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# Stabilization of the systems describing water waves and their interaction with a floating object

Pei Su

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École doctorale  
**Mathématiques  
et informatique**

 université  
de **BORDEAUX**

 Institut de  
Mathématiques de  
Bordeaux

THÈSE DE DOCTORAT

présentée par

**Pei SU**

pour obtenir le grade de

**DOCTEUR**

**DE L'UNIVERSITÉ DE BORDEAUX**

École Doctorale de Mathématiques et d'Informatique

Spécialité : Mathématiques appliquées

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# Stabilization of the systems describing water waves and their interaction with a floating object

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dirigée par Marius TUCSNAK et David LANNES

Soutenue le 9 December 2021 devant le jury composé de :

Bingyu ZHANG	Professeur	University of Cincinnati	Rapporteur
Lassi PAUNONEN	Professeur Associé	Tampere University	Rapporteur
Marius TUCSNAK	Professeur	Université de Bordeaux	Directeur
Matthieu HILLAIRET	Professeur	Université de Montpellier	Examineur
Birgit JACOB	Professeure	University of Wuppertal	Examinatrice
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# Stabilisation des systèmes décrivant le mouvement des vagues et leurs interactions avec un objet flottant

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Université de Bordeaux  
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# Résumé

Ce travail est consacré à l'étude des propriétés de contrôlabilité et de stabilisabilité pour des systèmes de vagues et leurs interactions avec un objet flottant. Les équations utilisées pour les vagues sont une version entièrement linéaire et entièrement dispersive de la formulation de Zakharov-Craig-Sulem. Pour l'interaction des vagues avec un objet flottant on utilise les équations des eaux peu profondes (Saint-Venant).

La première partie de cette thèse a pour objectif d'étudier la stabilisabilité d'une classe de système de commande avec générateur anti-adjoint. Ceci est motivé par le problème de contrôle du système linéarisé de vagues. Avec les hypothèses sur le spectre des opérateurs d'évolution impliqués dans le système de contrôle, nous obtenons un taux de décroissance non uniforme explicite de l'énergie de ce système, à condition que les données initiales soient lisses.

Nous considérons dans la deuxième partie suivantes le système de vagues avec amplitude faible dans un rectangle, où le contrôle agit sur une frontière latérale, en imposant la vitesse de l'eau. Nous étudions d'abord le caractère bien posé de l'ensemble du système, qui est abordé en formulant les équations comme un système de contrôle linéaire abstrait. Après, nous obtenons le taux de décroissance de l'énergie en utilisant les résultats généraux de la première partie. Ensuite, nous étudions le comportement asymptotique de la solution en régime d'eau peu profonde. Il s'avère que le système de vagues d'eau converge vers l'équation des vagues en 1D avec contrôle aux limites de Neumann, lors de la prise de la limite de faible profondeur.

Dans la dernière partie, nous nous intéressons à un objet rigide flottant dans un réservoir d'eau. L'objet se déplace uniquement dans la direction verticale et le contrôle est une force verticale imposée agissant sur l'objet. Nous dérivons les équations non linéaires pour ce système et étudions l'espace atteignable et la stabilisabilité des équations linéarisées dans divers cas.

## Mots-clefs

Système dimensionnel infini; équation des vagues linéariser; stabilisation forte; équations d'eau peu profonde; interactions fluide-structure.



# Abstract

This work is devoted to studying the controllability and stabilizability properties of the control model coming from water waves problem and floating body system in a bounded domain. The equations used for the water waves system in this work is a fully linear and fully dispersive version of the Zakharov-Craig-Sulem formulation. The floating body system here is described by the nonlinear shallow water equations.

The first part of this thesis aims to study the stability of a class of skew-adjoint control systems. This is motivated by the control problem of a linear water waves system. With the assumptions on the spectrum of the evolution operators involved in the control system, we obtain an explicit non-uniform decay rate of the energy of this system, provided that the initial data is smooth.

We consider in the second part the small-amplitude water waves system in a rectangular domain, where the control acts on one lateral boundary, by imposing the velocity of the water. We first study the well-posedness of the whole system, which is addressed by formulating the equations as an abstract linear control system. Afterwards, we obtain the decay rate of the energy by using the general results in the first part. Next we study the asymptotic behaviour of the solution in the shallow water regime. It turns out that the water waves system with boundary control converges to the wave equation in 1D with Neumann boundary control, when taking the shallowness limit.

In the last part, we are interested in a rigid object floating in a bounded water tank. The object only moves in the vertical direction and the control is a vertical force imposed from the bottom of the object. We derive the nonlinear equations for this system and study the reachability and stabilizability of the linearized equations in various cases.

## Keywords

Infinite dimensional system; linearized water waves equation; strong stabilization; shallow water equations; fluid-structure interactions.



# Contents

<b>Notation</b>	<b>xiii</b>
<b>1 Introduction (Français)</b>	<b>3</b>
1.1 Stabilisation d'un système de contrôle linéaire . . . . .	3
1.2 Contrôle d'un système de vagues . . . . .	6
1.2.1 Caractère bien posé et stabilisation du système linéarisé . . . . .	8
1.2.2 Analyse asymptotique des ondes de gravité . . . . .	8
1.3 Contrôle d'un système d'un objet flottant . . . . .	9
<b>2 Introduction (English)</b>	<b>13</b>
2.1 Stabilizability of a linear control system . . . . .	13
2.1.1 Conditions for exponential stability . . . . .	15
2.1.2 Non-uniform stability . . . . .	16
2.2 Control problem on free surface water waves . . . . .	19
2.2.1 Concepts on controllability and observability . . . . .	19
2.2.2 Boundary control problem imposed by a wave maker . . . . .	21
2.2.3 Moving water tank system . . . . .	23
2.3 Wave-structure interactions . . . . .	24
2.3.1 The water waves equations . . . . .	25
2.3.2 The water waves equations with a floating body . . . . .	28
2.3.3 Control problem on the floating body system . . . . .	30
2.4 Contributions of the thesis . . . . .	31
2.4.1 Stabilizability of a water waves system . . . . .	31
2.4.2 Asymptotic analysis of water waves equations in shallow water . . . . .	33
2.4.3 Control of floating body system . . . . .	35
2.4.4 List of publications . . . . .	38
2.5 Organization of the thesis . . . . .	38
<b>I Stabilization of a class of infinite dimensional systems</b>	<b>41</b>
<b>3 Stability of a class of skew-adjoint systems</b>	<b>43</b>
3.1 Problem setting and the main results . . . . .	43
3.2 Preliminaries . . . . .	45
3.3 Proof of the main results . . . . .	47
3.4 Some applications . . . . .	51
3.4.1 Some background on strictly positive operators on Hilbert spaces . . . . .	51

3.4.2	A system described by a fractional Schrödinger equation . . . . .	52
3.4.3	A system described by a fractional wave equation . . . . .	54
<b>II</b>	<b>Water waves system in a rectangular domain</b>	<b>57</b>
<b>4</b>	<b>Boundary control problem of a water waves system</b>	<b>59</b>
4.1	The equations of the water waves control system . . . . .	59
4.1.1	Derivation of the governing equations . . . . .	60
4.1.2	Linearized equations . . . . .	62
4.2	Dirichlet to Neumann and Neumann to Neumann operators in a rectangular domain . . . . .	63
4.2.1	Some background on the partial Dirichlet and Neumann maps . . . . .	64
4.2.2	Partial Dirichlet to Neumann and Neumann to Neumann maps . . . . .	69
4.3	Well-posedness of the linearized water waves system . . . . .	72
4.3.1	Well-posedness in weak sense . . . . .	74
4.3.2	Well-posedness in classical sense . . . . .	77
4.4	Stabilizability properties of the linear system . . . . .	83
4.5	The case of gravity-capillary waves . . . . .	87
4.5.1	Well-posedness and stabilizability issue . . . . .	89
<b>5</b>	<b>Asymptotic behaviour of the gravity waves in a rectangular domain</b>	<b>95</b>
5.1	Introduction and dimensionless equations . . . . .	95
5.2	Dimensionless Dirichlet to Neumann and Neumann to Neumann operators . . . . .	98
5.3	Operator form of the governing equations . . . . .	101
5.4	Operator convergence . . . . .	106
5.5	Main results and the proof . . . . .	111
5.6	Regular convergence . . . . .	115
5.7	Remark on zero mean condition . . . . .	122
5.7.1	A decomposition of the control system . . . . .	123
5.7.2	Stabilization using an observer . . . . .	125
<b>III</b>	<b>A rigid body floating in shallow water</b>	<b>129</b>
<b>6</b>	<b>Control of a floating body system in shallow water</b>	<b>131</b>
6.1	Introduction and problem setting . . . . .	131
6.1.1	Notation . . . . .	132
6.1.2	Observation and strategy . . . . .	133
6.2	Nonlinear modelling of floating object - shallow water interaction . . . . .	134
6.2.1	Governing equations of the floating body system with control . . . . .	135
6.2.2	Reformulation of the governing equations . . . . .	138
6.3	Well-posedness and spectral analysis for the linear system . . . . .	142
6.3.1	Well-posedness of the linearized system . . . . .	143
6.3.2	Spectral analysis . . . . .	147
6.4	Reachability and stabilizability of the linearized system . . . . .	152
6.4.1	Some background on controllability and reachable spaces . . . . .	152

6.4.2	Symmetric case . . . . .	153
6.4.3	Reachability and stabilizability . . . . .	157
6.5	The case when the object floats at one lateral boundary . . . . .	160
6.5.1	Derivation of the governing equations . . . . .	161
6.5.2	Linearized equations . . . . .	165
6.6	Conclusions, comments and open questions . . . . .	168
<b>7</b>	<b>Perspectives</b>	<b>171</b>
7.1	Control system in a bounded domain . . . . .	171
7.2	Shallow water convergence . . . . .	172
7.3	Control of floating object . . . . .	172





# Dedication

To my father with love



# Notation

We introduce here some notation which we frequently used throughout this thesis. For other notation, we introduce them in each part where they first appear.

The notation

$$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$$

stands for the sets of natural numbers (starting with 1), integers, real numbers and complex numbers, respectively. We denote  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ .

We shall denote by  $X$  a complex Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the corresponding norm  $\|\cdot\|$ . If  $X$  and  $Z$  are Hilbert spaces, then  $\mathcal{L}(X, Z)$  denotes the spaces of linear continuous operators from  $X$  to  $Z$ , with the usual (induced) norm. In particular, we denote  $\mathcal{L}(X) = \mathcal{L}(X, X)$ . For any  $T \in \mathcal{L}(X)$

$$\|T\|_{\mathcal{L}(X)} = \sup\{\|Tx\| : x \in X \text{ and } \|x\| \leq 1\}.$$

Sometimes we put a subscript near a norm or an inner product, such as in  $\|z\|_X$ , to indicate which norm or inner product we are using. The domain, range and kernel of an operator  $T$  will be denoted by  $\mathcal{D}(T)$ ,  $\text{Ran}(T)$  and  $\text{Ker}(T)$ , respectively. For any  $\alpha \in \mathbb{R}$ , we denote

$$\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \text{Re } s > \alpha\}.$$

In particular,  $\mathbb{C}_0$  represents the right half plane.

If  $n, k \in \mathbb{N}$  and  $\mathcal{O} \subset \mathbb{R}^n$  is an open set, then we use the notation  $\mathcal{H}^k(\mathcal{O})$  for the Sobolev space formed by the distributions  $f \in \mathcal{D}'(\mathcal{O})$  having the property that the derivatives  $\partial^\alpha f \in L^2(\mathcal{O})$  for every multi-index  $\alpha \in \mathbb{Z}^n$  with  $\alpha_j \geq 0$  and  $|\alpha| \leq k$ . For  $f \in \mathcal{H}^k(\mathcal{O})$  we set

$$\|f\|_k^2 = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2}^2.$$

In one dimension, let  $\mathcal{H}^0(\mathcal{O}) := L^2(\mathcal{O})$  and let  $\mathcal{H}^s(\mathcal{O})$ , with  $s > 0$ , denote the fractional-order Sobolev spaces obtained by interpolation via fractional powers of a positive operator (see, for instance, Lions and Magenes [74]). In one dimension, for a Banach space  $H$ ,  $C(\mathcal{O}, H)$  will denote the set of all continuous mappings from  $\mathcal{O}$  into  $H$ . For a closed bounded interval  $\mathcal{O} = [a, b]$ , the space  $C([a, b]; H)$  endowed with the norm

$$\|f\|_{C([a, b]; H)} = \sup\{\|f(t)\| : t \in [a, b]\}$$

is a Banach space. The spaces of  $k$ -times continuously differentiable mappings are denoted by  $C^k(\mathcal{O}; H)$  and  $C^k([a, b]; H)$ , for  $k \in \mathbb{N}$ . Functions in  $C^k(\mathcal{O}; H)$  are also called *functions of class  $C^k$* .

---

Let  $L^p([a, b]; X)$  be the Hilbert space of equivalent classes of strongly measurable mappings  $[a, b] \rightarrow X$  that are  $p$ -integrable with  $1 \leq p < \infty$  (resp. essentially bounded when  $p = \infty$ ), with norm

$$\|f\|_{L^p([a,b];X)} = \left\{ \int_a^b |f(s)|^p ds \right\}^{\frac{1}{p}},$$

$$\left( \text{resp. } \|f(t)\|_{L^\infty([a,b];X)} = \text{ess. sup } \{|f(t)|, t \in [a, b]\} \right).$$

Moreover, for an open interval  $J$  and any Hilbert space  $X$ , the Sobolev space  $\mathcal{H}^1(J; X)$  consists of those locally absolutely continuous function  $z : J \rightarrow X$  for which  $\frac{dz}{dt} \in L^2(J; X)$ . The space  $\mathcal{H}^k(J; X)$  is defined similarly, but now we require more, i.e.  $\frac{dz}{dt} \in \mathcal{H}^{k-1}(J; X)$ . The space  $\mathcal{H}_0^1(J; X)$  consists of those functions in  $\mathcal{H}^1(J; X)$  which vanish at the endpoints of  $J$  (i.e. they have limits equal to zero there).

For two functions  $u$  and  $v$  defined on  $[0, \infty)$  and for any  $\tau \geq 0$ , their  $\tau$ -concatenation, denoted by  $u \diamond_\tau v$ , is the function

$$u \diamond_\tau v = \begin{cases} u(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geq \tau. \end{cases}$$

If the operator  $A : \mathcal{D}(A) \rightarrow X$ , where  $\mathcal{D}(A) \subset X$ , then the *resolvent set* of  $A$ , denoted by  $\rho(A)$ , is the set of those points  $s \in \mathbb{C}$  for which the operator  $sI - A : \mathcal{D}(A) \rightarrow X$  is invertible and  $(sI - A)^{-1} \in \mathcal{L}(X)$ . The *spectrum* of  $A$ , denoted by  $\sigma(A)$ , is the complement of  $\rho(A)$  in  $\mathbb{C}$ . For every  $s \in \rho(A)$ ,  $(sI - A)^{-1}$  is called the *resolvent* of  $A$ . Throughout the thesis, we denote by  $R(s : A)$  the resolvent of  $A$  for  $s \in \rho(A)$  for simplicity.

For a diagonalizable operator (or Riesz-spectral operator)  $A : \mathcal{D}(A) \rightarrow X$ , we denote by  $(\lambda_k)_{k \in \Lambda}$  the eigenvalues of  $A$  and by  $(\phi_k)_{k \in \Lambda}$  the corresponding eigenvectors, which form an orthonormal basis in  $X$ . For every  $\alpha \in \mathbb{R}$ , we introduce a scale of Hilbert spaces  $X_\alpha$  associated with the operator  $A$ , which is defined by ( $X_0 = X$ )

$$X_\alpha = \left\{ z \in X \mid \sum_{k \in \Lambda} (1 + |\lambda_k|^{2\alpha}) |z_k|^2 < \infty \right\},$$

endowed with the inner product

$$\langle \eta, v \rangle_\alpha = \sum_{k \in \Lambda} (1 + |\lambda_k|^{2\alpha}) \eta_k v_k \quad \forall \eta, v \in X_\alpha,$$

where  $\eta_k = \langle \eta, \phi_k \rangle$  and  $v_k = \langle v, \phi_k \rangle$ . For every  $\alpha \in \mathbb{R}$ ,  $X_{-\alpha}$  is the dual space of  $X_\alpha$  with respect to the pivot space  $X$ . We use the notation  $\|\cdot\|_\alpha$  to represent the norm in  $X_\alpha$ . We will apply, in the following chapters, the above definition of a scale of Hilbert spaces to different operators.

Finally, if a function  $f$  only depends on time  $t$ , we denote by  $\dot{f}$  (or sometimes by  $\frac{d}{dt}f$ ) its derivative with respect to  $t$ . For a function with more than two variables, for instance  $f(x, y)$ , we denote the derivative of  $f$  with respect to  $x$  (or  $y$ ) by  $\partial_x f$  (or  $\partial_y f$ ) for the sake of simplicity. For a matrix  $M$ , we denote by  $M^\top$  the transpose of  $M$ . We use the notation  $X^\perp$  to represent the orthogonal complement of the space  $X$ . For a complex number  $\alpha \in \mathbb{C}$ , we use  $\bar{\alpha}$  to represent the complex conjugate of  $\alpha$ .

# Chapter 1

## Introduction (Français)

Cette thèse se concentre sur trois sujets concernant les problèmes de stabilisation des systèmes des vagues et des systèmes fluide-structure dans un domaine rectangulaire. Dans la première partie, nous étudions la stabilité non uniforme d'une classe de semi-groupe de contraction. Dans la deuxième partie, nous considérons la stabilisation et le comportement asymptotique d'un système d'ondes de gravité (capillaire) avec amplitude faible dans un domaine borné. Dans la troisième partie, nous nous intéressons à un système décrivant un objet rigide flottant en eau peu profonde.

### 1.1 Stabilisation d'un système de contrôle linéaire

Nous considérons la stabilisation d'une classe spéciale de systèmes linéaires de dimension infinie, qui est décrite par une équation différentielle abstraite de la forme suivante:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0. \end{cases} \quad (1.1)$$

Soient deux espaces de Hilbert  $U$  et  $X$ , où  $U$  est l'espace d'entrée et  $X$  l'espace d'état. La fonction  $u \in L^2_{\text{loc}}([0, \infty); U)$  dans (1.1) est appelée la fonction d'entrée et  $z \in X$  est la trajectoire d'état correspondante. L'opérateur  $A : \mathcal{D}(A) \rightarrow X$  est le générateur d'un semi-groupe fortement continu (ou  $C_0$ -semigroupe),  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  et l'opérateur  $B$  (éventuellement non-borné) est un opérateur de contrôle admissible pour  $\mathbb{T}$ . Dans le système ci-dessus, nous prenons la rétroaction colocalisée  $u(t) = -B^*z(t)$  (voir, par exemple, Liu [76], Curtain et Weiss [38], Ammari et Tucsnak [8] et leurs références), de sorte que les solutions contrôlées de (1.1) satisfont l'estimation de l'énergie suivante:

$$\|z(0)\|^2 = \|z(t)\|^2 + \int_0^t \|B^*z(\sigma)\|_U^2 d\sigma \quad (t \geq 0).$$

L'estimation ci-dessus implique en particulier que l'énergie  $\|z(t)\|^2$  est non croissante. Les principales questions envisagées dans ce sens consistent à donner quelques conditions suffisantes sur  $(A, B)$  afin d'avoir l'un des types de stabilité suivants:

1. Stabilité exponentielle, c'est-à-dire, il existe  $M, \alpha > 0$  tel que

$$\|z(t)\| \leq Me^{-\alpha t} \|z_0\| \quad (z_0 \in X, t \geq 0);$$

2. Stabilité faible, c'est-à-dire, pour chaque  $x \in X$  et  $z_0 \in X$ , on a

$$\lim_{t \rightarrow \infty} \langle z(t), x \rangle = 0;$$

3. Stabilité forte, c'est-à-dire, pour chaque  $z_0 \in X$ , on a

$$\lim_{t \rightarrow \infty} z(t) = 0;$$

4. Stabilité "polynomiale", c'est-à-dire, il existe une fonction  $f : [0, \infty) \rightarrow [0, \infty)$  satisfaisant  $\lim_{t \rightarrow \infty} f(t) = 0$ , tel que

$$\|z(t)\| \leq f(t) \|z_0\|_{\mathcal{D}(A)} \quad (z_0 \in \mathcal{D}(A), t \geq 0). \quad (1.2)$$

Le stabilité exponentielle est bien sûr la meilleur, mais il existe de nombreux systèmes de contrôle qui n'ont pas ce genre de propriété. Dans Le cas particulier où  $A$  est un opérateur anti-adjoint, la stabilité exponentielle est connue pour être équivalente à la contrôlabilité exacte en un temps fini  $\tau > 0$  de la paire  $(A, B)$  (voir par exemple, Haraux [55], Liu [76], Lasiecka et Triggiani [71], [70]). Avec la même hypothèse, la stabilité forte, est équivalente à l'observabilité en temps infini du couple  $(A, B^*)$ , ou de manière équivalente au fait que  $B^* \phi \neq 0$  pour tout vecteur propre  $\phi$  de  $A$  (voir par exemple, Tucsnak et Weiss [115]). Les propriétés de la stabilité faible sont étudiées par exemple dans Benchimol [18] et Weiss [118]. A notre connaissance, il n'y a pas de condition nécessaire et suffisante en termes de  $A$  et  $B$  pour la quatrième propriété (stabilité "polynomiale"). Le principal résultat théorique de la première partie de cette thèse donne des conditions suffisantes pour avoir la propriété de stabilité "polynomiale", sans avoir forcément recours à la une contrôlabilité approximative de  $(A, B)$  en temps fini.

Soit  $A - BB^* = \mathcal{A}$ , alors le système (1.1) avec  $u(t) = -B^*z(t)$  devient

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), \\ z(0) = z_0. \end{cases} \quad (1.3)$$

Cette classe de systèmes représente plusieurs types d'équations aux dérivées partielles avec amortissement, en particulier les équations d'onde, voir Ammari et Tucsnak [8], Anantharaman et Léautaud [9], et d'autres modèles hyperboliques PDE, par exemple Liu et Zhang [78], Dell'Oro et Pata [41].

Notez que la norme à droite de (1.2) ne peut pas être la norme de  $X$ . Sinon, par les propriétés classiques du semi-groupe, l'inégalité (1.2) implique que l'état du système de contrôle est exponentiellement stable. Par conséquent, nous appelons également la quatrième stabilité (1.2) *stabilité uniforme pour des données lisses*. Ensuite, nous citons quelques résultats classiques de Borichev et Tomilov [26], Ammari et Tucsnak [8], qui est l'idée principale de notre résultat principal dans cette partie.

**Proposition 1** (Borichev et Tomilov [26]). *Soit  $S_t$  un  $C_0$ -semigroupe borné sur un espace de Banach  $X$  de générateur  $\mathcal{A}$  tel que  $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$ . Alors pour  $\alpha > 0$  fixe, nous avons*

$$\|S_t \mathcal{A}^{-1}\| = O(t^{-\frac{1}{\alpha}}) \quad t \rightarrow \infty \quad (1.4)$$

si et seulement si

$$\|R(is : \mathcal{A})\| = O(|s|^{-\alpha}) \quad s \rightarrow \infty. \quad (1.5)$$

## 1.1. STABILISATION D'UN SYSTÈME DE CONTRÔLE LINÉAIRE

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Dans le cas où  $X$  est un espace de Hilbert, nous notons que le résultat de décroissance (1.4) est équivalent à la relation suivante:

$$\|S_t z\| \leq \frac{C}{(1+t)^{\frac{1}{\alpha}}} \|z\|_{\mathcal{D}(\mathcal{A})} \quad \forall z \in \mathcal{D}(\mathcal{A}).$$

De plus, il a été montré dans [26] qu'avec la condition résolvente (1.5) la norme de  $\|S_t z\|$  décroît toujours comme  $t^{-\frac{1}{\alpha}}$  pour des données plus régulières, c'est-à-dire  $z \in \mathcal{D}(\mathcal{A}^\alpha)$ .

Soient deux espace de Hilbert  $X$  et  $Y$ , et un paramètre  $\theta \in (0, 1)$ , on définit l'espace d'interpolation  $[X, Y]_\theta$  par

$$[X, Y]_\theta = \left\{ a : a \in X + Y, t^{-(\theta+1/2)} K(t, a; X, Y) \in L^2[0, \infty) \right\},$$

avec

$$K(t, a; X, Y) = \inf_{a_0 + a_1 = a} (\|a_0\|_X^2 + t^2 \|a_1\|_Y^2)^{1/2},$$

pour  $a_0 \in X$  et  $a_1 \in Y$ . La définition ci-dessus de l'espace d'interpolation  $[X, Y]_\theta$  est largement utilisée, par exemple, dans Lions et Magenes [75]. Cela coïncide avec la définition de l'espace d'interpolation  $(X, Y)_{\theta, 2}$  introduite dans Triebel [111].

Considérons maintenant le problème non amorti associé au système (1.3),

$$\begin{cases} \dot{w}(t) = Aw(t), \\ w(0) = z_0. \end{cases} \quad (1.6)$$

Soient  $Y$  et  $W$  deux espaces de Banach tels que

$$\mathcal{D}(\mathcal{A}) \subset Y \subset X \subset W,$$

$$\forall z \in \mathcal{D}(\mathcal{A}), \|z\|_{\mathcal{D}(\mathcal{A})} \sim \|z\|_Y$$

et

$$[Y, W]_\theta = X$$

pour un nombre réel fixe  $\theta \in (0, 1)$ , où  $[\cdot, \cdot]_\theta$  est l'espace d'interpolation introduit ci-dessus. La proposition suivante dit que le taux de décroissance explicite de la solution de (1.3) dépend d'une inégalité d'observabilité du problème non amorti (1.6).

**Proposition 2** (Ammari et Tucsnak [8]). *Soit pour tout  $z_0 \in \mathcal{D}(\mathcal{A})$  nous avons*

$$\int_0^T \|(B^* w)(t)\|_U^2 dt \geq C \|z_0\|_W^2,$$

alors il existe  $C > 0$ , telle que pour tout  $t > 0$  et pour tout  $z_0 \in \mathcal{D}(\mathcal{A})$ , on a

$$\|z(t)\| \leq \frac{C}{(1+t)^{\frac{\theta}{2(1-\theta)}}} \|z_0\|_{\mathcal{D}(\mathcal{A})},$$

où  $\theta \in (0, 1)$  tel que l'espace d'interpolation  $[Y, W]_\theta = X$ .



Après avoir introduit les deux théorèmes ci-dessus, nous pouvons présenter nos résultats principaux. Motivés par le système des vagues, nous considérons une classe de systèmes de contrôle avec rétroaction colocalisée, décrite par le modèle de contrôle (1.3), où  $A$  est anti-adjoint à résolvante compacte et  $B \in \mathcal{L}(U, X)$ . Dans ce cas, l'opérateur  $A$  est un opérateur spectrale de Riesz, ainsi, pour un ensemble  $J \subset \mathbb{Z}$ , on note ses valeurs propres par  $(i\mu_k)_{k \in J}$  et les vecteurs propres normalisés correspondants par  $(\phi_k)_{k \in J}$ , ce qui forme une base orthonormée dans  $X$ . Pour  $\alpha \in \mathbb{R}$ , on introduit une échelle d'espaces de Hilbert  $X_\alpha$  par

$$X_\alpha = \left\{ z \in X \mid \sum_{k \in J} (1 + |\mu_k|^{2\alpha}) |\langle z, \phi_k \rangle|^2 < \infty \right\}.$$

L'espace de Hilbert  $X_\alpha$  a en effet déjà été introduit dans la partie de la notation.

**Théorème 1** (Su et Tucsnak 2019). *Soit  $A : \mathcal{D}(A) \rightarrow X$  est anti-adjoint à résolvante compacte et  $B \in \mathcal{L}(U, X)$ . On note  $S = (S_t)_{t \geq 0}$  le semi-groupe engendré par l'opérateur  $\mathcal{A} = A - BB^*$ . Supposons que la paire propre de  $A$ , notée  $(i\mu_k, \phi_k)_{k \in J}$ , satisfasse (pour chaque  $k, l \in J$  et  $k \neq l$ )*

$$\begin{aligned} \mu_k \neq \mu_l \quad \text{and} \quad \mu_k = k^\alpha + O(k^q) \text{ as } k \rightarrow \infty; \\ \|B^* \phi_k\|_U \geq \frac{C}{k^\beta}, \end{aligned} \tag{1.7}$$

avec  $q < \alpha - 1$ . Si la condition (1.7) est satisfaite avec  $0 < \alpha < 1$  et  $\beta \geq 0$ , on a alors

$$\|S_t z_0\| \leq \frac{C}{(1+t)^{\frac{\alpha}{2(\beta-\alpha+1)}}} \|z_0\|_{\mathcal{D}(\mathcal{A})} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}).$$

Si la condition (1.7) est satisfait avec  $\alpha \geq 1$  et  $\beta > 0$ , on a l'inégalité d'observabilité pour tout  $z_0 \in X_1$ ,

$$\int_0^T \|(B^* w)(t)\|_U^2 dt \geq C \|z_0\|_{X_{-\frac{\alpha}{\beta}}}^2,$$

où  $w$  est la solution du système (1.6). De plus, nous avons le résultat de décroissance correspondant

$$\|S_t z_0\| \leq \frac{C}{(1+t)^{\frac{\alpha}{2\beta}}} \|z_0\|_{\mathcal{D}(\mathcal{A})} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}).$$

Il est connu que le système de contrôle est stabilisable de manière exponentielle lorsque  $\alpha \geq 1$  et  $\beta = 0$ , nous ne discutons donc pas ce cas ici. Par rapport à la condition suffisante introduite dans Chill et al. [32], la condition (1.7) est plus claire et plus facile à vérifier en utilisant la structure spectrale de  $A$ .

## 1.2 Contrôle d'un système de vagues

Notre intérêt dans cette partie réside dans le contrôle des vagues à la surface d'un fluide incompressible, non visqueux et irrotationnel. Le terme de contrôle dans le système peut être la vitesse du fluide ou une force externe, imposée par un générateur de vagues à partir d'un domaine borné ou produite par une frontière mobile. Le type de systèmes de vagues apparaît naturellement dans la pratique, par exemple un réservoir

## 1.2. CONTRÔLE D'UN SYSTÈME DE VAGUES

d'eau en mouvement et un système d'échange d'eau d'une piscine, qui a été étudié à la fois en ingénierie et en mathématiques depuis longtemps. En mathématiques, la question de la contrôlabilité et de la stabilisation pour le problème de contrôle sur les vagues sont toujours des sujets actifs dans ce domaine.

Nous nous intéressons à un système de contrôle des vagues dans un domaine rectangulaire  $\Omega$ . Le générateur à une frontière latérale induit de petites ondes sur la surface libre supérieure à travers une oscillation avec amplitude faible à cette frontière. En supposant que le domaine  $\Omega$  est formé par une surface d'eau libre  $\Gamma_s$  sur le dessus, un fond plat  $\Gamma_f$  et deux parois verticales, notées  $\Gamma_1$  et  $\Gamma_2$ . Plus précisément, le domaine  $\Omega$  (voir Figure 1.1) est

$$\Omega = \{(x, y) \mid 0 < x < \pi, -1 < y < 0\}.$$

De plus, nous supposons que le générateur de vagues est rigide dans le sens où la

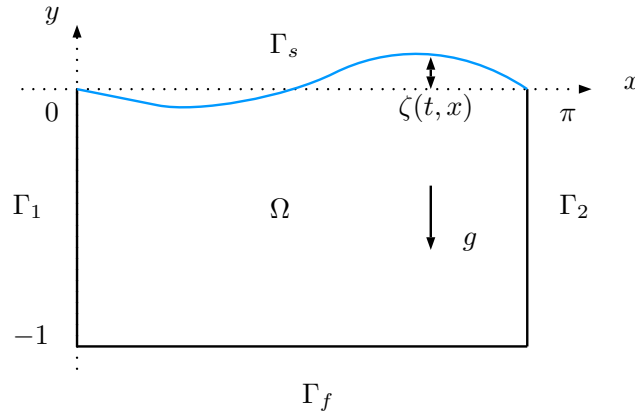


Figure 1.1: Un domaine rectangulaire  $\Omega$  rempli d'eau

vitesse est un multiple d'une fonction d'entrée scalaire  $v(t)$ , multipliée par une fonction donnée  $h$  de la hauteur le long de la frontière active. Pour chaque  $t \geq 0$  et  $(x, y) \in \Omega$ , le système linéarisé de vagues décrit ci-dessus est

$$\begin{cases} \Delta_{x,y} \phi = 0 & \text{in } \Omega, \\ \partial_t \phi(t, x, 0) + \zeta(t, x) = 0, \\ \partial_y \phi(t, x, 0) = \partial_t \zeta(t, x), \\ \partial_x \phi(t, 0, y) = -h(y)v(t), \\ \partial_y \phi(t, x, -1) = 0 = \partial_x \phi(t, \pi, y), \end{cases} \quad (1.8)$$

où  $\zeta(t, x)$  est l'altitude de la surface libre et  $\phi(t, x, y)$  est le potentiel de vitesse du fluide. Pour assurer la conservation du volume d'eau, on suppose généralement que  $\int_{-1}^0 h(y) dy = 0$ . Les propriétés de contrôlabilité et de stabilisabilité des systèmes dérivés de (1.8) sont d'abord étudiées dans Reid et Russell [96]. Pour le problème dans un domaine irrégulier et le cas des vagues avec tension de surface, voir par exemple Reid [94] et [95]. En outre, Mottelet a étudié dans [85] le système de contrôle avec le générateur flexible et rigide, respectivement. Dans ces références, ils considèrent directement le système de contrôle linéaire abstrait associé à (1.8) et il n'y a peu de travail

sur l'ensemble du système (1.8). De plus, il n'y a pas de formulation détaillée du système de contrôle et pas d'analyse complète des opérateurs d'évolution impliqués dans le modèle de contrôle.

### 1.2.1 Caractère bien posé et stabilisation du système linéarisé

Nous établissons dans Su et al. [110] et Su [108] la caractère bien posé de l'ensemble du système des vagues (1.8) via la construction des opérateurs partiels de Dirichlet à Neumann et de Neumann à Neumann. En étudiant le spectre de ces deux opérateurs et en utilisant les résultats généraux du Théorème 1, nous donnons le taux de décroissance explicite de l'énergie du système (1.8). Ensuite, nous présentons dans ce qui suit les principaux théorèmes.

**Théorème 2** (Su, Tucsnaak et Weiss [110]). *Le système d'ondes de gravité (1.8) peut être reformuler de manière en un système de contrôle linéaire abstrait (1.1) avec l'état  $z = \begin{bmatrix} \zeta \\ \xi \end{bmatrix}$  sur  $X$  et le contrôle scalaire  $u$  (l'accélération produite par le générateur de vagues). Pour  $u \in L^2_{\text{loc}}([0, \infty))$  et  $h \in L^2[-1, 0]$  avec  $\int_{-1}^0 h(y) dy = 0$ , les données initiales  $z_0 \in \mathcal{H}^{\frac{1}{2}}[0, \pi] \times L^2[0, \pi]$  et  $\phi_0 \in \mathcal{H}^1(\Omega)$ , le système (1.8) admet une unique solution  $(\zeta, \phi)$  satisfaisant*

$$\phi \in \mathcal{H}^1_{\text{loc}}([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); \mathcal{H}^{\frac{1}{2}}[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

*Ce système de contrôle est stabilisable fortement avec la rétroaction colocalisée  $u = -B^*z$ , et le semi-groupe en boucle fermée correspondant  $S_t$  généré par  $A - BB^*$  satisfait*

$$\|S_t z_0\|_X \leq \frac{C}{(1+t)^{\frac{1}{6}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0.$$

Les ondes à capillarité gravitationnelle ont été discutées dans [108] où nous avons besoin de l'opérateur de Sturm-Liouville pour traiter le terme de tension de surface. Le système de contrôle de gravité capillaire sous la forme de (1.1) est stabilisable fortement et satisfait

$$\|S_t z_0\|_X \leq \frac{C}{(1+t)^{\frac{3}{4}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0.$$

Nous voyons que le taux de décroissance le système de gravité capillaire est plus rapide que le cas de gravité, à cause de l'effet de tension de surface.

### 1.2.2 Analyse asymptotique des ondes de gravité

Nous considérons dans Su [107] le comportement asymptotique de la solution du système de contrôle des vagues gravité (1.8) en régime d'eau peu profonde, où l'échelle horizontale du domaine est beaucoup plus grande que la profondeur verticale. On introduit le paramètre de profondeur des vagues  $\mu$  par

$$\mu = \frac{h_0^2}{L^2}.$$

et quelques quantités sans dimension sont les suivantes:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{h_0}, \quad \bar{t} = \frac{t}{L/\sqrt{gh_0}}, \quad \bar{\zeta} = \frac{\zeta}{a}, \quad \bar{\phi} = \frac{\phi}{aL\sqrt{g/h_0}},$$

### 1.3. CONTRÔLE D'UN SYSTÈME D'UN OBJET FLOTTANT

où  $a$  est l'ordre de variation de la surface,  $L$  est l'échelle horizontale du domaine,  $h_0$  représente la profondeur d'eau typique,  $\bar{\zeta}$  et  $\bar{\phi}$  sont la version sans dimension de l'élevation de la surface libre  $\zeta$  et du potentiel de vitesse  $\phi$ , respectivement. Par conséquent, nous obtenons la version sans dimension de (1.8). Pour énoncer le résultat, nous introduisons l'équation d'onde suivante définie sur  $[0, \pi]$  avec un contrôle de Neumann à l'extrémité gauche, pour tout  $t \geq 0$  et  $x \in [0, \pi]$ , c'est-à-dire

$$\begin{cases} \partial_t^2 \zeta(t, x) - \partial_x^2 \zeta(t, x) = 0, \\ \partial_x \zeta(t, 0) = u(t), \quad \partial_x \zeta(t, \pi) = 0, \\ \zeta(0, x) = \zeta_0(x), \quad \partial_t \zeta(0, x) = \zeta_1(x). \end{cases} \quad (1.9)$$

**Théorème 3** (Su [107]). *Pour  $u \in L^2_{\text{loc}}[0, \infty)$  et pour toute donnée initiale  $\zeta_0 \in \mathcal{H}^1[0, \pi]$  et  $\zeta_1 \in L^2[0, \pi]$ , soit  $\zeta_\mu$  la solution du système de vagues dimensionless, satisfaisant*

$$\zeta_\mu \in C([0, \infty); \mathcal{H}^{\frac{1}{2}}[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

soit  $\zeta$  la solution faible du système (1.9) satisfaisant

$$\zeta \in C([0, \infty); \mathcal{H}^1[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

Alors, pour chaque  $\tau > 0$ , on a

$$\begin{aligned} \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\zeta_\mu - \zeta\|_{\mathcal{H}^{\frac{1}{2}}[0, \pi]} &= 0, \\ \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\partial_t \zeta_\mu - \partial_t \zeta\|_{L^2[0, \pi]} &= 0. \end{aligned}$$

D'après le Théorème 3, la solution du système de vagues converge vers la solution de l'équation des vagues unidimensionnel avec contrôle de Neumann, en prenant la limite de faible profondeur  $\mu \rightarrow 0$ .

### 1.3 Contrôle d'un système d'un objet flottant

Ici, nous considérons l'interaction des vagues avec un objet partiellement immergé dans un fluide modélisé par les équations de l'eau peu profonde. Le contrôle est maintenant une force verticale agissant sur l'objet déplacé verticalement. Nous supposons que l'objet flottant a des parois latérales verticales, avec un fond éventuellement non plat mais symétrique. De plus, soit le domaine intérieur  $\mathcal{I} = [-l, l]$  et le domaine extérieur  $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$ , avec  $\mathcal{E}^- = [-L, -l]$  et  $\mathcal{E}^+ = [l, L']$ . Tout d'abord, nous dérivons les équations régissant ce système de contrôle d'objet flottant dans un réservoir d'eau, toujours noté  $\Omega_t$ ,

$$\Omega_t = \{(x, y) \in [-L, L'] \times [-h_0, \zeta]\}.$$

Sur la base des équations d'eau peu profonde non linéaires avec une structure flottante dans une bande infinie introduites dans Lannes [67], nous avons besoin des conditions aux limites pour le débit horizontal  $q$  et des conditions de transmission aux points de contact du domaine intérieur  $\mathcal{I}$  et le domaine extérieur  $\mathcal{E}$ . Pour ce faire, dans [109] nous suivons les lois de conservation de l'énergie totale et du volume de l'eau. Les

équations gouvernantes complètes du système des objets flottants avec le terme de contrôle dans le domaine borné, pour tout  $t \geq 0$ , sont données par:

$$\partial_t \zeta + \partial_x q = 0 \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (1.10a)$$

$$\partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x P \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (1.10b)$$

$$\underline{P}_e(t, x) = 0 \quad x \in \mathcal{E}, \quad (1.10c)$$

$$\zeta_i(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad x \in \mathcal{I}, \quad (1.10d)$$

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + (\mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l)), \quad (1.10e)$$

$$m \ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (1.10f)$$

$$q_e(t, -L) = 0 = q_e(t, L'), \quad q_i(t, \pm l) = q_e(t, \pm l), \quad (1.10g)$$

avec les données initiales (connues):

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1 \quad \forall x \in \mathcal{I} \cup \mathcal{E},$$

où  $\delta$  est la variation du centre de gravité de l'objet,  $m$  est la masse de l'objet,  $g$  est l'accélération de la gravité. La fonction  $h_{\text{eq}}$  dans (1.10d) représentant la distance entre le bas de l'objet et le bas du domaine avec valeur paire et non négative. La quantité  $\mathfrak{B}$  dans (1.10e) est définie par

$$\mathfrak{B} = \frac{\rho q^2}{2h^2}.$$

De plus, le système (1.10) peut être formulé comme un système d'évolution du premier ordre défini uniquement dans le domaine extérieur  $\mathcal{E}$ , en termes de  $\zeta$ ,  $q$ , la décharge horizontale moyenne à deux points de contact  $\langle q_i \rangle$ , le déplacement et la vitesse de l'objet, avec les conditions de transmission et les conditions aux limites.

Ici, pour les propriétés de contrôle du système de l'objet flottant, nous considérons les équations linéarisées autour de l'état d'équilibre. On définit le saut et la moyenne d'une fonction  $f$  définie sur  $[-l, l]$  par  $\llbracket f \rrbracket = f(l) - f(-l)$  et  $\langle f \rangle = \frac{1}{2}(f(l) + f(-l))$ , respectivement. Pour chaque  $t \geq 0$  et  $x \in \mathcal{E}$ , le système d'objet flottant linéarisé est

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \frac{d}{dt} \langle q_i \rangle = -\frac{g}{2l\alpha} \llbracket \zeta \rrbracket, \\ \ddot{\delta} = -\frac{2\rho g l}{M} \delta + \frac{2\rho g l}{M} \langle \zeta \rangle + \frac{1}{M} u, \end{cases} \quad (1.11)$$

avec

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta}, \quad (1.12)$$

et conditions aux limites

$$q(t, -L) = 0 = q(t, L'), \quad (1.13)$$

où  $\alpha$  et  $M$  sont deux constantes dépendant de la masse  $m$  et de la fonction  $h_{\text{eq}}$ . Soit l'état et les données initiales de (1.11)

$$z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \dot{\delta}]^T, \quad z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^T. \quad (1.14)$$

Notre premier résultat est le caractère bien posé du système linéarisé (1.11)–(1.14).

### 1.3. CONTRÔLE D'UN SYSTÈME D'UN OBJET FLOTTANT

**Théorème 4** (Su et Tucsnak [109]). *Le système de l'objet flottant linéarisé (1.11)–(1.14) forme un système de contrôle linéaire avec l'état  $z$  dans l'espace d'état*

$$X = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right\}$$

et l'espace d'entrée  $U = \mathbb{C}$ . Pour  $u \in L^2_{\text{loc}}([0, \infty); U)$ , les données initiales  $z_0 \in X$ , le système (1.11)–(1.14) admet une solution unique  $z \in C([0, \infty); X)$ .

Pour chaque  $\tau > 0$ , l'entrée de la carte d'état  $\Phi_\tau : L^2([0, \infty); U) \rightarrow X$  du système (1.11)–(1.14) avec zéro donnée initiale (c'est-à-dire  $z_0 = 0$ ) définie par

$$\Phi_\tau u = z(\tau) \quad \forall u \in L^2_{\text{loc}}([0, \infty); U).$$

Le problème principal qui nous intéresse est l'espace atteignable de (1.11)–(1.14), c'est-à-dire  $\text{Ran } \Phi_\tau$  avec  $\tau > 0$ .

Remarquez que lorsque  $L' = L$  et que l'état initial est un état d'équilibre, l'ensemble du système de l'objet flottant conserve sa symétrie pour tout  $t \geq 0$ , dans le sens où  $\zeta$  et  $q$  satisfont

$$\zeta(t, -x) = \zeta(t, x) \quad q(t, -x) = -q(t, x) \quad \forall x \in \mathcal{E}.$$

Nous définissons l'espace de symétrie  $S$  par:

$$S = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \quad \text{and} \quad \begin{cases} \zeta(-x) = \zeta(x), & q(-x) = -q(x) \end{cases} \right\}.$$

Pour énoncer le résultat, nous introduisons l'espace de Hilbert  $W$  :

$$W = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \begin{cases} \llbracket q \rrbracket = -2l\eta, & \langle q \rangle = \langle q_i \rangle \quad \text{and} \quad q(-L) = 0 = q(L') \end{cases} \right\}.$$

**Théorème 5** (Su et Tucsnak [109]). *Supposons que l'objet flotte au milieu du domaine dans la direction horizontale, c'est-à-dire  $L' = L$ . Alors pour chaque  $\tau > \frac{2(L-l)}{\sqrt{g^{h_0}}}$ , on a*

$$(W \cap S) = \text{Ran } \Phi_\tau \subset (X \cap S), \quad (1.15)$$

où l'inclusion est dense avec injection continue.

Dans le cas symétrique décrit ci-dessus, le débit horizontal moyen  $\langle q_i \rangle$  et le saut de l'élévation  $\llbracket \zeta_e \rrbracket$  sont tous deux nuls, de sorte que l'état  $z$  et le système de contrôle linéaire (1.11)–(1.14) peuvent être simplifiés. On voit d'après l'égalité dans (1.15) que l'espace atteignable du système de contrôle (1.11)–(1.14) consiste en n'importe quelles ondes symétriques avec la régularité comme dans l'espace de Hilbert  $W$ . L'inclusion dans (1.15) signifie que le système n'est pas approximativement contrôlable en  $X$ , mais dans son sous-espace  $X \cap S$ . De plus, nous montrons au Chapitre 6 que si nous prenons l'espace d'états comme  $W \cap S$ , le système de contrôle est contrôlable exactement en temps fini. Plus de détails sur ce cas symétrique sont fournis au Chapitre 6, où nous analysons également le cas où l'objet flotte à une frontière latérale de  $\Omega$ .

**Théorème 6** (Su et Tucsnak [109]). *Supposons que l'objet se trouve au milieu du domaine, c'est-à-dire  $L' = L$ . Alors tout profil d'onde symétrique dans l'espace de Hilbert  $W$  peut être atteint par le système de contrôle (1.11)–(1.14). De plus, pour des données initiales régulières ce système est stabilisable fortement, c'est-à-dire*

$$\|z\|_X \leq \frac{C}{(1+t)^{\frac{1}{2}}} \|z_0\|_{W \cap S} \quad \forall z_0 \in W \cap S.$$

La situation en général est compliquée puisque l'espace propre du générateur impliqué dans le système de commande linéaire caractère bien posé, associé à (1.11)–(1.14), n'est peut-être pas simple. Dans ce cas, le système de contrôle avec un contrôle n'est pas observable. Nous étudions également le cas où l'objet est à l'une des extrémités du domaine, c'est-à-dire  $L = l$  ou  $L' = l$ , la dérivation de l'équation non linéaire est différente, mais le système linéarisé est similaire avec (1.11) dans le cas symétrique. Pour plus d'informations sur le cas général, nous donnons une discussion détaillée dans le Chapitre 6.

# Chapter 2

## Introduction (English)

This thesis focuses on three topics on stabilizability issues of some control models arising from water waves and fluid-structure systems in a rectangular fluid domain. In the first part, we study the non-uniform stability of a class of contraction semigroups. In the second part we consider the stabilizability and asymptotic behaviour of a small-amplitude gravity (capillary) water waves system in a bounded fluid domain. In the third part, we are interested in a system describing a rigid object floating in shallow water.

### 2.1 Stabilizability of a linear control system

We consider the stabilization of a special class of infinite dimensional linear systems, which have been extensively studied in the last decades. To specify our terminology and notation, we study the stability properties of abstract differential equations of the form

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0. \end{cases} \quad (2.1)$$

Let  $U$  be the input space and let  $X$  be the state space of (2.1), which are both Hilbert spaces. The function  $u \in L^2_{\text{loc}}([0, \infty); U)$  in (2.1) is called the input function and  $z \in X$  is the corresponding state trajectory. The operator  $A : \mathcal{D}(A) \rightarrow X$  is the generator of a strongly continuous semigroup (or  $C_0$ -semigroup)  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  and the operator  $B$  (possibly unbounded) is an admissible control operator for  $\mathbb{T}$ . We say that  $B$  is an *admissible control operator* for  $\mathbb{T}$ , i.e. the input map (also called the input to state map) defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma \quad \forall u \in L^2([0, \infty); U), \tau \geq 0,$$

satisfies  $\Phi_\tau \in \mathcal{L}(L^2([0, \infty); U); X)$ . If there is no ambiguity, the inner product and the norm in  $X$  will be simply denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ . During the last decades some important literature has been devoted to the stabilization of (2.1) by means of colocated feedback (see, for instance, Liu [76], Curtain and Weiss [38], Ammari and Tucsnak [8] and references therein); this means that the control function  $u$  is chosen in the feedback form  $u(t) = -B^*z(t)$  so that the controlled solutions of (2.1) satisfy the "energy



estimate"

$$\|z(0)\|^2 = \|z(t)\|^2 + \int_0^t \|B^*z(\sigma)\|_U^2 d\sigma \quad (t \geq 0).$$

The above estimate implies, in particular, that the map  $t \mapsto \|z(t)\|^2$  is non-increasing. Most of the references quoted above studied sufficient conditions for various decay types of  $\|z(t)\|$  when  $t \rightarrow \infty$ . The main questions considered in this direction consist in giving sufficient conditions on  $(A, B)$  in order to have one of the following stability types:

1. Exponential stability, i.e., existence of  $M, \alpha > 0$  such that

$$\|z(t)\| \leq Me^{-\alpha t} \|z_0\| \quad (z_0 \in X, t \geq 0);$$

2. Weak stability, i.e.  $\lim_{t \rightarrow \infty} \langle z(t), x \rangle = 0$  for every  $x \in X$  and  $z_0 \in X$ ;

3. Strong stability, i.e. showing that  $\lim_{t \rightarrow \infty} z(t) = 0$  for every  $z_0 \in X$ ;

4. "Polynomial" stability, which means finding a function  $f : [0, \infty) \rightarrow [0, \infty)$  related to a polynomial with  $\lim_{t \rightarrow \infty} f(t) = 0$ , such that

$$\|z(t)\| \leq f(t) \|z_0\|_{\mathcal{D}(A)} \quad (z_0 \in \mathcal{D}(A), t \geq 0). \quad (2.2)$$

The most desirable type of stability, of course, is exponential stability. But there are still many control systems does not satisfy this kind of property. For this, in the special case  $A^* + A = 0$  (i.e. the operator  $A$  is skew-adjoint), the exponential stability is known to be equivalent to the exact controllability in some finite time  $\tau$  of the pair  $(A, B)$ . This is the setup studied in Haraux [55], Liu [76], Lasiecka and Triggiani [71], [70] and others. Under the same assumption, the strong stability is equivalent to the infinite time observability of the pair  $(A, B^*)$  or, equivalently to the fact that  $B^*\phi \neq 0$  for every eigenvector  $\phi$  of  $A$  (for this, please refer to, for instance, Tucsnak and Weiss [115]). The properties of the weak stability are studied, for instance, in Benchimol [18] and Weiss [118], where Weiss provides a relationship between the weak stability (actually the weak  $L^p$ -stability) and the exponential stability in a special case. As far as we know, there is no necessary and sufficient condition in terms of  $A$  and  $B$  for the fourth property ("Polynomial" stability). The sufficient conditions we are aware of are some weak observability type inequalities (in finite time) for the pair  $(A, B)$  (see Russell [101] which gave the first results in this direction and Ammari and Tucsnak [8] for a more general case). The weak finite-time observability inequalities mentioned above imply, in particular, the approximate controllability of  $(A, B)$  in some finite time. The main theoretical result in the first part of this thesis gives sufficient conditions, not necessarily requiring the approximate controllability of  $(A, B)$  in finite time, in order to have the "Polynomial" stability property.

Let  $A - BB^* = \mathcal{A}$  and it is easy to see that the system (2.1) with the feedback  $u(t) = -B^*z(t)$  can be described by

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), \\ z(0) = z_0. \end{cases} \quad (2.3)$$

## 2.1. STABILIZABILITY OF A LINEAR CONTROL SYSTEM

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This class of dynamical systems represents several types of partial differential equations with damping, especially wave equations Ammari and Tucsnak [8], Anantharaman and Léautaud [9], and other hyperbolic PDE models, for instance Liu and Zhang [78], Dell’Oro and Pata [41].

### 2.1.1 Conditions for exponential stability

A number of references studied the stability of the model (2.3) by using the relationship between the resolvents of the operator  $\mathcal{A}$  and the imaginary axis. As we know, the first result on exponential stability of (2.3) is given in the following theorem. For the sake of simplicity, we use the notation  $R(\lambda, A)$  for the resolvent operator associated with a operator  $A$  with  $\lambda \in \rho(A)$ .

**Theorem 1** (Gearhart-Prüss Theorem [91]). *Assume that the operator  $\mathcal{A}$  is a generator of a bounded  $C_0$ -semigroup  $S_t$  on  $X$ , then  $S_t$  is exponentially stable if and only if*

$$\begin{aligned} i\mathbb{R} \cap \sigma(\mathcal{A}) &= \emptyset; \\ \sup_{\beta \in \mathbb{R}} \|R(i\beta : \mathcal{A})\|_X &< \infty. \end{aligned}$$

This result is usually called the Gearhart-Prüss theorem, although it was also obtained independently by Huang [57], Monauni [81] and [82]. It is also a direct consequence of the description of the resolvent set of  $S_t$  by means of the resolvent of the operator  $\mathcal{A}$ . There is another interesting approach to obtain exponential stability of (2.3) from the corresponding undamped system

$$\begin{cases} \dot{w} = Aw(t), \\ w(0) = z_0. \end{cases} \quad (2.4)$$

The main principle is based on HUM (Hilbert uniqueness method), and the sufficient and necessary condition for exponential stability is given by an observability inequality. This result was firstly given in Haraux [55] with the special assumptions on the operators  $A$  and  $B$ .

**Theorem 2** (Haraux [55]). *Let  $A$  be a skew-adjoint operator on  $X$ , let  $B$  be a linear continuous operator from  $U$  to  $X$ , i.e.  $B \in \mathcal{L}(U, X)$ . Assume that  $w$  is the solution of the undamped system (2.4), the semigroup  $S_t$  generated by  $\mathcal{A} = A - BB^*$  is exponentially stable if and only if there exists  $T \geq 0$  and  $C > 0$ , such that*

$$\int_0^T \|(B^*w)(t)\|_U^2 dt \geq C \|z_0\|_X^2.$$

Although we have the above criteria to determine if the system (2.3) is exponentially stable, in practice these conditions are not easy to be satisfied for a specific system. For example, when damping in a string system is located only on a subdomain or on part of the boundary, the energy of the system does not decay in a uniform way. Based on this observation, it was realized that the exponential stability is not achievable with a bounded  $B$  (i.e.  $B \in \mathcal{L}(U, X)$ ) in the case that the input space  $U$  is finite-dimensional, and  $A$  has infinitely many eigenvalues on the imaginary axis. The first result on this was illustrated in Russell [100] by a PDE model of an undamped

string. Similar results for a beam can be found in Slemrod [104]. More generally, we have the following theorem which is proved in Gibson [44], Triggiani [112] or Curtain and Zwart [39, Theorem 4.1.5 and Theorem 5.2.6].

**Theorem 3** (Curtain and Zwart [39]). *Let  $A$  be the generator of a  $C_0$ -semigroup  $\mathbb{T}_t$  on  $X$  and let  $\sigma(A) \cap i\mathbb{R}$  be an infinite set. If the control operator  $B \in \mathcal{L}(C^m, X)$  for some integer  $m \in \mathbb{N}$ , then the control system (2.1) is not exponentially stabilizable.*

According to the above theorem, we conclude that the semigroup generated by a compact perturbation of a generator is not exponentially stable. Now that these kind of systems are not exponentially (uniformly) stable, we still want to determine whether the corresponding solutions approach an equilibrium and how fast do the solutions approach it.

### 2.1.2 Non-uniform stability

For the system whose energy cannot decay in a uniform rate, other kind of decay rates have been introduced, for example, polynomial stability or more generally non-uniform stability. Note that the norm on the right side of (2.2) cannot be the  $X$ -norm. Otherwise, the inequality (2.2), by the classical semigroup properties, implies that the state of the control system is exponentially stable. Therefore, we also call the fourth stability (2.2) *uniformly stability for smooth data* (USSD). Early results giving sufficient conditions under which  $\mathcal{A}$  defined above (2.3) generates a weakly or strongly stable semigroup using LaSalle's principle can be found in Slemrod [103]. This result was improved by Benchimol [18] by using the canonical decomposition of contraction semigroups.

**Theorem 4** (Benchimol [18]). *Let the operator  $A$  generate a contraction semigroup  $\mathbb{T}_t$  and  $B \in \mathcal{L}(U, X)$ . The operator  $\mathcal{A} = A - BB^*$  generates a weakly stable semigroup if*

$$\{z \in X \mid B^* \mathbb{T}_t^* z = 0, \|\mathbb{T}_t z\| = \|\mathbb{T}_t^* z\| = \|z\| = 0, \forall t > 0\} = \{0\}. \quad (2.5)$$

Particularly, if  $A$  has compact resolvents, the condition (2.5) implies strong stability. A similar result was also obtained by Batty and Vu [16], where they improved the above sufficient condition (2.5) for strong stability. They obtained that if the spectrum of  $A$  has at most countably many points of intersection with the imaginary axis, then  $A - BB^*$  generates a strongly stable semigroup if and only if (2.5) holds. Although in this result there is a restriction condition  $B \in \mathcal{L}(U, X)$ , it does not exclude the boundary control problem for PDEs (for this, please refer to Slemrod [104] and Oostveen [87]). A more general result on strong stabilization has been obtained by Curtain and Weiss [39], where  $A$  needs not be skew-adjoint, the control operator  $B$  needs not be bounded (i.e.  $B \in \mathcal{L}(U, X)$ ), moreover with non-colocated feedback.

For the control model described by (2.3), similarly there are some strong stability results, firstly established in a Banach space  $X$ , derived from the information on the spectrum of  $\mathcal{A}$ . The following theorem was proved in Batty [13], see also [14], and it is actually implicitly contained already in Arendt and Batty [10].

**Theorem 5** (Batty [13]). *Let  $S_t$  be a bounded  $C_0$ -semigroup on a Banach space  $X$  with generator  $\mathcal{A}$ . Suppose that  $i\mathbb{R}$  is contained in the resolvent set  $\rho(\mathcal{A})$  of  $\mathcal{A}$  (i.e.  $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$ ). Then*

$$\lim_{t \rightarrow 0} \|S_t z\| = 0 \quad \forall z \in X. \quad (2.6)$$

## 2.1. STABILIZABILITY OF A LINEAR CONTROL SYSTEM

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Here we point out that the original result presented in [13] is a little different, where the convergence result is that the norm of  $S_t$ , as a bounded operator from  $\mathcal{D}(\mathcal{A})$  to  $X$ , tends to zero as  $t$  goes to infinity. By density of  $\mathcal{D}(\mathcal{A})$  in  $X$ , we directly have strong stability (2.6). In general, without any additional assumptions, the decay rate of the state trajectory can be arbitrary slow. However, in some special cases, e.g. damped wave equations, the decay rate in (2.6) corresponds to the decay rate of the energy of the control system described by the semigroup, and it is of interest to determine if this decay rate can be achieved. Normally the PDEs models can be rewritten in the abstract form (2.3), the decay rate of the sufficiently smooth trajectory can be also associated with the size of the resolvent operator  $R(\lambda : \mathcal{A})$  (for  $\lambda \in \rho(\mathcal{A})$ ) of  $\mathcal{A}$  on the imaginary axis. This approach was initially studied in Lebeau [72], and later pursued, in particular, in Burq and Hitrik [28], Christianson [33]. These applications motivated a systematic study for the decay rate of bounded  $C_0$ -semigroups in Liu and Rao [77], Batty and Duychaerts [15]. In the latter reference, they developed a new simplified approach for estimating the decay rate in terms of the growth of  $R(is : \mathcal{A})$  with  $s \in \mathbb{R}$ .

**Theorem 6** (Batty and Duychaerts [15]). *Let  $S_t$  be a bounded  $C_0$ -semigroup on a Banach space  $X$  with generator  $\mathcal{A}$ , such that  $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$ . We define the functions  $M$  and  $M_{\log}$ , for every  $\eta \geq 0$ , as*

$$M(\eta) := \max_{t \in [-\eta, \eta]} \|R(it : \mathcal{A})\|, \quad M_{\log}(\eta) := M(\eta)(\log(1 + M(\eta)) + \log(1 + \eta)).$$

Then there exists  $C, t_0 > 0$  such that, for every  $t \geq t_0$ ,

$$\|S_t \mathcal{A}^{-1}\| \leq \frac{C}{M_{\log}^{-1}(t/C)}.$$

If  $M$  defined above is polynomially growing with the power  $\alpha > 0$  (i.e.  $M(\eta) \leq C(1 + \eta^\alpha)$  for  $\eta > 0$ ), the corresponding decay rate becomes

$$\|S_t \mathcal{A}^{-1}\| \leq C \left( \frac{\log(t)}{t} \right)^{\frac{1}{\alpha}}. \quad (2.7)$$

Note that  $X$  in Theorem 6 is assumed to be a Banach space, it was conjectured in [15, Remark 9] that the logarithmic factor in (2.7) can be removed in the case when  $X$  is a Hilbert space. Based on the observation, Borichev and Tomilov achieved in [26] an optimal decay rate of (2.7) in the case of polynomially growing  $M$ . Moreover, they showed that the result in Theorem 6 is sharp on Banach spaces. The sufficient and necessary condition for the explicit decay rate is given in the following theorem.

**Theorem 7** (Borichev and Tomilov [26]). *Let  $S_t$  be a bounded  $C_0$ -semigroup on a Banach space  $X$  with generator  $\mathcal{A}$  such that  $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$ . Then for a fixed  $\alpha > 0$  we have*

$$\|S_t \mathcal{A}^{-1}\| = O(t^{-\frac{1}{\alpha}}) \quad t \rightarrow \infty \quad (2.8)$$

if and only if

$$\|R(is : \mathcal{A})\| = O(|s|^\alpha) \quad s \rightarrow \infty. \quad (2.9)$$

It is worth noting that the decay result (2.8) is equivalent to

$$\|\mathbb{S}_t z\| \leq \frac{C}{(1+t)^{\frac{1}{\alpha}}} \|z\|_{\mathcal{D}(A)} \quad \forall z \in \mathcal{D}(A),$$

when  $X$  is a Hilbert space. Moreover, it has been proved in [26] that with the resolvent condition (2.9) the norm of  $\|\mathbb{S}_t z\|$  still decays like  $t^{-\frac{1}{\alpha}}$  for more regular data, i.e.  $z \in \mathcal{D}(A^\alpha)$ .

In the case of  $\mathcal{A} = A - BB^*$ , Chill et al. presented in [32] a general sufficient condition for the stability of the semigroup generated by  $A - BB^*$  in terms of selected observability-type conditions of the pair  $(A, B^*)$ . To state the result, we quote some terminology from [32]. We say that  $(A, B^*)$  satisfies the *non-uniform Hautus test* if there exist functions  $M_0, m_0 : \mathbb{R} \rightarrow [r_0, \infty)$  with  $r_0 > 0$  such that

$$\|x\|^2 \leq M_0(s) \|(is - A)x\|^2 + m_0(s) \|B^*x\|, \quad \forall x \in \mathcal{D}(A), s \in \mathbb{R}.$$

When  $A$  is skew-adjoint, we say that  $(A, B^*)$  satisfies the *wavepacket condition* if there exist bounded functions  $\gamma, \delta : \mathbb{R} \rightarrow (0, \infty)$  such that

$$\|B^*x\|_U \geq \gamma(s) \|x\|_X \quad \forall x \in WP_{s, \delta(s)}(A), s \in \mathbb{R}.$$

The *wavepacket set*  $WP_{s, \delta(s)}(A)$  denotes the spectral subspace of  $A$  associated with the interval  $i(s - \delta(s), s + \delta(s)) \in i\mathbb{R}$ . The function  $M : [0, \infty) \rightarrow (0, \infty)$  is called positively increasing if there exist  $\alpha, c_\alpha, s_0 > 0$ , such that

$$\frac{M(\lambda s)}{M(s)} \geq c_\alpha \lambda^\alpha \quad \forall \lambda \geq 1, s \geq s_0.$$

We only present here the corresponding results for bounded  $B$ , i.e.  $B \in \mathcal{L}(U, X)$ . For the general case, please refer to [32] for more details.

**Theorem 8** (Chill et al. [32]). *Assume that  $A$  generates a contraction semigroup  $\mathbb{T}_t$  on Hilbert space  $X$  and that  $B \in \mathcal{L}(U, X)$ . If  $(A, B^*)$  satisfies the non-uniform Hautus test for continuous and even functions  $m_0(\cdot), M_0(\cdot)$ , such that  $M(\cdot) := M_0(\cdot) + m_0(\cdot)$  is strictly increasing and has positive increase on  $[0, \infty)$ , then the semigroup  $\mathbb{S}_t$  generated by  $\mathcal{A} = A - BB^*$  is non-uniformly stable and*

$$\|\mathbb{S}_t z\| \leq \frac{C}{M^{-1}(t)} \|z\|_{\mathcal{D}(A)} \quad \forall z \in \mathcal{D}(A), t \geq t_0, \quad (2.10)$$

for some  $C, t_0 > 0$ , where  $M^{-1}$  is the inverse function of  $M$ .

If  $A$  is skew-adjoint and  $(A, B^*)$  satisfies the wavepacket condition for continuous and even  $\gamma(\cdot), \delta(\cdot)$  such that  $\gamma(\cdot)^{-1} \delta(\cdot)^{-1}$  is strictly increasing and has positive increase, then  $\mathbb{S}_t$  is non-uniformly stable and (2.10) is satisfied for  $M(\cdot) = \gamma(\cdot)^{-2} \delta(\cdot)^{-2}$ .

When  $A$  is skew-adjoint and has compact resolvents, its wavepacket set consists of finite linear combinations of eigenfunctions. For some PDE models, there may exist a uniform gap between the eigenvalues of the evolution operator, and in this case the result in Theorem 8 gives the optimal decay rate for the solution of (2.3). In the same case, we notice that Ingham's inequality is available, which implies that we can derive an observability inequality with the initial data in a larger space. Based on a class

of a second-order evolution equations, this was firstly studied in Russell [101] in the case of bounded feedback controls, but it cannot be directly extended to unbounded feedbacks. This general case was addressed in Ammari and Tucsnak [8] by using the interpolation theory for the domain of a linear operator and additional assumptions for the control operator  $B$ . We do not go into the details about this result here.

## 2.2 Control problem on free surface water waves

Our interest in this part lies in the control of water waves on the surface of an incompressible, inviscid and irrotational fluid. The control term in the system can be the velocity of the fluid or an external force, imposed by a wave maker from one boundary of the fluid domain or produced by a movable boundary. This kind of water waves systems often appear in practice, for example a moving water tank and water exchange system of a pool, which has been studied in both engineering and mathematics for a long time. In mathematics, the controllability and stabilization issue for control problem on water waves are always hot topics in this field. Closely related to the main problem we investigate, we also introduce some similar interesting control system on water waves in this section. Next we quickly recall some basic concepts on controllability.

### 2.2.1 Concepts on controllability and observability

Still using the notation we introduced at the beginning of this section, for the infinite dimensional system (2.1), we first introduce some basic definitions and properties. We denote by  $\Sigma(A, B, -)$  the abstract linear control system (2.1), which is also called *state linear system* in Curtain and Zwart [40]. For simplicity, we only consider bounded control operator, i.e.  $B \in \mathcal{L}(U, X)$ . However, it is important to notice that the concepts for bounded  $B$  has a natural generalization to the general un-bounded case. Controllability is the property of being able to steer between two arbitrary points in the state space  $X$ . For finite-dimensional, time-invariant, linear systems, there is a simpler definition on controllability (see, for instance, Tucsnak and Weiss [115, Chapter 1]). In the infinite dimensional case, the situation becomes more complicated.

