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Dynamics without physical measures

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THÈSE

présentée par

Aminosadat TALEBI

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Systemes dynamiques non statistiques

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Systemes dynamiques non statistiques

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15 octobre 2020

Contents

Introduction	5
Part 1. Statistical instability and non-statistical dynamics	19
1. preliminaries	20
2. The L^1 convergence version	23
3. The essential convergence version	27
4. The version of convergence in law	32
5. Non-statistical Anosov-Katok maps	42
6. Comparison between different versions	47
Part 2. Non-statistical Rational maps	51
7. Proof of Main Theorem	52
8. Statistical bifurcation toward periodic measures	54
9. Periodic measures are dense in $\mathcal{M}_1(f)$	63
Bibliography	67

Introduction

Introduction en Français. Dans cette thèse, nous étudions les propriétés statistiques des systèmes dynamiques. Plus précisément, nous étudions si le comportement statistique asymptotique des orbites typiques via la convergence de la suite des mesures empiriques.

Nous étudions également la stabilité du comportement asymptotique sous l'effet de perturbations de la dynamique au sein d'une famille. Ce mémoire comporte deux parties. Dans la première partie, nous relierons ces deux études. La deuxième partie est consacrée à la preuve de l'existence d'un comportement statistique étrange, appelé l'oscillation maximale, dans l'espace des applications rationnelles de la sphère de Riemann.

Etant donné une mesure de référence sur l'espace des phases, en général, une application non conservative peut avoir un comportement statistique très compliqué, pour laquelle les orbites de presque chaque point définissent une suite divergente convergente de mesures empiriques. Une application à oscillation maximale est une application pour laquelle la suite de mesures empiriques de presque tout point s'accumule sur toutes les mesures invariantes de la dynamique. Notre résultat sur l'existence d'un tel comportement parmi les applications rationnelles est le suivant :

THÉORÈME. Dans le lieu de bifurcation maximale (qui est un fermé de mesure de Lebesgue positive) dans l'espace des fractions rationnelles de degré > 1 , une application topologiquement générique est d'oscillation maximale.

Comme toute application rationnelle de degré supérieur à un a au moins deux mesures invariantes, il vient qu'une application rationnelle oscillant maximale, la suite des mesures empiriques de presque tout point n'est pas convergente. La preuve de ce théorème utilise des arguments issus de la dynamique holomorphe et aussi de théorèmes généraux que nous avons proposés dans la première partie de la thèse.

Dans la première partie, nous avons développé un cadre abstrait qui vise à comprendre ce qui se cache derrière l'existence d'applications non statistiques dans une famille de dynamiques donnée. Une application est dite non statistique par rapport à la mesure de référence, s'il existe un ensemble de mesure positive de points qui ont leur suite de mesures empiriques divergente. Dans cette partie de la thèse, nous étendons et formalisons le concept d'instabilité statistique dans un sens général, indépendant du comportement statistique de la dynamique, et nous montrons comment l'abondance d'applications statistiquement instables, dans une famille donnée de dynamiques, implique l'existence d'applications non statistiques dans cette famille. Ce cadre abstrait est

développé en trois versions différentes (mais de nature similaire), sur la base desquelles la topologie est prise en compte pour la convergence de la suite de mesures empiriques. Une autre application de cette formalisation, et aussi un autre exemple d'applications à oscillation maximale:

THÉORÈME. *Dans l'ensemble des difféomorphismes dissipatif Anosov-Katok de l'anneau, une dynamique topologiquement générique est d'oscillation maximale avec exactement deux mesures ergodiques.*

Enfin, nous remarquons que bien que ce cadre abstrait ait été conçu pour étudier des dynamiques dissipatives présentant un comportement non-statistique statistique bizarre, il s'avère que ce cadre s'applique dans le monde de la dynamique conservative. En utilisant notre technologie, nous avons pu reprouver et développer un théorème d'Avila et Bochi concernant la continuité de l'opérateur qui associe à une dynamique conservative sa décomposition ergodique.

Introduction. In this thesis we deal with statistical properties of dynamical systems from a special point of view. We try to find more examples of those dynamical systems which display non-statistical behavior and also we give a general formalization of the concept of statistical (in)stability. Roughly speaking, displaying non-statistical behavior means that there is a set of positive measure in the phase space such that the orbit of each point in this set does not have a convergent distribution in the phase space. Here we have fixed a reference measure on the phase space. To be more precise, assume X is a compact Riemannian manifold and $f : X \rightarrow X$ is a continuous map and let μ be a probability measure whose density w.r.t. a Lebesgue measure is a smooth positive function. By $\mathcal{M}_1(X)$ we denote the space of probability measures on X . For a point $x \in X$ the n^{th} empirical measure

$$e_n^f(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)},$$

describes the distribution of the orbit of x up to the n^{th} iteration in the phase space, which asymptotically may or may not converge. If it converges, then by observing a finite number of iterations (but possibly large) one can predict how the orbit of x behaves approximately for larger iterations, from statistical point of view. However it may not converge. In this case we say that x is non-statistical. The non-statistical points exist in many well-known dynamics. For example, in the symbolic dynamics (see Example 1.2) and any other dynamics that has a shift map as a subdynamics. One of the first results in this area is the Baire genericity of non-statistical points within the phase space for

subshifts of finite type, Anosov diffeomorphisms and more generally any map with periodic specification property (see [41],[42] and [20, thm 1.1.11]). In these examples the set of points with non-statistical behavior is of zero measure with respect to the natural reference measure on the phase space. One can ask how large can be the set of non-statistical points. Can it be of positive Lebesgue measure? If yes, how large is the subset of maps having this behavior? In most of well-known examples, the non-statistical points have zero measure. For example if a dynamical system f preserves the measure μ , by Birkhoff ergodic theorem, we know that the sequence of empirical measures converge for almost every points, and hence, the non-statistical points, if they exist, have zero measure. There are also many examples of non-conservative maps that have convergent statistical behavior on a full measure set of points; for example in the context of logistic family $f_\lambda(x) = \lambda x(1 - x)$, Jakobson has proved that there is a set of parameters with positive measure such that for any parameter in this set, the corresponding map has a unique ergodic absolutely continuous probability invariant measure and the empirical measures of Lebesgue almost every point converges to this unique measure (see [24]).

However, there are a few examples of dynamical systems on smooth manifolds having a positive Lebesgue measure set of non-statistical points. Let us call any map with this kind of behavior a non-statistical map (In the first part of the thesis we introduce different versions of this definition).

One of the first examples of the non-statistical dynamical systems is the so called Bowen eye [43]. It is a vector field on \mathbb{R}^2 with an eye like open region such that Lebesgue almost every point in this region is non-statistical (see Example 1). This region is bounded by two saddle connections of two hyperbolic equilibrium points (see Figure 1.3). These kinds of examples are very non-persistence since the saddle connections are so. In [14], Colli and Vargas introduced a new kind of examples of non-statistical dynamics. Their example is a diffeomorphism of a two dimensional surface with a non-trivial wandering domain such that every point in this domain has non-statistical behavior. Such a diffeomorphism was obtained by doing a careful perturbation of an initial diffeomorphism with a thick horseshoe, having tangency between stable and unstable sets. In [28] Kiriki and Soma have shown that on any closed surface M and any open Newhouse domain \mathcal{N} in $\text{Diff}^r(M)$ for $2 \leq r < \infty$, the maps which have a non-trivial wandering domain with non-statistical behavior are dense in \mathcal{N} (see also [39], [27] and [25] for other related results). Let us mention the work of Crovisier

et al. [15] which contains examples of non-statistical maps in the context of partially hyperbolic diffeomorphisms. There is also an explicit example of a non-statistical diffeomorphism of the annulus introduced by Herman that can be found in [21].

There are also examples of non-statistical maps in the world of more specific families of dynamical systems (e.g. polynomial maps) where one loses the possibility of local perturbations as a possible mechanism to control the statistical behavior of the orbits. One of the recent examples of specific family of dynamics displaying non-statistical behavior is the recent work of Berger and Biebler [9]. They prove the existence of real polynomial automorphisms of \mathbb{C}^2 having some wandering Fatou component on which the dynamics has non-statistical behavior. Their work also contains a generalization of the result of Kiriki-Soma [29] to the case of $r = \infty$ or ω and also the result in [26]. There is another example of the non-statistical dynamics in the logistic family $f_\lambda : [0, 1] \rightarrow [0, 1]$ where $f_\lambda(x) = \lambda x(1 - x)$ and $\lambda \in [0, 4]$. Hofbauer and Keller have shown in [22] that there are uncountably many parameters $\lambda \in [0, 4]$ such that f_λ is non-statistical. Later, in [23] they proved that there are uncountably many parameters $\lambda \in [0, 4]$ such that the map f_λ has indeed maximal oscillation property:

THEOREM (Hofbauer-Keller [23]). *There exist uncountably many $\lambda \in [0, 4]$ for which f_λ has maximal oscillation:*

$$\text{for Lebesgue a.e. } x \in [0, 1], \quad \text{acc}(\{e_{f_\lambda}^n(x)\}_n) = \mathcal{M}_1(f_\lambda),$$

where $\mathcal{M}_1(f)$ is the set of all invariant probability measures of f .

Let Λ be the closure of the set of parameters found by Hofbauer and Keller, then we have proved the following improvement of their result in section 4:

THEOREM A. *The set of parameters λ for which the map f_λ has maximal oscillation is a Baire generic subset of Λ .*

Non-statistical rational maps. The existence of non-statistical maps in one dimensional real dynamics may lead us to ask if there is any non-statistical maps in one dimensional complex dynamics, or in other words, is there any rational map on the Riemann sphere displaying non-statistical behavior on a set of points with positive Lebesgue measure? We give an affirmative answer to this question and also we prove the Baire genericity of non-statistical behavior (and even maximal oscillation) within a “large” subset of rational maps. To provide more details let us give some definitions and notations.

We denote the space of degree d rational maps by Rat_d . In what follows we assume that $d > 1$. A rational map f is *strictly post-critically finite* if all of its critical points eventually land on some repelling periodic point. The closure of the set of degree d strictly post-critically finite rational maps is called *maximal bifurcation locus* (which has several other characterizations, see the main theorem of [12]). Let us note that by the work of Rees [38] and the work of Astorg, Gauthier, Mihalache and Vigny [6], the maximal bifurcation locus has positive measure as a subset of Rat_d equipped with its volume measure as a complex manifold. Here is our new result regarding the Baire genericity of maximally oscillating rational maps within the maximal bifurcation locus:

THEOREM B. *For a Baire generic map f in the maximal bifurcation locus, the set of accumulation points of the sequence of empirical measures is equal to the set of invariant measures of f for Lebesgue almost every point.*

We note that for a rational map f with degree larger than one, the set of invariant measures $\mathcal{M}_1(f)$ is a large set, and in particular has more than one element and hence a generic map in the maximal bifurcation locus is non-statistical.

Another part of the thesis is to show the existence of non-statistical dynamics within the Anosov-Katok diffeomorphisms of the annulus:

Maximally oscillating Anosov-Katok maps of the annulus.

Let us call the closure of the set of those C^r diffeomorphisms of the annulus which are C^r -conjugated to a rotation, the space of C^r Anosov-Katok maps, and denote it by \mathcal{AK}^r . Our next result shows the existence and Baire genericity of maximally oscillating dynamics in this space:

THEOREM C. *A Baire generic map in the set of Anosov-Katok maps \mathcal{AK}^r has exactly two ergodic invariant measures each of which is supported by a different boundary component of the annulus and moreover the map is maximally oscillating.*

Another part of the thesis is devoted to formalizing common aspects observed in the known examples of non-statistical dynamics and develop an abstract setting aimed to study the existence and abundance of this kind of behavior in a general family of dynamics:

Statistical instability and non-statistical dynamics. In this part we introduce a sufficient condition for a family of maps, that guarantees the existence of non-statistical dynamics within that family. This condition is related to the notion of *statistical instability* of

dynamical systems which roughly speaking is the possibility of making essential changes in statistical properties of a map under arbitrary small perturbations. Despite the structural stability which is more strict and sensitive to the topological structure of every point in the phase space, statistical stability sees the statistical behavior of almost every point and does not care about the dynamical behavior of a set of measure zero. The Axiom A maps are example of statistical stable dynamics since they are indeed structurally stable which is a stronger condition. Beyond structural stable dynamics, there are also other examples of the statistical stable maps. Alvez and Viana in [4] study a class of dynamics which is formed by statistically stable maps: In their work the dynamics which are studied have a unique physical measure with a full basin and stability or instability of a dynamics is equal to the continuity or discontinuity of the map sending the dynamics to its unique physical measure. There are several other works in this direction among which we can quote [2] where the statistical stability is proved for multidimensional piecewise expanding maps, the result of Baladi-Benedicks-Schnellmann [8] and also the results in [1], where the statistical stability is proved for the Hénon maps of Benedicks-Carleson type and the paper [46] where the statistical instability is proved for certain maps in the quadratic family (see also [3]). In this section of the thesis we generalize the notion of statistical (in)stability and define it in general case for any dynamical system independent of its statistical behavior. In particular, we do not assume that the system has a physical measure. And next we study the connection between this notion and existence of non-statistical maps in a given family of dynamics.

To be more precise, consider the n^{th} empirical function $e_n^f : X \rightarrow \mathcal{M}_1(X)$, sending each point $x \in X$ to its n^{th} empirical measure $e_n^f(x)$. We study three types of (non-)convergence of the sequence $\{e_n^f\}_n$; almost sure convergence, L^1 convergence and convergence in law. We define non-statistical maps and statistical instability for each kind of (non-)convergence and show how these two notions are related in each topology.

Let us start with explaining the results regarding the convergence in law. If we push-forward the reference measure μ on X to the space of probability measures on X using an empirical function, we obtain a probability measure on the space of probability measures on X which is denoted as follows:

$$\hat{e}_n^f := (e_n^f)_*(\mu) \in \mathcal{M}_1(\mathcal{M}_1(X)).$$

A map f is called non-statistical in law if the sequence $\{\hat{e}_n^f\}_n$ is not convergent. let us denote the set of accumulation points of this sequence

by $\text{acc}(\{\hat{e}_n^f\}_n)$ which is a compact subset of $\mathcal{M}_1(\mathcal{M}_1(X))$. Now let Λ be a closed subset of $C^0(X, X)$ which is endowed with a topology finer than C^0 topology. In general, the set valued map sending the dynamics $f \in \Lambda$ to the set $\text{acc}(\{\hat{e}_n^f\}_n)$ does not have any regularity for the Hausdorff topology on the space of compact subsets of $\mathcal{M}_1(\mathcal{M}_1(X))$. However a simple but important observation is that when you consider the sequence along with its accumulation points, we prove that this map enjoys from the semi-continuity property. Then as a consequence we obtain the following lemma which is the main lemma in this section:

MAIN LEMMA. *A Baire generic map $f \in \Lambda$ is a continuity point for the map \mathcal{E} where*

$$\mathcal{E}(f) = \overline{\{\hat{e}_n(f) | n \in \mathbb{N}\}}.$$

To define statistical instability of a map, we need to give some more definitions. The space $\mathcal{M}_1(\mathcal{M}_1(X))$ endowed with weak-* topology is a compact metric space. Let $\hat{\nu}$ be an element of $\mathcal{M}_1(\mathcal{M}_1(X))$. We say $f \in \Lambda$ *statistically bifurcates toward $\hat{\nu}$* if it can be approximated by elements of the form $\hat{e}_{n_k}^{f_k}$ where f_k approaches to f and n_k goes to infinity. Let $\mathcal{B}_{\Lambda, f}$ be the subset of those elements of $\mathcal{M}_1(\mathcal{M}_1(X))$ toward which f statistically bifurcates. We can think of the set $\mathcal{B}_{\Lambda, f}$ as the set of all asymptotic statistical behaviors that the family Λ can displays locally around the map f . We say $f \in \Lambda$ is *statistically unstable in law* iff $\#\mathcal{B}_{\Lambda, f} > 1$.

The following theorem is our main theorem regarding the connection between statistical instability and non-statistical maps in the level of convergence in law:

THEOREM D. *Baire generically, $B_{\Lambda, f}$ is equal to $\text{acc}(\{\hat{e}_n^f\}_n)$.*

We also investigate the statistical stability and non-statistical maps from the L^1 (non-)convergence point of view. We say a map f is L^1 *non-statistical* if the sequence of maps $e_n^f : X \rightarrow \mathcal{M}_1(X)$ is not convergent for L^1 topology (see 2.1 for definition of d_{L^1}):

$$\limsup_{m, n \rightarrow \infty} d_{L^1}(e_n^f, e_m^f) > 0.$$

We say a map $f \in \Lambda$ is L^1 statistically unstable if the following quantity is positive:

$$\limsup_{h, g \rightarrow f, m, n \rightarrow \infty} d_{L^1}(e_n^h, e_m^g).$$

This quantity measures how different are the statistical behavior of the maps approaching to f for iterations close to infinity. If a map is L^1

non-statistical then according to the definitions it is L^1 statistically unstable, but the existence of L^1 statistically unstable maps in a family of dynamics, does not necessarily imply the existence of L^1 non-statistical maps (see Example 2.4). However the following theorem states that if a family Λ contains sufficient L^1 statistically unstable maps, then we can conclude the existence of L^1 non-statistical maps within that family.

THEOREM E. *The L^1 non-statistical maps form a Baire generic subset of the interior of L^1 statistically unstable maps.*

The version of almost sure (non-)convergence of the above definitions and results is very similar to the L^1 version and we avoid explaining it in the introduction.

Our initial motivation for developing an abstract setting, was to capture those properties of the family of rational maps that imply the existence of non-statistical behavior within this family, and develop a setting that allows us to prove the existence of non-statistical dynamics in the other families having the same properties. But surprisingly, the theorems and lemmas which are proved in this abstract setting, turned out to have some applications in the world of conservative dynamics, where we know there is no non-statistical dynamics.

