_t} \left(\rho dW_t + \sqrt{1 - \rho^2} dB_t \right), \end{aligned}$$

où $(W_t)_{t \geq 0}$ et $(B_t)_{t \geq 0}$ sont deux mouvements browniens indépendants, $\kappa, \bar{V}, \sigma \geq 0$ et $\rho \in [-1, 1]$. Tandis que la construction de Dupire garantit par défaut les prix du marché, la calibration du modèle de Heston s'effectue à travers l'identification de ses paramètres, où chacun a une influence spécifique sur la courbe de la volatilité implicite du modèle.

La dernière pensée de classification constitue une classe hybride englobant les deux branches précédentes, où on suppose que la volatilité détient une partie 'locale' et une autre stochastique. Ce type de modélisation sert, aussi au delà du marché des actions, et est employé pour les taux d'intérêt. dans la modélisation LIBOR. Le modèle de Piterbarg [72] fait partie de ce type de modélisation. Il suppose que le taux SWAP, noté par $(S_t)_{t \geq 0}$, est donné par l'EDS suivante, sous sa mesure de probabilité appropriée :

$$\begin{aligned} dS_t &= \lambda(bS_t + (1-b)S_0)\sqrt{V_t}dW_t \\ dV_t &= \kappa(\bar{V} - V_t)dt + \sigma\sqrt{V_t}dB_t \end{aligned}$$

où $b \in [0, 1]$. On considère ainsi que la volatilité instantanée de l'actif $(S_t)_{t \geq 0}$ est exprimée en fonction d'une partie locale donnée par b et une partie stochastique représentée par $(V_t)_{t \geq 0}$.

Il est toujours utile de souligner que quel que soit le type du modèle employé pour l'évaluation du produit, mettre en place une stratégie de couverture reste la phase la plus délicate pour un trader. On considère que $(\phi_t)_{t \geq 0}$ est le prix d'une option européenne à l'instant t sur un sous-jacent $(S_t)_{t \geq 0}$. Dans le cadre du modèle de Dupire, et grâce à l'évaluation sous la probabilité risque neutre, on obtient que $\phi_t = \phi(t, S_t)$. On désigne alors par $(\Pi_t)_{t \geq 0}$ le prix d'un portefeuille autofinancé détenant l'option ϕ_t , une quantité $-\Delta_t$ de l'actif $(S_t)_{t \geq 0}$ et une quantité $-\Delta_t^0$ de l'actif sans risque noté par $(S_t^0)_{t \geq 0}$. Une stratégie de couverture consiste à exhiber des valeurs du couple (Δ_t, Δ_t^0) , tel que le portefeuille Π_t devient un portefeuille sans risque. Un calcul d'Itô est employé pour prouver que le poids risqué Δ_t est égal à $\partial_S \phi(t, S_t)$ pour une telle stratégie.

Cependant, dans un modèle de volatilité stochastique, la couverture delta-neutre en employant juste l'actif $(S_t)_{t \geq 0}$ reste insuffisante. En effet, une nouvelle source de risque apparaît qui est liée à la volatilité, le prix de cette option européenne est représentée par $\phi_t = \phi(t, S_t, V_t)$, où $(V_t)_{t \geq 0}$ est la volatilité instantanée de l'actif. Par conséquent, une stratégie de couverture fait intervenir une nouvelle option ϕ_t^1 sur le même sous-jacent :

$$d\Pi_t = d\phi_t - \Delta_t dS_t - \Delta_t^1 d\phi_t^1 - \Delta_t^0 dB_t^0 = r\Pi_t dt. \quad (1.1)$$

où les coefficients du portefeuille sont déterminés par un calcul d'Itô. Bien qu'une stratégie de couverture dans un modèle volatilité stochastique est moins évidente que celle d'un modèle à volatilité locale, ce dernier pourrait donner des résultats erronés. Hagan et al. [50] a bien montré que la dynamique de la courbe de la volatilité implicite, dans un modèle de type CEV, présente des anomalies en matière de couverture.

Dans un contexte multi sous-jacent, est ce que la volatilité reste le paramètre ultime pour comprendre le risque, ou bien y aurait-il d'autres risques ayant la même importance et permettant d'expliquer mieux les incertitudes du prix d'un portefeuille ?

La corrélation joue un rôle très important en finance de marché. Dans la littérature financière, Bollerslev, Engle et Wooldridge [11], Moskowitz [68], Engle et Sheppard [36] et d'autres ont prouvé que la corrélation entre actifs change à travers le temps. En outre, certaines crises financières sont souvent vues comme des périodes assez particulières, où la corrélation se manifeste très largement. Roll [76] a mis en évidence la présence de ce phénomène durant le crash

de 1987. Les études de Jorion[54] à propos de la crise Russie/LTCM approuvent ce constat. Longin et Solnik [64] ont utilisé la théorie des valeurs extrêmes pour prouver que la corrélation entre les actifs internationaux affichent des valeurs records pendant la période de crise. La crise de subprime en 2007/08 n'a pas été épargnée, d'autant plus que l'origine de la crise venait des produits dérivés de crédit (CDO...), étant considérés comme des produits phares de corrélation, à cette époque.¹ Ce phénomène peut être lié à la diversification. Quand la corrélation dans le marché est très grande, les outils de diversification d'un portefeuille disparaissent, les investisseurs donc deviennent plus réticent en terme d'investissement.

Historiquement, l'estimation de la corrélation s'effectue à travers les données historiques. Cependant, il s'est avéré que le marché des options détient une information supplémentaire sur la corrélation, qui est beaucoup plus riche que celle trouvée dans l'historique. En effet, Driessen, Maenhout et Vilkov [31] ont prouvé que la prime de risque sur la corrélation est négative, cela conduit à penser que le marché des options estime une corrélation beaucoup plus grande que l'historique. Ils ont aussi montré que la variation de l'indice S&P100 est due à la corrélation des actifs, plus qu'à la variance individuelle de chaque actif. Ce constat est aussi lié à la notion de la corrélation implicite. Dans le monde des taux d'intérêts, certains articles ont étudié la structure de la corrélation des taux d'intérêt de plusieurs maturités. Longstaff [65], Jong, Driessen et Pelsser [53] ont mis en évidence que la corrélation induite des prix des Cap et Swap options est différente de celle réalisée. Dans le même registre, mais cette fois-ci dans le monde des dérivés actions, Avellanda et al. [7] a bien expliqué l'incapacité des actifs individuels à capter le smile de la volatilité implicite d'un indice.

Dans la pratique, on montre que la corrélation vérifie certaines propriétés empiriques similaires à celles de la volatilité (dans un contexte unidimensionnel). Ces caractéristiques sont principalement étudiées par exemple dans Skintzi et Refenes [79]. On note particulièrement la persistance de la mémoire de la corrélation, où on observe empiriquement qu'une forte (basse) variation de la corrélation est généralement suivie d'une forte (basse) variation. On assiste ainsi à un regroupement des valeurs extrêmes en paquet de variation de la corrélation. On observe aussi un phénomène, dit de 'comouvement' entre la variation de l'indice et la corrélation. Enfin, une asymétrie de corrélation bien prononcée dans le marché des indices. Selon Skintzi et Refenes [79], la distribution du cours de l'indice est généralement en corrélation négative avec sa propre corrélation. En effet, dans un marché morose la corrélation tend à prendre des valeurs de plus en plus grandes, alors que si le marché est haussier, la rentabilité individuelle se dégage de chaque actif, ainsi la corrélation renvoie des valeurs petites. Par conséquent, comme les modèles financiers sont destinés à refléter au mieux l'état réel du marché, il est indispensable de retrouver ces effets 'stylisés' dans un modèle de corrélation.

1.2 Modélisation multi dimensionnel

Dans toute cette partie on suppose que $(S_t)_{t \geq 0}$ est un vecteur de dimension $d \in \mathbb{N}^*$, tel que chaque valeur de ses composantes constitue le prix d'un sous-jacent donné.

Pour l'évaluation des options sur indices (ou bien paniers) en dérivé action, Avellaneda [7] a montré qu'une corrélation aléatoire est indispensable pour mieux refléter la nappe de la volatilité implicite de l'indice. A l'instar de la modélisation unidimensionnelle, une corrélation

¹Par abus de langage, la corrélation dans le monde de la finance désigne d'une manière générale la dépendance entre les éléments, plutôt que sa formulation mathématique standard.

aléatoire pourrait être représentée soit par des modèles à corrélation locale, soit par des modèles à corrélation stochastique. Dans la littérature financière, introduire de la corrélation dans un modèle multi dimensionnel peut se faire de deux manières : on modélise ou bien la matrice de variance covariance du système, ou bien on spécifie la matrice de corrélation, accompagnée par le vecteur des variances individuelles. Le processus que nous étudions est principalement lié à la modélisation de la corrélation stochastique.

Quand la matrice de variance-covariance, notée par Σ , est modélisée, le modèle multi sous-jacent associé s'écrit généralement comme :

$$\forall i \in \{1, \dots, d\}, \quad \frac{dS_t^i}{S_t^i} = \mu_t^i dt + (\sqrt{\Sigma_t} dB_t)_i, \quad (1.2)$$

où $(B_t)_{t \geq 0}$ est un vecteur de mouvement brownien de dimension d , et $(\mu_t)_{t \geq 0} \in \mathbb{R}^d$ décrit la tendance du vecteur S_t . On note que le modèle n'est bien défini que sous la condition de positivité de la matrice Σ_t . Les variances instantanées de chaque élément de S_t sont données par les composantes diagonales de Σ_t .

Sous les mêmes notations que (1.2), on note par $(R_t)_{t \geq 0}$ un processus de corrélation. La deuxième classe de modèle vérifie l'équation stochastique suivante :

$$\forall i \in \{1, \dots, d\}, \quad \frac{dS_t^i}{S_t^i} = \mu^i dt + \sigma_t^i (\sqrt{R_t} dB_t)_i. \quad (1.3)$$

Les variances individuelles de chaque élément sont spécifiées par $(\sigma^i)_{1 \leq i \leq d}$, et le rôle unique de la matrice R est d'introduire la corrélation entre les mouvements browniens. La deuxième démarche se distingue clairement de la première, puisque R n'intervient que dans la loi jointe, tandis que la loi marginale de chaque élément n'est qu'en fonction de la tendance μ et de la variance σ associées. Cependant, bien que cette séparation de tâche est intéressante (du point de vue de modélisation), il est difficile de construire une matrice définie dans l'espace des matrices de corrélation.

Modélisation avec une matrice de variance-covariance

Dans ce cadre de modélisation, la matrice de Wishart apparaît régulièrement dans la littérature financière. Puisque le processus reste affine, i.e offre des outils d'évaluation par l'inversion de la transformée de Fourier et qui seront développés par la suite. En effet, la matrice de Wishart est une solution de l'EDS suivante

$$\Sigma_t = \Sigma_0 + \int_0^t (\alpha a a^t + b \Sigma_s + \Sigma_s b) ds + \int_0^t \left(\sqrt{\Sigma_s} dW_s a + a^T dW_s^T \sqrt{\Sigma_s} \right), \quad (1.4)$$

où $(W_t)_{t \geq 0}$ est une matrice, où toutes ses composantes sont des mouvements browniens indépendants, et a et b sont des matrices dans $\mathcal{M}_d(\mathbb{R})$ et α est un scalaire positif supérieur à $d - 1$. Cette dernière condition assure l'existence d'une telle solution dans l'espace des matrices symétriques positives $\mathcal{S}_d^+(\mathbb{R})$. Bien que la définition de la distribution (originale) de Wishart est connue depuis les années 20 [83], l'introduction du processus (1.4) revient aux travaux de thèse de Bru [16], où cette dernière a construit l'équation stochastique (1.4) dans le but d'étudier la résistance d'Escherichie coli aux antibiotiques. Elle s'est focalisée sur le sensibilité des analyses en composantes principales aux perturbations browniennes. La loi marginale du

processus correspond à la distribution de Wishart [17]. Par la suite, Cuchiero et al. [26] ont analysé une classe beaucoup plus large des processus affines avec saut définis dans l'espace $\mathcal{S}_d^+(\mathbb{R})$, qui inclut les processus de Wishart. Cette étude portait sur l'existence faible de la solution, tandis que Mayerhofer, Pfaffel et Stelzer [66] ont regardé les conditions de l'existence forte de cette classe affine. L'apparition de Wishart en finance revient au modèle Gourieroux et Sufana. Dans la première version de leur papier [47], ils ont souligné la flexibilité des spécifications du processus pour reconstituer une dynamique de dépendance. Dans la deuxième partie du papier [48], Gourieroux et Sufana ont proposé le modèle (1.2) en supposant que les mouvements browniens W_t et B_t définis dans (1.2), (1.4) sont indépendants. Ce modèle est considéré comme l'extension affine la plus naturelle du modèle de Heston [51] en multi-dimension. En effet, Duffie, Pan et Singleton, K. [32] et Dai et Singleton [29] ont présenté une classe générale des processus affines définis dans l'espace vectoriel (\mathbb{R}^d), néanmoins les utiliser pour élaborer un modèle multi-dimensionnel reste une tâche assez ardue. Prenons un cas simple où on suppose que pour tout $1 \leq i \leq d$, $(Y_t^i)_{t \geq 0}$ est le prix lognormal d'un actif $(S_t^i)_{t \geq 0}$. Sans perte de généralité, et sous les mêmes notations on considère l'EDS suivante :

$$\forall i \in \{1, \dots, d\}, dY_t^i = \left(\mu_t^i - \frac{(X_t^i)^2}{2} \right) dt + \sqrt{X_t^i} dB_t^i,$$

où $(X_t^i)_{1 \leq i \leq d}$ sont deux processus CIR. En prenant en considération le caractère affine développé dans Duffie et Singleton [32], on dit que le modèle est affine si et seulement si toutes les variations quadratique $d\langle X^i, Y^j \rangle$, $d\langle X^i, X^j \rangle$ et $d\langle Y^i, Y^j \rangle$ ont une forme affine en fonction de X_t et Y_t . Or, en supposant que la corrélation des mouvements browniens est constante, le modèle précédent donne, à une constante près, pour tout $1 \leq i, j \leq d$:

$$\begin{cases} d\langle X_t^i, Y_t^j \rangle &= \sqrt{X_t^i X_t^j} dt \\ d\langle X_t^i, X_t^j \rangle &= \sqrt{X_t^i X_t^j} dt \\ d\langle Y_t^i, Y_t^j \rangle &= \sqrt{X_t^i X_t^j} dt \end{cases} \quad (1.5)$$

Par conséquent, considérer que $X_t^i = X_t^j$ pour tout $1 \leq i, j \leq d$ présente la seule alternative pour garder cette propriété affine. Pour obtenir des dynamiques plus intéressantes, tout en gardant la propriété affine du modèle, il faut relaxer les hypothèses de la modélisation. En effet, dans ce cadre des vecteurs affines, on modélise plutôt les risques multi-facteurs au lieu des volatilités. On suppose par exemple que $d = 2$ et que les prix lognormaux suivent la dynamique suivante

$$dY^1 = \mu_t^1 dt + \sigma^{11} \sqrt{X_t^1} dB_t^1 + \sigma^{12} \sqrt{X_t^2} dB_t^2, \quad dY^2 = \mu_t^2 dt + \sigma^{21} \sqrt{X_t^1} dB_t^1 + \sigma^{22} \sqrt{X_t^2} dB_t^2 \quad (1.6)$$

où $(\sigma^{i,j})_{1 \leq i, j \leq 2}$ sont des constantes. On peut donc remarquer que le modèle reste affine si on admet que les deux CIR X^1 et X^2 sont indépendants. Il s'agit du modèle 'double Heston', présenté par Christoffersen, Heston et Jacob [24], où ce modèle est considéré comme une variation du modèle original de Heston [51]. Cette structure affine du modèle nous permet de résoudre les défauts du premier modèle. Par contre, cette configuration utilise le CIR non pas comme des variances, mais plutôt des variables d'état. Par conséquent, on perd la signification des processus manipulés.

Dans les deux cas, on trouve que la structure usuelle affine ne permet pas d'élaborer un modèle qui convient au contexte multi-dimensionnel. Le recours donc à des processus affines

définis dans l'espace des matrices positives devient important. Il est donc utile de mettre en place des méthodes numériques permettant d'évaluer les produits financiers.

Comme le système $(\ln(S_t), \Sigma_t)_{t \geq 0}$ est affine, une inversion de la transformée de Fourier est employée pour calculer le prix de certaines options. Sans perte de généralité et dans le cadre de dimension d , si on considère qu'un processus $(Z_t^z)_{t \geq 0} \in \mathbb{R}^d$, démarré de z , est affine alors il existe deux fonctions ψ et ζ tel que pour tout $u \in \mathbb{R}^d$

$$\mathbb{E} [\exp(i\langle u, Z_t^z \rangle)] = \exp [\psi(t, u) + \langle z, \zeta(t, u) \rangle].$$

Etant donné une fonction $f : \mathbb{R}^d \rightarrow R$, on s'intéresse à calculer $\mathbb{E} [f(Z_t^z)]$. Sans rentrer dans les détails techniques, on obtient par interversion sous le signe de l'espérance :

$$\begin{aligned} \mathbb{E} [f(Z_t^z)] &= \mathbb{E} \left[\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(i\langle u, Z_t^z \rangle) \tilde{f}(u) du \right] \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp [\psi(t, u) + \langle z, \zeta(t, u) \rangle] \tilde{f}(u) du \end{aligned}$$

où \tilde{f} est la transformée de Fourier de f , définie par $\tilde{f}(u) = \int_{\mathbb{R}^d} e^{-i\langle u, x \rangle} f(x) dx$. D'une manière générale, la fonction \tilde{f} est explicite, et le calcul de l'espérance revient à calculer une intégrale. En théorie, l'emploi direct de cette transformée semble être facile, mais dans la pratique, la troncature ainsi que la discretisation des intégrales demandent une couche technique supplémentaire pour garantir la stabilité de la technique [13]. Cette méthode reste aussi pertinente dans le cadre multi-dimension appliqué à des produits financiers particuliers. Cependant, lorsque le nombre de sous-jacent devient élevé, cette technique devient très onéreuse en temps de calcul, puisque elle demande le calcul d'une intégrale à grande dimension. En revanche, il existe des produits financiers très particulier portant sur deux ou trois dimensions, par exemple option sur spread et option de worst/best (Da Fonseca et al. [28]), où on se ramène à un calcul unidimensionnel, et l'emploi de cette technique devient opportun. Il est néanmoins utile de chercher d'autres moyens de calcul permettant de calculer les espérances dans un cadre plus général.

Ainsi, dans le cadre des matrices de Wishart, Bensoussan, BenAbid et El karoui [8] ont développé des approximations des prix des options vanilles, à maturité courte et longue terme. Possamaï et Gauthier [42] ont aussi proposé des méthodes de Monte Carlo, en s'inspirant des méthodes de Andersen et Andreasen [6] pour le modèle de Heston. Par ailleurs Teichmann [71] a développé aussi des méthodes d'évaluation qui se base sur les fonctions caractéristiques. Mon premier sujet de recherche s'inscrit dans ce cadre de travail, où on propose des schémas exacts et des approximations permettant par la suite d'utiliser la méthode de Monte Carlo. A notre connaissance, il s'agit des premiers schémas dédiés aux matrices de Wishart qui sont définis dans l'espace des matrices positives indépendamment des valeurs prises par ses paramètres.

Modélisation de la matrice de corrélation

La modélisation directe avec une matrice de corrélation reste la méthode la plus utilisée pour l'évaluation des produits sur panier. Cependant, la construction d'une matrice dynamique, étant bien définie dans l'espace de corrélation et ayant des propriétés similaires à celles des processus affines, reste une tâche difficile. Par contre, cette méthode est très employée puisque

elle assure une certaine compatibilité entre les dimensions. En effet, Dans le modèle (1.3), si on se place dans l'espace risque neutre (pour tout $i \in \{1, \dots, d\}$ la valeur de μ^i est le taux dans risque) et on suppose que par exemple, les variances individuelles σ^i sont des volatilités locales calibrées aux marchés vanilles. Alors, quelles que soient les valeurs prises par la matrice de corrélation R , le modèle (1.3) est déjà calibré par rapport aux marchés vanilles, et la matrice R n'intervient que dans le marché sur indice. Par conséquent, on peut comprendre l'utilité de ce modèle, même si la construction de la matrice R reste difficile à concevoir.

Dans la littérature financière, certains papiers ont exploré une nouvelle formulation, à la 'Dupire', d'une matrice locale de corrélation, permettant ainsi de calculer les prix des options sur indices. En effet, si on note par σ_I la volatilité locale Dupire associée à l'indice, alors cette volatilité doit vérifier la condition de Gyöngy

$$\forall x \in \mathbb{R}^+, \quad \sigma_I^2(t, x) = \mathbb{E} \left[\frac{\alpha_i \alpha_j S^i S^j \sigma^i \sigma^j R_{i,j}}{x^2} \middle| \sum_{i=1}^d \alpha_i S^i = x \right], \quad (1.7)$$

où $(\alpha_i)_{1 \leq i \leq d}$ représente les poids de chaque actif dans l'indice, noté par I . Par définition on considère que $I = \sum_{i=1}^d \alpha_i S^i$. Avellaneda, Boyer-Olson, Busca et Friz [7] ont proposé une méthode, nommée 'most likely price', pour évaluer les prix, ce résultat n'est pas lourd en terme de temps de calcul, mais l'approximation reste loin des valeurs exactes. Jourdain et Sbai [55] ont proposé un modèle qui prend en compte, à la fois, la volatilité implicite de l'indice et celles de ses composantes. A l'instar du calcul des options vanilles par inversion de la FFT, Lipton [63] et Carr et Laurence [20] ont présenté une méthode similaire en employant la transformation de Radon, notée par \mathcal{R} . Pour toute fonction f de \mathbb{R}^d à valeur dans \mathbb{R} , cette transformée est donnée par

$$\forall k \in \mathbb{R}, \quad \forall w \in \mathbb{R}^d \quad \|w\|_2^2 = 1, \quad \mathcal{R}(f)(w, k) = \int_{\mathbb{R}^d} f(u) \delta(\langle w, k \rangle - k) du.$$

On montre alors que la densité de l'indice est exprimée en fonction de la transformée de Radon inverse, étant une formule semi explicite. Bien que cette transformation semble être en adéquation avec la structure du panier, la dimension demeure un obstacle qui entrave la simplicité de l'implémentation. Regai [75] et Langnau [61] ont opté pour une autre démarche, pour le calcul de la matrice R . D'une part, Langnau [61] donne une structure très particulière de R , tel que l'espérance conditionnelle de (1.7) disparaît. D'autre part, Regai [75] propose une structure particulière de corrélation, il emploie par la suite une itération récursive pour vérifier la condition (1.7). L'avantage des modèles de corrélation locale réside dans l'efficacité de sa calibration. En revanche, ce modèle reste opaque en terme d'interprétation de paramètre. En outre, on pourrait observer numériquement que la dynamique de la vol implicite ne suit pas forcément la dynamique réelle. En effet, on retombe sur les mêmes anomalies observées et prouvées par Hagan et al. [50] concernant la volatilité locale. Par conséquent, le delta hedging de sa couverture pourrait induire à des erreurs d'interprétation.

Rebonato et Jäckel [74] ont employé une forme paramétrique générale à l'aide des coordonnées angulaires. Cette décomposition, dite 'hyper-sphère' considère que la matrice de cor-

r elation R s' ecrit sous la forme de $R = BB^T$, tel que B est une matrice triangulaire inf erieure

$$\forall 1 \leq i \leq j \leq d-1, B_{i,j} = \cos(\theta_{i,j}) \prod_{k=1}^{j-1} \sin(\theta_{i,k})$$

$$\forall 1 \leq i \leq d, B_{i,d} = \prod_{k=1}^{d-1} \sin(\theta_{i,k}).$$

Cette technique demande donc $d(d-1)/2$ coordonn ees angulaires, not ees par θ . Bien que cette technique permet de balayer un espace assez large des valeurs de matrice de corr elation, elle pr esente deux d efauts majeurs. D'une part,  etant donn e une matrice R , trouver les param etres θ associ es  a R demeure une t ache difficile. En plus, la d ependance de la corr elation en fonction de θ n'est pas intuitive. D'autre part, comme tout autre mod ele local, interpr eter les sensibilit es des param etres reste une tache on ereuse. Dans le cadre de la mod elisation LIBOR en swap option, Brigo [14] a pr esent e diff erentes structures de corr elations possibles.

Par ailleurs, Driessen et al. [31] proposent une matrice corr elation stochastique construite  a partir d'un processus de Wright-Fisher. Leur travail consiste  a consid erer que pour tous $1 \leq i \neq j \leq d$, $(R_t)_{i,j} = \rho_t$, o u $(\rho_t)_{t \geq 0}$ est un processus Wright-Fisher d efini dans $[0, 1]$ et r esout l'EDS suivante :

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sigma\sqrt{\rho_t(1 - \rho_t)}dW_t, \quad (1.8)$$

o u W_t est mouvement brownien, et κ , σ et $\bar{\rho}$ sont des constantes v erifiant $\bar{\rho} \in [0, 1]$ et $\kappa \geq 0$. Ces deux conditions assurent que le processus reste bien d efini dans l'espace $[0, 1]$. Le mod ele pr esente plusieurs propri et es int eressantes, mais l'utilisation d'un seul facteur de risque ρ_t reste une hypoth ese forte, puisque on suppose implicitement que tous les sous jacents d egagent le m eme risque de corr elation. Van Emmerich [35] a propos e un mod ele similaire avec des processus Wright-Fisher  etant d efinis dans $[-1, 1]$. Son  etude portait sur les  eventuelles applications dans deux cas : les option Quantos et des options sur deux actifs. Dans le m emoire de Kaya Boortz [12], ce dernier propose une am elioration du mod ele pr ec edent, en rajoutant d'autres facteurs et en utilisant la structure de la d ecomposition de Cholesky. Il consid ere ainsi que pour tout $1 \leq i \neq j \leq d$, $(R_t)_{i,j} = \rho_t^i \rho_t^j$, o u les processus $(\rho_t^i)_{1 \leq i \leq d}$ sont des Wright-Fisher d efinis dans $[-1, 1]$, et donn es par l'EDS suivante :

$$\forall 1 \leq i \leq d, d\rho_t^i = \kappa(\bar{\rho}^i - \rho_t^i)dt + \sigma\sqrt{1 - (\rho_t^i)^2}dB_t^i,$$

o u pour tout $1 \leq i \leq d$, les $(B_t^i)_{t \geq 0}$ forment un vecteur de mouvements browniens corr el es, et le reste des param etres sont constants. Ce mod ele permet donc d'avoir plus de param etres de risque et de disposer ainsi de plus de flexibilit e pour calibrer le prix du panier. Par contre, il suppose une structure tr es particuli ere de la condition initiale. En effet, Kaya Boortz souligne cette difficult e en pr esentant une m ethode it erative pour capter 'au mieux' la condition initiale.

L'objectif de la deuxi eme partie de ma th ese est de construire une matrice stochastique d efinie dans l'espace de corr elation, et donn ee par une  equation stochastique autonome. Il s'agit  a notre connaissance de la premi ere extension des processus de Wright-Fisher dans l'espace des matrices de corr elation. Il est important de noter que dans le cadre vectoriel, plusieurs extensions de Wright-Fisher ont  et e propos ees. En effet, Gouri eroux et Jasiak [46] ont construit un vecteur tel que chacune de ses composantes est un processus Wright-Fisher d efinie dans $[0, 1]$, et la somme des ses composantes est  egale  a 1. Un autre mod ele, bien connu dans la

théorie de population sous le nom Wright-Fisher multi-allèles, est considéré comme la première extension de ce processus.

Modélisation des risques multi-facteurs

Dans un autre aspect de modélisation financière, les processus à valeurs dans les matrices symétriques positives, notamment ceux affines, sont utilisés pour représenter le risque multi-facteur. Dans un souci de reproduire la dynamique du smile forward de la volatilité implicite pour les dérivés de la volatilité, cette conception offre une flexibilité plus riche que la modèle standard, celui du Heston [51]. Le modèle 'Double Heston', présenté par Christofferson et al. [24] et donné dans (1.6), en fait partie. Cette approche est complètement différente de celle employée par Bergomi [9] ou bien Büehler [18], où ces derniers spécifient directement la dynamique de toute la courbe forward de la variance forward, tout comme les modèles HJM pour les taux.

Dans le même esprit que les modèles multi-facteurs vectoriels, Da Fonseca et al. [27] ont proposé d'employer les matrices affines au lieu des vecteurs affines. D'une part, ce modèle présente une certaine richesse de corrélation entre les facteurs, et d'autre part, il offre une flexibilité supplémentaire des paramètres sur la nappe de la volatilité implicite. La modélisation des taux d'intérêt cours instantanés se fait à l'aide de cette classe affine des processus. Les prix de zéro-coupons sont alors presque explicites, par la résolution des équations de Riccati (Levin [62]). En outre, Cheridito et al. [23] ont montré l'utilité de la classe affine pour estimer la prime de risque de marché. Grâce à cette étude, ils dressent une estimation empirique de la prime de risque, via les prix des options et l'historique des zéros coupons, à plusieurs maturités. Burashi, Porchia et Trojani [19] ont élaboré la même technique appliquée au processus de Wishart, puisque ce dernier garde le même caractère affine. L'avantage de cette méthode réside dans l'utilisation de Wishart comme un risque multi-facteur, étant considéré comme une extension des classes affines usuelles.

Nous présentons ainsi dans le chapitre 2 des approximations, associées aux modèles affines définis dans $\mathcal{S}_d^+(\mathbb{R})$, et permettant ainsi de calculer le prix des options par la méthode de Monte-Carlo. Ensuite nous proposons dans le chapitre 3 une nouvelle classe de processus définis dans l'espace des matrices de corrélation. Nous montrons quelques propriétés liées aux modèles, ainsi que les méthodes numériques d'approximation. À la lumière de ces processus de corrélation, la dernière partie de ma thèse (dans le chapitre 4) portera sur la mise en oeuvre des modèles de corrélation stochastique, pour calibrer les options sur indices. L'objectif du travail est d'identifier les paramètres du processus permettant de caractériser la volatilité implicite des options sur indices d'une manière efficace.

1.3 Simulation des processus affines définis dans $\mathcal{S}_d^+(\mathbb{R})$

La première partie de mon travail traite de la discretisation des processus affines définis dans le cône des matrices symétriques positives. Cette classe des processus intègre aussi les matrices de Wishart. Sans perdre de généralité, ici nous choisirons une paramétrisation similaire à celle utilisée dans Cuchiero et al. [26] et Bru [17], mais qui est légèrement différentes de celle

employée dans la littérature financière. On considère ainsi l'EDS suivante :

$$X_t^x = x + \int_0^t (\bar{\alpha} + B(X_s^x)) ds + \int_0^t \left(\sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x} \right), \quad (1.9)$$

où $x \in \mathcal{S}_d^+(\mathbb{R})$, $a \in \mathcal{M}_d(\mathbb{R})$, $\bar{\alpha} \geq (d-1)a^T a$, B est une fonction linéaire vérifiant la condition suivante

$$\forall x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R}), \quad \text{Tr}(x_1 x_2) = 0 \implies \text{Tr}(B(x_1) x_2) \geq 0, \quad (1.10)$$

et $(W_t)_{t \geq 0}$ est une matrice carrée constituée de mouvements browniens indépendants. En outre, l'équation (1.9) admet une solution forte si $\bar{\alpha} \geq (d+1)a^T a$. On note par $AF F_d(x, \bar{\alpha}, B, a)$ la loi du processus $(X_t^x)_{t \geq 0}$ et par $AF F_d(x, \bar{\alpha}, B, a; t)$ la loi marginale de X_t^x . Le processus de Wishart correspond au cas particulier :

$$\exists \alpha \geq 0, \bar{\alpha} = \alpha a^T a \text{ and } \exists b \in \mathcal{M}_d(\mathbb{R}), \forall x \in \mathcal{S}_d(\mathbb{R}), B(x) = bx + xb^T.$$

Dans ce cas, on note par $WIS_d(x, \alpha, b, a)$ la loi du processus de Wishart $(X_t^x)_{t \geq 0}$ et par $WIS_d(x, \alpha, b, a; t)$ sa loi marginale à l'instant $t \geq 0$.

A cause de la dimension, l'analyse stochastique de ce type de processus reste une tâche ardue. Dans un premier temps, Bru [17] s'est intéressé à l'existence forte et faible d'une classe très spécifique des processus de Wishart. Cuchiero et al. [26] se sont intéressés par la suite à l'étude de l'existence faible de l'EDS (1.9) sous sa formulation la plus générale avec saut. En plus, l'étude des solutions fortes de (1.9) ont été traitées par Mayerhofer et al [66]. Par ailleurs, La contrainte $\bar{\alpha} \geq (d-1)a^T a$ liée à la solution forte peut être vue comme la condition de Feller observée dans le cadre des processus de CIR.

La difficulté de discrétiser ce processus réside dans le caractère non lipschitzien du coefficient de la diffusion et dans le domaine de définition de la solution. Du même que la fonction racine carrée (unidimensionnelle) n'est pas lipschitzienne autour de zéro, la fonction $x \in \mathcal{S}_d^+(\mathbb{R}) \rightarrow \sqrt{x}$ ne l'est pas non plus aux voisinages des matrices non inversibles. Cependant, on pourrait montrer que si $\bar{\alpha} \geq (d+1)aa^T$, alors la matrice reste toujours définie positive, et ses valeurs propres ne toucheront jamais zéro (Bru [17] et Mayerhofer et al. [66]), tandis que si $\bar{\alpha} < (d+1)aa^T$, on trouve des exemples où la matrice peut prendre des valeurs non inversibles. D'un point de vue heuristique, le comportement des valeurs propres ressemble clairement au processus CIR, où quand a est grand, la matrice a tendance à passer plus de temps près de la frontière de $\mathcal{S}_d^+(\mathbb{R})$. Par conséquent, la qualité des schémas usuels tend à se détériorer à cause de la fonction racine carrée.

Etant donné que les évaluations des produits financiers est réalisée à travers des espérances, élaborer des schémas exacts est utile. A notre connaissance, il s'agit de la première simulation exacte des matrices de Wishart valable quelles que soient les valeurs de ses paramètres. Par ailleurs, construire des schémas d'ordre faible reste toujours opportun, puisque les schémas exacts sont parfois lourds en terme de temps de calcul. L'emploi des schémas de discrétisation est donc efficace dès qu'il s'agit d'un calcul d'une espérance en fonction de la trajectoire $(X_s^x, s \in [0, t])$. Par ailleurs, dans le cas où on s'intéresse à la loi du couple $(X_t^x, \int_0^t X_s^x ds)$, ces approximations demeurent indispensables à cause de leurs rapidité. Alfonsi [5] a bien soulevé que dans le cadre de l'évaluation des options sous le modèle de Heston, les approximations numériques sont beaucoup plus rapide en temps de calcul que les techniques exactes usuelles

(Broadié et Kaya [15]). Par conséquent, il est opportun de disposer de schémas de discrétisation bien définis dans $\mathcal{S}_d^+(\mathbb{R})$ et associés à cette classe des processus affines.

Avant de présenter les différents schémas proposés, il est utile de préciser que le schéma usuel d'Euler-Maruyama n'est pas défini dans l'espace des matrices positives, pour tout $0 \leq i \leq N - 1$:

$$\hat{X}_{t_0^N}^N = x, \hat{X}_{t_{i+1}^N}^N = \hat{X}_{t_i^N}^N + (\bar{\alpha} + B(\hat{X}_{t_i^N}^N))\frac{T}{N} + \sqrt{\hat{X}_{t_i^N}^N}(W_{t_{i+1}^N} - W_{t_i^N})a + a^T(W_{t_{i+1}^N} - W_{t_i^N})^T \sqrt{\hat{X}_{t_i^N}^N}, \quad (1.11)$$

Sous cette discrétisation, on peut obtenir des composantes diagonales strictement négatives, car l'incrément brownien peut prendre des valeurs fortement négatives. Or, les composantes diagonales d'une matrice positive restent toujours positives. Pour pallier ce défaut, on peut utiliser un schéma d'Euler-Maruyama corrigé, pour tout $0 \leq i \leq N - 1$,

$$\hat{X}_{t_0^N}^N = x, \hat{X}_{t_{i+1}^N}^N = \hat{X}_{t_i^N}^N + (\bar{\alpha} + B(\hat{X}_{t_i^N}^N))\frac{T}{N} + \sqrt{(\hat{X}_{t_i^N}^N)^+}(W_{t_{i+1}^N} - W_{t_i^N})a + a^T(W_{t_{i+1}^N} - W_{t_i^N})^T \sqrt{(\hat{X}_{t_i^N}^N)^+}, \quad (1.12)$$

où on associe à toute matrice symétrique $x \in \mathcal{S}_d(\mathbb{R})$, une matrice positive $x^+ \in \mathcal{S}_d^+(\mathbb{R})$. Les vecteurs propres de x^+ sont les mêmes que ceux de x , tandis que chacune de ses valeurs propres est la partie positive de celle de x . Bien que le coefficient $\sqrt{x^+}$ est bien défini quel que soit la valeur de x , le schéma en entier peut prendre des valeurs en dehors des matrices symétriques positives. Ceci étant bien vérifié dans les résultats numériques, dès que $\bar{\alpha}$ est très proche de $(d - 1)aa^T$.

Les études mathématiques se sont intéressées à la simulation exacte de la distribution de Wishart, mais on observe clairement deux classifications : Wishart centré et non centré. On dit que la distribution de Wishart est centrée, si la valeur initiale du processus de Wishart est nulle. Dans ce cas, plusieurs méthodes de simulation ont été proposées. La plus fameuse est la décomposition de Bartlett. Cependant, dans le cas non centré, les méthodes de simulation ne se sont développées que dans le cas où $\alpha \in \mathbb{N}$. En effet, La simulation exacte réside en une simulation exacte des gaussiennes. En terme de distribution, ce résultat est présenté par Odell et Feiveson [70], Smith et Hocking [80], et Gleser [44], tandis que la version stochastique a été présentée par Bru [17]. On suppose que $\alpha \in \mathbb{N}$, et on considère une suite des processus d'Ornstein-Uhlenbeck, notée par $(Y_t^k)_{1 \leq k \leq \alpha}$, étant une solution de l'EDS suivante :

$$dY_t^k = bY_t^k + a^T d(Z_t^k)^T, \quad (1.13)$$

où $(Z_t^k)_{1 \leq k \leq \alpha}$ est une suite des vecteurs formés par des mouvements browniens indépendants. Si on pose $X_t = \sum_{k=1}^{\alpha} Y_t^k (Y_t^k)^T$, on déduit que ce processus est solution de cette EDS

$$dX_t = \alpha(a^T a + bX_t + X_t b^T)dt + a^T \left(\sum_{k=1}^{\alpha} dZ_t^T (Y_t^k)^T \right) + \left(\sum_{k=1}^{\alpha} Y_t^k dZ_t \right) a, \quad (1.14)$$

Sans rentrer dans les détails, on montre donc directement qu'il existe une matrice carrée de mouvement brownien notée tel que $(W_t)_{t \geq 0}$, tel que $(\sum_{k=1}^{\alpha} dZ_t^T (Y_t^k)^T) = \sqrt{X_t} dW_t$. Dans sa thèse, Bru [16], spécifie clairement la structure de $(W_t)_{t \geq 0}$ en utilisant le théorème de Lévy. Dans ce cas, la simulation de la matrice de Wishart se ramène à la simulation des processus d'Ornstein-Uhlenbeck, et donc des gaussiennes.

La littérature sur les études de discrétisation de ce type de processus est récente. Benabid, Benssouan et El Karoui [8] ont employé un changement de probabilité, pour passer d'un Wishart de paramètre α réel supérieur de $d + 1$, à un Wishart de paramètre $\alpha = d + 1$. Ce théorème

Girsanov reste toujours vrai dans le cadre de classe affine générale (Il suffit juste d'employer les même arguments théoriques que dans la partie de Girsanov dans le chapitre 3). Dans ce nouvel espace de probabilité, la simulation de Wishart revient à simuler des gaussiennes. Bien que la méthode est relativement simple à mettre en place, la ratio de Radon-Nikodym reste difficile à manipuler en terme de discrétisation. Gauthier et Possamaï [42] ont proposé une autre technique basée sur le schéma exponentiel quadratique d'Andersen. Même si l'approximation reste bien définie dans l'espace des matrices positives, la solution demande un 'matching' des moments qui reste lourd à mettre en oeuvre.

Dans le cadre des schémas de discrétisation, l'analyse mathématique des erreurs traite deux types d'erreurs : forte et faible. La première mesure l'écart entre la trajectoire du processus et de sa discretisation, tandis que la deuxième calcule juste l'erreur induite par la loi marginale d'un processus et de son approximation. Elle s'intéresse en particulier au comportement asymptotique de

$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(\hat{X}_{t_N}^N)] = \mathbb{E}[f(X_T)] - \mathbb{E}[f(\hat{X}_T^N)] \quad (1.15)$$

pour une classe de fonction f caractérisant la loi marginale de X . Dans ce travail, nous nous intéressons juste à l'erreur faible.

Dans un premier temps, on s'intéressera à la simulation exacte des matrices de Wishart. A l'instar de la relation liant les processus CIR et carrés de Bessel, on a établi une égalité en loi entre un processus de Wishart, ayant une structure générale, avec un processus Wishart particulier où la matrice b est égale à zéro. Cette égalité s'écrit sous la forme suivante :

$$WIS_d(x, \alpha, b, a; t) \stackrel{\text{Loi}}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, \underbrace{0}_{b=0}, I_d^n; t) \theta_t^T \quad (1.16)$$

où $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$ et $n = \text{Rk}(q_t)$, et la matrice $\theta_t \in \mathcal{G}_d(\mathbb{R})$ étant défini comme la solution de l'équation $q_t = t \theta_t I_d^n \theta_t^T$. On comprend alors que si on veut construire un schéma exact pour $WIS_d(x, \alpha, b, a; t)$, il est donc suffisant d'établir un schéma exact associé à $WIS_d(x, \alpha, 0, I_d^n; t)$. Pour simuler cette loi, nous utilisons la technique de décomposition d'opérateurs infinitésimaux.

En effet, on considère deux processus affines indépendants, noté par $X_t^{1,x}$ et $X_t^{2,x}$, démarant en un point x . Et on suppose qu'ils sont associés respectivement à deux opérateurs infinitésimaux L_1 et L_2 , tels que la condition de commutativité est vérifiée : $L_1 L_2 = L_2 L_1$. Si on note par Y_t^x un processus affine associé à $L_1 + L_2$, démarant en x , alors pour une classe de fonction f , on obtient

$$\mathbb{E}[f(Y_t^x)] = \sum_{k=0}^{+\infty} \frac{t^k}{k!} (L_1 + L_2)^k f(x). \quad (1.17)$$

Or, par un calcul très formel, on établit les égalités suivantes

$$\begin{aligned} \mathbb{E}[f(X_t^{2, X_t^{1,x}})] &= \mathbb{E}\left[\mathbb{E}\left[f(X_t^{2, X_t^{1,x}}) | X_t^{1,x}\right]\right] = \sum_{k_2=0}^{+\infty} \frac{t^{k_2}}{k_2!} \mathbb{E}\left[L_2^{k_2} f(X_t^{1,x})\right] \\ &= \sum_{k_1, k_2=0}^{+\infty} \frac{t^{k_2+k_1}}{k_1! k_2!} L_1^{k_1} L_2^{k_2} f(x) \\ &= \sum_{k=0}^{+\infty} \frac{t^k}{k!} (L_1 + L_2)^k f(x). \end{aligned} \quad (1.18)$$

Si la classe des fonctions f est assez large pour caractériser les lois marginales de tous ces processus, alors on conclut que pour tout $t \geq 0$, $Y_t \stackrel{\text{loi}}{=} X_t^{2, X_t^{1,x}}$. Cette propriété s'étend bien évidemment au cas où la composition fait intervenir plusieurs opérateurs infinitésimaux. Si on note maintenant par L l'opérateur infinitésimal de $WIS_d(x, \alpha, 0, I_d^n; t)$, nous démontrons que cet opérateur peut se décomposer en $L = \sum_{i=1}^n L_i$, où pour tout $1 \leq i \leq d$, L_i est l'opérateur associé à $WIS_d(x, \alpha, 0, e_d^i; t)$. On prouve aussi que la condition de commutativité est vérifiée :

$$\forall 1 \leq i, j, \leq d, L_i L_j = L_j L_i, \quad (1.19)$$

On construit alors d'une manière itérative le schéma suivant

$$\begin{aligned} X_t^{1,x} &\sim WIS_d(x, \alpha, 0, e_d^1; t), \\ X_t^{2, X_t^{1,x}} &\sim WIS_d(X_t^{1,x}, \alpha, 0, e_d^2; t), \\ &\dots \\ X_t^{n, \dots, X_t^{1,x}} &\sim WIS_d(X_t^{n-1, \dots, X_t^{1,x}}, \alpha, 0, e_d^n; t), \end{aligned}$$

Etant donné un temps t , chaque schéma intermédiaire $X_t^{i, \dots, X_t^{1,x}}$, pour tout $1 \leq i \leq d$, est construit par la distribution de Wishart ayant comme paramètre $(\alpha, 0, e_d^i)$ et démarrant en $X_t^{i-1, \dots, X_t^{1,x}}$. Et on déduit grâce à la technique de décomposition (1.18) que

$$X_t^{n, \dots, X_t^{1,x}} \sim WIS_d(x, \alpha, 0, I_d^n; t).$$

Finalement, l'objectif de la simulation exacte est achevé si on établit un schéma exact pour $WIS_d(x, \alpha, 0, e_d^i; t)$, pour tout $1 \leq i \leq d$. Par un argument de symétrie, il est juste suffisant de se focaliser sur $WIS_d(x, \alpha, 0, e_d^1; t)$. Avec une transformation adéquate et un calcul d'Itô direct, nous expliquons dans le chapitre 2 la technique employée pour réaliser cette dernière partie qui s'appuie sur la simulation des processus de Bessel carré et de loi gaussiennes.

La simulation des schémas de discrétisation faible se base étroitement sur les résultats précédents. En effet, on propose un schéma d'ordre 3 pour Wishart, en remplaçant les solutions exactes des processus de Bessel et les gaussiennes par leurs approximations correspondantes d'ordre faible 3. Nous présentons aussi un schéma d'ordre faible 2 associé à la classe générale des processus affines. Du même que pour l'approximation mentionnée d'ordre 3, ce schéma général se base aussi sur la décomposition des opérateurs. Etant donné que toutes les méthodes précédentes s'appuient sur la brique de base, le générateur L_1 , la complexité de ces méthodes revient donc au temps de calcul consacré à cet opérateur. En effet, sa complexité est d'ordre $O(d^3)$, puisque il fait intervenir une décomposition de Cholesky, et des multiplications matricielles. Comme ces schémas itèrent au plus d fois la solution associée à cet opérateur, alors ils demandent une complexité égale à $O(d^4)$. Le schéma d'Euler-Maruyama corrigé fait intervenir le calcul des valeurs et des vecteurs propres des matrices symétriques dans $\mathcal{S}_d(\mathbb{R})$, il a donc comme ordre de complexité $O(d^3)$. Bien que son ordre de complexité est plus faible que celui des schémas précédents, ces derniers sont dans les faits plus rapides, puisque ils ne font intervenir que des décompositions de Cholesky. En plus, nous observons dans la pratique que le schéma d'Euler-Maruyama est largement plus coûteux en terme de temps de calcul. Dans un souci d'optimiser nos schémas, si $\bar{\alpha} \geq (d-1)a^T a$, alors nous arrivons à mettre en place un schéma faible d'ordre 2 ayant un ordre de complexité égal à $O(d^3)$. Ce schéma exploite la relation liant le processus de Wishart avec les processus d'Ornstein-Uhlenbeck (1.14).

La dernière partie du chapitre est dédiée aux résultats numériques. Nous illustrons donc ces résultats en terme de temps de calcul et de qualité de convergence. On commence par mettre en évidence le temps occupé par chaque schéma, et nous concluons que le schéma d'Euler consomme plus de temps, tandis que si la condition $\bar{\alpha} \geq (d-1)a^T a$ est vérifiée le schéma modifié, à base des processus d'Ornstein-Uhlenbeck, est de loin le plus rapide. Nous observons aussi la qualité de convergence des schémas, et nous déduisons que le schéma d'ordre 3 est presque similaire au schéma exact, alors que le schéma d'Euler affiche une convergence d'ordre 1. Par ailleurs, tous les schémas d'ordre 2 proposés marquent le même ordre de convergence, mais avec des constantes de convergence différentes selon l'espérance calculée.

1.4 Des EDS retournant à la moyenne définies dans $\mathcal{C}_d(\mathbb{R})$

Le deuxième partie de ma thèse consiste à proposer des équations différentielles stochastiques définies dans l'espace de corrélation. A l'instar de l'extension des processus CIR dans les matrices variances-covariances, nous avons établi une extension des processus de Wright-Fisher dans l'espace de matrices de corrélation. Comme ces matrices décrivent la corrélation instantanée entre actif, une partie de ce chapitre se focalisera sur les schémas numériques pour le calcul des espérances pour le prix. Par ailleurs, ces processus pourraient naturellement être employés aussi pour modéliser une dépendance dynamique entre des quantités et être appliqués ainsi en dehors de la finance. Nous avons déjà vu que le processus de Wright-Fisher est une solution de l'EDS suivante :

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sigma\sqrt{1 - \rho_t^2}dB_t,$$

où $\kappa \geq 0$, $\bar{\rho} \in [-1, 1]$, $\sigma \geq 0$ et $(B_t)_{t \geq 0}$ est un mouvement brownien. Pour être précis, le processus de Wright-Fisher est défini dans $[0, 1]$ et donné par l'EDS (1.8). En effet, ce processus sert, dans la théorie de population, à identifier la fréquence d'apparition des gènes à travers les générations (Karlin Taylor [58]). Cependant, du point de vue de la corrélation, on préfère plutôt travailler avec le processus qui est défini dans $[-1, 1]$. Il existe bien évidemment une relation entre les deux définitions, puisque $\frac{1+\rho_t}{2}$ est une solution de (1.8).

La loi marginale du processus ρ_t est définie par ses moments, et sa densité s'exprime sous forme d'une expansion en fonction des polynômes orthogonaux de Jacobi (Kuznetsov [60] et Mazet [67]). Dans la littérature financière, ce processus est parfois appelé 'Jacobi'. Un processus de corrélation de dimension 2 peut donc s'écrire sous la forme suivante

$$X_t = \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}.$$

On suppose maintenant que $(W_t)_{t \geq 0}$ est une matrice carrée, où ses éléments sont des mouvements browniens indépendants, et on considère l'équation différentielle stochastique suivante :

$$\begin{aligned} X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds \\ + \sum_{n=1}^d a_n \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_s e_d^n X_s} \right), \end{aligned} \tag{1.20}$$

où $x, c \in \mathfrak{C}_d(\mathbb{R})$ et $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d)$ et $a = \text{diag}(a_1, \dots, a_d)$ sont des matrices diagonales positives telles que

$$\kappa c + c \kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R}) \text{ or } d = 2. \quad (1.21)$$

Sous ces hypothèses, nous avons montré que cette EDS admet une solution faible unique définie dans $\mathfrak{C}_d(\mathbb{R})$. En outre, l'unicité forte de l'EDS dans $\mathfrak{C}_d^*(\mathbb{R})$ est obtenue quand $x \in \mathfrak{C}_d^*(\mathbb{R})$ et

$$\kappa c + c \kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R}). \quad (1.22)$$

Dans toute cette partie, on note par $MRC_d(x, \kappa, c, a; t)$ la loi marginale X_t du processus et $MRC_d(x, \kappa, c, a)$ la loi du processus $(X_t)_{t \geq 0}$. L'abréviation *MRC* veut dire *Corrélation à Retour à la Moyenne* en anglais. En dimension $d = 2$, $(X_t)_{1,2}$ est la seule composante qui varie dans le temps. Nous observons qu'il existe un mouvement brownien $(B_t)_{t \geq 0}$ tel que

$$d(X_t)_{1,2} = (\kappa_1 + \kappa_2)(c_{1,2} - (X_t)_{1,2})dt + \sqrt{a_1^2 + a_2^2} \sqrt{1 - (X_t)_{1,2}^2} dB_t.$$

La composante $(X_t)_{1,2}$ suit tout simplement une diffusion de Wright-Fisher. Dans le cas de la dimension $d = 2$, la condition d'existence faible est toujours vérifiée. Si la condition d'existence forte est satisfaite alors le processus ne touche jamais $\{-1, 1\}$. En outre, en dimension supérieure, nous montrons que toute composante de cette matrice suit une diffusion de Wright-Fisher.

Loi ergodique

Dans cette étude, nous commençons par calculer les moments d'une solution de l'EDS (1.20). On considère que pour tout $x \in \mathcal{S}_d(\mathbb{R})$, $m \in \mathcal{S}_d(\mathbb{N})$,

$$x^m = \prod_{1 \leq i \leq j \leq d} x_{\{i,j\}}^{m_{\{i,j\}}} \text{ et } |m| = \sum_{1 \leq i \leq j \leq d} m_{\{i,j\}}.$$

Après un calcul direct, nous prouvons qu'il existe une constante positive K_m et une fonction polynomiale f_m , dépendant des moments de X strictement inférieurs à m , telles que

$$\mathbb{E}[X_t^m] = x^m \exp(-tK_m) + \exp(-tK_m) \int_0^t \exp(sK_m) \mathbb{E}[f_m(X_s^m)] ds.$$

Cette structure est similaire à celle constatée dans le cadre des processus affines, où chaque moment d'ordre donné s'écrit en fonction des moments d'ordre inférieur. En plus, on pourrait mener la même démarche pour calculer explicitement $\mathbb{E}[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}]$ pour tous $0 \leq t_1 \leq \dots \leq t_n$ et $m_1, \dots, m_n \in \mathcal{S}_d(\mathbb{N})$, ce qui donne l'unicité en loi du processus. Comme le processus est défini dans un espace compact, le calcul de la loi ergodique se résume dans le calcul de $\lim_{t \rightarrow +\infty} \mathbb{E}[X_t^m]$ pour tout moment $m \in \mathcal{S}_d(\mathbb{N})$, étant noté par $\mathbb{E}[X_\infty^m]$. Nous démontrons que

$$\begin{aligned} \mathbb{E}[X_\infty^m] &= x^m \text{ si } m \in \mathcal{S}_d(\mathbb{N}) \text{ tel que } m_{\{i,j\}} > 0 \iff \kappa_i = \kappa_j = 0, \\ \mathbb{E}[X_\infty^m] &= \mathbb{E}[f_m(X_\infty)]/K_m \text{ sinon.} \end{aligned}$$

Quand $t \rightarrow +\infty$, la loi marginale de X_t converge en loi, et ses moments $\mathbb{E}[X_\infty^m]$ sont uniques et définis par récurrence sur $|m|$. En outre, si pour tout $1 \leq i, j \leq d$ on vérifie $\kappa_i + \kappa_j > 0$, alors la loi X_∞ ne dépend pas de la condition initiale x et il s'agit de l'unique loi invariante.

Lien avec les processus de Wishart

Tout d'abord, pour toute matrice $x \in \mathcal{S}_d^+(\mathbb{R})$ telle que $x_{i,i} > 0$, $\forall 1 \leq i \leq d$, on définit la fonction $\mathbf{p}(x) \in \mathfrak{C}_d(\mathbb{R})$ par

$$(\mathbf{p}(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \quad 1 \leq i, j \leq d. \quad (1.23)$$

On suppose ainsi que $\alpha \geq \max(1, d-2)$ et $y \in \mathcal{S}_d^+(\mathbb{R})$ tel que $y_{i,i} > 0$ pour tout $1 \leq i \leq d$. On considère que $(Y_t^y)_{t \geq 0} \sim WIS_d(y, \alpha + 1, 0, e_d^1)$. On définit ainsi un changement de temps et un processus de corrélation : $X_t = \mathbf{p}(Y_t^y)$, $\phi(t) = \int_0^t \frac{1}{(Y_s^y)_{1,1}} ds$. Et on obtient donc l'égalité en loi suivante :

$$(X_{\phi^{-1}(t)}, t \geq 0) \stackrel{\text{loi}}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1). \quad (1.24)$$

Cette égalité explique qu'à travers un changement de temps, la matrice MRC n'est que la projection de la matrice de Wishart dans l'espace de corrélation. Par contre, elle est strictement valable que pour $a = e_d^i, \forall 1 \leq i \leq d$, et ne s'étend pas à tous les processus de Wishart.

Lien avec les processus de Wright-Fisher multi-allèles

Un autre lien qui nous semble intéressant à soulever est la connexion entre les processus MRC et les processus de Wright-Fisher multi-allèles. On considère que le processus $(X_t)_{t \geq 0} \sim MRC_d(y, \frac{\alpha}{2} e_d^1, I_d, e_d^1)$, où $(y)_{2 \leq i, j \leq d} = I_{d-1}$. Sous cette configuration, les seuls éléments qui diffusent dans le temps se trouvent dans la première colonne (ou la première ligne). En plus, à cause de la positivité de la matrice X_t , le vecteur $((X_t)_{1,i})_{2 \leq i \leq d}$ est défini dans la boule unité de dimension $d-1$: $\sum_{i=2}^d X_{1,i}^2 = 1$. Si on note par $(Z_t)_{t \geq 0}$ le vecteur de dimension $d-1$, vérifiant pour tout $1 \leq i \leq d-1$, $(Z_t)_i = (X_t)_{1,i+1}^2$, alors son opérateur infinitésimal s'écrit sous la forme :

$$\sum_{i=1}^{d-1} [1 - (1 + 2\alpha)z_i] \partial_{z_i} + 2 \sum_{1 \leq i, j \leq d-1} z_i (\mathbb{1}_{i=j} - z_j) \partial_{z_i} \partial_{z_j}$$

Il s'agit d'un cas particulier de Wright-Fisher multi-allèles (Etheridge [38]), où les deux modèles partagent exactement la même partie diffusion. Ce modèle ressemble aussi au modèle établi par Gourieroux et Jasiak [46], possédant aussi la même partie de diffusion.

Aussi le processus MRC apparaît comme une 'racine carrée' d'un processus Wright-Fisher multi-allèles. Nous avons vu dans le paragraphe précédent que toute matrice MRC admet une loi ergodique. Dans le cas particulier de X_t , une formule explicite de cette loi, notée par $((X_\infty)_{1,i})_{2 \leq i \leq d}$, est obtenue :

$$\mathbb{1}_{\sum_{i=2}^d z_i^2 \leq 1} \frac{\Gamma(\frac{\alpha+1}{2})}{(\sqrt{\pi})^{d-1} \Gamma(\frac{\alpha+2-d}{2})} \left(1 - \sum_{i=2}^d z_i^2 \right). \quad (1.25)$$

Sans surprise, nous constatons que le vecteur carré : $([(X_\infty)_{1,i}]^2)_{2 \leq i \leq d}$ suit bien un loi de Dirichlet, qui est connue comme la loi ergodique pour les processus Wright-Fisher.

L'existence faible et forte des processus MRC

La preuve de l'existence forte est similaire à celle développée par Bru [17] pour les processus de Wishart. En effet, nous supposons que la condition initiale est inversible, et nous définissons le temps d'arrêt de sortie de la solution de $\mathfrak{C}_d^*(\mathbb{R})$. Jusqu'à ce temps, on établit par un calcul d'Itô l'équation stochastique différentielle vérifiée par le déterminant de la solution. Sous les hypothèses de la solution forte (1.22) et en employant l'argument de McKean (Bru [16]), nous démontrons que le déterminant ne touche pas zéro, presque sûrement. Après, nous vérifions que les coefficients de la diffusion sont localement lipschitziennes sur $\mathfrak{C}_d^*(\mathbb{R})$, ce qui donne l'existence forte.

En ce qui concerne l'existence faible, Mentionnons tout d'abord qu'il existe dans la littérature plusieurs méthodes à employer dans un contexte matriciel. Bru [17] a choisi d'analyser directement les valeurs propres d'une solution donnée, cette méthode impose que les valeurs propres de la matrices ne coïncident jamais. Dans le cadre affine, une deuxième approche est employée par Cuchiero et al. [26], où ces derniers ont proposé d'étudier les fonctions caractéristiques. Néanmoins, Les deux approches ne peut pas être appliquées dans le cadre des matrices de corrélation, puisque d'une part, ses valeurs propres peuvent coïncider (au moins dans le cas $d = 2$), et d'autre part sa fonction caractéristique ne donne pas des caractérisations exploitable, comme il est le cas pour les processus affines. Nous proposons donc une preuve qui repose sur la construction d'une approximation qui converge vers une solution faible de l'EDS (1.20). la démonstration présente plusieurs aspects techniques, mais elle est résumée en trois étapes indispensables : construire une solution approchée à valeurs dans l'espace de corrélation, vérifier que cette approximation satisfait le critère de tension de Kolmogorov et enfin démontrer que sa limite est bien la solution du problème de la martingale associée à (1.20). L'unicité faible de l'équation revient au calcul que nous avons établi à priori, où nous avons montré que pour tout $0 \leq t_1 \leq \dots \leq t_n$ et $m_1, \dots, m_n \in \mathcal{S}_d(\mathbb{N})$ les moments $\mathbb{E}[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}]$ sont uniques et explicites.

Enfin, dans la dernière partie de ce paragraphe, nous considérons une extension de l'EDS, dans le cas où les coefficients dépendent du temps et de l'espace.

Théorème de Girsanov

Dans cette partie, nous avons étudié quelques transformations de Girsanov. Nous établissons les mêmes types de changement de probabilité que ceux utilisés dans le cadre des processus de Wishart (Donati-Martin et al. [30], Bru [17]) et des processus de CIR (Pitman et Yor [73]). Pour ce faire, nous changeons la structure de L'EDS, en la remplaçant par l'équation suivante

$$X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds + \sum_{n=1}^d a_n \int_0^t \left(\sqrt{x} \sqrt{I_d - \sqrt{x} e_d^n \sqrt{x}} (X_s) dW_s e_d^n + e_d^n dW_s^T \sqrt{I_d - \sqrt{x} e_d^n \sqrt{x}} \sqrt{x} \right), \quad (1.26)$$

Cette EDS a la même solution du problème de martingale que l'EDS (1.20), sous les mêmes conditions. On peut aussi prouver que la solution forte dans $\mathfrak{C}_d^*(\mathbb{R})$ est aussi donnée sous les mêmes critères que (1.20).

Sans rentrer dans les détails, le cadre matriciel nous demande d'établir une forme particulière pour appliquer le théorème de Girsanov. On suppose ainsi que $(H_t)_{t \geq 0}$ est un processus progressivement mesurable à valeurs dans $\mathcal{M}_d(\mathbb{R})$, et tel que

$$\mathcal{E}_t^H = \exp \left(\int_0^t \text{Tr}(H_s^T dW_s) - \frac{1}{2} \int_0^t \text{Tr}(H_s^T H_s) ds \right) \quad (1.27)$$

est une martingale. Si pour un horizon de temps finit $T > 0$, on note par \mathbb{Q} une probabilité définie par $\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \mathcal{E}_T^H$, alors $W_t^{\mathbb{Q}} = W_t - \int_0^t H_s ds$ est une matrice de mouvement brownien sous la probabilité \mathbb{Q} .

Dans un premier lieu, on suppose que $H_t = \sqrt{X_t} \lambda$, avec $\lambda = \text{diag}(\lambda_1, \dots, \lambda_d) \in \mathcal{S}_d(\mathbb{R})$. Nous obtenons que le processus $(X_t, t \geq 0)$ suit une loi $MRC_d(x, \kappa, c, a)$ sous la probabilité \mathbb{Q} . Cela veut dire que sous ce changement de probabilité, la loi du processus $(X_t)_{t \geq 0}$ (à horizon fixé $T > 0$) est préservée.

On considère également un autre changement de probabilité : $H_t = (\sqrt{X_t})^{-1} c^2 \lambda$, où c^2 est une matrice de corrélation et $\lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$, tel que pour tout $1 \leq i \leq d$, si $a_i = 0$ alors $\lambda_i = 0$. Sous des hypothèses d'unicité forte de la solution, nous montrons que $(X_t)_{t \geq 0}$ admet une nouvelle loi $MRC_d(x, \tilde{\kappa}, \tilde{c}, a)$, sous la probabilité \mathbb{Q} , où les matrices $\tilde{\kappa}$ et \tilde{c} sont bien spécifiées dans le chapitre.

Les changements de probabilité de ce type sont très fréquemment utilisés en finance, pour lier la probabilité historique et la probabilité risque neutre. En particulier, dans le cadre des processus affines, Cheridito et al. [23] montrent que ces changements de probabilités sont suffisants pour expliquer empiriquement le comportement de la prime de risque de marché des obligations zéro coupon.

Simulation des processus de corrélation à retour à la moyenne

La dernière partie du chapitre traite de la discrétisation des équations différentielles stochastiques données par (1.20). Comme l'objectif de ces études est d'appliquer ces matrices de corrélation en finance, il est important d'avoir à disposition des schémas de discrétisation permettant de calculer des espérances par Monte-Carlo. En plus, dans le chapitre, nous avons bien mis l'accent sur l'insuffisance du schéma d'Euler-Maruyama, puisque dans le cadre des matrices de corrélation, ce schéma est très coûteux en matière de temps de calcul et d'erreur engendrée. On propose ainsi deux types de discrétisation, la première est basée sur le lien avec les processus de Wishart en réutilisant les schémas pour le processus de Wishart. La deuxième est une construction directe et pourrait être utilisée pour construire des schémas dédiés au processus Wright-Fisher multi-allèles. Mentionnons que les deux méthodes utilisent la technique de décomposition des opérateurs infinitésimaux.

Avant d'aborder nos schémas, il est utile de présenter le schéma d'Euler-Maruyama corrigé dédié à l'EDS (1.20)

$$\begin{aligned} \tilde{X}_{t_{i+1}}^N &= \hat{X}_{t_i}^N + \left(\kappa(c - \hat{X}_{t_i}^N) + (c - \hat{X}_{t_i}^N)\kappa \right) \frac{T}{N} + \sum_{n=1}^d a_n \left(\sqrt{\hat{X}_{t_i}^N - \hat{X}_{t_i}^N e_d^n \hat{X}_{t_i}^N} (W_{t_{i+1}}^N - W_{t_i}^N) e_d^n \right. \\ &\quad \left. + e_d^n (W_{t_{i+1}}^N - W_{t_i}^N)^T \sqrt{\hat{X}_{t_i}^N - \hat{X}_{t_i}^N e_d^n \hat{X}_{t_i}^N} \right) \\ \hat{X}_{t_{i+1}}^N &= \mathbf{p}((\tilde{X}_{t_{i+1}}^N)^+). \end{aligned} \quad (1.28)$$

Ce schéma est bien défini dans l'espace des matrices de corrélation $\mathfrak{C}_d(\mathbb{R})$. La correction du schéma porte sur la positivité et la définition dans l'espace de corrélation, puisque par construction, la matrice $\tilde{X}_{t_{i+1}}^N$ dans (1.20) peut prendre des valeurs en dehors de $\mathfrak{C}_d(\mathbb{R})$. En plus, pour des considérations matricielles, la projection \mathbf{p} reste bien définie.