**Definition 1** (Controllability). *For the state linear system  $\Sigma(A, B, -)$ , for every  $\tau > 0$  we define the following concepts:*

1.  $\Sigma(A, B, -)$  is exactly controllable in time  $\tau$ , if for every initial data  $z_0 \in X$  and for every element  $z_1 \in X$ , there exists  $u \in L^2([0, \tau]; U)$  such that the solution of (2.1) satisfies  $z(\tau) = z_1$ ;
2.  $\Sigma(A, B, -)$  is null controllable in time  $\tau$ , if for every initial data  $z_0 \in X$ , there exists  $u \in L^2([0, \tau]; U)$  such that the solution of (2.1) satisfies  $z(\tau) = 0$ ;
3.  $\Sigma(A, B, -)$  is approximately controllable in time  $\tau$ , if for every  $\varepsilon > 0$ , for every initial data  $z_0 \in X$  and for every element  $z_1 \in X$ , there exists  $u \in L^2([0, \tau]; U)$  such that the solution of (2.1) satisfies  $\|z(\tau) - z_1\| < \varepsilon$ .

It is obvious to see that exact controllability implies null controllability, but in general the inverse claim is not true. For linear time-invariant control system, exact controllability is equivalent to null controllability. According to the classical semigroup

theory, for bounded control operator  $B$  and for initial data  $z_0 \in X$ , the system (2.1) admits a unique solution  $z \in C([0, \infty); X)$ , which is

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \quad \forall t \geq 0, u \in L^2_{\text{loc}}([0, \infty); U).$$

For every  $t \geq 0$ ,  $\Phi_t : L^2_{\text{loc}}([0, \infty); U) \rightarrow X$  is called the *input to state map* (or, briefly, input map, sometimes also called the controllability map) defined by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds.$$

Based on the properties of the semigroup, the controllability concepts can be equivalently defined in terms of the input map  $\Phi_t$ .

**Proposition 1.** *For every  $\tau > 0$ , the system  $\Sigma(A, B, -)$  is*

1. exactly controllable in time  $\tau$ , if and only if  $\text{Ran } \Phi_\tau = X$ ;
2. null controllable in time  $\tau$ , if and only if  $\text{Ran } \mathbb{T}_\tau \subset \text{Ran } \Phi_\tau$ ;
3. approximately controllable in time  $\tau$ , if and only if  $\overline{\text{Ran } \Phi_\tau} = X$ .

Although these exact controllability properties can be useful, large classes of PDEs will not be exactly controllable but only approximately controllable. It is possible to give some easily verifiable necessary and sufficient conditions for approximate controllability. To give these tests, we consider the dual control system of  $\Sigma(A, B, -)$ . According to the dual argument, we know that the controllability of the system  $\Sigma(A, B, -)$  is equivalent to the observability of its dual system  $\Sigma(A^*, -, B^*)$ . Therefore, we have the equivalent definition for controllability.

**Proposition 2** (Dual definition of controllability). *For every  $\tau > 0$ , we have the following equivalent definition for controllability.*

1.  $\Sigma(A, B, -)$  is exactly controllable in time  $\tau$ , if and only if there exists  $C_\tau > 0$  such that

$$\int_0^\tau \|B^* \mathbb{T}_t^* z\|_U^2 dt \geq C_\tau \|z\|^2 \quad \forall z \in X; \quad (2.11)$$

2.  $\Sigma(A, B, -)$  is null controllable in time  $\tau$ , if and only if there exists  $C_\tau > 0$  such that

$$\int_0^\tau \|B^* \mathbb{T}_t^* z\|_U^2 dt \geq C_\tau \|\mathbb{T}_t^* z\|^2 \quad \forall z \in X; \quad (2.12)$$

3.  $\Sigma(A, B, -)$  is approximately controllable in time  $\tau$ , if and only if we have the property

$$B^* \mathbb{T}_t z = 0 \quad t \in [0, \tau] \implies z = 0.$$

The inequalities (2.11) and (2.12) are called observability inequalities, in the sense that the observation quantity

$$\int_0^\tau \|B^* \mathbb{T}_t^* z\|_U^2 dt$$

gives the corresponding estimate for  $\|z\|$  and  $\|\mathbb{T}_t^* z\|$  when the system under consideration is exactly controllable and null controllable, respectively. Compared with Definition 1 and Proposition 1, the conditions proposed in Proposition 2 are more convenient to determine the controllability property for a certain control system.

### 2.2.2 Boundary control problem imposed by a wave maker

We consider an irrotational, incompressible, inviscid fluid with constant density in a two-dimensional bounded domain  $\Omega$ . We assume that the boundary  $\Gamma$  of  $\Omega$  consists of a top free surface  $\Gamma_s$ , a fixed boundary including a bottom  $\Gamma_f$  with finite arclength and two vertical walls of positive length, denoted by  $\Gamma_1$  and  $\Gamma_2$ , respectively (i.e. no "beaches"). Notice that we consider the small-amplitude water waves, the nonlinear

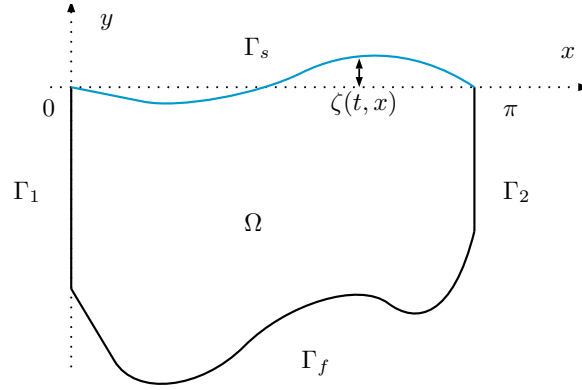


Figure 2.1: A bounded domain  $\Omega$  filled with water

terms are dropped during the linearization process. As we know that the dynamics of irrotational and incompressible fluid can be described by the Laplace equation, thereby the fluid motion can be expressed in terms of a velocity potential  $\phi(t, x, y)$  defined on  $\mathbb{R}^+ \times \Omega$ . Suppose that we control a small-amplitude movement of one end of the water tank, the one whose nominal equation is  $x = 0$ . We denote by  $\zeta$  the elevation of the free surface, then the governing equations, for every  $t \geq 0$ , read

$$\begin{cases} \Delta \phi(t, x, y) = 0 & \text{in } \Omega, \\ \partial_t \phi + g\zeta(t, x) = 0 & \text{on } \Gamma_s, \\ \partial_{\vec{n}} \phi = \partial_t \zeta(t, x) & \text{on } \Gamma_s, \\ \partial_{\vec{n}} \phi = v(t, y) & \text{on } \Gamma_1, \\ \partial_{\vec{n}} \phi = 0 & \text{on } \Gamma_2 \cup \Gamma_f, \end{cases} \quad (2.13)$$

where  $\vec{n}$  is the unit outer-normal vector on the corresponding part of the boundary  $\Gamma$ ,  $g$  represents the gravity acceleration and  $v$  is the velocity on the left boundary  $\Gamma_1$  produced by the small oscillation in the horizontal direction. This small oscillation can be imposed by a wave maker from that active boundary, which causes a small water waves on the free surface  $\Gamma_s$ . There are flexible and rigid wave makers used to generate the velocity, as the control signal, in the water waves system. There are two types of wave makers: flexible and rigid (please refer to Mottelet [85]). The flexible wave maker can produce the desired velocity, while the velocity produced by the rigid wave maker depending on a fixed "shape" described by a function  $h$ , thereby the corresponding velocity takes the form

$$v(t, y) = h(y)u(t).$$

Therefore, we have a scalar control  $u(t)$ . When  $h(y) = y$  the boundary condition on  $\Gamma_1$  represents a plane wave maker being able to rotate around an axis located at the bottom



of the fluid domain, with small angular velocity  $u(t)$ . The details on the derivation of the governing equations (2.13) will be provided in the main contents.

The control system (2.13), with rigid wave maker at one lateral boundary of  $\Omega$ , was firstly studied in Reid and Russell [96]. The fluid domain  $\Omega$  was assumed to be simple geometric in the sense that it is semi-infinite (bottomless) with straight sides. In this case, the evolution operator in the control model possesses explicit eigenvalues and eigenvectors. They showed that the problem of steering the system to zero is equivalent to a moment problem involving the expansion coefficients. It turned out that the system is null controllable in infinite time, while it fails in finite time. The corresponding null control is constructed via the Laplace transform of a set of functions which are bi-orthogonal to the exponentials, i.e.

$$\begin{aligned} \int_0^\infty e^{iw_k s} p_n(s) ds &= \delta_n^k, & \int_0^\infty e^{iw_k s} q_n(s) ds &= 0; \\ \int_0^\infty e^{iw_k s} q_n(s) ds &= \delta_n^k, & \int_0^\infty e^{iw_k s} p_n(s) ds &= 0, \end{aligned}$$

where  $\delta_n^k = 1$  if  $n = k$ , otherwise it is zero. The null control  $u$  takes the form

$$u(t) = \sum_{n=1}^{\infty} (c_n p_n(s) + d_n q_n(s)).$$

Following the semi-infinite fluid domain, Reid extended in [94] their controllability results to a bounded domain with irregular bottom contour, please refer to Figure 2.1. Moreover, the domain may be multi-connected, i.e. contain fixed objects. The approach developed in [96] requires the knowledge of the spectrum of the evolution operator, which cannot be applied directly to this general case. In this irregular case, the eigenvalues and the eigenvectors cannot be computed explicitly, so that the main difficulty lies in the eigenvalue estimate. To solve this, Reid employed a perturbation technique in [94], which comes from an observation that the velocity potential should decay exponentially with depth like the one in infinite bottom. The main idea is that we can first estimate the influence of a bottom by looking in the infinite depth tank at a flow through the proposed bottom, then computing the influence of placing a bottom there. He proved that actually the eigenvalues of the evolution operator are a bounded perturbation of the eigenvalues in infinite depth, and the difference behaves like  $1/k$ . This was addressed by extending the fluid domain to be symmetric with respect to  $x = 0$ ,  $x = \pi$  and  $y = 0$ . In this way, the open-loop null control for infinite time was built in a similar way as in [96]. Moreover, they proposed sufficient conditions for the convergence of the series describing the control.

Besides gravity water waves, Reid also studied in [95] the null controllability of gravity-capillary water waves where both gravity and surface tension are significant. The governing equations are of course a little different with the gravity case (2.13), because of the surface tension effect. The only difference lies in the boundary condition on the top derived from the free surface Bernoulli equation, which is

$$\partial_t \phi + g\zeta = T \partial_{xx} \zeta \quad \text{on } \Gamma_s. \quad (2.14)$$

The constant  $T$  in (2.14) represents a parameter related to the surface tension coefficient. Therefore, the governing equation for gravity-capillary waves system can be

## 2.2. CONTROL PROBLEM ON FREE SURFACE WATER WAVES

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obtained from (2.13) by replacing the second boundary condition with (2.14). Similarly, the control system was formulated as a first-order evolution equation, which is a little different with the one in gravity case, reducing the null controllability problem to a moment problem involving frequency exponentials. For the fluid domain  $\Omega$  with flat bottom, Reid proved in [95] that the gravity-capillary waves system is null controllable in arbitrary finite time.

The control term in the water waves system discussed above are all produced by the rigid wave maker. Assuming that the bottom of  $\Omega$  is flat, Mottelet considered in [85] the control problem on gravity waves with flexible and rigid wave maker, respectively. In [85], Mottelet firstly named the evolution operators used to formulate the control system, which are Dirichlet map and Neumann map. In the flexible case, the input space is infinite-dimensional and he gave a positive result for the approximate controllability and a counterexample to the exact controllability. In the rigid case, the input space is  $\mathbb{R}$  and the approximate controllability does not hold. He also studied the stability of the system with rigid wave maker at one boundary, where the state feedback was taken as the elevation of the free surface at the active boundary, saying  $x = 0$ . In practice, this requires a sensor installed on the wave maker. The control system is strongly stable, but not in a uniform way, by means of an ad hoc energy and a particular choice of the shape function  $h$ .

### 2.2.3 Moving water tank system

Different with the control imposed by the wave maker, there are also a lot of work considering the controllability and stabilization of a moving water tank system. The water tank is pushed by an external force in order to move it from one location to another, and the main problem during this process is the suppression of sloshing.

Based on the main framework used in [85], Mottelet studied in [86] a moving rectangular water tank which is controlled by means of a longitudinal acceleration in the  $x$  direction (please refer to Figure 2.1 with a flat bottom). For this system, we denote by  $\mathcal{U}$  the potential such that the volumic acceleration  $\vec{a}$  is given by  $\vec{a} = \nabla\mathcal{U}$ . Since the acceleration only occurs in the  $x$  direction, we have

$$\mathcal{U}(t, x, y) = (x + c)u(t),$$

where  $u$  is the amplitude of the acceleration of the tank and  $c$  is an arbitrary constant. We denote by  $P(t, x, y)$  the static pressure, then the Bernoulli equation reads

$$\frac{1}{2}|\nabla\phi|^2 + \partial_t\phi + gy + \mathcal{U} = \frac{P_{\text{atm}} - P}{\rho} \quad \text{in } \Omega,$$

where  $\rho$  is the density of the fluid,  $g$  is the acceleration of gravity and  $P_{\text{atm}}$  represents the atmospheric pressure. The corresponding linearized free surface Bernoulli equation is

$$\partial_t\phi + g\zeta + (x + c)u = 0 \quad \text{on } \Gamma_s. \quad (2.15)$$

Therefore, the governing equations of the moving water tank system are (2.13) by replacing the second equation with (2.15). These equations were formulated into a first-order evolution system in [86] where the control operator is simpler than the one in

[85]. By using the similar approach developed in [85], Mottelet showed that this control system is approximately controllable in infinite time and strongly stable in an ad hoc energy.

For completeness we mention that there are also many work on moving water tank system modeled by the nonlinear shallow water equation (also called the Saint-Venant equation), rather than the linear equations (2.13). In this case, the horizontal acceleration of the tank is assumed to be small compared to the gravity constant and that the height of the fluid is assumed to be small compared to the length of the tank. The governing equations (2.13) we introduced above are actually a fully linear and fully dispersive model, while the nonlinear water waves equations are a fully nonlinear and non-dispersive water waves model. We do not go into the details here on the approximation of water waves equation, which will be introduced in wave-structure part. This kind of moving water tank system was studied in, for instance, Coron [35], Prieur and De Halleux [90], Berger et al. [20]. In [35] this nonlinear control system was shown to be locally controllable around any steady state, and it can be moved from one steady state to any other steady state. The stabilization issue was considered in [90] and the control laws was constructed by means of Lyapunov approach. The model used in [20] is a linearized shallow water equations, and they studied the tracking control problem and proved that, through a funnel controller, the tracking error evolves within a pre-specified performance funnel.

### 2.3 Wave-structure interactions

In the final part, we consider the fluid-structure system which focuses on the study of the interaction of free surface water waves with, in particular, a floating body. The fluid under consideration, delimited by a top free surface and bottom, is assumed to be incompressible, irrotational and inviscid, see Figure 2.2. With these assumptions, the

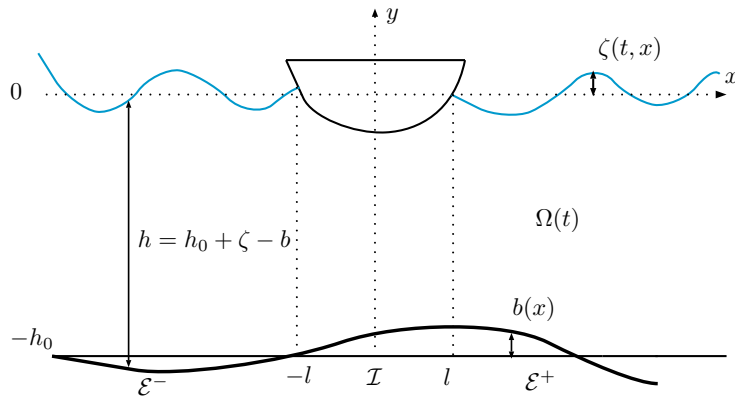


Figure 2.2: An object floating in shallow water

floating body problem was firstly studied in John [61, 62] in a simplified way, where the nonlinear effects and the evolution of the wetted surface were ignored. The governing equations were proposed in the exterior domain  $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$  (with  $\mathcal{E}^- = (-\infty, -l)$ ,  $\mathcal{E}^+ = (l, +\infty)$ ) and the interior domain  $\mathcal{I} = (-l, l)$ , where the exterior domain represents that the part of the fluid surface does not touch the object and the interior domain

### 2.3. WAVE-STRUCTURE INTERACTIONS

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is the remaining part (the fluid under the floating object). Afterwards, for the needs of the numerical studies some literature took into the nonlinear effects into consideration (see, for instance, Grilli et al. [51] and Hague and Swan [54]), while there were still various problems on the derivation of the transition equations. For this reason, Lannes proposed in [67] a new formulation for the floating body system, which is based on a new formulation of the standard water waves equations in terms of  $(\zeta, q)$ , where  $\zeta$  is the elevation of the free surface and  $q$  is the horizontal discharge.

What we are interested here is that the corresponding control problem of the floating body system in the shallow water regime. Before starting this topic, we first give a brief introduction to the governing equations of the floating body system. We begin from the water waves equations in a two dimensional domain, which is the case mainly concerned in this thesis. For the higher dimensional equations, please refer to the details in Lannes [66], [68] and [67].

#### 2.3.1 The water waves equations

As in Figure 2.2, let  $x \in \mathbb{R}$  be the horizontal variable, and let  $y \in \mathbb{R}$  be the vertical variable. We denote by  $\zeta(t, x)$  the elevation of the free surface of the water above the equilibrium position  $y = 0$ , and by  $P(t, x, y)$  the pressure at the position  $(x, y)$  at time  $t$ . Assume that the bottom of the fluid domain can be parametrized by  $y = -h_0 + b(x)$ , where  $h_0$  is the depth of the water level at the equilibrium state. The fluid domain denoted by  $\Omega_t$  at time  $t$  is

$$\Omega_t = \{(x, y) \in \mathbb{R}^2 \mid -h_0 + b(x) < y < \zeta(t, x)\}.$$

We denote by  $U(t, x, y) = (V(t, x, y), w(t, x, y))$  the velocity of the fluid particle at the position  $(x, y)$  at time  $t$ , where  $V(t, x, y)$  and  $w(t, x, y)$  is the horizontal and the vertical component of the velocity, respectively. The dynamics of the incompressible, irrotational and inviscid fluid with constant density  $\rho$  is described by the Euler equations

$$\partial_t U + U \cdot \nabla_{x,y} U = -\frac{1}{\rho} \nabla_{x,y} P - g \vec{e}_y \quad \text{in } \Omega_t, \quad (2.16)$$

where  $g$  is the gravity acceleration and  $\vec{e}_y$  is the unit upwards vertical vector. The incompressible and irrotational assumption imply that the velocity  $U$  satisfies

$$\nabla_{x,y} \cdot U = 0, \quad \nabla_{x,y} \times U = 0 \quad \text{in } \Omega_t. \quad (2.17)$$

The kinematic boundary condition means that the fluid particles always stay on the free surface, which reads

$$\begin{aligned} \partial_t \zeta - \underline{U} \cdot N &= 0 \quad \text{on } y = \zeta(t, x), \\ \text{with } \underline{U}(t, x, y) &= U(t, x, \zeta(t, x)), \quad N = \begin{bmatrix} -\partial_x \zeta \\ 1 \end{bmatrix}. \end{aligned} \quad (2.18)$$

The second boundary condition at the surface is the dynamic boundary condition for the pressure, i.e.

$$P = P_{\text{atm}} = \text{constant} \quad \text{on } y = \zeta(t, x), \quad (2.19)$$

where  $P_{\text{atm}}$  is the atmospheric pressure. In the above condition we do not consider the surface tension effect. The last boundary condition for the bottom is the impermeable condition

$$U_b \cdot N_b = 0 \quad \text{on} \quad y = -h_0 + b(x), \quad (2.20)$$

with

$$U_b(t, x) = U(t, x, -h_0 + b(x)), \quad N_b = \begin{bmatrix} -\partial_x b \\ 1 \end{bmatrix}.$$

The equations (2.16)–(2.20) are called the *free surface Euler equations*. For irrotational flow there exists a potential function  $\Phi(t, x, y)$ , such that  $U(t, x, y) = \nabla_{x,y}\Phi$ . Therefore, the equations (2.16)–(2.20) can be further written in terms of the velocity potential  $\Phi$ , and the resulting equations are called the *free surface Bernoulli equations*. For more details, please refer to [66] or [68]. Observe from the equations (2.17) and (2.18) that the velocity potential  $\Phi$  is fully determined by  $\psi(t, x) := \Phi(t, x, \zeta(t, x))$ , which implies that the free surface Bernoulli equations can be further reduced to two evolution equations on  $\zeta$  and  $\psi$ . We introduce the Dirichlet to Neumann operator  $A[\zeta, b]$  defined by

$$A[\zeta, b]\psi = N \cdot \nabla_{x,y}\Phi|_{y=\zeta}, \quad (2.21)$$

where  $\Phi$  solves

$$\begin{cases} \Delta_{x,y}\Phi = 0 & \text{in } \Omega_t, \\ \Phi|_{y=\zeta} = \psi, \quad N_b \cdot \nabla_{x,y}\Phi|_{y=-h_0+b} = 0. \end{cases}$$

With the Dirichlet to Neumann operator  $A[\zeta, b]$ , the water waves equations are formulated in terms of  $\zeta$  and  $\psi$  as follows:

$$\begin{cases} \partial_t \zeta - A[\zeta, b]\psi = 0, \\ \partial_t \psi + g\psi + \frac{1}{2}|\partial_x \psi|^2 - \frac{1}{2} \frac{(A[\zeta, b]\psi + \partial_x \zeta \partial_x \psi)^2}{1 + |\partial_x \zeta|^2} = 0. \end{cases} \quad (2.22)$$

The above  $(\zeta, \psi)$  system (2.22) defined on the free surface is called the *Zakharov-Craig-Sulem formulation* of the water waves equations. The local well-posedness of this formulation was provided in Lannes [65]. This local well-posedness has been extensively studied in two directions: low regularity in Alazard et al. [5] and uniform bounds of 3D shallow water in Alvarez-Samaniego and Lannes [6], Iguchi [58].

Note that the Zakharov-Craig-Sulem equations (2.22) do not depend on the vertical variable  $y$ , so that the water waves problem reduces one dimension. Another way to get rid of the vertical variable is to take integration of the free surface Euler equations, with respect to the variable  $y$ . Hence we introduce the *horizontal discharge*  $q$  as

$$q(t, x) = \int_{-h_0+b(x)}^{\zeta(t,x)} V(t, x, y) dy, \quad (2.23)$$

where  $V$  as defined earlier is the horizontal component of the velocity  $U$ . Combined with the boundary conditions (2.18) and (2.20), we vertically integrate the Euler equation (2.16) which gives

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \int_{-h_0+b}^{\zeta} V \otimes V \right) + \frac{1}{\rho} \nabla_{x,y} P = 0. \end{cases} \quad (2.24)$$

### 2.3. WAVE-STRUCTURE INTERACTIONS

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Observe that there are some quantities in expression (2.24) which are not explicitly determined by  $\zeta$  and  $q$ . For this reason, we introduce the decomposition technique from [68] for the horizontal velocity  $V$  and the pressure term  $P$ . We decompose  $V$  as

$$V(t, x, y) = \bar{V}(t, x) + V^*(t, x, y), \quad (2.25)$$

$$\bar{V}(t, x) = \frac{1}{h} \int_{-h_0+b}^{\zeta} V(t, x, y) dy \quad \text{and} \quad V^*(t, x, y) = V(t, x, y) - \bar{V}(t, x).$$

The quantity  $h = h_0 + \zeta - b$  is the water depth at time  $t$ . The pressure  $P$  is decomposed into the hydrostatic and the non-hydrostatic components as follows:

$$P = P_{\text{atm}} + \rho g(\zeta - y) + P_{\text{NH}}, \quad (2.26)$$

where  $P_{\text{NH}}$  is the non-hydrostatic pressure. Integrating the vertical component of the Euler equation (2.16), we obtain

$$P_{\text{NH}}(t, x, y) = \rho \int_y^{\zeta(t,x)} (\partial_t w + U \cdot \nabla_{x,y} w) dy.$$

Now combined with the two decomposition (2.25) and (2.26), the equations (2.24) becomes

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta + \partial_x R + ha_{\text{NH}} = 0, \end{cases} \quad (2.27)$$

where

$$R = \int_{-h_0+b}^{\zeta} V^* \otimes V^* \quad \text{and} \quad ha_{\text{NH}} = \int_{-h_0+b}^{\zeta} \partial_x \left( \int_y^{\zeta(t,x)} (\partial_t w + U \cdot \nabla_{x,y} w) dy \right).$$

The quantity  $a_{\text{NH}}$  in above expression is called the non-hydrostatic acceleration. The equations (2.27) are called the *discharge/elevation formulation* of the water waves equations. Although the last two terms in the second equation of (2.27) seem still complicated, the equations (2.27) form a close set of equations in  $\zeta$  and  $q$  and this has been fully explained in [67]. The discharge formulation (2.27) is convenient for deriving some asymptotic models for the water waves equation and for the floating body system. According to the definition of the horizontal discharge  $q$  in (2.23) and the vertically average horizontal velocity  $\bar{V}$  in (2.25), the system (2.27) can be equivalently written in terms of  $\zeta$  and  $\bar{V}$ . Based on the definition of  $V^*$  in (2.25), it is obvious to see that  $V^*$  represents the fluctuation of the horizontal velocity  $V$  with respect to its vertical average  $\bar{V}$ , which implies that the quantity  $R$  measures, in particular, the contribution to the rotational effects (see, for instance, Castro and Lannes [29], [30]). Observe that the non-hydrostatic pressure  $P_{\text{NH}}$  contains the linear dispersive terms, and it describes the property of wave propagation.

Note that the water waves equations (2.27) look pretty complicated, we usually study its properties through some asymptotic models. Assume that the horizontal scale of the fluid domain is  $L$  and the vertical water depth is  $h_0$ . We introduce the *shallowness parameter*  $\mu$  defined by

$$\mu = \frac{h_0^2}{L^2}. \quad (2.28)$$

In the shallow water regime, the shallowness parameter  $\mu$  is assumed to be far smaller than 1, i.e.  $\mu \ll 1$ . Based on the shallowness parameter  $\mu$  introduced in (2.28), the asymptotic models for the system (2.27) in the shallow water regime depend on the inner structure of the "turbulence" term  $R$  and the non-hydrostatic acceleration  $a_{\text{NH}}$ . In the case of the fluid domain with flat bottom, i.e.  $b = 0$  (see Figure 2.2), it turns out that the velocity  $U = (V, w)$ ,  $R$  and  $a_{\text{NH}}$  can be expanded in terms of  $\mu$  as

$$\begin{aligned} V &= \bar{V} + O(\mu), & w &= O(\mu), \\ R &= O(\mu^2), & a_{\text{NH}} &= O(\mu). \end{aligned}$$

Since the solution of the water waves equation was proved, in Lannes [6] (3D) and in Iguchi [58] (arbitrary space dimension), to be uniformly bounded with respect to the parameter  $\mu$ , the asymptotic models can be obtained by dropping high order terms on  $\mu$ . The *nonlinear shallow water equations* (NSW) are an approximation of the water waves equations (2.27) of order  $O(\mu)$  i.e. all the terms of order  $O(\mu)$  are dropped, which reads

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = 0. \end{cases} \quad (2.29)$$

With weak nonlinearity condition the second-order approximation of the water waves equations (2.27) is the *Boussinesq equations*, which are

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \left( 1 - \frac{h_0^2}{3} \partial_x^2 \right) \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = 0. \end{cases} \quad (2.30)$$

All the details on the derivation of (2.29) and (2.30) and other more precise approximate models can be found in [68]. Lannes also provided in [68] a detailed review on the well-posedness and some applications of the asymptotic models (2.29) and (2.30).

### 2.3.2 The water waves equations with a floating body

Let us now consider a rigid object floating in the water waves system. In the presence of the object, the bottom of  $\Omega_t$  in the horizontal direction is divided into the exterior domain  $\mathcal{E}$  and the interior domain  $\mathcal{I}$  which has been introduced earlier (see Figure 2.2). We observe that the acceleration of the free surface in this case will be affected by not only the dynamics of the fluid but also the motion of the object. Therefore, the surface pressure denoted by  $\underline{P}$  should emerge in the second equation of (2.27), and it is described by Newton's second law for the motion of the floating object. Let  $a_{\text{FS}} := \partial_t^2 \zeta$  be the acceleration of the free surface in absence of the floating object, and we derive from (2.27) that

$$a_{\text{FS}} = \partial_x \left( \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta + \partial_x R + h a_{\text{NH}} \right),$$

where  $a_{\text{NH}}$  has been introduced below (2.27). We denote by the notation  $f_e$  the restriction of the function  $f$  to the exterior domain  $\mathcal{E}$ , and by  $f_i$  the restriction of  $f$  to the interior domain  $\mathcal{I}$ . More precisely, we have the following governing equations for the floating object system.

### 2.3. WAVE-STRUCTURE INTERACTIONS

**Theorem 9** (Lannes [67]). *Let  $\zeta$  be the elevation of the free surface and let  $q$  be the horizontal discharge. The water waves system in the presence of a floating object is described by*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta + \partial_x R + ha_{\text{NH}} = -\frac{h}{\rho} \partial_x \underline{P}, \end{cases} \quad (2.31)$$

where the surface pressure  $\underline{P}$  satisfies

$$\underline{P}_e = P_{\text{atm}} \quad \text{and} \quad \begin{cases} -\partial_x \left( \frac{h}{\rho} \partial_x \underline{P}_i \right) = -\partial_t^2 \zeta_i + a_{\text{FS}}(h, q), \\ \underline{P}_i|_{\Gamma(t)} = P_{\text{atm}}. \end{cases} \quad (2.32)$$

The transition conditions at the contact line  $\Gamma(t)$  are

$$\zeta_e = \zeta_i \quad \text{and} \quad q_e = q_i \quad \text{on} \quad \Gamma(t). \quad (2.33)$$

In two dimensional case, the contact line  $\Gamma(t)$  is described by two moving contact points. In particular, when the object only moves in the vertical direction, these two points are fixed, which are  $x = \pm l$ . There are two types motion of the partially immersed object in the fluid: prescribed motion and freely floating. In each case the surface pressure  $\underline{P}$  is of course described in different way. In the prescribed motion, the velocity  $U$  and the position of the floating object are known, and the motion of the fluid is affected by the object. When the object is freely floating, the dynamics of the fluid-body system is governed by Newton's laws. These two situations are fully analysed in [67]. Moreover, the transition condition (2.33) at the contact line can be different, which depends on the shape of the object. The continuity condition for the surface elevation and for the horizontal discharge given in (2.33) obviously exclude the object with non-smooth contour in contact with the fluid, in particular, the object with vertical boundary. In the case of the object with vertical wall, the transition conditions at the contact line  $\Gamma(t)$  are

$$V \cdot \vec{n} = V_C \cdot \vec{n} \quad \text{on} \quad \Gamma(t), \quad (2.34)$$

$$\underline{P}_i = P_{\text{atm}} + \rho g(\zeta_e - \zeta_i) + \rho \int_{\zeta_i}^{\zeta_e} (\partial_t w + U \cdot \nabla_{x,y} w) \quad \text{on} \quad \Gamma(t), \quad (2.35)$$

where  $V_C$  is horizontal velocity of the object and  $\vec{n}$  is the unit normal vector on  $\Gamma(t)$  pointing towards the exterior domain. It is obvious that the condition (2.35) coincides with the transition condition for the object with smooth boundary contour, i.e. when  $\zeta_e = \zeta_i$  and  $\underline{P}_i = P_{\text{atm}}$  on  $\Gamma(t)$ .

There are also some interesting asymptotic models for the full water waves equations with a floating body system (2.31). In the shallow water approximation of order  $O(\mu)$ , the vertical variation of the horizontal velocity and the non-hydrostatic acceleration should be ignored. This means that the terms  $R$  and  $a_{\text{NH}}$  will be dropped in the system (2.31) and the expression for  $a_{\text{FS}}$ . Therefore, the *nonlinear shallow water equations with a floating structure* are

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P}, \end{cases} \quad (2.36)$$



where the surface pressure  $\underline{P}$  is determined by the simplified version of the equations (2.32) by dropping  $R$  and  $a_{\text{NH}}$  in  $a_{\text{FS}}$ . Similarly, for a more precise approximation, we take into account the dispersive effects which are ignored in the nonlinear shallow water model. To do this, with weak nonlinearity assumption on one hand we neglect the variation of the free surface and of the bottom, but keep the leading order term in the non-hydrostatic acceleration  $a_{\text{NH}}$ . As a second-order approximation of the system (2.31), the *Boussinesq equations with a floating structure* read

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \left(1 - \frac{h_0^2}{3} \partial_x^2\right) \partial_t q + \partial_x \left(\frac{q^2}{h}\right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P}. \end{cases} \quad (2.37)$$

The corresponding surface pressure  $\underline{P}$  in (2.37) satisfies (2.32) in the sense that  $R$  is ignored and the main term of  $a_{\text{NH}}$  is kept. Based on the nonlinear shallow water model, for the vertically moved object, Lannes also gave in [67] a detailed description of the surface pressure in the case of the prescribed motion and freely floating, respectively. In particular, he showed that the vertical motion of the freely floating body system finally reduce to a transmission problem and the transmission condition is given in terms of the vertical displacement of the object and the average horizontal discharge at two contact points. Recently, Beck and Lannes obtained in [17] the similar results for freely floating object in a fluid described by the Boussinesq equations.

### 2.3.3 Control problem on the floating body system

The main motivation of the control problem on the floating body system comes from Wave Energy Converters (WECs), which has been widely applied in ocean engineering. Ocean waves provide an enormous renewable and endless energy that can be converted into electricity by WECs. For this reason, a great number of different devices have been developed to harness wave energy by using wave-induced water motion.

According to the different working principle, the WECs are divided into several types: oscillating water columns (OWCs), overtopping devices (ODs), wave absorbing devices (WADs) and etc. The principle of the oscillating water columns is similar to a wind turbine, being based on the wave induced air pressurization. This device is set above the water level and it connects with a closed air chamber. The passage of the waves changes the water level inside the air housing and the rising and falling water level increases and decreases the air pressure at the same time. The change of the pressure forces the wind turbine to start running, thereby kinetic energy is converted into electricity. In order to improve the conversion efficiency, the oscillating water columns device is usually used in deep water area, since the shore dampens the large coming waves. The second type of WECs, the overtopping device, operates like a hydroelectric dam. Its floating arms focus waves onto a slope from which the wave overtops into a reservoir. The resulting difference in water elevation between the reservoir and the mean sea level then drives low-head hydro turbines. This device is also usually set in the offshore area. Another interesting WEC is the wave absorbing device, which absorbs the incoming waves directly. The most popular wave absorbing device is the so-called *Point Absorber*, which consists of a floater on the sea surface and a hydraulic system vertically installed below the floater. The floater is used to absorb the waves coming from all directions, and then this induces a motion of the piston, through the

hydraulic system, drives the hydraulic motor. This motor in turn powers the generator that can produce electricity. For other interesting WECs and their working principles, please refer to Drew et al. [42], López et al. [79], Babarit [11] and therein. Moreover, there are already some literature devoted the both engineering and mathematical problem for WECs, for instance, Li et al. [73], Cretel et al. [37] and Rahmati et al. [92] for WADs, in particular, point absorber device, Moretti et al. [84], Trivedi and Koley [113] and Bocchi et al. [25] for a certain OWCs system.

In this thesis, we are interested in the control system modeled by the point absorber device and the floater on the free surface is restricted to the vertical motion. From the engineering point of view, there are two fundamental problems in the working process of WECs: inefficient energy extraction and risk of device damage. For this reason, through a vertical force below the floater we expect to synchronize the motion of the floater and of the incoming waves, so that the energy is extracted in a relatively efficient way. At the same time, the WECs is protected by using this external force, although the adjustment in some sense weaken the large waves. In turn, the device described above can be also used as a wave maker (or wave generator), to produce the artificial waves for the needs of physical modelling experiments or entertainment facilities.

Mathematically speaking, we consider the control problem describing the interactions of water waves with a rigid body partially immersed in a bounded shallow water regime. Moreover, the body is allowed to move only in the vertical direction. The control signal is a vertical force, imposed from the bottom of the floating object, to adjust its motion. We shall derive the full governing equations of this floating body system and then study the controllability and stabilizability properties of this control model. Furthermore, we are interested in giving the detailed description for the reachable space, when the system begins from the equilibrium state.

## 2.4 Contributions of the thesis

We state here the main contributions of this thesis in order of the contents we introduced above. The results in this sections are taken from several papers by the author and her collaborators, including some conclusions which are not submitted and also some have been published or under review.

### 2.4.1 Stabilizability of a water waves system

Based on the abstract framework introduced around the model (2.1), here we introduce a non-uniform decay result with a more explicit sufficient condition on the spectrum structure of  $A$  and the properties of the control operator  $B$ . Motivated by the water waves system, we consider a class of control systems with colocated feedback, described by the control model (2.3), where  $A$  is skew-adjoint with compact resolvents and  $B \in \mathcal{L}(U, X)$ . In this case, the operator  $\mathcal{A} = A - BB^*$  is  $m$ -dissipative, so that it is the generator of a contraction semigroup (for this, please refer to [115, Chapter 3]). Moreover, the operator  $A$  is a Riesz-spectral operator, thereby, for a set  $J \subset \mathbb{Z}$ , we denote its eigenvalues by  $(i\mu_k)_{k \in J}$  and the corresponding normalized eigenvectors by  $(\phi_k)_{k \in J}$ , which forms an orthonormal basis in  $X$ . We introduce a scale of Hilbert spaces

$X_\alpha$ , for  $\alpha \in \mathbb{R}$ , defined by

$$X_\alpha = \left\{ z \in X \mid \sum_{k \in J} (1 + |\mu_k|^{2\alpha}) |\langle z, \phi_k \rangle|^2 < \infty \right\}.$$

The scale of Hilbert space  $X_\alpha$  actually has been already introduced in the notation part. For every  $\alpha \in \mathbb{R}$ ,  $X_{-\alpha}$  is the dual of  $X_\alpha$  with respect to the pivot space  $X$ . Moreover, for every  $\alpha \geq 0$ ,  $X_\alpha$  represents the domain of the operator  $A^\alpha$ , i.e.  $\mathcal{D}(A^\alpha)$ , endowed with the corresponding graph norm.

**Theorem 10** (Su and Tucsnak 2019). *Let  $A : \mathcal{D}(A) \rightarrow X$  be a skew-adjoint operator with compact resolvents and  $B \in \mathcal{L}(U, X)$ . We denote by  $\mathbf{S} = (\mathbf{S}_t)_{t \geq 0}$  the semigroup generated by  $\mathcal{A} = A - BB^*$ . Assume that the eigenpair of  $A$ , denoted by  $(i\mu_k, \phi_k)_{k \in J}$ , satisfy (for every  $k, l \in J$  and  $k \neq l$ )*

$$\begin{aligned} \mu_k &\neq \mu_l \quad \text{and} \quad \mu_k = k^\alpha + O(k^q) \text{ as } k \rightarrow \infty; \\ \|B^* \phi_k\|_U &\geq \frac{C}{k^\beta}, \end{aligned} \tag{2.38}$$

where  $q < \alpha - 1$ . If the condition (2.38) is satisfied with  $0 < \alpha < 1$  and  $\beta \geq 0$ , we have

$$\|\mathbf{S}_t z_0\| \leq \frac{C}{(1+t)^{\frac{\alpha}{2(\beta-\alpha+1)}}} \|z_0\|_{\mathcal{D}(\mathcal{A})} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}). \tag{2.39}$$

If (2.38) is satisfied with  $\alpha \geq 1$  and  $\beta > 0$ , for all  $z_0 \in X_1$ , we have the observability inequality

$$\int_0^T \|(B^* w)(t)\|_U^2 dt \geq C \|z_0\|_{X_{-\frac{\alpha}{\beta}}}^2,$$

where  $w$  is the solution of the undamped system (2.4). Moreover, we have the corresponding decay result

$$\|\mathbf{S}_t z_0\| \leq \frac{C}{(1+t)^{\frac{\alpha}{2\beta}}} \|z_0\|_{\mathcal{D}(\mathcal{A})} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}). \tag{2.40}$$

It is known that the control system is exponentially stabilizable when  $\alpha \geq 1$  and  $\beta = 0$ , so we do not discuss this case here. Compared with the sufficient condition introduced in Theorem 8, the condition (2.38) is more clear and easier verified by using the spectrum structure of  $A$ . Since the decay rate in Theorem 10 might be the same with the decay rate derived from Theorem 8 in some special PDEs, for example the linear water waves system, we did not submit this result which actually has been finished in 2019. The proof of Theorem 10 will be presented in Chapter 3.

Now we go back to the water waves control problem, described by the equations (2.13) with a rigid wave maker, in a two dimensional rectangular domain. All the work introduced before dealt with the first-order evolution system formulated from the original governing equations (2.13) directly, but the relationship between the first-order control system and the equations (2.13) is not very clear. Moreover, there is no strict definition and detailed analysis for the evolution operators used to formulate the control model in the above references. For this reason, we established in Su et al. [110] the well-posedness of the governing equations (2.13) by formulating them as an abstract linear control system (for this concept, please refer to Weiss [119] or Tucsnak and Weiss [115]) for scalar input  $u \in L_{\text{loc}}^2([0, \infty); U)$ . During this process, we introduced in

## 2.4. CONTRIBUTIONS OF THE THESIS

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[110] the *partial Dirichlet to Neumann* and the *partial Neumann to Neumann operators*, associated with certain boundaries of the rectangular domain, and studied their detailed properties. What is important is that we proved that the governing equations (2.13) can be equivalently recast as an abstract linear control system by using the Dirichlet to Neumann and the Neumann to Neumann operators. Moreover, we studied the stabilizability properties of the linear control system for gravity water waves.

**Theorem 11** (Su, Tucsnak and Weiss [110]). *The small-amplitude gravity water waves system (2.13) in rectangular domain  $\Omega$ , with rigid wave maker imposed at the left boundary  $\Gamma_1$ , can be equivalently recast to an abstract linear control system (2.1) with the state  $z = \begin{bmatrix} \zeta \\ \xi \end{bmatrix}$  on  $X$  and scalar control  $u$  (the acceleration produced by the wave maker). For  $u \in L^2_{\text{loc}}([0, \infty))$  and  $h \in L^2[-1, 0]$  with  $\int_{-1}^0 h(y) dy = 0$ , the initial data  $z_0 \in \mathcal{H}^{\frac{1}{2}}[0, \pi] \times L^2[0, \pi]$  and  $\phi_0 \in \mathcal{H}^1(\Omega)$ , the system (2.13) admits a unique solution  $(\zeta, \phi)$  satisfying*

$$\phi \in \mathcal{H}^1_{\text{loc}}([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); \mathcal{H}^{\frac{1}{2}}[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

The operator pair  $(A, B)$  in the model (2.1) are

$$\begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

where  $A_0$  and  $B_0$  is the Dirichlet to Neumann operator and the Neumann to Neumann operator, respectively. This control system is strongly stabilizable with the colocated feedback  $u = -B^*z$ , and the corresponding closed-loop semigroup  $S_t$  generated by  $A - BB^*$  satisfies

$$\|S_t z_0\|_X \leq \frac{C}{(1+t)^{\frac{1}{6}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0.$$

Similarly, the case of gravity-capillary water waves was discussed in [108] where we need the *Sturm-Liouville operator* to deal with the surface tension term. The corresponding governing equations for gravity-capillary waves can be also formulated into a first-order control system in form of (2.1) which is again strongly stabilizable and satisfies

$$\|S_t z_0\|_X \leq \frac{C}{(1+t)^{\frac{3}{4}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0.$$

As we expected, we observe that the decay rate of the trajectory of the control system for gravity-capillary waves is faster than the case of the gravity waves, because of the surface tension effect.

### 2.4.2 Asymptotic analysis of water waves equations in shallow water

Following the stabilization issue, we are interested in Su [107] the asymptotic behaviour of the solution of the gravity water waves control system (2.13) in the shallow water regime, where the horizontal scale of the fluid domain is much larger than the vertical depth. To state the result, we recall the shallowness parameter  $\mu$  introduced in (2.28). We define some dimensionless quantities as follows

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{h_0}, \quad \bar{t} = \frac{t}{L/\sqrt{gh_0}}, \quad \bar{\zeta} = \frac{\zeta}{a}, \quad \bar{\phi} = \frac{\phi}{aL\sqrt{g/h_0}}, \quad (2.41)$$

where  $a$  is the order of the surface variation,  $L$  is the horizontal scale of the fluid domain,  $h_0$  represents the typical water depth,  $\bar{\zeta}$  and  $\bar{\phi}$  are the dimensionless version of the elevation of the free surface  $\zeta$  and the velocity potential  $\phi$ , respectively. Based on the dimensionless quantities in (2.41), we derive the dimensionless version of the water waves equations (2.13), and denote the solution for the dimensionless system by  $\zeta_\mu$  and  $\phi_\mu$  to avoid any confusion. Omitting the overlines above the dimensionless quantities for the sake of simplicity, we take the dimensionless domain as

$$\Omega = \{(x, y) \mid x \in [0, \pi] \text{ and } y \in [-1, 0]\}.$$

The dimensionless linear water waves system with a rigid wave maker, for every  $t \geq 0$  and  $x \in [0, \pi]$ , reads

$$\begin{cases} (\mu \partial_x^2 + \partial_y^2) \phi_\mu(t, x, y) = 0, \\ \partial_t \zeta_\mu(t, x) - \frac{1}{\mu} \partial_y \phi_\mu(t, x, 0) = 0, \\ \partial_t \phi_\mu(t, x, 0) + \zeta_\mu(t, x) = 0, \\ \partial_x \phi_\mu(t, 0, y) = -h(y)v(t), \\ \partial_y \phi_\mu(t, x, -1) = 0 = \partial_x \phi_\mu(t, \pi, y), \end{cases} \quad (2.42)$$

where  $v$  is the velocity produced by the wave maker. According to Theorem 11, the system (2.42) is also equivalent to an abstract linear control system with the state  $[\zeta_\mu \ \dot{\zeta}_\mu]^\top$ , we thus propose the initial data as

$$\zeta_\mu(0, x) = \zeta_0(x), \quad \partial_t \zeta_\mu(0, x) = \zeta_1(x). \quad (2.43)$$

In this way, the input of the control model is the acceleration imposed by the wave maker, i.e.  $u = \dot{v}$ . To analyse the asymptotic behaviour of the system (2.42) in shallow water regime, we introduce the following wave equation defined on  $[0, \pi]$  with Neumann boundary control at the left endpoint, i.e. for all  $t \geq 0$ ,  $x \in [0, \pi]$ ,

$$\begin{cases} \partial_t^2 \zeta(t, x) - \partial_x^2 \zeta(t, x) = 0, \\ \partial_x \zeta(t, 0) = u(t), \quad \partial_x \zeta(t, \pi) = 0, \\ \zeta(0, x) = \zeta_0(x), \quad \partial_t \zeta(0, x) = \zeta_1(x). \end{cases} \quad (2.44)$$

With the above notation, we have the following asymptotic result.

**Theorem 12** (Su [107]). *For  $u \in L_{\text{loc}}^2[0, \infty)$  and for any initial data  $\zeta_0 \in \mathcal{H}^1[0, \pi]$  and  $\zeta_1 \in L^2[0, \pi]$ , let  $\zeta_\mu$  be the solution of the free surface equations of (2.42) with the initial data (2.43), satisfying*

$$\zeta_\mu \in C([0, \infty); \mathcal{H}^{\frac{1}{2}}[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

Let  $\zeta$  be the solution of the system (2.44) satisfying

$$\zeta \in C([0, \infty); \mathcal{H}^1[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

Then, for every  $\tau > 0$ , we have

$$\begin{aligned} \limsup_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\zeta_\mu - \zeta\|_{\mathcal{H}^{\frac{1}{2}}[0, \pi]} &= 0, \\ \limsup_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\partial_t \zeta_\mu - \partial_t \zeta\|_{L^2[0, \pi]} &= 0. \end{aligned}$$

According to Theorem 12, we know that the solution of the water waves system converges to the solution of the one-dimensional wave equation with Neumann boundary control, when taking the shallowness limit  $\mu \rightarrow 0$ . This is a natural result. Intuitively, the rectangular fluid domain becomes thinner and thinner in the horizontal direction and it reduces to a one-dimensional interval. From another point of view, the dispersive effect is dropped out during the limit process, and this is exactly the property of the one-dimensional wave equation. We find that the properties of the limit system is much better than the water waves system. The wave equation with Neumann boundary control is exactly controllable (see [19, Part III, Chapter 8] and [83] for the sufficiently large time, and [69] for finite time interval), while the water waves system (5.4) is even not approximately controllable (see [96] and [85]).

### 2.4.3 Control of floating body system

Instead of the boundary control problem of water waves, we consider the interaction of water waves with a partially immersed object in a fluid modeled by the shallow water equations. The object is assumed to move only in the vertical direction. The control now is a vertical force acting on the vertically moved object. We assume that the floating object has vertical lateral walls, with a possibly non-flat but symmetric bottom. Moreover, let the interior domain  $\mathcal{I} = [-l, l]$  and the exterior domain  $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$ , with  $\mathcal{E}^- = [-L, -l]$  and  $\mathcal{E}^+ = [l, L']$ . Firstly we derive the governing equations of this floating body control system in a bounded water tank, still denoted by  $\Omega_t$ ,

$$\Omega_t = \{(x, y) \in [-L, L'] \times [-h_0, \zeta]\}.$$

Based on the nonlinear shallow water equations with a floating structure in an infinite strip introduced in (2.36), we need the boundary conditions for the horizontal discharge  $q$  and the transmission conditions at the contact points of the interior domain  $\mathcal{I}$  and the exterior domain  $\mathcal{E}$ . To address this, in Su and Tucsnak [109] we follow the conservation laws of the total energy and of the volume of the water. The full governing equations of the floating body system with control term in bounded domain  $\Omega_t$ , for all  $t \geq 0$ , read

$$\partial_t \zeta + \partial_x q = 0 \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (2.45a)$$

$$\partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P} \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (2.45b)$$

$$\underline{P}_e(t, x) = 0 \quad x \in \mathcal{E}, \quad (2.45c)$$

$$\zeta_i(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad x \in \mathcal{I}, \quad (2.45d)$$

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + (\mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l)), \quad (2.45e)$$

$$m \ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (2.45f)$$

$$q_e(t, -L) = 0 = q_e(t, L'), \quad q_i(t, \pm l) = q_e(t, \pm l), \quad (2.45g)$$

with the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1 \quad \forall x \in \mathcal{I} \cup \mathcal{E},$$

where  $\delta$  is the variation of the center of gravity of the object,  $m$  is the mass of the object,  $g$  is the gravity acceleration. The function  $h_{\text{eq}}$  in (2.45d) is even and non-negative

representing the distance between the bottom of the object and the bottom of the fluid domain. The quantity  $\mathfrak{B}$  in (2.45e) is defined by

$$\mathfrak{B} = \frac{\rho q^2}{2h^2}.$$

Moreover, the equations (2.45) can be further formulated into a first-order evolution system in the exterior domain  $\mathcal{E}$ , in terms of the elevation  $\zeta$ , the horizontal discharge  $q$ , the average horizontal discharge at two contact points, the displacement and the velocity of the object, with transmission conditions and boundary conditions. It is worth noting that the surface pressure  $\underline{p}_i$  is determined by a second-order elliptic equation where the source term depends on the exterior functions  $\zeta_e, q_e$  and  $\delta$ . Combined with the boundary conditions (2.45d),  $\underline{p}_i$  is totally determined by the exterior quantities. For more details about this, please refer to the main contents of this thesis. Strictly speaking, the well-posedness of the full system (2.45) is open. But the main idea for the proof of the local well-posedness result probably can be done by using the approach developed in Iguchi and Lannes [59] where the problem is in an infinite strip and without control.

Here for the control properties of the floating body system, we consider the linearized equations around the equilibrium state. We define the jump and the average of a function  $f$  defined on  $[-l, l]$  by  $\llbracket f \rrbracket = f(l) - f(-l)$  and  $\langle f \rangle = \frac{1}{2}(f(l) + f(-l))$ , respectively. The linearized floating body system, for every  $t \geq 0$  and  $x \in \mathcal{E}$ , reads

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \frac{d}{dt} \langle q_i \rangle = -\frac{g}{2l\alpha} \llbracket \zeta \rrbracket, \\ \ddot{\delta} = -\frac{2\rho g l}{M} \delta + \frac{2\rho g l}{M} \langle \zeta \rangle + \frac{1}{M} u, \end{cases} \quad (2.46)$$

with transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta}, \quad (2.47)$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L'), \quad (2.48)$$

where  $\alpha$  and  $M$  are two constants depending on the mass  $m$  and the function  $h_{\text{eq}}$ . Let the state and the initial data of (2.46) be

$$z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \dot{\delta}]^\top, \quad z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \dot{\delta}_1]^\top. \quad (2.49)$$

Our first result is the well-posedness of the linear system (2.46)–(2.49).

**Theorem 13** (Su and Tucsnak [109]). *The linearized floating body system (2.46)–(2.49) forms a linear control system with the state  $z$  in the state space*

$$X = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right\}$$

and the input space  $U = \mathbb{C}$ . For  $u \in L^2_{\text{loc}}([0, \infty); U)$ , the initial data  $z_0 \in X$ , the system (2.46)–(2.49) admits a unique solution  $z \in C([0, \infty); X)$ .

## 2.4. CONTRIBUTIONS OF THE THESIS

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Our main interest is to study the reachable space of the control system (2.46)–(2.49), when the object is put in the middle of the fluid domain in the horizontal direction i.e.  $L = L'$ . This space is formed of all the states that can be reached from equilibrium by means of  $L^2$  controls  $u$ . For every  $\tau > 0$ , the input to state map  $\Phi_\tau : L^2([0, \infty); U) \rightarrow X$  of the system (2.46)–(2.49) with zero initial data (i.e.  $z_0 = 0$ ) defined by

$$\Phi_\tau u = z(\tau) \quad \forall u \in L^2_{\text{loc}}([0, \infty); U).$$

The reachable space is described by  $\text{Ran } \Phi_\tau$ , for every  $\tau \geq 0$ . Notice that when  $L' = L$  and the initial state is an equilibrium one, the whole floating body-fluid system preserves its symmetry for all  $t \geq 0$ , in the sense that  $\zeta$  and  $q$  satisfy

$$\zeta(t, -x) = \zeta(t, x) \quad q(t, -x) = -q(t, x) \quad \forall x \in \mathcal{E}.$$

We define the symmetry space  $S$  as follows:

$$S = \left\{ \begin{array}{l} [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \quad \text{and} \\ \zeta(-x) = \zeta(x), \quad q(-x) = -q(x) \end{array} \right\}.$$

To state the result, we introduce the Hilbert space  $W$ :

$$W = \left\{ \begin{array}{l} [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \\ \llbracket q \rrbracket = -2l\eta, \quad \langle q \rangle = \langle q_i \rangle \quad \text{and} \quad q(-L) = 0 = q(L') \end{array} \right\}.$$

**Theorem 14** (Su and Tucsnak [109]). *Assume that the object floats in the middle of the fluid domain in the horizontal direction, i.e.  $L' = L$ . Then for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ , we have*

$$(W \cap S) = \text{Ran } \Phi_\tau \subset (X \cap S), \quad (2.50)$$

where the inclusion is dense and with continuous embedding.

In the symmetric case described above, the average horizontal discharge  $\langle q_i \rangle$  and the jump of the elevation  $\llbracket \zeta_e \rrbracket$  are both zero, so that the state  $z$  and the linear control system (2.46)–(2.49) can be simplified. We see from the equality in (2.50) that the reachable space of the control system (2.46)–(2.49) consists of any symmetric waves with the regularity as in the Hilbert space  $W$ . The inclusion in (2.50) means that the system is not approximately controllable in  $X$ , but in its symmetric subspace  $X \cap S$ . Moreover, we show in Chapter 6 that if we take the state space as  $W \cap S$ , the control system is exactly controllable in finite time. More details on this symmetric case are provided in Chapter 6, where we also analyse the case when the object floats at one lateral boundary of  $\Omega$ .

**Theorem 15** (Su and Tucsnak [109]). *Assume that the object is in the middle of the fluid domain, i.e.  $L' = L$ . Then any symmetric wave profile in the Hilbert space  $W$  can be reached by the control system (2.46)–(2.49). Moreover, this system is strongly stabilizable with the colocated feedback for regular initial data, and we have*

$$\|z\|_X \leq \frac{C}{(1+t)^{\frac{1}{2}}} \|z_0\|_{W \cap S} \quad \forall z_0 \in W \cap S. \quad (2.51)$$



According to Theorem 15, taking the the colocated feedback, the symmetric waves with  $\mathcal{H}^1$ -regularity, generated by the motion of the object in the middle of the domain, decay like  $t^{-1/2}$ . As we already mentioned above, the solution  $z$  possess the symmetry condition in  $S$  when  $z_0$  is symmetric. Therefore, the left side of (2.51) actually is the norm of  $z$  in the Hilbert space  $X$  with the symmetric property. Moreover, we also show in [109] that, without symmetric initial data, the linear floating object system (2.46)–(2.49) is not controllable when the object is in the middle  $L' = L$ . The situation in the general case is complicated since the eigenspace of the evolution operator involved in the well-posed linear control system, associated with (2.46)–(2.49), is possibly not simple. In this case, the control system with one control is not observable.

When the object is at one of the end of the fluid domain, i.e.  $L = l$  or  $L' = l$ , the governing equation (2.45) is not available. Without loss of generality, we assume that the object floats at the right lateral boundary, i.e. the case when  $L' = l$ . As we can see from (2.45), the notation  $\zeta_e(t, l)$  and  $q_e(t, l)$  do not make sense since the right exterior domain  $\mathcal{E}^+ = (l, L')$  vanishes. Moreover, there are also some problems in the coupled PDE-ODE system formulated by the equations (2.45), where the two ODE equations are not compatible. For this reason, we go back to the model derivation by using the conservation of the energy and of the volume. The governing equations in this case become simpler and the surface pressure  $\underline{P}_i$  satisfies a first-order equation with one boundary condition at  $x = -l$ . But the linearized version is the same with the system (2.45) in symmetric case restricted in  $\mathcal{E}^-$ . Therefore, the results in Theorem 15 also hold when the object floats at one lateral boundary of the fluid domain. For more information about the general case, we give a detailed discussion in Chapter 6.

#### 2.4.4 List of publications

- [1 ] Su, P., Tucsnak, M., and Weiss, G., Stabilizability properties of a linearized water waves system, *Systems & Control Letters*, 139:104672, 2020.
- [2 ] Su, P., Asymptotic behaviour of a linearized water waves system in a rectangle, *arXiv preprint arXiv:2104.00286*.
- [3 ] Su, P., Strong stabilization of a linearized gravity-capillary water waves system in a tank, *2021 IEEE 60th Annual Conference on Decision and Control (CDC)*, accepted.
- [4 ] Su, P. and Tucsnak, M., Shallow water waves generated by a floating object: a control theoretical perspective, *arXiv preprint arXiv:2108.05088*, 2021

In addition, the short version of [1] entitled "Strong stabilization of small water waves in a pool" was published at *24th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Cambridge, UK, 2020.

## 2.5 Organization of the thesis

Following the order of the main topics stated in introduction, we divide the main contents of this thesis into five chapters. In Chapter 3, we study the stability of a class of infinite dimensional system. Under some appropriate assumptions on the spectrum of

## 2.5. ORGANIZATION OF THE THESIS

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the evolution operators in the control model, we give an explicit polynomial decay rate of the norm of the state trajectory for the smooth initial data, by taking a suitable state feedback. We consider in Chapter 4 a boundary control problem on water waves in a rectangular domain, where the control is imposed from one lateral boundary by a wave maker. We establish the well-posedness and stabilizability properties of this system by introducing a partial Dirichlet to Neumann and a Neumann to Neumann operators associated with certain edges. For the same control problem on gravity waves, Chapter 5 is devoted to studying the asymptotic behaviour of the solution of the control model in the shallow water regime. We prove that the solution of the linear water waves in rectangle converges to the solution of the 1-D wave equation with Neumann boundary control. Finally, in Chapter 6 we study a floating body system in shallow water, where the control is an external force acting on the object in the vertical direction. As we mentioned before, this is modeled by point absorber wave energy converter in engineering. We first derive the governing equations, then reformulate them as an initial boundary value problem of a first-order evolution system. Then we consider the well-posedness for the linearized equations and give the detailed description for the corresponding reachable space. Finally, we give in Chapter 7 some interesting remarks and comments on the related problems for future work.



## **Part I**

# **Stabilization of a class of infinite dimensional systems**



## Chapter 3

# Stability of a class of skew-adjoint systems

In this chapter we consider a class of systems which have been extensively studied in the last decades and which describe wave processes of various types going from linear elasticity to electromagnetics or to linearized water waves. This latter application was the original motivation of our work but we think that, for both the sake of clarity and for possible use in a different situations, it is preferable to present the results in an abstract context. We mention here that the results presented in this chapter are not submitted. There was a similar work done by R. Chill, L. Paunonen, D. Seifert, R. Stahn and Y. Tomilov [32], which appeared earlier on ArXiv 2019 and they discussed the non-uniform stability of the linear control model in a more general way. In the case we consider in this chapter, it turns out that the decay rate, at least for the water waves system, is the same with the one they obtained in [32].

### 3.1 Problem setting and the main results

The class of infinite dimensional systems considered in this chapter is motivated by mathematical models for controlled water waves, within the linearized theory. However, since we think that our main result should be applicable in other contexts, we choose to describe them in an abstract manner and then make precise the application to water waves systems in the following chapters. The statement of our main result concerning this application requires a good amount of notation. Therefore, in this section we just state the abstract result. To be more specific, the control systems considered here are described by a group of unitary operators and a bounded control operator. An important feature of the generator (which, by Stone's theorem, is a skew-adjoint operator) is that in the case of water waves its eigenvalues do not satisfy the gap conditions which are generally used to establish controllability and stabilizability properties in this type of situation.

To give a precise statement of our main results we need some notation. Let  $X$  (the state space) and  $U$  (the input space) be Hilbert spaces, let  $A : \mathcal{D}(A) \rightarrow X$  be a skew-adjoint operator with compact resolvents and let  $B \in \mathcal{L}(U, X)$  be the control operator (in this case,  $B$  is called bounded). In general, the control operator is linear continuous from  $U$  to a space larger than  $X$ , which is not our interest here. If there is no ambiguity

the inner product and the norm in  $X$  will be simply denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ . The systems considered here are described by the equation

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) & (t \geq 0), \\ z(0) = z_0, \end{cases} \quad (3.1)$$

where  $u \in L^2_{\text{loc}}([0, \infty); U)$  is the control function and  $z$  is the corresponding state trajectory. As we mentioned in introduction, we take colocated feedback  $u = -B^*z$ , so that the energy of the control system (3.1) is non-increasing. Therefore, the system (3.1) is transformed into

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), \\ z(0) = z_0, \end{cases} \quad (3.2)$$

where the new operator  $\mathcal{A} = A - BB^*$ . Note that the evolution operator  $A$  is skew-adjoint with compact resolvents and the control operator  $B \in \mathcal{L}(U, X)$ , it is not difficult to see that the operator  $\mathcal{A}$  in (3.2) is m-dissipative, so that it generates a contraction semigroup on  $X$ . We denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  the bounded strongly continuous semigroup (or  $C_0$ -semigroup) generated by  $\mathcal{A}$ . Moreover, according to the classical results for instance in [115], the operator  $A$  is diagonalizable (also called Riesz-spectral operator in [40]) with purely imaginary eigenvalues and its spectrum consists only of eigenvalues. We denote by  $J$  either  $\mathbb{N}$  or  $\mathbb{Z}^*$  and we assume that the eigenvalues of  $A$  can be organized in a sequence  $(i\mu_k)_{k \in J}$ , such that  $\mu_k \in \mathbb{R}$  for every  $k \in J$  and with a corresponding orthonormal basis in  $X$  denoted by  $(\phi_k)_{k \in J}$ . With the above notation, we shall give a detailed description for the non-uniform stability of the semigroup  $\mathbb{T}$  in terms of the spectral properties of  $A$  and of the operator  $B$ , without requiring the approximate controllability of  $(A, B)$  in any finite time. More precisely, our basic assumptions on the pair  $(A, B)$  are

[ $H_1$ ] The operator  $A$  is skew-adjoint with compact resolvents. With the above notation, assume that  $\mu_k \neq \mu_l$  for  $k \neq l$  and that

$$\mu_k = ck^\alpha + O(k^q), \quad (k \rightarrow \infty), \quad (3.3)$$

where  $c$  and  $\alpha$  are positive constants and  $q < \alpha$ .

[ $H_2$ ] There exist positive constants  $\beta, \tilde{c} > 0$ , such that the operator  $B \in \mathcal{L}(U, X)$  satisfies

$$\|B^*\phi_k\|_U \geq \frac{\tilde{c}}{|k|^\beta} \quad (k \in J). \quad (3.4)$$

In order to state the results clearly, we recall the definition of a scale of Hilbert spaces associated with the operator  $A$ . For every  $k \in J$  and  $z \in X$ , we denote by  $z_k$  the inner product  $\langle z, \phi_k \rangle$  in  $X$ . With the above notation, for every  $s \in \mathbb{R}$ ,  $X_s$  is

$$X_s = \left\{ z \in X \mid \sum_{k \in J} (1 + |\mu_k|^{2s}) |z_k|^2 < \infty \right\} \quad (3.5)$$

endowed with the inner product

$$\langle \zeta, \eta \rangle_s = \sum_{k \in J} (1 + |\mu_k|^{2s}) \zeta_k \eta_k \quad \forall \zeta, \eta \in X_s.$$

In particular,  $X_0 = X$ . It is worth noting that, for every  $s \geq 0$ , the Hilbert space  $X_s$  is equivalent to  $\mathcal{D}(A^s)$  endowed with the graph norm. Moreover, for negative subscript  $X_{-s}$  with  $s \geq 0$ , it is the dual of  $X_s$  with respect to the pivot space  $X$ .

Based on the above assumption  $[H_1]$  and  $[H_2]$ , the corresponding stability results are divided into three cases depending on the parameters  $\alpha$  and  $\beta$  in (3.3) and (3.4). The case  $\alpha \geq 1$ ,  $\beta = 0$  has already been studied in the literature (see, for instance, [55]) and the existing results yield finite time exact controllability of  $(A, B)$ , thus exponential stability of the system is obtained from (3.1) by using colocated feedback. In the case of  $\alpha \geq 1$  and  $\beta > 0$ , we have approximate controllability of the pair  $(A, B)$  and it has been shown, for instance in [8], that the closed-loop system (3.2) is polynomially stable where the decay rate is given in terms of the interpolation parameter of a scale of Hilbert spaces. Consequently, here we are mainly interested in giving the explicit decay rate, especially the case that  $0 < \alpha < 1$  and  $\beta \geq 0$ , of the norm of the solution of (3.2) in terms of the parameters  $\alpha$  and  $\beta$ .

The main abstract results are as follows:

**Theorem 3.1.1.** *Assume that  $A : \mathcal{D}(A) \rightarrow X$  and  $B \in \mathcal{L}(U, X)$  satisfy the assumptions  $[H_1]$  and  $[H_2]$  above for some  $0 < \alpha < 1$  and  $\beta \geq 0$ . Then the system described by (3.2) is polynomially stable. More precisely, there exists  $c_1 > 0$  such that*

$$\|z(t)\| \leq \frac{c_1}{(1+t)^{\frac{\alpha}{2(\beta-\alpha+1)}}} \|z_0\|_{X_1}, \quad (3.6)$$

for all  $t > 0$  and  $z(0) \in X_1$ .

**Theorem 3.1.2.** *Let the operator  $A : \mathcal{D}(A) \rightarrow X$  and  $B \in \mathcal{L}(U, X)$  satisfy the  $[H_1]$  and  $[H_2]$  with  $\alpha \geq 1$  and  $\beta > 0$ . Then there exists a constant  $c_2 > 0$  such that the state trajectory  $z(t)$  satisfies*

$$\|z(t)\| \leq \frac{c_2}{(1+t)^{\frac{\alpha}{2\beta}}} \|z_0\|_{X_1}, \quad (3.7)$$

for all  $t > 0$  and  $z_0 \in X_1$ .

In the following subsections, we focus on the proof of Theorem 3.1.1 and Theorem 3.1.2. Moreover, we introduce some simple applications of this result, for systems described by fractional versions of the Schrödinger and wave equations. For the main application on water waves system, which represents the main motivation of this work, we give a detailed discussion separately in the remaining chapters.

## 3.2 Preliminaries

In this section we recall some results which are by now classical (see, for instance, [26] and references therein) which shows that the polynomial stability of a semigroup follows from the estimate of the norms of the resolvents on the imaginary axis. Moreover, according to Hilbert uniqueness method, the stability result can be also obtained by an observability inequality, for instance in [8]. The first approach is available for any control system, but the second one requires that the spectrum of the evolution operator satisfies some conditions. We next recall some related conclusions.

Here the control model is described by the system (3.2) and the scale of Hilbert space  $X_\alpha$  introduced in (3.5). The following result is concluded from [77], [12], [26] and [99].