Application to Conservative Dynamics. . Let X be a smooth and compact manifold and $\Lambda \subset \text{Diff}_{Leb}^r(X)$. To each map $f \in \Lambda$ we can associate the ergodic decomposition $\hat{\mu}_f \in \mathcal{M}_1(\mathcal{M}_1(X))$ of the Lebesgue measure. Observe that

$$\hat{\mu}_f = \lim_n \hat{e}_n^f.$$

Using Theorem D we can conclude the following result regarding the continuity of the ergodic decomposition with respect to the dynamics. We should note that this theorem has been proved previously by Avila and Bochi in [7] but our approach for proof is different.

THEOREM (Avila-Bochi [7, thm B]). *A generic $f \in \text{Diff}_{Leb}^r$ is a continuity point of the map $f \mapsto \hat{\mu}_f$.*

Questions: This memoir provides tools to study the statistical behaviour of generic dynamical systems in an abstract class. When the class of dynamical systems is formed by dissipative C^r -diffeomorphisms of a compact manifold, this study is traditionally related to the notion of physical measures. We recall that an invariant probability measure ν is *physical* if its basin $B_\nu := \{x \in M : e_n(x) \rightarrow \nu\}$ has positive Lebesgue measure.

Question 0.1. *For $r \geq 2$, is it true for generic f in $\text{Diff}^r(M)$ that the union of the basins of the physical measures of f has full Lebesgue measure in M ?*

This question has been asked by Wilkinson and Shub in [40], but was in the mind of several other people (see [35], [10] and [37]).

Let us now relax the conditions on physical measures and develop some questions on the abundance of non-statistical dynamics.

It is known that the set of non statistical real quadratic maps has measure zero [30]. The following is still open:

Question 0.2. *Is the set of non-statistical rational maps of positive Lebesgue measure?*

We proved that non-statistical rational maps are generic in the maximal bifurcation locus, however the maximal bifurcation locus is a nowhere dense subset of the space of rational maps [31]. In a broader context we can ask:

Question 0.3. *Is there any non-trivial family of dynamics having a positive measure subset of non-statistical maps?*

Question 0.4. *Is there any open subset of dynamics in which the non-statistical maps are generic? In Newhouse domains?*

These questions are related to the following:

Question 0.5 (Takens' last problem, [44]). *Can non-statistical dynamics exist persistently within a non-trivial class of smooth dynamical systems?*

Organization of the thesis. The first section of the first part is devoted to some basic definitions and examples. In the next section we introduce the L^1 version of statistical (in)stability and non-statistical dynamics where theorem E is stated as the main theorem of this section. In section 3 we introduce the essential convergence version of what we have done in section 2. Theorems E' is the counterpart of theorem E. Section 4 of this note is devoted to the version of convergence in law. In the beginning of this section the set $B_{\Lambda, f}$ is introduced and some of its basic properties is studied. The main lemma in this section is an basic observation regarding the continuity and stability of statistical behavior of a generic map. Theorem D then is a consequence of this lemma together with the nice properties of the set $B_{\Lambda, f}$. In the rest of this section we (re)define the notion of maximally oscillating maps and prove theorem F and proposition 4.19 in this direction. At the end

of this section is theorem **A** which is a generalization of the result of Hofbauer and Keller [23]. Theorem **C** in section 5 is a result regarding the existence of non-statistical maps in the family of Anosov-Katok maps of the annulus. In the last section of the first part we give a comparison between different versions introduced in previous sections. The second part of this note is devoted to prove theorem **B**. The proof of this theorem is based on propositions 7.2 and 7.3 which are proved respectively in sections 8 and 9.

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I started to be advised by Meysam Nassiri in Iran in 2013. He helped me to have journey in different areas of dynamical systems to learn different subjects and also find my own interest for my PhD thesis. In this period I learned about different areas of dynamical systems among which I can name topological dynamics, dynamics of surface diffeomorphisms and complex dynamics. The latter had an important effect in my later research experience. He always encouraged me to participate in international mathematics conferences and socialize with other experts. In one of the conferences that was holding in Banach Center in Poland, I met Pierre Berger and he introduced me the concept of non-statistical dynamics and the problem of whether such dynamics exists in some explicit family of maps including rational maps on the Riemann sphere. This was the beginning of our joint degree program. Pierre brought me to the subject by giving me a list of papers that I read before I officially start my joint PhD in Paris in 2016. Then I started to work on the problem of the existence of non-statistical rational maps. Pierre thought me a geometric approach to answer the counterpart of this problem in one-dimensional real dynamics. This opened my eye to find a proof for existence of non-statistical maps in the family of rational maps. This was done during the first year of our joint program when I had several trips to Paris. The next step of the thesis was an attempt to formalize an abstract setting for dealing with non-statistical dynamics. It was a precious experience for me to do this under guidance of Pierre who asked me to develop more and more my first draft on this subject. Indeed his insistence on having a more general and mature section on the abstract setting resulted me to reach a version of the concept of statistical (in)stability that is very

satisfying to me and I think it is an starting point for my future researches. This was what I did not guess before I tried to change my first draft on the abstract setting. Pierre also gave me the idea of how it may be possible to have non-statistical dynamics within the family of Anosov-Katok diffeomorphisms. I am so thankful to him because of all he did for me during this thesis. He was so available both during the time that I spent in Paris and we had regular meetings and also when I was in Iran and we had online meetings. In the time of my stay in Iran I had the opportunity of having discussions on my subject with Meysam who eagerly listened to all my ideas and progresses and was always very supportive to me. I was working on another project under supervision of Meysam in Iran (which is not included in this thesis) that I feel it had indirect effect on me to have Baire generic results in the present note. He read my notes and helped me to develop it. I also made use of his comments in the section of abstract setting as well as in the other parts of the thesis. Once more I express my deepest thanks to both of my advisors.

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Part 1

Statistical instability and non-statistical dynamics

1. preliminaries

Let X be a compact metric space endowed with a reference (Borel) probability measure μ and Λ a subset of continuous self-mappings of X endowed with a topology finer than C^0 topology. For instance Λ can be a subset of C^r self-mappings of a smooth manifold, endowed with C^r topology and μ be a probability measure whose density w.r.t. a Lebesgue measure is a smooth positive function. For a compact metric space (X, d) , Let us denote the space of probability measures on X by $\mathcal{M}_1(X)$. This space can be endowed with weak-* topology which is metrizable, for instance with *Wasserstein metric* where the distance d_w between two probability measures ν_1 and ν_2 is defined as below:

$$d_w(\nu_1, \nu_2) := \inf_{\zeta \in \pi(\nu_1, \nu_2)} \int_{X \times X} d(x, y) d\zeta ,$$

where $\pi(\nu_1, \nu_2)$ is the set of all probability measures on $X \times X$ which their projections on the first coordinate is equal to ν_1 and on the second coordinate is equal to ν_2 . The Wasserstein distance induces the weak-* topology on $\mathcal{M}_1(X)$ and hence the compactness of (X, d) implies that $(\mathcal{M}_1(X), d_w)$ is a compact and complete metric space. We should note that our results and arguments in the rest of this note hold for any other metric inducing the weak-* topology on the space of probability measures.

For a point $x \in X$ and a map $f : X \rightarrow X$, the *empirical measure*

$$e_n^f(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$$

describes the distribution of the orbit of the point x up to the n^{th} iteration in the phase space, which asymptotically may or may not converge. If it converges, then by observing a finite number of iterations (but possibly large) one can predict how the orbit of x behaves approximately for larger iterations, from a statistical point of view. However it may not converge. In this case we fix the following terminology:

Definition 1.1. For a map $f : X \rightarrow X$ we say the orbit of a point x displays *non-statistical behavior*, or briefly x is *non-statistical* if the sequence $\{e_n^f(x)\}_n$ is divergent.

Example 1.2. This example shows the existence of non-statistical points for a well known dynamics; the shift map σ on $X = \{0, 1\}^{\mathbb{Z}}$.

Consider a point $\omega \in X$

$$\omega = 0.\underbrace{0\dots 0}_{n_1}\underbrace{0101\dots 0101}_{n_2}\underbrace{0\dots 0}_{n_3}\dots$$

made by putting together consecutive blocks of zero's and blocks of zero's and one's and suppose the length of i^{th} block is n_i satisfying

$$\lim_{i \rightarrow \infty} \frac{n_i}{n_{i+1}} = 0.$$

Then it can be checked easily that ω is a non-statistical point.

One can ask how large can be the set of points for which the empirical measures does not converge. Can it be of positive measure? Here is an example to answer this question:

Example 1.3 (The Bowen eye). One of the first examples of non-statistical maps was given by Bowen. It is a vector field in the plane with an eye-like region having two saddle fixed points in the corners with two saddle connections as the boundary of this region (see Figure 1). The vector field has a source equilibrium point inside this region and all of the points except this fixed point converge to the boundary. Let us denote the two equilibrium points in the corner by A and B and the unstable and stable eigenvalues of the linearization of the vector field at A by α_+ and $-\alpha_-$ and at B by β_+ and $-\beta_-$. For suitable choices of these numbers, the time one map of the vector field becomes a non-statistical diffeomorphism of \mathbb{R}^2 with respect to the Lebesgue measure restricted to the eye-like region.

Takens introduced in [43] the modulus associated with the upper and lower saddle connection which are denoted respectively by λ and σ . They are defined by

$$\lambda = \alpha_-/\beta_+ \text{ and } \sigma = \beta_-/\alpha_+.$$

The following theorem has been proved first by Gaunersdorfer in [19] and restated by Takens in [43]:

THEOREM. *If g is a continuous function on \mathbb{R}^2 with $g(A) > g(B)$, and $x(t)$ an orbit converging to the cycle, then we have:*

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(x(t)) dt &= \frac{\sigma}{1 + \sigma} g(A) + \frac{1}{1 + \sigma} g(B) \\ \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(x(t)) dt &= \frac{\lambda}{1 + \lambda} g(B) + \frac{1}{1 + \lambda} g(A). \end{aligned}$$

Let us denote the time- t map of the vector field by φ_t .

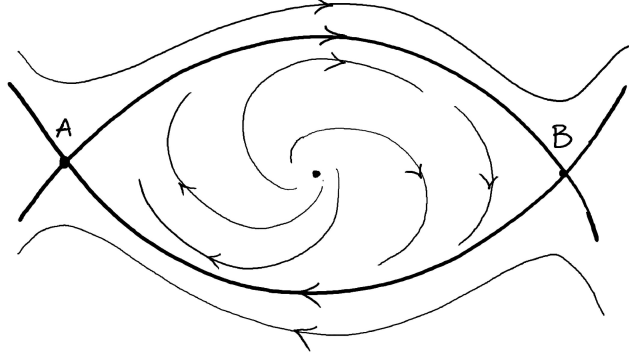


FIGURE 1. Bowen eye

Corollary 1.4. *The diffeomorphism φ_1 is non-statistical with respect to the restriction of the Lebesgue measure to the eye like region.*

PROOF. By suitable choices of the eigenvalues, we can make sure that the limsup and the liminf in the theorem are not equal. In fact this is the case if

$$\alpha_- \beta_- \neq \alpha_+ \beta_+,$$

In this case, the time averages of the map g along the orbit of almost every point in the eye like region oscillates between the limsup and the liminf and so is not convergent. Assume for the sake of contradiction that for a point x_0 in the eye like region, which is not the source, the sequence of empirical measures $\{e_n^{\varphi_1}(x_0)\}_n$ converges to a probability measure ν . So we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} g(z) d(e_n^{\varphi_1}(x_0))(z) = \int_{\mathbb{R}^2} g(x) d\nu.$$

On the other hand, the orbit of the point x_0 spends asymptotically most of its time around two fixed points A and B . This is because for any neighbourhood U of these two points, the time that the orbit of x_0 spends in U in each visit is more than the time it spends in the previous return and the time difference each two visit is uniformly bounded. As

a consequences we conclude that the time averages of the map φ_1 are asymptotically the same as the time averages of the continuous system φ_t and so:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} g(z) d(e_n^{\varphi_1}(x_0))(z) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\varphi_i(x_0)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n g(\varphi_t(x_0)) dt, \end{aligned}$$

which is a contradiction since according to our assumption, the last limit does not exist. So the sequence of empirical measures is not convergent for the point x_0 . \square

2. The L^1 convergence version

We define the n^{th} empirical function of a map f to be the map $e_n^f : X \rightarrow \mathcal{M}_1(X)$ sending a point $x \in X$ to the n^{th} empirical measure $e_n^f(x)$. We are going to study the L^1 (non-)convergence of the sequence of empirical functions. For this purpose, we need to give some definitions:

Let us denote the space of Borel measurable maps from X to $\mathcal{M}_1(X)$ by $L^1(X, \mathcal{M}_1(X))$. Note that since the empirical functions are continuous maps with respect to x , they are elements of $L^1(X, \mathcal{M}_1(X))$. We define a metric on this space where the distance between two elements $e, e' \in L^1(X, \mathcal{M}_1(X))$ is defined as follows:

$$(2.1) \quad d_{L^1}(e, e') = \int_X d_w(e(x), e'(x)) d\mu.$$

Let us study the convergence of the sequence of empirical functions with respect to this metric:

Definition 2.1. We say a map f is L^1 *non-statistical* if the sequence of maps $e_n^f : X \rightarrow \mathcal{M}_1(X)$ is not convergent for the L^1 topology:

$$\limsup_{m, n \rightarrow \infty} d_{L^1}(e_n^f, e_m^f) > 0.$$

In the following we introduce a condition, that if it is satisfied by a Baire space of dynamics Λ , then we can conclude the existence of L^1 non-statistical maps within Λ . Let us first, quantify the extent to which the statistical behavior of a map $f \in \Lambda$ can be changed by small perturbations. To this aim, we propose the following definition:

Definition 2.2. The *amplitude of L^1 statistical divergence w.r.t.* Λ of a map $f \in \Lambda$ is a real valued non-negative mapping which is defined as

follows

$$\Delta^1(f) := \limsup_{\substack{h,g \rightarrow f, m,n \rightarrow \infty \\ h,g \in \Lambda}} d_{L^1}(e_n^h, e_m^g).$$

Note that the definition of Δ^1 depends also on the set Λ , and not only on the map f . However for the sake of simplicity we hid this in the notation.

Observe that if Δ^1 is positive at f then the asymptotic behaviors of dynamics close to f are very sensitive to perturbations and in this sense the map f is statistically unstable. We introduce the following definition:

Definition 2.3. A map $f \in \Lambda$ is L^1 statistically unstable with respect to Λ if its amplitude of L^1 statistical divergence w.r.t. Λ is positive:

$$\Delta^1(f) > 0.$$

Example 2.4. Suppose Λ is the set of rigid rotations on \mathbb{S}^1 . The identity map $Id_{\mathbb{S}^1} \in \Lambda$ is L^1 statistically unstable, since the empirical measures of all of its points are atomic whereas we can approach the map $Id_{\mathbb{S}^1}$ by irrational rotations, and the empirical measures of any point are close to the Lebesgue measure for large enough iterations.

Now we want to investigate the relationship between L^1 statistical instability and existence of L^1 non-statistical maps. It is clear that if a map f is L^1 non-statistical then $\Delta^1(f) > 0$ and so f is L^1 statistically unstable, but the existence of a L^1 statistically unstable map does not necessarily imply the existence of L^1 non-statistical maps (see Example 2.4). However, if the interior of L^1 statistically unstable maps is non-empty, then the existence of “plenty” of L^1 non-statistical maps is guaranteed. We recall that Λ is a Baire space and its topology is finer than C^0 -topology.

THEOREM E. *The L^1 non-statistical maps form a Baire generic subset of the interior of L^1 statistically unstable maps.*

Before proving this theorem, let us give a particular application of this theorem:

Corollary 2.5. *If a Baire generic map $f \in \Lambda$ is L^1 -statistically unstable, then a generic map in Λ is L^1 non-statistical.*

PROOF. □

The proofs of Theorem E and corollary 2.5 use the following lemma:

Lemma 2.6. *The map Δ^1 is upper semi-continuous.*

PROOF. Let $\{f_k\}_k$ be a sequence of maps converging to f . For each k we can find two natural numbers n_k and m_k and two maps g_k and h_k near f_k such that

$$|\Delta^1(f_k) - d_{L^1}(e_{n_k}^{g_k}, e_{m_k}^{h_k})| < \frac{1}{k}.$$

Note that we can choose the sequences $\{n_k\}_k$ and $\{m_k\}_k$ both converging to infinity and also the sequence of maps $\{g_k\}_k$ and $\{h_k\}_k$ both converging to f . So we obtain

$$\Delta^1(f) \geq \limsup_k \Delta^1(f_k),$$

and this implies the upper semi-continuity of Δ^1 . \square

PROOF OF COROLLARY 2.5. If a generic map $f \in \Lambda$ is statistically unstable, then by definition $\Delta^1(f) > 0$. On the other hand since Δ^1 is semi-continuous, a generic map in Λ is a continuity point for it. Hence generic $f \in \Lambda$ has a neighbourhood in which all of the maps are statistically unstable. So there is an open and dense subset of statistically unstable maps and hence by theorem E, a generic map in Λ is non-statistical. \square

PROOF OF THEOREM E. Since Δ^1 is upper semi-continuous, there is a generic subset $\mathcal{G} \subset \Lambda$ on which the map Δ^1 is continuous. For a map $f \in \mathcal{G}$ which is also in the interior of L^1 statistically unstable maps, there exists a neighborhood $\mathcal{U}_f \subset \Lambda$ around f on which Δ^1 is uniformly positive:

$$(2.2) \quad \exists d > 0 \quad s.t. \quad \forall g \in \mathcal{U}_f, \quad \Delta^1(g) > d.$$

Now we construct a sequence of open and dense subsets in \mathcal{U}_f such that any map in the intersections of these sets is L^1 non-statistical. This will imply that L^1 non-statistical maps are Baire generic in \mathcal{U}_f and hence the L^1 non-statistical maps are locally generic in the interior of L^1 statistically unstable maps.

To construct such open and dense sets, first note that the function Δ^1 can be written as

$$\Delta^1(f) = \limsup_{g, h \rightarrow f, N \rightarrow \infty} \Delta_N^1(h, g),$$

where $\Delta_N^1(h, g) = \sup_{i, j \geq N} d_{L^1}(e_i^h, e_j^g)$.

Claim 2.7. The map Δ_N^1 is lower semi-continuous.

