*O*n s*o*me identities between special values *o*f L-functi*o*ns

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Sur quelques identités entre les valeurs spéciales des fonctions L

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Contents

Introduction (en français)

La fonction zêta de Riemann $\zeta(\cdot)$ a été étudiée sous de nombreuses formes différentes pendant des siècles. Ses valeurs spéciales $\zeta(n)$ aux entiers positifs $n \in \mathbb{N}^*$, $n \geq 2$ ont joué un rôle important en théorie des nombres. Ici $\mathbb{N}^*$ (respectivement $\mathbb{N}$) désigne l'ensemble des entiers positifs (resp. l'ensemble des entiers non négatifs). Les valeurs zêta aux entiers positifs pairs de $\zeta(\cdot)$ ont d'abord été calculées par Euler:

$$\frac{\zeta(n)}{(2i\pi)^n} = -\frac{1}{2}\frac{B_n}{n!} \text{ pour tout } n \geq 2, n \equiv 0 \pmod 2,$$

où B_n désigne le $n$ième nombre de Bernoulli. Euler a également travaillé sur des valeurs zêta multiples de profondeur r de la forme

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \ldots k_n^{n_r}}, \text{ où } n_1, \ldots, n_{r-1} \geq 1, n_r \geq 2.$$

Il a découvert l'identité suivante $\zeta(1,2) = \zeta(3)$. À la suite de Thakur, la valeur zêta multiple $\zeta(1,2)$ est appelée zeta-like.

En 1935, Carlitz [Car35] a considéré le cas des corps de fonctions et a introduit les valeurs zêta de Carlitz qui sont des analogues des valeurs zêta classiques $\zeta(n)$, $n \in \mathbb{N}^*$. Soit $\mathbb{F}_q$ un corps fini ayant q éléments, q étant une puissance d'un nombre premier p, et θ un indéterminée sur $\mathbb{F}_q$. Soit $A = \mathbb{F}_q[\theta]$, et soit $K = \mathbb{F}_q(\theta)$ équipé de la place rationnelle ∞. Soit $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ la complétion de K à ∞, et soit $\mathbb{C}_\infty$ la complétion d'un clôture algébrique fixe de K_∞ à ∞. Pour $d \in \mathbb{N}$, $A_{+,d}$ désigne l'ensemble des éléments unitaires dans A de degré d. Les valeurs zêta de Carlitz à $n \in \mathbb{N}$ sont définies par

$$\zeta_A(n) := \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

Les valeurs zêta aux entiers négatifs ont été étudiées par Goss. On peut montrer que $\zeta_A(n) \in A$ si $n \leq 0$ et même $\zeta_A(n) = 0$ si $n < 0$ et $n \equiv 0 \pmod{q-1}$ (voir [Gos96, Chapitre 8]). Dans le même article, Carlitz a également introduit la application exponentielle de Carlitz $\exp_C$ et la période de Carlitz $\tilde{\pi}$ (voir Section 1.2) qui sont des analogues de la application exponentielle et de $2i\pi$ respectivement. Il a introduit des analogues des nombres de Bernoulli, appelés les nombres de Bernoulli Carlitz $BC_n \in K$, et a prouvé (voir aussi [Gos96, Section 9.2])

$$\frac{\zeta_A(n)}{\tilde{\pi}^n} = \frac{BC_n}{\Pi_n} \text{ pout tout } n \geq 1, n \equiv 0 \pmod{q-1},$$

où $\Pi_n \in A$ est la n-ième factorielle de Carlitz (voir [Gos96, Chapitre 9]) qui est analogue à $n!$.

D'une part, en 2012, Pellarin [Pel12] a introduit des valeurs zêta de plusieurs variables dans les algèbres de Tate. Soit $s \geq 1$ un entier, et soit $t_1, \ldots, t_s$ des variables s sur K et on écrit $\underline{t}_s$ pour la famille de variables $\{t_1, \ldots, t_s\}$. Soit $\mathbb{T}_s$ l'algèbre de Tate dans les variables $\underline{t}_s$ avec des coefficients dans $\mathbb{C}_\infty$ (voir Section 3.2.1). Pour $n \in \mathbb{N}^*$, la valeur zêta dans les variables $\underline{t}_s$ est définie par

$$\zeta_A(n, \underline{t}_s) := \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a^n} \in \mathbb{T}_s^\times.$$

Pour $s \geq 2$ et $s \equiv 1 \pmod{q-1}$, on définit

$$\mathbb{B}_s := (-1)^{(s-1)/(q-1)} \frac{\zeta_A(1, \underline{t}_s)\omega(t_1)\ldots\omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s, \tag{0.0.1}$$

où ω est la fonction spéciale introduite par Anderson et Thakur dans [AT90] et donnée par

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{j\geq 0} \left(1 - \frac{t}{\theta^{q^j}}\right)^{-1} \tag{0.0.2}$$

pour un choix fixé de la $(q-1)$ième racine de $(-\theta)$ dans $\mathbb{C}_\infty$. Il existe un lien profond entre l'élément $\mathbb{B}_s$ et les valeurs zêta $\zeta_A(1, \underline{t}_s)$ (voir [AP14, APTR16]). En particulier, il s'avère que $\mathbb{B}_s$ est un polynôme en les variables $\underline{t}_s$. On appelle $\mathbb{B}_s$ le polynôme à plusieurs variables de type Bernoulli.

D'autre part, en 2004, Thakur a introduit en caractéristique p la valeur zêta multiple comme analogue aux valeurs zêta multiples d'Euler. Pour tout uple d'entiers positifs $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, Thakur [Tha04] a défini la valeur zêta multiple $\zeta_A(\mathfrak{s})$ par

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in K_\infty$$

où la somme parcoure par l'ensemble des uples $(a_1, \ldots, a_r) \in A_+^r$ avec $\deg a_1 > \ldots > \deg a_r$. On dit qu'un MZV $\zeta_A(s_1, \ldots, s_r)$ est zeta-like si $\zeta_A(s_1, \ldots, s_r)/\zeta_A(s_1 + \cdots + s_r)$ appartient à K.

L'objectif principal de cette thèse est d'étudier les relations entre les valeurs zêta ci-dessus.

Dans le premier chapitre, nous rappelons quelques définitions et propriétés de base des modules de Drinfeld, le module de Carlitz, ses objets associés (tels que la application exponentielle de Carlitz $\exp_C$ et la période de Carlitz $\tilde{\pi}$) et les algèbres de Tate en plusieurs variables.

Pour étudier la relation entre la valeur zêta dans les algèbres de Tate, nous sommes conduits à étudier un élément spécial $\mathbb{B}_s \in \mathbb{T}_s$. Ainsi, le Chapitre 2 est consacré à donner une introduction aux polynômes à plusieurs variables $\mathbb{B}_s$. Nous donnons une démonstration schématique du Théorème 2.0.2 qui dit que $\mathbb{B}_s$ est un polynôme dans $\mathbb{F}_q[\theta][\underline{t}_s]$. La preuve suit de près la technique et les idées utilisées dans [GAND+19, Théorème 4.6] et [AP14, Corollaire 21].

Dans le Chapitre 3, nous étudions les valeurs zêta dans les algèbres de Tate introduites par Pellarin. Le résultat principal est le Théorème 3.1.3 qui donne une réponse affirmative à la Conjecture 3.1.2 de Pellarin sur les identités pour ces valeurs zêta. Nous étudierons plus en détails les coefficients du polynôme $\mathbb{B}_s$ et montrerons comment utiliser ces propriétés pour prouver le résultat principal. Nous proposons également la Conjecture 3.5.1 sur une formule explicite des coefficients du polynôme $\mathbb{B}_s$.

Dans le Chapitre 4, nous examinons la Conjecture 3.5.1. Nous montrons que cette conjecture est vraie pour quelques petits cas ($i < 2q-1$ et $i < 3q-2$) (voir la Proposition 4.2.4) et suggérons un moyen de prouver la Conjecture 3.5.1 pour le cas $i < q^2$ (voir Section 4.3).

Dans le Chapitre 5, nous travaillons avec A plus général. Nous étudions les valeurs zêta de Goss associées à A général (voir Définition 5.1.29). Le résultat principal est le Théorème 5.3.17 qui est une généralisation du résultat de Speyer [Spe17]. Nous donnons une formule explicite du résultat principal et un resultat de non nullité dans certains cas particuliers dans la Section 5.3.4.

Enfin, dans le Chapitre 6, nous prouvons la Conjecture 6.1.2 de Lara Rodriguez et Thakur qui donne une liste complète de zeta-like de profondeur 2 de poids au plus q^2 (voir Théorème 6.1.3). Nous prouvons également un résultat similaire sur la détermination complète de tous les zeta-like de poids au plus q^2 (Théorème 6.1.5).

Introduction (in english)

The Riemann zeta function $\zeta(\cdot)$ has been studied in many different forms for centuries. Its special values $\zeta(n)$ at positive integer $n \in \mathbb{N}^*$, $n \geq 2$ have played an important role in number theory. Here $\mathbb{N}^*$ (respectively $\mathbb{N}$) denotes the set of positive integers (resp. the set of non-negative integers). The zeta values at even positive integers of $\zeta(\cdot)$ were first computed by Euler:

$$\frac{\zeta(n)}{(2i\pi)^n} = -\frac{1}{2}\frac{B_n}{n!} \text{ for all } n \geq 2, n \equiv 0 \pmod 2,$$

where B_n denotes the nth Bernoulli number. Euler also worked on multiple zeta values of depth r of the form

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \dots k_n^{n_r}}, \text{ where } n_1, \dots, n_{r-1} \geq 1, n_r \geq 2.$$

He discovered the following identity $\zeta(1,2) = \zeta(3)$. Following Thakur, the multiple zeta value $\zeta(1,2)$ is called zeta-like.

In 1935, Carlitz [Car35] considered the function field setting and introduced the Carlitz zeta values which are analogues of classical special zeta values $\zeta(n)$, $n \in \mathbb{N}^*$. Let $\mathbb{F}_q$ be a finite field having q elements, q being a power of a prime number p, and θ an indeterminate over $\mathbb{F}_q$. Let $A = \mathbb{F}_q[\theta]$, and let $K = \mathbb{F}_q(\theta)$ equipped with the rational place ∞. Let $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ be the completion of K at ∞, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞ at ∞. For $d \in \mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d. The Carlitz zeta values at $n \in \mathbb{N}$ is defined by

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

The zeta values at negative integers were studied by Goss. One can show that $\zeta_A(n) \in A$ if $n \leq 0$ and even $\zeta_A(n) = 0$ if $n < 0$ and $n \equiv 0 \pmod{q-1}$ (see [Gos96, Chapter 8]). In the same paper, Carlitz also introduced the Carlitz exponential map $\exp_C$ and the Carlitz period $\tilde{\pi}$ (see Section 1.2) which are analogues of the exponential map and $2i\pi$ respectively. He introduced analogues of the Bernoulli numbers, called the Bernoulli Carlitz numbers $BC_n \in K$, and proved (see also [Gos96, Section 9.2])

$$\frac{\zeta_A(n)}{\tilde{\pi}^n} = \frac{BC_n}{\Pi_n} \text{ for all } n \geq 1, n \equiv 0 \pmod{q-1},$$

where $\Pi_n \in A$ is the nth Carlitz factorial (see [Gos96, Chapter 9]) which is analogue to $n!$.

On the one hand, in 2012, Pellarin [Pel12] introduced several variables zeta values in Tate algebras. Let $s \geq 1$ be an integer, and let $t_1, \dots, t_s$ be s variables over K and we write $\underline{t}_s$ for the family of variables $\{t_1, \dots, t_s\}$. Let $\mathbb{T}_s$ be the Tate algebra in the variables $\underline{t}_s$ with coefficients in $\mathbb{C}_\infty$ (see Section 3.2.1). For $n \in \mathbb{N}^*$, the zeta value in the variables $\underline{t}_s$ is defined by

$$\zeta_A(n, \underline{t}_s) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a^n} \in \mathbb{T}_s^\times.$$

For $s \geq 2$ and $s \equiv 1 \pmod{q-1}$, we define

$$\mathbb{B}_s := (-1)^{(s-1)/(q-1)} \frac{\zeta_A(1, \underline{t}_s)\omega(t_1) \dots \omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s, \tag{0.0.3}$$

where ω is the special function introduced by Anderson and Thakur in [AT90] and given by

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{j\geq 0} \left(1 - \frac{t}{\theta^{q^j}}\right)^{-1} \tag{0.0.4}$$

for a fixed choice of the $(q-1)$th root of $(-\theta)$ in $\mathbb{C}_\infty$. There is a deep connection between the element $\mathbb{B}_s$ and the zeta values $\zeta_A(1, \underline{t}_s)$ (see [AP14, APTR16]). In particular, it turns out that $\mathbb{B}_s$ is a polynomial in variables $\underline{t}_s$. We call $\mathbb{B}_s$ the several variables Bernoulli-type polynomial.

On the other hand, in 2004, Thakur introduced the characteristic p multiple zeta value as analogues to the multiple zeta values of Euler. For any tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, Thakur [Tha04] defined the characteristic p multiple zeta value $\zeta_A(\mathfrak{s})$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \ldots, a_r) \in A_+^r$ with $\deg a_1 > \ldots > \deg a_r$. We say that a multiple zeta value $\zeta_A(s_1, \ldots, s_r)$ is zeta-like if $\zeta_A(s_1, \ldots, s_r)/\zeta_A(s_1 + \cdots + s_r)$ belongs to K.

The main goal of this thesis is to study the relations among the above zeta values.

In the first chapter, we recall some definitions and basic properties of Drinfeld modules, the Carlitz module, its related objects (such as the Carlitz exponential map $\exp_C$ and the Carlitz period $\tilde{\pi}$) and Tate algebras in several variables.

To study the relation between zeta value in Tate algebras, we are led to study a special element $\mathbb{B}_s \in \mathbb{T}_s$. Thus, Chapter 2 is devoted to give an introduction to the several variables polynomials $\mathbb{B}_s$. We give an outline proof of Theorem 2.0.2 which says that $\mathbb{B}_s$ is a polynomial in $\mathbb{F}_q[\theta][\underline{t}_s]$. The proof follows closely the technique and ideas used in [GAND+19, Theorem 4.6] and [AP14, Corollary 21].

In Chapter 3, we study the zeta values in Tate algebras which is introduced by Pellarin. The main result is Theorem 3.1.3 which give an affirmative answer to Conjecture 3.1.2 of Pellarin about identities for these zeta values. We will study more details on the coefficients of the polynomial $\mathbb{B}_s$ and show that how to use these properties to prove the main result. We also suggest Conjecture 3.5.1 about an explicit formula of the coefficients of the polynomial $\mathbb{B}_s$.

In Chapter 4 we investigate Conjecture 3.5.1. We show that this conjecture is true for some small cases ($i < 2q-1$ and $i < 3q-2$) (see Proposition 4.2.4) and suggest a way to prove Conjecture 3.5.1 for the case $i < q^2$ (see Section 4.3).

In Chapter 5, we work with a more general A. We study the Goss zeta values associated to the general A (see Definition 5.1.29). The main result is Theorem 5.3.17 which is a generalization of Speyer's result [Spe17]. We give an explicit formula of the main result and a non-vanishing result in some special cases in Section 5.3.4.

Finally, in Chapter 6, we prove Conjecture 6.1.2 of Lara Rodriguez and Thakur which gives a full list of depth 2 zeta-like of weight at most q^2 (see Theorem 6.1.3). We also prove a similar result about determining completely all zeta-like of weight at most q^2 (Theorem 6.1.5).

Chapter 1

Preliminaries

Let $\mathbb{F}_q$ be a finite field having q elements, q being a power of a prime number p, and θ an indeterminate over $\mathbb{F}_q$. Let $A = \mathbb{F}_q[\theta]$ and let $K = \mathbb{F}_q(\theta)$ equipped with the rational place ∞. Let $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ be the completion of K at ∞, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞ at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.

In this thesis, we restrict our attention to the case $A = \mathbb{F}_q[\theta]$ except Chapter 5. We will touch only a few aspects of the general theory in Chapter 5.

We recall the definition of Drinfeld $\mathbb{F}_q[\theta]$-modules in Section 1.1. In Section 1.2, we study the Carlitz module which is a special case of Drinfeld $\mathbb{F}_q[\theta]$-modules, and its related objects such as: the Carlitz exponential $\exp_C$, the Carlitz logarithm $\log_C$ and the Carlitz period $\tilde{\pi}$. Section 1.3 contains a brief summary of Tate algebras, which will be used when we study the zeta values in Tate algebras in the following chapters.

1.1 Drinfeld modules

Let L be a field containing $\mathbb{F}_q$ and let $\tau\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the map defined by $\tau(x) = x^q$.

Definition 1.1.1. The twisted polynomial ring $L\{\tau\}$ is defined as the set of polynomials in the variable τ and coefficients in L. The addition rule is the usual addition of polynomials. The multiplication rule is given by

$$\tau x = x^q \tau \text{ for all } x \in L.$$

Generally, the ring $L\{\tau\}$ is not commutative. For more details about this ring, we refer the reader to [Gos96, Chapter 1].

We say that L is an A-field if and only if there is a homomorphism of $\mathbb{F}_q$-algebras $\iota\colon A \to L$.

Definition 1.1.2. A Drinfeld A-module over an A-field L of rank $r \in \mathbb{N}^*$ is an $\mathbb{F}_q$-algebra homomorphism $\phi\colon A \to L\{\tau\}$ such that

$$\phi_\theta = \iota(\theta) + a_1\tau + \ldots + a_r\tau^r$$

for some $a_1, \ldots, a_r \in L$, $a_r \neq 0$.

Remark 1.1.3.

1. Since A is generated by θ, for $a \in A$, we have $\phi_a = \iota(a) + a_1\tau + \ldots + a_n\tau^n$ for some $a_1, \ldots, a_n \in L$ and $n = r \deg_\theta a$.
2. Via ϕ, L becomes an A-module with the action of A as follows: For all $a \in A$, $x \in L$, $a \cdot x = \phi_a(x)$. We denote this new A-module by $\phi(L)$ to distinct with the usual A-module L.

1.2 The Carlitz module

Definition 1.2.1. The Carlitz module over $\mathbb{C}_\infty$ is a homomorphism of $\mathbb{F}_q$-algebras $C\colon A \to \mathbb{C}_\infty\{\tau\}$ given by

$$C_\theta = \theta + \tau.$$

Proposition 1.2.2 (See [Gos96, Proposition 3.3.1])**.** *There exists a unique series* $\exp_C \in \mathbb{C}_\infty\{\{\tau\}\}$ *such that*

- $\exp_C \equiv 1 \pmod{\tau}$, *and*
- *for all* $a \in A$, *we have* $\exp_C a = C_a \exp_C$.

Proof. Since C is a homomorphism of $\mathbb{F}_q$-algebras and A is generated by θ, the following conditions are equivalent

$$\forall a \in A, \exp_C a = C_a \exp_C \iff \exp_C \theta = C_\theta \exp_\theta .$$

Let us write $\exp_C = \sum_{n\geq 0} e_n \tau^n$ with $e_n \in \mathbb{C}_\infty, n \geq 0$ and substitute in the second condition. Then we get

$$(\sum_{n\geq 0} e_n\tau^n)\theta = (\theta+\tau)\sum_{n\geq 0} e_n\tau^n.$$

Recall that for $x \in L$, $\tau x = x^q\tau$. It follows that

$$\sum_{n\geq 0} e_n\theta^{q^n}\tau^n = \sum_{n\geq 0}\theta e_n\tau^n + \sum_{n\geq 0} e_n^q\tau^{n+1}.$$

By comparing the coefficients of τ^n, we get

$$e_n(\theta^{q^n} - \theta) = e_{n-1}^q \text{ for } n = 1, 2, \ldots. \tag{1.2.1}$$

The condition $\exp_C \equiv 1 \pmod{\tau}$ implies that $e_0 = 1$. It follows that all the coefficients are uniquely determined by (1.2.1). It implies the existence and uniqueness of $\exp_C$. □

In the above proof, if we set $D_n := \frac{1}{e_n}$, we have

$$\exp_C = \sum_{n\geq 0} \frac{\tau^n}{D_n},$$

where

$$D_0 = 1, D_n = (\theta^{q^n} - \theta)D_{n-1}^q, \text{ for } n = 1, 2, \ldots. \tag{1.2.2}$$

The series $\exp_C \in \mathbb{C}_\infty\{\{\tau\}\}$ induces a morphism of A-modules $\mathbb{C}_\infty \to C(\mathbb{C}_\infty)$, also denoted by $\exp_C$.

Definition 1.2.3. The series $\exp_C$ is called the Carlitz exponential.

Similarly, we also have the notion of the Carlitz logarithm

$$\log_C = \sum_{n\geq 0} \frac{\tau^n}{l_n},$$

where

$$l_0 = 1, l_n = (\theta - \theta^{q^n})l_{n-1}, \text{ for } n = 1, 2, \ldots. \tag{1.2.3}$$

This series satisfies $\log_C \equiv 1 \pmod{\tau}$ and $a \log_C = \log_C C_a$ for all $a \in A$.

Now we recall a lemma which will be necessary in the sequel.

Lemma 1.2.4. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}_\infty$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to+\infty} v_\infty(a_n) = +\infty$.*

We list some properties of the Carlitz exponential and the Carlitz logarithm.

Proposition 1.2.5.

1. *The series $\exp_C$ converges for all $x \in \mathbb{C}_\infty$. Moreover, it is surjective.*
2. *The series $\log_C$ converges for every $x \in \mathbb{C}_\infty$ such that $v_\infty(x) > -\frac{q}{q-1}$.*
3. *On the disk $\{x \in \mathbb{C}_\infty : v_\infty(x) > -\frac{q}{q-1}\}$, we have $v_\infty(\exp_C(x)) = v_\infty(\log_C(x)) = v_\infty(x)$. Moreover, we also have $\exp_C(\log_C(x)) = \log_C(\exp_C(x)) = x$.*

Proof. Let us take $x \in \mathbb{C}_\infty$.

1. Recall that

$$\exp_C(x) = \sum_{n\geq 0} \frac{x^{q^n}}{D_n},$$

where D_n is defined in (1.2.2). Since $v_\infty(D_n) = v_\infty((\theta^{q^n} - \theta)(\theta^{q^n} - \theta^q)\cdots(\theta^{q^n} - \theta^{q^{n-1}})) = -nq^n$, we have $v_\infty(\frac{x^{q^n}}{D_n}) = q^n v_\infty(x) + nq^n \longrightarrow +\infty$ when $n \to +\infty$. By Lemma 1.2.4, $\exp_C(x)$ converges.

Let $y \in \mathbb{C}_\infty$. We use the fact that a nonconstant entire function (i.e., it is nonzero and it converges for every $x \in \mathbb{C}_\infty$) always has a zero (see [Gos96, Proposition 2.9 and 2.13]). Applying to the nonconstant entire function $\exp_C(x) - y$, we can always find an $x \in \mathbb{C}_\infty$ such that $\exp_C(x) - y = 0$.

2. Recall that

$$\log_C(x) = \sum_{n\geq 0} \frac{x^{q^n}}{l_n},$$

where l_n is defined in (1.2.3). Since $v_\infty(l_n) = v_\infty((\theta - \theta^{q^n})(\theta - \theta^{q^{n-1}})\cdots(\theta - \theta^q)) = -\frac{q^{n+1}-q}{q-1}$, we have $v_\infty(\frac{x^{q^n}}{l_n}) = q^n(v_\infty(x) + \frac{q}{q-1}) - \frac{q}{q-1} \longrightarrow +\infty$ if and only if $v_\infty(x) > -\frac{q}{q-1}$. By Lemma 1.2.4, the proof is done.

3. We fix $x \in \mathbb{C}_\infty$ such that $v_\infty(x) > -\frac{q}{q-1}$.

Firstly, for $n = 1$, we have $v_\infty(\frac{x^{q^n}}{D_n}) - v_\infty(x) = v_\infty(x)(q-1) + q > 0$ if and only if $v_\infty(x) > -\frac{q}{q-1}$. For $n \geq 2$, $v_\infty(\frac{x^{q^n}}{D_n}) - v_\infty(x) = (q^n - 1)(v_\infty(x) + n) + n \geq (q^n - 1)(-\frac{q}{q-1} + 2) + n > 0$. Thus $v_\infty(\exp_C(x)) = v_\infty(x + \frac{x^q}{D_1} + \frac{x^{q^2}}{D_2} + \ldots) = v_\infty(x)$.

Secondly, for $n \geq 1$, we have

$$v_\infty(\frac{x^{q^n}}{l_n}) - v_\infty(x) = (v_\infty(x) + \frac{q}{q-1})(q^n - 1) > 0.$$

Thus $v_\infty(\log_C(x)) = v_\infty(x + \frac{x^q}{l_1} + \frac{x^{q^2}}{l_2} + \ldots) = v_\infty(x)$.

Lastly, we claim that in $\mathbb{C}_\infty\{\{\tau\}\}$, $\exp_C \log_C = \log_C \exp_C = 1$. Indeed, we have $C_\theta \exp_C \log_C = \exp_C \theta \log_C = \exp_C \log_C C_\theta$. Let us write $\exp_C \log_C = \sum_{n\geq 0} a_n \tau^n \in \mathbb{C}_\infty\{\{\tau\}\}$ and note that $a_0 = 1$. By expanding the expression $C_\theta \exp_C \log_C = \exp_C \log_C C_\theta$ and comparing the coefficients of τ^n, $n \geq 1$, we deduce that $a_n = 0$ for $n \geq 1$. By a similar argument, we can show that $\log_C \exp_C = 1$. Last assertion follows immediately from the claim. □

In the end of this section, we recall the definition and basic properties of the Carlitz period (see [Gos96, Tha04, Car35] for more details).

Definition 1.2.6. The Carlitz period, denoted by $\tilde{\pi}$, is defined by

$$\tilde{\pi} = (-\theta)^{\frac{1}{q-1}} \theta \prod_{j\geq 1} (1 - \theta^{1-q^j})^{-1} \in \mathbb{C}_\infty^\times. \tag{1.2.4}$$

Remark 1.2.7. We have $v_\infty(\tilde{\pi}) = -\frac{q}{q-1}$.

We also have another formula of the Carlitz period (see [Gos96, Definition 3.2.7])

$$\tilde{\pi} = (\theta - \theta^q)^{\frac{1}{q-1}} \prod_{j\geq 1}(1 - \frac{\theta^{q^j} - \theta}{\theta^{q^{j+1}} - \theta}).$$

The following well-known result explains the term period of $\tilde{\pi}$.

Proposition 1.2.8 (See [Gos96, Corollary 3.2.9]). *We have*

$$\ker \exp_C = \tilde{\pi} A.$$

1.3 Tate algebras in several variables

Let $s \geq 1$ be an integer and let $t_1, \ldots, t_s$ be variables. We denote the set $\{t_1, \ldots, t_s\}$ by $\underline{t}_s$ or $\underline{t}$.

Let $L \subset \mathbb{C}_\infty$ be a complete field with respect to v_∞. The Gauss valuation v on $L[\underline{t}_s] := L[t_1, \ldots, t_s]$ is defined by: for all $f \in L[\underline{t}_s]$,

$$f = \sum_{\substack{\text{finite sum} \\ \underline{i}_s \in \mathbb{N}^s}} a_{\underline{i}_s} t_1^{i_1} \cdots t_s^{i_s},$$

we set $v(f) = \min_{\underline{i}_s}\{v_\infty(a_{\underline{i}_s})\}$. The Gauss valuation v is indeed a valuation on $L[\underline{t}_s]$ and extends the valuation v_∞ on L. We also denote v by v_∞.

Definition 1.3.1. The Tate algebra $\mathbb{T}_s(L)$ in the variables $\underline{t}_s$ with coefficients in L is the completion of $L[\underline{t}_s]$ with respect to the Gauss valuation. Explicitly, $\mathbb{T}_s(L)$ can be identified with the set of formal series

$$\{f \in L[\underline{t}_s], f = \sum_{\underline{i}_s \in \mathbb{N}^s} a_{\underline{i}_s} t_1^{i_1} \cdots t_s^{i_s} \text{ such that } \lim_{i_1+\ldots+i_s \to \infty} v_\infty(a_\infty) = +\infty\}.$$

Definition 1.3.2. When $L = \mathbb{C}_\infty$, we will write $\mathbb{T}_s$ instead of $\mathbb{T}_s(\mathbb{C}_\infty)$

The automorphism $\tau\colon x \mapsto x^q, \mathbb{C}_\infty \to \mathbb{C}_\infty$ extends to $\mathbb{T}_s$ by twisting the coefficients $a_{\underline{i}_s}$ and keeping the variables $t_1, \ldots, t_s$. In other word, we have

$$\tau(\sum_{\underline{i}_s \in \mathbb{N}^s} a_{\underline{i}_s} t_1^{i_1} \cdots t_s^{i_s}) = \sum_{\underline{i}_s \in \mathbb{N}^s} a_{\underline{i}_s}^q t_1^{i_1} \cdots t_s^{i_s}.$$

The map $\tau\colon \mathbb{T}_s \to \mathbb{T}_s$ is a continuous homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras.