Si on note par L l'opérateur infinitésimal associé à (1.20), alors cet opérateur se décompose en

$$L = L^{ODE} + \sum_{i=1}^d a_i^2 L_i,$$

où L^{ODE} est l'opérateur associé à une équation différentielle ordinaire ayant une solution explicite, et les générateurs infinitésimaux $(L_i)_{1 \leq i \leq d}$ sont associés respectivement aux processus $(MRC_d(y, \frac{\alpha}{2} e_d^i, I_d, e_d^i))_{1 \leq i \leq d}$. Ces opérateurs $(L_i)_{1 \leq i \leq d}$ vérifient la condition de commutativité. Grâce aux techniques de compositions, il suffit d'établir des schémas d'ordre 2 aux opérateurs $(L_i)_{1 \leq i \leq d}$, et pour des considérations de symétrie, le problème se résume à établir une solution pour L_1 . Ce dernier est construit de deux manières :

- La première méthode revient à l'idée du changement de temps développée en (1.24), où nous remplaçons à la fois, la matrice Wishart par son approximation d'ordre 2 (présentée dans le chapitre précédent) et le changement de temps ϕ (1.24) par le développement limité de son espérance jusqu'à l'ordre 3. Cela permettra par la suite d'obtenir une approximation de (1.20) d'ordre 2.
- La deuxième méthode trouve que $L_1 = \sum_{i=1}^d \mathcal{L}^i$, où $(\mathcal{L}^i)_{1 \leq i \leq d}$ sont aussi des opérateurs infinitésimaux bien définis dans l'espace de corrélation $\mathfrak{C}_d(\mathbb{R})$, mais ayant une structure plus simple à discrétiser. Nous proposons ainsi des approximations associées à chaque opérateur impliqué dans cette décomposition, à l'aide soit de la technique de la composition, soit de la technique de 'matching moment' développée par Alfonsi [5].

Par conséquent, nous réalisons un schéma faible d'ordre 2 pour l'EDS (1.20) par deux méthodes.

Le dernier paragraphe de cette partie est dédié aux illustrations numériques. De même que pour les schémas de discrétisation des processus affine à valeurs dans $\mathcal{S}_d^+(\mathbb{R})$, Les deux schémas associés aux processus MRC emploient comme brique de calcul de base l'opérateur L_1 . Ce dernier a pour ordre de complexité $O(d^3)$, sous les deux méthodes d'approximation. Comme les deux schémas utilisent au plus d fois l'opérateur L_1 , alors ils exigent au final une complexité d'ordre $O(d^4)$. Nous notons aussi que le schéma d'Euler-Maruyama corrigé demande à chaque pas de temps le calcul d fois des vecteurs et des valeurs propres d'une matrice symétrique dans $\mathcal{S}_d(\mathbb{R})$, cela implique une complexité d'ordre $O(d^4)$. Bien que cette méthode partage le même ordre de complexité que les deux évoqués précédemment, mais nous remarquons qu'elle reste dans la pratique la méthode la plus coûteuse, en terme de temps de calcul. Par ailleurs, dans le cas où $\kappa c + c\kappa \geq (d-1)a^2$, telle que la matrice a est proportionnelle à la matrice identité I_d , nous proposons une autre approximation d'ordre faible deux qui demande une complexité juste d'ordre $O(d^3)$. Un tableau est donné pour comparer le temps de calcul. Des figures sur la convergence asymptotique des schémas est aussi présentée, où nous observons que les résultats numériques correspondent bien aux convergences théoriques obtenues.

1.5 Modèles pour les options sur indice, une approche 'bottom up'

La dernière partie de ma thèse consiste à proposer un modèle multi sous-jacents permettant de calibrer à la fois le marché des options vanilles et celui de l'indice. Dans ce travail, les données du marché employées et analysées sont celles du *DAX*. Nous remercions Julien Guyon (Société Générale) de nous avoir fourni ces données. Avellaneda et al. [7] et Langnau [61] ont bien expliqué qu'une dépendance constante, dans un modèle multi sous-jacent, reste insuffisante pour capter les prix des options sur indices. En effet, on a vu précédemment que le risque de corrélation présente plusieurs caractéristiques typiques dans le marché. Entre autre, on trouve le phénomène de l'asymétrie de dépendance entre l'indice et sa matrice de corrélation, on verra bien dans le chapitre 4 que cette corrélation pourrait caractériser (au moins à très courte maturité) le skew à la monnaie de la volatilité implicite de l'indice. On rappelle que le skew à la monnaie définit la pente ou bien la dérivée de la volatilité implicite de l'indice par rapport au strike. L'incapacité des modèles simples à répondre aux exigences du marché pourrait conduire à des pertes et à des expositions très risquées. En tant que vendeur d'une option call sur indice, notée par $V(t, S_t)$, les profits et pertes, notés par $P\&L$, d'une stratégie de couverture delta neutre s'écrit sous la forme suivante

$$P\&L = -\frac{1}{2} \sum_{1 \leq i, j \leq d} \partial_i \partial_j V(t, S_t) S_t^i S_t^j \left(\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} - (R_t)_{i,j} \sigma_t^i \sigma_t^j \right) \Delta t,$$

où $(\sigma_t^i)_{1 \leq i \leq d, t \geq 0}$ et $(R_t)_{t \geq 0}$ sont respectivement les valeurs des volatilités locales et la corrélation données par le modèle. Durant les périodes baissières, la corrélation entre sous-jacent se manifeste de plus en plus souvent, et on déduit que le $P\&L$, à un facteur près du Gamma, est en mesure de prendre des valeurs très négatives si $\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} \gg (R_t)_{i,j} \sigma_t^i \sigma_t^j$. Ce qui justifie l'emploi d'une matrice de corrélation aléatoire permettant ainsi de réduire, voire même d'annuler, les pertes dans une stratégie de couverture. La motivation du travail est de mettre en oeuvre une structure de corrélation en mesure de répondre à ce besoin.

1.5.1 Modèle à corrélation locale

On se place dans un premier dans un modèle à corrélation locale pure. On considère alors que les composantes de l'indice suivent l'EDS suivante, sous la probabilité risque neutre

$$\forall 1 \leq i \leq d, \quad \frac{dS_t^i}{S_t^i} = r dt + (\sqrt{R(I_t)} dB_t)_i$$

$$I_t = \sum_{i=1}^d \alpha^i S_t^i,$$

où $(B_t)_{t \geq 0}$ est un vecteur de mouvement Brownien et $R(I_t)$ est une matrice de corrélation telle que

$$1 \leq i \neq j \leq d, \quad R_{i,j}(I_t) = \rho(I_t) := \frac{1}{1 + \eta \left(\frac{I_t}{I_0}\right)^\gamma} \wedge \rho_{\min}. \quad (1.29)$$

Tous les paramètres du modèle sont des constantes, telle que $\eta, \gamma \geq 0$ et $\rho_{\min} \in [0, 1]$. nous expliquerons dans le chapitre 4 les différentes sensibilités de ces paramètres sur la structure de la volatilité implicite. En particulier, nous montrons numériquement qu'ils jouent des rôles très distincts. En effet, le paramètre η a un impact direct sur le niveau de la volatilité implicite à la monnaie sans changer le skew, tandis que la valeur de γ influence directement la pente de la courbe (le skew) sans changer, pour autant, le niveau de la volatilité à la monnaie. Enfin, ρ_{\min} agit sur la partie droite de la courbe de volatilité implicite, ce qui joue plus sur la convexité de la courbe.

Par conséquent, grâce à cette séparation du rôle de chaque paramètre, nous avons mis en place un algorithme stable de calibration. Cette approche commence par établir des méthodes de calibration paramètre par paramètre sur un modèle réduit, de dimension 10. Une fois la calibration est obtenue sous ce modèle, on réalise ensuite une méthode générale d'optimisation pour le modèle de départ. Les résultats numériques montrent une certaine stabilité de paramètre entre le passage du modèle réduit au modèle général. En plus, la calibration est effectuée dans un temps raisonnable.

1.5.2 Modèle à corrélation stochastique

Bien que le modèle à corrélation locale peut calibrer correctement les données de marché, sa structure locale présente le même problème signalé par [50]. Ce dernier a montré que dans le cadre d'un modèle de type CEV, quand la valeur du sous-jacent à l'instant initial est perturbée, alors la dynamique de la courbe de la volatilité implicite du modèle prend la direction opposée de celle de la perturbation, ce qui n'est pas vérifié expérimentalement. Dans le contexte multi dimensionnel, on montre bien numériquement que la volatilité implicite du modèle précédent prend la direction opposée que celle de la perturbation. Cela veut dire que même si le modèle local calibre bien les options sur indices, la dynamique de sa volatilité implicite rend la stratégie de couverture du modèle moins efficace. Par conséquent, nous avons regardé d'autres types de modèle se basant sur une corrélation stochastique.

On suppose alors que les sous-jacents suivent l'EDS suivante sous la probabilité risque neutre :

$$\forall 1 \leq i \leq d, \frac{dS_t^i}{S_t^i} = rdt + (\sqrt{R_t}dB_t)_i \quad \forall 1 \leq i, \neq j \leq d, (R_t)_{i,j} = \rho_t. \quad (1.30)$$

Dans ce cas on suppose que $(\rho_t)_{t \geq 0}$ est un processus de Wright-Fisher défini dans $[0, 1]$ et donné par l'EDS suivante :

$$d\rho_t = \kappa(\rho_t - \rho(I_t))dt + \sigma\sqrt{\rho_t(1 - \rho_t)}dW_t,$$

où $(W_t)_{t \geq 0}$ est un mouvement Brownien indépendant du vecteur de Brownien $(B_t)_{t \geq 0}$ et $\kappa, \sigma \geq 0$ et $\rho(I_t)$ est défini par (1.29). L'introduction des modèles à corrélation stochastique s'inscrit dans le cadre d'investigation sur leurs possibilités potentielles d'offrir une paramétrisation plus riche. L'analyse des paramètres du processus $(\rho_t)_{t \geq 0}$ nous a permis d'identifier deux régimes de fonctionnement dépendant de la valeur de κ . Si ce dernier est très grand, le modèle se comporte comme le modèle local précédent. En revanche, quand la valeur de κ est petite nous obtenons un modèle ayant des valeurs de skew moins prononcées. Ce régime est plus influencé par les paramètres ρ_0 et σ .

Nos investigations ont aussi porté sur des matrices de corrélation ayant une structure plus générale, en employant les matrices MRC définies dans le chapitre 3. Dans ce cas de figure, et sous les mêmes notations que (1.30), on considère que la matrice de corrélation $(R_t)_{t \geq 0}$ est une solution de l'EDS suivante :

$$R_t = R_0 + \kappa \int_0^t (C(I_s) - R_s) ds + \sum_{n=1}^d \int_0^t \sigma(I_s) \left(\sqrt{R_s - R_s e_d^n R_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{R_s - R_s e_d^n R_s} \right). \quad (1.31)$$

Ici, $(W_t)_{t \geq 0}$ est une matrice de mouvement Brownien indépendant du vecteur $(B_t)_{t \geq 0}$, et $(C(I_t))_{t \geq 0}$ est une matrice de corrélation telle que pour tout $1 \leq i \neq j \leq d$, on a $R(I_t)_{i,j} = \rho(I_t)$ où la fonction $\rho(I_t)$ est donnée par (1.29). Enfin, la fonction $\sigma(I_t)$, étant bien explicitée dans le chapitre 4, a une forme spécifique pour vérifier les conditions d'existence du processus. Bien que la corrélation dans ce modèle balaye un espace plus riche, ce modèle MRC est défini par quatre paramètres. Les simulations numériques montrent qu'en terme d'interprétation, les paramètres jouent presque le même rôle. En effet, la valeur de κ identifie deux régimes, si κ est grand alors le modèle se comporte comme le modèle (LC), tel que la valeur de η détermine la valeur de la volatilité implicite à la monnaie et γ agit sur la pente de la courbe. Néanmoins, quand κ est petit alors le niveau de la volatilité est déterminée par R_0 et σ définit la convexité de la courbe.

Ce travail n'est pas encore achevé et nous continuons à tester différents modèles pour calibrer au mieux les données sur indices. Néanmoins, il présente certains résultats et certaines heuristiques que nous pensons utiles de mentionner dans cette thèse. Il donne également une motivation 'pratique' aux chapitres précédents sur la simulation des processus de Wishart et le processus de corrélation MRC .

Chapter 2

Exact and high order discretization schemes for Wishart processes and their affine extensions

Note: Ce chapitre a donné lieu un article [1] soumis à *Annals of Applied Probability*

Abstract This work deals with the simulation of Wishart processes and affine diffusions on positive semidefinite matrices. To do so, we focus on the splitting of the infinitesimal generator, in order to use composition techniques as Ninomiya and Victoir [69] or Alfonsi [5]. Doing so, we have found a remarkable splitting for Wishart processes that enables us to sample exactly Wishart distributions, without any restriction on the parameters. It is related but extends existing exact simulation methods based on Bartlett's decomposition. Moreover, we can construct high-order discretization schemes for Wishart processes and second-order schemes for general affine diffusions. These schemes are in practice faster than the exact simulation to sample entire paths. Numerical results on their convergence are given.

Introduction and first definitions

This paper focuses on simulation methods for Wishart processes and more generally for affine diffusions on positive semidefinite matrices. Before explaining our motivations and our main results, we start with a short introduction to these processes. Even though we use rather standard notations for matrices, they are recalled in Appendix 2.6.1, and we invite the reader to give first a quick look at it. Wishart processes have been initially introduced by Bru [16, 17]. They are also named because their marginal laws follow Wishart distributions. Very recently, Cuchiero et al. [26] have introduced a general framework for affine processes on positive semidefinite matrices $\mathcal{S}_d^+(\mathbb{R})$ that embeds Wishart processes and includes possible jumps. In this paper, we only consider continuous processes of this kind. Such processes solve the following SDE:

$$X_t^x = x + \int_0^t (\bar{\alpha} + B(X_s^x)) ds + \int_0^t \left(\sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x} \right). \quad (2.1)$$

Here, and throughout the paper, $(W_t, t \geq 0)$ denotes a d -by- d square matrix made of independent standard Brownian motions,

$$x, \bar{\alpha} \in \mathcal{S}_d^+(\mathbb{R}), a \in \mathcal{M}_d(\mathbb{R}) \text{ and } B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R})) \quad (2.2)$$

is a linear mapping on $\mathcal{S}_d(\mathbb{R})$. Wishart processes correspond to the case where

$$\exists \alpha \geq 0, \bar{\alpha} = \alpha a^T a \text{ and } \exists b \in \mathcal{M}_d(\mathbb{R}), \forall x \in \mathcal{S}_d(\mathbb{R}), B(x) = bx + xb^T. \quad (2.3)$$

When $d = 1$, (2.1) is simply the SDE of the Cox-Ingersoll-Ross process that has been broadly studied, and we will implicitly assume that $d \geq 2$ throughout the paper. Weak and strong uniqueness of SDE (2.1) has been studied by Bru [17], Cuchiero et al. [26] and Mayerhofer et al. [66]. We sum up here their results.

Theorem 1 — *If $x \in \mathcal{S}_d^+(\mathbb{R})$, $\bar{\alpha} - (d-1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$ and B satisfies the following condition*

$$\forall x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R}), \text{Tr}(x_1 x_2) = 0 \implies \text{Tr}(B(x_1) x_2) \geq 0, \quad (2.4)$$

there is a unique weak solution to the SDE (2.1). We denote by $AF F_d(x, \bar{\alpha}, B, a)$ the law of $(X_t^x)_{t \geq 0}$ and $AF F_d(x, \bar{\alpha}, B, a; t)$ the marginal law of X_t^x . If we assume moreover that $\bar{\alpha} - (d+1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$ and $x \in \mathcal{S}_d^{+,}(\mathbb{R})$, there is a unique strong solution to the SDE (2.1).*

Under the parametrization of Wishart processes (2.3), condition (2.4) is satisfied and weak uniqueness holds as soon as $\alpha \geq d-1$. In that case, we denote by $WIS_d(x, \alpha, b, a)$ the law of the Wishart process $(X_t^x)_{t \geq 0}$ and $WIS_d(x, \alpha, b, a; t)$ the law of X_t^x .

Throughout the paper, when we use the notation $AF F_d(x, \bar{\alpha}, B, a)$ or $AF F_d(x, \bar{\alpha}, B, a; t)$ (resp. $WIS_d(x, \alpha, b, a)$ or $WIS_d(x, \alpha, b, a; t)$), we implicitly assume that $\bar{\alpha} - (d-1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$ (resp. $\alpha \geq d-1$) and B satisfies (2.4) so that weak uniqueness holds.

In her Ph.D. thesis [16], Bru has introduced Wishart processes and used them in biology to study perturbed experimental data. Recently, a great attention has been paid to Wishart processes for applications in finance. Namely, Gourieroux and Sufana [47] and Da Fonseca et al. [28] have suggested to use these processes to model the instantaneous covariance matrix of d assets. It naturally extends stochastic volatility models for only one asset like the Heston model [51]. Obviously, processes on positive semidefinite matrices are really interesting to model the evolution of a dependence structure because they can describe a covariance matrix. However, when dealing with applications, it is in general crucial to be able to sample paths of such processes and make Monte-Carlo algorithms.

To the best of our knowledge, there is few literature on simulation methods for Wishart and general affine processes (2.1). Wishart distributions have been intensively studied in statistics when $\alpha \in \mathbb{N}$. In this case, exact simulation methods have proposed by Odell and Feiveson [70], Smith and Hocking [80] and Gleser [44] to mention a few. Concerning discretization schemes, the usual Euler-Maruyama scheme is not well-defined because of the square-root. This is what already happens for the Cox-Ingersoll-Ross process which corresponds to the case $d = 1$. One has then to find specific schemes. Recently, Benabid et al. [8] and Gauthier and Possamaï [42] have proposed numerical approximations for Wishart processes that are well defined under some restrictions on the parameters. However, there is no result on the accuracy of their methods. Currently, Teichmann [71] is working on dedicated schemes for general affine processes

by approximating their characteristic functions. Our study here is only dedicated to the diffusion (2.1).

Initially, our goal was to find high order discretization schemes for Wishart processes by splitting operators and using scheme compositions. Indeed, this approach has already proved to be very efficient for other affine diffusions, see [5]. Doing so, we incidentally have found a remarkable splitting for some canonical Wishart processes which enables us to sample exactly Wishart processes. In particular, our result extends the Bartlett's decomposition that is commonly used to sample central Wishart distributions when $\alpha \in \mathbb{N}$. This splitting also enables us to get high order discretization schemes for Wishart processes. Then, by using scheme composition, we also get a second order scheme for any affine diffusion (2.1). Contrary to the exact scheme, one has then to study the discretization error, and we provide a rigorous analysis of the weak error.

This paper is structured as follows. First, we present some general results on affine diffusions. We calculate their infinitesimal generator and obtain interesting identities in law that are intensively used next for the different simulation methods. Section 2.2 is devoted to the exact simulation of Wishart processes. It exhibits a remarkable splitting of the infinitesimal generator and shows how it can be used to sample exactly any Wishart distribution. Section 2.3 deals with high order schemes for affine diffusions. Thanks to the remarkable splitting, we are able to construct a third order scheme for Wishart processes and second order schemes for affine diffusions. Last, we give numerical illustrations of our convergence results in Section 2.4. We compare the time required by each method and also give a possible application of our results in finance.

2.1 General properties of affine processes on positive semidefinite matrices

2.1.1 The infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$

We start this subsection with a simple Lemma, which is useful to calculate the infinitesimal generator of affine processes. Its proof is basic and is left in Appendix 2.6.2.

Lemma 2 — *Let $(\mathcal{F}_t)_{t \geq 0}$ denote the filtration generated by $(W_t, t \geq 0)$. We consider a process $(Y_t)_{t \geq 0}$ valued in $\mathcal{S}_d(\mathbb{R})$, and we assume that there exist continuous (\mathcal{F}_t) -adapted processes $(A_t)_{t \geq 0}$, $(B_t)_{t \geq 0}$ and $(C_t)_{t \geq 0}$ that are respectively valued in $\mathcal{M}_d(\mathbb{R})$, $\mathcal{M}_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$ so that Y_t admits the following semimartingale decomposition:*

$$dY_t = C_t dt + B_t dW_t A_t + A_t^T dW_t^T B_t^T \quad (2.5)$$

Then, for $i, j, m, n \in \{1, \dots, d\}$, the quadratic covariation of $(Y_t)_{i,j}$ and $(Y_t)_{m,n}$ is given by:

$$\begin{aligned} d\langle (Y_t)_{i,j}, (Y_t)_{m,n} \rangle &= (B_t B_t^T)_{i,m} (A_t^T A_t)_{j,n} + (B_t B_t^T)_{i,n} (A_t^T A_t)_{j,m} \\ &\quad + (B_t B_t^T)_{j,m} (A_t^T A_t)_{i,n} + (B_t B_t^T)_{j,n} (A_t^T A_t)_{i,m}. \end{aligned} \quad (2.6)$$

It is worth to notice that the quadratic covariation given by (2.5) depends on A_t and B_t only through the matrices $A_t^T A_t$ and $B_t B_t^T$. Lemma 2 enables us to calculate easily the infinitesimal

generator for the affine process (2.1) which is defined by, for all $x \in \mathcal{S}_d^+(\mathbb{R})$:

$$L^{\mathcal{M}}f(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t}, \text{ with } f \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives.}$$

Proposition 3 — Infinitesimal Generator on $\mathcal{M}_d(\mathbb{R})$. Let $(X_t^x)_{t \geq 0} \sim AFF_d(x, \bar{\alpha}, B, a)$ be an affine process. Its infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$ is given by:

$$L^{\mathcal{M}} = \text{Tr}([\bar{\alpha} + B(x)]D^{\mathcal{M}}) + \frac{1}{2} \{2\text{Tr}(xD^{\mathcal{M}}a^T aD^{\mathcal{M}}) + \text{Tr}(x(D^{\mathcal{M}})^T a^T aD^{\mathcal{M}}) + \text{Tr}(xD^{\mathcal{M}}a^T a(D^{\mathcal{M}})^T)\},$$

where $D^{\mathcal{M}} = (\partial_{i,j})_{1 \leq i,j \leq d}$.

Proof : On the one hand, the drift part of the operator is given by $\sum_{k,l=1}^d \bar{\alpha}_{k,l} \partial_{k,l} + \sum_{k,l=1}^d (B(x))_{k,l} \partial_{k,l} = \text{Tr}((\bar{\alpha} + B(x))D^{\mathcal{M}})$. On the other hand, we get from (2.6) that the diffusion part of the operator is:

$$\begin{aligned} & \frac{1}{2} \sum_{n,m,i,j=1}^d [x_{i,m}(a^T a)_{j,n} + x_{i,n}(a^T a)_{j,m} + x_{j,m}(a^T a)_{i,n} + x_{j,n}(a^T a)_{i,m}] \partial_{i,j} \partial_{m,n} \\ &= \frac{1}{2} \sum_{n,m,i,j=1}^d x_{i,m}(a^T a)_{j,n} \partial_{i,j} \partial_{m,n} + \frac{1}{2} \sum_{n,m,i,j=1}^d x_{i,n}(a^T a)_{j,m} \partial_{i,j} \partial_{m,n} \\ &+ \frac{1}{2} \sum_{n,m,i,j=1}^d x_{j,m}(a^T a)_{i,n} \partial_{i,j} \partial_{m,n} + \frac{1}{2} \sum_{n,m,i,j=1}^d x_{j,n}(a^T a)_{i,m} \partial_{i,j} \partial_{m,n} \\ &= \frac{1}{2} [\text{Tr}(x(D^{\mathcal{M}})^T a^T a(D^{\mathcal{M}})^T) + 2\text{Tr}(xD^{\mathcal{M}}a^T aD^{\mathcal{M}}) + \text{Tr}(x(D^{\mathcal{M}})^T a^T aD^{\mathcal{M}})]. \end{aligned}$$

□

Here, we have given the infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$, while we know that the affine process $(X_t^x)_{t \geq 0}$ takes values in $\mathcal{S}_d^+(\mathbb{R}) \subset \mathcal{S}_d(\mathbb{R})$. Thus, we can also look at the infinitesimal generator of this diffusion on $\mathcal{S}_d(\mathbb{R})$, which is defined by:

$$x \in \mathcal{S}_d^+(\mathbb{R}), L^{\mathcal{S}}f(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives.}$$

For $x \in \mathcal{S}_d(\mathbb{R})$, we denote by $x_{\{i,j\}} = x_{i,j} = x_{j,i}$ the value of the coordinates (i, j) and (j, i) , so that $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})$ (see notations in Appendix 2.6.1). For $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, we then denote by $\partial_{\{i,j\}} f$ its derivative with respect to the coordinates $x_{\{i,j\}}$. We introduce

$$\begin{aligned} \pi : \mathcal{M}_d(\mathbb{R}) &\rightarrow \mathcal{S}_d(\mathbb{R}) \\ x &\mapsto (x + x^T)/2, \end{aligned}$$

that is such that $\pi(x) = x$ for $x \in \mathcal{S}_d(\mathbb{R})$. Obviously, $f \circ \pi \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R})$ and we have

$$L^{\mathcal{S}}f(x) = L^{\mathcal{M}}f \circ \pi(x).$$

By the chain rule, we have for $x \in \mathcal{S}_d(\mathbb{R})$, $\partial_{i,j} f \circ \pi(x) = (\mathbb{1}_{i=j} + \frac{1}{2} \mathbb{1}_{i \neq j}) \partial_{\{i,j\}} f(x)$ and get the following result.

Corollary 4 — Infinitesimal Generator on $\mathcal{S}_d(\mathbb{R})$. *The infinitesimal generator on $\mathcal{S}_d(\mathbb{R})$ associated to $AF F_d(x, \bar{\alpha}, B, a)$ is given by:*

$$L^S = \text{Tr}([\bar{\alpha} + B(x)]D^S) + 2\text{Tr}(xD^S a^T a D^S), \quad (2.7)$$

where D^S is defined by $D_{i,j}^S = (\mathbb{1}_{i=j} + \frac{1}{2}\mathbb{1}_{i \neq j})\partial_{\{i,j\}}$, for $1 \leq i, j \leq d$.

Of course, the generators L^M and L^S are equivalent: one can be deduced from the other. However, L^S already embeds the fact that the process lies in $\mathcal{S}_d(\mathbb{R})$, which reduces the dimension from d^2 to $d(d+1)/2$ and gives in practice shorter formulas. This is why we will mostly work in the sequel with infinitesimal generators on $\mathcal{S}_d(\mathbb{R})$. Unless it is necessary to make the distinction with L^M , we will simply denote $L = L^S$.

2.1.2 The characteristic function of Wishart processes

As for other affine processes, the characteristic function of affine processes on positive semidefinite matrices can be obtained by solving two ODEs. In the case of Wishart processes, it is possible to solve explicitly these ODEs by solving a matrix Riccati equation (see Levin [62]). Here, we give the closed formula for the Laplace transform and a precise description of its set of convergence.

Proposition 5 — *Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$, $q_t = \int_0^t \exp(sb)a^T a \exp(sb^T)ds$ and $m_t = \exp(tb)$. We introduce the set of convergence of the Laplace transform of X_t^x , $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\text{Tr}(vX_t^x))] < \infty\}$. This is a convex open set that is given explicitly by*

$$\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}. \quad (2.8)$$

Besides, the Laplace transform of X_t^x is well-defined for $v = v_R + iv_I$ with $v_R \in \mathcal{D}_{b,a;t}$, $v_I \in \mathcal{S}_d(\mathbb{R})$ and is given by:

$$\mathbb{E}[\exp(\text{Tr}(vX_t^x))] = \frac{\exp(\text{Tr}[v(I_d - 2q_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}. \quad (2.9)$$

The characteristic function corresponds to the case $v_R = 0$ that clearly belongs to $\mathcal{D}_{b,a;t}$. The proof of this result is given in Appendix 2.6.2. Let us remark that for $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$, the formula above becomes even simpler and we have for $v = v_R + iv_I$ such that $v_R \in \mathcal{D}_{b,a;t}$, $v_I \in \mathcal{S}_d(\mathbb{R})$:

$$\mathbb{E}[\exp(\text{Tr}(v\tilde{X}_t^x))] = \frac{\exp(\text{Tr}[v(I_d - 2tI_d^n v)^{-1} x])}{\det(I_d - 2tI_d^n v)^{\frac{\alpha}{2}}}. \quad (2.10)$$

2.1.3 Some identities in law for affine processes

This section presents simple but interesting identities in law for affine processes. We first state a preliminary lemma.

Lemma 6 — Let $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ and $q \in \mathcal{G}_d(\mathbb{R})$. We define $B_q \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ by $B_q(x) = (q^T)^{-1}B(q^T x q)q^{-1}$. Then,

$$B \text{ satisfies (2.4)} \iff B_q \text{ satisfies (2.4)}.$$

Proof: It is sufficient to prove the direct implication since $B(x) = q^T B_q((q^T)^{-1}x q^{-1})q$. Let $x, v \in \mathcal{S}_d^+(\mathbb{R})$ such that $\text{Tr}(xv) = 0$. We have

$$\text{Tr}(B_q(x)v) = \text{Tr}[B(q^T x q)q^{-1}v(q^T)^{-1}] \geq 0,$$

since B satisfies (2.4) and $\text{Tr}[(q^T)^{-1}x q^{-1}q v q^T] = \text{Tr}(xv) = 0$. \square

Proposition 7 — The following identities holds:

- $AF F_d(x, \bar{\alpha}, B, a) \underset{Law}{=} AF F_d(x, \bar{\alpha}, B, \sqrt{a^T a})$.
- (linear transformation) Let $q \in \mathcal{G}_d(\mathbb{R})$, we have

$$q^T AF F_d(x, \bar{\alpha}, B, a)q \underset{Law}{=} AF F_d(q^T x q, q^T \bar{\alpha} q, B_{q^{-1}}, aq),$$

where $B_{q^{-1}}$ is defined by $\forall y \in \mathcal{S}_d(\mathbb{R})$, $B_{q^{-1}}(y) = q^T B((q^T)^{-1}y q^{-1})q$.

Proof: From Proposition 3 or Corollary 4, we know that $(X_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, a)$ and $(\tilde{X}_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, \sqrt{a^T a})$ have the same infinitesimal generator. Therefore, they solve the same martingale problem. Thanks to the affine structure, one can show that this martingale problem has a unique solution by looking at the characteristic function. This is made in a much general case in Cuchiero et al. [26]. Similarly, we can check easily that $q^T AF F_d(y, \bar{\alpha}, B, a)q$ and $AF F_d(q^T y q, q^T \bar{\alpha} q, B_{q^{-1}}, aq)$ lead to the same martingale problem. Here, we remark that $B_{q^{-1}}$ satisfies (2.4) thanks to Lemma 6. \square

An interesting consequence of this linear transformation is given in the following corollary. It states that any affine process can be obtained as a linear transformation of an affine process for which we have $a = I_d^n$, where $I_d^n = (\mathbb{1}_{i=j \leq n})_{1 \leq i, j \leq d}$ (see notations in Appendix 2.6.1). Since our main goal here is to sample paths of such processes, this says us that it is sufficient to focus on this special case.

Corollary 8 — Let $(X_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, a)$ and $n = \text{Rk}(a)$ be the rank of $a^T a$. Then, there exist a diagonal matrix $\bar{\delta}$, and a non singular matrix $u \in \mathcal{G}_d(\mathbb{R})$ such that $\bar{\alpha} = u^T \bar{\delta} u$, and $a^T a = u^T I_d^n u$, and we have:

$$(X_t^x)_{t \geq 0} \underset{Law}{=} u^T AF F_d((u^{-1})^T x u^{-1}, \bar{\delta}, B_u, I_d^n) u,$$

where $\forall y \in \mathcal{S}_d(\mathbb{R})$, $B_u(y) = (u^{-1})^T B(u^T y u)u^{-1}$.

Proof: Once u is given, the identity in law comes directly from Proposition 7. We give now a constructive proof of the existence of u , which takes back the arguments given by Golub and Van Loan ([45], Theorem 8.7.1). Nonetheless, we explain it entirely since it gives a practical way to get u .

Let us consider $\bar{\alpha} + a^T a \in \mathcal{S}_d^+(\mathbb{R})$. From the extended Cholesky decomposition given in Lemma 33 there is a matrix $v \in \mathcal{G}_d(\mathbb{R})$ such that $v^T \bar{\alpha} v + v^T a^T a v = I_d^r$, where $r = \text{Rk}(\bar{\alpha} + a^T a)$. Since $v^T \bar{\alpha} v \in \mathcal{S}_d^+(\mathbb{R})$, $v^T a^T a v \in \mathcal{S}_d^+(\mathbb{R})$ and $z^T I_d^r z = 0$ for $z \in \mathbb{R}^d$ such that $z_1 = \dots = z_r = 0$, there are $s_1, s_2 \in \mathcal{S}_n^+(\mathbb{R})$ such that:

$$v^T \bar{\alpha} v = \begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } v^T a^T a v = \begin{pmatrix} s_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let o_2 be an orthogonal matrix such that $o_2^T s_2 o_2$ is a diagonal matrix. We assume without loss of generality that only the first n elements of this diagonal are positive: $o_2^T s_2 o_2 = \text{diag}(\eta_1, \dots, \eta_n, 0, \dots, 0)$. We set $o = \begin{pmatrix} o_2 & 0 \\ 0 & I_{d-r} \end{pmatrix}$ and get $I_d^r = o^T v^T \bar{\alpha} v o + o^T v^T a^T a v o$, which gives that $o^T v^T \bar{\alpha} v o$ is a diagonal matrix. Thus, we get the desired result by taking $u = \text{diag}(\sqrt{\eta_1}, \dots, \sqrt{\eta_n}, 1, \dots, 1) o^{-1} v^{-1}$. \square

Let us make few comments on the practical implementation to compute u . For Wishart processes, $\bar{\alpha} = \alpha a^T a$, and u can directly be obtained by using a single extended Cholesky decomposition (Lemma 33). In the general case, the computation of u mainly requires an extended Cholesky decomposition and the diagonalization of s_2 .

Up to now, we have stated identities for the law of affine processes. Thanks to the explicit characteristic function of Wishart processes, we are also able to get another interesting identity on the marginal laws.

Proposition 9 — *Let $t > 0$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\alpha \geq d - 1$. Let $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$ and $n = \text{Rk}(q_t)$. Then, there is $\theta_t \in \mathcal{G}_d(\mathbb{R})$ such that $q_t = t \theta_t I_d^n \theta_t^T$, and we have:*

$$WIS_d(x, \alpha, b, a; t) \underset{Law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t) \theta_t^T \quad (2.11)$$

This proposition plays a crucial role for the exact simulation of Wishart processes. Thanks to (2.11), we can sample any Wishart distribution if we are able to simulate exactly the distribution $WIS_d(x, \alpha, 0, I_d^n; t)$ for any $x \in \mathcal{S}_d^+(\mathbb{R})$. In Section 2.2, we focus on this and give a way to sample exactly $WIS_d(x, \alpha, 0, I_d^n; t)$. Let us stress here that we can compute the matrix θ_t by using the extended Cholesky decomposition of q_t/t , as it is explained in the proof below.

Proof: We apply Lemma 33 to $q_t/t \in \mathcal{S}_d^+(\mathbb{R})$ and consider (p, c_n, k_n) an extended Cholesky decomposition of q_t/t . We set $\theta_t = p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$. Then, θ_t is invertible and it is easy to check that $q_t = t \theta_t I_d^n \theta_t^T$. Now, let us observe that for $v \in \mathcal{S}_d(\mathbb{R})$,

$$\begin{aligned} \det(I_d - 2iq_t v) &= \det(\theta_t (\theta_t^{-1} - 2it I_d^n \theta_t^T v)) = \det(I_d - 2it I_d^n \theta_t^T v \theta_t), \\ \text{Tr}[iv(I_d - 2iq_t v)^{-1} m_t x m_t^T] &= \text{Tr}[i(\theta_t^{-1})^T \theta_t^T v (\theta_t \theta_t^{-1} - 2it \theta_t I_d^n \theta_t^T v \theta_t \theta_t^{-1})^{-1} m_t x m_t^T] \\ &= \text{Tr}[i \theta_t^T v \theta_t (I_d - 2it I_d^n \theta_t^T v \theta_t)^{-1} \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T]. \end{aligned}$$

Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$ and $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$. Then, from (2.9) and (2.10), we get that

$$\mathbb{E}[\exp(i\text{Tr}(vX_t^x))] = \mathbb{E}[\exp(i\text{Tr}(\theta_t^T v \theta_t \tilde{X}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T}))] = \mathbb{E}[\exp(i\text{Tr}(v \theta_t \tilde{X}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T))],$$

which gives the result. \square

Last, we have to mention that the identity (2.11) extends an usual identity between CIR and Bessel squared distribution. It gives when $d = 1$:

$$WIS_1(x, \alpha, b, a; t) \stackrel{\text{Law}}{=} a^2 \frac{e^{2bt} - 1}{2bt} WIS_1\left(\frac{2btx}{a^2(1 - e^{-2bt})}, \alpha, 0, 1; t\right).$$

In that case, this identity can also be obtained directly from the SDE. Let suppose that $(X_t^x)_{t \geq 0} \sim WIS_1(x, \alpha, b, a)$. We have $dX_t^x = (\alpha a^2 + 2bX_t)dt + 2a\sqrt{X_t^x}dW_t$. Then, $Y_t = e^{-2bt}X_t^x/a^2$ is a time-changed Bessel squared process since $dY_t = \alpha(e^{-2bt}dt) + 2\sqrt{Y_t}(e^{-bt}dW_t)$. We obtain $WIS_1(x, \alpha, b, a; t) \stackrel{\text{Law}}{=} a^2 e^{2bt} WIS_1(x/a^2, \alpha, 0, 1; \frac{1-e^{-2bt}}{2b})$. On the other and, a linear time-change gives that $WIS_1(x, \alpha, 0, 1; \lambda t) \stackrel{\text{Law}}{=} \lambda WIS_1(x/\lambda, \alpha, 0, 1; t)$, which leads to the same identity as (2.11) by taking $\lambda = (1 - e^{-2bt})/(2bt)$.

2.2 Exact simulation of Wishart processes

In this section, we present a new method to simulate exactly a Wishart process. It works without any restriction on the parameters. Wishart distributions have been thoroughly studied in statistics when $\alpha \in \mathbb{N}$ (which is then called the number of degrees of freedom). Exact simulation methods have already been proposed in that case. For instance, Odell and Feiveson [70] have given an exact simulation algorithm for central Wishart distributions based on the Bartlett's decomposition, and Gleser [44] extends it to any (non-central) Wishart distribution. Bru [17] explains also when $\alpha \in \mathbb{N}$ how Wishart processes can be obtained as a square of Ornstein-Uhlenbeck processes on matrices.

Here, our method relies on the identity in law (2.11) that enables us to focus on the case $b = 0, a = I_d^n$. Then, we show a remarkable splitting of the infinitesimal generator as the sum of commuting operators. These operators are associated to SDE that can be solved explicitly on $\mathcal{S}_d^+(\mathbb{R})$, which enables us to sample any Wishart distribution.

2.2.1 A remarkable splitting for $WIS_d(x, \alpha, 0, I_d^n)$

The following theorem explains how to split the infinitesimal generator of a $WIS_d(x, \alpha, 0, I_d^n)$ as the sum of commutative infinitesimal generators. It will play a crucial role in the sequel for the exact simulation and to get discretization schemes.

Theorem 10 — *Let L be the generator associated to the Wishart process $WIS_d(x, \alpha, 0, I_d^n)$ and L_i be the generator associated to $WIS_d(x, \alpha, 0, e_d^i)$ for $i \in \{1, \dots, d\}$. Then, we have*

$$L = \sum_{i=1}^n L_i \text{ and } \forall i, j \in \{1, \dots, d\}, L_i L_j = L_j L_i. \quad (2.12)$$

Proof: From (2.7), we easily get that $L = \sum_{i=1}^n L_i$ since $I_d^n = \sum_{i=1}^n e_d^i$. The commutativity property comes from a simple but tedious calculation which is left in Appendix 2.6.3. \square

Beyond the commutativity, two other features of (2.12) are important to notice.

- The operators L_i and L_j are the same up to the exchange of coordinates i and j .
- The processes $WIS_d(x, \alpha, 0, e_d^i)$ and $WIS_d(x, \alpha, 0, I_d^n)$ are well defined on $\mathcal{S}_d^+(\mathbb{R})$ under the same hypothesis, namely $\alpha \geq d - 1$ and $x \in \mathcal{S}_d^+(\mathbb{R})$.

This second property enables us the composition that we explain now. Let us consider $t > 0$ and $x \in \mathcal{S}_d^+(\mathbb{R})$. We define iteratively:

$$\begin{aligned} X_t^{1,x} &\sim WIS_d(x, \alpha, 0, e_d^1; t), \\ X_t^{2, X_t^{1,x}} &\sim WIS_d(X_t^{1,x}, \alpha, 0, e_d^2; t), \\ &\dots \\ X_t^{n, \dots, X_t^{1,x}} &\sim WIS_d(X_t^{n-1, \dots, X_t^{1,x}}, \alpha, 0, e_d^n; t). \end{aligned}$$

Thus, $X_t^{i, \dots, X_t^{1,x}}$ is sampled according to the distribution at time t of a Wishart process starting from $X_t^{i-1, \dots, X_t^{1,x}}$ and with parameters $(\alpha, 0, e_d^i)$. We have the following result.

Proposition 11 — *Let $X_t^{n, \dots, X_t^{1,x}}$ be defined as above. Then*

$$X_t^{n, \dots, X_t^{1,x}} \sim WIS_d(x, \alpha, 0, I_d^n; t).$$

Thanks to this proposition, we can generate a sample according to $WIS_d(x, \alpha, 0, I_d^n; t)$ as soon as we can simulate the laws $WIS_d(x, \alpha, 0, e_d^i)$. These laws are the same as $WIS_d(x, \alpha, 0, I_d^1; t)$, up to the permutation of the first and i^{th} coordinates. In the next subsection, it is explained how to draw such random variables.

It is really easy to give a formal proof of Proposition 11. Let f be a smooth function on $\mathcal{S}_d^+(\mathbb{R})$ and $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$. By iterating Itô's formula, we have that $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x) / k!$. Similarly, we also get by using the tower property of the conditional expectation that:

$$\mathbb{E} \left[f(X_t^{n, \dots, X_t^{1,x}}) \right] = \mathbb{E} \left[\mathbb{E} \left[f(X_t^{n, \dots, X_t^{1,x}}) | X_t^{n-1, \dots, X_t^{1,x}} \right] \right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!} \mathbb{E} \left[L_n^{k_n} f(X_t^{n-1, \dots, X_t^{1,x}}) \right]. \quad (2.13)$$

Simply by repeating this argument, we get that

$$\mathbb{E} \left[f(X_t^{n, \dots, X_t^{1,x}}) \right] = \sum_{k_1, \dots, k_n=0}^{+\infty} \frac{t^{\sum_{i=1}^n k_i}}{k_1! \dots k_n!} L_1^{k_1} \dots L_n^{k_n} f(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_1 + \dots + L_n)^k f(x) = \mathbb{E}[f(X_t^x)]. \quad (2.14)$$

To get the second equality, we identify a Cauchy product and use that the operators L_1, \dots, L_n commute. To make this formal proof correct, one has to check that the series are well defined and can be switched with the expectation. This check is made in the Appendix 2.6.3 for our framework and remains valid as soon as the operator L_i and L are of affine type.

2.2.2 Exact simulation for $WIS_d(x, \alpha, 0, I_d^1; t)$

In this subsection, we focus on the simulation of $WIS_d(x, \alpha, 0, I_d^1)$ with $\alpha \geq d - 1$ and $x \in \mathcal{S}_d^+(\mathbb{R})$. Thanks to Proposition 11, this enable us then to simulate samples of $WIS_d(x, \alpha, 0, I_d^n)$. For the sake of the clearness, we start with the case of $d = 2$ that avoids complexities due to the matrix decompositions. We deal with the general case just after.

The case $d = 2$

We start by writing explicitly the infinitesimal generator L_1 of $WIS_2(x, \alpha, 0, I_2^1)$. From (2.7), we get, for all $x \in \mathcal{S}_2^+(\mathbb{R})$,

$$L_1 f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{\{1,1\}} \partial_{\{1,1\}}^2 f(x) + 2x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} f(x) + \frac{x_{\{2,2\}}}{2} \partial_{\{1,2\}}^2 f(x). \quad (2.15)$$

We show now that this operator is in fact associated to an SDE that can be explicitly solved. We will denote by $(Z_t^1, t \geq 0)$ and $(Z_t^2, t \geq 0)$ two independent standard Brownian motions.

When $x_{\{2,2\}} = 0$, we also have $x_{\{1,2\}} = 0$ since x is nonnegative. In that case,

$$X_0^x = x, \quad d(X_t^x)_{\{1,1\}} = \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}}} dZ_t^1, \quad d(X_t^x)_{\{1,2\}} = 0, \quad d(X_t^x)_{\{2,2\}} = 0 \quad (2.16)$$

has the infinitesimal generator (2.15), which is the one of a Cox-Ingersoll-Ross SDE (or of a squared Bessel process of dimension α to be more precise). By using an algorithm that samples exactly a non central chi-square distribution (see for instance Glasserman [43]), we can then sample $WIS_2(x, \alpha, 0, I_2^1; t)$ when $x_{\{2,2\}} = 0$.

When $x_{\{2,2\}} > 0$, it easy to check that the SDE

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \frac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}} dZ_t^1 + 2\frac{(X_t^x)_{\{1,2\}}}{\sqrt{(X_t^x)_{\{2,2\}}}} dZ_t^2 \\ d(X_t^x)_{\{1,2\}} &= \sqrt{(X_t^x)_{\{2,2\}}} dZ_t^2 \\ d(X_t^x)_{\{2,2\}} &= 0 \end{aligned} \quad (2.17)$$

starting from $X_0^x = x$ has an infinitesimal generator equal to L_1 . To solve (2.17), we set:

$$(U_t^u)_{\{1,1\}} = (X_t^x)_{\{1,1\}} - \frac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}, \quad (U_t^u)_{\{1,2\}} = \frac{(X_t^x)_{\{1,2\}}}{\sqrt{x_{\{2,2\}}}}, \quad (U_t^u)_{\{2,2\}} = x_{\{2,2\}}. \quad (2.18)$$

Here, u stands for the initial condition, i.e. $u = U_0^u$. We get by using Itô calculus that

$$d(U_t^u)_{\{1,1\}} = (\alpha - 1)dt + 2\sqrt{(U_t^u)_{\{1,1\}}} dZ_t^1, \quad d(U_t^u)_{\{1,2\}} = dZ_t^2 \text{ and } d(U_t^u)_{\{2,2\}} = 0. \quad (2.19)$$

Therefore, $(U_t^u)_{\{1,2\}}$ and $(U_t^u)_{\{1,1\}}$ can be respectively be sampled by independent Gaussian and non-central chi-square variables. Then, we can get back X_t^x by inverting (2.18):

$$(X_t^x)_{\{1,1\}} = (U_t^u)_{\{1,1\}} + (U_t^u)_{\{1,2\}}^2, \quad (X_t^x)_{\{1,2\}} = (U_t^u)_{\{1,2\}} \sqrt{(U_t^u)_{\{2,2\}}}, \quad (X_t^x)_{\{2,2\}} = (U_t^u)_{\{2,2\}}. \quad (2.20)$$

The following proposition sums up and makes more precise the above result. Its proof is postponed to the general case (Theorem 13).

Proposition 12 — Let $x \in \mathcal{S}_2^+(\mathbb{R})$. Then, the process $(X_t^x)_{t \geq 0}$ defined by either (2.16) when $x_{\{2,2\}} = 0$ or (2.17) when $x_{\{2,2\}} > 0$ has its infinitesimal generator equal to L_1 . Moreover, the SDE (2.17) has a unique strong solution which is given by (2.20), where $(U_t^u, t \geq 0)$ is the solution of the SDE (2.19) starting from $u_{\{1,1\}} = x_{\{1,1\}} - \frac{x_{\{1,2\}}^2}{x_{\{2,2\}}} \geq 0$, $u_{\{1,2\}} = \frac{x_{\{1,2\}}}{\sqrt{x_{\{2,2\}}}}$, $u_{\{2,2\}} = x_{\{2,2\}}$.

This result gives an interesting way to figure out the dynamic associated to the operator L_1 , by using a change of variable. It is interesting to notice that the CIR process $(U_t^u)_{\{1,1\}}$ is well defined as soon as its degree $\alpha - 1$ is nonnegative, which coincides with the condition under which the Wishart process $WIS_2(x, \alpha, 0, I_2^1)$ is well-defined. Last, we notice that the solution of the operator L_1 involves a CIR process in the diagonal term and a Brownian motion in the non diagonal one. We will get a similar structure in the general d case.

The general case when $d \geq 2$

We present now a general way to sample exactly $WIS_d(x, \alpha, 0, I_d^1; t)$. We first write explicitly from (2.7) the infinitesimal generator of $WIS_d(x, \alpha, 0, I_d^1)$ for $x \in \mathcal{S}_d^+(\mathbb{R})$:

$$\begin{aligned} L_1 f(x) &= \alpha \partial_{\{1,1\}} f(x) + 2x_{\{1,1\}} \partial_{\{1,1\}}^2 f(x) + 2 \sum_{\substack{1 \leq m \leq d \\ m \neq 1}} x_{\{1,m\}} \partial_{\{1,m\}} \partial_{\{1,1\}} f(x) \\ &+ \frac{1}{2} \sum_{\substack{1 \leq m, l \leq d \\ m \neq 1, l \neq 1}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} f(x). \end{aligned} \quad (2.21)$$

As for $d = 2$ we will construct an SDE that has the same infinitesimal generator L_1 and that can be solved explicitly. To do so, we need however to use further matrix decomposition results. In the case $d = 2$, we have already noticed that we choose different SDEs whether $x_{2,2} = 0$ or not. Here, the SDE will depend on the rank of the submatrix $(x_{i,j})_{2 \leq i, j \leq d}$, and we set:

$$r = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) \in \{0, \dots, d - 1\}.$$

First, we consider the case where

$$\exists c_r \in \mathcal{G}_r \text{ lower triangular, } k_r \in \mathcal{M}_{d-1-r \times r}(\mathbb{R}), (x)_{2 \leq i, j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix} =: cc^T. \quad (2.22)$$

With a slight abuse of notation, we consider that this decomposition also holds when $r = 0$ with $c = 0$. When $r = d - 1$, $c = c_r$ is simply the usual Cholesky decomposition of $(x_{i,j})_{2 \leq i, j \leq d}$. As it is explained in Corollary 15, we can still get such a decomposition up to a permutation of the coordinates $\{2, \dots, d\}$.

Theorem 13 — Let us consider $x \in \mathcal{S}_d^+(\mathbb{R})$ such that (2.22) holds. Let $(Z_t^l)_{1 \leq l \leq r+1}$ be a vector of independent standard Brownian motions. Then, the following SDE (convention

$\sum_{k=1}^r(\dots) = 0$ when $r = 0$)

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \sum_{k=1}^r \left(\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}} \right)^2} dZ_t^1 \\ &\quad + 2 \sum_{k=1}^r \sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}} dZ_t^{k+1} \\ d(X_t^x)_{\{1,i\}} &= \sum_{k=1}^r c_{i-1,k} dZ_t^{k+1}, \quad i = 2, \dots, d \\ d((X_t^x)_{\{l,k\}})_{2 \leq k, l \leq d} &= 0 \end{aligned} \quad (2.23)$$

has a unique strong solution $(X_t^x)_{t \geq 0}$ starting from x . It takes values in $\mathcal{S}_d^+(\mathbb{R})$ and has the infinitesimal generator L_1 . Moreover, this solution is given explicitly by:

$$X_t^x = \bar{c} \begin{pmatrix} (U_t^u)_{\{1,1\}} + \sum_{k=1}^r ((U_t^u)_{\{1,k+1\}})^2 & ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{c}^T, \quad (2.24)$$

where

$$\begin{aligned} d(U_t^u)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}} dZ_t^1, \quad u_{\{1,1\}} = x_{\{1,1\}} - \sum_{k=1}^r (u_{\{1,k+1\}})^2 \geq 0, \\ d((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r} &= (dZ_t^{l+1})_{1 \leq l \leq r}, \quad (u_{\{1,l+1\}})_{1 \leq l \leq r} = c_r^{-1} (x_{\{1,l+1\}})_{1 \leq l \leq r}. \end{aligned} \quad (2.25)$$

and

$$\bar{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}.$$

Once again, we have made a slight abuse of notation when $r = 0$, and (2.24) should be simply read as $X_t^x = \begin{pmatrix} (U_t^u)_{\{1,1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in that case. In the statement above, it may seem weird that we use for u and U_t^u the same indexation as the one for symmetric matrices while we only use its first row (or column). The reason is that we can in fact see X_t^x as a function of U_t^u by setting:

$$(U_t^u)_{\{i,j\}} = u_{\{i,j\}} = x_{\{i,j\}} \text{ for } i, j \geq 2 \text{ and } (U_t^u)_{\{1,i\}} = u_{\{1,i\}} = 0 \text{ for } r+1 \leq i \leq d. \quad (2.26)$$

Thus, (c_r, k_r, I_{d-1}) is an extended Cholesky decomposition of $((U_t^u)_{i,j})_{2 \leq i, j \leq d}$ and can be seen as a function of U_t^u . We get from (2.24) that

$$X_t^x = h(U_t^u), \text{ with } h(u) = \sum_{r=\text{Rk}[(u_{i,j})_{2 \leq i, j \leq d}]}^{d-1} h_r(u) \text{ and} \quad (2.27)$$

$h_r(u) =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u) & 0 \\ 0 & k_r(u) & I_{d-r-1} \end{pmatrix} \begin{pmatrix} u_{\{1,1\}} + \sum_{k=1}^r (u_{\{1,k+1\}})^2 & (u_{\{1,l+1\}})_{1 \leq l \leq r}^T & 0 \\ (u_{\{1,l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u)^T & k_r(u)^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix}$$

, where $(c_r(u), k_r(u), I_{d-1})$ is the extended Cholesky decomposition of $(u_{i,j})_{2 \leq i, j \leq d}$ given by some algorithm (e.g. Golub and Van Loan [45], Algorithm 4.2.4). Equation (2.27) will play later an important role to analyse discretization schemes.

The proof of Theorem 13 is given in Appendix 2.6.3. It enables us to simulate exactly the distribution $WIS_d(x, \alpha, 0, I_d^1; t)$ simply by sampling one non-central chi-square distribution for $(U_t^u)_{\{1,1\}}$ (see Glasserman [43]) and r other independent Gaussian random variables. Like in the $d = 2$ case, we notice that the condition which ensures that the Cox-Ingersoll-Ross process $((U_t^u)_{\{1,1\}}, t \geq 0)$ is well defined for any $r \in \{0, \dots, d-1\}$, namely $\alpha - (d-1) \geq 0$, is the same as the one required for the definition of $WIS_d(x, \alpha, 0, I_d^1)$.

Remark 14 — From (2.24), we get easily by a calculation made in (2.45) that $\text{Rk}(X_t^x) = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) + \mathbb{1}_{(U_t^u)_{\{1,1\}} \neq 0}$, and therefore,

$$\text{Rk}(X_t^x) = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) + 1, \text{ a.s.}$$

Theorem 13 assumes that the initial value $x \in \mathcal{S}_d^+(\mathbb{R})$ satisfies (2.22). Now, we explain why it is still possible up to a permutation of the coordinates to be in such a case. This relies on the extended Cholesky decomposition which is stated in Lemma 33.

Corollary 15 — Let $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, 0, I_d^1)$ and (c_r, k_r, p) be an extended Cholesky decomposition of $(x_{i,j})_{2 \leq i, j \leq d}$ (Lemma 33). Then, $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ is a permutation matrix, and

$$(X_t^x)_{t \geq 0} \stackrel{\text{law}}{=} \pi^T WIS_d(\pi x \pi^T, \alpha, 0, I_d^1) \pi \text{ and } ((\pi x \pi^T)_{i,j})_{2 \leq i, j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix}$$

satisfies (2.22).

Proof: The result is a direct implication from Proposition 7, since $\pi^T = \pi^{-1}$ and $\pi I_d^1 \pi^T = I_d^1$.
□

Therefore, by a combination of Corollary 15 and Theorem 13, we get a simple way to construct explicitly a process that has the infinitesimal generator L_1 for any initial condition $x \in \mathcal{S}_d^+(\mathbb{R})$. In particular, this enables us to sample exactly the Wishart distribution $WIS_d(x, \alpha, 0, I_d^1; t)$. The algorithm below sums up the whole procedure.

Algorithm 1: Exact simulation for the operator L_1

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $d, \alpha \geq d - 1$ and $t > 0$.

Output: X , sampled according to $WIS_d(x, \alpha, 0, I_d^1; t)$

Compute the extended Cholesky decomposition (p, k_r, c_r) of $(x_{i,j})_{2 \leq i, j \leq d}$ given by Lemma 33, $r \in \{0, \dots, d - 1\}$ (see Golub and Van Loan [45] for an algorithm);

Set $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\tilde{x} = \pi x \pi^T$, $(u_{\{1, l+1\}})_{1 \leq l \leq r} = (c_r)^{-1}(\tilde{x}_{\{1, l+1\}})_{1 \leq l \leq r}$ and

$u_{\{1, 1\}} = \tilde{x}_{\{1, 1\}} - \sum_{k=1}^r (u_{\{1, k+1\}})^2 \geq 0$;

Sample independently r normal variables $G_2, \dots, G_{r+1} \sim \mathcal{N}(0, 1)$ and $(U_t^u)_{\{1, 1\}}$ as a

CIR process at time t starting from $u_{\{1, 1\}}$ solving

$d(U_t^u)_{\{1, 1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1, 1\}}}dZ_t^1$ (See Glasserman [43]). Set

$(U_t^u)_{\{1, l+1\}} = u_{\{1, l+1\}} + \sqrt{t}G_{l+1}$;

return

$$X = \pi^T \bar{c} \begin{pmatrix} (U_t^u)_{\{1, 1\}} + \sum_{k=1}^r ((U_t^u)_{\{1, k+1\}})^2 & ((U_t^u)_{\{1, l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t^u)_{\{1, l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{c}^T \pi,$$

where

$$\bar{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}$$

Let us discuss now the complexity of this algorithm. The number of operations required by the extended Cholesky decomposition is of order $O(d^3)$. From a computational point of view, the permutation is handled directly and does not require any matrix multiplication so that we can consider w.l.o.g. that $\pi = I_d$. Since c_r is lower triangular, the calculation of $u_{\{1, i\}}$, $i = 1, \dots, r + 1$ only requires $O(d^2)$ operations. Also, we do not perform in practice the matrix product (2.24), but only compute the values of $X_{\{1, i\}}$ for $i = 1, \dots, d$, which requires also $O(d^2)$ operations. Last, d samples are at most required. To sum up, it comes out that the complexity of the whole algorithm is of order $O(d^3)$.

2.2.3 Exact simulation for Wishart processes

We have now shown all the mathematical results that enable us to give an exact simulation method for general Wishart processes. This is made in two steps. First, by using the remarkable splitting (Proposition 11) and the exact scheme for $WIS_d(x, \alpha, 0, I_d^1; t)$ (Theorem 13 and Corollary 15), we get an exact simulation scheme for $WIS_d(x, \alpha, 0, I_d^n; t)$. As we will see, it is related but extends the Bartlett's decomposition of central Wishart distribution that dates back to 1933. Then, by using the identity in law (2.11), we are able to sample any Wishart distribution $WIS_d(x, \alpha, b, a; t)$.

Exact simulation for $WIS_d(x, \alpha, 0, I_d^n; t)$

By gathering the results obtained in the subsections 2.2.1 and 2.2.2, we get a way to sample exactly the distribution $WIS_d(x, \alpha, 0, I_d^n; t)$. Indeed, thanks to Theorem 13 and Corol-

lary 15 we know how to sample exactly $WIS_d(x, \alpha, 0, I_d^1; t)$. By a simple permutation of the first and k^{th} coordinates, we are then also able to sample according to $WIS_d(x, \alpha, 0, e_d^k; t)$ for $k \in \{1, \dots, d\}$. Thus, we get by Proposition 11 an exact simulation method to sample $WIS_d(x, \alpha, 0, I_d^n; t)$. It is given explicitly in the algorithm below.

Algorithm 2: Exact simulation for $WIS_d(x, \alpha, 0, I_d^n; t)$

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $n \leq d$, $\alpha \geq d - 1$ and $t > 0$.

Output: X , sampled according to $WIS_d(x, \alpha, 0, I_d^n; t)$

$y = x$

for $k = 1$ **to** n **do**

Set $p_{k,1} = p_{1,k} = p_{i,i} = 1$ for $i \notin \{1, k\}$, and $p_{i,j} = 0$ otherwise (permutation of the first and k^{th} coordinates).

$y = pYp$ where Y is sampled according to $WIS_d(pyp, \alpha, 0, I_d^1; t)$ by using Algorithm 1.

end

return $X = y$.

Since this algorithm basically runs n times Algorithm 1, it requires a complexity of order $O(nd^3)$ and therefore at most of order $O(d^4)$. As we have seen, the ‘‘bottleneck’’ of Algorithm 1 is the extended Cholesky decomposition which is in $O(d^3)$. All the other steps in Algorithm 1 require at most $O(d^2)$ operations. A natural question for Algorithm 2 is to wonder if we can reuse the Cholesky decomposition between the for loops instead of calculating it from scratch. For example, if it were possible to get the Cholesky decomposition of loop $k + 1$ from the one of loop k at a cost $O(d^2)$, the complexity of Algorithm 2 would then drop to $O(d^3)$. Despite our investigations, we have not been able to do so up to now.

Remark 16 — When $\alpha \geq 2d - 1$, it is possible to sample $WIS_d(x, \alpha, 0, I_d^n; t)$ in $O(d^3)$ by another mean. If $X_t^1 \sim WIS_d(x, d, 0, I_d^n; t)$ and $X_t^2 \sim WIS_d(0, \alpha - d, 0, I_d^n; t)$ are independent, we can check that $X_t^1 + X_t^2 \sim WIS_d(x, \alpha, 0, I_d^n; t)$. Then, X_t^1 can be sampled by using Proposition 30 and X_t^2 by using Corollary 18 since $X_t^2 \stackrel{\text{Law}}{=} tWIS_d(0, \alpha - d, 0, I_d^n; 1)$ from (2.10).

Remark 17 — Let $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$. From Remark 14, we get that for each $k \in \{1, \dots, n\}$, $X_t^{k, \dots, X_t^{1,x}} \in \mathcal{S}_d^{+,*}(\mathbb{R})$, a.s.. In that case, the extended Cholesky decomposition that has to be computed at each step is an usual Cholesky decomposition.

The Bartlett’s decomposition revisited

Now, we would like to illustrate our exact simulation method on the case $WIS_d(0, \alpha, 0, I_d^n; 1)$, which is known in the literature as the central Wishart distribution. In that case, we can perform explicitly the composition given by Proposition 11. We will show by an induction on n that:

$$X_1^{n, \dots, X_1^{1,0}} = \begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^T)_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix},$$

where $(L_{i,j})_{1 \leq j < i \leq d}$ and $L_{i,i}$ are independent random variables such that $L_{i,j} \sim \mathcal{N}(0, 1)$ and $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$, and $L_{i,j} = 0$ for $i < j$.

For $n = 1$, we know from Theorem 13 that $(X_1^{1,0})_{1,1} \sim \chi^2(\alpha)$ since $d(X_t^{1,0})_{1,1} = \alpha dt + 2\sqrt{(X_t^{1,0})_{1,1}}dZ_t^1$ with $(X_0^{1,0})_{1,1} = 0$, and all the other elements are equal to 0. Let us assume now that the induction hypothesis is satisfied for $n - 1$. Then, we can apply once again Theorem 13 (up to the permutation of the first and n^{th} coordinates). We have $\text{Rk}(X_1^{n-1, \dots, X_1^{1,0}}) = n - 1$, *a.s.*, and the Cholesky decomposition is directly given by $(L_{i,j})_{1 \leq i, j \leq n-1}$. Then, we get from (2.24) that there are independent variables $L_{n,n}^2 \sim \chi^2(\alpha - n + 1)$ and $L_{n,i} \sim \mathcal{N}(0, 1)$ for $i \in \{1, \dots, n - 1\}$ such that

$$X_1^{n, \dots, X_1^{1,0}} = \begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix} \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n-1}^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix},$$

Since

$$\begin{aligned} & \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & 0 & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ 0 & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

, we conclude by induction and get the following result which is known as the Bartlett's decomposition (see Kshirsagar [59] or Kabe [56]).

Corollary 18 — *Bartlett's Decomposition of central Wishart distributions.* *With the notations above,*

$$\begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^T)_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \sim WIS_d(0, \alpha, 0, I_d^n; 1).$$

Therefore, the identity given by Proposition 11 that we have obtained by a remarkable splitting of the infinitesimal generator extends in a natural way the Bartlett's decomposition to non-central Wishart distributions.

Exact simulation for $WIS_d(x, \alpha, b, a; t)$

Now, thanks to the identity in law (2.11), we get an exact simulation for $WIS_d(x, \alpha, b, a; t)$. The associated algorithm is given below. To the best of our knowledge, this is the first exact simulation method for non-central Wishart distributions that works for any $\alpha \geq d - 1$. The existing methods in the literature mainly focus on the case $\alpha \in \mathbb{N}$ that arises naturally in statistics. Namely, Odell and Feiveson [70] and Smith and Hocking [80] have proposed an exact simulation method for central Wishart distributions based on the Bartlett's decomposition. Gleser [44] has given an exact simulation method for non-central Wishart distributions, also when the degree α is an integer.

Algorithm 3: Exact simulation for $WIS_d(x, \alpha, b, a; t)$

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $\alpha \geq d - 1$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $t > 0$.

Output: X , sampled according to $WIS_d(x, \alpha, b, a; t)$.