PROOF. Note that

$$\Delta_N^1(h, g) = \sup_{M \geq N} \left\{ \sup_{N \leq i, j \leq M} d_{L^1}(e_i^h, e_j^g) \right\}.$$

But $\sup_{N \leq i, j \leq M} d_{L^1}(e_i^h, e_j^g)$ is continuous with respect to (h, g) , and supremum of a sequence of continuous functions is lower semi-continuous. \square

Next we show that for any $N \in \mathbb{N}$, the set

$$E(N) := \{h \in \mathcal{U}_f \mid \Delta_N^1(h, h) > \frac{d}{3}\},$$

is an open and dense subset of \mathcal{U}_f and moreover every map in the intersection $\bigcap_N E(N)$ is L^1 non-statistical.

The openness of $E(N)$ is guaranteed by lower semi-continuity of Δ_N^1 . Now we prove the denseness of $E(N)$. For any arbitrary map $h \in \mathcal{U}_f$, take a neighborhood V_h such that for any map $g \in V_h$ we have:

$$(2.3) \quad d_{L^1}(e_N^h, e_N^g) < \frac{d}{3}.$$

This is possible because N is fixed and e_N^g depends continuously on g . By 2.2 we know that $\Delta^1(h) > d$, and so we can choose $g_1, g_2 \in V_h$ such that for some integers $n, m > N$ it holds true that

$$(2.4) \quad d_{L^1}(e_n^{g_1}, e_m^{g_2}) > d.$$

But note that

$$d_{L^1}(e_n^{g_1}, e_m^{g_2}) \leq d_{L^1}(e_n^{g_1}, e_N^{g_1}) + d_{L^1}(e_N^{g_1}, e_N^{g_2}) + d_{L^1}(e_N^{g_2}, e_m^{g_2}).$$

Inequalities 2.3 and 2.4 imply that either

$$d_{L^1}(e_n^{g_1}, e_N^{g_1}) > \frac{d}{3} \text{ or } d_{L^1}(e_m^{g_2}, e_N^{g_2}) > \frac{d}{3}.$$

So at least one of the maps g_1 or g_2 is inside $E(N)$, and then recalling that h was chosen arbitrarily in \mathcal{U}_f and V_h , we conclude that $E(N)$ is dense in \mathcal{U}_f .

Now observe that for any map $h \in \bigcap_{N=1}^{\infty} E(N)$, the sequence of empirical functions is not a Cauchy sequence and hence h is L^1 non-statistical. The set $\bigcap_{N=1}^{\infty} E(N)$ is a Baire generic set in the open neighbourhood \mathcal{U}_f , so the L^1 non-statistical maps are generic in the set \mathcal{U}_f . Considering the fact that f is an arbitrary map in the generic set \mathcal{G} we can then conclude that L^1 non-statistical maps are indeed a generic subset of the interior of L^1 statistically unstable maps. \square

Definition 2.8. We say a map $f \in \Lambda$ is L^1 statistically stable w.r.t. Λ if $\Delta^1(f) = 0$.

Corollary 2.9. If Λ contains no L^1 non-statistical map, then a Baire generic map in Λ is L^1 statistically stable.

PROOF. Let $\mathcal{G} \subset \Lambda$ be the set of the continuity points of Δ^1 which is a Baire generic set as a consequence of Lemma 2.6. We can decompose \mathcal{G} into two subsets: the set \mathcal{G}_0 of maps $f \in \mathcal{G}$ such that $\Delta^1(f) = 0$, and the set \mathcal{G}_+ of maps $f \in \mathcal{G}$ such that $\Delta^1(f) > 0$. The set \mathcal{G}_+ is an open set and also it is in the interior of L^1 statistical unstable maps. Theorem E implies the existence of a generic subset of L^1 non-statistical maps in the interior of L^1 statistical unstable maps. But by our assumption, there is no L^1 non-statistical map in Λ . This implies that the interior of L^1 statistical unstable maps is empty and hence the set \mathcal{G}_+ is empty as well. So \mathcal{G}_0 is equal to \mathcal{G} , which is generic in Λ . \square

Let us introduce an application of Corollary 2.9 in the world of conservative dynamics where there is no possibilities to have L^1 non-statistical maps:

Corollary 2.10. *Suppose Λ is a set of μ -preserving dynamics, then a generic map $f \in \Lambda$ is L^1 statistically stable.*

PROOF. By Birkoff ergodic theorem, for any map $f \in \Lambda$, the sequence $\{e_n^f\}_n$ is L^1 convergent, so we have no L^1 non-statistical maps in Λ , and hence by corollary 2.9, a generic map in Λ is L^1 statistically stable. \square

3. The essential convergence version

In the previous section, we introduced the notion of L^1 statistical instability and investigated the relationship between this notion and existence of L^1 non-statistical maps. In this section we consider the pointwise (non-)convergence of the empirical functions $e_n^f : X \rightarrow \mathcal{M}_1(X)$ instead of L^1 (non)convergence. We show that the same statements hold true for pointwise convergence version of the results in the previous section, however, the arguments are a little bit more technical.

Definition 3.1 (Non-statistical dynamics). A map $f : X \rightarrow X$ is called *non-statistical* if the set of points that have non-statistical behavior is of positive measure.

First let us quantify how different is the statistical behavior of two arbitrary maps $h, g \in \Lambda$ for iterations larger than a fixed number $N \in \mathbb{N}$. To this aim we propose the following map Δ_N^e that associates to a couple of maps $h, g \in \Lambda$ a non-negative real number:

$$\Delta_N^e(h, g) := \int_X \sup_{N \leq n, m} d_w(e_n^h(x), e_m^g(x)) d\mu,$$

which can be interpreted as the average of the maximum difference between statistical behaviors that the orbit of a point can display under iterations of h and g for iterations larger than N . Note that Δ_N^e is not a distance. In particular if f is a non-statistical map, then $\Delta_N^e(f, f)$ is uniformly positive for every N :

Lemma 3.2. *A map f is non-statistical if and only if there is a real number $d > 0$ such that for each $N \in \mathbb{N}$ we have $\Delta_N^e(f, f) > d$.*

PROOF. Let f be a non-statistical map, and let $x \in X$ be a non-statistical point. Since the sequence of empirical measures of this point does not converge,

$$d_x := \inf_{N > 0} \sup_{N \leq n, m} d_w(e_n^f(x), e_m^f(x)) > 0$$

By definition, the set of non-statistical points has positive measure and $x \mapsto d_x$ is measurable, thus

$$\Delta_N^e(f, f) = \int_X \sup_{N \leq n, m} d_w(e_n^f(x), e_m^f(x)) d\mu \geq \int_X d_x d\mu > 0.$$

To prove the other side let f be a map for which the sequence of empirical measures of almost every point converges. So for almost every $x \in X$ we have

$$\lim_{N \rightarrow \infty} \sup_{N \leq n, m} d_w(e_n^f(x), e_m^f(x)) = 0.$$

Since the distance between empirical measures is bounded, we can then use Lebesgue dominated convergence theorem, to conclude

$$\begin{aligned} \lim_{N \rightarrow \infty} \Delta_N^e(f, f) &= \lim_{N \rightarrow \infty} \int_X \sup_{N \leq n, m} d_w(e_n^f(x), e_m^f(x)) d\mu \\ &= \int_X \lim_{N \rightarrow \infty} \sup_{N \leq n, m} d_w(e_n^f(x), e_m^f(x)) d\mu = 0. \end{aligned}$$

This finishes the proof. \square

We recall that Λ is a Baire space of maps endowed with a topology finer than the C^0 -topology. Now like the previous section, we quantify the difference in the statistical behaviors of maps converging to $f \in \Lambda$. For this purpose, we introduce the following definition:

Definition 3.3. The *amplitude of essential statistical divergence* of a map $f \in \Lambda$ is defined as bellow

$$\Delta^e(f) := \limsup_{\substack{h, g \rightarrow f, m, n \rightarrow \infty \\ h, g \in \Lambda}} \Delta_N^e(h, g).$$

Observe that if Δ^e is positive at f then the asymptotic behaviors of nearby maps are very sensitive to perturbations of f and so the map f is unstable from the statistical view point.

Definition 3.4. A map $f \in \Lambda$ is *statistically unstable with respect to* Λ if $\Delta^e(f) > 0$.

Example 3.5. Let X be the circle \mathbb{S}^1 , μ the normalized Lebesgue measure and Λ be the set of rotations of the circle. The map $f = Id_{\mathbb{S}^1}$ is statistically unstable, since the empirical measures of any point are the Dirac mass at that point, but for any arbitrarily close irrational rotation the sequence of empirical measures converges to the Lebesgue measure.

Example 3.6. Let X be the Riemann sphere $\hat{\mathbb{C}}$ and μ its normalized Lebesgue measure. Consider the set of quadratic maps:

$$\Lambda = \{f_c : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \mid f_c(x) = x^2 + c \text{ for } x \in \mathbb{C}, f_c(\infty) = \infty\}.$$

The map $f_{\frac{1}{4}}$ has a fixed point at $x = \frac{1}{2}$ which attracts the points in a non-empty open set U . For any $\varepsilon > 0$, the map $f_{\frac{1}{4}+\varepsilon}$ has a different dynamics: almost every point goes to infinity. So for any $\varepsilon > 0$ and every point $x \in U$ the sequence of empirical measures converges to $\delta_{\frac{1}{2}}$ and δ_∞ under iterating the maps $f_{\frac{1}{4}}$ and $f_{\frac{1}{4}+\varepsilon}$ respectively.

Hence the supremum in the definition of Δ_N^e is at least $d_w(\delta_{\frac{1}{2}}, \delta_\infty)$ for almost every point. So $\Delta_N^e(f_{\frac{1}{4}}, f_{\frac{1}{4}+\varepsilon}) > \mu(U)d_w(\delta_{\frac{1}{2}}, \delta_\infty)$, which is independent of ε and N . According to the definition, this means that $f_{\frac{1}{4}}$ is statistically unstable.

Proposition 3.7. *The map $\Delta^e : \Lambda \rightarrow \mathbb{R}$ is upper semi-continuous.*

PROOF. We recall that

$$\Delta^e(f) := \limsup_{h, g \rightarrow f, N \rightarrow \infty} \Delta_N^e(h, g).$$

Now let $\{f_k\}_k$ be a sequence of maps converging to f . For each k we can find a natural number N_k and two maps g_k and h_k near f_k such that

$$|\Delta^e(f_k) - \Delta_{N_k}^e(g_k, h_k)| < \frac{1}{k}.$$

Note that we can choose the sequence $\{N_k\}_k$ such that it converges to infinity and also the sequence of maps $\{g_k\}_k$ and $\{h_k\}_k$ such that both converge to f . So we obtain

$$\Delta^e(f) \geq \limsup_k \Delta^e(f_k),$$

and this implies the upper semi-continuity of Δ^e . \square

Now we want to investigate the relationship between statistical instability and existence of non-statistical maps. The following theorem is the counterpart of Theorem E:

THEOREM E'. *The non-statistical maps form a Baire generic subset of the interior of statistically unstable maps.*

PROOF. Since by Proposition 3.7 the map Δ^e is an upper semi-continuous map, there is a Baire generic set \mathcal{G} on which the map Δ^e is continuous. For a map $f \in \mathcal{G}$, which is also in the interior of statistically unstable maps, there exists a neighborhood \mathcal{U}_f around f on which Δ^e is uniformly positive:

$$\exists d > 0 \quad \text{s.t.} \quad \forall g \in \mathcal{U}_f, \quad \Delta^e(g) > d.$$

Now we construct a sequence of open and dense subsets in \mathcal{U}_f such that any map in the intersection of these sets is non-statistical. And then we can conclude that non-statistical maps are Baire generic in \mathcal{U}_f .

To construct such open and dense sets we need to show a semi-continuity property of the map Δ_N^e :

Lemma 3.8. *The map Δ_N^e is lower semi-continuous.*

PROOF. We recall that for $h, g \in \Lambda$

$$\Delta_N^e(h, g) := \int_X \sup_{N \leq n, m} d_w(e_n^h(x), e_m^g(x)) d\mu,$$

Now note that

$$\Delta_N^e(h, g) = \sup_{N \leq M} \int_X \sup_{N \leq n, m \leq M} d_w(e_n^h(x), e_m^g(x)) d\mu,$$

and we know that $\int_X \sup_{N \leq n, m \leq M} d_w(e_n^h(x), e_m^g(x)) d\mu$ is continuous with respect to h and g and supremum of a sequence of continuous functions is lower semi-continuous so we are done. \square

Next we show that for any $N \in \mathbb{N}$, the set

$$E(N) := \{h \in \mathcal{U}_f \mid \Delta_N^e(h, h) > \frac{d}{3}\},$$

is an open and dense subset of \mathcal{U}_f and moreover every map in the intersection $\bigcap_N E(N)$ is non-statistical.

The openness of $E(N)$ is guaranteed by lower semi-continuity of Δ_N^e . Now we prove the denseness of $E(N)$. For any arbitrary map

$h \in \mathcal{U}_f \subset \Lambda$, take a neighborhood V_h such that for any map $g \in V_h$ and any $x \in X$:

$$(3.1) \quad d_w(e_N^h(x), e_N^g(x)) < \frac{d}{3}.$$

This is possible because N is fixed, e_N^g depends continuously on g and X is compact. Now since $\Delta^e(h) > d$, we can choose $g_1, g_2 \in V_h$ such that for some integer $M > N$ it holds true that

$$(3.2) \quad \Delta_M^e(g_1, g_2) > d.$$

But since $\Delta_N^e(g_1, g_2)$ is decreasing in N we obtain:

$$(3.3) \quad \Delta_N^e(g_1, g_2) > d.$$

Now note that for each $x \in X$ we have

$$\begin{aligned} & \sup_{N \leq n, m} d_w(e_n^{g_1}(x), e_m^{g_2}(x)) \leq \\ & \sup_{N \leq n, m} d_w(e_n^{g_1}(x), e_m^{g_1}(x)) + \sup_{N \leq n, m} d_w(e_n^{g_2}(x), e_m^{g_2}(x)) + d_w(e_N^{g_1}(x), e_N^{g_2}(x)), \end{aligned}$$

and hence after integrating with respect to μ and using inequality [3.1](#) we obtain:

$$\Delta_N^e(g_1, g_2) \leq \Delta_N^e(g_1, g_1) + \Delta_N^e(g_2, g_2) + \frac{d}{3}.$$

Now using inequality [3.3](#) we conclude that at least one of the maps g_1 and g_2 is inside $E(N)$, and then recalling that h was chosen arbitrarily in \mathcal{U}_f and V_h arbitrary small, we conclude that $E(N)$ is dense in \mathcal{U}_f .

Observe that Lemma [3.2](#) implies that any map h in the set $\bigcap_{N=1}^{\infty} E(N)$, which is a Baire generic set inside \mathcal{U}_f , is non-statistical. So the non-statistical maps are generic in some open neighbourhood of f and f is an arbitrary map in the interior of statistically unstable maps intersected by the generic set \mathcal{G} . This implies that non-statistical maps are indeed a generic subset of the interior of statistically unstable maps as well. □

Definition 3.9. We say a map $f \in \Lambda$ is statistically stable with respect to Λ if $\Delta^e(f) = 0$.

Corollary 3.10. *If Λ contains no non-statistical map, then a generic map in Λ is statistically stable.*

Note that being statistically stable is a stronger condition than being L^1 statistically stable. So the conclusion of the previous theorem,

is a stronger version of the the conclusion in Corollary 3.10. We omit the proof of this theorem as well as the applications in the conservative setting since they are identical to what we had in the previous section.

4. The version of convergence in law

For a dynamical system $f : X \rightarrow X$ the map $e_n^f : X \rightarrow \mathcal{M}_1(X)$ associates to each point $x \in X$, its n^{th} empirical measure. Different points usually have different empirical measures. We can investigate how the empirical measures $e_n^f(x)$ are distributed in $\mathcal{M}_1(X)$ with respect to the reference measure μ on X and what is the asymptotic behavior of these distributions. To this aim, we can push forward the measure μ to the set of probability measures on X using the map e_n^f :

$$\hat{e}_n(f) := (e_n^f)_*(\mu).$$

The measure $\hat{e}_n(f)$ is a probability measure on the space of probability measures on X . We denote the space of probability measures on the space of probability measures by $\mathcal{M}_1(\mathcal{M}_1(X))$. We denote the Wasserstein metric on this space by \hat{d} . Note that the compactness of X implies the compactness of $\mathcal{M}_1(X)$ and hence the compactness of $\mathcal{M}_1(\mathcal{M}_1(X))$. So the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ lives in a compact space and have one or possibly more than one accumulation points.

Example 4.1. For any μ preserving map $f : X \rightarrow X$, the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ converges to a measure $\hat{\mu}$ which is the ergodic decomposition of the measure μ .

Example 4.2. If ν is a physical measure for the map $f : X \rightarrow X$ whose basin covers μ -almost every point, the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ converges to the Dirac mass concentrated on the point $\mu \in \mathcal{M}_1(X)$, which we denote by δ_μ .

In the next section we will see examples of the maps for which the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ does not converge. The following lemma provides some information about the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$:

Lemma 4.3. *For any $f \in \Lambda$ and any $n \in \mathbb{N}$ it holds true that*

$$\hat{d}_w(\hat{e}_n^f, \hat{e}_{n+1}^f) < \frac{\text{diam}(X)}{n+1},$$

where $\text{diam}(X)$ is the diameter of the space X .

PROOF. We recall that

$$\hat{e}_n^f = (e_n^f)_*(\mu).$$

First let us show for any $x \in X$ the following inequality holds true independent of the choice of $f \in \Lambda$:

$$d_w(e_n^f(x), e_{n+1}^f(x)) < \frac{\text{diam}(X)}{n+1}.$$

So according to the definition we should show that

$$\inf_{\gamma \in \pi(e_n^f(x), e_{n+1}^f(x))} \int_{X \times X} d(x, y) d\gamma(x, y) < \frac{\text{diam}(X)}{n+1}.$$

Consider the following element of $\pi(e_n^f(x), e_{n+1}^f(x))$:

$$\gamma = \frac{1}{n+1} \sum_{0 \leq i \leq n-1} \delta_{(f^i(x), f^i(x))} + \frac{1}{n(n+1)} \sum_{0 \leq i \leq n-1} \delta_{(f^i(x), f^n(x))}.$$

Note that

$$(\pi_1)_*(\gamma) = e_n^f(x) = \frac{1}{n} \sum_{0 \leq i \leq n-1} \delta_{f^i(x)},$$

$$(\pi_2)_*(\gamma) = e_{n+1}^f(x) = \frac{1}{n+1} \sum_{0 \leq i \leq n} \delta_{f^i(x)},$$

where π_1 and π_2 are the projection on the first and second coordinates.