Definition 1.3.3. We also have the definition of twisted polynomial ring $\mathbb{T}_s\{\tau\}$ over Tate algebras. The multiplication is defined by

$$\tau f = \tau(f)\tau, \quad \forall f \in \mathbb{T}_s.$$

We recall a property of Tate algebras.

Proposition 1.3.4 (See [FvdP04, Chapter 3, Theorem 3.2.1]). *With above notation, we have $\mathbb{T}_s(L)$ is a unique factorization domain.*

Chapter 2

Several variable polynomial $\mathbb{B}_s$

Let $A = \mathbb{F}_q[\theta]$. For $d \in \mathbb{N}$, we denote by A_+, $A_{+,d}$ and $A_{+,\leq d}$ respectively the set of monic polynomials in A, the set of monic polynomials of degree d in A and the set of monic polynomials in A of degree less than or equal d.

Through this chapter, we always assume that s is an integer such that $s \equiv 1 \pmod{q-1}$. We set $m := (s-1)/(q-1) \in \mathbb{N}$.

Let $\mathbb{T}_s$ be the Tate algebra in the variables $\underline{t}_s$ with coefficients in $\mathbb{C}_\infty$ (see Section 1.3). In 2012, Pellarin [Pel12] introduced the following element in $\mathbb{T}_s^\times$ called the zeta value in the variables $\underline{t}_s$

$$\zeta_A(1, \underline{t}_s) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

For $s = 1$, we write $t_1 = t$. Pellarin proved the following identity.

Theorem 2.0.1 (See [Pel12, Theorem 1]). *We have*

$$\frac{\zeta_A(1,t)\,\omega(t)}{\widetilde{\pi}} = \frac{1}{\theta - t}$$

where $\widetilde{\pi}$ is the Carlitz period given by Equation (1.2.4) and ω function is given by Equation (0.0.4).

We recall the definition of $\mathbb{B}_s$ in (0.0.3). For $s \geq 2$ and $s \equiv 1 \pmod{q-1}$, we define

$$\mathbb{B}_s := (-1)^m \frac{\zeta_A(1, \underline{t}_s)\omega(t_1)\ldots\omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s.$$

We have the following property of the element $\mathbb{B}_s$.

Theorem 2.0.2 (See [APTR16, Lemma 7.6] or [AP14, Corollary 21]). *The element $\mathbb{B}_s$ is a polynomial in $\mathbb{F}_q[\underline{t}_s, \theta]$. Moreover, it is a monic polynomial in the variable θ of degree $m-1$ and a symmetric polynomial in the variables $\underline{t}_s$.*

In this chapter, firstly, we present a proof of Theorem 2.0.1 in Section 2.1. Then we outline a proof of Theorem 2.0.2. This proof is divided into two steps. The first step (Section 2.2) is to show that $\mathbb{B}_s$ is a polynomial in $K_\infty[\underline{t}_s]$. The idea of the proof follows that of Pellarin's theorem in Section 2.1. The second step is to show that $\mathbb{B}_s$ is indeed a polynomial in $\mathbb{F}_q[\theta][\underline{t}_s]$.

2.1 Pellarin's theorem

In this Section, we present a proof Theorem 2.0.1 which follows closely that of [GAND+19, Theorem 4.6]. First we collect some lemmas which will be necessary in the sequel.

Lemma 2.1.1. *We have*

$$\frac{t^n}{\theta^n} \sum_{a \in A_{+,n}} \frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} = \sum_{\substack{a \in A, a(0)=1 \\ \deg_\theta a \leq n}} \frac{a(t)}{a(\theta)}.$$

Proof. For $a \in A_{+,n}$, we can write $a = a_0 + a_1\theta + \ldots + a_{n-1}\theta^{n-1} + \theta^n$, with $a_0, \ldots, a_{n-1} \in \mathbb{F}_q$. Applying to the left-hand side (LHS), we have

$$LHS = \sum_{a_0,\ldots,a_{n-1}\in\mathbb{F}_q} \frac{a_0t^n + a_1t^{n-1} + \ldots + a_{n-1}t + 1}{a_0\theta^n + a_1\theta^{n-1} + \ldots + a_{n-1}\theta + 1} = \sum_{\substack{a\in A, a(0)=1 \\ \deg_\theta a\leq n}} \frac{a(t)}{a(\theta)}.$$

The proof is done. □

Lemma 2.1.2. *For every $c \in \mathbb{F}_q\backslash\{0\}$, we have*

$$\sum_{\substack{a\in A, a(0)=1 \\ \deg_\theta a\leq n}} \frac{a(t)}{a} = \sum_{\substack{a\in A, a(0)=c \\ \deg_\theta a\leq n}} \frac{a(t)}{a} = -\sum_{\substack{a\in A, a(0)\neq 0 \\ \deg_\theta a\leq n}} \frac{a(t)}{a}.$$

Proof. The second equality is implied from the first equality and the fact that $|c \in \mathbb{F}_q\backslash\{0\}| = q - 1 = -1$. □

Lemma 2.1.3. *We have*

$$-\sum_{\substack{a\in A \\ \deg a\leq n}} \frac{a(t)}{a} = \sum_{d=0}^{n} \sum_{a\in A_{+,d}} \frac{a(t)}{a}.$$

Proof. By a similar method in the proof of Lemma 2.1.2, for every $c \in \mathbb{F}_q\backslash\{0\}$, we have

$$\sum_{\substack{a\in A, \deg_\theta a=d \\ a \text{ is monic}}} \frac{a(t)}{a} = \sum_{\substack{a\in A, \deg_\theta a=d \\ \text{leading coefficient of } a \text{ is } c}} \frac{a(t)}{a} = -\sum_{a\in A, \deg_\theta a=d} \frac{a(t)}{a}.$$

By summing from $d = 0$ to $d = n$, the proof follows. □

The following result is due to Carlitz [Gos96, Theorem 3.1.5].

Lemma 2.1.4 (See for example [GAD+19, Lemma 4.1])**.** *Let $d \geq 1$ be an integer. Then*

$$\sum_{a\in A_{+,d}} \frac{1}{a} = \frac{1}{l_d},$$

where $l_d = \prod_{k=1}^{d} (\theta - \theta^{q^k})$.

The following lemma was proved by Carlitz.

Lemma 2.1.5 (See [Car42, Page 688])**.** *Let $d \in \mathbb{N}, d \geq 1$. Then*

$$\sum_{a\in A_{+,d}} \frac{a(t)}{a} = \frac{1}{l_d} \prod_{k=0}^{d-1} (t - \theta^{q^k}), \tag{2.1.1}$$

where l_d is defined in (1.2.3).

Now, we are ready for a proof of Theorem 2.0.1.

Proof of Theorem 2.0.1.

We replace t by $\frac{1}{t}$, θ by $\frac{1}{\theta}$ and multiply both sides by $\frac{t^n}{\theta^n}$ in Equation (2.1.1). We have

$$\forall n \geq 1, \frac{t^n}{\theta^n} \sum_{a\in A_{+,n}} \frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} = \frac{t^n}{\theta^n} \prod_{k=1}^{n} (\frac{1}{\theta} - \frac{1}{\theta^{q^k}})^{-1} \prod_{k=0}^{n-1} (\frac{1}{t} - \frac{1}{\theta^{q^k}}),$$

i.e.,

$$\forall n \geq 1, \frac{t^n}{\theta^n} \sum_{a \in A_{+,n}} \frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} = \prod_{k=1}^{n} (1-\theta^{1-q^k})^{-1} \prod_{k=0}^{n-1} (1-\frac{t}{\theta^{q^k}}).$$

On the other hand, by Lemma 2.1.1 and Lemma 2.1.2, we obtain

$$\begin{aligned} \frac{t^n}{\theta^n} \sum_{a \in A_{+,n}} \frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} &= - \sum_{\substack{a \in A, a(0) \neq 0 \\ \deg_\theta a \leq n}} \frac{a(t)}{a} \\ &= - \left(\sum_{\substack{a \in A \\ \deg_\theta a \leq n}} \frac{a(t)}{a} - \frac{t}{\theta} \sum_{\substack{a \in A \\ \deg_\theta a \leq n-1}} \frac{a(t)}{a} \right). \end{aligned}$$

Hence, we have

$$- \left(\sum_{\substack{a \in A \\ \deg_\theta a \leq n}} \frac{a(t)}{a} - \frac{t}{\theta} \sum_{\substack{a \in A \\ \deg_\theta a \leq n-1}} \frac{a(t)}{a} \right) = \prod_{k=1}^{n} (1-\theta^{1-q^k})^{-1} \prod_{k=0}^{n-1} (1-\frac{t}{\theta^{q^k}}).$$

Let n tend to $+\infty$ and note that $\tilde{\pi} = (-\theta)^{\frac{1}{q-1}} \theta \prod_{j \geq 1} (1-\theta^{1-q^j})^{-1}$, $\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{j \geq 0} (1-\frac{t}{\theta^{q^j}})^{-1}$, we get

$$-(1-\frac{t}{\theta}) \lim_{n \to +\infty} \sum_{\substack{a \in A \\ \deg_\theta a \leq n}} \frac{a(t)}{a} = \frac{1}{\theta} \tilde{\pi} \omega(t)^{-1}.$$

By Lemma 2.1.3, it implies

$$(1-\frac{t}{\theta}) \lim_{n \to +\infty} \sum_{d=0}^{n} \sum_{a \in A_{+,d}} \frac{a(t)}{a} = \frac{1}{\theta} \tilde{\pi} \omega(t)^{-1}.$$

The proof follows. □

2.2 The first step of Proof of Theorem 2.0.2

The main statement of this section is Proposition 2.2.6. The idea of the proof is base on the proof of Pellarin's theorem in the previous section. We set

$$S_{d,s} := S_{d,s}(t_1, \ldots, t_s) = \sum_{a \in A_{+,d}} a(t_1) \cdots a(t_s).$$

Lemma 2.2.1 (See [AP14, Lemma 4]). *We have*

$$S_{d,s} \neq 0 \text{ if and only if } d(q-1) \leq s.$$

Lemma 2.2.2 (See [AP14, Lemma 5]). *Let $s' \in \mathbb{Z}^+$. Then $\sum_{d \geq 0} S_{d,s'} = 0$ if and only if $s' \equiv 0 \pmod{q-1}$.*

We have the following result due to B. Angles - F. Pellarin.

Proposition 2.2.3 (See [AP14, Proposition 10]). *Let $l, d \in \mathbb{N}$. Recall that $m = (s-1)/(q-1) \in \mathbb{N}$. If $l + m < d$ then*

$$\sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \equiv 0 \pmod{\prod_{j=1}^{s} (t_j - \theta^{q^l})}.$$

In addition, if n is an integer such that $n > l + m$ then we also have

$$\sum_{d=0}^{n} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \equiv 0 \pmod{\prod_{j=1}^{s} (t_j - \theta^{q^l})}.$$

Proof. We set

$$S(t_1,\dots,t_s) := \sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a}.$$

Let i be a non negative integer. Note that

$$q^i - 1 = (q-1)q^{i-1} + (q-1)q^{i-2} + \ldots + (q-1)q + (q-1).$$

We have

$$\begin{aligned} S(t_1,\dots,t_{s-1},\theta^{q^i}) &= \sum_{a\in A_{+,d}} a(t_1)\cdots a(t_{s-1})a(\theta)^{q^i-1} \\ &= \sum_{a\in A_{+,d}} a(t_1)\cdots a(t_{s-1})[a(\theta^{q^{i-1}})]^{q-1}\cdots[a(\theta)]^{q-1} \\ &= S_{d,s-1+i(q-1)}(t_1,\cdots,t_{s-1},\theta,\dots,\theta,\dots,\theta^{q^{i-1}},\dots,\theta^{q^{i-1}}). \end{aligned}$$

By Lemma 2.2.1, the above sum is zero if and only if $d(q-1) > s-1+i(q-1)$, i.e., $d > m+i$. It follows that for $i < d-m$,

$$S(t_1,\dots,t_s)|_{t_s=\theta^{q^i}} = 0.$$

Since $S(t_1,\dots,t_s)$ is symmetric in $t_1,\dots,t_s$, we have the first conclusion: for $l < d-m$,

$$\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} \equiv 0 \pmod{\prod_{j=1}^{s}(t_j-\theta^{q^l})}.$$

For the second part, we see that

$$\sum_{d\geq 0}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a}|_{t_j=\theta^{q^l}} = \sum_{d\geq 0}\sum_{a\in A_{+,d}} a(\theta)^{q^j-1}\prod_{\substack{1\leq i\leq s\\ i\neq j}} a(t_i) = 0.$$

The last equality follows from Lemma 2.2.2 and the fact that $s' = q^j - 1 + s - 1 \equiv 0 \pmod{q-1}$. Thus, we have

$$\sum_{d\geq 0}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} \equiv 0 \pmod{\prod_{j=1}^{s}(t_j-\theta^{q^l})}.$$

Combining with the first part, for $n > l+m$, we have

$$\begin{aligned} &\sum_{d=0}^{n}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} \\ &= \sum_{d\geq 0}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} - \sum_{d>n>l+m}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} \\ &\equiv 0 \pmod{\prod_{j=1}^{s}(t_j-\theta^{q^l})}. \end{aligned}$$

The proof is done. □

As a consequence, we have the following proposition.

Proposition 2.2.4. *We have*

$$\sum_{a\in A_{+,n}} \frac{a(t_1)\cdots a(t_s)}{a} = \frac{1}{l_n}\prod_{j=1}^{s}\prod_{l=0}^{n-m-1}(t_j-\theta^{q^l})\,B_{n,s},$$

where $B_{n,s}\in\mathbb{F}_q[\theta][t_1,\dots,t_s]$ is a polynomial of degree m in t_j, $j=1,\dots,s$ with coefficients in $A=\mathbb{F}_q[\theta]$ and the leading coefficient is 1.

Proof. In Proposition 2.2.3, for $n > m$, letting l run from 0 to $n-m-1$, we have

$$\sum_{a\in A_{+,n}} \frac{a(t_1)\cdots a(t_s)}{a} \equiv 0 \pmod{\prod_{j=1}^{s}\prod_{l=0}^{n-m-1}(t_j-\theta^{q^l})}.$$

Therefore we can write

$$\sum_{a\in A_{+,n}} \frac{a(t_1)\cdots a(t_s)}{a} = \frac{1}{l_n}\prod_{j=1}^{s}\prod_{l=0}^{n-m-1}(t_j-\theta^{q^l})\,B_{n,s},$$

where $B_{n,s}\in K_\infty[t_1,\dots,t_s]$. From this equality, we have some observations:

- Firstly, $B_{n,s}$ is of degree m in t_j for all $j=1,\dots,s$ since the degree of t_j on the left-hand side is n and degree of t_j in $\prod_{k=0}^{n-m-1}(t_j-\theta^{q^k})$ is $n-m$.
- Secondly, the coefficient of $t_1^m\cdots t_s^m$ in $B_{n,s}$ is 1 since the coefficient of $t_1^n\cdots t_s^n$ on the left-hand side, by Lemma 2.1.4, is
$$\sum_{a\in A_{+,n}}\frac{1}{a}=\frac{1}{l_n}.$$
- Lastly, the coefficients of $B_{n,s}\in K_\infty[t_1,\dots,t_s]$ are in fact in A. Indeed, if we multiply both sides of this equality with l_n and note that l_n is the least common multiple of all polynomial of degree n (see [Gos96, Proposition 3.1.6]), the left-hand side is in $A[t_1,\dots,t_s]$.

The proof is done. □

Remark 2.2.5. We see that

$$B_{n,s}\equiv\theta^{-\frac{q^{n+1}-1}{q-1}+s\frac{q^{n-m}-1}{q-1}}=\theta^{-(s-1)/(q-1)-(q^{n+1}-sq^{n-m})/(q-1)} \pmod{(t_1,\dots,t_s)}.$$

By Proposition 2.2.4, $t_1^m\cdots t_s^m B_{n,s}(\frac{1}{t_1},\dots,\frac{1}{t_s},\frac{1}{\theta})$ is a polynomial in t_j for $j=1,\dots,s$ of degree less than or equal m with coefficients in K_∞. By Remark 2.2.5, when $n\to+\infty$, this polynomial converges to a polynomial $B_s\in K_\infty[t_1,\dots,t_s]$ and $B_s\equiv\theta^{(s-1)/(q-1)}=\theta^m \pmod{(t_1,\dots,t_s)}$.

Set

$$B_s = t_1^m\cdots t_s^m\lim_{n\to+\infty}B_{n,s}(\frac{1}{t_1},\dots,\frac{1}{t_s},\frac{1}{\theta}).$$

Proposition 2.2.6. *With above notation, we have*

$$\frac{\zeta_A(1,\underline{t}_s)\omega(t_1)\cdots\omega(t_s)}{\tilde{\pi}}=\frac{B_s}{(\theta-t_1\cdots t_s)}.$$

Proof. We recall that

$$\sum_{a\in A_{+,n}}\frac{a(t_1)\cdots a(t_s)}{a}=\frac{1}{l_n}\prod_{k=0}^{n-m-1}\prod_{i=1}^{s}(t_i-\theta^{q^k})\,B_{n,s}.$$

Now, we apply the method using in the proof of Pellarin's theorem (Theorem 2.0.1): changing the variables and multiplying both sides with $\frac{t_1^n\cdots t_s^n}{\theta^n}$. It follows

$$\begin{aligned}\frac{t_1^n\cdots t_s^n}{\theta^n}\sum_{a\in A_{+,n}}\frac{a(\frac{1}{t_1})\cdots a(\frac{1}{t_s})}{a(\frac{1}{\theta})}&=\frac{t_1^n\cdots t_s^n}{\theta^n}\left(\prod_{k=1}^{n}(\frac{1}{\theta}-\frac{1}{\theta^{q^k}})^{-1}\right)\prod_{j=1}^{s}\prod_{k=0}^{n-m-1}(\frac{1}{t_j}-\frac{1}{\theta^{q^k}})B_{n,s}(\frac{1}{t_1},\dots,\frac{1}{t_s},\frac{1}{\theta})\\&=\prod_{k=1}^{n}(1-\theta^{1-q^k})^{-1}\prod_{j=1}^{s}\prod_{k=0}^{n-m-1}(1-\frac{t_j}{\theta^{q^k}})\times t_1^m\dots t_s^m B_{n,s}(\frac{1}{t_1},\dots,\frac{1}{t_s},\frac{1}{\theta}).\end{aligned}$$

On the other hand, we have

$$\frac{t_1^n\cdots t_s^n}{\theta^n}\sum_{a\in A_{+,n}}\frac{a(\frac{1}{t_1})\cdots a(\frac{1}{t_s})}{a(\frac{1}{\theta})}=\sum_{a\in A,a(0)=1,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_s)}{a}$$
$$=-\sum_{a\in A,a(0)\neq 0,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_n)}{a},$$

where the first equality (and the second equality) is a similar form of Lemma 2.1.1 (and Lemma 2.1.2).

Also,

$$\sum_{a\in A,a(0)\neq 0,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_n)}{a}=\sum_{a\in A,\deg_\theta a\le n}\frac{a(t_1\cdots a(t_s))}{a}-\frac{t_1\cdots t_s}{\theta}\sum_{a\in A,\deg_\theta a\le n-1}\frac{a(t_1)\cdots a(t_s)}{a}.$$

Finally, we have

$$-\left(\sum_{a\in A,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_s)}{a}-\frac{t_1\cdots t_s}{\theta}\sum_{a\in A,\deg_\theta a\le n-1}\frac{a(t_1)\cdots a(t_s)}{a}\right)$$
$$=\prod_{k=1}^{n}(1-\theta^{1-q^k})^{-1}\prod_{i=1}^{s}\prod_{k=0}^{n-m-1}(1-\frac{t_i}{\theta^{q^k}})\times t_1^m\dots t_s^m B_{n,s}(\frac{1}{t_1},\dots,\frac{1}{t_s},\frac{1}{\theta}).$$

Let $n\to\infty$, we get

$$-(1-\frac{t_1\cdots t_s}{\theta})\lim_{n\to+\infty}\sum_{a\in A,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_s)}{a}=\frac{1}{\theta}\tilde{\pi}\omega(t_1)^{-1}\cdots\omega(t_s)^{-1}B_s.$$

Note that, similar to Lemma 2.1.3, for $s\equiv 1 \pmod{q-1}$, we have

$$-\sum_{a\in A,\deg_\theta a\le n}\frac{a(t_1)\cdots a(t_s)}{a}=\sum_{d=0}^{n}\sum_{a\in A_{+,d}}\frac{a(t_1)\cdots a(t_s)}{a}.$$

Hence

$$(1-\frac{t_1\cdots t_s}{\theta})\zeta_A(1,\underline{t}_s)=\frac{1}{\theta}\tilde{\pi}\omega(t_1)^{-1}\cdots\omega(t_s)^{-1}B_s.$$

i.e.,

$$\frac{\zeta_A(1,\underline{t}_s)\omega(t_1)\cdots\omega(t_s)}{\tilde{\pi}}=\frac{B_s}{(\theta-t_1\cdots t_s)}.$$

The proof is done. □

As a consequence, we have

$$\mathbb{B}_s=\frac{B_s}{(\theta-t_1\cdots t_s)}.$$

By Remark 2.2.5, we have $\mathbb{B}_s\equiv\theta^{m-1} \pmod{(t_1,\dots,t_s)}$.

2.3 The second step of Proof of Theorem 2.0.2

In this section, firstly, we will collect some materials to prepare the proof of Theorem 2.3.6 (see [AP15, Theorem 2.9]). The main statement is Theorem 2.3.6 which is related to Gauss-Thakur sums (see [AP15] for more details). Then we outline the proof of showing that B_s is in $A[t_1,\dots,t_s]$ by using specialization as in [AP15, Section 2.5 Proof of Theorem 1].

We recall the notation in Section 1.2 and some properties. Let C be the Carlitz module and let $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the continuous surjective $\mathbb{F}_q$-linear map (the Carlitz exponential) satisfying $\exp_C a = C_a(\exp_C)$ for all $a \in A$.

We claim that

$$\{x \in \mathbb{C}_\infty, C_a(x) = 0\} = \{\exp_C(\frac{b\tilde{\pi}}{a}), b \in A, \deg_\theta(b) < \deg_\theta a\}.$$

Indeed, $\{\exp_C(\frac{b\tilde{\pi}}{a}), b \in A, \deg_\theta(b) < \deg_\theta a\} \subset \{x \in \mathbb{C}_\infty, C_a(x) = 0\}$ since $C_a(\exp_C(\frac{b\tilde{\pi}}{a})) = \exp_C(a\frac{b\tilde{\pi}}{a}) = 0$ by Proposition 1.2.8. For the inverse direction, let us take an $x \in \mathbb{C}_\infty$ such that $C_a(x) = 0$. Since $\exp_C$ is surjective (see Proposition 1.2.5), there exists $y \in \mathbb{C}_\infty$ such that $x = \exp_C(y)$. We have $0 = C_a(x) = C_a(\exp_C(y)) = \exp_C(ay)$. Again, by Proposition 1.2.8, we deduce that $y = \frac{b}{a}\tilde{\pi}$ for $b \in A$, $\deg b < \deg a$.

Definition 2.3.1. For $n \geq 0$, we define

$$\lambda_n = \exp_C(\frac{\widetilde{\pi}}{\theta^{n+1}}).$$

Since $C_a \exp_C = \exp_C a$, we have

$$\sqrt[q-1]{-\theta} = \lambda_0, \forall n \geq 1, C_\theta(\lambda_n) = \lambda_{n-1}.$$

We recall the definition of ω in (0.0.4)

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{j \geq 0} \left(1 - \frac{t}{\theta^{q^j}}\right)^{-1}.$$

Lemma 2.3.2. *We have*

$$\omega(t) = \sum_{n \geq 0} \lambda_n t^n.$$

Proof. The idea of the proof is base on the fact that

$$\{g(t) \in \mathbb{T}, \tau(g(t)) = g(t)\} = \mathbb{F}_q[t]. \tag{2.3.1}$$

This fact is a result of the following fact: $\{x \in \mathbb{C}_\infty, \tau(x) = x\} = \mathbb{F}_q$.

Now, set

$$f(t) = \sum_{n \geq 0} \lambda_n t^n \in \mathbb{T}.$$

We recall that for $x \in \mathbb{C}_\infty, v_\infty(x) > \frac{-q}{q-1}$, $v_\infty(\exp_C(x)) = v_\infty(x)$. Hence $v_\infty(\lambda_n) = v_\infty(\frac{\tilde{\pi}}{\theta^{n+1}}) = n + 1 - \frac{q}{q-1}$. Therefore

$$f(t) \in \mathbb{T}^\times.$$

We see that

$$\tau(f(t)) = \sum_{n \geq 0} (C_\theta(\lambda_n) - \theta\lambda_n)t^n = (t - \theta)f(t).$$

Hence

$$\tau(\frac{\omega(t)}{f(t)}) = \frac{\omega(t)}{f(t)}.$$

Using (2.3.1), we obtain

$$\frac{\omega(t)}{f(t)} \in \mathbb{F}_q[t] \setminus \{0\}.$$

It remains to show that

$$v_\infty(\frac{\omega(t)}{f(t)} - 1) > 0.$$

This is true by the following equality

$$\begin{aligned}\omega(t) - f(t) &= \lambda_0 \left(\prod_{k\geq 0} (1 - \frac{t}{\theta^{q^k}})^{-1} - 1 \right) + \lambda_1 t + \lambda_2 t^2 + \cdots \\ &= \lambda_0 \, (\text{ positive valuation }) + \lambda_1 t + \lambda_2 t^2 + \cdots .\end{aligned}$$

The proof is done. □

Now, we consider the following subfield of $\mathbb{C}_\infty$

$$\Omega = \overline{\mathbb{F}}_q((\lambda_0^{-1})).$$

We observe that Ω is complete, contains $\tilde{\pi}$. Its valuation ring is $\overline{\mathbb{F}}_q[[\lambda_0^{-1}]]$, its maximal ideal is $\lambda_0^{-1}\overline{\mathbb{F}}_q[[\lambda_0^{-1}]]$ and its residue field is $\overline{\mathbb{F}}_q$. Let $\sigma\colon \Omega \to \Omega$ be the continuous morphism of $\overline{\mathbb{F}}_q$-algebras such that

$$\sigma(\lambda_0) = \lambda_0^q.$$

Lemma 2.3.3. *We have*

$$\Omega^\sigma := \{x \in \Omega, \sigma(x) = x\} = \overline{\mathbb{F}}_q.$$

Proof. It is clear that $\overline{\mathbb{F}}_q \subset \Omega^\sigma$. Let $x \in \Omega^\sigma \setminus \{0\}$. We have

$$q v_\infty(x) = v_\infty(\sigma(x)) = v_\infty(x).$$

Therefore $x \in \overline{\mathbb{F}}_q[[\lambda_0^{-1}]]^\times$. If we write $x = \sum_{i\geq 0} \zeta_i \lambda_0^{-i}, \zeta_i \in \overline{\mathbb{F}}_q, \zeta_0 \neq 0$. We get

$$\sum_{i\geq 0} \zeta_i \lambda_0^{-i} = \sum_{i\geq 0} \zeta_i \lambda_0^{-qi}.$$

Therefore $x = \zeta_0 \in \overline{\mathbb{F}}_q$. □

Definition 2.3.4. For $a \in A_+$, we set

$$\lambda_a = \exp_C(\frac{\widetilde{\pi}}{a}) \in \Omega.$$

Remark 2.3.5. With notation in Definition 2.3.1, we have $\lambda_n = \lambda_{\theta^{n+1}}$.

We have the following theorem due to B. Angles and F. Pellarin.

Theorem 2.3.6 (See [AP15, Theorem 2.9])**.** *Let $\zeta \in \overline{\mathbb{F}}_q$ and let P be an irreducible polynomial such that $P(\zeta) = 0$. Then*

$$\omega(\zeta) = P'(\zeta)(- \sum_{a\in A, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} C_a(\lambda_P)).$$

Proof. First, we show that the equality is true for $\zeta = 0$. Suppose that $\zeta = 0$. Since P is irreducible and $P(0) = P(\zeta) = 0$, it implies $P(\theta) = \theta$. The left-hand side equals $\omega(0) = \lambda_0 = \lambda_\theta$. The right-hand side equals

$$- \sum_{a\in A, \deg a < 1} a(0)^{-1} C_a(\exp_C(\frac{\tilde{\pi}}{\theta})) = - \sum_{a\in \mathbb{F}_q^\times} a^{-1} \exp_C(\frac{a\tilde{\pi}}{\theta}) = - \sum_{a\in \mathbb{F}_q^\times} \exp_C(\frac{\tilde{\pi}}{\theta}) = \lambda_0.$$

The equality follows.