Calculate $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$ and (p, c_n, k_n) an extended Cholesky decomposition of q_t/t .

Set $\theta_t = p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ and $m_t = \exp(tb)$.

return $X = \theta_t Y \theta_t^T$, where $Y \sim WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)$ is sampled by Algorithm 2.

2.3 High order discretization schemes for Wishart and semi definite positive affine processes

Up to now we have given an exact simulation method for Wishart processes. It relies on a remarkable splitting of the infinitesimal generator given in Theorem 10. When dealing with discretization schemes, splitting operators is a powerful technique to construct schemes for SDEs from other schemes obtained on simpler SDEs. This idea of splitting originates from the seminal work of Strang [81] in the field of ODEs. As pointed by Ninomiya and Victoir [69] or Alfonsi [5], it is rather easy to analyse the weak error of schemes obtained by splitting, simply by using the same arguments as Talay and Tubaro [82] for the Euler-Maruyama scheme. Thus, we will only focus on the analysis of the weak error, i.e. the error made on marginal distributions.

Up to our knowledge, there are very few papers in the literature that deal with discretization schemes for Wishart processes. Recently, Benabid et al. [8] have proposed a Monte-Carlo method to calculate expectations on Wishart processes which is based on a Girsanov change of probability. Gauthier and Possamaï [42] introduce a moment-matching scheme for Wishart processes. Both methods are well defined under some restrictions on the parameters, and there is no theoretical result on their accuracy. Currently, Teichmann [71] is working on dedicated schemes for general affine processes by approximating their characteristic functions.

This section is structured as follows. First, we recall basic results on the splitting technique to get discretization schemes for SDEs. We will take the same framework as Alfonsi [5] since it is somehow designed for affine processes. Then, we will explain how to get high order schemes for $WIS_d(x, \alpha, 0, I_d^n)$ from the remarkable splitting (2.12). From this result, we will be able to get a second order scheme for any semi definite positive affine processes and a third order scheme for Wishart processes.

2.3.1 General results on discretization schemes for SDEs

In this paragraph, we try to present the minimal knowledge that we will use later to get schemes for affine processes. We take back the framework developed in Alfonsi ([5], Section 1) and refer to this paper for further details and proofs.

Let us start with some notations. We consider a domain $\mathbb{D} \subset \mathbb{R}^\zeta$, $\zeta \in \mathbb{N}^*$. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_\zeta) \in \mathbb{N}^\zeta$, we define $\partial_\gamma = \partial_1^{\gamma_1}, \dots, \partial_\zeta^{\gamma_\zeta}$ and $|\gamma| = \sum_{i=1}^\zeta \gamma_i$ and set:

$$\mathcal{C}_{\text{pol}}^\infty(\mathbb{D}) = \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall x \in \mathbb{D}, |\partial_\gamma f(x)| \leq C_\gamma (1 + \|x\|^{e_\gamma})\},$$

where $\|\cdot\|$ is a norm on \mathbb{R}^ζ . We will say that $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ is a *good sequence* for $f \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$ if one has $|\partial_\gamma f(x)| \leq C_\gamma(1 + \|x\|^{e_\gamma})$. Mainly (but not only), we will consider in this paper $\mathbb{D} = \mathcal{S}_d^+(\mathbb{R})$ as a subspace of $\mathcal{S}_d(\mathbb{R}) \simeq \mathbb{R}^{d(d+1)/2}$. In that case, it is natural to keep the same indexation as for matrices, and we rather use the notation $\partial_\gamma = \prod_{1 \leq i \leq j \leq d} \partial_{\{i,j\}}^{\gamma_{\{i,j\}}}$ for $\gamma = (\gamma_{\{i,j\}})_{1 \leq i \leq j \leq d} \in \mathbb{N}^{d(d+1)/2}$.

Definition 19 — Let $b : \mathbb{D} \rightarrow \mathbb{R}^\zeta$, $\sigma : \mathbb{D} \rightarrow \mathcal{M}_\zeta(\mathbb{R})$. The operator L defined for $f \in \mathcal{C}^2(\mathbb{D}, \mathbb{R})$ by:

$$Lf(x) = \sum_{i=1}^{\zeta} b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{\zeta} (\sigma \sigma^T)_{i,j}(x) \partial_i \partial_j f(x) \quad (2.28)$$

is said to satisfy the required assumptions on \mathbb{D} if the following conditions hold:

- $\forall i, j \in \{1, \dots, \zeta\}, b_i(x), (\sigma \sigma^T)_{i,j}(x) \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$,
- for any $x \in \mathbb{D}$, the SDE $X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s$ has a unique weak solution defined for $t \geq 0$, and thus $\mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$.

In the case of affine diffusions, $b_i(x)$ and $(\sigma \sigma^T)_{i,j}(x)$ are affine functions of x and the operator satisfies the required assumption on the appropriated domain. Let us stress that if L satisfies the required assumption on \mathbb{D} and $f \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$, all the iterated functions $L^k f(x)$ are well defined on \mathbb{D} and belong to $\mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$ for any $k \in \mathbb{N}$.

Let us turn now to discretization schemes. We will consider in this paper a final time horizon $T > 0$ and the regular time grid defined by $t_i^N = iT/N, i = 0, \dots, N$. When discretizing a Markovian process, a scheme is usually described as a way to sample the process at the next time step from its current position. Thus, we will say that $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ is a *family of transition probabilities on \mathbb{D}* if $\hat{p}_x(t)$ is a probability law on \mathbb{D} for any time-step $t > 0$ and any position $x \in \mathbb{D}$. Then, a *discretization scheme* on the regular time grid associated to this family starting from $x_0 \in \mathbb{D}$ is simply a sequence $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$ of \mathbb{D} -valued random variables such that:

- $\hat{X}_{t_0^N}^N = x_0$,
- the law of $\hat{X}_{t_{i+1}^N}^N$ is sampled according to $\hat{p}_{\hat{X}_{t_i^N}^N}(T/N)(dz)$ independently from the previous samples, i.e. $\mathbb{E}[f(\hat{X}_{t_{i+1}^N}^N) | (\hat{X}_{t_j^N}^N, 0 \leq j \leq i)] = \int_{\mathbb{D}} f(z) \hat{p}_{\hat{X}_{t_i^N}^N}(T/N)(dz)$ for any bounded measurable function $f : \mathbb{D} \rightarrow \mathbb{R}$.

Up to the initial condition, the discretization scheme $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$ is entirely characterized by $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$. With a slight abuse of language, we will then say that $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ is a *scheme on \mathbb{D}* . We will denote by \hat{X}_t^x a random variable on \mathbb{D} which is sampled according to $\hat{p}_x(t)$.

Definition 20 — Let L be an operator that satisfies the required assumption on \mathbb{D} . A scheme $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ is a *potential weak ν -th-order scheme for the operator L* if for any

function $f \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$ with a good sequence $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$, there exist positive constants C , E , and η depending only on $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ such that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x)] - \left[f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) \right] \right| \leq Ct^{\nu+1}(1 + \|x\|^E). \quad (2.29)$$

When the coefficients of the SDE have a sublinear growth, we can check that the exact scheme (i.e. $\hat{X}_t^x = X_t^x$, where $(X_t^x, t \geq 0)$ solves the SDE associated to L) is a potential ν th-order scheme for any order $\nu \in \mathbb{N}$ ([5], Proposition 1.12). In that case, (2.29) is then clearly equivalent to:

$$\exists C, E, \eta > 0, \forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)] \right| \leq Ct^{\nu+1}(1 + \|x\|^E).$$

In practice, having a potential weak ν th-order scheme for the operator L is the main requirement to get a weak error of order ν . This is precised by the following theorem that relies on the idea developed by Talay and Tubaro [82].

Theorem 21 — *Let us consider an operator L that satisfies the required assumptions on \mathbb{D} and a discretization scheme $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$ with transition probabilities $\hat{p}_x(t)(dz)$ on \mathbb{D} that starts from $\hat{X}_{t_0^N}^N = x_0 \in \mathbb{D}$. We assume that*

1. $\hat{p}_x(t)(dz)$ is a potential weak ν th-order scheme for the operator L ,
2. the scheme has bounded moments, i.e.:

$$\forall q \in \mathbb{N}^*, \exists N(q) \in \mathbb{N}, \sup_{N \geq N(q), 0 \leq i \leq N} \mathbb{E}[\|\hat{X}_{t_i^N}^N\|^q] < \infty, \quad (2.30)$$

3. $f : \mathbb{D} \rightarrow \mathbb{R}$ is a function such that $u(t, x) = \mathbb{E}[f(X_{T-t}^x)]$ is defined on $[0, T] \times \mathbb{D}$, \mathcal{C}^∞ , solves $\forall t \in [0, T], \forall x \in \mathbb{D}, \partial_t u(t, x) = -Lu(t, x)$, and satisfies:

$$\forall l \in \mathbb{N}, \gamma \in \mathbb{N}^\zeta, \exists C_{l,\gamma}, e_{l,\gamma} > 0, \forall x \in \mathbb{D}, t \in [0, T], |\partial_t^l \partial_\gamma u(t, x)| \leq C_{l,\gamma}(1 + \|x\|^{e_{l,\gamma}}). \quad (2.31)$$

Then, there is $K > 0$, $N_0 \in \mathbb{N}$, such that $|\mathbb{E}[f(\hat{X}_{t_i^N}^N)] - \mathbb{E}[f(X_T^{x_0})]| \leq K/N^\nu$ for $N \geq N_0$.

Let us mention that condition (2.30) is slightly weakened with respect to Theorem 1.9 in [5]. However, we can check that (2.30) is in fact sufficient to make work the proof of this theorem given in [5].

Let us comment briefly the hypothesis of this theorem. The boundedness of the moments (2.30) holds as soon as the coefficients of the SDE have a sublinear growth (see Lemma 36 in Appendix 2.6.4). It is instead much more technical to get the bounds (2.31) on the derivatives of u . In their original paper, Talay and Tubaro have considered stronger assumptions on b and σ (namely, \mathcal{C}^∞ with bounded derivatives) to get such bounds. In the case of Wishart processes, we are able to get (2.31) when $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ by using the explicit formula of the characteristic function (2.9). This is stated in Proposition 43. We however deem that this result still holds for the general affine case.

Now, we present results that explain how the property of being a potential weak ν th-order scheme can be preserved when we use the splitting technique. To do so, we need to introduce first the composition of schemes. Let us consider two transition probabilities $\hat{p}_x^1(t)(dz)$ and $\hat{p}_x^2(t)(dz)$ on \mathbb{D} . Then, we define:

$$\hat{p}^2(t_2) \circ \hat{p}_x^1(t_1)(dz) := \int_{\mathbb{D}} \hat{p}_y^1(t_2)(dz) \hat{p}_x^1(t_1)(dy),$$

which is the law obtained when one uses first use scheme 1 with a time step t_1 and then scheme 2 with a time step t_2 with independent samples. More generally, if one has m transition probabilities $\hat{p}_x^1, \dots, \hat{p}_x^m$ on \mathbb{D} , we define

$$\hat{p}^m(t_m) \circ \dots \circ \hat{p}_x^1(t_1)(dz) := \hat{p}^m(t_m) \circ (\hat{p}^{m-1}(t_{m-1}) \circ \dots \circ \hat{p}_x^1(t_1)(dz)).$$

Proposition 22 — *Let L_1, L_2 be two operators satisfying the required assumption on \mathbb{D} . Let \hat{p}_x^1 and \hat{p}_x^2 be respectively two potential weak ν th-order schemes on \mathbb{D} for L_1 and L_2 .*

- *If $L_1 L_2 = L_2 L_1$, $\hat{p}^2(t) \circ \hat{p}_x^1(t)(dz)$ is a potential weak ν th-order discretization scheme for $L_1 + L_2$.*
- *If $\nu \geq 2$, $\hat{p}^2(t/2) \circ \hat{p}_x^1(t) \circ \hat{p}_x^2(t/2)$ and $\frac{1}{2} (\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}_x^1(t) \circ \hat{p}_x^2(t))$ are potential weak second order schemes for $L_1 + L_2$.*

2.3.2 High order schemes for Wishart processes

In this paragraph, we will give a way to get weak ν th-order schemes for any Wishart processes. The construction of these schemes is the same as the one used for the exact scheme. First we obtain a ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^1)$. Then, we get a ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^n)$ by scheme composition. Last, we use the identity in law (2.11) to get a weak ν th-order scheme for any Wishart processes.

A potential weak ν th-order scheme associated to L_1

In this paragraph, we present a potential weak ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^1)$. Roughly speaking, we obtain this scheme from the exact scheme given by Theorem 13 and Corollary 15 by replacing the Gaussian random variables with moment matching variables and the exact CIR distribution with a sample according to a potential weak ν -th order scheme for the CIR.

Theorem 23 — *Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and (c_r, k_r, p) be an extended Cholesky decomposition of $(x_{i,j})_{2 \leq i,j \leq d}$. We set $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\tilde{x} = \pi x \pi^T$, so that $\tilde{x} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix}$. Like in Theorem 13, we have*

$$u_{\{1,1\}} = \tilde{x}_{\{1,1\}} - \sum_{k=1}^r (u_{\{1,k+1\}})^2 \geq 0, \text{ where } (u_{\{1,l+1\}})_{1 \leq l \leq r} = c_r^{-1}(\tilde{x}_{\{1,l+1\}})_{1 \leq l \leq r},$$

and we set $u_{\{1,i\}} = 0$ if $r + 2 \leq i \leq d$ and $u_{\{i,j\}} = \tilde{x}_{\{i,j\}}$ if $i, j \geq 2$. Let $(\hat{G}^i)_{1 \leq i \leq r}$ be a sequence of independent real variables with finite moments of any order such that:

$$\forall i \in \{1, \dots, r\}, \forall k \leq 2\nu + 1, \mathbb{E}[(\hat{G}^i)^k] = \mathbb{E}[G^k], \text{ where } G \sim \mathcal{N}(0, 1).$$

Let $(\hat{U}_t^u)_{\{1,1\}}$ be sampled independently according to a potential weak ν th-order scheme for the CIR process $d(U_t^u)_{\{1,1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1$ starting from $u_{\{1,1\}}$ and that satisfies the immersion property (Definition 37). We set:

$$(\hat{U}_t^u)_{\{1,i\}} = u_{\{1,i\}} + \sqrt{t}\hat{G}^i, \quad 2 \leq i \leq r+1, \quad (\hat{U}_t^u)_{\{1,i\}} = 0, \quad r+2 \leq i \leq d, \quad (\hat{U}_t^u)_{\{i,j\}} = u_{\{i,j\}} \text{ if } i, j \geq 2.$$

Then, the scheme defined by $\hat{X}_t^x = \pi^T h_r(\hat{U}_t^u) \pi$ is a potential ν th-order scheme for L_1 , where the function h_r is defined by (2.27).

The proof of this result is technical and requires further definitions. It is left in Appendix 23. Second and third order schemes for the CIR process that satisfy the immersion property can be found in Alfonsi [5] (see Corollary 40). We can therefore get second (resp. third) order schemes for L_1 by taking any variables that matches the five (resp. the seven) first moments of $\mathcal{N}(0, 1)$. This can be obtained by taking

$$\mathbb{P}(\hat{G}^i = \sqrt{3}) = \mathbb{P}(\hat{G}^i = -\sqrt{3}) = \frac{1}{6} \text{ and } \mathbb{P}(\hat{G}^i = 0) = \frac{2}{3} \quad (2.32)$$

$$\text{(resp. } \mathbb{P}(\hat{G}^i = \varepsilon\sqrt{3 + \sqrt{6}}) = \frac{\sqrt{6} - 2}{4\sqrt{6}}, \mathbb{P}(\hat{G}^i = \varepsilon\sqrt{3 - \sqrt{6}}) = \frac{1}{2} - \frac{\sqrt{6} - 2}{4\sqrt{6}}, \varepsilon \in \{-1, 1\}). \quad (2.33)$$

A potential weak ν th-order scheme associated to $WIS_d(x, \alpha, 0, I_d^n)$

From Theorem 10, we know that the infinitesimal generator of $WIS_d(x, \alpha, 0, I_d^n)$ is given by $L = \sum_{i=1}^n L_i$, where the operators L_i are the same as L_1 up to the permutation of the first and i th coordinate. Moreover, these operators are commuting. Thus, if $\pi^{1 \leftrightarrow i}$ denotes the associated permutation matrix and \hat{X}_t^x is the potential ν th-order scheme defined by Theorem 23,

$$\pi^{1 \leftrightarrow i} \hat{X}_t^{\pi^{1 \leftrightarrow i} x \pi^{1 \leftrightarrow i}} \pi^{1 \leftrightarrow i} \text{ is a potential } \nu\text{th-order scheme for } L_i.$$

We then get a potential ν th-order scheme for L thanks to Proposition 22.

Corollary 24 — Let $\hat{p}_x^i, i \in \{1, \dots, d\}$ be potential weak ν th-order scheme for L_i . Then for any $n \leq d$,

$$\hat{p}^n(t) \circ \dots \circ \hat{p}_x^1(t)(dz)$$

defines a potential weak ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^n)$.

Remark 25 — Let $\varepsilon > 0$ and \hat{X}_t^x be a potential weak ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^n)$. Since $x + \varepsilon t^{\nu+1} I_d$ is a potential weak ν th-order scheme for the operator $L = 0$, we easily get by scheme composition (Proposition 22) that $\hat{X}_t^{x + \varepsilon t^{\nu+1} I_d}$ is also a potential weak ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^n)$. This scheme starts from an invertible initial condition, and by Remark 14, we only make in that case usual Cholesky decompositions on invertible matrices.

A weak ν th-order scheme for Wishart processes $WIS_d(x, \alpha, b, a)$

Now that we have a potential ν th-order scheme for $WIS_d(x, \alpha, 0, I_d^n)$, we are in position to construct a scheme for any Wishart process $WIS_d(x, \alpha, b, a)$ thanks to the identity (2.11). Unfortunately, we need to make some technical restrictions on a and b (namely, $a \in \mathcal{G}_d(\mathbb{R})$ or $ba^T a = a^T a b$) to show that we get like this a potential ν th-order scheme. We however believe that this is rather due to our analysis of the error and that the scheme converges as well without this restriction. We mention in addition that we give in the next section a second order scheme based on Corollary 8 for which we can make our error analysis for any parameters.

Proposition 26 — *Let $t > 0$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\alpha \geq d - 1$. Let $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$ and $n = \text{Rk}(a^T a)$. We assume that either $a \in \mathcal{G}_d(\mathbb{R})$ or b and $a^T a$ commute. We define*

- if $n = d$, θ_t as the (usual) Cholesky decomposition of q_t/t ,
- if $n < d$, $\theta_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds} p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ where (c_n, k_n, p) is the extended Cholesky decomposition of $a^T a$ otherwise.

In both cases, $\theta_t \in \mathcal{G}_d(\mathbb{R})$.

Let \hat{Y}_t^y denote a potential weak ν th-order scheme for $WIS_d(y, \alpha, 0, I_d^n)$. Then, the scheme defined by

$$\hat{X}_t^x = \theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T, \quad (2.34)$$

is a potential weak ν th-order scheme for $WIS_d(x, \alpha, b, a)$.

Proof: First, let us check that $\theta_t \in \mathcal{G}_d(\mathbb{R})$ is well defined, such that $q_t/t = \theta_t I_d^n \theta_t^T$ and satisfies:

$$\exists K, \eta > 0, \forall t \in (0, \eta), \max(\|\theta_t\|, \|\theta_t\|^{-1}) \leq K. \quad (2.35)$$

When $n = d$, q_t/t is definite positive as a convex combination of definite positive matrices and the usual Cholesky decomposition is well defined. Moreover, (2.35) holds since q_t/t goes to $a^T a$ which is invertible when $t \rightarrow 0^+$. When $n < d$, we have assumed in addition that b and $a^T a$ commute. Therefore, $q_t = a^T a (\int_0^t \exp(sb) \exp(sb^T) ds/t)$. Since $a^T a$ and $(\int_0^t \exp(sb) \exp(sb^T) ds/t)$ are positive semidefinite matrices that commute, we have

$$q_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds} a^T a \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}.$$

Once again, $\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds$ is definite positive as a convex combination of definite positive matrices and we get that $\theta_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds} p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$ satisfies $q_t/t = \theta_t I_d^n \theta_t^T$ by Lemma 33. Similarly, (2.35) holds since $p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ does not depend on t and $\sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}$ goes to I_d when $t \rightarrow 0^+$.

Let $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$. Since the exact scheme is a potential ν th-order scheme, there are constants $C, E, \eta > 0$ depending only on a good sequence of f such that

$$\forall t \in (0, \eta), |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x)| \leq Ct^{\nu+1}(1 + \|x\|^E). \quad (2.36)$$

On the other hand we have from Proposition 9,

$$\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)] = \mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)]. \quad (2.37)$$

Let us introduce $f_{\theta_t}(y) := f(\theta_t y \theta_t^T) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. By the chain rule, we have $\partial_{\{i,j\}} f_{\theta_t}(y) = \text{Tr}[\theta_t (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i}) \theta_t^T \partial f(\theta_t y \theta_t^T)]$, where $(\partial f(x))_{k,l} = (\mathbb{1}_{k=l} + \frac{1}{2} \mathbb{1}_{k \neq l}) \partial_{\{k,l\}} f(x)$ and $e_d^{i,j} = (\mathbb{1}_{k=i, l=j})_{1 \leq k, l \leq d}$. From (2.35), we see that there is a good sequence $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^{d(d+1)/2}}$ that can be obtained from a good sequence of f such that:

$$\forall t \in (0, \eta), \forall y \in \mathcal{S}_d^+(\mathbb{R}), |\partial_\gamma f_{\theta_t}(y)| \leq C_\gamma (1 + \|y\|^{e_\gamma}).$$

Therefore, we get that there are constants still denoted by $C, E, \eta > 0$ such that for every $t \in (0, \eta)$,

$$\left| \mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] \right| \leq Ct^{\nu+1} (1 + \|\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T\|^E). \quad (2.38)$$

From (2.35), we get that there is a constant $K' > 0$ such that $\|\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T\|^E \leq K' \|x\|^E$ for $t \in (0, \eta)$. Thus, we get the result by gathering (2.36), (2.37) and (2.38). \square

Now, we want to show thanks to Theorem 21 that the scheme given by Proposition 26 gives indeed a weak error of order ν . However, if we exclude the exact simulation of the CIR, we only have at our disposal (up to our knowledge) at most a third order scheme for the CIR (see [5]). It seems thus fair to state the result on the weak error in that case.

Theorem 27 — *Let $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$ such that either $a \in \mathcal{G}_d(\mathbb{R})$ or $a^T a b = b a^T a$, and $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$. Let $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$ be sampled with the scheme defined by Proposition 26 with the third order scheme for the CIR given in [5] and starting from $x \in \mathcal{S}_d^+(\mathbb{R})$. Then,*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_i^N}^N)] - \mathbb{E}[f(X_t^x)]| \leq C/N^3.$$

Proof : The conditions of Theorem 21 are satisfied thanks to Propositions 26, 43 and Lemma 36. \square

2.3.3 Second order schemes for affine diffusions on $\mathcal{S}_d^+(\mathbb{R})$

In this part, we present two different potential second order schemes for general affine processes. The first one is well-defined without any restriction on the parameters. The second one is faster but requires to assume in addition that $\bar{\alpha} - da^T a \in \mathcal{S}_d^+(\mathbb{R})$.

Let $(X_t^x)_{t \geq 0} \sim AFF_d(x, \bar{\alpha}, B, a)$. Thanks to Corollary 8, there is $u \in \mathcal{G}_d(\mathbb{R})$ and a diagonal matrix $\bar{\delta}$ such that $\bar{\alpha} = u^T \bar{\delta} u$, $a^T a = u^T I_d^n u$ and we have:

$$(X_t^x)_{t \geq 0} \stackrel{law}{=} (u^T Y_t^{(u^{-1})^T x u^{-1}})_{t \geq 0}, \text{ where } Y_t^y \sim AFF_d(y, \bar{\delta}, B_u, I_d^n).$$

Lemma 28 — *Let \hat{Y}_t^y be a potential ν th-order scheme for $AFF_d(y, \bar{\delta}, B_u, I_d^n)$. It follows then that $u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u$ is a potential ν th-order scheme for $AFF_d(x, \bar{\alpha}, B, a)$.*

Proof: Let $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. We then have $x \mapsto f(u^T x u) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. Since u is fixed, there are constants C, η, E depending only on a good sequence of f such that for $t \in (0, \eta)$, $|\mathbb{E}[f(u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u)] - \mathbb{E}[f(X_t^x)]| = |\mathbb{E}[f(u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u)] - \mathbb{E}[f(u^T Y_t^{(u^{-1})^T x u^{-1}} u)]| \leq C t^{\nu+1} (1 + \|(u^{-1})^T x u^{-1}\|^E) \leq C' t^{\nu+1} (1 + \|x\|^E)$, for some constant $C' > C$. \square

We now focus on finding a scheme for $AFF_d(y, \bar{\alpha}, B_u, I_d^n)$. Since $\bar{\delta}$ is a diagonal matrix such that $\bar{\delta} - (d-1)I_d^n \in \mathcal{S}_d^+(\mathbb{R})$, we have

$$\delta_{\min} := \min_{1 \leq i \leq n} \bar{\delta}_{i,i} \geq d-1.$$

From Corollary 4, we can write the infinitesimal generator of Y_t^y

$$\begin{aligned} L &= \text{Tr}([\bar{\delta} + B(x)]D^S) + 2\text{Tr}(xD^S I_d^n D^S) \\ &= \underbrace{\text{Tr}([\bar{\delta} - \delta_{\min} I_d^n + B_u(x)]D^S)}_{L_{ODE}} + \underbrace{\delta_{\min} \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d(x, \delta_{\min}, 0, I_d^n)}} \end{aligned} \quad (2.39)$$

as the sum of the infinitesimal generator of $WIS_d(x, \delta_{\min}, 0, I_d^n)$ and of the generator of the affine ODE $x'(t) = \bar{\delta} - \delta_{\min} I_d^n + B_u(x(t))$. Of course, this ODE can be solved explicitly. Moreover, we know by Lemma 45 that if $x(0) = x \in \mathcal{S}_d^+(\mathbb{R})$, the solution of this ODE remains in $\mathcal{S}_d^+(\mathbb{R})$ since Assumption (2.4) holds for B_u and $\bar{\delta} - \delta_{\min} I_d^n \in \mathcal{S}_d^+(\mathbb{R})$. We denote by $p_x^{ODE}(t)$ the Dirac mass at $x(t)$: this is an exact scheme for L_{ODE} . Thanks to Proposition 22, we get the following result.

Proposition 29 — *Let $\hat{p}_x^W(t)$ denote a potential second order scheme for $WIS_d(x, \delta_{\min}, 0, I_d^n)$. With the notations above, the following schemes*

$$\frac{1}{2} p^{ODE}(t) \circ \hat{p}_x^W(t) + \frac{1}{2} \hat{p}_x^W(t) \circ p_x^{ODE}(t) \text{ and } p^{ODE}(t/2) \circ \hat{p}_x^W(t) \circ p_x^{ODE}(t/2)$$

are potential second order schemes for the affine process $AFF_d(x, \bar{\delta}, B_u, I_d^n)$.

By combining Lemma 28 and Proposition 29, we get a potential second order scheme for any affine process. In the numerical experiments in Section 2.4, we have used the composition $p^{ODE}(t/2) \circ \hat{p}_x^W(t) \circ p_x^{ODE}(t/2)$ even though the other one would have worked as well.

Let us stress now that different splittings from (2.39) are possible. For example, we could have chosen instead $L = \text{Tr}([\bar{\delta} - \beta I_d^n + B_u(x)]D^S) + \beta \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)$ for any

$\beta \in [d-1, \delta_{\min}]$. When $\bar{\alpha} - da^T a \in \mathcal{S}_d^+(\mathbb{R})$ (or equivalently when $\bar{\delta} - dI_d^n \in \mathcal{S}_d^+(\mathbb{R})$), the following splitting

$$L = \underbrace{\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)}_{\tilde{L}_{ODE}} + \underbrace{d\text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d(x,d,0,I_d^n)}} \quad (2.40)$$

is really interesting. Indeed it is known from Bru [17] that Wishart processes can be seen as the square of an Ornstein-Uhlenbeck process on matrices and can be simulated very efficiently. More precisely, we will use the following result.

Proposition 30 — *Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and $c \in \mathcal{M}_d(\mathbb{R})$ be such that $c^T c = x$. Then, the process $X_t^x = (c + W_t I_d^n)^T (c + W_t I_d^n)$ satisfies*

$$(X_t^x)_{t \geq 0} \stackrel{\text{law}}{=} WIS_d(x, d, 0, I_d^n).$$

If \hat{G} denote a d -by- d matrix with independent elements sampled according to (2.32), $\hat{X}_t^x = (c + \sqrt{t}\hat{G}I_d^n)^T (c + \sqrt{t}\hat{G}I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$.

Proof: We have by using Itô calculus $dX_t^x = (c + W_t I_d^n)^T dW_t I_d^n + I_d^n dW_t^T (c + W_t I_d^n) + dI_d^n dt$. By using Lemma 2, The quadratic covariation of $(X_t^x)_{i,j}$ and $(X_t^x)_{m,n}$ is given by $d\langle (X_t^x)_{i,j}, (X_t^x)_{m,n} \rangle = (X_t^x)_{i,m} (I_d^n)_{j,n} + (X_t^x)_{i,n} (I_d^n)_{j,m} + (X_t^x)_{j,m} (I_d^n)_{i,n} + (X_t^x)_{j,n} (I_d^n)_{i,m}$. Therefore, $(X_t^x)_{t \geq 0}$ solves the same martingale problem as $WIS_d(x, d, 0, I_d^n)$, which is known to have a unique solution from Cuchiero et al. [26].

Let us show now that \hat{X}_t^x is a potential second order scheme. We can see $c + \sqrt{t}\hat{G}I_d^n$ as the Ninomiya-Victoir scheme with moment-matching variables (see Theorem 1.18 in [5]) associated to $\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^n \partial_{i,j}^2$ on $\mathcal{M}_d(\mathbb{R})$. Let $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. Then, $x \in \mathcal{M}_d(\mathbb{R}) \mapsto f(x^T x) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{M}_d(\mathbb{R}))$ and there are constants $C, E, \eta > 0$ depending only on a good sequence of f such that, for all $\forall t \in (0, \eta)$

$$|\mathbb{E}[f((c + \sqrt{t}\hat{G}I_d^n)^T (c + \sqrt{t}\hat{G}I_d^n))] - \mathbb{E}[f((c + W_t I_d^n)^T (c + W_t I_d^n))] | \leq Ct^{\nu+1}(1 + \|c\|^E).$$

Let us observe now that the Frobenius norm of c is $\sqrt{\text{Tr}(c^T c)} = \sqrt{\text{Tr}(x)} \leq \sqrt{d + \text{Tr}(x^2)} \leq \sqrt{d} + \sqrt{\text{Tr}(x^2)}$. Therefore, for any norm, there is a constant $K > 0$ such that $\|c\| \leq K(1 + \|x\|)$, which gives the result. \square

Corollary 31 — *We assume that $\bar{\delta} - dI_d^n \in \mathcal{S}_d^+(\mathbb{R})$. Let $\hat{p}^{W_{bis}}(t)_x$ denote the second order scheme of Proposition 30, and $p_x^{ODE}(t)$ be the exact scheme for the ODE associated to $\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)$. Then, $\frac{1}{2}p^{ODE}(t) \circ \hat{p}_x^{W_{bis}}(t) + \frac{1}{2}\hat{p}^{W_{bis}}(t) \circ p_x^{ODE}(t)$ and $p^{ODE}(t/2) \circ \hat{p}^{W_{bis}}(t) \circ p_x^{ODE}(t/2)$ are potential second-order scheme for $AF F_d(x, \bar{\delta}, B_u, I_d^n)$.*

The scheme given by Lemma 28 and Corollary 31 is not well-defined when $\bar{\alpha} - da^T a \notin \mathcal{S}_d^+(\mathbb{R})$. When it is defined, it is however more efficient than the scheme given by Proposition 29 and Lemma 28. We have already mentioned in subsection 2.2.3 that the exact scheme has a cost of $O(d^4)$. Similarly, the schemes given by Propostions 26 and 29 have a cost of $O(d^4)$

operations. On the contrary, the scheme given by Proposition 30 only requires one Cholesky decomposition. Thus, the scheme given by Lemma 28 and Corollary 31 has a complexity in $O(d^3)$. We will illustrate this in the next section.

We conclude this section by mentioning that in the case of Wishart processes, we can show that the weak error is indeed of order 2 thanks to Theorem 21, Lemma 36 and Proposition 43.

Theorem 32 — *Let $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$. Let $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$ be the scheme starting from x which is sampled either by the scheme defined by Lemma 28 and Proposition 29 (with the second order scheme for the CIR given in [5]) or, if we have in addition $\alpha \geq d$, by the scheme given by Lemma 28 and Corollary 31. Then,*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_i^N}^N)] - \mathbb{E}[f(X_t^x)]| \leq C/N^2.$$

2.4 Numerical results on the simulation methods

The scope of this section is to compare the different simulation methods given in this paper. We still consider a time horizon T and the regular time-grid $t_i^N = iT/N$, for $i = 0, \dots, N$. In addition, we want to compare our schemes to a standard one, and we will consider the following corrected Euler-Maruyama scheme for $AF F_d(x, \bar{\alpha}, B, a)$, for every $0 \leq i \leq N - 1$:

$$\begin{aligned} \hat{X}_{t_{i+1}^N}^N &= \hat{X}_{t_i^N}^N + (\bar{\alpha} + B(\hat{X}_{t_i^N}^N)) \frac{T}{N} + \sqrt{(\hat{X}_{t_i^N}^N)^+} (W_{t_{i+1}^N} - W_{t_i^N}) a + a^T (W_{t_{i+1}^N} - W_{t_i^N})^T \sqrt{(\hat{X}_{t_i^N}^N)^+} \\ \hat{X}_{t_0^N}^N &= x. \end{aligned} \tag{2.41}$$

Here, x^+ denotes the matrix that has the same eigenvectors as x with the same eigenvalue if it is positive and a zero eigenvalue otherwise. Namely, we set $x^+ = \text{odiag}(\lambda_1^+, \dots, \lambda_d^+) o^T$ for $x = \text{odiag}(\lambda_1, \dots, \lambda_d) o^T$. Thus, x^+ is by construction a positive semidefinite matrix and its square root is well defined. Without this positive part, the scheme above is not well defined for any realization of W .

First, we compare the time required by the different schemes and the exact simulation. Then, we present numerical results on the convergence of the different schemes. Last, we give an application of our scheme to the Gourieroux-Sufana model in finance.

2.4.1 Time comparison between the different algorithms

In this paragraph, we compare the time required by the different schemes given in this paper. As it has already been mentioned, the complexity of the exact scheme as well as the one of the second order scheme (given by Lemma 28 and Proposition 29) and the third order scheme (given by Proposition 26) is in $O(d^4)$ for one time-step. To be more precise, they require $O(d^4)$ operations that mainly corresponds to d Cholesky decompositions, $O(d^2)$ generations of Gaussian (or moment-matching) variables and $O(d)$ generations of noncentral chi-square distributions (or second or third order schemes for the CIR). The time saved by the second and third order schemes with respect to the exact scheme only comes from the generation of random

variables. For example, the generation of the moment-matching variables (2.32) and (2.33) is 2.5 faster than the generation of $\mathcal{N}(0, 1)$ on our computer. The gain between the second or third order schemes for the CIR given in Alfonsi [5] and the exact sampling of the CIR given by Glasserman [43] is much greater, but it depends on the parameters of the CIR. When the dimension d gets larger, the absolute gain in time between the discretization schemes and the exact scheme is of course increased. However, the relative gain instead decreases to 1, because more and more time is devoted to matrix operations and Cholesky decompositions that are the same in both cases. Let us now quickly analyse the complexity of the other schemes. The second order scheme given by Lemma 28 and Corollary 31 (called “second order bis” later) has a complexity in $O(d^3)$ operations for one Cholesky decomposition and matrix multiplications, with $O(d^2)$ generations of Gaussian variables. The complexity of the corrected Euler scheme is of the same kind. At each time-step, $O(d^3)$ operations are needed for matrix multiplications and for diagonalizing the matrix in order to compute the square-root of its positive part. However, diagonalizing a symmetric matrix is in practice much longer than computing a Cholesky decomposition even though both algorithms are in $O(d^3)$. Also, one has to sample $O(d^2)$ Gaussian variables for the Brownian increments. In Table 2.4.1, we have calculated by a Monte-Carlo method one value of the characteristic function of a Wishart process. It is also known analytically thanks to (2.9), and we have indicated in each case the exact value. We have considered dimensions $d = 3$ and $d = 10$. We have given in each case an example where $\alpha \geq d$ and another one where $d - 1 \leq \alpha < d$. We have used the different algorithms presented in this paper: “2nd order bis” stands for the scheme given by Lemma 28 and Corollary 31 (with the moment-matching variables (2.32)), “2nd order” stands for the scheme given by Lemma 28 and Proposition 29 (with (2.32) and the second order scheme for the CIR given by [5]), “3rd order” stands for the scheme given by Proposition 26 (with (2.33) and the third order scheme for the CIR given by [5]), and “Corrected Euler” stands for the corrected Euler-Maruyama scheme presented in the beginning of this section. For the exact scheme, we have both considered the cases with one time-step T and N time-steps T/N . Of course, the first case is sufficient to calculate an expectation that only depends on X_T , but the second case allows to compute also pathwise expectations. For each method, we have given the value obtained and the time needed on our computer (3000 MHz CPU).

First, let us mention that the exact value is in each case in the confidence interval except for the corrected Euler scheme. As one can expect, the exact method with one time-step is by far the quickest method to compute an expectation that only depends on the final value. We put aside this case and focus now on the generation of the whole path. We see from Table 2.4.1 that the second and the third order schemes require roughly the same computation time. As expected, the second order scheme bis is much faster when it is defined (i.e. when $\alpha \geq d$). On the contrary, the Euler scheme is much slower than the second and third order scheme, even though it should be faster for large d . This is due to the cost of the matrix diagonalization. Let us mention that the time required by the discretization schemes is proportional to N and do not depend on the parameters when the dimension is given. On the contrary, the time needed by the exact scheme may change according to α and can increase considerably when α is close to $d - 1$. To be more precise, the exact simulation method for the CIR given by Glasserman [43] uses a rejection sampling when the degree of freedom is lower than 1, which corresponds to the case $d - 1 \leq \alpha < d$. The rejection rate can in fact be rather high, notably when the time-step gets smaller. For $N = 30$, $d = 3$ and $\alpha = 2.2$, the exact scheme is four times slower than the second order scheme and 2.5 slower than the exact scheme with $\alpha = 3.5$.

Schemes	$N = 10$			$N = 30$		
	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	-0.526852	-0.227962	12			
2^{nd} order bis	-0.526229	-0.228663	41	-0.526486	-0.229078	125
2^{nd} order	-0.526577	-0.228923	76	-0.526574	-0.228133	229
3^{rd} order	-0.527021	-0.227286	82	-0.527613	-0.228376	244
Exact (N steps)	-0.526963	-0.228303	123	-0.526891	-0.227729	369
Corrected Euler	-0.525627*	-0.233863*	225	-0.525638*	-0.231449*	687
$\alpha = 3.5, d = 3, \Delta_R = 1, \Delta_{Im} = 1$, exact value R. = -0.527090 and Im.= -0.228251						
Exact (1 step)	-0.591579	-0.037651	12			
2^{nd} order	-0.590444	-0.037024	77	-0.590808	-0.036487	229
3^{rd} order	-0.591234	-0.034847	82	-0.590818	-0.036210	246
Exact (N steps)	-0.591169	-0.036618	174	-0.592145	-0.037411	920
Corrected Euler	-0.589735*	-0.042002*	223	-0.590079*	-0.039937*	680
$\alpha = 2.2, d = 3, \Delta_R = 0.9, \Delta_{Im} = 1.3$, exact value R. = -0.591411 and Im.= -0.036346						
Exact (1 step)	0.062712	-0.063757	181			
2^{nd} order bis	0.064237	-0.063825	921	0.064573	-0.062747	2762
2^{nd} order	0.064922	-0.064103	1431	0.063534	-0.063280	4283
3^{rd} order	0.064620	-0.064543	1446	0.064120	-0.063122	4343
Exact (N steps)	0.063418	-0.064636	1806	0.063469	-0.064380	5408
Corrected Euler	0.068298*	-0.058491*	2312	0.061732*	-0.056882*	7113
$\alpha = 10.5, d = 10, \Delta_R = 1.4, \Delta_{Im} = 1.3$, exact value R. = 0.063960 and Im.= -0.063544						
Exact (1 step)	-0.036869	-0.094156	177			
2^{nd} order	-0.036246	-0.094196	1430	-0.035944	-0.092770	4285
3^{rd} order	-0.035408	-0.093479	1441	-0.036277	-0.093178	4327
Exact (N steps)	-0.036478	-0.092860	1866	-0.036145	-0.093003	6385
Corrected Euler	-0.028685*	-0.094281*	2321	-0.030118*	-0.088988*	7144
$\alpha = 9.2, d = 10, \Delta_R = 1.4, \Delta_{Im} = 1.4$, exact value R. = -0.036064 and Im.= -0.093275						

Table 2.1: $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t^N}^N))]$ calculated by a Monte-Carlo with 10^6 samples for a Wishart process with $a = I_d, b = 0, x = 10I_d, v = 0.09I_d$ and $T = 1$. The starred numbers are those for which the exact value is outside the 95% confidence interval, and Δ_R (resp. Δ_I) gives the two standard deviations value on the real (resp. imaginary) part in 10^{-3} .

Let us draw a conclusion from this time comparison between the different schemes. Obviously, we recommend to use the exact scheme when calculating expectations that depend on one or few dates. Instead, when calculating pathwise expectations of affine processes by Monte-Carlo, we would recommend to use in general the second order bis scheme when $\alpha \geq d$ and the second order (or third order for Wishart processes) when $d - 1 \leq \alpha < d$.

2.4.2 Numerical results on the convergence

Now, we want to illustrate the theoretical results of convergence obtained in this paper for the different schemes. To do so, we have plotted for each scheme $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t_N}^N))]$ in function of the time step T/N . This expectation is calculated by a Monte-Carlo method. As for the time comparison, we illustrate the convergence for $d = 3$ in Figure 2.1 and $d = 10$ in Figure 2.2. Each time, we consider a case where $\alpha \geq d$ and a case where $d - 1 \leq \alpha < d$, which is in general tougher. In these figures,

- scheme 1 denotes the value obtained by the exact scheme with one time-step,
- scheme 2 stands for the second order scheme given by Lemma 28 and Proposition 29,
- scheme 3 denotes the third order scheme given by Theorem 27,
- scheme 4 is the corrected Euler scheme (2.41).

Here, we have not plotted the convergence of the second order (bis) scheme given by Lemma 28 and Corollary 31 because it would have given almost the same convergence as the other second order scheme.

N	2	4	8	10	16	30
Figure 2.1, right	-0.000698	0.000394	0.033193	0.111991	0.185128	0.210201
Figure 2.2, right	0.494752	-0.464121	0.657041	0.643042	0.637585	0.619553

Table 2.2: Values obtained by the Euler scheme in the numerical experiments of Figures 2.1 and 2.2.

As expected, we observe in both Figures 2.1 and 2.2 convergences that fit our theoretical results. Namely, Scheme 2 converges in $O(1/N^2)$ and Scheme 3 converges faster in $O(1/N^3)$. In some cases such as Figure 2.2, Scheme 3 already matches the exact value from $N = 2$. Even though it seems to converge at a $O(1/N)$ speed, the corrected Euler scheme is clearly not competitive with respect to the other schemes. In the tough case $d - 1 \leq \alpha \leq d$, the values obtained by the Euler scheme are in fact outside the figures, and we have put the corresponding values in Table 2.2.

We want to conclude this section by testing numerically the convergence of our schemes when we calculate pathwise expectations. Of course, our theoretical results only bring on the weak error, but we may hope that our schemes converge also quickly when considering more intricate expectations. In Figure 2.4.2, we approximate $\mathbb{E}[\max_{0 \leq t \leq T} \text{Tr}(X_t^x)]$ with the different schemes by computing the maximum on the time-grid. The convergence seems to be roughly in $O(1/\sqrt{N})$ for all the schemes (see Figure 2.4.2, left), including the exact scheme. However, the main error seems to come from the approximation of $\max_{0 \leq t \leq T} \text{Tr}(X_t^x)$ by $\max_{0 \leq k \leq N} \text{Tr}(X_{t_k}^x)$.

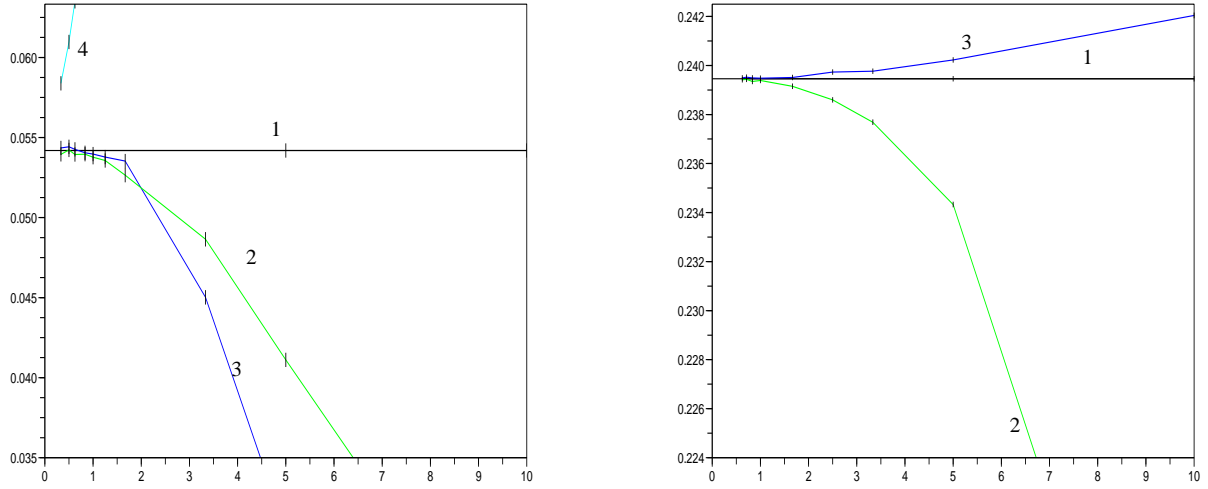


Figure 2.1: $d = 3$, 10^7 Monte-Carlo samples, $T = 10$. The real value of $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))]$ in function of the time-step T/N . Left: $v = 0.05I_d$ and Wishart parameters $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Exact value: 0.054277. Right: $v = 0.2I_d + 0.04q$ and Wishart parameters $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Exact value: 0.239836. Here, q is the matrix defined by: $q_{i,j} = \mathbb{1}_{i \neq j}$. The width of each point represents the 95% confidence interval.

In fact, we have plotted in Figure 2.4.2 (right) the difference between $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k}^N)]$ and $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(X_{t_k}^x)]$. Then, we find convergences that are very similar to those obtained for the weak error: schemes 2 and 3 converge at a speed which is respectively compatible with $O(1/N^2)$ and $O(1/N^3)$. Scheme 4 seems also to give a $O(1/N)$ convergence. It would be hasty to draw a global conclusion from this simple example. Nonetheless, the convergence of schemes 2 and 3 is really encouraging on pathwise expectations, if we put aside the problem of approximating a function of $(X_t^x, 0 \leq t \leq T)$ by a function of $(X_{t_k}^x, 0 \leq k \leq N)$.

2.4.3 An application in finance to the Gouriéroux and Sufana model

In this paragraph, we want to give a possible application of our schemes in finance. More precisely, we will consider the model introduced by Gouriéroux and Sufana [47]. This is a model for d risky assets S_t^1, \dots, S_t^d . Let $(B_t, t \geq 0)$ denote a standard Brownian motion on \mathbb{R}^d that is independent from $(W_t, t \geq 0)$. Then, we consider the following dynamics for the assets:

$$t \geq 0, 1 \leq l \leq d, S_t^l = S_0^l + r \int_0^t S_u^l du + \int_0^t S_u^l (\sqrt{X_u} dB_u)_l, \quad (2.42)$$

where $X_t = X_0 + \int_0^t (\alpha a^T a + b X_u + X_u b^T) du + \int_0^t (\sqrt{X_u} dW_u a + a^T dW_u^T \sqrt{X_u})$ is a Wishart process. Here, $(\sqrt{X_u} dB_u)_l$ is simply the l^{th} coordinates of the vector $\sqrt{X_u} dB_u$. We can easily check that the instantaneous quadratic covariation matrix between the log-prices of the assets is X_t . Last, r denotes the instantaneous interest rate.

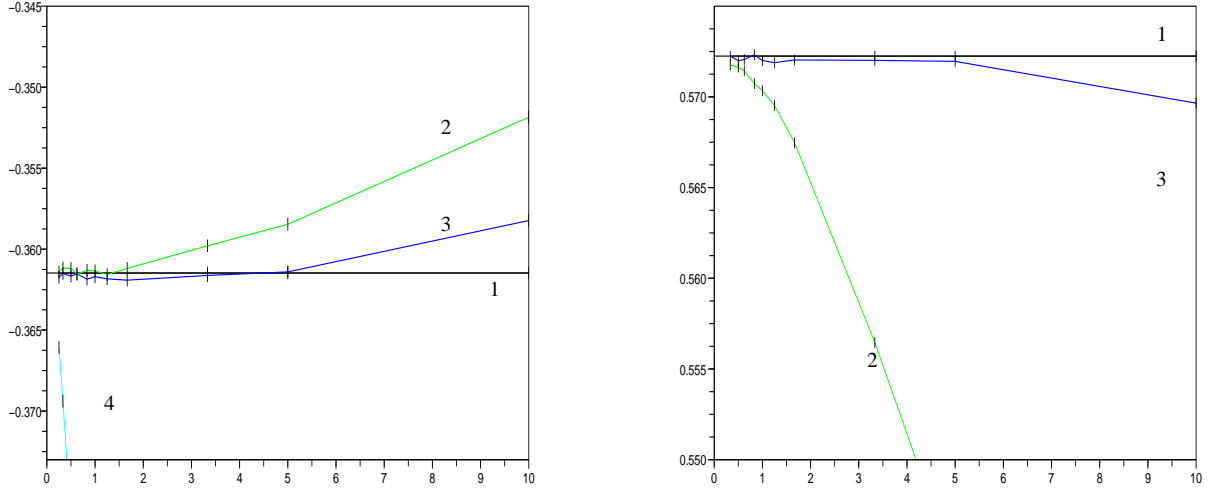


Figure 2.2: $d = 10$, 10^7 Monte-Carlo samples, $T = 10$. Left: imaginary value of $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))]$ with $v = 0.009I_d$ in function of the time-step T/N . Wishart parameters: $x = 0.4I_d$, $\alpha = 12.5$, $b = 0$ and $a = I_d$. Exact value: -0.361586 . Right: real value of $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))]$ with $v = 0.009I_d$ in function of T/N . Wishart parameters: $x = 0.4I_d$, $\alpha = 9.2$, $b = -0.5I_d$ and $a = I_d$. Exact value 0.572241 . The width of each point represents the 95% confidence interval.

To simulate both assets and the Wishart matrix, we proceed as follows. We observe that the generator of (S_t, X_t) can be written as

$$L = L^S + L^X, \text{ where } L^S = \sum_{i=1}^d r s_i \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^d s_i s_j x_{i,j} \partial_{s_i} \partial_{s_j},$$

and L^X is the generator of the Wishart process $WIS_d(x, \alpha, b, a)$. The operator L^S is associated to the SDE $dS_t^l = r S_t^l + S_t^l (\sqrt{x} dB_t)_l$ that can be solved explicitly. We have indeed $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (\sqrt{x} B_t)_l]$. Let us also remark that $\sqrt{x} B_t \stackrel{\text{Law}}{=} c B_t$ if we have $cc^T = x$: both are centred Gaussian vectors with the same covariance matrix. In practice, it is more efficient to use $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (c B_t)_l]$ where c is computed with an extended Cholesky decomposition of x rather than calculating \sqrt{x} , which requires a diagonalization. Let us then denote by $p_{(s,x)}^S(t)$ the exact scheme for L^S (x is unchanged) and $\hat{p}_{(s,x)}^X(t)$ a second order scheme for the Wishart process $WIS_d(x, \alpha, b, a)$ (s is unchanged). Then, we consider the following scheme

$$\frac{1}{2}(\hat{p}^X(t) \circ \hat{p}_{(s,x)}^S(t) + \hat{p}^S(t) \circ \hat{p}_{(s,x)}^X(t)),$$

i.e. we draw a Bernoulli variable of parameter $1/2$ to decide whether we use first $\hat{p}_{(s,x)}^X(t)$ or $\hat{p}_{(s,x)}^S(t)$. Under some conditions that we do not check here, this construction is known from Proposition 22 to preserve the second-order convergence. To be consistent with Subsection 2.4.2, this scheme is denoted by scheme 2 later in this paragraph. To compare this scheme

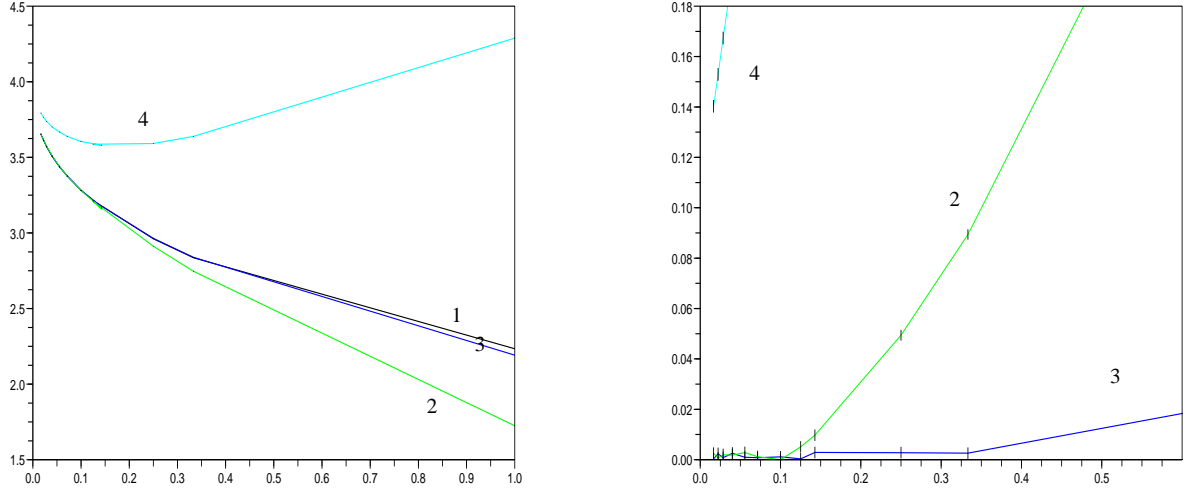


Figure 2.3: $d = 3$, 10^7 Monte-Carlo samples, $T = 1$. Wishart parameters $x = 0.4I_d + 0.2q$ with $q_{i,j} = \mathbb{1}_{i \neq j}$, $\alpha = 2.2$, $b = 0$ and $a = I_d$. Left, $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k^N}^N)]$, right: $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k^N}^N)] - \mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(X_{t_k^N}^x)]$ in function of T/N . The width of each point gives the precision up to two standard deviations.

with a more basic one, we consider the Euler-Maruyama scheme defined by (2.41) and

$$\hat{S}_{t_0^N}^{l,N} = S_0^l, \hat{S}_{t_{i+1}^N}^{l,N} = \hat{S}_{t_i^N}^{l,N} \left(1 + rT/N + (\sqrt{\hat{X}_{t_i^N}^N} (B_{t_{i+1}^N} - B_{t_i^N}))_l \right), 0 \leq i \leq N-1.$$

It is denoted by scheme 4 like in Subsection 2.4.2.

We have plotted in Figure 2.4.3 the price of a put option on the maximum of two risky assets ($d = 2$). The Gourieroux and Sufana model is an affine model, and the characteristic function of S_t is explicitly known (see [47]). Thus, it is possible to adapt the method proposed by Carr and Madan [21] and to calculate by numerical integration (which is possible for small dimensions) the value of this put option. We have given in Figure 2.4.3 the exact value obtained by this method. As one might have guessed, we observe a quadratic convergence for scheme 2 and a linear convergence for scheme 4. The benefit of using scheme 2 is clear since it already fits with the exact value from $N = 5$ in both cases: its convergence is really satisfactory.

2.5 Conclusion and prospects

Let us draw a brief summary of this paper. Thanks to a remarkable splitting of the infinitesimal generator of Wishart processes, we have been able to sample exactly any Wishart distribution. We have also proposed a third order scheme for Wishart processes and a second order scheme for general affine diffusions. We have confirmed these rates of convergence with numerical tests and analysed the time complexity of each method. It comes out that we recommend to use the exact scheme to compute expectations that depend on one (or few) times. To calculate pathwise expectations, we instead recommend generally to use discretization schemes. More

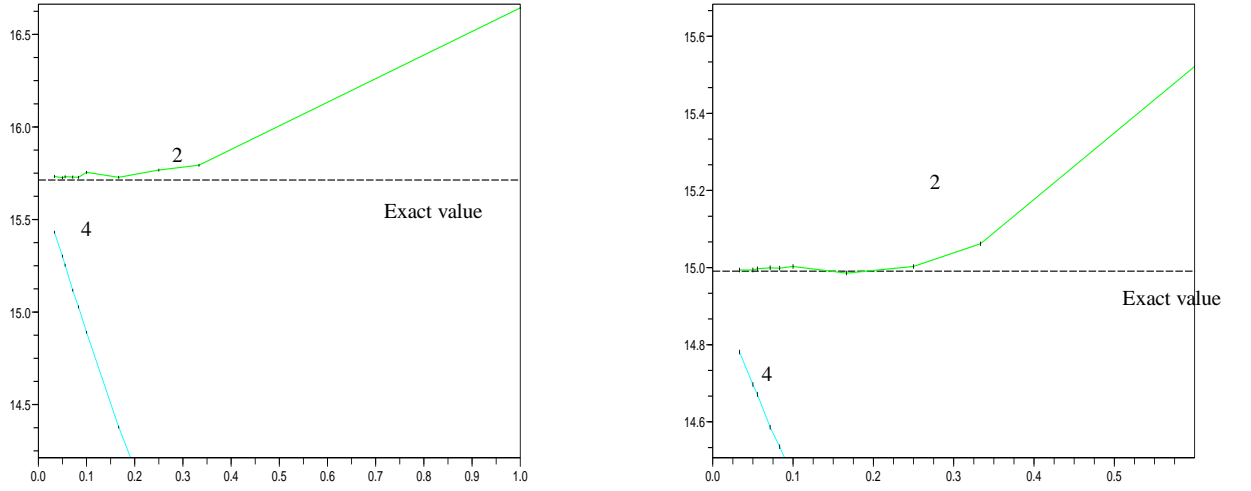


Figure 2.4: $\mathbb{E}[e^{-rT}(K - \max(\hat{S}_{t_N}^{1,N}, \hat{S}_{t_N}^{2,N}))^+]$ in function of T/N . $d = 2$, $T = 1$, $K = 120$, $S_0^1 = S_0^2 = 100$, and $r = 0.02$. Wishart parameters: $x = 0.04I_d + 0.02q$ with $q_{i,j} = \mathbb{1}_{i \neq j}$, $a = 0.2I_d$, $b = 0.5I_d$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). The width of each point gives the precision up to two standard deviations (10^6 Monte-Carlo samples).

precisely, the second order scheme given by Lemma 28 and Corollary 31 has to be preferred when $\alpha \geq d$. Otherwise, we recommend to use the third order scheme given by Theorem 27 for Wishart processes or the second order scheme given by Lemma 28 and Proposition 29 for general affine diffusions.

Let us give now some prospects of this work. As a possible continuation of this paper, it is natural to wonder if it is possible to extend our schemes to affine diffusions on positive semidefinite matrices that include jumps (see Cuchiero et al.[26]). From a modelling point of view, we believe that Wishart processes could be used in a wide range of applications. In fact, they can be used as soon as one has to model dependence dynamics. Thus, we hope that the possibility of sampling such processes will stimulate different kinds of dependence models.

2.6 Appendix

2.6.1 Notations and some results on matrices

Notations for real matrices :

- For $d, d' \in \mathbb{N}^*$, $\mathcal{M}_d(\mathbb{R})$ denotes the real d square matrices and $\mathcal{M}_{d \times d'}(\mathbb{R})$ the real matrices with d rows and d' columns.
- $\mathcal{S}_d(\mathbb{R})$, $\mathcal{S}_d^+(\mathbb{R})$, $\mathcal{S}_d^{+,*}(\mathbb{R})$, and $\mathcal{G}_d(\mathbb{R})$ denote respectively the set of symmetric, symmetric positive semidefinite, symmetric positive definite and non singular matrices.
- For $x \in \mathcal{M}_d(\mathbb{R})$, x^T , $\text{adj}(x)$, $\det(x)$, $\text{Tr}(x)$ and $\text{Rk}(x)$ are respectively the transpose, the adjugate, the determinant, the trace and the rank of x .
- For $x \in \mathcal{S}_d^+(\mathbb{R})$, \sqrt{x} denotes the unique symmetric positive semidefinite matrix such that $(\sqrt{x})^2 = x$
- The identity matrix is denoted by I_d and we set for $n \leq d$, $I_d^n = (\mathbb{1}_{i=j \leq n})_{1 \leq i, j \leq d}$ and $e_d^n = (\mathbb{1}_{i=j=n})_{1 \leq i, j \leq d}$, so that $I_d^n = \sum_{i=1}^n e_d^i$. We also set for $1 \leq i, j \leq d$, $e_d^{i,j} = (\mathbb{1}_{k=i, l=j})_{1 \leq k, l \leq d}$.
- For $x \in \mathcal{S}_d(\mathbb{R})$, we denote by $x_{\{i,j\}}$ the value of $x_{i,j}$, so that $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})$. We use both notations in the paper: notation $(x_{i,j})_{1 \leq i, j \leq d}$ is of course more convenient for matrix calculations while $(x_{\{i,j\}})_{1 \leq i \leq j \leq d}$ is preferred to emphasize that we work on symmetric matrices and that we have $x_{i,j} = x_{j,i}$.
- For $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, $\text{diag}(\lambda_1, \dots, \lambda_d)$ denotes the diagonal matrix such that for all $1 \leq i \leq d$, $\text{diag}(\lambda_1, \dots, \lambda_d)_{i,i} = \lambda_i$.

Now, we present different results that are used in the paper. We first recall the extended Cholesky decomposition of positive semidefinite matrices which is used intensively and then give two other technical results.

Lemma 33 — *Let $q \in \mathcal{S}_d^+(\mathbb{R})$ be a matrix with rank r . Then there is a permutation matrix p , an invertible lower triangular matrix $c_r \in \mathcal{G}_r(\mathbb{R})$ and $k_r \in \mathcal{M}_{d-r \times r}(\mathbb{R})$ such that:*

$$pqp^T = cc^T, \quad c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.$$

The triplet (c_r, k_r, p) is called an extended Cholesky decomposition of q .

Besides, $\tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$, and we have:

$$q = (\tilde{c}^T p)^T I_d^r \tilde{c}^T p.$$

The proof of this result and a numerical procedure to get such a decomposition can be found in Golub and Van Loan ([45], Algorithm 4.2.4). When $r = d$, we can take $p = I_d$, and c_r is the usual Cholesky decomposition.

Lemma 34 — Let $b, c \in \mathcal{S}_d(\mathbb{R})$. If either $b \in \mathcal{S}_d^+(\mathbb{R})$ or $c \in \mathcal{S}_d^+(\mathbb{R})$, then $I_d + ibc$ is invertible. In particular, if $b \in \mathcal{S}_d^{+,*}(\mathbb{R})$, $b + ic$ is invertible.