So we have $\gamma \in \pi(e_n^f(x), e_{n+1}^f(x))$ and hence

$$\begin{aligned} d_w(e_n^f(x), e_{n+1}^f(x)) &\leq \int_{X \times X} d(x, y) d\gamma(x, y) \\ &= \sum_{0 \leq i \leq n-1} \frac{1}{n(n+1)} d(f^i(x), f^n(x)) \\ &\leq \frac{\text{diam}(X)}{n+1}. \end{aligned}$$

Now consider the following measure on $\mathcal{M}_1(\mathcal{M}_1(X)) \times \mathcal{M}_1(\mathcal{M}_1(X))$:

$$\hat{\gamma} = \int_X \delta_{(e_n^f(x), e_{n+1}^f(x))} d\mu.$$

Obviously $\hat{\gamma} \in \pi(\hat{e}_n^f, \hat{e}_{n+1}^f)$, and so

$$\begin{aligned} \hat{d}_w(\hat{e}_n^f, \hat{e}_{n+1}^f) &\leq \int_{\mathcal{M}_1(\mathcal{M}_1(X)) \times \mathcal{M}_1(\mathcal{M}_1(X))} d(x, y) d\hat{\gamma}(x, y) \\ &= \int_X d_w(e_n^f(x), e_{n+1}^f(x)) d\mu \leq \frac{\text{diam}(X)}{n+1}. \end{aligned}$$

□

Now let Λ be a Baire space of self-mappings of X endowed with a topology finer than C^0 -topology. For each $f \in \Lambda$ the accumulation points of the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ form a compact subset of

$\mathcal{M}_1(\mathcal{M}_1(X))$ which we denote it by $acc(\{\hat{e}_n(f)\}_{n \in \mathbb{N}})$. This set can vary dramatically by small perturbations of f in Λ :

Example 4.4. Let Λ be the set of rigid rotations on \mathbb{S}^1 and Lebesgue as the reference measure. For the identity map id on \mathbb{S}^1 , the sequence $\{\hat{e}_n(id)\}_{n \in \mathbb{N}}$ is a constant sequence. Indeed for every $n \in \mathbb{N}$ we have:

$$(4.1) \quad \hat{e}_n(f) = \int_{\mathbb{S}^1} \delta_{\delta_x} dLeb.$$

So $acc(\{\hat{e}_n(f)\}_{n \in \mathbb{N}})$ is equal to $\{\int_{\mathbb{S}^1} \delta_{\delta_x} dLeb\}$. But for any irrational rotation R_θ (arbitrary close to the identity map), the sequence $\{\hat{e}_n(R_\theta)\}_{n \in \mathbb{N}}$ converges to δ_{Leb} which is a different accumulation point.

In the previous example, for an irrational rotation R_θ close to the identity map, the empirical measures of almost every point start to go toward the Lebesgue measure, and hence the sequence $\{\hat{e}_n(R_\theta)\}_{n \in \mathbb{N}}$ goes toward δ_{Leb} . To study the same phenomenon for the other dynamical systems, we propose the following definition. We recall that Λ is a Baire space of self-mappings of X endowed with a topology finer than C^0 -topology and μ is a reference measure on X .

Definition 4.5. For a map $f \in \Lambda$ and a probability measure $\hat{\nu} \in \mathcal{M}_1(\mathcal{M}_1(X))$, we say f *statistically bifurcates toward $\hat{\nu}$ through perturbations in Λ* , if there is a sequence of maps $\{f_k\}_k$ in Λ converging to f and a sequence of natural numbers $\{n_k\}_k$ converging to infinity such that the sequence $\{\hat{e}_{n_k}(f_k)\}_k$ converges to $\hat{\nu} \in \mathcal{M}_1(\mathcal{M}_1(X))$.

For the sake of simplicity, when the space Λ in which we are allowed to perturb our dynamics is fixed, we say f statistically bifurcates toward $\hat{\nu}$.

For any map $f \in \Lambda$, by $\mathcal{B}_{\Lambda, f}$ we denote the set of those measures $\hat{\nu} \in \mathcal{M}_1(\mathcal{M}_1(X))$ that f statistically bifurcates toward $\hat{\nu}$ through perturbations in Λ .

Remark 4.6. By definition, it holds true that

$$acc(\{\hat{e}_n^f\}_n) \subset \mathcal{B}_{\Lambda, f}.$$

Here are some nice properties of the set $\mathcal{B}_{\Lambda, f}$:

Lemma 4.7. *The set $\mathcal{B}_{\Lambda, f}$ is a compact subset of $\mathcal{M}_1(\mathcal{M}_1(X))$.*

PROOF. By the definition it is clear that the set $\mathcal{B}_{\Lambda, f}$ is closed. The compactness is a consequence of compactness of $\mathcal{M}_1(\mathcal{M}_1(X))$. \square

Lemma 4.8. *The set $\mathcal{B}_{\Lambda, f}$ is a connected subset of $\mathcal{M}_1(\mathcal{M}_1(X))$.*

PROOF. For the sake of contrary assume that $B_{\Lambda, f}$ is not connected, and can be decomposed to two non-empty disjoint closed sets A and B . Therefore there is some real number $d > 0$ such that $\hat{d}_w(A, B) > d$. Take two elements $\hat{\nu} \in A$ and $\hat{\eta} \in B$ and let N in lemma 4.3 is chosen so that

$$\forall n > N, \hat{d}_w(\hat{e}_n^f, \hat{e}_{n+1}^f) < \frac{d}{3}.$$

We can find a neighborhood U of f so that

$$\forall g \in U, \hat{d}_w(\hat{e}_n^f, \hat{e}_n^g) < \frac{d}{3}.$$

This is possible since the map sending f to \hat{e}_N^f is continuous. Now take two maps $h, g \in U$ such that for some integers $n_1, n_2 > N$ it holds true that

$$\hat{d}_w(\hat{e}_{n_1}^g, \hat{\nu}) < \frac{d}{3} \quad \text{and} \quad \hat{d}_w(\hat{e}_{n_2}^h, \hat{\eta}) < \frac{d}{3}.$$

Consider the following sequence of elements of $\mathcal{M}_1(\mathcal{M}_1(X))$:

$$\hat{\nu}, \hat{e}_{n_1}^g, \hat{e}_{n_1-1}^g, \dots, \hat{e}_N^g, \hat{e}_N^h, \dots, \hat{e}_{n_2-1}^h, \hat{e}_{n_2}^h, \hat{\eta}.$$

The distance between two consecutive elements of this sequence is less than $\frac{d}{3}$, and hence there is an element of this sequence which lies out of $\frac{d}{3}$ neighborhood of $A \cup B = B_{\Lambda, f}$. By taking N larger, we obtain another element of $\mathcal{M}_1(\mathcal{M}_1(X))$ out of $\frac{d}{3}$ neighborhood of $B_{\Lambda, f}$. So there is a sequence like $\hat{e}_{n_k}^{f_k}$ out of $\frac{d}{3}$ neighborhood of $B_{\Lambda, f}$ and because of the compactness of $\mathcal{M}_1(\mathcal{M}_1(X))$ this sequence has a convergent subsequence converging to an element out of $B_{\Lambda, f}$. By definition any accumulation point of this sequence is an element of $B_{\Lambda, f}$ which is a contradiction. \square

Lemma 4.9. *For any $\hat{\nu} \in \mathcal{B}_{\Lambda, f}$, any measure ν in the support of $\hat{\nu}$ is invariant under iteration of f .*

PROOF. By definition there is a sequence of maps $\{f_k\}_k$ in Λ converging to f and a sequence of natural numbers $\{n_k\}_k$ converging to infinity such that

$$\lim_{k \rightarrow \infty} \hat{d}_w(\hat{e}_{n_k}(f_k), \hat{\nu}) = 0.$$

If ν is in the support of $\hat{\nu}$ then for any neighbourhood \mathcal{U} of ν and for k large enough, we have

$$\hat{e}_{n_k}(f_k)(\mathcal{U}) > 0.$$

Recalling that

$$\hat{e}_{n_k}(f_k)(\mathcal{U}) = \int_X \delta_{e_{n_k}^{f_k}(x)}(\mathcal{U}) d\mu,$$

we conclude that the integrand of the integral above is non-zero for a subset of X with positive measure and hence in particular for each k there is a point $x_k \in X$ such that $e_{n_k}^{f_k}(x_k) \in \mathcal{U}$. Since \mathcal{U} is an arbitrary neighbourhood of ν we can choose x_k such that

$$\lim_{k \rightarrow \infty} e_{n_k}^{f_k}(x_k) = \nu.$$

On the other hand note that for a large k the map f_k is close to the map f so the measure $e_{n_k}^{f_k}(x_k)$ is close to $f_*(e_{n_k}^{f_k}(x_k))$. So we have

$$\lim_{k \rightarrow \infty} d_w(e_{n_k}^{f_k}(x_k), f_*(e_{n_k}^{f_k}(x_k))) = 0,$$

which together with the continuity of f_* imply that $f_*(\nu) = \nu$ and so we are done. \square

The set $\mathcal{B}_{\Lambda, f}$ depends on the choice of the set of dynamics Λ in which we are allowed to perturb the map f . If Λ is replaced by a larger set of maps, then we may have more elements in $\mathcal{M}_1(\mathcal{M}_1(X))$ toward which f statistically bifurcates. This is what we can see in the following example:

Example 4.10. If Λ is the set of rigid rotations of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, the elements of $\mathcal{M}_1(\mathcal{M}_1(X))$ toward which the identity map statistically bifurcates, are exactly the following ones:

$$\hat{\nu}_s := \int_X \delta_{Leb[x, x+s]} dLeb,$$

where $s \in \mathbb{R}$ is arbitrary and $Leb[x, x+s]$ denotes the normalized Lebesgue measure on the interval $[x, x+s]$. When s is larger than one, we choose an interval of length one in the universal cover of \mathbb{S}^1 starting from a point in the fiber above x and we push forward the normalized Lebesgue measure on this interval to the circle by the projection map. When $s = 0$, we set $Leb[x, x]$ to be the Dirac mass on x . To prove that the identity map statistically bifurcates toward $\hat{\nu}_s$, the sequence $\{f_k = R_{\frac{1}{k}}\}_k$ of rotations converging to the identity and the sequence of times $\{n_k = \lfloor sk \rfloor\}_k$ work:

$$\lim_{k \rightarrow \infty} e_{n_k}^{f_k}(x) = Leb[x, x+s].$$

It is not hard to check that these are the only measures that the identity map statistically bifurcates toward through perturbations in Λ . But if Λ is the set of all smooth diffeomorphisms of \mathbb{S}^1 the set $\mathcal{B}_{\Lambda, f}$ contains all these measures together with other measures, in particular the measure δ_{δ_x} (the Dirac mass on the Dirac mass on x), for any point $x \in \mathbb{S}^1$. To see this, note that we can approach the identity map by Morse-Smale

maps having two fixed points and the point x as their only attracting fixed point.

Let us remind some definitions that we need in the rest of this section.

Let X and Y be two topological spaces with Y compact. Denote the set of all compact subsets of Y by $C(Y)$.

Definition 4.11. A map $\varphi : X \rightarrow C(Y)$ is called lower semi-continuous if for any $x \in X$ and any V open subset of Y with $\varphi(x) \cap V \neq \emptyset$, there is a neighbourhood U of x such that for any $y \in U$ the intersection $\varphi(y) \cap V$ is non-empty. The map φ is called upper semi-continuous if for any $x \in X$ and any V open subset of Y with $\varphi(x) \subset V$, there is a neighbourhood U of x such that for any $y \in U$ the set $\varphi(y)$ is contained in V . And finally φ is called continuous at x if it is both upper and lower semi-continuous at x .

Remark 4.12. To say x is a continuity point of a set valued map $\varphi : X \rightarrow C(Y)$ with the above definition, is indeed equal to say x is a continuity point of φ with considering $C(Y)$ as a topological space endowed with Hausdorff topology.

We also recall the following theorem of Fort [18] which generalizes the well known theorem about real valued semi-continuous maps to the set valued semi-continuous maps:

THEOREM (Fort). *For any Baire topological space X and compact topological space Y , the set of continuity points of a semi-continuous map from X to $C(Y)$ is a Baire generic subset of X .*

Now fixing a set of dynamics Λ we reprove the following fact on semi-continuity property of the map sending the dynamics to its set of invariant probability measures.

Lemma 4.13. *The map sending $f \in \Lambda$ to its set of invariant probability measures is upper semi-continuous.*

PROOF. We need to prove that if we have a sequence of dynamics $\{f_n\}_n$ converging to f and a sequence of invariant measures $\{\mu_n\}_n$ for these maps converging to a measure μ then μ is an f invariant measure. But this is an special case of Proposition 5.9 in [48] where this fact is proved in the context of stationary measures for locally constant skew products. \square

We recall that by Lemma 4.7, the set $\mathcal{B}_{\Lambda, f}$ is compact. We can ask about dependence of the set $\mathcal{B}_{\Lambda, f}$ on the map f . The following lemma shows that this dependency is semi-continuous:

Lemma 4.14. *The map sending $f \in \Lambda$ to the set $\mathcal{B}_{\Lambda, f}$ is upper semi-continuous.*

PROOF. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence converging to $f \in \Lambda$. We need to prove that if for each $n \in \mathbb{N}$, the map f_n statistically bifurcates toward a measure $\hat{\nu}_n \in \mathcal{M}_1(\mathcal{M}_1(X))$ through perturbations in Λ and the sequence $\{\hat{\nu}_n\}_n$ is convergent to a measure $\hat{\nu}$, then the map f statistically bifurcates toward $\hat{\nu}$ through perturbations in Λ . Then the proof is finished by observing that for n large enough, small perturbations of the map f_n are small perturbations of the map f , and $\hat{\nu}_n$ is close to $\hat{\nu}$. \square

To each map $f \in \Lambda$ one can associate the set of accumulation points of the sequence $\{\hat{e}_n(f)\}_{n \in \mathbb{N}}$ which is a compact subset of $\mathcal{M}_1(\mathcal{M}_1(X))$. By looking more carefully at the Example 4.4, we see that this map is neither upper semi-continuous nor lower semi-continuous. However if we add the points of this sequence to its accumulation points and consider the map sending $f \in \Lambda$ to the closure $\overline{\{\hat{e}_n(f) | n \in \mathbb{N}\}}$, we obtain a semi-continuous map:

Lemma 4.15. *The map $\mathcal{E} : \Lambda \rightarrow C(\mathcal{M}_1(\mathcal{M}_1(X)))$ defined as*

$$\mathcal{E}(f) := \overline{\{\hat{e}_n(f) | n \in \mathbb{N}\}},$$

is lower semi-continuous.

PROOF. Let V be an open subset of $\mathcal{M}_1(\mathcal{M}_1(X))$ intersecting $\mathcal{E}(f)$. So there is $n \in \mathbb{N}$ such that $\hat{e}_n(f) \in V$. But note that the map $f \mapsto \hat{e}_n(f)$ is continuous and so there is a neighborhood U of f so that for any $g \in U$, we have $\hat{e}_n(g) \in V$ and so $\mathcal{E}(g)$ intersects the set V . This shows that \mathcal{E} is lower semi-continuous. \square

The following lemma is an interesting consequence of Lemma 4.15 that shows how the set $\mathcal{E}(f)$ depends on the dynamics f .

MAIN LEMMA. *A Baire generic map $f \in \Lambda$ is a continuity point for the map \mathcal{E} .*

This lemma gives a view to the statistical behaviors of generic maps in any Baire space of dynamics: for a generic map, the statistical behavior that can be observed for times close to infinity can not be changed dramatically by small perturbations.

PROOF. Using Lemma 4.15, this is a direct consequence of Fort's theorem. \square

The following theorem reveals how two notion of statistical instability in law and being non-statistical in law are connected. There is another proof of this theorem which is communicated by Pierre Berger that can be found in [45] (Theorem 1.14).

THEOREM D. *Baire generically, $B_{\Lambda,f}$ is equal to $\text{acc}(\{\hat{e}_n^f\}_n)$.*

PROOF. Take a generic map f from the main lemma above. By remark 4.6, $\text{acc}(\{\hat{e}_n^f\}_n) \subset B_{\Lambda,f}$. So if $B_{\Lambda,f}$ has only one element, since $\text{acc}(\{\hat{e}_n^f\}_n)$ is non-empty, the equality holds. Now suppose that $B_{\Lambda,f}$ has more than one element. For the sake of contradiction suppose there is a measure $\hat{\nu} \in B_{\Lambda,f}$ which is not in $\text{acc}(\{\hat{e}_n^f\}_n)$. Then there is a number $n \in \mathbb{N}$ such that $\hat{\nu} = \hat{e}_n^f$ and \hat{e}_n^f is an isolated point of the sequence $\{\hat{e}_n^f\}_n$. Recalling that for generic f we have $B_{\Lambda,f} \subset \mathcal{E}(f)$, we can conclude that $B_{\Lambda,f}$ can be written as a union of two disjoint and non-empty closed set:

$$B_{\Lambda,f} = \{\hat{e}_n^f\} \cup \{\hat{e}_n^f\}^c.$$

This is in contrary to the connectedness of the set $B_{\Lambda,f}$. \square

If we have any information about the set $B_{\Lambda,f}$ then by using theorem D we may translate it to some information about $\text{acc}(\{e_n^f(x)\}_{n \in \mathbb{N}})$ for generic $f \in \Lambda$. In particular we obtain the following corollary:

Corollary 4.16. *The set Λ contains a Baire generic subset of maps that are statistically unstable in law iff it contains a Baire generic subset of maps which are non-statistical in law.*

Now we are going to study a special statistical bifurcation scenario for which this lemma can be used to deduce information about the behavior of generic maps. Suppose the initial map $f \in \Lambda$ has an invariant measure ν such that by a small perturbation of the map, the empirical measures of arbitrary large subset of points is close to ν for an iteration close to infinity. For instance you can think of the identity map on \mathbb{S}^1 which can be perturbed to an irrational rotation for which the empirical measures of almost every point converges to the Lebesgue measure or it can be perturbed to a Morse-Smale map having one attracting fixed point and so the empirical measures of almost every point converges to the Dirac mass on that attracting fixed point. These measures could be interpreted as a potential physical measure with full basin for our initial dynamics; a measure that for some small perturbation of the initial map and for some large iteration, the empirical measures for a large set of points is close to that measure.