**Proposition 3.2.1.** *Let  $X$  be a Hilbert space and  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow X$  be the generator of a bounded  $C_0$ -semigroup  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on  $X$ . Moreover, assume that for fixed  $s > 0$ :*

$$\begin{aligned} & i) \quad i\mathbb{R} \subset \rho(\mathcal{A}); \\ & ii) \quad \|R(i\omega : \mathcal{A})\| = O(|\omega|^s) \quad (\omega \rightarrow \infty). \end{aligned} \quad (3.8)$$

Then we have

$$\|\mathbb{T}_t z_0\| \leq \frac{C}{(1+t)^{\frac{1}{s}}} \|z_0\|_{\mathcal{D}(\mathcal{A})} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}), t \geq 0. \quad (3.9)$$

Actually, under the assumption  $i\mathbb{R} \subset \rho(\mathcal{A})$ , the asymptotic condition of the resolvent operator in (3.8) is equivalent to the non-uniform result (3.9). For more regular initial data in the sense of  $z_0$  is in the domain of  $\mathcal{A}^s$  with  $s > 1$ , there are also corresponding estimate for the norm of the solution  $z$  in [26], which is

$$\|\mathbb{T}_t z_0\| \leq \frac{C}{1+t} \|z_0\|_{\mathcal{D}(\mathcal{A}^s)} \quad \forall z_0 \in \mathcal{D}(\mathcal{A}^s). \quad (3.10)$$

Moreover, it turns out that with the same assumption on  $\rho(\mathcal{A})$  the estimate (3.9) is equivalent to (3.10). As we expected, the more regular the initial data, the faster the decay of the norm of the solution. For more details about the non-uniform decay results of the operator semigroup, please refer to [12], [15], [26], [99] and also therein.

The result below appears in various forms in the literature, so we provide its precise statement and a very short proof only for completeness purposes, with no claim of originality.

**Proposition 3.2.2.** *Under the assumptions on the pair  $(A, B)$  in  $[H_1]$  and  $[H_2]$ , the operator  $\mathcal{A}$  defined in (3.2) generates a contraction semigroup on  $X$ . Moreover, we have  $i\mathbb{R} \subset \rho(\mathcal{A})$ .*

*Proof.* Using the fact that  $A$  is skew-adjoint it follows by applying Stone's theorem that  $A$  generates a  $C_0$ -group of unitary operators on  $X$ . Moreover, since  $BB^* \in \mathcal{L}(X)$ , according to a classical perturbation theorem (see, for instance, [115, Theorem 2.11.2]), the operator  $\mathcal{A}$  generates a  $C_0$ -semigroup on  $X$ . It is not difficult to see that this semigroup is contractive, since  $\mathcal{A}$  is actually a  $m$ -dissipative operator. To show that  $i\mathbb{R} \subset \rho(\mathcal{A})$ , we first remark that the embedding  $\mathcal{D}(\mathcal{A}) \subset X$  is compact, so that the spectrum of  $\mathcal{A}$  consists only of eigenvalues. If  $iv \in \sigma(\mathcal{A})$  for some  $v \in \mathbb{R}$ , there exists  $\phi \in \mathcal{D}(\mathcal{A}) \setminus \{0\}$  such that

$$(A - BB^*)\phi = iv\phi.$$

Taking the inner product of both sides in the above equation with  $\phi$  and then taking real parts, the resulting equation implies that  $B^*\phi = 0$  and  $\phi$  is an eigenvector of  $A$ , which contradicts the assumption (3.4). We thus conclude that  $i\mathbb{R} \subset \rho(\mathcal{A})$ .  $\square$

Next we introduce the second approach to derive the explicit non-uniform stability by using an observability inequality of the undamped system. Firstly, we introduce the interpolation space  $[X, Y]_\theta$  for two Hilbert spaces  $X$  and  $Y$  with the parameter  $\theta \in (0, 1)$ , which is defined as

$$[X, Y]_\theta = \left\{ a : a \in X + Y, t^{-(\theta+1/2)} K(t, a; X, Y) \in L^2[0, \infty) \right\},$$

### 3.3. PROOF OF THE MAIN RESULTS

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where

$$K(t, a; X, Y) = \inf_{a_0 + a_1 = a} (\|a_0\|_X^2 + t^2 \|a_1\|_Y^2)^{1/2},$$

for  $a_0 \in X$  and  $a_1 \in Y$ . The above definition for the interpolation space  $[X, Y]_\theta$  is widely used, for instance, in Lions and Magenes [75]. This coincides with the definition of  $(X, Y)_{\theta, 2}$  introduced in Triebel [111].

Let us consider the undamped problem associated with the system (3.1), which is

$$\begin{cases} \dot{w}(t) = Aw(t) \\ w(0) = z_0 \end{cases} \quad (3.11)$$

It is well-known that (3.11) is well-posed in  $X_1$  and  $X$ . Now we quote a result from [8] which is proposed through a second-order evolution equation. Here in order to keep consistency with the context, we state it by using the first-order system (3.2). Together with the state space  $X$ , let  $Y$  and  $W$  be two Banach spaces such that

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &\subset Y \subset X \subset W, \\ \forall z \in \mathcal{D}(\mathcal{A}), \quad \|z\|_{\mathcal{D}(\mathcal{A})} &\sim \|z\|_Y \end{aligned}$$

and

$$[Y, W]_\theta = X$$

for a fixed real number  $\theta \in (0, 1)$ , where  $[\cdot, \cdot]_\theta$  is the interpolation space introduced above. The following proposition says that the explicit decay rate of the solution of (3.2) depends on an observability inequality of the undamped problem (3.11).

**Proposition 3.2.3.** *If for all  $z_0 \in \mathcal{D}(\mathcal{A})$  we have*

$$\int_0^T \|(B^*w)(t)\|_U^2 dt \geq C \|z_0\|_W^2, \quad (3.12)$$

*then there exists a constant  $C > 0$  such that for all  $t > 0$  and for all  $z_0 \in \mathcal{D}(\mathcal{A})$  we have*

$$\|z(t)\| \leq \frac{C}{(1+t)^{\frac{\theta}{2(1-\theta)}}} \|z_0\|_{\mathcal{D}(\mathcal{A})}.$$

*where  $\theta \in (0, 1)$  is such that the interpolation space  $[Y, W]_\theta = X$ .*

**Remark 1.** To obtain the observability inequality (3.12), we usually need Ingham theorem (see, for instance, [115]) that requires that the existence of a uniform gap between the eigenvalues of the evolution operator. However, not every control system possesses this property, for instance, a fractional Schrödinger equation introduced later. In this case, the result in Proposition 3.2.1 is still available.

### 3.3 Proof of the main results

Based on some classical results introduced in Section 3.2, we present in what follows the proofs of Theorem 3.1.1 and Theorem 3.1.2.

When  $\alpha \in (0, 1)$  and  $\beta \geq 0$ , in order to prove Theorem 3.1.1 we follow the ideas in [77] and first introduce the following lemma, which plays an important role in the main proof of Theorem 3.1.1.

**Lemma 3.3.1.** *Let  $(\mu_k)_{k \in J}$  be a sequence satisfying (3.3) for some  $\alpha \in (0, 1)$  and  $q < \alpha$ . Then there exist constants  $M, \gamma > 0$  such that for every  $\omega \in \mathbb{R}$  with  $|\omega| \geq M$ , the interval  $\left[\omega - \gamma\omega^{\frac{\alpha-1}{\alpha}}, \omega + \gamma\omega^{\frac{\alpha-1}{\alpha}}\right]$  contains at most one element of the sequence  $(\mu_k)_{k \in J}$ .*

*Proof.* For the sake of simplicity we assume that  $J = \mathbb{N}$  and that the elements of sequence  $(\mu_k)_{k \in \mathbb{N}}$  are positive. In this case (3.3) becomes

$$\mu_k = ck^\alpha + O(k^q) \quad (k \rightarrow \infty),$$

and we can assume, after possibly excluding a finite number of terms, that the sequence  $(\mu_k)_{k \in \mathbb{N}}$  is strictly increasing. Let us assume, by absurd, that the conclusion of the lemma is false. This would imply the existence of a sequence  $(\omega_n)_{n \in \mathbb{N}}$  of positive real numbers with  $\omega_n \rightarrow \infty$  (as  $n \rightarrow \infty$ ) and of a sequence of positive integers  $(k_n)_{n \in \mathbb{N}}$  with  $k_n \rightarrow \infty$  such that

$$\{\mu_{k_n}, \mu_{k_n+1}\} \subset \left[\omega_n - \frac{1}{n}\omega_n^{\frac{\alpha-1}{\alpha}}, \omega_n + \frac{1}{n}\omega_n^{\frac{\alpha-1}{\alpha}}\right]. \quad (3.13)$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{\mu_{k_n+1} - \mu_{k_n}}{\omega_n^{\frac{\alpha-1}{\alpha}}} = 0. \quad (3.14)$$

Combining (3.13) and (3.14) we obtain that

$$\lim_{n \rightarrow \infty} \frac{\mu_{k_n+1} - \mu_{k_n}}{\mu_{k_n}^{\frac{\alpha-1}{\alpha}}} = 0. \quad (3.15)$$

On the other hand, it is not difficult to check that from (3.3) we have

$$\liminf_{k \rightarrow \infty} \frac{\mu_{k+1} - \mu_k}{\mu_k^{\frac{\alpha-1}{\alpha}}} > 0. \quad (3.16)$$

Indeed, note that the parameter  $\alpha \in (0, 1)$ , we know that there is no uniform gap between  $(\mu_k)_{k \in \mathbb{N}}$ . In this case, we shall show that this is true for the sequence  $(\mu_k^{1/\alpha})_{k \in \mathbb{N}}$ . Based on the formula for  $\mu_k$  in (3.3), we obtain that

$$\begin{aligned} \left|\mu_{k+1}^{1/\alpha} - \mu_k^{1/\alpha}\right| &\geq \frac{1}{\alpha} |\mu_{\bar{\zeta}}|^{\frac{1}{\alpha}-1} \left(\alpha \bar{\zeta}^{\alpha-1} - cq \bar{\zeta}^{q-1}\right) \\ &\geq \frac{1}{\alpha} (\bar{\zeta}^\alpha - cq \bar{\zeta}^q)^{\frac{1}{\alpha}-1} \left(\alpha \bar{\zeta}^{\alpha-1} - cq \bar{\zeta}^{q-1}\right), \end{aligned} \quad (3.17)$$

where  $\bar{\zeta}$  is between  $k+1$  and  $k$ . Notice that  $q < \alpha < 1$ , it is not difficult to obtain from (3.17) that there exists  $\delta > 0$  such that

$$\inf_{k \in \mathbb{N}} \left|\mu_{k+1}^{1/\alpha} - \mu_k^{1/\alpha}\right| \geq \delta.$$

Moreover, we know that  $\left|\mu_{k+1}^{1/\alpha} - \mu_k^{1/\alpha}\right| = \frac{1}{\alpha} (\mu_{k+1} - \mu_k) |\eta|^{\frac{1}{\alpha}-1}$ ,  $\eta$  is in the middle of  $\mu_{k+1}$  and  $\mu_k$ . Hence, we derive that

$$\mu_k^{\frac{1-\alpha}{\alpha}} (\mu_{k+1} - \mu_k) = \frac{\mu_k^{\frac{1-\alpha}{\alpha}} \left|\mu_{k+1}^{1/\alpha} - \mu_k^{1/\alpha}\right|}{\frac{1}{\alpha} |\eta|^{\frac{1-\alpha}{\alpha}}} \geq \delta \alpha,$$

where we used the strictly increasing of the sequence  $(\mu_k)_{k \in \mathbb{N}}$ . Therefore, we conclude that (3.16) holds, which contradicts (3.15). The proof is finished.  $\square$

### 3.3. PROOF OF THE MAIN RESULTS

We are now in a position to prove, following the methodology used, in particular, in [77] and Ramdani, Takahashi and Tucsnak [93], our main abstract result.

*Proof of Theorem 3.1.1.* According to Proposition 3.2.1, we need to prove that  $\|(R(i\omega : \mathcal{A}))\| = O(|\omega|^{2(\frac{\beta-\alpha+1}{\alpha})})$  as  $\omega \rightarrow \infty$ . It clearly suffices to prove that

$$\sup_{|\omega| \geq M} \frac{1}{|\omega|^{2(\frac{\beta-\alpha+1}{\alpha})}} \|R(i\omega : \mathcal{A})\| < \infty, \quad (3.18)$$

where  $M$  is the constant in Lemma 3.3.1. To this aim we adopt a contradiction argument. Assuming that (3.18) is false we obtain the existence of the sequences  $(\omega_n)_{n \in \mathbb{N}}$  of real numbers and  $(z_n)_{n \in \mathbb{N}}$  of elements of  $\mathcal{D}(\mathcal{A})$  such that  $|\omega_n| \geq M$  and

$$\|z_n\| = 1 \quad (n \in \mathbb{N}), \quad (3.19)$$

and

$$|\omega_n|^{2(\frac{\beta-\alpha+1}{\alpha})} (i\omega_n I - \mathcal{A})z_n \rightarrow 0 \quad \text{in } X. \quad (3.20)$$

Note that, since the map  $\omega \mapsto \|R(i\omega : \mathcal{A})\|$  is real analytic on  $\mathbb{R}$  (thus bounded on compact sets), we necessarily have  $|\omega_n| \rightarrow \infty$ . Since for each  $n \in \mathbb{N}$  we have

$$\operatorname{Re} \left\langle |\omega_n|^{2(\frac{\beta-\alpha+1}{\alpha})} (i\omega_n I - \mathcal{A})z_n, z_n \right\rangle = |\omega_n|^{2(\frac{\beta-\alpha+1}{\alpha})} \|B^* z_n\|_U^2,$$

from (3.20) it follows that

$$\begin{aligned} |\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} B^* z_n &\rightarrow 0 && \text{in } U, \\ |\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} B B^* z_n &\rightarrow 0 && \text{in } X. \end{aligned} \quad (3.21)$$

Observing that

$$|\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} (i\omega_n I - \mathcal{A})z_n = |\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} (i\omega_n I - A)z_n + |\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} B B^* z_n,$$

and combining with (3.20) and (3.21) we obtain that

$$|\omega_n|^{\frac{\beta-\alpha+1}{\alpha}} (i\omega_n I - A)z_n \rightarrow 0 \quad \text{in } X.$$

Since  $z_n = \sum_{k \in J} \langle z_n, \phi_k \rangle \phi_k$  and  $A\phi_m = i\mu_m \phi_m$  for each  $m \in J$  the last formula implies that

$$\lim_{n \rightarrow \infty} \sum_{k \in J} |\omega_n|^{2(\frac{\beta-\alpha+1}{\alpha})} |(\omega_n - \mu_k) \langle z_n, \phi_k \rangle|^2 = 0. \quad (3.22)$$

At this stage we introduce the set

$$\mathcal{F} = \left\{ n \in \mathbb{N} \mid \exists k(n) \in J \text{ such that } |\omega_n - \mu_{k(n)}| < \gamma |\omega_n|^{\frac{\alpha-1}{\alpha}} \right\}, \quad (3.23)$$

where  $\gamma$  is defined in Lemma 3.3.1.

We distinguish two cases.

**First case.** Suppose that the set  $\mathcal{F}$  defined in (3.23) is finite. Then, we can assume, without loss of generality, that  $\mathcal{F}$  is empty, i.e., that

$$|\omega_n - \mu_k| \geq \gamma |\omega_n|^{\frac{\alpha-1}{\alpha}} \quad (n \in \mathbb{N}, k \in J).$$

Then (3.22) implies that  $\lim_{n \rightarrow \infty} z_n = 0$  which contradicts (3.19) and thus shows that (3.18) holds in the case when the set  $\mathcal{F}$  defined in (3.23) is finite.

**Second case.** Assume that the set  $\mathcal{F}$  is infinite. Then, for the sake of simplicity, we can suppose, without loss of generality, that  $\mathcal{F} = \mathbb{N}$ . In this case (3.22) and the fact that, according to Lemma 3.3.1

$$|\omega_n - \mu_k| \geq \gamma |\omega_n|^{\frac{\alpha-1}{\alpha}} \quad (n \in \mathbb{N}, k \neq k(n)), \quad (3.24)$$

imply that

$$\lim_{n \rightarrow \infty} \left( |\omega_n|^{2(\frac{\beta-\alpha+1}{\alpha})} \left| (\omega_n - \mu_{k(n)}) \langle z_n, \phi_{k(n)} \rangle \right|^2 + \sum_{k \neq k(n)} |\omega_n|^{\frac{2\beta}{\alpha}} |\langle z_n, \phi_k \rangle|^2 \right) = 0.$$

Denoting  $\psi_n = \langle z_n, \phi_{k(n)} \rangle \phi_{k(n)}$ , the last formula and (3.24) imply that

$$\lim_{n \rightarrow \infty} |\omega_n|^{\frac{2\beta}{\alpha}} \|z_n - \psi_n\|^2 = 0. \quad (3.25)$$

The above estimate and (3.19) clearly imply that

$$\lim_{n \rightarrow \infty} \left| \langle z_n, \phi_{k(n)} \rangle \right| = \lim_{n \rightarrow \infty} \|\psi_n\| = 1. \quad (3.26)$$

Moreover, from (3.25) we have

$$\lim_{n \rightarrow \infty} \left\| \omega_n^{\frac{\beta}{\alpha}} B^*(z_n - \psi_n) \right\|_U = 0,$$

which, combined with (3.21), implies that

$$\lim_{n \rightarrow \infty} \left\| \omega_n^{\frac{\beta}{\alpha}} B^* \psi_n \right\|_U = 0. \quad (3.27)$$

On the other hand, from the inequality in the definition (3.23) of the set  $\mathcal{F}$  it follows that there exists  $c_0 > 0$  such that  $|\omega_n|^{\frac{\beta}{\alpha}} \geq c_0 |k(n)|^\beta$  for  $n$  large enough. This fact, combined with (3.4) and (3.26) implies that

$$\liminf_{n \rightarrow \infty} \left\| \omega_n^{\frac{\beta}{\alpha}} B^* \psi_n \right\|_U > 0,$$

which contradicts (3.27). We have thus shown that (3.18) also holds in the case when the set  $\mathcal{F}$  defined in (3.23) is infinite.

Therefore, we obtain from Proposition 3.2.1 that (3.6) holds.  $\square$

Now we consider the case when  $[H_1]$  and  $[H_2]$  are satisfied with  $\alpha \geq 1$  and  $\beta > 0$ . In this case, there is a uniform gap between the eigenvalues  $(\mu_k)_{k \in J}$ . Based on Proposition 3.2.3, we present the proof of Theorem (3.1.2) in what follows.

*Proof of Theorem 3.1.2.* According to standard Riesz-spectral theory in [115] or [40] we know that the solution of the undamped system (3.11) has the representation as follows:

$$w(t) = \sum_{k \in J} e^{i\mu_k t} \langle z_0, \phi_k \rangle \phi_k, \quad (3.28)$$

### 3.4. SOME APPLICATIONS

where, for every  $k \in J$ ,  $i\mu_k$  is the eigenvalue of the operator  $A$  and the corresponding eigenvector is  $\phi_k$ . Notice that there exists  $\tau > 0$ , such that

$$\inf_{k,l \in J, k \neq l} |\mu_k - \mu_l| > \tau,$$

when the sequence  $(\mu_k)_{k \in J}$  satisfy (3.3) with  $\alpha \geq 1$ . Using the formula (3.28), we have, for every  $T > 0$ ,

$$\int_0^T \|(B^*w)(t)\|_U^2 dt = \int_0^T \left| \sum_{k \in J} e^{i\mu_k t} \langle z_0, \phi_k \rangle B^* \phi_k \right|^2 dt.$$

By using the assumption (3.4) and Ingham theorem (a generalization of Parsevals equality, for instance, in [115, Chapter 8]), there exists  $c > 0$ , such that, for every  $T > \frac{\pi}{\tau}$ ,

$$\int_0^T \|(B^*w)(t)\|_U^2 dt \geq C_T \sum_{k \in J} |\langle z_0, \phi_k \rangle|^2 \|B^* \phi_k\|_U^2 \geq C_T \sum_{k \in J} |\langle z_0, \phi_k \rangle|^2 \cdot \frac{c}{|k|^{2\beta}}.$$

Using the assumption (3.3) again, it is not difficult to see that

$$\sum_{k \in J} |\langle z_0, \phi_k \rangle|^2 \cdot \frac{c}{|k|^{2\beta}} \geq C \sum_{k \in J} |\langle z_0, \phi_k \rangle|^2 \frac{1}{|\mu_k|^{\frac{2\beta}{\alpha}}}. \quad (3.29)$$

Recalling the definition of a scale of Hilbert spaces  $X_s$  with  $s \in \mathbb{R}$  in (3.5), we conclude from (3.29) that

$$\int_0^T \|(B^*w)(t)\|_U^2 dt \geq C_T \|z_0\|_{X_{-\frac{\beta}{\alpha}}}^2.$$

In view of the definition of the space  $Y$  introduced in Proposition 3.2.3, we see that actually  $Y = X_1$ . For  $W = X_{-\frac{\beta}{\alpha}}$ ,  $Y = X_1$  and after a simple calculation we obtain that

$$\left[ X_1, X_{-\frac{\beta}{\alpha}} \right]_{\theta} = X \quad \text{with } \theta = \frac{\alpha}{\alpha + \beta}.$$

Consequently, with the help of Proposition 3.2.3, we arrive at (3.7) and finish the proof.  $\square$

**Remark 2.** Actually we can also get the explicit decay rate by estimating the upper-bound of resolvent operator as in the proof of Theorem 3.1.1. The only difference lies in Lemma 3.3.1 and finally we obtain (3.7) again. The approach presented in Proposition 3.2.3 only available in the case when  $[H_1]$  is satisfied with  $\alpha \geq 1$ .

## 3.4 Some applications

### 3.4.1 Some background on strictly positive operators on Hilbert spaces

In this section we discuss two simple examples, which can be seen as toy models, illustrating the possible applications of Theorem 3.1.1. To formulate these problems, involving fractional versions of the Schrödinger and wave equations, we first remind in this subsection some basic facts on strictly positive operators on general Hilbert

spaces and their fractional powers. The framework introduced in this subsection will be further used in the remaining parts.

Let  $H$  be a Hilbert space, with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ , and let  $A_0 : \mathcal{D}(A_0) \rightarrow H$  be a strictly positive operator (which means that  $A_0$  is self-adjoint and there exists  $m > 0$  such that  $\langle A_0 f, f \rangle \geq m \|f\|^2$  for every  $f \in \mathcal{D}(A_0)$ ). We assume that  $A_0$  has compact resolvents, so that, by classical spectral theory, there exists an orthonormal basis of  $H$ , denoted by  $(\varphi_k)_{k \in \mathbb{N}}$ , formed by eigenvectors of  $A_0$ . The corresponding sequence of eigenvalues is denoted by  $(\lambda_k)_{k \in \mathbb{N}}$ . We clearly have that  $\lambda_k > 0$  for each  $k \in \mathbb{N}$ .

Recalling the Hilbert spaces introduced in (3.5), in this context we denote by  $(H_s)_{s \in \mathbb{R}}$  a scale of Hilbert spaces associated with the operator  $A_0$  and the corresponding eigenvalues  $(\lambda_k)_{k \in \mathbb{N}}$ . Note that  $A_0$  maps  $H_s$  onto  $H_{s-1}$  for every  $s \geq 1$ . Moreover, for every  $s \geq 0$  we define the fractional powers of the operator  $A_0$  by setting  $A_0^s : H_s \rightarrow H$  and

$$A_0^s f = \sum_{k \in \mathbb{N}} (1 + |\lambda_k|^{2s}) \langle f, \varphi_k \rangle \varphi_k \quad \forall f \in H_s. \quad (3.30)$$

It is easily checked that  $A_0^s$  is a strictly positive operator on  $H$  with domain  $H_s$ .

Operators  $A_0$  as above (and their fractional powers) appear naturally in the study of systems described by the Schrödinger or wave equations (possibly fractional). For Schrödinger type equations the corresponding semigroup generator is  $iA_0$  (or  $iA_0^s$ ), in the case of wave type equations the semigroup generator is a matrix operator in which  $A_0$  appears in some of its blocks. More precisely (see, for instance [115, Section 3.7]), we have:

**Proposition 3.4.1.** *Let the Hilbert space  $H$  and the strictly positive operator  $A_0 : \mathcal{D}(A_0) \rightarrow H$  be as above. Define  $X = H_{\frac{1}{2}} \times H$ , with the scalar product*

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle w_1, w_2 \rangle_{\frac{1}{2}} + \langle v_1, v_2 \rangle.$$

Define a dense subspace of  $X$  by  $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$  and the linear operator  $A : \mathcal{D}(A) \rightarrow X$  by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -A_0 \varphi \end{bmatrix}.$$

Then  $A$  is skew-adjoint on  $X$  and  $0 \in \rho(A)$ .

Moreover,  $A$  has compact resolvents and it is diagonalizable, with the eigenvalues  $(i\mu_k)_{k \in \mathbb{Z}^*}$  corresponding to the orthonormal basis of eigenvectors

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \quad \forall k \in \mathbb{Z}^*,$$

where, for every  $k \in \mathbb{N}$  we define  $\mu_k = \sqrt{\lambda_k}$ ,  $\varphi_{-k} = -\varphi_k$  and  $\mu_{-k} = -\mu_k$ .

### 3.4.2 A system described by a fractional Schrödinger equation

In this section we consider the particular case of the framework introduced in Section 3.4.1 by choosing  $H = L^2[0, \pi]$  and the operator  $A_0$  to be the Dirichlet Laplacian on  $[0, \pi]$ . This means that  $A_0 : \mathcal{D}(A_0) \rightarrow H$  is defined by

$$\mathcal{D}(A_0) = \mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi), \quad (3.31)$$

### 3.4. SOME APPLICATIONS

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$$A_0\varphi = -\frac{d^2\varphi}{dx^2} \quad \forall \varphi \in \mathcal{D}(A_0), \quad (3.32)$$

where, for every  $m \in \mathbb{N}$ ,  $\mathcal{H}^m(0, \pi)$  stands, as usual, for the Sobolev space formed by the functions in  $L^2[0, \pi]$  which have  $m$  derivatives, in the distribution sense, in  $L^2[0, \pi]$ , whereas  $\mathcal{H}_0^m(0, \pi)$  stands for the subspace of  $\mathcal{H}^m(0, \pi)$  formed by those functions in  $\mathcal{H}^m(0, \pi)$  which vanish, together with their derivatives up to order  $m - 1$ , at  $x = 0$  and  $x = \pi$ . It is well-known that  $A_0$  is a strictly positive operator on  $H$  and that the family  $(\varphi_k)_{k \in \mathbb{N}}$  defined by

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx) \quad (x \in [0, \pi]),$$

form an orthonormal basis in  $X$  consisting of the eigenvectors of  $A_0$ . Moreover, we have

$$A_0\varphi_k = k^2\varphi_k \quad \forall k \in \mathbb{N}.$$

For  $s \in (0, \frac{1}{2})$  we consider the control system described by the equation

$$\begin{cases} \dot{z} = iA_0^s z + B_0 u, \\ z(0) = z_0, \end{cases} \quad (3.33)$$

where  $B_0 \in \mathcal{L}(H)$  is defined by

$$B_0 v = v\chi_I \quad \forall v \in H, \quad (3.34)$$

with  $\chi_I$  the indicator function of some non-empty open interval  $I \subset (0, \pi)$ . Choosing the control function  $u$  in (3.33) in the feedback form  $u = -B_0^* z$  we obtain the closed-loop problem

$$\begin{cases} \dot{z} = iA_0^s z - z\chi_I & (t \geq 0), \\ z(0) = z_0. \end{cases} \quad (3.35)$$

The main result of this subsection is

**Proposition 3.4.2.** *With the above notation, for every  $s \in (0, \frac{1}{2})$  and  $z_0 \in H_s$  the initial value problem (3.35) admits a unique solution*

$$z \in C([0, \infty); H_s) \cap C^1([0, \infty); H).$$

Moreover, there exists a constant  $c_s > 0$  such that

$$\|z(t)\| \leq \frac{c_s}{(1+t)^{\frac{s}{1-2s}}} \|z_0\|_s \quad \forall z_0 \in H_s, t \geq 0. \quad (3.36)$$

*Proof.* It suffices to apply Theorem 3.1.1 with  $X = U = H$ ;  $A = iA_0^s$  and  $B = B_0$ , where  $A_0$  and  $B_0$  are defined in (3.31), (3.32) and (3.34), respectively. Indeed, with these choices  $A$  is clearly skew-adjoint and the eigenvalues of  $A$  are  $(i\mu_k)_{k \in \mathbb{N}}$  with  $\mu_k = k^{2s}$  for every  $k \in \mathbb{N}$ . Moreover, the corresponding orthonormal basis formed of the eigenvectors of  $A$  is  $(\phi_k)_{k \in \mathbb{N}}$ , with  $\phi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$  for each  $k \in \mathbb{N}$ . The assumptions of Theorem 3.1.1 are thus satisfied with  $\alpha = 2s$  and  $\beta = 0$ , so we obtain the estimate (3.36).  $\square$



### 3.4.3 A system described by a fractional wave equation

In this subsection we continue to use the notation and the assumptions in Subsection 3.4.2 for the spaces  $H$ ,  $U$  and the operators  $A_0$  and  $B_0$ . This means, in particular, that we take  $H = L^2[0, \pi]$ , that  $A_0$  is defined by (3.31), (3.32) and that  $B_0$  is given by (3.34).

For  $s > 0$ , we consider the control system described by the second-order differential equation

$$\begin{cases} \ddot{w}(t) + A_0^s w(t) = B_0 u(t) & (t \geq 0), \\ w(0) = w_0, \quad \dot{w}(0) = w_1. \end{cases} \quad (3.37)$$

For  $s = 1$  the above system describes the vibrations of a string occupying the interval  $[0, \pi]$  with control  $u$  acting on an interval  $I \subset [0, \pi]$ , whereas for  $s = 2$  it describes the controlled vibrations of hinged Euler-Bernoulli beam (please refer to, for instance, [7] and [114]). Using the feedback law  $u = -B_0^* \dot{w}$ , corresponding to colocated actuators and sensors, the obtained closed-loop system can be written in the form

$$\begin{cases} \ddot{w}(t) + A_0^s w(t) + \dot{w}(t) \chi_I = 0 & (t \geq 0), \\ w(0) = w_0, \quad \dot{w}(0) = w_1. \end{cases} \quad (3.38)$$

The main result in this subsection is:

**Proposition 3.4.3.** *With the above notation, for every  $s \in (0, 1)$ ,  $w_0 \in H_s$  and  $w_1 \in H_{\frac{s}{2}}$  the initial value problem (3.38) admits a unique solution*

$$w \in C([0, \infty); H_s) \cap C^1([0, \infty), H_{\frac{s}{2}}).$$

Moreover, there exists a constant  $c_s > 0$  such that for every  $t \geq 0$ ,  $w_0 \in H_s$  and  $w_1 \in H_{\frac{s}{2}}$  we have

$$\|w(t)\|_{\frac{s}{2}}^2 + \|\dot{w}(t)\|^2 \leq \frac{c_s^2}{(1+t)^{1-s}} (\|w_0\|_s^2 + \|w_1\|_{\frac{s}{2}}^2). \quad (3.39)$$

*Proof.* Let  $X = H_{\frac{s}{2}} \times H$  and  $A : \mathcal{D}(A) \rightarrow X$  be defined by

$$\mathcal{D}(A) = H_s \times H_{\frac{s}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0^s & 0 \end{bmatrix}. \quad (3.40)$$

According to Proposition 3.4.1 (with  $A_0$  replaced by  $A_0^s$ ), the operator  $A$  is skew-adjoint on  $X$ . Moreover, setting  $\mu_k = k^s$ ,  $\mu_{-k} = -\mu_k$ ,  $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$  and  $\varphi_{-k} = -\varphi_k$  for every  $k \in \mathbb{N}$ , the eigenvalues of  $A$  are  $(i\mu_k)_{k \in \mathbb{Z}^*}$  corresponding to the orthonormal basis of eigenvectors

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \quad (k \in \mathbb{Z}^*).$$

Setting next

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad (3.41)$$

where  $B_0$  has been defined in (3.34) we see that  $B \in \mathcal{L}(U, X)$  and the initial value problem (3.38) can be written

$$\dot{z}(t) = (A - BB^*)z(t) \quad \text{for } t \geq 0, \quad z(0) = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix},$$

### 3.4. SOME APPLICATIONS

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with  $z(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$  and  $A, B$  defined in (3.40), (3.41). Using the above properties of  $A$  and  $B$ , we see that they satisfy the assumptions in Theorem 3.1.1 with  $\alpha = s$  and  $\beta = 0$ . Thus there exists a constant  $c_s > 0$  such that

$$\|z(t)\|_X \leq \frac{c_s}{(1+t)^{\frac{s}{2(1-s)}}} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{\mathcal{D}(A)} \quad \left( \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(A) \right),$$

which clearly implies the conclusion (3.39).  $\square$

**Remark 3.** Most of the literature devoted to the analysis and the control of systems involving fractional differential operators is based on the so called *integral definition* of the fractional Laplacian, which is

$$\left( -\frac{d^2}{dx^2} \right)^s z(x) := c_s \lim_{\varepsilon \rightarrow 0^+} \int_{x-\varepsilon}^{x+\varepsilon} \frac{z(x) - z(y)}{|x-y|^{1+2s}} dy \quad (s \in (0, 1), x \in [0, \pi]), \quad (3.42)$$

with  $c_s$  a normalization constant given by (for instance [97])

$$c_s = \frac{s 2^{2s} \Gamma\left(\frac{1+2s}{2}\right)}{\sqrt{2} \Gamma(1-s)},$$

where  $\Gamma$  is the Gamma function.

As shown, for instance, in [102] the above formula (3.42) defines a strictly positive operator on  $L^2[0, \pi]$ , which is different with the operator  $A_0^s$  defined in (3.30) (with  $A_0$  the Dirichlet Laplacian). As far as we know, the controllability and stabilizability for the fractional Schrödinger equation and the wave equation involving the operator defined in (3.42) are just studied, for instance, in [117], [22] and [23], in the case of  $s \geq \frac{1}{2}$  for the Schrödinger equation and  $s \geq 1$  for the wave equation. Our results are valid for every  $s \in (0, 1)$ , in particular, due to the fact that the eigenvectors of  $A_0^s$  coincide with the eigenvectors of  $A_0$ , which is not the case of the operator defined in (3.42). For  $s < \frac{1}{2}$ , the eigenvectors of the fractional Laplace operator in integral definition (3.42) are not necessarily smooth and we are not aware whether they satisfy assumptions of type (3.4), allowing the application of Theorem 3.1.1.



## **Part II**

# **Water waves system in a rectangular domain**



## Chapter 4

# Boundary control problem of a water waves system

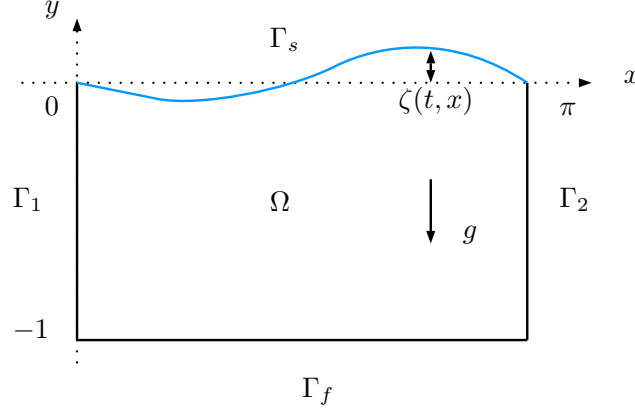
In this chapter, we study a boundary control system describing water waves in a rectangular fluid domain. We first derive the full governing equations of the water waves equations with boundary control, based on the *Zakharov-Craig-Sulem formulation* (2.22). Then we linearize the governing equations and study its well-posedness through an abstract linear control system. Moreover, we consider the stabilizability of the linear system by state feedback and give an explicit decay rate for the norm of the state trajectory. We analyse the case of gravity waves and gravity-capillary waves, respectively. Actually, the stabilization issue discussed in this chapter is an interesting application of the results in Chapter 3. Part of contents in this chapter are based on Su et al. [110] and Su [108].

### 4.1 The equations of the water waves control system

We are interested in a control system of the water waves in a rectangular domain, in the presence of a wave maker. The control acts on one of lateral edges of the fluid domain, by imposing the horizontal velocity of the water along that boundary. We assume that the domain  $\Omega_t$  is delimited at its top by a free water surface  $\Gamma_s$  and that the bottom  $\Gamma_f$  is flat. The other two components of the boundary of the fluid domain, denoted by  $\Gamma_1$  and  $\Gamma_2$ , are supposed to be vertical. Let  $x \in [0, \pi]$  the horizontal variable and let  $y$  be the vertical variable. When the free surface is at rest, we assume that the position of free surface is at  $y = 0$  and the typical water depth is 1, so that the bottom of the domain is at  $y = -1$ . We denote by  $\zeta(t, x)$  the elevation of the free surface at time  $t$ . As shown in Figure 4.1, the domain  $\Omega_t$  is described by

$$\Omega_t = \{(x, y) \mid 0 < x < \pi, -1 < y < \zeta(t, x)\}.$$

We assume that the fluid fills  $\Omega_t$  that it is homogeneous, incompressible, inviscid and that it undergoes irrotational flows. There is a wave maker that acts at, without loss of generality, the left boundary of  $\Omega_t$ , by injecting (or extracting) fluid in the horizontal direction, at a velocity determined by the control signal. We denote by  $\mathcal{V}(t, y)$  the velocity produced by this wave maker. With the above notation, we derive in what follows the equations for the control problem of water waves in  $\Omega_t$ .


 Figure 4.1: A rectangular domain  $\Omega_t$  filled with water

### 4.1.1 Derivation of the governing equations

Here we give a detailed derivation of the governing equations for the control problem of the water waves system described above. We denote by  $U(t, x, y) \in \mathbb{R}^2$  the velocity of the fluid particle located at the position  $(x, y)$  at time  $t$ , and by  $V(t, x, y)$  and  $w(t, x, y)$  the horizontal and the vertical component of  $U(t, x, y)$ , respectively. The pressure at  $(x, y)$  at time  $t$  is denoted by  $P(t, x, y)$ . As we know that the motion of a non-viscous fluid with constant density  $\rho$  is governed by the Euler equation

$$\partial_t U + U \cdot \nabla_{x,y} U = -\frac{1}{\rho} \nabla_{x,y} P - g e_y \quad \text{in } \Omega_t, \quad (4.1)$$

where  $g$  is the acceleration of gravity and  $e_y$  is the unit upwards vector in the vertical direction. For incompressible fluid, we have

$$\nabla_{x,y} \cdot U = 0 \quad \text{in } \Omega_t. \quad (4.2)$$

Moreover, the fluid is assumed to be irrotational, i.e.

$$\nabla_{x,y} \times U = 0 \quad \text{in } \Omega_t, \quad (4.3)$$

which implies that there exists a potential function, denoted by  $\phi$ , such that

$$U(t, x, y) = \nabla_{x,y} \phi(t, x, y) \quad \text{in } \Omega_t.$$

This, together with (4.1), gives that

$$\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + \frac{P}{\rho} + gy = C(t),$$

Without loss of generality, we take  $\phi = \tilde{\phi} + f(t)$  with  $f'(t) = C(t) - \frac{P_{\text{atm}}}{\rho}$ . The notation  $P_{\text{atm}}$  represents the pressure of atmosphere. Hence, we obtain the Bernoulli equation

$$\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy = -\frac{1}{\rho} (P - P_{\text{atm}}) \quad \text{in } \Omega_t. \quad (4.4)$$

#### 4.1. THE EQUATIONS OF THE WATER WAVES CONTROL SYSTEM

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Combining with (4.2) and (4.3), we have

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega_t. \quad (4.5)$$

Next we need boundary conditions of the velocity potential  $\phi(t, x, y)$ . The first free surface condition is called the kinematic boundary condition, which means that the fluid particles does not cross the surface. We denoted by  $\vec{n}$  the outer normal vector on the free surface  $\Gamma_s$ . We know that the equation of the free surface is  $\gamma(t, x, y) = y - \zeta(t, x) = 0$ . The fluid particle  $M(t) = (x(t), y(t))$  always stays on  $\Gamma_s$ , i.e. we have

$$\frac{d}{dt}\gamma(t, M(t)) = \partial_t \gamma + U \cdot \nabla_{x,y} \gamma = 0,$$

where we used the relation  $\frac{d}{dt}M(t) = U$ . Note that  $\partial_t \gamma = -\partial_t \zeta$  and the normal vector  $\vec{n} = \nabla_{x,y} \gamma$  on  $\Gamma_s$ , we obtain the first free surface condition for  $\phi$

$$\partial_{\vec{n}} \phi = \partial_t \zeta \quad \text{on } \Gamma_s. \quad (4.6)$$

The second free surface equation is the restriction of the Bernoulli equation (4.4) on  $\Gamma_s$ , that is

$$\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + g\zeta = 0 \quad \text{on } \Gamma_s. \quad (4.7)$$

Because of the existence of the wave maker on the left boundary  $\Gamma_1$ , then we have

$$\partial_{\vec{n}} \phi = \mathcal{V}(t, y) \quad \text{on } \Gamma_1, \quad (4.8)$$

where  $\vec{n}$  is the outer normal vector on  $\Gamma_1$ . Note that the flat bottom  $\Gamma_f$  is impermeable, the normal derivative of  $\phi$  are zero on  $\gamma_f$ , i.e.

$$\partial_{\vec{n}} \phi = 0 \quad \text{on } \Gamma_f. \quad (4.9)$$

Similarly, we have the boundary condition on  $\Gamma_2$ ,

$$\partial_{\vec{n}} \phi = 0 \quad \text{on } \Gamma_2. \quad (4.10)$$

Therefore, we obtain the governing equations of the water waves system with wave maker on the left boundary, which is given by (4.5)–(4.10). Let  $\psi(t, x) = \phi(t, x, \zeta(t, x))$  be the trace of the velocity potential  $\phi$  on  $\Gamma_s$ . Note that the water waves equations (4.5)–(4.10) reduce to a Laplace equation with mixed boundary conditions defined in  $\Omega_t$ , if  $\psi(t, x)$  is known on  $\Gamma_s$ . Hence, the whole water waves system actually fully depends on the equations on  $\zeta$  and  $\psi$ . Based on this observation, we introduce two important evolution operators to rewrite the governing equations.

The first operator is called the *Dirichlet to Neumann operator*, denoted by  $A[\zeta]$ , which is defined by

$$A[\zeta] : \psi \mapsto (\partial_{\vec{n}} \Psi) \Big|_{y=\zeta(t,x)}, \quad (4.11)$$

where  $\Psi(t, x, y)$  satisfies

$$\begin{cases} \Delta_{x,y} \Psi = 0 & \text{in } \Omega_t, \\ \Psi \Big|_{y=\zeta(t,x)} = \psi(t, x), \\ (\partial_{\vec{n}} \Psi) \Big|_{x=0} = 0 = (\partial_{\vec{n}} \Psi) \Big|_{x=\pi} = (\partial_{\vec{n}} \Psi) \Big|_{y=-1}. \end{cases}$$



The second operator is called the *Neumann to Neumann operator*, denoted by  $B[\zeta]$ , which is defined by

$$B[\zeta] : \mathcal{V}(t, y) \mapsto (\partial_{\bar{n}}\Phi) \Big|_{y=\zeta(t,x)}, \quad (4.12)$$

and  $\Phi(t, x, y)$  is determined by

$$\begin{cases} \Delta_{x,y}\Phi = 0 & \text{in } \Omega_t, \\ \Phi|_{y=\zeta(t,x)} = 0, \\ (\partial_{\bar{n}}\Phi)|_{x=0} = \mathcal{V}(t, y), \\ (\partial_{\bar{n}}\Phi)|_{x=\pi} = 0 = (\partial_{\bar{n}}\Phi)|_{y=-1} \end{cases}$$

It is interesting to see that the two Laplace equations used to define the Dirichlet to Neumann operator  $A[\zeta]$  and the Neumann to Neumann operator  $B[\zeta]$  is a decomposition of the water waves system (4.5)–(4.10) in the sense that

$$\phi(t, x, y) = \Psi(t, x, y) + \Phi(t, x, y) \quad \text{in } \Omega_t.$$

For the properties of the operator  $A[\zeta]$  and  $B[\zeta]$ , please refer to [66] for details. By using the operators  $A[\zeta]$  and  $B[\zeta]$ , we obtain the equations of  $\zeta$  and  $\psi$ , for every  $t \geq 0$  and  $x \in [0, \pi]$ , as follows:

$$\begin{cases} \partial_t \zeta - A[\zeta]\psi = B[\zeta]\mathcal{V}, \\ \partial_t \psi + g\zeta + \frac{1}{2} |\partial_x \psi|^2 - \frac{1}{2} \frac{(A[\zeta]\psi + B[\zeta]\mathcal{V} + \partial_x \zeta \partial_x \psi)^2}{1 + |\partial_x \zeta|^2} = 0. \end{cases} \quad (4.13)$$

We see that the above equation is fully nonlinear and fully dispersive, which is called the *Zakharov-Craig-Sulem formulation* (or  $(\zeta, \psi)$  formulation). As far as we know, there is no results for the equations (4.13) up to now. In the case when the control term  $\mathcal{V} = 0$  and the equations are defined in the whole real axis, i.e.  $x \in \mathbb{R}$ , the local well-posedness of this formulation is proved in Lannes [65] for arbitrary dimension. The local well-posedness is further studied in Alazard et al. [5] for low regularity and in Alvarez and Lannes [6], Iguchi [58] for uniform bounds in 3D shallow water. For the control problem on  $(\zeta, \psi)$  formulation, Alazard discussed in [2] the stabilization issue in a rectangle where the external pressure as the control signal acts on a part of the free surface, by absorbing the waves coming from the left. For more details on water waves models we refer to Whitham's book [123, Chapter 13] and to Lannes [66, Chapter 1].

#### 4.1.2 Linearized equations

In the remaining part of this chapter, we consider the small-amplitude water waves in a rectangular fluid domain with a wave maker imposed from the left boundary. We linearize the water waves equations (4.5)–(4.10) around the rest state. The linearized fluid domain, denoted by  $\Omega$ , is

$$\Omega = \{(x, y) \mid 0 < x < \pi, -1 < y < 0\}. \quad (4.14)$$

Moreover, we assume that the wave maker is rigid in the sense that the velocity  $\mathcal{V}(t, y)$  is a multiple of a scalar input function  $v(t)$ , times a given function  $h$  of the height along the active boundary, i.e.

$$\mathcal{V}(t, y) = h(y)v(t).$$

## 4.2. DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS IN A RECTANGULAR DOMAIN

For the sake of simplicity, we take the gravity acceleration  $g = 1$  in the following systems. The linearized version of the water waves system (4.5)–(4.10), for every  $t \geq 0$ ,  $(x, y) \in \Omega$ , reads

$$\begin{cases} \Delta_{x,y} \phi = 0 & \text{in } \Omega, \\ \partial_t \phi(t, x, 0) + \zeta(t, x) = 0, \\ \partial_y \phi(t, x, 0) = \partial_t \zeta(t, x), \\ \partial_x \phi(t, 0, y) = -h(y)v(t), \\ \partial_y \phi(t, x, -1) = 0 = \partial_x \phi(t, \pi, y), \end{cases} \quad (4.15)$$

To describe the control system in terms of the elevation  $\zeta$ , we denote the derivative of the velocity by

$$\varphi(t, x, y) := \partial_t \phi(t, x, y).$$

We write the linearized equations (4.15) in terms of  $\varphi$ , which, for every  $t \geq 0$  and  $(x, y) \in \Omega$ , are

$$\begin{cases} \Delta_{x,y} \varphi = 0 & \text{in } \Omega, \\ \varphi(t, x, 0) + \zeta(t, x) = 0, \\ \partial_y \varphi(t, x, 0) = \partial_t^2 \zeta(t, x), \\ \partial_x \varphi(t, 0, y) = -h(y)u(t), \\ \partial_y \varphi(t, x, -1) = 0 = \partial_x \varphi(t, \pi, y), \end{cases} \quad (4.16)$$

where the control  $u = \dot{v}$  is the acceleration imposed by the wave maker. The function  $h$  is given and it represents the profile of the acceleration field imposed by the wave maker. Usually we assume that  $\int_{-1}^0 h(y)dy = 0$ , to ensure the conservation of the volume of water. As far as we know, the controllability and stabilizability properties of systems derived from (4.15) have been first studied in Reid and Russell [96]. For the problem in an irregular domain and the case of the water waves with surface tension, please refer to Reid [94] and [95]. Further Mottelet studied in [85] the control system with the flexible and the rigid wave maker, respectively.

Next we shall study the well-posedness of the linearized system (4.16). To do this, we give in the following a detailed analysis of the linear Dirichlet to Neumann operator and the linear Neumann to Neumann operator in the fluid domain  $\Omega$  introduced in (4.14).

## 4.2 Dirichlet to Neumann and Neumann to Neumann operators in a rectangular domain

In this section we consider two boundary value problems for the Laplacian in the rectangular domain  $\Omega$  defined in (4.14) and we define the corresponding solution operators. Note that,  $\Omega$  being a rectangle, we are able to construct these solution operators, as well as the Dirichlet to Neumann and Neumann to Neumann operators in an elementary and explicit way, using the separation of variables and analysis of Fourier or Dirichlet series. Another possible approach to these issues, pursued in [85], is the use of the much more sophisticated theory of elliptic problems in polygonal domains as described, for instance, in Grisvard [52].

We introduce in what follows some notation used in this section. We set

$$H = \left\{ \eta \in L^2[0, \pi] \mid \int_0^\pi \eta(x) dx = 0 \right\}, \quad (4.17)$$

which is a Hilbert space when endowed with the inner product inherited from  $L^2[0, \pi]$ . It is known that the family  $(\varphi_k)_{k \in \mathbb{N}}$  defined by

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx) \quad \forall x \in [0, \pi], \quad (4.18)$$

forms an orthonormal basis in  $H$ . For any  $\eta \in H$ , we denote  $\eta_k = \langle \eta, \varphi_k \rangle$ . The scale of Hilbert spaces  $(H_\alpha)_{\alpha \in \mathbb{R}}$  are defined by  $H_0 = H$  and

$$H_\alpha = \left\{ \eta \in H \mid \sum_{k \in \mathbb{N}} (1 + k^{2\alpha}) |\eta_k|^2 < \infty \right\} \quad (\alpha \in \mathbb{R}), \quad (4.19)$$

with the inner products  $(\langle \cdot, \cdot \rangle_\alpha)_{\alpha \in \mathbb{R}}$  defined by  $\langle \eta, \psi \rangle_\alpha = \sum_{k \in \mathbb{N}} (1 + k^{2\alpha}) \eta_k \overline{\psi_k}$ , for all  $\eta, \psi \in H_\alpha$ . It is not difficult to check that

$$H_1 = \left\{ \eta \in \mathcal{H}^1(0, \pi) \mid \int_0^\pi \eta(x) dx = 0 \right\}.$$

By interpolation theory (see, for instance, Lions and Magenes [74], Bensoussan et al. [19, Part II] and Chandler-Wilde et al. [31]) actually we have

$$H_s = \left\{ \eta \in \mathcal{H}^s(0, \pi) \mid \int_0^\pi \eta(x) dx = 0 \right\} \quad \forall s \in (0, 1).$$

We set

$$\mathcal{H}_{\text{top}}^1(\Omega) = \{f \in \mathcal{H}^1(\Omega) \mid f(x, 0) = 0, x \in (0, \pi)\}, \quad (4.20)$$

where the values at the top boundary are defined in the sense of the Dirichlet trace, as in [74], [115, Section 13.6].

#### 4.2.1 Some background on the partial Dirichlet and Neumann maps

We first consider the partial Dirichlet and Neumann maps. We begin by introducing a self-adjoint operator on  $L^2(\Omega)$  which plays an important role in our arguments in this section.

**Proposition 4.2.1.** *With  $\Omega$  as in (4.14), we consider the operator  $A_1 : \mathcal{D}(A_1) \rightarrow L^2(\Omega)$  defined by*

$$\mathcal{D}(A_1) = \left\{ f \in \mathcal{H}^2(\Omega) \mid \begin{array}{l} f(x, 0) = 0, \quad \partial_y f(x, -1) = 0 \quad x \in (0, \pi) \\ \partial_x f(0, y) = 0, \quad \partial_x f(\pi, y) = 0 \quad y \in (-1, 0) \end{array} \right\},$$

$$A_1 f = -\Delta f \quad \forall f \in \mathcal{D}(A_1).$$

Then  $A_1$  is a strictly positive operator on  $L^2(\Omega)$ .

## 4.2. DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS IN A RECTANGULAR DOMAIN

*Proof.* The operator  $A_1$  is obviously symmetric. Moreover, the family

$$\Psi_{kl}(x, y) = \frac{2}{\sqrt{\pi}} \cos(kx) \sin \left[ (2l-1) \frac{\pi}{2} y \right] \quad \forall k, l \in \mathbb{N}, (x, y) \in \Omega, \quad (4.21)$$

is an orthonormal basis for  $L^2(\Omega)$  formed of eigenvectors of  $A_1$ , corresponding to the eigenvalues

$$\lambda_{kl} = k^2 + (2l-1)^2 \frac{\pi^2}{4} \quad \forall k, l \in \mathbb{N}.$$

Let  $g \in L^2(\Omega)$ , so that  $g = \sum_{k,l \in \mathbb{N}} c_{kl} \Psi_{kl}$ , with  $c_{kl} \in l^2(\mathbb{N}^2)$ . This implies that  $f$  defined by

$$f = \sum_{k,l \in \mathbb{N}} \frac{c_{kl}}{k^2 + (2l-1)^2 \frac{\pi^2}{4}} \Psi_{kl},$$

satisfies  $f \in \mathcal{D}(A_1)$  and  $A_1 f = g$ . Thus the operator  $A_1$  is onto so that (see, for instance, [115, Proposition 3.2.4])  $A_1$  is self-adjoint. Finally, it follows from the first Green formula that

$$\langle A_1 f, f \rangle_{L^2(\Omega)} = \|\nabla f\|_{L^2(\Omega)}^2 \quad \forall f \in \mathcal{D}(A_1).$$

This, together with a version of the Poincaré inequality (see [115, Theorem 13.6.9]), implies that  $A_1$  is strictly positive.  $\square$

**Proposition 4.2.2.** *For every  $\eta \in L^2[0, \pi]$  there exists a unique function  $D\eta \in L^2(\Omega)$  such that*

$$\int_{\Omega} (D\eta)(x, y) \overline{g(x, y)} dx dy = - \int_0^{\pi} \eta(x) \overline{\partial_y(A_1^{-1}g)}(x, 0) dx \quad \forall g \in L^2(\Omega). \quad (4.22)$$

Moreover, the operator  $\eta \mapsto D\eta$  (called the partial Dirichlet map) is bounded from  $L^2[0, \pi]$  into  $L^2(\Omega)$ .

*Proof.* We first note from Proposition 4.2.1 that  $A_1^{-1} \in \mathcal{L}(L^2(\Omega), \mathcal{H}^2(\Omega))$ . Thus, by a standard trace theorem the map  $g \mapsto \partial_y(A_1^{-1}g)(\cdot, 0)$  is bounded from  $L^2(\Omega)$  to  $L^2[0, \pi]$ . Consequently, the right-hand side of (4.22) defines an anti-linear functional of the argument  $g \in L^2(\Omega)$ , and the result follows by applying the Riesz representation theorem. (See also [115, Section 10.6].)  $\square$

**Remark 4.** For every  $\eta \in H$ , we have  $D\eta \in C^\infty(\Omega)$  and  $\Delta(D\eta) = 0$ . Indeed, this follows by an argument that is similar to the one used in the proof of [115, Proposition 10.6.2]: We take  $g = \Delta\varphi$  with  $\varphi \in \mathcal{D}(\Omega)$  in (4.22) to see that  $\Delta(D\eta) = 0$  in the sense of distributions. It follows from [115, Remark 13.5.6] that  $D\eta \in \mathcal{H}_{loc}^n(\Omega)$  for every  $n \in \mathbb{N}$ . Then we use the embedding  $\mathcal{H}_{loc}^n(\Omega) \subset C^m(\Omega)$  for  $n > 1 + m$  ( $m \in \mathbb{N}$ ) (see [115, Remark 13.4.5]), so that indeed  $D\eta \in C^\infty(\Omega)$ , and hence  $\Delta(D\eta) = 0$ . Moreover, if  $D\eta \in C^1(\overline{\Omega})$ , then  $D\eta$  is the unique function in  $C^2(\Omega) \cap C(\overline{\Omega})$  that satisfies, in the classical sense, the following boundary value problem:

$$\begin{cases} \Delta(D\eta)(x, y) = 0 & (x, y) \in \Omega, \\ (D\eta)(x, 0) = \eta(x), \quad \partial_y(D\eta)(x, -1) = 0 & x \in (0, \pi), \\ \partial_x(D\eta)(0, y) = 0, \quad \partial_x(D\eta)(\pi, y) = 0 & y \in (-1, 0). \end{cases} \quad (4.23)$$

To see this, we take in (4.22)  $g = \Delta f$ , where  $f \in \mathcal{D}(A_1)$ , and use integration by parts, which yields that

$$\begin{aligned} \int_0^\pi \eta(x) \overline{\partial_y f(x,0)} dx &= \int_0^\pi [(D\eta) \overline{\partial_y f}] (x,0) dx + \int_0^\pi [(\partial_y D\eta) \bar{f}] (x,-1) dx \\ &\quad + \int_{-1}^0 [(\partial_x D\eta) \bar{f}] (0,y) dy - \int_{-1}^0 [(\partial_x D\eta) \bar{f}] (\pi,y) dy. \end{aligned}$$

If we choose  $f \in \mathcal{D}(A_1)$  such that  $f = 0$  on the lateral boundaries and the bottom of  $\Omega$ , we obtain that  $(D\eta)(x,0) = \eta(x)$  for almost every  $x \in [0, \pi]$ . By choosing suitable other test functions  $f \in \mathcal{D}(A_1)$  (we omit the details), we can obtain also the remaining three equalities in (4.23).

**Remark 5.** The term ‘‘partial Dirichlet map’’ comes from the fact that  $D$  acts on the upper boundary of  $\Omega$  rather than the entire boundary  $\partial\Omega$ .

**Lemma 4.2.3.** *For every  $\eta \in H$ ,  $D\eta$  is given by*

$$(D\eta)(x,y) = \sum_{k \in \mathbb{N}} \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \varphi_k(x) \cosh [k(y+1)] \quad \forall x, y \in \Omega, \quad (4.24)$$

where the functions  $\varphi_k$  have been introduced in (4.18). Moreover, for every  $\eta \in H_3$  we have  $D\eta \in C^2(\overline{\Omega})$ .

*Proof.* Using Remark 4 it is easily checked that, for every  $k \in \mathbb{N}$ , we have

$$(D\varphi_k)(x,y) = \sqrt{\frac{2}{\pi}} \frac{\cos(kx) \cosh[k(y+1)]}{\cosh(k)} \quad \forall x, y \in \Omega. \quad (4.25)$$

On the other hand, we can see that the right-hand side of (4.24) converges in  $L^2(\Omega)$ . This fact, together with (4.25) clearly implies (4.24).

Moreover, for every  $\alpha \in \{0, 1, 2\}$  we have

$$\left| \frac{\partial^{\alpha, 2-\alpha}}{\partial x^\alpha \partial y^{2-\alpha}} \left( \frac{\cos(kx) \cosh(k(y+1))}{\cosh(k)} \right) \right| \leq k^2 \quad \forall k \in \mathbb{N}, x, y \in \Omega.$$

Using the Cauchy-Schwarz inequality and the fact that  $\sum_{k \in \mathbb{N}} 1/k^2 = \pi^2/6$ ,

$$\begin{aligned} \sum_{k \in \mathbb{N}} \left| \frac{\partial^{\alpha, 2-\alpha}}{\partial x^\alpha \partial y^{2-\alpha}} \left( \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \varphi_k(x) \cosh(k(y+1)) \right) \right| \\ \leq \sum_{k \in \mathbb{N}} \frac{1}{k} \cdot k^3 |\langle \eta, \varphi_k \rangle| \leq \frac{\pi}{\sqrt{6}} \|\eta\|_{H_3} \quad \forall \eta \in H_3, x, y \in \Omega. \end{aligned}$$

Combining the last estimate with (4.24), we obtain that indeed  $D\eta \in C^2(\overline{\Omega})$  for every  $\eta \in H_3$ .  $\square$

**Corollary 4.2.4.** *Let  $\gamma_0 : C(\overline{\Omega}) \rightarrow C[-1, 0]$  be the partial Dirichlet trace operator defined by*

$$(\gamma_0 g)(y) = g(0, y) \quad \forall g \in C(\overline{\Omega}), y \in [-1, 0],$$

and let  $D$  be the map defined in Proposition 4.2.2. Then  $\tilde{C}_0$  defined by

$$\tilde{C}_0 \eta = \gamma_0 D\eta \quad \forall \eta \in H_3$$

can be uniquely extended to a bounded operator  $C_0 \in \mathcal{L}(H, L^2[-1, 0])$ .

## 4.2. DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS IN A RECTANGULAR DOMAIN

*Proof.* According to Lemma 4.2.3, we have

$$(\tilde{C}_0\eta)(y) = \sum_{k \in \mathbb{N}} \sqrt{\frac{2}{\pi}} \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \cosh[k(y+1)] \quad \forall \eta \in H_3, y \in [-1, 0],$$

which implies that there exists a constant  $K > 0$  such that

$$\|\tilde{C}_0\eta\|_{L^2[-1,0]}^2 \leq K \sum_{k \in \mathbb{N}} |\langle \eta, \varphi_k \rangle|^2 = K \|\eta\|_H^2 \quad \forall \eta \in H_3,$$

which shows that  $\tilde{C}_0$  can be extended as claimed.  $\square$

**Lemma 4.2.5.** *The partial Dirichlet map  $D$  defined in Proposition 4.2.2 is bounded from  $H_{\frac{1}{2}}$  to  $\mathcal{H}^1(\Omega)$ , i.e.  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ . Moreover,*

$$(D\eta)(x, 0) = \eta(x) \quad \forall \eta \in H_{\frac{1}{2}}, \text{ equality in } L^2[0, \pi], \quad (4.26)$$

$$\int_{\Omega} \nabla(D\eta) \cdot \overline{\nabla \Psi} \, dx \, dy = 0 \quad \forall \eta \in H_{\frac{1}{2}}, \Psi \in \mathcal{H}_{\text{top}}^1(\Omega), \quad (4.27)$$

where  $\mathcal{H}_{\text{top}}^1(\Omega)$  has been introduced in (4.20).

*Proof.* According to Lemma 4.2.3,  $D\eta$  is given by (4.24). Since  $\{\sqrt{\frac{2}{\pi}} \sin(kx)\}_{k \in \mathbb{N}}$  is an orthonormal basis in  $L^2[0, \pi]$ , we have that for every  $\eta \in H_{\frac{1}{2}}$ ,

$$\begin{aligned} \|\partial_x(D\eta)\|_{L^2(\Omega)}^2 &= \int_{-1}^0 \int_0^\pi \left| \sum_{k \in \mathbb{N}} \sqrt{\frac{2}{\pi}} \frac{k \langle \eta, \varphi_k \rangle}{\cosh k} \cosh[k(y+1)] \sin(kx) \right|^2 dx \, dy \\ &\leq \sum_{k \in \mathbb{N}} \frac{|k \langle \eta, \varphi_k \rangle|^2}{\cosh^2 k} \int_{-1}^0 \cosh^2[k(y+1)] \, dy \\ &= \sum_{k \in \mathbb{N}} \frac{k^2 |\langle \eta, \varphi_k \rangle|^2}{\cosh^2 k} + \sum_{k \in \mathbb{N}} \frac{k |\langle \eta, \varphi_k \rangle|^2}{2 \cosh^2 k} \sinh(2k), \end{aligned}$$

which clearly implies that there exists  $K_1 > 0$  such that

$$\|\partial_x(D\eta)\|_{L^2(\Omega)} \leq K_1 \|\eta\|_{\frac{1}{2}} \quad \forall \eta \in H_{\frac{1}{2}}.$$

A similar estimate for  $\|\partial_y(D\eta)\|_{L^2}$  can be obtained in a completely similar manner. Moreover, we know from Proposition 4.2.2 that  $\|D\eta\|_{L^2}$  is also bounded by a similar estimate. Therefore, we conclude that  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ . Formula (4.26) in the lemma follows from the last part of Lemma 4.2.3 together with Remark 4 and the density of  $H_3$  in  $H_{\frac{1}{2}}$ .

To prove (4.27) first we assume that  $\eta \in H_3$  so that, according to Remark 4,  $D\eta$  is the unique classical solution of (4.23). Multiplying the first equation in (4.23) by  $\overline{\Psi} \in \mathcal{H}_{\text{top}}^1(\Omega)$  and integrating by parts, it follows that (4.27) holds for  $\eta \in H_3$ . Using the density of  $H_3$  in  $H_{\frac{1}{2}}$  and the fact that  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ , it follows that indeed (4.27) holds for all  $\eta \in H_{\frac{1}{2}}$ .  $\square$

The second important map constructed in this section is a partial Neumann map. To this aim, recall the space  $\mathcal{H}_{\text{top}}^1(\Omega)$  introduced in (4.20) and notice that, due the version of the Poincaré inequality in [115, Theorem 13.6.9], the sesquilinear form on  $\mathcal{H}_{\text{top}}^1(\Omega)$  given by

$$a[f, g] = \int_{\Omega} \nabla f \cdot \overline{\nabla g} \, dx \, dy \quad \forall f, g \in \mathcal{H}_{\text{top}}^1(\Omega) \quad (4.28)$$

defines an inner product on  $\mathcal{H}_{\text{top}}^1(\Omega)$  which is equivalent to the one inherited from  $\mathcal{H}^1(\Omega)$ . These facts, combined with the continuity of the Dirichlet trace (as an operator from  $\mathcal{H}^1(\Omega)$  to  $L^2(\partial\Omega)$ ), imply the following:

**Proposition 4.2.6.** *For every  $v \in L^2[-1, 0]$  there exists a unique function  $Nv \in \mathcal{H}_{\text{top}}^1(\Omega)$  such that*

$$\int_{\Omega} \nabla(Nv) \cdot \overline{\nabla g} \, dx \, dy = \int_{-1}^0 v(y) \overline{g(0, y)} \, dy \quad \forall g \in \mathcal{H}_{\text{top}}^1(\Omega).$$

Moreover, the operator  $N$ , called a partial Neumann map, is linear and bounded from  $L^2[-1, 0]$  to  $\mathcal{H}_{\text{top}}^1(\Omega)$ .

*Proof.* The results follow from the Lax-Milgram theorem by using the sesquilinear form  $a[\cdot, \cdot]$  introduced in (4.28) (see also [121, Proposition 7.1]).  $\square$

**Remark 6.** The above proposition can be formulated also as follows: for every  $v \in L^2[-1, 0]$  the boundary value problem

$$\begin{cases} \Delta f(x, y) = 0 & ((x, y) \in \Omega), \\ f(x, 0) = 0, \quad \partial_y f(x, -1) = 0 & (x \in (0, \pi)), \\ \partial_x f(0, y) = -v, \quad \partial_x f(\pi, y) = 0 & (y \in (-1, 0)), \end{cases} \quad (4.29)$$

admits a unique weak solution  $f = Nv \in \mathcal{H}_{\text{top}}^1(\Omega)$ . If  $f \in C^2(\overline{\Omega})$  and  $v \in C[-1, 0]$ , then  $f = Nv$  is the unique classical solution of (4.29).

We note that the sequence  $(\psi_k)_{k \in \mathbb{N}}$  defined by

$$\psi_k(y) = \sqrt{2} \cos \left[ (2k-1) \frac{\pi}{2} (y+1) \right] \quad \forall k \in \mathbb{N}, y \in [-1, 0], \quad (4.30)$$

is an orthonormal basis in  $L^2[-1, 0]$  (see [115, [Section 2.6]). We can use this basis to construct the scale of Hilbert spaces  $(\mathcal{U}_\beta)_{\beta \geq 0}$  defined by  $\mathcal{U}_0 = L^2[-1, 0]$  and (for  $\beta > 0$ )

$$\mathcal{U}_\beta = \left\{ v \in \mathcal{U}_0 \mid \sum_{k \in \mathbb{N}} (2k-1)^{2\beta} \left| \int_{-1}^0 v(y) \psi_k(y) \, dy \right|^2 < \infty \right\},$$

with the inner products  $(\langle \cdot, \cdot \rangle_\beta)_{\beta \geq 0}$  given, for every  $v, \chi \in \mathcal{U}_\beta$ , by

$$\langle v, \chi \rangle_{\mathcal{U}_\beta} = \sum_{k \in \mathbb{N}} (2k-1)^{2\beta} \left( \int_{-1}^0 v(y) \psi_k(y) \, dy \right) \overline{\left( \int_{-1}^0 \chi(y) \psi_k(y) \, dy \right)}.$$

## 4.2. DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS IN A RECTANGULAR DOMAIN

**Lemma 4.2.7.** *Let  $N$  be the operator defined in Proposition 4.2.6. Then for every  $v \in L^2[-1, 0]$  and every  $(x, y) \in \Omega$  we have*

$$(Nv)(x, y) = \sum_{k \in \mathbb{N}} a_k \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] \cos \left[ (2k-1) \frac{\pi}{2} (y+1) \right], \quad (4.31)$$

with convergence in  $\mathcal{H}_{\text{top}}^1(\Omega)$ , where

$$a_k = \frac{2\sqrt{2} \langle v, \psi_k \rangle}{(2k-1)\pi \sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \quad \forall k \in \mathbb{N}.$$

Moreover, for every  $v \in \mathcal{U}_2$  we have  $Nv \in C^2(\overline{\Omega})$ .