Now, we can assume that $\zeta \neq 0$. Set

$$G = - \sum_{a\in A, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} C_a(\lambda_P) \in \Omega.$$

We have

$$G = - \sum_{a\in A, a\neq 0, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} \exp_C(\frac{a\widetilde{\pi}}{P}).$$

We observe that

$$\sum_{a\in A, a\neq 0, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} a \equiv 0 \pmod{\prod_{k=1}^{d-1}(\theta - \zeta^{q^k})},$$

where $d = \deg_\theta P$. We therefore get

$$\sum_{a\in A, a\neq 0, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} a = -P'(\zeta)^{-1} \prod_{k=1}^{d-1}(\theta - \zeta^{q^k}).$$

We have

$$v_\infty(\frac{G}{\lambda_0 P'(\zeta)^{-1}} - 1) > 0,$$

and

$$v_\infty(\frac{G}{\omega(\zeta) P'(\zeta)^{-1}} - 1) > 0.$$

We have

$$\sigma(\omega(\zeta)) = \tau(\omega(t)) \mid_{t=\zeta} = (\theta - \zeta)\omega(\zeta).$$

Furthermore , we have

$$\sigma(G) = - \sum_{a\in A, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} (C_a(\lambda_P))^q.$$

Thus

$$\sigma(G) = - \sum_{a\in A, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} (C_{\theta a}(\lambda_P)) - \theta G.$$

Now, we have (since $P \neq \theta$)

$$- \sum_{a\in A, \deg_\theta a < \deg_\theta P} a(\zeta)^{-1} (C_{\theta a}(\lambda_P)) = \zeta G.$$

Thus, we get

$$\sigma(G) = (\theta - \zeta)G.$$

We conclude that $\frac{G}{\omega(\zeta)} \in \overline{\mathbb{F}}_q$. But recall that $v_\infty(\frac{G}{\omega(\zeta)} - P'(\zeta)^{-1}) > 0$, thus $G = P'(\zeta)^{-1}\omega(\zeta)$. □

Now, we list some propositions without proof to complete the proof of Theorem 2.0.2. We will need the following result due to B. Angles - F. Pellarin - L. Taelman.

Proposition 2.3.7 (See [AP14, Section 2.5]). *Let $s \geq q, s \equiv 1 \pmod{q-1}$. Let $\zeta_1, \ldots, \zeta_s \in \overline{\mathbb{F}}_q$ and let $P_1, \ldots, P_s$ be irreducible polynomials such that $P_i(\zeta_i) = 0$. We assume that $P_i \neq P_j$ if $i \neq j$. Then*

$$\frac{\zeta_A(t_1, \ldots, t_s)\omega(t_1)\cdots\omega(t_s)}{\widetilde{\pi}} \mid_{t_1=\zeta_1,\ldots,t_s=\zeta_s} \in \mathbb{F}_q[\zeta_1, \ldots, \zeta_s][\theta].$$

Combining with the following proposition, we have the conclusion.

Proposition 2.3.8 (See [AP14, Lemma 20]). *Let $f(t_1, \ldots, t_s) \in K_\infty[t_1, \ldots, t_s]$ such that for all $\zeta_1, \ldots, \zeta_s \in \overline{F}_q$ pairwise not conjugate over $\mathbb{F}_q$,*

$$f(\zeta_1, \ldots, \zeta_s) \in \mathbb{F}_q[\theta](\zeta_1, \ldots, \zeta_s).$$

Then $f(t_1, \ldots, t_s) \in \mathbb{F}_q[\theta][t_1, \ldots, t_s]$.

Chapter 3

Pellarin's conjectures

This chapter is taken from the paper published in Transactions of American Mathematics Society (see [LND21a]). It is available at https://doi.org/10.1090/tran/8357.

In Section 3.1, we present some of Pellarin's conjectures (Conjecture 3.1.1, Conjecture 3.1.2) and the statement of the main result (Theorem 3.1.3). In Section 3.2 we study the several variable Bernoulli-type polynomial $\mathbb{B}_s$. We introduce a notion of weight for polynomials and explain how to deduce Pellarin's conjectures from a lower bound on the weight of $\mathbb{B}_s$ (see Theorem 3.2.10). Section 3 is devoted to prove a key result, Theorem 3.3.1, which gives an explicit expression of $\mathbb{B}_s$ in terms of symmetric polynomials. Putting all together, we prove Theorem 3.1.3 in Section 3.4. Then we discuss some interesting questions in Section 3.5, which we will investigate a bit in Chapter 4.

3.1 Introduction

3.1.1 Background

A classical topic in number theory is the study of the Riemann zeta function $\zeta(.)$ and its special values $\zeta(n)$ for $n \in \mathbb{N}$ and $n \geq 2$. Here $\mathbb{N}$ (resp. $\mathbb{N}^*$) denotes the set of non-negative integers (resp. the set of positive integers). By a well-known analogy between the arithmetic of number fields and global function fields, Carlitz suggested to transport the classical results to the function field setting in positive characteristic. In [Car35] he considered the rational function field equipped with the infinity place and introduced the Carlitz zeta values $\zeta_A(n)$ which are considered as the analogues of $\zeta(n)$. Let $\mathbb{F}_q$ be a finite field having q elements, q being a power of a prime number p, and θ an indeterminate over $\mathbb{F}_q$. Let $A = \mathbb{F}_q[\theta]$, and let $K = \mathbb{F}_q(\theta)$ equipped with the rational place ∞. Let $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ be the completion of K at ∞, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞ at ∞. For $d \in \mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d. For $n \in \mathbb{Z}$, the value at n of the Carlitz-Goss zeta function is given by

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

One can show that $\zeta_A(n) \in A$ if $n \leq 0$ and even $\zeta_A(n) = 0$ if $n < 0$ and $n \equiv 0 \pmod{q-1}$ (see [Gos96, Chapter 8]).

We now move to the context of Tate algebras. Let $s \geq 1$ be an integer, and let $t_1, \ldots, t_s$ be s variables over K and we write $\underline{t}_s$ for the family of variables $\{t_1, \ldots, t_s\}$. Let $\mathbb{T}_s$ be the Tate algebra in the variables $\underline{t}_s$ with coefficients in $\mathbb{C}_\infty$ (see Section 3.2.1). In 2012 Pellarin [Pel12] introduced the following element in $\mathbb{T}_s^\times$ called *the zeta value in the variables* $\underline{t}_s$

$$\zeta_A(1, \underline{t}_s) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

For $s = 1$, he proved the remarkable identity (see [Pel12, Theorem 1])

$$\frac{\zeta_A(1, t_1)\,\omega(t_1)}{\widetilde{\pi}} = \frac{1}{\theta - t_1}$$

where $\widetilde{\pi}$ is the Carlitz fundamental period (see [Gos96, Tha04]), and $\omega(t_1)$ is the special function introduced by Anderson and Thakur in [AT90] and given by

$$\omega(t_1) = (-\theta)^{\frac{1}{q-1}} \prod_{j \geq 0} \left(1 - \frac{t_1}{\theta^{q^j}}\right)^{-1}$$

for a fixed choice of the $(q-1)$th root of $(-\theta)$ in $\mathbb{C}_\infty$.

Since their introduction various works have revealed the importance of these zeta values for both their proper interest and their applications to values of the Goss L-functions, characteristic p multiple zeta values, Anderson's log-algebraicity identities, Taelman's units, and Drinfeld modular forms in Tate algebras (see for example [ANDTR19, AP15, APTR16, APTR18, ATR17, Dem15a, Dem15b, Gez19, GP19, PP18a, PP18b, Tha17]). We should mention that generalizations of these zeta values to various settings have been also conducted (see for example [ANDTR17a, ANDTR17b, Gre19, Gre17, GP18]).

3.1.2 Conjectures of Pellarin and statement of the main result

From now on we will always suppose that $s \equiv 1 \pmod{q-1}$ and set

$$\Sigma := \{1, \dots, s\}, \tag{3.1.1}$$

and

$$m := \frac{s-1}{q-1} \in \mathbb{N}. \tag{3.1.2}$$

In a recent work [Pel21] Pellarin revisited the theory of Drinfeld modular forms which were initially developed by Goss in [Gos80b, Gos80c, Gos80a] and Gekeler in [Gek88]. In his investigation he proposed several conjectures for the zeta value $\zeta_A(1, \underline{t}_s)$ which would lead to new identities for Eisenstein series. We refer the reader to [Pel21, Section 9] for more details.

Conjecture 3.1.1 ([Pel21], Conjecture 9.1)**.** *We have*

$$\zeta_A(1, \underline{t}_s) \in \mathbb{F}_p\left[\tau^k(\zeta_A(1, t_i)) : \ 1 \leq i \leq s, \ k \in \mathbb{Z}\right].$$

As Pellarin mentioned in his paper (see the discussion just before [Pel21, Conjecture 9.1]), the central point of this conjecture is that negative twists are allowed, and that the coefficients belong to $\mathbb{F}_p$. Further, Pellarin suggested an explicit formula for $\zeta_A(1, \underline{t}_s)$ when q is large enough. More precisely, letting $k \in \mathbb{Z}$ and U be a subset of Σ, we set

$$\mathcal{L}_U^{(k)} := \tau^k\left(\prod_{i \in U} \zeta_A(1, t_i)\right). \tag{3.1.3}$$

Conjecture 3.1.2 ([Pel21], Conjecture 9.4)**.** *Let Σ and m be defined as in (3.1.1) and (3.1.2), respectively. Suppose that q is large enough, depending on m. Then we have the following formula*

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ of Σ (see Definition 3.2.1) satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

The aim of the present chapter is to give an affirmative answer to this conjecture with an explicit bound for q.

Theorem 3.1.3. *Conjecture 3.1.2 holds for $q > m$.*

A proof of Theorem 3.1.3 will be given in Section 3.4. Let us outline the main ideas of the proof.

1. First, using the link between zeta values in Tate algebras and Taelman's class formula due to Anglès, Pellarin and Tavares Ribeiro in [AP14, APTR16], we state an equivalent statement of Conjecture 3.1.2 (see Conjecture 3.2.5). Instead of identities on zeta values, it gives conjectural expressions on a certain several variable Bernoulli-type polynomial $\mathbb{B}_s \in \mathbb{F}_q[\underline{t}_s, \theta]$.

2. Next, using the polynomial $\mathbb{B}_s$ we give an expression of the zeta value $\zeta_A(1, \underline{t}_s)$ in terms of products of twists of zeta values $\zeta_A(1, t_i)$ for $i \in \Sigma$ with coefficients in $\mathbb{C}_\infty$ (see Proposition 3.2.8). Using some specialization arguments we are able to compute explicitly some coefficients of this expression (see Lemma 3.2.9). Furthermore, we introduce a notion of weight for polynomials in Definition 3.2.7 and show that if the weight of $\mathbb{B}_s$ is bounded below by 1, then the other coefficients vanish which implies Conjecture 3.1.2 (see Theorem 3.2.10).

3. Finally, we succeed in proving the previous bound for q large enough (see Section 3.3). In order to do so we express $\mathbb{B}_s$ as a linear combination of symmetric polynomials in $\underline{t}_s$. For q large enough we then compute explicitly this expression of $\mathbb{B}_s$ (see Theorem 3.3.1), which implies immediately the desired estimation of its weight (see Section 3.4). We mention that the proof of Theorem 3.3.1 is of combinatorial nature and that combinatorial properties of $\mathbb{B}_s$ have already had important applications in function field arithmetic (see [ANDTR19, GAND+19, PP18a] for more details).

3.2 The several variable Bernoulli-type polynomial

In this section we study the several variable Bernoulli-type polynomial $\mathbb{B}_s$. In Section 3.2.2 we recall its definition, basic properties, and connection with zeta values in Tate algebras. In Section 3.2.3 we use this polynomial to formulate a conjecture equivalent to Conjecture 3.1.2 (see Conjecture 3.2.5). Section 3.2.4 is devoted to express the zeta value $\zeta_A(1, \underline{t}_s)$ in terms of products of twists of zeta values in one variable $\zeta_A(1, t_i)$ for $i \in \Sigma$ (see Proposition 3.2.8). The key result states that under some mild condition on $\mathbb{B}_s$, Conjecture 3.1.2 holds (see Theorem 3.2.10).

3.2.1 Preliminaries

In this chapter we will work with the set of all (finite) sequences of integers $\underline{\ell}$. When we consider a sequence $\underline{\ell} = (\ell_1, \dots, \ell_d)$ of integers, the reader should keep in mind that d depends on the sequence $\underline{\ell}$, and that ℓ_i may be 0.

Definition 3.2.1. An *ordered set partition* of Σ defined as in (3.1.1) is a set partition $U_1 \sqcup \ldots \sqcup U_d$ of Σ equipped with a total order on its blocks $U_1 \prec \cdots \prec U_d$. Here we require that $U_d \neq \emptyset$ but the other blocks may be empty.

We will denote this ordered set partition of Σ by

$$\underline{U} = (U_1 \mid U_2 \mid \ldots \mid U_d).$$

Recall that $\{t_1, t_2, \dots, t_s\}$ denotes a family of s variables, and we will also denote this family by $\underline{t}_s$. For any ring R we set $R[\underline{t}_s] := R[t_1, \dots, t_s]$.

Let L be an extension of K_∞ in $\mathbb{C}_\infty$ such that L is complete with respect to v_∞. Then the polynomial ring $L[\underline{t}_s] = L[t_1, \dots, t_s]$ is equipped with the Gauss valuation: For a polynomial $f \in L[\underline{t}_s]$, if we write

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in L,$$

then the Gauss valuation of f is defined by

$$v_\infty(f) := \inf\{v_\infty(a_{i_1, \dots, i_s}),\ i_1, \dots, i_s \in \mathbb{N}\}.$$

We define the Tate algebra $\mathbb{T}_s(L)$ in the variables $\underline{t}_s$ with coefficients in L as the completion of $L[\underline{t}_s]$ with respect to the Gauss valuation. Explicitly, $\mathbb{T}_s(L)$ is the set of formal series

$$f = \sum_{i_1,\dots,i_s \in \mathbb{N}} a_{i_1,\dots,i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1,\dots,i_s} \in L,$$

such that

$$\lim_{i_1+\dots+i_s \to +\infty} v_\infty(a_{i_1,\dots,i_s}) = +\infty.$$

When $L = \mathbb{C}_\infty$, we will write $\mathbb{T}_s$ instead of $\mathbb{T}_s(\mathbb{C}_\infty)$. Let $\tau\colon \mathbb{T}_s \to \mathbb{T}_s$ be the continuous homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras such that for a formal series $f \in \mathbb{T}_s$, if we write

$$f = \sum_{i_1,\dots,i_s \in \mathbb{N}} a_{i_1,\dots,i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1,\dots,i_s} \in \mathbb{C}_\infty,$$

then

$$\tau(f) = \sum_{i_1,\dots,i_s \in \mathbb{N}} a_{i_1,\dots,i_s}^q t_1^{i_1} \dots t_s^{i_s}.$$

With this action of τ on $\mathbb{T}_s$, we have the non-commutative rings $\mathbb{T}_s\{\tau\}$ and $\mathbb{T}_s\{\{\tau\}\}$. The latter set consists of the formal series $\sum_{i \geq 0} f_i \tau^i$ with $f_i \in \mathbb{T}_s$ for all i, and the elements of the former are the polynomials in τ with coefficients in $\mathbb{T}_s$. The commutation rule defining the product is given by $\tau f = \tau(f)\tau$ for $f \in \mathbb{T}_s$.

3.2.2 The several variable polynomial $\mathbb{B}_s$

We briefly recall the deep connection between the zeta value $\zeta_A(1, \underline{t}_s)$ and the several variable Bernoulli-type polynomial $\mathbb{B}_s$ as explained in [AP14, APTR16].

Recall that for $s = 1$, Pellarin proved the following identity (see [Pel12, Theorem 1]):

$$\frac{\zeta_A(1, t_1)\, \omega(t_1)}{\widetilde{\pi}} = \frac{1}{\theta - t_1}. \tag{3.2.1}$$

For $s \geq 2$ and $s \equiv 1 \pmod{q-1}$, we define

$$\mathbb{B}_s := (-1)^m \frac{\zeta_A(1, \underline{t}_s)\omega(t_1) \dots \omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s \tag{3.2.2}$$

where m is given by (3.1.2). Then by [APTR16, Lemma 7.6] (see also [AP15, Corollary 21]), we have

Proposition 3.2.2. *The element $\mathbb{B}_s$ is a polynomial in $\mathbb{F}_q[\underline{t}_s, \theta]$. Moreover, it is a monic polynomial in the variable θ of degree $m-1$ and a symmetric polynomial in the variables $\underline{t}_s$.*

Inspired by Taelman's theory in [Tae10, Tae12b], Anglès, Pellarin, and Tavares Ribeiro showed that the polynomial $\mathbb{B}_s$ is closely connected to the class module H_ϕ of a certain Drinfeld $A[\underline{t}_s]$-module ϕ of rank one as follows (see [APTR16, Section 7] for more details). Let $\phi\colon A[\underline{t}_s] \to \mathbb{T}_s\{\tau\}$ be the Drinfeld $A[\underline{t}_s]$-module over $\mathbb{T}_s$ given by a homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras such that

$$\phi_\theta = \theta + (t_1 - \theta) \cdots (t_s - \theta)\tau.$$

There exists a unique formal series $\exp_\phi \in \mathbb{T}_s\{\{\tau\}\}$ called the exponential series attached to ϕ such that

$$\exp_\phi \equiv 1 \pmod{\tau},$$

and

$$\phi_a \exp_\phi = \exp_\phi a, \quad a \in A[\underline{t}_s].$$

One can show that the exponential series induces a natural $\mathbb{F}_q[\underline{t}_s]$-linear map

$$\exp_\phi\colon \mathbb{T}_s \to \mathbb{T}_s.$$

Following Taelman [Tae10, Tae12b], we define the class module H_ϕ by

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A[\underline{t}_s])}$$

where $\phi(A[\underline{t}_s])$ is the $\mathbb{F}_q[\underline{t}_s]$-module $A[\underline{t}_s]$ equipped with the $A[\underline{t}_s]$-module structure induced by ϕ. By [APTR16, Proposition 7.2] the class module H_ϕ is a finitely generated $\mathbb{F}_q[\underline{t}_s]$-module of rank $m-1$. The importance of the polynomials $\mathbb{B}_s$ is explained in the following theorem.

Theorem 3.2.3 ([APTR16], Theorem 7.7)**.** *We denote by* $\mathrm{Fitt}_{A[\underline{t}_s]}(H_\phi)$ *the Fitting ideal of the torsion* $A[\underline{t}_s]$*-module* H_ϕ *of finite type. Then*

$$\mathrm{Fitt}_{A[\underline{t}_s]}(H_\phi) = \mathbb{B}_s A[\underline{t}_s].$$

In particular,

$$\mathbb{B}_s = \det_{\mathbb{F}_q[\underline{t}_s][Z]} \left(Z \cdot \mathrm{Id} - \phi_\theta \mid_{H_\phi \otimes_{\mathbb{F}_q[\underline{t}_s]} \mathbb{F}_q[\underline{t}_s][Z]} \right) \mid_{Z=\theta}.$$

A few explicit examples of the polynomials $\mathbb{B}_s$ are given in [ANDTR19, APTR16] (see also [PP18a]). We need to introduce some more notation.

Definition 3.2.4. For any sequence $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$, we set

$$\ell_0 := s - (\ell_1 + \cdots + \ell_d) \in \mathbb{Z},$$

and define

$$\sigma_s(\underline{\ell}) := \sigma_s(\ell_1, \dots, \ell_d) = \sum \prod_{k=1}^{d} \prod_{i \in U_{k+1}} t_i^k,$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_{d+1})$ of Σ such that $|U_{k+1}| = \ell_k$ for $0 \le k \le d$. Here by convention, empty products are one and empty sums are equal to zero.

In particular, $\sigma_s(\underline{\ell}) = 0$ if $\ell_1 + \cdots + \ell_d > s$, which is equivalent to the condition $\ell_0 < 0$. The reader should keep in mind that ℓ_i may be 0. For example,

$$\sigma_s(0, 0, 1) = \sum_{i=1}^{s} t_i^3.$$

Here are some more explicit examples that will appear in the explicit formulas of $\mathbb{B}_s$ for small values of s:

$$\begin{aligned}
\sigma_{2q-1}(q) &= \sum_{1 \le i_1 < \cdots < i_q \le 2q-1} t_{i_1} \cdots t_{i_q}, \\
\sigma_{3q-2}(q) &= \sum_{1 \le i_1 < \cdots < i_q \le 3q-2} \prod_{j=1}^{q} t_{i_j}, \\
\sigma_{3q-2}(2q-1) &= \sum_{1 \le i_1 < \cdots < i_{2q-1} \le 3q-2} \prod_{j=1}^{2q-1} t_{i_j}, \\
\sigma_{3q-2}(2q) &= \sum_{1 \le i_1 < \cdots < i_{2q} \le 3q-2} \prod_{j=1}^{2q} t_{i_j}, \\
\sigma_{3q-2}(q-1, q) &= \sum_{1 \le i_1 < \cdots < i_{q-1} \le 3q-2} \sum_{\substack{1 \le k_1 < \cdots < k_q \le 3q-2 \\ k_\ell \ne i_j}} \prod_{j=1}^{q-1} t_{i_j} \prod_{\ell=1}^{q} t_{k_\ell}^2.
\end{aligned}$$

By [ANDTR19, Lemma 3.4] we have

$$\mathbb{B}_q = 1, \tag{3.2.3}$$

$$\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q), \tag{3.2.4}$$

$$\mathbb{B}_{3q-2} = \theta^2 - \theta[\sigma_{3q-2}(q) + \sigma_{3q-2}(2q-1)] + [\sigma_{3q-2}(q-1, q) + \sigma_{3q-2}(2q)]. \tag{3.2.5}$$

3.2.3 A conjecture equivalent to Conjecture 3.1.2

In this section we use the several variable polynomial $\mathbb{B}_s$ to formulate a conjecture equivalent to Conjecture 3.1.2 (see Conjecture 3.2.5).

Let $k \in \mathbb{N}^*$. Since $\tau\omega(t_1) = (t_1 - \theta)\omega(t_1)$, we get

$$\tau^{-k}\left(\frac{1}{(t_1-\theta)\omega(t_1)}\right) = \frac{(t_1 - \theta^{\frac{1}{q^{k-1}}})\cdots(t_1-\theta^{\frac{1}{q}})}{\omega(t_1)}.$$

By Equation (3.2.1), we know that

$$\zeta_A(1,t_1) = \frac{\widetilde{\pi}}{(\theta - t_1)\,\omega(t_1)} = -\frac{\widetilde{\pi}}{(t_1-\theta)\omega(t_1)}.$$

It follows that

$$\tau^{-k}(\zeta_A(1,t_1)) = \tau^{-k}\left(-\frac{\widetilde{\pi}}{(t_1-\theta)\omega(t_1)}\right) = -\frac{\widetilde{\pi}^{\frac{1}{q^k}}(t_1 - \theta^{\frac{1}{q^{k-1}}})\cdots(t_1-\theta^{\frac{1}{q}})}{\omega(t_1)}$$

since $(-1)^{q^k} = -1$.

Similarly, for $1 \le i \le s$, we obtain

$$\tau^{-k}(\zeta_A(1,t_i)) = -\frac{\widetilde{\pi}^{\frac{1}{q^k}}(t_i - \theta^{\frac{1}{q^{k-1}}})\cdots(t_i-\theta^{\frac{1}{q}})}{\omega(t_i)} = -\frac{\widetilde{\pi}^{\frac{1}{q^k}} b_k^*(t_i)}{\omega(t_i)} \tag{3.2.6}$$

where we set

$$b_k^*(t_i) := (t_i - \theta^{\frac{1}{q^{k-1}}})\cdots(t_i-\theta^{\frac{1}{q}}). \tag{3.2.7}$$

Note that $b_1^*(t_i) = 1$.

For a subset U of Σ, we define

$$B_k^*(\underline{t}_U) := \prod_{i\in U} b_k^*(t_i).$$

By the previous discussion we deduce that Conjecture 3.1.2 is equivalent to the following conjecture.

Conjecture 3.2.5 ([Pel21], Conjecture 9.7)**.** *Suppose that q is large enough, depending on m. Then the following formula holds*

$$\mathbb{B}_s = (-1)^{m-1}\sum B_1^*(\underline{t}_{U_1})\cdots B_d^*(\underline{t}_{U_d}) \tag{3.2.8}$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ of Σ (see Definition 3.2.1) satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We now present the cases $m = 1$ and $m = 2$ to illustrate combinatorial computations which we may encounter. We follow the presentation of Pellarin given in [Pel21, Section 9.1.1] and see that by direct calculations Conjecture 3.2.5 holds in these cases.

1. For $m = 1$, Conjecture 3.2.5 holds since both sides of (3.2.8) are equal to 1 (see (3.2.3) for the left-hand side).

2. For $m = 2$, by (3.2.4) the left-hand side of (3.2.8) equals

   $$\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q).$$

   Since $|\Sigma| = s = 2q-1$, we see that the only ordered set partitions appearing on the right-hand side of (3.2.8) are $(U_1 \mid U_2)$ with $|U_1| = q-1$ and $|U_2| = q$. It follows that the right-hand side of (3.2.8) is equal to

   $$-\sum_{U_2\subset\Sigma,\,|U_2|=q}\ \prod_{i\in U_2}\left(t_i - \theta^{1/q}\right).$$

We claim that this expression is equal to $\theta - \sigma_{2q-1}(q)$, which confirms Conjecture 3.2.5 for $m = 2$. In fact, it is easy to see that all the terms defined over $\mathbb{F}_q[\theta^{1/q}]$ but not over $\mathbb{F}_q[\theta]$ cancel. Further, the terms over $\mathbb{F}_q[\theta]$ give exactly the polynomial $\theta - \sigma_{2q-1}(q)$ as desired.

3. More generally, our strategy follows that given in the case $m = 2$. On the one hand, we show that on the right-hand side of (3.2.13) all the terms not defined over $\mathbb{F}_q[\theta]$ cancel, which is exactly explained in the rest of this Section. On the other hand, we compute the terms over $\mathbb{F}_q[\theta]$ and prove that they give exactly $\mathbb{B}_s$, which will be done in Section 3.3.

Remark 3.2.6. Let $\underline{U} = (U_1 \mid \cdots \mid U_d)$ be an ordered set partition of Σ satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We set $\ell_j = |U_j|$ for $1 \leq j \leq d$. Then the sequence $\underline{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$ with $\ell_d \geq 1$ is a solution of the system

$$\begin{cases} \ell_1 + \ldots + \ell_d = s \\ \frac{\ell_1}{q} + \cdots + \frac{\ell_d}{q^d} = 1. \end{cases} \tag{3.2.9}$$

We assume further that $m < q$ where m is defined as in (3.1.2). Then one can show easily that if $\underline{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$ with $\ell_d \geq 1$ is a solution of the above system, then we can write $\ell_1 = q - s_1, \ell_2 = s_1 q - s_2, \ldots, \ell_d = s_{d-1} q$ for a sequence $(s_1, \ldots, s_{d-1}) \in (\mathbb{N}^*)^{d-1}$ with $s_1 + \ldots + s_{d-1} = m - 1$. In fact, the map

$$(\ell_1, \ldots, \ell_d) \mapsto (s_1, \ldots, s_{d-1})$$

gives rise to a bijection between the set $\mathcal{U}_m$ of solutions $\underline{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$ with $\ell_d \geq 1$ of the system (3.2.9) and that of sequences $(s_1, \ldots, s_{d-1}) \in (\mathbb{N}^*)^{d-1}$ with $s_1 + \ldots + s_{d-1} = m - 1$. In particular, for $m \geq 2$, the cardinal of the set $\mathcal{U}_m$ equals 2^{m-2}.

We will give, for $m = 1, 2, 3, 4$ and $m < q$, the explicit list of the elements of $\mathcal{U}_m$.

- $m = 1$: $\mathcal{U}_1 = \{(q)\}$;
- $m = 2$: $\mathcal{U}_2 = \{(q-1, q)\}$;
- $m = 3$: $\mathcal{U}_3 = \{(q-1, q-1, q), (q-2, 2q)\}$;
- $m = 4$: $\mathcal{U}_4 = \{(q-1, q-1, q-1, q), (q-1, q-2, 2q), (q-2, 2q-1, q), (q-3, 3q)\}$.

3.2.4 Twists of zeta values in one variable

In this section we will first give an expression for the zeta value $\zeta_A(1, \underline{t}_s)$ in terms of products of twists of zeta values in one variable $\zeta_A(1, t_i)$ for $i \in \Sigma$ with coefficients in $\mathbb{C}_\infty$ (see Proposition 3.2.8). Next, using specialization properties we determine some coefficients of this expression (see Lemma 3.2.9). Finally, under some mild condition on $\mathbb{B}_s$, we deduce Conjecture 3.1.2 (and its equivalent form, Conjecture 3.2.5) from the previous calculations (see Theorem 3.2.10).

We start introducing a notion of weight for polynomials.