We denote these measures by $\mathcal{M}_{\Lambda, f}$ which are defined more precisely as bellow:

$$\mathcal{M}_{\Lambda, f} := \{\nu \in \mathcal{M}_1(X) \mid \delta_\nu \in \mathcal{B}_{\Lambda, f}\}.$$

THEOREM F. *Let Λ be a Baire space of self-mappings of X endowed with a topology finer than C^0 -topology. For a Baire generic map $f \in \Lambda$ the empirical measures of μ almost every point $x \in X$, accumulates to each measure in $\mathcal{M}_{\Lambda, f}$ or in other words:*

$$(4.2) \quad \text{for } \mu - \text{a.e. } x \in X, \quad \mathcal{M}_{\Lambda, f} \subset \text{acc}(\{e_n^f(x)\}_{n \in \mathbb{N}}).$$

PROOF. To prove the theorem it suffices to show that if $f \in \Lambda$ is a continuity point of the map \mathcal{E} it satisfies condition (4.2). This is because by Corollary the continuity points of the map \mathcal{E} form a Baire generic subset of Λ .

Take any measure ν inside $\mathcal{M}_{\Lambda, f}$. Lemma D implies that $\delta_\nu \in \mathcal{E}(f)$. Now there are two possibilities, either there is a number $n \in \mathbb{N}$ such that $\hat{e}_n(f) = \delta_\nu$ or not. If not, there is a sequence $\{n_i\}_{i \in \mathbb{N}}$ converging to infinity such that

$$\lim_{i \rightarrow \infty} \hat{e}_{n_i}(f) = \delta_\nu.$$

So for a small neighbourhood $U \subset \mathcal{M}_1(X)$ of ν , we have:

$$\lim_{i \rightarrow \infty} \hat{e}_{n_i}(f)(U) = \delta_\nu(U) = 1,$$

and by equation (4.1), in the definition of $\hat{e}_n(f)$, we obtain:

$$\lim_{i \rightarrow \infty} \left(\int_X \delta_{e_{n_i}^f(x)} d\mu \right)(U) = 1.$$

So for μ -almost every point $x \in X$ we have:

$$\lim_{i \rightarrow \infty} \delta_{e_{n_i}^f(x)}(U) = 1.$$

Since U is an arbitrary neighbourhood, we can conclude that for μ -almost every point $x \in X$, the measure δ_ν is contained in the accumulation points of the sequence $\{\delta_{e_{n_i}^f(x)}\}_i$. But this is equal to say that ν is in the accumulation points of the sequence $\{e_{n_i}^f(x)\}_i$. So the measure ν is an accumulation point of the sequence $\{e_n^f(x)\}_{n \in \mathbb{N}}$, which is what we sought.

It remains to check the case that there is a number $n \in \mathbb{N}$ such that $\hat{e}_n(f) = \delta_\nu$. In this case, again by using (4.1) in the definition of $\hat{e}_n(f)$ we obtain:

$$\int_X \delta_{e_n^f(x)} d\mu = \delta_\nu,$$

so μ -almost every $x \in X$ has the property that the measure $e_n^f(x)$ is equal to ν . Recalling that ν is an f -invariant measure, every point x with this property should be a periodic point and ν should be the invariant probability measure supported on the orbit of x . So obviously the measure ν lies in the accumulation points of the sequence $\{e_n^f(x)\}_{n \in \mathbb{N}}$. This finishes the proof. \square

If one can find any information about the set $\mathcal{M}_{\Lambda, f}$ for a generic map f in Λ then by Theorem [F](#), we can translate this information to information about the statistical behavior of μ -almost every point for a generic subset of maps.

The following lemma shows how the set $\mathcal{M}_{\Lambda, f}$ depends on the map f :

Lemma 4.17. *The map sending $f \in \Lambda$ to the set $\mathcal{M}_{\Lambda, f}$ is upper semi-continuous.*

PROOF. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence converging to $f \in \Lambda$. We need to prove that if for each $n \in \mathbb{N}$, the map f_n statistically bifurcates toward a measure $\nu_n \in \mathcal{M}_{\Lambda, f_n}$ through perturbations in Λ and the sequence $\{\nu_n\}_n$ is convergent to a measure ν , then the map f statistically bifurcates toward ν through perturbations in Λ . Considering the fact that for n large enough, small perturbations of the map f_n are small perturbations of the map f , the rest of the proof is straight forward. \square

Now let us see what is the consequence of this lemma and Theorem [F](#) when we know the maps in a dense subset bifurcates toward each Dirac mass at an invariant measure. Before that we introduce the following definition which was used for the first time by Hofbauer and Keller in [\[23\]](#):

Definition 4.18. A map $f \in \Lambda$ is said to have *maximal oscillation* if the empirical measures of almost every point accumulates to all of the invariant measures in $\mathcal{M}_1(f)$.

Proposition 4.19 (Maximal oscillation). *If there is $D \subset \Lambda$ dense such that any map $f \in D$ bifurcates toward the Dirac mass at each invariant measure through perturbations in Λ , or in other word $M_{\Lambda, f} = \mathcal{M}_1(f)$, then a generic $f \in \Lambda$ has maximal oscillation.*

PROOF. By Proposition [4.17](#) the map sending f to $\mathcal{M}_{\Lambda, f}$ is semi-continuous. By lemma [4.13](#), The map sending f to $\mathcal{M}_1(f)$ is also upper semi-continuous. So by applying Fort's theorem we can find a Baire generic subset $\mathcal{B} \subset \Lambda$ such that any f in this set is a continuity point

for each of these maps. Now we can approach each map f in \mathcal{B} by maps g in D , for which we know $\mathcal{M}_1(g)$ and $\mathcal{M}_{\Lambda,g}$ co-inside. So $\mathcal{M}_1(f)$ and $\mathcal{M}_{\Lambda,f}$ co-inside as well. By Theorem F we know there is a Baire generic subset of Λ that for any map f in this set the empirical measures of μ almost every point $x \in X$ accumulates to each of measures in $\mathcal{M}_{\Lambda,f}$. The intersection of this Baire generic set with \mathcal{B} is still a Baire generic set and for a map f in this intersection the empirical measures of μ almost every point $x \in X$ accumulates to each of measures in $\mathcal{M}_1(f)$. \square

Hofbauer and Keller proved in [23] that there is an uncountable set of parameters λ for which the logistic map $f_\lambda(x) = \lambda x(1-x)$ restricted to the interval $[0, 1]$ has maximal oscillation. Let us denote the closure of this set of parameters by Λ . It can be shown that this is indeed the closure of the set of parameters for which the critical point of the map is in the preimages of some repelling periodic points, and by Jacobson theorem [24] we know this set is of positive Lebesgue measure. As a corollary of Proposition 4.19 we can give the following improvement of the Hofbauer and Keller result:

THEOREM A. *The set of parameters λ for which the map f_λ has maximal oscillation is a Baire generic subset of Λ .*

PROOF. Take the set D in the proposition 4.19 to be the set of Hofbauer and Keller parameters. \square

Will see in the next chapter that the scenario which is described in Theorem 4.19 actually happens in the context of complex dynamics where Λ is a subset of the space of rational maps on the Riemann sphere with a fixed degree, which is called “maximal bifurcation locus”. In the next section we present an example for application of Theorem F to a special class of maps.

5. Non-statistical Anosov-Katok maps

In [5] Anosov and Katok introduced a method for obtaining Lebesgue conservative ergodic maps with unexpected metric properties on manifolds which admit a \mathbb{S}^1 free action. They considered a class of conservative maps on such a manifold that can be approximated by periodic maps (like rational rotations of the torus) and prove that the set of ergodic transformations is a Baire generic subset of this space (which is an intersection of countably many open and dense subsets). Herman and Fathi in [16] pushed forward their method to construct minimal and uniquely ergodic maps. They also proved that these maps form

a second category subset of the space of maps that can be approximated by periodic ones. Being a G_δ subset had been previously known for properties like ergodicity, minimality and transitivity. The main aspect of the mentioned works in proving the genericity of these properties was to use this new method of Anosov and Katok to conclude the density of such properties. Herman could also apply the Anosov-Katok method to construct exotic invariant sets for holomorphic maps of the Riemann sphere. Here we use the Anosov-Katok method to construct and prove the genericity of diffeomorphisms of the annulus with unexpected statistical properties.

Let us denote the annulus $[0, 1] \times \mathbb{R}/\mathbb{Z}$ by \mathbb{A} and for $r \in [0, \infty]$ the space of all C^r orientation preserving diffeomorphisms of \mathbb{A} by $\text{Diff}_+^r(\mathbb{A})$ endowed with the C^r -topology. We denote the closure of the set of all C^r diffeomorphisms of the annulus which are C^r -conjugate to a rotation by \mathcal{AK}^r and call it the space of C^r Anosov-Katok maps. We also use \mathcal{AK}_{vol}^r to denote the closure of the set of C^r volume preserving diffeomorphisms conjugate to a rotation with a conjugacy fixing every point of the boundary. The spaces \mathcal{AK}^r and \mathcal{AK}_{vol}^r endowed with the induced C^r -topology are Baire spaces. We remind that for any measure $\nu \in \mathcal{M}_1(X)$, the Dirac mass on ν , which is an element of $\mathcal{M}_1(\mathcal{M}_1(X))$, is denoted by $\hat{\delta}_\nu$.

Before stating our main result in this section let us recall a related result of Fayad and Katok that we will use later in our proof:

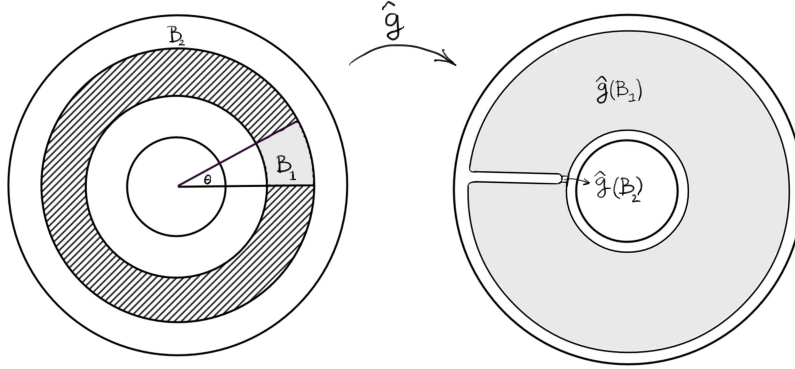
THEOREM 5.1 (Fayad-Katok [17, thm 3.3]). *A Baire generic map in the space $\mathcal{AK}_{vol}^\infty$ has only three ergodic measures, two one dimensional Lebesgue measures on the boundary components and the volume measure of the annulus.*

Now we are going to prove the Baire genericity of non-statistical behavior and indeed maximal oscillation in the set of Anosov-Katok maps \mathcal{AK}^r .

THEOREM C. *A Baire generic map in the set of Anosov-Katok maps \mathcal{AK}^r has exactly two ergodic invariant measures each of which is supported by a different boundary component of the annulus and moreover the map is maximally oscillating.*

Remark 5.2. Note that an Anosov-Katok map has at least two invariant measures which are supported on different boundary components of the annulus.

PROOF. We need the following lemma to prove the theorem:

FIGURE 2. The map \hat{g}

Lemma 5.3. *Let C be one of the connected components of the boundary of \mathbb{A} , and f be an arbitrary map in \mathcal{AK} . Then there is a measure ν supported on the set C such that f statistically bifurcates towards $\hat{\delta}_\nu$.*

□

PROOF. Since f can be approximated by maps which are conjugate to rotation, there is a rational number $\frac{p}{q}$ and a C^r diffeomorphism h such that the map $h^{-1}R_{\frac{p}{q}}h$ is close to f . Let

$$B := [r_1, r_2] \times \mathbb{R}/\mathbb{Z},$$

for some distinct real numbers $r_1, r_2 \in (0, 1)$. Take a real number $\theta > 0$ and define

$$B_1 := [r_1, r_2] \times [0, \theta)$$

and

$$B_2 := [r_1, r_2] \times [\theta, 1).$$

Sublemma 5.4. *For any $\sigma < 1$ close to 1 and $\varepsilon > 0$ small, there is a map $\hat{g} \in \text{Diff}_+^r(\mathbb{A})$ with the following properties:*

- \hat{g} is identity on a neighborhood of the set C ,
- $\text{Leb}(\hat{g}(B_1)) > \sigma$,
- $\hat{g}(B_2)$ is included in the ε -neighborhood $N_\varepsilon(C)$ of C .

PROOF. Using bump functions, we construct a map \hat{g} as depicted in Figure 2. The technical details are left to the reader. □

Now let \hat{g} be a map found in the sublemma. This map can be lifted using the covering map $\pi : \mathbb{A} \rightarrow \mathbb{A}$, $\pi(r, \theta) = (r, q\theta)$. Let g be the lift of \hat{g} which is identity around the set C . The diffeomorphism g has similar properties:

- g is identity on a neighbourhood of the set C ,
- $Leb(g(\pi^{-1}(B_1))) > \sigma$,
- $g(\pi^{-1}(B_2)) \subset N_\varepsilon(C)$.

Now note that g commutes with $R_{\frac{p}{q}}$ so:

$$h \circ g \circ R_{\frac{p}{q}} \circ g^{-1} \circ h^{-1} = h \circ R_{\frac{p}{q}} \circ h^{-1}$$

Choose α' irrational and small enough so that $h \circ g \circ R_{\alpha'} \circ g^{-1} \circ h^{-1}$ is arbitrary close to $h \circ R_{\frac{p}{q}} \circ h^{-1}$. Indeed $h \circ g$ is C^r and the map sending α to $h \circ g \circ R_{\alpha} \circ g^{-1} \circ h^{-1}$ is hence continuous. Since α' is irrational, the orbit closure of any point in \mathbb{A} under iterating the map $h \circ g \circ R_{\alpha'} \circ g^{-1} \circ h^{-1} =: f'$ is $h \circ g$ -image of the orbit closure of a point under iterating the map $R_{\alpha'}$, which is a circle in \mathbb{A} . So for any point $x \in h \circ g(\pi^{-1}(B_2))$, the orbit closure of x is the $h \circ g$ image of a vertical circle C' intersecting $\pi^{-1}(B_2) \subset B$ and so contained in $\pi^{-1}(B)$. The map $h \circ g \circ R_{\alpha'} \circ g^{-1} \circ h^{-1}|_{h \circ g(C')}$ is conjugate to $R_{\alpha'}|_{C'}$. Now note that if $R_{\alpha'}^n \circ g^{-1} \circ h^{-1}(x) \in \pi^{-1}(B_i)$ then $(f')^n(x) \in h \circ g(\pi^{-1}(B_i))$ for $i \in \{1, 2\}$. The orbit of each point in C , in average, spends θ portion of times in $\pi^{-1}(B_1)$ and $1 - \theta$ portion of times in $\pi^{-1}(B_2)$. So the orbit of the point x spends θ portion of times in $h \circ g(\pi^{-1}(B_1))$ and $1 - \theta$ portion of times in $h \circ g(\pi^{-1}(B_2))$. By choosing θ and ε sufficiently small, we can guarantee that the asymptotic averages of any point in $h \circ g(\pi^{-1}(B_1))$ is arbitrary close to a measure $\nu_{f'}$ which is the pushforward of the Lebesgue measure on the boundary component C by the map $h \circ g$. If σ is chosen sufficiently close to one, then map g is so that $Leb(h \circ g(\pi^{-1}(B_1)))$ is sufficiently close to one and so since for a large number n the n^{th} empirical measures of points in $h \circ g(\pi^{-1}(B_1))$ is close to $\nu_{h \circ g}$ and so the measure $\hat{e}_n^{f'}$ is close to $\hat{\delta}_{\nu_{f'}}$. Now taking ν as any accumulation point of measures like $\nu_{f'}$ where f' approaches f , according to the definition we can see that f statistically bifurcates toward $\hat{\delta}_\nu$ and so we are done. \square

PROOF OF THEOREM C. Lemma 5.3 implies that for any map $f \in \mathcal{AK}^r$ there are two measures $\nu_{1,f}$ and $\nu_{2,f}$ which are supported on different connected components of the boundary of \mathbb{A} such that f statistically bifurcates toward both $\hat{\delta}_{\nu_{1,f}}$ and $\hat{\delta}_{\nu_{2,f}}$. Now using Theorem F we conclude that for a generic map $f \in \mathcal{AK}^r$ and for almost every point x in the phase space X , the set of accumulation points of the sequence $\{e_n^f(x)\}_n$ contains at least two measures $\nu_{1,f}$ and $\nu_{2,f}$. We are going to show that generically these two measures are the only ergodic invariant measures of the map f and the empirical measures of almost every point accumulates to any convex combination of these two ergodic

measures (which is indeed the whole space of invariant measures) and hence f is maximally oscillating. Note that since the map which sends each dynamics to its set of invariant measures is semi-continuous, and the continuity points of a semi-continuous map is a generic set, we can choose f to be a continuity point of this mapping. Now approximate f by a map like $h \circ g \circ R_{\alpha'} \circ g^{-1} \circ h^{-1}$ coming from lemma 5.3. Using theorem 5.1, we know that the map $R_{\alpha'}$ can be approximated in C^∞ -topology (and hence in C^r -topology) by a map $e \in \mathcal{AK}_{vol}^\infty$ which has only three ergodic measures, two one dimensional Lebesgue measures on the boundaries and the volume measure of the annulus. The map $h \circ g \circ e \circ g^{-1} \circ h^{-1}$ is close to the initial map f and has only three ergodic invariant measures which are the push forward of three ergodic measures of e by the map $h \circ g$. Note that if in sublemma 5.4 the numbers r_1 and r_2 are chosen close to 0 and 1 then set B_2 has measure close to one. In this case observe that the pushforward of the volume measure by the map $h \circ g$ is a measure which is close to the pushforward of the one dimensional Lebesgue measure of one of the boundary components (which is denoted by C in the lemma). Hence the set of invariant measures for the map $h \circ g \circ R_{\alpha'} \circ g^{-1} \circ h^{-1}$ is a triangle that have two of its vertices very close to each other. We know that the map sending the dynamics to its set of invariant measures is upper semi-continuous (see 4.13) and hence it is continuous for maps in a Baire generic set. Hence we can assume that f is a continuity point of this mapping and the set of invariant measures of f is in an arbitrary neighbourhood of triangle which is arbitrary close to a segment. So the set of invariant measures of f is a segment. But this means that f has exactly two ergodic invariant measures. Note that these measures are supported on different boundary components of the annulus. So on each boundary component of the annulus there is only one ergodic measure and hence any invariant measures on one of the boundary components is equal to the corresponding ergodic measure on that component. So two measures $\nu_{1,f}$ and $\nu_{2,f}$ toward which f statistically bifurcates are exactly two ergodic measures of f . Moreover since these two measures are in the accumulation points of the sequence of empirical measures for almost every points, and the set of invariant measures is the line segment between these two measures, the sequence of empirical measures of almost every points have to accumulate to every point in this line segment and this finishes the proof. \square

6. Comparison between different versions

In this section we compare different versions of defining statistical instability and non-statistical dynamics and show how they are related. The first proposition describes the relation between different versions of defining non-statistical maps.