Proof : We start with the first assertion. Since $(I_d + ibc)^T = I_d + icb$, it is sufficient to check the case where $c \in \mathcal{S}_d^+(\mathbb{R})$. By a way of contradiction, let us assume that there is $x \in \mathbb{C}^d \setminus \{0\}$ such that $x + ibcx = 0$. We respectively denote by $x_R \in \mathbb{R}^d$ and $x_I \in \mathbb{R}^d$ the real and imaginary part of x . One gets easily that $x_R = bcx_I$ and $x_I = -bcx_R$. Since $x \neq 0$, we have necessarily $x_R \neq 0$, $cx_R \neq 0$, $bcx_R \neq 0$ and $cbcx_R \neq 0$. Since c is nonnegative, we get by decomposing on an orthonormal basis that $cx_R \cdot x_R > 0$ and $cbcx_R \cdot bcx_R > 0$. However, we also have $cx_R \cdot x_R = -cx_R \cdot bcbcx_R$, which leads to a contradiction. The second assertion is now obvious since $b + ic = b(I_d + ib^{-1}c)$. \square

2.6.2 Proofs of Section 2.1

Proof of Lemma 2

Proof : From the SDE (2.5), we have

$$\begin{aligned} (dY_t)_{i,j} &= (C_t)_{i,j}dt + (B_t dW_t A_t + A_t^T dW_t^T B_t^T)_{i,j} \\ &= (C_t)_{i,j}dt + \sum_{k,l=1}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})(dW_t)_{k,l} \end{aligned}$$

Then we get the following formula for the quadratic covariation

$$\begin{aligned} \frac{d\langle (Y_t)_{i,j}, (Y_t)_{m,n} \rangle}{dt} &= \sum_{k,l}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})((B_t)_{m,k}(A_t)_{l,n} + (B_t)_{n,k}(A_t)_{l,m}) \\ &= \sum_{l,k}^d (B_t)_{i,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,j} + \sum_{l,k}^d (B_t)_{i,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,j} \\ &\quad + \sum_{l,k}^d (B_t)_{j,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,i} + \sum_{l,k}^d (B_t)_{j,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,i} \\ &= (B_t B_t^T)_{i,m}(A_t^T A_t)_{n,j} + (B_t B_t^T)_{i,n}(A_t^T A_t)_{m,j} \\ &\quad + (B_t B_t^T)_{j,m}(A_t^T A_t)_{n,i} + (B_t B_t^T)_{j,n}(A_t^T A_t)_{m,i}. \end{aligned}$$

\square

Proof of Proposition 5

Proof : Let $v \in \mathcal{S}_d(\mathbb{R})$ such that $\forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})$. As it is usual for affine diffusions, the Laplace transform can be formulated with ODE solutions. Namely, we will show that $\mathbb{E}[\exp(\text{Tr}(vX_t^x))] = \exp[\phi(t, v) + \text{Tr}(\psi(t, v)x)]$, where ψ and ϕ solve following ODEs (see for example Cuchiero et al. [26]):

$$\begin{aligned} \partial_t \psi(t, v) &= \psi(t, v)b + b^T \psi(t, v) + 2\psi(t, v)a^T a \psi(t, v) & ; & \psi(0, v) = v \\ \partial_t \phi(t, v) &= \alpha \text{Tr}(\psi(t, v)) & ; & \phi(0, v) = 0. \end{aligned}$$

The function ψ solves an usual matrix Riccati ODE. As shown by Levin [62], ψ can be obtained explicitly by the mean of the following exponential matrix :

$$\begin{aligned} \begin{pmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{pmatrix} &:= \exp \left(t \begin{pmatrix} b & -2a^T a \\ 0 & -b^T \end{pmatrix} \right) \\ &= \begin{pmatrix} \exp(tb) & -2 \exp(tb) \int_0^t \exp(-sb) a^T a \exp(-sb^T) ds \\ 0 & \exp(-tb^T) \end{pmatrix}. \end{aligned}$$

From [62], we obtain that

$$\begin{aligned}
\psi(t, v) &= (\psi(0, v)A_{1,2}(t) + A_{2,2}(t))^{-1}(\psi(0, v)A_{1,1}(t) + A_{2,1}(t)) \\
&= \left(-2v \exp(tb) \int_0^t \exp(-sb) a^T a \exp(-sb^T) ds + \exp(-tb^T) \right)^{-1} v \exp(tb) \\
&= \left(-2v \int_0^t \exp(sb) a^T a \exp(sb^T) ds \exp(-tb^T) + \exp(-tb^T) \right)^{-1} v \exp(tb) \\
&= \exp(tb^T) (I_d - 2vq_t)^{-1} v \exp(tb),
\end{aligned}$$

provided that $I_d - 2q_s v$ is invertible for $s \in [0, t]$, which holds by assumption. Therefore we get for $x \in \mathcal{S}_d(\mathbb{R})$,

$$\begin{aligned}
\text{Tr}(\psi(t, v)x) &= \text{Tr} \left(\exp(tb^T) (I_d - 2vq_t)^{-1} v \exp(tb)x \right) \\
&= \text{Tr} \left((I_d - 2vq_t)^{-1} v \exp(tb)x \exp(tb^T) \right) \\
&= \text{Tr} \left(v (I_d - 2q_t v)^{-1} \exp(tb)x \exp(tb^T) \right),
\end{aligned}$$

since $v(I_d - 2q_t v)^{-1} = (I_d - 2vq_t)^{-1} v$. As explained by Grasselli and Tebaldi ([49], section 4.2), ϕ can also be calculated explicitly by the mean of the exponential matrix above, and we get:

$$\phi(t, v) = -\frac{\alpha}{2} \text{Tr} \left(\log[(I_d - 2vq_t) \exp(tb^T)] - t \text{Tr}(b) \right).$$

By using that $\exp(\text{Tr}(\log(A))) = \det(A)$ for $A \in \mathcal{G}_d(\mathbb{R})$, we deduce then that

$$\begin{aligned}
\exp(\phi(t, v)) &= \exp\left(\frac{\alpha}{2} t \text{Tr}(b)\right) \left(\exp(\text{Tr}(\log\{(I_d - 2vq_t) \exp(tb^T)\})) \right)^{\frac{-\alpha}{2}} \\
&= \exp\left(\frac{\alpha}{2} t \text{Tr}(b)\right) \left(\det\{(I_d - 2vq_t)\} \det\{\exp(tb^T)\} \right)^{\frac{-\alpha}{2}} \\
&= \exp\left(\frac{\alpha}{2} t \text{Tr}(b) - \frac{\alpha}{2} t \text{Tr}(b)\right) \left(\det\{(I_d - 2vq_t)\} \right)^{\frac{-\alpha}{2}} \\
&= \frac{1}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}.
\end{aligned}$$

For the last equality, we have used that $\det(I_d - 2vq_t) = \det((I_d - 2vq_t)^T) = \det(I_d - 2q_t v)$.

Now, it remains to show that (2.9) indeed holds. By Itô calculus, we get that for $s \in (0, t)$:

$$\begin{aligned}
&d \exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v)X_s^x)] \\
&= \exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v)X_s^x)] \text{Tr}[\psi(t-s, v)(\sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x})].
\end{aligned} \tag{2.43}$$

Thus, $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v)X_s^x)]$ is a positive local martingale and therefore a supermartingale, which gives that $\mathbb{E}[\exp(\text{Tr}(vX_t^x))] \leq \exp[\phi(t, v) + \text{Tr}(\psi(t, v)x)] < \infty$, i.e. $\mathcal{D}_{b,a;t} \subset \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$, where $\mathcal{D}_{b,a;t} := \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}$ and $\tilde{\mathcal{D}}_{x,\alpha,b,a;t} := \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\text{Tr}(vX_t^x))] < \infty\}$.

On the other hand, when $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$, we can check that $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v)X_s^x)] \leq 1$ by observing that $\det(I_d - 2q_t v) = \det(I_d + 2\sqrt{-v}q_t\sqrt{-v}) \geq 1$ and

$$\begin{aligned}
&\text{Tr} \left(v (I_d - 2q_t v)^{-1} \exp(tb)x \exp(tb^T) \right) \\
&= -\text{Tr} \left(\sqrt{-v} (I_d + 2\sqrt{-v}q_t\sqrt{-v})^{-1} \sqrt{-v} \exp(tb)x \exp(tb^T) \right) \leq 0.
\end{aligned}$$

In that case, $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v)X_s^x)]$ is a martingale from (2.43), and (2.9) holds.

Let us observe now that $\mathcal{D}_{b,a;t}$ is convex. In fact, we have $\det(I_d - 2q_s v) = \det(I_d - 2\sqrt{q_s}v\sqrt{q_s})$, and therefore $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2\sqrt{q_s}v\sqrt{q_s} \in \mathcal{S}_d^{+,*}(\mathbb{R})\}$ which

is obviously convex. The Laplace transform $v \mapsto \mathbb{E}[\exp(\text{Tr}(vX_t^x))]$ is an analytic function on $\mathcal{D}_{b,a;t}$ (see for example Lemma 10.8 in [40]). The RHS of (2.9) is also analytic on $\mathcal{D}_{b,a;t}$ and coincides with the Laplace transform when $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Therefore, (2.9) holds for $v \in \mathcal{D}_{b,a;t}$ since $\mathcal{D}_{b,a;t}$ is convex. Now, we can extend to complex values of v . Indeed, the RHS of (2.9) is well defined for $v = v_R + iv_I$ with $v_R \in \mathcal{D}_{b,a;t}$, thanks to Lemma 34. Since both hand sides are analytic functions of v , (2.9) holds for $v = v_R + iv_I$.

Last, we want to show that $\mathcal{D}_{b,a;t} = \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$. We first consider the case $b = 0$ and assume by a way of contradiction that there is $v \in \tilde{\mathcal{D}}_{x,\alpha,0,a;t} \setminus \mathcal{D}_{0,a;t}$ for some x, α, a and $t > 0$. Let $\tilde{t} = \min\{s \in [0, t], I_d - 2q_s v \notin \mathcal{G}_d(\mathbb{R})\} \in (0, t]$. On the one hand, we have $v \notin \mathcal{D}_{0,a;\tilde{t}}$ and $v \in \mathcal{D}_{0,a;s}$ for $s \in [0, \tilde{t})$. On the other hand, we have by Jensen's inequality:

$$s \in [0, t], \exp((t-s)\text{Tr}(va^T a)) \exp(\text{Tr}(vX_s^x)) \leq \mathbb{E}[\exp(\text{Tr}(vX_t^x)) | \mathcal{F}_s],$$

which gives $s \in [0, t] \mapsto \exp(-s\text{Tr}(va^T a)) \mathbb{E}[\exp(\text{Tr}(vX_s^x))]$ is nondecreasing and finite. Since (2.9) holds for $s < \tilde{t}$, we get that $\mathbb{E}[\exp(\text{Tr}(vX_{\tilde{t}}^x))] = +\infty$, which leads to a contradiction. Let us now consider the case $b \neq 0$. From Proposition 9 (which is a consequence of the characteristic function obtained above), we have

$$v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \iff \theta_t^T v \theta_t \in \mathcal{D}_{0,I_d^{\tilde{t}}} \iff \forall s \in [0, t], \det(I_d - 2(s/t)q_t v) \neq 0.$$

In particular, $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ is an open set. For $v \in \mathcal{G}_d(\mathbb{R})$, we have $\det(I_d - 2(s/t)q_t v) \neq 0 \iff \det(v^{-1} - 2(s/t)q_t) \neq 0$ (resp. $\det(I_d - 2q_s v) \neq 0 \iff \det(v^{-1} - 2q_s) \neq 0$). Since $sq_t \leq s'q_t$ (resp. $q_s \leq q_{s'}$) for $s \leq s'$, we know from Theorem 8.1.5 in [45] that the (real) eigenvalues of $v^{-1} - 2(s/t)q_t$ (resp. $v^{-1} - 2q_s$) are nonincreasing w.r.t. s . Since they are also continuous, and $v^{-1} - 2(s/t)q_t = v^{-1} - 2q_s$ for $s \in \{0, t\}$, we get that $\forall s \in [0, t], \det(v^{-1} - 2(s/t)q_t) \neq 0 \iff \forall s \in [0, t], \det(v^{-1} - 2q_s) \neq 0$, and thus $\tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R}) = \mathcal{D}_{b,a;t} \cap \mathcal{G}_d(\mathbb{R})$. Let $v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$. Since $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ is an open set, there is $\varepsilon > 0$ such that $v \pm \varepsilon I_d \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R})$. Since $\mathcal{D}_{b,a;t}$ is convex, $v = (v + \varepsilon I_d + v - \varepsilon I_d)/2 \in \mathcal{D}_{b,a;t}$. \square

2.6.3 Proofs of Section 2.2

Proof of Theorem 10

Proof: From (2.7), we get:

$$L_i = \alpha \partial_{\{i,i\}} + 2x_{\{i,i\}} \partial_{\{i,i\}}^2 + 2 \sum_{\substack{1 \leq m \leq d \\ m \neq i}} x_{\{i,m\}} \partial_{\{i,m\}} \partial_{\{i,i\}} + \frac{1}{2} \sum_{\substack{1 \leq m, l \leq d \\ m \neq i, l \neq i}} x_{\{m,l\}} \partial_{\{i,m\}} \partial_{\{i,l\}} \quad (2.44)$$

We want to show that $L_i L_j = L_j L_i$ for $i \neq j$. Up to a permutation of the coordinates, L_i and L_j are the same operators as L_1 and L_2 . It is therefore sufficient to check that $L_1 L_2 = L_2 L_1$. By a straightforward but tedious calculation, we get

$$L_1 L_2 =$$

$$\begin{aligned}
& \underbrace{\alpha^2 \partial_{\{1,1\}} \partial_{\{2,2\}}}_{(0)} + \underbrace{2\alpha x_{\{2,2\}} \partial_{\{1,1\}} \partial_{\{2,2\}}^2}_{(1)} + \underbrace{2\alpha \sum_{j \neq 2} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(2)} + \\
& \frac{\alpha}{2} \underbrace{(\partial_{\{1,2\}}^2)}_{(3)} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(4)} + \underbrace{2\alpha x_{\{1,1\}} \partial_{\{1,1\}}^2 \partial_{\{2,2\}}}_{(1)} + \underbrace{4x_{\{1,1\}} x_{\{2,2\}} \partial_{\{1,1\}}^2 \partial_{\{2,2\}}^2}_{(5)} + \\
& 4 \underbrace{\sum_{j \neq 2} x_{\{1,1\}} x_{\{2,j\}} \partial_{\{1,1\}}^2 \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(6)} + x_{\{1,1\}} \underbrace{(2\partial_{\{1,1\}} \partial_{\{1,2\}}^2)}_{(7)} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial_{\{1,1\}}^2 \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(8)} + \\
& 2\alpha \underbrace{\sum_{m \neq 1} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(2)} + 4 \underbrace{\sum_{m \neq 1} x_{\{1,m\}} x_{\{2,2\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,2\}}^2}_{(6)} + \\
& 4 \left(\underbrace{\sum_{m \neq 1, j \neq 2} x_{\{1,m\}} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(9)} + \underbrace{x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(10)} \right) \\
& + \underbrace{\sum_{m \neq 1, k \neq 2, j \neq 2} x_{\{1,m\}} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(11)} + \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial_{\{1,2\}}^2 \partial_{\{1,m\}}}_{(12)} \\
& + 2 \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,m\}}}_{(13)} + \underbrace{x_{\{1,2\}} \partial_{\{1,2\}}^3 + 2x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(14)} \\
& \frac{\alpha}{2} \underbrace{\sum_{m \neq 1, l \neq 1} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}}}_{(4)} + \underbrace{\sum_{m \neq 1, l \neq 1} x_{\{2,2\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}}^2}_{(8)} + \\
& \underbrace{\sum_{m \neq 1, l \neq 1, j \neq 2} x_{\{2,j\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(11)} + 2 \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{2,m\}} \partial_{\{1,2\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(13)} \\
& + 2x_{\{2,2\}} \partial_{\{1,2\}}^2 \partial_{\{2,2\}} + \frac{1}{4} \underbrace{\sum_{\substack{m \neq 1, l \neq 1 \\ j \neq 2, k \neq 2}} x_{\{m,l\}} x_{\{j,k\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,k\}} \partial_{\{2,j\}}}_{(15)} \\
& + \underbrace{\sum_{\substack{m \neq 1, l \neq 1 \\ m \neq 2, l \neq 2}} x_{\{m,l\}} \partial_{\{1,l\}} \partial_{\{1,2\}} \partial_{\{2,m\}}}_{(16)} + \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{2,m\}} \partial_{\{1,2\}}^2 \partial_{\{2,m\}}}_{(12)}.
\end{aligned}$$

Obviously, we have the same formula for $L_2 L_1$ simply by exchanging the index 1 and 2. It is then sufficient to check that the above formula remains unchanged when we exchange the two index. Each term above is marked with a number. If this number is in the form (\bar{r}) ,

then the associated term is symmetric. Otherwise, there exist in the formula its corresponding symmetric term which is marked with the same number. \square

Proof of Proposition 11

Proof: Let $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$. We will check that for any polynomial function f of the matrix elements, we have $\mathbb{E}[f(X_t^x)] = \mathbb{E}[f(X_t^{n, \dots, X_t^{1,x}})]$. Let us consider a polynomial function f of degree m :

$$x \in \mathcal{S}_d(\mathbb{R}), f(x) = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} a_\gamma \bar{x}^\gamma,$$

where $|\gamma| = \sum_{1 \leq i \leq j \leq d} |\gamma_{\{i,j\}}|$ and $\bar{x}^\gamma = \prod_{1 \leq i \leq j \leq d} x_{\{i,j\}}^{\gamma_{\{i,j\}}}$. Since the operator are affine, it is easy to check that $Lf(x)$ and $L_i f(x)$ are also polynomial functions of degree m . We set:

$$\|f\|_{\mathbb{P}} = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} |a_\gamma| \text{ and } |L| = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \|L\bar{x}^\gamma\|_{\mathbb{P}},$$

so that $\|L^k f\|_{\mathbb{P}} \leq |L|^k \|f\|_{\mathbb{P}}$ for any $k \in \mathbb{N}$. Therefore, the series $\sum_{k=0}^{\infty} t^k L^k f(x)/k!$ converges absolutely. By using $l + 1$ times Itô's formula, we get:

$$\mathbb{E}[f(X_t^x)] = \sum_{k=0}^l \frac{t^k}{k!} L^k f(x) + \int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds.$$

Wishart processes have bounded moments since the drift and diffusion coefficients have a sub-linear growth. Thus, $C = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \sup_{s \in [0, t]} \mathbb{E}[|\overline{X_s^x}^\gamma|] < \infty$ and we obtain that $|\int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds| \leq C \|f\|_{\mathbb{P}} (t|L|)^{l+1}/(l+1)! \xrightarrow{t \rightarrow +\infty} 0$. Thus, we have $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x)/k!$ and similarly we get that

$$\mathbb{E} \left[f(X_t^{n, \dots, X_t^{1,x}}) | X_t^{n-1, \dots, X_t^{1,x}} \right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!} L_n^{k_n} f(X_t^{n-1, \dots, X_t^{1,x}}).$$

Now, we remark that $\tilde{C} = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \sup_{s \in [0, t]} \max(\mathbb{E}[|\overline{X_t^{1,x}}^\gamma|], \dots, \mathbb{E}[|\overline{X_t^{n, \dots, X_t^{1,x}}}^\gamma|]) < \infty$ by using once again that Wishart processes have bounded moments. As we have already proven that $\mathbb{E}[|L_n^{k_n} f(X_t^{n-1, \dots, X_t^{1,x}})|] \leq \tilde{C} \|f\|_{\mathbb{P}} |L_n|^{k_n}$, we can then switch the expectation with the series and get (2.13). Then, since $L_n^{k_n} f(x)$ are polynomial function of degree m , we can iterate this argument and finally get (2.14), which gives the result. \square

Proof of Theorem 13

The proof is divided into two parts. First, we prove that the SDE (2.23) has a unique strong solution which is given by (2.24) and is well defined on $\mathcal{S}_d^+(\mathbb{R})$. Second, we show that its infinitesimal generator is equal to the operator L_1 defined in (2.15).

First step. Let us assume that $(X_t^x)_{t \geq 0}$ is a solution to (2.23). We use the matrix decomposition of $(x_{i,j})_{2 \leq i, j \leq d}$ given by (2.22) and set:

$$\begin{aligned} (U_t)_{\{1, l+1\}} &= \sum_{i=1}^r (c_r^{-1})_{l,i} (X_t^x)_{\{1, i+1\}}, \quad l \in \{l, \dots, r\}, \\ (U_t)_{\{1, 1\}} &= (X_t^x)_{\{1, 1\}} - \sum_{l=1}^r \left(\sum_{i=1}^r (c_r^{-1})_{l,i} (X_t^x)_{\{1, i+1\}} \right)^2 = (X_t^x)_{\{1, 1\}} - \sum_{l=1}^r ((U_t)_{\{1, l+1\}})^2. \end{aligned}$$

We get by using Lemma 35 that:

$$\begin{aligned} &\bar{c} \begin{pmatrix} (U_t)_{\{1, 1\}} + \sum_{k=1}^r ((U_t)_{\{1, k+1\}})^2 & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{c}^T \\ &= \begin{pmatrix} (U_t)_{\{1, 1\}} + \sum_{k=1}^r ((U_t)_{\{1, k+1\}})^2 & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T c_r^T & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T k_r^T \\ c_r ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & c_r c_r^T & c_r k_r^T \\ k_r ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & k_r c_r^T & 0 \end{pmatrix} = X_t^x. \end{aligned}$$

Since $\bar{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}$ is invertible, $X_t^x \in \mathcal{S}_d^+(\mathbb{R})$ if, and only if for all $z \in \mathbb{R}^d$

$$\begin{aligned} &z^T \begin{pmatrix} (U_t)_{\{1, 1\}} + \sum_{i=1}^r ((U_t)_{\{1, i+1\}})^2 & ((U_t)_{\{1, l\}})_{2 \leq l \leq r+1} & 0 \\ ((U_t)_{\{1, l\}})_{2 \leq l \leq r+1} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} z \\ &= z_1^2 (U_t)_{\{1, 1\}} + \sum_{i=1}^r (z_{i+1} + (U_t)_{\{1, i+1\}} z_1)^2 \geq 0, \iff (U_t)_{\{1, 1\}} \geq 0. \end{aligned} \tag{2.45}$$

In particular, we get that $(U_0)_{\{1, 1\}} = u_{\{1, 1\}} \geq 0$ since $x \in \mathcal{S}_d^+(\mathbb{R})$. Now, by Itô calculus, we get from (2.23) that $d(U_t)_{\{1, l+1\}} = \sum_{i=1}^r \sum_{k=1}^r (c_r^{-1})_{l,i} (c_r)_{i,k} dZ_t^{k+1} = dZ_t^{l+1}$ and

$$\begin{aligned} d(U_t)_{\{1, 1\}} &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1, 1\}}} dW_t^1 + 2 \sum_{l=1}^r \sum_{k=1}^r (c_r^{-1})_{l,k} (X_t)_{\{1, k+1\}} dW_t^{l+1} \\ &\quad - \sum_{l=1}^r 2((U_t)_{\{1, l+1\}}) dW_t^{l+1} \\ &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1, 1\}}} dW_t^1. \end{aligned}$$

Thus, the solution $(X_t^x)_{t \geq 0}$ is necessarily the one given by (2.24) (pathwise uniqueness holds for $((U_t^u)_{\{1, l\}})_{1 \leq l \leq r+1}$, and especially for the CIR diffusion $(U_t^u)_{\{1, 1\}}$ since $\alpha \geq d - 1 \geq r$). Reciprocally, it is easy to check by Itô calculus that (2.24) solves (2.23).

Second step. Now, we want to show that L_1 is the infinitesimal operator associated to the process $(X_t^x)_{t \geq 0}$. It is sufficient to compare the drift and the quadratic covariation of the process X_t^x with L_1 . Since the drift part of $(X_t^x)_{t \geq 0}$ clearly corresponds to the first order of L_1 ,

we study directly the quadratic part. From (2.23), we have for $i, j \in \{2, \dots, d\}^2$:

$$\begin{aligned}
d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,1\}} \rangle / dt &= 4\langle (X_t^x)_{\{1,1\}}, \sum_{k=1}^r \left(-[\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}}] \right. \\
&\quad \left. + [\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}}]^2 \right) \rangle \\
&= 4\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,1\}} \rangle, \\
d\langle (X_t^x)_{\{1,i\}}, (X_t^x)_{\{1,j\}} \rangle / dt &= \sum_{k=1}^r (c_r)_{i-1,k} (c_r)_{j-1,k} = (cc^T)_{i-1,j-1} = \langle (X_t^x)_{\{i,j\}}, \\
d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,i\}} \rangle / dt &= 2 \sum_{k=1}^r \sum_{l=1}^r (c_r)_{i-1,k} (c_r^{-1})_{k,l} \langle (X_t^x)_{\{1,l+1\}} \rangle \\
&= 2\langle (X_t^x)_{\{1,i\}}, (X_t^x)_{\{1,i\}} \rangle, \text{ if } i \leq r+1, \\
d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,i\}} \rangle / dt &= 2 \sum_{k=1}^r \sum_{l=1}^r (k_r)_{i-1-r,k} (c_r^{-1})_{k,l} \langle (X_t^x)_{\{1,l+1\}} \rangle \\
&= 2 \sum_{l=1}^r (k_r c_r^{-1})_{i-1-r,l} \langle (X_t^x)_{\{1,l+1\}} \rangle \\
&= 2\langle (X_t^x)_{\{1,i\}}, (X_t^x)_{\{1,i\}} \rangle \text{ if } i > r+1, \text{ by Lemma 35.}
\end{aligned}$$

Thus, we deduce that L_1 is the infinitesimal generator of $(X_t^x)_{t \geq 0}$.

Lemma 35 — Let $y \in \mathcal{S}_d^+(\mathbb{R})$. We set $r = \text{Rk}((y_{i,j})_{2 \leq i,j \leq d})$, $y_1^r = (y_{1,i+1})_{1 \leq i \leq r}$ and $y_1^{r,d} = (y_{1,i+1})_{r+1 \leq i \leq d}$. We assume that there are an invertible matrix c_r and a matrix k_r defined on $\mathcal{M}_{d-r-1 \times r}(\mathbb{R})$, such that

$$(y_{i,j})_{2 \leq i,j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix}.$$

Then, we have $y_1^{r,d} = k_r c_r^{-1} y_1^r$.

Proof: We set $p = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{array} \right)$ and have $p^{-1} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c_r^{-1} & 0 \\ 0 & -k_r c_r^{-1} & I_{d-r-1} \end{array} \right)$. Since the matrix

$$p^{-1} y (p^{-1})^T = \left(\begin{array}{c|cc} y_{1,1} & (c_r^{-1} y_1^r)^T & (y_1^{r,d} - k_r c_r^{-1} y_1^r)^T \\ \hline c_r^{-1} y_1^r & I_r & 0 \\ y_1^{r,d} - k_r c_r^{-1} y_1^r & 0 & 0 \end{array} \right)$$

is positive semidefinite, we necessarily have $y_1^{r,d} - k_r c_r^{-1} y_1^r = 0$. \square

2.6.4 Proofs of Section 2.3

Lemma 36 — Let us consider an operator L on \mathbb{D} that satisfies the required assumption and such that b and σ have a sublinear growth, i.e. $\exists K > 0, \|b(x)\|_2 + \|\sigma(x)\|_2 \leq K(1 + \|x\|_2)$. Let us consider \hat{X}_t^x a potential ν th-order scheme for L , with $\nu \geq 1$. Then, condition (2.30) holds.

Proof: We have $\mathbb{D} \subset \mathbb{R}^\zeta$, and we set for $p \in \mathbb{N}^*$, $f_p(x) = \left(\sum_{i=1}^\zeta x_i^2 \right)^p = (\|x\|_2)^{2p}$, $x \in \mathbb{D}$. Clearly, $f_p \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$, and since \hat{X}_t^x is a potential ν th-order scheme for L , we have

$$\lim_{t \rightarrow 0^+} [\mathbb{E}[f_p(\hat{X}_t^x)] - f_p(x)]/t = Lf_p(x). \quad (2.46)$$

Since we have $\partial_i f_p(x) = 2px_i f_{p-1}(x)$ and $\partial_i \partial_j f_p(x) = 4p(p-1)x_i x_j f_{p-2}(x) + \mathbb{1}_{i=j}(2p f_{p-1}(x))$, it is easy to check that

$$\begin{aligned} |Lf_p(x)| &\leq \sum_{i=1}^{\zeta} 2pK(1 + \|x\|_2) \|x\|_2^{2p-1} + \sum_{i,j=1}^{\zeta} 4p^2 K^2 (1 + \|x\|_2)^1 \|x\|_2^{2p-2} \\ &< C_p(1 + \|x\|_2^{2p}) = C_p(1 + f_p(x)), \end{aligned}$$

for some constant $C_p > 0$. From (2.46), we get that:

$$\exists \eta > 0, \forall t \in (0, \eta), \mathbb{E}[f_p(\hat{X}_t^x)] \leq f_p(x) + tC_p(1 + f_p(x)).$$

We consider now the scheme $(\hat{X}_{t_i^N}^N, i = 0, \dots, N)$ with $N > T/\eta$ that starts from $x_0 \in \mathbb{D}$. We have $\mathbb{E}[f_p(\hat{X}_{t_{i+1}^N}^N)] \leq \mathbb{E}[f_p(\hat{X}_{t_i^N}^N)] + C_p(T/N)(1 + \mathbb{E}[f_p(\hat{X}_{t_i^N}^N)])$. Let $u_0 = f_p(x_0)$ and $u_{i+1} = u_i(1 + C_p T/N) + C_p T/N$. We have $\mathbb{E}[f_p(\hat{X}_{t_{i+1}^N}^N)] \leq u_i$ and $u_i = (1 + C_p T/N)^i (f_p(x_0) + 1) - 1 \leq e^{C_p T} (f_p(x_0) + 1)$. Therefore we have

$$\sup_{N > T/\eta} \sup_{0 \leq i \leq N} \mathbb{E}[\|\hat{X}_{t_i^N}^N\|_2^{2p}] \leq e^{C_p T} (\|x_0\|_2^{2p} + 1) < \infty,$$

which gives (2.30). □

Proof of Theorem 23

Theorem 23 defines the scheme as $\pi^T h_r(\hat{U}_t^u) \pi$. We can prove (see later) that \hat{U}_t^u is a potential ν th-order scheme for some operator, and it would be then easy to analyze the error if h_r were in $\mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. Unfortunately, h_r is only smooth w.r.t. to the coordinates $u_{\{1,1\}}, \dots, u_{\{1,d\}}$ and is not a priori in $\mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. In fact, this is sufficient to show that $\pi^T h_r(\hat{U}_t^u) \pi$ is a potential ν th-order scheme because these coordinates are the only one that are modified by the scheme. This requires however some further analysis which is made below.

Let $\mathbb{D} \subset \mathbb{R}^\zeta$ be a domain. We introduce for any domain $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$, $\xi \in \mathbb{N}^*$ the set

$$\begin{aligned} \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}}) &= \left\{ f \in \mathcal{C}^\infty(\mathbb{D} \times \tilde{\mathbb{D}}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall (x, \tilde{x}) \in \mathbb{D} \times \tilde{\mathbb{D}}, \right. \\ &\quad \left. |\partial_\gamma f(x, \tilde{x})| \leq C_\gamma (1 + \|(x, \tilde{x})\|^{e_\gamma}) \right\}, \end{aligned}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^{\zeta+\xi}$ and $\partial_\gamma = \partial_1^{\gamma_1}, \dots, \partial_\zeta^{\gamma_\zeta}$ denotes the derivatives w.r.t. the coordinates of \mathbb{D} . For $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$, we will say that $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ is a *good sequence* for f if it is such that $|\partial_\gamma f(x, \tilde{x})| \leq C_\gamma (1 + \|(x, \tilde{x})\|^{e_\gamma})$ for any $(x, \tilde{x}) \in \mathbb{D} \times \tilde{\mathbb{D}}$.

Let us now consider an operator L that satisfies the required assumption on \mathbb{D} . It is easy to check that all the iterated functions $L^k f$ are well defined and belong to $\mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ as soon as $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. Let us fix $\tilde{x} \in \tilde{\mathbb{D}}$ and consider \hat{X}_t^x sampled according to a potential weak ν th-order scheme for L . Since $x \mapsto f(x, \tilde{x})$ belongs to $\mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$, we know by (2.29) that there are constants $C_{\tilde{x}}, E_{\tilde{x}}, \eta_{\tilde{x}}$ such that

$$\forall t \in (0, \eta_{\tilde{x}}), \left| \mathbb{E}[f(\hat{X}_t^x)] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \leq C_{\tilde{x}} t^{\nu+1} (1 + \|x\|^{E_{\tilde{x}}}).$$

In practice, one would like instead to get some bounds where the dependence with respect to \tilde{x} is more tractable. This is why we introduce the following definition.

Definition 37 — *Let L be an operator that satisfies the required assumption on \mathbb{D} . We will say that a potential weak ν th-order scheme for L satisfies the immersion property if for any $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$, $\xi \in \mathbb{N}^*$ and any function $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ with a good sequence $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$, there exist positive constants C , E , and η depending only on $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ such that*

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x, \tilde{x})] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \leq Ct^{\nu+1} (1 + \|(x, \tilde{x})\|^E).$$

In practice, most of the usual schemes satisfy this property. In fact, to prove that a scheme is a potential ν th-order scheme, it is common to use a Taylor expansion that gives generally at the same time the immersion property. We illustrate this for the exact scheme below.

Proposition 38 — *Let $\mathbb{D} \subset \mathbb{R}^\zeta$, $b : \mathbb{D} \rightarrow \mathbb{R}^\zeta$ and $\sigma : \mathbb{D} \rightarrow \mathcal{M}_\zeta(\mathbb{R})$ such that $\|b(x)\| + \|\sigma(x)\| \leq C(1 + \|x\|)$ for some $C > 0$, and assume that $Lf(x) = \sum_{i=1}^{\zeta} b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{\zeta} (\sigma \sigma^T(x))_{i,j} \partial_i \partial_j f(x)$ satisfies the required assumption on \mathbb{D} . Then, for any $\nu \in \mathbb{N}$, the exact scheme is a potential weak ν th-order scheme for L and it satisfies the immersion property.*

Proof: Let $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. We know from the sublinear growth condition that we have bounds on the moments of X_t^x : $\forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in [0, 1], \mathbb{E}[\|(X_t^x, \tilde{x})\|^q] \leq C_q(1 + \|(x, \tilde{x})\|^q)$. By iterating Itô's Formula, we get then easily for $t \in [0, 1]$,

$$\mathbb{E}[f(X_t^x, \tilde{x})] = \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x, \tilde{x}) + \int_0^t \frac{(t-s)^\nu}{\nu!} \mathbb{E}[L^{\nu+1} f(X_s^x, \tilde{x})] ds.$$

Since $L^{\nu+1} f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$, there are constants $C > 0$ and $q \in \mathbb{N}^*$ depending only on a good sequence of f such that $|L^{\nu+1} f(x, \tilde{x})| \leq C(1 + \|(x, \tilde{x})\|^q)$. Thus, we deduce that $|\mathbb{E}[f(X_t^x, \tilde{x})] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x, \tilde{x})| \leq \frac{t^{\nu+1}}{(\nu+1)!} C(1 + C_q(1 + \|(x, \tilde{x})\|^q))$. \square

Besides, the immersion property is easily preserved by scheme composition.

Proposition 39 — *Let L_1 and L_2 be two operators that satisfy the required assumptions on \mathbb{D} , and assume that $\hat{p}_x^1(t)(dz)$ and $\hat{p}_x^2(t)(dz)$ are respectively potential weak ν th-order discretization schemes on \mathbb{D} for these operators that satisfy the immersion property. Let $\lambda_1, \lambda_2 > 0$ and $\hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x} \sim \hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz)$. Let $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. Then, there are constants C, E, ν that only depend on a good sequence of f such that*

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x}, \tilde{x})] - \sum_{l_1 + l_2 \leq \nu} \frac{\lambda_1^{l_1} \lambda_2^{l_2}}{l_1! l_2!} t^{l_1 + l_2} L_1^{l_1} L_2^{l_2} f(x, \tilde{x}) \right| \leq Ct^{\nu+1} (1 + \|(x, \tilde{x})\|^E).$$

Therefore,

- If $L_1 L_2 = L_2 L_1$, $\hat{p}^2(t) \circ \hat{p}_x^1(t)(dz)$ is a potential weak ν th-order discretization scheme for $L_1 + L_2$ satisfying the immersion property.
- If $\nu \geq 2$, $\hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}_x^2(t/2)$ and $\frac{1}{2} (\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}^1(t) \circ \hat{p}_x^2(t))$ are potential weak second order schemes for $L_1 + L_2$ satisfying the immersion property.

This proposition is a straightforward extension of Proposition 1.15, Corollary 1.16 and Theorem 1.17 of Alfonsi [5], and we do not repeat the proof here. Thanks to this result, we can prove the immersion property of the schemes that are obtained by splitting. The Ninomiya-Victoir scheme [69] which is obtained by a composition of exact schemes naturally satisfies this property. By looking at the proof of Theorem 1.18 in [5], it still satisfies this property if we replace the Gaussian samples by moment matching variables. Also, we can check that the second and third order schemes for the CIR process presented in Alfonsi [5] satisfy the immersion property.

Corollary 40 — *The Ninomiya-Victoir scheme and the second and third order scheme for the Cox-Ingersoll-Ross process given in [5] satisfy the immersion property.*

Corollary 41 — *Let L_1 (resp. L_2) be an operator that satisfies the required assumptions on \mathbb{D}_1 (resp. \mathbb{D}_2). Let \hat{X}_t^{1,x_1} and \hat{X}_t^{2,x_2} be potential weak ν th-order schemes for L_1 and L_2 sampled independently. Then, $(\hat{X}_t^{1,x_1}, \hat{X}_t^{2,x_2})$ is a potential weak ν th-order schemes on $\mathbb{D}_1 \times \mathbb{D}_2$ that satisfies the immersion property.*

Proof: From the immersion property, it is easy to check that (\hat{X}_t^{1,x_1}, x_2) (resp. (x_1, \hat{X}_t^{2,x_2})) is a potential ν th order scheme for L_1 (resp. L_2) on $\mathbb{D}_1 \times \mathbb{D}_2$ that satisfies the immersion property. The composition of these schemes is simply $(\hat{X}_t^{1,x_1}, \hat{X}_t^{2,x_2})$. Since L_1 and L_2 operate on different domains, we have $L_1 L_2 = L_2 L_1$, which gives the result. \square

Proof of Theorem 23.

Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and $f \in C_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ and $r = \text{Rk}((x_{i,j})_{2 \leq i,j \leq d})$. Since the operator L_1 satisfies the required assumption, we know that the exact scheme is a potential ν th-order scheme (Proposition 38), and there are constants $C, E, \eta > 0$ depending on a good sequence of f such that

$$\forall t \in (0, \eta), |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L_1^k f(x)| \leq C(1 + \|x\|^E). \quad (2.47)$$

On the other hand, we know from Theorem 13, equation (2.27) and Corollary 15 that we have

$$X_t^x = \pi^T h_r(U_t^u) \pi,$$

where $(U_t^u)_{\{1,l\}}$ solves the SDEs (2.25) starting from the initial condition $u_{1,l}$ for $1 \leq l \leq r + 1$, and $(U_t^u)_{\{i,j\}} = u_{\{i,j\}}$ for the other coordinates. We have also $\hat{X}_t^x = \pi^T h_r(\hat{U}_t^u) \pi$ by construction, and it is natural to focus on the function $u \mapsto f(\pi^T h_r(u) \pi)$.

Let us consider the set

$$\{x \in \mathcal{S}_d(\mathbb{R}), \text{ s.t. } (x_{i,j})_{2 \leq i,j \leq d} \in \mathcal{S}_{d-1}^+(\mathbb{R}), x_{1,1} \geq 0\}.$$

It is isomorphic to $(\mathbb{R}_+ \times \mathbb{R}^{d-1}) \times \mathcal{S}_{d-1}^+(\mathbb{R})$ by the map $x \mapsto ((x_{1,1}, \dots, x_{1,d}), (x_{i,j})_{2 \leq i, j \leq d})$. We have to notice now that the function h_r defined by (2.27) is such that $h_r \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{R}_+ \times \mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$. It is indeed a polynomial function with respect to $u_{1,1}, \dots, u_{1,d}$. Then, it is easy to check that $u \mapsto f(\pi^T h_r(u) \pi) \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{R}_+ \times \mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$ since $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. Moreover, by the chain rule, we can get a good sequence for this function that only depend on a good sequence of f since π and h_r are fixed.

By assumption, $(\hat{U}_t^u)_{\{1,1\}}$ is a potential ν th-order scheme for the operator $(\alpha - r)\partial_{\{1,1\}} + 2u_{\{1,1\}}\partial_{\{1,1\}}^2$ and satisfy the immersion property. The schemes $(\hat{U}_t^u)_{\{1,i\}}$ ($2 \leq i \leq r+1$) can be seen as a Ninomiya-Victoir scheme with moment matching variables. They are therefore potential ν th-order scheme for the operator $\frac{1}{2}\partial_{\{1,i\}}^2$ ([5], Theorem 1.18) and satisfy the immersion property from Corollary 40. Therefore, from Corollary 41, $((\hat{U}_t^u)_{\{1,1\}}, \dots, (\hat{U}_t^u)_{\{1,d\}})$ is a potential ν th order scheme for $(\alpha - r)\partial_{\{1,1\}} + 2u_{\{1,1\}}\partial_{\{1,1\}}^2 + \frac{1}{2}\sum_{i=1}^r \partial_{\{1,i\}}^2$ satisfying the immersion property. Thus, there are constants that we still denote by $C, E, \eta > 0$ depending on a good sequence of f such that:

$$\forall t \in (0, \eta), |\mathbb{E}[f(\pi^T h_r(U_t^u) \pi)] - \mathbb{E}[f(\pi^T h_r(\hat{U}_t^u) \pi)]| \leq Ct^{\nu+1}(1 + \|u\|^E). \quad (2.48)$$

Now, one has to notice that $\|u\| \leq C'(1 + \|x\|)$ for some constant $C' > 0$ since we have $u_{\{1,1\}} + \sum_{k=1}^r (u_{\{1,k+1\}})^2 = \tilde{x}_{\{1,1\}}$ and \tilde{x} and x have the same Frobenius norm. We get then the result by gathering (2.47) and (2.48). \square

Remark 42 — *We can check by looking at the proof above that the scheme obtained for L_1 also satisfies the immersion property. Since we do not need it for the construction of the further schemes for $WIS_d(x, \alpha, b, a)$ or $AF F_d(x, \bar{\alpha}, B, a)$, we will no longer mention it.*

Proof of the third condition of Theorem 21 for Wishart processes

The scope of this part is to show the following result.

Proposition 43 — *Let $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$, $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$, $x \in \mathcal{S}_d^+(\mathbb{R})$ and $T > 0$. Then, $\tilde{u}(t, x) = \mathbb{E}[f(X_t^x)]$ is \mathcal{C}^∞ on $[0, T] \times \mathcal{S}_d^+(\mathbb{R})$, solves $\partial_t \tilde{u}(t, x) = L\tilde{u}(t, x)$ and its derivatives satisfy*

$$\forall l \in \mathbb{N}, \forall n \in \mathbb{N}^{\frac{d(d+1)}{2}} \exists C_{l,n}, e_{l,n} > 0, \forall x \in \mathcal{S}_d^+(\mathbb{R}), \forall t \in [0, T], \left| \partial_t^l \prod_{1 \leq i \leq j \leq d} \partial_{\{i,j\}}^{n_{\{i,j\}}} \tilde{u}(t, x) \right| \leq C_{l,n}(1 + \|x\|^{e_{l,n}}). \quad (2.49)$$

For technical reasons on the boundary of $\mathcal{S}_d^+(\mathbb{R})$, it is convenient here to work with $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ rather than $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$. The proof of Proposition 43 is based on a remarkable identity on the derivatives of the Laplace transform which is stated below.

Lemma 44 — *Let $(X_t^x)_{t \geq 0} \stackrel{\text{Law}}{\sim} WIS_d(x, \alpha, b, a)$ and $v = v_R + iv_I$ such that $v_R \in \mathcal{D}_{b,a,t}$ and $v_I \in \mathcal{S}_d(\mathbb{R})$. We denote by $\phi(t, \alpha, x, v)$ the Laplace transform of X_t^x given by (2.9), the other*

parameters a, b being fixed. Then, the derivative w.r.t $x_{\{k,l\}}$ satisfies the following equality

$$\partial_{\{k,l\}}\phi(t, \alpha, x, v) = \phi(t, \alpha + 2, x, v)p_t^{\{k,l\}}(v),$$

where $p_t^{\{k,l\}}$ is a polynomial function of the matrix elements of degree d defined by :

$$\begin{aligned} p_t^{\{k,l\}}(v) &= \text{Tr} \left[v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right] \\ &=: \sum_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} a_t^{\gamma, \{k,l\}} \bar{v}^\gamma, \text{ where } \bar{v}^\gamma = \prod_{\{i,j\}} v_{\{i,j\}}^{\gamma_{\{i,j\}}}. \end{aligned}$$

Moreover, its coefficients are bounded uniformly in time:

$$\exists K_t > 0, \forall s \in [0, t], \max_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} (|a_s^{\gamma, \{k,l\}}|) \leq K_t.$$

Proof: We get from (2.9),

$$\begin{aligned} \partial_{\{k,l\}}\phi(t, \alpha, x, v) &= \frac{\text{Tr} \left[v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right]}{\det(I_d - 2q_t v)} \\ &\times \frac{\exp \left(\text{Tr} \left[v (I_d - 2q_t v)^{-1} m_t x m_t^T \right] \right)}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}} \\ &= \phi(t, \alpha + 2, x, v) \text{Tr} \left[v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right]. \end{aligned}$$

Since $s \mapsto \|m_s\|$ and $s \mapsto \|q_s\|$ are continuous functions on $[0, t]$, we obtain the bounds on the polynomial coefficients. \square

Proof: Let $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$. First, let us observe that (2.49) is obvious when $l = |n| = 0$. Since we have $\forall l \in \mathbb{N}, L^l f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$, and $\partial_t^l \tilde{u}(t, x) = \mathbb{E}(L^l f(X_t^x))$, it is sufficient to prove (2.49) only for the derivatives w.r.t x .

We first focus on the case $|n| = 1$ and want to show that $\partial_{\{k,l\}} \tilde{u}(t, x)$ satisfies (2.49). The sketch of this proof is to write f as the inverse Fourier transform of its Fourier transform and then use Lemma 44. Unfortunately, f has not a priori the required integrability to do that, and we have to introduce an auxiliary function f_ρ .

Definition of the new function f_ρ . Since $\mathcal{D}_{b,a;T}$ given by (2.8) is an open set and $0 \in \mathcal{D}_{b,a;T}$, there is $\rho > 0$ such that $\rho I_d \in \mathcal{D}_{b,a;T}$. Let $\mu : \mathbb{R} \rightarrow \mathbb{R}$ be the function such that $\mu(x) = 0$ if $x \leq -1$ or $x \geq 0$, $\mu(x) = \exp(\frac{1}{x(x+1)})$ if $-1 < x < 0$. We have $\mu \in \mathcal{C}^\infty(\mathbb{R})$. We consider then the cutoff function $\zeta : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$ defined as $\forall x \in \mathbb{R}, \zeta(x) = \frac{\int_{-\infty}^x \mu(y) dy}{\int_{\mathbb{R}} \mu(y) dy}$. It is nondecreasing, such that $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 0$ if $x \leq -1$ and $\zeta(x) = 1$ if $x \geq 0$. Besides, we have $\zeta \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{R})$ since all its derivatives have a compact support. Now, we define a $\vartheta \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ as

$$\begin{aligned} \vartheta : \mathcal{S}_d(\mathbb{R}) &\rightarrow \mathbb{R} \\ x &\mapsto \prod_{i=1}^d \zeta(x_{\{i,i\}}) \prod_{i \neq j} \zeta(x_{\{j,j\}} x_{\{i,i\}} - x_{\{i,j\}}^2). \end{aligned}$$

It is important to notice that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ if $x \in \mathcal{S}_d(\mathbb{R})$ and $\vartheta(x) = 0$ if there is $i \in \{1, \dots, d\}$ such that $x_{\{i,i\}} < -1$ or $i < j \in \{1, \dots, d\}$ such that $x_{\{i,j\}}^2 > 1 + x_{\{i,i\}}x_{\{j,j\}}$. Let $\gamma \in \mathbb{N}^{d(d-1)/2}$. Since $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$, there are constants $K, E > 0$ and $K', E' > 0$ such that, $\forall x \in \mathcal{S}_d(\mathbb{R})$

$$\begin{aligned} |\partial^\gamma(\vartheta f)(x)| &\leq K(1 + \|x\|^E) \prod_{i=1}^d \left(\mathbb{1}_{\{|x_{\{i,i\}}| > -1\}} \right) \prod_{1 \leq i < j \leq d} \left(\mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}} \right) \\ &\leq K'(1 + \|(x_{\{i,i\}})_{1 \leq i \leq d}\|^{E_1}) \prod_{i=1}^d \left(\mathbb{1}_{\{|x_{\{i,i\}}| > -1\}} \right) \prod_{1 \leq i < j \leq d} \left(\mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}} \right). \end{aligned}$$

Here, the upper bound only involves the diagonal coefficients. We define

$$x \in \mathcal{S}_d(\mathbb{R}), f_\rho(x) := \vartheta(x)f(x) \exp(-\text{Tr}(\rho x)),$$

and obtain from the last inequality that f_ρ belongs to the Schwartz space of rapidly decreasing functions since $\rho > 0$. Thus, its Fourier transform also belongs to the Schwartz space and we have

$$f_\rho(x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(-\text{Tr}(ivx)) \mathcal{F}(f_\rho)(v) dv$$

where $\mathcal{F}(f_\rho)(v) = \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\text{Tr}(ivx)) f_\rho(x) dx$, and in particular $f_\rho, \mathcal{F}(f_\rho) \in L^1(\mathcal{S}_d(\mathbb{R})) \cap L^\infty(\mathcal{S}_d(\mathbb{R}))$.

A new representation of $\tilde{u}(t, x)$. We have $f(x) = \exp(\rho \text{Tr}(x)) f_\rho(x)$ for $x \in \mathcal{S}_d^+(\mathbb{R})$, and therefore

$$\begin{aligned} \tilde{u}(t, x) &= \mathbb{E}[\exp(\text{Tr}(\rho X_t^x)) f_\rho(X_t^x)] \\ &= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \mathbb{E} \left[\int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\text{Tr}((-iv + \rho I_d) X_t^x)) \mathcal{F}(f_\rho)(v) dv \right] \\ &= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \mathbb{E}[\exp(\text{Tr}((-iv + \rho I_d) X_t^x))] \mathcal{F}(f_\rho)(v) dv. \end{aligned}$$

The last equality holds since

$$\int_{\mathbb{R}^{\frac{d(d+1)}{2}}} |\mathbb{E}[\exp(\text{Tr}((-iv + \rho I_d) X_t^x))] \times |\mathcal{F}(f_\rho)(v)| dv \leq \phi(t, \alpha, x, \rho I_d) \|\mathcal{F}(f_\rho)\|_1 < \infty.$$

Here, we have used that $\rho I_d \in \mathcal{D}_{b,a;T}$ to get $\phi(t, \alpha, x, \rho I_d) < \infty$.

Derivation with respect to $x_{\{k,l\}}$, $k, l \in \{1, \dots, d\}$. From Lemma 44, we have by Lebesgue's theorem

$$\partial_{\{k,l\}}^x \tilde{u}(t, x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \phi(t, \alpha + 2, x, -iv + \rho I_d) p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v) dv \quad (2.50)$$

since $|\partial_{\{k,l\}}^x \phi(t, \alpha, x, -iv + \rho I_d) \mathcal{F}(f_\rho)(v)| \leq |\phi(t, \alpha + 2, x, \rho I_d)| |p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v)|$ and $p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v)$ is a rapidly decreasing function.

Let $1 \leq k', l' \leq d$. An integration by part gives

$$\begin{aligned} & \int_{\mathbb{R}} (\rho I_d - iv)_{\{k', l'\}} \exp(\text{Tr}[x(iv - \rho I_d)]) \vartheta(x) f(x) dx_{\{k', l'\}} \\ &= \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k' = l'} \right) \int_{\mathbb{R}} \exp(\text{Tr}[x(iv - \rho I_d)]) \partial_{\{k', l'\}} (\vartheta(x) f(x)) dx_{\{k', l'\}}, \end{aligned}$$

and thus

$$\begin{aligned} & (\rho I_d - iv)_{\{k', l'\}} \mathcal{F}(\exp[-\rho \text{Tr}(x)] \vartheta(x) f(x))(v) \\ &= \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k' = l'} \right) \mathcal{F}(\exp[-\rho \text{Tr}(x)] \partial_{\{k', l'\}} [\vartheta(x) f(x)])(v). \end{aligned}$$

We set $\varphi(\gamma) = \prod_{1 \leq k' \leq l' \leq d} \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k' = l'} \right)^{\gamma_{\{k', l'\}}}$ for $\gamma \in \mathbb{N}^{d(d+1)/2}$ and get by iterating the argument that:

$$\prod_{1 \leq k' \leq l' \leq d} (\rho I_d - iv)_{\{k', l'\}}^{\gamma_{\{k', l'\}}} \mathcal{F}(f_\rho)(v) = \varphi(\gamma) \mathcal{F}(\exp[-\rho \text{Tr}(x)] \partial_\gamma (\vartheta \times f)(x))(v). \quad (2.51)$$

Since $p_t^{\{k, l\}}(\rho I_d - iv) = \sum_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} a_t^{\gamma, \{k, l\}} \prod_{1 \leq k' \leq l' \leq d} (\rho I_d - iv)_{\{k', l'\}}^{\gamma_{\{k', l'\}}}$, we get from (2.50) and (2.51):

$$\partial_{\{k, l\}} u(t, x) = \sum_{|\gamma| \leq d} a_t^{\gamma, \{k, l\}} \varphi(\gamma) \mathbb{E}(\partial_\gamma (f \times \vartheta)(Y_t^x)) = \sum_{|\gamma| \leq d} a_t^{\gamma, \{k, l\}} \varphi(\gamma) \mathbb{E}(\partial_\gamma f(Y_t^x)) \quad (2.52)$$

where $(Y_t^x)_{t \geq 0} \underset{\text{Law}}{\sim} WIS_d(x, \alpha + 2, b, a)$. Here, we have used that $\partial_\gamma (\vartheta \times f)(y) = \partial_\gamma f(y)$ for $y \in \mathcal{S}_d^+(\mathbb{R})$. From Lemma 44 $(a_t^{\gamma, \{k, l\}})_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d}$ is bounded for $t \in [0, T]$, and we get (2.49) when $|n| = 1$ since $\partial_\gamma f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$. Thanks to (2.52), a derivative of order $|n|$, can be seen as a (bounded) linear combination of derivatives of order $|n| - 1$, and we easily get (2.49) by an induction on $|n|$.

It remains to check that we have indeed $\partial_t \tilde{u}(t, x) = Lu(t, x)$. Let $t, h > 0$. By the Markov property, we have $\tilde{u}(t+h, x) = \mathbb{E}[\tilde{u}(t, X_h^x)]$. From (2.49) and Itô's formula, we get $[\tilde{u}(t+h, x) - u(t, x)]/h \xrightarrow{h \rightarrow 0^+} Lu(t, x)$. \square

Lemma 45 — Let $\alpha, x \in \mathcal{S}_d^+(\mathbb{R})$, $B \in \mathcal{L}(\mathcal{S}_d^+(\mathbb{R}))$ that satisfies (2.4), and $x(t)$ be the solution of the following ODE

$$x(t) = x + \int_0^t (\alpha + B(x(s))) ds. \quad (2.53)$$

Then, we have $x(t) \in \mathcal{S}_d^+(\mathbb{R})$ for $t \geq 0$.

Proof: The ODE (2.53) is affine and has unique solution on $\mathcal{S}_d^+(\mathbb{R})$ which is given by

$$t \geq 0, x(t) = \exp(tB)(x) + \int_0^t \exp(sB)(\alpha) ds, \quad (2.54)$$

where $\forall t \in \mathbb{R}^+, \forall x \in \mathcal{S}_d(\mathbb{R})$, $\exp(tB)(x) = \sum_{k=0}^{\infty} \frac{t^k B^k(x)}{k!}$, $B^k(x) = \underbrace{B \circ \dots \circ B}_{k \text{ times}}(x)$ such that $B^0(x) = x$.

We assume first that $\alpha, x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ and consider $\tau = \inf\{t \geq 0, x(t) \notin \mathcal{S}_d^+(\mathbb{R})\}$, with the convention $\inf \emptyset = +\infty$. We have $\tau > 0$. Let us assume by a way of contradiction that $\tau < \infty$. Then, $x(\tau)$ cannot be invertible and there is $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y \neq 0$ and $\text{Tr}(yx(\tau)) = 0$. From (2.54) and (2.4), we get

$$\text{Tr}(x'(\tau)y) = \text{Tr}([B(x(\tau)) + \alpha]y) > 0,$$

since α is positive definite. Therefore, there is $\epsilon \in (0, \tau)$ such that $\text{Tr}(yx(\tau - \epsilon)) < 0$. Let us recall now that $z \in \mathcal{S}_d^+(\mathbb{R}) \iff \forall y \in \mathcal{S}_d^+(\mathbb{R}), \text{Tr}(yz) \geq 0$. Thus, $x(\tau - \epsilon) \notin \mathcal{S}_d^+(\mathbb{R})$, which contradicts the definition of τ .

In the general case $\alpha, x \in \mathcal{S}_d^+(\mathbb{R})$, we observe that the solution (2.54) is continuous w.r.t. x and α , and thus $\forall t \geq 0, x(t) \in \mathcal{S}_d^+(\mathbb{R})$ since $\mathcal{S}_d^+(\mathbb{R})$ is a closed set. \square

Chapter 3

A Mean-Reverting SDE on Correlation matrices

Note: Ce chapitre a donné lieu un article [3] soumis à *Stochastic Processes and their Applications*

Abstract We introduce a mean-reverting SDE whose solution is naturally defined on the space of correlation matrices. This SDE can be seen as an extension of the well-known Wright-Fisher diffusion. We provide conditions that ensure weak and strong uniqueness of the SDE, and describe its ergodic limit. We also shed light on a useful connection with Wishart processes that makes understand how we get the full SDE. Last, we focus on the simulation of this diffusion and present discretization schemes that achieve a second-order weak convergence.

Introduction

The scope of this paper is to introduce an SDE that is well defined on the set of correlation matrices. Our main motivation comes from an application to finance, where the correlation is commonly used to describe the dependence between assets. More precisely, a diffusion on correlation matrices can be used to describe the instantaneous correlation between the log-prices of different stocks. Thus, it is also very important for practical purpose to be able to sample paths of this SDE in order to compute expectations (for prices or greeks). This is why an entire part of this paper is devoted to get an efficient simulation scheme. More generally, processes on correlation matrices can naturally be used to model the dynamic of the dependence between some quantities and can be applied to a much wider range of applications. In this paper, we focus on the definition, the mathematical properties and the sampling of this SDE. In a further work, we will investigate a possible implementation in finance to model an index and its stock components.

There are works on particular Stochastic Differential Equations that are defined on positive semidefinite matrices such as Wishart processes (Bru [17]) or their Affine extensions (Cuchiero et al. [26]). On the contrary, there is to the best of our knowledge very few literature dedicated to some stochastic differential equations that are valued on correlation matrices. Of course, general results are known for stochastic differential equations on manifolds. However, no particular SDE defined on correlation matrices has been studied in detail. In dimension $d = 2$,

correlation matrices are naturally described by a single real $\rho \in [-1, 1]$. The probably most famous SDE on $[-1, 1]$ is the following Wright-Fisher diffusion:

$$dX_t = \kappa(\bar{\rho} - X_t)dt + \sigma\sqrt{1 - X_t^2}dB_t, \quad (3.1)$$

where $\kappa \geq 0$, $\bar{\rho} \in [-1, 1]$, $\sigma \geq 0$, and $(B_t)_{t \geq 0}$ is a real Brownian motion. Here, we make a slight abuse of language. Strictly speaking, Wright-Fisher diffusions are defined on $[0, 1]$ and this is in fact the process $(\frac{1+X_t}{2}, t \geq 0)$ that is a Wright-Fisher one. They have originally been used to model gene frequencies (see Karlin and Taylor [58]). The marginal law of X_t is known explicitly with its moments, and its density can be written as an expansion with respect to the Jacobi orthogonal polynomial basis (see Mazet [67]). This is why the process $(X_t, t \geq 0)$ is sometimes also called Jacobi process in the literature. In higher dimension ($d \geq 3$), no similar SDE has been yet considered. To get processes on correlation matrices, it is instead used parametrization of subsets of correlation matrices. For example, one can consider X_t defined by $(X_t)_{i,j} = \rho_t$ for $1 \leq i \neq j \leq d$, where ρ_t is a Wright-Fisher diffusion on $[-1/(d-1), 1]$. More sophisticated examples can be found in [12]. The main purpose of this paper is to propose a natural extension of the Wright-Fisher process (3.1) that is defined on the whole set of correlation matrices.

Let us now introduce the process. We first advise the reader to have a look at our notations for matrices located at the end of this introduction, even though they are rather standard. We consider $(W_t, t \geq 0)$, a d -by- d square matrix process whose elements are independent real standard Brownian motions, and focus on the following SDE on the correlation matrices $\mathfrak{C}_d(\mathbb{R})$:

$$\begin{aligned} X_t &= x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds \\ &\quad + \sum_{n=1}^d a_n \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_s e_d^n X_s} \right), \end{aligned} \quad (3.2)$$

where $x, c \in \mathfrak{C}_d(\mathbb{R})$ and $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d)$ and $a = \text{diag}(a_1, \dots, a_d)$ are nonnegative diagonal matrices such that

$$\kappa c + c \kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R}) \text{ or } d = 2. \quad (3.3)$$

Under these assumptions, we will show in Section 3.2 that this SDE has a unique weak solution which is well-defined on correlation matrices, i.e. $\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})$. We will also show that strong uniqueness holds if we assume moreover that $x \in \mathfrak{C}_d^*(\mathbb{R})$ and

$$\kappa c + c \kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R}). \quad (3.4)$$

Looking at the diagonal coefficients, conditions (3.3) and (3.4) imply respectively $\kappa_i \geq (d-2)a_i^2/2$ and $\kappa_i \geq da_i^2/2$. This heuristically means that the speed of the mean-reversion has to be high enough with respect to the noise in order to stay in $\mathfrak{C}_d(\mathbb{R})$. Throughout the paper, we will denote $MRC_d(x, \kappa, c, a)$ the law of the process $(X_t)_{t \geq 0}$ and $MRC_d(x, \kappa, c, a; t)$ the law of X_t . Here, *MRC* stands for Mean-Reverting Correlation process. When using these notations, we implicitly assume that (3.3) holds.

In dimension $d = 2$, the only non trivial component is $(X_t)_{1,2}$. We can show easily that there is a real Brownian motion $(B_t, t \geq 0)$ such that

$$d(X_t)_{1,2} = (\kappa_1 + \kappa_2)(c_{1,2} - (X_t)_{1,2})dt + \sqrt{a_1^2 + a_2^2} \sqrt{1 - (X_t)_{1,2}^2} dB_t.$$

Thus, the process (3.2) is simply a Wright-Fisher diffusion. Our parametrization is however redundant in dimension 2, and we can assume without loss of generality that $\kappa_1 = \kappa_2$ and $a_1 = a_2$. Then, the condition $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ is always satisfied, while assumption (3.4) is the condition that ensures $\forall t \geq 0, (X_t)_{1,2} \in (-1, 1)$. In larger dimensions $d \geq 3$, we can also show that each non-diagonal element of (3.2) follows a Wright-Fisher diffusion (3.1).

The paper is structured as follows. In the first Section, we present first properties of Mean-Reverting Correlation processes. We calculate the infinitesimal generator and give explicitly their moments. In particular, this enables us to describe the ergodic limit. We also present a connection with Wishart processes that clarifies how we get the SDE (3.2). It is also useful later in the paper to construct discretization schemes. Last, we show a link between some MRC processes and the multi-allele Wright-Fisher model. Then, Section 3.2 is devoted to the study of the weak existence and strong uniqueness of the SDE (3.2). We discuss the extension of these results to time and space dependent coefficients κ, c, a . Also, we exhibit a change of probability that preserves the family of MRC processes. The last and third Section is devoted to obtain discretization schemes for (3.2). This is a crucial issue if one wants to use MRC processes effectively. To do so, we use a remarkable splitting of the infinitesimal generator as well as standard composition technique. Thus, we construct discretization schemes with a weak error of order 2. This can be done either by reusing the second order schemes for Wishart processes obtained in [1] or by an ad-hoc splitting (see Appendix 3.4.4). All these schemes are tested numerically and compared with a (corrected) Euler-Maruyama scheme.

Notations for real matrices :

- For $d \in \mathbb{N}^*$, $\mathcal{M}_d(\mathbb{R})$ denotes the real d square matrices; $\mathcal{S}_d(\mathbb{R})$, $\mathcal{S}_d^+(\mathbb{R})$, $\mathcal{S}_d^{+,*}(\mathbb{R})$, and $\mathcal{G}_d(\mathbb{R})$ denote respectively the set of symmetric, symmetric positive semidefinite, symmetric positive definite and non singular matrices.
- The set of correlation matrices is denoted by $\mathfrak{C}_d(\mathbb{R})$:

$$\mathfrak{C}_d(\mathbb{R}) = \{x \in \mathcal{S}_d^+(\mathbb{R}), \forall 1 \leq i \leq d, x_{i,i} = 1\}$$

We also define $\mathfrak{C}_d^*(\mathbb{R}) = \mathfrak{C}_d(\mathbb{R}) \cap \mathcal{G}_d(\mathbb{R})$, the set of the invertible correlation matrices.

- For $x \in \mathcal{M}_d(\mathbb{R})$, x^T , $\text{adj}(x)$, $\det(x)$, $\text{Tr}(x)$ and $\text{Rk}(x)$ are respectively the transpose, the adjugate, the determinant, the trace and the rank of x .
- For $x \in \mathcal{S}_d^+(\mathbb{R})$, \sqrt{x} denotes the unique symmetric positive semidefinite matrix such that $(\sqrt{x})^2 = x$
- The identity matrix is denoted by I_d . We set for $1 \leq i, j \leq d$, $e_d^{i,j} = (\mathbb{1}_{k=i, l=j})_{1 \leq k, l \leq d}$ and $e_d^i = e_d^{i,i}$. Last, we define $e_d^{\{i,j\}} = e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i}$.

- For $x \in \mathcal{S}_d(\mathbb{R})$, we denote by $x_{\{i,j\}}$ the value of $x_{i,j}$, so that $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} e_d^{\{i,j\}}$. We use both notations in the paper: notation $(x_{i,j})_{1 \leq i,j \leq d}$ is of course more convenient for matrix calculations while $(x_{\{i,j\}})_{1 \leq i \leq j \leq d}$ is preferred to emphasize that we work on symmetric matrices and that we have $x_{i,j} = x_{j,i}$.
- For $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, $\text{diag}(\lambda_1, \dots, \lambda_d) \in \mathcal{S}_d(\mathbb{R})$ denotes the diagonal matrix such that $\text{diag}(\lambda_1, \dots, \lambda_d)_{i,i} = \lambda_i$.
- For $x \in \mathcal{S}_d^+(\mathbb{R})$ such that $x_{i,i} > 0$ for all $1 \leq i \leq d$, we define $\mathbf{p}(x) \in \mathfrak{C}_d(\mathbb{R})$ by

$$(\mathbf{p}(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \quad 1 \leq i, j \leq d. \quad (3.5)$$

- For $x \in \mathcal{S}_d(\mathbb{R})$ and $1 \leq i \leq d$, we denote by $x^{[i]} \in \mathcal{S}_{d-1}(\mathbb{R})$ the matrix defined by $x_{k,l}^{[i]} = x_{k+1_{k \geq i}, l+1_{l \geq i}}$ and $x^i \in \mathbb{R}^{d-1}$ the vector defined by $x_k^i = x_{i,k}$ for $1 \leq k < i$ and $x_k^i = x_{i,k+1}$ for $i \leq k \leq d-1$. For $x \in \mathfrak{C}_d(\mathbb{R})$, we have $(x - x e_d^i x)^{[i]} = x^{[i]} - x^i (x^i)^T$.