Definition 3.2.7. 1) Let $\underline{\ell} = (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$ be an s-tuple of integers. We consider the monomial $\underline{t}_s^{\underline{\ell}} := \prod_{i=1}^s t_i^{\ell_i}$ and define its weight by

$$\mathbf{w}(\underline{t}_s^{\underline{\ell}}) := \sum_{i=1}^{s} \frac{1}{q^{\ell_i + 1}}.$$

2) Let $P(\underline{t}_s) \in \mathbb{C}_\infty[\underline{t}_s]$ be a non-zero polynomial. If we express

$$P(\underline{t}_s) = \sum_{\underline{\ell}} a_{\underline{\ell}} \underline{t}_s^{\underline{\ell}}, \quad a_{\underline{\ell}} \in \mathbb{C}_\infty,$$

where the sum runs through the set of s-tuples $\underline{\ell} = (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$, then we define its weight by

$$\mathbf{w}(P) := \min\{\mathbf{w}(\underline{t}_s^{\underline{\ell}}) : a_{\underline{\ell}} \neq 0\}.$$

Proposition 3.2.8. *With the notation as above, we can express*

$$\zeta_A(1,\underline{t}_s) = (-1)^{m-1} \sum_{\underline{\ell}\in(\mathbb{N}^*)^s} \widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^{s} \tau^{-\ell_i}\left(\zeta_A(1,t_i)\right), \quad \gamma_{\underline{\ell}} \in \mathbb{C}_\infty, \tag{3.2.10}$$

where the sum runs through a finite set of s-tuples $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$ such that if $\gamma_{\underline{\ell}} \neq 0$, then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}.$$

Proof. The proof is divided into several steps.

Step 1. First, we will express the polynomial $\mathbb{B}_s \in \mathbb{F}_q[\underline{t}_s,\theta]$ as a sum of products of $b_k^*(t_i)$ defined as in (3.2.7).

Recall that for $k \in \mathbb{N}^*$ and for $1 \leq i \leq s$, we have set in (3.2.7)

$$b_k^*(t_i) = (t_i - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_i - \theta^{\frac{1}{q}}) \in \mathbb{C}_\infty[t_i]$$

which is a polynomial in the variable t_i of degree $k-1$. It follows that for $n \in \mathbb{N}$, we can write

$$t_i^n = \sum_{k=1}^{n+1} a_{k,n} b_k^*(t_i), \quad a_{k,n} \in \mathbb{C}_\infty. \tag{3.2.11}$$

We note that the coefficients $a_{k,n}$ do not depend on $i \in \Sigma$.

For the polynomial $\mathbb{B}_s \in \mathbb{F}_q[\underline{t}_s,\theta]$, we write

$$\mathbb{B}_s = \sum_{\underline{j}} \beta_{\underline{j}} \underline{t}_s^{\underline{j}} = \sum_{\underline{j}} \beta_{\underline{j}} t_1^{j_1} \dots t_s^{j_s}, \quad \text{with } \beta_{\underline{j}} \in \mathbb{F}_q[\theta],$$

where the sum runs through a finite set of s-tuples $\underline{j} = (j_1,\dots,j_s) \in \mathbb{N}^s$.

For any s-tuple of positive integers $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$, we set

$$\gamma_{\underline{\ell}} := \sum_{\underline{j}} \beta_{\underline{j}} \prod_{i=1}^{s} a_{\ell_i,j_i} \in \mathbb{C}_\infty, \tag{3.2.12}$$

where the coefficients a_{ℓ_i,j_i} are defined as in (3.2.11), and the sum runs through the set of s-tuples $\underline{j} = (j_1,\dots,j_s) \in \mathbb{N}^s$ such that $j_i + 1 \geq \ell_i$ for $1 \leq i \leq s$.

By (3.2.11), we get

$$\begin{aligned} \mathbb{B}_s &= \sum_{\underline{j}} \beta_{\underline{j}} t_1^{j_1} \dots t_s^{j_s} \\ &= \sum_{\underline{j}} \beta_{\underline{j}} \prod_{i=1}^{s} \sum_{\ell_i=1}^{j_i+1} a_{\ell_i,j_i} b_{\ell_i}^*(t_i) \\ &= \sum_{\underline{\ell}} \gamma_{\underline{\ell}} b_{\ell_1}^*(t_1) \cdots b_{\ell_s}^*(t_s). \end{aligned} \tag{3.2.13}$$

Here

- the first and second sum run through a finite set of s-tuples $\underline{j} = (j_1,\dots,j_s) \in \mathbb{N}^s$.
- the third sum runs through a finite set of s-tuples $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$.

Step 2. Next, letting $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$ be an s-tuple of positive integers, we claim that if $\gamma_{\underline{\ell}} \neq 0$, then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}.$$

In fact, if $\gamma_{\underline{\ell}} \neq 0$, then by (3.2.12), there exists an s-tuple $\underline{j} = (j_1, \dots, j_s) \in \mathbb{N}^s$ such that $\ell_i \leq j_i + 1$ for $1 \leq i \leq s$ and

$$\beta_{\underline{j}} \prod_{i=1}^{s} a_{\ell_i, j_i} \neq 0.$$

It implies that $\beta_{\underline{j}} \neq 0$. Thus we obtain

$$\mathbf{w}(\mathbb{B}_s) \leq \mathbf{w}(t_1^{j_1} \dots t_s^{j_s}) = \sum_{i=1}^{s} \frac{1}{q^{j_i+1}} \leq \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}.$$

Here the first inequality and the second equality follow from the fact that $\beta_{\underline{j}} \neq 0$ and Definition 3.2.7, respectively. The last inequality comes from the fact that $\ell_i \leq j_i + 1$ for all $1 \leq i \leq s$.

Step 3. We now switch to zeta values in Tate algebras. We have

$$\begin{aligned}
\zeta_A(1, \underline{t}_s) &= \frac{(-1)^m \widetilde{\pi} \mathbb{B}_s}{\omega(t_1) \dots \omega(t_s)} \quad \text{by (3.2.2)} \\
&= \frac{(-1)^m \widetilde{\pi} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \gamma_{\underline{\ell}} b^*_{\ell_1}(t_1) \cdots b^*_{\ell_s}(t_s)}{\omega(t_1) \dots \omega(t_s)} \quad \text{by (3.2.13)} \\
&= (-1)^m \widetilde{\pi} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \gamma_{\underline{\ell}} \prod_{i=1}^{s} \frac{(-1)\tau^{-\ell_i}(\zeta_A(1, t_i))}{\widetilde{\pi}^{\frac{1}{q^{\ell_i}}}} \quad \text{by (3.2.6)} \\
&= (-1)^{m-1} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \widetilde{\pi}^{1 - \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^{s} \tau^{-\ell_i}(\zeta_A(1, t_i)) \quad (\text{since } (-1)^s = -1)
\end{aligned}$$

where the sum runs through a finite set of s-tuples of positive integers $\underline{\ell} \in (\mathbb{N}^*)^s$. The proof of Proposition 3.2.8 is finished. □

We now calculate some coefficients of the expression (3.2.10) using specialization arguments. Let $\underline{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$ be an s-tuple of non-negative integers. We study the following specialization of $(t_1, \dots, t_s)$:

$$t_i = \theta^{q^{-k_i}} = \theta^{\frac{1}{q^{k_i}}}, \quad i = 1, \dots, s.$$

Let $i \in \Sigma$. For an s-tuple $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$, we have

$$\begin{aligned}
\tau^{-\ell_i}(\zeta_A(1, t_i))\big|_{t_i = \theta^{q^{-k_i}}} &= \left(\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_i)}{a^{q^{-\ell_i}}} \right)\Big|_{t_i = \theta^{q^{-k_i}}} \\
&= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(\theta^{q^{-k_i}})}{a^{q^{-\ell_i}}} \\
&= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^{q^{-k_i}}}{a^{q^{-\ell_i}}} \\
&= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^{\left(\frac{1}{q^{\ell_i}} - \frac{1}{q^{k_i}}\right)}}.
\end{aligned}$$

Recall that $\zeta_A(n) = 0$ if $n < 0$ and $n \equiv 0 \pmod{q-1}$, $\zeta_A(0) = 1$ and $\zeta_A(n) \neq 0$ if $n > 0$ (see for example [Gos96, Chapter 8]). It follows that

$$\tau^{-\ell_i}(\zeta_A(1, t_i))\big|_{t_i = \theta^{q^{-k_i}}} = \begin{cases} 0 & \text{if } \ell_i > k_i, \\ 1 & \text{if } \ell_i = k_i, \\ \neq 0 & \text{otherwise.} \end{cases} \tag{3.2.14}$$

We now analyze the term $\zeta_A(1,\underline{t}_s)_{\big|t_i=\theta^{q^{-k_i}}}$. We write

$$\begin{aligned}\zeta_A(1,\underline{t}_s)_{\big|t_i=\theta^{q^{-k_i}}} &= \left(\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(t_1)\dots a(t_s)}{a}\right)_{\big|t_i=\theta^{q^{-k_i}}}\\ &= \sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(\theta^{q^{-k_1}})\dots a(\theta^{q^{-k_s}})}{a}\\ &= \sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{1}{a^{\left(1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}\right)}}.\end{aligned}$$

Since $s\equiv 1 \pmod{q-1}$, we can write

$$1-\sum_{i=1}^{s}\frac{1}{q^{k_i}} = \frac{u}{q^k}$$

with $u\equiv 0 \pmod{q-1}$ and $k\in\mathbb{N}^*$. Again, since $\zeta_A(n)=0$ if $n<0$ and $n\equiv 0 \pmod{q-1}$, $\zeta_A(0)=1$ and $\zeta_A(n)\neq 0$ if $n>0$, we deduce

$$\zeta_A(1,\underline{t}_s)_{\big|t_i=\theta^{q^{-k_i}}} = \begin{cases}0 & \text{if } \sum_{i=1}^{s}\frac{1}{q^{k_i}}>1,\\ 1 & \text{if } \sum_{i=1}^{s}\frac{1}{q^{k_i}}=1,\\ \neq 0 & \text{otherwise.}\end{cases} \tag{3.2.15}$$

Lemma 3.2.9. *We continue with the notation of Proposition 3.2.8. Then for any s-tuple $\underline{k}=(k_1,\dots,k_s)\in(\mathbb{N}^*)^s$, we have*

$$\gamma_{\underline{k}} = \begin{cases}0 & \text{if } \sum_{i=1}^{s}\frac{1}{q^{k_i}}>1,\\ (-1)^{m-1} & \text{if } \sum_{i=1}^{s}\frac{1}{q^{k_i}}=1.\end{cases}$$

Proof. We divide the proof into two steps.

Step 1. Recall that the coefficients $\gamma_{\underline{\ell}}$ are defined as in Proposition 3.2.8. We consider the set $\mathfrak{B}$ of s-tuples $\underline{\ell}=(\ell_1,\dots,\ell_s)\in(\mathbb{N}^*)^s$ such that $\gamma_{\underline{\ell}}\neq 0$. We choose one s-tuple $\underline{k}=(k_1,\dots,k_s)\in\mathfrak{B}$ such that the sum $\sum_{i=1}^{s}\frac{1}{q^{k_i}}$ is maximal. Thus $\gamma_{\underline{k}}\neq 0$.

We claim that $\sum_{i=1}^{s}\frac{1}{q^{k_i}}\leq 1$. In fact, suppose that $\sum_{i=1}^{s}\frac{1}{q^{k_i}}>1$. We consider (3.2.10) and study the specialization of $(t_1,\dots,t_s)$ given as above:

$$t_i=\theta^{q^{-k_i}}=\theta^{\frac{1}{q^{k_i}}},\quad i=1,\dots,s.$$

Since $\sum_{i=1}^{s}\frac{1}{q^{k_i}}>1$, Equation (3.2.15) implies

$$\zeta_A(1,\underline{t}_s)_{\big|t_i=\theta^{q^{-k_i}}}=0. \tag{3.2.16}$$

Thus the specialization value of the left-hand side of (3.2.10) equals 0.

We now analyze the right-hand side of (3.2.10). First, we consider the term corresponding to the s-tuple $\underline{k}=(k_1,\dots,k_s)\in(\mathbb{N}^*)^s$. By Equation (3.2.14), we get

$$(-1)^{m-1}\left(\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}}\gamma_{\underline{k}}\prod_{i=1}^{s}\tau^{-k_i}(\zeta_A(1,t_i))\right)_{\big|t_i=\theta^{q^{-k_i}}} = (-1)^{m-1}\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}}\gamma_{\underline{k}}.$$

Next, for other s-tuple $\underline{\ell} \in \mathfrak{B}$, that means $\gamma_{\underline{\ell}} \neq 0$, we claim that there exists $1 \leq i \leq s$ such that $\ell_i > k_i$. Suppose that $\ell_i \leq k_i$ for $1 \leq i \leq s$ and $\ell_j < k_j$ for some $1 \leq j \leq s$. Thus we get $\sum_{i=1}^{s} \frac{1}{q^{k_i}} < \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}$, which contradicts with the fact that $\sum_{i=1}^{s} \frac{1}{q^{k_i}}$ is maximal.

Since $\ell_i > k_i$ for some $1 \leq i \leq s$, by (3.2.14) we have

$$\tau^{-\ell_i}\left(\zeta_A(1,t_i)\right)_{\big|t_i=\theta^{q^{-k_i}}} = 0.$$

Thus we obtain

$$(-1)^{m-1}\left(\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{\ell_i}}}\gamma_{\underline{\ell}}\prod_{i=1}^{s}\tau^{-\ell_i}(\zeta_A(1,t_i))\right)_{\big|t_i=\theta^{q^{-k_i}}} = 0.$$

Putting all together, the specialization value of the right-hand side of (3.2.10) equals

$$(-1)^{m-1}\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}}\gamma_{\underline{k}}. \tag{3.2.17}$$

By (3.2.16) and (3.2.17) we conclude that $(-1)^{m-1}\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}}\gamma_{\underline{k}} = 0$. Thus $\gamma_{\underline{k}} = 0$, which is a contradiction.

To summarize we have proved that for any s-tuple $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$, if $\sum_{i=1}^{s} \frac{1}{q^{\ell_i}} > 1$, then $\gamma_{\underline{\ell}} = 0$.

Step 2. We consider an s-tuple $\underline{k} = (k_1, \dots, k_s) \in (\mathbb{N}^*)^s$ such that the sum $\sum_{i=1}^{s} \frac{1}{q^{k_i}} = 1$.

We claim that $\gamma_{\underline{k}} = (-1)^{m-1}$. As before, we consider (3.2.10) and study the specialization of $(t_1, \dots, t_s)$ given as above:

$$t_i = \theta^{q^{-k_i}} = \theta^{\frac{1}{q^{k_i}}}, \quad i = 1, \dots, s.$$

Since $\sum_{i=1}^{s} \frac{1}{q^{k_i}} = 1$, Equation (3.2.15) implies that

$$\zeta_A(1, \underline{t}_s)_{\big|t_i=\theta^{q^{-k_i}}} = 1. \tag{3.2.18}$$

Thus the specialization value of the left-hand side of (3.2.10) equals 1.

We now analyze the right-hand side of (3.2.10). For any s-tuple $\underline{\ell} \in (\mathbb{N}^*)^s$ such that $\gamma_{\underline{\ell}} \neq 0$, we know that

$$\sum_{i=1}^{s} \frac{1}{q^{\ell_i}} \leq 1 = \sum_{i=1}^{s} \frac{1}{q^{k_i}}.$$

Thus the arguments given in Step 1 can be applied so that the specialization value of the right-hand side of (3.2.10) equals

$$(-1)^{m-1}\widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{k_i}}}\gamma_{\underline{k}} = (-1)^{m-1}\gamma_{\underline{k}}. \tag{3.2.19}$$

Here the equality comes from the fact that $\sum_{i=1}^{s} \frac{1}{q^{k_i}} = 1$.

By (3.2.18) and (3.2.19) we get $(-1)^{m-1}\gamma_{\underline{k}} = 1$. Thus $\gamma_{\underline{k}} = (-1)^{m-1}$ as required.

The proof of Lemma 3.2.9 is complete. □

As a consequence of Proposition 3.2.8 and Lemma 3.2.9, we prove the key result of this section.

Theorem 3.2.10. *Suppose that* $\mathbf{w}(\mathbb{B}_s) \geq 1$. *Then*

1) We have $\mathbf{w}(\mathbb{B}_s) = 1$.

2) Conjecture 3.1.2 holds, that means we have

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ *of* Σ *such that*

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

Proof. Proposition 3.2.8 states that we can write

$$\zeta_A(1,\underline{t}_s) = (-1)^{m-1} \sum_{\underline{\ell}\in(\mathbb{N}^*)^s} \widetilde{\pi}^{1-\sum_{i=1}^{s}\frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^{s} \tau^{-\ell_i}\left(\zeta_A(1,t_i)\right), \quad \gamma_{\underline{\ell}} \in \mathbb{C}_\infty,$$

where the sum runs through a finite set of s-tuples $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$ such that if $\gamma_{\underline{\ell}} \neq 0$, then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}.$$

Thus the hypothesis $\mathbf{w}(\mathbb{B}_s) \geq 1$ implies that if $\gamma_{\underline{\ell}} \neq 0$, then $\sum_{i=1}^{s} \frac{1}{q^{\ell_i}} \geq \mathbf{w}(\mathbb{B}_s) \geq 1$. Combining this property with Lemma 3.2.9, we conclude that $\mathbf{w}(\mathbb{B}_s) = 1$, and that

$$\zeta_A(1,\underline{t}_s) = \sum_{\underline{\ell}\in(\mathbb{N}^*)^s} \prod_{i=1}^{s} \tau^{-\ell_i}\left(\zeta_A(1,t_i)\right)$$

where the sum runs through the set of s-tuples $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$ satisfying

$$\sum_{i=1}^{s} \frac{1}{q^{\ell_i}} = 1.$$

We wish to re-index the above sum by ordered set partitions of Σ. For an s-tuple $\underline{\ell} = (\ell_1,\dots,\ell_s) \in (\mathbb{N}^*)^s$, we can associate an ordered set partition $\underline{U} = (U_1 \mid \dots \mid U_d)$ of Σ as follows. We put $d = \max\{\ell_i : i \in \Sigma\}$ and for $1 \leq j \leq d$,

$$U_j = \{i \in \Sigma : \ell_i = j\}.$$

In fact, we see that this association gives rise to a bijection between the set of s-tuple $\underline{\ell} \in (\mathbb{N}^*)^s$ and the set of ordered set partitions of Σ. Furthermore, it is clear that

$$\sum_{j=1}^{d} \frac{|U_j|}{q^j} = \sum_{i=1}^{s} \frac{1}{q^{\ell_i}}.$$

Using this bijection we conclude that

$$\zeta_A(1,\underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where $\mathcal{L}_U^{(k)}$ are defined as in (3.1.3), and the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ of Σ satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

The proof is complete. □

3.3 Coefficients of the Bernoulli-type polynomial

In this (long) section we study the expression of $\mathbb{B}_s$ as a linear combination of symmetric polynomials in $\underline{t}_s$. We will give explicit formulas for some coefficients of this expression (see Theorem 3.3.1). To do so we need to write down similar expressions of zeta values (see Propositions 3.3.10 and 3.3.12) and Anderson-Thakur's special functions. We then deduce such an expression for $\mathbb{B}_s$ (see Sections 3.3.3 and 3.3.4). For the desired coefficients we are able to compute them by using combinatorial tools (see Section 3.3.4).

3.3.1 The key result

We recall that for any sequence $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$, we have defined the symmetric polynomial $\sigma_s(\underline{\ell})$ as in Definition 3.2.4.

In what follows, we define

$$\mathfrak{A}^+ := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1} : \sum_{j=0}^{d} a_j = m\}, \tag{3.3.1}$$

$$\mathfrak{A} := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d} \times \mathbb{N} : \sum_{j=0}^{d} a_j = m\},$$

$$\mathfrak{N}_\ell^+ := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d : n_1 \le \dots \le n_d, \sum_{j=1}^{d} n_j = \ell\}, \quad \ell \in \mathbb{N}^*,$$

$$\mathfrak{N}_\ell := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d : \sum_{j=1}^{d} n_j = \ell\}, \quad \ell \in \mathbb{N}^*.$$

It is clear that $\mathfrak{A}^+ \subset \mathfrak{A}$ and $\mathfrak{N}_\ell^+ \subset \mathfrak{N}_\ell$ for all $\ell \in \mathbb{N}^*$.

We now state the main result of this section whose proof will be given in Section 3.3.4.

Theorem 3.3.1. *Recall that (see Proposition 3.2.2)*

$$\mathbb{B}_s = \theta^{m-1} + B_1\theta^{m-2} + \dots + B_{m-1}, \quad B_\ell \in \mathbb{F}_q[\underline{t}_s].$$

Let $\ell \in \mathbb{N}^*$ such that $1 \le \ell \le q-2$, and let $\mathfrak{N}_\ell$ and $\mathfrak{A}^+$ be defined as in (3.3.1). Then we have

$$B_\ell = \sum_{\underline{n} \in \mathfrak{N}_\ell} \sum_{\underline{a} \in \mathfrak{A}^+} B(\underline{n}, \underline{a})\, \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_{d-1}(q-1) + n_{d-1} - n_d, a_d(q-1) + n_d)$$

where

- *the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ satisfying $\sum_{j=1}^{d} n_j = \ell$, that means $\underline{n} \in \mathfrak{N}_\ell$,*
- *the second sum runs through the set of sequences $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$ satisfying $\sum_{j=0}^{d} a_j = m$, that means $\underline{a} \in \mathfrak{A}^+$,*
- *the coefficient $B(\underline{n}, \underline{a}) \in \mathbb{F}_p$ is given by*

$$B(\underline{n}, \underline{a}) = (-1)^\ell \prod_{j=1}^{d} \binom{a_j - 1}{n_j - 1}.$$

Remark 3.3.2. 1) We note that if the coefficient $B(\underline{n}, \underline{a}) \neq 0$, then $a_j \ge n_j$ for $1 \le j \le d$.

2) The reader may compare the above expression with formulas given in (3.2.3), (3.2.4) and (3.2.5). We leave the reader to write down explicitly the polynomial $\mathbb{B}_{4q-3}$ for $q > 4$.

Remark 3.3.3. We now present a heuristic explanation for the formulas given in Theorem 3.3.1. We assume that $m < q$ (see the discussion after Conjecture 3.2.5 for $m = 1, 2$). By Conjecture 3.2.5 we write

$$\mathbb{B}_s = (-1)^{m-1} \sum B_1^*(\underline{t}_{U_1}) \cdots B_d^*(\underline{t}_{U_d})$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ of Σ satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

By Remark 3.2.6 we have an explicit description of the set of such partitions when $m < q$. Using this description we can write down all the terms defined over $\mathbb{F}_q[\theta]$ of the right-hand side. By this way we obtain a nice formula for $\mathbb{B}_s$ as given in Theorem 3.3.1.

3.3.2 Preparatory lemmas

We first collect several combinatorial lemmas which will be necessary in the sequel.

Lemma 3.3.4. *Let $n \in \mathbb{N}^*$ with $n < q$, and let $a \in \mathbb{N}^*$. Then we have*

$$\binom{a(q-1)+n-1}{n-1} = (-1)^{n-1}\binom{a-1}{n-1} \pmod{p}.$$

Proof. This lemma is an application of Lucas's theorem (see for example [Gra97]). We write down completely the proof for the convenience of the reader.

We always work in $\mathbb{F}_p$. Since $1 \leq n < q$, by Lucas's theorem we can assume that $1 \leq a \leq q$. By Lucas's theorem and the fact $1 \leq n < q$ again, we get

$$\begin{aligned}
\binom{a(q-1)+n-1}{n-1} &= \binom{(a-1)q+q-a+n-1}{n-1} \\
&= \binom{q-a+n-1}{n-1} \\
&= \frac{(q-a+n-1)\dots(q-a+1)}{(n-1)!} \\
&= \frac{(-a+n-1)\dots(-a+1)}{(n-1)!} \\
&= (-1)^{n-1}\frac{(a-n+1)\dots(a-1)}{(n-1)!} \\
&= (-1)^{n-1}\binom{a-1}{n-1}
\end{aligned}$$

as required. □

The next lemma follows from standard combinatorial arguments and the details of the proof will be left to the reader.

Lemma 3.3.5. *For any integer $j \geq 0$ and any sequence $(\ell_1, \dots, \ell_d) \in \mathbb{N}^d$, we have*

$$\begin{aligned}
\sigma_s(j)\sigma_s(\ell_1, \dots, \ell_d) = \sum_{\underline{j}=(j_1,\dots,j_{d+1})\in\mathbb{N}^{d+1}} &\binom{\ell_1+j_1-j_2}{j_1}\dots\binom{\ell_d+j_d-j_{d+1}}{j_d}\times \\
&\times \sigma_s(\ell_1+j_1-j_2, \dots, \ell_d+j_d-j_{d+1}, j_{d+1})
\end{aligned}$$

where the sum runs through the set of sequences $\underline{j} = (j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}$ such that $j_1+\dots+j_{d+1} = j$.

In what follows, we fix x to be an indeterminate over K. We recall that for any $k \in \mathbb{N}$, the binomial polynomial

$$\binom{x}{k} := \frac{x(x-1)\dots(x-k+1)}{k!} \in \mathbb{Q}[x]$$

represents a polynomial in the variable x with rational coefficients. Note that its value at $\ell \in \mathbb{N}$ is equal to the binomial coefficient $\binom{\ell}{k}$.

Lemma 3.3.6. *For $M, N \in \mathbb{N}$, we have the following equality in $\mathbb{Q}[x]$:*

$$\sum_{k=0}^{M}(-1)^k\binom{x+N-k}{M-k}\binom{x}{k} = \binom{N}{M}.$$

Proof. For $M, N \in \mathbb{N}$, we define

$$P_{M,N}(x) := \sum_{k=0}^{M}(-1)^k\binom{x+N-k}{M-k}\binom{x}{k} \in \mathbb{Q}[x].$$

We claim that

$$P_{M,N}(x) = \binom{N}{M}.$$

The proof is by induction on $M \in \mathbb{N}$. For $M = 0$, the assertion is clear. Suppose that we have proved the claim for $M-1$ with $M \in \mathbb{N}^*$, i.e., for all $N \in \mathbb{N}$,

$$P_{M-1,N}(x) = \binom{N}{M-1}.$$

We now show that the claim is true for M. In other words, we have to show that for all $N \in \mathbb{N}$, the following equality holds

$$P_{M,N}(x) = \binom{N}{M}.$$

For $N \in \mathbb{N}$, we have the following equality in $\mathbb{Q}[x]$:

$$P_{M,N+1}(x) = P_{M,N}(x) + P_{M-1,N}(x),$$

which implies

$$P_{M,N+1}(x) - \binom{N+1}{M} = \left(P_{M,N}(x) - \binom{N}{M}\right) + \left(P_{M-1,N}(x) - \binom{N}{M-1}\right).$$

By the induction hypothesis, we know that the second term in the above sum vanishes. Thus

$$P_{M,N+1}(x) - \binom{N+1}{M} = P_{M,N}(x) - \binom{N}{M}. \tag{3.3.2}$$

Since (3.3.2) holds for all $N \in \mathbb{N}$, we deduce

$$P_{M,N}(x) - \binom{N}{M} = P_{M,N-1}(x) - \binom{N-1}{M} = \ldots = P_{M,0}(x).$$

To conclude, it suffices to prove that $P_{M,0}(x) = 0$. In fact, we have

$$P_{M,0}(x) = \sum_{k=0}^{M} (-1)^k \binom{x-k}{M-k}\binom{x}{k} = \binom{x}{M}\sum_{k=0}^{M}(-1)^k\binom{M}{k} = 0.$$

The proof is finished. □

Lemma 3.3.7. *For $M, N \in \mathbb{N}$ with $M \leq N$, we have the following equality in $\mathbb{Q}[x]$:*

$$\sum_{k=0}^{M}(-1)^k\binom{x-N+k-1}{k}\binom{x}{M-k} = \binom{N}{M}.$$

Proof. We consider the polynomial in $\mathbb{Q}[x]$ defined by

$$P(x) := \sum_{k=0}^{M}(-1)^k\binom{x-N+k-1}{k}\binom{x}{M-k}.$$

We write

$$\begin{aligned} P(x) &= \sum_{k=0}^{M}(-1)^k\binom{x-N+k-1}{k}\binom{x}{M-k} \\ &= \sum_{k=0}^{M}(-1)^k\frac{(x-N)\ldots(x-N+k-1)}{k!} \times \frac{x\ldots(x-(M-k)+1)}{(M-k)!} \\ &= \sum_{k=0}^{M}\frac{(N-x)\ldots(N-x-k+1)}{k!} \times \frac{x\ldots(x-(M-k)+1)}{(M-k)!}. \end{aligned}$$

Since $M \leq N$, we have $\deg P \leq M \leq N$. We know that for any integer ℓ with $0 \leq \ell \leq N$, we have the equality

$$P(x=\ell) = \sum_{k=0}^{M}\binom{N-\ell}{k}\binom{\ell}{M-k} = \binom{N}{M}.$$

It follows that $P(x)$ is the constant polynomial $\binom{N}{M}$. The proof is finished. □

3.3.3 An expression of the zeta value $\zeta_A(1, \underline{t}_s)$

The main goal of this section is to express the zeta value $\zeta_A(1, \underline{t}_s)$ as a series in θ^{-1} whose coefficients are symmetric polynomials in $\underline{t}_s$. We make use of the notion of basic sums introduced in [ANDTR19, Section 5.2] to obtain such an expression (see Proposition 3.3.10).