Proposition 6.1. *Suppose $f : X \rightarrow X$ is a continuous map of a compact metric space.*

- i) If f is non-statistical in law, then it is L^1 non-statistical.*
- ii) If f is L^1 non-statistical, then it is non-statistical.*

PROOF. i) Let f be non-statistical in law. So by definition the sequence $\{\hat{e}_n(f) = (e_n^f)_*\mu\}_n$ is not convergent. We recall that

$$(e_n^f)_*(\mu) = \int_X \hat{\delta}_{e_n^f(x)} d\mu(x).$$

where $\hat{\delta}_{e_n^f(x)} \in \mathcal{M}_1(\mathcal{M}_1(X))$ is the Dirac mass supported on the point $e_n^f(x) \in \mathcal{M}_1(X)$.

Suppose to the contrary that f is not L^1 non-statistical. So the sequence of maps $\{e_n^f : X \rightarrow \mathcal{M}_1(X)\}$ is convergent in the L^1 topology. Let us call the limit point of this sequence by e_∞^f . Now we show that $(e_n^f)_*\mu$ converges to $(e_\infty^f)_*\mu$ which is a contradiction. For simplicity we denote $(e_n^f)_*\mu$ by ν_n and $(e_\infty^f)_*\mu$ by ν_∞ . We recall that

$$\hat{d}_w(\nu_n, \nu_\infty) = \min_{\xi \in \pi(\nu_n, \nu_\infty)} \int_{\mathcal{M}_1(X) \times \mathcal{M}_1(X)} d_w(e, e') d\xi(e, e').$$

where $\pi(\nu_n, \nu_\infty)$ is the set of all probability measures on $\mathcal{M}_1(X) \times \mathcal{M}_1(X)$ which projects to ν_n and ν_∞ under the projections to the first and second coordinate respectively. Consider the following element of $\pi(\nu_n, \nu_\infty)$:

$$\xi := \int_X d_w(\delta_{e_n^f(x)}, \delta_{e_\infty^f(x)}) d\mu$$

We have

$$\hat{d}_w(\nu_n, \nu_\infty) \leq \int_{\mathcal{M}_1(X) \times \mathcal{M}_1(X)} d_w(e, e') d\xi_n(e, e').$$

On the other hand,

$$\int_{\mathcal{M}_1(X) \times \mathcal{M}_1(X)} d_w(e, e') d\xi_n(e, e') = \int_X d_w(e_n^f(x), e_\infty^f(x)) d\mu = d_{L^1}(e_n^f, e_\infty^f).$$

So we obtain $\hat{d}_w(\nu_n, \nu_\infty) \leq d_{L^1}(e_n^f, e_\infty^f)$, which implies that ν_n is converging to ν_∞ and is a contradiction.

ii) Let f be L^1 non-Statistical. We show that f is non-statistical and so the maps $e_n^f : X \rightarrow \mathcal{M}_1(X)$ do not converge almost surely. By contrary, suppose the maps $e_n^f : X \rightarrow \mathcal{M}_1(X)$ converge almost surely to a map $e_\infty^f : X \rightarrow \mathcal{M}_1(X)$. Hence the map $d_w(e_n^f(\cdot), e_\infty^f(\cdot)) : X \rightarrow \mathbb{R}$ converges to zero almost surely and by dominated convergence theorem we obtain that

$$d_{L^1}(e_n^f, e_\infty^f) = \int_X d_w(e_n^f(x), e_\infty^f(x)) d\mu(x) \rightarrow 0 \quad (n \rightarrow \infty).$$

Which is a contradiction. \square

The next proposition, shows that the same hierarchy holds for different versions of defining statistically unstable maps.

Proposition 6.2. *Suppose Λ is a set of continuous self mappings of a compact metric space X . It holds true that:*

i) *If $f \in \Lambda$ is statistically unstable in law, then it is L^1 statistically unstable.*

ii) *If $f \in \Lambda$ is L^1 statistically unstable, then it is statistically unstable.*

PROOF. i) If f is statistically in law, then by definition there is at least two different elements $\hat{\nu}_1$ and $\hat{\nu}_2$ in $\mathcal{M}_1(\mathcal{M}_1(X))$ toward which f statistically bifurcates. This means that there are two sequences of maps $\{f_k^1\}_k$ and $\{f_k^2\}_k$ converging to f , and two sequences of positive integers $\{n_k^1\}_k$ and $\{n_k^2\}_k$ converging to infinity such that

$$\lim_{k \rightarrow \infty} \hat{e}_{n_k^1}(f_k^1) = \hat{\nu}_1, \quad \lim_{k \rightarrow \infty} \hat{e}_{n_k^2}(f_k^2) = \hat{\nu}_2$$

Now on the contrary suppose f is L^1 statistically stable. So the sequence $\{e_{n_k^1}^{f_k^1} : X \rightarrow X\}_k$ and $\{e_{n_k^2}^{f_k^2} : X \rightarrow X\}_k$ both converge to a map $e_\infty^f : X \rightarrow X$ in the L^1 topology. Using the same arguments in the proof of part (i) in the previous proposition we conclude that both sequences $\{(e_{n_k^1}^{f_k^1})_* \mu\}_k$ and $\{(e_{n_k^2}^{f_k^2})_* \mu\}_k$ converge to $(e_\infty^f)_* \mu$. Hence we have $\hat{\nu}_1 = \hat{\nu}_2 = (e_\infty^f)_* \mu$ and since $\hat{\nu}_1$ and $\hat{\nu}_2$ were distinct elements $\mathcal{M}_1(\mathcal{M}_1(X))$, this is a contradiction.

ii) Suppose f is not statistically stable, so there is a map $e_\infty^f : X \rightarrow \mathcal{M}_1(X)$ such that for any sequence $\{f_k\}_k$ converging to f and any sequence of natural numbers $\{n_k\}_k$ converging to infinity the sequence of maps $\{e_{n_k}^{f_k} : X \rightarrow \mathcal{M}_1(X)\}_k$ converge almost surely to the map e_∞^f . Using dominated convergence theorem, we conclude the convergence of this sequence in the L^1 topology to the map e_∞^f and hence f can not be L^1 statistically unstable. \square

Let us mention that Proposition 6.2 implies a theorem of Avila and Bochi [7, Theorem B]:

THEOREM (Avila-Bochi 09). *Assume X is a smooth compact connected manifold and m a smooth volume measure on X . For any conservative map f of X , denote the ergodic decomposition of m by κ_f . Fix an integer $r \geq 0$. The points of continuity of the map*

$$f \in \text{Diff}_m^r(X) \mapsto \kappa_f \in \mathcal{M}_1(\mathcal{M}_1(X)),$$

form a residual set.

Here we can give a short proof of this theorem using Corollary 2.10 and Proposition 6.2:

PROOF. We prove that every map f in the set of L^1 statistically stable maps, which by Corollary 2.10 is residual in $\text{Diff}_m^r(X)$, is a continuity point of ergodic decomposition. Let f be L^1 statistically stable. By Proposition 6.2, f is statistically stable in law. So for any sequence $\{f_k\}_k$ converging to f and $\{n_k\}_k$ converging to infinity the sequence $\{\hat{e}_{n_k}(f_k)\}_k$ converges to the ergodic decomposition $\kappa_f = \lim_{n \rightarrow \infty} \hat{e}_n(f)$, and hence we can conclude that κ_{f_k} converges to κ_f . So f is a continuity point of the map sending f to κ_f . \square

Remark 6.3. In fact if $f \in \text{Diff}_m^r$ is L^1 statistically stable, then for any positive number $\varepsilon > 0$, there is a neighbourhood \mathcal{U} of f such that not only for any map $g \in \mathcal{U}$, the limit e_∞^g of the sequence of empirical functions is ε -close to e_∞^f , but also for large enough integer n , the empirical functions e_n^g are ε -close to e_∞^f as well. This is because

$$0 = \Delta^1(f) \geq \limsup_{n \rightarrow \infty} \limsup_{g \rightarrow f} d_{L^1}(e_\infty^f, e_n^g).$$

Part 2

Non-statistical Rational maps

Denote by Rat_d the space of rational maps of degree d on the Riemann sphere $\hat{\mathbb{C}}$. A rational map is called *postcritically finite* if all of its critical points have finite orbit. A map in Rat_d is called *strictly postcritically finite* if each of its critical points is eventually mapped to a repelling periodic orbit (this is equivalent to say that this postcritically finite map has no periodic critical point). The closure of strictly postcritically finite maps is a subset of bifurcation locus and is called *maximal bifurcation locus*. In this part of the thesis we show that in the family of degree d rational maps of the Riemann sphere the maximal bifurcation locus contains a (relatively) Baire generic subset of maps displaying maximal non-statistical behavior:

THEOREM B (Main theorem). *For a Baire generic map f in the maximal bifurcation locus, the set of accumulation points of the sequence of empirical measures is equal to the set of invariant measures of f for Lebesgue almost every point.*

Although the set of strictly postcritically finite rational functions is a countable union of 3-dimensional sub-varieties of Rat_d , its closure – the maximal bifurcation locus – has recently been shown to have positive measure w.r.t. the volume measure of the space of rational maps as a complex manifold (see [6]).

The proofs are based on the abstract setting we developed in section 4 and also a transversality argument allowing us to control the behavior of the orbit of critical points for maps close to postcritically finite rational maps.

Let us say a few words about the organization of this part of the thesis. In Section 7 we prove the main theorem of this part using two propositions 7.2 and 7.3. In Proposition 7.2 we show that any map in the maximal bifurcation locus statistically bifurcates toward the Dirac mass on an arbitrary periodic measure. This proposition is proved in Section 8. In the last section, Proposition 7.3 is proved in which we show the periodic measures are dense in the set of invariant measures for strictly postcritically finite maps.

7. Proof of Main Theorem

First let us introduce the following definitions and notations that we use while dealing with the dynamics of rational maps. We say a point $x \in X$ is *preperiodic* if it is mapped to a periodic point p after some iterations. In this case we may say the point x is *preperiodic to* the periodic point p . We say a measure $\mu \in \mathcal{M}_1(f)$ is an *invariant periodic measure* if it is supported on the orbit of a periodic point.

The space of degree d rational maps Rat_d is a $2d + 1$ dimensional complex manifold. To see this, note that we can parametrize it around any element $\frac{P}{Q} \in \text{Rat}_d$ using the coefficients of the polynomials P and Q . These two polynomials have terms up to degree d so there is $2d + 2$ coefficients. But note that multiplying both P and Q by a constant does not change the rational map, so the dimension is $2d + 1$.

Remark 7.1. Any degree d rational map has $2d - 2$ critical points counting with multiplicity.

Here are some notations:

- $\text{Per}(f)$ for the set of the periodic points of a map f .
- $\mathcal{C}(f)$ for the set of critical points of a rational map f .
- $\mathcal{P}(f)$ for the postcritical set of a rational map which is defined as follows:

$$\mathcal{P}(f) := \bigcup_{n \geq 1} f^n(\mathcal{C}).$$

- κ_d for the set of those maps in Rat_d for which every critical point eventually land on a repelling periodic point (these maps are called strictly postcritically finite maps).
- κ_d^* for the set of those maps in κ_d for which all the critical points are simple and the postcritical set does not contain any critical point.

To prove the main theorem we show that the maps in κ_d enjoy from two nice properties stated in the following propositions. The proofs of these propositions is postponed to the next sections.

The first proposition is related to the statistical behavior of perturbations of the maps in κ_d within the maximal bifurcation locus $\overline{\kappa_d}$.

Proposition 7.2. *Assume f is a map in κ_d . Then for any periodic point $p \in \text{Per}(f)$, f statistically bifurcates toward $\delta_{e_{\infty}^f(p)}$ through perturbations in $\overline{\kappa_d}$.*

Note that a rational map of degree greater than one, has always (infinitely) many different periodic orbits, and in fact, the set of periodic points is dense in the Julia set. So the set of periodic measures contains infinitely many elements, each one corresponds to a periodic cycle. The next proposition states that for a map in κ_d , the set of periodic measures is in some sense maximal.

Proposition 7.3. *For any strictly postcritically finite rational map f , the set of invariant probability measures which are supported on the*

orbit of a periodic point, is dense in the set of all invariant measures $\mathcal{M}_1(f)$.

Remark 7.4. In the proof of Proposition 7.3 we will see that every periodic point for a strictly postcritically finite map is repelling.

From these two propositions we conclude that for a map in κ_d the set of measures to which f statistically bifurcates is maximal.

Corollary 7.5. *Any map in κ_d statistically bifurcates toward the Dirac mass on any of its invariant measures through perturbations in $\overline{\kappa_d}$ or in other word:*

$$\forall f \in \kappa_d, \quad \{\nu \in \mathcal{M}_1(X) \mid \delta_\nu \in \mathcal{B}_{\Lambda, f}\} = \mathcal{M}_1(f).$$

Remark 7.6. We use the word maximal because a map f cannot statistically bifurcate toward the Dirac mass on a measure that is not f -invariant.

Let us show how this corollary together with Proposition 4.19 implies the main theorem.

END OF PROOF OF MAIN THEOREM. By Corollary 7.5, every map in κ_d bifurcates toward the Dirac mass on each of its invariant measures. So by Proposition 4.19, for a generic f in $\overline{\kappa_d}$, the set of accumulation points of the sequence of empirical measures of Leb-almost every point, is equal to the whole set of invariant measures. This finishes the proof. \square

8. Statistical bifurcation toward periodic measures

The aim of this section is to prove Proposition 7.2. First let us recall the following two theorems from [12] and [13].

We recall that a *Lattès map* f is a postcritically finite map which is semi-conjugated to an affine map $A : z \mapsto az + b$ on a complex torus \mathcal{T} , via a finite to one holomorphic semi conjugacy Θ :

$$\Theta \circ A = f \circ \Theta .$$

A lattès map f is *flexible* if we can choose Θ with degree 2 and A with $a > 1$ integer.

We denote by \mathcal{L}_d the set of flexible Lattès maps of degree d . We refer the reader to the paper of Milnor [34] for further discussion on Lattès maps.

We observe that:

$$\mathcal{L}_d \subset \kappa_d^* \subset \kappa_d .$$

On the other hand:

THEOREM (Buff-Epstein). *The following inclusion holds true:*

$$\kappa_d \setminus \mathcal{L}_d \subset \overline{\kappa_d^* \setminus \mathcal{L}_d}.$$

This theorem is a part of the main theorem of [12], whereas the following one is the main theorem of [13].

THEOREM (Buff-Gauthier). *A flexible Lattès map can be approximated by strictly postcritically finite rational maps which are not a flexible Lattès map:*

$$\mathcal{L}_d \subset \overline{\kappa_d \setminus \mathcal{L}_d}.$$

These two theorems imply:

Corollary 8.1. *Any strictly postcritically finite rational map $f \in \kappa_d$ can be approximated by maps in κ_d^* which are not flexible Lattès map:*

$$\kappa_d \subset \overline{\kappa_d^* \setminus \mathcal{L}_d}.$$

PROOF. By Proposition , if f is not a flexible Lattès map we are done. If f is a flexible Lattès map, then first by Proposition , it can be approximated by a strictly postcritically finite map which is not a flexible Lattès map. Now using Proposition again, we are done. \square

Corollary 8.1 enables us to transfer the following property of maps in $\kappa_d^* \setminus \mathcal{L}_d$ to those in κ_d , in order to deduce Proposition 7.2.

Lemma 8.2 (Main lemma). *Let f be a map in $\kappa_d^* \setminus \mathcal{L}_d$. Then for any periodic point $q \in \text{Per}(f)$, f statistically bifurcates toward $\delta_{e_\infty^f(q)}$ through perturbations in $\overline{\kappa_d}$.*

We will prove this lemma below, before this let us prove Proposition 7.2.

PROOF OF PROPOSITION 7.2. For any map f in κ_d , any periodic point p is repelling, and its hyperbolic continuation is well defined and so the periodic measure supported on its cycle has a well defined continuation for f' close to f .