*Proof.* By using Remark 6 and separation of variables, we see that

$$(N\psi_k)(x, y) = \frac{2\psi_k(y) \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right]}{(2k-1)\pi \sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \quad \forall k \in \mathbb{N}, \quad (4.32)$$

for all  $(x, y) \in \Omega$ . Since  $Nv = \sum_{k \in \mathbb{N}} \langle v, \psi_k \rangle N\psi_k$ , this clearly implies (4.31), with convergence in  $\mathcal{H}_{\text{top}}^1(\Omega)$  due to Proposition 4.2.6. For every  $j \in \{0, 1, 2\}$ ,

$$\left| \frac{\partial^{j, 2-j}}{\partial x^j \partial y^{2-j}} (N\psi_k)(x, y) \right| \leq \frac{\sqrt{2}}{2} \pi (2k-1) \quad \forall k \in \mathbb{N}, (x, y) \in \Omega,$$

so that for every  $v \in \mathcal{U}$ , the series  $Nv = \sum_{k \in \mathbb{N}} \langle v, \psi_k \rangle N\psi_k$  converges in  $C^2(\overline{\Omega})$  if the sequence  $k \langle v, \psi_k \rangle$  is in  $l^1$ . For this (by an argument similar to the one in the proof of Lemma 4.2.3) it is sufficient if the sequence  $k^2 \langle v, \psi_k \rangle$  is in  $l^2$ , which is precisely the condition  $v \in \mathcal{U}_2$ .  $\square$

### 4.2.2 Partial Dirichlet to Neumann and Neumann to Neumann maps

In this section we give an explicit construction of the operators allowing us to recast (4.16) as a well-posed linear control system. Recall the orthonormal basis  $(\varphi_k)_{k \in \mathbb{N}}$  in  $H$  introduced in (4.17) and the corresponding spaces  $H_\alpha$  in (4.19). First we note a direct consequence of Proposition 4.2.2 and of Lemma 4.2.3.

**Corollary 4.2.8.** *Let  $\gamma_1 : C^1(\overline{\Omega}) \rightarrow C[0, \pi]$  be the partial Neumann trace operator defined by*

$$(\gamma_1 f)(x) = \partial_y f(x, 0) \quad \forall f \in C^1(\overline{\Omega}), x \in [0, \pi].$$

Then  $\tilde{A}_0$  defined by

$$\tilde{A}_0 \eta = \gamma_1 D \eta \quad \forall \eta \in H_3,$$

where  $D$  is the Dirichlet map defined in Proposition 4.2.2, is a linear bounded map from  $H_3$  to  $C[0, \pi]$ . Moreover, we have

$$\tilde{A}_0 \varphi_k = k \tanh(k) \varphi_k \quad \forall k \in \mathbb{N}.$$

We are now in a position to define a partial Dirichlet to Neumann map.



**Proposition 4.2.9.** *The operator  $\tilde{A}_0$  introduced in Corollary 4.2.8 has a unique continuous extension to an operator  $A_0 : H_1 \rightarrow H$ . This extension is strictly positive and  $\mathcal{D}(A_0^{\frac{1}{2}}) = H_{\frac{1}{2}}$ . For each  $k \in \mathbb{N}$ , we have  $A_0 \varphi_k = \lambda_k \varphi_k$ , where*

$$\lambda_k = k \tanh(k) \quad \forall k \in \mathbb{N} \quad (4.33)$$

and

$$A_0 \eta = \sum_{k \in \mathbb{N}} \lambda_k \langle \eta, \varphi_k \rangle \varphi_k \quad \forall \eta \in H_1. \quad (4.34)$$

*Proof.* It is clear from the previous proposition that  $A_0$  fits into the class of diagonalizable operators discussed in [115] in Proposition 3.2.9 and the remarks after it, and in Proposition 3.4.8 of the same book.  $\square$

**Proposition 4.2.10.** *Let  $A_0$  and  $D$  be the operators introduced in Propositions 4.2.9 and 4.2.2, respectively. Let  $\gamma \in H_{\frac{1}{2}}$  and  $\Psi \in \mathcal{H}^1(\Omega)$  be such that*

$$\Psi(x, 0) = \gamma(x), \text{ equality in } L^2[0, \pi].$$

*Then for every  $\eta \in H_{\frac{1}{2}}$  we have  $D\eta \in \mathcal{H}^1(\Omega)$  and*

$$\langle A_0^{\frac{1}{2}} \eta, A_0^{\frac{1}{2}} \gamma \rangle = \langle \nabla(D\eta), \nabla \Psi \rangle_{L^2(\Omega)}. \quad (4.35)$$

*Proof.* First we assume that  $\eta \in H_3$ , so that according to Lemma 4.2.3 we have  $D\eta \in C^2(\bar{\Omega})$ . Then (4.35) follows by a simple integration by parts and Proposition 4.2.9. The fact that  $D\eta \in \mathcal{H}^1(\Omega)$  for every  $\eta \in H_{\frac{1}{2}}$  has already been proved in Lemma 4.2.5. Finally, to prove that (4.35) still holds for  $\eta \in H_{\frac{1}{2}}$  it suffices to use the density of  $H_3$  in  $H_{\frac{1}{2}}$ , combined with Lemma 4.2.5.  $\square$

**Corollary 4.2.11.** *With  $\gamma_1$  as in Corollary 4.2.8, define the operator  $\tilde{B}_1$  by*

$$\tilde{B}_1 v = \gamma_1 N v \quad \forall v \in \mathcal{U}_2,$$

*where  $N$  is the Neumann map introduced in Proposition 4.2.6. Then  $\tilde{B}_1$  is a bounded linear operator from  $\mathcal{U}_2$  to  $C[0, \pi]$ . Moreover, we have*

$$(\tilde{B}_1 \psi_k)(x) = \frac{(-1)^k \sqrt{2}}{\sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right], \quad (4.36)$$

*for all  $k \in \mathbb{N}$ ,  $x \in [0, \pi]$ , where the functions  $\psi_k$  have been defined in (4.30).*

*Proof.* This follows from Lemma 4.2.7 and the formula (4.32) for  $N\psi_k$ .  $\square$

We are now ready to define a Neumann to Neumann map.

**Theorem 4.2.12.** *The operator  $\tilde{B}_1$  introduced in Corollary 4.2.11 can be extended in a unique manner to a linear bounded operator  $B_1 : L^2[-1, 0] \rightarrow L^2[0, \pi]$ . Moreover, for every  $v \in L^2[-1, 0]$  with  $\int_{-1}^0 v(y) dy = 0$  we have that  $B_1 v \in H$ , where  $H$  is defined in (4.17). Finally,*

$$\begin{aligned} \int_0^\pi (B_1 v)(x) \overline{\Psi(x, 0)} dx &= \int_\Omega \nabla(Nv)(x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy \\ &\quad - \int_{-1}^0 v(y) \overline{\Psi(0, y)} dy \quad \forall \Psi \in \mathcal{H}^1(\Omega). \end{aligned} \quad (4.37)$$

## 4.2. DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS IN A RECTANGULAR DOMAIN

*Proof.* For any  $v \in L^2[-1, 0]$  we set

$$b_k = \frac{(-1)^k \sqrt{2}}{\sinh \left[ (2k-1) \frac{\pi^2}{2} \right]}, \quad v_k = \langle v, \psi_k \rangle,$$

and notice that these sequences are in  $l^2$  and  $\|(v_k)\|_{l^2} = \|v\|_{L^2[-1,0]}$ . From (4.36) it follows that if  $v \in \mathcal{U}_2$  then for every  $x \in [0, \pi]$  we have

$$(\tilde{B}_1 v)(x) = \sum_{k \in \mathbb{N}} b_k v_k \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] = \frac{f(x) + g(x)}{2}, \quad (4.38)$$

where

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{N}} b_k v_k \exp \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right], \\ g(x) &= \sum_{k \in \mathbb{N}} b_k v_k \exp \left[ (2k-1) \frac{\pi}{2} (\pi - x) \right]. \end{aligned} \quad (4.39)$$

On one hand, from  $0 \leq \exp \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] \leq 1$  for all  $x \in [0, \pi]$ , by using Cauchy-Schwarz we obtain that there exists  $C_1 > 0$  such that

$$\int_0^\pi |f(x)|^2 dx \leq C_1 \|v\|_{L^2[-1,0]}^2 \quad \forall v \in \mathcal{U}_2. \quad (4.40)$$

On the other hand, from (4.39) it follows that

$$\int_0^\pi |g(x)|^2 dx = \frac{1}{\pi} \sum_{k,l \in \mathbb{N}} \frac{c_k \bar{c}_l v_k \bar{v}_l}{k+l-1},$$

where  $c_k = b_k \exp \left[ (2k-1) \frac{\pi^2}{2} \right]$  for all  $k \in \mathbb{N}$ . Using that  $|c_k| \leq |c_1| < \sqrt{10}$  for all  $k \in \mathbb{N}$ , together with Hilbert's inequality, see for instance [56, Chapter IX] or the nice survey [60], we obtain that

$$\int_0^\pi |g(x)|^2 dx \leq 10 \sum_{k \in \mathbb{N}} |v_k|^2 \quad \forall v \in \mathcal{U}_2. \quad (4.41)$$

Putting together (4.38), (4.40) and (4.41), it follows that there exists  $C > 0$  such that

$$\|\tilde{B}_1 v\|_{L^2[0,\pi]}^2 \leq C \|v\|_{L^2[-1,0]}^2 \quad \forall v \in \mathcal{U}_2.$$

The above estimate, combined with the density of  $\mathcal{U}_2$  in  $L^2[-1, 0]$ , implies that indeed  $\tilde{B}_1$  admits an unique extension  $B_1 \in \mathcal{L}(L^2[-1, 0], L^2[0, \pi])$ .

Assume again that  $v \in \mathcal{U}_2$ . Then, according to Remark 6 and to Lemma 4.2.7 we have that  $f = Nv$  is a classical solution of (4.29), so that for every  $v \in \mathcal{U}_2$  we have

$$\begin{aligned} 0 &= \int_\Omega \Delta(Nv)(x, y) \overline{\Psi(x, y)} dx dy \\ &= \int_{-1}^0 v(y) \overline{\Psi(0, y)} dy + \int_0^\pi (B_1 v)(x) \overline{\Psi(x, 0)} dx - \int_\Omega \nabla(Nv) \cdot \overline{\nabla \Psi} dx dy. \end{aligned}$$

Thus (4.37) holds for  $v \in \mathcal{U}_2$  and by density for  $v \in L^2[-1, 0]$ . Using the assumption  $\int_{-1}^0 v(y) dy = 0$  and taking  $\Psi = 1$  in (4.37) we obtain that  $B_1 v$  indeed satisfies the condition  $\int_0^\pi (B_1 v)(x) dx = 0$ , which implies that  $B_1 v \in H$ .  $\square$

**Remark 7.** An alternative proof of (4.41) can be given using the Carleson measure criterion for admissibility, see for instance [115, Sect. 5.3]. To this aim, consider the Hilbert space  $\tilde{X} = l^2$ , the strictly negative operator  $\tilde{A} = \text{diag} \left( -\frac{(2k-1)\pi}{2} \right)$  and the observation functional  $\tilde{C} = [c_1 \ c_2 \ c_3 \ \dots]$ . Then according to the aforementioned criterion,  $\tilde{C}$  is an admissible observation operator for the operator semigroup generated by  $\tilde{A}$ , and (4.41) follows.

The above theorem clearly implies the following result:

**Corollary 4.2.13.** *Let  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) \, dy = 0$  and let  $B_0$  be the operator defined by*

$$B_0 u = u B_1 h \quad \forall u \in \mathbb{C}.$$

Then  $B_0 \in \mathcal{L}(\mathbb{C}, H)$ . Moreover, we have

$$\int_0^\pi (B_0 u)(x) \overline{\Psi(x, 0)} \, dx = u \int_\Omega \nabla(Nh) \cdot \overline{\nabla \Psi} \, dx \, dy - u \int_{-1}^0 h(y) \overline{\Psi(0, y)} \, dy, \quad (4.42)$$

for all  $u \in \mathbb{C}$  and  $\Psi \in \mathcal{H}^1(\Omega)$ . In particular,

$$B_0^* \eta = - \int_{-1}^0 h(y) \overline{(C_0 \eta)(y)} \, dy \quad \forall \eta \in H, \quad (4.43)$$

where  $C_0 = \gamma_0 D$  is the operator introduced in Corollary 4.2.4.

*Proof.* The fact that  $B_0 \in \mathcal{L}(\mathbb{C}, H)$  and (4.42) follows from Theorem 4.2.12 (in particular from (4.37)) with  $v = uh$ . Moreover, taking  $\Psi = D\eta$  with  $\eta \in H_{\frac{1}{2}}$  (see Lemma 4.2.5) in (4.42), we see that for every  $u \in \mathbb{C}$ ,

$$\langle B_0 u, \eta \rangle = u \langle B_1 h, \eta \rangle = -u \int_{-1}^0 h(y) \overline{(C_0 \eta)(y)} \, dy + u \int_\Omega \nabla(Nh) \cdot \overline{\nabla(D\eta)} \, dx \, dy.$$

Using (4.27) it follows that the last term in the right-hand side of the above equation is zero, so that we obtain (4.43).  $\square$

### 4.3 Well-posedness of the linearized water waves system

In this section, we study the well-posedness of the linearized water waves system (4.16). The analysis is based on formulating the system (4.16) as a well-posed linear control system. The construction of this system is completed through the partial Dirichlet to Neumann operator  $A_0$  defined in Proposition 4.2.9, and the partial Neumann to Neumann operator  $B_0$  introduced in Corollary 4.2.13. Before stating the main result for (4.16), we need some background and more notation. First we recall the concept of a well-posed linear control system, following Weiss [119] (where these systems have been called *abstract linear control systems*), see also Tucsnak and Weiss [115].

**Definition 4.3.1.** *Let  $U$  and  $X$  be Hilbert spaces. A well-posed linear control system with the state space  $X$  and the input space  $U$  is a couple  $(\mathbb{T}, \Phi)$  of families of operators such that*

1.  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  is a strongly continuous operator semigroup (also called a  $C_0$ -semigroup) on  $X$ .

### 4.3. WELL-POSEDNESS OF THE LINEARIZED WATER WAVES SYSTEM

2.  $\Phi = (\Phi_t)_{t \geq 0}$  is a family of bounded linear operators from  $L^2([0, \infty); U)$  to  $X$  (called input maps) such that for every  $u, v \in L^2([0, \infty); U)$ ,

$$\Phi_{\tau+t}(u \diamond_{\tau} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v \quad \forall t, \tau \geq 0, \quad (4.44)$$

where we used the concatenation of functions, see Notation part.

For any  $\tau \geq 0$ , let  $\mathbf{P}_{\tau} u$  denote the truncation of  $u : [0, \infty) \rightarrow U$  to  $[0, \tau]$ , setting  $(\mathbf{P}_{\tau} u)(t) = 0$  for  $t > \tau$ . It follows from (4.44) that  $\Phi_{\tau} \mathbf{P}_{\tau} = \Phi_{\tau}$  (causality), and hence  $\Phi_{\tau}$  has a natural extension to  $L^2_{\text{loc}}([0, \infty); U)$ .

Still using the notation from the above definition, if  $z_0 \in X$  and  $u \in L^2_{\text{loc}}([0, \infty); U)$ , then we call the function  $z(t) = \mathbb{T}_t z_0 + \Phi_t u$  the *state trajectory* of the system corresponding to the initial state  $z_0$  and the input  $u$ . Let  $A : \mathcal{D}(A) \rightarrow X$  denote the generator of  $\mathbb{T}$ . For every well-posed linear control system, there exists a (usually unbounded) operator  $B$  defined on  $U$  and with values in an extrapolation space that contains  $X$ , with the following property: For any  $z_0 \in X$  and  $u \in L^2_{\text{loc}}([0, \infty); U)$ , the corresponding state trajectory is the unique solution (in the extrapolation space) of the abstract differential equation

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0. \end{cases} \quad (4.45)$$

For details on this see [115, Chapter 4]. The above operator  $B$  is called the *control operator* of the system. This operator is called *bounded* if  $B \in \mathcal{L}(U, X)$  (this is the case of interest in this part).

We would like to formulate the system of equations (4.16) as a well-posed linear control system. This is not obvious, because the equations (4.16) do not even resemble (4.45). We have to define what we mean by the state of our system at some time  $t \geq 0$ : this should be

$$z(t) = \begin{bmatrix} \tilde{\zeta}(t, \cdot) \\ \zeta(t, \cdot) \end{bmatrix}. \quad (4.46)$$

Throughout this section we denote by  $X$  the Hilbert space  $H_{\frac{1}{2}} \times H$ , where  $H$  and  $(H_{\alpha})_{\alpha > 0}$  have been defined in (4.17) and (4.19), respectively. Here we specify the linear operator  $A : \mathcal{D}(A) \rightarrow X$  with  $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$  and

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -A_0 \varphi \end{bmatrix} \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A), \quad (4.47)$$

where  $A_0 = \gamma_1 D$  is the strictly positive operator on  $H$ , with domain  $H_1$ , which has been introduced in Proposition 4.2.9. We redefine the inner product on  $H_{\frac{1}{2}}$  as

$$\langle x, z \rangle_{\frac{1}{2}} = \left\langle A_0^{\frac{1}{2}} x, A_0^{\frac{1}{2}} z \right\rangle,$$

which is equivalent to the original inner product on  $H_{\frac{1}{2}}$ . Then  $A$  is skew-adjoint on  $X$  (see, for instance, [115, Proposition 3.7.6]), so that, according to Stone's theorem (see, for instance [115, Section 3.7]),  $A$  generates a group  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  of unitary operators on  $X$ . Moreover, we recall that  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$  and that we have introduced the input space  $U = \mathbb{C}$ . Let  $B \in \mathcal{L}(U, X)$  be given by

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad (4.48)$$

where  $B_0 \in \mathcal{L}(U, H)$  is as in Corollary 4.2.13.

### 4.3.1 Well-posedness in weak sense

Based on the abstract framework and the notation introduced above, we first consider the well-posedness of the water wave equations (4.16) in weak sense. In the following, we define what we mean by a solution of (4.16) in this case.

**Definition 4.3.2.** Given  $u \in L^2_{\text{loc}}[0, \infty)$  and  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$ , a couple  $(\varphi, \zeta)$  is called a solution of (4.16) if

$$\begin{aligned} \varphi \in L^2_{\text{loc}}([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H), \\ \varphi(t, \cdot, 0) + \zeta(t, \cdot) = 0, \text{ equality in } L^2_{\text{loc}}([0, \infty); L^2[0, \pi]), \end{aligned} \quad (4.49)$$

and for every  $\Psi \in \mathcal{H}^1(\Omega)$  and every  $t > 0$  we have

$$\begin{aligned} \int_0^\pi \dot{\zeta}(t, x) \overline{\Psi(x, 0)} dx - \int_0^\pi \dot{\zeta}(0, x) \overline{\Psi(x, 0)} dx = \\ \int_0^t \int_\Omega \nabla \varphi(\sigma, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy d\sigma - \int_0^t u(\sigma) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy d\sigma. \end{aligned} \quad (4.50)$$

**Remark 8.** We explain the connection between the water waves equations (4.16) and their variational formulation (4.49)-(4.50). In one direction, assume that  $(\varphi, \zeta)$  is a classical solution of (4.16), having the smoothness

$$\varphi \in C([0, \infty); \mathcal{H}^2(\Omega)), \quad \zeta \in C([0, \infty); H_{\frac{1}{2}}) \cap C^2([0, \infty); H). \quad (4.51)$$

(If  $u \not\equiv 0$ , this implies that  $h \in \mathcal{H}^{\frac{1}{2}}(-1, 0)$  and  $u$  is continuous.) The equation (4.49) is simply copied from (4.16). We multiply the first equation in (4.16) with  $\overline{\Psi}$  and apply the first Green formula (integration by parts), taking into account the last three lines of (4.16). After this we do simple integration with respect to  $t$ , and we obtain (4.50).

In the opposite direction, let us assume that  $(\varphi, \zeta)$  is a solution of (4.49)-(4.50) with the additional regularity (4.51), and  $u$  is continuous. Then (4.50) can be differentiated with respect to the time  $t$ , and after using the first Green formula we obtain

$$\begin{aligned} \int_0^\pi \ddot{\zeta}(t, x) \overline{\Psi(x, 0)} dx = \int_{\partial\Omega} \left( \frac{\partial}{\partial \nu} \varphi \right) \overline{\Psi} d\sigma - \int_\Omega \Delta \varphi(t, x, y) \cdot \overline{\Psi(x, y)} dx dy \\ - u(t) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy \quad \forall t \geq 0, \end{aligned}$$

where  $\frac{\partial}{\partial \nu}$  denotes the Neumann trace on the entire boundary  $\partial\Omega$ . Considering only functions  $\Psi$  with compact support in  $\Omega$ , we see from the above that we must have  $\Delta \varphi = 0$ . After this, we consider test functions  $\Psi$  whose trace is supported on one of the four segments of  $\partial\Omega$ , knowing that these traces are dense in the  $L^2$  space of the relevant segment, see [115, Theorem 13.6.10]. From here we can get that  $\varphi$  and  $\zeta$  satisfy also the last three equations in (4.16). (It also follows that  $h \in \mathcal{H}^{\frac{1}{2}}(-1, 0)$ .)

### 4.3. WELL-POSEDNESS OF THE LINEARIZED WATER WAVES SYSTEM

**Remark 9.** Recalling that the function  $\varphi(t, x, y)$  involved in the equation (4.16) is the time derivative of the velocity potential  $\phi(t, x, y)$ , we derive from Definition 4.3.2 that

$$\phi \in \mathcal{H}_{\text{loc}}^1([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H),$$

if  $(\phi, \zeta)$  is a solution of (4.15), the initial data  $\phi(0, x, y) \in \mathcal{H}^1(\Omega)$  and the velocity  $v \in \mathcal{H}_{\text{loc}}^1[0, \infty)$ .

The following result establishes the existence of a well-posed linear control system corresponding to (4.16).

**Theorem 4.3.3.** *Let  $h \in L^2[-1, 0]$  be such that  $\int_{-1}^0 h(y) dy = 0$ . Then for every  $u \in L_{\text{loc}}^2[0, \infty)$ ,  $\zeta_0 \in H_{\frac{1}{2}}$  and  $w_0 \in H$ , there exists a unique solution of (4.16) with  $\zeta(0) = \zeta_0$  and  $\dot{\zeta}(0) = w_0$ . Moreover, there exists a well-posed linear control system  $(\mathbb{T}, \Phi)$  with state space  $X = H_{\frac{1}{2}} \times H$  and input space  $U = \mathbb{C}$  such that, setting  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix}$  and using the state from (4.46), we have*

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u \quad \forall \tau \geq 0. \quad (4.52)$$

Finally, the generator  $A$  of  $\mathbb{T}$  is skew-adjoint, with domain  $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ , and there exists  $B \in \mathcal{L}(\mathbb{C}, X)$  such that for any  $\tau \geq 0$ ,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma \quad \forall u \in L_{\text{loc}}^2[0, \infty). \quad (4.53)$$

*Proof.* With the above notation for  $X$ ,  $A$ ,  $U$  and  $B$  we consider, for each  $\tau \geq 0$ , the map  $\Phi_\tau$  defined by (4.53), which is clearly linear and bounded from  $L^2([0, \infty); U)$  into  $X$ . Let  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix} \in X$ , let  $u \in L_{\text{loc}}^2([0, \infty); U)$  and define  $z(t) = \begin{bmatrix} \zeta \\ w \end{bmatrix} \in C([0, \infty); X)$  by (4.52). Then, according to a classical result (see, for instance, [115, Remark 4.1.2]), for every  $t \geq 0$  and  $\psi \in \mathcal{D}(A)$  we have

$$\langle z(t) - z_0, \psi \rangle_X = \int_0^t [-\langle z(\sigma), A\psi \rangle_X + \langle Bu(\sigma), \psi \rangle_X] d\sigma.$$

Setting  $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ , with  $\psi_1 \in H_1$  and  $\psi_2 \in H_{\frac{1}{2}}$  and using the specific structure (4.47), (4.48) of  $A$  and  $B$ , the last formula implies that

$$\begin{aligned} \langle A_0^{\frac{1}{2}}(\zeta(t) - \zeta_0), A_0^{\frac{1}{2}}\psi_1 \rangle + \langle w(t) - w_0, \psi_2 \rangle &= - \left\langle \int_0^t A_0^{\frac{1}{2}}\zeta(\sigma) d\sigma, A_0^{\frac{1}{2}}\psi_2 \right\rangle \\ &\quad + \left\langle \int_0^t w(\sigma) d\sigma, A_0\psi_1 \right\rangle + \int_0^t \langle B_0 u(\sigma), \psi_2 \rangle d\sigma, \end{aligned} \quad (4.54)$$

for every  $t \geq 0$ ,  $\psi_1 \in H_1$ ,  $\psi_2 \in H_{\frac{1}{2}}$ . The above formula holds, in particular, for  $\psi_2 = 0$  and arbitrary  $\psi_1 \in H_1$ , which yields that

$$\zeta(t) - \zeta_0 = \int_0^t w(\sigma) d\sigma \quad \forall t \geq 0,$$

so that  $w(t) = \dot{\zeta}(t)$ , for all  $t \geq 0$ . Inserting the last two formulas in (4.54), we obtain that

$$\langle \dot{\zeta}(t) - w_0, \psi_2 \rangle = - \int_0^t \langle A_0^{\frac{1}{2}}\zeta(\sigma), A_0^{\frac{1}{2}}\psi_2 \rangle d\sigma + \int_0^t \langle B_0 u(\sigma), \psi_2 \rangle d\sigma, \quad (4.55)$$

where  $t \geq 0$ ,  $w_0 = \zeta(0)$  and  $\psi_2 \in H_{\frac{1}{2}}$ . (This formula (4.55) is the weak form of the equation  $\dot{\zeta} = -A_0\zeta + B_0u$ .) Let  $\Psi \in \mathcal{H}^1(\Omega)$  be such that  $\int_0^\pi \Psi(x, 0) dx = 0$  and then  $\psi_2(x) = \Psi(x, 0)$  is a function in  $H_{\frac{1}{2}}$ . By combining (4.55) and (4.42) it follows that

$$\begin{aligned} \langle \dot{\zeta}(t) - w_0, \psi_2 \rangle &= - \int_0^t \langle A_0^{\frac{1}{2}} \zeta(\sigma), A_0^{\frac{1}{2}} \psi_2 \rangle d\sigma \\ &\quad + \int_0^t u(\sigma) \int_{\Omega} \nabla(Nh) \cdot \overline{\nabla \Psi} dx dy d\sigma - \int_0^t u(\sigma) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy d\sigma. \end{aligned} \quad (4.56)$$

On the other hand, from Proposition 4.2.10 it follows that  $D\zeta(\sigma) \in \mathcal{H}^1(\Omega)$  and

$$\langle A_0^{\frac{1}{2}} \zeta(\sigma), A_0^{\frac{1}{2}} \psi_2 \rangle = \langle \nabla(D\zeta(\sigma)), \nabla \Psi \rangle \quad \forall \Psi \in \mathcal{H}^1(\Omega), \psi_2(x) = \Psi(x, 0).$$

The above formula, when combined with (4.56), and setting

$$\varphi(t, \cdot, \cdot) = - [D\zeta(t)](\cdot, \cdot) + u(t)(Nh)(\cdot, \cdot) \quad \forall t \geq 0, \quad (4.57)$$

implies that  $\varphi(t, \cdot, \cdot) \in \mathcal{H}^1(\Omega)$  and  $(\varphi, \zeta)$  satisfies (4.50) for every  $\Psi \in \mathcal{H}^1(\Omega)$  with  $\int_0^\pi \Psi(x, 0) dx = 0$ . On the other hand,  $(\varphi, \zeta)$  obviously satisfies (4.50) if  $\Psi$  is a constant function, thus  $(\varphi, \zeta)$  satisfies (4.50) for every  $\Psi \in \mathcal{H}^1(\Omega)$ . Moreover, according to Lemma 4.2.5, Proposition 4.2.6 and the above definition of  $\varphi$ , we have that  $\varphi \in L_{\text{loc}}^2([0, \infty), \mathcal{H}^1(\Omega))$  and (4.49) holds, so that  $(\varphi, \zeta)$  is a solution of (4.16) in the sense of Definition 4.3.2.

Conversely, assume that  $(\varphi, \zeta)$  is a solution of (4.16) in the sense of Definition 4.3.2, with  $\zeta(0) = \zeta_0 \in H_{\frac{1}{2}}$  and  $\dot{\zeta}(0) = w_0 \in H$ . Using the fact that (4.50) holds, in particular, for  $\Psi \in \mathcal{H}_{\text{top}}^1(\Omega)$  it follows that for every  $t \geq 0$  and every  $\Psi \in \mathcal{H}_{\text{top}}^1(\Omega)$  we have

$$\int_{\Omega} \nabla \varphi(t, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy - u(t) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy = 0.$$

Using the notation

$$\bar{\varphi}(t, \cdot, \cdot) = \varphi(t, \cdot, \cdot) - u(t)(Nh)(\cdot, \cdot), \quad (4.58)$$

where  $N$  is the Neumann map defined in Proposition 4.2.6, it follows that

$$\int_{\Omega} \nabla \bar{\varphi}(t, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy = 0 \quad \forall \Psi \in \mathcal{H}_{\text{top}}^1(\Omega).$$

The last formula holds, in particular, for  $\Psi \in \mathcal{D}(A_1)$ , where  $\mathcal{D}(A_1)$  has been defined in Proposition 4.2.1, so that an integration by parts yields that

$$\int_{\Omega} \bar{\varphi}(t, x, y) \overline{\Delta \Psi(x, y)} dx dy = \int_0^\pi \bar{\varphi}(t, x, 0) \overline{\partial_y \Psi(x, 0)} dx \quad \forall \Psi \in \mathcal{D}(A_1). \quad (4.59)$$

According to Definition 4.3.2 and Proposition 4.2.6, we have (4.49) and  $(Nh)(x, 0) = 0$  for  $x \in [0, \pi]$ , so that from (4.59) it follows that

$$\int_{\Omega} \bar{\varphi}(t, x, y) \overline{\Delta \Psi(x, y)} dx dy = - \int_0^\pi \zeta(t, x) \overline{\partial_y \Psi(x, 0)} dx \quad \forall \Psi \in \mathcal{D}(A_1).$$

### 4.3. WELL-POSEDNESS OF THE LINEARIZED WATER WAVES SYSTEM

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Comparing the above formula with the definition (4.22) of the Dirichlet map, with  $g = \Delta\Psi = -\overline{A_1\Psi}$ , and recalling that  $A_1$  is onto, it follows that

$$\bar{\varphi}(t, \cdot, \cdot) = -[D\zeta(t)](\cdot, \cdot) \quad \forall t \geq 0.$$

The last formula and (4.58) yield that again (4.57) holds. Now we take  $\psi_2 \in H_{\frac{1}{2}}$  and we recall from Lemma 4.2.5 that  $D\psi_2 \in \mathcal{H}^1(\Omega)$  and that  $(D\psi_2)(x, 0) = \psi_2(x)$  for  $x \in [0, \pi]$ . We can thus choose  $\Psi = D\psi_2$  in (4.50) and using Proposition 4.2.10 and Corollary 4.2.13, it follows that  $\zeta$  satisfies (4.55). This easily implies that  $z = \begin{bmatrix} \zeta \\ \zeta \end{bmatrix}$  satisfies (4.52).  $\square$

**Remark 10.** We mention that, according to the above theorem and what we have said around (4.45), the state trajectories of our system are solutions of (4.45), in the sense of [115, Section 4.1-4.2], and our control operator  $B$  is bounded. Moreover, we see from the above Theorem 4.3.3 that the whole water waves system (4.16) is equivalent to the well-posed linear control system (4.45) with the operator  $A$  and  $B$  introduced in (4.47) and (4.48), respectively.

#### 4.3.2 Well-posedness in classical sense

This section is a continuation and extension of the recent work [110], in the sense that we consider the strong solution of the linearized water waves system in Section 4.1, of course with more regular initial data and control function. In Section 4.3, we studied the weak well-posedness of the linear equation (4.16), which is decomposed into two boundary value problems for the Laplace operator. In this framework, we cannot expect to obtain the strong solution, since it causes a singular point in the rectangular domain  $\Omega$  defined in (4.14). To improve the regularity, we first need to consider if the singularity can be removed and if the corresponding solution is smoother (possibly not in the domain of the generator to some power) when we choose a smoother input function and initial data. Now we give a precise description of the problem that we shall investigate.

The linearized water waves system (4.15), as discussed in Section 4.2, can be divided into two Laplace equations of  $D\psi$  and  $N(hv)$  with mixed boundary conditions in (4.23) and (4.29), i.e.

$$\phi(t, x, y) = (D\psi)(t, x, y) + v(t)(Nh)(x, y) \quad \forall t \geq 0, (x, y) \in \Omega,$$

where  $D$  is the Dirichlet map introduced in Proposition 4.2.2 and  $N$  is the Neumann map introduced in Proposition 4.2.6. The function  $\psi(t, x)$  is the free surface velocity potential  $\phi(t, x, \zeta(t, x))$ . Obviously, the main problem of this decomposition is that there exists a singular point at the left upper corner of the domain  $\Omega$ , i.e.  $(x, y) = (0, 0)$ . Therefore, the solution of (4.15) cannot be a strong solution since the velocity is not continuous at this corner, i.e.  $\partial_x \psi(t, 0) \neq -v(t)$  if  $v(t)$  is non-zero, no matter how regular the control function and the initial data are. Here for the sake of simplicity, we assume that the shape function  $h = 1$ , which does not matter since we do not use the Neumann to Neumann map in this section.



In order to obtain the strong solution of (4.15), we shall remove the singular point by using a transform. Let  $\chi$  be a smooth function on  $[0, \pi]$  satisfying

$$\chi(x) = \begin{cases} 1 & x \in (0, \frac{1}{4}), \\ 0 & x \in (\pi - \frac{1}{4}, \pi). \end{cases} \quad (4.60)$$

By using the function  $\chi$  in (4.60), we introduce the new function  $\tilde{\phi}(t, x, y)$  defined by

$$\tilde{\phi}(t, x, y) := \phi(t, x, y) + x\chi(x)v(t). \quad (4.61)$$

After doing the transform (4.61), we obtain a non-homogeneous Laplace equation with mixed boundary conditions for  $\tilde{\phi}$ , which, for all  $t \geq 0$  and  $(x, y) \in \Omega$ , read

$$\begin{cases} \Delta_{x,y}\tilde{\phi}(t, x, y) = v(t)(x\chi)'' & \text{in } \Omega, \\ \partial_y\tilde{\phi}(t, x, 0) - \partial_t\zeta(t, x) = 0, \\ \partial_t\tilde{\phi}(t, x, 0) + \zeta(t, x) = x\chi(x)u(t), \\ \partial_x\tilde{\phi}(t, 0, y) = 0, \\ \partial_y\tilde{\phi}(t, x, -1) = 0 = \partial_x\tilde{\phi}(t, \pi, y), \end{cases} \quad (4.62)$$

where  $u = \dot{v}$ . Observe from (4.62) that the non-homogeneous Neumann boundary value in (4.15) is shifted to the source term of the Laplace equation for  $\tilde{\phi}$ . In this way, we obtain a Poisson equation with one non-homogeneous Dirichlet condition on the top and three homogeneous Neumann conditions on the other boundaries. Therefore, we expect to obtain the solution of (4.15) with good regularity through the new Laplace system (4.62) for  $\tilde{\phi}$ . We notice that the equations (4.62) can be formulated into a new second-order evolution equation in terms of a perturbation of the elevation  $\zeta$ , thereby we have a new decomposition of (4.62) where the singularity is eliminated.

For any  $\tau > 0$  and  $m \in \mathbb{N}$ , we set

$$\mathcal{H}_L^m[0, \tau] = \left\{ u \in \mathcal{H}^m[0, \tau] \mid u(0) = \frac{du}{dt}(0) = \dots = \frac{d^{m-1}u}{dt^{m-1}}(0) = 0 \right\}.$$

As explained earlier, here we consider a second-order evolution equation of  $\zeta$ , which is equivalent to the equations (4.15) in the sense of Theorem 4.3.3. We thus propose the corresponding initial data

$$\zeta(0, x) = \zeta_0(x), \quad \partial_t\zeta(0, x) = \zeta_1(x). \quad (4.63)$$

Therefore, we take the acceleration  $u = \dot{v}$  as the input signal.

To study the well-posedness of (4.62), we need more properties of the Dirichlet map  $D$  and the Dirichlet to Neumann operator  $A_0$  introduced in Proposition 4.2.2 and Proposition 4.2.9, respectively.

**Lemma 4.3.4.** *The partial Dirichlet map  $D$  defined in Proposition 4.2.2 is bounded from  $H_{\frac{3}{2}}$  to  $\mathcal{H}^2(\Omega)$ , i.e.  $D \in \mathcal{L}(H_{\frac{3}{2}}, \mathcal{H}^2(\Omega))$ .*

### 4.3. WELL-POSEDNESS OF THE LINEARIZED WATER WAVES SYSTEM

*Proof.* According to Lemma 4.2.3,  $D\eta$  is given by (4.24) for every  $\eta \in H$ . Note that  $\{\cos(kx)\}_{k \in \mathbb{N}}$  is an orthogonal set in  $H$ , we have that for every  $\eta \in H_{\frac{3}{2}}$ ,

$$\begin{aligned} \|\partial_x^2(D\eta)\|_{L^2(\Omega)}^2 &= \int_{-1}^0 \int_0^\pi \left| \sum_{k \in \mathbb{N}} \frac{k^2 \langle \eta, \varphi_k \rangle}{\cosh(k)} \cosh[k(y+1)] \cos(kx) \right|^2 dx dy \\ &\leq \sum_{k \in \mathbb{N}} \frac{|k^2 \langle \eta, \varphi_k \rangle|^2}{\cosh^2(k)} \int_{-1}^0 \cosh^2[k(y+1)] dy \\ &= \sum_{k \in \mathbb{N}} \frac{k^4 |\langle \eta, \varphi_k \rangle|^2}{\cosh^2(k)} + \sum_{k \in \mathbb{N}} \frac{k^3 |\langle \eta, \varphi_k \rangle|^2}{2 \cosh^2(k)} \sinh(2k), \end{aligned}$$

which clearly implies that there exists  $K_1 > 0$  such that

$$\|\partial_x^2(D\eta)\|_{L^2(\Omega)} \leq K_1 \|\eta\|_{H_{\frac{3}{2}}} \quad \forall \eta \in H_{\frac{3}{2}}.$$

A similar estimate for  $\|\partial_y^2(D\eta)\|_{L^2}$  and  $\|\partial_x \partial_y(D\eta)\|_{L^2}$  can be obtained in a completely similar manner. Moreover, we know from Proposition 4.2.2 that  $\|D\eta\|_{L^2}$  is also bounded by a similar estimate. Furthermore, according to Lemma 4.2.5, we see that  $\|\partial_x(D\eta)\|_{L^2}$  and  $\|\partial_y(D\eta)\|_{L^2}$  can be controlled by  $\|\eta\|_{H_{\frac{3}{2}}}$ . Therefore, we obtain that  $D$  is linear continuous from  $H_{\frac{3}{2}}$  to  $\mathcal{H}^2(\Omega)$ .  $\square$

**Lemma 4.3.5.** *Let  $A_0$  be the Dirichlet to Neumann map introduced in Proposition 4.2.9. For every  $\alpha \geq 0$ ,  $A_0\eta \in H_\alpha$  if and only if  $\eta \in H_{\alpha+1}$ .*

*Proof.* According to (4.34), we have

$$\|A_0\eta\|_{H_\alpha}^2 = \sum_{k \in \mathbb{N}} k^{2\alpha} |k \tanh(k)|^2 |\langle \eta, \varphi_k \rangle|^2.$$

The right hand side of the above equality is equivalent to  $\|\eta\|_{H_{\alpha+1}}^2$ , which clearly implies the result.  $\square$

Recalling the definition of the operator  $A_1$  in Proposition 4.2.1, we have the following lemma.

**Lemma 4.3.6.** *With the operator  $A_1$  defined in Proposition 4.2.1, for every  $v \in \mathbb{C}$ , the equation*

$$A_1\Phi = -v(x\chi(x))'',$$

*admits a unique solution  $\Phi \in \mathcal{H}^2(\Omega)$ , which is given by*

$$\Phi = \sum_{k, l \in \mathbb{N}} \frac{v a_{kl}}{k^2 + (2l-1)^2 \frac{\pi^2}{4}} \Psi_{kl}, \quad (4.64)$$

*where  $\Psi_{kl}$  has been introduced in (4.21) and*

$$a_{kl} = \langle (x\chi(x))'', \Psi_{kl} \rangle_{L^2(\Omega)} \quad \forall k, l \in \mathbb{N}.$$

**Remark 11.** Observe that the double series (4.64) is absolutely convergent and the series of derivatives with respect to  $y$  converges uniformly on  $[-1, 0]$ , so that we define  $\partial_y \Phi$  by differentiating  $\Phi$  term by term. Moreover, the series  $\partial_y \Phi$ , as a function of  $y$ , is continuous on  $[-1, 0]$ , hence  $\partial_y \Phi(t, x, 0)$  can be defined by taking  $y = 0$  term by term. Based on this observation, we introduce the following result.

For the sake of the simplicity, we use the notation  $a_k$  and  $b_k$  defined by

$$a_k = \langle (x\chi)'', \varphi_k \rangle, \quad b_k = \langle x\chi, \varphi_k \rangle \quad \forall k \in \mathbb{N}, \quad (4.65)$$

which are the Fourier coefficient of the Fourier cosine series of  $(x\chi)''$  and  $x\chi$ , respectively.

**Remark 12.** The cosine expansion of  $(x\chi)''$  is infinitely differentiable since  $(x\chi)''$  is smooth with compact support on  $(0, \pi)$ . Moreover, note that the Fourier cosine expansion of  $x\chi$  is only once differentiable, we thus conclude that  $x\chi \in H_1$  and  $(x\chi)'' \in H_\alpha$  for all  $\alpha \geq 0$ , i.e.  $a_k$  and  $b_k$  defined in (4.65) satisfy

$$\sum_{k \in \mathbb{N}} k^2 |b_k|^2 < \infty, \quad \sum_{k \in \mathbb{N}} k^{2\alpha} |a_k|^2 < \infty \quad \forall \alpha \geq 0.$$

Based on the structure of the system (4.62) and Remark 11, we introduce a new operator  $S$  in the following proposition.

**Proposition 4.3.7.** *For every  $\alpha \geq 0$ , the operator  $S$  defined by*

$$(Sv)(t, x) := \partial_y \Phi(t, x, 0)$$

*is linear continuous from  $\mathbb{C}$  to  $H_\alpha$ , i.e.  $S \in \mathcal{L}(\mathbb{C}, H_\alpha)$ .*

*Proof.* According to (4.64) and Remark 11, we have

$$S = \sum_{k, l \in \mathbb{N}} \frac{-2a_k}{(k^2 + (2l-1)^2 \frac{\pi^2}{4})} \varphi_k, \quad (4.66)$$

where  $a_k$  and  $\varphi_k$  have been defined in (4.65) and (4.18), respectively. The formula (4.66) obviously implies that  $S \in H$ . For every  $\alpha \geq 0$ , we have

$$\|Sv\|_{H_\alpha}^2 = \sum_{k \in \mathbb{N}} k^{2\alpha} |\langle S, \varphi_k \rangle|^2 \leq \sum_{k \in \mathbb{N}} k^{2\alpha} |a_k|^2 < \infty,$$

which, together with Remark 12, implies that  $S \in H_\alpha$ . □

Recalling the Dirichlet to Neumann map  $A_0$  introduced in Proposition 4.2.9 and the function  $S$  defined in Proposition 4.3.7, we immediately obtain from the structure of the governing equations (4.62) that

$$\partial_y \tilde{\phi}(t, x, 0) = A_0 \tilde{\psi}(t, x) + Sv(t), \quad (4.67)$$

where  $t \geq 0$ ,  $x \in (0, \pi)$  and  $\tilde{\psi}(t, x) = \tilde{\phi}(t, x, 0)$ . To derive the second-order evolution equation, we define

$$\tilde{\zeta}(t, x) = \zeta(t, x) - x\chi(x)u(t), \quad (4.68)$$

### 4.3. WELL-POSEDNESS OF THE LINEARIZED WATER WAVES SYSTEM

which belongs to the domain of the operator  $A_0$ . With the initial data in (4.63), we derive from the system (4.62) that the second-order evolution equation for  $\tilde{\zeta}$  is

$$\begin{cases} \partial_t^2 \tilde{\zeta}(t, x) + A_0 \tilde{\zeta}(t, x) = Su - x\chi(x)\ddot{u}, \\ \tilde{\zeta}(0, x) = \tilde{\zeta}_0(x), \quad \partial_t \tilde{\zeta}(0, x) = \tilde{\zeta}_1(x), \end{cases} \quad (4.69)$$

where

$$\tilde{\zeta}_0(x) = \zeta_0(x) - x\chi(x)u(0) \quad \tilde{\zeta}_1(x) = \zeta_1(x) - x\chi(x)\dot{u}(0).$$

**Remark 13.** The equation (4.67) is obviously based on the relation:

$$\tilde{\phi}(t, x, y) = (D\tilde{\psi})(t, x, y) + \Phi(t, x, y),$$

where  $D$  is the Dirichlet map introduced in Proposition 4.2.2 and the function  $\Phi$  is defined in Lemma 4.3.6. According to the definition of  $\tilde{\phi}$  in (4.61), we have a corresponding decomposition for the original velocity potential  $\phi$ .

Denoting

$$\tilde{z}(t) = \begin{bmatrix} \tilde{\zeta}(t, \cdot) \\ \dot{\tilde{\zeta}}(t, \cdot) \end{bmatrix}, \quad \tilde{z}(0) = \begin{bmatrix} \tilde{\zeta}_0(\cdot) \\ \tilde{\zeta}_1(\cdot) \end{bmatrix},$$

we have a first-order evolution equation corresponding to the system (4.69) with the state  $\tilde{z}$  as follows:

$$\begin{cases} \dot{\tilde{z}}(t) = A\tilde{z}(t) + \mathfrak{B}u(t), \\ \tilde{z}(0) = \tilde{z}_0, \end{cases} \quad (4.70)$$

where  $A$  has been introduced in (4.47) and

$$\mathfrak{B}u = \begin{bmatrix} 0 \\ Su - x\chi\ddot{u} \end{bmatrix}. \quad (4.71)$$

The state of system (4.70) is still  $X = H_{\frac{1}{2}} \times H$ . It is not difficult to see from Remark 12 and Proposition 4.3.7 that  $\mathfrak{B}$  is an admissible control operator on  $X$  (for this concept, please refer to [115, Chapter 4]). For every  $m \in \mathbb{N}$ , we denote  $(X_0 = X)$

$$X_m = \mathcal{D}(A^m) = H_{\frac{m+1}{2}} \times H_{\frac{m}{2}},$$

endowed with the corresponding graph norm, so that it forms a Hilbert space.

**Remark 14.** The Hilbert spaces  $X_m$ , for all  $m \in \mathbb{N}$ , satisfy

$$\cdots X_3 \subset X_2 \subset X_1 \subset X.$$

Each inclusion is dense and with continuous embedding. The Hilbert spaces  $X_{-m}$ , of course, also make sense, which is defined by the completion of  $X$  with respect to the norm  $\|(sI - A)^{-m}z\|$  with  $z \in X$  (see, for instance, [115, Section 2.10]). As we already mentioned in notation part,  $X_{-m}$  is the dual of  $X_m$  with respect to the pivot space  $X$ , for every  $m \in \mathbb{N}$ . Here we are interested in the Hilbert spaces  $X_m$  with positive integer subscript.

Using the above notation, we have the regularity result of the second-order evolution equation (4.69).

**Theorem 4.3.8.** *For every  $\tau > 0$ , let  $u \in \mathcal{H}_L^5[0, \tau]$ ,  $\zeta_0 \in H_2$  and  $\zeta_1 \in H_{\frac{3}{2}}$ . Then the system (4.69) admits a unique solution  $\tilde{\zeta}$  satisfying*

$$\tilde{\zeta} \in C([0, \tau]; H_2) \cap C^1([0, \tau]; H_{\frac{3}{2}}). \quad (4.72)$$

*Proof.* Recalling that  $A$  is skew-adjoint, we denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  the  $\mathbb{C}_0$ -semigroup generated by the operator  $A$ . The mild solution of (4.70) is given by

$$\tilde{z}(t) = \mathbb{T}_t \tilde{z}_0 + \tilde{\Phi}_t u,$$

where  $\tilde{\Phi}_t$  is the input map defined by

$$\tilde{\Phi}_t u = \int_0^t \mathbb{T}_{t-\sigma} \mathfrak{B} u(\sigma) d\sigma$$

with  $\mathfrak{B}u$  introduced in (4.71). Since  $u \in \mathcal{H}_L^5[0, \tau]$ , actually we have  $\tilde{\zeta}_0 = \zeta_0$  and  $\tilde{\zeta}_1 = \zeta_1$ . For the initial data  $\tilde{z}_0 \in H_2 \times H_{\frac{3}{2}} = X_3$ , it is obvious that  $\mathbb{T}_t \tilde{z}_0 \in X_3$ . Moreover, Proposition 4.3.7 implies that  $\mathfrak{B}$  is an admissible control operator, that is  $\tilde{\Phi}_t u \in X$ . Next we prove that  $\tilde{\Phi}_t u$  belongs to  $X_3$  if the input  $u \in \mathcal{H}_L^5[0, \tau]$  for every  $\tau > 0$ .

Note that  $\mathfrak{B}u$  contains the second derivative of  $u$  with respect to time, we obtain, according to [116, Proposition 2.3], that

$$\tilde{\Phi}_t \mathcal{H}_L^5[0, \tau] \subset X_3 + (\beta I - A)^{-1} \mathfrak{B} \mathbb{C} \quad \forall \beta \in \rho(A),$$

which is independent of the choice of  $\beta \in \rho(A)$ . Notice that  $1 \in \rho(A)$ , we assume that  $(I - A) \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \end{bmatrix} = \mathfrak{B}u$ , which gives that

$$\tilde{\phi} = \tilde{\psi} = (I + A_0)^{-1} (Su - x\chi\ddot{u}).$$

It remains to prove that  $\tilde{\phi} \in H_2$ . Since  $A_0$  is diagonalizable, we have

$$\sum_{k \in \mathbb{N}} |\lambda_k|^4 |\langle \tilde{\phi}, \varphi_k \rangle|^2 = \sum_{k \in \mathbb{N}} \frac{|\lambda_k|^4}{|1 + \lambda_k|^2} |\langle Su - x\chi\ddot{u}, \varphi_k \rangle|^2,$$

where  $\lambda_k = k \tanh(k)$ . Recalling the formula of  $S$  given in Proposition 4.3.7, we further have the estimate,

$$\sum_{k \in \mathbb{N}} |\lambda_k|^4 |\langle \tilde{\phi}, \varphi_k \rangle|^2 \leq C \sum_{k \in \mathbb{N}} |\lambda_k|^2 \left[ |a_k|^2 + |b_k|^2 \right],$$

where  $a_k$  and  $b_k$  have been introduced in 4.65. Note that  $\lambda_k$  is equivalent to  $k$ , we obtain from Remark 12 that the above series is convergent, i.e.  $\tilde{\phi} \in H_2$ . Therefore, we conclude that  $\begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \end{bmatrix} \in H_2 \times H_{\frac{3}{2}} = X_3$  and then  $\tilde{z}(t) \in C([0, \tau]; X_3)$ , which further implies (4.72).  $\square$

**Remark 15.** The semigroup  $\mathbb{T}$  in the above proof can be regarded as its restriction on  $X_3$ . For each  $m \in \mathbb{N}$ , the original semigroup  $\mathbb{T}$  on  $X$  generated by the operator  $A$  has a restriction to  $X_m$ , that is the image of  $\mathbb{T}$  through the unitary operator  $(sI - A)^{-m} \in \mathcal{L}(X, X_m)$ , where  $s \in \rho(A)$ . We refer to [115, Proposition 2.10.4 and Remark 2.10.5] for the details.

#### 4.4. STABILIZABILITY PROPERTIES OF THE LINEAR SYSTEM

**Theorem 4.3.9.** *For every  $\tau > 0$ ,  $u \in \mathcal{H}_L^5[0, \tau]$ ,  $\zeta_0 \in H_2$  and  $\zeta_1 \in H_{\frac{3}{2}}$ , the system (4.15) admits a unique solution  $(\zeta, \phi)$  satisfying*

$$\zeta \in C([0, \tau]; \mathcal{H}^2[0, \pi]) \cap C^1([0, \tau]; \mathcal{H}^{3/2}[0, \pi]),$$

and

$$\phi \in \mathcal{H}_{\text{loc}}^1([0, \infty); \mathcal{H}^2(\Omega)).$$

*Proof.* Since the well-posedness of the weak solution of (4.15) has been established, here we just need to improve the regularity of the pair  $(\zeta, \phi)$ . We first note that, for every  $\alpha \geq 0$ ,  $H_\alpha \subset \mathcal{H}^\alpha[0, \pi]$ . According to Theorem 4.3.8 and the relation (4.68), we immediately have

$$\zeta \in C([0, \tau]; \mathcal{H}^2[0, \pi]) \cap C^1([0, \tau]; \mathcal{H}^{3/2}[0, \pi]),$$

since  $x\chi$  is smooth on  $[0, \pi]$  and  $u \in \mathcal{H}_L^5[0, \tau]$ . Moreover, Remark 13 implies that we have the decomposition

$$\phi(t, x, y) = (D\tilde{\psi})(t, x, y) + \Phi(t, x, y) - x\chi(x)v(t), \quad (4.73)$$

where  $D$  is the Dirichlet map defined in Proposition 4.2.2 and the function  $\Phi$  is introduced in Lemma 4.3.6. Putting the second equation of (4.62), (4.67) and (4.68) together, we have

$$A_0\tilde{\psi} = \partial_t\tilde{\zeta} + x\chi\dot{u} - Sv.$$

Recalling the properties of the function  $x\chi$  discussed in Remark 12, we obtain from Theorem 4.3.8 and Proposition 4.3.7 that  $A_0\tilde{\psi} \in H_1$ . Moreover, using Lemma 4.3.5, it follows that  $\tilde{\psi} \in H_2 \subset H_{\frac{3}{2}}$ . We thus obtain from Lemma 4.3.4 and Lemma 4.3.6 that  $\phi \in L^2([0, \tau]; \mathcal{H}^2(\Omega))$ .

Taking the derivative of  $\phi$  in (4.73) with respect to time and using the third equation of (4.62), we have

$$\partial_t\phi(t, x, y) = -(D\tilde{\zeta})(t, x, y) + \partial_t\Phi(t, x, y) - x\chi(x)u(t).$$

Using again Lemma 4.3.4 and Lemma 4.3.6, it follows that  $\partial_t\phi \in L^2([0, \tau]; \mathcal{H}^2(\Omega))$ . Therefore, we conclude that  $\phi \in \mathcal{H}^1([0, \tau]; \mathcal{H}^2(\Omega))$ .  $\square$

#### 4.4 Stabilizability properties of the linear system

Based on the well-posedness results in Theorem 4.3.3 and Remark 10, we are interested in the stabilization of the linear control system (4.45) with state feedback. We recall some commonly used stabilizability concepts, for the particular situation of bounded control and feedback operators.

**Definition 4.4.1.** *Let  $\Sigma = (\mathbb{T}, \Phi)$  be a well-posed linear control system with state space  $X$  and input space  $U$ . Let  $A$  be the generator of  $\mathbb{T}$  and assume that there exists  $B \in \mathcal{L}(U, X)$  such that (4.53) holds. For some feedback operator  $F \in \mathcal{L}(X, U)$  we denote by  $\mathbb{T}^{\text{cl}}$  the (closed-loop) operator semigroup on  $X$  generated by  $A + BF$ . Then the system  $(\mathbb{T}, \Phi)$  is:*

1. Exponentially stabilizable with bounded feedback, if there exists  $F \in \mathcal{L}(X, U)$  such that the semigroup  $\mathbb{T}^{\text{cl}}$  is exponentially stable;

2. Strongly stabilizable with bounded feedback, if there exists  $F \in \mathcal{L}(X, U)$  such that the semigroup  $\mathbb{T}^{cl}$  is strongly stable;
3. Uniformly stabilizable for smooth data (USSD), if there exists  $F \in \mathcal{L}(X, U)$  and  $f : [0, \infty) \rightarrow [0, \infty)$ , with  $\lim_{t \rightarrow \infty} f(t) = 0$ , such that

$$\|\mathbb{T}_t^{cl} z_0\|_X \leq f(t) \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0. \quad (4.74)$$

If  $f$  in (4.74) can be chosen such that  $\lim_{t \rightarrow \infty} t^m f(t) = 0$  for some  $m \in \mathbb{N}$ , then the USSD property is called polynomial stabilizability.

In (4.74) and also later,  $\|\cdot\|_{\mathcal{D}(A)}$  denotes the graph norm on  $\mathcal{D}(A)$ .

**Remark 16.** Note that the property (4.74) does not imply that the semigroup  $\mathbb{T}^{cl}$  is strongly stable. Indeed, consider  $\mathbb{T}_t^{cl}$  to be  $e^{-0.7t}$  times the semigroup from [120, Example 2.3] (based on Zabczyk [124]), with  $\lambda_n = 2^n$ , then it satisfies (4.74) with  $f(t) = Me^{-0.2t}$  (for some  $M > 0$ ) but  $\mathbb{T}^{cl}$  is exponentially growing:  $\|\mathbb{T}_t^{cl}\| = e^{0.3t}$ . However, if  $\mathbb{T}^{cl}$  is a bounded semigroup and (4.74) holds, then it is easy to see that  $\mathbb{T}^{cl}$  is strongly stable.

Here is our main result:

**Theorem 4.4.2.** Let  $\Sigma = (\mathbb{T}, \Phi)$  be the well-posed linear control system introduced in Theorem 4.3.3. Then

1.  $\Sigma$  is not exponentially stabilizable with bounded feedback;
2.  $\Sigma$  is strongly stabilizable with bounded feedback if and only if  $h$  is a strategic profile, in the sense that

$$\int_{-1}^0 h(y) \cosh[k(y+1)] dy \neq 0 \quad \forall k \in \mathbb{N}; \quad (4.75)$$

In this case, one strongly stabilizing feedback operator is  $F = -B^*$ .

3. If

$$\inf_{k \in \mathbb{N}} \frac{k}{\cosh k} \left| \int_{-1}^0 h(y) \cosh[k(y+1)] dy \right| > 0, \quad (4.76)$$

then the system  $\Sigma$  is USSD. More precisely, the feedback operator  $F = -B^*$  leads to the closed-loop semigroup  $\mathbb{T}^{cl}$  (with generator  $A - BB^*$ ) which is strongly stable and has the following property: there exists  $M > 0$  such that

$$\|\mathbb{T}_t^{cl} z_0\|_X \leq \frac{M}{(1+t)^{\frac{1}{6}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0. \quad (4.77)$$

**Remark 17.** It is not difficult to check (by integration by parts) that condition (4.76) is satisfied, for instance, if there exists  $\varepsilon \in (0, 1)$  such that

$$\|h'\|_{L^\infty[-1,0]} < \frac{(1-\varepsilon) \tanh 1}{1 - \frac{2}{e}} |h(0)|, \quad (4.78)$$

where  $e = 2.71828\dots$  is the basis of the natural logarithm. Indeed, there are many functions satisfying (4.78) and  $\int_{-1}^0 h(y) dy = 0$ , such as the linear function  $h_1(y) =$

#### 4.4. STABILIZABILITY PROPERTIES OF THE LINEAR SYSTEM

$y + \frac{1}{2}$ , the trigonometric function  $h_2(y) = \cos\left[\frac{1}{2}\pi\left(y + \frac{3}{2}\right)\right]$  and some slightly modified step functions. Compared with other strategic conditions, for instance the constraint condition at rational points in [7], the condition (4.78) is easier to satisfy in practice.

**Remark 18.** The first two conclusions in Theorem 4.4.2 appear partially in [85], with some steps of the proof not given. (For instance the operators  $A_0$  and  $B_0$  that we introduce in Section 4.2.2 are used without a detailed construction and proof of their main properties.) As far as we know, the property of the water waves system described in the third point of Theorem 4.4.2 is new, and gives us more detailed information on the stability of the closed-loop system.

In order to prove Theorem 4.4.2, we need the following preliminary result on the eigenvalues and the eigenvectors of the operator  $A$  introduced in (4.47). According to Proposition 4.2.9, we know that the eigenvalues of  $A_0$  behaves like  $k$  with  $k \in \mathbb{N}$ , and the corresponding eigenvectors is  $\varphi_k$  introduced in (4.18). This, together with the definition of  $A$ , we have the following conclusion.

**Lemma 4.4.3.** *Let  $(\lambda_k)_{k \in \mathbb{N}}$  and  $(\varphi_k)_{k \in \mathbb{N}}$  be the sequences defined in (4.33) and (4.18), respectively. We extend the sequences  $\mu_k = (\sqrt{\lambda_k})_{k \in \mathbb{N}}$  and  $(\varphi_k)_{k \in \mathbb{N}}$  to  $\mathbb{Z}^*$  by setting*

$$\mu_{-k} = -\mu_k, \quad \varphi_{-k} = -\varphi_k \quad \forall k \in \mathbb{N}.$$

Then the family  $\{\phi_k\}_{k \in \mathbb{Z}^*}$  defined by

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \quad \forall k \in \mathbb{Z}^* \quad (4.79)$$

is an orthonormal basis in  $X$  formed of eigenvectors of the operator  $A$  defined in (4.47). Moreover, for each  $k \in \mathbb{Z}^*$ ,  $A\phi_k = i\mu_k\phi_k$ . Finally, there exists  $\varepsilon > 0$  such that for every  $\omega \in \mathbb{R}$  with  $|\omega| \geq 1$ , the interval  $\left[\omega - \frac{\varepsilon}{|\omega|}, \omega + \frac{\varepsilon}{|\omega|}\right]$  contains at most one element of the sequence  $(\mu_k)_{k \in \mathbb{Z}^*}$ .

*Proof.* According to Proposition 4.2.9, the family  $(\varphi_k)_{k \in \mathbb{N}}$  defined in (4.18) is an orthonormal basis in  $H$  formed of eigenvectors of  $A_0$  and for  $k \in \mathbb{N}$ ,  $A_0\varphi_k = \lambda_k\varphi_k$ , where  $(\lambda_k)_{k \in \mathbb{N}}$  have been defined in (4.33). Using the structure (4.47) of  $A$  and a classical result (see, for instance, [115, Section 3.7]), it follows that  $A$  is diagonalizable, with the eigenvalues  $(i\mu_k)_{k \in \mathbb{Z}^*}$  corresponding to the orthonormal basis of eigenvectors  $(\phi_k)_{k \in \mathbb{Z}^*}$  defined in (4.79). Note that for  $k \in \mathbb{N}$ ,  $\mu_k \approx \sqrt{k}$ , with exponentially vanishing approximation error. Therefore, we obtain the desired result by taking  $\alpha = \frac{1}{2}$  in Lemma 3.3.1.  $\square$

We are now in a position to prove Theorem 4.4.2 for the stabilizability result.

*Proof of Theorem 4.4.2.* 1. The first assertion follows directly from Curtain and Zwart [40, Theorem 5.2.6], since  $A$  has infinitely many unstable eigenvalues. Alternatively, we can apply the main result of Gibson [44] or Guo, Guo and Zhang [53, Theorem 3].

2. To prove the second assertion, notice that, since the adjoint of the operator  $B$  defined in (4.48) is  $B^* = \begin{bmatrix} 0 & B_0^* \end{bmatrix}$ , we see from (4.43) and (4.79) that

$$B^*\phi_k = \frac{-1}{\sqrt{2}} \int_{-1}^0 h(y) \overline{(C_0\varphi_k)(y)} dy \quad \forall k \in \mathbb{Z}^*,$$



where  $C_0 = \gamma_0 D$ . Using (4.25), we get from the above that

$$B^* \phi_k = -\frac{1}{\sqrt{\pi}} \int_{-1}^0 h(y) \frac{\cosh[k(y+1)]}{\cosh k} dy \quad \forall k \in \mathbb{Z}^*. \quad (4.80)$$

Assume now that  $h$  is a strategic profile, i.e., (4.75) holds. Then clearly

$$B^* \phi_k \neq 0 \quad \forall k \in \mathbb{Z}^*.$$

According to [115, Proposition 6.9.1] the pair  $(A^*, B^*)$  is approximately observable in infinite time (we have used that the eigenvalues of  $A$  are distinct). Now it follows from the main result of Benchimol [18] that the semigroup generated by  $A - BB^*$  is strongly stable.

Conversely, let us assume that  $h$  is not a strategic profile, i.e., that  $h$  does not satisfy assumption (4.75). Then from (4.80) there exists a  $k \in \mathbb{N}$  such that  $B^* \phi_k = 0$ . Since  $A^* = -A$ , it follows that for every  $F \in \mathcal{L}(X, U)$ ,

$$(A^* + F^* B^*) \phi_k = -i\mu_k \phi_k.$$

Let  $\mathbb{T}^{cl}$  denote the semigroup generated by  $A + BF$ , we have

$$\left(\mathbb{T}_t^{cl}\right)^* \phi_k = e^{-i\mu_k t} \phi_k \quad \forall t \geq 0,$$

which implies that  $|\langle \mathbb{T}_t^{cl} \phi_k, \phi_k \rangle| = 1$  for all  $t \geq 0$ , so that  $\mathbb{T}^{cl}$  is not strongly stable. We have thus shown that if  $(A, B)$  is strongly stabilizable, then  $h$  satisfies (4.75), which ends the proof of the second assertion.

3. To prove the third assertion, first we notice that by combining (4.76) and (4.80) it follows that there exists  $M_0 > 0$  such that

$$|B^* \phi_k| \geq \frac{M_0}{|k|} \quad \forall k \in \mathbb{Z}^*. \quad (4.81)$$

According to Lemma 4.4.3 and Proposition 4.2.9, the eigenvalues of the operator  $A$  behaves like  $k^{1/2}$ . Hence, the assumptions  $[H_1]$  and  $[H_2]$  introduced in (3.3) and (3.4) are satisfied with  $\alpha = \frac{1}{2}$  and  $\beta = 1$ . Now we apply Theorem 3.1.1 and obtain the stability result (4.77), for the initial data  $z_0 \in \mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ .  $\square$

**Remark 19.** Another approach to obtain the decay rate (4.77) is a result presented in Chill et al. [32]. To verify it, we introduce, for every  $s \in \mathbb{R}$  and  $\delta > 0$ , the vector space  $WP_{s,\delta}(A)$ , called *wave package of frequency  $s$  and width  $\delta$  associated with the operator  $A$* , which is defined by

$$WP_{s,\delta}(A) = \begin{cases} \{0\} & \text{if } |\mu_k - s| \geq \delta \text{ for all } k \in \mathbb{Z}^*, \\ \text{span } \{\phi_k \mid k \in \mathbb{Z}^* \text{ and } |\mu_k - s| < \delta\} & \text{else.} \end{cases}$$

According to Lemma 4.4.3 there exists  $\varepsilon > 0$  such that, setting

$$\delta(s) = \frac{\varepsilon}{|s| + 1} \quad \forall s \in \mathbb{R}, \quad (4.82)$$

## 4.5. THE CASE OF GRAVITY-CAPILLARY WAVES

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we either have that  $WP_{s,\delta(s)}(A) = \{0\}$  or

$$WP_{s,\delta(s)}(A) = \text{span} \{\phi_{k(s)}\},$$

where  $k(s)$  is the unique element of  $\mathbb{Z}^*$  such that

$$s - \delta(s) < \mu_{k(s)} < s + \delta(s).$$

Using the fact that  $\mu_k = \sqrt{k \tanh(k)}$  and  $\mu_{-k} = -\mu_k$  for  $k \in \mathbb{N}$ , together with (4.81), it follows that there exists  $M_1 > 0$  such that

$$|B^* \phi| \geq \frac{M_1}{(|s| + 1)^2} \|\phi\|_X \quad \forall \phi \in WP_{s,\delta(s)}(A), s \in \mathbb{R}.$$

We have thus obtained that the pair  $(A, B)$  satisfies the assumptions of Theorem 1.1 in [32] with  $\delta$  given by (4.82) and

$$\gamma(s) = \frac{M_1}{(|s| + 1)^2} \quad \forall s \in \mathbb{R}.$$

We can apply Theorem 1.1 in [32] to conclude that the semigroup  $\mathbb{T}^{cl}$  generated by  $A - BB^*$  satisfies (4.77).