3.1 Some properties of MRC processes

3.1.1 The infinitesimal generator

We first calculate the quadratic covariation of $MRC_d(x, \kappa, c, a)$. By Lemma 72, we get:

$$\begin{aligned} \langle d(X_t)_{i,j}, d(X_t)_{k,l} \rangle &= \left[a_i^2 (\mathbb{1}_{i=k} (X_t - X_t e_d^i X_t)_{j,l} + \mathbb{1}_{i=l} (X_t - X_t e_d^i X_t)_{j,k}) \right. \\ &\quad \left. + a_j^2 (\mathbb{1}_{j=k} (X_t - X_t e_d^j X_t)_{i,l} + \mathbb{1}_{j=l} (X_t - X_t e_d^j X_t)_{i,k}) \right] dt \\ &= \left[a_i^2 (\mathbb{1}_{i=k} ((X_t)_{j,l} - (X_t)_{i,j} (X_t)_{i,l}) + \mathbb{1}_{i=l} ((X_t)_{j,k} - (X_t)_{i,j} (X_t)_{i,k})) \right. \\ &\quad \left. + a_j^2 (\mathbb{1}_{j=k} ((X_t)_{i,l} - (X_t)_{j,i} (X_t)_{j,l}) + \mathbb{1}_{j=l} ((X_t)_{i,k} - (X_t)_{j,i} (X_t)_{j,k})) \right] dt. \end{aligned} \quad (3.6)$$

We remark in particular that $d\langle (X_t)_{i,j}, d(X_t)_{k,l} \rangle = 0$ when i, j, k, l are distinct.

We are now in position to calculate the infinitesimal generator of $MRC_d(x, \kappa, c, a)$. The infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$ is defined by:

$$x \in \mathfrak{C}_d(\mathbb{R}), \quad L^{\mathcal{M}} f(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \quad \text{for } f \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives.}$$

By straightforward calculations, we get from (3.6) that:

$$\begin{aligned} L^{\mathcal{M}} &= \sum_{\substack{1 \leq i, j \leq d \\ j \neq i}} (\kappa_i + \kappa_j) (c_{i,j} - x_{i,j}) \partial_{i,j} \\ &\quad + \frac{1}{2} \sum_{\substack{1 \leq i, j, k \leq d \\ j \neq i, k \neq i}} a_i^2 (x_{j,k} - x_{i,j} x_{i,k}) [\partial_{i,j} \partial_{i,k} + \partial_{i,j} \partial_{k,i} + \partial_{j,i} \partial_{i,k} + \partial_{j,i} \partial_{k,i}]. \end{aligned} \quad (3.7)$$

Here, $\partial_{i,j}$ denotes the derivative with respect to the element at the i^{th} line and j^{th} column. We know however that the process that we consider is valued in $\mathfrak{C}_d(\mathbb{R}) \subset \mathcal{S}_d(\mathbb{R})$. Though it is equivalent, it is often more convenient to work with the infinitesimal generator on $\mathcal{S}_d(\mathbb{R})$, which is defined by:

$$x \in \mathfrak{C}_d(\mathbb{R}), Lf(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives,}$$

since it eliminates redundant coordinates. For $x \in \mathcal{S}_d(\mathbb{R})$, we denote by $x_{\{i,j\}} = x_{i,j} = x_{j,i}$ the value of the coordinates (i, j) and (j, i) , so that $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})$. For $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, $\partial_{\{i,j\}} f$ denotes its derivative with respect to $x_{\{i,j\}}$. For $x \in \mathcal{M}_d(\mathbb{R})$, we set $\pi(x) = (x + x^T)/2$. It is such that $\pi(x) = x$ for $x \in \mathcal{S}_d(\mathbb{R})$, and we have $Lf(x) = L^{\mathcal{M}} f \circ \pi(x)$. By the chain rule, we have for $x \in \mathcal{S}_d(\mathbb{R})$, $\partial_{i,j} f \circ \pi(x) = (\mathbb{1}_{i=j} + \frac{1}{2} \mathbb{1}_{i \neq j}) \partial_{\{i,j\}} f(x)$ and we get:

$$L = \sum_{i=1}^d \left(\sum_{\substack{1 \leq j \leq d \\ j \neq i}} \kappa_i (c_{\{i,j\}} - x_{\{i,j\}}) \partial_{\{i,j\}} + \frac{1}{2} \sum_{\substack{1 \leq j, k \leq d \\ j \neq i, k \neq i}} a_i^2 (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}} \right). \quad (3.8)$$

Then, we will say that a process $(X_t, t \geq 0)$ valued in $\mathfrak{C}_d(\mathbb{R})$ solves the martingale problem of $MRC_d(x, \kappa, c, a)$ if for any $n \in \mathbb{N}^*$, $0 \leq t_1 \leq \dots \leq t_n \leq t \leq s$, $g_1, \dots, g_n \in \mathcal{C}(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ we have:

$$\mathbb{E} \left[\prod_{i=1}^n g_i(X_{t_i}) \left(f(X_t) - f(X_s) - \int_s^t Lf(X_s) ds \right) \right] = 0, \text{ and } X_0 = x \quad (3.9)$$

Now, we state simple but interesting properties of mean-reverting correlation processes. Each non-diagonal coefficient follows a Wright-Fisher type diffusion and any principal submatrix is also a mean-reverting correlation process. This result is a direct consequence of the calculus above and the weak uniqueness of the SDE (3.2) obtained in Corollary 48.

Proposition 46 — *Let $(X_t)_{t \geq 0} \sim MRC_d(x, \kappa, c, a)$. For $1 \leq i \neq j \leq d$, there is Brownian motion $(\beta_t^{i,j}, t \geq 0)$ such that*

$$d(X_t)_{i,j} = (\kappa_i + \kappa_j)(c_{i,j} - (X_t)_{i,j})dt + \sqrt{a_i^2 + a_j^2} \sqrt{1 - (X_t)_{i,j}^2} d\beta_t^{i,j}. \quad (3.10)$$

Let $I = \{k_1 < \dots < k_{d'}\} \subset \{1, \dots, d\}$ such that $1 < d' < d$. For $x \in \mathcal{M}_d(\mathbb{R})$, we define $x^I \in \mathcal{M}_{d'}(\mathbb{R})$ by $(x^I)_{i,j} = x_{k_i, k_j}$ for $1 \leq i, j \leq d'$. We have:

$$(X_t^I)_{t \geq 0} \stackrel{\text{law}}{=} MRC_{d'}(x^I, \kappa^I, c^I, a^I).$$

3.1.2 Calculation of moments and the ergodic law

We first introduce some notations that are useful to characterise the general form for moments. For every $x \in \mathcal{S}_d(\mathbb{R})$, $m \in \mathcal{S}_d(\mathbb{N})$, we set:

$$x^m = \prod_{1 \leq i \leq j \leq d} x_{\{i,j\}}^{m_{\{i,j\}}} \text{ and } |m| = \sum_{1 \leq i \leq j \leq d} m_{\{i,j\}}.$$

A function $f : \mathcal{S}_d(\mathbb{R}) \rightarrow \mathbb{R}$ is a polynomial function of degree smaller than $n \in \mathbb{N}$ if there are real numbers a_m such that $f(x) = \sum_{|m| \leq n} a_m x^m$, and we define the norm of f by $\|f\|_{\mathbb{P}} = \sum_{|m| \leq n} |a_m|$.

We want to calculate the moments $\mathbb{E}[X_t^m]$ of $(X_t, t \geq 0) \sim MRC_d(x, \kappa, c, a)$. Since the diagonal elements are equal to 1, we will take $m_{\{i,i\}} = 0$. Let us also remark that for $i \neq j$ such that $\kappa_i = \kappa_j = 0$, we have from (3.3) that $a_i = a_j = 0$. Therefore we get $(X_t)_{i,j} = x_{i,j}$ by (3.10).

Proposition 47 — *Let $m \in \mathcal{S}_d(\mathbb{N})$ such that $m_{i,i} = 0$ for $1 \leq i \leq d$. Let $(X_t)_{t \geq 0} \sim MRC_d(x, \kappa, c, a)$. For $m \in \mathcal{S}_d(\mathbb{N})$, $Lx^m = -K_m x^m + f_m(x)$, with*

$$K_m = \sum_{i=1}^d \sum_{j=1}^d \kappa_i m_{\{i,j\}} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}} m_{\{i,k\}}$$

and

$$f_m(x) = \sum_{i=1}^d \sum_{j=1}^d \kappa_i c_{\{i,j\}} m_{\{i,j\}} x^{m - e_d^{\{i,j\}}} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}} m_{\{i,k\}} x^{m - e_d^{\{i,j\}} - e_d^{\{i,k\}} + e_d^{\{j,k\}}}$$

is a polynomial function of degree smaller than $|m| - 1$. We have

$$\mathbb{E}[X_t^m] = x^m \exp(-tK_m) + \exp(-tK_m) \int_0^t \exp(sK_m) \mathbb{E}[f_m(X_s)] ds. \quad (3.11)$$

Proof: The calculation of Lx^m is straightforward from (3.8). By using Itô's formula, we get easily that $\frac{d\mathbb{E}[X_t^m]}{dt} = -K_m \mathbb{E}[X_t^m] + \mathbb{E}[f_m(X_t)]$, which gives (3.11). \square

Equation (3.11) allows us to calculate explicitly any moment by induction on $|m|$. Here are the formula for moments of order 1 and 2:

$$\forall 1 \leq i \neq j \leq d, \mathbb{E}[(X_t)_{i,j}] = x_{i,j} e^{-t(\kappa_i + \kappa_j)} + c_{i,j} (1 - e^{-t(\kappa_i + \kappa_j)}),$$

and for given $1 \leq i \neq j \leq d$ and $1 \leq k \neq l \leq d$ such that $\kappa_i + \kappa_j > 0$ and $\kappa_k + \kappa_l > 0$,

$$\begin{aligned} \mathbb{E}[(X_t)_{i,j} (X_t)_{k,l}] &= x_{i,j} x_{k,l} e^{-tK_{i,j,k,l}} + (\kappa_i + \kappa_j) c_{i,j} \gamma_{k,l}(t) + (\kappa_k + \kappa_l) c_{k,l} \gamma_{i,j}(t) \\ &\quad + a_i^2 (\mathbb{1}_{i=k} \gamma_{j,l}(t) + \mathbb{1}_{i=l} \gamma_{j,k}(t)) + a_j^2 (\mathbb{1}_{j=k} \gamma_{i,l}(t) + \mathbb{1}_{j=l} \gamma_{i,k}(t)), \end{aligned}$$

where $K_{i,j,k,l} = \kappa_i + \kappa_j + \kappa_k + \kappa_l + a_i^2 (\mathbb{1}_{i=k} + \mathbb{1}_{i=l}) + a_j^2 (\mathbb{1}_{j=k} + \mathbb{1}_{j=l})$ and

$$\forall m, n \in \{i, j, k, l\}, \gamma_{m,n}(t) = c_{m,n} \frac{1 - e^{-tK_{i,j,k,l}}}{K_{i,j,k,l}} + (x_{m,n} - c_{m,n}) \frac{e^{-t(\kappa_m + \kappa_n)} - e^{-tK_{i,j,k,l}}}{K_{i,j,k,l} - \kappa_m - \kappa_n}.$$

Let f be a polynomial function of degree smaller than $n \in \mathbb{N}$. From Proposition 47, L is a linear mapping on the polynomial functions of degree smaller than n , and there is a constant $C_n > 0$ such that $\|Lf\|_{\mathbb{P}} \leq C_n \|f\|_{\mathbb{P}}$. On the other hand, we have by Itô's formula $\mathbb{E}[f(X_t)] = f(x) +$

$\int_0^t \mathbb{E}[L^i f(X_s)] ds$, and by iterating $\mathbb{E}[f(X_t)] = \sum_{i=0}^k \frac{t^i}{i!} L^i f(x) + \int_0^t \frac{(t-s)^k}{k!} \mathbb{E}[L^{k+1} f(X_s)] ds$. Since $\|L^i f\|_{\mathbb{P}} \leq C_n^i \|f\|_{\mathbb{P}}$, the series converges and we have

$$\mathbb{E}[f(X_t)] = \sum_{i=0}^{\infty} \frac{t^i}{i!} L^i f(x) \quad (3.12)$$

for any polynomial function f . We also remark that the same iterated Itô's formula gives

$$\forall f \in C^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R}), \forall k \in \mathbb{N}^*, \exists C > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), |\mathbb{E}[f(X_t)] - \sum_{i=0}^k \frac{t^i}{i!} L^i f(x)| \leq C t^{k+1}, \quad (3.13)$$

since $L^{k+1} f$ is a bounded functions on $\mathfrak{C}_d(\mathbb{R})$.

Let us discuss some interesting consequences of Proposition 47. Obviously, we can calculate explicitly in the same manner $\mathbb{E}[X_{t_1}^{m_1} \dots X_{t_n}^{m_n}]$ for $0 \leq t_1 \leq \dots \leq t_n$ and $m_1, \dots, m_n \in \mathcal{S}_d(\mathbb{N})$. Therefore, the law of $(X_{t_1}, \dots, X_{t_n})$ is entirely determined and we get the weak uniqueness for the SDE (3.2).

Corollary 48 — *Every solution $(X_t, t \geq 0)$ to the martingale problem (3.9) have the same law.*

Proposition 47 allows us to compute the limit $\lim_{t \rightarrow +\infty} \mathbb{E}[X_t^m]$ that we note $\mathbb{E}[X_\infty^m]$ by a slight abuse of notation. Let us observe that $K_m > 0$ if and only if there is i, j such that $\kappa_i + \kappa_j > 0$ and $m_{i,j} > 0$. We have

$$\begin{aligned} \mathbb{E}[X_\infty^m] &= x^m \text{ if } m \in \mathcal{S}_d(\mathbb{N}) \text{ is such that } m_{\{i,j\}} > 0 \iff \kappa_i = \kappa_j = 0, \\ \mathbb{E}[X_\infty^m] &= \mathbb{E}[f_m(X_\infty)]/K_m \text{ otherwise.} \end{aligned} \quad (3.14)$$

Thus, X_t converges in law when $t \rightarrow +\infty$, and the moments $\mathbb{E}[X_\infty^m]$ are uniquely determined by (3.14) with an induction on $|m|$. In addition, if $\kappa_i + \kappa_j > 0$ for any $1 \leq i, j \leq d$ (which means that at most only one coefficient of κ is equal to 0), the law of X_∞ does not depend on the initial condition and is the unique invariant law. In this case the ergodic moments of order 1 and 2 are given by:

$$\begin{aligned} \mathbb{E}[(X_\infty)_{i,j}(X_\infty)_{k,l}] &= \frac{a_i^2(\mathbb{1}_{i=k}c_{j,l} + \mathbb{1}_{i=l}c_{j,k}) + a_j^2(\mathbb{1}_{j=k}c_{i,l} + \mathbb{1}_{j=l}c_{i,k})}{K_{i,j,k,l}} \\ &= \frac{(\kappa_i + \kappa_j + \kappa_k + \kappa_l)c_{i,j}c_{k,l}}{K_{i,j,k,l}}, \\ \mathbb{E}[(X_\infty)_{i,j}] &= c_{i,j}. \end{aligned}$$

3.1.3 The connection with Wishart processes

Wishart processes are affine processes on positive semidefinite matrices. They have been introduced by Bru [17] and solves the following SDE:

$$Y_t^y = y + \int_0^t ((\alpha + 1)a^T a + bY_s^y + Y_s^y b^T) ds + \int_0^t \left(\sqrt{Y_s^y} dW_s a + a^T dW_s^T \sqrt{Y_s^y} \right), \quad (3.15)$$

where $a, b \in \mathcal{M}_d(\mathbb{R})$ and $y \in \mathcal{S}_d^+(\mathbb{R})$. Strong uniqueness holds when $\alpha \geq d$ and $y \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Weak existence and uniqueness holds when $\alpha \geq d - 2$. This is in fact very similar to the results that we obtain for mean-reverting correlation processes. The parameter $\alpha + 1$ is called the number of degrees of freedom, and we denote by $WIS_d(y, \alpha + 1, b, a)$ the law of $(Y_t^y, t \geq 0)$.

Once we have a positive semidefinite matrix $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$, a trivial way to construct a correlation matrix is to consider $\mathbf{p}(y)$, where \mathbf{p} is defined by (3.5). Thus, it is somehow natural then to look at the dynamics of $\mathbf{p}(Y_t^y)$, provided that the diagonal elements of the Wishart process do not vanish. In general, this does not lead to an autonomous SDE. However, the particular case where the Wishart parameters are $a = e_d^1$ and $b = 0$ is interesting since it leads to the SDE satisfied by the mean-reverting correlation processes, up to a change of time. Obviously, we have a similar property for $a = e_d^i$ and $b = 0$ by a permutation of the i th and the first coordinates.

Proposition 49 — *Let $\alpha \geq \max(1, d - 2)$ and $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$. Let $(Y_t^y)_{t \geq 0} \sim WIS_d(y, \alpha + 1, 0, e_d^1)$. Then, $(Y_t^y)_{i,i} = y_{i,i}$ for $2 \leq i \leq d$ and $(Y_t^y)_{1,1}$ follows a squared Bessel process of dimension $\alpha + 1$ and a.s. never vanishes. We set*

$$X_t = \mathbf{p}(Y_t^y), \quad \phi(t) = \int_0^t \frac{1}{(Y_s^y)_{1,1}} ds.$$

The function ϕ is a.s. one-to-one on \mathbb{R}_+ and defines a time-change such that:

$$(X_{\phi^{-1}(t)}, t \geq 0) \stackrel{\text{law}}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1).$$

In particular, there is a weak solution to $MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1)$. Besides, the processes $(X_{\phi^{-1}(t)}, t \geq 0)$ and $((Y_t^y)_{1,1}, t \geq 0)$ are independent.

Proof: From (3.15), $a = e_d^1$ and $b = 0$, we get $d(Y_t^y)_{i,j} = 0$ for $2 \leq i, j \leq d$ and

$$d(Y_t^y)_{1,1} = (\alpha + 1)dt + 2 \sum_{k=1}^d (\sqrt{Y_t^y})_{1,k} (dW_t)_{k,1}, \quad d(Y_t^y)_{1,i} = \sum_{k=1}^d (\sqrt{Y_t^y})_{i,k} (dW_t)_{k,1}. \quad (3.16)$$

In particular, $d\langle (Y_t^y)_{1,1} \rangle = 4(Y_t^y)_{1,1} dt$ and $(Y_t^y)_{1,1}$ is a squared Bessel process of dimension $\alpha + 1$. Since $\alpha + 1 \geq 2$ it almost surely never vanishes. Thus, $(X_t, t \geq 0)$ is well defined, and we get:

$$d(X_t)_{1,i} = -\frac{\alpha}{2} (X_t)_{1,i} \frac{dt}{(Y_t^y)_{1,1}} + \sum_{k=1}^d \left(\frac{(\sqrt{Y_t^y})_{i,k}}{\sqrt{(Y_t^y)_{1,1}} y_{i,i}} - (X_t)_{1,i} \frac{(\sqrt{Y_t^y})_{1,k}}{(Y_t^y)_{1,1}} \right) (dW_t)_{k,1} \quad (3.17)$$

By Lemma 75, $\phi(t)$ is a.s. one-to-one on \mathbb{R}_+ , and we consider the Brownian motion $(\tilde{W}_t, t \geq 0)$ defined by $(\tilde{W}_{\phi(t)})_{i,j} = \int_0^t \frac{(dW_s)_{i,j}}{\sqrt{(Y_s^y)_{1,1}}} ds$. We have by straightforward calculus

$$d(X_{\phi^{-1}(t)})_{1,i} = -\frac{\alpha}{2} (X_{\phi^{-1}(t)})_{1,i} dt + \sum_{k=1}^d \left(\frac{(\sqrt{Y_{\phi^{-1}(t)}^y})_{i,k}}{\sqrt{y_{i,i}}} - (X_{\phi^{-1}(t)})_{1,i} \frac{(\sqrt{Y_{\phi^{-1}(t)}^y})_{1,k}}{\sqrt{(Y_{\phi^{-1}(t)}^y)_{1,1}}} \right) (d\tilde{W}_t)_{k,1}$$

$$d\langle (X_{\phi^{-1}(t)})_{1,i}, (X_{\phi^{-1}(t)})_{1,j} \rangle = [(X_{\phi^{-1}(t)})_{i,j} - (X_{\phi^{-1}(t)})_{1,i} (X_{\phi^{-1}(t)})_{1,j}] dt, \quad (3.18)$$

which shows by uniqueness of the solution of the martingale problem (Corollary 48) that $(X_{\phi^{-1}(t)}, t \geq 0) \stackrel{law}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2}e_d^1, I_d, e_d^1)$.

Let us now show the independence. We can check easily that

$$d\langle (X_t)_{1,i}, (X_t)_{1,j} \rangle = \frac{1}{(Y_t^y)_{1,1}} [(X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j}] \text{ and } d\langle (X_t)_{1,i}, (Y_t^y)_{1,1} \rangle = 0. \quad (3.19)$$

We define $\Psi(y) \in \mathcal{S}_d(\mathbb{R})$ for $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ by $\Psi(y)_{1,i} = \Psi(y)_{i,1} = y_{1,i}/\sqrt{y_{1,1}y_{i,i}}$ and $\Psi(y)_{i,j} = y_{i,j}$ otherwise. By (3.16) and (3.17), $(\Psi(Y_t), t \geq 0)$ solves an SDE on $\mathcal{S}_d(\mathbb{R})$. This SDE has a unique weak solution. Indeed, we can check that for any solution $(\tilde{Y}_t, t \geq 0)$ starting from $\Psi(y)$, $(\Psi^{-1}(\tilde{Y}_t), t \geq 0) \sim WIS_d(y, \alpha + 1, 0, e_d^1)$, which gives our claim since Ψ is one-to-one and weak uniqueness holds for $WIS_d(y, \alpha + 1, 0, e_d^1)$ (see [17]). Let $(B_t, t \geq 0)$ denote a real Brownian motion independent of $(W_t, t \geq 0)$. We consider a weak solution to the SDE, for all $2 \leq i, j \leq d$

$$\begin{aligned} d(\bar{Y}_t)_{1,1} &= (\alpha + 1)dt + 2\sqrt{(\bar{Y}_t)_{1,1}}dB_t, & d(\bar{Y}_t)_{i,j} &= 0, \\ d(\bar{Y}_t)_{1,i} &= -\frac{\alpha}{2}(\bar{Y}_t)_{1,i}\frac{dt}{(\bar{Y}_t)_{1,1}} + \sum_{k=1}^d \left(\frac{(\sqrt{\bar{Y}_t})_{i,k}}{\sqrt{(\bar{Y}_t)_{1,1}y_{i,i}}} - (\bar{Y}_t)_{1,i}\frac{(\sqrt{\bar{Y}_t})_{1,k}}{(\bar{Y}_t)_{1,1}} \right) (dW_t)_{k,1}, \end{aligned}$$

that starts from $\bar{Y}_0 = \Psi(y)$. It solves the same martingale problem as $\Psi(Y_t)$, and therefore $(\Psi(Y_t), t \geq 0) \stackrel{law}{=} (\bar{Y}_t, t \geq 0)$. We set $\bar{\phi}(t) = \int_0^t \frac{1}{(Y_s)_{1,1}} ds$. As above, $((\bar{Y}_{\bar{\phi}^{-1}(t)})_{1,i}, i = 2 \dots, d)$ solves an SDE driven by $(W_t, t \geq 0)$ and is therefore independent of $((\bar{Y}_t)_{1,1}, t \geq 0)$, which gives the desired independence. \square

Remark 50 — *There is a connection between squared-Bessel processes and one-dimensional Wright-Fisher diffusions that is similar to Proposition 49. Let us consider $Z_t^i = z_i + \beta_i t + \int_0^t \sigma \sqrt{Z_s^i} dB_s^i, i = 1, 2$ two squared Bessel processes driven by independent Brownian motions. We assume that $\beta_1, \beta_2, \sigma \geq 0$ and $\sigma^2 \leq 2(\beta_1 + \beta_2)$ so that $Y_t = Z_t^1 + Z_t^2$ is a squared Bessel processes that never reaches 0. By using Itô calculus, there is a real Brownian $(B_t, t \geq 0)$ motion such that $X_t = Z_t^1/Y_t$ satisfies*

$$dX_t = (\beta_1 + \beta_2) \left(\frac{\beta_1}{\beta_1 + \beta_2} - X_t \right) \frac{dt}{Y_t} + \sigma \sqrt{X_t(1 - X_t)} \frac{dB_t}{\sqrt{Y_t}},$$

and we have $\langle dX_t, dY_t \rangle = 0$. Thus, we can use the same argument as in the proof above: we set $\phi(t) = \int_0^t 1/(Y_s) ds$ and get that $(X_{\phi^{-1}(t)}, t \geq 0)$ is a one-dimensional Wright-Fisher diffusion that is independent of $(Y_t, t \geq 0)$. This property obviously extends the well known identity between Gamma and Beta laws. This kind of change of time have also been considered in the literature by [39] or [47] for similar but different multi-dimensional settings.

3.1.4 A remarkable splitting of the infinitesimal generator

In this section, we present a remarkable splitting for the mean-reverting correlation matrices. This result will play a key role in the simulation part. In fact, we have already obtained in [1]

very similar properties for Wishart processes. Of course, these properties are related through Proposition 49, which is illustrated in the proof below.

Theorem 51 — *Let $\alpha \geq d - 2$. Let L be the generator associated to the $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$ on $\mathfrak{C}_d(\mathbb{R})$ and L_i be the generator associated to $MRC_d(x, \frac{\alpha}{2}e_d^i, I_d, e_d^i)$, for $i \in \{1, \dots, d\}$. Then, we have*

$$L = \sum_{i=1}^d a_i^2 L_i \text{ and } \forall i, j \in \{1, \dots, d\}, L_i L_j = L_j L_i. \quad (3.20)$$

Proof: The formula $L = \sum_{i=1}^d a_i^2 L_i$ is obvious from (3.8). The commutativity property can be obtained directly by a tedious but simple calculus, which is made in Appendix 3.4.3. Here, we give another proof that uses the link between Wishart and Mean-Reverting Correlation processes given by Proposition 49.

Let L_i^W denotes the generator of $WIS_d(x, \alpha + 1, 0, e_d^i)$. From [1], we have $L_i^W L_j^W = L_j^W L_i^W$ for $1 \leq i, j \leq d$. Let us consider $\alpha \geq \max(5, d - 2)$ and $x \in \mathfrak{C}_d(\mathbb{R})$. We set for $i = 1, 2$ ($Y_t^{i,x}, t \geq 0$) $\sim WIS_d(x, \alpha + 1, 0, e_d^i)$, and we assume that the Brownian motions of their associated SDEs are independent. Since $L_1^W L_2^W = L_2^W L_1^W$, we know from [1] that $Y_t^{1, Y_t^{2,x}} \stackrel{law}{=} Y_t^{2, Y_t^{1,x}}$ and thus

$$\mathbb{E}[f(\mathbf{p}(Y_t^{1, Y_t^{2,x}}))] = \mathbb{E}[f(\mathbf{p}(Y_t^{2, Y_t^{1,x}}))],$$

for any polynomial function f . By Proposition 49, $\mathbf{p}(Y_t^{1, Y_t^{2,x}}) \stackrel{law}{=} X_{(\phi^1)^{-1}(\phi^1(t))}^{1, \mathbf{p}(Y_t^{2,x})}$, where the process $(X_{(\phi^1)^{-1}(u)}^{1, \mathbf{p}(Y_t^{2,x})}, u \geq 0)$ is a mean-reverting correlation process independent of $\phi^1(t) = \int_0^t \frac{1}{(Y_s^{1, Y_t^{2,x}})_{1,1}} ds$. Since $(Y_t^{2,x})_{1,1} = 1$, $(Y_s^{1, Y_t^{2,x}})_{1,1}$ follows a squared Bessel of dimension $\alpha + 1$ starting from 1. Using the independence, we get by (3.13)

$$\mathbb{E}[f(\mathbf{p}(Y_t^{1, Y_t^{2,x}})) | Y_t^{2,x}, \phi^1(t)] = f(\mathbf{p}(Y_t^{2,x})) + \phi^1(t) L_1 f(\mathbf{p}(Y_t^{2,x})) + \frac{\phi^1(t)^2}{2} L_1^2 f(\mathbf{p}(Y_t^{2,x})) + O(\phi^1(t)^3).$$

By Lemma 76, we have $\mathbb{E}[\phi^1(t)] = t + \frac{3-\alpha}{2}t^2 + O(t^3)$, $\mathbb{E}[\phi^1(t)^2] = t + O(t^3)$, $\mathbb{E}[\phi^1(t)^3] = O(t^3)$. Thus, we get:

$$\begin{aligned} & \mathbb{E}[f(\mathbf{p}(Y_t^{2, Y_t^{1,x}})) | Y_t^{2,x}] \\ &= f(\mathbf{p}(Y_t^{2,x})) + t L_1 f(\mathbf{p}(Y_t^{2,x})) + \frac{t^2}{2} [L_1^2 f(\mathbf{p}(Y_t^{2,x})) + (3 - \alpha) L_1 f(\mathbf{p}(Y_t^{2,x}))] + O(t^3). \end{aligned}$$

Once again, we use Proposition 49 and (3.13) to get similarly that $E[f(\mathbf{p}(Y_t^{2,x}))] = f(x) + t L_2 f(x) + \frac{t^2}{2} [L_2^2 f(x) + (3 - \alpha) L_2 f(x)] + O(t^3)$ for any polynomial function f . We finally get:

$$\begin{aligned} & \mathbb{E}[f(\mathbf{p}(Y_t^{1, Y_t^{2,x}}))] = f(x) + t(L_1 + L_2)f(x) \\ &+ \frac{t^2}{2} [L_1^2 f(x) + 2L_2 L_1 f(x) + L_2^2 f(x) + (3 - \alpha)(L_1 + L_2)f(x)] + O(t^3). \end{aligned}$$

Similarly, we also have

$$\begin{aligned} & \mathbb{E}[f(\mathbf{p}(Y_t^{2, Y_t^{1,x}}))] = f(x) + t(L_1 + L_2)f(x) \\ &+ \frac{t^2}{2} [L_1^2 f(x) + 2L_1 L_2 f(x) + L_2^2 f(x) + (3 - \alpha)(L_1 + L_2)f(x)] + O(t^3), \end{aligned} \quad (3.21)$$

and since both expectations are equal, we get $L_1 L_2 f(x) = L_2 L_1 f(x)$ for any $\alpha \geq \max(5, d - 2)$. However, we can write $L_i = \frac{1}{2}(\alpha L_i^D + L_i^M)$, with

$$L_i^D = \sum_{\substack{1 \leq j \leq d \\ j \neq i}} x_{\{i,j\}} \partial_{\{i,j\}} \quad \text{and} \quad L_i^M = \sum_{\substack{1 \leq j, k \leq d \\ j \neq i, k \neq i}} (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}}.$$

Thus, we have $\alpha^2 L_1^D L_2^D + \alpha(L_1^D L_2^M + L_1^M L_2^D) + L_1^M L_2^M = \alpha^2 L_2^D L_1^D + \alpha(L_2^D L_1^M + L_2^M L_1^D) + L_2^M L_1^M$ for any $\alpha \geq \max(5, d - 2)$. This gives $L_1^D L_2^D = L_2^D L_1^D$, $L_1^D L_2^M + L_1^M L_2^D = L_2^D L_1^M + L_2^M L_1^D$, $L_1^M L_2^M = L_2^M L_1^M$, and therefore $L_1 L_2 = L_2 L_1$ holds without restriction on α . \square

Remark 52 — Let $x \in \mathfrak{C}_d(\mathbb{R})$, $(Y_t^{1,x}, t \geq 0) \sim WIS_d(x, \alpha + 1, 0, e_d^1)$ and L_1^W its infinitesimal generator. Equation (3.21) and the formula $E[f(\mathbf{p}(Y_t^{1,x}))] = f(x) + t L_1 f(x) + \frac{t^2}{2}[L_1^2 f(x) + (3 - \alpha)L_1 f(x)] + O(t^3)$ used in the proof above lead formally to the following identities for $x \in \mathfrak{C}_d(\mathbb{R})$ and $f \in C^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$,

$$\begin{aligned} L_1^W(f \circ \mathbf{p})(x) &= L_1 f(x), \quad (L_1^W)^2(f \circ \mathbf{p})(x) = L_1^2 f(x) + (3 - \alpha)L_1 f(x), \\ L_1^W L_2^W(f \circ \mathbf{p})(x) &= L_1 L_2 f(x), \end{aligned}$$

that can be checked by basic calculations.

The property given by Theorem 51 will help us to prove the weak existence of mean-reverting correlation processes. It plays also a key role to construct discretization scheme for these diffusions. In fact, it gives a simple way to sample the law $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a, ;t)$. Let $x \in \mathfrak{C}_d(\mathbb{R})$. We construct iteratively:

- $X_t^{1,x} \sim MRC_d(x, \frac{\alpha}{2}a_1^2 e_d^1, I_d, a_1 e_d^1, ;t)$,
- For $2 \leq i \leq d$, the distribution $X_t^{i, \dots, X_t^{1,x}} \sim MRC_d(X_t^{i-1, \dots, X_t^{1,x}}, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i, ;t)$ is sampled, conditionally to $X_t^{i-1, \dots, X_t^{1,x}}$, independently according to the distribution of a mean-reverting correlation process at time t with parameters $(\frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$ starting from $X_t^{i-1, \dots, X_t^{1,x}}$.

Proposition 53 — Let $X_t^{d, \dots, X_t^{1,x}}$ be defined as above. Then, $X_t^{d, \dots, X_t^{1,x}} \sim MRC_d(x, \frac{\alpha}{2}a^2, I_d, a, ;t)$.

Let us notice that $MRC_d(x, \frac{\alpha}{2}, a_i^2 e_d^i, I_d, a_i e_d^i; t) \stackrel{\text{law}}{=} MRC_d(x, \frac{\alpha}{2}, e_d^i, I_d; e_d^i) a_i^2 t$. Moreover, we also remark that $MRC_d(x, \frac{\alpha}{2}, e_d^i, I_d, e_d^i; t)$ and $MRC_d(x, \frac{\alpha}{2}, e_d^i, I_d, e_d^1; t)$ are the same law up to the permutation of the first and the i -th coordinate. Thus, it is sufficient to be able to sample this latter law in order to sample $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a, ;t)$ by Proposition 53.

Proof: Let f be a polynomial function and $X_t^x \sim MRC_d(x, \frac{\alpha}{2}, a^2, I_d, a; t)$. By (3.12), $\mathbb{E}[f(X_t^x)] = \sum_{j=0}^{\infty} \frac{t^j}{j!} L^j f(x)$. Using once again (3.12),

$$\mathbb{E}[f(X_t^{d, \dots, X_t^{1,x}})] = \mathbb{E}[\mathbb{E}[f(X_t^{d, \dots, X_t^{1,x}}) | X_t^{d-1, \dots, X_t^{1,x}}]] = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbb{E}[L_d^j f(X_t^{d-1, \dots, X_t^{1,x}})],$$

and we finally obtain by iterating

$$\mathbb{E}[f(X_t^{d,\dots,X_t^{1,x}})] = \sum_{j_1,\dots,j_d=0}^{\infty} \frac{t^{j_1+\dots+j_d}}{j_1! \dots j_d!} L_1^{j_1} \dots L_d^{j_d} f(x) = \sum_{j=0}^d \frac{t^j}{j!} (L_1 + \dots + L_d)^j f(x) = \mathbb{E}[f(X_t^x)],$$

since the operators commute. \square

We can also extend Proposition 53 to the limit laws. More precisely, let us denote by $MRC_d(x, \kappa, c, a; \infty)$ the law characterized by (3.14). We define similarly for $x \in \mathfrak{C}_d(\mathbb{R})$, $X_\infty^{1,x} \sim MRC_d(x, \frac{\alpha}{2} a_1^2 e_d^1, I_d, a_1 e_d^1; \infty)$ and, conditionally to $X_\infty^{i-1,\dots,X_\infty^{1,x}}$, we obtain $X_\infty^{i,\dots,X_\infty^{1,x}} \sim MRC_d(X_\infty^{i-1,\dots,X_\infty^{1,x}}, \frac{\alpha}{2} a_i^2 e_d^i, I_d, a_i e_d^i; \infty)$ for $2 \leq i \leq d$. We have:

$$X_\infty^{d,\dots,X_\infty^{1,x}} \sim MRC_d(x, \frac{\alpha}{2} a^2, I_d, a; \infty). \quad (3.22)$$

To check this we consider $(X_t, t \geq 0) \sim MRC_d(x, \frac{\alpha}{2} a^2, I_d, a)$ and $m \in \mathcal{S}_d(\mathbb{N})$ such that $m_{i,i} = 0$. By Proposition 47, $\mathbb{E}[X_t^m]$ is a polynomial function of x that we write $\mathbb{E}[X_t^m] = \sum_{m' \in \mathcal{S}_d(\mathbb{N}), |m'| \leq |m|} \gamma_{m,m'}(t) x^{m'}$. From the convergence in law (3.14), we get that the coefficients $\gamma_{m,m'}(t)$ go to a limit $\gamma_{m,m'}(\infty)$ when $t \rightarrow +\infty$, and $\mathbb{E}[X_\infty^m] = \sum_{|m'| \leq |m|} \gamma_{m,m'}(\infty) x^{m'}$. Similarly, the moment m of $MRC_d(x, \frac{\alpha}{2} a_i^2 e_d^i, I_d, a_i e_d^i; t)$ can be written as $\sum_{|m'| \leq |m|} \gamma_{m,m'}^i(t) x^{m'}$. We get from Proposition 53:

$$\mathbb{E}[X_t^m] = \sum_{|m_1| \leq \dots \leq |m_d| \leq |m|} \gamma_{m,m_d}^d(t) \gamma_{m_d,m_{d-1}}^{d-1}(t) \dots \gamma_{m_2,m_1}^1(t) x^{m_1},$$

which gives (3.22) by letting $t \rightarrow +\infty$.

3.1.5 A link with the multi-allele Wright-Fisher model

Theorem 51 and Proposition 53 have shown that any law $MRC_d(x, \frac{\alpha}{2} a^2, I_d, a; t)$ can be obtained by composition with the elementary law $MRC_d(x, \frac{\alpha}{2}, I_d, e_d^1; t)$. By the next proposition, we can go further and focus on the case where $(x_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$.

Proposition 54 — *Let $x \in \mathfrak{C}_d(\mathbb{R})$. Let $u \in \mathcal{M}_{d-1}(\mathbb{R})$ and $\check{x} \in \mathfrak{C}_d(\mathbb{R})$ such that $x = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \check{x} \begin{pmatrix} 1 & 0 \\ 0 & u^T \end{pmatrix}$ and $(\check{x})_{2 \leq i,j \leq d} = I_{d-1}$ (Lemma 71 gives a construction of such matrices). Then, for $\alpha \geq 2$,*

$$MRC_d(x, \frac{\alpha}{2} e_d^1, I_d, e_d^1) \stackrel{\text{law}}{=} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} MRC_d(\check{x}, \frac{\alpha}{2} e_d^1, I_d, e_d^1) \begin{pmatrix} 1 & 0 \\ 0 & u^T \end{pmatrix}.$$

Proof: Let $(\check{X}_t, t \geq 0) \sim MRC_d(\check{x}, \frac{\alpha}{2} e_d^1, I_d, e_d^1)$. We set $X_t = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \check{X}_t \begin{pmatrix} 1 & 0 \\ 0 & u^T \end{pmatrix}$. Clearly, $((\check{X}_t)_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$ and the matrix $((X_t)_{i,j})_{2 \leq i,j \leq d}$ is constant and equal to $uu^T =$

$(x_{i,j})_{2 \leq i,j \leq d}$. We have for $2 \leq i \leq d$, $(X_t)_{1,i} = \sum_{k=2}^d u_{i-1,k-1}(\check{X}_t)_{1,k}$. By (3.6), we get $\langle d(\check{X}_t)_{1,k}, d(\check{X}_t)_{1,l} \rangle = [\mathbb{1}_{k=l} - (\check{X}_t)_{1,k}(\check{X}_t)_{1,l}]dt$. Therefore, the quadratic variations

$$\begin{aligned} \langle d(X_t)_{1,i}, d(X_t)_{1,j} \rangle &= \left(\sum_{k=2}^d u_{i-1,k-1} u_{j-1,k-1} - \sum_{k,l=2}^d u_{i-1,k-1}(\check{X}_t)_{1,k} u_{j-1,l-1}(\check{X}_t)_{1,l} \right) dt \\ &= ((X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j}) dt, \end{aligned}$$

are by (3.6) the one of $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$. This gives the claim by using the weak uniqueness (Corollary 48). \square

For $x \in \mathcal{S}_d(\mathbb{R})$ such that $(x_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$ and $x_{1,1} = 1$, we have $\det(x) = 1 - \sum_{i=2}^d x_{1,i}^2$ and therefore

$$x \in \mathfrak{C}_d(\mathbb{R}) \iff \sum_{i=2}^d x_{1,i}^2 \leq 1. \quad (3.23)$$

The process $(X_t)_{t \geq 0} \sim MRC_d(x, \frac{\alpha}{2}, I_d, e_d^1; t)$ is such that $((X_t)_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$. In this case, the only non constant elements are on the first row (or column). More precisely, $((X_t)_{1,i})_{i=2,\dots,d}$ is a vector process on the unit ball in dimension $d-1$ such that

$$d\langle (X_t)_{1,i}, (X_t)_{1,j} \rangle = (\mathbb{1}_{i=j} - (X_t)_{1,i}(X_t)_{1,j})dt.$$

For $i = 1, \dots, d-1$, we set $\zeta_t^i = (X_t)_{1,i+1}^2$. We have $\langle d\zeta_t^i, d\zeta_t^j \rangle = 4\zeta_t^i(\mathbb{1}_{i=j} - \zeta_t^j)dt$ and the drift of ζ_t^i is $(1 - (1 + 2\alpha)\zeta_t^i)dt$. Thus, $(\zeta_t^i)_{1 \leq i \leq d-1}$ satisfies $\sum_{i=1}^{d-1} \zeta_t^i \leq 1$ and has the following infinitesimal generator

$$\sum_{i=1}^{d-1} [1 - (1 + 2\alpha)z_i] \partial_{z_i} + 2 \sum_{1 \leq i,j \leq d-1} z_i (\mathbb{1}_{i=j} - z_j) \partial_{z_i} \partial_{z_j}$$

This is a particular case of the multi-allele Wright-Fisher diffusion (see for example Etheridge [38]), where $(\zeta_t^1, \dots, \zeta_t^{d-1}, 1 - \sum_{i=1}^{d-1} \zeta_t^i)$ describes population ratios along the time. Similar diffusions have also been considered by Gourieroux and Jasiak [46] in a different context. Roughly speaking, $((X_t)_{1,i})_{2 \leq i \leq d}$ can be seen as a square-root of a multi-allele Wright-Fisher diffusion that is such that its drift coefficient remains linear.

Also, the identity in law given by Proposition 54 allows us to compute more explicitly the ergodic limit law. Let $x \in \mathfrak{C}_d(\mathbb{R})$ such that $(x_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$, $(X_t^x)_{t \geq 0} \sim MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$ and $(Y_t^x)_{t \geq 0} \sim WIS_d(x, \alpha + 1, 0, e_d^1)$. We know by [1] that $((Y_t^x)_{i,j})_{1 \leq i,j \leq d} = I_{d-1}$ and

$$((Y_t^x)_{1,i})_{1 \leq i \leq d} \stackrel{law}{=} (Z_t^{x_{1,1}} + \sum_{i=2}^d (x_{1,i} + \sqrt{t}N_i)^2, x_{1,2} + \sqrt{t}N_2, \dots, x_{1,d} + \sqrt{t}N_d),$$

where $N_i \sim \mathcal{N}(0, 1)$ are independent standard Gaussian variables and $Z_t^{x_{1,1}} = x_{1,1} + (\alpha + 2 - d)t + 2 \int_0^t \sqrt{Z_u^{x_{1,1}}} d\beta_u$ is a Bessel process independent of the Gaussian variables starting from $x_{1,1}$. By a time scaling, we have $Z_t^{x_{1,1}} \stackrel{law}{=} tZ_1^{x_{1,1}/t}$, and thus:

$$(\mathbf{p}(Y_t^x)_{1,i})_{2 \leq i \leq d} \stackrel{law}{=} \frac{\left(\frac{x_{1,2}}{\sqrt{t}} + N_2, \dots, \frac{x_{1,d}}{\sqrt{t}} + N_d \right)}{\sqrt{Z_1^{x_{1,1}/t} + \sum_{i=2}^d \left(\frac{x_{1,i}}{\sqrt{t}} + N_i \right)^2}} \xrightarrow{t \rightarrow +\infty} \frac{(N_2, \dots, N_d)}{\sqrt{Z_1^0 + \sum_{i=2}^d N_i^2}}.$$

On the other hand, we know that X_t^x converges in law when $t \rightarrow +\infty$, and Proposition 49 immediately gives, with the help of Lemma 75 that $((X_\infty^x)_{1,i})_{2 \leq i \leq d} \stackrel{\text{law}}{=} \frac{(N_2, \dots, N_d)}{\sqrt{Z_1^0 + \sum_{i=2}^d N_i^2}}$. By simple calculations, we get that $((X_\infty^x)_{1,i})_{2 \leq i \leq d}$ has the following density:

$$\mathbb{1}_{\sum_{i=2}^d z_i^2 \leq 1} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{(\sqrt{\pi})^{d-1} \Gamma\left(\frac{\alpha+2-d}{2}\right)} \left(1 - \sum_{i=2}^d z_i^2\right). \quad (3.24)$$

In particular, we can check that $((X_\infty^x)_{1,i}^2)_{2 \leq i \leq d}$ follows a Dirichlet law, which is known as the ergodic limit of multi-allele Wright-Fisher models. Last, let us mention that we can get an explicit but cumbersome expression of the density of the law $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; \infty)$ by combining (3.22), Proposition 54 and (3.24).

3.2 Existence and uniqueness results for MRC processes

In this section we show weak and strong existence results for the SDE (3.2), respectively under assumptions (3.3) and (3.4). These assumptions are of the same nature as the one known for Wishart processes. To prove the strong existence and uniqueness, we make assumptions on the coefficients that ensures that X_t remains in the set of the invertible correlation matrices where the coefficients are locally Lipschitz. This is similar to the proof given by Bru [17] for Wishart processes. Then, we prove the weak existence by introducing a sequence of processes defined on $\mathfrak{C}_d(\mathbb{R})$, which is tight such that any subsequence limit solves the martingale problem (3.9). Next, we extend our existence results when the parameters are no longer constant. Last, we exhibit some change of probability that preserves the global dynamics of our Mean-Reverting Correlation processes.

3.2.1 Strong existence and uniqueness

Theorem 55 — *Let $x \in \mathfrak{C}_d^*(\mathbb{R})$. We assume that (3.4) holds. Then, there is a unique strong solution of the SDE (3.2) that is such that $\forall t \geq 0, X_t \in \mathfrak{C}_d^*(\mathbb{R})$.*

Proof: By Lemma 68, we have $(\sqrt{x - xe_d^n x})^{[n]} = \sqrt{x^{[n]} - x^n(x^n)^T}$ and $x^{[n]} - x^n(x^n)^T \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$ when $x \in \mathfrak{C}_d^*(\mathbb{R})$. For $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ such that $x^{[n]} - x^n(x^n)^T \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$, we define $f^n(x) \in \mathcal{S}_d^+(\mathbb{R})$ by $(f^n(x))_{n,j} = 0$ for $1 \leq j \leq d$ and $(f^n(x))^{[n]} = \sqrt{x^{[n]} - x^n(x^n)^T}$. The function f^n is well defined on an open set of $\mathcal{S}_d(\mathbb{R})$ that includes $\mathfrak{C}_d^*(\mathbb{R})$, and is such that $f^n(x) = \sqrt{x - xe_d^n x}$ for $x \in \mathfrak{C}_d^*(\mathbb{R})$. Since the square-root of a positive semi-definite matrix is locally Lipschitz on the positive definite matrix set, we get that the SDE

$$X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds + \sum_{n=1}^d a_n \int_0^t (f^n(X_s) dW_s e_d^n + e_d^n dW_s^T f^n(X_s)),$$

has a unique strong solution for $0 \leq t < \tau$, where

$$\tau = \inf\{t \geq 0, X_t \notin \mathcal{S}_d^{+,*}(\mathbb{R}) \text{ or } \exists i \in \{1, \dots, d\}, X_t^{[i]} - X_t^i (X_t^i)^T \notin \mathcal{S}_{d-1}^{+,*}(\mathbb{R})\}, \inf \emptyset = +\infty.$$

For $1 \leq i \leq d$, we have $(f^n(X_s)dW_s e_d^n)_{i,i} = \mathbb{1}_{i=n} \sum_{j=1}^d f^n(X_s)_{n,j} (dW_s)_{j,n} = 0$ and then:

$$d(X_t)_{i,i} = 2\kappa_{i,i}(1 - (X_t)_{i,i})dt,$$

which immediately gives $(X_t)_{i,i} = 1$ for $0 \leq t < \tau$. Thus, $X_t \in \mathfrak{C}_d^*(\mathbb{R})$ for $0 \leq t < \tau$ and $\tau = \inf\{t \geq 0, X_t \notin \mathfrak{C}_d^*(\mathbb{R})\}$ by Lemma 68, and the process X_t is solution of (3.2) up to time τ . We set $Y_t = \log(\det(X_t)) + \text{Tr}(2\kappa - a^2)t$. By Lemma 73, we have

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \text{Tr}[X_s^{-1}(\kappa c + c\kappa - da^2)]ds + 2 \int_0^t \sqrt{\text{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_s \\ &\geq Y_0 + 2 \int_0^t \sqrt{\text{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_s, \end{aligned}$$

since $\kappa c + c\kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R})$ by Assumption (3.4). Now, we use the McKean argument exactly like Bru [17] did for Wishart processes: on $\{\tau < \infty\}$, $Y_t \xrightarrow{t \rightarrow \tau} -\infty$, and the local martingale $\int_0^t \sqrt{\text{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_s \xrightarrow{t \rightarrow \tau} -\infty$, which is almost surely not possible. We deduce that $\tau = +\infty$, a.s. \square

3.2.2 Weak existence and uniqueness

The weak uniqueness has already been obtained in Proposition 47, and we provide in this section a constructive proof of a weak solution to the SDE (3.2). In the case $d = 2$, this result is already well-known. In fact, by Proposition 46, the associated martingale problem is the one of a one-dimensional Wright-Fisher process. For this SDE, strong (and therefore weak) existence and uniqueness holds since the diffusion coefficient is 1/2-Hölderian.

Thus, we can assume without loss of generality that $d \geq 3$. The first step is to focus on the existence when $a = \text{diag}(a_1, \dots, a_d) \in \mathcal{S}_d^+(\mathbb{R})$, $\alpha \geq d - 2$, $\kappa = \frac{\alpha}{2}a^2$ and $c = I_d$. By Proposition 49, we know that weak existence holds for $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$, and thus for $MRC_d(x, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$ for $i = 1, \dots, d$ and $a_i \geq 0$, by using a permutation of the coordinates and a linear time-scaling. Therefore, by using Proposition 53, the distribution $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a, \cdot; t)$ is also well-defined on $\mathfrak{C}_d(\mathbb{R})$ for any $t \geq 0$. Let $T > 0$ be a time-horizon, $N \in \mathbb{N}^*$, and $t_i^N = iT/N$. We define $(\hat{X}_t^N, t \in [0, T])$ as follows.

- We set $\hat{X}_0^N = x$.
- For $i = 0, \dots, N-1$, $\hat{X}_{t_{i+1}^N}^N$ is sampled according to the law $MRC_d(\hat{X}_{t_i^N}^N, \frac{\alpha}{2}a^2, I_d, a; T/N)$, conditionally to $\hat{X}_{t_i^N}^N$.
- For $t \in [t_i^N, t_{i+1}^N]$, $\hat{X}_t^N = \frac{t-t_i^N}{T/N} \hat{X}_{t_i^N}^N + \frac{t_{i+1}^N-t}{T/N} \hat{X}_{t_{i+1}^N}^N = \hat{X}_{t_i^N}^N + \frac{t-t_i^N}{T/N} (\hat{X}_{t_{i+1}^N}^N - \hat{X}_{t_i^N}^N)$.

The process $(\hat{X}_t^N, t \in [0, T])$ is continuous and such that almost surely, $\forall t \in [0, T]$, $\hat{X}_t^N \in \mathfrak{C}_d(\mathbb{R})$. We endow the set of matrices with the norm $\|x\| = \left(\sum_{i,j=1}^d x_{i,j}^A\right)^{1/4}$. The sequence of processes $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$ satisfies the following Kolmogorov tightness criterion.

Lemma 56 — *Under the assumptions above, there is a constant $K > 0$ such that:*

$$\forall 0 \leq s \leq t \leq T, \mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] \leq K(t-s)^2. \quad (3.25)$$

Proof: We first consider the case $s = t_k^N$ and $t = t_l^N$ for some $0 \leq k \leq l \leq N$. Then, by Proposition 53, we know that conditionally on $\hat{X}_{t_k^N}^N$, $\hat{X}_{t_l^N}^N$ follows $MRC_d(\hat{X}_{t_k^N}^N, \frac{\alpha}{2}a^2, I_d, a)$. In particular, each element $(\hat{X}_{t_l^N}^N)_{i,j}$ follows the marginal law of a one-dimensional Wright-Fisher process with parameters given by equation (3.10). Thus, by Proposition 74 there is a constant still denoted by $K > 0$ such that for any $1 \leq i, j \leq d$, $\mathbb{E}[(\hat{X}_{t_l^N}^N)_{i,j} - (\hat{X}_{t_k^N}^N)_{i,j}]^4] \leq K(t_l^N - t_k^N)^2$, and therefore

$$\mathbb{E}[\|\hat{X}_{t_l^N}^N - \hat{X}_{t_k^N}^N\|^4] \leq Kd^2(t_l^N - t_k^N)^2.$$

Let us consider now $0 \leq s \leq t \leq T$. If there exists $0 \leq k \leq N-1$, such that $s, t \in [t_k^N, t_{k+1}^N]$, then $\mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] = \left(\frac{s-t}{T/N}\right)^4 \mathbb{E}[\|\hat{X}_{t_{k+1}^N}^N - \hat{X}_{t_k^N}^N\|^4] \leq Kd^2(s-t)^2$. Otherwise, there are $k \leq l$ such that $t_k^N - T/N < s \leq t_k^N \leq t_l^N \leq t < t_l^N + T/N$, and $\mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] \leq Kd^2[(t_k^N - s)^2 + (t - t_l^N)^2 + (t_l^N - t_k^N)^2] \leq K'(t-s)^2$ for some constant $K' > 0$. \square

The sequence $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$ is tight, and we will show that any limit of subsequence solves the the martingale problem (3.9). More precisely, we will show that for any $n \in \mathbb{N}^*$, $0 \leq t_1 \leq \dots \leq t_n \leq t \leq s \leq T$, $g_1, \dots, g_n \in \mathcal{C}(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, $f \in \mathcal{C}^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ we have:

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N) \left(f(\hat{X}_t^N) - f(\hat{X}_s^N) - \int_s^t Lf(\hat{X}_u^N) du \right) \right] = 0. \quad (3.26)$$

We set $k^N(s)$ and $l^N(t)$ the indices such that $t_{k^N(s)}^N - T/N < s \leq t_{k^N(s)}^N$ and $t_{l^N(t)}^N \leq t < t_{l^N(t)}^N + T/N$. Clearly, f is Lipschitz and Lf is bounded on $\mathfrak{C}_d(\mathbb{R})$. It is therefore sufficient to show that

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N) \left(f(\hat{X}_{t_{l^N(t)}^N}^N) - f(\hat{X}_{t_{k^N(s)}^N}^N) - \int_{t_{k^N(s)}^N}^{t_{l^N(t)}^N} Lf(\hat{X}_u^N) du \right) \right] = 0. \quad (3.27)$$

We decompose the expectation as the sum of

$$\begin{aligned} & \mathbb{E} \left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N) \int_{t_{k^N(s)}^N}^{t_{l^N(t)}^N} (Lf(\hat{X}_{t_{l^N(t)}^N}^N) - Lf(\hat{X}_u^N)) du \right] \\ & + \mathbb{E} \left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N) \left(\sum_{j=k^N(s)}^{l^N(t)-1} f(\hat{X}_{t_{j+1}^N}^N) - f(\hat{X}_{t_j^N}^N) - \frac{T}{N} Lf(\hat{X}_{t_j^N}^N) \right) \right] \end{aligned} \quad (3.28)$$

To get that the first expectation goes to 0, we claim that:

$$\mathbb{E} \left[\int_{t_{k^N(s)}^N}^{t_{l^N(t)}^N} |\beta(u, \hat{X}_u^N) - \beta(t_{l^N(t)}^N, \hat{X}_{t_{l^N(t)}^N}^N)| du \right] \rightarrow 0 \quad (3.29)$$

when $\beta : (t, x) \in [0, T] \times \mathfrak{C}_d(\mathbb{R}) \rightarrow \mathbb{R}$ is continuous. This formulation will be reused later on. By Lemma 56, (3.29) holds when β is Lipschitz with respect to (t, x) . If β is not Lipschitz, we can still approximate it uniformly on the compact set $[0, T] \times \mathfrak{C}_d(\mathbb{R})$ by using for example the Stone-Weierstrass theorem, which gives (3.29).

On the other hand, we know by (3.13) that the second expectation goes to 0. To be precise, (3.13) has been obtained by using Itô's formula while we do not know yet at this stage that the process $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$ exists. It is nevertheless true: (3.13) holds for $MRC_d(x, \frac{\alpha}{2}a_i^2 e_i^i, I_d, e_i^i)$ since this process is already known to be well defined, and we get by using Proposition 53 and Proposition 63 that $\exists K > 0, |f(\hat{X}_{t_{j+1}^N}^N) - f(\hat{X}_{t_j^N}^N) - (T/N)Lf(\hat{X}_{t_j^N}^N)| \leq K/N^2$. Thus, $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$ converges in law to a solution of the martingale problem (3.9). This concludes the existence of $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$.

Now, we are in position to show the existence of $MRC_d(x, \kappa, c, a)$ under Assumption (3.3). We denote by $\xi(t, x)$ the solution to the linear ODE:

$$\xi'(t, x) = \kappa(c - x) + (c - x)\kappa - \frac{d-2}{2}[a^2(I_d - x) + (I_d - x)a^2], \quad \xi(0, x) = x \in \mathfrak{C}_d(\mathbb{R}). \quad (3.30)$$

By Lemma 67, we know that $\forall t \geq 0, \xi'(t, x) \in \mathfrak{C}_d(\mathbb{R})$. It is also easy to check that:

$$\exists K > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), \|\xi(t, x) - x\| \leq Kt.$$

Now, we define $(\hat{X}_t^N, t \in [0, T])$ as follows.

- We set $\hat{X}_0^N = x \in \mathfrak{C}_d(\mathbb{R})$.
- For $i = 0, \dots, N-1$, and conditionally to $\hat{X}_{t_i^N}^N$, the distribution $\hat{X}_{t_{i+1}^N}^N$ is sampled according to $MRC_d(\xi(T/N, \hat{X}_{t_i^N}^N), \frac{d-2}{2}a^2, I_d, a; T/N)$. More precisely, we denote by $(\bar{X}_t^N, t \in [t_i^N, t_{i+1}^N])$ a solution to

$$\begin{aligned} \bar{X}_t^N &= \xi(T/N, \hat{X}_{t_i^N}^N) + \frac{d-2}{2} \int_{t_i^N}^t [a^2(I_d - \bar{X}_u^N) + (I_d - \bar{X}_u^N)a^2] du \\ &\quad + \sum_{n=1}^d a_n \int_{t_i^N}^t \left(\sqrt{\bar{X}_u^N - \bar{X}_u^N e_d^n \bar{X}_u^N} dW_u e_d^n + e_d^n dW_u^T \sqrt{\bar{X}_u^N - \bar{X}_u^N e_d^n \bar{X}_u^N} \right), \end{aligned}$$

and we set $\hat{X}_{t_{i+1}^N}^N = \bar{X}_{t_{i+1}^N}^N$.

- For $t \in [t_i^N, t_{i+1}^N]$, $\hat{X}_t^N = \hat{X}_{t_i^N}^N + \frac{t-t_i^N}{T/N}(\hat{X}_{t_{i+1}^N}^N - \hat{X}_{t_i^N}^N)$.

We proceed similarly and show that the Kolmogorov criterion (3.25) holds for $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$. As already shown in Lemma 56, it is sufficient to check that this criterion holds for $s = t_k^N \leq t = t_l^N$. We have

$$\begin{aligned} \|\hat{X}_{t_l^N}^N - \hat{X}_{t_k^N}^N\|^4 &= \left\| \sum_{j=k}^{l-1} \hat{X}_{t_{j+1}^N}^N - \xi(T/N, \hat{X}_{t_j^N}^N) + \xi(T/N, \hat{X}_{t_j^N}^N) - \hat{X}_{t_j^N}^N \right\|^4 \\ &\leq 2^3 \left(\left\| \sum_{j=k}^{l-1} \bar{X}_{t_{j+1}^N}^N - \bar{X}_{t_j^N}^N \right\|^4 + (l-k)^4 \left(\frac{KT}{N} \right)^4 \right). \end{aligned}$$

Since $(\bar{X}_t^N, t \in [0, T])$ is valued in the compact set $\mathfrak{C}_d(\mathbb{R})$, we get easily by using Burkholder-Davis-Gundy inequality that $\mathbb{E}[\|\sum_{j=k}^{l-1} \bar{X}_{t_{j+1}}^N - \bar{X}_{t_j}^N\|^4] \leq K(t_l - t_k)^2$ and then $\mathbb{E}[\|\hat{X}_{t_l}^N - \hat{X}_{t_k}^N\|^4] \leq K(t_l - t_k)^2$ for some constant $K > 0$ that does not depend on N .

Thus, $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$ satisfies the Kolmogorov criterion and is tight. It remains to show that any subsequence converges in law to the solution of the martingale problem (3.9). We proceed as before and reuse the same notations. From (3.28), it is sufficient to show that

$$\exists K > 0, |f(\hat{X}_{t_{j+1}}^N) - f(\hat{X}_{t_j}^N) - (T/N)Lf(\hat{X}_{t_j}^N)| \leq K/N^2.$$

Once again, we cannot directly use (3.13) since we do not know at this stage that the process $MRC_d(x, \kappa, c, a)$ exists. We have $L = L^\xi + \tilde{L}$, where L^ξ is the operator associated to $\xi(t, x)$ and \tilde{L} is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$. We have: $\exists K > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), |f(\xi(t, x)) - f(x) - tL^\xi f(x)| \leq Kt^2$, and (3.13) holds for \tilde{L} . By Proposition 63, we get: $\exists K > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), |f(\xi(t, x)) - f(x) - tf(x)| \leq Kt^2$, which gives (3.26) and concludes the proof of the weak existence.

Theorem 57 — *Under assumption (3.3), there is a unique weak solution $(X_t, t \geq 0)$ to SDE (3.2) such that $\mathbb{P}(\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})) = 1$.*

Remark 58 — *Assumption (3.3) has only be used in the proof of Theorem 57 to ensure that ξ defined by (3.30) satisfies*

$$\forall t \geq 0, x \in \mathfrak{C}_d(\mathbb{R}), \xi(t, x) \in \mathfrak{C}_d(\mathbb{R}). \quad (3.31)$$

As pointed by Remark 66, this is a sufficient but not necessary condition. In fact, a weak solution of (3.2) exists under (3.31), which is more general but less tractable condition than (3.3).

3.2.3 Extension to non-constant coefficients

In this paragraph, we consider the SDE (3.2) with time and space dependent coefficients:

$$\begin{aligned} X_t = x + \int_0^t [\kappa(s, X_s)(c(s, X_s) - X_s) + (c(s, X_s) - X_s)\kappa(s, X_s)] ds \\ + \sum_{n=1}^d a_n(s, X_s) \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_s e_d^n X_s} \right), \end{aligned} \quad (3.32)$$

where $\kappa(t, x)$, $c(t, x)$ and $a(t, x)$ are measurable functions such that for any $t \geq 0$ and $x \in \mathfrak{C}_d(\mathbb{R})$, $\kappa(t, x)$ and $a(t, x)$ are nonnegative diagonal matrices and $c(t, x) \in \mathfrak{C}_d(\mathbb{R})$. Then, under the following assumption

$$\begin{aligned} \forall T > 0, \sup_{t \in [0, T]} |\kappa(t, I_d)| < \infty, \forall t \in [0, T], \exists K > 0, \|f(t, x) - f(t, y)\| \leq K\|x - y\| \text{ for } f \in \{\kappa, c, a\}, \\ \forall t \geq 0, x \in \mathfrak{C}_d(\mathbb{R}), \kappa(t, x)c(t, x) + c(t, x)\kappa(t, x) - da^2(t, x) \in \mathcal{S}_d^+(\mathbb{R}) \text{ and } X_0 \in \mathfrak{C}_d^*(\mathbb{R}), \end{aligned} \quad (3.33)$$

strong existence and uniqueness holds for (3.32). To get this result, we observe that $\mathbf{p}(x)$ is Lipschitz on $\{x \in \mathcal{S}_d^+(\mathbb{R}) \text{ s.t. } \forall 1 \leq i \leq d, 1/2 \leq x_{i,i} \leq 2\}$. Therefore, the SDE

$$X_t = x + \int_0^t (\kappa(s, \mathbf{p}(X_s))[c(s, \mathbf{p}(X_s)) - X_s] + [c(s, \mathbf{p}(X_s)) - X_s]\kappa(s, \mathbf{p}(X_s))) ds \\ + \sum_{n=1}^d \int_0^t a_n(s, \mathbf{p}(X_s)) (f^n(X_s) dW_s e_d^n + e_d^n dW_s^T f^n(X_s))$$

has a unique solution up to time $\tau = \inf\{t \geq 0, X_t \notin \mathcal{S}_d^{+,*}(\mathbb{R}) \text{ or } \exists i \in \{1, \dots, d\}, X_t^{[i]} - X_t^i (X_t^i)^T \notin \mathcal{S}_{d-1}^{+,*}(\mathbb{R}) \text{ or } (X_t)_{i,i} \notin [1/2, 2]\}$, and we proceed then exactly as for the proof of Theorem 55.

Also, weak existence holds for (3.32) if we assume that:

$$\kappa(t, x), c(t, x), a(t, x) \text{ are continuous on } \mathbb{R}_+ \times \mathfrak{C}_d(\mathbb{R}) \\ \forall t \geq 0, x \in \mathfrak{C}_d(\mathbb{R}), \kappa(t, x)c(t, x) + c(t, x)\kappa(t, x) - (d-2)a^2(t, x) \in \mathcal{S}_d^+(\mathbb{R}).$$

To get this result, we proceed as in Section 3.2.2 and define $(\hat{X}_t^N, t \in [0, T])$ as follows.