Hence, to show that f statistically bifurcates toward $\delta_{e_\infty^f(p)}$ through perturbations in $\overline{\kappa_d}$, it is enough to show that there is some map f' in $\overline{\kappa_d}$ arbitrary close to f that statistically bifurcates toward the Dirac mass on the continuation $e_\infty^{f'}(p)$ of this measure. But by Corollary 8.1, arbitrary close to f we can find elements of $\kappa_d^* \setminus \mathcal{L}_d$, and by Main lemma, these maps statistically bifurcate toward the Dirac mass on any of their periodic measures, in particular, to the Dirac mass on the continuation $e_\infty^{f'}(p)$. This finishes the proof of Proposition 7.2. \square

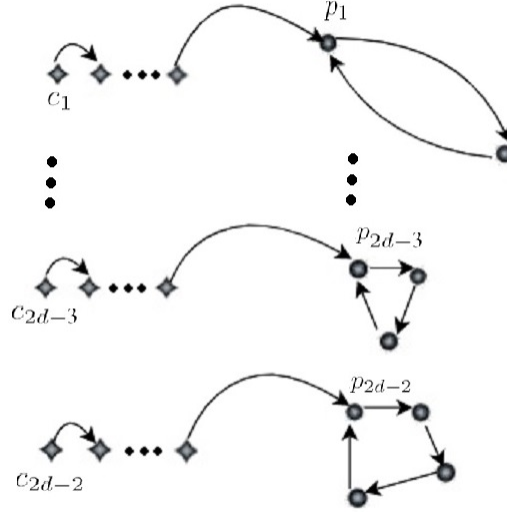


FIGURE 3. The dynamics of the initial map $f \in \kappa_d^* \setminus \mathcal{L}_d$.

PROOF OF LEMMA 8.2. Denote by $c_1(f), \dots, c_{2d-2}(f)$ the $2d-2$ distinct critical points of f . There are repelling periodic points $p_1(f), \dots, p_{2d-2}(f)$ and positive integers n_1, \dots, n_{2d-2} such that as it is shown in Figure 1

$$f^{n_i}(c_i(f)) = p_i(f), \quad 1 \leq i \leq 2d-2$$

The critical points are simple and periodic points are repelling so by the implicit function theorem, for any $1 \leq i \leq 2d-2$ there are

- analytic germ $c_i : (\text{Rat}_d, f) \rightarrow \hat{\mathbb{C}}$ following the critical point of f as g ranges in a neighbourhood of f in Rat_d and
- analytic germ $p_i : (\text{Rat}_d, f) \rightarrow \hat{\mathbb{C}}$ following the periodic point of f as g range in a neighbourhood of f in Rat_d .

Let $F : (\text{Rat}_d, f) \rightarrow \mathbb{C}^{2d-2}$ and $G : (\text{Rat}_d, f) \rightarrow \mathbb{C}^{2d-2}$ be defined by:

$$F(g) = \begin{pmatrix} F_1(g) \\ \vdots \\ F_{2d-2}(g) \end{pmatrix} \text{ with } F_j(g) := f^{n_j}(c_j(g)) \text{ and } P(g) = \begin{pmatrix} p_1(g) \\ \vdots \\ p_{2d-2}(g) \end{pmatrix}.$$

Denote by $D_f F$ and $D_f P$ the differentials of F and P at f . The following transversality result has been proved many times, see for example [47]. We recall a version which is presented in [12]:

Proposition 8.3 (Epstein). *The linear map*

$$D_f F - D_f P : T_f \text{Rat}_d \rightarrow T_{p_1(f)} \hat{\mathbb{C}} \times \dots \times T_{p_{2d-2}(f)} \hat{\mathbb{C}}$$

has rank $2d - 2$. The kernel of $D_f F - D_f P$ is the tangent space to the subset of Rat_d which is formed by those maps that are conjugate to f by a Möbius transformation.

This nice property enables us to have control on the orbits of the critical points while perturbing the dynamics.

Proposition 8.4. *For any map in $f \in \kappa_d^*$ which is not a flexible Lattès map, there is a holomorphic, one-dimensional family $\{f_\lambda\}_{\lambda \in \mathbb{D}}$ such that $f_0 = f$, and except $c_1(f_0)$, the other critical points are persistently preperiodic through this family.*

PROOF. For any $1 \leq i \leq 2d - 2$ let the map $\varphi_i : U_i \rightarrow \mathbb{C}$ be a local coordinate around $p_i(f)$, such that $\varphi_i(p_i(f)) = 0$. Then by the previous proposition the derivative of the map

$$\Phi := \begin{pmatrix} \varphi_1 \circ F_1 - \varphi_1 \circ p_1 \\ \vdots \\ \varphi_{2d-2} \circ F_{2d-2} - \varphi_{2d-2} \circ p_{2d-2} \end{pmatrix}.$$

at f has full rank, so if we denote the ε -neighbourhood of zero in the complex plane by \mathbb{C}_ε , then by Rank theorem there is a one dimensional holomorphic family $\{f_\lambda\}_{\lambda \in \mathbb{D}_\varepsilon}$ for $\varepsilon > 0$ sufficiently small, such that $\Phi(f_\lambda) = (\lambda, 0, 0, \dots, 0)$. So for any $\lambda \in \mathbb{D}_\varepsilon$ and for any $j \neq 1$ we have $f_\lambda^{n_j}(c_j(f_\lambda)) = p_j(f_\lambda)$. And obviously this equality does not hold true for critical point $c_1(f_\lambda)$. By reparametrizing the family, we obtain a family $\{f_\lambda\}_{\lambda \in \mathbb{D}}$ enjoying the desired properties. \square

Let us consider a family $\{f_\lambda\}_{\lambda \in \mathbb{D}}$ coming from Proposition 8.4, and denote the bifurcation locus of this family by B recalling that:

Definition 8.5. The bifurcation locus of a family consists of those parameters that the dynamics is not structurally stable within that family.

Remark 8.6. The bifurcation locus B is non-empty and in particular contains 0.

PROOF. The family we are considering is so that $c_1(f_\lambda)$ is sent to $p_1(f_\lambda)$ by n_1 iteration for $\lambda = 0$, but this does not happen for $\lambda \neq 0$. So f_0 is not structurally stable in this family. \square

Remark 8.7. Since for every λ sufficiently close to zero the orbit of each critical point other than $c_1(f_0)$ is finite, $c_1(f_\lambda)$ is disjoint from the orbit of the other critical points. So by reparameterizing the maps associated to the parameters close to zero, we can assume that every map in the family satisfies this property. This is a technical assumption that we will use later.

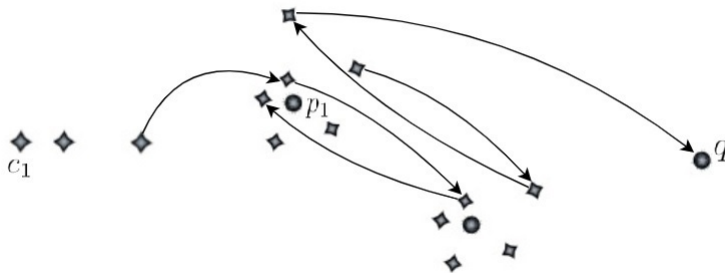


FIGURE 4. For the map f_{λ^*} which is a suitable perturbation of $f_0 = f$, the orbit of the critical point c_1 is repelled by the cycle of the periodic point p_1 and eventually land on q (see lemma 8.8).

Lemma 8.8. *For every periodic point $q(f_0)$ of the map f_0 , there is a parameter λ^* in the bifurcation locus B arbitrary close to zero, such that $c_1(f_{\lambda^*})$ is preperiodic to $q(f_{\lambda^*})$.*

PROOF. The proof uses the well known normal family argument. Let U be a small neighbourhood around $0 \in \mathbb{D}$. Recalling that the parameter zero is in the bifurcation locus, by Theorem 4.2 of McMullen's paper [32], there is j for which the family $\{\lambda \in U \mapsto f_{\lambda}^n(c_j(f_{\lambda}))\}_{n \in \mathbb{N}}$ is not normal. But by Proposition 8.4, this family is eventually periodic for $j \neq 1$ and hence it is normal. So for $j = 1$, it is not normal. Using this we are going to find λ^* in U such that $c_1(f_{\lambda^*})$ is preperiodic to $q(f_{\lambda^*})$. If this holds for $\lambda^* = 0$ we are done. If not:

Claim 8.9. *If $c_1(f_0)$ is not preperiodic to $q(f_0)$, then any pre-image of $q(f_0)$ depends holomorphically on the parameter in a neighbourhood of zero.*

PROOF. Take $q'(f_0)$ to be a pre-image of $q(f_0)$. If $q'(f_0)$ does not meet any critical point before landing on $q(f_0)$, obviously it depends analytically on the parameter. Otherwise there exists $j \neq 1$ such that $q'(f_0)$ is sent to $c_j(f_0)$ and $c_j(f_0)$ is sent to $q(f_0)$. Proposition 8.4 implies that for every parameter $\lambda \in \mathbb{D}$, $c_j(f_{\lambda})$ is preperiodic to $q(f_{\lambda})$, and so $q'(f_{\lambda})$ is indeed a preimage of $c_j(f_{\lambda})$. Since the latter depends analytically on the parameter, its preimage depends analytically as well. \square

Now take $q_1(f_0), q_2(f_0)$ and $q_3(f_0)$ to be three distinct preimages of $q(f_0)$. There exists an analytic family of Möbius maps Γ_{λ} sending back

the continuation of these three preimages to themselves:

$$\Gamma_\lambda(q_m(f_\lambda)) = q_m(f_\lambda) \quad m \in \{1, 2, 3\}.$$

Since composing with Möbius maps does not affect normality, the family $\{\lambda \in U \mapsto \Gamma_\lambda^{-1}(f_\lambda^n(c_1(f_\lambda)))\}_{n \in \mathbb{N}}$ is not a normal family as well, so by Montel's theorem, it cannot avoid all of the three points $q_1(f_0)$, $q_2(f_0)$ and $q_3(f_0)$. Hence, there is a parameter λ^* , a natural number $l \in \mathbb{N}$ and $m \in \{1, 2, 3\}$ such that the following equality holds:

$$(8.1) \quad \Gamma_{\lambda^*}^{-1}(f_{\lambda^*}^l(c_1(f_{\lambda^*}))) = q_m(f_0).$$

So $f_{\lambda^*}^l(c_1(\lambda^*)) = q_m(f_{\lambda^*})$ which means the critical point $c_1(f_{\lambda^*})$ is preperiodic to $q(f_{\lambda^*})$.

To prove that the parameter λ^* is in the bifurcation locus, note that the equation (8.1) cannot hold true in a neighbourhood of λ^* , since otherwise it holds true for any parameter in U but we have assumed that $c_1(f_0)$ is not preperiodic to $q(f_0)$. \square

Now let the parameter λ^* is chosen so that $c_1(f_{\lambda^*})$ is preperiodic to the periodic point $q(f_{\lambda^*})$ which is the continuation of the periodic point q in the statement of the main lemma.

Lemma 8.10. *Arbitrary close to the parameter λ^* , there is a parameter $\hat{\lambda}$ such that $f_{\hat{\lambda}}$ has a parabolic periodic point $\hat{q}(f_{\hat{\lambda}})$ and the invariant probability measure supported on the orbit of $\hat{q}(f_{\hat{\lambda}})$ is arbitrary close to the invariant probability measure supported on the orbit of $q(f_{\lambda^*})$.*

PROOF. For simplicity, after reparametrizing the family, we may assume that λ^* is equal to zero. Without loss of generality, we may also assume that the period of $q(f_\lambda)$ is equal to one and so it is a fixed point. Otherwise we can repeat the following arguments for a family formed by an iteration of f_λ . Conjugating the family by Möbius maps, we can assume that q remains a fixed point for all maps in this family. Up to a holomorphic change of local coordinates we can also assume that f_λ is linear in a neighbourhood of q and has the following form:

$$(8.2) \quad f_\lambda(q + z) = \gamma_\lambda z + q,$$

where γ_λ is the multiplier of the repelling fixed point q for the map f_λ . Next note that since for the map f_0 the pre-images of any point accumulates to any point in the Julia set, and the Julia set is the whole Riemann sphere, arbitrary close to q , there are pre-images of the critical point $c_1(f)$. We choose one of this pre-images \tilde{c} , which is in the linearization domain of q . We can also assume the change of coordinates around q is so that the point \tilde{c} stays a preimage of $c_1(f_\lambda)$ for λ close to zero.

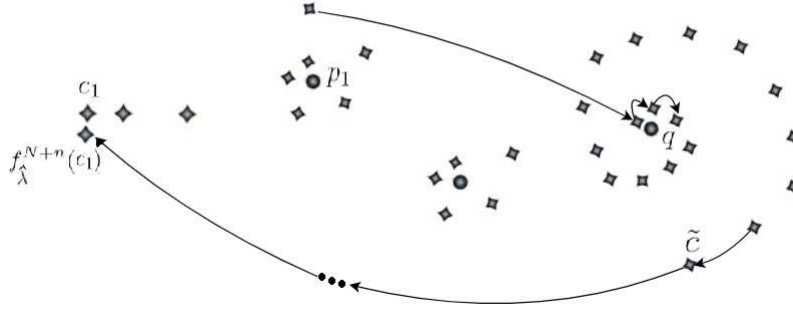


FIGURE 5. For the map $f_{\tilde{\lambda}}$ which is a suitable perturbation of f_{λ^*} , the orbit of the critical point c_1 , after staying a long time close to q , returns to a small neighbourhood of itself and a parabolic periodic point appears which shadows the orbit of the critical point (see lemma 8.10).

Since $c_1(f_0)$ is preperiodic to q , there is a natural number $N \in \mathbb{N}$ such that $f_0^N(\tilde{c}) = q$, and also since \tilde{c} meets only one critical point $c_1(f_0)$ (which is simple) before landing on q , the Taylor expansion of $f_{\tilde{\lambda}}^N(z)$ around $z = \tilde{c}$ and $\lambda = 0$ has the following form:

$$(8.3) \quad f_{\tilde{\lambda}}^N(\tilde{c} + z) = q + A_{\lambda} + B_{\lambda}z^2 + z^3\varepsilon_{\lambda}(z),$$

where A_{λ} , B_{λ} and $\varepsilon_{\lambda}(z)$ depend holomorphically on λ and z . A_{λ} is zero at $\lambda = 0$ but it is not identically zero in a neighbourhood of $\lambda = 0$. This is true because $c_1(f_{\tilde{\lambda}})$ is not persistently preperiodic to q , and so $A_{\lambda} = \lambda^j \hat{A}(\lambda)$, for some holomorphic map \hat{A} with $\hat{A}(0) \neq 0$ and for some natural number $j \in \mathbb{N}$. On the other hand since \tilde{c} meets only one critical point, which is simple, before landing on q , there is no first order term in equation 8.3 and also $B_0 \neq 0$.

By equation 8.2

$$(8.4) \quad f_{\tilde{\lambda}}^{N+n}(\tilde{c} + z) = q + \gamma_{\tilde{\lambda}}^n A_{\lambda} + \tilde{\gamma}_{\tilde{\lambda}}^n B_{\lambda} z^2 + \tilde{\gamma}_{\tilde{\lambda}}^n z^3 \varepsilon_{\lambda}(z).$$

Now for each $n \gg 1$, we are going to find a parameter λ_n close to zero such that the map f_{λ_n} has a parabolic periodic point close to \tilde{c} with period $n + N$ and multiplier equal to one. We find this parameter so that the parabolic periodic point spends most of its time close to the fixed point q . For this purpose we need to solve the following system

of equations:

$$(8.5) \quad f_\lambda^{N+n}(\tilde{c} + z) = \tilde{c} + z,$$

$$(8.6) \quad (f_\lambda^{n+N})'(\tilde{c} + z) = 2\gamma_\lambda^n B_\lambda z + 3\gamma_\lambda^n z^2 \varepsilon_\lambda(z) + \gamma_\lambda^n z^3 (\varepsilon_\lambda)'(z) = 1.$$

From the second equation we obtain

$$(8.7) \quad z = \frac{1}{2\gamma_\lambda^n B_\lambda} - \frac{3z^2 \varepsilon_\lambda(z) - z^3 \varepsilon'_\lambda(z)}{2B_\lambda} := G_{n,\lambda}(z).$$

Using this equation we can find z implicitly in terms of λ . Fix a sufficiently small neighbourhood \mathcal{U} of $\lambda = 0$ and a small neighbourhood W of $z = 0$ such that for large n and for any $\lambda \in \mathcal{U}$ the map $G_{n,\lambda}$ is uniformly contracting on W . So for each n and λ the map $G_{n,\lambda}$ has a unique fixed point $z_n(\lambda)$. To estimate the norm of this fixed point, using the equation 8.7 we obtain

$$z_n(\lambda) \left(1 + \frac{3z_n(\lambda) \varepsilon_\lambda(z_n(\lambda)) - z_n^2(\lambda) \varepsilon'_\lambda(z_n(\lambda))}{2B_\lambda} \right) = \frac{1}{2\gamma_\lambda^n B_\lambda},$$

so the norm of $z_n(\lambda)$ is of $O(\frac{1}{|\gamma_\lambda^n|})$. Now to find λ_n we insert $z_n(\lambda)$ into equation 8.5:

$$\frac{z_n(\lambda) + \tilde{c} - q}{\gamma_\lambda^n} - B_\lambda z_n^2(\lambda) - z_n^3(\lambda) \varepsilon_\lambda(z_n(\lambda)) = A_\lambda = \hat{A}_\lambda \lambda^j.$$

So

$$(8.8) \quad \lambda^j = \frac{1}{\hat{A}_\lambda} \left(\frac{\tilde{c} - q}{\gamma_\lambda^n} + \frac{z_n(\lambda)}{\gamma_\lambda^n} - B_\lambda z_n^2(\lambda) - z_n^3(\lambda) \varepsilon_\lambda(z_n(\lambda)) \right) := H_n(\lambda).$$

since the sequence of maps $\lambda^j - H_n(\lambda)$ converges uniformly on \mathcal{U} to the map λ^j , by Hurwitz theorem we conclude that for n large enough, the equation $\lambda^j - H_n(\lambda) = 0$, has j solutions counted with multiplicity. Let λ_n be one of these solutions. The pair $(\lambda_n, z(\lambda_n))$ solves both equations 8.5 and 8.6 so $z_n(\lambda_n)$ is a parabolic periodic point of f_{λ_n} with period $n + N$. It remains to show that this periodic point spends most of its time close to the fixed point q .