**Remark 20.** For the proof of the second assertion we could use (instead of Benchimol [18]) the stronger result of Batty and Vu [16], where  $A$  generates a contraction semigroup and  $B$  is still bounded. An even more general result is in the recently published [39], where  $A$  generates a contraction semigroup and  $B$  may be very unbounded (not even admissible).

## 4.5 The case of gravity-capillary waves

In this section, we consider a similar boundary control system of the gravity-capillary water waves system in a water tank where the gravity and the surface tension are both significant. As in the previous sections, we first present the full governing equations for this control system, then we analyse the well-posedness and the stabilizability of the linearized equations. The contents for the gravity-capillary waves discussed in this section are mainly based on the work by Su [108].

Still using the notation introduced in Section 4.1, we derive the non-linear governing equations. The surface tension effect arises from the forces exerted by the air and the water on the surface of the wave, which prevent the interface from becoming deformed. These forces cause the surface to behave like an elastic membrane, although their effect is small when dealing with large masses of water. For more about the effect of surface tension, please refer to [45], [66] or [123]. To derive the governing equations, we begin from the Bernoulli equation introduced in (4.4). More precisely, the surface tension describes a discontinuity in the pressure along the interface, and the pressure on the free surface is given by the formula:

$$P(t, x, \zeta(t, x)) = P_{\text{atm}} - \sigma \kappa, \tag{4.83}$$

where  $\sigma$  is the surface tension coefficient,  $\kappa$  is the mean curvature of the surface. Since the free surface is described by the equation  $F(t, x, y) = y - \zeta(t, x) = 0$ , the mean curvature  $\kappa$  is given by

$$\kappa(\zeta) = -(\partial_x, \partial_y) \cdot \frac{(-\partial_x \zeta, 1)}{\sqrt{1 + \partial_x^2 \zeta}} = \partial_x \left( \frac{\partial_x \zeta}{\sqrt{1 + \partial_x^2 \zeta}} \right) = \frac{\partial_x^2 \zeta}{(1 + \partial_x^2 \zeta)^{3/2}}.$$

Using the pressure on the free surface (4.83), we derive from (4.4) that the free surface Bernoulli equation with the surface tension reads

$$\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + g \zeta = T \kappa \quad \text{on } \Gamma_s, \quad (4.84)$$

where the parameter  $T = \frac{\sigma}{\rho}$ . The other boundary conditions are the same with the gravity case. Therefore, the governing equations of the control system on the gravity-capillary water waves are (4.5)–(4.6), (4.8)–(4.10) and (4.84). Similarly, we denote by  $\psi(t, x) = \phi(t, x, \zeta(t, x))$  the trace of the velocity potential  $\phi$  on the surface  $\Gamma_s$ . Recalling the non-linear Dirichlet to Neumann operator  $A[\zeta]$  and the non-linear Neumann to Neumann operator  $B[\zeta]$  introduced in Section 4.1, we obtain the  $(\zeta, \psi)$  formulation of the governing equations, which, for every  $t \geq 0$  and  $x \in [0, \pi]$ , are

$$\begin{cases} \partial_t \zeta - A[\zeta] \psi = B[\zeta] \mathcal{V}, \\ \partial_t \psi + g \zeta + \frac{1}{2} |\partial_x \psi|^2 - \frac{1}{2} \frac{(A[\zeta] \psi + B[\zeta] \mathcal{V} + \partial_x \zeta \partial_x \psi)^2}{1 + |\partial_x \zeta|^2} = T \kappa, \end{cases} \quad (4.85)$$

where  $\mathcal{V}(t, y)$  is the velocity produced by the wave maker. The boundary control system (4.85) is also still completely open. Different with the boundary control described above, as we have already mentioned in Section 4.1, the  $(\zeta, \psi)$  formulation for the gravity-capillary waves with a surface pressure control is setup studied by Alazard et al., for instance, in [4] and [3].

Now we go back to the equations (4.5)–(4.6), (4.8)–(4.10) and (4.84). Similarly, we assume that the wave maker is rigid in the sense that  $\mathcal{V}(t, y) = h(y)v(t)$ . Therefore, the linearized gravity-capillary water waves system, for every  $t \geq 0$  and  $(x, y) \in \Omega$  (see (4.14)), is

$$\begin{cases} \Delta_{x,y} \phi = 0 & \text{in } \Omega, \\ \partial_t \phi(t, x, 0) + g \zeta(t, x) = T \partial_x^2 \zeta, \\ \partial_y \phi(t, x, 0) = \partial_t \zeta(t, x), \\ \partial_x \phi(t, 0, y) = -h(y)v(t), \\ \partial_y \phi(t, x, -1) = 0 = \partial_x \phi(t, \pi, y). \end{cases} \quad (4.86)$$

Similarly, we shall formulate the system (4.86) into an evolution equation in terms of the elevation  $\zeta$ . Taking the derivative of (4.86) with respect to  $t$ , we obtain the equation for  $\varphi = \partial_t \phi$  as follows:

$$\begin{cases} \Delta_{x,y} \varphi = 0 & \text{in } \Omega, \\ \varphi(t, x, 0) + g \zeta(t, x) = T \partial_x^2 \zeta, \\ \partial_y \varphi(t, x, 0) = \partial_t^2 \zeta(t, x), \\ \partial_x \varphi(t, 0, y) = -h(y)u(t), \\ \partial_y \varphi(t, x, -1) = 0 = \partial_x \varphi(t, \pi, y), \end{cases} \quad (4.87)$$

## 4.5. THE CASE OF GRAVITY-CAPILLARY WAVES

where  $u = \dot{v}$  is the acceleration imposed by the rigid wave maker from the left boundary  $\Gamma_1$ .

### 4.5.1 Well-posedness and stabilizability issue

As in the gravity case, we establish in this section the well-posedness of the system (4.87) with regular control  $u$  and initial data, which is given in terms of the elevation  $\zeta$ . It implies that the whole system (4.87) is equivalent to an infinite dimensional control system. Moreover, we prove that the linear control system is strongly stabilizable and, for regular initial data, we give the explicit decay rate for the energy of the state.

Recalling the scale of Hilbert spaces  $H_\alpha$  introduced in (4.19), for every  $\alpha \in \mathbb{R}$  and  $\alpha > \frac{3}{2}$ , we define

$$H_\alpha^b = \left\{ \eta \in H_\alpha \mid \frac{d\eta}{dx}(0) = \frac{d\eta}{dx}(\pi) = 0 \right\}. \quad (4.88)$$

Now we give the definition of a solution of the system (4.87).

**Definition 4.5.1.** Given  $u \in \mathcal{H}_{\text{loc}}^1[0, \infty)$  and  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$ , a couple  $(\varphi, \zeta)$  is a solution of (4.87) if

$$\varphi \in C([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); H_3^b) \cap C^1([0, \infty); H_{\frac{3}{2}}),$$

$$\varphi(t, \cdot, 0) + g\zeta(t, \cdot) = T\partial_x^2\zeta(t, \cdot), \text{ in } C([0, \infty); L^2[0, \pi]),$$

and for every  $\Psi \in \mathcal{H}^1(\Omega)$  and every  $t > 0$  we have

$$\begin{aligned} & \int_0^\pi \dot{\zeta}(t, x)\Psi(x, 0) dx - \int_0^\pi \dot{\zeta}(0, x)\Psi(x, 0) dx \\ &= \int_0^t \int_\Omega \nabla\varphi(\tau, x, y) \cdot \nabla\Psi(x, y) dx dy d\tau - \int_0^t u(\tau) \int_{-1}^0 h(y)\Psi(0, y) dy d\tau. \end{aligned}$$

Compared with the gravity case [110], here we need more regular control and initial data to define the weak solution of (4.87) because of the existence of the surface tension. The boundary condition in the definition (4.88) is for the conservation of the energy. For this, we give a detailed explanation in the following lemma.

**Lemma 4.5.2.** If the elevation of the free surface  $\zeta(t, x)$  satisfies the boundary condition

$$\partial_x\zeta(t, 0) = 0 = \partial_x\zeta(t, \pi) \quad \forall t \geq 0,$$

then the total energy of the gravity-capillary waves system is conserved.

*Proof.* The total energy of the gravity-capillary waves system (4.86), denoted by  $E_{\text{tot}}(t)$ , consists of the kinetic energy  $E_{\text{kin}}(t)$ , the gravitational potential energy  $E_{\text{gra}}(t)$  and the elastic potential energy  $E_{\text{ela}}(t)$ . Assuming that there is no wave maker at the left boundary  $\Gamma_1$  for simplicity, it is not difficult to see that the kinetic energy is

$$E_{\text{kin}}(t) = \frac{1}{2} \int_\Omega |\nabla\phi|^2 = \frac{1}{2} \int_{\partial\Omega} \phi \cdot \partial_n\phi = \frac{1}{2} \int_0^\pi \phi \cdot \partial_n\phi dx = \frac{1}{2} \int_0^\pi \psi \cdot A_0\psi dx,$$

where  $\psi(t, x) = \phi(t, x, \zeta(t, x))$  and  $A_0$  is the Dirichlet to Neumann operator introduced in Proposition 4.2.9. We consider a water column with a height of  $\zeta(t, x)$  and a width

of  $dx$ , then the position of the center of gravity is at  $\frac{1}{2}\zeta$ . Hence, the potential energy of this column is  $\frac{1}{2}g\zeta^2 dx$  (assuming that the density  $\rho = 1$ ) and we get the gravitational potential energy

$$E_{\text{gra}}(t) = \frac{1}{2} \int_0^\pi g\zeta^2 dx.$$

Moreover, according to Hooke's law, the elastic energy induced by the surface tension  $E_{\text{ela}}(t)$  is given by

$$E_{\text{ela}} = \frac{T}{2} \int_0^\pi |\partial_x \zeta|^2 dx,$$

where  $T = \frac{\sigma}{\rho}$ . For more about the elastic energy, please refer to, for instance, [45]. Therefore, the total energy of the system  $E_{\text{tot}}(t)$  is as follows:

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{gra}} + E_{\text{ela}} = \frac{1}{2} \int_0^\pi (\psi \cdot A_0 \psi + g\zeta^2 + T|\partial_x \zeta|^2) dx. \quad (4.89)$$

Taking the derivative of  $E_{\text{tot}}(t)$  with respect to  $t$ , we obtain that

$$\begin{aligned} \frac{d}{dt} E_{\text{tot}}(t) &= \int_0^\pi (\partial_t \psi A_0 \psi + g\zeta \partial_t \zeta + T \partial_x \zeta \partial_{tx} \zeta) dx \\ &= \int_0^\pi (T \partial_x^2 \zeta - g\zeta) A_0 \psi + \int_0^\pi g\zeta A_0 \psi + T \int_0^\pi \partial_x (A_0 \psi) \cdot \partial_x \zeta \\ &= \int_0^\pi T \partial_x^2 \zeta A_0 \psi + T \int_0^\pi \partial_x (A_0 \psi) \cdot \partial_x \zeta = T \int_0^\pi \partial_x (A_0 \psi \partial_x \zeta) dx \\ &= T (A_0 \psi \partial_x \zeta)(t, \pi) - T (A_0 \psi \partial_x \zeta)(t, 0), \end{aligned}$$

where we used the self-adjointness of the operator  $A_0$  and the free surface equations in (4.86). Therefore, it is obvious that the total energy  $E_{\text{tot}}$  is conserved if  $\partial_x \zeta(t, 0) = 0 = \partial_x \zeta(t, \pi)$ .  $\square$

For the well-posedness of the system (4.87), we shall prove this by formulating them as a well-posed linear control system through the evolution operators. Besides the Dirichlet to Neumann operator  $A_0$  and the Neumann to Neumann operator  $B_0$  introduced in Section 4.2, we still need the *Sturm-Liouville operator*  $L$  defined by

$$Lf = -Tf''.$$

**Proposition 4.5.3.** *Let  $A_0$  be the operator defined in Proposition 4.2.9 and  $L$  be the Sturm-Liouville operator. Then  $\tilde{A} = A_0 + A_0 L : H_3^b \rightarrow H$  is strictly positive and  $\mathcal{D}(\tilde{A}^{\frac{1}{2}}) = H_{\frac{3}{2}}$ . For every  $k \in \mathbb{N}$ , we have  $\tilde{A}\varphi_k = \omega_k \varphi_k$ , where*

$$\omega_k = (Tk^3 + k) \tanh(k) \quad \forall k \in \mathbb{N}, \quad (4.90)$$

and  $\varphi_k$  has been introduced in (4.18). Moreover, we have

$$\tilde{A}\eta = \sum_{k \in \mathbb{N}} \omega_k \langle \eta, \varphi_k \rangle \varphi_k \quad \forall \eta \in H_3^b.$$

*Proof.* It is not difficult to verify that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  introduced in (4.18) forms an orthonormal basis in  $H_3^b$  and in  $H_2^b$ , respectively. Moreover, note that the operator  $L$  is well-defined from  $H_2^b$  to  $H$  (self-adjoint, strictly positive with compact resolvents),

## 4.5. THE CASE OF GRAVITY-CAPILLARY WAVES

we obtain, according to Proposition 4.2.9, that  $\mathcal{D}(\tilde{A}) = H_3^b$  and  $\tilde{A} = A_0 + A_0L$  is diagonalizable (see, for instance, [115, Chapter 3]) with the eigenvalues  $\omega_k$  (4.90) and the corresponding eigenvectors  $\varphi_k$ . Finally,  $D(\tilde{A}^{\frac{1}{2}}) = H_{\frac{3}{2}}$  follows from [115, Proposition 3.4.8].  $\square$

**Proposition 4.5.4.** *Let  $\tilde{A}$  be the operator introduced in Propositions 4.5.3. Let  $\psi \in H_{\frac{1}{2}}$  and  $\Psi \in \mathcal{H}^1(\Omega)$  be such that*

$$\Psi(x, 0) = \psi(x), \text{ equality in } L^2[0, \pi].$$

Then for every  $\eta \in H_3^b$  we have

$$\langle \tilde{A}\eta, \psi \rangle = \int_{\Omega} \nabla (D(\eta - T\partial_x^2\eta(t, x))) \cdot \nabla \Psi dx dy. \quad (4.91)$$

*Proof.* For every  $\eta \in H_3$ , we first note that  $\eta - T\partial_x^2\eta \in H_1$ , which implies that  $D(\eta - T\partial_x^2\eta) \in \mathcal{H}^1(\Omega)$  (see Lemma 4.2.5). To prove (4.91), we assume that  $\eta \in H_5$ , so that  $\eta - T\partial_x^2\eta \in H_3$  and according to Lemma 4.2.3 we have  $D(\eta - T\partial_x^2\eta) \in C^2(\overline{\Omega})$ , which satisfies Laplace equation with one Dirichlet condition on the top  $D(\eta - T\partial_x^2\eta)(x, 0) = \eta - T\partial_x^2\eta$  and three homogeneous Neumann conditions, in the classical sense. Then (4.91) follows by a simple integration by parts and Proposition 4.5.3. Finally, we conclude, by a density argument, that (4.91) still holds for  $\eta \in H_3^b$ .  $\square$

**Remark 21.** For every  $\eta \in H_1$ ,  $D\eta$  is not in  $\mathcal{H}^2(\Omega)$ . Actually, using the formula 4.24 it is not difficult to see that  $D \in \mathcal{L}(H_{\frac{3}{2}}, \mathcal{H}^2(\Omega))$ . On the other hand, for every  $\eta \in H_{\frac{1}{2}}$ , we do not have  $D\partial_x^2\eta \in \mathcal{H}^1(\Omega)$ . This is also the reason why we need more regular initial data here, so that Definition 4.5.1 makes sense.

Based on Definition 4.5.1, we establish in what follows the well-posedness result via the existence of a well-posed linear control system (for this concept, please refer to Definition 4.3.1) associated with (4.87). We still use the notation  $X$  to represent the state space of the control system, and denote

$$X = H_{\frac{3}{2}} \times H,$$

where  $H$  and  $(H_{\alpha})_{\alpha>0}$  have been defined in (4.17) and (4.19), respectively. We also introduce the linear operator  $A : \mathcal{D}(A) \rightarrow X$  with  $\mathcal{D}(A) = H_3^b \times H_{\frac{3}{2}} = X_1$  and

$$A = \begin{bmatrix} 0 & I \\ -\tilde{A} & 0 \end{bmatrix}, \text{ i.e., } A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -\tilde{A}\varphi \end{bmatrix}, \quad (4.92)$$

for every  $\forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A)$ , and the operator  $\tilde{A}$  in (4.92) is a strictly positive operator on  $H$ , with domain  $H_3^b$ , which has been introduced in Proposition 4.5.3. Therefore,  $A$  is skew-adjoint on  $X$  (see, for instance, [115, Proposition 3.7.6]), so that, according to Stone's theorem (see, for instance [115, Section 3.7]),  $A$  generates a group  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  of unitary operators on  $X$ . Moreover, we define  $B$  by

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

where  $B_0 \in \mathcal{L}(\mathbb{C}, H)$  is as in Corollary 4.2.13. Clearly  $B \in \mathcal{L}(\mathbb{C}, X)$ .

**Remark 22.** In the abstract framework described above, we identify the inner product  $\langle x, z \rangle_{3/2}$  with  $\langle \tilde{A}^{\frac{1}{2}}x, \tilde{A}^{\frac{1}{2}}z \rangle$  for every  $x, z \in H_{\frac{3}{2}}$ , since the eigenvalues of  $\tilde{A}$  are equivalent to  $k^3$  (see (4.90)). Moreover, the inner product of the state space  $X$  we used here is

$$\left\langle \begin{bmatrix} \zeta_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} \zeta_2 \\ \eta_2 \end{bmatrix} \right\rangle_X = \langle \zeta_1, \zeta_2 \rangle_{\frac{3}{2}} + \langle \eta_1, \eta_2 \rangle, \quad (4.93)$$

which is not induced by the natural energy (4.89). Actually, (4.93) is a higher order energy and it is convenient for calculation here.

**Theorem 4.5.5.** *Let  $h$  be as in Definition 4.5.1. Given  $u \in \mathcal{H}_{\text{loc}}^1[0, \infty)$ ,  $\zeta_0 \in H_3^b$  and  $w_0 \in H_{\frac{3}{2}}$ , there exists a unique solution of (4.87) with  $\zeta(0) = \zeta_0$  and  $\dot{\zeta}(0) = w_0$ . Moreover, there exists a well-posed linear control system  $(\mathbb{T}, \Phi)$  with state space  $X = H_{\frac{3}{2}} \times H$  and input space  $\mathbb{C}$  such that, setting  $z = \begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix}$  and  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix}$ , we have*

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u \quad \forall \tau \geq 0. \quad (4.94)$$

Finally, the generator  $A$  of  $\mathbb{T}$  is skew-adjoint, with domain  $\mathcal{D}(A) = H_3^b \times H_{\frac{3}{2}}$ , and there exists  $B \in \mathcal{L}(\mathbb{C}, X)$  such that for any  $\tau \geq 0$ ,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma \quad \forall u \in L_{\text{loc}}^2[0, \infty).$$

*Proof.* With the above notation for  $X$ ,  $A$  and  $B$  we consider, for each  $\tau \geq 0$ , the map  $\Phi_\tau$  is clearly linear and bounded from  $L^2[0, \infty)$  into  $X$ . Let  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix} \in X_1$ , let  $u \in \mathcal{H}_{\text{loc}}^1[0, \infty)$  and define  $z(t) = \begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix}$  by (4.52). Then, according to a classical result (see, for instance, [115, Section 4]),  $z(t) \in C([0, \infty); X_1) \cap C^1([0, \infty); X)$  and it satisfies

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0. \end{cases} \quad (4.95)$$

in the classical sense. For every  $t \geq 0$  and  $\psi \in X$ , we then take an inner product of (4.95) and  $\psi$ . The strategy of the remaining proof of Theorem 4.5.5 is similar with the proof of Theorem 4.3.3, and we omit the details here for simplicity. The only difference lies in Proposition 4.5.4.  $\square$

Now we introduce the stabilization result for the control system on the gravity-capillary waves.

**Theorem 4.5.6.** *Let  $\Sigma = (\mathbb{T}, \Phi)$  be the well-posed linear control system introduced in Theorem 4.5.5. Then  $\Sigma$  is not exponentially stabilizable, but it is strongly stabilizable iff  $h$  is a strategic profile, in the sense that*

$$\int_{-1}^0 h(y) \cosh[k(y+1)] dy \neq 0 \quad \forall k \in \mathbb{N}.$$

*In this case, one strongly stabilizing feedback operator is  $F = -B^*$ . Moreover, if*

$$\inf_{k \in \mathbb{N}} \frac{k}{\cosh k} \left| \int_{-1}^0 h(y) \cosh[k(y+1)] dy \right| > 0, \quad (4.96)$$

## 4.5. THE CASE OF GRAVITY-CAPILLARY WAVES

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the feedback operator  $F = -B^*$  leads to the closed-loop semigroup  $\mathbb{T}^{cl}$  (with generator  $A - BB^*$ ) which is strongly stable and there exists  $M > 0$  such that

$$\|\mathbb{T}_t^{cl} z_0\|_X \leq \frac{M}{(1+t)^{\frac{3}{4}}} \|z_0\|_{\mathcal{D}(A)} \quad z_0 \in \mathcal{D}(A). \quad (4.97)$$

*Proof.* The first part of the proof is similar with the corresponding part in Theorem 4.3.3. Here we just show the proof of the decay rate (4.97). According to the structure described in Lemma 4.4.3, we know from Proposition 4.5.3 that the eigenvalues of the operator  $A$  introduced in (4.92) are  $k^{\frac{3}{2}}$  for  $k \in \mathbb{Z}^*$ . With the condition (4.96), we have

$$|B^* \phi_k| \geq \frac{M_0}{|k|} \quad \forall k \in \mathbb{Z}^*,$$

where  $(\phi_k)_{k \in \mathbb{Z}^*}$  is an orthonormal basis in  $X$  formed of the eigenvectors of  $A$ . Note that there is a gap between the eigenvalues of the operator  $A$ , i.e.  $\inf_{k \in \mathbb{Z}^*} |\mu_{k+1} - \mu_k| > \delta > 0$ . Now we see that the assumptions  $[H_1]$  and  $[H_2]$  in Section 3.1 are satisfied with  $\alpha = \frac{3}{2}$  and  $\beta = 1$ . We can apply Theorem 3.1.2 in Chapter 3 to conclude that the solution of the control system (4.94) with the bounded feedback  $u = -B^*z$  satisfies (4.97).  $\square$

**Remark 23.** Compared with Theorem 4.4.2 and Theorem 4.5.6, as we expected, the energy of the gravity-capillary waves decays faster than the gravity case, because of the surface tension effect.





## Chapter 5

# Asymptotic behaviour of the gravity waves in a rectangular domain

In this chapter, we are interested in the asymptotic analysis of the boundary control system on the gravity waves described in Chapter 4 in the shallow water regime, where the water depth is much smaller than the horizontal scale. To study the asymptotic behaviour of the solution, we derive the corresponding dimensionless governing equations by using the shallowness parameter and some dimensionless quantities, which are introduced below. Combined with the well-posedness result in Chapter 4, we justify the convergence of the solution of the water waves system in terms of the elevation in appropriate sense when taking the shallowness limit. The contents in this chapter are mainly based on the work in Su [107].

### 5.1 Introduction and dimensionless equations

We study the asymptotic behaviour of a system describing small-amplitude water waves in a rectangular domain, in the presence of a wave maker, where the horizontal scale  $L$  is much larger than the typical water depth  $h_0$ . The construction of the water waves model begins from the so-called *Zakharov-Craig-Sulem formulation* (ZCS), which is a fully nonlinear and fully dispersive model in terms of the elevation of the free surface and the free surface velocity potential (see, for instance, Lannes' book [66]). Based on some assumptions on the nonlinearity and the topography of the fluid domain, described by the *shallowness parameter*

$$\mu = \frac{h_0^2}{L^2}, \quad (5.1)$$

there are many asymptotic models in the shallow water regime. *The nonlinear shallow water equations* is an approximation of ZCS where all the terms of order  $O(\mu)$  are dropped, so that it is a fully nonlinear and non-dispersive model. Moreover, *the Boussinesq equations* is an approximation of ZCS of order  $O(\mu^2)$  with the weak nonlinearity assumption. The full justification (convergence) of the shallow water approximation of ZCS models mentioned above are provided in [66, Chapter 5 and Chapter 6] by considering the corresponding Cauchy problem in a strip domain that is unbounded in the horizontal direction. For other interesting asymptotic models, please refer to Lannes [66], [68] and also therein.

Here, instead of considering a fluid filling an infinite strip, we consider the similar topic on the linearized water waves equation in a rectangular domain with a wave maker applied from one lateral boundary. Our aim is to describe the dynamics of this system when the shallowness parameter  $\mu$  tends to zero. Now let us precisely state the problem.

The domain  $\Omega$  is bounded by a top free surface  $\Gamma_s$  and a flat bottom  $\Gamma_f$ . The other two components of the fluid domain, denoted by  $\Gamma_1$  and  $\Gamma_2$ , are vertical walls, see Figure 5.1. The fluid filling the rectangular domain

$$\Omega = \{(x, y) \mid (x, y) \in (0, \pi L) \times (-h_0, 0)\}$$

is assumed to be homogeneous, incompressible, inviscid and irrotational. There is a wave maker that acts at the left boundary of  $\Omega$ , by imposing the acceleration of the fluid in the horizontal direction, as a scalar input signal  $u$ .

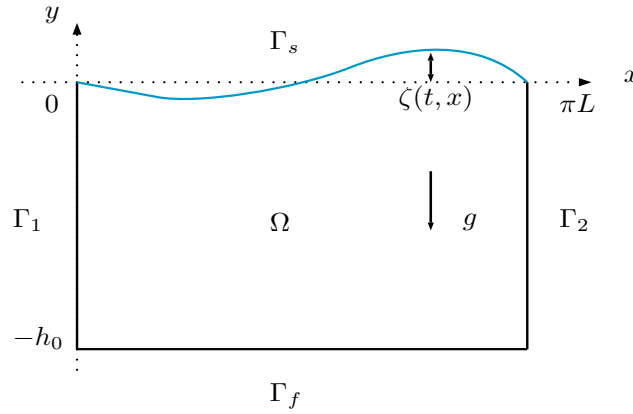


Figure 5.1: A rectangular domain  $\Omega$  filled with water

We consider the water waves system in the shallow water configurations, in the sense that  $\mu \ll 1$ . The governing equations of the above water waves system has been introduced in (4.15). In order to study the asymptotic behaviour of this system, we define the following dimensionless quantities,

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{h_0}, \quad \bar{t} = \frac{t}{L/\sqrt{gh_0}}, \quad \bar{\zeta} = \frac{\zeta}{a}, \quad \bar{\phi} = \frac{\phi}{aL\sqrt{g/h_0}}, \quad (5.2)$$

where  $a$  is the order of the surface variation,  $\phi$  is the velocity potential of the fluid,  $\zeta$  is the elevation of the top free surface and  $g$  represents the gravity acceleration. The quantities in (5.2) marked with a overline are their corresponding dimensionless version. With the variables  $\bar{x}$  and  $\bar{y}$ , the dimensionless domain, denoted by  $\bar{\Omega}$ , is

$$\bar{\Omega} = \{(\bar{x}, \bar{y}) \mid (\bar{x}, \bar{y}) \in (0, \pi) \times (-1, 0)\}. \quad (5.3)$$

For the sake of simplicity, we omit the overlines in what follows and from now we always use the dimensionless quantities. Moreover, to avoid any confusion we use the notation  $\zeta_\mu$  and  $\phi_\mu$ , instead of  $\zeta$  and  $\phi$ , to represent the unknown functions in the dimensionless equation. The dimensionless governing equations of the water waves

system, for all  $t \geq 0$  and for  $(x, y) \in \Omega$ , are

$$\begin{cases} \Delta_\mu \phi_\mu(t, x, y) = 0, \\ \partial_t \zeta_\mu(t, x) - \frac{1}{\mu} \partial_y \phi_\mu(t, x, 0) = 0, \\ \partial_t \phi_\mu(t, x, 0) + \zeta_\mu(t, x) = 0, \\ \partial_x \phi_\mu(t, 0, y) = -h(y)v(t), \\ \partial_y \phi_\mu(t, x, -1) = 0 = \partial_x \phi_\mu(t, \pi, y), \end{cases} \quad (5.4)$$

where  $v$  is the velocity produced by the wave maker. In the above equations  $\Delta_\mu$  defined by

$$\Delta_\mu = \mu \partial_x^2 + \partial_y^2$$

is called the "twisted" Laplace operator (see [66]), and the function  $h$  represents the profile of the velocity imposed by the wave maker. The system (5.4) is actually a fully linear and fully dispersive approximation of ZCS constrained in a rectangle.

The controllability properties of the system derived by (5.4), as far as we know, are firstly studied in Russell and Reid [96] and further in Mottelet [85]. Now we recall here some recent works on the similar problem. Different with the control introduced in the system (5.4), Alazard discussed in [2] the stabilization of the nonlinear water waves system in a rectangle where the external pressure as the control signal acts on a part of the free surface, by absorbing the waves coming from the left. For the problem in a cubic domain, in an irregular domain and the case of the water waves with surface tension, please refer to Reid [94] and [95], Craig et al. [36], Alazard et al. [4] and [1]. Recently, for  $u \in L^2_{\text{loc}}[0, \infty)$ , we established in our paper [110] the well-posedness of the system (5.4), and further showed that it can be recast as a well-posed linear control system (for this concept, please refer to [110], Weiss [119] or Tucsnak and Weiss [115]).

Observe that the free surface equations of (5.4) determine the whole system, which means that if we know  $\psi_\mu(t, x) = \phi_\mu(t, x, 0)$ , thereby the velocity potential  $\phi_\mu$  can be obtained by solving a boundary value problem for Laplacian. As explained in Chapter 4, with the help of the Dirichlet to Neumann and the Neumann to Neumann operators, the system (5.4) reduces to a second-order evolution equation in terms of  $\zeta_\mu$ . We thus propose the corresponding initial data

$$\zeta_\mu(0, x) = \zeta_0(x), \quad \partial_t \zeta_\mu(0, x) = \zeta_1(x). \quad (5.5)$$

Therefore, we take the acceleration  $u = \dot{v}$  as the input signal. We will provide in the following sections more details about the formulation of the dimensionless governing equations (5.4).

Intuitively, the vertical dependence of the velocity potential becomes negligible when the fluid domain becomes thinner and thinner in the vertical direction. From another point of view, the dispersion relation (that is the relation between  $\omega$  and  $\kappa$  when the solution takes the form  $e^{i(\kappa x - \omega t)}$ ) of the linearized water waves is  $\omega^2 = g\kappa \tanh \kappa h_0$ , where  $\omega$  is the angular frequency,  $\kappa$  is the wave number and  $h_0$  is the typical depth of the fluid domain. For more details about this, we refer to [66, Chapter 1] and to Whitham [123, Chapter 13]). It is obvious that the dispersion relation is approximately  $\omega^2 \sim gh_0 \kappa^2$  as  $\kappa h_0 \rightarrow 0$  and the phase speed  $c_0 = \sqrt{gh_0}$  becomes independent of  $\kappa$ . The dispersive effects drop out in this limit, and in one dimension, this is exactly the property of the wave equation.

In Chapter 4 we assumed that the shape function  $h$  satisfies zero mean condition, i.e.

$$\int_{-1}^0 h(y) dy = 0,$$

to ensure the conservation of the volume of the water. This condition should be removed in this part since the system under consideration is in the shallow water regime, where the velocity of the fluid is independent of the vertical variable (see, for instance, [66] and [123]). In this case,  $h$  should be a constant, which means that the velocity or the acceleration is homogeneously imposed by the wave maker from the left edge. Without loss of generality, we might as well take  $h = 1$ .

The main contribution brought in by this chapter is that we justify the passage to the limit from the linear water waves system (5.4) to the one dimensional wave equation with boundary control (i.e. showing that, in an appropriate sense,  $\zeta_\mu \rightarrow \zeta$ ) with the same initial data  $\zeta_0$  and  $\zeta_1$ , as the shallowness parameter  $\mu$  goes to zero. Our approach is based on the famous Trotter-Kato approximation theorem (see, for instance, [88, Chapter 3]) and a special change of variables, as well as a detailed analysis of Fourier series. Moreover, a scattering semigroup discussed in [105] and [50] provides the possibility for us to apply the Trotter-Kato theorem to control systems.

We derive, in Section 5.2, the dimensionless Dirichlet to Neumann and Neumann to Neumann operators. Next we do some preparations in Section 5.3 and propose a change of variables to rewrite the control system, which allows us to apply the Trotter-Kato theorem. In Section 5.4 we prove two important convergence results on the resolvent of the evolution operators. Finally, in Section 5.5 we state the main results and focus on the proof of the main results.

## 5.2 Dimensionless Dirichlet to Neumann and Neumann to Neumann operators

In this section we derive the dimensionless form of the Dirichlet to Neumann and Neumann to Neumann maps, using the dimensionless quantities introduced in (5.2). We provided in Chapter 4 a detailed construction of these two important operators allowing us to recast (5.4) as a well-posed linear control system. Following Section 4.2 in Chapter 4, we go back to the definition of these two operators, which is closely related to two boundary value problems for the Laplace operator in the rectangular domain  $\Omega$ . Note that,  $\Omega$  being a rectangle, we use separation of variables and detailed analysis of Fourier series to construct the dimensionless version of all related operators.

Recalling the dimensionless quantities introduced in (5.2), it is not difficult to see that we have

$$\partial x = \frac{1}{L} \partial \bar{x}, \quad \partial y = \frac{1}{h_0} \partial \bar{y}, \quad \partial t = \frac{\sqrt{gh_0}}{L} \partial \bar{t}.$$

Based on the above relations, we define the "twisted" gradient and Laplace operators as follows ( $\mu$  is given by (5.1)):

$$\nabla_\mu = (\sqrt{\mu} \partial \bar{x}, \partial \bar{y}), \quad \Delta_\mu = \mu \partial \bar{x}^2 + \partial \bar{y}^2.$$

**Remark 24.** The domain in this section is the one defined in (5.3) and we still denote it by  $\Omega$  for simplicity.

## 5.2. DIMENSIONLESS DIRICHLET TO NEUMANN AND NEUMANN TO NEUMANN OPERATORS

We present in this part the definition and some important remarks on the Dirichlet and Neumann maps in dimensionless version. Moreover, we state several results on the properties of these maps. The proofs of these results can be obtained by slight variations of the proofs of the corresponding results in Chapter 4, so that we omit the details here.

Here we still use the notation introduced in Section 4.2. In this chapter we set  $H = L^2[0, \pi]$  (without zero mean condition now). It is known that the family  $(\varphi_k)_{k \geq 0}$  introduced in (4.18) forms an orthonormal basis in  $H$ . The inner product in  $H$  is denoted by  $\langle \cdot, \cdot \rangle$  and the associated norm by  $\|\cdot\|$ . The scale of Hilbert spaces  $(H_\alpha)_{\alpha \in \mathbb{R}}$  are defined as in (4.19) with the inner products  $\langle \eta, \psi \rangle_\alpha = \sum_{k \geq 0} (1 + k^{2\alpha}) \langle \eta, \varphi_k \rangle \langle \psi, \varphi_k \rangle$ , for all  $\eta, \psi \in H_\alpha$ . Moreover, for every  $\alpha \in \mathbb{R}$ ,  $H_{-\alpha}$  is the dual space of  $H_\alpha$  with respect to the pivot space  $H$ .

We note that, for fixed  $\mu \ll 1$ , the properties of the evolution operators introduced in Section 4.2 still hold. In this section, we denoted by  $D_\mu$  the dimensionless version of the partial Dirichlet map and by  $N_\mu$  the dimensionless version of the partial Neumann map introduced in Proposition 4.2.2 and Proposition 4.2.6, respectively. We give in what follows the formula of  $D_\mu$  and  $N_\mu$ , as well as the properties of the dimensionless Dirichlet to Neumann and the dimensionless Neumann to Neumann operators.

**Lemma 5.2.1.** *For every  $\eta \in H$ ,  $D_\mu \eta$  is given, for every  $x, y \in \Omega$ , by*

$$(D_\mu \eta)(x, y) = \sum_{k \geq 0} \frac{\langle \eta, \varphi_k \rangle}{\cosh(\sqrt{\mu}k)} \varphi_k(x) \cosh[\sqrt{\mu}k(y+1)],$$

where the functions  $\varphi_k$  have been defined in (4.18).

**Lemma 5.2.2.** *For every  $v \in L^2[-1, 0]$  and every  $(x, y) \in \Omega$  we have*

$$(N_\mu v)(x, y) = \sum_{k \in \mathbb{N}} a_k \cosh \left[ \frac{(2k-1)}{2\sqrt{\mu}} \pi(x-\pi) \right] \cos \left[ (2k-1) \frac{\pi}{2} (y+1) \right],$$

where

$$a_k = \frac{2\sqrt{2\mu} \langle v, \psi_k \rangle}{(2k-1)\pi \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi^2 \right]} \quad \forall k \in \mathbb{N}.$$

**Proposition 5.2.3.** *Let  $A_\mu$  be the dimensionless Dirichlet to Neumann operator. Then  $A_\mu : H_1 \rightarrow H$  and we have  $A_\mu \varphi_k = \lambda_{\mu,k} \varphi_k$  with  $(\lambda_{\mu,0} = 0)$*

$$\lambda_{\mu,k} = \sqrt{\mu}k \tanh(\sqrt{\mu}k) \quad \forall k \in \mathbb{N},$$

and

$$A_\mu \eta = \sum_{k \in \mathbb{N}} \lambda_{\mu,k} \langle \eta, \varphi_k \rangle \varphi_k \quad \forall \eta \in H_1.$$

**Remark 25.** The dimensionless Dirichlet to Neumann map  $A_\mu$  introduced in Proposition 5.2.3 is positive, but not strictly positive, which is the difference with the one discussed in Section 4.2.2. This is induced, as explained in the introduction, by removing zero mean condition from the state space, so that the system fits in the shallow water configurations.

**Proposition 5.2.4.** *Let  $B_\mu$  be the dimensionless Neumann to Neumann operator. Then the operator  $B_\mu$  belongs to  $\mathcal{L}(\mathbb{C}, H)$  and for every  $u \in \mathbb{C}$*

$$(B_\mu u)(x) = u \sum_{k \in \mathbb{N}} b_{\mu,k} \cosh \left[ \frac{(2k-1)}{2\sqrt{\mu}} \pi(x-\pi) \right],$$

where

$$b_{\mu,k} = \frac{-4\sqrt{\mu}}{(2k-1)\pi \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi^2 \right]}.$$

The proofs of Proposition 5.2.3 and Proposition 5.2.4 are completely similar with the corresponding ones for the usual Dirichlet to Neumann and Neumann to Neumann maps (with dimension) in Section 4.2. Therefore, we omit the details here. Next we introduce a convergence property on the Neumann to Neumann map  $B_\mu$ , which plays an important role in our arguments.

**Theorem 5.2.5.** *Let  $B_w = -\delta_0$ , where  $\delta_0$  is the Dirac mass concentrated at  $x = 0$  and let  $B_\mu$  be the Neumann to Neumann map defined in Proposition 5.2.4. Then we have*

$$\lim_{\mu \rightarrow 0} \left\| \frac{1}{\mu} B_\mu u - B_w u \right\|_{H_{-1}} = 0 \quad \forall u \in \mathbb{C}, \quad (5.6)$$

where  $H_{-1}$  is the dual of  $H_1$  with respect to the pivot space  $H$ .

*Proof.* One readily sees that, equivalently, we need to show that for every  $u \in \mathbb{C}$  and for every  $\phi \in H_1$  with  $\|\phi\|_{H_1} \leq 1$ ,

$$\lim_{\mu \rightarrow 0} \sup_{\|\phi\|_{H_1} \leq 1} \left| \left\langle \frac{1}{\mu} B_\mu u - B_w u, \phi \right\rangle \right| = 0. \quad (5.7)$$

According to Proposition 5.2.4, we have

$$\frac{1}{\mu} B_\mu u = u \sum_{k \in \mathbb{N}} c_{\mu,k} \cosh \left[ \frac{(2k-1)}{2\sqrt{\mu}} \pi(x-\pi) \right], \quad (5.8)$$

where

$$c_{\mu,k} = \frac{-4}{\sqrt{\mu}(2k-1)\pi \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi^2 \right]}.$$

We denote

$$f_{\mu,k}(x) = \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi x \right],$$

and obtain by using integration by parts that

$$\begin{aligned} \int_0^\pi \cosh \left[ \frac{(2k-1)}{2\sqrt{\mu}} \pi(x-\pi) \right] \phi(x) dx = \\ \frac{2\sqrt{\mu}}{(2k-1)\pi} \left\{ \phi(0) f_{\mu,k}(\pi) - \int_0^\pi \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi(x-\pi) \right] \phi'(x) dx \right\}. \end{aligned} \quad (5.9)$$

Furthermore, note that

$$\int_0^\pi \sinh \left[ \frac{2k-1}{2\sqrt{\mu}} \pi(x-\pi) \right] \phi'(x) dx = e^{-\frac{2k-1}{2\sqrt{\mu}} \pi^2} \int_0^\pi f_{\mu,k}(x) \phi'(x) dx - f_{\mu,k}(\pi) \int_0^\pi e^{-\frac{2k-1}{2\sqrt{\mu}} \pi x} \phi'(x) dx,$$

we thus have the following estimate (using (5.9))

$$\left| \left\langle \frac{1}{\mu} B_\mu u - B_w u, \phi \right\rangle \right| \leq \sum_{k \in \mathbb{N}} \frac{32\mu|u|}{(2k-1)^4 \pi^6} e^{-\frac{2k-1}{2\sqrt{\mu}} \pi^2} \int_0^\pi |f_{\mu,k}(x) \phi'(x)| dx + \sum_{k \in \mathbb{N}} \frac{8|u|}{(2k-1)^2 \pi^2} \int_0^\pi \left| e^{-\frac{2k-1}{2\sqrt{\mu}} \pi x} \phi'(x) \right| dx. \quad (5.10)$$

In the above estimate, we used the fact  $\sum_{k \in \mathbb{N}} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$  and  $\sinh x \geq x^2$  for large  $x$ .

Note that there exists a constant  $C > 0$ , such that  $e^{-\frac{2k-1}{2\sqrt{\mu}} \pi^2} f_{\mu,k}(\pi) \leq C$  uniformly with respect to  $\mu$  and  $k$ , we immediately obtain that

$$\sum_{k \in \mathbb{N}} \frac{\mu}{(2k-1)^4} e^{-\frac{2k-1}{2\sqrt{\mu}} \pi^2} \int_0^\pi |f_{\mu,k}(x) \phi'(x)| dx \leq C \sum_{k \in \mathbb{N}} \frac{\mu}{(2k-1)^4} \|\phi'\| \leq C\mu.$$

Moreover, since  $\left\| e^{-\frac{2k-1}{2\sqrt{\mu}} \pi x} \right\|^2 \leq C \frac{2\sqrt{\mu}}{(2k-1)\pi}$  we have

$$\sum_{k \in \mathbb{N}} \frac{1}{(2k-1)^2} \int_0^\pi \left| e^{-\frac{2k-1}{2\sqrt{\mu}} \pi x} \phi'(x) \right| dx \leq C \sum_{k \in \mathbb{N}} \frac{\mu^{\frac{1}{4}}}{(2k-1)^{\frac{5}{2}}} \|\phi'\| \leq C\mu^{\frac{1}{4}}.$$

Therefore, we conclude that, for every fixed  $u \in \mathbb{C}$ , the right-hand side of (5.10) can be controlled by  $C\mu^{\frac{1}{4}}$ , which clearly implies (5.7).  $\square$

### 5.3 Operator form of the governing equations

In this section, we formulate the governing equations (5.4) as a well-posed LTI (linear time-invariant) system in an appropriate Hilbert space. To this aim, we first define a scale of Hilbert spaces associated with a certain operator and then derive the dimensionless control system related to (5.4), finally formulate the control system into the one that allows us to apply the Trotter-Kato approximation theorem in Section 5.5.

For a self-adjoint positive operator  $A : \mathcal{D}(A) \rightarrow H$  with compact resolvents, according to the classical results (see, for instance, [115, Chapter 3]), the operator  $A$  is diagonalizable, also called Riesz-spectral operator in some literature (for instance in [40]), with an orthonormal basis  $(\varphi_k)_{k \geq 0}$  of eigenvectors and the corresponding positive eigenvalues  $(\lambda_k)_{k \geq 0}$ . For any  $z \in H$ , we denote  $z_k = \langle z, \varphi_k \rangle$ . Now we use the framework introduced in Section 3.4.1 for the strictly positive operator  $A$ . It is obvious to see that, for every  $\alpha \geq 0$ , Hilbert space  $H_\alpha$  is actually the domain of the operator  $A^\alpha$  with its graph norm  $\|\cdot\|_{gr}$ . Furthermore, for every  $\alpha \in \mathbb{R}$ ,  $H_{-\alpha}$  is the dual space of  $H_\alpha$  with respect to the pivot space  $H$ . We will apply, in the following part, the definition of a scale of Hilbert spaces to different operators.



Next we formulate the equations (5.4) into a second-order evolution equation in terms of  $\zeta_\mu$ . Recalling the definition of the Dirichlet to Neumann map  $A_\mu$  and the Neumann to Neumann map  $B_\mu$  in Section 5.2, we immediately obtain from the structure of the governing equations (5.4) that

$$\partial_y \phi_\mu(t, x, 0) = A_\mu \psi_\mu(t, x) + B_\mu v(t),$$

where  $t \geq 0$ ,  $x \in (0, \pi)$  and  $\psi_\mu(t, x) = \phi_\mu(t, x, 0)$ . Taking the derivative of the second equation in (5.4) with respect to time and eliminating  $\psi_\mu(t, x)$  by using the third equation of (5.4), we get the second-order control system associated with (5.4), i.e. for all  $t \geq 0$ ,  $x \in (0, \pi)$ ,

$$\begin{cases} \partial_t^2 \zeta_\mu(t, x) + \frac{1}{\mu} A_\mu \zeta_\mu(t, x) = \frac{1}{\mu} B_\mu u(t), \\ \zeta_\mu(0, x) = \zeta_0(x), \quad \partial_t \zeta_\mu(0, x) = \zeta_1(x), \end{cases} \quad (5.11)$$

where  $u = \dot{v}$  is the input signal, the operators  $A_\mu$  and  $B_\mu$  are defined in Proposition 5.2.3 and Proposition 5.2.4, respectively.

With the same initial data as in (5.11), we introduce the following wave equation defined on  $(0, \pi)$  with Neumann boundary control, i.e. for all  $t \geq 0$ ,  $x \in (0, \pi)$ ,

$$\begin{cases} \partial_t^2 \zeta(t, x) - \partial_x^2 \zeta(t, x) = 0, \\ \partial_x \zeta(t, 0) = u(t), \quad \partial_x \zeta(t, \pi) = 0, \\ \zeta(0, x) = \zeta_0(x), \quad \partial_t \zeta(0, x) = \zeta_1(x). \end{cases} \quad (5.12)$$

Moreover, we introduce the operator  $A_w : \mathcal{D}(A_w) \rightarrow H$  as follows:

$$A_w = -\frac{d^2}{dx^2} \quad \mathcal{D}(A_w) = \left\{ f \in \mathcal{H}^2[0, \pi] \mid \frac{df}{dx}(0) = \frac{df}{dx}(\pi) = 0 \right\}. \quad (5.13)$$

With the operators  $B_w$  defined in Theorem 5.2.5, we write the equations (5.12) in operator form as follows, i.e. for all  $t \geq 0$ ,  $x \in (0, \pi)$ ,

$$\begin{cases} \partial_t^2 \zeta(t, x) + A_w \zeta(t, x) = B_w u(t), \\ \zeta(0, x) = \zeta_0(x), \quad \partial_t \zeta(0, x) = \zeta_1(x). \end{cases} \quad (5.14)$$

It is known that the operator  $A_w$  defined in (5.13) is diagonalizable with the eigenvalues  $k^2$  and the corresponding eigenvectors  $\varphi_k$  are given in (4.18). For the operator  $A_w$ , we denote by  $\mathbb{H}_\alpha$  with  $\alpha \in \mathbb{R}$  the scale of Hilbert spaces which has been introduced in Section 3.4.1. Notice that the Dirichlet to Neumann operator  $A_\mu$  in Proposition 5.2.3 is also diagonalizable, so that, for  $\alpha \in \mathbb{R}$ , we denote by  $H_{\mu, \alpha}$  the scale of Hilbert spaces associated with the operator  $\frac{1}{\mu} A_\mu$ . More precisely, for every  $\alpha \in \mathbb{R}$ , we have

$$\begin{aligned} H_{\mu, \alpha} &= \left\{ f \in H \mid \sum_{k \geq 0} \left( 1 + \left| \frac{k \tanh(\sqrt{\mu k})}{\sqrt{\mu}} \right|^{2\alpha} \right) |\langle f, \varphi_k \rangle|^2 < \infty \right\}, \\ \mathbb{H}_\alpha &= \left\{ f \in H \mid \sum_{k \geq 0} \left( 1 + |k|^{4\alpha} \right) |\langle f, \varphi_k \rangle|^2 < \infty \right\}. \end{aligned} \quad (5.15)$$

### 5.3. OPERATOR FORM OF THE GOVERNING EQUATIONS

Therefore, we have  $\mathbb{H}_0 = \mathbb{H} = H = H_{\mu,0}$  and  $\mathbb{H}_{-\alpha}$  (or  $H_{\mu,-\alpha}$ ) is the dual space of  $\mathbb{H}_\alpha$  (or  $H_{\mu,\alpha}$ ) with respect to the pivot space  $H$ . It is not difficult to see that actually we have  $\mathbb{H}_{\frac{1}{2}} = \mathcal{H}^1[0, \pi]$ . For more details on a scale of Hilbert space, please refer to [115, Chapter 2].

We mention that the operator  $B_\mu$  is bounded (see Proposition 5.2.4), i.e.  $B_\mu \in \mathcal{L}(\mathbb{C}, H)$ , and  $B_w$  induces an admissible control operator in the first-order system associated to (5.14) with the state  $\begin{bmatrix} \zeta \\ \partial_t \zeta \end{bmatrix}$  (please refer to [115, Proposition 6.2.5]), although it is unbounded (not contained in the state space), i.e.  $B_w \in \mathcal{L}(\mathbb{C}, \mathbb{H}_{-\frac{1}{2}})$ .

**Remark 26.** According to Proposition 5.2.3, the eigenvalues of  $\frac{1}{\mu}A_\mu$  denoted by  $\lambda_{\mu,k}$  are  $\frac{1}{\sqrt{\mu}}k \tanh(\sqrt{\mu}k)$ , which is equivalent to  $k$  for fixed  $\mu \in (0, 1)$ . Therefore, for every  $\alpha \geq 0$ , Hilbert space  $H_{\mu,\alpha}$  is actually equivalent to  $H_\alpha$  introduced in (4.19). Moreover, according to interpolation theory (see, for instance, [74], [19, Part II] and [31]), for  $\alpha \in (0, 1)$ , the scale of Hilbert space  $H_\alpha$  is exactly the classical Sobolev space  $\mathcal{H}^\alpha[0, \pi]$ .

**Remark 27.** The initial boundary value problem (5.12) is a well-posed boundary control system (for this concept, see for instance [115, Chapter 10]), which is equivalent to (5.14) in weak sense, that is, for every  $u \in L^2_{\text{loc}}[0, \infty)$ , for every  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$  and  $\zeta_1 \in \mathbb{H}$ , there exists a unique function

$$\zeta \in C([0, \infty); \mathbb{H}_{\frac{1}{2}}) \cap C^1([0, \infty); \mathbb{H}),$$

such that  $\zeta(0, x) = \zeta_0$  and it satisfies, for every  $t \geq 0$  and every  $\psi \in \mathbb{H}_{\frac{1}{2}}$ ,

$$\begin{aligned} \int_0^\pi \partial_t \zeta(t, x) \overline{\psi(x)} dx - \int_0^\pi \zeta_1(x) \overline{\psi(x)} dx \\ = - \int_0^t \int_0^\pi \partial_x \zeta(\sigma, x) \overline{\frac{d\psi}{dx}(x)} dx d\sigma - \int_0^t u(\sigma) \overline{\psi(0)} d\sigma. \end{aligned}$$

In what follows, we are ready to study the asymptotic behaviour of the system (5.11) when  $\mu$  goes to zero. We shall consider the relationship between the solutions of (5.11) and (5.14). Normally, the state of the control system is taken as  $\begin{bmatrix} \zeta \\ \partial_t \zeta \end{bmatrix}$ , but the main problem lies in the difference of the energy space of (5.11) and (5.14), one is  $H_{\mu, \frac{1}{2}} \times H$  and the other is  $\mathbb{H}_{\frac{1}{2}} \times \mathbb{H}$ . It means that we cannot apply the Trotter-Kato theorem directly. According to the classical semigroup theory (see, for instance, [115] and [40]), for  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$  and  $\zeta_1 \in \mathbb{H}$ , (5.11) and (5.14) admit unique solutions  $\zeta_\mu$  and  $\zeta$ , respectively, which satisfy

$$\zeta_\mu \in C([0, \infty); H_{\mu, \frac{1}{2}}) \cap C^1([0, \infty); H),$$

and

$$\zeta \in C([0, \infty); \mathbb{H}_{\frac{1}{2}}) \cap C^1([0, \infty); \mathbb{H}).$$

We thus consider the following new variables:

$$\alpha_\mu := \partial_t \zeta_\mu \quad \beta_\mu := \left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_\mu, \quad (5.16)$$

and

$$\alpha := \partial_t \zeta \quad \beta := A_w^{1/2} \zeta, \quad (5.17)$$

where  $A_w$  and  $A_\mu$  are introduced in (5.13) and Proposition 5.2.3, respectively. In this way, we have  $\alpha_\mu, \beta_\mu \in C([0, \infty); H)$  and  $\alpha, \beta \in C([0, \infty); \mathbb{H})$ . Setting

$$w_\mu(t) = \begin{bmatrix} \alpha_\mu(t, \cdot) \\ \beta_\mu(t, \cdot) \end{bmatrix} \quad \text{and} \quad w(t) = \begin{bmatrix} \alpha(t, \cdot) \\ \beta(t, \cdot) \end{bmatrix},$$

we obtain from (5.11) and (5.14) that

$$\begin{cases} \frac{dw_\mu}{dt}(t) = \mathcal{A}_\mu w_\mu(t) + \mathcal{B}_\mu u(t), \\ w_\mu(0) = w_{\mu,0}, \end{cases} \quad (5.18)$$

and

$$\begin{cases} \frac{dw}{dt}(t) = \mathcal{A}_0 w(t) + \mathcal{B}_0 u(t), \\ w(0) = w_0, \end{cases} \quad (5.19)$$

where

$$\mathcal{A}_\mu = \begin{bmatrix} 0 & -\left(\frac{1}{\mu} A_\mu\right)^{1/2} \\ \left(\frac{1}{\mu} A_\mu\right)^{1/2} & 0 \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} 0 & -A_w^{1/2} \\ A_w^{1/2} & 0 \end{bmatrix}, \quad (5.20)$$

$$\mathcal{B}_\mu = \begin{bmatrix} \frac{1}{\mu} B_\mu \\ 0 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad (5.21)$$

and

$$w_{\mu,0} = \begin{bmatrix} \zeta_1 \\ \left(\frac{1}{\mu} A_\mu\right)^{1/2} \zeta_0 \end{bmatrix}, \quad w_0 = \begin{bmatrix} \zeta_1 \\ A_w^{1/2} \zeta_0 \end{bmatrix}. \quad (5.22)$$

Let  $X = H \times H$ , then the operator  $\mathcal{A}_\mu : \mathcal{D}(\mathcal{A}_\mu) \rightarrow X$  with  $\mathcal{D}(\mathcal{A}_\mu) = H_{\mu, \frac{1}{2}} \times H_{\mu, \frac{1}{2}}$  and  $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \rightarrow X$  with  $\mathcal{D}(\mathcal{A}_0) = \mathbb{H}_{\frac{1}{2}} \times \mathbb{H}_{\frac{1}{2}}$ . Furthermore, it is not difficult to see that  $\mathcal{B}_\mu \in \mathcal{L}(\mathbb{C}, X)$  and  $\mathcal{B}_0 \in \mathcal{L}(\mathbb{C}, \mathbb{H}_{-\frac{1}{2}} \times \mathbb{H}_{-\frac{1}{2}})$ . With the help of the new variables defined in (5.16) and (5.17), the control systems we are now focusing on are (5.18) and (5.19), which possess the same state space  $X$  and provide the possibility to apply the Trotter-Kato theorem.

**Remark 28.** Note that the operator  $A_\mu$  is not onto, the constant part will be removed after doing the transform (5.16) and (5.17). But this is not a problem for justifying the limit since we only need the convergence of the resolvent at a nonzero resolvent point. Moreover, the water waves system and the wave equation have the same initial data, we could obtain the convergence of  $\zeta_\mu$  from the convergence of  $\alpha_\mu$  and  $\beta_\mu$ . For more details about this, please refer to the proof of the main result.

Based on the structure of the operators  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$ , we introduce the following lemma, which is probably known in the semigroup community. However, for the sake of completeness (and with no claim of originality) we give here its precise statement and a short proof. For simplicity, we denote by  $R(\lambda : A) = (\lambda I - A)^{-1}$  the resolvent of  $A$  with  $\lambda \in \rho(A)$  (resolvent set of  $A$ ).

### 5.3. OPERATOR FORM OF THE GOVERNING EQUATIONS

**Lemma 5.3.1.** *Let the operator  $A : \mathcal{D}(A) \rightarrow H$  be positive (i.e.  $A \geq 0$ ) with compact resolvents. Then the operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow X$  defined by*

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}),$$

$$\mathcal{A} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -A^{1/2}\psi \\ A^{1/2}\varphi \end{bmatrix}, \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(\mathcal{A}),$$

generates a unitary group on  $X$ .

*Proof.* The operator  $\mathcal{A}$  is obviously skew-symmetric since  $\operatorname{Re}\langle \mathcal{A}w, w \rangle_X = 0$  for all  $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{D}(\mathcal{A})$ . Note that, for every  $\begin{bmatrix} f \\ g \end{bmatrix} \in X$ , there exists  $\varphi$  and  $\psi$  defined by

$$\varphi = -R(-1 : A)(f + A^{1/2}g), \quad \psi = R(-1 : A)(-g + A^{1/2}f),$$

satisfy  $\varphi, \psi \in \mathcal{D}(A^{1/2})$  and

$$(I + \mathcal{A}) \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \quad (5.23)$$

Indeed, note that since  $A$  is positive, then  $\sigma(A) \subset [0, \infty)$ , which implies  $-1 \in \rho(A)$ , so that the operator  $I + A$  is invertible. Next we show that  $\varphi, \psi \in \mathcal{D}(A^{1/2})$ . The positive operator  $A^{1/2} : H_{\frac{1}{2}} \rightarrow H$  has a unique extension (still denoted by  $A^{1/2}$ ) such that  $A^{1/2} \in \mathcal{L}(H, H_{-\frac{1}{2}})$ , where the Hilbert spaces  $H_s$  ( $s \in \mathbb{R}$ ) is the scale of Hilbert space associated with the operator  $A$ . Moreover,  $A : H_1 \rightarrow H$  also has a unique extension such that  $A \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ , which implies that  $R(-1 : A) \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$ . Thus, for every  $g \in H$ ,  $R(-1 : A)A^{1/2}g \in H_{\frac{1}{2}}$ . Since  $A$  is positive with compact resolvents we obtain that  $A$  is diagonalizable. According to the properties of diagonalizable operator (see, for instance, [115, Section 3.6]), it is straight to verify that  $R(-1 : A)$  commutes with  $A^{1/2}$ , i.e.  $R(-1 : A)A^{1/2}f = A^{1/2}R(-1 : A)f$ , for every  $f \in H$ . Therefore,  $\varphi$  and  $\psi$  defined in the above formally satisfy (5.23). It follows that  $I + \mathcal{A}$  is onto.

Similarly, we get  $I - \mathcal{A}$  is also onto. Then  $\mathcal{A}$  is skew-adjoint on  $X$  (see, for instance, [115, Proposition 3.7.3]), so that, according to Stone's theorem,  $\mathcal{A}$  generates a group of unitary operators on  $X$ .  $\square$

**Remark 29.** The proof of Lemma 5.3.1 we present here, according to the definition in [115], is a direct way to show that the block operator  $\mathcal{A}$  is a skew-adjoint operator. There are of course some simplified proofs, for instance, we do a unitary similarity transformation  $Q \in \mathcal{L}(H \times H)$  defined by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}.$$

It is not difficult to check that

$$Q^{-1} = Q^* = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -iI \\ I & iI \end{bmatrix},$$

and

$$Q^* \begin{bmatrix} 0 & -A^{1/2} \\ A^{1/2} & 0 \end{bmatrix} Q = \begin{bmatrix} -iA^{1/2} & 0 \\ 0 & iA^{1/2} \end{bmatrix}. \quad (5.24)$$

Note that the matrix operator on the right side of (5.24) is skew-adjoint, Lemma 5.3.1 follows directly from the above transformation. Alternatively, according to the first part in the proof, we see that  $\mathcal{A}$  is dissipative. Since  $A$  has compact resolvents, the block operator  $\mathcal{A}$  also has compact resolvents. Therefore, the result follows from the Lumer-Phillips theorem.

**Remark 30.** The scale of Hilbert spaces  $H_s$  ( $s \in \mathbb{R}$ ) associated with the positive operator  $A$ , where  $H_{-\alpha}$  is the dual of  $H_\alpha$  ( $\alpha \geq 0$ ) with the pivot space  $H$ , have the dense and continuous embeddings (see [115, Section 3.4])

$$H_1 \subset H_{\frac{1}{2}} \subset H \subset H_{-\frac{1}{2}} \subset H_{-1}.$$

The operators  $A^{1/2}$  and  $A$  have unique extensions such that  $\tilde{A}^{1/2} \in \mathcal{L}(H, H_{-\frac{1}{2}})$  and  $\tilde{A} \in \mathcal{L}(H, H_{-1})$ . Moreover, according to [115, Section 2.10], for every  $\lambda \in \rho(A)$ , the resolvent also have the corresponding extensions  $R(\lambda : \tilde{A}) \in \mathcal{L}(H_{-1}, H)$  and  $R(\lambda : \tilde{A}^{1/2}) \in \mathcal{L}(H_{-\frac{1}{2}}, H)$ , which are unitary. In addition, as we already mentioned in the proof of Lemma 5.3.1, the resolvent  $R(\lambda : A)$  commutes with  $A^{1/2}$ , then we could use directly the expression  $A^{1/2}R(\lambda : A)$  instead of  $R(\lambda : A)A^{1/2}$ . In this case, we do not need to consider the extension of the operator.

## 5.4 Operator convergence

This section contains the main ingredients of the convergence results, based on appropriate decomposition of the Fourier series describing the operators  $\mathcal{A}_\mu, \mathcal{A}_0$  introduced in (5.20) and the control operators  $\mathcal{B}_\mu, \mathcal{B}_0$  in (5.21), which play an important role in the proof of the main result in Section 5.5.

We use the notation  $A \in G(M, \omega)$  in what follows for an operator  $A$ , which is the generator of a  $C_0$ -semigroup  $\mathbb{T}(t)$  satisfying  $\|\mathbb{T}(t)\| \leq Me^{\omega t}$  for every  $t \geq 0$ . With this notation, Lemma 5.3.1 implies that  $\mathcal{A}_\mu, \mathcal{A}_0 \in G(1, 0)$  since  $\frac{1}{\mu}A_\mu$  and  $A_w$  are positive.

**Lemma 5.4.1.** *With the operators  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$  defined in (5.20), for every  $\begin{bmatrix} f \\ g \end{bmatrix} \in X$  we have*

$$\lim_{\mu \rightarrow 0} R(1 : \mathcal{A}_\mu) \begin{bmatrix} f \\ g \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in } X. \quad (5.25)$$

*Proof.* According to Lemma 5.3.1, the operators  $\mathcal{A}_\mu, \mathcal{A}_0 \in G(1, 0)$ , which implies that  $1 \in \rho(\mathcal{A}_\mu) \cap \rho(\mathcal{A}_0)$ . We denote, for every  $\begin{bmatrix} f \\ g \end{bmatrix} \in X$ ,

$$\begin{bmatrix} \varphi_\mu \\ \psi_\mu \end{bmatrix} = R(1 : \mathcal{A}_\mu) \begin{bmatrix} f \\ g \end{bmatrix}, \quad \begin{bmatrix} \varphi_0 \\ \psi_0 \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} f \\ g \end{bmatrix}.$$

It follows that

$$\begin{aligned} \varphi_\mu &= -R\left(-1 : \frac{1}{\mu}A_\mu\right) \left[ \left(\frac{1}{\mu}A_\mu\right)^{1/2} f + g \right], \\ \psi_\mu &= R\left(-1 : \frac{1}{\mu}A_\mu\right) \left[ -f + \left(\frac{1}{\mu}A_\mu\right)^{1/2} g \right], \end{aligned}$$

## 5.4. OPERATOR CONVERGENCE

and

$$\varphi_0 = -R(-1 : A_w)(A_w^{1/2}f + g), \quad \psi_0 = R(-1 : A_w)(-f + A_w^{1/2}g).$$

As explained in the proof of Lemma 5.3.1 and Remark 30, we have  $\begin{bmatrix} \varphi_\mu \\ \psi_\mu \end{bmatrix} \in X$  and  $\begin{bmatrix} \varphi_0 \\ \psi_0 \end{bmatrix} \in X$ , which means that the expression in (5.25) makes sense.

Next we prove the convergence of each component of (5.25) in  $H$  as  $\mu$  goes to zero. Since  $\frac{1}{\mu}A_\mu$  and  $A_w$  are diagonalizable operators, according to [115, Proposition 2.6.2], we obtain from (4.34) and (5.13) that

$$R\left(-1 : \frac{1}{\mu}A_\mu\right)g - R(-1 : A_w)g = \sum_{k \in \mathbb{N}} F_\mu(k) \langle g, \varphi_k \rangle \varphi_k,$$

with

$$F_\mu(k) = \frac{1}{1+k^2} - \frac{1}{1 + \frac{k \tanh(\sqrt{\mu}k)}{\sqrt{\mu}}}. \quad (5.26)$$

Denoting  $h(x) = \frac{\tanh x}{x}$  ( $h(0) := 1$ ), we have

$$F_\mu(k) = - \int_0^{\sqrt{\mu}k} \left( \frac{1}{1+k^2h(x)} \right)' dx.$$

Note that

$$\left| \left( \frac{1}{1+k^2h} \right)' \right| \leq \frac{-h'}{k^2h^2},$$

and  $\frac{-h'}{h^2} \leq 1$  on  $[0, \infty)$ , which implies that  $|F_\mu(k)| \leq \frac{\sqrt{\mu}}{k}$ . We thus arrive at

$$\left\| R\left(-1 : \frac{1}{\mu}A_\mu\right)g - R(-1 : A_w)g \right\|^2 \leq \mu \|g\|^2. \quad (5.27)$$

Similarly, for every  $f \in H$  we have

$$R\left(-1 : \frac{1}{\mu}A_\mu\right) \left( \frac{1}{\mu}A_\mu \right)^{1/2} f - R(-1 : A_w)A_w^{1/2}f = \sum_{k \in \mathbb{N}} G_\mu(k) \langle f, \varphi_k \rangle \varphi_k,$$

with

$$G_\mu(k) = \frac{k}{1+k^2} - \frac{\left( \frac{k \tanh(\sqrt{\mu}k)}{\sqrt{\mu}} \right)^{1/2}}{1 + \frac{k \tanh(\sqrt{\mu}k)}{\sqrt{\mu}}}. \quad (5.28)$$

We similarly have

$$G_\mu(k) = - \int_0^{\sqrt{\mu}k} \left( \frac{kh^{1/2}(x)}{1+k^2h(x)} \right)' dx.$$

It is not difficult to see that

$$\left| \left( \frac{kh^{1/2}}{1+k^2h} \right)' \right| \leq \frac{-h'}{2kh^{3/2}}.$$

Moreover, note that  $\frac{-h'}{h^{3/2}}$  is bounded on  $[0, \infty)$ , we thus obtain that

$$|G_\mu(k)| \leq C\sqrt{\mu}, \quad (5.29)$$

which yields that

$$\left\| R\left(-1 : \frac{1}{\mu}A_\mu\right) \left(\frac{1}{\mu}A_\mu\right)^{1/2} f - R(-1 : A_w)A_w^{1/2}f \right\|^2 \leq C\mu \|f\|^2.$$

This, together with (5.27), implies that  $\lim_{\mu \rightarrow 0} \varphi_\mu = \varphi_0$  in  $H$  and  $\lim_{\mu \rightarrow 0} \psi_\mu = \psi_0$  in  $H$ , which ends the proof.  $\square$

**Lemma 5.4.2.** *Let  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$  be the same as in Lemma 5.4.1. For the operators  $\mathcal{B}_\mu$  and  $\mathcal{B}_0$  defined in (5.21), for every  $u \in \mathbf{C}$  we have*

$$\lim_{\mu \rightarrow 0} R(1 : \mathcal{A}_\mu)\mathcal{B}_\mu u = R(1 : \mathcal{A}_0)\mathcal{B}_0 u \quad \text{in } X.$$

*Proof.* For every  $u \in \mathbf{C}$ , it is obvious that  $R(1 : \mathcal{A}_\mu)\mathcal{B}_\mu u \in X$  since  $\mathcal{B}_\mu \in \mathcal{L}(\mathbf{C}, X)$ . Note that  $\mathcal{B}_0 \in \mathcal{L}(\mathbf{C}, \mathbb{H}_{-\frac{1}{2}} \times \mathbb{H}_{-\frac{1}{2}})$ , according to Remark 30, then we have  $R(1 : \mathcal{A}_0)\mathcal{B}_0 u \in X$ . For every  $u \in \mathbf{C}$ , let

$$\begin{bmatrix} \tilde{\varphi}_\mu \\ \tilde{\psi}_\mu \end{bmatrix} = R(1 : \mathcal{A}_\mu)\mathcal{B}_\mu u, \quad \begin{bmatrix} \tilde{\varphi}_0 \\ \tilde{\psi}_0 \end{bmatrix} = R(1 : \mathcal{A}_0)\mathcal{B}_0 u.$$

We immediately have

$$\begin{aligned} \tilde{\varphi}_\mu &= -R\left(-1 : \frac{1}{\mu}A_\mu\right) \frac{1}{\mu}B_\mu u, \\ \tilde{\psi}_\mu &= -R\left(-1 : \frac{1}{\mu}A_\mu\right) \left(\frac{1}{\mu}A_\mu\right)^{1/2} \frac{1}{\mu}B_\mu u, \end{aligned}$$

and

$$\tilde{\varphi}_0 = -R(-1 : A_w)B_w u, \quad \tilde{\psi}_0 = -R(-1 : A_w)A_w^{1/2}B_w u.$$

For the sake of clarity, we split the remaining proof into two steps.