- We set $\hat{X}_0^N = x$.
- For $i = 0, \dots, N-1$, we denote by $(\bar{X}_t^N, t \in [t_i^N, t_{i+1}^N])$ a solution to

$$\bar{X}_t^N = \hat{X}_{t_i^N}^N + \int_{t_i^N}^t \left[\kappa(t_i^N, \hat{X}_{t_i^N}^N)(c(t_i^N, \hat{X}_{t_i^N}^N) - \bar{X}_u^N) + (c(t_i^N, \hat{X}_{t_i^N}^N) - \bar{X}_u^N)\kappa(t_i^N, \hat{X}_{t_i^N}^N) \right] du \\ + \sum_{n=1}^d a_n(t_i^N, \hat{X}_{t_i^N}^N) \int_{t_i^N}^t \left(\sqrt{\bar{X}_u^N - \bar{X}_u^N e_d^n \bar{X}_u^N} dW_u e_d^n + e_d^n dW_u^T \sqrt{\bar{X}_u^N - \bar{X}_u^N e_d^n \bar{X}_u^N} \right),$$

and we set $\hat{X}_{t_{i+1}^N}^N = \bar{X}_{t_{i+1}^N}^N$.

- For $t \in [t_i^N, t_{i+1}^N]$, $\hat{X}_t^N = \hat{X}_{t_i^N}^N + \frac{t-t_i^N}{T/N}(\hat{X}_{t_{i+1}^N}^N - \hat{X}_{t_i^N}^N)$.

We can check that $(\hat{X}_t^N, t \in [0, T])$ satisfies the Kolmogorov criterion and is tight. To obtain (3.26), we proceed as in Section 3.2.2. More precisely, let us denote for $u \in [0, T]$ L_u the infinitesimal generator of (3.32), and \hat{L}_u the infinitesimal generator with frozen coefficient at $(t_i, \hat{X}_{t_i}^N)$ when $u \in [t_i^N, t_{i+1}^N]$. In (3.28), the first term tends to zero :

$$\mathbb{E} \left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N) \int_{t_{k^N(s)}^N}^{t_{l^N(t)}^N} (\hat{L}_u f(\hat{X}_{t_{l^N(u)}^N}^N) - L_u f(\hat{X}_u^N)) du \right] \rightarrow 0$$

thanks to (3.29), and the second term goes to 0 as before.

To sum up, it is rather easy to extend our results of strong existence and uniqueness, and weak existence when the coefficients are not constant. However, we can no longer get explicit formulas for the moments in this case. Thus, if the coefficients satisfy (3.34) but not (3.33), the weak uniqueness remains an open question, which is beyond the scope of this paper.

3.2.4 A Girsanov Theorem

In this section, we will use an alternative writing of the SDE (3.2). In fact, by Lemma (72), the SDE

$$X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds + \sum_{n=1}^d a_n \int_0^t (h_n(X_s) dW_s e_d^n + e_d^n dW_s^T h_n(X_s)^T), \quad (3.34)$$

is associated to the same martingale problem as $MRC_d(x, \kappa, c, a)$ for any functions $h_n : \mathcal{S}_d(\mathbb{R}) \rightarrow \mathcal{M}_d(\mathbb{R})$ such that $h_n(x)h_n(x)^T = x - xe_d^n x$ for $x \in \mathfrak{C}_d(\mathbb{R})$. In this paper, we have arbitrarily decided to take the symmetric version $h_n(x) = \sqrt{x - xe_d^n x}$. Obviously, other choices are possible. An interesting choice is the following one:

$$x \in \mathcal{S}_d^+(\mathbb{R}), h_n(x) = \sqrt{x} \sqrt{I_d - \sqrt{x} e_d^n \sqrt{x}} = \sqrt{x} (I_d - \sqrt{x} e_d^n \sqrt{x}), \quad (3.35)$$

where the second equality comes from Lemma 69. Obviously, our weak existence and uniqueness results (Theorem 57) applies to (3.34) since (3.2) and (3.34) solve the same martingale problem. However, we have to show again that strong uniqueness holds for (3.34) under Assumption (3.4) and $x \in \mathfrak{C}_d^*(\mathbb{R})$. The proof is in fact very similar to Theorem 55. We know that there is one strong solution to

$$\begin{aligned} X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds \\ + \sum_{n=1}^d a_n \int_0^t \left(\sqrt{X_s} (I_d - \sqrt{X_s} e_d^n \sqrt{X_s}) dW_s e_d^n + e_d^n (I_d - \sqrt{X_s} e_d^n \sqrt{X_s}) \sqrt{X_s} dW_s^T \right) \end{aligned}$$

up to time $\tau = \inf\{t \geq 0, X_t \notin \mathcal{S}_d^+(\mathbb{R})\}$. On $t \in [0, \tau)$, there are real Brownian motions β_t^i such that

$$d(X_t)_{i,i} = 2\kappa_i(1 - (X_t)_{i,i})dt + 2a_i(1 - (X_t)_{i,i})\sqrt{(X_t)_{i,i}}d\beta_t^i,$$

which gives $(X_t)_{i,i} = 1$ by strong uniqueness of this SDE. We then conclude as in the proof of Theorem 55 and get in particular that $X_t \in \mathfrak{C}_d^*(\mathbb{R})$ for $t \geq 0$.

We consider now a solution to (3.34), and a progressively measurable process $(H_s)_{s \geq 0}$, valued in $\mathcal{M}_d(\mathbb{R})$, such that

$$\mathcal{E}_t^H = \exp \left(\int_0^t \text{Tr}(H_s^T dW_s) - \frac{1}{2} \int_0^t \text{Tr}(H_s^T H_s) ds \right) \quad (3.36)$$

is a martingale. For a given time horizon $T > 0$, we denote by \mathbb{Q} the probability measure, if it exists, defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \mathcal{E}_T^H, \quad (3.37)$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of the process $(X_t)_{t \geq 0}$. Then, $W_t^{\mathbb{Q}} = W_t - \int_0^t H_s ds$ is a

$d \times d$ Brownian matrix under \mathbb{Q} , and the process $(X_t)_{t \geq 0}$ satisfies

$$\begin{aligned}
X_t &= x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) ds \\
&+ \int_0^t \left(\sum_{i=1}^d a_i \left\{ \sqrt{X_s} \left[I_d - \sqrt{X_s} e_d^i \sqrt{X_s} \right] H_s e_d^i + e_d^i H_s^T \left[I_d - \sqrt{X_s} e_d^i \sqrt{X_s} \right] \sqrt{X_s} \right\} \right) ds \\
&+ \sum_{i=1}^d a_i \int_0^t \left(\sqrt{X_s} \left[I_d - \sqrt{X_s} e_d^i \sqrt{X_s} \right] dW_s^{\mathbb{Q}} e_d^i + e_d^i d(W_s^{\mathbb{Q}})^T \left[I_d - \sqrt{X_s} e_d^i \sqrt{X_s} \right] \sqrt{X_s} \right).
\end{aligned} \tag{3.38}$$

We present now changes of probability such that $(X_t, t \geq 0)$ is also a mean-reverting correlation process under \mathbb{Q} .

Proposition 59 — *We assume (3.3). We consider $(X_t, t \geq 0) \sim MRC_d(x, \kappa, c, a)$ and take $H_t = \sqrt{X_t} \lambda$, with $\lambda = \text{diag}(\lambda_1, \dots, \lambda_d) \in \mathcal{S}_d(\mathbb{R})$. Then, (3.36) is a martingale and $(X_t, t \geq 0) \sim MRC_d(x, \kappa, c, a)$ under \mathbb{Q} .*

Proof: Since the process $(X_t, t \geq 0)$ is bounded, (3.36) is clearly a martingale. For $y \in \mathfrak{C}_d(\mathbb{R})$, $e_d^i y e_d^i = e_d^i$ and we have $(\sqrt{y}(I_d - \sqrt{y} e_d^i \sqrt{y})) \sqrt{y} \lambda e_d^i = \lambda_i (y - y e_d^i y) e_d^i = 0$, which gives the result by (3.38). \square

Proposition 60 — *Let $x \in \mathfrak{C}_d^*(\mathbb{R})$. We consider $(X_t, t \geq 0) \sim MRC_d(x, \kappa^1, c^1, a)$ and assume that κ^1, c^1, a satisfy (3.4). Let $c^2 \in \mathfrak{C}_d(\mathbb{R})$ and κ^2 be a real diagonal matrix such that $a_i = 0 \implies \kappa_i^2 = 0$ and $\kappa^1 c^1 + c^1 \kappa^1 + \kappa^2 c^2 + c^2 \kappa^2 - da^2 \in \mathcal{S}_d^+(\mathbb{R})$. We set:*

$$\lambda = \text{diag}(\lambda_1, \dots, \lambda_d) \text{ with } \lambda_i = \begin{cases} \kappa_i^2 / a_i & \text{if } a_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } H_t = (\sqrt{X_t})^{-1} c^2 \lambda$$

This defines with (3.36) and (3.37) a change of probability such that

$$(X_t, t \geq 0) \sim MRC_d(x, \kappa, c, a) \text{ under } \mathbb{Q},$$

where $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d) \in \mathcal{S}_d^+(\mathbb{R})$ and $c \in \mathfrak{C}_d(\mathbb{R})$ are defined as in Lemma 67.

Proof: We have $a_i^1 \sqrt{y}(I_d - \sqrt{y} e_d^i \sqrt{y}) \sqrt{y}^{-1} c^2 \lambda e_d^i = \kappa_i^2 (c^2 e_d^i - y e_d^i c^2 e_d^i) = \kappa_i^2 (c^2 - y) e_d^i$, which gives the claim by (3.38), provided that $\mathbb{E}[\mathcal{E}_T^H] = 1$ for any $T > 0$. We prove now this martingale property with an argument already used in Rydberg [78] and Cheridito, Filipovic, and Yor ([84], Theorem 2.4).

Let $(X_t, t \geq 0)$ (resp. $(\bar{X}_t, t \geq 0)$) be a strong solution to (3.34) with parameters κ^1, c^1, a (resp. κ, c, a) and Brownian motion $(W_t, t \geq 0)$. For $\varepsilon > 0$, we define:

$$\tau^\varepsilon = \inf\{t \geq 0, \det(X_t) \leq \varepsilon\}, \quad H_t^\varepsilon = \mathbb{1}_{\tau^\varepsilon \geq t} (\sqrt{X_t})^{-1} c^2 \lambda.$$

We have $\lim_{\varepsilon \rightarrow 0^+} \tau^\varepsilon = +\infty$, a.s. and therefore

$$\mathbb{E}[\mathcal{E}_T^H] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{E}_T^H \mathbb{1}_{\tau^\varepsilon \geq T}].$$

On the other hand, we have $\mathbb{E}[\mathcal{E}_T^H \mathbb{1}_{\tau^\varepsilon \geq T}] = \mathbb{E}[\mathcal{E}_T^{H^\varepsilon} \mathbb{1}_{\tau^\varepsilon \geq T}]$. We clearly have $\mathbb{E}[\mathcal{E}_T^{H^\varepsilon}] = 1$ and $W_t^\varepsilon = W_t - \int_0^t H_s^\varepsilon ds$ is a Brownian motion under $\frac{d\mathbb{Q}^\varepsilon}{d\mathbb{P}} = \mathcal{E}_T^{H^\varepsilon}$. Let $(\bar{X}_t^\varepsilon, t \in [0, T])$ be the strong solution to (3.34) with the Brownian motion W_t^ε and parameters κ, c, a . By construction, $\bar{X}_t^\varepsilon = X_t$ for $0 \leq t \leq T \wedge \tau^\varepsilon$ and thus $\mathbb{1}_{\tau^\varepsilon \geq T} = \mathbb{1}_{\bar{\tau}^\varepsilon \geq T}$, where $\bar{\tau}^\varepsilon = \inf\{t \geq 0, \det(\bar{X}_t^\varepsilon) \leq \varepsilon\}$. We deduce that $\mathbb{E}[\mathcal{E}_T^{H^\varepsilon} \mathbb{1}_{\tau^\varepsilon \geq T}] = \mathbb{Q}^\varepsilon(\bar{\tau}^\varepsilon \geq T) = \mathbb{P}(\inf\{t \geq 0, \det(\bar{X}_t) \leq \varepsilon\} \geq T) \xrightarrow{\varepsilon \rightarrow 0^+} 1$, since $\kappa c + c\kappa - da^2 = \kappa^1 c^1 + c^1 \kappa^1 + \kappa^2 c^2 + c^2 \kappa^2 - da^2 \in \mathcal{S}_d^+(\mathbb{R})$. \square

Let us assume now that $a_i > 0$ for any $1 \leq i \leq d$. A consequence of Proposition 60 is that the probability measures induced by $MRC_d(x, \kappa, c, a)$ and $MRC_d(x, \kappa', c', a)$ are equivalent as soon as (3.4) holds for κ, c, a and κ', c', a . By transitivity, it is in fact sufficient to check this for $\kappa' = \frac{d}{2}a^2$ and $c' = I_d$. By Lemma 67, there is a diagonal nonnegative matrix $\tilde{\kappa}$ and $\tilde{c} \in \mathfrak{C}_d(\mathbb{R})$ such that $\tilde{\kappa}\tilde{c} + \tilde{c}\tilde{\kappa} = \kappa c + c\kappa - da^2$. We get then the probability equivalence by using twice Proposition 60 with $\kappa^1 = \frac{d}{2}a^2, c^1 = I_d, \kappa^2 = \tilde{\kappa}, c^2 = \tilde{c}$ and $\kappa^1 = \kappa, c^1 = c, \kappa^2 = -\tilde{\kappa}, c^2 = \tilde{c}$.

3.3 Second order discretization schemes for MRC processes

In the previous sections, we focused on the existence of Mean-Reverting Correlation processes (3.2) and some of their mathematical properties. From a practical perspective, it is also very important to be able to sample such processes. By sampling, we mean here that we have an algorithm to generate the process on a given time-grid. Through this section, we will consider for sake of simplicity a regular time grid $t_i^N = iT/N, i = 0, \dots, N$ for a given time horizon $T > 0$. Despite our investigations, the sampling of the exact distribution does not seem trivial, and we will focus on discretization schemes. Anyway, discretization schemes are in practice equally or more efficient than exact sampling, at least in the case of square-root diffusions such as Cox-Ingersoll-Ross process and Wishart process (see respectively [5] and [1]). First, let us say that usual schemes such as Euler-Maruyama fail to be defined for (3.2) as well as for other square-root diffusions. Indeed, this scheme is given by

$$\begin{aligned} \hat{X}_{t_{i+1}^N}^N &= \hat{X}_{t_i^N}^N + \left(\kappa(c - \hat{X}_{t_i^N}^N) + (c - \hat{X}_{t_i^N}^N)\kappa \right) \frac{T}{N} + \sum_{n=1}^d a_n \left(\sqrt{\hat{X}_{t_i^N}^N - \hat{X}_{t_i^N}^N e_d^n \hat{X}_{t_i^N}^N} (W_{t_{i+1}^N} - W_{t_i^N}) e_d^n \right. \\ &\quad \left. + e_d^n (W_{t_{i+1}^N} - W_{t_i^N})^T \sqrt{\hat{X}_{t_i^N}^N - \hat{X}_{t_i^N}^N e_d^n \hat{X}_{t_i^N}^N} \right). \end{aligned} \quad (3.39)$$

Thus, even if $\hat{X}_{t_i^N}^N \in \mathfrak{C}_d(\mathbb{R})$, $\hat{X}_{t_{i+1}^N}^N$ can no longer be in $\mathfrak{C}_d(\mathbb{R})$ and the matrix square-root can no longer be defined at the next time-step. It is possible to consider the following modification of the Euler scheme:

$$\hat{X}_{t_{i+1}^N}^N = \mathbf{p}((\tilde{X}_{t_{i+1}^N}^N)^+), \quad (3.40)$$

where $\tilde{X}_{t_{i+1}^N}^N$ denotes the right hand side of (3.39). Here, $x^+ \in \mathcal{S}_d^+(\mathbb{R})$ is defined by $x^+ = \text{oddiag}(\lambda_1^+, \dots, \lambda_d^+)o$ for $x \in \mathcal{S}_d(\mathbb{R})$ such that $x = \text{oddiag}(\lambda_1^+, \dots, \lambda_d^+)o$ where o is an orthogonal

matrix. Let us check that this scheme is well defined if we start from $\hat{X}_{t_0^N}^N \in \mathfrak{C}_d(\mathbb{R})$. By Lemma 68, the square-roots are well defined, we have $(\tilde{X}_{t_1^N})_{i,i} = 1$ and thus $(\tilde{X}_{t_{i+1}^N})_{i,i}^+ \geq 1$ and $\mathbf{p}((\tilde{X}_{t_1^N})^+)$ is well defined. By induction, this modified Euler scheme is always defined and takes values in the set of correlation matrices. However, as we will see in the numerical experiments, it is time-consuming and converges rather slowly.

In this section, we present discretization schemes that are obtained by composition, thanks to a splitting of the infinitesimal generator. This technique has already been used for square-root type diffusions such as the Cox-Ingersoll-Ross model [5] and Wishart processes [1], leading to accurate schemes. The strength of this approach is that we can, by an ad-hoc splitting of the operator, decompose the sampling of the whole diffusion into pieces that are more tractable and that we can simulate by preserving the domain (here, the set of correlation matrices). Besides, it is really easy to analyze the weak error of these schemes.

3.3.1 Some results on the weak error of discretization schemes

We present now the main results on the splitting technique that can be found in [5] and [1] for the framework of Affine diffusions. Here, we have in addition further simplifications that comes from the fact that the domain that we consider $\mathbb{D} \subset \mathbb{R}^\zeta$ is compact (typically $\mathfrak{C}_d(\mathbb{R})$ or $\mathbb{D} = \{x \in \mathbb{R}^{d-1}, \sum_{i=1}^{d-1} x_i^2\}$ in Appendix 3.4.4). For $\gamma \in \mathbb{N}^\zeta$, we set $\partial_\gamma f = \partial_{\gamma_1}^1 \dots \partial_{\gamma_\zeta}^\zeta$ and $|\gamma| = \sum_{i=1}^\zeta \gamma_i$. We denote by $\mathcal{C}^\infty(\mathbb{D})$ the set of infinitely differentiable functions on \mathbb{D} and say that that $(C_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ is a *good sequence* for $f \in \mathcal{C}^\infty(\mathbb{D})$ if we have $\max_{x \in \mathbb{D}} |\partial_\gamma f(x)| \leq C_\gamma$. A differential operator $Lf(x) = \sum_{0 < |\gamma| \leq 2} a_\gamma(x) \partial_\gamma f(x)$ satisfies the *required assumption* if we have $a_\gamma \in \mathcal{C}^\infty(\mathbb{D})$ for any γ . This property is of course satisfied by the infinitesimal generator (3.8) of $MRC_d(x, \kappa, c, a)$ since the functions a_γ are either affine or polynomial functions of second degree. Since we are considering Markovian processes on \mathbb{D} , we will by a slight abuse of notation represent a discretization scheme by a probability measure $\hat{p}_x(t)(dz)$ on \mathbb{D} that describes the law of the scheme starting from $x \in \mathbb{D}$ with a time step $t > 0$. Also, we denote by \hat{X}_t^x a random variable that follows this law. Then, the discretization scheme on the full time grid $(t_i^N, i = 0, \dots, N)$ will be obtained by:

- $\hat{X}_{t_0^N}^N = x \in \mathbb{D}$,
- conditionally to $\hat{X}_{t_i^N}^N, \hat{X}_{t_{i+1}^N}^N$ is sampled according to the probability law $\hat{p}_{\hat{X}_{t_i^N}^N}(T/N)(dz)$,

and we write with a slight abuse of notation $\hat{X}_{t_{i+1}^N}^N = \hat{X}_{T/N}^{\hat{X}_{t_i^N}^N}$.

A discretization scheme \hat{X}_t^x is said to be a *potential ν -th order scheme for the operator L* if for a sequence $(C_\gamma)_{\gamma \in \mathbb{N}^\zeta} \in (\mathbb{R}_+)^{\mathbb{N}^\zeta}$, there are constants $C, \eta > 0$ such that for any function $f \in \mathcal{C}^\infty(\mathbb{D})$ that admits $(C_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ as a good sequence, we have:

$$\forall t \in (0, \eta), x \in \mathbb{D} \left| \mathbb{E}[f(\hat{X}_t^x)] - \left[f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) \right] \right| \leq Ct^{\nu+1}. \quad (3.41)$$

This is the main assumption that a discretization scheme should satisfy to get a weak error of order ν . This is precised by the following theorem given in [5] that relies on the idea developed by Talay and Tubaro [82] for the Euler-Maruyama scheme.

Theorem 61 — Let L be an operator satisfying the required assumptions on a compact domain \mathbb{D} . We assume that:

1. \hat{X}_t^x is a potential weak ν th-order scheme for L ,
2. $f : \mathbb{D} \rightarrow \mathbb{R}$ is a function such that $u(t, x) = \mathbb{E}[f(X_{T-t}^x)]$ is defined and C^∞ on $[0, T] \times \mathbb{D}$, and solves $\forall t \in [0, T], \forall x \in \mathbb{D}, \partial_t u(t, x) = -Lu(t, x)$.

Then, there is $K > 0, N_0 \in \mathbb{N}$, such that $|\mathbb{E}[f(\hat{X}_{t/N}^x)] - \mathbb{E}[f(X_T^x)]| \leq K/N^\nu$ for $N \geq N_0$.

The mathematical analysis of the Cauchy problem for Mean-Reverting Correlation processes is beyond the scope of this paper. This issue has recently been addressed for the case of one-dimensional Wright-Fisher processes by Epstein and Mazzeo [37], and Chen and Stroock [22] for the absorbing boundary case. In this setting, Epstein and Mazzeo have shown that $u(t, x)$ is smooth for $f \in C^\infty([0, 1])$. However, since we have an explicit formula for the moments (3.11), we obtain easily that for any polynomial function f , the second point of Theorem 61 is satisfied. By the Stone-Weierstrass theorem, we can approximate for the supremum norm any continuous function by a polynomial function and get the following interesting corollary.

Corollary 62 — Let \hat{X}_t^x be potential weak ν th-order scheme for $MRC_d(x, \kappa, c, a)$. Let f be a continuous function on $\mathfrak{C}_d(\mathbb{R})$. Then,

$$\forall \varepsilon > 0, \exists K > 0, |\mathbb{E}[f(\hat{X}_{t/N}^x)] - \mathbb{E}[f(X_T^x)]| \leq \varepsilon + K/N^\nu.$$

Let us now focus on the first assumption of Theorem 61. The property of being a potential weak order scheme is easy to handle by using scheme composition. This technique is well known in the literature and dates back to Strang [81] the field of ODEs. In our framework, we recall results that are stated in [5].

Proposition 63 — Let L_1, L_2 be the generators of SDEs defined on \mathbb{D} that satisfies the required assumption on \mathbb{D} . Let $\hat{X}_t^{1,x}$ and $\hat{X}_t^{2,x}$ denote respectively two potential weak ν th-order schemes on \mathbb{D} for L_1 and L_2 .

1. The scheme $\hat{X}_t^{2, \hat{X}_t^{1,x}}$ is a potential weak first order discretization scheme for $L_1 + L_2$. Besides, if $L_1 L_2 = L_2 L_1$, this is a potential weak ν th-order scheme for $L_1 + L_2$.
2. Let B be an independent Bernoulli variable of parameter $1/2$. If $\nu \geq 2$,

$$(a) B\hat{X}_t^{2, \hat{X}_t^{1,x}} + (1 - B)\hat{X}_t^{1, \hat{X}_t^{2,x}} \quad \text{and} \quad (b) \hat{X}_{t/2}^{2, \hat{X}_t^{1, \hat{X}_{t/2}^{2,x}}}$$

are potential weak second order schemes for $L_1 + L_2$.

Here, the composition $\hat{X}_{t_2}^{2, \hat{X}_{t_1}^{1,x}}$ means that we first use the scheme 1 with time step t_1 and then, conditionally to $\hat{X}_{t_1}^{1,x}$, we sample the scheme 2 with initial value $\hat{X}_{t_1}^{1,x}$ and time step t_2 .

3.3.2 A second-order scheme for MRC processes

First, we split the infinitesimal generator of $MRC_d(x, \kappa, c, a)$ as the sum

$$L = L^\xi + \tilde{L},$$

where \tilde{L} is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$ and L^ξ is the operator associated to $\xi(t, x)$ given by (3.30). Obviously, the ODE (3.30) can be solved explicitly and we have to focus on the sampling of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$. We use now Theorem 51 and consider the splitting

$$\tilde{L} = \sum_{i=1}^d a_i^2 \tilde{L}_i,$$

where \tilde{L}_i is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}e_d^i, I_d, e_d^i)$. We claim now that it is sufficient to have a potential second order scheme for $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1)$ in order to get a potential second order scheme for $MRC_d(x, \kappa, c, a)$. Indeed, if we have such a scheme, we also get by a permutation of the coordinates a potential second order scheme $\hat{X}_t^{i,x}$ for $MRC_d(x, \frac{d-2}{2}e_d^i, I_d, e_d^i)$. Then, by time-scaling, $\hat{X}_{a_i^2 t}^{i,x}$ is a potential second order scheme for $MRC_d(x, \frac{d-2}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$. Thanks to the commutativity, we get by Proposition 63 that $\hat{X}_{a_d^2 t}^{d,\dots, \hat{X}_{a_1^2 t}^{1,x}}$ is a potential second order scheme for \tilde{L} . Last, still by using Proposition 63 we obtain that

$$\xi(t/2, \hat{X}_{a_d^2 t}^{d,\dots, \hat{X}_{a_1^2 t}^{1,x}}) \text{ is a potential second order scheme for } MRC_d(x, \kappa, c, a). \quad (3.42)$$

Now, we focus on getting a second order scheme for $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1)$. It is possible to construct such a scheme by using an ad-hoc splitting of the infinitesimal generator. This is made in Appendix 3.4.4. Here, we achieve this task by using the connection between Wishart and MRC process and the existing scheme for Wishart processes. In Ahdida and Alfonsi [1], we have obtained a potential second order scheme $\hat{Y}_t^{1,x}$ for $WIS_d(x, d-1, 0, e_d^1)$. Besides, this scheme is constructed with discrete random variables, and we can check that there is a constant $K > 0$ such that for any $1 \leq i \leq d$, $|(\hat{Y}_t^{1,x})_{i,i} - 1| \leq K\sqrt{t}$ holds almost surely (we even have $(\hat{Y}_t^{1,x})_{i,i} = 1$ for $2 \leq i \leq d$). Therefore, we have $1/2 \leq (\hat{Y}_t^{1,x})_{i,i} \leq 3/2$ for $t \leq 1/(4K^2)$. Let $f \in \mathcal{C}^\infty(\mathfrak{C}_d(\mathbb{R}))$. Then $f(\mathbf{p}(y))$ is \mathcal{C}^∞ with bounded derivatives on $\{y \in \mathcal{S}_d^+(\mathbb{R}), 1/2 \leq y_{i,i} \leq 3/2\}$. Since $\hat{Y}_t^{1,x}$ is a potential second order scheme, it comes that there are constants $C, \eta > 0$ that only depend on a good sequence of f such that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_t^{1,x}))] - f(x) - t\tilde{L}_1^W(f \circ \mathbf{p})(x) - \frac{t^2}{2}(\tilde{L}_1^W)^2(f \circ \mathbf{p})(x) \right| \leq Ct^3, \quad (3.43)$$

where \tilde{L}_1^W is the generator of $WIS_d(x, d-1, 0, e_d^1)$. Thanks to Remark 52, we get that there are constants C, η depending only on a good sequence of f such that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_t^{1,x}))] - f(x) - \left(t + (5-d)\frac{t^2}{2} \right) \tilde{L}_1 f(x) - \frac{t^2}{2}(\tilde{L}_1)^2 f(x) \right| \leq Ct^3. \quad (3.44)$$

In particular, $\mathbf{p}(\hat{Y}_t^{1,x})$ is a potential first order scheme for L_1 and even a second order scheme when $d = 5$. We can improve this by taking a simple time-change. We set:

$$\phi(t) = \begin{cases} t - (5-d)\frac{t^2}{2} & \text{if } d \geq 5 \\ \frac{-1 + \sqrt{1+2(5-d)t}}{5-d} & \text{otherwise,} \end{cases}$$

so that in both cases, $\phi(t) = t - (5-d)\frac{t^2}{2} + O(t^3)$. Then, we have that there are constants C, η still depending only on a good sequence of f such that $\forall t \in (0, \eta)$,

$$\left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_{\phi(t)}^{1,x}))] - f(x) - t\tilde{L}_1 f(x) - \frac{t^2}{2}(\tilde{L}_1)^2 f(x) \right| \leq Ct^3, \text{ and therefore}$$

$$\mathbf{p}(\hat{Y}_{\phi(t)}^{1,x}) \text{ is a potential second order scheme for } MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1). \quad (3.45)$$

3.3.3 A faster second-order scheme for MRC processes under Assumption (3.46)

We would like to discuss on the time complexity of the scheme given by (3.42) and (3.45) with respect to the dimension d . The second order scheme given in Ahdida and Alfonsi [1] for $WIS_d(x, d-1, 0, e_d^1)$ requires $O(d^3)$ operations. Since it is used d times in (3.42) to generate a sample, the overall complexity is in $O(d^4)$. In the same manner, the second order given in Appendix 3.4.4 requires $O(d^4)$ operations. However, it is possible to get a faster second order scheme with complexity $O(d^3)$ if we make the following assumption:

$$a_1 = \dots = a_d \text{ (i.e. } a = a_1 I_d) \text{ and } \kappa c + c\kappa - (d-1)a^2 \in \mathcal{S}_d^+(\mathbb{R}). \quad (3.46)$$

This latter assumption is stronger than (3.3) but weaker than (3.4), which respectively ensures weak and strong solutions to the SDE. Under (3.46), we can check by Lemma 67 that

$$\zeta'(t, x) = \kappa(c-x) + (c-x)\kappa - \frac{d-1}{2}[a^2(I_d-x) + (I_d-x)a^2], \quad \zeta(0, x) = x \in \mathfrak{C}_d(\mathbb{R}) \quad (3.47)$$

takes values in $\mathfrak{C}_d(\mathbb{R})$. Then, we split the infinitesimal generator of $MRC_d(x, \kappa, c, a)$ as the sum

$$L = L^\zeta + a_1^2 \bar{L},$$

where L^ζ is the operator associated to the ODE ζ , and \bar{L} is the infinitesimal generator of $MRC_d(x, \frac{d-1}{2}I_d, I_d, I_d)$. In [1], it is given a second order scheme \hat{Y}_t^x for $WIS_d(x, d, 0, I_d)$ that has a time-complexity in $O(d^3)$. We then consider $f \in \mathcal{C}^\infty(\mathfrak{C}_d(\mathbb{R}))$ and get by using the same arguments as before that there are constants $C, \eta > 0$ depending only on a good sequence of f such that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_t^x))] - f(x) - t\bar{L}^W(f \circ \mathbf{p})(x) - \frac{t^2}{2}(\bar{L}^W)^2(f \circ \mathbf{p})(x) \right| \leq Ct^3,$$

where \bar{L}^W is the infinitesimal generator of $WIS_d(x, d, 0, I_d)$. Thanks to Remark 52, we get that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_t^x))] - f(x) - \left(t + (4-d)\frac{t^2}{2} \right) \bar{L}f(x) - \frac{t^2}{2}\bar{L}^2 f(x) \right| \leq Ct^3.$$

	$d = 3$	$d = 10$
2^{nd} order “fast”	19	224
2^{nd} order	65	1677
2^{nd} order “direct”	90	3105
1^{st} order “fast”	19	224
Corrected Euler	400	14322

Table 3.1: Computation time in seconds to generate 10^6 paths up to $T = 1$ with $N = 10$ time-steps of the following MRC process: $\kappa = 1.25I_d$, $c = I_d$, $a = I_d$, and $x_{i,j} = 0.7$ for $i \neq j$.

In particular, $\mathbf{p}(\hat{Y}_t^x)$ is a first order scheme for $MRC_d(x, \frac{d-1}{2}I_d, I_d, I_d)$ and by Proposition 63,

$$\zeta(t, \mathbf{p}(\hat{Y}_{a_1^2 t}^x)) \text{ is a potential first order scheme for } MRC_d(x, \kappa, c, a). \quad (3.48)$$

As before, we can improve this by using the following time-change: $\psi(t) = t - (4-d)\frac{t^2}{2}$ if $d \geq 4$ and $\psi(t) = \frac{-1 + \sqrt{1+2(4-d)t}}{4-d}$ otherwise, so that $\psi(t) = t - (4-d)\frac{t^2}{2} + O(t^3)$ in both cases. We get that $\mathbf{p}(\hat{Y}_{\psi(t)}^x)$ is a potential second order scheme for $MRC_d(x, \frac{d-1}{2}I_d, I_d, I_d)$. Then, we obtain that

$$\zeta(t/2, \mathbf{p}(\hat{Y}_{a_1^2 \psi(t)}^{\zeta(x,t/2)})) \text{ is a potential second order scheme for } MRC_d(x, \kappa, c, a) \quad (3.49)$$

by using Proposition 63. Its time complexity is in $O(d^3)$.

3.3.4 Numerical experiments on the discretization schemes

In this part, we discuss briefly the time needed by the different schemes presented in the paper. We also illustrate the weak convergence of the schemes to check that it is in accordance with Corollary 62. In Table 3.1, we have indicated the time required to sample 10^6 scenarios for different time-grids in dimension $d = 3$ and $d = 10$. These times have been obtained with a 2.50 GHz CPU computer. As expected, the modified Euler scheme given by (3.40) is the most time consuming. This is mainly due to the computation of the matrix square-roots that require several diagonalizations. Between the second order schemes that are defined for any parameters satisfying (3.3), the second order scheme given by (3.42) and (3.45) is rather faster than the “direct” one presented in Appendix 3.4.4. However, it has a larger bias on our example in Figure (3.1), and their overall efficiency is similar. Nonetheless, both are as expected overtaken by the fast second order scheme (3.49). Let us recall that it is only defined under Assumption (3.46) which is satisfied by our set of parameters. Also, the fast first order scheme given by (3.48) requires roughly the same computation time.

Let us switch now to Figure 3.1 that illustrates the weak convergence of the different schemes. To be more precise, we have plotted the following combinations the moments of order 3 and 1 (i.e. respectively

$$\mathbb{E} \left[\sum_{\substack{1 \leq i \neq j \leq 3 \\ 1 \leq k \neq l \leq 3}} \left[(\hat{X}_T^N)_{i,j} (\hat{X}_T^N)_{k,l}^2 \right] + (\hat{X}_T^N)_{1,2} (\hat{X}_T^N)_{2,3} (\hat{X}_T^N)_{1,3} \right], \quad (3.50)$$

and $\mathbb{E} \left[\sum_{1 \leq i \neq j \leq d} (\hat{X}_T^N)_{i,j} \right]$ in function of the time-step T/N . These expectations can be calculated exactly for the MRC process thanks to Proposition 47, and the exact value is reported in both graphics. As expected, we observe a quadratic convergence for the second order schemes, and a linear convergence for the first order scheme. In particular, this demonstrates numerically the gain that we get by considering the simple change of time ψ between the schemes (3.48) and (3.49). Last, the modified Euler scheme shows a roughly linear convergence. It has however a much larger bias and is clearly not competitive.

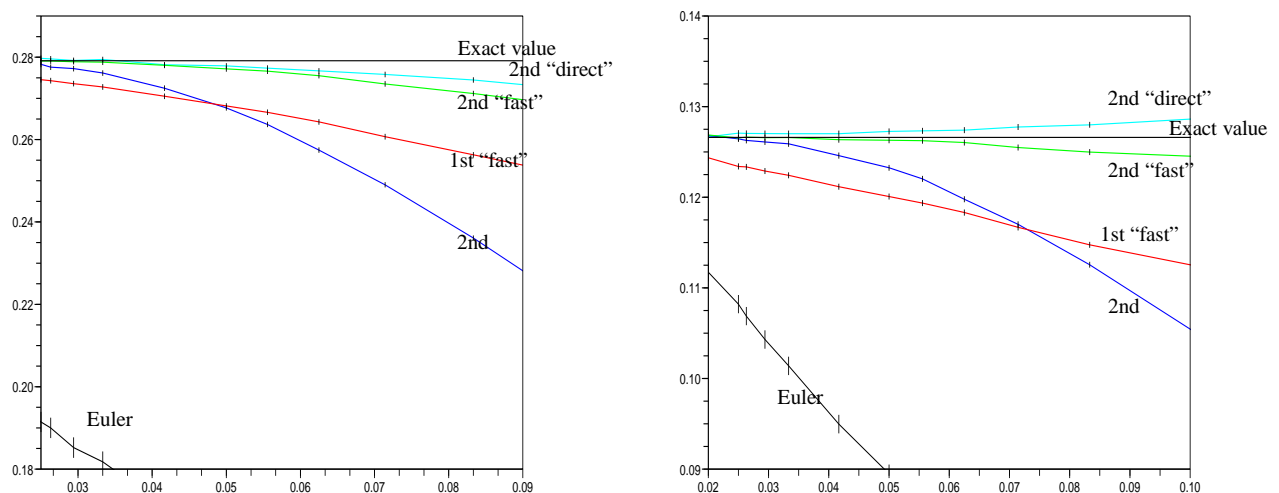


Figure 3.1: $d = 3$, same parameters as for Table 3.1. In the left (resp. right) side is plotted (3.50) (resp. $\mathbb{E} \left[\sum_{1 \leq i \neq j \leq d} (\hat{X}_T^N)_{i,j} \right]$) in function of the time step $1/N$. The width of each point represents the 95% confidence interval (10^7 scenarios for the modified Euler scheme and 10^8 for the others).

3.4 Appendix

3.4.1 Some results on correlation matrices

Linear ODEs on correlation matrices

Let $b \in \mathcal{S}_d(\mathbb{R})$ and $\kappa \in \mathcal{M}_d(\mathbb{R})$. In this section, we consider the following linear ODE

$$x'(t) = b - (\kappa x(t) + x(t)\kappa^T), \quad x(0) = x \in \mathfrak{C}_d(\mathbb{R}), \quad (3.51)$$

and we are interested in necessary and sufficient conditions on κ and b such that

$$\forall x \in \mathfrak{C}_d(\mathbb{R}), \forall t \geq 0, x(t) \in \mathfrak{C}_d(\mathbb{R}). \quad (3.52)$$

Let us first look at necessary conditions. We have for $1 \leq i, j \leq d$:

$$x'_{i,j}(t) = b_{i,j} - \sum_{k=1}^d \kappa_{i,k} x_{k,j}(t) + x_{i,k}(t) \kappa_{j,k}.$$

In particular, we necessarily have $x'_{i,i}(t) = 0$. This gives for $t = 0$, $l \neq i$ and $x(0) = I_d + \rho(e_d^{i,l} + e_d^{l,i})$ that $b_{i,i} - 2\kappa_{i,i} - 2\rho\kappa_{i,l} = 0$ for any $\rho \in [-1, 1]$. It comes out that:

$$\kappa_{i,l} = 0 \text{ if } l \neq i, \quad b_{i,i} = 2\kappa_{i,i}.$$

Thus, the matrix κ is diagonal and we denote $\kappa_i = \kappa_{i,i}$. We get $x'_{i,j}(t) = b_{i,j} - (\kappa_i + \kappa_j)x_{i,j}(t)$ for $i \neq j$. If $\kappa_i + \kappa_j = 0$, we have $x_{i,j}(t) = x_{i,j} + b_{i,j}t$, which implies that $b_{i,j} = 0$. Otherwise, $\kappa_i + \kappa_j \neq 0$ and we get:

$$x_{i,j}(t) = x_{i,j} \exp(-(\kappa_i + \kappa_j)t) + \frac{b_{i,j}}{\kappa_i + \kappa_j} [1 - \exp(-(\kappa_i + \kappa_j)t)].$$

Once again, this implies that $\kappa_i + \kappa_j > 0$ since the initial value $x \in \mathfrak{C}_d(\mathbb{R})$ is arbitrary. We set for $1 \leq i, j \leq d$,

$$c_{i,i} = 1, \text{ and for } i \neq j, \quad c_{i,j} = \begin{cases} \frac{b_{i,j}}{\kappa_i + \kappa_j} & \text{if } \kappa_i + \kappa_j > 0 \\ 0 & \text{if } \kappa_i + \kappa_j = 0. \end{cases} \quad (3.53)$$

We have $b = \kappa c + c \kappa$ and for $x = I_d$, $c = \lim_{t \rightarrow +\infty} x(t) \in \mathfrak{C}_d(\mathbb{R})$, and deduce the following result.

Proposition 64 — *Let $b \in \mathcal{S}_d(\mathbb{R})$ and $\kappa \in \mathcal{M}_d(\mathbb{R})$. If the linear ODE (3.51) satisfies (3.52), then we have necessarily:*

$$\exists c \in \mathfrak{C}_d(\mathbb{R}), \exists \kappa_1, \dots, \kappa_d \in \mathbb{R}, \forall i \neq j, \kappa_i + \kappa_j \geq 0, \kappa = \text{diag}(\kappa_1, \dots, \kappa_d) \text{ and } b = \kappa c + c \kappa. \quad (3.54)$$

Conversely, let us assume that (3.54) holds and $b \in \mathcal{S}_d^+(\mathbb{R})$. We get that $\kappa_i = b_{i,i}/2 \geq 0$ and for $t \geq 0$, $\exp(\kappa t)x(t)\exp(\kappa t) = x + \int_0^t \exp(\kappa s)b\exp(\kappa s)ds$ is clearly positive semidefinite. Therefore, (3.52) holds. We get the following result.

Proposition 65 — Let $\kappa_1, \dots, \kappa_d \geq 0$, $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d)$ and $c \in \mathfrak{C}_d(\mathbb{R})$. If $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ or $d = 2$, the ODE

$$x'(t) = \kappa(c - x) + (c - x)\kappa, \quad x(0) = x \in \mathfrak{C}_d(\mathbb{R}) \quad (3.55)$$

satisfies (3.52).

Let us note here that the parametrization of the ODE (3.55) is redundant when $d = 2$, and we can assume without loss of generality that $\kappa_1 = \kappa_2$ for which $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ is clearly satisfied.

Remark 66 — The condition given by Proposition 64 is necessary but not sufficient, and the condition given by Proposition 65 is sufficient but not necessary. Let $d = 3$ and $c = I_3$. We can check that for $\kappa = (1, \frac{1}{2}, -\frac{1}{2})$, (3.54) holds but (3.52) is not true. Also, we can check that for $\kappa = (1, 1, -\frac{1}{2})$, (3.52) holds.

Lemma 67 — Let κ^1, κ^2 be diagonal matrices and $c^1, c^2 \in \mathfrak{C}_d(\mathbb{R})$ such that $\kappa^1 c^1 + c^1 \kappa^1 + \kappa^2 c^2 + c^2 \kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$. Then, the ODE

$$x' = \kappa^1(c^1 - x) + (c^1 - x)\kappa^1 + \kappa^2(c^2 - x) + (c^2 - x)\kappa^2$$

satisfies (3.52). Besides, $x' = \kappa(c - x) + (c - x)\kappa$ with $\kappa = \kappa^1 + \kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$ and $c \in \mathfrak{C}_d(\mathbb{R})$ defined by:

$$c_{i,i} = 1, \text{ and for } i \neq j, \quad c_{i,j} = \begin{cases} \frac{(\kappa_i^1 + \kappa_j^1)c_{i,j}^1 + (\kappa_i^2 + \kappa_j^2)c_{i,j}^2}{\kappa_i + \kappa_j} & \text{if } \kappa_i + \kappa_j > 0 \\ 0 & \text{if } \kappa_i + \kappa_j = 0. \end{cases}$$

Proof: Since $b = \kappa^1 c^1 + c^1 \kappa^1 + \kappa^2 c^2 + c^2 \kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$, (3.52) holds for $x' = b - \kappa x + x\kappa$. Then, we know by (3.53) that c is a correlation matrix. \square

Some algebraic results on correlation matrices

Lemma 68 — Let $c \in \mathfrak{C}_d(\mathbb{R})$ and $1 \leq i \leq d$. Then we have: $c - ce_d^i c \in \mathcal{S}_d^+(\mathbb{R})$, $(c - ce_d^i c)_{i,j} = 0$ for $1 \leq j \leq d$, $(c - ce_d^i c)^{[i]} = c^{[i]} - c^i (c^i)^T$ and:

$$\left(\sqrt{c - ce_d^i c} \right)^{[i]} = \sqrt{c^{[i]} - c^i (c^i)^T} \text{ and } \left(\sqrt{c - ce_d^i c} \right)_{i,j} = 0.$$

Besides, if $c \in \mathfrak{C}_d^*(\mathbb{R})$, $c^{[i]} - c^i (c^i)^T \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$.

Proof: Up to a permutation, it is sufficient to prove the result for $i = 1$. We have

$$c - ce_d^1 c = \begin{pmatrix} 0 & 0_{d-1}^T \\ 0_{d-1} & c^{[1]} - c^1 (c^1)^T \end{pmatrix} = aca^T, \text{ with } a = \begin{pmatrix} 0 & 0_{d-1} \\ -c^1 & I_{d-1} \end{pmatrix} \in \mathcal{S}_d^+(\mathbb{R}).$$

Besides, we have $\text{Rk}(aca^T) = \text{Rk}(a\sqrt{c}) = d - 1$ when $c \in \mathfrak{C}_d^*(\mathbb{R})$, which gives $c^{[i]} - c^i(c^i)^T \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$. \square

Lemma 69 — Let $c \in \mathfrak{C}_d(\mathbb{R})$ and $1 \leq n \leq d$. Then $I_d - \sqrt{c}e_d^n\sqrt{c} \in \mathcal{S}_d^+(\mathbb{R})$ and is such that

$$\sqrt{I_d - \sqrt{c}e_d^n\sqrt{c}} = I_d - \sqrt{c}e_d^n\sqrt{c}.$$

Proof: The matrix $(\sqrt{c}e_d^n\sqrt{c})_{i,j} = (\sqrt{c})_{i,n}(\sqrt{c})_{j,n}$ is of rank 1 and $\sum_{j=1}^d (\sqrt{c}e_d^n\sqrt{c})_{i,j}(\sqrt{c})_{j,n} = (\sqrt{c})_{i,n}$ since $\sum_{j=1}^d (\sqrt{c})_{j,n}^2 = c_{j,j} = 1$. Therefore $((\sqrt{c})_{i,n})_{1 \leq i \leq d}$ is an eigenvector, and the eigenvalues of $I_d - \sqrt{c}e_d^n\sqrt{c}$ are 0 and 1 (with multiplicity $d - 1$). \square

Lemma 70 — Let $q \in \mathcal{S}_d^+(\mathbb{R})$ be a matrix with rank r . Then there is a permutation matrix p , an invertible lower triangular matrix $m_r \in \mathcal{G}_r(\mathbb{R})$ and $k_r \in \mathcal{M}_{d-r \times r}(\mathbb{R})$ such that:

$$pqp^T = mm^T, \quad m = \begin{pmatrix} m_r & 0 \\ k_r & 0 \end{pmatrix}.$$

The triplet (m_r, k_r, p) is called an extended Cholesky decomposition of q .

The proof of this result and a numerical procedure to get such a decomposition can be found in Golub and Van Loan ([45], Algorithm 4.2.4). When $r = d$, we can take $p = I_d$, and m_r is the usual Cholesky decomposition.

Lemma 71 — Let $c \in \mathfrak{C}_d(\mathbb{R})$, $r = \text{Rk}((c_{i,j})_{2 \leq i,j \leq d})$ and (m_r, k_r, \tilde{p}) an extended Cholesky decomposition of $(c_{i,j})_{2 \leq i,j \leq d}$. We set $p = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{p}^T \end{pmatrix}$, $m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_r & 0 \\ 0 & k_r & 0 \end{pmatrix}$ and $\check{c} =$

$$\begin{pmatrix} 1 & (m_r^{-1}c_1^r)^T & 0 \\ m_r^{-1}c_1^r & I_r & 0 \\ 0 & 0 & I_{d-r-1} \end{pmatrix}, \quad \text{where } c_1^r \in \mathbb{R}^r, \text{ with } (c_1^r)_i = (p^T c p)_{1,i+1} \text{ for } 1 \leq i \leq r.$$

We have:

$$c = pm\check{c}m^T p^T \text{ and } \check{c} \in \mathfrak{C}_d(\mathbb{R}).$$

Proof: By straightforward block-matrix calculations, on has to check that the vector $c_1^{r,d} \in \mathbb{R}^{d-(r+1)}$ defined by $(c_1^{r,d})_i = (p^T c p)_{1,i}$ for $r + 1 \leq i \leq d$ is equal to $k_r m_r^{-1} c_1^r$. To get this, we

introduce the matrix $q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}$ and have $q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_r^{-1} & 0 \\ 0 & -k_r m_r^{-1} & I_{d-r-1} \end{pmatrix}$.

Since the matrix

$$q^{-1} p^T c p (q^{-1})^T = \begin{pmatrix} 1 & (m_r^{-1}c_1^r)^T & (c_1^{r,d} - k_r m_r^{-1}c_1^r)^T \\ m_r^{-1}c_1^r & I_r & 0 \\ c_1^{r,d} - k_r m_r^{-1}c_1^r & 0 & 0 \end{pmatrix}$$

is positive semidefinite, we have $c_1^{r,d} = k_r m_r^{-1} c_1^r$, $\begin{pmatrix} 1 & (m_r^{-1} c_1^r)^T \\ m_r^{-1} c_1^r & I_r \end{pmatrix} \in \mathcal{S}_{r+1}^+(\mathbb{R})$ and thus $\check{c} \in \mathfrak{C}_d(\mathbb{R})$. \square

3.4.2 Some auxiliary results

Calculation of quadratic variations

Lemma 72 — *Let $(\mathcal{F}_t)_{t \geq 0}$ denote the filtration generated by $(W_t, t \geq 0)$. We consider a process $(Y_t)_{t \geq 0}$ valued in $\mathcal{S}_d(\mathbb{R})$ such that*

$$dY_t = B_t dt + \sum_{n=1}^d (A_t^n dW_t e_d^n + e_d^n dW_t^T (A_t^n)^T),$$

where $(A_t^n)_{t \geq 0}$, $(B_t)_{t \geq 0}$ are continuous (\mathcal{F}_t) -adapted processes respectively valued in $\mathcal{M}_d(\mathbb{R})$, and $\mathcal{S}_d(\mathbb{R})$. Then, we have for $1 \leq i, j, k, l \leq d$:

$$d\langle Y_{i,j}, Y_{k,l} \rangle_t = \left[\mathbb{1}_{i=k} (A_t^i (A_t^i)^T)_{j,l} + \mathbb{1}_{i=l} (A_t^i (A_t^i)^T)_{j,k} + \mathbb{1}_{j=k} (A_t^j (A_t^j)^T)_{i,l} + \mathbb{1}_{j=l} (A_t^j (A_t^j)^T)_{i,k} \right] dt \quad (3.56)$$

Proof: Since $(A_t^n dW_t e_d^n)_{i,j} = \mathbb{1}_{j=n} (A_t^j dW_t)_{i,j}$ and $(e_d^n dW_t^T (A_t^n)^T)_{i,j} = \mathbb{1}_{i=n} (A_t^i dW_t)_{j,i}$, we get:

$$d(Y_t)_{i,j} = (B_t)_{i,j} dt + \sum_{n=1}^d (A_t^j)_{i,n} (dW_t)_{n,j} + (A_t^i)_{j,n} (dW_t)_{n,i}.$$

Then,

$$\frac{d\langle Y_{i,j}, Y_{k,l} \rangle_t}{dt} = \sum_{n=1}^d \left(\mathbb{1}_{j=l} (A_t^j)_{i,n} (A_t^j)_{k,n} + \mathbb{1}_{j=k} (A_t^j)_{i,n} (A_t^j)_{l,n} + \mathbb{1}_{i=l} (A_t^i)_{j,n} (A_t^i)_{k,n} + \mathbb{1}_{i=k} (A_t^i)_{j,n} (A_t^i)_{l,n} \right),$$

which precisely gives (3.56). \square

Lemma 73 — *Let us consider $x \in \mathfrak{C}_d^*(\mathbb{R})$, and $(X_t)_{t \geq 0}$ a solution of the SDE (3.2). Let τ denote the stopping time defined as $\tau = \{t \geq 0, X_t \notin \mathfrak{C}_d^*(\mathbb{R})\}$. Then, there exists a real Brownian motion $(\beta_t)_{t \geq 0}$ such that for $0 \leq t < \tau$,*

$$\frac{d(\det(X_t))}{\det(X_t)} = \text{Tr}[X_t^{-1}(\kappa c + c\kappa - (d-2)a^2)]dt - \text{Tr}(2\kappa + a^2)dt + 2\sqrt{\text{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_t, \quad (3.57)$$

$$d \log(\det(X_t)) = \text{Tr}[X_t^{-1}(\kappa c + c\kappa - da^2)]dt - \text{Tr}(2\kappa - a^2)dt + 2\sqrt{\text{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_t. \quad (3.58)$$

Proof: First, let us recall that $\forall i, j, k, l \in \{1, \dots, d\}$, $\forall x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ $\partial_{i,j} \det(x) = (\text{adj}(x))_{i,j} = \det(x)x_{i,j}^{-1}$, $\partial_{k,l} \partial_{i,j}(\det(x)) = \det(x)(x_{l,k}^{-1}x_{i,j}^{-1} - x_{l,j}^{-1}x_{i,k}^{-1})$. Since x is symmetric, we have in particular that $\partial_{k,l} \partial_{i,j}(\det(x)) = 0$ if $i = l$ or $j = k$. Itô's Formula gives for $t < \tau$:

$$\begin{aligned} \frac{d(\det(X_t))}{\det(X_t)} &= \sum_{1 \leq i, j \leq d} (X_t^{-1})_{i,j} d(X_t)_{i,j} \\ &\quad + \frac{1}{2} \sum_{\substack{1 \leq i, j \leq d \\ 1 \leq k, l \leq d}} ((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}) \langle d(X_t)_{i,j}, d(X_t)_{k,l} \rangle. \end{aligned}$$

On the one hand we have

$$\sum_{1 \leq i, j \leq d} (X_t^{-1})_{i,j} d(X_t)_{i,j} = (\text{Tr}[X_t^{-1}(\kappa c + c\kappa)] - \text{Tr}(2\kappa)) dt + 2 \sum_{i=1}^d a_i \text{Tr} \left[X_t^{-1} e_d^i dW_s^T \sqrt{X_t - X_t e_d^i X_t} \right].$$

On the other hand we get by (3.6):

$$\begin{aligned} &\sum_{\substack{1 \leq i, j \leq d \\ 1 \leq k, l \leq d}} ((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}) \langle d(X_t)_{i,j}, d(X_t)_{k,l} \rangle \\ &= \sum_{\substack{1 \leq i, j \leq d \\ 1 \leq k, l \leq d}} ((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}) \times \{ a_j^2 \mathbb{1}_{j=k} (X_t - X_t e_d^j X_t)_{i,l} \\ &\quad + a_j^2 \mathbb{1}_{j=l} (X_t - X_t e_d^j X_t)_{i,k} + a_i^2 \mathbb{1}_{i=l} (X_t - X_t e_d^i X_t)_{j,k} + a_i^2 \mathbb{1}_{i=k} (X_t - X_t e_d^i X_t)_{j,l} \} \\ &= \sum_{j=1}^d \left(\sum_{1 \leq i, k \leq d} a_j^2 (X_t - X_t e_d^j X_t)_{i,k} ((X_t^{-1})_{i,j}(X_t^{-1})_{k,j} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,j}) \right) \\ &\quad + \sum_{i=1}^d \left(\sum_{1 \leq j, l \leq d} a_i^2 (X_t - X_t e_d^i X_t)_{j,l} ((X_t^{-1})_{i,j}(X_t^{-1})_{i,l} - (X_t^{-1})_{i,i}(X_t^{-1})_{j,l}) \right) \\ &= 2 \sum_{i=1}^d a_i^2 (\text{Tr} [(X_t - X_t e_d^i X_t) X_t^{-1} e_d^i X_t^{-1}] - (X_t^{-1})_{i,i} \text{Tr} [(X_t - X_t e_d^i X_t) X_t^{-1}]). \end{aligned}$$

Since $X_t \in \mathfrak{C}_d^*(\mathbb{R})$, we obtain that $\text{Tr} [(X_t - X_t e_d^i X_t) X_t^{-1} e_d^i X_t^{-1}] = (X_t^{-1})_{i,i} - 1$ and that $\text{Tr} [X_t^{-1} (X_t - X_t e_d^i X_t)] = d - (X_t)_{i,i} = d - 1$. We finally get:

$$\begin{aligned} d(\det(X_t)) &= \det(X_t) (\text{Tr}[X_t^{-1}(\kappa c + c\kappa - (d-2)a^2)] - \text{Tr}(2\kappa + a^2)) dt \\ &\quad + 2 \det(X_t) \sum_{i=1}^d a_i \text{Tr} \left[X_t^{-1} e_d^i dW_s^T \sqrt{X_t - X_t e_d^i X_t} \right]. \end{aligned} \tag{3.59}$$

Now, we compute the quadratic variation of $\det(X_t)$ by using (3.6):

$$\begin{aligned}
\frac{d\langle \det(X) \rangle_t}{\det(X_t)^2} &= \sum_{\substack{1 \leq i, j \leq d \\ 1 \leq k, l \leq d}} (X_t^{-1})_{i,j} (X_t^{-1})_{k,l} \{ a_j^2 \mathbb{1}_{j=k} (X_t - X_t e_d^j X_t)_{i,l} + a_j^2 \mathbb{1}_{j=l} (X_t - X_t e_d^j X_t)_{i,k} \\
&\quad + a_i^2 \mathbb{1}_{i=l} (X_t - X_t e_d^i X_t)_{j,k} + a_i^2 \mathbb{1}_{i=k} (X_t - X_t e_d^i X_t)_{j,l} \} dt \\
&= 4 \sum_{i=1}^d a_i^2 \text{Tr} [X_t^{-1} e_d^i X_t^{-1} (X_t - X_t e_d^i X_t)] dt \\
&= 4 \sum_{i=1}^d a_i^2 ((X_t^{-1})_{i,i} - 1) dt = 4 [\text{Tr}(a^2 X_t^{-1}) - \text{Tr}(a^2)] dt.
\end{aligned}$$

It is indeed nonnegative: we can show by diagonalizing and using the convexity of $x \mapsto 1/x$ that $x_{i,i}^{-1} \geq 1/x_{i,i} = 1$. Then, there is a Brownian motion $(\beta_t, t \geq 0)$ such that (3.57) holds (see Theorem 3.4.2 in [57]). \square

Proposition 74 — Let $k, \theta, \eta \geq 0$. For a given $x \in [-1, 1]$, let us consider a process $(X_t^x)_{t \geq 0}$, starting from x , and defined as the solution of the following SDE

$$dX_t^x = k(\theta - X_t^x)dt + \eta \sqrt{1 - (X_t^x)^2} dB_t, \quad (3.60)$$

where $(B_t)_{t \geq 0}$ is a real Brownian motion. Then there exists a positive constant $K > 0$, such that

$$\forall t \geq 0, \forall x \in [-1, 1], \mathbb{E} [(X_t^x - x)^4] \leq Kt^2$$

Proof: For a given $x \in [-1, 1]$, we set $f^x(y) = (y - x)^4$. If we denote L the infinitesimal operator of the process X_t^x , then we notice that $f^x(x) = Lf^x(x) = 0$. Besides, $(x, y) \in [-1, 1]^2 \mapsto L^2 f^x(y)$ is continuous and therefore bounded:

$$\exists K > 0, \forall x, y \in [-1, 1], |L^2 f^x(y)| \leq 2K. \quad (3.61)$$

Since the process $(X_t^x)_{t \geq 0}$ is defined on $[-1, 1]$, we get by applying twice Itô's formula:

$$\mathbb{E} [f^x(X_t^x)] = \int_0^t \int_0^s \mathbb{E} [L^2 f^x(X_u^x)] dud s.$$

From (3.61), one can deduce that $\left| \int_0^t \int_0^s \mathbb{E} [L^2 f^x(X_u^x)] dud s \right| \leq Kt^2$, and obtain the final result. \square

Some basic results on squared Bessel processes

Lemma 75 — Let $\beta \geq 2$ and $Z_t = z + \beta t + 2 \int_0^t \sqrt{Z_s} dB_s$ be a squared Bessel process of dimension β starting from $z > 0$. Then we have

$$\mathbb{P}(\forall t \geq 0, \int_0^t \frac{ds}{Z_s} < \infty) = 1 \quad \text{and} \quad \int_0^{+\infty} \frac{ds}{Z_s} = +\infty \text{ a.s.}$$

Proof: The first claim is obvious, since the square Bessel process does never touch zero under the condition of $\beta \geq 2$. (see for instance [52], part 6.1.3). By using a comparison theorem ($\forall t \geq 0, Z_t \leq Z'_t$ a.s. if $\beta \leq \beta'$), it is sufficient to prove the second claim for $\beta \in \mathbb{N}$. In this case, it is well known that $(W_t^1 + \sqrt{z})^2 + \sum_{k=2}^n (W_t^k)^2$ follows a square Bessel process of dimension n , where $(W_t^k, t \geq 0)$ are independent Brownian motion. By the law of the iterated logarithm, $\limsup_{t \rightarrow +\infty} \frac{(W_t^k)^2}{2t \log(\log(t))} = 1$, which gives the desired result since $\int_1^\infty \frac{dt}{t \log(\log(t))} = +\infty$. \square

Lemma 76 — Let $\beta \geq 6$. Let $Z_t = 1 + \beta t + 2 \int_0^t \sqrt{Z_s} dB_s$ be a squared Bessel process of dimension β starting from 1 and $\phi(t) = \int_0^t \frac{1}{Z_s} ds$. Then we have

$$\mathbb{E}[\phi(t)] = t + \frac{4 - \beta}{2} t^2 + O(t^3), \quad \mathbb{E}[\phi(t)^2] = t^2 + O(t^3), \quad \mathbb{E}[\phi(t)^3] = O(t^3).$$

Proof: For a fixed time $t > 0$, the density of Z_t is given by:

$$z > 0, p(t, z) = \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^k}{k!} \frac{1}{2t \Gamma(k + \frac{\beta}{2})} \left(\frac{z}{2t}\right)^{k-1 + \frac{\beta}{2}} e^{-\frac{z}{2t}}.$$

Let us consider that $\gamma \in \{1, 2, 3\}$, then all negative moments can be written as

$$\mathbb{E} \left[\frac{1}{Z_t^\gamma} \right] = \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+\gamma} \Gamma(k + \frac{\beta}{2} - \gamma)}{k! \Gamma(k + \frac{\beta}{2})} = \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+\gamma}}{k!} \frac{1}{(k + \frac{\beta}{2} - 1) \times \cdots \times (k + \frac{\beta}{2} - \gamma)}.$$

We have $\frac{1}{(k + \frac{\beta}{2} - 1)} = \frac{1}{k+1} - \frac{\beta-4}{2(k+2)(k+1)} + O(\frac{1}{k^3})$, which yields to the following expansion:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{Z_t} \right] &= \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+1}}{(k+1)!} - (\beta-4)t \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+2}}{(k+2)!} + O\left(\frac{t^2}{2} \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+3}}{(k+3)!}\right) \\ &= 1 - (\beta-4)t + O(t^2) \end{aligned} \quad (3.62)$$

The first equality is thus obtained. We use the same argument to get:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{Z_t^2} \right] &= \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+2}}{(k+2)!} + O\left(t \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+3}}{(k+3)!}\right) = 1 + O(t) \\ \mathbb{E} \left[\frac{1}{Z_t^3} \right] &= O\left(\sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{k+3}}{(k+3)!}\right) = O(1). \end{aligned} \quad (3.63)$$

By Jensen's inequality, one can deduce that $\mathbb{E} \left[\left(\int_0^t \frac{ds}{Z_s} \right)^3 \right] \leq t^2 \mathbb{E} \left[\int_0^t \frac{ds}{(Z_s)^3} \right]$. Thanks to the moment expansion in (3.63), we find the third equality. Finally, by Jensen's equality, we obtain that

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t \left[\frac{1}{Z_s} - 1 \right] ds \right)^2 \right] &\leq t \mathbb{E} \left[\int_0^t \left(\frac{1}{Z_s} - 1 \right)^2 ds \right] = t \mathbb{E} \left[\int_0^t \frac{ds}{(Z_s)^2} \right] - 2t \mathbb{E} \left[\int_0^t \frac{ds}{Z_s} \right] + t^2 \\ &= t^2 - 2t^2 + t^2 + O(t^3) = O(t^3). \end{aligned}$$

It yields that

$$\mathbb{E} \left[\left(\int_0^t \left[\frac{1}{Z_s} \right] ds \right)^2 \right] = \mathbb{E} \left[\left(\int_0^t \left[\frac{1}{Z_s} - 1 \right] ds \right)^2 \right] - t^2 + 2t \int_0^t \mathbb{E} \left[\frac{1}{Z_s} \right] ds = t^2 + O(t^3).$$

□

3.4.3 A direct proof of Theorem 51

Proof: From (3.8) we have $2L_i = -\alpha L_i^D + L_i^M$, with:

$$L_i^D = \sum_{\substack{1 \leq j \leq d \\ j \neq i}} x_{\{i,j\}} \partial_{\{i,j\}}, \quad L_i^M = \sum_{\substack{1 \leq j,k \leq d \\ j \neq i, k \neq i}} (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}}.$$

We want to show that $L_i L_j = L_j L_i$ for $i \neq j$. Up to a permutation of the coordinates, L_i and L_j are the same operators as L_1 and L_2 . It is therefore sufficient to check that $L_1 L_2 = L_2 L_1$. Since $L_1 L_2 = L_1^M L_2^M - \alpha(L_1^D L_2^M + L_1^M L_2^D) + \alpha^2 L_1^D L_2^D$, it is sufficient to check that the three terms remain unchanged when we exchange indices 1 and 2. To do so we write:

$$\begin{aligned} L_1^M &= \sum_{3 \leq i,j \leq d} (x_{\{i,j\}} - x_{\{1,i\}} x_{\{1,j\}}) \partial_{\{1,i\}} \partial_{\{1,j\}} + 2 \sum_{3 \leq i \leq d} (x_{\{2,i\}} - x_{\{1,2\}} x_{\{1,i\}}) \partial_{\{1,2\}} \partial_{\{1,i\}} \\ &\quad + (1 - x_{\{1,2\}}^2) \partial_{\{1,2\}}^2 \\ L_2^M &= \sum_{3 \leq k,l \leq d} (x_{\{k,l\}} - x_{\{2,k\}} x_{\{2,l\}}) \partial_{\{2,k\}} \partial_{\{2,l\}} + 2 \sum_{3 \leq l \leq d} (x_{\{1,l\}} - x_{\{1,2\}} x_{\{2,l\}}) \partial_{\{1,2\}} \partial_{\{2,l\}} \\ &\quad + (1 - x_{\{1,2\}}^2) \partial_{\{1,2\}}^2 \\ L_1^D &= x_{\{1,2\}} \partial_{\{1,2\}} + \sum_{3 \leq i \leq d} x_{\{1,i\}} \partial_{\{1,i\}}, \quad L_2^D = x_{\{1,2\}} \partial_{\{1,2\}} + \sum_{3 \leq l \leq d} x_{\{2,l\}} \partial_{\{2,l\}}. \end{aligned}$$

By a straightforward but tedious calculation, we get :

$$\begin{aligned} L_1^M L_2^M &= \underbrace{\sum_{3 \leq i,j,k,l \leq d} (x_{\{i,j\}} - x_{\{1,i\}} x_{\{1,j\}}) (x_{\{k,l\}} - x_{\{2,k\}} x_{\{2,l\}}) \partial_{\{1,i\}} \partial_{\{1,j\}} \partial_{\{2,k\}} \partial_{\{2,l\}}}_{\text{1}} \\ &+ \underbrace{\sum_{3 \leq i,j \leq d} (x_{\{i,j\}} - x_{\{1,i\}} x_{\{1,j\}}) (2\partial_{\{1,2\}} \partial_{\{2,i\}} \partial_{\{1,j\}} + 2\partial_{\{1,2\}} \partial_{\{2,j\}} \partial_{\{1,i\}})}_{\text{2}} \\ &+ 2 \underbrace{\sum_{3 \leq i,j,l \leq d} (x_{\{i,j\}} - x_{\{1,i\}} x_{\{1,j\}}) (x_{\{1,l\}} - x_{\{1,2\}} x_{\{2,l\}}) \partial_{\{1,2\}} \partial_{\{2,l\}} \partial_{\{1,i\}} \partial_{\{1,j\}}}_{\text{3}} \\ &+ \underbrace{\sum_{3 \leq i,j \leq d} (x_{\{i,j\}} - x_{\{1,i\}} x_{\{1,j\}}) (1 - x_{\{1,2\}}^2) \partial_{\{1,i\}} \partial_{\{1,j\}} \partial_{\{1,2\}}^2}_{\text{4}} \\ &+ 2 \underbrace{\sum_{3 \leq i,k,l \leq d} (x_{\{2,i\}} - x_{\{1,2\}} x_{\{1,i\}}) (x_{\{k,l\}} - x_{\{2,k\}} x_{\{2,l\}}) \partial_{\{2,k\}} \partial_{\{2,l\}} \partial_{\{1,2\}} \partial_{\{1,i\}}}_{\text{3}} \end{aligned}$$

$$\begin{aligned}
& +4 \sum_{3 \leq i \leq d} (x_{\{2,i\}} - x_{\{1,2\}}x_{\{1,i\}}) \left(\underbrace{\partial_{\{1,2\}}^2 \partial_{\{2,i\}}}_{\bar{5}} - \underbrace{\sum_{3 \leq l \leq d} x_{\{2,l\}} \partial_{\{1,2\}} \partial_{\{2,l\}} \partial_{\{1,i\}}}_{\bar{6}} \right) \\
& +4 \underbrace{\sum_{3 \leq i \leq d} \sum_{3 \leq l \leq d} (x_{\{1,l\}} - x_{\{1,2\}}x_{\{2,l\}}) \partial_{\{1,2\}}^2 \partial_{\{2,l\}} \partial_{\{1,i\}}}_{\bar{7}} \\
& +2 \sum_{3 \leq i \leq d} (x_{\{2,i\}} - x_{\{1,2\}}x_{\{1,i\}}) \left(\underbrace{-2x_{\{1,2\}} \partial_{\{1,2\}}^2 \partial_{\{1,i\}}}_{\bar{8}} + \underbrace{(1 - x_{\{1,2\}}^2) \partial_{\{1,2\}}^3 \partial_{\{1,i\}}}_{\bar{9}} \right) \\
& + \underbrace{\sum_{3 \leq k, l \leq d} (x_{\{k,l\}} - x_{\{2,k\}}x_{\{2,m\}}) (1 - x_{\{1,2\}}^2) \partial_{\{2,k\}} \partial_{\{2,m\}} \partial_{\{1,2\}}^2}_{\bar{4}} \\
& + (1 - x_{\{1,2\}}^2) \left(\underbrace{\sum_{3 \leq l \leq d} 2(x_{\{1,l\}} - x_{\{1,2\}}x_{\{2,l\}}) \partial_{\{1,2\}}^3 \partial_{\{2,l\}}}_{\bar{9}} - 4 \underbrace{\sum_{3 \leq l \leq d} x_{\{2,l\}} \partial_{\{1,2\}}^2 \partial_{\{2,l\}}}_{\bar{10}} \right) \\
& + \underbrace{(1 - x_{\{1,2\}}^2) \partial_{\{1,2\}}^2 ((1 - x_{\{1,2\}}^2) \partial_{\{1,2\}}^2)}_{\bar{11}}
\end{aligned}$$

In this formula, the terms \bar{n} are already symmetric by exchanging 1 and 2. The terms n are paired with the corresponding symmetric term. To analyse the terms \tilde{n} , we have to do further calculations. On the one hand,

$$\begin{aligned}
\tilde{10} + \tilde{5} &= \sum_{3 \leq l \leq d} 4x_{\{1,2\}} (x_{\{1,2\}}x_{\{2,l\}} - x_{\{1,l\}}) \partial_{\{1,2\}}^2 \partial_{\{2,l\}} \\
\tilde{8} &= \sum_{3 \leq l \leq d} 4x_{\{1,2\}} (x_{\{1,2\}}x_{\{1,l\}} - x_{\{2,l\}}) \partial_{\{1,2\}}^2 \partial_{\{1,l\}},
\end{aligned}$$

are symmetric together. On the other hand we have

$$\tilde{2} + \tilde{6} = \sum_{\substack{1 \leq i, j \leq d \\ i \neq 1, 2, j \neq 1, 2}} \{4x_{\{i,j\}} - 4x_{\{1,i\}}x_{\{1,j\}} - 4x_{\{2,i\}}x_{\{2,j\}} + 4x_{\{1,i\}}x_{\{2,j\}}x_{\{1,2\}}\} \partial_{\{1,2\}} \partial_{\{1,i\}} \partial_{\{2,j\}},$$

which is symmetric.