Considering the fact that the norm of $z_n(\lambda)$ is of $O(\frac{1}{|\gamma_\lambda^n|})$ the equation 8.8 implies that the norm of λ_n^j and hence the norm of A_{λ_n} are of $O(\frac{1}{|\gamma_\lambda^n|})$ and so the distance between $f_{\lambda_n}^N(\tilde{c} + z_n(\lambda_n))$ and the fixed point q is of this order as well. This shows that the orbit of $z_n(\lambda_n)$ stays $n - O(1)$ iterations close to q . Note that since N is fixed, by increasing n the proportion of times that this parabolic periodic point spends close to q tends to 1 and so we are done. \square

The following lemma describes the statistical behavior of Lebesgue a.e. point for the dynamics $f_{\hat{\lambda}}$, where the parameter $\hat{\lambda}$ is given by Lemma 8.10.

Lemma 8.11. *Under the iteration of the map $f_{\hat{\lambda}}$ the empirical measures of Lebesgue almost every point converges to the invariant probability measure supported on the orbit of the parabolic periodic point $\hat{q}(f_{\hat{\lambda}})$.*

PROOF. Let \tilde{U} be an immediate basin of attraction of the parabolic periodic point $\hat{q}(f_{\hat{\lambda}})$. By Theorem 10.15 in [33], the domain \tilde{U} contains a critical point of the map $f_{\hat{\lambda}}$. The only critical point which can live in \tilde{U} is $c_1(\hat{\lambda})$, because the other ones are preperiodic to repelling periodic points and so are in the Julia set. Assume for the sake of contradiction that there exists a Fatou component \tilde{V} of $f_{\hat{\lambda}}$ which has an orbit disjoint from \tilde{U} . By Sullivan's classification of Fatou components for rational maps, the domain \tilde{V} should be a preimage of a periodic Fatou component \tilde{W} . The component \tilde{W} cannot be neither a component of the immediate attracting basin of an attracting periodic point nor a component of an immediate attracting basin of a parabolic periodic point, because otherwise it should contain a critical point other than $c_1(\hat{\lambda})$ in its forward orbit which is not possible. Since the boundary of a Siegel disk or a Herman ring is accumulated by the orbit of a critical point, the component \tilde{W} cannot be neither of these cases as well. But these are the only possible cases, which is a contradiction.

Consequently, the set $\bigcup_{n \geq 0} f_{\hat{\lambda}}^{-n}(\tilde{U})$ is the whole Fatou set. Next note that every critical point of $f_{\hat{\lambda}}$ is non-recurrent. In [36] it is proved that a rational map with no recurrent critical point has a Julia set with Hausdorff dimension less than two or a Julia set equal to $\hat{\mathbb{C}}$. As the Fatou set of $f_{\hat{\lambda}}$ is non empty, the Julia set of $f_{\hat{\lambda}}$ has Hausdorff dimension less than two and in particular has zero Lebesgue measure. This means that almost every point $x \in \hat{\mathbb{C}}$ eventually fall into \tilde{U} , and will be attracted by the orbit of $\hat{q}(f_{\hat{\lambda}})$. \square

Remark 8.12. The map $f_{\hat{\lambda}}$ is in the set $\bar{\kappa}_d$.

PROOF. Since the parabolic periodic point for $f_{\hat{\lambda}}$ is not persistent, the parameter $\hat{\lambda}$ is in the bifurcation locus B . So again using a normal family argument as in Lemma 8.8, it can be shown that $f_{\hat{\lambda}}$ is approximated by maps f_{λ} , for which the critical point $c_1(f_{\lambda})$ is preperiodic to a repelling periodic point. This means that $f_{\lambda} \in \kappa_d$ and hence $f_{\hat{\lambda}} \in \bar{\kappa}_d$. \square

To finish the proof of the main lemma, note that by Lemma 8.11 and Lemma 8.10, the limit of the empirical measures of almost every point for the map $f_{\hat{\lambda}}$ is close to $e_{\infty}^f(q)$. And moreover, by the previous remark, $f_{\hat{\lambda}}$ is in $\bar{\kappa}_d$. So the map f statistically bifurcates toward $\delta_{e_{\infty}^f(q)}$ with perturbations in $\bar{\kappa}_d$. \square

9. Periodic measures are dense in $\mathcal{M}_1(f)$

The aim of this section is to prove Proposition 7.3. Through out this section we assume that f is a strictly postcritically finite rational map of degree $d \geq 2$. Since f has no periodic critical point, it has at least one critical point $c \in \mathcal{C}(f)$ which is not in the post critical set $\mathcal{P}(f)$. So the set $f^{-1}(\{c\})$ has d elements, and since $d \geq 2$, the set $\mathcal{A} := \mathcal{P}(f) \cup \mathcal{C}(f) \cup f^{-1}(\{c\})$ has at least three elements. The Riemann surface $\hat{\mathbb{C}} \setminus \mathcal{A}$ is hence a hyperbolic Riemann surface and has the Poincaré disk \mathbb{D} as a universal cover. Let us fix a covering map $\pi : \mathbb{D} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{A}$.

For any point $x \in \hat{\mathbb{C}} \setminus \mathcal{A}$ and any of its d preimages y , the map f is a local diffeomorphism from a neighborhood of y onto a neighborhood of x . Thus its inverse branch is well defined and can be locally lifted to the universal covering. We claim that this map can be extended to a map $F : \mathbb{D} \rightarrow \mathbb{D}$ satisfying the following property:

$$(9.1) \quad f \circ \pi \circ F = \pi.$$

To see this, choose $\tilde{x} \in \pi^{-1}(\{x\})$ and $\tilde{y} \in \pi^{-1}(\{y\})$, and define $F(\tilde{x}) = \tilde{y}$. To define F on an arbitrary point $\tilde{z} \in \mathbb{D}$, consider a curve $\gamma : [0, 1] \rightarrow \mathbb{D}$ with $\gamma(0) = \tilde{x}$ and $\gamma(1) = \tilde{z}$. Then by projecting this curve to $\hat{\mathbb{C}} \setminus \mathcal{A}$ and using the continuation of the inverse branch sending x to y , we obtain a curve in $\hat{\mathbb{C}} \setminus \mathcal{A}$ starting at y and ending at a point in $f^{-1}(\{\pi(\tilde{z})\})$. This new curve has a lift to the universal cover, which starts at \tilde{y} . We define $F(\tilde{z})$ as the endpoint of the latter curve. The map F is well defined since for any other curve γ' joining \tilde{x} to \tilde{z} , the loop $(\gamma')^{-1} \circ \gamma$ is contractible in \mathbb{D} , so its projection $\pi((\gamma')^{-1} \circ \gamma)$, is a contractible loop in $\hat{\mathbb{C}} \setminus \mathcal{A}$ as well. The inverse image of this loop under the continuation of the branch of f^{-1} sending x to y is then contractible in $\hat{\mathbb{C}} \setminus \mathcal{A}$, and so lifts to a closed loop in \mathbb{D} , starting from \tilde{y} . This Shows that we obtain the same points for $F(\tilde{z})$ using both γ and γ' , and hence F is well defined. By definition, it is obvious that equation (9.1) holds for F .

We denote the hyperbolic metric on the Poincaré disk \mathbb{D} by \tilde{d}_h . Recall that any Deck transformation of the covering $\pi : \mathbb{D} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{A}$ is a biholomorphism, and so it leaves invariant the Poincaré metric \tilde{d}_h .

Thus we can push forward the metric \tilde{d}_h and obtain a metric d_h on $\hat{\mathbb{C}} \setminus \mathcal{A}$.

Lemma 9.1. *For the metric \tilde{d}_h , the derivative $DF(z)$ is contracting at every $z \in \mathbb{D}$.*

PROOF. Schwarz lemma implies that if F is not an isomorphism of the Poincaré disk, then $DF(z)$ is \tilde{d}_h -contracting for every $z \in \mathbb{D}$. We are going to show that f is not surjective and hence can not be an isomorphism. Choose a point $x \in \mathcal{A}$ which is a preimage of the critical point c . Let y be a preimage of x . We recall that c is not in the postcritical set, so y cannot be in \mathcal{A} . Now take any point $\tilde{y} \in \pi^{-1}(y)$. Since we have

$$f \circ \pi \circ F(\mathbb{D}) = \pi(\mathbb{D}) = \hat{\mathbb{C}} \setminus \mathcal{A},$$

\tilde{y} cannot be in the range of F . □

The following corollaries are immediate consequences of the previous lemma:

Corollary 9.2. *At every point $x \in \hat{\mathbb{C}} \setminus \mathcal{A}$, any inverse branch of f has a contracting derivative for the metric d_h .*

Corollary 9.3. *Any periodic point of f is repelling.*

PROOF OF PROPOSITION 7.3. We shall prove that every probability measure of f can be approximated by invariant probability measures supported on the orbit of a periodic point. First let us show this for the case where the probability measure is ergodic.

Lemma 9.4. *Any ergodic invariant probability measure $\mu \in \mathcal{M}_1(f)$, can be approximated by invariant probability measures supported on the orbit of a periodic point.*

PROOF. Since μ is ergodic, we can find a point x in the support of μ which is *regular* for μ meaning that the sequence of the empirical measures $\{e_n^f(x)\}_{n \in \mathbb{N}}$ converges to μ . If the orbit of x intersects the set \mathcal{A} , the point x is eventually periodic and in fact is a periodic point in \mathcal{A} . In this case, the measure μ is itself a measure supported on the orbit of the periodic point x . So let us assume that the orbit of x is disjoint from \mathcal{A} . For small $r > 0$, let $B_r(x)$ be the ball of radius r about x with respect to the metric d_h . Since the metric d_h is complete, the closure of $B_r(x)$ is included in $\hat{\mathbb{C}} \setminus \mathcal{A}$. Note that there are only finite inverse branches of f , and we can use Corollary 9.2 to conclude that there is a number $0 < \alpha < 1$ such that any inverse branch of f over $B_r(x)$ is at least α -contracting.

On the other hand, since x is in the support of μ , and also a regular point for this measure, its orbit returns infinitely many times to its hyperbolic $\frac{r}{4}$ -neighbourhood. Let $m \in \mathbb{N}$ be such that $\alpha^m < \frac{1}{2}$. Choose $n \in \mathbb{N}$ such that the orbit of x up to n iterations contains at least $m+1$ points inside $B_{\frac{r}{4}}(x)$, including $f^n(x)$. Let $U_0 := B_{r/2}(f^n(x))$, and for each $1 \leq i \leq n$, denote the connected component of $f^{-i}(U_0)$ containing $f^{n-i}(x)$ by U_i . Since any inverse branch of f is non-expanding, any U_i is contained in a ball of radius $\frac{r}{2}$ around $f^{n-i}(x)$. And so when $f^{n-i}(x)$ is $\frac{r}{4}$ close to x , U_i is contained in $B_r(x)$. This implies that f^{-1} sending U_i to U_{i+1} is α -contracting and so the branch g of f^{-n} from U_0 to U_n is α^m -contracting. Recalling that $\alpha^m < \frac{1}{2}$, this implies that U_n is in $\frac{r}{4}$ -neighbourhood of x . But U_0 covers the $\frac{r}{4}$ -neighbourhood of x , so g sends U_0 into itself, and is α^m -contracting. Thus there is a fixed point p of g in the closure of U_0 . This fixed point is an n -periodic point of f satisfying:

$$(9.2) \quad \forall i \in \{0, \dots, n\}, \quad d_h(f^i(x), f^i(p)) < \frac{r}{2}.$$

But there is a constant $C > 0$ (depending only on \mathcal{A}) such that for any two points x and y in $\hat{\mathbb{C}} \setminus \mathcal{A}$ we have:

$$d(x, y) < C d_h(x, y),$$

where $d(x, y)$ is the standard spherical metric between x and y in $\hat{\mathbb{C}}$. We refer the reader to [11]. So the orbit of x and the periodic point p are close to each other in the spherical metric:

$$\forall i \in \{0, \dots, n\}, \quad d(f^i(x), f^i(p)) < C \frac{r}{2},$$

and hence

$$d_w(e_n^f(x), e_n^f(p)) < C \frac{r}{2}.$$

By choosing r small enough and n large enough, we can guarantee that $e_n^f(x)$ is close to μ . This shows that μ can be approximated by the invariant measures supported on the orbit of periodic points. \square

The final step in the proof of Proposition 7.3 is to show that every invariant measure of f can be approximated by the invariant measures supported on the orbit of only one periodic point. For this we show that any finite convex combination of ergodic invariant measures of f can be approximated by such measures, and since, the finite convex combinations of ergodic invariant measures are dense in the set of invariant

measures of f (according to ergodic decomposition theorem, any invariant measure can be written as an integral of erg), Proposition 7.3 follows.

Let μ_1, \dots, μ_k be k ergodic invariant measures, and $\mu = \sum_{i=1}^k c_i \mu_i$ a convex combination of these measures for some $0 \leq c_i \leq 1$ with $\sum_{i=1}^k c_i = 1$. By lemma 9.4 for each $1 \leq i \leq k$, there exists a periodic point p_i such that $d_w(\mu_i, e_\infty^f(p_i))$ is arbitrary small and hence $d_w(\mu, \sum_{i=1}^k c_i e_\infty^f(p_i))$ is small. So for our purpose, it is enough to show that the measure $\sum_{i=1}^k c_i e_\infty^f(p_i)$ can be approximated by invariant probability measures which are supported on the orbit of only one periodic point. To show this, For technical reasons it is better to bring into play another repelling periodic point p_0 , which is not in the post critical set $\mathcal{P}(f)$.

Since the Julia set of f is the whole Riemann sphere, the set of all preimages of each periodic point p_i is dense in $\hat{\mathbb{C}}$, and in particular has a point in the linearization domain of the other k periodic points. So we can find $\varepsilon > 0$ such that the preimages of ε -neighbourhood of p_i has a connected component in the linearization domain of p_{i+1} (for $i=k$, consider p_0 instead of p_{i+1}). Let us denote the ε -neighbourhood of p_i by U_i . Now note that preimages of U_i has indeed a connected component in U_{i+1} because any subset of the linearization domain, has preimages converging to the periodic point p_i . Take $l_i \in \mathbb{N}$ such that $f^{-l_i}(U_i)$ has a connected component in U_{i+1} (in U_0 , for $i = k$).

Now we find a periodic point, in a backward orbit of U_0 which returns to itself. For each set of natural numbers $\{n_1, \dots, n_k\} \subset \mathbb{N}$ such that n_i is divisible by the period of p_i , consider the following backward orbit of U_0 : the set U_0 is sent by f^{-l_0} into U_1 . Then for each $1 \leq i \leq k$ spends n_i backward iterations in the linearization domain of p_i , and then by f^{-l_i} goes from U_i to U_{i+1} (to U_0 for $i = k$). So finally, we will obtain a preimage \tilde{U}_0 of U_0 in itself. Since U_0 does not intersect the post critical set, there is no critical point in the preimages of this set, and the inverse branch sending U_0 to \tilde{U}_0 is a homeomorphism, and in particular, by Brouwer fixed point theorem, it has a fixed point p . This fixed point is a periodic point for the map f with the period equal to $N := l_0 + \sum_{i=1}^k l_i + n_i$. This periodic point spends n_i iteration close to the orbit of p_i , so since the sum $\sum_{i=0}^k l_i$ is bounded, by choosing very large integers n_i such that for each i , the number $\frac{n_i}{N}$ is close c_i , we can guarantee that $e_N^f(p)$ is arbitrarily close to $\sum_{i=1}^k c_i e_\infty^f(p_i)$. This finishes the proof of Proposition 7.3.

□

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Résumé

Dans cette thèse, nous étudions les propriétés statistiques des systèmes dynamiques. Plus précisément, nous étudions si le comportement statistique asymptotique des orbites typiques via la convergence de la suite des mesures empiriques. Nous étudions également la stabilité du comportement asymptotique sous l'effet de perturbations de la dynamique au sein d'une famille.

Ce mémoire comporte deux parties. Dans la première partie, nous relierons ces deux études. La seconde partie est consacrée à la preuve de l'existence de dynamiques non statistiques dans l'espace des applications rationnelles de la sphère de Riemann. Une application est dite non statistique par rapport à la mesure de référence, s'il existe un ensemble de mesure positive de points qui ont leur suite de mesures empiriques divergente.

Dans la première partie, nous avons développé un cadre abstrait qui vise à comprendre ce qui se cache derrière l'existence d'applications non statistiques dans une famille de dynamiques donnée. Dans cette partie de la thèse, nous étendons et formalisons le concept d'instabilité statistique dans un sens général, indépendant du comportement statistique de la dynamique, et nous montrons comment l'abondance d'applications statistiquement instables, dans une famille donnée de dynamiques, implique l'existence d'applications non statistiques dans cette famille. Nous proposons une autre application de cette formalisation, ainsi qu'un autre exemple de cartes non statistiques: les difféomorphismes Anosov-Katok non statistiques de l'anneau.

Mots-clés: Dynamiques non statistiques, instabilité statistique, applications rationnelles, difféomorphismes Anosov-Katok.

Abstract

In this thesis, we study the statistical properties of dynamical systems. More precisely, we study the asymptotic statistical behavior of typical orbits via the convergence of the sequence of empirical measures. We also study the stability of asymptotic behavior under the effect of small perturbations of the dynamics within a family of maps.

This note has two parts. In the first part, we link the two subjects mentioned above. The second part is devoted to the proof of the existence of non-statistical dynamics in the space of rational maps of the Riemann sphere. A map is said to be non-statistical with respect to the reference measure, if there is a set of positive measures of points which have a divergent sequence of empirical measures.

In the first part, we have developed an abstract setting which aims to understand what lies behind the existence of non-statistical maps in a given family of dynamics. In this part, we extend and formalize the concept of statistical instability in a general sense, independent of the statistical behavior of the dynamics, and we show how the abundance of statistically unstable maps, in a given family of dynamics, implies the existence of non-statistical maps in this family. We provide another application of this formalization, and also another example of non-statistical maps: non-statistical Anosov-Katok diffeomorphisms of the annulus.

Keywords: Non-statistical dynamics, statistical instability, rational maps, Anosov-Katok diffeomorphisms.