Following [ANDTR19, Section 5.2], we recall some facts of basic sums. For a sequence $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$, we set

$$\begin{aligned} w(\underline{k}) &:= dk_0 + (d-1)k_1 + \dots + k_{d-1}, \\ |\underline{k}| &:= k_0 + \dots + k_{d-1}, \\ C_{\underline{k}} &:= (-1)^{|\underline{k}|} \frac{|\underline{k}|!}{k_0! \dots k_{d-1}!} \in \mathbb{F}_p. \end{aligned}$$

Letting $a \in A_{+,d}$, we write $a = a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1} + \theta^d$. Thus we get

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\underline{k}=(k_0,\dots,k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} a^{\underline{k}} \frac{1}{\theta^{w(\underline{k})}}$$

where we put $a^{\underline{k}} = \prod_{j=0}^{d-1} a_j^{k_j}$.

It follows that

$$\begin{aligned} &\sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} \\ &= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0,\dots,k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} a^{\underline{k}} a(t_1) \dots a(t_s) \\ &= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0,\dots,k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} \sum_{\substack{\underline{\ell}=(\ell_0,\dots,\ell_d) \in \mathbb{N}^{d+1}, \\ |\underline{\ell}|=s}} a^{\underline{k}} a^{\underline{\ell}} \sigma_s(\ell_1, \dots, \ell_d) \\ &= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0,\dots,k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\substack{\underline{\ell}=(\ell_0,\dots,\ell_d) \in \mathbb{N}^{d+1}, \\ |\underline{\ell}|=s}} \sigma_s(\ell_1, \dots, \ell_d) \sum_{a \in A_{+,d}} a^{\underline{k}+\underline{\ell}}. \end{aligned}$$

Here we put $a^{\underline{\ell}} = \prod_{j=0}^{d-1} a_j^{\ell_j}$ and $a^{\underline{k}+\underline{\ell}} = \prod_{j=0}^{d-1} a_j^{k_j+\ell_j}$.

Letting $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ and $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$ two sequences of integers, we say that $\underline{\ell}$ is $\underline{k}$-*admissible* if $(k_0 + \ell_0, \dots, k_{d-1} + \ell_{d-1}) \in ((q-1)\mathbb{N}^*)^d$. We see that if $\underline{\ell}$ is $\underline{k}$-admissible, then the sum $\sum_{a \in A_{+,d}} a^{\underline{k}+\underline{\ell}}$ is equal to $(-1)^d$. Otherwise, this sum is equal to 0.

Given a sequence $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ as above, we define another sequence $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ by

$$\begin{aligned} n_1 &:= k_0 + 1, \\ n_2 &:= k_0 + k_1 + 1, \\ &\vdots \\ n_d &:= k_0 + \dots + k_{d-1} + 1. \end{aligned}$$

This sequence satisfies

i) $n_1 \le \dots \le n_d$,

ii) $\sum_{j=1}^{d} n_j = w(\underline{k}) + d$.

We observe that the sequence $\underline{k}$ is completely determined by the associated sequence $\underline{n}$. In fact, we have $k_0 = n_1 - 1$ and $k_j = n_{j+1} - n_j$ for $1 \le j \le d-1$.

Let $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ satisfying $n_1 \leq \dots \leq n_d$ as above, and let $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ be the associated sequence. We set

$$\mathfrak{L}_{\underline{n},s} := \{\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1} : \ \underline{\ell} \text{ is } \underline{k}\text{-admissible and } |\underline{\ell}| = s\}. \tag{3.3.3}$$

Then

$$\sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} = \frac{(-1)^d}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\underline{\ell} \in \mathfrak{L}_{\underline{n},s}} \sigma_s(\ell_1, \dots, \ell_d).$$

Let $\underline{\ell} = (\ell_0, \dots, \ell_d)$ be a sequence in $\mathfrak{L}_{\underline{n},s}$ defined as above. Then there exist $a_0, \dots, a_{d-1} \in \mathbb{N}^*$ such that

$$\begin{aligned} \ell_0 &= a_0(q-1) - n_1 + 1, \\ \ell_1 &= a_1(q-1) + n_1 - n_2, \\ &\vdots \\ \ell_{d-1} &= a_{d-1}(q-1) + n_{d-1} - n_d. \end{aligned}$$

Thus we get

$$\ell_d = s - \sum_{j=0}^{d-1} \ell_j = a_d(q-1) + n_d$$

where we put $a_d := m - \sum_{j=0}^{d-1} a_j$ and recall that m is defined as in (3.1.2). Since $\underline{\ell} \in \mathbb{N}^{d+1}$, we deduce the following lemma.

Lemma 3.3.8. *The set $\mathfrak{L}_{\underline{n},s}$ consists of the elements $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$ of the form*

$$\begin{aligned} \ell_0 &= a_0(q-1) - n_1 + 1, \\ \ell_1 &= a_1(q-1) + n_1 - n_2, \\ &\vdots \\ \ell_{d-1} &= a_{d-1}(q-1) + n_{d-1} - n_d, \\ \ell_d &= a_d(q-1) + n_d, \end{aligned}$$

where $a_0, \dots, a_d$ are integers such that

- $a_0 > 0, \dots, a_{d-1} > 0$,
- $\sum_{j=0}^{d} a_j = m$.

Remark 3.3.9. We note that a_d may be negative.

To summarize we have proved the following proposition.

Proposition 3.3.10. *We have*

$$\zeta_A(1, \underline{t}_s) = \sum_{\ell \geq 0} \alpha_{\ell,s} \theta^{-\ell}$$

with

$$\alpha_{\ell,s} = \sum_{\underline{n} \in \mathfrak{N}_\ell^+} \widetilde{C}(\underline{n}) \sum_{\underline{\ell} \in \mathfrak{L}_{\underline{n},s}} \sigma_s(\underline{\ell})$$

where

- *the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell^+$ defined as in (3.3.1),*
- *the second sum runs through the set of sequences $\underline{\ell} \in \mathfrak{L}_{\underline{n},s}$ defined as in (3.3.3),*
- *the coefficient $\widetilde{C}(\underline{n}) \in \mathbb{F}_p$ equals*

$$(-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

3.3.4 Proof of Theorem 3.3.1

This section is devoted to prove Theorem 3.3.1 which compute the first $q-2$ coefficients of the polynomial $\mathbb{B}_s$. We start proving intermediate results and give a proof of Theorem 3.3.1 at the end of this section.

We first need the following consequence of Lemma 3.3.8.

Lemma 3.3.11. *Let ℓ be an integer with $1 \le \ell \le q-2$, and let $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ satisfying $n_1 \le \dots \le n_d$ and $\sum_{j=1}^{d} n_j = \ell$. Then the set $\mathfrak{L}_{\underline{n},s}$ defined as in (3.3.3) consists of the sequences $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$ of the form*

$$\begin{aligned} \ell_0 &= a_0(q-1) - n_1 + 1, \\ \ell_1 &= a_1(q-1) + n_1 - n_2, \\ &\vdots \\ \ell_{d-1} &= a_{d-1}(q-1) + n_{d-1} - n_d, \\ \ell_d &= a_d(q-1) + n_d, \end{aligned}$$

where $a_0, \dots, a_d$ are integers such that

- *$a_0 > 0, \dots, a_{d-1} > 0$ and $a_d \ge 0$,*
- *$\sum_{j=0}^{d} a_j = m$.*

Proof. We have to prove that $a_d \ge 0$. In fact, the fact that $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ satisfies $n_1 \le \dots \le n_d$ and $\sum_{j=1}^{d} n_j = \ell$ implies $n_d < \ell$. Thus $n_d < q-2$ since $\ell \le q-2$. It follows immediately that $a_d \ge 0$ since $\ell_d = a_d(q-1) + n_d$ and $\ell_d \in \mathbb{N}$. □

As an immediate consequence of Lemma 3.3.11 we obtain

Proposition 3.3.12. *Let ℓ be an integer with $1 \le \ell \le q-2$, and let $\mathfrak{N}_\ell^+$ and $\mathfrak{A}$ be defined as in (3.3.1). Then we have*

$$\alpha_{\ell,s} = \sum_{\underline{n} \in \mathfrak{N}_\ell^+} \sum_{\underline{a} \in \mathfrak{A}} \widetilde{C}(\underline{n})\, \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)$$

where

- *the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell^+$,*
- *the second sum runs through the set of sequences $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}$,*
- *the coefficient $\widetilde{C}(\underline{n}) \in \mathbb{F}_p$ equals*

$$\widetilde{C}(\underline{n}) = (-1)^{d+n_d-1} \frac{(n_d-1)!}{(n_1-1)!(n_2-n_1)! \dots (n_d-n_{d-1})!}.$$

Remark 3.3.13. In the above formula, we could take the first sum over the bigger set of sequences $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$ defined as in (3.3.1) since for any sequence $\underline{n} \in \mathfrak{N}_\ell \setminus \mathfrak{N}_\ell^+$, we have $\widetilde{C}(\underline{n}) = 0$.

We set $B_0 := 1$ and write

$$\mathbb{B}_s = \theta^{m-1}(B_0 + B_1\theta^{-1} + \dots + B_{m-1}\theta^{-(m-1)}), \quad B_\ell \in \mathbb{F}_q[\underline{t}_s].$$

Recall that (see (3.2.2))

$$\mathbb{B}_s := (-1)^m \frac{\zeta_A(1, \underline{t}_s)\omega(t_1) \dots \omega(t_s)}{\widetilde{\pi}}$$

Dividing this equality by θ^{m-1} yields an equality between formal series of the form $\sum_{j\geq 0} f_i\theta^{-j}$ with $f_j \in \mathbb{F}_q[\underline{t}_s]$. Thus to compute the first coefficients $B_1, \dots, B_{q-2}$ of $\mathbb{B}_s$, it suffices to look at both sides modulo $\theta^{-(q-1)}$, i.e., by forgetting the terms θ^{-j} for $j \geq q-1$.

On the left-hand side, we obtain

$$B_0 + B_1\theta^{-1} + \dots + B_{q-2}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}$$

where we put $B_k = 0$ for $k \geq m$.

On the right-hand side, for the zeta value $\zeta_A(1, \underline{t}_s)$, Proposition 3.3.10 gives

$$\alpha_{0,s} + \alpha_{1,s}\theta^{-1} + \dots + \alpha_{q-2,s}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}.$$

For other factors, we write

$$\prod_{j\geq 1}\left(1 - \frac{\theta}{\theta^{q^j}}\right) \equiv 1 \pmod{\theta^{-(q-1)}},$$

and

$$\begin{aligned}\prod_{i=1}^{s}\prod_{j\geq 0}\left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} &\equiv \prod_{i=1}^{s}\left(1 - \frac{t_i}{\theta}\right)^{-1} \pmod{\theta^{-(q-1)}} \\ &\equiv \left(1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^{q-2}\sigma_s(q-2)\theta^{-(q-2)}\right)^{-1} \pmod{\theta^{-(q-1)}}.\end{aligned}$$

Putting all together, we get

$$\begin{aligned}&\left(B_0 + B_1\theta^{-1} + \dots + B_{q-2}\theta^{-(q-2)}\right)\left(1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^{q-2}\sigma_s(q-2)\theta^{-(q-2)}\right) \\ &= \alpha_{0,s} + \alpha_{1,s}\theta^{-1} + \dots + \alpha_{q-2,s}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}.\end{aligned}$$

In other words, for all $1 \leq \ell \leq q-2$, we have

$$B_\ell - \sigma_s(1)B_{\ell-1} + \dots + (-1)^\ell\sigma_s(\ell)B_0 = \alpha_{\ell,s}. \tag{3.3.4}$$

Hence B_ℓ is completely determined by $B_0, \dots, B_{\ell-1}$.

We now prove one of the key results of this section.

Proposition 3.3.14. *Let ℓ be an integer with $1 \leq \ell \leq q-2$, and let $\mathfrak{N}_\ell$ and $\mathfrak{A}^+$ be defined as in (3.3.1). Then*

$$B_\ell = \sum_{\underline{n}\in\mathfrak{N}_\ell}\sum_{\underline{a}\in\mathfrak{A}^+} C(\underline{n}, \underline{a})\,\sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d) \tag{3.3.5}$$

where

- *the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$,*
- *the second sum runs through the set of sequences $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$,*
- *the coefficient $C(\underline{n}, \underline{a}) \in \mathbb{F}_p$ is equal to*

$$C(\underline{n}, \underline{a}) = (-1)^d\prod_{j=1}^{d}\binom{a_j(q-1) + n_j - 1}{n_j - 1}.$$

Proof. Let ℓ be an integer with $1 \leq \ell \leq q-2$. It suffices to prove (3.3.4) where B_ℓ is given by (3.3.5) and $\alpha_{\ell,s}$ is given in Proposition 3.3.12.

If we set

$$S := B_\ell - \sigma_s(1)B_{\ell-1} + \dots + (-1)^\ell\sigma_s(\ell)B_0 = \sum_{k=0}^{\ell}(-1)^k\sigma_s(k)B_{\ell-k},$$

then we replace $B_{\ell-k}$ by (3.3.5) for $1 \le k \le \ell \le q-2$ to get

$$S = \sum_{k=0}^{\ell} (-1)^k \sigma_s(k) \sum_{\underline{\ell} \in \mathfrak{N}_{\ell-k}} \sum_{\underline{a} \in \mathfrak{A}^+} C(\underline{\ell}, \underline{a})\ \sigma_s(a_1(q-1)+\ell_1-\ell_2, \dots, a_d(q-1)+\ell_d),$$

where the second sum (resp. the third sum) is over the set of sequences $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathfrak{N}_{\ell-k}$ (resp. $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$). By Lemma 3.3.5 we develop the above expression to get

$$\begin{aligned} S = \sum_{k=0}^{\ell} (-1)^k \sum_{\underline{\ell} \in \mathfrak{N}_{\ell-k}} \sum_{\underline{a} \in \mathfrak{A}^+} & C(\underline{\ell}, \underline{a})\ \sigma_s(a_1(q-1)+\ell_1-\ell_2+j_1-j_2, \dots, a_d(q-1)+\ell_d+j_d-j_{d+1}, j_{d+1}) \\ & \times \sum_{\substack{(j_1,\dots,j_{d+1}) \in \mathbb{N}^{d+1} \\ j_1+\dots+j_{d+1}=k}} \binom{a_1(q-1)+\ell_1-\ell_2+j_1-j_2}{j_1} \cdots \binom{a_d(q-1)+\ell_d+j_d-j_{d+1}}{j_d}. \end{aligned}$$

For sequences $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathfrak{N}_{\ell-k}$ and $(j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}$ with $j_1 + \dots + j_{d+1} = k$ as appeared in the above sum, we put

$$n_1 = \ell_1 + j_1, \quad \dots, \quad n_d = \ell_d + j_d, \quad n_{d+1} = j_{d+1}.$$

Then the sequence $\underline{n} = (n_1, \dots, n_{d+1})$ belongs to $(\mathbb{N}^*)^d \times \mathbb{N}$ and satisfies $n_1 + \dots + n_{d+1} = \ell$.

Using this notation and the formula for $C(\underline{\ell}, \underline{a})$ we can re-index the sums in S to get

$$\begin{aligned} S &= \sum_{\underline{n}} \sum_{\underline{a} \in \mathfrak{A}^+} \sigma_s(a_1(q-1)+n_1-n_2, \dots, a_d(q-1)+n_d, n_{d+1}) \\ &\quad \times (-1)^{d+n_{d+1}} \sum_{\underline{\ell}} \prod_{j=1}^{d} (-1)^{n_j-\ell_j} \binom{a_j(q-1)+\ell_j-1}{\ell_j-1} \binom{a_j(q-1)+n_j-n_{j+1}}{n_j-\ell_j} \\ &= \sum_{\underline{n}} \sum_{\underline{a} \in \mathfrak{A}^+} S(\underline{n}, \underline{a})\, \sigma_s(a_1(q-1)+n_1-n_2, \dots, a_d(q-1)+n_d, n_{d+1}) \end{aligned}$$

where

- the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$ such that $n_1 + \dots + n_{d+1} = \ell$,
- the second sum runs through the set of sequences $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$,
- the third sum of the first equality runs through the set of sequences $\underline{\ell} = (\ell_1, \dots, \ell_d) \in (\mathbb{N}^*)^d$ such that $\ell_j \le n_j$ for all $1 \le j \le d$,
- the coefficients $S(\underline{n}, \underline{a})$ are given by

$$S(\underline{n}, \underline{a}) = (-1)^{d+n_{d+1}} \sum_{\underline{\ell}} \prod_{j=1}^{d} (-1)^{n_j-\ell_j} \binom{a_j(q-1)+\ell_j-1}{\ell_j-1} \binom{a_j(q-1)+n_j-n_{j+1}}{n_j-\ell_j}. \tag{3.3.6}$$

  where the sum runs through the set of sequences $\underline{\ell} = (\ell_1, \dots, \ell_d) \in (\mathbb{N}^*)^d$ such that $\ell_j \le n_j$ for all $1 \le j \le d$.

In Lemma 3.3.15 below we compute explicitly the coefficients $S(\underline{n}, \underline{a})$. Combining it with Proposition 3.3.12, we deduce immediately Proposition 3.3.14. □

Lemma 3.3.15. *Let ℓ be an integer with $1 \le \ell \le q-2$. Let $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$ be a sequence satisfying $n_1 + \dots + n_{d+1} = \ell$, and let $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$ be a sequence of positive integers.*

We recall that $S(\underline{n}, \underline{a})$ is defined as in (3.3.6). Then we have

1. *If $n_{d+1} > 0$, then*

$$S(\underline{n}, \underline{a}) = \widetilde{C}(n_1, \dots, n_{d+1}) = (-1)^{d+n_{d+1}} \frac{(n_{d+1}-1)!}{(n_1-1)!(n_2-n_1)! \dots (n_{d+1}-n_d)!}.$$

2. *If* $n_{d+1}=0$, *then*

$$S(\underline{n},\underline{a})=\widetilde{C}(n_1,\dots,n_d)=(-1)^{d+n_d-1}\frac{(n_d-1)!}{(n_1-1)!(n_2-n_1)!\dots(n_d-n_{d-1})!}.$$

Here recall that the coefficients $\widetilde{C}(\underline{n})$ *are defined as in Proposition 3.3.12.*

Proof. In fact, we write

$$S(\underline{n},\underline{a})=(-1)^{d+n_{d+1}}\prod_{j=1}^{d}\sum_{1\le\ell_j\le n_j}(-1)^{n_j-\ell_j}\binom{a_j(q-1)+\ell_j-1}{\ell_j-1}\binom{a_j(q-1)+n_j-n_{j+1}}{n_j-\ell_j}.$$

We consider separately each factor of the above product and distinguish three cases.

Case 1: the jth factor for $1\le j\le d-1$.

We apply Lemma 3.3.6 to $x=a_j(q-1)+n_j-n_{j+1}$, $M=n_j-1$ and $N=n_{j+1}-1$ to obtain

$$\begin{aligned}&\sum_{1\le\ell_j\le n_j}(-1)^{n_j-\ell_j}\binom{a_j(q-1)+\ell_j-1}{\ell_j-1}\binom{a_j(q-1)+n_j-n_{j+1}}{n_j-\ell_j}\\&=\sum_{k=0}^{n_j-1}(-1)^k\binom{a_j(q-1)+n_j-1-k}{n_j-1-k}\binom{a_j(q-1)+n_j-n_{j+1}}{k}\quad\text{where } k=n_j-\ell_j\\&=\binom{n_{j+1}-1}{n_j-1}.\end{aligned}$$

Case 2: the dth factor with $n_{d+1}>0$.

We apply Lemma 3.3.6 to $x=a_d(q-1)+n_d-n_{d+1}$, $M=n_d-1$ and $N=n_{d+1}-1$ to obtain

$$\begin{aligned}&\sum_{1\le\ell_d\le n_d}(-1)^{n_d-\ell_d}\binom{a_d(q-1)+\ell_d-1}{\ell_d-1}\binom{a_d(q-1)+n_d-n_{d+1}}{n_d-\ell_d}\\&=\sum_{k=0}^{n_d-1}(-1)^k\binom{a_d(q-1)+n_d-1-k}{n_d-1-k}\binom{a_d(q-1)+n_d-n_{d+1}}{k}\quad\text{where } k=n_d-\ell_d\\&=\binom{n_{d+1}-1}{n_d-1}.\end{aligned}$$

Case 3: the dth factor with $n_{d+1}=0$.

Note that $n_d\ge 1$. We apply Lemma 3.3.7 to $x=a_d(q-1)+n_d$ and $M=N=n_d-1$ to obtain

$$\begin{aligned}&\sum_{1\le\ell_d\le n_d}(-1)^{n_d-\ell_d}\binom{a_d(q-1)+\ell_d-1}{\ell_d-1}\binom{a_d(q-1)+n_d-n_{d+1}}{n_d-\ell_d}\\&\sum_{1\le\ell_d\le n_d}(-1)^{n_d-\ell_d}\binom{a_d(q-1)+\ell_d-1}{\ell_d-1}\binom{a_d(q-1)+n_d}{n_d-\ell_d}\\&=\sum_{k=0}^{n_d-1}(-1)^{n_d-1+k}\binom{a_d(q-1)+k}{k}\binom{a_d(q-1)+n_d}{n_d-1-k}\quad\text{where } k=\ell_d-1\\&=(-1)^{n_d-1}.\end{aligned}$$

Putting all together, we obtain Lemma 3.3.15. The proof is finished. □

We are now ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. By Proposition 3.3.14, letting $\mathfrak{N}_\ell$ and $\mathfrak{A}^+$ be defined as in (3.3.1), for $1\le\ell\le q-2$, we have

$$B_\ell=\sum_{\underline{n}}\sum_{\underline{a}}C(\underline{n},\underline{a})\,\sigma_s(a_1(q-1)+n_1-n_2,\dots,a_d(q-1)+n_d)$$

where

- the first sum runs through the set of sequences $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ satisfying $\sum_{j=1}^{d} n_j = \ell$ (e.g. $\underline{n} \in \mathfrak{N}_\ell$),
- the second sum runs through the set of sequences $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$ satisfying $\sum_{j=0}^{d} a_j = m$ (e.g. $\underline{a} \in \mathfrak{A}^+$),
- the coefficient $C(\underline{n}, \underline{a}) \in \mathbb{F}_p$ is equal to

$$C(\underline{n}, \underline{a}) = (-1)^d \prod_{j=1}^{d} \binom{a_j(q-1) + n_j - 1}{n_j - 1}.$$

To prove Theorem 3.3.1, it suffices to prove that for $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$ and $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$ as above, the coefficients $C(\underline{n}, \underline{a})$ and $B(\underline{n}, \underline{a})$ given in Theorem 3.3.1 are the same. In fact, by Lemma 3.3.4 we have

$$\begin{aligned} C(\underline{n}, \underline{a}) &= (-1)^d \prod_{j=1}^{d} \binom{a_j(q-1) + n_j - 1}{n_j - 1} \\ &= (-1)^d \prod_{j=1}^{d} (-1)^{n_j - 1} \binom{a_j - 1}{n_j - 1} \\ &= (-1)^{n_1 + \dots + n_d} \prod_{j=1}^{d} \binom{a_j - 1}{n_j - 1}. \end{aligned}$$

Since $\sum_{j=1}^{d} n_j = \ell$, it follows that

$$C(\underline{n}, \underline{a}) = (-1)^\ell \prod_{j=1}^{d} \binom{a_j - 1}{n_j - 1} = B(\underline{n}, \underline{a})$$

as desired. The proof is finished. □

3.4 Proof of the main result

In this section we present a proof of Theorem 3.1.3. We have to show that if $m < q$ where m is defined as in (3.1.2), then the following formula holds

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions $\underline{U} = (U_1 \mid \cdots \mid U_d)$ of Σ satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We assume that $m < q$. Then the polynomial $\mathbb{B}_s$ is completely determined by Theorem 3.3.1. We claim that $\mathbf{w}(\mathbb{B}_s) \geq 1$. In fact, by Theorem 3.3.1 and Definition 3.2.7 it suffices to prove that

$$\mathbf{w}(\sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)) \geq 1$$

where

- $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$,
- $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$ such that $\sum_{j=0}^{d} a_j = m$,

satisfying $a_j \geq n_j$ for $1 \leq j \leq d$ (see Remark 3.3.2).

We see that

$$\begin{aligned}&\mathbf{w}(\sigma_s(a_1(q-1)+n_1-n_2,\dots,a_d(q-1)+n_d))\\&=\frac{a_0q-n_1}{q}+\frac{a_1(q-1)+n_1-n_2}{q^2}+\dots+\frac{a_d(q-1)+n_d}{q^{d+1}}.\end{aligned}$$

Since $a_j \geq n_j$ for $1 \leq j \leq d$ and $a_0 \geq 1$, we get

$$\mathbf{w}(\sigma_s(a_1(q-1)+n_1-n_2,\dots,a_d(q-1)+n_d)) \geq \frac{q-n_1}{q}+\frac{n_1q-n_2}{q^2}+\dots+\frac{n_dq}{q^{d+1}}=1$$

as required.

Now we know that $\mathbf{w}(\mathbb{B}_s) \geq 1$. Thus Theorem 3.1.3 follows immediately from Theorem 3.2.10.

3.5 Remarks

In this chapter we have succeeded in proving Conjecture 3.1.2 and thus get a partial answer to Conjecture 3.1.1. We expect that Conjecture 3.1.1 always holds. Thus it is tempting to ask whether Theorem 3.3.1 holds in full generality so that we can remove the restriction $1 \leq \ell \leq q-2$.

Conjecture 3.5.1. *Recall that (see Proposition 3.2.2)*

$$\mathbb{B}_s = \theta^{m-1} + B_1\theta^{m-2} + \dots + B_{m-1}, \quad B_\ell \in \mathbb{F}_q[\underline{t}_s].$$

Let $\ell \in \mathbb{N}^*$ such that $1 \leq \ell \leq m-1$. Then we have

$$B_\ell = \sum_{\underline{n}} \sum_{\underline{a}} B(\underline{n},\underline{a})\,\sigma_s(a_1(q-1)+n_1-n_2,\dots,a_d(q-1)+n_d)$$

where

- *the first sum runs through the set of sequences $\underline{n}=(n_1,\dots,n_d) \in (\mathbb{N}^*)^d$ satisfying $\sum_{j=1}^{d} n_j = \ell$,*
- *the second sum runs through the set of sequences $\underline{a}=(a_0,\dots,a_d) \in (\mathbb{N}^*)^{d+1}$ satisfying $\sum_{j=0}^{d} a_j = m$,*
- *the coefficients $B(\underline{n},\underline{a}) \in \mathbb{F}_p$ are given by*

$$B(\underline{n},\underline{a}) = (-1)^\ell \prod_{j=1}^{d} \binom{a_j-1}{n_j-1}.$$

By similar arguments as before we prove the following result.

Proposition 3.5.2. *Conjecture 3.5.1 implies Conjecture 3.1.1.*

Proof. Suppose that Conjecture 3.5.1 holds. From the explicit formula for $\mathbb{B}_s$, by similar arguments as those given in Section 3.4, we see that $\mathbf{w}(\mathbb{B}_s) \geq 1$. Combined with Theorem 3.2.10, it implies immediately Conjecture 3.1.1. □

Remark 3.5.3. 1) For $m = 1, 2, 3$, we have explicit formulas for $\mathbb{B}_s$ (see Section 3.2.2) and see easily that Conjecture 3.5.1 holds for these small values. They provide the first evidence to support our conjecture.

Chapter 4

Beyond Pellarin's conjectures

Let $s \in \mathbb{N}$, $s \geq 2$, $s \equiv 1 \pmod{q-1}$ and $m := (s-1)/(q-1) \in \mathbb{N}$. We write (see Proposition 3.2.2)

$$\mathbb{B}_s = \theta^{m-1} + B_1\theta^{m-2} + \ldots + B_{m-1}, B_\ell \in \mathbb{F}_q[t].$$

In previous chapter, we give an explicit formula of B_ℓ with $1 \leq \ell < q-1$ in Theorem 3.3.1. It means that Conjecture 3.5.1 holds for $\ell < q-1$. The goal of this chapter is to go beyond this bound and to investigate this conjecture for $\ell < q(q-1)$.

The steps that we investigate are as follows: First, we formulate an "approximate" equation which we need to prove, i.e., if we substitute the formula of B_ℓ from Conjecture 3.5.1, it should true. Then, in the next step, by using this Equation, we prove Conjecture 3.5.1 in some cases, i.e., $\ell < 2q-2$, $\ell < 3q-3$. This step is done by purely combinatorial arguments. The last step is to formulate a conjecture based on step two.

4.1 Step 1

The aim of this step is to construct an "approximate" equation: see Equation (4.1.3).