Now we focus on $L_1^D L_2^M + L_1^M L_2^D$. We number the terms with the same rule as above, and

get:

$$\begin{aligned}
L_1^D L_2^M + L_1^M L_2^D &= \underbrace{\sum_{3 \leq k, l \leq d} (x_{\{k, l\}} - x_{\{2, l\}} x_{\{2, k\}}) x_{\{1, 2\}} \partial_{\{2, l\}} \partial_{\{2, k\}} \partial_{\{1, 2\}}}_{1} + \\
&+ \underbrace{2 \sum_{3 \leq l \leq d} x_{\{1, 2\}} (x_{\{1, l\}} - x_{\{1, 2\}} x_{\{2, l\}}) \partial_{\{1, 2\}}^2 \partial_{\{2, l\}}}_{2} - \underbrace{2 \sum_{3 \leq l \leq d} x_{\{1, 2\}} x_{\{2, l\}} \partial_{\{2, l\}} \partial_{\{1, 2\}}}_{3} \\
&+ \underbrace{x_{\{1, 2\}} \partial_{\{1, 2\}} \left\{ (1 - x_{\{1, 2\}}^2) \partial_{\{1, 2\}}^2 \right\}}_{4} + \underbrace{\sum_{3 \leq i, k, l \leq d} x_{\{1, i\}} (x_{\{l, k\}} - x_{\{2, k\}} x_{\{2, l\}}) \partial_{\{2, k\}} \partial_{\{2, l\}} \partial_{\{1, i\}}}_{5} \\
&+ \underbrace{2 \sum_{3 \leq i, l \leq d} x_{\{1, i\}} (x_{\{1, l\}} - x_{\{1, 2\}} x_{\{2, l\}}) \partial_{\{1, 2\}} \partial_{\{2, l\}} \partial_{\{1, i\}}}_{6} + \underbrace{2 \sum_{3 \leq i \leq d} x_{\{1, i\}} \partial_{\{1, 2\}} \partial_{\{2, i\}}}_{7} \\
&+ \underbrace{\sum_{3 \leq i \leq d} x_{\{1, i\}} (1 - x_{\{1, 2\}}^2) \partial_{\{1, 2\}}^2 \partial_{\{1, i\}}}_{8} + \underbrace{\sum_{3 \leq i, j \leq d} (x_{\{i, j\}} - x_{\{1, i\}} x_{\{1, j\}}) x_{\{1, 2\}} \partial_{\{1, 2\}} \partial_{\{1, i\}} \partial_{\{1, j\}}}_{1} \\
&+ \underbrace{\sum_{3 \leq i, j, l \leq d} x_{\{2, l\}} (x_{\{i, j\}} - x_{\{1, i\}} x_{\{1, j\}}) \partial_{\{1, i\}} \partial_{\{1, j\}} \partial_{\{2, l\}}}_{5} \\
&+ \underbrace{2 \sum_{3 \leq i \leq d} x_{\{1, 2\}} (x_{\{2, i\}} - x_{\{1, 2\}} x_{\{1, i\}}) \partial_{\{1, 2\}}^2 \partial_{\{1, i\}}}_{2} + \underbrace{2 \sum_{3 \leq i \leq d} (x_{\{2, i\}} - x_{\{1, i\}} x_{\{1, 2\}}) \partial_{\{1, i\}} \partial_{\{1, 2\}}}_{7 \quad 3} \\
&+ \underbrace{2 \sum_{3 \leq i, l \leq d} x_{\{2, l\}} (x_{\{2, i\}} - x_{\{1, 2\}} x_{\{1, i\}}) \partial_{\{1, i\}} \partial_{\{1, 2\}} \partial_{\{2, l\}}}_{6} + \underbrace{(1 - x_{\{1, 2\}}^2) \partial_{\{1, 2\}}^2 \{x_{\{1, 2\}} \partial_{\{1, 2\}}\}}_{9} \\
&+ \underbrace{\sum_{3 \leq l \leq d} (1 - x_{\{1, 2\}}^2) x_{\{2, l\}} \partial_{\{1, 2\}}^2 \partial_{\{2, l\}}}_{8}.
\end{aligned}$$

Therefore, $L_1^D L_2^M + L_1^M L_2^D$ is symmetric when we exchange 1 and 2. Last, it is easy to check that $L_1^D L_2^D = L_2^D L_1^D$, which concludes the proof. \square

3.4.4 A direct construction of a second order scheme for MRC processes

In Section 3.3, we have presented a second order scheme for Mean-Reverting Correlation processes that is obtained from a second order scheme for Wishart processes. In this section, we propose a second order scheme that is constructed directly by a splitting of the generator of Mean-Reverting Correlation processes. As pointed in (3.42), it is sufficient to construct a potential second order scheme for $MRC_d(x, \frac{d-2}{2} e_d^1, I_d, e_d^1; t)$. Thanks to the transformation given by Proposition 54, it is even sufficient to construct such a scheme when $(x)_{2 \leq i, j \leq d} = I_{d-1}$.

Consequently, in the rest of this section, we focus on getting a potential second order scheme for $MRC_d(x, \frac{d-2}{2} e_d^1, I_d, e_d^1, ; t)$ where $(x)_{2 \leq i, j \leq d} = I_{d-1}$. By (3.23), the matrix x

is a correlation matrix if $\sum_{i=2}^d x_{1,i}^2 \leq 1$. Besides, the only non constant elements are on the first row (or the first column) and the vector $((X_t)_{1,i})_{2\dots d}$ is thus defined on the unit ball \mathbb{D} :

$$\mathbb{D} = \left\{ x \in \mathbb{R}^{d-1}, \sum_{i=1}^{d-1} x_i^2 \leq 1 \right\}. \quad (3.64)$$

With a slight abuse of notation, the process $((X_t)_i)_{1\dots d-1}$ will denote the vector $((X_t)_{1,i+1})_{1\dots d-1}$. Its quadratic covariance is given by $d\langle (X_t)_i, d(X_t)_j \rangle = (\mathbb{1}_{i=j} - (X_t)_i(X_t)_j) dt$, and the infinitesimal generator L^1 of $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1)$ can be rewritten on \mathbb{D} , as

$$L^1 = -\frac{d-2}{2} \sum_{i=1}^{d-1} x_i \partial_i + \frac{1}{2} \sum_{1 \leq i, j \leq d-1} (\mathbb{1}_{i=j} - x_i x_j) \partial_i \partial_j. \quad (3.65)$$

One can prove that for all $\forall 1 \leq i \leq d-1$, the following stochastic differential equation

$$dM_t^i = -\frac{d-2}{2} M_t^i + M_t^i \sqrt{1 - \sum_{j=1}^{d-1} (M_t^j)^2} dB_t^1 + (1 - (M_t^i)^2) dB_t^{i+1} - M_t^i \sum_{\substack{1 \leq j \leq d-1 \\ j \neq i}} M_t^j dB_t^{j+1} \quad (3.66)$$

is associated to the martingale problem of L^1 , where $(B_t)_{t \geq 0}$ denotes a standard Brownian motion in dimension d . By Theorem (57), there is a unique weak solution $(M_t)_{t \geq 0}$ that is defined on \mathbb{D} .

The scope of this section is to derive a potential second order discretization for the operator L^1 , by using an ad-hoc splitting and the results of Proposition 63. We consider the following splitting

$$L^1 = \mathcal{L}^1 + \sum_{m=1}^{d-1} \mathcal{L}^{m+1}, \quad (3.67)$$

where we have, for $1 \leq m \leq d-1$:

$$\begin{aligned} \mathcal{L}^1 &= \frac{1}{2} \left(1 - \sum_{i=1}^{d-1} x_i^2 \right) \sum_{1 \leq l, k \leq d-1} x_k x_l \partial_k \partial_l, \\ \mathcal{L}^{m+1} &= \frac{1}{2} \left(- \sum_{1 \leq k \neq m \leq d-1} x_k \partial_k + (1 - x_m^2) \partial_m^2 - 2x_m(1 - x_m^2) \sum_{1 \leq k \neq m \leq d-1} x_k \partial_k \partial_m \right. \\ &\quad \left. + \sum_{\substack{1 \leq k \neq m \leq d-1 \\ 1 \leq l \neq m \leq d-1}} x_k x_l x_m^2 \partial_k \partial_l \right). \end{aligned}$$

Thanks to Proposition 63, it is sufficient to focus on getting potential second-order schemes for the operators $\mathcal{L}^1, \dots, \mathcal{L}^d$.

Potential second order schemes for $\mathcal{L}^2, \dots, \mathcal{L}^d$

All the generators \mathcal{L}^{l+1} , $l = 1, \dots, d-1$ have the same solution as \mathcal{L}^2 up to the permutation of the first coordinate and the l -th one. It is then sufficient to focus on the first operator \mathcal{L}^2 . By

straightforward calculus, we find that the following SDE

$$d(X_t)_1 = (1 - (X_t)_1^2)dB_t, \quad \forall 2 \leq i \leq d-1 \quad d(X_t)_i = -(X_t)_i \left(\frac{dt}{2} + (X_t)_1 dB_t \right), \quad (3.68)$$

$$X_0 = x \in \mathbb{D},$$

is well a solution of the martingale problem for the generator \mathcal{L}^2 . The SDE that defines $(X_t)_1$ is autonomous. Since $x_1 \in [-1, 1]$, it has clearly a unique strong valued in $[-1, 1]$. It yields that the SDE (3.68) has a unique strong solution on \mathbb{R}^d . To prove that $(X_t)_{t \geq 0}$ takes values in \mathbb{D} we consider $V_t = \sum_{i=1}^d (X_t)_i^2$. By Itô calculus, it follows that

$$dV_t = (1 - V_t)(1 - (X_t)_1^2)dt + 2(X_t)_1(1 - V_t)dB_t.$$

Thus, $1 - V_t$ can be written as a stochastic exponential starting from $1 - V_0 \geq 0$ and is therefore nonnegative. We now introduce the Ninomiya-Victoir scheme for the SDE (3.68).

Proposition 77 — *Let us consider $x \in \mathbb{D}$. Let Y be sampled according to $\mathbb{P}(Y = \sqrt{3}) = \mathbb{P}(Y = -\sqrt{3}) = \frac{1}{6}$, so that it fits the first five moments of a standard Gaussian variable. Then $\hat{X}_t^x = X^0(\frac{t}{2}, X^1(\sqrt{t}Y, X^0(\frac{t}{2}, x)))$ is well defined on \mathbb{D} and is a potential second order scheme for the infinitesimal operator \mathcal{L}^2 , where for all $2 \leq l \leq d-1$, $t \geq 0$, $x \in \mathbb{D}$, $y \in \mathbb{R}$:*

$$\begin{aligned} X_1^0(t, x) &= \frac{x_1 e^t}{\sqrt{e^{2t}x_1^2 + (1-x_1^2)}}, & X_l^0(t, x) &= \frac{x_l}{\sqrt{e^{2t}x_1^2 + (1-x_1^2)}}, \\ X_1^1(y, x) &= \frac{e^{2y}(1+x_1) - (1-x_1)}{e^{2y}(1+x_1) + (1-x_1)}, & X_l^1(y, x) &= \frac{2e^y x_l}{e^{2y}(1+x_1) + (1-x_1)}. \end{aligned}$$

Proof: The proof is a direct application of the Ninomiya-Victoir's scheme [69] and we introduce for all $2 \leq l \leq d-1$, the following ODEs:

$$\begin{aligned} \partial_t X_1^0(t, x) &= X_1^0(t, x)(1 - (X_1^0(t, x))^2), & \partial_t X_l^0(t, x) &= -X_l^0(t, x)(X_1^0(t, x))^2 \\ \partial_y X_1^1(y, x) &= (1 - (X_1^1(y, x))^2), & \partial_y X_l^1(y, x) &= -X_l^1(y, x)X_1^1(y, x). \end{aligned}$$

These ODEs can be solved explicitly as stated above. We have to check that they are well defined on \mathbb{D} . This can be checked with the explicit formulas or by observing both equations $\partial_t(\sum_{l=1}^{d-1}(X_l^0(t, x))^2) = 2(X_1^0(t, x))^2(1 - \sum_{l=1}^{d-1}(X_l^0(t, x))^2)$, $\partial_t(\sum_{l=1}^{d-1}(X_l^1(t, x))^2) = 2X_1^1(t, x)(1 - \sum_{l=1}^{d-1}(X_l^1(t, x))^2)$. Last, Theorem 1.18 in Alfonsi [5] ensures that \hat{X}_t^x is a potential second order scheme for \mathcal{L}^2 . \square

Potential second order scheme for \mathcal{L}^1

Let $(B_t)_{t \geq 0}$ be a real a Brownian motion. We consider the following SDE:

$$\forall 1 \leq i \leq d-1, \quad d(X_t)_i = (X_t)_i \sqrt{1 - \sum_{m=1}^{d-1} (X_t)_m^2} dB_t, \quad X_0 = x \in \mathbb{D} \quad (3.69)$$

Its infinitesimal generator is \mathcal{L}^1 , and we claim that it has a unique strong solution. To check this, we set $Z_t = \sqrt{\sum_{i=1}^{d-1} (X_t)_i^2}$. By Itô calculus, we get that the process $(Z_t)_{t \geq 0}$ is solution of the following SDE

$$dZ_t = Z_t \sqrt{1 - Z_t^2} dB_t, \quad Z_0 = \sqrt{\sum_{i=1}^{d-1} x_i^2}. \quad (3.70)$$

Since the SDE (3.70) satisfies the Yamada-Watanabe conditions (Proposition 2.13, Chapter 5 of [57]), it has a unique strong solution defined on $[0, 1]$. If $Z_0 = 0$, we necessarily have $Z_t = 0$ and thus $(X_t)_i = 0$ for any $t \geq 0$. Otherwise, we have by Itô calculus $d \ln((X_t)_i) = d \ln(Z_t)$, and then

$$\forall 1 \leq i \leq d-1, (X_t)_i = \begin{cases} 0, & \text{if } Z_0 = 0 \\ \frac{x_i}{Z_0} Z_t & \text{otherwise.} \end{cases} \quad (3.71)$$

Conversely, we check easily that (3.71) is a strong solution of (3.70), which proves our claim. The explicit solution (3.71) indicates that the SDE (3.70) is one-dimensional up to a basic transformation. Thanks to the next proposition, it is sufficient to construct a potential second order scheme for Z_t in order to get a potential second order scheme for (3.70).

Proposition 78 — *Let us consider $x \in \mathbb{D}$, and \hat{Z}_t^z denote the second potential order scheme for $(Z_t)_{t \geq 0}$, starting from a given value $z \in [0, 1]$. Then the following scheme \hat{X}_t^x*

$$\forall 1 \leq i \leq d-1, (\hat{X}_t^x)_i = \begin{cases} 0 & \text{if } \sum_{j=1}^{d-1} x_j^2 = 0, \\ \frac{x_i}{\sqrt{\sum_{j=1}^{d-1} x_j^2}} \hat{Z}_t^{\sqrt{\sum_{j=1}^{d-1} x_j^2}} & \text{otherwise,} \end{cases}$$

is a second potential order scheme for \mathcal{L}^1 which is well defined on \mathbb{D} .

Proof: For a given $x \in \mathbb{D}$ and $f \in \mathcal{C}^\infty(\mathbb{D})$, let $(X_t^x)_{t \geq 0}$ denote a process defined by (3.71) and starting from $x \in \mathbb{D}$. It is sufficient to prove that

$$\left| \mathbb{E} [f(X_t^x)] - \mathbb{E} [f(\hat{X}_t^x)] \right| \leq Kt^3.$$

The case where $x = 0$ is trivial, and we assume thus that $\sum_{i=1}^{d-1} x_i^2 > 0$. Let $f \in \mathcal{C}^\infty(\mathbb{D})$. We define $g^x : [0, 1] \rightarrow \mathbb{R}$ by $\forall y \in [0, 1]$, $g^x(y) = f\left(\frac{x_1}{\sqrt{\sum_{j=1}^{d-1} x_j^2}} y, \dots, \frac{x_{d-1}}{\sqrt{\sum_{j=1}^{d-1} x_j^2}} y\right)$. Since for every $1 \leq i \leq d-1$, $\left| \frac{x_i}{\sqrt{\sum_{j=1}^{d-1} x_j^2}} \right| \leq 1$, it follows we can construct from a good sequence of f a good sequence for g^x that does not depend on x . By the definition of the second potential scheme, there exist positive constants $K > 0$ and $\eta > 0$, depending only on a good sequence of f such that $\forall t \in [0, \eta]$

$$\left| \mathbb{E} \left[g^x \left(Z_t \sqrt{\sum_{j=1}^{d-1} x_j^2} \right) \right] - \mathbb{E} \left[g^x \left(\hat{Z}_t^{\sqrt{\sum_{j=1}^{d-1} x_j^2}} \right) \right] \right| \leq Kt^3,$$

which gives the desired result. \square

We now focus on finding a potential second order scheme for $(Z_t)_{t \geq 0}$. To do so, we try the Ninomiya-Victoir's scheme [69] and consider the following ODEs for $z \in [0, 1]$,

$$\forall t \geq 0, \partial_t Z_0(t, z) = Z_0(t, z)(Z_0(t, z) - \frac{1}{2}), \quad \forall x \in \mathbb{R}, \partial_x Z_1(x, z) = Z_1(x, z) \sqrt{1 - Z_1(x, z)^2}.$$

These ODEs can be solved explicitly. On the one hand, it follows that for every $t \geq 0$ and $z \in [0, 1]$

$$Z_0(t, z) = \frac{z \exp(-t/2)}{\sqrt{1 - 2z^2(1 - \exp(-t))}}.$$

On the other hand, we get by considering the change of variable $\sqrt{1 - Z_1^2}$ that for every $x \in \mathbb{R}$ and $z \in [0, 1]$,

$$Z_1(x, z) = \begin{cases} \frac{2z \exp(-x)}{1 - \sqrt{1 - z^2} + \exp(-2x)(1 + \sqrt{1 - z^2})} & \text{if } x \leq \frac{1}{2} \ln\left(\frac{1 + \sqrt{1 - z^2}}{1 - \sqrt{1 - z^2}}\right), \\ 1 & \text{otherwise.} \end{cases}$$

Then, the Ninomiya-Victoir scheme is given by $Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$, where Y is a random variable that matches the five first moments of the standard Gaussian variable. Unfortunately, the composition $Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$ may not be defined if z is close to 1. To correct this, we proceed like Alfonsi [5] for the CIR diffusion. First, we consider Y that has a bounded support so that $Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$ is well defined when z is far enough from 1 (namely when $0 \leq z \leq K(t) \leq 1$ with $K(t) = 1 + O(t)$). When the initial value z is close to 1, we instead use a moment-matching scheme, and then we prove that the whole scheme is potentially of order 2 (Propositions 79 and 80).

Ninomiya-Victoir's scheme for $(Z_t)_{t \geq 0}$ away from 1

Proposition 79 — *Let us consider a discrete random variable Y that follows $\mathbb{P}(Y = \sqrt{3}) = \mathbb{P}(Y = -\sqrt{3}) = \frac{1}{6}$, and $\mathbb{P}(Y = 0) = \frac{2}{3}$, so that it matches the five first moments of a standard Gaussian.*

- *For a given $z \in [0, 1]$, the map $z \mapsto Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$ is well defined on $[0, 1]$, if and only if $z \in [0, K(t)]$, where the threshold function $K(t)$ is given in (3.73).*
- *For a given function $f \in \mathcal{C}^\infty([0, 1])$, there are constants $\eta, C > 0$ depending only on a good sequence of f such that $\forall t \in [0, \eta], \forall z \in [0, K(t)]$,*

$$\left| \mathbb{E} \left[Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z))) \right] - \left(f(z) - tL_Z f(z) + \frac{t^2}{2} L_Z^2 f(z) \right) \right| \leq Ct^3, \quad (3.72)$$

where L_Z is the infinitesimal operator associated to the SDE (3.70).

For every $t \geq 0$ the function $K(t)$ is valued on $[0, 1]$ such that

$$K(t) = \sqrt{\frac{1}{2 - e^{-t/2}}} \wedge \frac{\sqrt{1 - D(t, \sqrt{3})^2}}{\sqrt{e^{-t/2} + 2(1 - D(t, \sqrt{3})^2)(1 - e^{-t/2})}}, \quad \lim_{t \rightarrow 0} \frac{1 - K(t)}{t} = \frac{\sqrt{3}}{2}(1 + \sqrt{3}), \quad (3.73)$$

$$\text{with } \forall y \in \mathbb{R}^+ D(t, y) = \frac{1 - e^{-2\sqrt{t}y} + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}}(1 + e^{-2\sqrt{t}y})}{e^{-2\sqrt{t}y} + 1 + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}}(1 - e^{-2\sqrt{t}y})}.$$

Proof : The main technical thing here is to check the first point. Then, (3.72) is a direct consequence of Theorem 1.18 in Alfonsi [5]. By construction, we have $Z_0(t/2, z) \in [0, 1] \Leftrightarrow z \leq \frac{1}{\sqrt{2 - \exp(t/2)}}$. We conclude that the whole scheme $Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$ is well defined on $[0, 1]$, if and only if $Z_1(\sqrt{t}Y, Z_0(t/2, z)) \leq \frac{1}{\sqrt{2 - \exp(t/2)}}$. By slight abuse of notation, we denote in the following $Z_0(t/2, z)$ by the shorthand Z_0 . Let us assume for a while that we have:

$$\sqrt{1 - Z_0^2(1 + e^{-2\sqrt{t}Y})} + e^{-2\sqrt{t}Y} - 1 \geq 0, \text{ a.s.} \quad (3.74)$$

It yields then to

$$\begin{aligned} & Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z))) \in [0, 1] \\ & \Leftrightarrow \sqrt{1 - [Z_1(\sqrt{t}Y, Z_0(t/2, z))]^2} \geq \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}} \\ & \Leftrightarrow \sqrt{\left(\frac{e^{-2\sqrt{t}Y}(1 + \sqrt{1 - Z_0^2}) - (1 - \sqrt{1 - Z_0^2})}{e^{-2\sqrt{t}Y}(1 + \sqrt{1 - Z_0^2}) + (1 - \sqrt{1 - Z_0^2})} \right)^2} \geq \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}} \\ & \stackrel{\text{By (3.74)}}{\Rightarrow} \frac{\sqrt{1 - Z_0^2}(1 + e^{-2\sqrt{t}Y}) + e^{-2\sqrt{t}Y} - 1}{\sqrt{1 - Z_0^2}(e^{-2\sqrt{t}Y} - 1) + 1 + e^{-2\sqrt{t}Y}} \geq \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}} \\ & \Leftrightarrow \sqrt{1 - Z_0^2} \geq \frac{1 - e^{-2\sqrt{t}Y} + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}}(1 + e^{-2\sqrt{t}Y})}{e^{-2\sqrt{t}Y} + 1 + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}}(1 - e^{-2\sqrt{t}Y})} := D(t, Y). \end{aligned} \quad (3.75)$$

We can check that the mapping $D : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \mapsto D(t, x) = -1 + \frac{2(1 + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}})}{e^{-2\sqrt{t}x} + 1 + \sqrt{\frac{1 - e^{-t/2}}{2 - e^{-t/2}}}(1 - e^{-2\sqrt{t}x})}$

is non decreasing on x , and $D(t, x) \leq 1$. Since $Y \in \{-\sqrt{3}, 0, \sqrt{3}\}$, it yields thus that the last condition is equivalent to:

$$Z_0(t/2, z) \leq \sqrt{1 - D(t, \sqrt{3})^2} \Leftrightarrow z \leq \frac{\sqrt{1 - D(t, \sqrt{3})^2}}{\sqrt{e^{t/2} + 2(1 - e^{-t/2})(1 - D(t, \sqrt{3})^2)}}. \quad (3.76)$$

Conversely, if (3.76) is satisfied, we can check that $D(t, Y)(1 + e^{-2\sqrt{t}Y}) + e^{-2\sqrt{t}Y} - 1 \geq 0$. Therefore (3.74) holds. To sum up, when $z \in [0, K(t)]$, both $Z_0(t/2, Z_1(\sqrt{t}Y, Z_0(t/2, z)))$ and $Z_0(t/2, z)$ are defined in $\in [0, 1]$.

Last, it remains to compute the limit of $(1 - K(t))/t$. First, it is obvious that $\lim_{t \rightarrow 0} K(t) = 1$. We can check that $\sqrt{1 - D^2(t, \sqrt{3})} = \frac{2e^{-\sqrt{3}\sqrt{t}}}{\sqrt{2 - e^{-t/2}} + \sqrt{1 - e^{-t/2}}(1 - e^{-\sqrt{3}\sqrt{t}})} = 1 + t(\frac{1}{4} - \frac{\sqrt{3}}{2}(1 + \sqrt{3})) + o(t)$, and therefore $1 - \frac{\sqrt{1 - D(t, \sqrt{3})^2}}{\sqrt{1 + 2(1 - D(t, \sqrt{3})^2)(1 - e^{-t/2})}} = t(\frac{\sqrt{3}}{2}(1 + \sqrt{3})) + o(t)$. It yields that $\lim_{t \rightarrow 0} \frac{1 - K(t)}{t} = \frac{\sqrt{3}}{2}(1 + \sqrt{3}) \vee \frac{1}{2}$.

□

Potential second order scheme for $(Z_t)_{t \geq 0}$ in a neighbourhood of 1

Let $(Z_t)_{t \geq 0}$ be solution of the SDE (3.70). By Itô calculus, its moments satisfy the following induction:

$$\forall k \geq 2, \mathbb{E} [Z_t^k] = \left(z^k - \int_0^t \frac{k(k-1)}{2} e^{-\frac{k(k-1)}{2}s} \mathbb{E} [Z_s^{k+2}] ds \right) \exp\left(\frac{k(k-1)}{2}t\right).$$

We obtain first that $\mathbb{E} [Z_t^6] = z^6 + O(t)$, then $\mathbb{E} [Z_t^4] = z^4 + 6z^4t(1 - z^2) + O(t^2)$ and last

$$\mathbb{E} [Z_t^2] = z^2 + tz^2(1 - z^2) + \frac{t^2}{2}z^2(1 - z^2)(1 - 6z^2) + \mathcal{O}(t^3). \quad (3.77)$$

Moreover, by straightforward calculus, one can check that if $t \leq \frac{2}{5}$ and for every $z \in [0, 1]$

$$z^2 + tz^2(1 - z^2) + \frac{t^2}{2}z^2(1 - z^2)(1 - 6z^2) \leq 1, \quad tz^2(1 - z^2) + \frac{t^2}{2}z^2(1 - z^2)(1 - 6z^2) \geq 0. \quad (3.78)$$

Since $\mathbb{E}(Z_t) = z$, the right hand side of (3.78) corresponds to the asymptotic variance of Z_t . To approximate the process $(Z_t)_{t \geq 0}$ near to one, we use a discrete random variable, denoted by \hat{Z}_t^z , that fits both the exact first moment and the asymptotic second given by (3.77). We assume that \hat{Z}_t^z takes two possible values $0 \leq z^+ < z^- \leq 1$, with probability $p(t, z)$ and $1 - p(t, z)$ respectively. We introduce two positive variables (m^+, m^-) , defined as $z^+ = z + m^+$ and $z^- = z - m^-$. Since we are looking to match the moment, we get the following equations:

$$\begin{aligned} \mathbb{E} [\hat{Z}_t^z] &= z & \Leftrightarrow m^+p(t, z) &= m^-(1 - p(t, z)) \\ \mathbb{E} [(\hat{Z}_t^z)^2] &= z^2 + tz^2(1 - z^2) + \frac{t^2}{2}z^2(1 - z^2)(1 - 6z^2) & \Leftrightarrow (m^+)^2 \frac{p}{1-p} &= tz^2(1 - z^2) + \frac{t^2}{2}z^2(1 - z^2)(1 - 6z^2) \end{aligned} \quad (3.79)$$

We choose $m^+ = z(1 - z)$ and then have $p(t, z) = 1 - \frac{1}{1 + \frac{t(1+z)(1 + \frac{t}{2}(1 - 6z^2))}{1 - z}}$, $m^- = tz(1 + z)(1 + \frac{t}{2}(1 - 6z^2))$.

The random variable \hat{Z}_t^z is well defined on $[0, 1]$ if and only if $z^+ \leq 1$ and $z^- \geq 0$, which is respectively equivalent to $z(1 - z) \leq (1 - z)$ and $t(1 + z)(1 + \frac{t}{2}(1 - 6z^2)) \leq 1$. By straightforward calculus, we can check that these conditions are satisfied. Since $1 - K(t) \underset{t \rightarrow 0}{=} O(t)$ by Proposition 79, we deduce that there is $C > 0$ such that

$$\forall t \in [0, \frac{2}{5}], \forall z \in [K(t), 1], \forall q \in \mathbb{N}^*, \mathbb{E} [(1 - \hat{Z}_t^z)^q] \leq C^q t^q \quad (3.80)$$

Proposition 80 — *Let $U \sim \mathcal{U}([0, 1])$. The scheme $\hat{Z}_t^z = z^+ \mathbb{1}_{\{U \leq p(t, z)\}} + z^- \mathbb{1}_{\{U > p(t, z)\}}$ is a potential second order scheme on $z \in [K(t), 1]$: for any function $f \in C^\infty([0, 1])$, there are positive constants C and μ that depend on a good sequence of the function f , such that*

$$\forall t \in [0, \eta \wedge \frac{2}{5}], \forall z \in [K(t), 1], \left| \mathbb{E} [f(\hat{Z}_t^z)] - f(z) - tL_Z f(z) - \frac{t^2}{2}(L_Z)^2 f(z) \right| \leq Ct^3, \quad (3.81)$$

where L_Z is the infinitesimal operator associated to the SDE (3.70).

Proof : Let us consider a function $f \in C^\infty([0, 1])$. Since the exact scheme is a potential second order scheme (see Alfonsi [5]), there exist then two positive constants η and C , such that $\forall t \in [0, \mu], \forall z \in [0, 1], |\mathbb{E}[f(Z_t^z)] - f(z) - tL_Z f(z) - \frac{t^2}{2}(L_Z)^2 f(z)| \leq Ct^3$. We conclude that it is sufficient to prove that $\forall z \in [K(t), 1], |\mathbb{E}[f(Z_t^z)] - \mathbb{E}[f(\hat{Z}_t^z)]| \leq Ct^3$, for a constant positive variable C . By a third order Taylor expansion of f near to one, we obtain that

$$\forall z \in [0, 1], \left| f(z) - \left(f(1) - f'(1)(1-z) + \frac{(1-z)^2}{2} f''(1) \right) \right| \leq \|f^{(3)}\|_\infty (1-z)^3.$$

Thus, there is a constant $C > 0$ depending on a good sequence of f such that

$$|\mathbb{E}[f(Z_t^z)] - \mathbb{E}[f(\hat{Z}_t^z)]| \leq C \left(\mathbb{E}[(1 - \hat{Z}_t^z)^3] + \mathbb{E}[(1 - Z_t^z)^3] + \left| \mathbb{E}[(1 - Z_t^z)^2] - \mathbb{E}[(1 - \hat{Z}_t^z)^2] \right| \right)$$

By (3.80), the first term is of order $O(t^3)$. The last term is equal to $|\mathbb{E}[(Z_t^z)^2] - z^2 - tz^2(1-z^2) - \frac{t^2}{2}z^2(1-z^2)(1-6z^2)|$ and is also of order $O(t^3)$ by (3.77). Last, we have by Itô calculus that $\forall q \geq 2, \mathbb{E}[(1 - Z_t^z)^q] \leq (1-z)^q + q(q-1) \int_0^t \mathbb{E}[(1 - Z_s^z)^{q-1}] ds$. By induction, we get that there is a constant $R_q > 0$, such that $\forall z \in [K(t), 1], \mathbb{E}[(1 - Z_t^z)^q] \leq R_q t^q$, which finally gives the claimed result. \square

Chapter 4

Index modeling: a bottom up approach

Note: Contrairement aux parties précédentes, ce chapitre ne présente pas de nouveaux résultats mathématiques mais traite la modélisation multi dimensionnel en finance et étudie numériquement les données du marché. Il a donné lieu l'article [2]

Abstract In this paper we investigate different class of models in order to calibrate simultaneously the index option and Vanillas option of its components. It is known that option market for equity indices show a non-trivial dependency structure between its constitutions. Moreover, a simple correlation structure fails usually to explain correctly the steepness of the observed volatility skew of the index. We explain how this skew is strongly related to the instantaneous correlation between the variance of the log index price and the price itself, and we describe also the contribution of single-stock skew market. We propose a parametric local correlation model for the pricing of multi asset derivative, taking into consideration the term structure of the index implied volatility. Finally, we propose two class of stochastic correlation model. The first kind is considered as a perturbation around a local correlation model, whereas the second one combines between local and stochastic factor. Some numerical methods are described in order to get an efficient simulation and calibration results.

4.1 Introduction

Correlation plays a crucial role in financial market, especially during crisis. In the literature many authors, Bollerslev, Engle and Wooldridge [11], Moskowitz [68], Engle and Sheppard [36], demonstrated the existence of such risk factor which changes over time. Correlation measures the risk of multi asset portfolio in quite similar way as the volatility does in the case of single stocks. In time of crisis, correlation seems to be an important indicator of the real state of the market. Indeed, asset return correlations usually peak during financial crisis while asset value decreases. In this chapter, we will focus on financial product depending on multi asset, especially basket options, which can be seen as the multi-dimension version of standard European options. Avellaneda et al. [7] has shown that considering a trivial way for correlation risk factors is very inadequate for the basket options pricing, even tough individual stock option are taking into in account. In the most cases of pricing, the multi asset modelling is done via a vector of correlated Brownian motions, with constant correlation, and local volatilities well calibrated to the single market. Despite the fact that it is by construction calibrated to data

single market. Avellaneda et al. [7] and Langnau [61] gives several examples to show that this constant correlation model is not able to catch the information incorporated in the basket option prices, especially the skew of the index implied volatility.

In the modelling point of view, there exists two principal types of correlation modelling. The first one considers that the dependency is derived by the variance-covariance matrix. Let $(S_t^i)_{1 \leq i \leq d, t \geq 0}$ denote the price of d underlyings. Under the risk neutral measure, all underlyings are supposed to be solution of the following SDE

$$\forall 1 \leq i \leq d, \frac{dS_t^i}{S_t^i} = rdt + (\sqrt{\Sigma_t} dB_t)_i,$$

where r is the risk-free rate, $(B_t)_{t \geq 0}$ is a vector of Brownian motion and $(\Sigma_t)_{t \geq 0}$ is a process defined in the semi positive matrices $\mathcal{S}_d^+(\mathbb{R})$. Under this consideration, the challenge is to create such matrix in a way to be able to characterize all basket options features. Wishart process remains the most known model used for this purpose. It has been introduced by Gouriou and Sufana [47] and enhanced by Da Fonseca et al. [28] in order to get a negative implied volatility skew. This model will be presented in Section 4.2 to explain its Pro and cons in term of Basket option pricing.

The second method is the most widely used in literature. It characterizes the dependency via a correlation matrix and a vector of individual volatilities, denoted respectively by $(R_t)_{t \geq 0}$ and $(\sigma_t^i)_{1 \leq i \leq d, t \geq 0}$. Under the neutral risk measure, the multi-asset model becomes solution of the following SDE

$$\forall 1 \leq i \leq d, \frac{dS_t^i}{S_t^i} = rdt + \sigma_t^i (\sqrt{R_t} dB_t)_i. \quad (4.1)$$

The greatest advantage of this approach is its compatibility with the single equity market. Indeed, if we suppose that $(\sigma_t^i)_{1 \leq i \leq d}$ are local volatilities calibrated into the single equity market, then the whole model, by construction, fits single equity market. It is then sufficient to specifies only the dynamic of R_t to fit the index market. Under these considerations, Avellaneda et al [7] use a method based on steepest descents and the most-likely price configuration, to derive an approximation in short maturity of Basket option prices. While the objective of Langnau [61] is to present an extension of the methodology of the Dupire's local volatility, in order to provide in a consistent way the local correlation structure. Under the same hypothesis, Reghai [75] presents an other local correlation structure for index, and highlights the methodology calibrate the model.

Lipton [63] and Carr and Laurence [20] have presented an other approach based on Radon transform. Theoretically, if we suppose that there exists a continuum of European basket option, then the density of the index can be represented as the inverse of the Radon transform. Jourdain and Sbai [55] consider a top-down model, because they specify explicitly the dynamic of the correlation with respect to index local volatility, which is calibrated to the market.

Numerous stylized facts have been observed from correlation in the literature. In Skintzi and Refenes [79], these features have been summarized into three major properties: Correlation persistence, co-movement between volatility and correlation and the asymmetry dependency between the correlation and its stock return. The first property is associated to the persistence of the memory which means that periods of high (low) correlation are likely followed by high (low) correlation again. The second characteristic is mainly related to a strong positive relationship between index correlation changes and index variance change. As in single equity worlds, the last property corresponds to the negative correlation between the index and its own

correlation. We will see in this work, that this asymmetry property is connected to the negative skew of the implied volatility of the index. An adequate correlation model has to meet and incorporate all these features and constraints to be able to reflect the right correlation dynamic.

Moreover, in term of hedging perspectives, it is important that the multi asset derivative models incorporate these effects, because an increase of correlation has the unpleasant effect of increasing the cost of risk management hedging, especially for short cross-gamma positions. Indeed, under the direct correlation modelling (4.1) and for a given multi underlying option $(V(t, S_t))_{t \geq 0}$, the infinitesimal daily $P\&L$ of delta neutral portfolios can be written as

$$P\&L = \frac{1}{2} \sum_{1 \leq i, j \leq d} \partial_i \partial_j V(t, S_t) S_t^i S_t^j \left(\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} - (R_t)_{i,j} \sigma_t^i \sigma_t^j \right) \Delta t.$$

Consequently, if the implicit correlation $(R_t)_{t \geq 0}$, derived by the model, does not well predict the behavior of the realized correlation, then during bullish market, underlyings are strongly correlated and the realized covariance might exceed the covariance of the model, i.e. $\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} \gg (R_t)_{i,j} \sigma_t^i \sigma_t^j$, which can damage the risk-management of trading book.

This part is structured as follows. The first section is devoted to the description of the general the model that it will be used. The second part focuses on the market data analysis, and brings some answers about the skew of the implied volatility related to an index. the last part shows the behavior of index implied volatility under basic models, and proposes two models in order to calibrate DAX implied volatility. We start by studying a pure local correlation model, and after that we give an other alternative for index modelling based on stochastic correlation process. Numerical results are given in each part to put in evidence all assumptions related to each model.

4.2 Model description

In a previous work, we have introduced a new SDE that is defined on correlation matrices. Namely, we have shown that the following SDE

$$C_t = C_0 + \int_0^t (\kappa(c - C_s) + (c - C_s)\kappa) ds + \sum_{n=1}^d a_n \int_0^t \left(\sqrt{C_s - C_s e_d^n C_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{C_s - C_s e_d^n C_s} \right), \quad (4.2)$$

is well defined on the space of correlation matrices $\mathfrak{C}_d(\mathbb{R})$, when $C_0, c \in \mathfrak{C}_d(\mathbb{R})$, κ and a are diagonal matrices such that $\kappa c + c \kappa - (d - 2)a^2$ is a positive semidefinite matrix. Here, W_t denote a d -square Brownian matrix whose elements are independent real Brownian motions. Also, we have provided a way to sample such processes.

This work is a first investigation to use these processes in a financial context. More precisely, we will study a bottom-up model where the individual stocks are modelled with local stochastic volatility models [4] and are correlated through a correlation process. Our motivation is to get some unified model that could price in a consistent manner any payoff defined on a basket of assets.

We directly assume that we are under a risk-neutral measure. We consider d stocks S_t^1, \dots, S_t^d that are assumed to follow local stochastic volatility models. Namely we assume that

$$\begin{aligned} dS_t^i &= rS_t^i + \sigma^i(t, S_t^i) f^i(\nu_t^i) dB_t^i \\ d\nu_t^i &= b^i(t, \nu_t^i) dt + \gamma^i(t, \nu_t^i) (\rho_i dB_t^i + \sqrt{1 - \rho_i^2} d\hat{B}_t^i), \end{aligned} \quad (4.3)$$

where dB_t^i and $d\hat{B}_t^i$ are real Brownian increments. Local stochastic volatility models are tractable and can be individually calibrated to the single stock market. They obviously embed both local and stochastic volatility models. Let us explain now how the assets are correlated. We assume that $(\tilde{B}_t^1, \dots, \tilde{B}_t^d, \hat{B}_t^1, \dots, \hat{B}_t^d)$ is a standard $2d$ -dimensional Brownian motion, and that we have at our disposal an (adapted) process C_t , so that we set:

$$dB_t = \sqrt{C_t} d\tilde{B}_t. \quad (4.4)$$

We have $dB_t^i = \sum_{k=1}^d (\sqrt{C_t})_{i,k} d\tilde{B}_t^k$ and thus C_t is the instantaneous correlation of B_t , i.e.

$$\langle dB_t^i, dB_t^j \rangle = (C_t)_{i,j} dt.$$

Let us note that this property is preserved if we take $dB_t = L_t d\tilde{B}_t$, where L_t is such that $L_t L_t^T = C_t$. In addition, we have $\langle dB_t^i, d\hat{B}_t^j \rangle = 0$, and the instantaneous covariance of the log-stocks is given by

$$\langle d \log(S_t^i), d \log(S_t^j) \rangle = (C_t)_{i,j} \sigma^i(t, S_t^i) f^i(\nu_t^i) \sigma^j(t, S_t^j) f^j(\nu_t^j).$$

We consider an index on the basket which is defined by:

$$I_t = \sum_{i=1}^d \alpha_t^i S_t^i,$$

where the weights α_t^i are assumed to be positive.

Let us turn now to the modelling of the joint dynamics. A first possible choice is to consider a ‘‘local correlation model’’ (denoted (LC) thereafter), where C_t is simply a deterministic function of the time and the Index:

$$C_t = C(t, I_t), \text{ where } C : (\mathbb{R}_+^*)^2 \rightarrow \mathfrak{C}_d(\mathbb{R}). \quad (4.5)$$

This type of model has been considered by Reghai [75] and Langnau [61] typical choice is to consider correlation matrices such that all non diagonal elements are equal, i.e.

$$C_{i,j}(t, I_t) = \rho(t, I_t), \text{ where } -\frac{1}{d-1} \leq \rho(t, I_t) \leq 1. \quad (4.6)$$

To extend the local correlation model, one would like to add stochasticity to the correlation. There is a first simple extension, if we restrict correlation matrices to $\{c \in \mathfrak{C}_d(\mathbb{R}), \exists \rho \in [-1/(d-1), 1] \text{ s.t. } i \neq j \implies c_{i,j} = \rho\}$. In fact, we can consider the following scalar diffusion

$$d\rho_t = k(\rho_t - \rho(t, I_t)) dt + \gamma \sqrt{\rho_t(1 - \rho_t)} dW_t, \quad (4.7)$$

where $k, \gamma > 0$. The process ρ_t is then defined on $[0, 1]$, and we then set

$$(C_t)_{i,j} = \rho_t \text{ for } i \neq j. \quad (4.8)$$

This model will be denoted by **(SC1D)** in the sequel. This extension has the huge advantage to be rather simple and to involve only a one-dimensional process. It is thus rather convenient for pricing, by using Monte-Carlo simulations. A possible drawback is however that all the instantaneous correlations are equal, and this kind of model may induce some detectable arbitrage. Also, if one has more precise views on the market (for example, correlations between main firms, or between sectors), it is impossible to take them into account in one single parameter. Let us say here that the Index has a “mean effect” on the correlations, and as far as the Index is concerned, one single correlation may be sufficient to catch the main lines of the Index dynamics. However, we believe that this becomes different if one consider financial products on sub-baskets of the Index. At least from a theoretical perspective, having more tractability on the instantaneous correlation is interesting to propose a model that price consistently every assets on the basket.

This motivates us to consider the following model, denoted by **(SC)** thereafter:

$$\begin{aligned} C_t = & C_0 + \int_0^t (\kappa(s, I_s)(C(s, I_s) - C_s) + (C(s, I_s) - C_s)\kappa(s, I_s)) ds \\ & + \sum_{n=1}^d a_n(s, I_s) \int_0^t \left(\sqrt{C_s - C_s e_d^n C_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{C_s - C_s e_d^n C_s} \right). \end{aligned} \quad (4.9)$$

In the full generality, this SDE will admit a weak solution as soon as $\kappa(t, x)C(t, x) + C(t, x)\kappa(t, x) - (d-2)a(t, x)^2 \in \mathcal{S}_d^+(\mathbb{R})$, for every $t \geq 0$, $x > 0$. However, for a practical implementation, we will restrict some of the parameters. Namely, we will assume that the diagonal matrices $\kappa(s, I_s)$ and $a(s, I_s)$ are proportional to the identity (all their diagonal elements are equal). By a slight abuse of notation, we still denote by $\kappa(s, I_s) \in \mathbb{R}$ and $a(s, I_s) \in \mathbb{R}_+$ the corresponding value. Thus, with these simplifications, the SDE on C_t is

$$\begin{aligned} C_t = & C_0 + 2 \int_0^t \kappa(s, I_s)(C(s, I_s) - C_s) ds \\ & + \int_0^t a(s, I_s) \left(\sqrt{C_s - C_s e_d^n C_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{C_s - C_s e_d^n C_s} \right). \end{aligned} \quad (4.10)$$

The meaning of the parameters is clear: C_t is a mean-reverting diffusion toward the $C(t, I_t)$ which is perturbed by a noise driven by a . The speed of the mean-reversion is tuned by κ . For a first implementation, we will moreover assume that all the non diagonal elements of $C(t, I_t)$ are equal, i.e. $C_{i,j}(t, I_t) = \rho(t, I_t)$. Let us set $J_d \in \mathcal{S}_d^+(\mathbb{R})$ such that $(J_d)_{i,j} = 1$ for $1 \leq i, j \leq d$. Then the condition on the weak existence can be rewritten $2\kappa(\rho(t, x)J_d + (1 - \rho(t, x))I_d) - (d-2)a^2 I_d \in \mathcal{S}_d^+(\mathbb{R})$, and is satisfied if and only if:

$$2\kappa(t, x)(1 - \rho(t, x)) \geq (d-2)a^2(t, x).$$

The main advantages of such a model are the following. First, one can still calibrate individual stocks by using one’s favorite model. Then, the correlation process that describes the dependence between the stocks has a clear meaning and intuitive parameters. Thus, it provides

a model that prices and hedges in a consistent manner stocks and every financial products on the basket Index. As we will see, pricing by using discretization schemes and Monte-Carlo method can be achieved in an acceptable time. Even if calibrating the correlation process is (as we will see) still an hard task, we believe that the study of this type of model is relevant and could lead to a better understanding of the dependence between stocks.

Pro and cons with respect to affine multidimensional models

There are few existing multidimensional stochastic volatility models in the literature. Here, we want to compare some features of the model presented above with the affine model introduced by Gourieroux and Sufana [47] and enhanced by Da Fonseca et al. [28]. They propose an affine dynamics for the stocks, whose instantaneous covariance is described by a Wishart process. Namely, they consider the following diffusions:

$$\begin{aligned} d \log(S_t) &= \left(r \mathbb{1} - \frac{1}{2} \begin{bmatrix} (\Sigma_t)_{1,1}^2 \\ \vdots \\ (\Sigma_t)_{d,d}^2 \end{bmatrix} \right) dt + \sqrt{\Sigma_t} [(1 - \|\rho\|_2^2) dZ_t + dW_t \rho], \\ d\Sigma_t &= (\alpha aa^T + b\Sigma_t + \Sigma_t b^T) dt + \sqrt{\Sigma_t} dW_t a + a^T dW_t^T \sqrt{\Sigma_t}. \end{aligned} \quad (4.11)$$

Here, $\log(S_t)$ is the vector of the log-stocks, a and b are matrices in $\mathcal{M}_d(\mathbb{R})$, r is the risk-free rate, $(W_t)_{t \geq 0}$ is a matrix made up of independent Brownian motions and $(Z_t)_{t \geq 0}$ is a vector of independent of Brownian motions, α is a real value greater than $d - 1$, and $\rho \in \mathbb{R}^d$ is a correlation vector such that $\|\rho\|_2^2 \leq 1$. Under this model, the instantaneous covariance matrix is given by:

$$\langle d \log(S_t^i), d \log(S_t^j) \rangle = (\Sigma_t)_{i,j} dt, \quad (4.12)$$

while the vector ρ tunes the dependence between the stocks and their instantaneous covariance:

$$\langle d \log(S_t^i), d(\Sigma_t)_{k,l} \rangle = (\Sigma_t)_{i,k} \sum_{1 \leq m \leq d} \rho_m a_{k,m} + (\Sigma_t)_{i,l} \sum_{1 \leq m \leq d} \rho_m a_{l,m} \quad (4.13)$$

Moreover, this dynamics ensures that the couple $(\log(S_t), \Sigma_t)$ is an affine process, and its characteristic function can be obtained by solving Riccati equations (see Da Fonseca et al. [28]).

We would like now to focus on the dynamics of the individual stocks. From (4.12) and (4.13), we get that there are Brownian motions β_t^i , $i = 1, \dots, d$ such that:

$$d(\Sigma_t)_{i,i} = (\alpha(aa^T)_{i,i} + 2 \sum_{j=1}^d b_{i,j}(\Sigma_t)_{i,j} dt + 2\sqrt{(aa^T)_{i,i}}\sqrt{(\Sigma_t)_{i,i}}d\beta_t^i.$$

The non diagonal elements $(\Sigma_t)_{i,j}$ for $i \neq j$ can be seen as factors that drives the volatility of each stock. In its full form, the dynamics of one stock and its volatility is not autonomous. This means in practice that it is not possible to calibrate this model separately to single stock market (mainly, European options on single stocks). This calibration has to be done at the same for all the stocks, which is a priori a very challenging task since there are many parameters and a lot of data.

Then, one may want to recover autonomous dynamics for each stock in order to calibrate separately each stocks. Within this model, the only possible choice is to assume that b is a

diagonal matrix. In this case, $(\Sigma_t)_{i,i}$ follows a CIR diffusion and the couple $((\Sigma_t)_{i,i}, S_t^i)$ follows an Heston model: there are independent Brownian motions β^i, γ^i such that

$$\begin{aligned} d \log(S_t^i) &= \left(r - \frac{(\Sigma_t)_{i,i}^2}{2} \right) dt + \sqrt{(\Sigma_t)_{i,i}} \left(\tilde{\rho}^i d\beta_t^i + \sqrt{1 - (\tilde{\rho}^i)^2} d\gamma_t^i \right) \\ d(\Sigma_t)_{i,i} &= \left(\alpha (a^T a)_{i,i} + 2b_{i,i}(\Sigma_t)_{i,i} \right) dt + 2\sqrt{(a^T a)_{i,i}} \sqrt{(\Sigma_t)_{i,i}} d\beta_t^i, \\ \tilde{\rho}^i &= \frac{(a^T \rho)^i}{\sqrt{(a^T a)_{i,i}}} = \frac{\sum_{k=1}^d a_{k,i} \rho^k}{\sqrt{\sum_{k=1}^d a_{k,i}^2}}. \end{aligned}$$

In practice, each individual stock could be then calibrated like in the Heston model. Unfortunately, there are further restrictions implied by this model and especially that $\alpha \geq d - 1$, which is the condition that ensures the existence of the Wishart process. This is unlikely because when calibrating Heston to market data, it is typical to get values of α around or below 1. When modelling many stocks together (say $d \geq 10$), it is then not possible to fit conveniently stocks data because of this restriction on α .

One of the main feature of the model (4.11) is the Affine property. It allows to obtain the characteristic function of the stocks by solving Riccati differential equations. Then, the pricing of European style options can be made by using Fourier inversion. This approach is known to be very efficient in a one-dimensional framework (see Carr and Madan [21]) and has been used successfully by Da Fonseca et al.[28] to price Best-of options with $d = 2$ assets. However, when the number of assets d is much larger, the Fourier inversion requires an integration in dimension d and can no longer be computed quickly. Unless the payoff has a very particular structure (for example if it depends on 2 or 3 assets or on a linear combination of the log-assets), the pricing by Fourier inversion is no longer competitive with respect to Monte-Carlo methods. Thus, we believe that the Affine property does not really give a real advantage in terms of computational methods when the number of assets involved gets large (say 5 or more).

To sum up, we believe that the model introduced by Da Fonseca et al. is relevant when considering a few number of assets. In this case, the restriction on α does not prevent from a good calibration of the individual stocks, and one can take advantage of the Affine structure to get fast pricing methods. If the goal is instead to model a large basket of assets, we believe that the bottom-up approach that consists in modelling each stock individually and then plug a correlation structure is much more tractable. It allows to keep one's preferred model for each stock, and to calibrate them separately. Besides, the parameters that tune dependence between the assets are clearly identified.

4.3 Market data and index option analysis

In this part, we want to investigate how the bottom-up approach can be used to fit the Index prices. To do so, we will make further assumption with respect to (4.3): we will assume that the stocks follow local volatility model, that is $f_i \equiv 1$ for any $1 \leq i \leq d$. We have been kindly given by Julien Guyon at Société Générale market data on the DAX Index at the 4th October 2010. Here is the composition of the DAX Index at this date.

We will assume for sake of simplicity that the weights α_t^i does not depend on the time t . The data that we have at our disposal are the following. For each stock, we have a local volatility

SIEMENS	BASF	BAYER	E-ON	DAIMLER-CHRYSLER	ALLIANZ	SAP
9.91	8.03	7.97	7.67	7.35	6.97	6.01
DT-TELEKOM	DEUTSCHE-BANK	RWE	M-RUECK	LINDE	BMW	VOLKSWAGEN
5.6	5.06	3.86	3.23	3.06	2.92	2.26
DEUT-POST	ADIDAS	DEUT-BOERSE	FRESENIUS-MC	THYSSEN-KRUPP	MAN	HENKEL-VZ
2.05	1.76	1.65	1.63	1.52	1.47	1.29
SDF	LUFTHANSA	METRO	INFINEON-TECH	HEIDEL-Z	FRESENIUS	BEIERSDORF
1.17	1.16	1.12	1.02	0.94	0.9	0.86
COMMERZBANK	MERCK-DEM					
0.84	0.74					

Table 4.1: The composition of Dax index in percentage. 4/10/2010 .

function that is already calibrated to European options. For the DAX Index, we also have a local volatility function that is calibrated to European options on the DAX, which enables us to compute any European option on the DAX. Our scope is to find a correlation process C_t that enables us to recover Index European option prices. Here, we will first target to calibrate one single maturity. Obviously, we would like in practice to calibrate simultaneously all the maturity. Nonetheless, we have to make this choice in order to understand the behavior of the model and the impact of each parameter. We hope (even if this is not the case yet!) that we will be able later to propose a calibrated model across the maturities.

In order to explain the motivation to derive a random correlation, let us first assume that the correlation is constant, and consider the (LC) model such that all components $(C_t)_{1 \leq i \neq j \leq d}$ are equal to a constant ρ . We suppose also that all volatilities σ^i are exactly the local volatilities well calibrated to the single market, which means that the multi-underlyings model is already calibrated to the single equity market. Avellaneda et al. [7] and Langnau [61] have already explained that this Gaussian copula modelling is unable to fully fit the volatility skew of the market. Figure 4.1 shows a comparison between the implied volatility of the DAX market and model one. Roughly speaking, we conclude that this model does not incorporate completely the negative skew market. In the rest of this chapter, we will explain why this model produces a negative implied volatility skew, and how can improve it to be able to match the non trivial dependency formed by the market.

We will proceed as follows. First, we will investigate index pricing option under some basic models. We will identify some features that characterize the pricing issues. After that, we will discuss the calibration of a local correlation model (LC). Then, we will discuss the impact of the noise and the mean-reversion in the stochastic correlation models (SC1D) and (SC).

First of all, since we are interested on models that fit the market data, we are then looking for ways to take into account the implied volatility features of options. As in each part of this section we will study certain models following these issues, let $\Sigma_{T,K}$ denote the index implied volatility associated to these models, for a given a strike $K > 0$ and maturity $T > 0$. Moreover, we consider also that $\Sigma_{T,K}^{Mkt}$ is the index implied volatility related to the market. In this chapter, the implied volatility skew of a model (market) is defined as the first derivative of the implied volatility of the model (market) w.r.t the strike at-the-money: $\partial_K \Sigma_{T,K}$ ($\partial_K \Sigma_{T,K}^{Mkt}$). We define also the kurtosis of the model (market) as the second derivative one: $\partial_K^2 \Sigma_{T,K}$ ($\partial_K^2 \Sigma_{T,K}^{Mkt}$). More generally, the skew will denote the slope of the implied volatility around the money, while the kurtosis give information about the convexity related to the curve of the implied volatility.

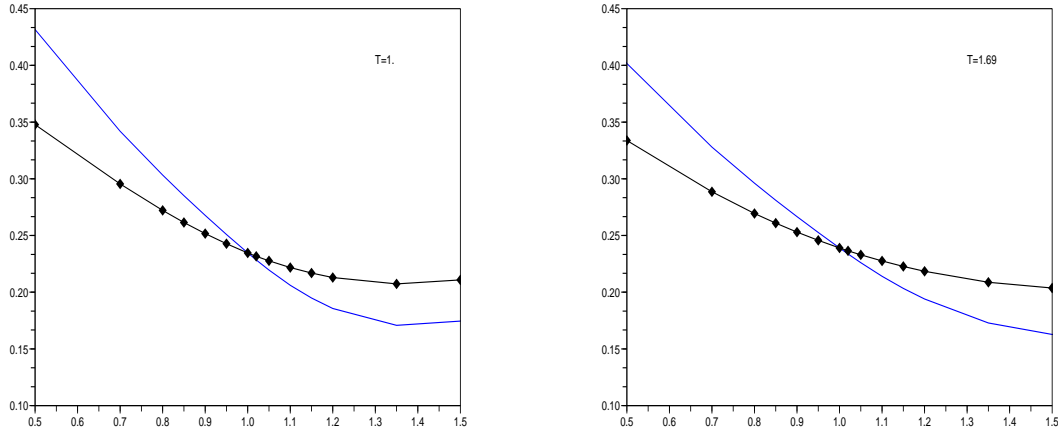


Figure 4.1: $d = 10, 10^6$ Monte-Carlo samples, and discretization time step 0.025. Graphs of index implied volatility for both DAX market (solid line) and the model with constant correlation ρ (dash line), for different maturities $T = \{1, 1.69\}$. All local volatilities are calibrated in Vanillas market, and ρ fits the ATM implied volatility of the market.