**Step 1:** We prove that  $\lim_{\mu \rightarrow 0} \tilde{\varphi}_\mu = \tilde{\varphi}_0$  in  $H$ . To this aim, we first note, using a triangle inequality, that

$$\begin{aligned} \|\tilde{\varphi}_\mu - \tilde{\varphi}_0\| &\leq \left\| \left[ R\left(-1 : \frac{1}{\mu}A_\mu\right) - R(-1 : A_w) \right] \frac{1}{\mu}B_\mu u \right\| \\ &\quad + \left\| R(-1 : A_w) \left( \frac{1}{\mu}B_\mu u - B_w u \right) \right\|, \end{aligned} \quad (5.30)$$

where we used the fact that  $R(-1 : A_w)\frac{1}{\mu}B_\mu u \in H$ . We note that  $R(-1 : A_w) \in \mathcal{L}(\mathbb{H}_{-\frac{1}{2}}, \mathbb{H}_{\frac{1}{2}})$ . Indeed, by the Riesz representation theorem, for every  $f \in \mathbb{H}_{-\frac{1}{2}}$ , there exists a unique  $\varphi \in \mathbb{H}_{\frac{1}{2}}$  such that

$$\langle \varphi, \psi \rangle + \langle A_w^{1/2}\varphi, A_w^{1/2}\psi \rangle = \langle f, \psi \rangle_{\mathbb{H}_{-\frac{1}{2}}, \mathbb{H}_{\frac{1}{2}}} \quad \forall \psi \in \mathbb{H}_{\frac{1}{2}}, \quad (5.31)$$

## 5.4. OPERATOR CONVERGENCE

which is  $(I + A_w)\varphi = f$  in  $\mathbb{H}_{-\frac{1}{2}}$ . Taking  $\psi = \varphi$  in (5.31), it follows that

$$\|\varphi\|_{\mathbb{H}_{\frac{1}{2}}}^2 \leq \|f\|_{\mathbb{H}_{-\frac{1}{2}}} \cdot \|\varphi\|_{\mathbb{H}_{\frac{1}{2}}},$$

which implies that  $\|\varphi\|_{\mathbb{H}_{\frac{1}{2}}} \leq \|f\|_{\mathbb{H}_{-\frac{1}{2}}}$ , i.e.  $\|R(-1 : A_w)\|_{\mathcal{L}(\mathbb{H}_{-\frac{1}{2}}, \mathbb{H}_{\frac{1}{2}})} \leq 1$ . This, together with (5.6), yields that the last term on the right-hand side of (5.30) converges to zero, where we used the fact that  $\mathbb{H}_{\frac{1}{2}} = \mathcal{H}^1[0, \pi]$  and the continuous dense embedding  $\mathbb{H}_{\frac{1}{2}} \hookrightarrow H$ .

Next we estimate the square of the first norm on the right side of (5.30). According to [115, Proposition 2.6.2], we write it in form of Fourier series, which reads

$$\sum_{k \in \mathbb{N}} |F_\mu(k)|^2 \cdot \left| \left\langle \frac{1}{\mu} B_\mu u, \varphi_k \right\rangle \right|^2, \quad (5.32)$$

where  $F_\mu(k)$  has been defined in (5.26). According to (5.8), it is not difficult to see that

$$\left\langle \frac{1}{\mu} B_\mu u, \varphi_k \right\rangle = \frac{-2\sqrt{2}u}{\mu\sqrt{\pi}} \sum_{l \in \mathbb{N}} H_\mu(k, l), \quad (5.33)$$

with

$$H_\mu(k, l) = \frac{1}{\left(\frac{2l-1}{2\sqrt{\mu}}\pi\right)^2 + k^2}.$$

Moreover, we readily see that

$$\sum_{l \in \mathbb{N}} H_\mu(k, l) \leq \sum_{l \in \mathbb{N}} \frac{4\mu}{(2l-1)^2\pi^2} = \frac{\mu}{2}. \quad (5.34)$$

Recalling that  $|F_\mu(k)| \leq \frac{\sqrt{\mu}}{k}$ , we thus conclude that (5.32) can be controlled by  $C\mu$ . Therefore, we obtain that (5.30) converges to zero as  $\mu \rightarrow 0$ .

**Step 2:** We prove that  $\lim_{\mu \rightarrow 0} \tilde{\psi}_\mu = \tilde{\psi}_0$  in  $H$ . Since  $R(-1 : A_w)A_w^{1/2} \frac{1}{\mu} B_\mu u \in H$  for fixed  $\mu$ , we consider the following triangle inequality:

$$\begin{aligned} \|\tilde{\psi}_\mu - \tilde{\psi}_0\| &\leq \left\| R(-1 : A_w)A_w^{1/2} \left( \frac{1}{\mu} B_\mu u - B_w u \right) \right\| \\ &\quad + \left\| \left[ R\left(-1 : \frac{1}{\mu} A_\mu\right) \left(\frac{1}{\mu} A_\mu\right)^{1/2} - R(-1 : A_w)A_w^{1/2} \right] \frac{1}{\mu} B_\mu u \right\|. \end{aligned} \quad (5.35)$$

We first, using Fourier series, prove that  $R(-1 : A_w)A_w^{1/2} \in \mathcal{L}(\mathbb{H}_{-\frac{1}{2}}, H)$ . For every  $f \in \mathbb{H}_{-\frac{1}{2}}$ , we have  $f = \langle f, 1 \rangle \frac{1}{\pi} + \sum_{k \in \mathbb{N}} (1+k^2)^{-1/2} \langle f, \varphi_k \rangle \varphi_k$  and then, according to [115, Proposition 2.6.2],

$$R(-1 : A_w)A_w^{1/2} f = \sum_{k \in \mathbb{N}} \frac{-k}{(1+k^2)^{3/2}} \langle f, \varphi_k \rangle \varphi_k.$$

It follows that

$$\left\| R(-1 : A_w)A_w^{1/2} f \right\|^2 \leq \|f\|_{\mathbb{H}_{-\frac{1}{2}}}^2.$$



This, combined with (5.6), implies that the first norm on the right side of (5.35) converges to zero.

Note that the square of the second norm on the right side of (5.35) is

$$\frac{8u^2}{\mu^2\pi} \sum_{k \in \mathbb{N}} |G_\mu(k)|^2 \left| \sum_{l \in \mathbb{N}} H_\mu(k, l) \right|^2,$$

where  $G_\mu(k)$  and  $H_\mu(k, l)$  are defined in (5.28) and (5.33), respectively. Different with Step 1 we need here a more precise estimate for  $G_\mu(k)$  and  $H_\mu(k, l)$ , such that the double series goes to zero as  $\mu \rightarrow 0$ . Besides (5.34), we have the following alternative estimate

$$\sum_{l \in \mathbb{N}} H_\mu(k, l) = \sum_{l \in \mathbb{N}} \frac{\mu}{l^2 + (\sqrt{\mu}k)^2} \leq \sum_{l \leq \sqrt{\mu}k} \frac{1}{k^2} + \sum_{l \geq \sqrt{\mu}k} \frac{\mu}{l^2} \leq 2 \frac{\sqrt{\mu}}{k}, \quad (5.36)$$

where we used the fact  $\sum_{k \geq a} \frac{1}{k^2} \leq \frac{1}{a}$ . Recalling the definition of  $G_\mu(k)$  and the function  $h$  introduced in the proof of Lemma 5.4.1, we have

$$|G_\mu(k)| = \left| \frac{k(1 - h_{\mu,k})(1 - k^2 h_{\mu,k})}{(1 + k^2 h_{\mu,k}^2)(1 + k^2)} \right|,$$

where  $h_{\mu,k} = (h(\sqrt{\mu}k))^{1/2}$ . If  $\sqrt{\mu}k \leq \delta < 1$ , we still use the estimate (5.29)  $|G_\mu(k)| \leq C\sqrt{\mu}$ . It is not difficult to see that in this case we further have  $|G_\mu(k)| \leq C \frac{\mu^{1/4}}{k^{1/2}}$ . If  $\sqrt{\mu}k > \delta$ , there exists  $c > 0$  such that  $\tanh(\sqrt{\mu}k) \geq c$ , which yields that

$$1 > h_{\mu,k} \geq \frac{C}{\mu^{1/4}k^{1/2}}.$$

It follows that we have

$$|G_\mu(k)| \leq \frac{k|1 - k^2 h_{\mu,k}|}{k^4 h_{\mu,k}^2} \leq C \frac{\mu^{1/4}}{k^{1/2}}. \quad (5.37)$$

If  $k^2 h_{\mu,k} \geq 1$ , (5.37) is a direct consequence. Otherwise, we still have

$$|G_\mu(k)| \leq \frac{1}{k^3 h_{\mu,k}^2} \leq C \frac{\mu^{1/2}}{k^2} \leq C \frac{\mu^{1/4}}{k^{1/2}},$$

since  $\mu < 1$ . We thus conclude that (5.37) holds for every  $\mu \in (0, 1)$  and  $k \in \mathbb{N}$ . Putting together (5.29), (5.34), (5.36) and (5.37) we thus arrive at

$$\frac{1}{\mu^2} \sum_{k \in \mathbb{N}} |G_\mu(k)|^2 \left| \sum_{l \in \mathbb{N}} H_\mu(k, l) \right|^2 \leq C \mu^{1/4}.$$

Therefore, the proof of Lemma 5.4.2 is completed. □

## 5.5 Main results and the proof

Combined with what we discussed in Section 5.3 and the preliminary convergence results in Section 5.4, in this section we state the relationship between the control system (5.18) and (5.19), which is equivalent to the water waves system (5.11) and the wave equation (5.14), respectively.

For any  $\omega \in \mathbb{R}$ , we introduce the weighted Hilbert space  $L_\omega^2[0, \infty) := e_\omega L^2[0, \infty)$ , where  $(e_\omega v)(t) = e^{\omega t} v(t)$  for every  $t \geq 0$ , with the norm  $\|v\|_{L_\omega^2} = \|e_{-\omega} v\|_{L^2}$ . Similarly, the Hilbert space  $\mathcal{H}_\omega^1[0, \infty) := e_\omega \mathcal{H}^1[0, \infty)$  contains the elements  $(e_\omega v)(t) = e^{\omega t} v(t)$  for every  $v \in \mathcal{H}^1[0, \infty)$ , with the norm  $\|v\|_{\mathcal{H}_\omega^1} = \|e_{-\omega} v\|_{\mathcal{H}^1}$ . For every  $\tau > 0$ , we consider the zero extension of the input space  $L^2[0, \tau]$  on  $(\tau, \infty)$ , which gives a subspace of  $L_\omega^2[0, \infty)$ .

Here is our main result:

**Theorem 5.5.1.** *For  $u \in L_{\text{loc}}^2[0, \infty)$  and for any initial data  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$  and  $\zeta_1 \in \mathbb{H}$ , let  $\zeta_\mu$  be the solution of the gravity water waves system (5.4) with the initial data (5.5), satisfying*

$$\zeta_\mu \in C([0, \infty); \mathcal{H}^{\frac{1}{2}}[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

Let  $\zeta$  be the weak solution of the wave equation (5.12) satisfying

$$\zeta \in C([0, \infty); \mathcal{H}^1[0, \pi]) \cap C^1([0, \infty); L^2[0, \pi]).$$

Then, for every  $\tau > 0$ , we have

$$\begin{aligned} \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\zeta_\mu - \zeta\|_{\mathcal{H}^{\frac{1}{2}}[0, \pi]} &= 0, \\ \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\partial_t \zeta_\mu - \partial_t \zeta\|_{L^2[0, \pi]} &= 0. \end{aligned}$$

*Proof.* To present the proof clearly, we divide it into the following three steps.

**Step 1:** The convergence of a scattering semigroup. For every  $u \in \mathcal{U} = L_\omega^2[0, \infty)$ , according to Lemma 5.3.1, we denote by  $\mathbb{T}_\mu = (\mathbb{T}_{\mu, t})_{t \geq 0}$  the  $C_0$ -semigroup generated by the operator  $\mathcal{A}_\mu$ , and by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  the  $C_0$ -semigroup generated by  $\mathcal{A}_0$ . Since  $\mathcal{A}_\mu, \mathcal{A}_0 \in G(1, 0)$ , then the growth bound  $\omega(\mathbb{T}) = 0 = \omega(\mathbb{T}_\mu)$ . The solutions of the differential equations (5.18) and (5.19) are

$$w_\mu(t) = \mathbb{T}_{\mu, t} w_{\mu, 0} + \Phi_{\mu, t} u(t),$$

and

$$w(t) = \mathbb{T}_t w_0 + \Phi_t u(t),$$

where the initial data  $w_{\mu, 0}$  and  $w_0$  are introduced in (5.22). Note that it is not difficult to check that  $\mathcal{B}_0$  is an admissible control operator, then the controllability map  $\Phi_t$  defined by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} \mathcal{B}_0 u(\sigma) d\sigma,$$

is bounded from  $\mathcal{U}$  to  $X = H \times H$ . Similarly, we have  $\Phi_{\mu, t} \in \mathcal{L}(\mathcal{U}, X)$  since  $\mathcal{B}_\mu \in \mathcal{L}(\mathbb{C}, X)$ . To justify the limit  $\lim_{\mu \rightarrow 0} w_\mu = w$  in  $X$ , we first define bounded operators  $\mathfrak{T}_{\mu, t}$  and  $\mathfrak{T}_t$  by

$$\mathfrak{T}_{\mu, t} = \begin{bmatrix} \mathbb{T}_{\mu, t} & \Phi_{\mu, t} \\ 0 & S_t \end{bmatrix}, \quad \mathfrak{T}_t = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ 0 & S_t \end{bmatrix},$$

where  $(S_t)_{t \geq 0}$  is the unilateral left shift semigroup on  $\mathcal{U}$ , i.e.  $S_t u(\xi) = u(\xi + t)$  for every  $\xi \geq 0$ . Then  $(\mathfrak{T}_{\mu,t})_{t \geq 0}$  and  $(\mathfrak{T}(t))_{t \geq 0}$  form  $C_0$ -semigroups on  $X \times \mathcal{U}$ , respectively, with the same growth bound  $\omega > \omega(\mathbb{T}) = 0$  (Such semigroups were used in [115, Section 4.1], [105] and [50]). The generators of  $\mathfrak{T}_{\mu,t}$  and  $\mathfrak{T}_t$  are

$$\mathcal{A}_\mu = \begin{bmatrix} \mathcal{A}_\mu & \mathcal{B}_\mu \delta_0 \\ 0 & \frac{d}{d\xi} \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} \mathcal{A}_0 & \mathcal{B}_0 \delta_0 \\ 0 & \frac{d}{d\xi} \end{bmatrix},$$

where  $\delta_0 u(\xi) = u(0)$  for every  $u \in \mathcal{U}$ , and

$$\mathcal{D}(\mathcal{A}_\mu) = \left\{ \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in X \times \mathcal{H}_\omega^1[0, \infty) \mid \mathcal{A}_\mu x_0 + \mathcal{B}_\mu u_0(0) \in X \right\}.$$

Similarly,  $\mathcal{D}(\mathcal{A}_0)$  can be defined by using  $\mathcal{A}_0$  and  $\mathcal{B}_0$  in the above set. Here for simplicity we choose  $\omega \in (0, 1)$  such that  $1 \in \rho(\mathcal{A}_\mu) \cap \rho(\mathcal{A}_0)$ . Setting, for every  $\begin{bmatrix} x \\ u \end{bmatrix} \in X \times \mathcal{U}$ ,

$$\begin{bmatrix} x_{\mu,0} \\ u_0 \end{bmatrix} = R(1 : \mathcal{A}_\mu) \begin{bmatrix} x \\ u \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} x \\ u \end{bmatrix},$$

we have

$$\begin{cases} x_0 - \mathcal{A}_0 x_0 - \mathcal{B}_0 u_0(0) = x, \\ u_0 - \frac{du_0}{d\xi} = u. \end{cases} \quad (5.38)$$

The second equation in (5.38) admits a unique solution  $u_0$  given via its Laplace transform

$$\widehat{u}_0(s) = \frac{\widehat{u}(s) - u_0(0)}{1 - s}.$$

According to the Paley-Wiener theorem (see, for instance, [115, Theorem 12.4.2]),  $u_0(0) = \widehat{u}(1)$  is the only choice such that  $\widehat{u}_0(s) \in \mathcal{H}^2(\mathbb{C}_0)$ , where  $\mathcal{H}^2(\mathbb{C}_0)$  is the Hardy space with  $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ . We obtain from (5.38) that

$$x_0 = R(1 : \mathcal{A}_0)(x + \mathcal{B}_0 u_0(0)).$$

Similarly, we have

$$x_{\mu,0} = R(1 : \mathcal{A}_\mu)(x + \mathcal{B}_\mu u_0(0)).$$

Recalling Lemma 5.4.1 and Lemma 5.4.2 we thus conclude that  $\lim_{\mu \rightarrow 0} x_{\mu,0} = x_0$  in  $X$ . It yields that, for every  $\begin{bmatrix} x \\ u \end{bmatrix} \in X \times \mathcal{U}$ ,

$$\lim_{\mu \rightarrow 0} R(1 : \mathcal{A}_\mu) \begin{bmatrix} x \\ u \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{in } X \times \mathcal{U}.$$

By applying the Trotter-Kato theorem (see, for instance, [88, Chapter 3]), it follows that

$$\lim_{\mu \rightarrow 0} \mathfrak{T}_{\mu,t} \begin{bmatrix} x \\ u \end{bmatrix} = \mathfrak{T}_t \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{in } X \times \mathcal{U},$$

uniformly with respect to  $t$  on compact intervals. Thus we have

$$\lim_{\mu \rightarrow 0} (\mathbb{T}_{\mu,t} x + \Phi_{\mu,t} u) = \mathbb{T}_t x + \Phi_t u \quad \text{in } X \times \mathcal{U}.$$

## 5.5. MAIN RESULTS AND THE PROOF

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In particular (when  $u = 0$ ), for every  $x \in X$ , we have

$$\lim_{\mu \rightarrow 0} \mathbb{T}_{\mu,t} x = \mathbb{T}_t x \quad \text{in } X. \quad (5.39)$$

**Step 2:** We prove that  $\lim_{\mu \rightarrow 0} w_\mu = w$  in  $X$ . In order to justify this limit, it suffices to prove  $\lim_{\mu \rightarrow 0} \mathbb{T}_{\mu,t} w_{\mu,0} = \mathbb{T}_t w_0$ , where  $w_{\mu,0}$  and  $w_0$  are introduced in (5.22). We first show that for every  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$ , we have

$$\lim_{\mu \rightarrow 0} \left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_0 = A_w^{1/2} \zeta_0 \quad \text{in } H.$$

Since  $\frac{1}{\mu} A_\mu$  and  $A_w$  are diagonalizable we have

$$\left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_0 - A_w^{1/2} \zeta_0 = \sum_{k \in \mathbb{N}} k I_\mu(k) \langle \zeta_0, \varphi_k \rangle \varphi_k,$$

with

$$I_\mu(k) = \left( \frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k} \right)^{1/2} - 1.$$

Just like estimating  $F_\mu(k)$  in Lemma 5.4.1, we use the similar argument here and obtain that  $|I_\mu(k)| \leq \sqrt{\mu}k$ . It implies that for every  $\bar{\zeta} \in \mathbb{H}_1$ ,

$$\left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} \bar{\zeta} - A_w^{1/2} \bar{\zeta} \right\|^2 \leq \mu \|\bar{\zeta}\|_{\mathbb{H}_1}^2.$$

Note that embedding  $\mathbb{H}_1 \hookrightarrow \mathbb{H}_{\frac{1}{2}}$  is dense and continuous, for every  $\varepsilon > 0$ , there exists  $\bar{\zeta} \in \mathbb{H}_1$ , such that  $\|\bar{\zeta} - \zeta_0\|_{\mathbb{H}_{\frac{1}{2}}} < \frac{\varepsilon}{3}$ . Moreover, it is not difficult to check that

$$\left( \frac{1}{\mu} A_\mu \right)^{1/2} \in \mathcal{L}(\mathbb{H}_{\frac{1}{2}}, \mathbb{H}), \quad A_w^{1/2} \in \mathcal{L}(\mathbb{H}_{\frac{1}{2}}, \mathbb{H}),$$

and their operator norms are uniformly bounded. We thus have the following estimate

$$\begin{aligned} \left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_0 - A_w^{1/2} \zeta_0 \right\| &\leq \left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} (\zeta_0 - \bar{\zeta}) \right\| \\ &\quad + \left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} \bar{\zeta} - A_w^{1/2} \bar{\zeta} \right\| + \left\| A_w^{1/2} (\bar{\zeta} - \zeta_0) \right\|. \end{aligned}$$

Hence, for every  $\varepsilon > 0$ , there exists  $\mu_0 = \left(\frac{\varepsilon}{3}\right)^2$ , such that for every  $\mu < \mu_0$ , we have

$$\left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_0 - A_w^{1/2} \zeta_0 \right\| \leq C\varepsilon.$$

Now we estimate the following difference by using a trangle inequality,

$$\|\mathbb{T}_{\mu,t} w_{\mu,0} - \mathbb{T}_t w_0\|_X \leq \|\mathbb{T}_{\mu,t} w_{\mu,0} - \mathbb{T}_{\mu,t} w_0\|_X + \|\mathbb{T}_{\mu,t} w_0 - \mathbb{T}_t w_0\|_X.$$

Note that  $\mathbb{T}_{\mu,t}$  is unitary and

$$\|\mathbb{T}_{\mu,t}w_{\mu,0} - \mathbb{T}_{\mu,t}w_0\|_X \leq \|\mathbb{T}_{\mu,t}\|_{\mathcal{L}(X)} \left\| \left( \frac{1}{\mu} A_\mu \right)^{1/2} \zeta_0 - A_w^{1/2} \zeta_0 \right\|,$$

we have  $\lim_{\mu \rightarrow 0} \mathbb{T}_{\mu,t}(w_{\mu,0} - w_0) = 0$  for every  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$  and  $\zeta_1 \in \mathbb{H}$ . This, together with (5.39), implies that  $\lim_{\mu \rightarrow 0} \mathbb{T}_{\mu,t}w_{\mu,0} = \mathbb{T}_t w_0$  in  $X$ . We thus achieve that  $\lim_{\mu \rightarrow 0} w_\mu = w$  in  $X$  (that is, for every  $t \geq 0$ ,  $\lim_{\mu \rightarrow 0} \alpha_\mu = \alpha$  and  $\lim_{\mu \rightarrow 0} \beta_\mu = \beta$  hold in  $H$ ).

**Step 3:** We prove that  $\lim_{\mu \rightarrow 0} \zeta_\mu = \zeta$  in  $C^1([0, \tau]; H) \cap C([0, \tau]; H_{\frac{1}{2}})$ . Recalling the definition of  $\alpha_\mu, \beta_\mu$  in (5.16), and  $\alpha, \beta$  in (5.17), we need to translate the convergence results in Step 2 in form of the original variables  $\zeta_\mu$  and  $\zeta$ . Note that  $\zeta_\mu$  and  $\zeta$  have the same initial data, according to Leibniz formula, we obtain from the first convergence,  $\lim_{\mu \rightarrow 0} \alpha_\mu = \alpha$  in  $H$ , that

$$\lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\zeta_\mu - \zeta\| \leq \lim_{\mu \rightarrow 0} \tau \sup_{t \in [0, \tau]} \|\alpha_\mu - \alpha\|.$$

We thus arrive at  $\lim_{\mu \rightarrow 0} \zeta_\mu = \zeta$  in  $C^1([0, \tau]; H)$ . Moreover, taking the second convergence  $\lim_{\mu \rightarrow 0} \beta_\mu = \beta$  into account we further have

$$\lim_{\mu \rightarrow 0} \left[ I + \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right] \zeta_\mu = (I + A_w^{1/2}) \zeta \quad \text{in } H. \quad (5.40)$$

since

$$\left\| \left[ I + \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right] \zeta_\mu - (I + A_w^{1/2}) \zeta \right\| \leq \|\zeta_\mu - \zeta\| + \|\beta_\mu - \beta\|.$$

Notice that, for every  $x \geq 0$ ,  $\frac{\tanh x}{x} \sim \frac{1}{1+x}$  (that is, each function can be controlled by the other one multiplied by a positive constant), we obtain that

$$R \left( -1 : \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right) \in \mathcal{L}(H, H_{\frac{1}{2}}),$$

and its operator norm is uniformly bounded. It follows from (5.40) that

$$\lim_{\mu \rightarrow 0} \left[ \zeta_\mu + R \left( -1 : \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right) (I + A_w^{1/2}) \zeta \right] = 0 \quad \text{in } H_{\frac{1}{2}}.$$

Furthermore, we have

$$\begin{aligned} \|\zeta_\mu - \zeta\|_{H_{\frac{1}{2}}} &\leq \left\| \zeta_\mu + R \left( -1 : \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right) (I + A_w^{1/2}) \zeta \right\|_{H_{\frac{1}{2}}} \\ &\quad + \left\| R \left( -1 : \left( \frac{1}{\mu} A_\mu \right)^{1/2} \right) (I + A_w^{1/2}) \zeta + \zeta \right\|_{H_{\frac{1}{2}}}. \end{aligned} \quad (5.41)$$

## 5.6. REGULAR CONVERGENCE

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Observing that it remains to prove that the second norm on the right side of (5.41) converges to zero, we estimate its square, i.e.

$$\sum_{k \in \mathbb{N}} k |J_\mu(k)|^2 |\langle \zeta, \varphi_k \rangle|^2,$$

where

$$J_\mu(k) = \frac{1+k}{1+k \left( \frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k} \right)^{1/2}} - 1.$$

Still using the function  $h_{\mu,k}$  defined in the proof of Lemma 5.4.2, we have

$$|J_\mu(k)| = \frac{k(1-h_{\mu,k})}{1+h_{\mu,k}},$$

since  $h_{\mu,k} \in (0, 1)$ . If  $\sqrt{\mu}k \leq \delta < 1$ , there exists  $c > 0$  such that  $h_{\mu,k} \geq c$ . According to the Taylor expansion of  $h_{\mu,k}$ , we obtain that  $1 - h_{\mu,k} \leq C\sqrt{\mu}k$ , so that

$$|J_\mu(k)| \leq C\sqrt{\mu}k \leq C\mu^{1/4}k^{1/2}.$$

If  $\sqrt{\mu}k > \delta$ , we have  $h_{\mu,k} \geq \frac{c}{\mu^{1/4}k^{1/2}}$ , which clearly implies that  $|J_\mu(k)| \leq C\mu^{1/4}k^{1/2}$ . Hence we have

$$\sum_{k \in \mathbb{N}} k |J_\mu(k)|^2 |\langle \zeta, \varphi_k \rangle|^2 \leq \sqrt{\mu} \|\zeta\|_{\mathbb{H}_{\frac{1}{2}}}^2 \leq C\sqrt{\mu},$$

where we used  $\zeta \in C([0, \infty); \mathbb{H}_{\frac{1}{2}})$  for every  $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$  and  $\zeta_1 \in \mathbb{H}$ .

Therefore, we finish the proof of Theorem 5.5.1.  $\square$

**Remark 31.** The scattering semigroup  $(\mathfrak{T}_t)_{t \geq 0}$  used in Step 1 is actually a part of the so-called *Lax-Phillips semigroup* of index  $\omega$  introduced in Staffans and Weiss [105].

**Remark 32.** As we expected, according to the Theorem 5.5.1, the elevation of the water waves system behaves like the displacement of a string in one dimension. Although we have this relationship between the water waves system and the wave equation, their controllability properties are very different. As we know, the wave equation with Neumann boundary control is exactly controllable (see [19, Part III, Chapter 8] and [83] for the sufficiently large time, and [69] for finite time interval), while the water waves system (5.4) is even not approximately controllable (see [96] and [85]).

## 5.6 Regular convergence

Recalling that we established the strong well-posedness of the water waves system in Section 4.3.2, in this part we shall consider the regular convergence of the solution of the water waves system with regular initial data and control term. To address this, we need some preparation on the derivation of the dimensionless version of the evolution operator associated with the new decomposition of the water waves system proposed in Section 4.3.2, as well as the related properties of these operators.

We first note that, by using the dimensionless quantities defined in (5.2), the dimensionless version of the system (4.62), for every  $t \geq 0$  and  $(x, y) \in \Omega$  (see (5.3)), reads

$$\begin{cases} \Delta_\mu \tilde{\phi}_\mu(t, x, y) = \mu v(t)(x\chi)'' , \\ \partial_t \zeta_\mu(t, x) - \frac{1}{\mu} \partial_y \tilde{\phi}_\mu(t, x, 0) = 0, \\ \partial_t \tilde{\phi}_\mu(t, x, 0) + \zeta_\mu(t, x) = x\chi(x)u(t), \\ \partial_x \tilde{\phi}_\mu(t, 0, y) = 0, \\ \partial_y \tilde{\phi}_\mu(t, x, -1) = 0 = \partial_x \tilde{\phi}_\mu(t, \pi, y), \end{cases} \quad (5.42)$$

where the smooth function  $\chi$  has been introduced in (4.60) and  $\tilde{\phi}_\mu = \phi_\mu + x\chi(x)v(t)$ . Here we still denote the dimensionless quantities by  $x, y, t$  and  $\zeta_\mu, \phi_\mu$  without overlines for simplicity.

**Proposition 5.6.1.** *Let  $\Phi_\mu$  be the dimensionless version of  $\Phi$  introduced in Lemma 4.3.6, then  $\Phi_\mu \in \mathcal{H}^2(\Omega)$  and*

$$\Phi_\mu = \sum_{k \geq 0, l \in \mathbb{N}} \frac{\mu v a_{k,l}}{\mu k^2 + (2l-1)^2 \frac{\pi^2}{4}} \Psi_{kl},$$

where  $\Psi_{kl}$  and  $a_{kl}$  have been defined in (4.21) and Lemma 4.3.6, respectively. For every  $\alpha \geq 0$ , the operator  $S_\mu$  defined by  $S_\mu v := \partial_y \Phi_\mu(t, x, 0)$  belongs to  $\mathcal{L}(\mathbb{C}, H_\alpha)$ . Moreover, we have the limit

$$\lim_{\mu \rightarrow 0} \frac{1}{\mu} S_\mu = (x\chi)'' \quad \text{in } H_\alpha. \quad (5.43)$$

*Proof.* We first note that it is not difficult to derive the formula of  $\Phi_\mu$  from Lemma 4.3.6. According to (4.64) and Remark 11, we have

$$S_\mu = -\frac{\mu}{\pi} + \sum_{k, l \in \mathbb{N}} \frac{2\mu a_k}{(\mu k^2 + (2l-1)^2 \frac{\pi^2}{4})} \varphi_k, \quad (5.44)$$

where  $a_k$  and  $\varphi_k$  have been defined in (4.65) and (4.18), respectively. The formula (5.44) obviously implies that  $S_\mu \in H$ . For every  $\alpha \geq 0$ , we have

$$\sum_{k \in \mathbb{N}} k^{2\alpha} \left| \left\langle \frac{1}{\mu} S_\mu, \varphi_k \right\rangle \right|^2 \leq \sum_{k \in \mathbb{N}} k^{2\alpha} |a_k|^2 < \infty,$$

which directly implies that  $S_\mu \in H_\alpha$  since the function  $(x\chi)''$  is smooth enough with compact support on  $(0, \pi)$ . Moreover, using the expansion  $(x\chi)'' = -\frac{1}{\pi} + \sum_{k \in \mathbb{N}} a_k \varphi_k$ , we obtain that

$$\left\| \frac{1}{\mu} S_\mu - (x\chi(x))'' \right\|_{H_\alpha}^2 \leq C\mu^2 \|(x\chi)''\|_{H_2}^2 \leq C\mu^2.$$

Therefore, the proof is completed. □

Recalling the transform

$$\tilde{\zeta}_\mu(t, x) = \zeta_\mu(t, x) - x\chi(x)u(t),$$

## 5.6. REGULAR CONVERGENCE

we derive the dimensionless version of the system (4.69), which is

$$\begin{cases} \partial_t^2 \tilde{\zeta}_\mu(t, x) + \frac{1}{\mu} A_\mu \tilde{\zeta}_\mu(t, x) = \frac{1}{\mu} S_\mu u - x\chi(x)\ddot{u}, \\ \tilde{\zeta}_\mu(0, x) = \tilde{\zeta}_0(x), \quad \partial_t \tilde{\zeta}_\mu(0, x) = \tilde{\zeta}_1(x), \end{cases} \quad (5.45)$$

where  $A_\mu$  is the dimensionless Dirichlet to Neumann operator  $A_0$  in Proposition 4.2.9 and  $S_\mu$  is the dimensionless version of  $S$  in Proposition 4.3.7. The initial data  $\tilde{\zeta}_0$  and  $\tilde{\zeta}_1$  in (5.45) are the same with the ones in (4.69). For the system (5.45), we have discussed in Section 4.3.2 the strong well-posedness with smooth initial data and control  $u$ . More precisely, please refer to Theorem 4.3.8 for the details. Let  $\zeta$  satisfy the wave equation (5.12). To study the regular convergence of the system (5.45), we denote

$$\tilde{\zeta} = \zeta - x\chi(x)u(t).$$

Therefore, we derive from (5.12) the following control system in terms of  $\tilde{\zeta}$

$$\begin{cases} \partial_t^2 \tilde{\zeta}(t, x) + A_w \tilde{\zeta}(t, x) = (x\chi(x))''u(t) - x\chi(x)\ddot{u}, \\ \tilde{\zeta}(0, x) = \tilde{\zeta}_0(x), \quad \partial_t \tilde{\zeta}(0, x) = \tilde{\zeta}_1(x), \end{cases} \quad (5.46)$$

where the operator  $A_w$  has been introduced in (5.13).

Recalling the scale of Hilbert spaces  $\mathbb{H}_\alpha$  (associated with the operator  $A_w$ ) introduced in (5.15), we have the following regularity result for the system (5.46).

**Theorem 5.6.2.** *For every  $\tau > 0$ , let  $u \in \mathcal{H}_L^4[0, \tau]$ ,  $\zeta_0 \in \mathbb{H}_{\frac{3}{2}}$  and  $\zeta_1 \in \mathbb{H}_1$ . Then the system (5.46) admits a unique solution  $\tilde{\zeta}$  satisfying*

$$\tilde{\zeta} \in C([0, \tau]; \mathbb{H}_{\frac{3}{2}}) \cap C^1([0, \tau]; \mathbb{H}_1). \quad (5.47)$$

*Proof.* The proof is similar with the proof of Theorem 4.3.8. Let  $\tilde{z}(t) = \begin{bmatrix} \tilde{\zeta} \\ \partial_t \tilde{\zeta} \end{bmatrix}$  and denote

$$\mathfrak{A} = \begin{bmatrix} 0 & I \\ -A_w & 0 \end{bmatrix}, \quad \mathfrak{B}u = \begin{bmatrix} 0 \\ (x\chi)''u - x\chi\ddot{u} \end{bmatrix}.$$

Let  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  be the  $\mathbf{C}_0$ -semigroup generated by the operator  $\mathfrak{A}$  (In this case, the operator  $\mathfrak{A}$  is no longer skew-adjoint, since  $A_w$  is not strictly positive. But  $\mathfrak{A}$  is still a generator of a  $\mathbf{C}_0$ -semigroup, for this please refer to, for instance, [40, Section 2.3]). For every  $\tau > 0$ , assuming that  $u \in \mathcal{H}_L^4[0, \tau]$  and let  $(I - \mathfrak{A}) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathfrak{B}u$ , we obtain that  $\phi = \psi$  and

$$\phi = \psi = (I + A_w)^{-1} ((x\chi)''u - x\chi\ddot{u}).$$

We notice that

$$\sum_{k \in \mathbb{N}} |\lambda_k|^3 |\langle \phi, \varphi_k \rangle|^2 \leq C \sum_{k \in \mathbb{N}} |\lambda_k| (|a_k|^2 + |b_k|^2),$$

where  $\lambda_k = k^2$  is the eigenvalue of the operator  $A_w$  corresponding to the eigenvector  $\varphi_k$  defined in (4.18). According to Remark 12, we obtain that  $\phi \in \mathbb{H}_{\frac{3}{2}}$ . For every  $m \in \mathbb{N}$ , we denote the Hilbert space

$$\mathbb{X}_m = \mathcal{D}(\mathfrak{A}^m) = \mathbb{H}_{\frac{m+1}{2}} \times \mathbb{H}_{\frac{m}{2}}.$$



Hence, according to [116, Proposition 2.3], we have

$$\Phi_t \mathcal{H}_L^4[0, \tau] \subset \mathbb{X}_2 + (sI - \mathfrak{A})^{-1} \mathfrak{B} \mathbb{C} \subset \mathbb{X}_2,$$

where  $\Phi$  is the input map for the pair  $(\mathfrak{A}, \mathfrak{B})$ . For  $\zeta_0 \in \mathbb{H}_{\frac{3}{2}}$  and  $\zeta_1 \in \mathbb{H}_1$ , we thus arrive at

$$\tilde{z}(t) = \mathcal{T}_t \tilde{z}_0 + \Phi_t u \in C([0, \tau]; \mathbb{X}_2).$$

Therefore, we obtain the result (5.47).  $\square$

**Remark 33.** The Hilbert spaces  $\mathbb{X}_m$  defined above are similar to  $X_m$  discussed in Remark 14 and the related results can be directly applied to  $\mathbb{X}_m$ . According to Theorem 5.6.2 and using the relation (4.68), the system (5.46) is equivalent to the wave equation with boundary control (5.12) in the classical sense.

Note that, for every  $\alpha \geq 0$ ,  $\mathbb{H}_\alpha \subset \mathcal{H}^{2\alpha}[0, \pi]$ , then Theorem 5.6.3 is direct consequence of Theorem 5.6.2.

**Theorem 5.6.3.** *For every  $\tau > 0$ ,  $u \in \mathcal{H}_L^4[0, \tau]$ ,  $\zeta_0 \in \mathbb{H}_{\frac{3}{2}}$  and  $\zeta_1 \in \mathbb{H}_1$ , the system (5.12) admits a unique solution  $\zeta$  satisfying*

$$\zeta \in C([0, \tau]; \mathcal{H}^3[0, \pi]) \cap C^1([0, \tau]; \mathcal{H}^2[0, \pi]).$$

Here we recall that the Hilbert spaces  $H_{\mu, \alpha}$  and  $\mathbb{H}_\alpha$  introduced under (5.14), which is the scale of the Hilbert spaces associated with the operator  $\frac{1}{\mu} A_\mu$  and  $A_w$ , respectively. As in Section 5.3, we do the following change of variables,

$$\tilde{\alpha}_\mu := \partial_t \tilde{\zeta}_\mu \quad \tilde{\beta}_\mu := \left( \frac{1}{\mu} A_\mu \right)^{1/2} \tilde{\zeta}_\mu,$$

and

$$\tilde{\alpha} := \partial_t \tilde{\zeta} \quad \tilde{\beta} := A_w^{1/2} \tilde{\zeta}.$$

For every  $\tau > 0$ , let  $u \in \mathcal{H}_L^5[0, \tau]$ ,  $\zeta_0 \in \mathbb{H}_{\frac{3}{2}}$  and  $\zeta_1 \in \mathbb{H}_1$ , we obtain from Theorem 4.3.8 and Theorem 5.6.2 that

$$\tilde{\alpha}_\mu, \tilde{\beta}_\mu \in C([0, \tau]; H_{\mu, \frac{3}{2}}) \quad \tilde{\alpha}, \tilde{\beta} \in C([0, \tau]; \mathbb{H}_1).$$

Setting

$$\tilde{w}_\mu(t) = \begin{bmatrix} \tilde{\alpha}_\mu(t, \cdot) \\ \tilde{\beta}_\mu(t, \cdot) \end{bmatrix} \quad \text{and} \quad \tilde{w}(t) = \begin{bmatrix} \tilde{\alpha}(t, \cdot) \\ \tilde{\beta}(t, \cdot) \end{bmatrix},$$

we obtain from (4.69) and (5.46) that

$$\begin{cases} \frac{d\tilde{w}_\mu}{dt}(t) = \mathcal{A}_\mu \tilde{w}_\mu(t) + \tilde{\mathcal{B}}_\mu u(t), \\ \tilde{w}_\mu(0) = \tilde{w}_{\mu, 0}, \end{cases}$$

and

$$\begin{cases} \frac{d\tilde{w}}{dt}(t) = \mathcal{A}_0 \tilde{w}(t) + \tilde{\mathcal{B}}_0 u(t), \\ \tilde{w}(0) = \tilde{w}_0, \end{cases}$$

## 5.6. REGULAR CONVERGENCE

where  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$  have been introduced in (5.20),  $\tilde{\mathcal{B}}_\mu$  and  $\tilde{\mathcal{B}}_0$  are

$$\tilde{\mathcal{B}}_\mu u = \begin{bmatrix} \frac{1}{\mu} S_\mu u - x\chi\ddot{u} \\ 0 \end{bmatrix}, \quad \tilde{\mathcal{B}}_0 u = \begin{bmatrix} (x\chi)''u - x\chi\ddot{u} \\ 0 \end{bmatrix}, \quad (5.48)$$

and

$$\tilde{w}_{\mu,0} = \begin{bmatrix} \tilde{\zeta}_1 \\ \left(\frac{1}{\mu} A_\mu\right)^{1/2} \tilde{\zeta}_0 \end{bmatrix}, \quad \tilde{w}_0 = \begin{bmatrix} \tilde{\zeta}_1 \\ A_w^{1/2} \tilde{\zeta}_0 \end{bmatrix}.$$

Notice that  $\mathbb{H}_1 \subset H_{\mu, \frac{3}{2}}$  and  $H_{\mu, \frac{3}{2}}$  is equivalent to  $H_{\frac{3}{2}}$  for fixed  $\mu$ , we shall apply the Trotter-Kato theorem on the space  $\mathcal{X} = H_{\frac{3}{2}} \times H_{\frac{3}{2}}$ .

**Remark 34.** We have proved in Lemma 5.3.1 that the operators  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$  defined in (5.20) is skew-adjoint on  $H \times H$  and hence they generate unitary groups  $\mathbb{T}_\mu = (\mathbb{T}_{\mu,t})_{t \in \mathbb{R}}$  and  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  on  $H \times H$ , respectively. As explained in Remark 15, we still denote by  $\mathbb{T}_\mu$  the restriction of  $\mathbb{T}_\mu$  to  $\mathcal{X}$  (considered as an operator in  $\mathcal{L}(\mathcal{X})$ ) is the image of  $\mathbb{T}_\mu \in \mathcal{L}(H \times H)$  through the unitary operator  $(sI - \mathcal{A}_\mu)^{-3} \in \mathcal{L}(H \times H, \mathcal{X})$  with  $s \in \rho(\mathcal{A}_\mu)$ . More precisely,

$$\mathbb{T}_{\mu,t} z = (sI - \mathcal{A}_\mu)^{-3} \mathbb{T}_{\mu,t} (sI - \mathcal{A}_\mu)^3 z \quad \forall z \in \mathcal{X}.$$

Therefore, these operators  $(\mathbb{T}_{\mu,t})_{t \in \mathbb{R}}$  form a strongly continuous group on  $\mathcal{X}$ , whose generator is the restriction of  $\mathcal{A}_\mu$  to  $\mathcal{D}(\mathcal{A}_\mu^4)$ . Similarly, we still denote by  $\mathbb{T}$  the restriction of  $\mathbb{T}$  to  $\mathcal{X}$ . For more details about this, please refer to [115, Section 2.10].

To obtain the regular convergence of the system (5.45), we still need to show some properties of the resolvent operator on a smooth space. For this reason, let us introduce the following lemma.

**Lemma 5.6.4.** *Let  $A : \mathcal{D}(A) \rightarrow H$  be positive with compact resolvents and its non-zero eigenvalues satisfy  $\lambda_k \geq 1$  for  $k \in \mathbb{N}$ . Then we have*

$$R(s : A) \in \mathcal{L}(\mathcal{D}(A^{\frac{m-1}{2}})),$$

$$R(s : A) A^{1/2} \in \mathcal{L}(\mathcal{D}(A^{\frac{m-1}{2}})),$$

where  $s \in \mathbb{R}^-$  and  $m \in \mathbb{N}$ .

*Proof.* Since  $A$  is positive, we have  $(-\infty, 0) \subset \rho(A)$ . We prove the main result by induction. The case  $m = 1$ , i.e.  $R(s : A) \in \mathcal{L}(H)$ , is obviously true. Still using the notation  $\mathcal{D}(A^\alpha) = H_\alpha$  for any  $\alpha \geq 0$ , we suppose that the statement is true for  $m - 1$ , i.e.

$$R(s : A) \in \mathcal{L}(H_{\frac{m-2}{2}}).$$

Now we show that  $R(s : A) H_{\frac{m-1}{2}} \subset H_{\frac{m-1}{2}}$ . Note that the operator  $A$  is diagonalizable, the eigenvectors denoted by  $\varphi_k$  forms an orthonormal basis in  $H$ . For every  $f \in H_{\frac{m-1}{2}}$ , we estimate the norm of  $R(s : A)f$  in  $H_{\frac{m-1}{2}}$ , i.e.

$$\|R(s : A)f\|_{H_{\frac{m-1}{2}}}^2 = \sum_{k \in \mathbb{N}} \frac{|\lambda_k|^{m-1}}{|s - \lambda_k|^2} |\langle f, \varphi_k \rangle|^2 \leq \sum_{k \in \mathbb{N}} |\lambda_k|^{m-3} |\langle f, \varphi_k \rangle|^2 \leq \|f\|_{H_{\frac{m-1}{2}}}^2,$$

where we used  $\lambda_k \geq 1$  and  $s \leq 0$ . Since the embedding  $H_{\frac{m-1}{2}} \hookrightarrow H_{\frac{m-2}{2}}$  is continuous and  $R(s : A)H_{\frac{m-1}{2}} \subset H_{\frac{m-2}{2}}$ , we obtain from [115, Proposition 2.9.3] that  $R(s : A) \in \mathcal{L}(H_{\frac{m-1}{2}})$ .

Similarly, one readily sees that the second statement is true when  $m = 1$ . Suppose that it holds for  $m - 1$ , i.e.  $R(s : A)A^{1/2} \in \mathcal{L}(H_{\frac{m-2}{2}})$ . It is straightforward to verify that

$$\left\| R(s : A)A^{1/2}g \right\|_{H_{\frac{m-2}{2}}}^2 = \sum_{k \in \mathbb{N}} \frac{|\lambda_k|^m}{|s - \lambda_k|^2} |\langle g, \varphi_k \rangle|^2 \leq \|g\|_{H_{\frac{m-1}{2}}}^2,$$

for every  $g \in H_{\frac{m-1}{2}}$ . By using the similar argument, the second statement holds for every  $m \in \mathbb{N}$ .

Therefore, the proof is finished.  $\square$

Based on the above preparation, we have the operator convergence results for the operator  $\mathcal{A}_\mu$  and  $\mathcal{A}_0$ . The corresponding proof is similar as Section 5.4, hence we just explain in what follows that the notation make sense.

**Lemma 5.6.5.** *With the operators  $\mathcal{A}_\mu, \mathcal{A}_0$  defined in (5.20), for every  $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{X}$  we have*

$$\lim_{\mu \rightarrow 0} R(1 : \mathcal{A}_\mu) \begin{bmatrix} f \\ g \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in } \mathcal{X}. \quad (5.49)$$

*Proof.* As explained in Remark 34, we have  $\mathcal{A}_\mu, \mathcal{A}_0 \in G(1, 0)$  and then  $1 \in \rho(\mathcal{A}_\mu) \cap \rho(\mathcal{A}_0)$ . We denote, for every  $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{X}$ ,

$$\begin{bmatrix} \zeta_\mu \\ \eta_\mu \end{bmatrix} = R(1 : \mathcal{A}_\mu) \begin{bmatrix} f \\ g \end{bmatrix}, \quad \begin{bmatrix} \zeta_0 \\ \eta_0 \end{bmatrix} = R(1 : \mathcal{A}_0) \begin{bmatrix} f \\ g \end{bmatrix}.$$

It follows that

$$\begin{aligned} \zeta_\mu &= -R\left(-1 : \frac{1}{\mu}A_\mu\right) \left[ \left(\frac{1}{\mu}A_\mu\right)^{1/2} f + g \right], \\ \eta_\mu &= R\left(-1 : \frac{1}{\mu}A_\mu\right) \left[ -f + \left(\frac{1}{\mu}A_\mu\right)^{1/2} g \right], \end{aligned}$$

and

$$\zeta_0 = -R(-1 : A_w)(A_w^{1/2}f + g), \quad \eta_0 = R(-1 : A_w)(-f + A_w^{1/2}g).$$

We first obtain from Lemma 5.6.4 that  $\zeta_\mu, \eta_\mu \in H_{\frac{3}{2}}$ , for every  $f, g \in H_{\frac{3}{2}}$ . Moreover, note that the eigenvalues of the operator  $\frac{1}{\mu}A_\mu$  are equivalent to  $k$  for fixed  $\mu \in (0, 1)$ , using the Fourier series we can directly verify that

$$\|R(-1 : A_w)f\|_{H_{\frac{3}{2}}}^2 \leq \|f\|_{H_{\frac{3}{2}}}^2 \quad \forall f \in H_{\frac{3}{2}},$$

and

$$\left\| R(-1 : A_w)A_w^{1/2}g \right\|_{H_{\frac{3}{2}}}^2 \leq \|g\|_{H_{\frac{3}{2}}}^2 \quad \forall g \in H_{\frac{3}{2}}.$$

It follows that  $\zeta_0, \eta_0 \in H_{\frac{3}{2}}$ . By slightly modifying the corresponding proof of the convergence result in  $H \times H$ , we have  $\lim_{\mu \rightarrow 0} \zeta_\mu = \zeta_0$  in  $H_{\frac{3}{2}}$  and  $\lim_{\mu \rightarrow 0} \eta_\mu = \eta_0$  in  $H_{\frac{3}{2}}$ , that is (5.49).  $\square$

## 5.6. REGULAR CONVERGENCE

**Lemma 5.6.6.** *With the operators  $\tilde{\mathcal{B}}_\mu u$ ,  $\tilde{\mathcal{B}}_0 u$  defined in (5.48), we have*

$$\lim_{\mu \rightarrow 0} R(1 : \mathcal{A}_\mu) \tilde{\mathcal{B}}_\mu u = R(1 : \mathcal{A}_0) \tilde{\mathcal{B}}_0 u \quad \text{in } \mathcal{X}.$$

*Proof.* We first denote

$$\begin{bmatrix} \tilde{\zeta}_\mu \\ \tilde{\eta}_\mu \end{bmatrix} = R(1 : \mathcal{A}_\mu) \tilde{\mathcal{B}}_\mu u, \quad \begin{bmatrix} \tilde{\zeta}_0 \\ \tilde{\eta}_0 \end{bmatrix} = R(1 : \mathcal{A}_0) \tilde{\mathcal{B}}_0 u,$$

where  $\tilde{\mathcal{B}}_\mu u$  and  $\tilde{\mathcal{B}}_0 u$  are introduced in (5.48). We immediately have

$$\begin{aligned} \tilde{\zeta}_\mu &= -R \left( -1 : \frac{1}{\mu} A_\mu \right) \left( \frac{1}{\mu} S_\mu u - x\chi \ddot{u} \right), \\ \tilde{\eta}_\mu &= -R \left( -1 : \frac{1}{\mu} A_\mu \right) \left( \frac{1}{\mu} A_\mu \right)^{1/2} \left( \frac{1}{\mu} S_\mu u - x\chi \ddot{u} \right), \end{aligned}$$

and

$$\begin{aligned} \tilde{\zeta}_0 &= -R(-1 : A_w) ((x\chi)'' u - x\chi \ddot{u}), \\ \tilde{\eta}_0 &= -R(-1 : A_w) A_w^{1/2} ((x\chi)'' u - x\chi \ddot{u}). \end{aligned}$$

Based on the properties of the Fourier cosine expansion of  $x\chi$  and  $(x\chi)''$ , Remark 12, together with Proposition 5.6.1 and Lemma 5.6.4, implies that

$$R \left( -1 : \frac{1}{\mu} A_\mu \right) \frac{1}{\mu} S_\mu \in H_{\frac{3}{2}}, \quad R \left( -1 : \frac{1}{\mu} A_\mu \right) \left( \frac{1}{\mu} A_\mu \right)^{1/2} \frac{1}{\mu} S_\mu \in H_{\frac{3}{2}},$$

and

$$R(-1 : A_w)(x\chi)'' \in H_{\frac{3}{2}}, \quad R(-1 : A_w) A_w^{1/2} (x\chi)'' \in H_{\frac{3}{2}}.$$

Furthermore, using Fourier series it is not difficult to see that for fixed  $\mu \in (0, 1)$ ,

$$\begin{aligned} \left\| R \left( -1 : \frac{1}{\mu} A_\mu \right) x\chi \right\|_{H_{\frac{3}{2}}}^2 &\leq \|x\chi\|_{H_1}^2, \\ \left\| R \left( -1 : \frac{1}{\mu} A_\mu \right) \left( \frac{1}{\mu} A_\mu \right)^{1/2} x\chi \right\|_{H_{\frac{3}{2}}}^2 &\leq \|x\chi\|_{H_1}^2, \end{aligned}$$

and

$$\begin{aligned} \|R(-1 : A_w) x\chi\|_{H_{\frac{3}{2}}}^2 &\leq \|x\chi\|_{H_1}^2, \\ \|R(-1 : A_w) A_w^{1/2} x\chi\|_{H_{\frac{3}{2}}}^2 &\leq \|x\chi\|_{H_1}^2. \end{aligned}$$

We thus obtain that  $\begin{bmatrix} \tilde{\zeta}_\mu \\ \tilde{\eta}_\mu \end{bmatrix} \in \mathcal{X}$  and  $\begin{bmatrix} \tilde{\zeta}_0 \\ \tilde{\eta}_0 \end{bmatrix} \in \mathcal{X}$ . The remaining proof is similar with Lemma 5.4.2 by using triangle inequality and the estimate (5.26) and (5.37).  $\square$

Now we are in a position to state the regular convergence of the system (5.42) in terms of the elevation  $\zeta_\mu$ . The proof is similar with the proof of Theorem 5.5.1, so that we omit the details here.

**Theorem 5.6.7.** *For any  $\tau > 0$ , let  $u \in \mathcal{H}_L^5[0, \tau]$ ,  $\zeta_0 \in \mathbb{H}_{\frac{3}{2}}$  and  $\zeta_1 \in \mathbb{H}_1$ . Then we have*

$$\begin{aligned} \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\zeta_\mu - \zeta\|_{\mathcal{H}^2[0, \pi]} &= 0, \\ \lim_{\mu \rightarrow 0} \sup_{t \in [0, \tau]} \|\partial_t \zeta_\mu - \partial_t \zeta\|_{\mathcal{H}^{\frac{3}{2}}[0, \pi]} &= 0. \end{aligned}$$

**Remark 35.** In the proof of Theorem 5.6.7, we still regard  $L_\omega^2[0, \infty)$  as the input space, which does not contradict  $u \in \mathcal{H}_L^5[0, \tau]$ . The condition  $u \in \mathcal{H}_L^5[0, \tau]$  is just used in the part of the regularity of the solution of the evolution equations.

## 5.7 Remark on zero mean condition

As we already noticed, we made in Chapter 4 an assumption that the shape function  $h \in L^2[-1, 0]$  satisfies zero mean condition, i.e.

$$\int_{-1}^0 h(y) dx = 0,$$

which ensures the conservation of the volume of water. More precisely, it is not difficult to see that the conservation of mass of the water waves system in the domain  $\Omega$  (see (4.14)) implies

$$\partial_t \zeta + \partial_x \bar{V} = 0, \tag{5.50}$$

where  $V(t, x, y)$  is the horizontal component of the velocity of the fluid  $U(t, x, y)$  and  $\bar{V}(t, x, y)$  is vertically average horizontal velocity defined by

$$\bar{V}(t, x) = \int_{-1}^0 V(t, x, y) dy.$$

The conservation of the volume means that

$$\partial_t \int_0^\pi \zeta dx = 0,$$

which, together with (5.50), implies that

$$\int_0^\pi \partial_x \bar{V}(t, x, y) dx = \bar{V}(t, 0) - \bar{V}(t, \pi) = -v(t) \int_{-1}^0 h(y) dy.$$

Based on the above detailed analysis, we see that the volume of water is conserved if and only if  $h$  satisfies zero mean condition. Moreover, zero mean condition excludes the constant from the space  $L^2[0, \pi]$  such that the Dirichlet to Neumann operator  $A_0$  is strictly positive. In the first-order control system, this makes that the evolution operator  $A$  in (4.47) is skew-adjoint, so that it is the generator of a unitary group.

In this Chapter, as we already mentioned at the beginning, we should remove zero mean condition for the asymptotic analysis issue, since the whole system is considered in the shallow water regime. In this case shape  $h$  should be independent of the vertical variable  $y$ , and obviously the conservation of the volume fails when taking shallowness limit. The state space of the control system will not only contain zero mean functions but also the other elements in  $L^2[0, \pi]$ , which are actually constant part. Although the

## 5.7. REMARK ON ZERO MEAN CONDITION

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operator  $A_0$  is no longer strictly positive, the matrix  $A$  still is a generator of a strongly continuous semigroup. Therefore, the well-posedness result in Chapter 4 holds without zero mean condition. It is natural to ask a question:

Do the stabilizability properties still hold?

In the remaining part of this section, we focus on the discussion of this question.

### 5.7.1 A decomposition of the control system

We employ here a decomposition technique introduced, for instance, in [40, Section 5.2] to rewrite the boundary control system on the water waves. What we need to point out next is that kernel space of the Dirichlet-Neumann operator  $A_0$ , denoted by  $\text{Ker } A_0$ , is a one-dimensional constant space. Recalling the definition of  $A_0$ , for each  $\psi \in L^2[0, \pi]$ ,

$$\langle A_0\psi, \psi \rangle_{L^2[0, \pi]} = \langle \partial_{\bar{n}} D\psi, D\psi \rangle_{\partial\Omega} = \|\nabla D\psi\|^2 + \langle \Delta D\psi, \psi \rangle = \|\nabla D\psi\|^2,$$

where  $D$  is the partial Dirichlet map introduced in Proposition 4.2.2. Thus,  $A_0\psi = 0$  if and only if  $D\psi = \psi$  is constant. Therefore, we denote

$$H_k = \ker A_0 = \{\lambda 1 \mid \lambda \in \mathbb{C}\} \quad H_R = (\ker A_0)^\perp = \text{Ran } A_0,$$

and we have an orthogonal decomposition of  $L^2[0, \pi]$ :

$$L^2[0, \pi] = H_R \oplus H_k.$$

Note that  $H_R$  is exactly the space  $H$  defined in (4.17) and  $H_k$  consists of all complex numbers, which is a one-dimensional space. Equivalently,  $H_R$  and  $H_k$  represents the zero mean part and the constant part of  $L^2[0, \pi]$ , respectively. Next, for every function  $f \in L^2[0, \pi]$  we define a projection of  $f$  onto the vector space  $\text{span}\{\frac{1}{\sqrt{\pi}}\}$  as

$$(\Pi f)(x) = \left[ \Pi_{\text{span}\{\frac{1}{\sqrt{\pi}}\}} f \right](x) = \frac{\langle f, \frac{1}{\sqrt{\pi}} \rangle}{\|\frac{1}{\sqrt{\pi}}\|^2} \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^\pi f(x) dx.$$

The water waves system on the state space  $X = H_{\frac{1}{2}} \times H$  reads

$$\frac{d}{dt} \begin{bmatrix} \zeta \\ \check{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \check{\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ B_0 \end{bmatrix} u(t),$$

where  $A_0$  and  $B_0$  is the Dirichlet to Neumann operator and the Neumann to Neumann operator introduced in Proposition 4.2.9 and Corollary 4.2.13, respectively. Based on the above notation we now decompose the state space

$$X = H_{\frac{1}{2}} \times H \implies H_{\frac{1}{2}, R} \times H_R \times H_{\frac{1}{2}, k} \times H_k = X_R \times X_k,$$

where  $H_{\frac{1}{2}, k}$  and  $H_k$  are actually one-dimensional complex space,  $H_{\frac{1}{2}, R}$  and  $H_R$  are the same with zero mean spaces defined in (4.19). We thus have the corresponding decomposition for the state of control system as follows:

$$z^\top = [\zeta \quad \check{\zeta}]^\top \implies [\zeta_R \quad \check{\zeta}_R \quad \zeta_k \quad \check{\zeta}_k]^\top = [z_R \quad z_k]^\top,$$

Note that, according to Corollary 4.2.13,  $B_0$  preserves zero mean property, we define the operators

$$B_k = \Pi B_0 : \mathbb{C} \rightarrow H_k \quad B_R = (I - \Pi)B_0 : \mathbb{C} \rightarrow H_R,$$

and extend the matrix  $A$  and  $B$  in the following way:

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \implies \begin{bmatrix} 0 & I & 0 & 0 \\ -A_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ B_R & 0 \\ 0 & 0 \\ 0 & B_k \end{bmatrix}.$$

For every input  $u \in U = \mathbb{C}$ , we assume that  $u = u_R + u_k$ , where  $u_R$  is the input for "R" part and  $u_k$  is the input for "K" part. Therefore, we have

$$\frac{d}{dt} \begin{bmatrix} \zeta_R \\ \dot{\zeta}_R \\ \zeta_k \\ \dot{\zeta}_k \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ -A_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_R \\ \dot{\zeta}_R \\ \zeta_k \\ \dot{\zeta}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_R & 0 \\ 0 & 0 \\ 0 & B_k \end{bmatrix} \begin{bmatrix} u_R \\ u_k \end{bmatrix}. \quad (5.51)$$

Up to now, we successfully decompose the original system into an infinite-dimensional system ("R" part) and a finite-dimensional system ("K" part), which are un-coupled sub-control system. We know from Chapter 4 that "R" is strongly stabilizable with colocated feedback  $F_R z_R = -B_R^* \dot{\zeta}_R$ . The remaining work is just to find a bounded feedback operator  $F_k \in \mathcal{L}(X_k, \mathbb{C})$  to stabilize "K" part. Assuming that  $F_k = [f_1 \ f_2]$ , we need to stabilize the matrix

$$\mathcal{A}_k = A_k + \tilde{B}_k F_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_k \end{bmatrix} [f_1 \ f_2] = \begin{bmatrix} 0 & 1 \\ B_k f_1 & B_k f_2 \end{bmatrix}.$$

In finite-dimensional case,  $\mathcal{A}_k$  is stable if and only if its eigenvalues is strictly negative, which implies that

$$B_k f_2 < 0 \quad B_k f_1 < 0. \quad (5.52)$$

The finite dimensional system "K" is stabilizable with the feedback  $F_k z_k = f_1 \zeta_k + f_2 \dot{\zeta}_k$ , where  $f_1$  and  $f_2$  satisfy the above constraints (5.52). Thus we conclude that (5.51) is strongly stabilizable with the bounded feedback

$$u = \begin{bmatrix} u_R \\ u_k \end{bmatrix} = Fz = \begin{bmatrix} -B_R^* \dot{\zeta}_R \\ f_1 \zeta_k + f_2 \dot{\zeta}_k \end{bmatrix}, \quad (5.53)$$

which implies that

$$F = \begin{bmatrix} 0 & -B_R^* & 0 & 0 \\ 0 & 0 & f_1 & f_2 \end{bmatrix}.$$

Based on the above analysis, we see that finally the stability of the whole system depends on "R" part, thereby the stability properties we obtained in Chapter 4 still hold without zero mean condition.

### 5.7.2 Stabilization using an observer

According to the state feedback  $Fz$  in (5.53), we only need to measure the time derivative of the elevation of the free surface  $\dot{\zeta}_R$  (i.e. the normal velocity of the surface according to the kinematic condition) to stabilize "R" system, while for "K" system we need the knowledge of both  $\zeta_k$  and  $\dot{\zeta}_k$ . If  $\zeta_k$  or  $\dot{\zeta}_k$  is not available to the controller or it is not easy to measure one of these quantities in practice, we consider stabilizing "K" by using an observer, which requires the detectability of the control system. An observer for "K" is another system that receives the input  $u$  and the output  $y$  of "K" as inputs, and the output of the observer is an estimate  $z_{k,e}$  of the state  $z_k$  of "K", such that

$$\lim_{t \rightarrow \infty} \|z_{k,e} - z_k\|_{X_k} = 0.$$

The principle of an observer is that by combining a measured feedback signal with knowledge of the control-system components (primarily the plant and feedback system), the behaviour of the plant can be known with greater precision than using the feedback signal alone. For more details about the feedback controllers and observers, please refer to, for instance, Ellis [43] and Besançon [21]. To do this, we choose  $\zeta_k$  as the output of "K" system, then we have

$$\begin{cases} \frac{d}{dt} \begin{bmatrix} \zeta_k \\ \dot{\zeta}_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_k \\ \dot{\zeta}_k \end{bmatrix} + \begin{bmatrix} 0 \\ B_k \end{bmatrix} u_k = A_k z_k + \tilde{B}_k u_k, \\ y = \zeta_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_k \\ \dot{\zeta}_k \end{bmatrix} = C_k z_k. \end{cases} \quad (5.54)$$

To make this approach work, we first need to verify that  $(A_k, C_k)$  is detectable (or saying that  $(A_k^*, C_k^*)$  is stabilizable), i.e. there exists  $H_k \in \mathbb{C}^{2 \times 1}$  such that  $A_k + H_k C_k$  is stable. Let  $H_k$  take the form

$$H_k = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

Then the matrix  $A_k + H_k C_k$  is

$$A_k + H_k C_k = \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix}. \quad (5.55)$$

We assume that  $\lambda_1$  and  $\lambda_2$  are two eigenvalues of  $A_k + H_k C_k$ , it is clear to see from (5.55) that  $\lambda_1 + \lambda_2 = h_1$  and  $\lambda_1 \lambda_2 = -h_2$ . This implies that the matrix  $A_k + H_k C_k$  is stable if we choose  $h_1 < 0$  and  $h_2 < 0$ . This, together with the stabilizable of the pair  $(A_k, \tilde{B}_k)$  discussed in Section 5.7.1, implies the existence of a stabilizing controller, denoted by  $\Sigma_c$ , for the control system  $\Sigma = (A_k, \tilde{B}_k, C_k, -)$ . The principle of the observer is described by

$$\dot{z}_{k,e} = (A_k + H_k C_k) z_{k,e} + \tilde{B}_k u_k - H_k y,$$

where  $u$  and  $y$  are the input of the above observer and the corresponding output is the state estimate  $z_{k,e}$ . Moreover, we have  $\dot{z}_k = A_k z_k + \tilde{B}_k u_k$ . Now subtracting the equations for  $z_{k,e}$  and  $z_k$  and taking into account that  $y = C_k z_k$ , we obtain the equation for the difference  $\delta = z_{k,e} - z_k$ , which is

$$\dot{\delta} = (A_k + H_k C_k) \delta. \quad (5.56)$$



Therefore, the error system (5.56) is stable i.e.  $\delta \rightarrow 0$  (since the matrix  $A_k + H_k C_k$  is stable), regardless of the input  $u_k$  and regardless of the initial state. In this case, we only need the knowledge of  $\zeta_R$  and  $\zeta_k$  as the state feedback to stabilize the original system.