Recall that

$$\mathbb{B}_s := (-1)^m \frac{\zeta_A(1, \underline{t}_s)\omega(t_1)\ldots\omega(t_s)}{\tilde{\pi}}.$$

We divide both sides by θ^{m-1} and note that $\zeta_A(1, \underline{t}_s) = \sum_{\ell \geq 0} \alpha_{\ell,s}\theta^{-\ell}$ with $\alpha_{\ell,s}$ is defined in Proposition 3.3.10. We obtain

$$\sum_{\ell=0}^{s} B_\ell\theta^{-\ell} = \frac{\left(\sum_{\ell\geq 0}\alpha_{\ell,s}\theta^{-\ell}\right)\prod_{i=1}^{s}\prod_{j\geq 0}(1-\frac{t_i}{\theta^{q^j}})^{-1}}{\prod_{j\geq 1}(1-\theta^{-(q^j-1)})^{-1}}.$$

Note that $\prod_{i=1}^{s}(1-\frac{t_i}{\theta^{q^j}}) = \sum_{\ell=0}^{s}(-1)^\ell\sigma_s(\ell)\theta^{-\ell q^j}$, where $\sigma_s(\ell)$ is defined in Definition 3.2.4. Thus

$$\left(\sum_{\ell=0}^{s} B_\ell\theta^{-\ell}\right)\prod_{j\geq 0}\sum_{\ell=0}^{s}(-1)^\ell\sigma_s(\ell)\theta^{-\ell q^j} = \left(\sum_{\ell\geq 0}\alpha_{\ell,s}\theta^{-\ell}\right)\prod_{j\geq 1}(1-\theta^{-(q^j-1)}). \tag{4.1.1}$$

To compute the coefficient B_ℓ for $1 \leq \ell < q(q-1)$, it suffices to look at both sides of Equation (4.1.1) modulo $\theta^{-q(q-1)}$, i.e., by forgetting the term θ^{-j} for $j \geq q(q-1)$. We get

$$\left(\sum_{\ell=0}^{s} B_\ell\theta^{-\ell}\right)\left(\sum_{\ell=0}^{s}(-1)^\ell\sigma_s(\ell)\theta^{-\ell}\right)\left(\sum_{\ell=0}^{s}(-1)^\ell\sigma_s(\ell)\theta^{-\ell q}\right) \equiv \left(\sum_{\ell\geq 0}\alpha_{\ell,s}\theta^{-\ell}\right)(1-\theta^{-(q-1)}) \pmod{\theta^{-q(q-1)}}. \tag{4.1.2}$$

Remark 4.1.1. 1. To remind the reader, in Section 3.3.1, to prove Theorem 3.3.1, we consider Equation (4.1.1) module $\theta^{-(q-1)}$. When we consider modulo $\theta^{-q(q-1)}$, we have extra factors $\left(\sum_{\ell=0}^{s}(-1)^{\ell}\sigma_s(\ell)\theta^{-\ell q}\right)$ and $(1-\theta^{-(q-1)})$ respectively on the left-hand side and on the right-hand side. These extra factors make the computation much more involved.

2. In Equation (4.1.2) and later, we still keep some terms $\theta^{-\ell}$ for $\ell \geq q(q-1)$ when consider modulo $\theta^{-q(q-1)}$. It does not effect the equation but creates a clear formula.

Similar to Section 3.3.4, we define the following sum with index (assume that $B_k = 0$ for $k < 0$)

$$S_i := \sum_{\ell \geq 0}(-1)^{\ell}\sigma_s(\ell)B_{i-\ell} = B_i - \sigma_s(1)B_{i-1} + \ldots + (-1)^i\sigma_s(i)B_0.$$

We note that in Section 3.3.4, we do not have the index "i" in S. Equation (4.1.2) becomes

$$\left(\sum_{i\geq 0} S_i\theta^{-i}\right)\left(\sum_{\ell=0}^{s}(-1)^{\ell}\sigma_s(\ell)\theta^{-\ell q}\right) \equiv \left(\sum_{\ell\geq 0}\alpha_{\ell,s}\theta^{-\ell}\right)(1-\theta^{-(q-1)}) \pmod{\theta^{-q(q-1)}},$$

i.e.,

$$\sum_{i\geq 0}\left(\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}\right)\theta^{-i} = \sum_{i\geq 0}(\alpha_{i,s}-\alpha_{i-q+1,s})\theta^{-i} \pmod{\theta^{-q(q-1)}}.$$

Here, we assume that $\alpha_{i,s} = 0$ for $i < 0$. By comparing the coefficients of θ^{-i} for $1 \leq i \leq q(q-1)$, we get

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq} = \alpha_{i,s} - \alpha_{i-q+1,s}. \tag{4.1.3}$$

Now, for $1 \leq \ell \leq m-1$, we set

$$B_{\ell}^{*} = \sum_{\underline{n}}\sum_{\underline{a}} B(\underline{n},\underline{a})\,\sigma_s(a_1(q-1)+n_1-n_2,\ldots,a_d(q-1)+n_d)$$

where

- the first sum runs through the set of sequences $\underline{n} = (n_1,\ldots,n_d) \in (\mathbb{N}^*)^d$ satisfying $\sum_{j=1}^{d} n_j = \ell$,
- the second sum runs through the set of sequences $\underline{a} = (a_0,\ldots,a_d) \in (\mathbb{N}^*)^{d+1}$ satisfying $\sum_{j=0}^{d} a_j = m$,
- the coefficients $B(\underline{n},\underline{a}) \in \mathbb{F}_p$ are given by

$$B(\underline{n},\underline{a}) = (-1)^{\ell}\prod_{j=1}^{d}\binom{a_j-1}{n_j-1}.$$

Conjecture 3.5.1 is equivalent to show that $B_i = B_i^*$ for $1 \leq i \leq m-1$. We also set

$$S_i^* := \sum_{\ell\geq 0}(-1)^{\ell}\sigma_s(\ell)B_{i-\ell}^* = B_i^* - \sigma_s(1)B_{i-1}^* + \ldots + (-1)^i\sigma_s(i)B_0^*.$$

By the definition of S_i and S_i^* and by induction, the following statements are equivalent

$$\begin{aligned} B_i = B_i^* \text{ for all } 1 \leq i \leq m-1 &\iff S_i = S_i^* \text{ for all } 1 \leq i \leq m-1 \\ &\iff \sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq} = \sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}^*. \end{aligned}$$

We need to show that

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}^* = \alpha_{i,s} - \alpha_{i-q+1,s}.$$

In the following Sections, we will write explicitly formulas of both sides and show that they are equal for some cases.

4.2 Step 2

We begin with some notation and then we compute both sides of Equation (4.1.3).

4.2.1 Notation

Since this chapter contains a lot of long equations and repeating forms, we assume some "short" notation as follows.

Recall the notation in (3.3.1).

$$\mathfrak{A}^+ := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1} : \sum_{j=0}^{d} a_j = m\},$$

$$\mathfrak{A} := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d} \times \mathbb{N} : \sum_{j=0}^{d} a_j = m\},$$

$$\mathfrak{N}_\ell^+ := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^{d} : n_1 \le \dots \le n_d, \sum_{j=1}^{d} n_j = \ell\}, \quad \ell \in \mathbb{N}^*,$$

$$\mathfrak{N}_\ell := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^{d} : \sum_{j=1}^{d} n_j = \ell\}, \quad \ell \in \mathbb{N}^*.$$

Let $\underline{X} = (X_0, \dots, X_d) \in \mathfrak{A}$. It is equivalent to $(X_1, \dots, X_d) \in (\mathbb{N}^*)^{d-1} \times \mathbb{N}$ and $X_1 + \dots + X_d < m$. For short, we only write $X_d \ge 0$ instead (We do not write the condition $X_1 + \dots + X_d < m$ and $X_1, \dots, X_{d-1} \ge 1$). For example

$$\sum_{X_d \ge 0} f := \sum_{\substack{(X_1,\dots,X_d) \in (\mathbb{N}^*)^{d-1} \times \mathbb{N} \\ X_1 + \dots + X_d < m}} f.$$

For $(b_1, \dots, b_d) \in (\mathbb{N})^d$, we define

$$\sigma(\underline{X}, \underline{b}) = \sigma(\underline{X}, \underline{b})_{(1,0)\dots(d,0)} := \sigma_s(X_1(q-1) + b_1 - b_2, \dots, X_d(q-1) + b_d).$$

For $a_k \ge 0$, we define an operator that acts on $\sigma(\underline{X}, \underline{b})$ as follows

$$f_{(k,a_k)}(\sigma(\underline{X}, \underline{b})) = \sigma_{(k,a_k)}$$
$$:= \sigma_s(X_1(q-1) + b_1 - b_2, \dots, X_{k-1}(q-1) + b_{k-1} - b_k - a_k, X_k(q-1) + b_k - b_{k+1} + a_k, \dots, X_d(q-1) + b_d)$$

and

$$\sigma(\underline{X}, \underline{b})_{(1,a_1)\dots(d,a_d)} := f_{(d,a_d)}(\sigma(\underline{X}, \underline{b})_{(1,a_1)\dots(d-1,a_{d-1})}).$$

For $k = d+1$, we note that

$$\begin{aligned}\sigma(\underline{X}, \underline{b})_{(d,a_d)a_{d+1}} &:= \sigma(\underline{X}, \underline{b})_{(d,a_d)(d+1,a_{d+1})} \\ &= \sigma_s(X_1(q-1) + b_1 - b_2, \dots, X_d(q-1) + b_d + a_d - a_{d+1}, a_{d+1}).\end{aligned}$$

4.2.2 The left-hand side of Equation (4.1.3)

In this section, we compute the left-hand side of Equation (4.1.3).

Proposition 4.2.1. *We have*

$$\sum_{k \ge 0} (-1)^k \sigma_s(k) S_{i-kq} \tag{4.2.1}$$

$$= \sum_{K \ge 0} \sum_{X_d \ge 0} \sum_{\substack{b_1 + \dots + b_d = i - Kq \\ a_1 + \dots + a_d + a_{d+1} = K \\ 1 \le b_j \le q \\ a_j \ge 0}} (-1)^{d + b_d - 1 + a_d + a_{d+1}} \prod_{j=1}^{d-1} \binom{b_{j+1} - 1}{b_j - 1} \binom{b_{j+1} - b_j + a_{j+1}}{a_j} \times \binom{b_d + a_d - a_{d+1}}{a_d} \times$$

$$\times \sigma(\underline{X}, \underline{b})_{(1,a_1)\dots(d,a_d)a_{d+1}}.$$

Proof. The proof uses similar arguments as in that of Proposition 3.3.14. We omit the details. □

We write

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}=\sum_{k\geq 0}A_k,$$

where

$$A_k=\sum_{X_d\geq 0}\sum_{\substack{b_1+\ldots+b_d=i-Kq\\ a_1+\ldots+a_d+a_{d+1}=K\\ 1\leq b_j\leq q\\ a_j\geq 0}}(-1)^{d+b_d-1+a_d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\binom{b_{j+1}-b_j+a_{j+1}}{a_j}\times\binom{b_d+a_d-a_{d+1}}{a_d}\times$$

$$\times\,\sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}.$$

4.2.3 The right-hand side of Equation (4.1.3)

The author formulated a formula of α_i and α_{i-q+1} which is "similar" to that of $\sum(-1)^k\sigma_s(k)S_{i-kq}$ in Equation (4.2.1). The following proposition is a simple reformulation of Proposition 3.3.12.

Proposition 4.2.2. *For $i<q^2$, we have*

$$\alpha_i=\sum_{k\geq 0}\sum_{X_d\geq 0}\sum_{\substack{b_1+\ldots+b_d=i-kq+q-b_{d+1}\\ a_1+\ldots+a_d+a_{d+1}=k+b_{d+1}-q\\ 1\leq b_j\leq q,a_j\geq 0}}(-1)^{d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\binom{a_{j+1}}{a_j}\times$$

$$\times\binom{a_{d+1}+q-1-b_{d+1}}{a_d}\binom{b_{d+1}-1}{b_d-1}\times\sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}$$

and

$$\alpha_{i-q+1}=\sum_{k\geq 0}\sum_{X_d\geq 0}\sum_{\substack{b_1+\ldots+b_d=i-kq+q+1-b_{d+1}\\ a_1+\ldots+a_d+a_{d+1}=k-1+b_{d+1}-q\\ 1\leq b_j\leq q,a_j\geq 0}}(-1)^{d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\binom{a_{j+1}}{a_j}\times$$

$$\times\binom{a_{d+1}+q-1-b_{d+1}}{a_d}\binom{b_{d+1}-1}{b_d-1}\times\sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}.$$

Remark: In both formulas of α_i and α_{i-q+1}, the sums $(b_1+\ldots+b_d)+(a_1+\ldots+a_{d+1})$ are equal and the values are $k(q+1)$, which do not depend on b_{d+1}. We write

$$\alpha_i=\sum_{k\geq 0}B_k$$

and

$$\alpha_{i-q+1}=\sum_{k\geq 0}C_k.$$

4.2.4 Step 2 for some cases

We need to show that

Conjecture 4.2.3. *For $i<q^2$, we have*

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}=\alpha_i-\alpha_{i-q+1},$$

where

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}$$

$$=\sum_{K\geq 0}\sum_{X_d\geq 0}\sum_{\substack{b_1+\ldots+b_d=i-Kq\\ a_1+\ldots+a_d+a_{d+1}=K\\ 1\leq b_j\leq q\\ a_j\geq 0}}(-1)^{d+b_d-1+a_d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\binom{b_{j+1}-b_j+a_{j+1}}{a_j}\times\binom{b_d+a_d-a_{d+1}}{a_d}$$

$$\times\,\sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}$$

and

$$\alpha_i=\sum_{\substack{b_1+\ldots+b_d=i\\ b_j\geq 1}}\sum_{\substack{X_1+\ldots+X_d<m\\ X_1,\ldots,X_{d-1}\geq 1\\ \text{no condition on } X_d}}(-1)^{d+b_d-1}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\sigma(\underline{X},\underline{b})_{(1,0)\ldots(d,0)}.$$

Conjecture 4.2.3 is equivalent to show that

$$\sum_{k\geq 0}A_k=\sum_{k\geq 0}B_k-\sum_{k\geq 0}C_k.$$

For small values of k, we have the following proposition.

Proposition 4.2.4. *For $k=1,2$, we have*

$$A_k=B_k-C_k.$$

Proof. The idea of the proof is as follows:

We find all value of $(a_1,\ldots,a_d,a_{d+1})$ corresponding to A_k,B_k,C_k.

1. For A_k, the number of solutions $(a_1,\ldots,a_{d+1})$ of $a_1+\ldots+a_{d+1}=k\leq 2$ is small. After expanding A_k, we get the terms $\sigma(\underline{X},\underline{b}),\sigma(\underline{X},\underline{b})_{(k,1)},\sigma(\underline{X},\underline{b})_{(k_1,1)(k_2,1)}$, $1\leq k,k_1,k_2\leq d+1$. Then, we apply the following lemma whose proof is by direct calculations.

   Lemma 4.2.5. *For any $i\in\mathbb{N}$, for any function $g(b_k)$, we have*

   $$\sum_{\substack{b_1+\ldots+b_d=i\\ 1\leq b_j\leq q}}(b_k-b_{k+1})\tilde{C}(\underline{b})g(b_k)\sigma(\underline{X},\underline{b})_{(k,1)}=\sum_{\substack{b_1+\ldots+b_d=i+1\\ 1\leq b_j\leq}}(b_{k-1}-b_k)\tilde{C}(\underline{b})g(b_k-1)\sigma(\underline{X},\underline{b})$$

   where

   $$\tilde{C}(\underline{b})=(-1)^{d+b_d-1}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}.$$

   By applying this lemma, with suitable coefficients, we can transform $\sigma(\underline{X},\underline{b})_{(k,1)},\sigma(\underline{X},\underline{b})_{(k_1,1)(k_2,1)}$ to $\sigma(\underline{X},\underline{b})$, then we can gather the coefficients of these σ's.

2. For B_k, C_k, the number of solutions $(a_1,\ldots,a_{d+1})$ of $a_1+\ldots+a_{d+1}=k+b_{d+1}-q\geq 0$ is also small since $b_{d+1}\leq q$ (hence, the possible values of b_{d+1} are $q,q-1,q-2,q-3$). Also, since there are term like $\binom{a_{j+1}}{a_j}$, we have an additional condition that $0\leq a_1\leq\ldots\leq a_d$.

3. After expanding A_k and gathering by Lemma 4.2.5, the results obtained coincide with B_k-C_k.

□

Corollary 4.2.6. *Conjecture 4.2.3 is true with $i<2q-2$, $i<3q-3$.*

Proof. For $i<3q-3$, we have $b_1+\ldots+b_d=i-kq\geq 0$, hence $k\leq 2$. By Proposition 4.2.4, it implies that

$$\sum_{k\geq 0}(-1)^k\sigma_s(k)S_{i-kq}=A_1+A_2=B_1+B_2-C_1-C_2=\alpha_i-\alpha_{i-q+1}.$$

□

4.3 Step 3 and comment

By Proposition 4.2.4, we suggest the following conjecture.

Conjecture 4.3.1. *For all $k < q$, we have*

$$A_k = B_k - C_k,$$

i.e.,

$$\sum_{X_d\geq 0} \sum_{\substack{b_1+\ldots+b_d=i-kq \\ a_1+\ldots+a_{d+1}=k \\ 1\leq b_j\leq q \\ a_j\geq 0}} (-1)^{d+b_d-1+a_d+a_{d+1}} \prod_{j=1}^{d-1} \binom{b_{j+1}-1}{b_j-1}\binom{b_{j+1}-b_j+a_{j+1}}{a_j} \times \binom{b_d+a_d-a_{d+1}}{a_d} \times$$

$$\times \sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}$$

$$= \sum_{X_d\geq 0} \sum_{\substack{b_1+\ldots+b_d=i-kq+q-b_{d+1} \\ a_1+\ldots+a_{d+1}=k+b_{d+1}-q \\ 1\leq b_j\leq q \\ a_j\geq 0}} (-1)^{d+a_{d+1}} \prod_{j=1}^{d-1} \binom{b_{j+1}-1}{b_j-1}\binom{a_{j+1}}{a_j} \times \binom{a_{d+1}+q-1-b_{d+1}}{a_d}\binom{b_{d+1}-1}{b_d-1} \times$$

$$\times \sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}$$

$$- \sum_{X_d\geq 0} \sum_{\substack{b_1+\ldots+b_d=i-kq+q+1-b_{d+1} \\ a_1+\ldots+a_{d+1}=k-1+b_{d+1}-q \\ 1\leq b_j\leq q \\ a_j\geq 0}} (-1)^{d+a_{d+1}} \prod_{j=1}^{d-1} \binom{b_{j+1}-1}{b_j-1}\binom{a_{j+1}}{a_j} \times \binom{a_{d+1}+q-1-b_{d+1}}{a_d}\binom{b_{d+1}-1}{b_d-1} \times$$

$$\times \sigma(\underline{X},\underline{b})_{(1,a_1)\ldots(d,a_d)a_{d+1}}.$$

Comment: To prove Conjecture 4.2.3 for the case $i < 4q-4$, we need to prove Conjecture 4.3.1 for $k = 3$. There are some possibilities:

1. Do as the method in Proposition 4.2.4. The difficult part is after expanding A_k and applying Lemma 4.2.5. To apply Lemma 4.2.5, we need a suitable coefficients.
   (a) From $\sigma_{(k,1)}$ to σ, applying Lemma 4.2.5 in the k-position, to get a suitable coefficient, we have an extra term (this term can not apply Lemma 4.2.5).
   (b) From $\sigma_{(k_1,1)(k_2,1)}$ to $\sigma_{(k_1,1)(k_2,0)}$, applying Lemma 4.2.5 in the k_2-position, to get a suitable coefficient, we have an extra term. In this extra term and main term, we can also apply Lemma 4.2.5 (in k_1 position). But it will generate some "extra term". Roughly speaking, finally, the coefficient of $\sigma_{(k_1,1)(k_2,1)}$ can divide into three terms, one term with coefficient help us to apply Lemma 4.2.5 two times, one term with coefficient help us to apply Lemma 4.2.5 one time, and the remaining term that can not apply Lemma 4.2.5.
   (c) From $\sigma_{(k_1,1)(k_2,1)(k_3,1)}$ to σ, we do the same.
   Question: Using this method to prove Conjecture 4.3.1, Lemma 4.2.5 is not enough. Could we get a generation of Lemma 4.2.5?
   Answer: The author has some generation of Lemma 4.2.5, that are Lemma 4.3.2 and 4.3.3.
2. Maybe induction works in this situation. We can change variable as follows:

$$b_d + q \to b_d$$
$$a_d - 1 \to a_d$$
$$X_d - 1 \to X_d$$
$$X_{d-1} + 1 \to X_{d-1}.$$

Note that

$$\begin{aligned}&\sigma_s(X_1(q-1)+b_1-b_2+a_1-a_2,\dots,X_{d-1}(q-1)+b_{d-1}-b_d+a_{d-1}-a_d,X_d(q-1)+b_d+a_d-a_{d+1},a_{d+1})\\&=\sigma_s(X_1(q-1)+b_1-b_2+a_1-a_2,\dots,\\&\qquad,(X_{d-1}+1)(q-1)+b_{d-1}-(b_d+q)+a_{d-1}-(a_d-1),\\&\qquad,(X_d-1)(q-1)+b_d+q+a_d-1-a_{d+1},a_{d+1}).\end{aligned}$$

The following lemmas are some generalizations of Lemma 4.2.5 which may helpful to prove Conjecture 4.3.1. The proofs are by direct calculations and we omit the details.

Lemma 4.3.2. *We have*

$$\begin{aligned}&\sum_{\substack{b_1+\dots+b_d=i\\1\le b_1\le\dots\le b_d\le q}}(b_k-b_{k+1})\dots(b_k-b_{k+1}+a-1)f(b_k)\tilde{C}(\underline{b})\sigma(\underline{X},\underline{b})_{(k,a)}\\&=\sum_{\substack{b_1+\dots+b_d=i+a\\1\le b_1\le\dots\le b_d\le q}}(b_{k-1}-b_k)\dots(b_{k-1}-b_k+a-1)f(b_k-a)\tilde{C}(\underline{b})\sigma(\underline{X},\underline{b})_{(k,0)}\end{aligned}$$

i.e.,

$$\sum_{\substack{b_1+\dots+b_d=i\\1\le b_1\le\dots\le b_d\le q}}\binom{b_k-b_{k+1}+a-1}{a}\tilde{C}(\underline{b})f(b_k)\sigma(\underline{X},\underline{b})_{(k,a)}=\sum_{\substack{b_1+\dots+b_d=i+a\\1\le b_1\le\dots\le b_d\le q}}\binom{b_{k-1}-b_k+a-1}{a}\tilde{C}(\underline{b})f(b_k-a)\sigma(\underline{X},\underline{b})_{(k,0)}$$

i.e.,

$$\sum_{\substack{b_1+\dots+b_d=i\\1\le b_1\le\dots\le b_d\le q}}\binom{b_{k+1}-b_k}{a}\tilde{C}(\underline{b})f(b_k)\sigma(\underline{X},\underline{b})_{(k,a)}=\sum_{\substack{b_1+\dots+b_d=i+a\\1\le b_1\le\dots\le b_d\le q}}\binom{b_k-b_{k-1}}{a}\tilde{C}(\underline{b})f(b_k-a)\sigma(\underline{X},\underline{b})_{(k,0)}.$$

Lemma 4.3.3.

$$\begin{aligned}&\sum_{K\ge0}\sum_{X_d\ge0}\sum_{\substack{b_1+\dots+b_d=i-Kq\\a_1+\dots+a_d+a_{d+1}=K\\1\le b_j\le q\\a_j\ge0}}(-1)^{d+b_d-1+a_d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\binom{b_{j+1}-b_j+a_{j+1}}{a_j}\times\binom{b_d+a_d-1}{a_d}\\&\qquad\times\sigma(\underline{X},\underline{b})_{(1,a_1)\dots(d,a_d)a_{d+1}}\\&=\sum_{K\ge0}\sum_{X_d\ge0}\sum_{\substack{b_1+\dots+b_d=i-Kq+a_1+\dots+a_d\\a_1+\dots+a_d+a_{d+1}=K\\1\le b_j\le q\\a_j\ge0}}(-1)^{d+b_d-1+a_d+a_{d+1}}\prod_{j=1}^{d-1}\binom{b_{j+1}-1}{b_j-1}\times\\&\times\binom{b_1-1}{a_1}\binom{b_2-b_1+a_1}{a_2}\binom{b_3-b_2+a_2}{a_3}\dots\binom{b_d-b_{d-1}+a_{d-1}}{a_d}\\&\qquad\times\sigma(\underline{X},\underline{b})_{(1,0)\dots(d,0)a_{d+1}}.\end{aligned}$$

Chapter 5

Generalization of Speyer's results

In 2017, Speyer (see [Spe17]) proved some conjectures due to D. Thakur for $A = \mathbb{F}_q[\theta]$ (see also Section 5.2). The aim of this chapter is to generalize Speyer's results in the context of rank one Drinfeld modules.

We begin with some preliminaries in Section 5.1. In Section 5.2 we briefly recall some of Speyer's results (see [Spe17]) in the case $A = \mathbb{F}_q[\theta]$. Section 5.3 is devoted to some generalization of Speyer's results. The main result is Theorem 5.3.17. We also give some examples of this theorem in some special cases (see Section 5.3.4).

5.1 Preliminaries

In this section, we recall basic definitions and properties of rank one Drinfeld modules. We refer the reader to [Gos96, Chapter 4, Chapter 7] for more details. From Subsection 5.1.2 to 5.1.5, we follows closely [Gos96, Chapter 4, Chapter 7]. Subsection 5.1.6 is devoted to construct the most important definition: the "Goss" map. We define the zeta values of Goss associated to A in Subsection 5.1.7. We also add some combinatorial materials in Subsection 5.1.8.

5.1.1 Notation

- $K/\mathbb{F}_q$: a global function field ($\mathbb{F}_q$ is algebraically closed in K).
- ∞: a place of K.
- A: the ring of element of K which are regular outside ∞.
- K_∞: the ∞-adic completion of K. Let $\mathbb{F}_\infty$ be the residue field of K_∞ and $d_\infty = [\mathbb{F}_\infty : \mathbb{F}_q]$.
- $\overline{K_\infty}$: a fixed algebraically closure of K.
- v_∞: the discrete valuation on K corresponding to the place ∞ normalized that $v_\infty(K_\infty^\times) = \mathbb{Z}^\times$.
- $\pi \in K_\infty$: a uniformizer.

Example: $K = \mathbb{F}_q(\theta)$, ∞: the unique pole of θ, $A = \mathbb{F}_q[\theta]$, $v_\infty(\theta) = -1$, $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$.

- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure $\overline{K_\infty}$ of K_∞. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.
- I_A: the group of non-zero fractional ideals of A.
- $\mathcal{A}$: the set of non-zero ideals of A.
- $\mathcal{M}$: the set of maximal ideals of A.

- $\deg : I_A \to \mathbb{Z}$: the natural homomorphism of groups that for every prime ideal $P \in \mathcal{A}$, we have

$$\deg(P) = \dim_{\mathbb{F}_q}(A/P).$$

We recall the fact that for $a \in A\backslash\{0\}, \deg a = \dim_{\mathbb{F}_q} A/aA$ and further for $x \in K^\times$, $\deg(xA) = -v_\infty(x)d_\infty$.

- In this chapter, we always fix a sign function $\operatorname{sgn}\colon K_\infty^\times \to \mathbb{F}_\infty^\times$ given by $\operatorname{sgn}(\sum_{i\geq i_0} a_i\pi^i) = a_{i_0}$ for $i_0 \in \mathbb{Z}$, $a_{i_0} \neq 0$, $a_i \in \mathbb{F}_\infty$.

- Set

$$\operatorname{Pic}(A) := \frac{I_A}{\{xA, x \in K^\times\}}.$$

- Set

$$\operatorname{Pic}^+(A) := \frac{I_A}{\{xA, x \in K^\times, \operatorname{sgn}(x) = 1\}}. \tag{5.1.1}$$

We set

$$h := \left|\operatorname{Pic}^+(A)\right|. \tag{5.1.2}$$

5.1.2 Drinfeld modules, Exponential map

This Subsection follows closely [Gos96, Section 4.5, Section 4.6].

We consider Drinfeld modules over $\mathbb{C}_\infty$.

Definition 5.1.1. A Drinfeld A-module over $\mathbb{C}_\infty$ is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathbb{C}_\infty\{\tau\}$ such that

$$\phi_a = a + a_1\tau + \ldots + a_d\tau^d,$$

for some $d \geq 1$, $a_1, \ldots, a_d \in \mathbb{C}_\infty$, $a_d \neq 0$.

Proposition 5.1.2 (See [Gos96, Lemma 4.5.1])**.** *Let ϕ be a Drinfeld A-module. Then there exists $r \in \mathbb{N}^*$ such that*

$$\deg_\tau \phi_a = r \deg a \text{ for all } a \in A.$$

Definition 5.1.3. The number r in Proposition 5.1.2 is called the rank of ϕ.

Proposition 5.1.4 (See [Gos96, Section 4.6])**.** *Let ϕ be a Drinfeld A-module over $\mathbb{C}_\infty$. There exists a unique element $\exp_\phi \in \mathbb{C}_\infty\{\{\tau\}\}$ such that*

- $\exp_\phi \equiv 1 \pmod{\tau}$,
- $\exp_\phi a = \phi_a \exp_\phi$ *for all* $a \in A$.

Definition 5.1.5. The series $\exp_\phi$ is called the exponential of Drinfeld module ϕ.

We can write $\exp_\phi = \sum_{i\geq 0} e_i\tau^i$. The element $\exp_\phi$ induces a homomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$ such that for all $x \in \mathbb{C}_\infty$, we have

$$\exp(x) = \sum_{i\geq 0} e_i x^{q^i}.$$

Definition 5.1.6. We denote by $\Lambda_\phi \subset \mathbb{C}_\infty$ the kernel of $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$.