Since the index is defined as $I_t = \sum_{i=1}^d \alpha^i S_t^i$, then it follows that

$$dI_t = rdt + \sum_{i=1}^d \alpha^i S_t^i \sigma^i(t, S_t^i) dB_t^i$$

$$\langle dI_t, dI_t \rangle = \sum_{1 \leq i, j \leq d} \alpha^i \alpha^j S_t^i S_t^j \sigma^i(t, S_t^i) \sigma^j(t, S_t^j) (R_t)_{i,j},$$

let $\bar{\sigma}_t^2$ denote the instantaneous variance for the lognormal index, it yields then that

$$\bar{\sigma}_t^2 = \frac{1}{I_t^2} \frac{\langle dI_t, dI_t \rangle}{dt} = \frac{\sum_{1 \leq i, j \leq d} \alpha^i \alpha^j S_t^i S_t^j \sigma^i(t, S_t^i) \sigma^j(t, S_t^j) (R_t)_{i,j}}{I_t^2}$$

As the skew effect in index option pricing has been highlighted many times in the literature, we shall study in each model the instantaneous covariance between the index and its variance. Under some hypothesis, Durrleman [34] has proved that the value of this instantaneous covariance will define in short time the value (especially the sign) of the skew implied volatility model. Indeed, Durrleman has proved in [34] that at-the-money level and skew of the implied volatility, in short maturity can be expressed as

$$\Sigma_{t, I_t} = \sqrt{\bar{\sigma}_t^2},$$

$$\partial_K \Sigma_{t, I_t} = \frac{\nu_t}{4I_t \bar{\sigma}_t^2}$$

where for every $t \geq 0$, $\nu_t = \frac{1}{I_t \bar{\sigma}_t^{2/2}} \frac{\langle dI_t, d\bar{\sigma}_t^2 \rangle}{dt}$. It is clear then that the sign of the implied volatility in short term is derived by the instantaneous covariance between the index I_t and its variance $\bar{\sigma}_t^2$. By straightforward Itô calculus, it follows that in model (4.3)

$$\begin{aligned}
\frac{\langle dI_t, \overline{\sigma}_t^2 \rangle}{dt} &= \underbrace{\frac{2}{I_t^3} (u_t^T [R_t \text{diag} \{ \alpha^1 (\sigma_t^1)^2 S_t^1, \dots, \alpha^d (\sigma_t^d)^2 S_t^d \} R_t] u_t I_t - (\overline{\sigma}_t^2 I_t)^4)}_{P1} \\
&+ \underbrace{\frac{2}{I_t^2} u_t^T [R_t \text{diag} \{ \alpha^1 \sigma_t^1 (S_t^1)^2 \partial_1 \sigma_t^1, \dots, \alpha^d \sigma_t^d (S_t^d)^2 \partial_d \sigma_t^d \} R_t] u_t}_{P2} \\
&+ \underbrace{\partial_I \rho(I_t) \left(\sum_{1 \leq i \neq j \leq d} \alpha^i S_t^i \sigma_t^i \alpha^j S_t^j \sigma_t^j \right) \frac{\overline{\sigma}_t^2}{I_t^2}}_{P3},
\end{aligned} \tag{4.14}$$

where for every $t \geq 0$, $(u_t)_{t \geq 0}$ denote the vector defined on \mathbb{R}^d as $u_t = (\alpha^1 \sigma_t^1 S_t^1, \dots, \alpha^d \sigma_t^d S_t^d) T$. To simplify notations, all σ_t^i for every $1 \leq i \leq d$ denote here respectively functions $\sigma^i(t, S_t^i)$ for all $t \geq 0$. The main idea of next part is to identify the three term involved in (4.14): $\{P1, P2, P3\}$, in order to derive the sign of this instantaneous covariance.

4.4 Calibration to Market data

The main idea of this section is to derive an intuitive procedure to explain the impact of all terms in (4.14), by presenting basic models, and to explain the structure of (LC) that has been taking into account within this section.

4.4.1 Calibration of basic models

To illustrate the role played by each term $\{P1, P2, P3\}$ in formula (4.14) and to explain how to get a negative implied volatility skew, we start by studying some basic models. All the models involved in this section assume that the correlation $(\rho_t)_{t \geq 0}$ in (4.6) is constant.

Black & Scholes Index model

First of all, if we assume that all local volatilities $(\sigma^i(t, S_t))_{1 \leq i \leq d, t \geq 0}$ and local correlation $(\rho_t)_{t \geq 0}$ are constants and denoted respectively by $(\sigma^i)_{1 \leq i \leq d}$ and ρ , then the instantaneous covariance obtained in (4.6) becomes

$$\frac{\langle dI_t, d\overline{\sigma}_t^2 \rangle}{dt} = \frac{2}{I_t^3} (u_t^T [R \text{diag} \{ \alpha^1 (\sigma^1)^2 S_t^1, \dots, \alpha^d (\sigma^d)^2 S_t^d \} R] u_t I_t - (u^T R u)^2),$$

where $\forall 1 \leq i, j \leq d$, $R_{i,j} = \mathbb{1}_{i=j} + \mathbb{1}_{i \neq j} \rho$. In the case of dimension $d = 1$, the index value becomes $I_t = S_t^1$ and the covariance is by construction equal to zero, we fall thus on Black & Scholes model, characterized by a constant implied volatility. However, the previous instantaneous covariance will not be equal to zero as soon as the dimension d is greater than 1, which means that the skew would appear even though all volatilities and correlation are constants.

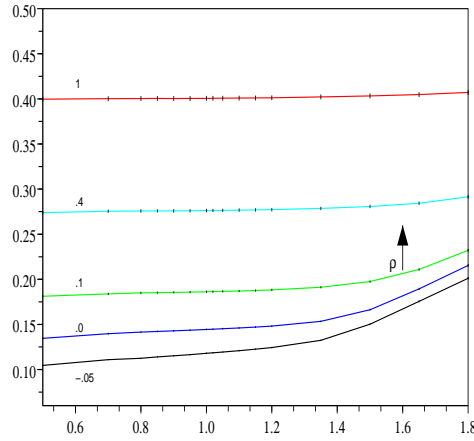


Figure 4.2: $d = 30$, 10^6 Monte-Carlo samples, $T = 1.3$ and discretization time step 0.025. The index implied volatility w.r.t to strike moneyness ($\frac{K}{T_0}$) for the basic model. The smile value for $\sigma = 0.4$ and for different choices of $\rho = \{-0.05, 0., 0.1, 0.4, 1\}$. The width of each point represents the 95% confidence interval.

The sign of this skew can take positive value, if we suppose, for example, that all components σ^i share the same value σ , then by Hölder inequality we obtain that

$$\frac{\langle dI_t, d\bar{\sigma}_t^2 \rangle}{dt} = \frac{2\sigma^4(1-\rho)^4}{I_t^3} \left(I_t \sum_{i=1}^d (\alpha^i S_t^i)^3 - \left[\sum_{i=1}^d (\alpha^i S_t^i)^2 \right]^2 \right) \geq 0,$$

which means that the index and its variance move in the same direction independently on the value of ρ and σ . From Figure 4.2 one can observe that for small value of correlation, a positive skew value is noticed especially in high strikes. However, as soon as the value of correlation ρ takes value near to one, the implied volatility becomes almost constant for every value of the strike. These observations are exactly related to the term $(1-\rho)^4$ written in the formula.

Since the implied volatility skew of the market is negative, this model is not adequate for the market fitting purposes, we need then that both term $P2$ and $P3$ in (4.14) have to take negative values. Consequently, the next step is to consider more general assumptions in modelling involving these terms.

Pure local volatility model - Displaced Model

We still consider a constant correlation, and we assume now that all volatilities are no longer constants, but follow for example a displaced diffusion function [77], such that for every $1 \leq i \neq j \leq d$, $(R_t)_{i,j} = \rho$. We suppose therefore that

$$\forall t \geq 0, \forall 1 \leq i \leq d, \sigma^i(t, S_t^i) = \frac{(\beta^i S_t^i + (1-\beta^i) S_0^i e^{rt})}{S_t^i}. \quad (4.15)$$

In the context of pricing vanillas options, it is known that for each underlying, each value of $\beta^i \in [0, 1]$ characterizes the skew of the implied volatility. In the index pricing situation, while

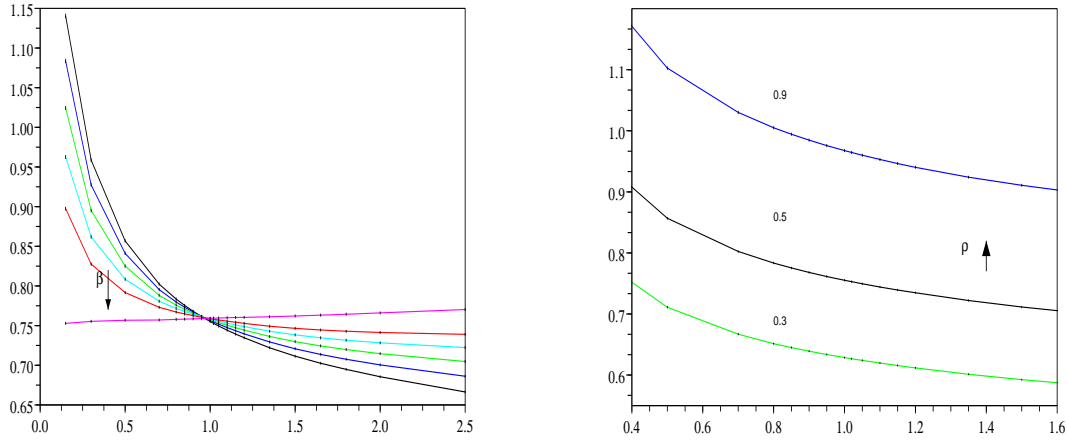


Figure 4.3: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the index implied volatility w.r.t to strike moneyness for displaced model. all constants $(\beta^i)_{1 \leq i \leq d}$ are equal to β . Right: $\beta = 0.7$ and $\rho = \{0.3, 0.5, 0.9\}$. Left: $\beta = \{0, 0.1, 0.5, 0.6, 0.7, 0.9\}$ and $\rho = 0.5$. The width of each point represents the 95% confidence interval.

the third term $P3$ in (4.14) is still equal to zero by definition, the second term $P2$ is given by

$$\begin{aligned} & \frac{2}{I_t^2} u_t^T [R_t \text{diag} \{ \alpha^1 \sigma^1 (S_t^1)^2 \partial_1 \sigma^1, \dots, \alpha^d \sigma^d (S_t^d)^2 \partial_d \sigma^d \} R_t] u_t \\ & = -\frac{2}{I_t^2} u_t^T [R_t \text{diag} \{ \alpha^1 \sigma^1 S_0^1 (1 - \beta^1), \dots, \alpha^d \sigma^d S_0^d (1 - \beta^d) \} R_t] u_t. \end{aligned}$$

By construction, it yields that $P2$ is negative if all $(\beta^i)_{1 \leq i \leq d}$ are defined on $[0, 1]$. For the sake of simplicity for our numerical experiments, we suppose that $\beta^i = \beta$ for every $1 \leq i \leq d$. The right left side of Figure 4.3 shows the index implied volatility skew for different choices of β . As in the single name context, we notice that the skew of the index depends inversely on β . When β reaches the value 1, we fall on the previous model, where all parameter are constants and the skew takes a positive value. The right hand side of Figure 4.3 shows how implied volatilities behave with respect to the correlation value ρ . The three curves almost share the same skew value at the money. However, one can notice that the value of ρ modifies systematically the level of the index implied volatility.

Pure local volatility model - CEV Model

Under the hypothesis that the correlation is a constant parameter, we can study also the case when the local volatility follow a CEV structure (Cox [25]). We recall that CEV model suppose that for every $t \geq 0$ and $1 \leq i \leq d$, $\sigma^i(t, S_t^i) = (S_t^i)^{\beta_i - 1}$. In the single name world, in order to be able to fit a negative single implied volatility, all constants $(\beta^i)_{1 \leq i \leq d}$ have to take values less than one. Going back to formula (4.14), we can notice that under the hypothesis of the existence of the process, the second term $P2$ in (4.14) takes by construction a negative value, because all

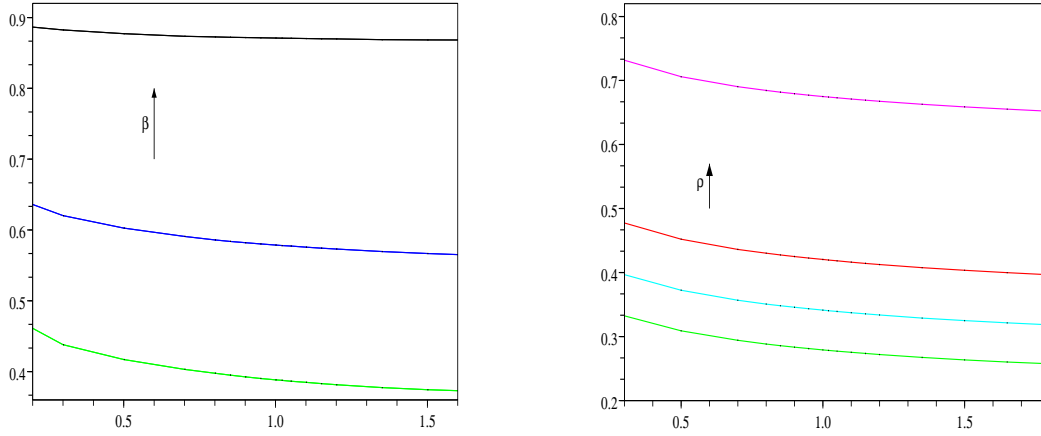


Figure 4.4: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. All graphs correspond to the index implied volatility w.r.t to strike moneyness for displaced model. all constants $(\beta^i)_{1 \leq i \leq d}$ are equal to β . Left: $\rho = 0.3$ and $\beta = \{0.8, 0.9, 1\}$. Right: $\rho = \{0.55, 0.65, 0.7, 0.75\}$ and $\beta = 0.3$. The width of each point represents the 95% confidence interval.

derivative of the local volatilities w.r.t underlyings are negative:

$$\begin{aligned} & \frac{2}{I_t^2} u_t^T \left[R_t \text{diag} \left\{ \alpha^1 \sigma^1 (S_t^1)^2 \partial_1 \sigma^1, \dots, \alpha^d \sigma^d (S_t^d)^2 \partial_d \sigma^d \right\} R_t \right] u_t \\ & = -\frac{2}{I_t^2} u_t^T \left[R_t \text{diag} \left\{ \alpha^1 \sigma^1 S_0^1 (S_t^1)^{\beta^1} (1 - \beta^1), \dots, \alpha^d \sigma^d S_0^d (S_t^d)^{\beta^d} (1 - \beta^d) \right\} R_t \right] u_t. \end{aligned}$$

We can state then that if the local volatilities are well calibrated to the vanilla markets, then they can help to compensate the lack of a negative value in the instantaneous covariance (4.14). In order to study the behavior of the implied volatility model w.r.t these parameter, we consider that all $(\beta^i)_{1 \leq i \leq d}$ share the same value, denoted by β . In Figure 4.4, we illustrate the impact of both β and ρ in terms of the implied volatility.

As in the displaced model, the left side of Figure 4.4 shows that the skew of the index implied volatility is negative and highly dependent on the value of β , That is why we illustrate this impact in the center of Figure 4.4 by normalizing the previous curves. The parameter ρ plays the same role as in the displaced model, because it is strongly linked to the level of the at-the-money implied volatility model (on the right of Figure 4.4).

Moreover, empirical studies about the dynamics, at-the-money, of the volatility skew show that it approximately independent of the volatility level, over time [41]. This property is easily translated in the displaced model, because in Figure 4.3 when β takes different values, the index implied volatility skew changes, but the level of its implied volatility at-the-money remains almost constant. However, for the CEV model (Figure 4.4), as soon as the value of β changes, both the level and skew of the index implied volatility take different values.

We conclude that the local volatility functions can help for index pricing issues, but the impact of these functions is related to their structure. Furthermore, since the local volatilities are supposed to be calibrated to vanillas market, they are already frozen by the market. We have then to use an other way to put the smile on the index. This is why it is interesting to take

into account a correlation that depends on the index in order to have a negative sign of the skew via the term $P3$ in (4.14).

4.4.2 Calibration of a local correlation model (LC)

We have already seen that local volatilities can be used to adjust the skew of the index implied volatility. However, in order to take into consideration vanillas market, these local volatilities are supposed to fit each single name market. For the sequel, we suppose that all local volatilities are well calibrated to the equity market. Figure 4.1 shows that the only use of this local volatilities with a constant correlation will not lead to calibrate the implied volatility market. Avellaneda et al. [7] have stressed the lack of these structure to fit the market. Consequently, Reghai [75] and Langnau [61] consider that the correlation has a local structure, in order to calibrate the skew index market.

The first approach of correlation modelling is to consider a pure local correlation model (LC). We consider thus a typical class, which derives the dynamic correlation through a power function. The local correlation in (4.6) is given by the following equation

$$\forall 1 \leq i \neq j \leq d, (C_t)_{i,j} := \rho(I_t) = \frac{1}{1 + \eta \left(\frac{I_t}{I_0}\right)^\gamma} \vee \rho_{\min}, \quad (4.16)$$

where $\gamma \geq 0$, $\eta \geq 0$ and $\rho_{\min} \in [0, 1]$.

The choice of this structure comes from the special features of the correlation (derived already in the the introduction), especially concerning the asymmetry moves between the index and the correlation of its components. Indeed, from a modelling point of view, the local correlation increases (decreases) as soon as the value of $(I_t)_{t \geq 0}$ decreases (increases). In term of implied volatility skew, since the derivative of $\rho(t, I_t)$ w.r.t the index is negative, then the third term in (4.14) takes negative value too, and we can therefore expect that this structure is sufficient to fit the skew given by the market. First of all, it is interesting in itself to understand the behavior of the model parameters in term of implied volatility curve.

Figure 4.5 shows that the value of η is directly related to the level of the index implied volatility index, especially it derives a direct dependency with the at-the-money implied volatility, because this value increases as soon as the η takes higher value. Intuitively, η is strongly linked to the expected value of the correlation matrix. When η is near to zero, the value of the correlation ρ_t is almost equal to $\mathbb{1}_{\mathcal{M}_d(\mathbb{R})}$, and all underlyings are therefore strongly correlated. Consequently, the price of option tends to take a higher value. While, if the η becomes bigger, then all components tends to be more independent, and the implied volatility will have then a smaller value.

Moreover, Figure 4.5 shows that the at-the-money implied volatility skew is highly dependent on the value of the parameter γ . Indeed, the bigger γ is, the bigger is the implied volatility skew. Going back to the instantaneous covariance formula (4.14), its third term is explicitly related to

$$\partial_I \rho(I_t) = -\frac{\gamma}{I_0} \frac{\eta \left(\frac{I_t}{I_0}\right)^{\gamma-1}}{\left(1 + \eta \left(\frac{I_t}{I_0}\right)^\gamma\right)^2}.$$

The parameter γ plays thus an important role to set a negative sign for the whole formula. The advantage of this model is its ability to separate two features related to the level and skew of

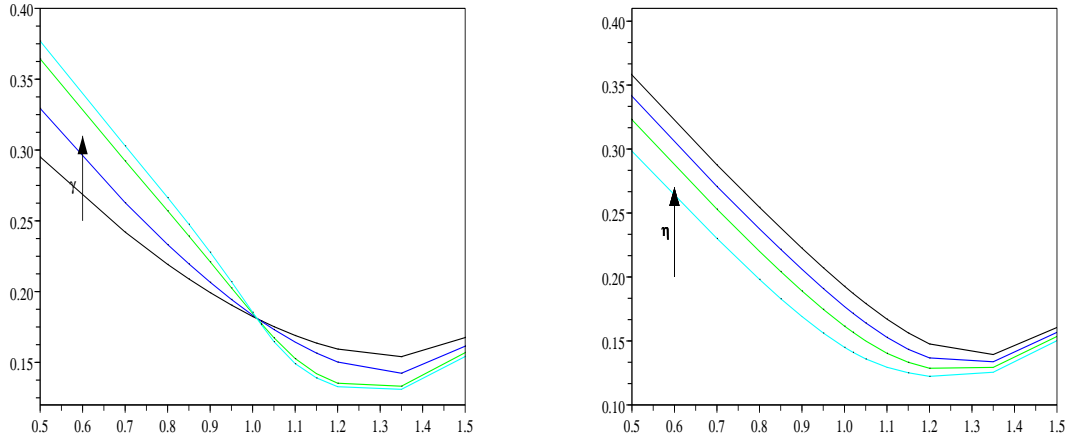


Figure 4.5: $d = 10$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the index implied volatility w.r.t to strike moneyness under (LC) model, such that $\rho_{\min} = 0.1$. Left: $\eta = 2$, and $\gamma = \{2, 5, 15, 30\}$. Right: $\gamma = 8$ and $\eta = \{1.6, 2.4, 3.6, 6\}$. The width of each point represents the 95% confidence interval.

the implied volatility. This property will help to do calibration efficiently knowing the value of parameters.

The parameter ρ_{\min} is used to characterize the queue distribution of the index on the right side (in-the-money region). Indeed, it corresponds intuitively to the common correlation in a wealthy market, which is related to the case $\frac{I_t}{I_0} \gg 1$. Figure 4.6 (LHS) shows that for a given level and skew, ρ_{\min} can enhance the sensibility of the curve in term of convexity, for small value of ρ_{\min} . However, when this parameter takes lower values, it has the same effect as η : it becomes strongly correlated to the level of the implied volatility at-the-money.

In the light of these results, we are able to propose the calibration algorithm. The optimization routine is divided into five steps.

- **Step 1: Dimension reduction**, since the dimension d is equal to 30 for DAX context, we will consider a new basket $\tilde{I}_t = \sum_{i=1}^{10} \tilde{\alpha}^i S_t^i$ by taking the 10 first biggest underlying (in terms of composition α , see Table (4.1)). Let us assume that $\alpha_1 \geq \dots \geq \alpha_d$. We consider that

$$1 \leq i \leq 10, \tilde{\alpha}_i = \frac{\alpha^i}{\sum_{i=1}^{10} \alpha^i}.$$

Since these components represents over 70% of the total DAX index, it is then interesting to do such approximation for time reduction purposes. On the right hand side of Figure 4.6, we present the value of the implied volatility for both case $d = 10$ and $d = 30$ and under the same parameter. We notice that the change of dimension mainly modifies only the at-the-money level of the implied volatility, while the skew remains the same under both considerations.

Moreover, all pricing are performed by Monte-Carlo. Before beginning optimization, the algorithm starts by saving in memory all random variables possibly involved in pricing procedure. Consequently, at each iteration, we use the same random variables.

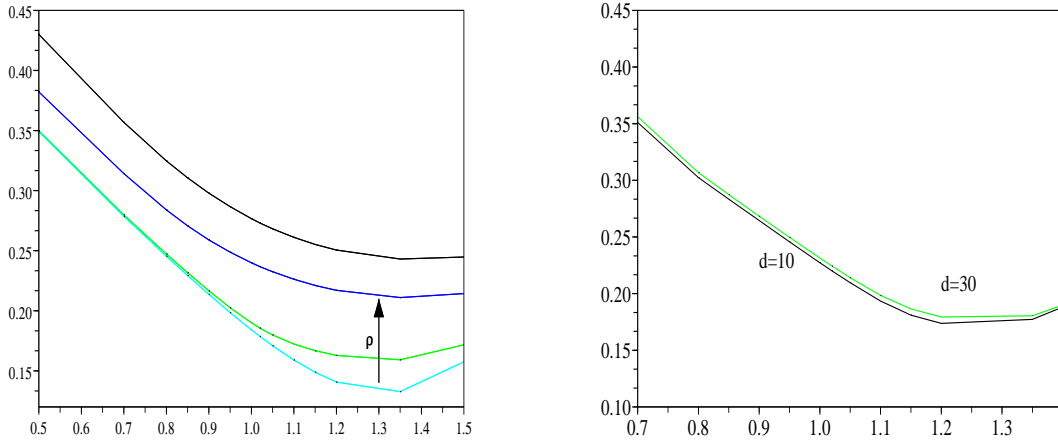


Figure 4.6: 30^6 Monte-Carlo samples, and discretization time step 0.025. Plot of the index implied volatility w.r.t to strike moneyness, under (LC) model. Left: $T = 1$, $\gamma = 8$, and $\eta = 2$, for different values of $\rho_{min} = \{1, 0.7, 0.3, 0.001\}$. Right: $T = 0.64$, $\gamma = 14.1$, $\eta = 0.55$, and $\rho_{min} = 0.35$ for both case $d = 10$ and $d = 30$. The width of each point represents the 95% confidence interval.

- **Step 2: At-the-money implied volatility level target**, we work on a the new model (LC) derived by the index \tilde{I}_t . The main goal of the second step is to look for the adequate value related to the parameter η that matches the market at-the-money level. Because, we have already seen, that this parameter is strongly correlated to this level target. The routine is performed by a simple dichotomy algorithm.
- **Step 3 : At-the-money implied volatility skew target**, it consists on fitting the smile of the market, it is done by the γ adjustment. We have already observed that γ is highly connected to the skew of the market, and is less linked to the at-the-money level. The research routine is based on the Newton-Raphson algorithm.
- **Step 4 : Right queue distribution**, during this step, we take into account just some point on the the right side of the implied volatility. The left side of Figure 4.6 shows that this can be handled by small value ρ_{min} . Since, the change can be reasonably be monotone, then Newton-Raphson algorithm is sufficient to deal with this.
- **Step 5 : General research proceeding**, we go back finally to the index I_t under the model (LC) and by taking into consideration the previous calibrated parameterization. The last task uses a descent gradient algorithm on the three parameter.

One can notice that second, third and fourth steps are achieved in one dimension, because we focus on just one parameter. However, the optimizer in the fifth step takes into account all parameters. In general, all calibration of each maturity are obtained over 4 minutes, per maturity.

Before beginning both pricing and optimization procedures, we keep in memory all possible random variables involved in the simulation. Consequently, at each iteration, and at each step in optimization we use the same random variable. On the one hand, this method can enhance

Table 4.2: The error of calibration: $\sum_i \left(\frac{\Sigma_{T,K^i} - \Sigma_{T,K^i}^{Mkt}}{\Sigma_{T,K^i}^{Mkt}} \right)^2$ for different maturities, under the (SC1D) model,

with $\kappa = 15$ and $\sigma = 0$.

T	0.17	0.28	0.64	1	1.35	1.67	2	2.34	3.1	3.8	4.5	6
γ	22.30	15.14	11.907	9.88	9.08	8.99	9.30	8.52	7.15	6.43	4.60	4.15
η	6.00	4.43	2.52	1.83	1.45	1.37	1.014	0.88	0.66	0.54	0.53	0.38
ρ_{\min}	0.0756	0.0752	0.0755	0.0752	0.0752	0.0752	0.0757	0.0755	0.0753	0.0757	0.0758	0.0753
Error	0.00004	0.00001	0.0008	0.0008	0.0009	0.0006	0.0006	0.0007	0.0008	0.0008	0.0007	0.0006

the simulation in term of time computation. On the other hand, it ensures certain stability for optimizer. Moreover, since we are interested in the simulation in the case of a special case of correlation $(R_t)_{i,j} = \rho_t$ forever $1 \leq i, j \leq d$ and $t \geq 0$, then we use a method that do not involve the compute of Cholesky decomposition for the process $(R_{t \geq 0})$, at each time step. Instead of adding new Brownian motions as in [75], we remark that

$$\sqrt{R_t} = \lambda_1 \mathbb{1}_{\mathcal{M}_d(\mathbb{R})} + \lambda_2 I_d, \quad \lambda_1 = \frac{\sqrt{1 + (d-1)\rho_t} - \lambda_2}{d}, \quad \lambda_2 = \sqrt{1 - \rho_t} \quad (4.17)$$

One has to notice that this optimization procedure will depend on whether the initial guess of parameters, obtained by the approximation $(\tilde{I}_t)_{t \geq 0}$, will be or not able a pertinent starting point for the real index $(I_t)_{t \geq 0}$ during the fifth step. We remark from Figure 4.6 that in the last we usually deal with just the level of the implied volatility, because the skew of the implied of volatility under the model (\tilde{I}_t) remains the same for the real index $(I_t)_{t \geq 0}$

Table 4.2 illustrates the behavior of all parameters over maturities calibrated. It shows that γ tends to take high value, when the maturity is small, it can be explained naturally by the fact that the skew at-the-money, in short time, of option is high in the market data. Moreover, the level of the implied volatility of DAX option becomes small when the maturity increases, that is why the η calibrated takes small value over time. We note also some stability over time with respect to the parameter ρ_{\min} .

In Figure 4.7, we illustrate several results about calibration algorithm that has been run for different maturities. We notice that the model is able to fit the market in all kind of maturities. Indeed, in short maturity, the model has to deal with both high skew and kurtosis, while in long maturity the convexity is less important.

Even though, the (LC) model is able take into consideration all features of the implied volatility, it has a drawback related to the dynamic of its implied volatility once the spot is perturbed. The main idea has been already studied by Hagan et al. [50] in single market. He shows that, in the case of CEV models, once the spot is perturbed, the implied volatility moves in the opposite direction of the spot. The result is still valid under the local correlation assumption. Figure 4.8 shows this undesirable feature in the case of negative perturbation, which gives wrong hedges. To see this, let us consider $BS(S, K, \sigma, T)$ be the Black and Scholes formula. Under the (LC) model, the value of call option, denoted by $V(t, S_t)$ is given by the Black and Scholes formula

$$V(t, S_t) = BS(S_t, K, \Sigma_{T,K}(I_t), T),$$

where $\Sigma_{T,K}(I_t)$ is the implied volatility of the model with initial spot I_t . The following $(\Delta)_{1 \leq i \leq d}^i$ hedging strategies

$$\forall 1 \leq i \leq d, \quad \Delta^i = \partial_{S^i} V(t, S_t) = \partial_{S^i} BS + [\partial_{\Sigma} BS] \partial_{S^i} \Sigma_{T,K}(I_t)$$

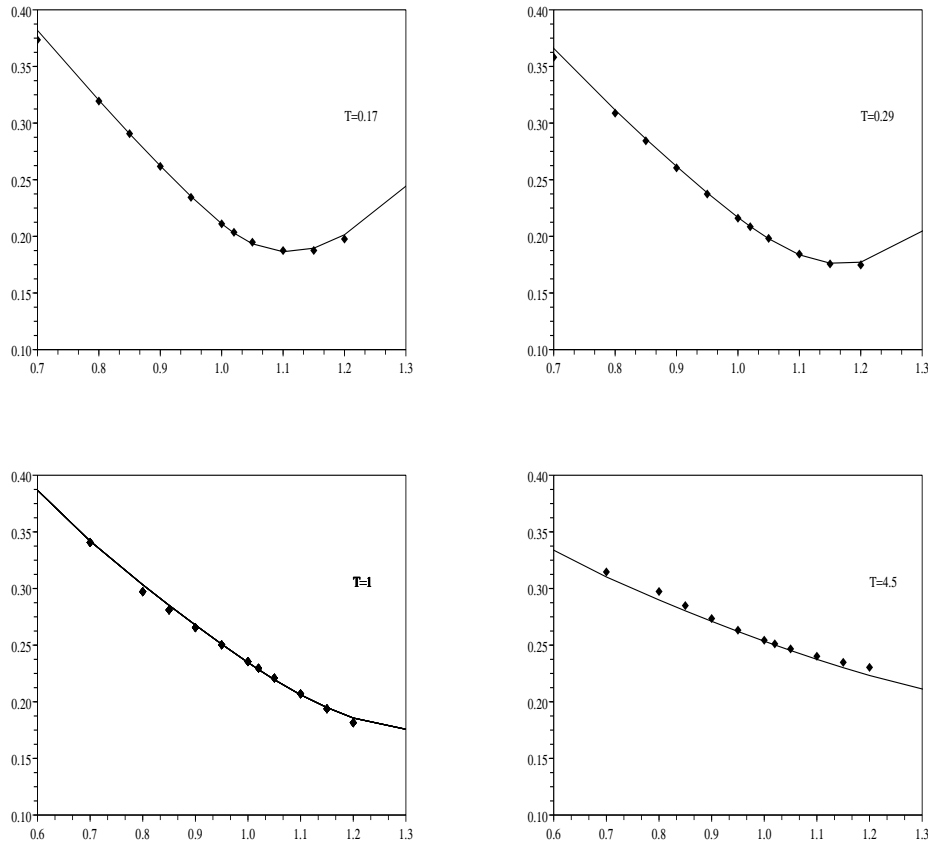


Figure 4.7: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Calibration results with respect to different maturities $T = \{0.17, 0.29, 1, 4.5\}$, under the (LC) model compared to the implied volatility (Dash line) given by the market: DAX (solid line).

is the one given by the model (LC). The first term is clearly the Δ risk in the Black and Scholes model. The second term corresponds to the term correction by the local (LC) model. It is divided into two part, the first one is the risk of Vega, defined as $\partial_{\Sigma} BS$, multiplied by the sensitivity of the implied volatility model with respect to the change of stock value. Under a negative perturbation, we see then that since the real market moves in the opposite direction of the one derived by the model, the term $\partial_{S^i} \Sigma_{T,K}(I_t)$ will take the wrong value in term of hedge, even though it calibrates the smile of the market. However, we also notice in Figure 4.8 that, under a positive perturbation, the implied volatility model moves in the right direction as the index changes, which means that under the (LC), we are highly exposed to the Vega risk if the index decreases.

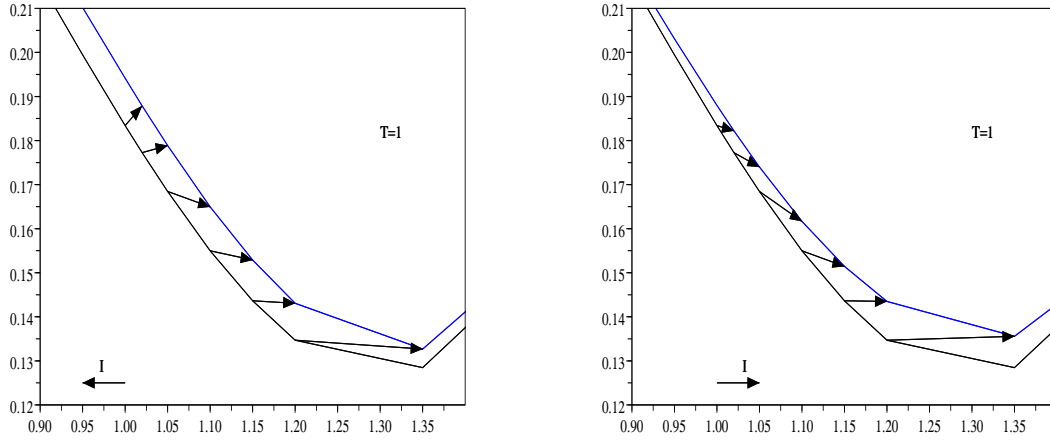


Figure 4.8: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the shape of index implied volatility w.r.t to strike moneyness for (LC) model after perturbation, $\gamma = 11$, $\eta = 2$ and $\rho_{min} = 0$.

index implied volatility dynamic under local correlation

In this section, we try to investigate whether the opposite dynamic direction is strongly related to the local structure. We take into consideration the following model, for all $1 \leq i \leq d$:

$$\begin{aligned} \frac{dS_t^i}{S_t^i} &= rdt + \sqrt{\nu_t^i} dB_t^i \\ dv_t^i &= \kappa_\nu^i (\bar{\nu}^i - \nu_t^i) dt + \sigma^i \sqrt{\nu_t^i} \left(\rho_i dB_t^i + \sqrt{1 - \rho_i^2} d\hat{B}_t^i \right). \end{aligned} \quad (4.18)$$

Under the same notation as in (4.3), we have $(\tilde{B}_t^1, \dots, \tilde{B}_t^1, B_t^1, \dots, B_t^d)$ is a $2d$ standard Brownian motions, such that for all $0 \leq i \leq d$ we assume that $B_t^i = (\sqrt{C_i} B_t)_i$, where the local correlation matrix $(C_t)_{t \geq 0}$ is given by (4.16). We note that each component of the vector $(S_t)_{t \geq 0}$ follows a Heston model. Here, we are not interested into calibration performance, we just study the dynamic of the implied volatility. We plot therefore the index implied volatility curve with respect to the change of the initial vector value $S_0 \rightarrow S_0(1 \mp \epsilon)$.

Figure (4.9) shows the behavior of the index implied volatility. We can see that under these modelling consideration, its dynamic follows the movement of index change better than (LC). We can deduce implicitly, like as in Hagan [50], that the use of a stochastic structure, can enhance the dynamic properties of the implied volatility. Back again to the Heston model, Figure (4.10) shows the behavior of the implied volatility model associated to one underlying. We recall that Heston model is given by the following SDE

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{\nu_t} dW_t \\ dv_t &= \kappa\nu(\bar{\nu} - \nu_t)dt + \sigma\sqrt{\nu_t} \left(\rho dW_t + \sqrt{1 - \rho^2} dZ_t \right), \end{aligned} \quad (4.19)$$

where $(Z_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ are an independent Brownian motion.

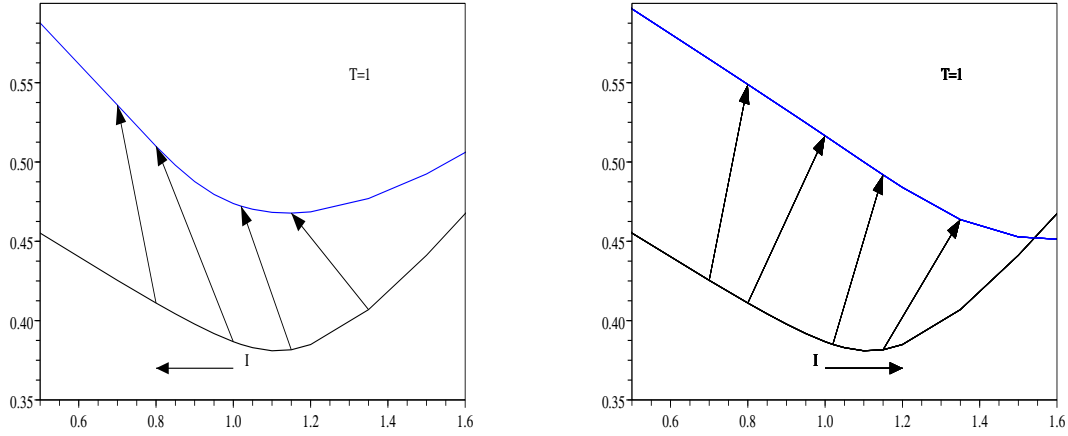


Figure 4.9: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the shape of index implied volatility after perturbation w.r.t to strike moneyness under the multi dimension Heston model in 4.18. We assume that all instantaneous volatilities share the same parameterization $\gamma = 14.1$, $\eta = 0.55$, $\rho_{min} = 0.26$, $\epsilon = 0.2$ and for all $1 \leq i \leq d$, $\rho_i = -0.8$, $\kappa_\nu^i = 1.2$, $\sigma^i = 0.38$, $\nu_0^i = 0.0174$ and $\bar{v}^i = 0.03$.

In Figure (4.10), shows that Heston model moves exactly by the same order of perturbation as the initial stock value changes. We call this kind of model a 'Sticky Delta' one. Indeed, by simple calculus, one can prove That the Heston implied volatility depends only on S_0/K . Indeed, the price of the call option is given by

$$\begin{aligned} \mathbb{E} [e^{-rT}(K - S_T)^+] &= K \mathbb{E} \left[e^{-rT} \left(1 - \frac{S_0}{K} e^{\int_0^T (r - \frac{\nu_s^2}{2}) ds + \int_0^T \nu_s dW_s} \right)^+ \right] \\ &= K \left(e^{-rT} \mathcal{N}(-d_2) - \frac{S_0}{K} \mathcal{N}(-d_1) \right), \end{aligned}$$

where $d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\Sigma(T, K, S_0)^2}{2})T}{\Sigma(T, K, S_0)}$ and $d_2 = d_1 - \Sigma(T, K, S_0)\sqrt{T}$, it follows $\Sigma(T, K, S_0) := \Sigma(T, \frac{S_0}{K})$.

One has to wonder if it is possible to derive a stochastic correlation, in order to prevent the drawback of local framework in term of the dynamic of the implied volatility. In the sequel, we discuss two types of stochastic correlation processes, the first one is defined by (SC1D) and (SC) models, which are considered as local models perturbed by stochastic factor. The second type of modelling assume a pure stochastic correlation, which derives from *LSV* (joint Local Stochastic structure) models developed for one dimension context.

Performing local correlation structure for calibration of different maturities

From Table 4.2, one can deduce that the previous local correlation model, defined in (4.16) is not adequate in order to calibrate simultaneously all maturities options. Looking closer to the evolution of the parameter γ , we have noticed that γ takes value from 22 to 4. Since, in short maturity the smile of the implied volatility for index options takes high value, the parameter γ

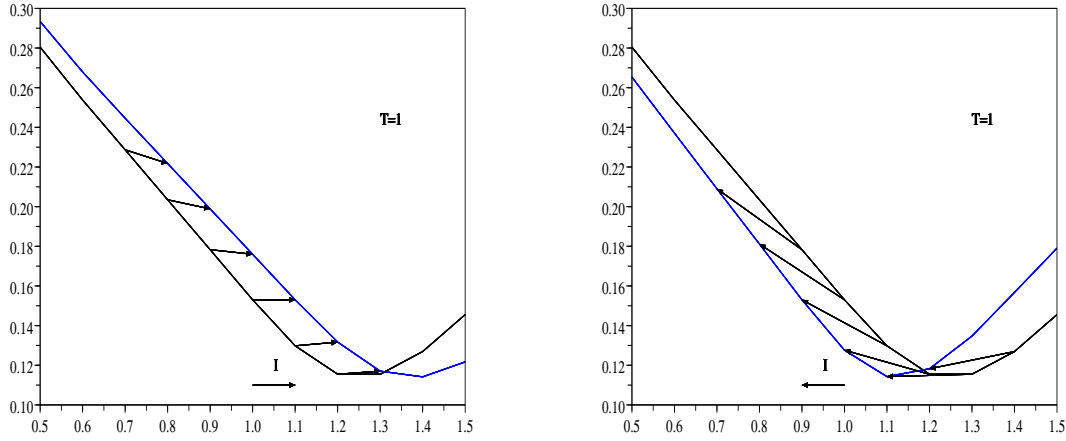


Figure 4.10: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the shape of index implied volatility w.r.t to strike moneyness for displaced model after perturbation. We use the same calibration parameter as in [41]: $\rho = -0.71$, $\kappa_\nu^i = 1.32$, $\bar{\nu} = 0.0354$, $\nu_0 = 0.0174$ and $\sigma = 0.38$.

takes naturally big value to compensate this phenomena. Moreover, the parameter η changes also value with respect to different maturities (from 0.3 to 6), because this parameter is strongly related to the level of the implied volatility.

In order to calibrate in the same time all options, one can derive, as a first alternative, a non parametric function, and assume that both $\eta := (\eta_t)_{t \geq 0}$, $\gamma := (\gamma_t)_{t \geq 0}$ depend on time t . However, since η and γ variate highly with respect to maturities, it leads us to consider a new local correlation structure, which its parameters are more stable than the first model.

The idea is to consider a new instantaneous correlation, which is given by

$$\forall 1 \leq i \neq j \leq d, (C_t)_{i,j} = \frac{1}{1 + \eta \left[\left(\frac{I_t e^{-rt} - I_0}{\sqrt{t} I_0} + 1 \right)^+ \right]^\gamma}, \theta = \frac{\sum_{k=1}^d \alpha^k \sigma_0^k S_0^k}{I_0}. \quad (4.20)$$

It follows from Table 4.3, that this new correlation structure detains more stable parameter with

Table 4.3: The error of calibration: $\sum_i \left(\frac{\Sigma_{T,K^i} - \Sigma_{T,K^i}^{Mkt}}{\Sigma_{T,K^i}^{Mkt}} \right)^2$ for different maturities, under (LC) model defined in (4.20)

T	0.17	0.28	0.64	1	1.35	1.67	2	2.34	3.1	3.8	4.5	6
γ	1.50	2.57	3.29	3.76	3.72	4.00	4.33	4.20	3.95	4.53	3.96	3.29
η	1.29	0.95	0.61	0.48	0.40	0.37	0.35	0.32	0.24	0.21	0.20	0.17
Error	0.0003	0.00011	0.0002	0.0005	0.0007	0.0008	0.0008	0.0009	0.0010	0.0009	0.0005	0.0002

respect to maturities, it is then a potential candidate to calibrate all option market. Indeed, we suppose now that both η and γ are no more constants, and they depend on time : $\eta := (\eta_t)_{t \geq 0}$ and $\gamma := (\gamma_t)_{t \geq 0}$. Let $(T_i)_{i \in \mathbb{N}}$ denote a sequence related to all market maturities, then we assume that structure of the parameter is the following

$$\forall i \in \mathbb{N}, \forall t \in [T_i, T_{i+1}[, \gamma_t = \gamma_{T_i} \text{ and } \eta_t = \eta_{T_i}. \quad (4.21)$$

Table 4.4: The error of calibration: $\sum_i \left(\frac{\Sigma_{T, \kappa^i} - \Sigma_{T, \kappa^i}^{Mkt}}{\Sigma_{T, \kappa^i}^{Mkt}} \right)^2$ for different maturities, under time dependent model (LC) model defined in (4.20) and (4.21)

T	0.17	0.28	0.64	1	1.35	1.67	2
γ_t	1.50	4.26	5.09	5.00	4.68	11.56	78.35
η_t	1.29	1.06	0.47	0.27	0.12	0.23	0.14
Error	0.0003	0.00009	0.0006	0.0009	0.0013	0.0011	0.0012

Table 4.4 shows the calibration result over maturities, under the consideration that all parameter in model (4.20) depend on time. We remark that till one year, the model calibrate correctly market data. However, the model is less performing beyond one maturity. Indeed, since square root function w.r.t time (in (4.20)) has no effect when maturity is big, then the parameter $(\gamma_t)_{t \geq 0}$ has to take big high value in order to catch the skew of the market.

4.5 Discussion of the parameters in the models (SC1D) and (SC)

Under both models (SC1D) and (SC), we no longer assume that the correlation modelling is done via a local correlation, but instead, we replaced it by a pure stochastic diffusion one. The idea of this approach is to investigate whether these models are able (or not) to characterize better the term structure of the implied volatility index. Moreover, we would like to see if they have the same drawback as the local correlation in term of hedging.

Let us go back to Formula (4.14), it yields that the third term P^3 disappears because we are placed in pure stochastic correlation. We assume the following model

$$d\rho_t = \kappa \left(\rho_t - \frac{1}{1 + \eta \left(\frac{t}{T_0} \right) \gamma} \right) dt + \sigma \sqrt{\rho_t (1 - \rho_t)} dW_t.$$

Intuitively, in long time this model behaves like the one (4.16) with a local correlation, where κ is large and σ is small. Therefore, depending on the value of the mean-reverting κ the model is characterized by two regimes, that are defined by its parameters. Indeed, we remark that when κ takes high value, the model becomes most likely independent of both ρ_0 and σ . In this regime, both γ and η play the same role as in (LC) model: γ characterize the skew of the implied volatility and η defines its level at-the-money. Figure 4.11 shows these results for different choices.

However, when the value of κ remains small the impact of both ρ_0 and σ become significant. Indeed, ρ_0 plays the same role as η , which means that the at-the-money increases (decreases) as soon as, the value of ρ_0 increases (decreases). Intuitively, in short maturity, ρ_0 is closed to the expected value of ρ_t , which is highly correlated to the level of the implied volatility. Moreover, σ has an impact on the right queue distribution of implied volatility. All these results are given in Figure 4.12

Since when κ takes high value, (SC1D) behaves like (LC) model. Then, the calibration under this regime of each maturity can be obtained by the same optimization method. Table 4.5 illustrates the behavior of parameter calibration, for a fixed value of κ and σ .

We are also interested in a second type of pure stochastic correlation related to the *MRC* problem. The model can be seen as the extension of (SC1D) model. We consider therefore that the whole correlation $(R_t)_{t \geq 0}$ is solution of the following SDE

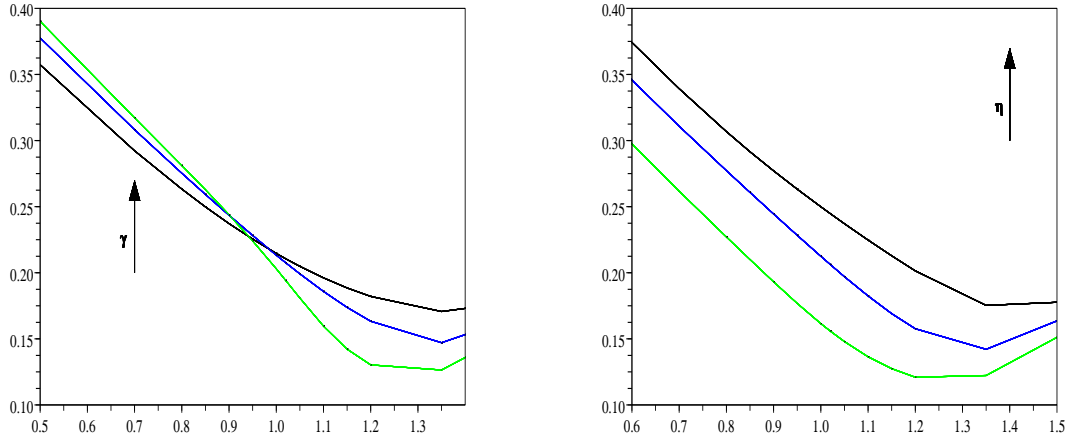


Figure 4.11: $d = 30$, 10^6 Monte-Carlo samples, $T = 1$ and discretization time step 0.025. Both graphs correspond to the index implied volatility w.r.t to strike moneyness under (SC1D) model, such that $\sigma = 0$. Left: $\eta = 2$, and $\gamma = \{4, 10, 20\}$. Right: $\gamma = 8$ and $\eta = \{0.2, 0.9, 4\}$. The width of each point represents the 95% confidence interval.

Table 4.5: The error of calibration: $\sum_i \left(\frac{\Sigma_{T,K^i} - \Sigma_{T,K^i}^{Mkt}}{\Sigma_{T,K^i}^{Mkt}} \right)^2$ for different maturities, under the (SC1D) model, with $\kappa = 15$ and $\sigma = 0$.

T	0.17	0.28	0.64	1	1.35	1.67	2	2.34	3.1	3.8	4.5	6
γ	22.13	17.55	11.13	9.66	8.89	8.33	7.53	6.67	6.07	4.32	4.11	3.60
η	1.32	1.03	0.64	0.48	0.40	0.37	0.34	0.31	0.23	0.20	0.19	0.15
Error	0.00009	0.00005	0.0004	0.0006	0.0008	0.0009	0.0009	0.0009	0.001	0.001	0.0006	0.0002

$$C_t = C_0 + \int_0^t \kappa(\bar{c}_s - C_s)tds + \sum_{n=1}^d \int_0^t a_s \left(\sqrt{C_s - C_s e_d^n C_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{C_s - C_s e_d^n C_s} \right). \quad (4.22)$$

The mean reversion correlation $(\bar{c}_t)_{t \geq 0}$ and the correlation volatility $(a_t)_{t \geq 0}$ are given by the following

$$\forall 1 \leq i \neq j \leq d, (\bar{c}_t)_{i,j} = \rho_t \text{ and } a_t = \sigma \sqrt{\frac{\kappa(1 - \rho_t)}{d - 1}},$$

where $\rho_t = \frac{1}{1 + \eta \left(\frac{t}{T_0} \right)^\gamma}$ and $\sigma \in [0, 1]$, $\kappa \geq 0$ and $\eta, \gamma \geq 0$. The sensibility of parameters are quite similar to (LC) model. On the one hand, one can have two regimes, with respect to the value of κ . In large value, we are supposed to fall on the (LC) model, while in the low value of κ , the skew implied volatility of the model seems to be less visible, because we supposed that all Brownian motions in both SDE (4.3) and (4.22) are independent. In other word, for a given value of both γ and η , the most expected value of the skew will be obtained when κ takes a great value. On the other hand, both parameter γ and η play the same role as in (LC) and (SC1D) models, because they are respectively related to the skew and the level of the implied

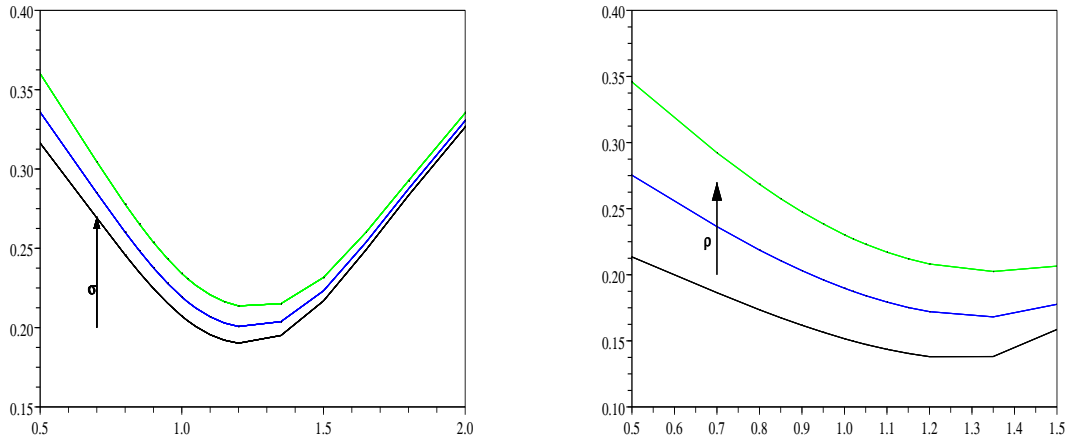


Figure 4.12: $d = 30$, 10^6 Monte-Carlo samples, $T = 0.64$ and discretization time step 0.025. Both graphs correspond to the index implied volatility w.r.t to strike moneyness under (SC1D) model, such that $\gamma = 20$, $\kappa = 1$ and $\eta = 5$. Left: $\rho_0 = 0.55$, $\sigma = \{0, 0.1, 0.2\}$. Right: $\rho_0 = \{0, 0.3, 0.7\}$ and $\sigma = 0.1$. The width of each point represents the 95% confidence interval.

volatility model at-the-money. In Figure 4.13, we illustrate the behavior of (SC) model with respect to these three parameters κ , γ and η .

Moreover, the parameter C_0 keeps the same impact on the implied volatility as the parameter η , but it is more visible when κ takes a low value. Even though, we observe that all these parameters, in certain sense, plays the same role as in (LC) model, but σ becomes the only one whose behavior has changed. Indeed, Figure 4.14, shows that σ is strongly related to the left side of the queue distribution. The left queue distribution is high as soon as the parameter σ take higher values. Finally, one can deduce that in order to calibrate all maturities, this model is strongly related to (LC) calibration, which means that even if we consider that both parameter η and γ are time dependent, it is very hard to obtain an adequate calibration result for the term structure.

4.6 Stochastic local correlation model

In the previous section, the stochastic correlation model is seen as a perturbation around a local correlation model. We have concluded that in term of calibration, the previous model has the same performance as its local version one: (LC), which means that in short maturity, the parameter γ takes very high value, and the term structure calibration becomes much more hard. In this section, we will try to use the stochastic effect in order to compensate the high value of γ in short maturities. We will try to derive a model that is quite similar to the so known *LSV* model in the single name world. Indeed, we take into consideration both local and stochastic correlation. Under the same notation as in Section 4.2, we suppose that the stock

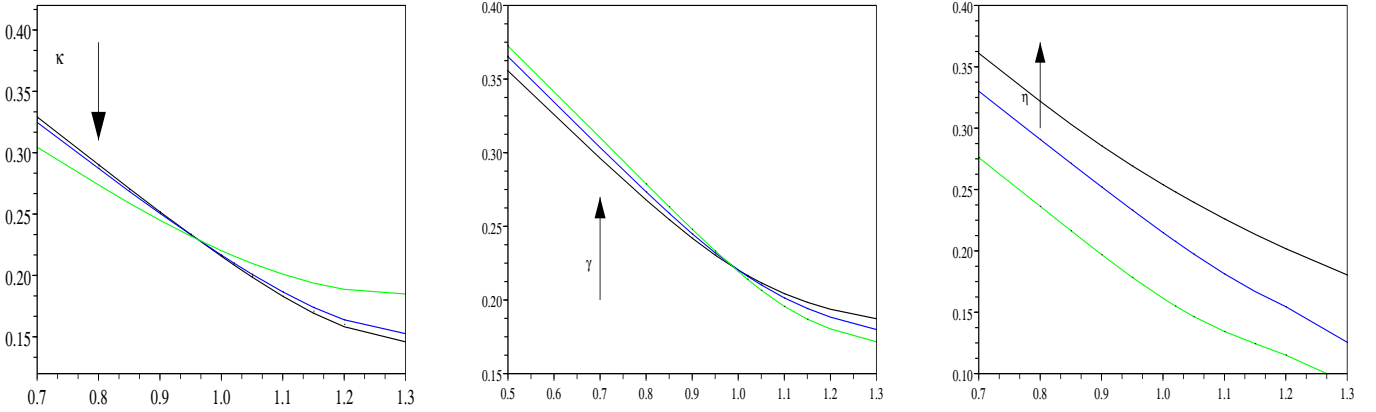


Figure 4.13: $d = 30$, 10^6 Monte-Carlo samples, $T = 0.64$ and discretization time step 0.025., $\sigma = 0$ and $\rho_0 = 0.55$. Both graphs correspond to the index implied volatility w.r.t to strike moneyness under (SC) model. Left: $\kappa = \{15, 8, 1\}$. Center: $\gamma = \{4, 10, 20\}$. Right: $\eta = \{0.2, 1, 4\}$. The width of each point represents the 95% confidence interval.

vector $(S_t^i)_{1 \leq i \leq d}$ is solution of the following SDE, for all $1 \leq i \leq d$

$$\frac{dS_t^i}{S_t^i} = rdt + \sigma^i(t, S_t^i) \left(\sqrt{\lambda_t} (\sqrt{C_t} d\tilde{B}_t)^i + (1 - \lambda_t) dZ_t^1 + \sqrt{\lambda_t(1 - \lambda_t)(1 - \varrho^2)} dZ_t^2 + \varrho \sqrt{(1 - \lambda_t)\lambda_t} dB_t^\lambda \right), \quad (4.23)$$

where $(\tilde{B}_t)_{t \geq 0}$ and $(Z_t^i)_{i \in \{1,2\}, t \geq 0}$ are vector of Brownian motions, whereas the process $(\lambda_t)_{t \geq 0}$ is given by the following Wright-Fisher SDE:

$$d\lambda_t = \theta(\bar{\lambda} - \lambda_t)dt + \beta \sqrt{\lambda_t(1 - \lambda_t)} dB_t^\lambda,$$

where $(B_t^\lambda)_{t \geq 0}$ is a Brownian motion, $\theta \geq 0$, $\bar{\lambda} \in [0, 1]$, $\beta \geq 0$ and ϱ is real value defined on $[-1, 1]$. Finally, we suppose that for every $1 \leq i \neq j \leq d$, $(C_t)_{i,j} = \rho(I_t)$, where $\rho(I_t)$ is defined in (4.16).

The instantaneous variance is given by

$$\bar{\sigma}_t^2 = \frac{\sum_{2 \leq i,j \leq d} \alpha^i \alpha^j \sigma_t^i \sigma_t^j S_t^i S_t^j [\lambda_t (C_t)_{i,j} + (1 - \lambda_t)]}{I_t^2}.$$

The choice of the sign of ϱ depends on the new term that will occur when we compute again the quadratic covariance $\frac{\langle dI_t, \bar{\sigma}_t^2 \rangle}{dt}$. Therefore, under the same notations as in (4.14)

$$\begin{aligned} \frac{\langle dI_t, \bar{\sigma}_t^2 \rangle}{dt} &= P1 + P2 + \lambda_t P3 \\ &\quad - \underbrace{\varrho \frac{\beta \lambda_t (1 - \lambda_t)}{I_t^2} \left(\sum_{1 \leq i,j \leq d} \alpha^i S_t^i \sigma_t^i [1 - (C_t^1)_{i,j}] \alpha^j S_t^j \sigma_t^j \right)}_{P4} \left(\sum_{1 \leq k \leq d} \alpha^k S_t^k \sigma_t^k \right), \end{aligned} \quad (4.24)$$

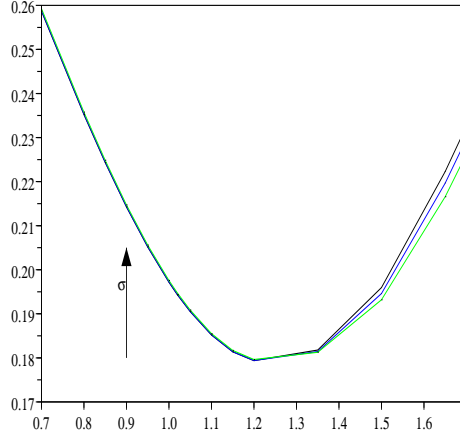


Figure 4.14: $d = 30$, 10^6 Monte-Carlo samples, $T = 0.64$ and discretization time step 0.025. Graph correspond to the index implied volatility w.r.t to strike moneyness under (SC) model, such that $\gamma = 20, \eta = 5, \rho_0 = 0.55$ and $\kappa = 1$. $\sigma = \{0, 0.1, 0.2\}$. The width of each point represents the 95% confidence interval.

We note by $P4$ the new term appearing in (4.14), and Looking closely to its right hand side, we remark that by construction the sign of $P4$ is related to sign of the parameter ϱ . We conclude that the negative value of instantaneous correlation (4.24) can be derive by a local structure effect (the term $P3$) or by a pure stochastic effect (the new term $P4$). We notice also that the term $P3$ is multiplied by λ_t , which means that the stochastic correlation effect will just compensate the role of the local structure that is weakened by the factor λ_t . Before deriving sensitivities numerical result, we explain briefly how simulate this model efficiently without involving Cholesky decomposition at each step time.

Looking closer to Equation 4.23, one can notice that by construction $\sqrt{(1 - \lambda_t)\lambda_t}dB_t^\lambda = \frac{1}{\beta} (d\lambda_t - \theta(\bar{\lambda} - \lambda_t)dt)$, and the square root matrix $\sqrt{C_t}$ can be obtained explicitly from Equation 4.17. We consider the time horizon $T > 0$, and the regular time-grid $t_i^N = iT/N$, for $j = 0, \dots, N$. The price stock are given by the following scheme, for all $1 \leq i \leq d$:

$$\begin{aligned} \hat{S}_{t_{j+1}}^i &= \hat{S}_{t_j}^i \left(r(t_{j+1} - t_j) + \frac{\varrho \sigma^i(t_j, \hat{S}_{t_j}^i)}{\beta} \left[\hat{\lambda}_{t_{j+1}} - \hat{\lambda}_{t_j} - \theta(\bar{\lambda} - \hat{\lambda}_{t_j})(t_{j+1} - t_j) \right] + \right. \\ &\left. \sqrt{(t_{j+1} - t_j)\hat{\lambda}_{t_j}\sigma^i(t_j, \hat{S}_{t_j}^i)} \left\{ \left(\frac{\sqrt{1 + (d-1)\rho(\hat{I}_{t_j})} - \sqrt{1 - \rho(\hat{I}_{t_j})}}{d} \mathbb{1}_{\mathcal{M}_d(\mathbb{R})} + \sqrt{1 - \rho(\hat{I}_{t_j})} I_d \right) N_{j+1} \right\}^i \right. \\ &\left. + \sigma^i(t_j, \hat{S}_{t_j}^i) \left[(1 - \hat{\lambda}_{t_j})\sqrt{t_{j+1} - t_j} G_{j+1}^1 + \sqrt{(1 - \varrho^2)\hat{\lambda}_{t_j}(1 - \hat{\lambda}_{t_j})(t_{j+1} - t_j)} G_{j+1}^2 \right] \right) \end{aligned}$$

where for all $1 \leq j \leq N$ and $1 \leq i \leq d$, N_j^i, G_j^1 and G_j^2 are independent Gaussian variables. The simulation of the Jacobi process $(\hat{\lambda}_{t_j})_{1 \leq j \leq N}$ is done by the same techniques explained in [3].

Since the process $(\lambda_t)_{t \geq 0}$ is a mean reverting process, which means that in long maturity, one can assume that the process tends to the value $\bar{\lambda}$. One can assume intuitively that the model behaves in long maturity as a pure (LC) model.

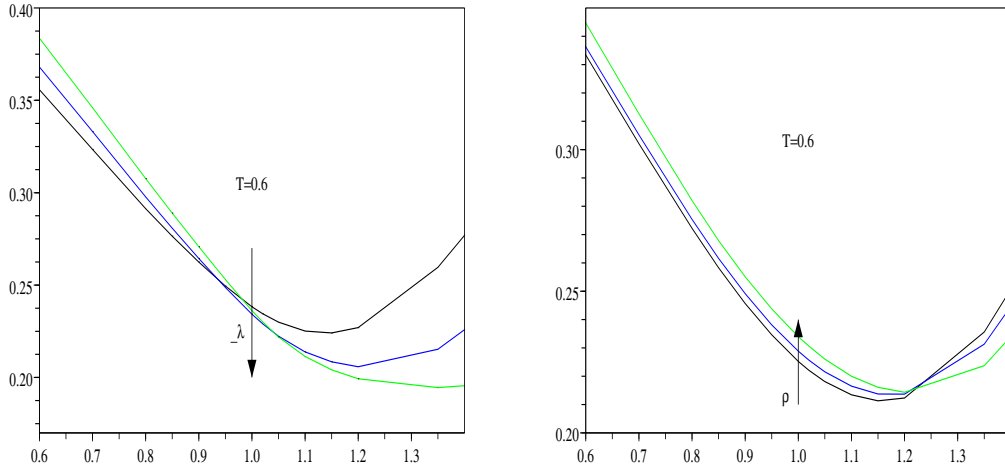


Figure 4.15: $d = 30$, 10^6 Monte-Carlo samples, $T = 0.64$ and discretization time step 0.025 $\eta = 0.55$ $\gamma = 15$, $\beta = 0.2$, and $\lambda_0 = 0.9$ Both graphs correspond to the index implied volatility w.r.t to strike moneyness under the model in 4.23. Left: $\rho = 0.5$ and $\bar{\lambda} \in \{0.1, 0.5, 0.9\}$. Right: $\rho = \{0.1, 0.5, 0.8\}$ and $\bar{\lambda} = 0.9$.

In Figure 4.15, we show the behavior of the index implied volatility with respect to the value of ρ , $\bar{\lambda}$. On the right hand side of this figure, one can conclude that ρ is strongly related to the skew of the curve, which corresponds to the term $P4$ in (4.24). On the left hand side, we observe a change of the level related to the implied volatility at the money. One has to remember that the instantaneous local volatility is written as : $\lambda_t C(I_t) + (1 - \lambda_t) \mathbb{1}_{\mathcal{M}_d(\mathbb{R})}$. When the value of $\bar{\lambda}$ decreases, it leads implicitly to decrease the probability that λ_t takes low value, and the instantaneous local volatility tends therefore to $\mathbb{1}_{\mathcal{M}_d(\mathbb{R})}$. It follows then level of the implied volatility increases.

4.7 Summary and conclusion

This paper investigate different models that can be used in order to calibrate both Vanillas and index option markets. We choose to consider a correlation modelling instead of variance covariance modelling. First of all, we highlight the major drawback associated to the adoption of a covariance modelling such as Wishart process. We start the paper by presenting a pure local correlation model (LC). We were able to identify by a few parameters the behavior of the index implied volatility. We explain how one can enhance the term structure calibration under the local structure hypothesis.

Moreover, we propose a different model taking into account the stochastic effect of certain factors. The first one (denoted by (SC1D) and (SC)) is considered as a perturbation around the (LC) model. The second one tries to mix both local structure and stochastic effect one. In both models, we presents several efficient discretization method to enhance time computation. We conclude that in term of calibration the first one has the same result as (LC) model, and is not well appropriate to calibrate the term structure. The second model has better chance to calibrate, because the stochastic factors can compensate the high value of γ in short maturity

in order to keep a regular model parameter value with respect to the maturity.

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