To understand this process clearly, we use the block diagram to explain the principle of the observer described above. The original closed-loop system  $(A_k, \tilde{B}_k)$ , with state feedback  $u_k = F_k z_k$  has been shown in Figure 5.2. As we can see from this block diagram that the output is the state  $z_k$ , which requires the knowledge of  $\zeta_k$  and  $\dot{\zeta}_k$ . To improve this control system in practice, we consider constructing a controller

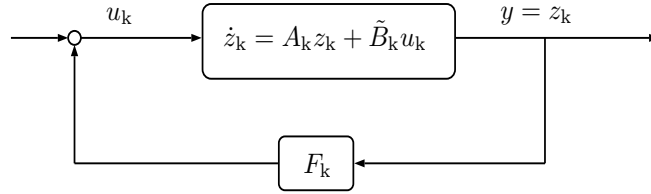


Figure 5.2: The original closed-loop system

by using an observer to estimate only one of the state component  $\zeta_k$  or  $\dot{\zeta}_k$ . For this reason, we choose  $C_k = [1 \ 0]$  and as in (5.54) the corresponding output  $y$  becomes  $y = C_k z_k = \zeta_k$ . The input of the observer are  $u_k$  and  $y$ , and its output is the estimate of the state  $z_{k,e}$ . The details of the control system with the controller described above has been shown in Figure 5.3, where the big dotted frame is the structure of the controller  $\Sigma_c$ . It is not difficult to see that this system is stable. Indeed, according to Figure 5.3, we have

$$\dot{z}_k = A_k z_k + \tilde{B}_k u_k = A_k z_k + \tilde{B}_k (F_k z_{k,e}) = (A_k + \tilde{B}_k F_k) z_k + \tilde{B}_k F_k \delta.$$

Now choosing the state variables of the closed-loop dynamics to be  $z_k$  and  $\delta$ , we obtain

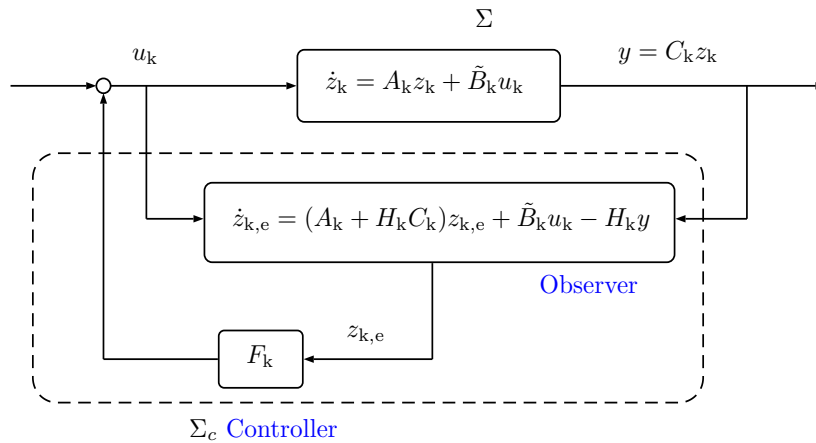


Figure 5.3: The closed-loop system with controller  $\Sigma_c$

$$\frac{d}{dt} \begin{bmatrix} z_k \\ \delta \end{bmatrix} = \begin{bmatrix} A_k + \tilde{B}_k F_k & \tilde{B}_k F_k \\ 0 & A_k + H_k C_k \end{bmatrix} \begin{bmatrix} z_k \\ \delta \end{bmatrix}, \quad (5.57)$$

## 5.7. REMARK ON ZERO MEAN CONDITION

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where we used the equation for  $\delta$  in (5.56). Obviously, we see that the system (5.57) is stable since  $(A_k, \tilde{B}_k)$  is stabilizable and  $(A_k, C_k)$  is detectable. Moreover, the above process for decomposing of the closed-loop eigenvalues into the sets  $\sigma(A_k + \tilde{B}_k F_k)$  and  $\sigma(A_k + H_k C_k)$  is called the separation principle.

**Remark 36.** We expect that the state feedback only depends on  $\dot{\zeta}_k$  so that we only need to measure the velocity of the free surface  $\dot{\zeta}$  without knowing the information about  $\zeta$ . In this case, the observer is called the *Luenberger Observer* (see, for instance, [43]). This is impossible since  $(A_k, \tilde{C}_k)$  is not detectable with  $\tilde{C}_k = [0 \ 1]$ .



## **Part III**

# **A rigid body floating in shallow water**



## Chapter 6

# Control of a floating body system in shallow water

In this chapter, we consider a control system describing the interaction of water waves with a partially immersed rigid body in a bounded container. The fluid is modeled by the shallow water equations. The motion of the object obeys Newton's laws. The control signal is a vertical force acting on the floating body. We give a detailed derivation for the full governing equations of the floating body system in shallow water, in particular, with a control term. Then we consider the linearized system for the corresponding control issue, for instance, the reachability and stabilizability, by analysing the spectral properties of the related evolution operators. The situation, of course, is different when the object floats in different position of the water tank. Part of contents in this chapter are based on recent work by Su and Tucsnak [109].

### 6.1 Introduction and problem setting

We study a rigid object floating in a water tank, which is delimited by a top free surface and a flat bottom with two vertical walls. Assuming that the rigid body is restricted to move only in the vertical direction and that it floats in a rectangular fluid domain which fits in the shallow water regime (for this concept, please refer to Lannes [66, 68] or Whitham [123]). We also assume that the floating object has vertical lateral walls, with a possibly non-flat but symmetric bottom. Moreover, the body is actuated by a vertical external force from its bottom and this force is regarded as the control signal.

The system we consider is also of interest for modelling and controlling a class of *Wave Energy Converters* (WECs) where all devices are used to capture the variations of the free surface waves and convert this kinetic energy into electricity. The most popular WECs is the so-called *Point Absorber*, which consists of a floater on the sea surface and hydraulic cylinders vertically installed below the floater (for more details, please refer to Li et al. [73] and Cretel et al. [37]). Mathematically speaking, this device acting from the bottom of the floating body produces a vertical force, as a control signal, to synchronize the motion of the body and of incoming waves and so maximize the energy production or generate desired waves.

There are a number of works which are devoted to the subject of fluid-structure interaction systems. For instance, the case of the body completely immersed in the

fluid is studied in Glass et al. [48], Lacave and Takahashi [64] and the corresponding control problem is considered in Roy and Takahashi [98], Glass et al. [47]. The case when the body is floating i.e. only partially immersed in the fluid, is setup studied in John [61, 62] under simplified assumptions. Recently, Lannes gave in [67] a new formulation of the governing equations and proposed a formulation of the problem as a coupling between a standard wave model (in which the surface elevation is free and the pressure is constrained) and a congested model containing an object (where the pressure is free and the surface elevation is constrained); this method can be implemented with various asymptotic models: non-viscous 1D shallow water model in Iguchi and Lannes [59], viscous 1D shallow water model in Maity et al. [80], 2D radial symmetric shallow water equations in Bocchi [24], Boussinesq equations in Bresch et al. [27] and also in Beck and Lannes [17]. We also refer to Godlewski et al. [49] where the constraint for the equations with the object is released, using a typical "low Mach" technique. For other interesting formulations and asymptotic models (depending on the shallowness parameter) for the water waves system we refer to Lannes [66, 68] and references therein. As far as we know, all the references on floating bodies mentioned above are only concerned with the object freely floating in the fluid and there are almost no work on the control issue.

Here we are interested in the following problem: given a rigid body floating in a fluid at rest in a bounded container, determine the control force acting on the body in order to obtain a prescribed wave profile. The main contribution in this chapter consists in showing that, within the linearized shallow water regime and in a spatially symmetric geometry, we can find controls steering the system from rest to any symmetric wave profile having an appropriate space regularity. In order to achieve this goal we pass through the following preliminary steps:

- Deriving the full nonlinear control model and reformulate it as a first-order evolution system;
- Establishing the well-posedness of the linearized control system.

### 6.1.1 Notation

To state things clearly, we introduce here, constantly referring to Figure 6.1, some notation which is used throughout this paper. We take the coordinate system as in Figure 6.1, where the ordinate axis passes through the center of the floating object. The set  $\mathcal{I} := [-l, l]$ , called the *interior region* in the remaining part of this chapter, is the projection of the object on the bottom of the fluid domain  $\Omega(t)$ . The *exterior region* is denoted by  $\mathcal{E} := \mathcal{E}^- \cup \mathcal{E}^+$  with  $\mathcal{E}^- = (-L, -l)$  and  $\mathcal{E}^+ = (l, L')$ .

Let  $h_0$  denote the water depth when the object is at equilibrium state. In the same situation of equilibrium, let  $(0, y_{G,\text{eq}})$  denote the coordinate of the center of gravity of the object and let  $h_{\text{eq}}(x)$  denote the distance between the point of abscissa  $x$  of the bottom of the object and the bottom of the fluid domain. We assume that the bottom of the object is symmetric with respect to  $x = 0$ , which implies that  $h_{\text{eq}}(x)$  is a positive single-valued even function. We denote by  $m$  the mass of the object, by  $\rho$  the constant density of the fluid. We also denote by  $\zeta(t, x)$  the elevation of the water surface with respect to the rest state, by  $h(t, x) = h_0 + \zeta(t, x)$  the total height of the water column. Moreover, we introduce the horizontal discharge, denoted by  $q(t, x)$ , that is the vertical

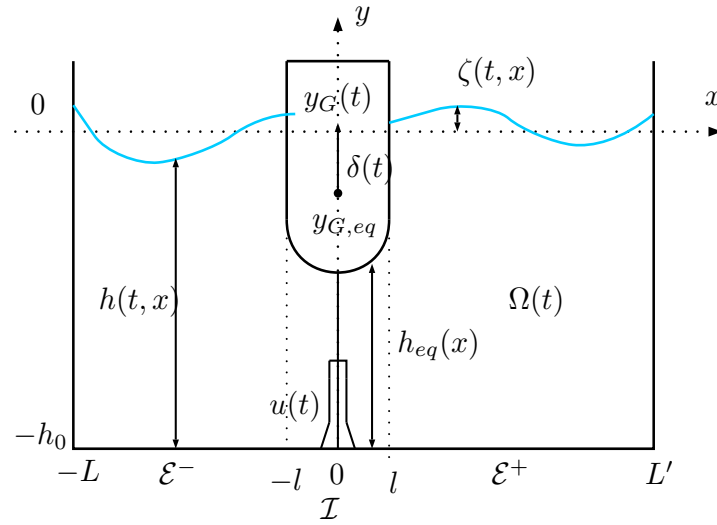


Figure 6.1: Floating body in a tank filled with water

integral of the horizontal velocity of the fluid (in the shallow water regime, it is  $h$  times the velocity of the fluid). We use the notation  $\underline{P}$  to represent the pressure on the water surface. When the object moves in the vertical direction, let  $(0, y_G(t))$  be the position of the center of gravity at time  $t$ , and  $\delta(t) = y_G(t) - y_{G,eq}$  be the variation of the position of the center of mass. Furthermore, the vertical control force acting on the object at time  $t$  is denoted by  $u(t)$ .

We define the jump and the average of a function  $f$  defined on  $[-l, l]$  by  $\llbracket f \rrbracket = f(l) - f(-l)$  and  $\langle f \rangle = \frac{1}{2}(f(l) + f(-l))$ , respectively. Moreover,  $f_i = f|_{\mathcal{I}}$  stands for the restriction of  $f$  to the interior domain  $\mathcal{I}$  and  $f_e = f|_{\mathcal{E}}$  denotes the restriction of  $f$  to the exterior domain  $\mathcal{E}$ .

### 6.1.2 Observation and strategy

The departure point of our derivation of the control system describing the interaction of the floating body with the fluid is a nonlinear model introduced in Lannes [67], where the fluid fills an infinite strip in the horizontal direction. Taking the control term into account, the governing equations of the floating body system in the fluid domain  $\Omega(t)$  can be obtained from the conservation laws of the total energy and of the volume of the water. In this case, the interior surface pressure  $\underline{P}_i$  is not only determined by the fluid dynamics, but also by the external vertical force below the floater. We show that  $\underline{P}_i$  satisfies a second-order elliptic equation, and its source term and boundary term are given in terms of  $\delta$ ,  $\langle q_i \rangle$  and the exterior functions  $\zeta_e$ ,  $q_e$ . Based on the nonlinear shallow water equations and Newton's equation, we derive the equations for  $\delta$  and  $\langle q_i \rangle$  and find that their source terms again consist of the exterior functions, respectively. In this way, the whole system can be converted to an initial boundary value problem defined only in the exterior domain  $\mathcal{E}$ . Furthermore, it can be reformulated as a first-order evolution equation, which is good for deriving the linearized version for the control topic in the following sections.

Our main interest is to study the reachable space of the linearized control system.



This space is formed of all the states that can be reached from equilibrium by means of  $L^2$  controls  $u$ . Assuming that the state of the control system is  $z$  with the state space  $X$ , for every  $\tau > 0$ , the bounded linear map  $\Phi_\tau : L^2([0, \infty); U) \rightarrow X$  is called an *input-to-state map* (briefly, input map) with zero initial data (i.e.  $z_0 = 0$ ) defined by

$$\Phi_\tau u = z(\tau) \quad \forall u \in L^2_{\text{loc}}([0, \infty); U). \quad (6.1)$$

To describe the reachable space  $\text{Ran } \Phi_\tau$ , for every  $\tau > 0$ , we could consider proving an observability inequality and then apply the closed graph theorem. For the stability properties, we use the general stability results presented in Chapter 3. All these analysis are based on a detailed description of the spectral structure of the evolution operators involved in the control system.

## 6.2 Nonlinear modelling of floating object - shallow water interaction

In this section, we derive the nonlinear governing equations describing the motion of the floating object in the fluid domain  $\Omega(t)$ , in the presence of a control applied from the bottom of the object. We follow the approach developed in [17, 67] with modifications to include the external force  $u(t)$  and the presence of the vertical boundaries of the water tank. Here we assume that the fluid fills the domain  $\Omega(t)$ , that it is homogeneous, incompressible, inviscid and irrotational. We also assume that we are here in a configuration where wave motion is correctly described by the nonlinear shallow water equations. We first know from [67] that the *nonlinear shallow water equations with a floating structure* are given, for every  $t \geq 0$  and  $x \in \mathbb{R}$ , by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P}, \end{cases} \quad (6.2)$$

where  $\zeta$ ,  $q$  and  $h$  have been introduced in Section 6.1.1,  $g$  is the gravity acceleration. The surface pressure  $\underline{P}(t, x)$  in (6.2) restricted to the exterior domain  $\mathcal{E}$  is zero, i.e.

$$\underline{P}_e = 0,$$

while the interior pressure  $\underline{P}_i(t, x)$  is unknown and it is determined by the motion of the fluid below the object and also the control signal  $u$ . We denote by  $\zeta_w(t, x)$  the parameterization of the part of the bottom of the object in contact with the fluid (the subscript "w" represents the "wetted" part of the object). Therefore, we have the water surface in the interior domain  $\mathcal{I}$  that matches the bottom of the object, i.e.

$$\zeta_w(t, x) = \zeta_i(t, x) \quad \forall x \in \mathcal{I}. \quad (6.3)$$

It is not difficult to see that we have the relation

$$\zeta_w(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad \forall x \in \mathcal{I}. \quad (6.4)$$

Moreover, we obtain from (6.4) that  $\partial_t \zeta_w = \dot{\delta}$ , which is the kinematic condition on the water surface (i.e. the vertical velocity of the water surface below the object is consistent with the velocity of the object in the vertical direction). We consider in what

## 6.2. NONLINEAR MODELLING OF FLOATING OBJECT - SHALLOW WATER INTERACTION

follows restricting the model (6.2) to the interval  $[-L, L']$ ,  $-L$  and  $L'$  being the horizontal coordinates of the water tank  $\Omega(t)$ , in particular with the control term. To this end, we observe that the following conditions need to be satisfied:

- *The conservation of the volume of the water.*

We first notice that the two vertical boundary of  $\Omega(t)$  are impermeable, which implies that

$$q(t, -L) = 0 = q(t, L'). \quad (6.5)$$

Based on this condition, the conservation of the volume of the water means that

$$\partial_t \int_{\mathcal{E} \cup \mathcal{I}} \zeta(t, x) dx = 0,$$

which implies by a simple calculation that

$$q_i(t, \pm l) = q_e(t, \pm l). \quad (6.6)$$

- *The conservation of the total energy of the fluid-structure system.*

We denote by  $E_f$  and  $E_s$  the mechanical energy of the fluid and the mechanical energy of the solid, respectively. Because of the existence of the vertical force  $u$ , the total energy of the floating object system  $E_{\text{tot}}(t) = E_f(t) + E_s(t)$  should satisfy

$$\frac{d}{dt} E_{\text{tot}}(t) = u(t) \dot{\delta}(t). \quad (6.7)$$

To ensure the conservation of the energy, we can obtain from the equation (6.7) the boundary conditions for the interior pressure  $\underline{P}_i$ . We give a detailed analysis about this in the following subsection.

### 6.2.1 Governing equations of the floating body system with control

Based on the second conservation law on the energy of the fluid-structure system, we derive the boundary conditions of the surface pressure  $\underline{P}_i$  at the two contact points  $x = \pm l$ . To do this, we first note that the mechanical energy of the object  $E_s$  is

$$E_s(t) = mg\delta(t) + \frac{1}{2}m\dot{\delta}^2(t).$$

Recalling the definition of the horizontal discharge  $q(t, x)$ , it is not difficult to see that the mechanical energy of the fluid  $E_f$  is

$$E_f(t) = \frac{\rho}{2} \int_{\mathcal{E} \cup \mathcal{I}} \left( g\zeta^2(t, x) + \frac{q^2}{h}(t, x) \right) dx.$$

Note that we have Newton's law for the motion of the object with the control  $u$  imposed from its bottom,

$$m\ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (6.8)$$

which means that the motion of the object is determined by its weight, the hydrodynamic force and the external force. Moreover, as the object at equilibrium satisfies *Archimedes' principle*, we have

$$m = \rho \int_{-l}^l (h_0 - h_{\text{eq}}(x)) \, dx. \quad (6.9)$$

Therefore, combining with Newton's law (6.8) we obtain

$$m\ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) + \rho g (h_{\text{eq}}(x) - h_0) \, dx + u(t).$$

For the sake of convenience, we introduce the *hydrodynamic interior pressure*

$$\Pi_i := \underline{P}_i + \rho g \zeta_w. \quad (6.10)$$

Finally, taking the relation (4.68) into account we arrive at

$$m\ddot{\delta}(t) + 2l\rho g \delta(t) = \int_{-l}^l \Pi_i(t, x) \, dx + u(t). \quad (6.11)$$

Based on the above analysis, we give in the following proposition the boundary condition of the interior pressure. To present it clearly, we define the energy flux  $\mathfrak{F}$  as

$$\mathfrak{F}(\zeta, q) = q (\rho g \zeta + \underline{P}_i + \mathfrak{B}) \quad \text{with} \quad \mathfrak{B} = \frac{\rho q^2}{2h^2}. \quad (6.12)$$

**Remark 37.** Actually, the energy density  $\mathfrak{E}$  of the system (6.2) is defined as

$$\mathfrak{E}(\zeta, q) = \frac{\rho}{2} \left( g \zeta^2(t, x) + \frac{q^2}{h}(t, x) \right).$$

According to the structure of the system (6.2), we have

$$\partial_t \mathfrak{E} + \partial_x \mathfrak{F} = \underline{P} \partial_x q. \quad (6.13)$$

**Proposition 6.2.1.** *Assume that the quantities  $\zeta$ ,  $q$ ,  $h$ ,  $\delta$  and  $\underline{P}$  introduced above are smooth on  $\mathcal{I}$  and  $\mathcal{E}$ . Then the total energy of the fluid-structure system  $E_{\text{tot}}$  satisfies*

$$\frac{d}{dt} E_{\text{tot}}(t) = [\mathfrak{F}_e - \mathfrak{F}_i] + u(t) \dot{\delta}(t).$$

Therefore, if the hydrodynamic pressure  $\Pi_i$  satisfies the boundary condition

$$\Pi_i(t, \pm l) = \rho g \zeta_e(t, \pm l) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \quad (6.14)$$

the total energy is conserved in the sense of (6.7).

*Proof.* We first note that the variation of the mechanical energy of the solid  $E_s$  satisfies

$$\frac{d}{dt} E_s(t) = \dot{\delta}(t) \int_{-l}^l \underline{P}_i(t, x) \, dx + u(t) \dot{\delta}(t),$$

## 6.2. NONLINEAR MODELLING OF FLOATING OBJECT - SHALLOW WATER INTERACTION

where we used the fact (6.8). For the mechanical energy of the fluid  $E_f$ , using the relation (6.13) we have

$$\frac{d}{dt}E_f(t) = \int_{\mathcal{E} \cup \mathcal{I}} (-\partial_x \mathfrak{F} + \underline{P} \partial_x q) dx = \llbracket \mathfrak{F}_e - \mathfrak{F}_i \rrbracket - \dot{\delta}(t) \int_{-l}^l \underline{P}_i(t, x) dx.$$

In the above calculation, we used the boundary conditions (6.5), which implies that  $\mathfrak{F}(t, -L) = 0 = \mathfrak{F}(t, L')$ . It follows that

$$\frac{d}{dt}E_{\text{tot}}(t) = \frac{d}{dt}(E_f(t) + E_s(t)) = \llbracket \mathfrak{F}_e - \mathfrak{F}_i \rrbracket + u(t)\dot{\delta}(t).$$

Moreover, combined with the continuity of the horizontal discharge (6.6), the boundary equations (6.14) are equivalent to  $\llbracket \mathfrak{F}_e \rrbracket = \llbracket \mathfrak{F}_i \rrbracket$ . Therefore, if  $\Pi_i$  satisfies (6.14), the mechanical energy of the fluid-structure system is conserved.  $\square$

**Remark 38.** According to the definition of the hydrodynamic pressure  $\Pi_i$  in (6.10), we obtain from (6.14) the corresponding boundary conditions for the pressure  $\underline{P}_i$ ,

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + (\mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l)).$$

It is worthwhile noting that actually  $\mathfrak{B}_i(t, \pm l)$  is fully determined by  $\delta$  and  $\langle q_i \rangle$ . Indeed, we denote by  $h_w(t, x)$  the height of the water column in the interior domain  $\mathcal{I}$ . By the definition of  $h$  and (4.68) we know that

$$h_w(t, x) = h_0 + \zeta_w(t, x) = h_{\text{eq}}(x) + \delta(t) \quad \forall x \in \mathcal{I}. \quad (6.15)$$

Together with (6.3) and the kinematic condition  $\partial_t \zeta_w = \dot{\delta}$ , we obtain that the system (6.2) restricted to the interior domain, for all  $t \geq 0$  and  $x \in \mathcal{I}$ , reads

$$\begin{cases} \partial_x q_i = -\dot{\delta}, \\ \partial_t q_i + \partial_x \left( \frac{q_i^2}{h_w} \right) + g h_w \partial_x \zeta_w = -\frac{h_w}{\rho} \partial_x \underline{P}_i. \end{cases} \quad (6.16)$$

The first equation in (6.16) implies that

$$q_i(t, x) = -x \dot{\delta}(t) + \langle q_i \rangle \quad \forall x \in \mathcal{I}. \quad (6.17)$$

Recalling the definition of  $\mathfrak{B}$  in (6.12) we have

$$\mathfrak{B}_i(t, \pm l) = \frac{\rho}{2} \left( \frac{q_i(t, \pm l)}{h_w(t, \pm l)} \right)^2 = \frac{\rho}{2} \left( \frac{\mp l \dot{\delta}(t) + \langle q_i \rangle}{h_{\text{eq}}(\pm l) + \delta(t)} \right)^2.$$

Up to now, we obtain the governing equations describing the dynamics of the floating object in the rectangular domain  $\Omega(t)$  with the control term  $u$ . For the sake of

convenience, we put all the equations together as follows, for all  $t \geq 0$ ,

$$\partial_t \zeta + \partial_x q = 0 \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (6.18a)$$

$$\partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P} \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (6.18b)$$

$$\underline{P}_e(t, x) = 0 \quad x \in \mathcal{E}, \quad (6.18c)$$

$$\zeta_i(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad x \in \mathcal{I}, \quad (6.18d)$$

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \quad (6.18e)$$

$$m \ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (6.18f)$$

$$q_e(t, -L) = 0 = q_e(t, L'), \quad q_i(t, \pm l) = q_e(t, \pm l), \quad (6.18g)$$

with the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1 \quad \forall x \in \mathcal{I} \cup \mathcal{E}.$$

**Remark 39.** There is another interesting formulation for the governing equations (6.18). As in [80], we can define the *Langrangian*  $\mathcal{L}$  and the *action functional*  $\mathcal{S}$  as

$$\mathcal{L}(\zeta, q, \delta) = (K_f + K_s) - (U_f + U_s),$$

$$\mathcal{S}(\zeta, q, \delta) = \int_0^\tau (\mathcal{L}(\zeta, q, \delta) + u \delta) dt \quad \forall \tau > 0,$$

where  $K_f$  and  $U_f$  are the kinetic energy and the potential energy of the fluid, respectively. Similarly,  $K_s$  and  $U_s$  denote the corresponding energies for the solid. The equations (6.18) can be alternatively obtained by using the *Hamiltonian principle* (see, for instance, [89]) with the equations (6.18a) and (6.18d) as constraints. The approach presented here is more convenient to generalize the model to a fluid described by the *Boussinesq equations* (which are not necessarily Hamiltonian) or even by the *Serre-Green-Naghdi equations*, which are more precise approximation of the water waves system (please refer to [66, 67, 68] and also therein).

## 6.2.2 Reformulation of the governing equations

In this part, we reformulate the governing equations (6.18) as a first-order evolution system, which will be convenient for the control issue in Section 6.3. Observe that we already have (6.17) for the interior horizontal discharge  $q_i$ . In what follows we shall derive the equations for  $\delta$  and  $\langle q_i \rangle$ . To this end, we first show that the hydrodynamic pressure  $\Pi_i$  satisfies a boundary value problem of a second-order elliptic equation.

**Proposition 6.2.2.** *The hydrodynamic interior pressure  $\Pi_i$  defined in (6.10) is the unique solution of the boundary problem*

$$\begin{cases} -\partial_x \left( \frac{h_w}{\rho} \partial_x \Pi_i \right) = -\delta + \partial_x^2 \left( \frac{q_i^2}{h_w} \right), \\ \Pi_i(t, \pm l) = \rho g \zeta_e(t, \pm l) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \end{cases} \quad (6.19)$$

where  $t \geq 0$  and  $x \in \mathcal{I}$ .

## 6.2. NONLINEAR MODELLING OF FLOATING OBJECT - SHALLOW WATER INTERACTION

*Proof.* The boundary condition of  $\Pi_i$  has been given in (6.14). Taking the derivative of the second equation in (6.16) with respect to  $x$  and using the definition (6.10) and the first equation of (6.16), we obtain (6.19) directly.  $\square$

Recall that we assumed that the bottom of the object is possibly non-flat but symmetric with respect to  $x = 0$ , which implies that  $x \mapsto h_{\text{eq}}(x)$  is an even function. To derive the equations for  $\delta$  and  $\langle q_i \rangle$ , Proposition 6.2.2 and the interior equations (6.16) will play an important role in the following analysis.

**Proposition 6.2.3.** *Assume that  $\underline{P}_i$ ,  $q_i$  and  $\delta$  are smooth solutions of the interior equations in (6.18), then  $\langle q_i \rangle$  satisfies*

$$\alpha(\delta) \frac{d}{dt} \langle q_i \rangle + \alpha'(\delta) \delta \langle q_i \rangle = -\frac{1}{2\rho l} \llbracket \rho g \zeta_e + \mathfrak{B}_e \rrbracket, \quad (6.20)$$

where  $\mathfrak{B}$  is defined in (6.12) and

$$\alpha(\delta) = \frac{1}{2l} \int_{-l}^l \frac{1}{h_w} dx, \quad \alpha'(\delta) = -\frac{1}{2l} \int_{-l}^l \frac{1}{h_w^2} dx, \quad (6.21)$$

with  $h_w$  in (6.15).

*Proof.* The proof presented here is a simplification of the proof provided in [17], in the presence of additional dispersive terms, where the system is described by the Boussinesq equations rather than the nonlinear shallow water equations. Firstly, note that (6.15) and (6.17) imply that

$$\frac{1}{2l} \int_{-l}^l \frac{q_i}{h_w} dx = \frac{1}{2l} \int_{-l}^l \frac{\langle q_i \rangle}{h_w} dx = \alpha \langle q_i \rangle,$$

where we used the property  $h_{\text{eq}}(x) = h_{\text{eq}}(-x)$ , for every  $x \in (-l, l)$ . Recalling the definition of the hydrodynamic pressure  $\Pi_i$  in (6.10), we take an integration of the second equation in (6.16) with respect to  $x$ , which gives

$$2l\alpha \frac{d}{dt} \langle q_i \rangle + \left\llbracket \frac{q_i^2}{h_w^2} \right\rrbracket + \int_{-l}^l \frac{q_i^2}{h_w^3} \partial_x h_w dx = -\frac{1}{\rho} \llbracket \Pi_i \rrbracket. \quad (6.22)$$

By using (6.17) we find that

$$\int_{-l}^l \frac{q_i^2}{h_w^3} \partial_x h_w dx = -2\delta \langle q_i \rangle \int_{-l}^l \frac{x \partial_x h_w}{h_w^3} dx.$$

According to the boundary condition of  $\Pi_i$  in Proposition 6.2.2, we have

$$\llbracket \Pi_i \rrbracket = \llbracket \rho g \zeta_e + \mathfrak{B}_e - \mathfrak{B}_i \rrbracket.$$

Moreover, we obtain from (6.17) that

$$\llbracket q_i^2 \rrbracket = q_i^2(t, l) - q_i^2(t, -l) = -4l \delta \langle q_i \rangle,$$

which yields that

$$\left\llbracket \frac{q_i^2}{h_w^2} \right\rrbracket = -4l \left\langle \frac{1}{h_w^2} \right\rangle \delta \langle q_i \rangle, \quad \llbracket \mathfrak{B}_i \rrbracket = \frac{\rho}{2} \left\llbracket \frac{q_i^2}{h_w^2} \right\rrbracket = -2l\rho \left\langle \frac{1}{h_w^2} \right\rangle \delta \langle q_i \rangle.$$

Therefore, the equation (6.22) becomes

$$\alpha \frac{d}{dt} \langle q_i \rangle - \left\langle \frac{1}{h_w^2} \right\rangle \delta \langle q_i \rangle - \delta \langle q_i \rangle \frac{1}{l} \int_{-l}^l \frac{x \partial_x h_w}{h_w^3} dx = -\frac{1}{2\rho l} [\rho g \zeta_e + \mathfrak{B}_e],$$

which implies (6.20) by using integration by parts for  $\alpha'$  introduced in (6.21).  $\square$

Based on Newton's laws (6.11) and the second-order elliptic system (6.19), next we derive the equations for  $\delta$ . It is worthwhile noting that there is an external force  $u$  on the right side of (6.23), which does not appear in the equations obtained in [17, 67].

**Proposition 6.2.4.** *For smooth solutions of the system (6.18), the displacement of the center of gravity of the object  $\delta$  satisfies*

$$M(\delta) \ddot{\delta} - 2\rho l \beta(\delta) \delta^2 + 2\rho g l \delta = 2l \langle \rho g \zeta_e + \mathfrak{B}_e \rangle + \rho l \alpha'(\delta) \langle q_i \rangle^2 + u, \quad (6.23)$$

where  $\alpha'(\delta)$  has been introduced in (6.21) and

$$M(\delta) = m + \int_{-l}^l \frac{\rho x^2}{h_w} dx, \quad \beta(\delta) = \frac{1}{4l} \int_{-l}^l \frac{x^2}{h_w^2} dx. \quad (6.24)$$

*Proof.* To derive (6.23), we first obtain from Newton's laws presented in (6.11), by doing integration by parts, that

$$m \ddot{\delta} + 2l \rho g \delta = 2l \langle \Pi_i \rangle - \int_{-l}^l x \partial_x \Pi_i(t, x) dx + u. \quad (6.25)$$

Taking an integration of the first equation of (6.19) with respect to  $x$ , we have

$$-\frac{h_w}{\rho} \partial_x \Pi_i = -x \delta + \partial_x \left( \frac{q_i^2}{h_w} \right) + C_0(t),$$

where  $C_0(t)$  is an arbitrary function of  $t$ . Observe that the integration of an odd function on  $(-l, l)$  vanishes, which implies that

$$-\int_{-l}^l x \partial_x \Pi_i dx = -\int_{-l}^l \frac{\rho x^2}{h_w} \delta dx + \rho \int_{-l}^l \frac{x}{h_w} \partial_x \left( \frac{q_i^2}{h_w} \right) dx. \quad (6.26)$$

Moreover, using integration by parts the second term on the right side of (6.26) gives

$$\rho \int_{-l}^l \frac{x}{h_w} \partial_x \left( \frac{q_i^2}{h_w} \right) dx = 2\rho l \left\langle \frac{1}{h_w^2} \right\rangle \langle q_i^2 \rangle - \int_{-l}^l \frac{\rho q_i^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx.$$

Using (6.17) we immediately have

$$\begin{aligned} \langle q_i^2 \rangle &= l^2 \delta^2 + \langle q_i \rangle^2, \\ -\int_{-l}^l \frac{\rho q_i^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx &= -\int_{-l}^l \left( \frac{\rho x^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) \delta^2 + \frac{\rho \langle q_i \rangle^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) \right) dx, \end{aligned}$$

which yields that

$$\begin{aligned} -\int_{-l}^l x \partial_x \Pi_i dx &= -\delta \int_{-l}^l \frac{\rho x^2}{h_w} dx + \delta^2 \left( 2\rho l \left\langle \frac{x^2}{h_w^2} \right\rangle - \int_{-l}^l \frac{\rho x^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx \right) \\ &\quad + \langle q_i \rangle^2 \left( 2\rho l \left\langle \frac{1}{h_w^2} \right\rangle - \int_{-l}^l \frac{\rho}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx \right). \quad (6.27) \end{aligned}$$

## 6.2. NONLINEAR MODELLING OF FLOATING OBJECT - SHALLOW WATER INTERACTION

Furthermore, according to the definition of  $\Pi_i$  in (6.10) and  $\mathfrak{B}$  in (6.12), together with the boundary condition in (6.19), we have

$$\langle \Pi_i \rangle = \langle \rho g \zeta_e + \mathfrak{B}_e - \mathfrak{B}_i \rangle, \quad \langle \mathfrak{B}_i \rangle = \left\langle \frac{1}{h_w^2} \right\rangle \left( \frac{\rho l^2}{2} \delta^2 + \frac{\rho}{2} \langle q_i \rangle^2 \right). \quad (6.28)$$

Based on the structure of the right side of (6.27), it is interesting to see, by using integration by parts, that

$$\begin{aligned} - \int_{-l}^l \frac{\rho}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx &= - \int_{-l}^l \frac{\rho}{h_w^2} dx - \frac{1}{2} \int_{-l}^l \rho x \partial_x \left( \frac{1}{h_w^2} \right) dx \\ &= - \frac{1}{2} \int_{-l}^l \frac{\rho}{h_w^2} dx - \rho l \left\langle \frac{1}{h_w^2} \right\rangle, \end{aligned} \quad (6.29)$$

and

$$\begin{aligned} - \int_{-l}^l \frac{\rho x^2}{h_w} \partial_x \left( \frac{x}{h_w} \right) dx &= - \int_{-l}^l \frac{\rho x^2}{h_w^2} dx - \frac{1}{2} \int_{-l}^l \rho x^3 \partial_x \left( \frac{1}{h_w^2} \right) dx \\ &= \frac{1}{2} \int_{-l}^l \frac{\rho x^2}{h_w^2} dx - \rho l^3 \left\langle \frac{1}{h_w^2} \right\rangle. \end{aligned} \quad (6.30)$$

Now we go back to (6.25), putting together (6.27), (6.28), (6.29) and (6.30), and we obtain (6.23) immediately.  $\square$

According to the equations (6.20) and (6.23), we see that, for given initial data, the average horizontal discharge  $\langle q_i \rangle$  and the displacement  $\delta$  are totally determined by the functions in the exterior domain  $\mathcal{E}$ . Therefore, the governing equations (6.18) can be reduced to a non-linear shallow water system restricted to the exterior domain. Here again, in the absence of the external force  $u(t)$ , the corresponding result can be deduced from [17] by neglecting the dispersive terms.

**Theorem 6.2.5.** *For smooth solutions, the system (6.18) can be equivalently rewritten into the following transmission system defined in the exterior domain  $\mathcal{E}$ , i.e. for every  $t \geq 0$ ,  $x \in \mathcal{E}$ ,*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad (6.31)$$

with transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta}, \quad (6.32)$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L').$$

Moreover, the discharge  $\langle q_i \rangle$  and the displacement  $\delta$  are determined, for every  $t \geq 0$  and  $x \in \mathcal{E}$ , by

$$\begin{cases} \alpha(\delta) \frac{d}{dt} \langle q_i \rangle + \alpha'(\delta) \dot{\delta} \langle q_i \rangle = - \frac{1}{2\rho l} \llbracket \rho g \zeta_e + \mathfrak{B}_e \rrbracket, \\ M(\delta) \ddot{\delta} - 2\rho l \beta(\delta) \dot{\delta}^2 + 2\rho g l \delta - \rho l \alpha'(\delta) \langle q_i \rangle^2 = 2l \langle \rho g \zeta_e + \mathfrak{B}_e \rangle + u, \end{cases} \quad (6.33)$$

where  $\mathfrak{B}$  is introduced in (6.12),  $\alpha(\delta)$  and  $\alpha'(\delta)$ ,  $M(\delta)$  and  $\beta(\delta)$  are defined in (6.21) and (6.24), respectively.



*Proof.* Recalling the continuity condition (6.6) and the equation (6.17), we first obtain the transmission conditions (6.32). For given initial data of  $\zeta$ ,  $q$ ,  $\langle q_i \rangle$ ,  $\delta$  and  $\dot{\delta}$ , the coupled system (6.31) and (6.33), with transmission and boundary conditions for  $q$ , form a closed initial boundary value problem.  $\square$

According to Theorem 6.2.5, we can further rewrite the system (6.31)-(6.33) into a first-order evolution system in terms of  $\zeta$ ,  $q$ ,  $\langle q_i \rangle$ ,  $\delta$  and  $\dot{\delta}$ . This is straightforward and we have the following corollary.

**Corollary 6.2.6.** *For smooth solutions, the system (6.18) is equivalent to the following first-order evolution system, i.e. for every  $t \geq 0$  and  $x \in \mathcal{E}$ ,*

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -\partial_x \left( \frac{q^2}{h} \right) - gh \partial_x \zeta, \\ \frac{d}{dt} \begin{bmatrix} \langle q_i \rangle \\ \delta \\ \dot{\delta} \end{bmatrix} = \mathcal{Q}^{-1}(\alpha, M) \begin{bmatrix} -\alpha' \dot{\delta} \langle q_i \rangle - \frac{1}{2\rho l} \llbracket \rho g \zeta + \mathfrak{B} \rrbracket \\ \dot{\delta} \\ 2\rho l \beta \dot{\delta}^2 - 2\rho g l \delta + \rho l \alpha' \langle q_i \rangle^2 + 2l \langle \rho g \zeta + \mathfrak{B} \rangle + u \end{bmatrix}, \end{cases} \quad (6.34)$$

with transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l \dot{\delta},$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L'),$$

where the matrix  $\mathcal{Q}(\alpha, M)$  in (6.34) is defined as

$$\mathcal{Q}(\alpha, M) = \begin{bmatrix} \alpha(\delta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M(\delta) \end{bmatrix},$$

the quantity  $\mathfrak{B}$  is introduced in (5.26),  $\alpha(\delta)$  and  $\alpha'(\delta)$ ,  $M(\delta)$  and  $\beta(\delta)$  are defined in (5.17) and (6.24), respectively.

**Remark 40.** The well-posedness theory for (6.31)-(6.33) is a delicate question, due to the nonlinear couplings: the boundary conditions (6.32) of the hyperbolic problem (6.31) require the knowledge of  $\langle q_i \rangle$  and  $\delta$ . Conversely, the equations (6.33) require the knowledge of the trace of  $\zeta$  and  $q$  at the contact points  $x = \pm l$ . An interesting question, which lies outside the scope of the present work, is to adapt to our case the local existence theory developed in [59] in the case of an unbounded fluid domain and without control.

### 6.3 Well-posedness and spectral analysis for the linear system

In this section, we shall work on the linearized version of the first-order evolution system associated with (6.31)-(6.33). Before studying the control problem in Section 6.4, we first present the linearized model and establish its well-posedness. In the second

part of this section, we focus on the spectral analysis of the semigroup generator associated with this linearized equation.

Linearizing the system (6.31)-(6.33) in Theorem 6.2.5 around the equilibrium state  $[\zeta \ q \ \langle q_i \rangle \ \delta \ \dot{\delta}]^T = [0 \ 0 \ 0 \ 0 \ 0]^T$ , we obtain, for all  $t \geq 0$  and  $x \in \mathcal{E}$ ,

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \frac{d}{dt} \langle q_i \rangle = -\frac{g}{2l\alpha} \llbracket \zeta \rrbracket, \\ \ddot{\delta} = -\frac{2\rho g l}{M} \delta + \frac{2\rho g l}{M} \langle \zeta \rangle + \frac{1}{M} u, \end{cases} \quad (6.35)$$

with transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta},$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L'),$$

and the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \langle q_i \rangle(0) = \langle q_i \rangle_0, \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1.$$

The constants  $\underline{\alpha}$  and  $\underline{M}$  in (6.35) are

$$\underline{\alpha} = \alpha(0) \quad \underline{M} = M(0), \quad (6.36)$$

where  $\alpha(\delta)$  and  $M(\delta)$  have been defined in (6.21) and (6.24), respectively.

**Remark 41.** A more intuitive and more general way to linearize the floating body system (6.31)-(6.33) is to write them in dimensionless form by using the nonlinearity parameter  $\varepsilon$  defined by

$$\varepsilon = \frac{a}{h_0},$$

where  $a$  is the typical amplitude of the waves. Then the associated linear system can be obtained by ignoring all the terms including  $\varepsilon$  in the dimensionless version of (6.31)-(6.33). This commonly used approach has been introduced in [66] and [68].

### 6.3.1 Well-posedness of the linearized system

Observe that our system has been recast in the exterior domain  $\mathcal{E}$ , so we need to rewrite the energy of the whole system in terms of the exterior functions. Recalling that the mechanical energy for the fluid and for the object are presented in Section 6.2, we decompose the total energy of the linearized system (6.35) into the interior part  $\underline{E}_{\text{int}}$  and the exterior part  $\underline{E}_{\text{ext}}$  as follows:

$$\begin{aligned} \underline{E}_{\text{ext}}(t) &= \frac{\rho}{2} \int_{\mathcal{E}} \left( \frac{q^2}{h_0}(t, x) + g\zeta^2(t, x) \right) dx, \\ \underline{E}_{\text{int}}(t) &= \frac{\rho}{2} \int_{\mathcal{I}} \left( \frac{q^2}{h_{\text{eq}}}(t, x) + g\zeta^2(t, x) \right) dx + \frac{1}{2} m \dot{\delta}^2 + mg\delta. \end{aligned}$$

The underline in the notation  $E$  represents the corresponding energy for linear system. Using the relation (4.68), (6.15) and (6.17), together with Archimedes' principle (6.9), we obtain that

$$\underline{E}_{\text{int}}(t) = \frac{1}{2} \delta^2 \left( m + \int_{\mathcal{I}} \frac{\rho x^2}{h_{\text{eq}}} dx \right) + \langle q_i \rangle^2 \int_{\mathcal{I}} \frac{\rho}{2h_{\text{eq}}} dx + \frac{\rho g}{2} \int_{\mathcal{I}} (h_{\text{eq}}(x) - h_0)^2 dx + \rho g l \delta^2.$$

Therefore, we conclude that the total energy for (6.35), denoted by  $\underline{E}_{\text{tot}}$ , is

$$\begin{aligned} \underline{E}_{\text{tot}}(t) = \frac{\rho}{2} \int_{\mathcal{E}} \left( \frac{q^2}{h_0}(t, x) + g \zeta^2(t, x) \right) dx + \frac{1}{2} \underline{M} \delta^2 + \langle q_i \rangle^2 \rho l \underline{\alpha} \\ + \rho g l \delta^2 + \frac{\rho g}{2} \int_{\mathcal{I}} (h_{\text{eq}}(x) - h_0)^2 dx, \end{aligned} \quad (6.37)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are introduced in (6.36).

Based on the formula of the total energy  $\underline{E}_{\text{tot}}$  in (6.37), we introduce the Hilbert space  $X$  defined by

$$X = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^T \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right\}, \quad (6.38)$$

endowed with the inner product

$$\left\langle \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix}, \begin{bmatrix} \tilde{\zeta} \\ \tilde{q} \\ \langle \tilde{q}_i \rangle \\ \tilde{\delta} \\ \tilde{\eta} \end{bmatrix} \right\rangle_X = \frac{\rho g}{2} \langle \zeta, \tilde{\zeta} \rangle_{L^2(\mathcal{E})} + \frac{\rho}{2h_0} \langle q, \tilde{q} \rangle_{L^2(\mathcal{E})} + \rho l \underline{\alpha} \langle q_i \rangle \overline{\langle \tilde{q}_i \rangle} + \rho g l \delta \bar{\tilde{\delta}} + \frac{\underline{M}}{2} \eta \bar{\tilde{\eta}}. \quad (6.39)$$

**Remark 42.** We can see from (6.37) that the total energy only depends on the functions  $\delta$ ,  $\langle q_i \rangle$ ,  $\zeta$  and  $q$  with the space variable  $x \in \mathcal{E}$ . The condition

$$\int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0$$

in the definition of the space  $X$  is motivated by the conservation of the volume.

Equations (6.35) determine a well-posed linear control system (also called *abstract linear control system* in Weiss [119] or Tucsnak and Weiss [115]), with state space  $X$  defined in (6.38) and control space  $U = \mathbb{C}$ , by choosing the appropriate spaces and operators. More precisely, let  $A : \mathcal{D}(A) \rightarrow X$  and  $B \in \mathcal{L}(U, X)$  be defined by

$$A = \begin{bmatrix} 0 & -\frac{d}{dx} & 0 & 0 & 0 \\ -gh_0 \frac{d}{dx} & 0 & 0 & 0 & 0 \\ -\frac{g}{2l\underline{\alpha}} [\cdot] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{\underline{M}} \rho g l \langle \cdot \rangle & 0 & 0 & -\frac{2}{\underline{M}} \rho g l & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\underline{M}} \end{bmatrix}, \quad (6.40)$$

with

$$\mathcal{D}(A) = \left\{ [\zeta \ q \ \langle q_i \rangle \ \delta \ \eta]^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \right. \\ \left. \begin{aligned} \llbracket q \rrbracket &= -2l\eta, \quad \langle q \rangle = \langle q_i \rangle \quad \text{and} \quad q(-L) = 0 = q(L') \end{aligned} \right\}. \quad (6.41)$$

In other words, with the above choice of spaces and operators, the initial boundary value problem of the system (6.35) can be rewritten as

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0, \end{cases} \quad (6.42)$$

where  $z$  and  $z_0$  are

$$z = [\zeta \ q \ \langle q_i \rangle \ \delta \ \dot{\delta}]^\top, \quad z_0 = [\zeta_0 \ q_0 \ \langle q_i \rangle_0 \ \delta_0 \ \delta_1]^\top.$$

The well-posedness of the linearized fluid-structure system (6.42) is a direct consequence of the fact that  $B \in \mathcal{L}(U, X)$  and of following result:

**Proposition 6.3.1.** *The operator  $A : \mathcal{D}(A) \rightarrow X$  defined in (6.40)–(6.41) is skew-adjoint. Therefore, it generates a group of unitary operators on the Hilbert space  $X$ . Moreover,  $A$  has compact resolvents.*

*Proof.* We first show that  $A$  is skew-symmetric. For the sake of simplicity the computations leading to the property are performed looking to  $X$  as a Hilbert space over  $\mathbb{R}$ . For every  $z = [\zeta \ q \ \langle q_i \rangle \ \delta \ \eta]^\top \in \mathcal{D}(A)$ , using the inner product defined in (6.39) we have

$$\langle Az, z \rangle_X = -\frac{\rho g}{2} \left( \left\langle \frac{dq}{dx}, \zeta \right\rangle_{L^2(\mathcal{E})} + \left\langle \frac{d\zeta}{dx}, q \right\rangle_{L^2(\mathcal{E})} + \llbracket \zeta_e \rrbracket \langle q_i \rangle - 2l \langle \zeta_e \rangle \eta \right).$$

By using an integration by parts, we get

$$-\left\langle \frac{dq}{dx}, \zeta \right\rangle_{L^2(\mathcal{E})} = \llbracket (\zeta q)_e \rrbracket + \left\langle q, \frac{d\zeta}{dx} \right\rangle_{L^2(\mathcal{E})}.$$

Note that the boundary conditions in  $\mathcal{D}(A)$  implies that

$$q(l) = \langle q_i \rangle - l\eta, \quad q(-l) = \langle q_i \rangle + l\eta,$$

which, by a simple calculation, gives that

$$\langle Az, z \rangle_X = 0.$$

According to [115, Section 3.7], we thus obtain that the operator  $A$  is skew-symmetric.

Secondly, we prove that  $A$  is onto. For every  $f = [f_1 \ f_2 \ f_3 \ f_4 \ f_5]^\top \in X$ , let us solve the equation

$$A \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix} = \begin{bmatrix} -\frac{dq}{dx} \\ -gh_0 \frac{d\zeta}{dx} \\ -\frac{g}{2l\alpha} \llbracket \zeta_e \rrbracket \\ \eta \\ \frac{2}{M} \rho g l \langle \zeta_e \rangle - \frac{2}{M} \rho g l \delta \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix} \in \mathcal{D}(A), \quad (6.43)$$

which immediately implies that  $\eta = f_4$ . Solving the equation from the first component of (6.43), i.e.  $-\frac{dq}{dx} = f_1$  with the boundary conditions  $q(-L) = 0$  and  $q(L') = 0$ , we obtain

$$q(x) = \begin{cases} -\int_{-L}^x f_1(\xi) d\xi & \forall x \in (-L, -l), \\ \int_x^{L'} f_1(\xi) d\xi & \forall x \in (l, L'). \end{cases} \quad (6.44)$$

Similarly, from the second equation we get

$$\zeta(x) = \begin{cases} -\frac{1}{gh_0} \int_{-L}^x f_2(\xi) d\xi + c_1 := F(x) + c_1 & \forall x \in (-L, -l), \\ \frac{1}{gh_0} \int_x^{L'} f_2(\xi) d\xi + c_2 := G(x) + c_2 & \forall x \in (l, L'), \end{cases} \quad (6.45)$$

where the constants  $c_1$  and  $c_2$  are to be determined. The above formula, together with the last component of (6.43), gives the expression for  $\delta$ :

$$\delta = \frac{1}{2} (F(-l) + G(l) + c_1 + c_2) - \frac{M}{2\rho g l} f_5.$$

Moreover, we derive from the third equation of (6.43) that

$$-\frac{g}{2l\alpha} (G(l) - F(-l) + c_2 - c_1) = f_3. \quad (6.46)$$

Note that the functions  $\zeta$  and  $\delta$  must satisfy the condition for the conservation of the volume

$$\int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0,$$

which implies that

$$Lc_1 + L'c_2 = \frac{M}{\rho g} f_5 - \int_{-L}^{-l} F(x) dx - \int_l^{L'} G(x) dx - G(l)l - F(-l)l. \quad (6.47)$$

Combining (6.46) and (6.47), we can determine the constants  $c_1$  and  $c_2$  in (6.45). According to the continuity of the discharge (6.6) and (6.44), we have  $\langle q_i \rangle = \langle q \rangle = \frac{1}{2}(q(l) + q(-l))$ . Finally, we still need to verify that  $\llbracket q \rrbracket = -2l\eta$ . Since  $f \in X$ , we have  $\int_{\mathcal{E}} f_1(x) dx + 2lf_4 = 0$ , which, together with (6.44), implies that  $\llbracket q \rrbracket = -2lf_4 = -2l\eta$ . Thus we have found  $z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^T \in \mathcal{D}(A)$ , so that (6.43) holds.

According to a classical result [115, Proposition 3.7.2], we conclude that  $A$  is skew-adjoint and  $0 \in \rho(A)$ . By Stone's theorem (see, for instance, [115, Theorem 3.8.6]),  $A$  generates a unitary group on  $X$ . Moreover, it is not difficult to see that  $\mathcal{D}(A)$  is compactly embedded in the state space  $X$ , which implies that the operator  $A$  has compact resolvents.  $\square$

Based on Proposition 6.3.1, we denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  the *strongly continuous group* (also called  $C_0$ -group) generated by the operator  $A$ . Note that  $B \in \mathcal{L}(\mathbb{C}, X)$ , which is of course an admissible control operator (for this concept, see, for instance, [115, Chapter 4]). Therefore,  $(A, B)$  forms a well-posed linear control system (see Definition 4.3.1). According to the classical semigroup theory, we have the following conclusion.

**Theorem 6.3.2.** For  $u \in L^2_{\text{loc}}[0, \infty)$ , the initial data  $z_0 = [\zeta_0 \ q_0 \ \langle q_i \rangle_0 \ \delta_0 \ \delta_1]^\top \in X$ , the linear system (6.42) admits a unique mild solution  $z$ . This solution is given by

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma,$$

and it satisfies

$$z \in C([0, \infty); X).$$

### 6.3.2 Spectral analysis

In this part, we focus on the study of the spectral structure of the operator  $A$  introduced in (6.40)–(6.41). Note that the operator  $A$  is skew-adjoint, the eigenvalues of  $A$  are purely imaginary, i.e.  $\sigma(A) \subset i\mathbb{R}$ . We give in the following proposition the characteristic equation for the eigenvalues and the formula for the corresponding eigenvectors.

**Proposition 6.3.3.** For the operator  $A$  introduced in (6.40)–(6.41),  $i\omega$  with  $\omega \in \mathbb{R}$  is the eigenvalues of  $A$  if and only if  $\omega$  satisfies

$$\begin{aligned} -\sqrt{\frac{g}{h_0}} \left[ 2\rho l^2 \omega + (\underline{M}\omega^2 - 2\rho g l) \frac{1}{l\underline{\alpha}\omega} \right] (f_\omega(L)g_\omega(L') + f_\omega(L')g_\omega(L)) \\ + 2(\underline{M}\omega^2 - 2\rho g l) f_\omega(L)f_\omega(L') + \frac{4\rho g l}{h_0\underline{\alpha}} g_\omega(L)g_\omega(L') = 0, \end{aligned} \quad (6.48)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are given in (6.36),  $f_\omega$  and  $g_\omega$  are defined as

$$f_\omega(x) = \sin\left(\frac{\omega}{\sqrt{gh_0}}(x-l)\right), \quad g_\omega(x) = \cos\left(\frac{\omega}{\sqrt{gh_0}}(x-l)\right). \quad (6.49)$$

Moreover,  $\phi = [\varphi \ \psi \ c \ a \ b]^\top$  is an eigenvector corresponding to the eigenvalue  $i\omega$  if and only if

$$\varphi(x) = \begin{cases} \frac{iK_1}{\sqrt{gh_0}} \cos\left(\frac{\omega}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ -\frac{iK_2}{\sqrt{gh_0}} \cos\left(\frac{\omega}{\sqrt{gh_0}}(L'-x)\right) & \forall x \in (l, L'), \end{cases} \quad (6.50)$$

$$\psi(x) = \begin{cases} K_1 \sin\left(\frac{\omega}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ K_2 \sin\left(\frac{\omega}{\sqrt{gh_0}}(L'-x)\right) & \forall x \in (l, L'), \end{cases} \quad (6.51)$$

and

$$\begin{aligned} c &= \frac{1}{2}(\psi(l) + \psi(-l)), \\ a &= \frac{i}{2\omega l}(\psi(l) - \psi(-l)), \quad b = -\frac{1}{2l}(\psi(l) - \psi(-l)), \end{aligned} \quad (6.52)$$

where  $K_1, K_2$  are not simultaneously vanishing real numbers (not necessarily independent).

*Proof.* Let  $\phi = [\varphi \ \psi \ c \ a \ b]^T \in \mathcal{D}(A)$  be the eigenvector of the operator  $A$  corresponding to the eigenvalue  $i\omega$  with  $\omega \in \mathbb{R}$ . To obtain the formula of  $\phi$ , we solve the equation

$$A \begin{bmatrix} \varphi \\ \psi \\ c \\ a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{d\psi}{dx} \\ -gh_0 \frac{d\varphi}{dx} \\ -\frac{g}{2l\alpha} \llbracket \varphi \rrbracket \\ b \\ \frac{2}{M} \rho g l \langle \varphi \rangle - \frac{2}{M} \rho g l a \end{bmatrix} = i\omega \begin{bmatrix} \varphi \\ \psi \\ c \\ a \\ b \end{bmatrix}, \quad (6.53)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are introduced in (6.36). Recalling the definition of  $\mathcal{D}(A)$  in (6.41), we have

$$\psi(-L) = 0 = \psi(L'), \quad \int_{\mathcal{E}} \varphi(x) dx + 2la = 0, \quad (6.54)$$

and

$$\llbracket \psi \rrbracket = -2lb, \quad \langle \psi \rangle = c. \quad (6.55)$$

Combining the first two equations in (6.53), we obtain a second-order differential equation for  $\psi$

$$\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{gh_0} \psi \quad \forall x \in \mathcal{E},$$

which, together with the boundary condition in (6.54), implies that  $\psi$  takes the form (6.51). In (6.51),  $K_1$  and  $K_2$  are not simultaneously zero. Notice that  $-\frac{d\psi}{dx} = i\omega \varphi$ , we further obtain (6.50). Using the relation between  $\varphi$  and  $a$  in (6.54), we derive that

$$a = \frac{i}{2\omega l} (\psi(l) - \psi(-l)),$$

which further, by using the fourth equation of (6.53), implies that

$$b = -\frac{1}{2l} (\psi(l) - \psi(-l)).$$

Taking the conditions (6.55) into account, we have

$$c = \frac{1}{2} (\psi(l) + \psi(-l)).$$

This, together with the third and the last components of (6.53), imply that the imaginary part of the eigenvalue  $\omega$  satisfies

$$\begin{cases} \frac{ig}{l\underline{\alpha}\omega} (\varphi(l) - \varphi(-l)) = \psi(l) + \psi(-l), \\ \frac{\rho g l}{\underline{M}} (\varphi(l) + \varphi(-l)) = \left( \frac{i\rho g}{\underline{M}\omega} - \frac{i\omega}{2l} \right) (\psi(l) - \psi(-l)), \end{cases} \quad (6.56)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are given in (6.36). Using the formula (6.50) and (6.51), the system (6.56) yields that

$$\left[ \sqrt{\frac{g}{h_0}} \frac{1}{l\underline{\alpha}\omega} g_{\omega}(L) - f_{\omega}(L) \right] K_1 + \left[ \sqrt{\frac{g}{h_0}} \frac{1}{l\underline{\alpha}\omega} g_{\omega}(L') - f_{\omega}(L') \right] K_2 = 0, \quad (6.57)$$

$$\begin{aligned} & \left[ \sqrt{\frac{g}{h_0}} 2\rho l^2 \omega g_\omega(L) - (\underline{M}\omega^2 - 2\rho g l) f_\omega(L) \right] K_1 \\ & + \left[ (\underline{M}\omega^2 - 2\rho g l) f_\omega(L') - \sqrt{\frac{g}{h_0}} 2\rho l^2 \omega g_\omega(L') \right] K_2 = 0, \end{aligned} \quad (6.58)$$

where  $f_\omega$  and  $g_\omega$  are introduced in (6.49). According to the knowledge of linear algebra, the equations (6.57) and (6.58) admit non-trivial solutions  $[K_1 \ K_2]^\top$ , if the determinant of their coefficient matrix is zero. Therefore, we obtain the characteristic equation (6.48).  $\square$

Since  $A$  is skew-adjoint with compact resolvents (see Proposition 6.3.1), according to a classical result (see, for instance, [115, Chapter 3]), we know that  $A$  is diagonalizable, also called Riesz-spectral operator, for instance, in [40]. We denote by  $(\phi_k)_{k \in \mathbb{Z}^*}$  an orthonormal basis in  $X$  consisting of eigenvectors of  $A$  and by  $(i\omega_k)_{k \in \mathbb{Z}^*}$  the corresponding purely imaginary eigenvalues. Observe that the coefficient matrix of the system (6.57)–(6.58) can be zero, which implies that the roots of (6.48), i.e. the eigenvalues  $(i\omega_k)_{k \in \mathbb{Z}^*}$ , are not necessarily simple. We specify this situation in what follows.

**Remark 43.** Assume that  $\kappa := \underline{M} - 2\rho l^3 \underline{\alpha} > 0$  and that the parameters  $L, L', l$  and  $h_0$  satisfy

$$\sqrt{\frac{2\rho l}{\kappa h_0} \frac{L' - L}{\pi}} \in \mathbb{Z}, \quad (6.59)$$

and

$$\tan \left( \sqrt{\frac{2\rho l}{\kappa h_0}} (L - l) \right) = \frac{1}{l \underline{\alpha}} \sqrt{\frac{\kappa}{2\rho l h_0}}. \quad (6.60)$$

(Recall that the constants  $\underline{M}$  and  $\underline{\alpha}$  have been introduced in (6.36)). Then there exist two double eigenvalues of  $A$ , denoted by  $i\omega^+$  and  $i\omega^-$ , with

$$\omega^\pm = \pm \sqrt{\frac{2\rho g l}{\kappa}}.$$

We are not able to confirm or to inform the existence of  $L, L' > 0, l < \min\{L, L'\}$  and of a function  $h_{\text{eq}}$  to simultaneously satisfying the assumptions at the beginning of this remark. However, it is clear that these conditions are, generically with respect to the parameters listed above, not satisfied, so that the eigenvalues are generically simple. Note that if there is at least one double eigenvalue then the system cannot be controlled (even approximately) by a scalar input. The result below provides a sufficient condition in a special case ensuring that all the eigenvalues of  $A$  are simple.

**Proposition 6.3.4.** *Assume that the bottom of the floating object is flat. Let  $h_0 > 2\sqrt{\frac{2}{3}}l$  and the function  $h_{\text{eq}}$  satisfies*

$$h_0 > h_{\text{eq}} \geq \frac{1}{2} \left( h_0 + \sqrt{h_0^2 - \frac{8}{3}l^2} \right) \quad \text{or} \quad 0 < h_{\text{eq}} \leq \frac{1}{2} \left( h_0 - \sqrt{h_0^2 - \frac{8}{3}l^2} \right). \quad (6.61)$$

*The all the eigenvalues of  $A$  are simple.*



*Proof.* Recalling the definition of  $h_{\text{eq}}$ , the flat bottom of the object implies that  $h_{\text{eq}}$  is a positive constant function. Using (6.36), (6.21) and (6.24), it is not difficult to see that if  $h_{\text{eq}}$  satisfies the condition (6.61) then  $\underline{M} - 2\rho l^3 \underline{\alpha} \leq 0$ . This excludes the situation of the double eigenvalues discussed in Remark 43.  $\square$

In order to study the reachability and stabilizability properties of the linearized floating-body system in Section 6.4, it is necessary to make the inner structure of the eigenvalues (the distance between the eigenvalues) clear for the explicit decay rate of the solution of the control system (6.42).

**Proposition 6.3.5.** *Assume that the eigenvalues  $(i\omega_k)_{k \in \mathbb{Z}^*}$  of the operator  $A$  are simple. Then  $(\omega_k)_{k \in \mathbb{Z}^*}$  form a strictly increasing sequence, i.e.  $\lim_{|k| \rightarrow \infty} |\omega_k| = \infty$ . Moreover, we assume that  $\frac{L'-l}{L-l}$  is a real algebraic number of degree  $n$  with  $n \in \mathbb{N}$  (i.e. it is a root of a non-zero polynomial of degree  $n$  in one variable with rational coefficients), then there exists  $C_0 > 0$  such that*

$$\inf_{k \in \mathbb{Z}^*} |\omega_{k+1} - \omega_k| \geq C_0 \quad \text{if } \frac{L'-l}{L-l} \in \mathbb{Q} \text{ and } \frac{L'-l}{L-l} \neq \frac{r+1}{r} \quad \forall r \in \mathbb{Z}^*, \quad (6.62)$$

$$\inf_{k \in \mathbb{Z}^*} |k(\omega_{k+1} - \omega_k)| \geq C_0 \quad \text{otherwise.} \quad (6.63)$$

*Proof.* Since  $A$  is skew-adjoint with compact resolvents, according to [115, Proposition 3.2.12], the imaginary part of the eigenvalues  $(\omega_k)_{k \in \mathbb{Z}^*}$  can be ordered to form a strictly increasing sequence such that  $\lim_{|k| \rightarrow \infty} |\omega_k| = \infty$ . Therefore, it suffices to show that (6.62) and (6.63) holds for  $|k|$  large enough. Noting that the functions  $f_{\omega_k}$  and  $g_{\omega_k}$  defined in (6.49) are bounded for large values of  $|k|$ , we rewrite the equation (6.48) as

$$\begin{aligned} \sqrt{\frac{g}{h_0}} \left( \frac{M}{l\underline{\alpha}} + 2\rho l^2 \right) (f_{\omega_k}(L)g_{\omega_k}(L') + f_{\omega_k}(L')g_{\omega_k}(L)) \omega_k + r_{\omega_k} \\ = 2\underline{M}f_{\omega_k}(L)f_{\omega_k}(L')\omega_k^2, \end{aligned} \quad (6.64)$$

where  $r_{\omega_k}$  represents the remaining bounded terms. As  $|k|$  approaches to infinity, we observe that the right hand side of (6.64) grows faster than the left side, thus we must have

$$\lim_{|k| \rightarrow \infty} f_{\omega_k}(L)f_{\omega_k}(L') = 0.$$

Based on this observation, the eigenvalues of  $A$  can be split into two subsequences  $(i\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $(i\omega_{n_k})_{k \in \mathbb{Z}^*}$ , which is induced by  $f_{\omega_k}(L) \rightarrow 0$  and  $f_{\omega_k}(L') \rightarrow 0$  as  $|k| \rightarrow \infty$ , respectively. Therefore, there are two subsequences of  $\mathbb{Z}^*$ :  $(m_k)_{k \in \mathbb{Z}^*}$  and  $(n_k)_{k \in \mathbb{Z}^*}$  such that, for  $|k|$  large enough, we have

$$\begin{aligned} \omega_{m_k} &= \mu m_k \pi + O(\varepsilon_{m_k}) & \text{with } \lim_{|k| \rightarrow \infty} \varepsilon_{m_k} &= 0, \\ \omega_{n_k} &= \nu n_k \pi + O(\tilde{\varepsilon}_{n_k}) & \text{with } \lim_{|k| \rightarrow \infty} \tilde{\varepsilon}_{n_k} &= 0, \end{aligned}$$

where  $\mu = \frac{\sqrt{gh_0}}{L-l}$  and  $\nu = \frac{\sqrt{gh_0}}{L'-l}$ . For large  $|k|$ , substituting the first subsequence

$(\omega_{m_k})_{k \in \mathbb{Z}^*}$  into the equation (6.64), we have

$$\begin{aligned} & -\sqrt{\frac{g}{h_0}} \left( \frac{M}{l\alpha} + 2\rho l^2 \right) \left[ \varepsilon_{m_k} \cos \left( \frac{L'-l}{L-l} m_k \pi \right) + \sin \left( \frac{L'-l}{L-l} m_k \pi \right) + O(\varepsilon_{m_k}^2) \right] \omega_{m_k} \\ & + 2M \left[ \varepsilon_{m_k} \sin \left( \frac{L'-l}{L-l} m_k \pi \right) + O(\varepsilon_{m_k}^2) \right] \omega_{m_k}^2 + \text{lower order terms} = 0, \end{aligned}$$

which implies that  $\omega_{m_k} = O(\varepsilon_{m_k}^{-1})$  and thus we derive that  $\varepsilon_{m_k} = O(m_k^{-1})$  for large  $|k|$ . Similarly, we also obtain that  $\tilde{\varepsilon}_{n_k} = O(n_k^{-1})$ . Notice that there is a gap between every two elements both from the sequence  $(\omega_{m_k})_{k \in \mathbb{Z}^*}$  or  $(\omega_{n_k})_{k \in \mathbb{Z}^*}$ . Now we consider the distance between  $(\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $(\omega_{n_k})_{k \in \mathbb{Z}^*}$ . Since the eigenvalues are strictly increasing, we estimate the difference

$$|\omega_{p+1} - \omega_p| = \left| pv\pi \left( \frac{\mu}{\nu} - \frac{p+1}{p} \right) + O\left(\frac{1}{p}\right) \right|, \quad (6.65)$$

where  $\omega_p \in (\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $\omega_{p+1} \in (\omega_{n_k})_{k \in \mathbb{Z}^*}$  correspond to different type of the eigenvalues. If  $\frac{\mu}{\nu} = \frac{L'-l}{L-l}$  is a rational number but different with  $\frac{k+1}{k}$  for any  $k \in \mathbb{Z}^*$ , we see that there is a uniform gap between the eigenvalues of  $A$ . If  $\frac{\mu}{\nu} = \frac{k_0+1}{k_0}$  for some  $k_0 \in \mathbb{Z}^*$ , we obtain from (6.65) that the distance between the eigenvalues is of order  $\frac{1}{k}$ . If  $\frac{\mu}{\nu}$  is not a rational number, then it is an irrational algebraic number of degree  $n \geq 2$ . According to Liouville's approximation theorem (see, for instance, Stolarsky's book [106, Chapter 3]), there exists a constant  $C > 0$  such that

$$\left| \frac{\mu}{\nu} - \frac{q}{p} \right| \geq \frac{C}{p^n},$$

for all rational numbers  $\frac{q}{p}$ . Hence, we derive from (6.65) that  $|\omega_{p+1} - \omega_p| \geq \frac{c}{p}$ . Putting all the cases together, we finish the proof.  $\square$

**Remark 44.** We remark that the set of real algebraic numbers of degree  $n$  with  $n \in \mathbb{N}$  contains all rational numbers and some irrational numbers. All rational numbers form the real algebraic numbers of degree 1, and the other part of the real algebraic numbers are irrational algebraic numbers with  $n \geq 2$ . In particular, the irrational algebraic numbers of degree 2 are called *quadratic irrational numbers*.