The important property of Λ_ϕ is the following proposition.

Proposition 5.1.7 (See [Gos96, Theorem 4.6.9]). *We have Λ_ϕ is an A-lattice i.e., Λ_ϕ is discrete in $\mathbb{C}_\infty$ and Λ_ϕ is a finitely generated A-module of rank r, where r is the rank of ϕ.*

By [Gos96, Theorem 2.14], we have the following formula of the exponential $\exp_\phi$ of Drinfeld module ϕ.

Proposition 5.1.8. *For $x \in \mathbb{C}_\infty$, we have*

$$\exp_\phi(x) = x \prod_{\lambda \in \Lambda_\phi \setminus \{0\}} (1 - \frac{x}{\lambda}).$$

5.1.3 Action of ideals on Drinfeld modules

This Subsection follows closely [Gos96, Section 4.9].

Let $\phi : A \to \overline{K_\infty}\{\tau\}$ be a Drinfeld A-module of rank r.

Definition 5.1.9. Let I be a nonzero ideal of A. We denote by I_ϕ the left ideal generated by $\{\phi_b\}_{b \in I}$. Since $\mathbb{C}_\infty\{\tau\}$ is a principal ideal domain, the ideal I_ϕ is generated by one element. We denote by $\phi_I \in \mathbb{C}_\infty\{\tau\}$ the monic polynomial in τ generating I_ϕ, i.e.,

$$\sum_{b \in I} \mathbb{C}_\infty\{\tau\}\phi_b = \mathbb{C}_\infty\{\tau\}\phi_I.$$

We denote by $\psi(I)$ the constant coefficient of ϕ_I.

By "monic" property of ϕ_I, it is stable under the right multiplication with ϕ_a for all $a \in A$. Thus there exists a unique element in $\mathbb{C}_\infty\{\tau\}$, denoted by $I * \phi_a$, such that

$$\phi_I \phi_a = (I * \phi_a)\phi_I.$$

We have constructed the following map

$$\begin{aligned} I * \phi \colon A &\longrightarrow \mathbb{C}_\infty\{\tau\} \\ a &\longmapsto I * \phi_a. \end{aligned}$$

We list some properties of this map.

Proposition 5.1.10 (see [Gos96, Section 4.9]). *We have*

1. *$I * \phi$ is a Drinfeld module of rank r.*
2. *We have $\Lambda_{I*\phi} = \psi(I)I^{-1}\Lambda_\phi$, where $\Lambda_{I*\phi}$, Λ_ϕ are respectively the kernel of the exponential map of $I * \phi$ and ϕ.*
3. *The kernel of the exponential of $I * \phi$ have the following property*

$$\exp_{I*\phi} \psi(I) = \phi_I \exp \phi.$$

By [Gos96, Theorem 2.14], we have the following formula of the exponential of $I * \phi$.

Corollary 5.1.11. *We have*

$$\exp_{I*\phi}(X) = X \prod_{\lambda \in I^{-1}\Lambda_\phi \setminus \{0\}} (1 - \frac{X}{\psi(I)\lambda}). \tag{5.1.3}$$

5.1.4 Standard sgn-normalized Drinfeld modules of rank one

This subsection follows closely [Gos96, Chapter 7].

Definition 5.1.12. A sign function on $K_\infty^\times$ is a homomorphism $\mathrm{sgn} : K_\infty^\times \to \mathbb{F}_\infty^\times$ which is the identity on $\mathbb{F}_\infty^\times$. We make a convention that $\mathrm{sgn}(0) = 0$.

Recall that we always fix a sign function as follows (see Section 5.1.1)

$$\begin{aligned} \mathrm{sgn}: \quad & K^\times \longrightarrow \mathbb{F}_\infty^\times \\ x = \sum_{i \geq i_0} a_i \pi^i & \longmapsto \mathrm{sgn}(x) := a_{i_0}. \end{aligned}$$

Definition 5.1.13. A sgn-normalized Drinfeld module of rank one is a homomorphism of $\mathbb{F}_q$-algebras $\rho : A \to \mathbb{C}_\infty\{\tau\}$ such that

$$\exists i \in \mathbb{Z}, \forall a \in A\backslash\{0\}, \rho_a = a + \ldots + \cdots + \mathrm{sgn}(a)^{q^i} \tau^{\deg(a)}.$$

Recall that $\ker \exp_\rho$ is an A-lattice of rank one (see Proposition 5.1.7).

Definition 5.1.14. We say that a sign-normalized rank one Drinfeld module ρ is standard if $\ker \exp_\rho$ is a free A-module of rank one.

Proposition 5.1.15. *There always exists a standard Drinfeld module of rank one.*

Proof. Let ϕ be a Drinfeld module of rank one and Λ_ϕ be the kernel of its exponential map. Then by Proposition 5.1.7, Λ_ϕ is an A-lattice of rank one. We recall the fact that A is a Dedekind ring. Thus $\Lambda_\phi \cong I$ with I is an ideal of A. Hence, $\Lambda_\phi = Ih$ for some $h \in \mathbb{C}_\infty$. We have

$$\ker(I * \phi) = \psi(I) I^{-1} \Lambda_\phi = A\psi(I)h.$$

It means that the Drinfeld module $I * \phi$ is standard. □

Remark 5.1.16. If $d_\infty = 1$ then the standard module is unique. In general, this is not true.

5.1.5 A little bit of class field theory

This subsection follows closely [Gos96, Chapter 7]. In this subsection, we recall a little bit of class field theory.

Let ϕ be a standard sgn-normalized Drinfeld A-module of rank one and $\exp_\phi$ be the exponential of ϕ. We can write

$$\exp_\phi = \sum_{n \geq 0} e_n \tau^n, \text{ with } e_n \in \mathbb{C}_\infty.$$

Definition 5.1.17. We set

$$H_A^+ := K(e_n, n \geq 0),$$

and

$$\mathbb{T} := H_A^+([I], I \in I_A). \tag{5.1.4}$$

We have another way to define H_A^+. For any $a \in A\backslash\mathbb{F}_q$, we can write $\phi_a = \sum_{i=0}^{\deg a} (a,i)\tau^i$, where (a,i) is the coefficients depend on a and i.

Definition 5.1.18 (See [Gos96, Defintion 7.4.1])**.** We define

$$H_A^+ = K((a,i), 0 \leq i \leq \deg a).$$

We have the following property of H_A^+.

Proposition 5.1.19 (See [Gos96, Proposition 7.4.4])**.** *The extension H_A^+/K is a finite abelian extension which is ramified outside ∞.*

Definition 5.1.20. Set $G := \mathrm{Gal}(H_A^+/K)$ and let $\sigma : I_A \to G$ be the Artin map.

Recall that if $P \in \mathcal{M}$, then the Frobenious map $\sigma_P \in G$ is the map such that

$$\forall b \in B, \sigma_P(b) \equiv b^{q^{\deg(P)}} \pmod{PB},$$

where B is the integral closure of A in H_A^+. The Artin map induces an isomorphism of groups

$$\mathrm{Pic}^+(A) \cong G.$$

Let $I \in \mathcal{A}$ be a nonzero ideal of A. We define $\sigma_I(\exp_\phi) := \sum_{n \geq 0} \sigma_I(e_i)\tau^i$.

Proposition 5.1.21 (See [Gos96, Theorem 7.4.8]). *Let $I \in \mathcal{A}$ and $I * \phi$ be the Drinfeld constructed in Section 5.1.3. Then we have*

$$\exp_{I*\phi} = \sigma_I(\exp_\phi).$$

5.1.6 Goss' map

We recall that h is defined by (5.1.2) and $\pi \in K_\infty$ is a uniformizer. We first collect a lemma which will be necessary in the sequel.

Lemma 5.1.22 (See [Gos96, Lemma 8.2.2]). *For $x \in 1 + \pi\mathbb{F}_\infty[[\pi]]$, there exists a unique $y \in \overline{K_\infty}$ such that $v_\infty(y-1) > 0$ and $y^h = x$.*

Proof. Suppose that $h = h_1 p^n$ where $h_1 \nmid p$, $n \geq 0$. We have

$$X^{h_1} - x \equiv X^{h_1} - 1 \pmod{\pi}.$$

On the other hand, its formal derivative is non zero, i.e., $(X^{h_1}-1)' = h_1 X^{h_1-1} \neq 0$. Hence, 1 is a simple root of the polynomial $X^{h_1} - 1$. By Hensel's Lemma, there exist a unique lift $y \in \mathbb{F}_\infty[[\pi]]$ of 1 such that $(X^{h_1} - x)(y) = 0$ i.e., there exists a unique $y \in 1 + \pi\mathbb{F}_\infty[[\pi]]$ such that $y^{h_1} = x$.

Let us take $z \in \overline{K_\infty}$ such that $z^{p^n} = y$. We note that $z^h = x$ and $v_\infty(z^{p^n} - 1) = v_\infty(y-1) > 0$. On the other hand, $v_\infty(z^{p^n} - 1) = p^n v_\infty(z-1)$. It follows that $v_\infty(z-1) > 0$. This imply the existence.

Now, for the uniqueness, if there exists z' such that $z^{p^n} = z'^{p^n} = y$, then $0 = z^{p^n} - z'^{p^n} = (z - z')^{p^n}$. Hence, $z = z'$. The proof is done. □

Definition 5.1.23. By Lemma 5.1.22, we define $x^{1/h} := y$.

Let $\pi' \in \overline{K_\infty}$ such that

$$(\pi')^{d_\infty} = \pi.$$

We are read to define the "Goss map"

$$[.] : I_A \to \overline{K_\infty}.$$

For $I \in I_A$, there exists a unique $x \in K^\times$ such that $\mathrm{sgn}(x) = 1$ and $I^h = xA$. Note that $\frac{x}{\pi^{v_\infty(x)}} \in 1 + \pi\mathbb{F}_\infty[[\pi]]$. Thus, $\left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}}$ is well-defined by Lemma 5.1.22.

Definition 5.1.24. We define

$$[I] := (\pi')^{-\deg I}\left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}}.$$

Remark 5.1.25. We have $v_\infty([I]) = -\deg I v_\infty(\pi') = -\frac{\deg I}{d_\infty}$ tends to $-\infty$ when $\deg I$ tends to ∞.

We list some basic properties of the Goss map.

Proposition 5.1.26 (See [Gos96, Section 8.2]). *For all $I, J \in I_A$, we have*

1. $[I][J] = [IJ]$.

2. For $a \in K^\times$ and $I = aA$ then

$$[I] := \frac{a}{\operatorname{sgn}(a)}.$$

3. $[.]$ is injective.

Proof. 1. For $I, J \in I_A$, there exist unique $x, y \in K^\times$ such that $\operatorname{sgn}(x) = \operatorname{sgn}(y) = 1$ and $I^h = x, J^h = y$. We can see that $xy \in K^\times$ satisfies $\operatorname{sgn}(xy) = 1$ and $(IJ)^h = xy$. We have

$$[I][J] = (\pi')^{-\deg I - \deg J}\left(\frac{xy}{\pi^{v_\infty(x)+v_\infty(y)}}\right)^{\frac{1}{h}} = [IJ].$$

2. Assume $I = aA$. There exists a unique $x \in K^\times$ such that $\operatorname{sgn}(x) = 1$ and $I^h = xA$. Thus $a^h A = xA$. We claim that

$$x = \left(\frac{a}{\operatorname{sgn}(a)}\right)^h.$$

Indeed, we have $\operatorname{sgn}(a) \in \mathbb{F}_\infty^\times$. Hence, $\operatorname{sgn}(a)^{\frac{q^d_\infty - 1}{q-1}} = \operatorname{Nm}_{\mathbb{F}_\infty/\mathbb{F}_q}(\operatorname{sgn}(a)) \in \mathbb{F}_q^\times$, where Nm is the norm over $\mathbb{F}_q$. On the other hand, $h := \left|\operatorname{Pic}^+(A)\right| = |\operatorname{Pic}(A)|\frac{q^{d_\infty}-1}{q-1}$. It follows that $\operatorname{sgn}(a)^h = \operatorname{sgn}(a)^{\frac{q^{d_\infty}-1}{q-1}|\operatorname{Pic}(A)|} = (\operatorname{Nm}_{\mathbb{F}_\infty/\mathbb{F}_q}(\operatorname{sgn}(a)))^{|\operatorname{Pic}(A)|} \in \mathbb{F}_q^\times$. Thus $xA = a^h A = a^h \operatorname{sgn}(a)^{-h} A = (\frac{a}{\operatorname{sgn}(a)})^h A$. It implies $x = (\frac{a}{\operatorname{sgn}(a)})^h \lambda$ with $\lambda \in \mathbb{F}_q^\times$. Note that $\operatorname{sgn}(x) = \operatorname{sgn}(\frac{a}{\operatorname{sgn}(a)}) = 1$. It follows that $\lambda = 1$, i.e., $x = (\frac{a}{\operatorname{sgn}(a)})^h$.

Since $xA = a^h A$, we have $v_\infty(x) = h v_\infty(a)$. It implies

$$\frac{x}{\pi^{v_\infty(x)}} = \left(\frac{a/\operatorname{sgn}(a)}{\pi^{v_\infty(a)}}\right)^h.$$

It is clear that $v_\infty(\frac{a/\operatorname{sgn}(a)}{\pi^{v_\infty(a)}} - 1) > 0$. Hence, by Lemma 5.1.22, we have

$$\left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}} = \frac{a/\operatorname{sgn}(a)}{\pi^{v_\infty(a)}}.$$

Recall that $\deg I = \deg aA = -d_\infty v_\infty(a)$. Hence, we have

$$[I] = (\pi')^{-\deg I}\left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}} = (\pi')^{d_\infty v_\infty(a)}\frac{a/\operatorname{sgn}(a)}{\pi^{v_\infty(a)}} = \frac{a}{\operatorname{sgn}(a)}.$$

3. Suppose that $I, J \in I_A$ such that $[I] = [J]$. Suppose that $I^h = xA$, $J^h = yA$ for $x, y \in K^\times$, $\operatorname{sgn}(x) = \operatorname{sgn}(y) = 1$. We get $[xA] = [yA]$. By part 2, we have

$$\frac{x}{\operatorname{sgn}(x)} = [xA] = [yA] = \frac{y}{\operatorname{sgn}(y)}.$$

It implies that $x = y$. Hence $I^h = J^h$. Since every nonzero fractional ideal has a unique decomposition into primes ideals, it follows that $I = J$. □

5.1.7 Zeta function

We recall the following important fact.

Proposition 5.1.27. *Let $n \in \mathbb{N}$. Then*

$$|\{I \in \mathcal{A} : \deg I = n\}| < +\infty.$$

Proposition 5.1.28. *The sum $\sum_{I \in \mathcal{A}} \frac{1}{[I]^n}$ converges in $\mathbb{C}_\infty$.*

Proof. The proof follows immediately from Proposition 5.1.27 and Remark 5.1.25. □

Definition 5.1.29. The value at n of the Goss zeta function associated to A is defined by

$$\zeta_A(n) := \sum_{I \in \mathcal{A}} \frac{1}{[I]^n} \in \mathbb{C}_\infty.$$

For $n \geq 1$, we also have

$$\zeta_A(n) = \prod_{P \in \mathcal{M}} (1 - \frac{1}{[P]^n})^{-1}. \tag{5.1.5}$$

Remark 5.1.30. By Equality (5.1.5), we have $\zeta_A(n) \neq 0$.

5.1.8 Elementary symmetric polynomials and polynomial $\mathcal{P}_n$, g_p

In this subsection, we will discuss some results in combinatorics about elementary symmetric polynomials. We will represent some homogeneous symmetric polynomials as expressions of elementary symmetric polynomials (see Proposition 5.1.32 and Corollary 5.1.33). In the end, we consider the values of these symmetric polynomials at the zeta values of Goss.

Notation

We first define these homogeneous symmetric polynomials in finite variables. Then we will define its values at the zeta values of Goss in "infinite variables". These definition is well-defined by Proposition 5.1.27 and Remark 5.1.25.

Let S be a finite set of variables and $s := |S|$.

Definition 5.1.31. The elementary symmetric polynomials in the variables $\{X\}_{X \in S}$ of degree n, denoted by E_n, is defined by

$$E_n = E_n(X)_{X \in S} := \begin{cases} 1 & \text{if } n = 0, \\ \sum_{S' \subset S, |S'| = n} \prod_{X \in S'} X & \text{if } 1 \leq n \leq s, \\ 0 & \text{if } n > s. \end{cases}$$

Let $l \in \mathbb{N}^*$. Let $f(X)_{X \in S}$ (or f, for short) be a homogeneous symmetric polynomial of degree l in the variables $\{X\}_{X \in S}$. Recall that the ring of symmetric polynomial is generated by elementary symmetric polynomials E_k, $k \geq 0$. Hence, a homogeneous symmetric polynomial f of degree l can always be written as

$$f = \sum_{|\underline{\lambda}| = l} c_{\underline{\lambda}} E_{\underline{\lambda}},$$

where $|\underline{\lambda}| := \lambda_1 + \ldots + \lambda_r$ with $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r$ and

$$E_{\underline{\lambda}} := E_{\lambda_1} \cdots E_{\lambda_r}.$$

We consider the following symmetric polynomials

$$g_p = g_p(X)_{X \in S} := \frac{1}{p} \left((\sum_{X \in S} X)^p - \sum_{X \in S} X^p \right) \in \mathbb{Z}[X]$$

and

$$\mathcal{P}_n = \mathcal{P}_n(X)_{X \in S} := \sum_{X \in S} X^n.$$

We will represent these polynomials in terms of elementary symmetric polynomials E_n, $n \geq 0$.

Polynomial $\mathcal{P}_n$

The function $\mathcal{P}_n$ is a homogeneous symmetric polynomial of degree n. Hence, it can always be written as

$$\mathcal{P}_n = \sum d_{\underline{\lambda}} E_{\underline{\lambda}}(X)_{X\in S}$$

where the sum runs through $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{N})^r$, $r \in \mathbb{N}$, $1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$ and $\lambda_1 + \cdots + \lambda_r = n$.

The method of computing the coefficients d_λ follows [Spe17, page 1240]. The coefficients $d_{\underline{\lambda}}$ are described in the following proposition.

Proposition 5.1.32. *We have*

$$\mathcal{P}_n(X)_{X\in S} = (-1)^{n-1} n \sum_{\substack{i_1,\ldots,i_s\geq 0 \\ i_1+2i_2+\ldots+si_s=n}} \frac{(-1)^{i_1+\ldots+i_s-1}}{i_1+\ldots+i_s} \binom{i_1+\ldots+i_s}{i_1, i_2, \ldots, i_s} E_1^{i_1}\cdots E_s^{i_s}. \tag{5.1.6}$$

Proof. Let U be an indeterminate. We have

$$\sum_{X\in S}\sum_{j\geq 1} \frac{(-1)^{j-1}X^jU^j}{j} = \sum_{X\in S} \log(1+XU).$$

Hence

$$\sum_{j\geq 1} \frac{(-1)^{j-1}\mathcal{P}_jU^j}{j} = \log(\prod_{X\in S}(1+XU)).$$

Note that $\prod_{X\in S}(1+XU) = 1 + \sum_{i=1}^{s} E_iU^i$. Expanding the log on the right-hand side as a Taylor series, we get

$$\begin{aligned}\sum_{j\geq 1} \frac{(-1)^{j-1}\mathcal{P}_jU^j}{j} &= \sum_{j\geq 1} \frac{(-1)^{j-1}}{j}(\sum_{i=1}^{s} E_iU^i)^j \\ &= \sum_{j\geq 1} \frac{(-1)^{j-1}}{j} \sum_{\substack{i_1,\ldots,i_s\geq 0 \\ i_1+\ldots+i_s=j}} \binom{j}{i_1,\ldots,i_s} E_1^{i_1}E_2^{i_2}\ldots E_s^{i_s}U^{i_1+2i_2+\cdots+si_s}.\end{aligned}$$

We consider the coefficients of U^n in both sides. It follow that

$$\frac{(-1)^{n-1}\mathcal{P}_n}{n} = \sum_{\substack{i_1,\ldots,i_s\geq 0 \\ i_1+2i_2+\ldots+si_s=n}} \frac{(-1)^{i_1+\ldots+i_s-1}}{i_1+\ldots+i_s} \binom{i_1+\ldots+i_s}{i_1,\ldots,i_s} E_1^{i_1}\cdots E_s^{i_s}.$$

Hence, we have

$$\mathcal{P}_n = (-1)^{n-1} n \sum_{\substack{i_1,\ldots,i_s\geq 0 \\ i_1+2i_2+\ldots+si_s=n}} \frac{(-1)^{i_1+\ldots+i_s-1}}{i_1+\ldots+i_s} \binom{i_1+\ldots+i_s}{i_1,i_2,\ldots,i_s} E_1^{i_1}\cdots E_s^{i_s}.$$

□

Polynomial g_p

As a consequence of Proposition 5.1.32, we have the following corollary.

Corollary 5.1.33. *We have*

$$g_p(X)_{X\in S} = (-1)^p \sum_{\substack{i_1,\ldots,i_s\geq 0 \\ i_1+2i_2+\ldots+si_s=p \\ i_1<p}} \frac{(-1)^{i_1+\ldots+i_s-1}}{i_1+\ldots+i_s} \binom{i_1+\ldots+i_s}{i_1,i_2,\ldots,i_s} E_1^{i_1}\cdots E_s^{i_s}.$$

In particular, we have $g_2 = E_2$.

Proof. We see that $g_p = \frac{1}{p}(E_1^p - \mathcal{P}_p)$. The coefficient of E_1^p in $\mathcal{P}_p$ in (5.1.6) corresponds to $i_1 = p$, $i_2 = \ldots = i_s = 0$. The proof is done. □

Now, we will describe the values of a homogeneous symmetric polynomials f at the value of Goss map.

We denote by $\mathcal{A}_{\leq n}$ the set nonzero ideals of A of degree less than or equal n. The sequence $\{f(\frac{1}{[I]^{q^{d_\infty}-1}})\}_{I\in\mathcal{A}_{\leq n}} \subset \mathbb{C}_\infty$ converges in $\mathbb{C}_\infty$ by Proposition 5.1.27 and Remark 5.1.25. Hence, we define

$$f(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}} := \lim_{n\to\infty} f(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}_{\leq n}} \in \mathbb{C}_\infty.$$

By applying the same arguments, we can define

$$E_l(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}} := \lim_{n\to+\infty} E_l(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}_{\leq n}},$$

and

$$g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}} := \lim_{n\to\infty} g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}_{\leq n}}.$$

Thus, as a consequence of Corollary 5.1.33, when passing to limit, we obtain the following proposition.

Proposition 5.1.34. *We have*

$$g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}} = (-1)^p \sum_{\substack{i_1,\ldots,i_s>0\\ i_1+2i_2+\ldots+si_s=p\\ i_1<p}} \frac{(-1)^{i_1+\ldots+i_s-1}}{i_1+\ldots+i_s}\binom{i_1+\ldots+i_s}{i_1,i_2,\ldots,i_s} E_1^{i_1}(\frac{1}{[I]^k})_{I\in\mathcal{A}}\cdots E_s^{i_s}(\frac{1}{[I]^k})_{I\in\mathcal{A}}.$$

Remark: s in this Proposition is not the same as the one that $s := |S|$.

5.2 Speyer's results

In this section, we recall some of Speyer's results. His main results is Theorem 5.2.3.

Let $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$. Denote by A^+ ($A_{\leq n}$) the set of monic polynomials in A (the set of polynomials in A of degree less than or equal n) and $\mathcal{P}^+$ the set of monic irreducible polynomials in A. Set

$$G_p(X) = \frac{(1-X^p)-(1-X)^p}{p(1-X)^p} \in \mathbb{F}_p(X).$$

Let S be a set of s variables. We set

$$g_p(X)_{X\in S} := \frac{1}{p}\left((\sum_{X\in S} X)^p - \sum_{X\in S} X^p\right).$$

Denote by $A_{+,\leq n}$ the set of monic polynomials in A of degree less than or equal n. We define

$$g_p(\frac{1}{a^k})_{a\in A_+} := \lim_{n\to+\infty} g_p(X)_{X\in A_{+,\leq n}}.$$

The above sum is well-defined, since only finitely many terms contribute to the coefficient of any particular power of $\frac{1}{\theta}$.

Speyer formulates the following formula.

Proposition 5.2.1. *For all $k \in \mathbb{Z}^+$, we have*

$$\sum_{P\in\mathcal{P}_1} G_p(\frac{1}{P^k}) = \frac{g_p(\frac{1}{a^k})_{a\in A_+}}{\zeta(kp)}.$$

Then, he uses the arithmetic properties of the Carlitz exponential $\exp_C$ to get some interesting results. By using the following identity

$$\frac{\exp_C(\tilde{\pi}X)}{\tilde{\pi}X} = \prod_{a\in A\setminus\{0\}}(1+\frac{X}{a}),$$

he shows that

$$E_k(\frac{1}{a})_{a\in A} = \begin{cases} \frac{\tilde{\pi}^k}{D_j}, k = q^j - 1, \\ 0, \text{ otherwise,} \end{cases}$$

where E_k is the elementary symmetric polynomial of degree k and

$$E_k(\frac{1}{a})_{a\in A} = \lim_{n\to\infty} E_k(X)_{X\in A_{\leq n}}.$$

Since the ring of symmetric polynomial is generated by the E_k, we deduce the following proposition.

Proposition 5.2.2. *If f is a homogeneous symmetric polynomial of degree k then $f(\frac{1}{a})_{a\in A} \in \tilde{\pi}^k K$.*

Note that $g_p(X)$ is a homogeneous symmetric polynomials of degree p, it implies that $g_p(\frac{1}{a^k})_{a\in A_+} \in \tilde{\pi}^{kp}K$. On the other hand, for $k \in \mathbb{Z}^+$ and $k \equiv 0 \pmod{q-1}$, $\zeta(kp) \in \tilde{\pi}^{kp}K$ is a well-known result. By Proposition 5.2.1, we get the main theorem as follows.

Theorem 5.2.3. *For any $k \in \mathbb{Z}^+$ and $k \equiv 0 \pmod{q-1}$, we have*

$$\sum_{P\in\mathcal{P}^+} G_p(\frac{1}{P^k}) \in K.$$

5.3 Generalization of Speyer's results

For the rest of this section, we always suppose that ϕ is a standard sgn-normalized Drinfeld A-module of rank one. There exists $\tilde{\pi} \in \mathbb{C}_\infty$ such that

$$\ker \exp_\phi = \tilde{\pi}A. \tag{5.3.1}$$

Let us fix $\tilde{\pi}$ for a fix standard sgn-normalized Drinfeld A-module ϕ of rank one.

In this section, we generalize Speyer's reults. The main result is Theorem 5.3.17. The idea of the proof is based on an equality in $\mathbb{C}_\infty$ (see Proposition 5.3.7) which is a consequence of Speyer's lemma. We will show that both the numerator and denominator on the right hand side of this equation are in $\tilde{\pi}^{kp}\mathbb{T}$ (where $\mathbb{T}$ is defined in (5.1.4)) by using a key lemma (see Lemma 5.3.10). It implies the rationality of of left hand side.

5.3.1 Speyer's lemma

Let p be an odd prime number. Recall

$$G_p(X) := \frac{1-X^p-(1-X)^p}{p(1-X)^p} \in \mathbb{Z}[[X]].$$

Set

$$f(x) := \left|\{(m_1,\dots,m_p) \in \mathbb{Z}^p : \sum_{i=1}^p m_i = n, \min_{i=1,\dots,p} m_i = 0\}\right|.$$

Also set

$$g(x) := \left|\{(m_1,\dots,m_p) \in \mathbb{Z}^p : \sum_{i=1}^p m_i = n, \min_{i=1,\dots,p} m_i \geq 0\}\right|,$$

$$h(x) := \left|\{(m_1,\dots,m_p) \in \mathbb{Z}^p : \sum_{i=1}^p m_i = n, \min_{i=1,\dots,p} m_i \geq 1\}\right|.$$

We see that $f(n) = g(n) - h(n)$ and for $0 \leq n \leq q$, $h(n) = 0$.

Lemma 5.3.1. *For $n \geq 0$, we have*

$$g(n) = \binom{n+p-1}{p-1}.$$

Proof. There is a bijection between the set $\{(m_1, \ldots, m_p) \in \mathbb{Z}^p : \sum_{i=1}^{p} m_i = n, \min_{i=1,\ldots,p} m_i \geq 0\}$ and the set of binary sequences of length $n+p-1$ which have exactly $p-1$ number 0.

$$(m_1, \ldots, m_p) \longleftrightarrow \underbrace{\underbrace{11\ldots1}_{m_1 \text{ times}}\, 0\, \underbrace{11\ldots1}_{m_2 \text{ times}}\, 0 \cdots 0\, \underbrace{11\ldots1}_{m_p \text{ times}}}_{n+p-1}.$$

The number of such binary sequences is $\binom{n+p-1}{p-1}$. The proof follows. □

Lemma 5.3.2. *We have*

$$h(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq q-1, \\ g(n-p) & \text{if } p \leq n. \end{cases}$$

Proof. The first part is obvious. For the second part, we see that $(m_1, \ldots, m_p) \in S_n^3$ if any only if $(m_1 - 1, \ldots, m_p - 1) \in S_{n-p}^2$. Hence, $h(n) = g(n-p)$. □

Lemma 5.3.3. *For $n = 0$, $f(n) = 1$. For $n \geq 1$, we have*

$$f(n) = \binom{n+p-1}{p-1} - \binom{n-1}{p-1}.$$

Proof. For $1 \leq n \leq q-1$, $h(n) = 0$, hence $f(n) = g(n) - h(n) = g(n)$. By Lemma 5.3.1, we are done. For $n \geq q$, by Lemma 5.3.2, $f(n) = g(n) - h(n) = g(n) - g(n-p)$. By Lemma 5.3.1, we are done. □

Lemma 5.3.4. *For $n \geq 1$, we have $f(n) \equiv 0 \pmod{p}$.*

Remark: $f(0) = 1 \not\equiv 0 \pmod{p}$.