**Remark 45.** In the proof of Proposition 6.3.3, we have obtained the specific expression for the eigenvectors  $\phi_k = [\varphi_k \ \psi_k \ c_k \ a_k \ b_k]^\top$ , which is, for every  $k \in \mathbb{Z}^*$ , given by (6.50)–(6.52). Now we normalize  $\phi_k$  in the Hilbert space  $X$  introduced in (6.38). By using (6.50)–(6.52) and after elementary but tedious calculations, we check that for every  $k \in \mathbb{Z}^*$  we have

$$\begin{aligned} \|\phi_k\|_X^2 &= \left( \frac{\rho l \alpha}{4} + \frac{M}{8l^2} + \frac{\rho g}{4\omega_k^2 l} \right) (K_2^2 f_{\omega_k}(L')^2 + K_1^2 f_{\omega_k}(L)^2) \\ &+ \left( \frac{\rho l \alpha}{2} - \frac{M}{4l^2} - \frac{\rho g}{2\omega_k^2 l} \right) K_1 K_2 f_{\omega_k}(L) f_{\omega_k}(L') + \frac{\rho}{2h_0} (K_1^2 (L-l) + K_2^2 (L'-l)), \end{aligned}$$

where  $\alpha$ ,  $M$  and  $f_{\omega_k}$  are defined in (6.36) and (6.49), respectively. Therefore, we obtain the normalized engenvectors  $\hat{\phi}_k := (\gamma_k \phi_k)_{k \in \mathbb{Z}^*}$  with  $\|\hat{\phi}_k\|_X = 1$ , where  $\gamma_k$  is defined by

$$\begin{aligned} \gamma_k^{-2} = & \left( \frac{\rho l \alpha}{4} + \frac{M}{8l^2} + \frac{\rho g}{4\omega_k^2 l} \right) (K_2^2 f_{\omega_k}(L')^2 + K_1^2 f_{\omega_k}(L)^2) \\ & + \left( \frac{\rho l \alpha}{2} - \frac{M}{4l^2} - \frac{\rho g}{2\omega_k^2 l} \right) K_1 K_2 f_{\omega_k}(L) f_{\omega_k}(L') \\ & + \frac{\rho}{2h_0} (K_1^2(L-l) + K_2^2(L'-l)) \quad (k \in \mathbb{Z}^*). \end{aligned}$$

## 6.4 Reachability and stabilizability of the linearized system

### 6.4.1 Some background on controllability and reachable spaces

We begin by recalling some definitions on the controllability of general infinite dimensional systems. We consider the abstract differential equation of the form

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0, \end{cases} \quad (6.66)$$

where  $A$  is an infinitesimal generator of a strongly continuous semigroup  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on a Hilbert space  $X$ , and  $B$  is an admissible control operator of the system (6.66) from the input space  $U$  to the state space  $X$ . This operator is called *bounded* if  $B \in \mathcal{L}(U, X)$ , which is the case of interest in this paper. At a given time  $t$ , the control  $u(t)$  belongs to the input space  $U$ .

Using the semigroup  $\mathbb{T}$  and the control operator  $B$  we can define the input maps  $(\Phi_\tau)_{\tau \geq 0}$  (already appearing in (6.1)) by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-s} B u(s) ds \quad \forall \tau > 0, u \in L_{\text{loc}}^2([0, \infty); U). \quad (6.67)$$

An important role in control theory is played by the range of the operators  $(\Phi_\tau)_{\tau \geq 0}$  defined in (6.67) and denoted, for every  $\tau > 0$ , by  $\text{Ran } \Phi_\tau$ . For each  $\tau > 0$ ,  $\text{Ran } \Phi_\tau$  is called *the reachable space of the system (6.66) in time  $\tau$* . These spaces appear, in particular, in the definition of exact and approximate controllability which are recalled below (see, for instance, [115, Chapter 11] or [40, Chapter 4]).

**Definition 6.4.1.** Let  $\tau > 0$ .

1. The system (6.66) is exactly controllable in time  $\tau$  if every element of  $X$  can be reached from the origin at time  $\tau$ , i.e. if

$$\text{Ran } \Phi_\tau = X;$$

2. The system (6.66) is approximately controllable in time  $\tau$  if

$$\overline{\text{Ran } \Phi_\tau} = X;$$

It is well known, see, for instance, [115, Chapter 6,8], that approximate controllability can be characterized by duality as follows:

**Proposition 6.4.2.** *Let  $\tau > 0$ .*

1. *The system (6.66) is approximately controllable in time  $\tau$  if and only if*

$$B^* \mathbb{T}_t^* z = 0 \quad \forall t \in [0, \tau] \implies z = 0.$$

2. *Assume that  $A$  is skew-adjoint and with compact resolvents, so that there exists an orthonormal basis  $(\phi_k)_{k \in \mathbb{Z}^*}$  in  $X$  consisting of eigenvectors of  $A$  and let  $(i\omega_k)_{k \in \mathbb{Z}^*}$ , with  $\omega_k \in \mathbb{R}$  be the corresponding eigenvalues. Moreover, assume that the eigenvalues of  $A$  are simple and that there exists  $m, \gamma > 0$  such that*

$$|\omega_k - \omega_l| \geq \gamma \quad (k, l \in \mathbb{Z}^*, k \neq l, |k| \geq m, |l| \geq m).$$

*Then the following conditions are equivalent:*

- *The system (6.66) is approximately controllable in any time  $\tau > \frac{2\pi}{\gamma}$ ;*
- *$B^* \phi_k \neq 0$  for every  $k \in \mathbb{Z}^*$ .*

### 6.4.2 Symmetric case

In this section we come back to the system (6.42), in the particular case of a symmetric geometry and of initial data satisfying appropriate symmetry conditions. We show that in this case the state trajectories of (6.42) coincide with those of a "reduced" system whose state space is a closed subspace of  $X$  defined in (6.38) and we study the reachable spaces of this reduced system.

Let the floating object be in the middle of the fluid domain  $\Omega$  in the horizontal direction, i.e.  $L = L'$ , see Figure 6.1. We assume that, at the initial state, the floating body system is at equilibrium state, i.e. for every  $x \in \mathcal{E}$ ,

$$z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^T = [0 \quad 0 \quad 0 \quad 0 \quad 0]^T.$$

In this case, when the object moves in the vertical direction, the fluid on two sides of the object goes in opposite directions. To describe this more clearly, we define the Hilbert space  $X_{\text{sym}}$  by

$$X_{\text{sym}} = \left\{ \begin{bmatrix} \zeta & q & 0 & \delta & \eta \end{bmatrix}^T \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right. \\ \left. \zeta(-x) = \zeta(x), \quad q(-x) = -q(x) \right\}, \quad (6.68)$$

with the inner product

$$\left\langle \begin{bmatrix} \zeta \\ q \\ 0 \\ \delta \\ \eta \end{bmatrix}, \begin{bmatrix} \tilde{\zeta} \\ \tilde{q} \\ 0 \\ \tilde{\delta} \\ \tilde{\eta} \end{bmatrix} \right\rangle_{X_{\text{sym}}} = \frac{\rho g}{2} \langle \zeta, \tilde{\zeta} \rangle_{L^2(\mathcal{E})} + \frac{\rho}{2h_0} \langle q, \tilde{q} \rangle_{L^2(\mathcal{E})} + \rho g l \delta \tilde{\delta} + \frac{M}{2} \eta \tilde{\eta},$$

where  $\underline{M}$  has been introduced in (6.36).

**Proposition 6.4.3.** *The Hilbert space  $X_{\text{sym}}$  introduced in (6.68) is  $\mathbb{T}$ -invariant i.e.*

$$\mathbb{T}_t z \in X_{\text{sym}} \quad \forall t \geq 0, z \in X_{\text{sym}},$$

where  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  is the unitary group generated by the operator  $A$  defined in (6.40).

*Proof.* Note that it suffices to show that the system (6.35) preserves the symmetry condition in the Hilbert space  $X_{\text{sym}}$ . Assume that the elevation  $\zeta$  and the horizontal discharge  $q$  satisfy (6.35) and have the following properties

$$\zeta(t, -x) = \zeta(t, x), \quad q(t, -x) = -q(t, x) \quad \forall t \geq 0, x \in \mathcal{E}. \quad (6.69)$$

We define  $\hat{\zeta}$  and  $\hat{q}$  as

$$\hat{\zeta}(t, x) = \zeta(t, -x), \quad \hat{q}(t, x) = -q(t, -x) \quad \forall t \geq 0, x \in \mathcal{E},$$

which implies that

$$\langle q_i \rangle = -\langle \hat{q}_i \rangle, \quad \llbracket \zeta_e \rrbracket = -\llbracket \hat{\zeta}_e \rrbracket, \quad \langle \zeta_e \rangle = \langle \hat{\zeta}_e \rangle.$$

It is not difficult to obtain the corresponding equation for  $\hat{\zeta}$  and  $\hat{q}$ , which implies that  $\hat{\zeta}$  and  $\hat{q}$  also satisfy the system (6.35).  $\square$

Note that the symmetric property (6.69) implies

$$\llbracket \zeta_e \rrbracket = 0 = \langle q_e \rangle = \langle q_i \rangle \quad \text{and} \quad \langle \zeta_e \rangle = \zeta_e(t, l),$$

which simplify the linear control system (6.42). Since  $X_{\text{sym}}$  is a closed subspace of  $X$  introduced in (6.38), we have the following decomposition

$$X = X_{\text{sym}} \oplus X_{\text{sym}}^\perp. \quad (6.70)$$

**Remark 46.** The word "symmetric" in this section means that not only that the object is in the center of the domain in the horizontal direction ( $L' = L$ ), but also that the functions  $\zeta$  and  $q$  satisfy the symmetry condition (6.69).

We thus obtain a new linear system on the spatial domain  $\mathcal{E}$ . In this symmetric case, the system (6.42) with zero initial data reduces to the following equations defined on  $\mathcal{E}$ , i.e. for all  $t \geq 0, x \in \mathcal{E}$ ,

$$\begin{cases} \dot{w} = A_{\text{sym}} w + Bu, \\ w(0) = w_0, \end{cases} \quad (6.71)$$

where  $w$  and  $w_0$  are

$$w = [\zeta \quad q \quad 0 \quad \delta \quad \dot{\delta}]^\top, \quad w_0 = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top.$$

The operator  $A_{\text{sym}} : \mathcal{D}(A_{\text{sym}}) \rightarrow X_{\text{sym}}$  is densely defined as

$$A_{\text{sym}} = \begin{bmatrix} 0 & -\frac{d}{dx} & 0 & 0 & 0 \\ -gh_0 \frac{d}{dx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{M} \rho g l \langle \cdot \rangle & 0 & 0 & -\frac{2}{M} \rho g l & 0 \end{bmatrix}, \quad (6.72)$$

with the domain

$$\mathcal{D}(A_{\text{sym}}) = \left\{ \begin{array}{l} [\zeta \quad q \quad 0 \quad \delta \quad \eta]^T \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid [\zeta \quad q \quad 0 \quad \delta \quad \eta]^T \in X_{\text{sym}}, \\ \llbracket q \rrbracket = -2l\eta, \quad q(-L) = 0 = q(L) \end{array} \right\}, \quad (6.73)$$

where  $\underline{M}$  is introduced in (6.36). The control operator  $B$  has been defined in (6.40) and we clearly have  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ .

Note that  $A_{\text{sym}}$  is the part of  $A$  in the closed subspace  $X_{\text{sym}}$  of  $X$ , so it inherits from  $A$  the properties of being skew-adjoint and has compact resolvents. Therefore, it is diagonalizable and generates a group of unitary operators, denoted by  $\mathbb{T}_{\text{sym}} = (\mathbb{T}_{\text{sym},t})_{t \in \mathbb{R}}$ , on the Hilbert space  $X_{\text{sym}}$  defined in (6.68). Moreover, according to [115, Section 2.4], it is interesting to see from Proposition 6.4.3 that  $\mathbb{T}_{\text{sym}}$  is the restriction of  $\mathbb{T}$  to  $X_{\text{sym}}$ . Therefore, for  $u \in L_{\text{loc}}^2[0, \infty)$ , the linear system (6.71) is well-posed and the solution  $w \in C([0, \infty); X_{\text{sym}})$ .

**Remark 47.** Since  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ , it is clear that the input maps of  $(A, B)$  and of  $(A_{\text{sym}}, B)$ , the latter being defined by

$$\Phi_{\text{sym},\tau} u = \int_0^\tau \mathbb{T}_{\text{sym},\tau-s} B u(s) ds \quad \forall u \in L_{\text{loc}}^2([0, \infty); U),$$

have the same range, i.e., that

$$\text{Ran } \Phi_\tau = \text{Ran } \Phi_{\text{sym},\tau} \quad \forall \tau > 0.$$

This means, in particular, that the orthogonal complement space  $X_{\text{sym}}^\perp$  in (6.70) is out of control, justifying the fact that we concentrate on the reachability of the pair  $(A_{\text{sym}}, B)$ .

The spectrum of the operator  $A_{\text{sym}}$  can be obtained, by using the properties (6.69), from the spectrum of  $A$  discussed in Proposition 6.3.3. More precisely, we have:

**Proposition 6.4.4.** *Assume that the object is in the middle of the fluid domain which has the symmetry geometry in the sense (6.69). The eigenvalues of the operator  $A_{\text{sym}}$ , denoted by  $i\omega_{\text{sym},k}$ , and the corresponding eigenvectors  $\phi_{\text{sym},k} = [\varphi_{\text{sym},k} \quad \psi_{\text{sym},k} \quad 0 \quad a_{\text{sym},k} \quad b_{\text{sym},k}]^T \in \mathcal{D}(A_{\text{sym}})$ , for all  $x \in \mathcal{E}$  and  $k \in \mathbb{Z}^*$ , are*

$$\varphi_{\text{sym},k}(x) = \begin{cases} \frac{iK}{\sqrt{gh_0}} \cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ \frac{iK}{\sqrt{gh_0}} \cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-x)\right) & \forall x \in (l, L), \end{cases} \quad (6.74)$$

$$\psi_{\text{sym},k}(x) = \begin{cases} K \sin\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ -K \sin\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-x)\right) & \forall x \in (l, L), \end{cases} \quad (6.75)$$

and

$$a_{\text{sym},k} = \frac{i}{\omega_{\text{sym},k} l} \psi_{\text{sym},k}(l), \quad b_{\text{sym},k} = -\frac{1}{l} \psi_{\text{sym},k}(l), \quad (6.76)$$

where  $K$  is an arbitrary constant and the imaginary part of the eigenvalues  $\omega_{\text{sym},k}$  with  $k \in \mathbb{Z}^*$  satisfies

$$(\underline{M}\omega_{\text{sym},k}^2 - 2\rho gl)f_{\omega_{\text{sym},k}}(L) = \sqrt{\frac{g}{h_0}}2\rho l^2\omega_{\text{sym},k}g_{\omega_{\text{sym},k}}(L), \quad (6.77)$$

with  $f_{\omega_{\text{sym},k}}$  and  $g_{\omega_{\text{sym},k}}$  introduced in (6.49). Moreover, the eigenvalues  $(i\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  are simple and  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  form a strictly increasing sequence, with

$$\lim_{k \in \mathbb{Z}^*, |k| \rightarrow \infty} |\omega_{\text{sym},k+1} - \omega_{\text{sym},k}| = \frac{\sqrt{gh_0}}{L-l}\pi.$$

*Proof.* Let  $\phi_{\text{sym}} = [\varphi_{\text{sym}} \ \psi_{\text{sym}} \ 0 \ a_{\text{sym}} \ b_{\text{sym}}]^\top \in \mathcal{D}(A_{\text{sym}})$  be an eigenvector of  $A_{\text{sym}}$  corresponding to the eigenvalue  $i\omega_{\text{sym}}$  ( $\omega_{\text{sym}} \in \mathbb{R}$ ), we solve the equation

$$A_{\text{sym}}\phi_{\text{sym}} = i\omega_{\text{sym}}\phi_{\text{sym}}.$$

According to Proposition 6.3.3, using the symmetry condition (6.69) we obtain that  $\phi_{\text{sym}}$  take the form (6.74)–(6.76), in particular, the third component of  $\phi_{\text{sym}}$  vanishes. In this case, the constants  $K_1$  and  $K_2$  in Proposition 6.3.3 have the relation  $K_1 = -K_2 = K$ . The equation for  $\omega_{\text{sym}}$  thus becomes

$$\frac{2\rho gl}{\underline{M}}\varphi_{\text{sym}}(l) = i \left( \frac{2\rho g}{\underline{M}\omega_{\text{sym}}} - \frac{\omega_{\text{sym}}}{l} \right) \psi_{\text{sym}}(l),$$

which gives the characteristic equation (6.77). Clearly, the solutions of (6.77), denoted by  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$ , form a strictly increasing sequence. According to the proof of Proposition 6.3.5, there is one type of the eigenvalues in the symmetric case and for large  $|k|$

$$\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l) = k\pi + O\left(\frac{1}{k}\right). \quad (6.78)$$

Moreover, (6.78) implies that there exists  $M > 0$  such that

$$|\omega_{\text{sym},k+1} - \omega_{\text{sym},k}| > \frac{\sqrt{gh_0}}{L-l}\pi \quad \forall k \in \mathbb{Z}^* \text{ and } |k| > M, \quad (6.79)$$

which ends the proof.  $\square$

**Remark 48.** Without using Proposition 6.3.5, the asymptotic behaviour of the eigenvalues in (6.78) can be obtained in an alternative way. By using the characteristic equation (6.77), without loss of generality, we assume that  $\cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l)\right)$  is non-zero. It follows that

$$\tan\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l)\right) = \sqrt{\frac{g}{h_0}}2\rho l^2 \frac{\omega_{\text{sym},k}}{\underline{M}\omega_{\text{sym},k}^2 - 2\rho gl} = O\left(\frac{1}{\omega_{\text{sym},k}}\right),$$

for large  $k \in \mathbb{Z}^*$ . Based on the above expression, we assume that

$$\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l) = k\pi + \theta_k,$$

with  $\theta_k \rightarrow 0$  as  $k \rightarrow \infty$ . By using the fixed point method introduced in, for instance, the book [46, Chapter 7] or [34, Lemma A.3], we derive that  $\theta_k = O(k^{-1})$ .

## 6.4. REACHABILITY AND STABILIZABILITY OF THE LINEARIZED SYSTEM

By using (6.74)–(6.76), we do some trivial calculations and obtain for every  $k \in \mathbb{Z}^*$  that

$$\|\phi_{\text{sym},k}\|_{X_{\text{sym}}}^2 = \left( \frac{M}{2l^2} + \frac{\rho g}{\omega_{\text{sym},k}^2 l} \right) K^2 f_{\omega_{\text{sym},k}}^2(L) + \frac{\rho}{h_0} K^2 (L-l),$$

where  $f_{\text{sym},k}$  and  $\underline{M}$  are introduced in (6.49) and (6.36) respectively. Now, for every  $k \in \mathbb{Z}^*$ , we define  $\gamma_{\text{sym},k}$  by

$$(\gamma_{\text{sym},k})^{-2} = \left( \frac{M}{2l^2} + \frac{\rho g}{\omega_{\text{sym},k}^2 l} \right) K^2 f_{\omega_{\text{sym},k}}^2(L) + \frac{\rho}{h_0} K^2 (L-l). \quad (6.80)$$

We therefore obtain the normalized eigenvectors  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*} := (\gamma_{\text{sym},k} \phi_{\text{sym},k})_{k \in \mathbb{Z}^*}$  that form an orthonormal basis in  $X_{\text{sym}}$ .

**Remark 49.** As we already realized, the symmetry property (6.69) excludes the case of the double eigenvalues discussed in Section 6.3.2. Based on the decomposition (6.70), we notice that  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*}$  is a proper subset of  $(\widehat{\phi}_k)_{k \in \mathbb{Z}^*}$  introduced in Remark 45. Moreover, we have

$$\omega_{\text{sym},k} = \omega_{j(k)} \quad \forall k \in \mathbb{Z}^*,$$

where  $\omega_{j(k)}$  is the eigenvalue of  $A$  and the subscript  $j(k) \in \mathbb{Z}^*$  can be easily found.

### 6.4.3 Reachability and stabilizability

In this section, we begin the main topic on the reachability and stabilizability of the control system (6.71) in the symmetric case. The adjoint  $B^* \in \mathcal{L}(X_{\text{sym}}, \mathbb{C})$  of the control operator  $B$  defined in (6.40) is

$$B^* = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}. \quad (6.81)$$

We have the following results on the description of the reachable space  $\text{Ran } \Phi_\tau$  with  $\tau > 0$ .

**Theorem 6.4.5.** *Assume that the object floats in the middle of the fluid domain in the horizontal direction, i.e.  $L' = L$ . Then for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ , we have*

$$\mathcal{D}(A_{\text{sym}}) = \text{Ran } \Phi_\tau \subset X_{\text{sym}}, \quad (6.82)$$

where each inclusion is dense and with continuous embedding.

*Proof.* According to a classical result (see, for instance, [115, Chapter 4]), we know that for every  $\tau > 0$  and every  $z \in X_{\text{sym}}$ ,

$$(\Phi_{\text{sym},\tau}^* z)(t) = \begin{cases} B^* \mathbb{T}_{\text{sym},\tau-t}^* z & \text{for } t \in [0, \tau], \\ 0 & \text{for } t > \tau, \end{cases}$$

where  $B^*$  is introduced in (6.81) and  $\mathbb{T}_{\text{sym}}$  is the  $C_0$ -group generated by  $A_{\text{sym}}$ . This implies that for every  $\tau > 0$  we have

$$\left\| (\Phi_{\text{sym},\tau}^* z) \right\|_{L^2([0,\tau];U)}^2 = \int_0^\tau \left\| B^* \mathbb{T}_{\text{sym},\tau-t}^* z \right\|_U^2 dt.$$



Notice that  $0 \in \rho(A_{\text{sym}})$  and the imaginary part of the eigenvalues  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  is strictly increasing, there exists  $c > 0$  such that  $|\omega_{\text{sym},k}| \geq c$ , which implies that  $\gamma_{\text{sym},k}$  defined in (6.80) is lower bounded by a positive constant. Combining (6.81) and Proposition 6.4.4, we have

$$|B^*(\widehat{\phi}_{\text{sym},k})| = \left| \frac{\gamma_{\text{sym},k}}{2l} \psi_{\text{sym},k}(l) \right| \sim \left| \sin \left( \frac{\omega_{\text{sym},k}}{\sqrt{gh_0}} (L-l) \right) \right|, \quad (6.83)$$

for  $k \in \mathbb{Z}^*$ . The notation " $\sim$ " means that each function can be controlled by the other one multiplied by a positive constant. Putting (6.78) and (6.83) together, we obtain that

$$|B^*(\widehat{\phi}_{\text{sym},k})| \sim \frac{1}{k} \quad \forall k \in \mathbb{Z}^*. \quad (6.84)$$

Since the operator  $A_{\text{sym}}$  is diagonalizable and skew-adjoint on  $X_{\text{sym}}$ , we have

$$\mathbb{T}_{\text{sym},t} z = \sum_{k \in \mathbb{Z}^*} e^{i\omega_{\text{sym},k} t} \langle z, \widehat{\phi}_{\text{sym},k} \rangle \widehat{\phi}_{\text{sym},k} \quad \forall z \in X_{\text{sym}},$$

where  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*}$ , an orthonormal basis of  $X_{\text{sym}}$ , is introduced around (6.80). Hence, for every  $\tau > 0$  we have

$$\int_0^\tau \left\| B^* \mathbb{T}_{\text{sym},\tau-t}^* z \right\|_U^2 dt = \int_0^\tau \left| \sum_{k \in \mathbb{Z}^*} e^{-i\omega_{\text{sym},k} t} \langle z, \widehat{\phi}_{\text{sym},k} \rangle B^* \widehat{\phi}_{\text{sym},k} \right|^2 dt.$$

Recalling (6.79) and using the Ingham theorem (a generalization of Parseval's equality, see, for instance, in [115, Chapter 8] or [63]), there exists  $c_1 > 0$  depending only on  $h_0$ ,  $L$ ,  $l$  and  $\tau$ , such that

$$\int_0^\tau \left\| B^* \mathbb{T}_{\text{sym},\tau-t}^* z \right\|_U^2 dt \leq c_1 \sum_{k \in \mathbb{Z}^*} |\langle z, \widehat{\phi}_{\text{sym},k} \rangle|^2 |B^* \widehat{\phi}_{\text{sym},k}|^2.$$

This, together with (6.84), implies that, for every  $\tau > 0$ ,

$$\left\| \Phi_{\text{sym},\tau}^* z \right\|_{L^2([0,\tau];U)}^2 \leq c_1 \|z\|_{\mathcal{D}(A_{\text{sym}})'}^2 \quad \forall z \in X_{\text{sym}},$$

where  $\mathcal{D}(A_{\text{sym}})'$  is the dual of  $\mathcal{D}(A_{\text{sym}})$  with respect to the pivot space  $X_{\text{sym}}$ . Now we introduce the identity function on  $\mathcal{D}(A_{\text{sym}})$ , denoted by  $\text{id}_{\mathcal{D}(A_{\text{sym}})}$ , then we have  $\text{id}_{\mathcal{D}(A_{\text{sym}})} \in \mathcal{L}(\mathcal{D}(A_{\text{sym}}), X_{\text{sym}})$ . Since, for every  $\tau > 0$ ,  $\Phi_{\text{sym},\tau} \in \mathcal{L}(L^2([0,\tau];U); X_{\text{sym}})$ , we apply next a classical consequence of the closed graph theorem (see, for instance, [115, Proposition 12.1.2]), which follows that

$$\text{Ran } \Phi_{\text{sym},\tau} \subset \mathcal{D}(A_{\text{sym}}) \quad \forall \tau > 0. \quad (6.85)$$

Moreover, using the second part of Ingham theorem (see again [115, Chapter 8]), there exists  $\tau_0 := \frac{2(L-l)}{\sqrt{gh_0}}$  and  $c_2 > 0$  such that, for every  $\tau > \tau_0$ ,

$$\int_0^\tau \left\| B^* \mathbb{T}_{\text{sym},\tau-t}^* z \right\|_U^2 dt \geq c_2 \sum_{k \in \mathbb{Z}^*} |\langle z, \widehat{\phi}_{\text{sym},k} \rangle|^2 |B^* \widehat{\phi}_{\text{sym},k}|^2. \quad (6.86)$$

## 6.4. REACHABILITY AND STABILIZABILITY OF THE LINEARIZED SYSTEM

Similarly, we obtain from (6.84) the inclusion:

$$\text{Ran } \Phi_{\text{sym},\tau} \supset \mathcal{D}(A_{\text{sym}}) \quad \forall \tau > \tau_0. \quad (6.87)$$

Combined with (6.85), (6.87) and Remark 47, we conclude that  $\text{Ran } \Phi_\tau = \mathcal{D}(A_{\text{sym}})$  for every  $\tau > \tau_0$ . Recalling that  $A_{\text{sym}}$  is densely defined, we immediately conclude that (6.82) holds.  $\square$

**Remark 50.** We see from the equality in (6.82) that all symmetric state with the regularity as in  $\mathcal{D}(A_{\text{sym}})$  form the reachable space of the control system (6.35). The second inclusion in (6.82) means that the system is not approximately controllable in  $X$ , but in its symmetric subspace  $X_{\text{sym}}$ .

We give below, as a consequence of our main theorem, the following result on the controllability and stabilizability properties of the system (6.71)–(6.73).

**Corollary 6.4.6.** *Let  $L' = L$  and the initial data  $\zeta_0$  and  $q_0$  satisfy the symmetry condition (6.69). Then the linear system defined by (6.71)–(6.73) on  $X_{\text{sym}}$  (briefly designed by  $(A_{\text{sym}}, B)$ ), has the following properties*

1.  $(A_{\text{sym}}, B)$  is not exactly controllable in time  $\tau$  for any finite  $\tau > 0$ ;
2.  $(A_{\text{sym}}, B)$  is approximately controllable on  $X_{\text{sym}}$  in time  $\tau$  for any  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ ;
3.  $(A_{\text{sym}}, B)$  is strongly stabilizable with the feedback operator  $F = -B^*$ . More precisely, there exists  $C > 0$  such that the closed-loop semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  satisfies

$$\|\mathbb{T}_{\text{sym},t}^{\text{cl}} w_0\|_{X_{\text{sym}}} \leq \frac{C}{(1+t)^{\frac{1}{2}}} \|w_0\|_{\mathcal{D}(A_{\text{sym}})} \quad \forall w_0 \in \mathcal{D}(A_{\text{sym}}), t \geq 0. \quad (6.88)$$

*Proof.* (1) Note that the operator  $A_{\text{sym}}$  is skew-adjoint and  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ , then the first assertion follows directly from Curtain and Zwart [40, Theorem 4.1.5] or [40, Theorem 5.2.6] in the same book, since  $A_{\text{sym}}$  has infinitely many unstable eigenvalues. Equivalently, we know that the system  $(A_{\text{sym}}, B)$  is not exponentially stabilizable (see, for instance, Haraux [55] and Liu [76]). Alternatively, we can apply the main result of Gibson [44] or Guo, Guo and Zhang [53, Theorem 3].

(2) The second assertion is a direct consequence of Theorem 6.4.5. By duality it suffices to show that there exists  $\tau_0 > 0$ , such that for every  $\tau > \tau_0$ ,

$$B^* \mathbb{T}_{\text{sym},t}^* z = 0 \quad \text{on } [0, \tau] \implies z = 0. \quad (6.89)$$

Let  $B^* \mathbb{T}_{\text{sym},t}^* z = 0$  on  $[0, \tau]$  with  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ , we obtain from (6.86) that  $\langle z, \hat{\phi}_{\text{sym},k} \rangle = 0$  for every  $k \in \mathbb{Z}^*$ , which implies that  $z = 0$ . This, together with Proposition 6.4.2, gives the result.

(3) The approximate controllability of the system  $(A_{\text{sym}}, B)$  is equivalent to the fact that the semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  is strongly stable (for this, please refer to Benchimol [18], Batty and Vu [16]). To obtain the explicit decay rate, we notice that (6.78) and (6.84) imply that the assumptions  $[H_1]$ ,  $[H_2]$  in Chapter 3 are satisfied with  $\alpha = 1$  and  $\beta = 1$ . Therefore, we apply Theorem 3.1.2 and conclude that the semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  satisfies (6.88).  $\square$

Based on the above analysis, now we expect to have the exact controllability of the control system (6.71)–(6.73) with  $L^2$  controls with different spaces. There are two ways to achieve this in general: one is to expand the input signal space (bringing in distributions) and the other one is to shrink the state space. According to the reachable space presented in Theorem 6.4.5, it is convenient to apply the second strategy here. Thus we take the domain of the operator  $A_{\text{sym}}$ , i.e.  $\mathcal{D}(A_{\text{sym}})$ , as the new state space. In this way, we have the following conclusion.

**Corollary 6.4.7.** *With the same assumptions as in Corollary 6.4.6, the linear system (6.71)–(6.73) with the state space  $\mathcal{D}(A_{\text{sym}})$  is exactly controllable in finite time  $\tau$ , for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ .*

*Proof.* Recalling that  $\mathbb{T}_{\text{sym}}$  is the  $C_0$ -semigroup generated by the operator  $A_{\text{sym}}$ , we denote by  $\mathbb{T}_{\text{sym}}|_{\mathcal{D}(A_{\text{sym}})}$  its restriction to  $\mathcal{D}(A_{\text{sym}})$ . According to [115, Proposition 2.10.4], the operators  $(\mathbb{T}_{\text{sym},t}|_{\mathcal{D}(A_{\text{sym}})})_{t \geq 0}$  are linear continuous on  $\mathcal{D}(A_{\text{sym}})$  and they form a  $C_0$ -semigroup on  $\mathcal{D}(A_{\text{sym}})$ . More precisely, the fact that  $\mathbb{T}_{\text{sym},t}$ , as an operator in  $\mathcal{L}(\mathcal{D}(A_{\text{sym}}))$ , is the image of  $\mathbb{T}_{\text{sym},t} \in \mathcal{L}(X)$  through the unitary resolvent operator  $(sI - A_{\text{sym}})^{-1} \in \mathcal{L}(X, \mathcal{D}(A_{\text{sym}}))$ , which can be written as follows:

$$\mathbb{T}_{\text{sym},t}|_{\mathcal{D}(A_{\text{sym}})}z = (sI - A_{\text{sym}})^{-1}\mathbb{T}_{\text{sym},t}(sI - A_{\text{sym}})z \quad \forall z \in \mathcal{D}(A_{\text{sym}}).$$

Moreover, the generator of the  $C_0$ -semigroup  $\mathbb{T}_{\text{sym}}|_{\mathcal{D}(A_{\text{sym}})}$  is the restriction of  $A_{\text{sym}}$  to  $\mathcal{D}(A_{\text{sym}}^2)$ , which is defined by

$$\mathcal{D}(A_{\text{sym}}^2) = \{z \in \mathcal{D}(A_{\text{sym}}) | A_{\text{sym}}z \in \mathcal{D}(A_{\text{sym}})\}.$$

In this case, it is not difficult to see that the control operator  $B$  (introduced in (6.40)) is unbounded, i.e. for every  $u \in \mathbb{C}$ ,  $Bu$  is not in the state space  $\mathcal{D}(A_{\text{sym}})$  but in a larger space  $X$ . Using the result in (6.82), the control system  $(A_{\text{sym}}, B)$  with the state space  $\mathcal{D}(A_{\text{sym}})$  is exactly controllable in time  $\tau$ , for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ .  $\square$

## 6.5 The case when the object floats at one lateral boundary

As we have already noticed that the model (6.18) we derived and discussed in the previous sections is only suitable for the case that the object does not touch one of the lateral boundaries of the water tank (i.e.  $L \neq l$  and  $L' \neq l$ , see Figure 6.1), since at least the notation in the system (6.18) does not make sense in these two special situations. Intuitively, the exterior domain becomes one interval and one of the transmission conditions becomes an impermeable boundary condition. In this section, we derive the new governing equations for the control system when the object floats at one lateral boundary of  $\Omega$ . Without loss of generality, let us say it floats at the right boundary i.e.  $L' = l$ , see Figure 6.2. Moreover, we assume that the internal wall of that boundary is smooth enough, so that there is no friction when the object moves in the vertical direction. In this case, the external vertical force  $u$  can be also imposed from the top of the object by using a string, since it is located at the boundary of the tank. In addition to the WECs, this of course represents a class of model used in various applications.

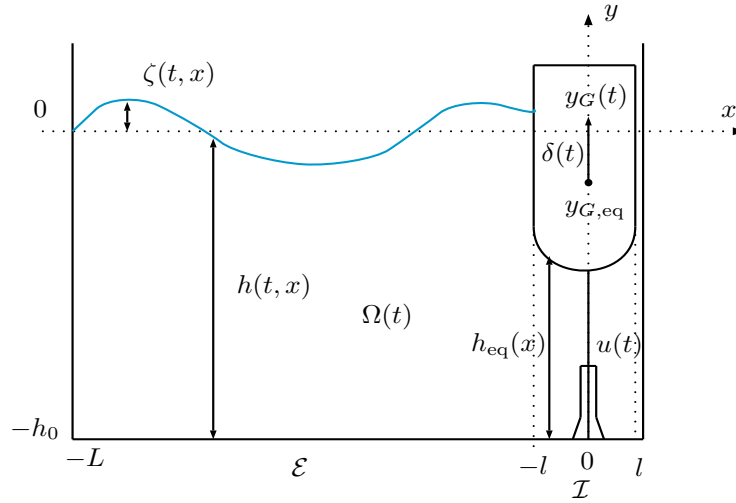


Figure 6.2: The object floats at the right boundary of the tank

### 6.5.1 Derivation of the governing equations

Still using the notation introduced in Section 6.2, once again, we begin from the non-linear shallow water equations in a strip presented in (6.2). We restrict this model to our bounded interval  $[-L, l]$  by finding the boundary conditions of the horizontal discharge  $q$  and of the surface pressure  $\underline{P}_i$ . As shown in Figure 6.2, the exterior domain  $\mathcal{E}$  and the interior  $\mathcal{I}$  are

$$\mathcal{E} = (-L, -l), \quad \mathcal{I} = (-l, l).$$

Similarly, we consider the following two conservation conditions:

- *The conservation of the volume of the water.*

In this case, we have two impermeable boundaries at  $x = -L$  and  $x = l$ , which are

$$q_e(t, -L) = 0 = q_i(t, l). \quad (6.90)$$

Using (6.90) and the first equation in (6.2), we have

$$0 = \partial_t \int_{\mathcal{E} \cup \mathcal{I}} \zeta(t, x) dx = - \int_{-L}^l (\partial_x q)(t, x) dx = -q_e(t, -l) + q_i(t, -l),$$

which clearly implies that

$$q_i(t, -l) = q_e(t, -l). \quad (6.91)$$

- *The conservation of the total energy of the fluid-structure system.*

The main principle is the same with the one we proposed in Section 6.2. For the sake of simplicity, we recall it here again.

$$\frac{d}{dt} E_{\text{tot}}(t) = u(t) \dot{\delta}(t). \quad (6.92)$$

Based on the condition (6.92) for the conservation of the total energy, we present in the following proposition the boundary condition of the surface pressure  $\underline{P}_i$  at the contact point  $x = -l$ .

**Proposition 6.5.1.** *Assume that the quantities  $\zeta$ ,  $q$ ,  $h$ ,  $\delta$  and  $\underline{P}$  are smooth on  $\mathcal{I}$  and  $\mathcal{E}$ . Then the total energy of the fluid-structure system  $E_{\text{tot}}$  satisfies*

$$\frac{d}{dt}E_{\text{tot}}(t) = \mathfrak{F}_i(t, -l) - \mathfrak{F}_e(t, -l) + u(t)\dot{\delta}(t).$$

Therefore, if the surface pressure  $\underline{P}_i$  satisfies the boundary condition

$$\underline{P}_i(t, -l) = \rho g(\zeta_e(t, -l) - \zeta_i(t, -l)) + \mathfrak{B}_e(t, -l) - \mathfrak{B}_i(t, -l), \quad (6.93)$$

the total energy is conserved in the sense of (6.92). In the above expression,  $\mathfrak{F}$  and  $\mathfrak{B}$  have been introduced in (6.12).

The proof of Proposition 6.5.1 is similar with the proof of Proposition 6.2.1, so we omit the details here.

Now we consider the interior shallow water equations (6.16). We derive from the first equation of (6.16) that

$$q_i(t, x) = -x\dot{\delta} + \frac{1}{2}q_i(t, -l) \quad \forall x \in \mathcal{I}.$$

This, together with the boundary condition (6.90) and (6.91), implies that

$$q_i(t, x) = (-x + l)\dot{\delta} \quad \forall x \in \mathcal{I}. \quad (6.94)$$

Therefore, we further obtain that

$$q_e(t, -l) = q_i(t, -l) = 2l\dot{\delta}.$$

Similarly, we obtain the governing equations for the floating object system when the object floats at the right boundary (i.e.  $L' = l$ ), which, for every  $t \geq 0$  and  $x \in [-L, l]$ , are

$$\partial_t \zeta + \partial_x q = 0 \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (6.95a)$$

$$\partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P} \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (6.95b)$$

$$\underline{P}_e(t, x) = 0 \quad x \in \mathcal{E}, \quad (6.95c)$$

$$\zeta_i(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad x \in \mathcal{I}, \quad (6.95d)$$

$$\underline{P}_i(t, -l) = \rho g(\zeta_e(t, -l) - \zeta_i(t, -l)) + \mathfrak{B}_e(t, -l) - \mathfrak{B}_i(t, -l), \quad (6.95e)$$

$$m\ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (6.95f)$$

$$q_e(t, -L) = 0 = q_i(t, l), \quad q_e(t, -l) = 2l\dot{\delta} = q_i(t, -l), \quad (6.95g)$$

with the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1 \quad \forall x \in \mathcal{I} \cup \mathcal{E}.$$

## 6.5. THE CASE WHEN THE OBJECT FLOATS AT ONE LATERAL BOUNDARY

Note that in the above equations (6.95), the transmission condition of  $q$  at  $x = -l$  is clearly given (only depends on  $\delta$ ) and there is only one boundary condition for the surface pressure  $\underline{P}_i$ , which is the difference with the system (6.18).

Next we prove that the surface pressure  $\underline{P}_i$  is fully determined by a first-order evolution equation and thereby give its explicit formula which only depends on  $\delta$  and  $\zeta_e(t, -l)$ . This is also a difference with the case  $L, L' \neq l$ , where  $\underline{P}_i$  is determined by a second-order elliptic equation (see Proposition 6.2.2).

**Proposition 6.5.2.** *With the same assumptions as in Proposition 6.5.1, the interior surface pressure  $\underline{P}_i$  satisfies*

$$\begin{cases} \partial_x \underline{P}_i = -\frac{\rho}{h_w} \left( (l-x)\ddot{\delta} + \delta^2 \partial_x \left( \frac{(l-x)^2}{h_w} \right) + gh_w \partial_x h_w \right), \\ \underline{P}_i(t, -l) = \rho g (\zeta_e(t, -l) - \zeta_i(t, -l)) + \mathfrak{B}_e(t, -l) - \mathfrak{B}_i(t, -l), \end{cases}$$

where  $h_w$  and  $\mathfrak{B}$  is introduced in (6.15) and (6.12), respectively. Therefore, we obtain

$$\begin{aligned} \underline{P}_i(t, x) = \rho \ddot{\delta} \int_{-l}^x \frac{s-l}{h_w} ds + \rho \delta^2 \left( \int_{-l}^x \frac{l-s}{h_w^2} ds - \frac{(l-x)^2}{2h_w^2} + \frac{2l^2}{(h_0 + \zeta_e(t, -l))^2} \right) \\ - \rho g \delta - \rho g h_{\text{eq}}(x) + \rho g \zeta_e(t, -l) + \rho g h_0. \end{aligned} \quad (6.96)$$

*Proof.* We first obtain from the second equation (6.95b) in the interior domain  $\mathcal{I}$  that

$$\partial_x \underline{P}_i = -\frac{\rho}{h_w} \left( \partial_t q_i + \partial_x \left( \frac{q_i^2}{h_w} \right) + gh_w \partial_x \zeta_i \right). \quad (6.97)$$

Using the formula of  $q_i$  in (6.94) and the relation (6.95d), we have

$$\begin{aligned} \partial_x q_i = -\dot{\delta}, \quad \partial_x \zeta_i = \partial_x h_{\text{eq}} = \partial_x h_w, \quad q_i^2(t, -l) = 4l^2 \delta^2, \\ \partial_t q_i = (l-x)\ddot{\delta}, \quad \mathfrak{B}_e(t, -l) = \frac{2\rho l^2 \delta^2}{(h_0 + \zeta_e(t, -l))^2}, \quad \mathfrak{B}_i(t, -l) = \frac{2\rho l^2 \delta^2}{h_w^2(t, -l)}, \end{aligned}$$

which, together with (6.97), gives that

$$\partial_x \underline{P}_i = -\frac{\rho}{h_w} \left( (l-x)\ddot{\delta} + \delta^2 \partial_x \left( \frac{(l-x)^2}{h_w} \right) + gh_w \partial_x h_w \right),$$

with the boundary condition

$$\begin{aligned} \underline{P}_i(t, -l) = \rho g \zeta_e(t, -l) - \rho g h_{\text{eq}}(-l) - \rho g \delta + \rho g h_0 \\ + 2\rho l^2 \delta^2 \left( \frac{1}{(h_0 + \zeta_e(t, -l))^2} - \frac{1}{h_w^2(t, -l)} \right). \end{aligned}$$

According to Newton-Leibniz theorem, we have the expression for  $\underline{P}_i$ , for every  $t \geq 0$  and  $x \in \mathcal{I}$ ,

$$\begin{aligned} \underline{P}_i(t, x) = \rho \ddot{\delta} \int_{-l}^x \frac{s-l}{h_w} ds - \rho \delta^2 \int_{-l}^x \frac{1}{h_w} \partial_s \left( \frac{(l-s)^2}{h_w} \right) ds - \rho g \int_{-l}^x \partial_s h_w ds \\ + \rho g \zeta_e(t, -l) - \rho g h_{\text{eq}}(-l) - \rho g \delta + \rho g h_0 + 2\rho l^2 \delta^2 \left( \frac{1}{(h_0 + \zeta_e(t, -l))^2} - \frac{1}{h_w^2(t, -l)} \right). \end{aligned} \quad (6.98)$$

We use an integration by parts and find that

$$-\int_{-l}^x \frac{1}{h_w} \partial_x \left( \frac{(l-s)^2}{h_w} \right) ds = \int_{-l}^x \frac{l-s}{h_w^2} ds - \frac{(l-x)^2}{2h_w^2} + \frac{2l^2}{h_w^2(t, -l)},$$

which, together with (6.15) and (6.98), implies (6.96) directly.  $\square$

As in Theorem 6.2.5, we consider in what follows reformulating the governing equations (6.95) into an initial boundary problem of a first-order evolution system in terms of  $\zeta$ ,  $q$  and  $\delta$ .

**Theorem 6.5.3.** *For smooth solutions, the system (6.95) can be equivalently rewritten into the following coupled PDE-ODE system defined in the exterior domain  $\mathcal{E}$ , i.e. for every  $t \geq 0$ ,  $x \in \mathcal{E}$ ,*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad (6.99)$$

with boundary conditions

$$q(t, -L) = 0, \quad q(t, -l) = 2l\delta. \quad (6.100)$$

Moreover, the displacement  $\delta$  are determined, for every  $t \geq 0$  and  $x \in \mathcal{E}$ , by

$$\mathcal{M}(\delta)\ddot{\delta} - \left( \frac{1}{2}\gamma(\delta) + \frac{4l^3}{(h_0 + \zeta_e(t, -l))^2} \right) \rho\delta^2 + 2\rho gl\delta = 2\rho gl\zeta_e(t, -l) + U(t) \quad (6.101)$$

where  $\mathfrak{B}$  is introduced in (6.12),  $\gamma(\delta)$  and  $\mathcal{M}(\delta)$  are defined as

$$\gamma(\delta) = \int_{-l}^l \frac{(l-x)^2}{h_w^2(t, x)} dx, \quad \mathcal{M}(\delta) = m + \rho\kappa(\delta), \quad \kappa(\delta) = \int_{-l}^l \frac{(l-x)^2}{h_w(t, x)} dx. \quad (6.102)$$

The initial data, as in (6.95), are given by  $\zeta_0$ ,  $q_0$ ,  $\delta_0$  and  $\delta_1$ .

*Proof.* Based on the above analysis, we only need to prove the second-order nonlinear ODE for  $\delta$ , i.e. (6.101). Recalling Newton's equation (6.95f), we calculate the integration of  $\underline{P}_i$  on  $\mathcal{I}$ . According to Proposition 6.5.2, we have

$$\begin{aligned} \int_{-l}^l \underline{P}_i(t, x) dx &= \rho\ddot{\delta} \int_{-l}^l \int_{-l}^x \frac{s-l}{h_w(t, s)} ds dx + \rho\delta^2 \int_{-l}^l \int_{-l}^x \frac{l-s}{h_w^2(t, s)} ds dx \\ &\quad - \rho\delta^2 \int_{-l}^l \frac{(l-x)^2}{2h_w^2} dx - \rho g \int_{-l}^l h_{\text{eq}}(x) dx + 2\rho gl\zeta_e(t, -l) \\ &\quad - 2\rho gl\delta + 2\rho glh_0 + \rho\delta^2 \frac{4l^3}{(h_0 + \zeta_e(t, -l))^2}. \end{aligned} \quad (6.103)$$

To simplify the notation, we introduce  $\kappa(\delta)$ ,  $\gamma(\delta)$  and  $\mathcal{M}(\delta)$  as in (6.102). Taking an integration by parts, we obtain from (6.103) that

$$\begin{aligned} \int_{-l}^l \underline{P}_i(t, x) dx &= -\rho\kappa(\delta)\ddot{\delta} + \frac{1}{2}\rho\gamma(\delta)\delta^2 - 2\rho gl\delta - \rho g \int_{-l}^l h_{\text{eq}}(x) dx \\ &\quad + 2\rho gl\zeta_e(t, -l) + 2\rho glh_0 + \rho\delta^2 \frac{4l^3}{(h_0 + \zeta_e(t, -l))^2} \end{aligned}$$

## 6.5. THE CASE WHEN THE OBJECT FLOATS AT ONE LATERAL BOUNDARY

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which, combined with Newton's equation (6.95f), implies that

$$\begin{aligned} \mathcal{M}(\delta)\ddot{\delta} = \rho\delta^2 \left( \frac{1}{2}\gamma(\delta) + \frac{4l^3}{(h_0 + \zeta_e(t, -l))^2} \right) - 2\rho gl\delta + 2\rho gl\zeta_e(t, -l) \\ - \rho g \int_{-l}^l h_{\text{eq}}(x)dx + 2\rho glh_0 - mg + u(t). \end{aligned}$$

Based on the above expression, we denote  $U(t)$  as the new control term, which is

$$U(t) = -\rho g \int_{-l}^l h_{\text{eq}}(x)dx + 2\rho glh_0 - mg + u(t).$$

Therefore, we obtain (6.101) immediately.  $\square$

**Remark 51.** We see that all the interior functions  $\zeta_i$ ,  $q_i$ ,  $P_i$  are fully determined by the variation of the position of the object  $\delta$ ,  $\zeta_e(t, -l)$  and the control  $U(t)$  in the case that the object floats one lateral boundary. Compared with Theorem 6.2.5, we have simpler boundary condition in Theorem 6.5.3 for the horizontal discharge  $q$  and, in particular, only one ODE of  $\delta$  coupled with the boundary of the nonlinear shallow water equations.

### 6.5.2 Linearized equations

In this section, we linearize the coupled system (6.99)–(6.101) around the equilibrium state  $[\zeta \ q \ \delta \ \dot{\delta}]^\top = [0 \ 0 \ 0 \ 0]^\top$  and we obtain, for every  $t \geq 0$  and  $x \in \mathcal{E} = (-L, -l)$  (see Figure 6.2),

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \ddot{\delta} = -\frac{2\rho gl}{\mathcal{M}}\delta + \frac{2\rho gl}{\mathcal{M}}\zeta(t, -l) + \frac{1}{\mathcal{M}}U, \end{cases} \quad (6.104)$$

with boundary conditions

$$q(t, -L) = 0, \quad q(t, -l) = 2l\dot{\delta}, \quad (6.105)$$

and the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1.$$

The constants  $\underline{\mathcal{M}}$  in (6.104) is

$$\underline{\mathcal{M}} = \mathcal{M}(0), \quad (6.106)$$

where  $\mathcal{M}(\delta)$  has been defined in (6.102).

We still denote by  $\underline{E}_{\text{tot}}$  the total energy of the linearized system (6.104), which includes the interior energy  $\underline{E}_{\text{int}}$  and the exterior energy  $\underline{E}_{\text{ext}}$ . After doing some trivial calculations as in Section 6.3.1, we have

$$\underline{E}_{\text{int}}(t) = \frac{1}{2}\underline{\mathcal{M}}\dot{\delta}^2 + \rho gl\delta^2 + \frac{\rho g}{2} \int_{-l}^l (h_{\text{eq}}(x) - h_0)^2 dx,$$



$$\underline{E}_{\text{ext}}(t) = \frac{\rho}{2} \int_{-L}^{-l} \left( g\zeta^2 + \frac{q^2}{h_0} \right) dx.$$

Therefore, the total energy  $\underline{E}_{\text{tot}}(t)$  is

$$\underline{E}_{\text{tot}}(t) = \frac{\rho}{2} \int_{-L}^{-l} \left( g\zeta^2 + \frac{q^2}{h_0} \right) dx + \frac{1}{2} \mathcal{M} \delta^2 + \rho g l \delta^2 + \frac{\rho g}{2} \int_{-l}^l (h_{\text{eq}}(x) - h_0)^2 dx.$$

**Remark 52.** Note that the linearized model (6.104) with the objects at the right boundary takes the same form with the symmetric version of the linearized equation (6.35), i.e. the case we discussed in Subsection 6.4.2. But this does not mean that we could derive the nonlinear governing equations (6.99)–(6.101) directly from Theorem 6.2.5. On one hand, as we already mentioned, not only does the notation  $\zeta_e(t, l)$  and  $\mathfrak{B}_e(t, l)$  not make sense, but the surface pressure  $\underline{p}_i$  also satisfies different equations (see Proposition 6.2.2 and Proposition 6.5.2). On the other hand, the nonlinear ODE system for  $\langle q_i \rangle$  and  $\delta$  in (6.33) are not compatible. More precisely, taking  $L' = l$  in (6.32) and (6.5), it is not difficult to obtain that

$$q_i(t, -l) = 2l\dot{\delta}, \quad \langle q_i \rangle = \frac{1}{2} q_i(t, -l) = l\dot{\delta}.$$

Substituting this into the first equation in (6.33), we have

$$\alpha(\delta) l \ddot{\delta} + \alpha'(\delta) l \dot{\delta}^2 = -\frac{1}{2\rho l} \llbracket \rho g \zeta_e + \mathfrak{B}_e \rrbracket,$$

which is clearly different with the second equation in (6.33).

Since the linearized equation is the same with the one in symmetric case, we conclude the following proposition for the reachability of the system (6.104). The strategy of the proof is similar with Theorem 6.4.5, so we omit the details here. Now we denote the linear system in this case by the pair  $(A_r, B_r)$ , which can be written directly from (6.104).

**Proposition 6.5.4.** *When the object is located at the right boundary of the domain  $\Omega$ , i.e.  $L' = l$ , the reachable space  $\text{Ran } \Phi_\tau$ , for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$  satisfies*

$$\mathcal{D}(A_r) = \text{Ran } \Phi_\tau \subset X_r,$$

where  $X_r$  and  $\mathcal{D}(A_r)$  are defined as

$$X_r = \left\{ [\zeta \quad q \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^2 \mid \int_{-L}^{-l} \zeta(x) dx + 2l\delta = 0 \right\},$$

$$\mathcal{D}(A_r) = \left\{ \begin{array}{l} [\zeta \quad q \quad \delta \quad \eta]^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^2 \mid [\zeta \quad q \quad \delta \quad \eta]^\top \in X_r, \\ q(-L) = 0 \quad \text{and} \quad q(-l) = 2l\eta \end{array} \right\}.$$

Therefore, all the free surface waves with the regularity as in  $X_r$  form the reachable space of the control system (6.104).

**Remark 53.** Similarly, the system  $(A_r, B_r)$  on  $X_r$  is not exactly controllable but approximately controllable in finite time. Moreover, if we take  $\mathcal{D}(A_r)$  as the state space, the system is exactly controllable in finite time.

## 6.5. THE CASE WHEN THE OBJECT FLOATS AT ONE LATERAL BOUNDARY

**Remark 54.** Note that the linear equations (6.104)–(6.105) is a coupled control system, there is another approach introduced in Weiss and Zhao [122] is available for the exact controllability. The author has tried this approach and decompose the control system (6.104)–(6.105) as a finite-dimensional control system  $\Sigma_f$  and an infinite-dimensional system  $\Sigma_d$ . The system  $\Sigma_d$  with the state  $z = [\zeta \ q]^\top$ , for every  $t \geq 0$  and  $x \in (-L, -l)$ , is

$$\Sigma_d : \begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ q(t, -L) = 0, \quad q(t, -l) = v, \\ y = -2\rho gl \zeta(t, -l), \end{cases} \quad (6.107)$$

where  $v$  and  $y$  is the input function and the output function, respectively. The finite system  $\Sigma_f$  with  $x = [\delta \ \dot{\delta}]^\top$ , for every  $t \geq 0$ , is

$$\Sigma_f : \begin{cases} \ddot{\delta} = -\frac{2\rho gl}{\mathcal{M}} \delta - \frac{1}{\mathcal{M}} y + \frac{1}{\mathcal{M}} U, \\ v = 2l \dot{\delta}, \end{cases} \quad (6.108)$$

where  $U - y$  and  $v$  is the input and the output, respectively. It is obvious to see that the system  $\Sigma_d$  (6.107) is coupled with the system  $\Sigma_f$  (6.108) in the sense that the output of (6.107) is a part of the input of (6.108) and the output of (6.108) is the input of (6.107). More clearly, we have the following block diagram. It is not difficult to verify that the

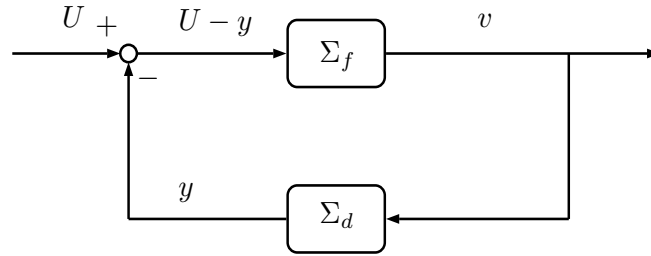


Figure 6.3: The coupled system  $\Sigma_c$  consisting of  $\Sigma_f$  and  $\Sigma_d$

system  $\Sigma_d$  is a well-posed boundary control system. It can be formulated to a standard linear control model with the state  $z = [\zeta \ q]^\top$ , for every  $t \geq 0$  and  $x \in (-L, -l)$  as

$$\begin{cases} \dot{z} = Az + Bv, \\ z(0) = z_0, \\ y = Cz, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & -\frac{d}{dx} \\ -gh_0 \frac{d}{dx} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{\rho g}{2} \delta_{-l} \\ 0 \end{bmatrix}, \quad C = [-2\rho gl \gamma_{-l} \ 0].$$

The bold  $\delta_{-l}$  in above expression represents Dirac mass concentrated at  $x = -l$ , and  $\gamma_{-l}$  is the trace operator at  $x = -l$ . Moreover,  $\Sigma_d$  is a *Strictly Proper with an Integrator system* (SPI) with input space  $U = \mathbb{C}$ , state space  $X = L^2(-L, -l) \times L^2(-L, -l)$  (with

suitable compatibility condition associated with the conservation of the volume) and output space  $Y = \mathbb{C}$ . For the definition of SPI system, please refer to [122] for more details. The finite-dimensional system  $\Sigma_f$  with the state  $x = [\delta \ \delta']^\top$  can be written as

$$\begin{cases} \dot{x} = \mathbf{a}x + \mathbf{b}(U - y), \\ v = \mathbf{c}x, \end{cases}$$

where  $\mathbf{a} \in \mathbb{C}^{2 \times 2}$ ,  $\mathbf{b} \in \mathbb{C}^{2 \times 1}$  and  $\mathbf{c} \in \mathbb{C}^{1 \times 2}$ . It turns out that the system  $(A, B)$  is exactly controllable in any time  $\tau$  with  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ . For the finite-dimensional system  $\Sigma_f$ ,  $(\mathbf{a}, \mathbf{b})$  is controllable and  $\mathbf{c}\mathbf{b}$  is invertible. According to [122, Theorem 1.2], we still need to verify that there exists  $\beta \in \rho(A)$  such that the operators  $A^*$  and  $(\alpha^\times(\beta))^*$  with  $\alpha^\times(\beta) = \mathbf{a} + \mathbf{b}(\mathbf{c}\mathbf{b})^{-1}\mathbf{c}(\beta I - \mathbf{a})$  have no common eigenvalues. Since 0 is one of the eigenvalues of the operators  $(\alpha^\times(\beta))^*$ , we have to exclude 0 from the spectrum of  $A$ . As we mentioned above, we have to take the condition for the conservation of the volume, i.e.

$$\int_{-L}^{-l} \zeta(x) dx + 2l\delta = 0$$

into account and put this into the state space in a suitable way. This seems not direct and obvious, but the author believe that this can be achieved by a clever modification of the decomposition. Finally the result should be the same with Corollary 6.4.7. For the details on how to use this decomposition argument, following their general work, Zhao and Weiss provide in [125] an application of this approach to the SCOLE model.

## 6.6 Conclusions, comments and open questions

In this chapter, we investigated a coupled PDE-ODE system describing the motion of a floating body in a free boundary ideal fluid, within the linearized shallow water regime. The floating body is constrained to move vertically and it is actuated by a control force applied from the bottom of the object. Our main result asserts that, provided that, in a symmetric geometrical configuration, the system can be steered from rest to any smooth enough symmetric wave profile.

The main question left open in our work is the description of the reachable space of the considered system without symmetry conditions. Using the properties of the eigenvalues of the generator (see Subsection 6.3.2) this could be accomplished provided that one has lower bounds on  $|B^*\hat{\phi}_k|$ , where  $B^* \in \mathcal{L}(X, \mathbb{C})$  is defined in (6.81), and  $(\hat{\phi}_k)_{k \in \mathbb{Z}^*}$  is the orthonormal basis introduced in Remark 45. Obtaining such lower bounds does not seem an easy task. Indeed, combining (6.50)–(6.52) and (6.81) we obtain that for every  $k \in \mathbb{Z}^*$ ,

$$|B^*\hat{\phi}_k| = \frac{1}{4l} |\gamma_k (K_2 f_{\omega_k}(L') - K_1 f_{\omega_k}(L))|, \quad (6.109)$$

where  $\hat{\phi}_k$  and  $\gamma_k$  are introduced in Remark 45,  $f_{\omega_k}$  is defined in (6.49); with constant  $K_1$  and  $K_2$  which we are unable to express in a simple manner in terms of  $\omega_k$ . We also recall from Remark 43 that we are, in the general case, unable to confirm or to inform the existence of double eigenvalues.

## 6.6. CONCLUSIONS, COMMENTS AND OPEN QUESTIONS

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Another open question of interest are the study of the system obtained by adding a viscosity term in the shallow water equations, in the spirit of Maity et al. [80]. This could lead, in particular, to a description of the reachable space for nonlinear systems in which the fluid is modeled by the nonlinear shallow water equations. Finally, let us mention that an interesting question could be to consider the corresponding boundary control problems, in the spirit of [110], [107] and [108].



# Chapter 7

## Perspectives

### 7.1 Control system in a bounded domain

**Control of non-linear system.** For the boundary control problem of the water waves system in a rectangular domain, the model we used is actually the fully linear and fully dispersive approximation of the Zakharov-Crag-Sulem (ZCS) equations (2.22), in particular, with the control term. The intuitive question is that

"how about the control system of the nonlinear water waves equations?"

In this case, the control not only appears in the source term of the kinematic condition on the free surface, but also induces another nonlinear term in the free surface Bernoulli equation (see (4.13)). As far as we know, this boundary control problem is completely open. Instead of imposing a wave maker from the lateral boundary of the fluid domain, it might be easier to take the pressure as the control term acting on the small part of the free surface with appropriate property. In this case the control signal as an external force only appears in the source term of the free surface Bernoulli equation. There are some related work on the nonlinear water waves system with the pressure as control, for instance, Alazard et al. [4], [5] and [2]. To weaken the difficulties, it might be better to begin from some asymptotic models of the full water waves system, for instance the nonlinear shallow water equations and the Boussinesq equations. For the well-posedness issue, we might first consider constructing an iterative sequences with good regularity, (i.e. a linearization of the nonlinear term), then justify the limit in a low regularity. During this process, the properties of the corresponding linearized system plays an important role. The main idea here is the application of the fixed point argument. This technique is used, for instance, in the work [80] by Maity et al.

**Water waves system in a general convex domain.** Another interesting direction is to consider the control system in a general convex water tank, which is a more practical problem. To tackle the irregular fluid domain, one way is to estimate the spectrum of the evolution operator like the idea proposed in Reid [94]. Another possible technique is to employ a conformal mapping to straighten the boundary and make it regular. The control problem of the water waves system in a general convex domain, even the linear case, is almost open up to now. Anyway, to set up the new problem on water waves we cannot avoid studying the Dirichlet to Neumann and the Neumann to Neumann operator in various cases.

## 7.2 Shallow water convergence

**Dispersive effect and the controllability.** In the second part of the thesis, we justified the passage of the limit from the linear water waves system to the wave equation in 1 D, by taking the shallowness limit. During this process, we see that the dispersive effect becomes weaker and weaker, and finally it vanishes in the wave equation. As we have already mentioned, the water waves system for the boundary control problem in Chapter 4 is the fully linear and fully dispersive model, thereby we also call it the linearized ZCS equations. After taking the dispersive limit, the control properties of the wave equation becomes much better than the linear ZCS system. It is important to clarify that how the dispersive term affects the control properties. Based on this observation, it is also interesting to study the control problem of the asymptotic models for the water waves. For example, we consider the control problem of a fluid described by the linear Boussinesq equations, whose dispersive term is weaker than the linear ZCS system. Without the control term, we know that the linearized Boussinesq equation is a second-order approximation of the linearized ZCS, with respect to the shallowness parameter  $\mu$  introduced in (5.1). Therefore, for the corresponding control system, we probably have the similar relation as follows:

$$\text{Linear ZCS} \xrightarrow{O(\mu^2)} \text{Linear Boussinesq} \xrightarrow{O(1)} \text{wave equation.}$$

According to the above analysis, the controllability properties of the linear Boussinesq equations is probably better than the linear ZCS system, but worse than the wave equation. This phenomenon can be valuable for engineering applications.

**Higher regularity and regular convergence.** As we discussed in Section 4.3.2 and Section 5.6, the higher regularity requires the more regular input function and the initial data. What's more, the geometry of the fluid domain determines the properties of the solution directly. For irregular domain, there are some singularities where the solution is not smooth. This is the main problem to improve the regularity and may lead to new difficulties.

## 7.3 Control of floating object

**Boundary control of the floating object system.** As we already mentioned, there is almost no result on the control issue of the floating body system. Except for adjusting the motion of the floater in the vertical direction, it is also interesting to consider the boundary control problem in the shallow water regime. Still using the  $(\zeta, q)$  formulation for instance, we take the value of the horizontal discharge at the left boundary, i.e.  $q(t, -L)$ , as the control term. In this case, we need to notice that the volume of the water is no longer conserved. To fix this, it might need to change the boundary condition on the right. In addition, we can also study the system describing the interaction of a floating object and a fluid modeled by the *Boussinesq equations*, which is a second-order shallow water approximation of the water waves system with weak nonlinearity. Of course, there will be new challenge and difficulties induced by the nonlinearity and dispersion term.

**Optimal control for the efficiency of energy extraction.** Since we focus on the stability of the control system, the feedback control we used is colocated feedback, which

is a very natural state feedback and make the energy non-increasing. As a wave energy converter, according to practical applications, this control might not be a good choice. Therefore, it is necessary to construct appropriate control to maximize the energy extraction. For the sake of economy, we have to protect the wave energy converter from the impact of large coming waves. Hence, the problem finally is an optimal control design with the above constraints.

**Modelling and numerical simulation.** Besides the control model described by the point absorber device, there are various wave energy converters applied in engineering, for instance, oscillating water columns (OWCs), overtopping devices (ODs), etc. These devices are the suitable for different environment and the corresponding control system is a pretty new project. Moreover, according to the specific situation, the system possesses different boundary conditions (see, for instance, [68]):

- Generating boundary condition. The elevation of the free surface is known at the entrance of the fluid domain and it is described by a prescribe function  $f(t)$ , i.e.

$$\zeta(t, -L) = f(t).$$

In practice, this function can be measured by putting a buoy at the left boundary. In this case, the boundary  $x = -L$  has no physical sense. This boundary condition appears, for instance, in a work by Bocchi et al. [25], where the system is modeled by OWCs wave energy converter.

- Impermeable boundary condition. There is a fixed wall at the boundary  $x = -L$ , the fluid cannot go through this wall, which is

$$U(t, x, y) \cdot \vec{n} = 0,$$

where  $U(t, x, y)$  is the velocity of the fluid particle and  $\vec{n}$  is the normal vector along that boundary. This boundary condition has been frequently used in the main part.

- Transparent boundary condition. This boundary condition is important for numerical simulations in the cases where there is no physical boundary, for instance, at  $x = -L$ . To discrete the domain and compile the algorithm, we impose a boundary condition there that does not induce any artificial reflection. Based on the hyperbolic governing equations (for instance the nonlinear shallow water equations), the boundary condition at  $x = -L$  can be derived from the *Riemann invariant*. In particular, if  $x = -L$  is the left boundary, the boundary condition there is obtained by taking

$$R_r(\zeta, q) = 0,$$

where  $R_r$  represents the right-going Riemann invariant.

Independent of these specific topics, some numerical simulation can be carried out to verify the corresponding result and the related properties. Based on the structure of the governing equations, some important schemes are required during this process. Moreover, there are some complicated situation where the boundary is a superposition of the above several boundary conditions. In these cases, modelling is the first task to start the mathematical analysis.





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