Proof. We recall a "small version" Lucas' theorem. Let $0 \leq y \leq q-1$ and $x \in \mathbb{Z}$. We have

$$\binom{x+p}{y} \equiv \binom{x}{y} \pmod{p}.$$

The proof is a direct consequence of Lucas' theorem. □

By Lemma 5.3.4, $\sum_{n\geq 1} \frac{f(n)}{p} X^n \in \mathbb{Z}[[X]]$.

Lemma 5.3.5 (Speyer's Lemma, see [Spe17, page 1237])**.** *For $n \geq 1$, we have*

$$G_p(X) = \sum_{n\geq 1} \frac{f(n)}{p} X^n \text{ in } \mathbb{Z}[[X]].$$

Proof. We have following formal series

$$\begin{aligned}
\frac{1}{1-X} &= \sum_{i\geq 0} X^i, \\
\frac{1}{(1-X)^p} &= (\sum_{i\geq 0} X^i)^p = \sum_{m_i \geq 0} X^{m_1} \ldots X^{m_p} = \sum_{n\geq 0} g(n) X^n, \\
\frac{X^p}{(1-X)^p} &= \sum_{n\geq 0} g(n) X^{n+p} = \sum_{n\geq p} g(n-p) X^n = \sum_{n\geq p} h(n) X^n = \sum_{n\geq 0} h(n) X^n.
\end{aligned}$$

It follows that

$$\frac{1}{(1-X)^p} - \frac{X^p}{(1-X)^p} = \sum_{n\geq 0}(g(n)-h(n))X^n = \sum_{n\geq 0} f(n)X^n.$$

Hence

$$\frac{1}{(1-X)^p} - \frac{X^p}{(1-X)^p} - 1 = \sum_{n\geq 1} f(n)X^n.$$

The left-hand side is exactly $pG_p(X)$. The proof is done. □

For $I \in \mathcal{A}\backslash\{A\}$, we set

$$F(I) := |\{(I_1,\dots,I_p) \in \mathcal{A}^p, \gcd(I_1,\dots,I_p) = A, I_1\cdots I_p = I\}|. \tag{5.3.2}$$

Remark: Since $\mathcal{A}$ is Dedekind domain, every ideals have an unique factorization into prime ideals. Hence we can define gcd as usual.

We see that if $I = P_1^{k_1}\cdots P_r^{k_r}$ is the decomposition of I in product of maximal ideals then

$$F(I) = f(k_1)\cdots f(k_r). \tag{5.3.3}$$

As a consequence of Speyer's Lemma, we have the following corollary.

Corollary 5.3.6. *We have*

$$\sum_{I\in\mathcal{A}\backslash\{A\}} \frac{F(I)/p}{[I]^k} = \sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) \text{ in } \overline{K_\infty}.$$

Proof. We consider Equation (5.3.3). If $r \geq 2$, by Lemma 5.3.4, $p^2 | f(k_1)f(k_2)$. Hence $F(I)/p = 0$ in $\mathbb{F}_p$. It implies

$$\sum_{I\in\mathcal{A}\backslash\{A\}} \frac{F(I)/p}{[I]^k} = \sum_{P\in\mathcal{M}}\sum_{n\geq 0} \frac{F(P^n)/p}{[P^n]^k} = \sum_{P\in\mathcal{M}}\sum_{n\geq 0} \frac{f(n)/p}{[P^n]^k} = \sum_{P\in\mathcal{M}}\sum_{n\geq 0} \frac{f(n)/p}{[P^k]^n} = \sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}).$$

The last equality is from Speyer's lemma (Lemma 5.3.5). □

We have the following equality in $\mathbb{C}_\infty$.

Proposition 5.3.7. *Let $k \geq 1$ be an integer, we have*

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) = \frac{g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}}}{\zeta_A(kp)} \text{ in } \overline{K_\infty}. \tag{5.3.4}$$

Proof. Set $\Delta := \{(I,\dots,I) \in \mathcal{A}_n^p\}$ and C is the cyclic group of degree p. C acts on $\mathcal{A}_n^p$ by rotating coordinates. We have

$$\begin{aligned} g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}_n} &= \sum_{(I_1,\dots,I_p)\in(\mathcal{A}_n\backslash\Delta)/C} \frac{1}{[I_1]^k\cdots[I_p]^k} \\ &= \sum_{D\in\mathcal{A}_n} \frac{1}{[D]^{kp}} \times \sum_{\substack{(I_1,\dots,I_p)\in(\mathcal{A}_n\backslash\{(A,\dots,A)\})/C \\ \gcd(I_1,\dots,I_p)=A}} \frac{1}{[I_1]^k\cdots[I_p]^k}. \end{aligned}$$

Let $n \to \infty$, we have

$$\begin{aligned} g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}} &= \zeta_A(kp) \times \sum_{\substack{(I_1,\dots,I_p)\in(\mathcal{A}\backslash\{(A,\dots,A)\})/C \\ \gcd(I_1,\dots,I_p)=A}} \frac{1}{[I_1]^k\cdots[I_p]^k} \\ &= \zeta_A(kp) \times \sum_{I\in\mathcal{A}\backslash\{A\}} \frac{F(I)/p}{[I]^k} \quad (\text{ where } F(I) \text{ is defined in (5.3.3)}) \\ &= \zeta(kp) \sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) \quad (\text{ by Corollary 5.3.6}) \end{aligned}$$

Note that $\zeta(kp) \neq 0$ by the form (5.1.5). Thus the proof follows. □

5.3.2 Key lemma

In this section, we will firstly prove a key lemma (Lemma 5.3.10). Then we use this lemma to show that both numerator and denominator of Equation (5.3.4) are in $\tilde{\pi}^{kp}\mathbb{T}$ (see Proposition 5.3.16) where $\mathbb{T}$ is defined in (5.1.4).

We use the notation in Definition 5.1.9. The following proposition will be necessary in the sequel.

Proposition 5.3.8. *Let $I \in \mathcal{A}$. We use the notation of $\psi(I)$ in Definition 5.1.9. Then, the element $\frac{\psi(I)}{[I]}$ is in $\mathcal{O}_{\mathbb{T}}^{\times}$, where $\mathcal{O}_{\mathbb{T}}$ is the integral closure of A in $\mathbb{T}$. It depends only on the class of I in $\mathrm{Pic}^{+}(A)$.*

Proof. This proposition is a combination of results in [Gos96, Lemma 4.9.2, Theorem 7.4.8 and 7.6.2, Proposition 8.2.10]. □

Definition 5.3.9. We use the notation in Definition 5.1.20. Let $g \in G$ and $I \in \mathcal{A}$ such that $g = \sigma_I$. By Proposition 5.3.8, the following element is well-defined

$$\alpha(g) := \frac{\psi(I)}{[I]} \in \mathcal{O}_{\mathbb{T}}^{\times}.$$

Lemma 5.3.10 (Key Lemma)**.** *For $I \in \mathcal{A}$, we denote the class of I in $\mathrm{Pic}(A)$ by $\overline{I}$. Let $g \in G$ and $I \in \mathcal{A}$ such that $\sigma_I = g$. Let X be an indeterminate. We have*

$$\prod_{\xi \in \frac{\mathbb{F}_{\infty}^{\times}}{\mathbb{F}_q^{\times}}} \frac{g(\exp_{\phi})(\xi\alpha(g)X)}{\xi\alpha(g)X} = \prod_{J \in \mathcal{A} \cap \overline{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]}\right)^{q^{d_\infty}-1}\right).$$

where $\tilde{\pi}$ is defined in (5.3.1).

Proof. Set

$$F_g(X) := \frac{g(\exp_{\phi}(\alpha(g)X))}{\alpha(g)X}.$$

Note that $g = \sigma_I$. By Proposition 5.1.21 and Equation 5.1.3, we have

$$F_g(X) = \prod_{a \in I^{-1}\setminus\{0\}} \left(1 - \frac{\alpha(g)X}{\tilde{\pi}\psi(I)a}\right).$$

Note that $\alpha(g) = \frac{\psi(I)}{[I]}$. Hence we have

$$F_g(X) = \prod_{a \in I^{-1}\setminus\{0\}} \left(1 - \frac{X}{\tilde{\pi}[I]a}\right).$$

Note that $\sigma : I_A \to G$ is an isomorphism. We see that if $\overline{I} = \overline{J}$ in $\mathrm{Pic}^{+}(A)$ (i.e., $\sigma_I = \sigma_J$) then $J = aI$ for some $a \in K^{\times}$, $\mathrm{sgn}(a) = 1$. On the other hand, suppose $a, b \in I^{-1}\setminus\{0\}$, then $Ia = Ib \Leftrightarrow \frac{a}{b} \in \mathbb{F}_q^{\times}$. Hence

$$F_g(X) = \prod_{J \subset A, J = Ia} \prod_{\delta \in \mathbb{F}_q^{\times}} (1 - \frac{X}{\tilde{\pi}[I]\delta a}).$$

It follows that

$$\begin{aligned}\prod_{\xi \in \frac{\mathbb{F}_{\infty}^{\times}}{\mathbb{F}_q^{\times}}} F_g(\xi X) &= \prod_{J \in A, J = Ia} \prod_{\delta \in \mathbb{F}_{\infty}^{\times}} (1 - \delta\frac{X}{\tilde{\pi}[J]a}) \\ &= \prod_{J \in \mathcal{A} \cap \overline{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]}\right)^{q^{d_\infty}-1}\right).\end{aligned}$$

□

Proposition 5.3.11. *With above notation, we have*

$$\prod_{I\in\mathcal{A}}\left(1-\left(\frac{X}{\tilde{\pi}[I]}\right)^{q^{d_\infty}-1}\right)\in\mathbb{T}[[X]].$$

Proof. We have

$$\prod_{I\in\mathcal{A}}\left(1-\left(\frac{X}{\tilde{\pi}[I]}\right)^{q^{d_\infty}-1}\right)=\prod_{\overline{I}\in\mathrm{Pic}(A)}\prod_{J\in\overline{I}\cap\mathcal{A}}\left(1-\left(\frac{X}{\tilde{\pi}[J]}\right)^{q^{d_\infty}-1}\right).$$

By Lemma 5.3.10, the proof follows. □

Corollary 5.3.12. *For all $l\in\mathbb{N}$, we have*

$$E_l(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}}\in\tilde{\pi}^{l(q^{d_\infty}-1)}\mathbb{T}.$$

Proof. We have

$$\prod_{I\in\mathcal{A}}\left(1-\left(\frac{X}{\tilde{\pi}[I]}\right)^{q^{d_\infty}-1}\right)=\sum_{l\geq 0}(-1)^l\frac{1}{\tilde{\pi}^{l(q^{d_\infty}-1)}}E_l\left(\frac{1}{[I]^{q^{d_\infty}-1}}\right)_{I\in\mathcal{A}}X^{l(q^{d_\infty}-1)}. \tag{5.3.5}$$

By Proposition 5.3.11, the left-hand side of Equation (5.3.5) is in $\mathbb{T}[[X]]$. Hence, we have

$$(-1)^l\frac{1}{\tilde{\pi}^{l(q^{d_\infty}-1)}}E_l(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}}\in\mathbb{T}.$$

The proof is done. □

We collect some well-known results which will be used to prove a more general results of Corollary 5.3.12.

Lemma 5.3.13 (Newton identity, see [Zei84])**.** *For $n\in\mathbb{N}^*$, we have*

$$nE_n=\sum_{i=1}^{n}(-1)^{i-1}E_{n-i}\mathcal{P}_i.$$

We also have

$$\mathcal{P}_n=(-1)^{n-1}nE_n+\sum_{i=1}^{n-1}(-1)^{n-1+i}E_{n-i}\mathcal{P}_i.$$

Proposition 5.3.14 (See [Gos96, Theorem 8.19.4])**.** *For $n\in\mathbb{Z}, n\geq 1, n\equiv 0 \pmod{q^{d_\infty}-1}$, we have*

$$\zeta_A(n)\in\tilde{\pi}^n\mathbb{T}.$$

We have the following result:

Proposition 5.3.15. *For $1\leq l<p$, $1\leq i$, we have*

$$E_l(\frac{1}{[I]^{i(q^{d_\infty}-1)}})_{I\in\mathcal{A}}\in\tilde{\pi}^{li(q^{d_\infty}-1)}\mathbb{T}.$$

Proof. Note that $E_1=\mathcal{P}_1$. Corollary 5.3.12 gives an initial value of l and i. By using Newton identity and induction on l and i, the proof follows. □

Proposition 5.3.16. *Let $k\geq 1, k\equiv 0 \pmod{q^{d_\infty}-1}$ If f is a homogeneous symmetric polynomial of degree l then $f(\frac{1}{[I]^k})_{I\in A}$ is in $\tilde{\pi}^{kl}\mathbb{T}$.*

Proof. Recall that for $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{N}^s$,

$$|\underline{\lambda}| := \lambda_1 + \ldots + \lambda_s$$

and

$$E_{\underline{\lambda}} := E_{\lambda_1} \cdots E_{\lambda_s}.$$

A homogeneous symmetric polynomial f of degree l can always be written as

$$f = \sum_{|\lambda|=l} c_{\underline{\lambda}} E_{\underline{\lambda}}.$$

Pass to limit and note that $f(\frac{1}{[I]^{q^{d_\infty}-1}})_{I\in\mathcal{A}}$ is always defined, by Corollary 5.3.15, we have $E_{\underline{\lambda}}(\frac{1}{[I]^k})_{I\in\mathcal{A}}$ is in $\tilde{\pi}^{|\underline{\lambda}|k}\mathbb{T}$. Thus

$$f(\frac{1}{[I]^k})_{I\in\mathcal{A}} \in \tilde{\pi}^{lk}\mathbb{T}.$$

The proof is done. □

5.3.3 Main result

Theorem 5.3.17. *Let k be integer such that $k \geq 1, k \equiv 0 \pmod{q^{d_\infty}-1}$. We have*

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) \in \mathbb{T}.$$

Proof. Recall that from Proposition 5.3.7, we have

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) = \frac{g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}}}{\zeta_A(kp)}.$$

For $k \equiv 0 \pmod{q^{d_\infty}-1}$, we know that by Proposition 5.3.14,

$$\zeta_A(kp) \in \tilde{\pi}^{kp}\mathbb{T}.$$

Also, g_p is a homogeneous symmetric polynomial of degree p, by Proposition 5.3.16, we have

$$g_p(\frac{1}{[I]^k})_{I\in\mathcal{A}} \in \tilde{\pi}^{kp}\mathbb{T}.$$

Hence

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^k}) \in \mathbb{T}.$$

The proof is done.

□

5.3.4 Examples

In this section, we will find an explicit formula of Theorem 5.3.17 in the case $d_\infty = 1$, $k = q-1$ (see Equation 5.3.6). Moreover, in the end, we will give a non-zero sum in the case that $d_\infty = 1$, $q = 3$ and $h = 2$.

An explicit formula in the case $d_\infty = 1$, $k = q-1$

For $g \in G$ and $I \in \mathcal{A}$ such that $\sigma_I = g$, Lemma 5.3.10 implies

$$\frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \prod_{J \in \mathcal{A} \cap \bar{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]}\right)^{q-1}\right).$$

Hence

$$\prod_{g \in G} \frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \prod_{I \in \mathcal{A}} \left(1 - \left(\frac{X}{\tilde{\pi}[I]}\right)^{q^{d_\infty}-1}\right).$$

Expanding the right-hand side, we have

$$\prod_{g \in G} \frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \sum_{l \geq 0} (-1)^l \frac{1}{\tilde{\pi}^{l(q-1)}} E_l\left(\frac{1}{[I]^{q-1}}\right)_{I \in \mathcal{A}} X^{l(q-1)}.$$

Recall that $\exp_\phi = \sum\limits_{i \geq 0} e_i \tau^i$. We have

$$\prod_{g \in G} \sum_{i \geq 0} g(e_i)\alpha(g)^{q^i-1} X^{q^i-1} = \sum_{l \geq 0} (-1)^l \frac{1}{\tilde{\pi}^{l(q-1)}} E_l\left(\frac{1}{[I]^{q-1}}\right)_{I \in \mathcal{A}} X^{l(q-1)}.$$

Comparing the coefficients of X^n in both sides, we have the following proposition.

Proposition 5.3.18. *We have*

$$E_l(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} = (-1)^l \tilde{\pi}^{l(q-1)} \sum_{\substack{i_1, \ldots, i_h \geq 0 \\ (q^{i_1}-1)+\ldots+(q^{i_h}-1)=l(q-1)}} \prod_{j=1}^{h} g_j(e_{i_j})\alpha(g_j)^{q^{i_j}-1}.$$

In particular, we have

$$E_1(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} = -\tilde{\pi}^{q-1} \sum_{g \in G} g(e_1)\alpha(g)^{q-1}.$$

If $h = 2$, $l \geq 3$ and $i_1 + 2i_2 + \ldots + li_l = p$, we have

$$E_l(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} = 0.$$

Recall that

$$g_p(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} = \sum_{\substack{i_1, \ldots, i_s > 0 \\ i_1 + \ldots + s i_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\ldots+i_s-1}}{i_1 + \ldots + i_s} \binom{i_1 + \ldots + i_s}{i_1, i_2, \ldots, i_s} E_1^{i_1}(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} \cdots E_s^{i_s}(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}}.$$

On the other hand,

$$\zeta_A(p(q-1)) = \mathcal{P}_p(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}} = \sum_{\substack{i_1, \ldots, i_s > 0 \\ i_1 + 2i_2 + \ldots + s i_s = p}} \frac{(-1)^{p+i_1+\ldots+i_s-1} p}{i_1 + \ldots + i_s} \binom{i_1 + \ldots + i_s}{i_1, i_2, \ldots, i_s} E_1^{i_1}(\frac{1}{[I]^{q-1}}) \cdots E_s^{i_s}(\frac{1}{[I]^{q-1}}).$$

The only non-zero term in the above sum corresponds to the indices $(i_1, \ldots, i_s) \in (\mathbb{N}^*)^s$ such that $i_1 + \ldots + i_s = p$ (otherwise, $\binom{i_1+\ldots+i_s}{i_1,\ldots,i_s}=0$). Since $i_1 + 2i_2 + \ldots + si_s = p$, it implies that $i_1 = p$, $i_2 = \ldots = i_s = 0$. Hence

$$\zeta_A(p(q-1)) = E_1^p(\frac{1}{[I]^{q-1}})_{I \in \mathcal{A}}.$$

Thus we have

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^{q-1}}) = \frac{g_p(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}}{\zeta_A(p(q-1))} \tag{5.3.6}$$
$$= \sum_{\substack{i_1,\ldots,i_s>0\\ i_1+\ldots+si_s=p\\ i_1<p}} \frac{(-1)^{p+i_1+\ldots+i_s-1}}{i_1+\ldots+i_s}\binom{i_1+\ldots+i_s}{i_1,i_2,\ldots,i_s}\frac{E_2^{i_2}(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}\cdots E_s^{i_s}(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}}{E_1^{p-i_1}(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}}.$$

A non-zero result in the case $d_\infty = 1$, $q = 3$ and $h = 2$

There exits a global function field K such that $h = 2$, $q = 3$ and $d_\infty = 1$. An example is described in [GP18, Example 9.2]. For details, we take $K = \mathbb{F}_q(\theta, y)$ where $y^2 = \theta^3 - \theta^2 - 1$.

Recall that for $q = 3$, we have

$$G_3(X) = \frac{1 - X^3 - (1-X)^3}{3(1-X)^3} = \frac{X}{(1-X)^2}.$$

Proposition 5.3.19. *If $h = 2$, $d_\infty = 1$ and $q = 3$, we have*

$$\sum_{P\in\mathcal{M}} G_3(\frac{1}{[P]^2}) = \frac{E_2(\frac{1}{[I]^2})_{I\in\mathcal{A}}}{E_1^2(\frac{1}{[I]^2})_{I\in\mathcal{A}}} = \frac{g_1(e_1)\alpha(g_1)^2 g_2(e_1)\alpha(g_2)^2}{(g_1(e_1)\alpha(g_1)^2 + g_2(e_1)\alpha(g_2)^2)^2} \neq 0.$$

Proof. For $h = 2$, we have $G = \{g_1, g_2\}$ and

$$E_2(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}} = \tilde{\pi}^{2(q-1)} g_1(e_1)\alpha(g_1)^{q-1} g_2(e_1)\alpha(g_2)^{q-1}$$

and

$$E_1(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}} = -\tilde{\pi}^{q-1}(g_1(e_1)\alpha(g_1)^{q-1} + g_2(e_1)\alpha(g_2)^{q-1}).$$

In addition, if $q = 3$, we have

$$E_2(\frac{1}{[I]^2})_{I\in\mathcal{A}} = \tilde{\pi}^4 g_1(e_1)\alpha(g_1)^2 g_2(e_1)\alpha(g_2)^2$$

and

$$E_1(\frac{1}{[I]^2})_{I\in\mathcal{A}} = -\tilde{\pi}^2(g_1(e_1)\alpha(g_1)^2 + g_2(e_1)\alpha(g_2)^2).$$

Thus we have

$$\sum_{P\in\mathcal{M}} G_p(\frac{1}{[P]^{q-1}}) = -\sum_{\substack{i_1,i_2>0\\ i_1+2i_2=p\\ i_1<p}} \frac{(-1)^{i_1+i_2-1}}{i_1+i_2}\binom{i_1+i_2}{i_1,i_2}\frac{E_2^{i_2}(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}}{E_1^{p-i_1}(\frac{1}{[I]^{q-1}})_{I\in\mathcal{A}}}.$$

Replacing $q = 3$ we obtain

$$\sum_{P\in\mathcal{M}} G_3(\frac{1}{[P]^2}) = \frac{g_1(e_1)\alpha(g_1)^2 g_2(e_1)\alpha(g_2)^2}{(g_1(e_1)\alpha(g_1)^2 + g_2(e_1)\alpha(g_2)^2)^2}.$$

Note that the denominator in the above equation is $\left(\frac{\zeta_A(\frac{1}{2})}{\tilde{\pi}^2}\right)^2$, which is non-zero and rational. The first coefficient e_1 of $\exp_\phi$ is non-zero (see [Tha93, Page 565]). Hence the above sum is non-zero. □

Remark 5.3.20. For the case $A = \mathbb{F}_q[\theta]$ and $q = 3$, by [Spe17, Theorem 1.7], this sum is 0.

Chapter 6

Multizeta values

This chapter is taken from preprint paper (see [LND21b]). It is available at https://hal.archives-ouvertes.fr/hal-03093398v2/

This chapter is organized as follows. In Section 6.1, we present a conjecture of Lara Rodriguez and Thakur (Conjecture 6.1.2) and statement of main results (Subsection 6.1.4). In §6.2 we briefly review the CPY criterion deciding whether a MZV is zeta-like or Eulerian. We introduce the notion of dual t-motives and recall the work of Anderson and Thakur [AT09] connecting dual t-motives and MZV's. After recalling the Anderson-Brownawell-Papanikolas criterion in [ABP04] we state the key CPY criterion deciding whether a MZV is zeta-like (resp. Eulerian). The rest of the chapter is devoted to the proofs of the main results (see §6.3 for Theorem 6.1.3, §6.4 for Theorem 6.1.4, and §6.5 for Theorem 6.1.5, respectively). At the end we give some remarks in §6.6.

6.1 Introduction

6.1.1 Classical multiple zeta values

Multiple zeta values of Euler (MZV's for short) are real numbers of the form

$$\zeta(n_1,\dots,n_r)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\dots k_r^{n_r}},\quad \text{where } n_i\geq 1, n_r\geq 2.$$

Here r is called the depth and $w=n_1+\cdots+n_r$ is called the weight of the presentation $\zeta(n_1,\dots,n_r)$. For $r=1$ we recover the special values $\zeta(n)$ for $n\geq 2$ of the Riemann zeta function. These values have been studied in different contexts with deep connections to mathematical physics, knot theory, mixed Tate motives, and modular forms (see the survey of Zagier [Zag94] and the book of Burgos Gil and Fresan [IGF] for more details and more complete references).

Relations among MZV's have been studied extensively for the last three decades. Of particular interest, we are interested in two special relations that were discovered by Euler.

- The first one states that

$$\frac{\zeta(n)}{(2i\pi)^n}=-\frac{1}{2}\frac{B_n}{n!}\quad \text{for all } n\geq 2,\ n\equiv 0 \pmod 2,$$

  where B_n denotes the nth Bernoulli number. We say that $\zeta(n)$ for $n\geq 2$ and n even is Eulerian.

- The second one is the following identity

$$\zeta(1,2)=\zeta(3).$$

  We say that $\zeta(1,2)$ is zeta-like.

More generally, we say that a MZV $\zeta(n_1,\dots,n_r)$ is *Eulerian* (resp. *zeta-like*) if $\zeta(n_1,\dots,n_r)/(2i\pi)^{n_1+\cdots+n_r}$ (resp. $\zeta(n_1,\dots,n_r)/\zeta(n_1+\cdots+n_r)$) is rational.

Until now, we have extremely limited knowledge about zeta-like MZV's. We refer the reader to [LRT14, Remark after Conjecture 4.3] and [Tha17, §7.5] for a discussion about the known Eulerian and zeta-like MZV's. We also mention that a sufficient condition for Eulerian MZV's in terms of motivic MZV's was given by Brown (see [Bro12, Theorem 3.3]), but this condition is not completely effective (see [CPY19, §1] for a detailed discussion).

6.1.2 Characteristic p multiple zeta values (MZV)

By a well-known analogy between the arithmetic of number fields and that of global function fields conceived in the 1930s by Carlitz, we now switch to the function field setting.

Let $A=\mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p>0$. Let $K=\mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞. Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta)=-1$, and by $|\cdot|_\infty=q^{-v_\infty}$ the associated absolute value on K. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.

In [Car35] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n\in\mathbb{N}$ given by

$$\zeta_A(n):=\sum_{a\in A_+}\frac{1}{a^n}\in K_\infty$$

which are analogues of classical special zeta values in the function field setting. Here A_+ denotes the set of monic polynomials in A. For any tuple of positive integers $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$, Thakur [Tha04] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1,\dots,s_r)$ by

$$\zeta_A(\mathfrak{s}):=\sum\frac{1}{a_1^{s_1}\dots a_r^{s_r}}\in K_\infty$$

where the sum runs through the set of tuples $(a_1,\dots,a_r)\in A_+^r$ with $\deg a_1>\dots>\deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s})=s_1+\cdots+s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's. Thakur [Tha09a] showed that all the MZV's do not vanish.

Since their introduction many works have revealed the importance of these values for both their independent interest and for their applications to a wide variety of arithmetic applications, see for example [AT90, AT09, APTR18, ANDTR20, CPY19, Pap15, Pel12, Tae12b, Tae12a, Tha09b, Tod18, Yu91]. We refer the reader to the excellent surveys of Thakur [Tha17, Tha20] for more details and more complete references.

As in the classical setting one can argue that the main goal of this theory is to determine all algebraic relations over K among MZV's. It is worth noting that analogues of the aforementioned identities of Euler were proved:

- In 1935 Carlitz [Car35] introduced analogues of the Bernoulli numbers BC_n and proved (see also [Gos96, §9.2])

$$\frac{\zeta_A(n)}{\widetilde{\pi}^n}=\frac{BC_n}{\Gamma_n}\qquad\text{for all } n\geq 1,\ n\equiv 0\pmod{q-1}.$$

  Here $\widetilde{\pi}$ is the Carlitz period which is the analogue of $2i\pi$ (see [Gos96, Tha04]), and $\Gamma_n\in A$ is the nth Carlitz factorial (see §6.2.3 for more details).

- In [Tha09b] Thakur proved

$$(\theta^q-\theta)\zeta_A(1,q-1)=\zeta_A(q).$$

More precisely, we say that a MZV $\zeta_A(s_1,\dots,s_r)$ is *Eulerian* (resp. *zeta-like*) if $\zeta_A(s_1,\dots,s_r)/\widetilde{\pi}^{s_1+\cdots+s_r}$ (resp. $\zeta_A(s_1,\dots,s_r)/\zeta_A(s_1+\cdots+s_r)$) belongs to K.

In [LRT14, Tha17] Lara Rodriguez and Thakur proved some families of zeta-like MZV's and made several conjectures on zeta-like MZV's based on the numerical evidence, which will be discussed below.

6.1.3 A conjecture of Lara Rodriguez and Thakur

In [LRT14] Lara Rodriguez and Thakur showed (see [LRT14, Remark p. 796]):

Theorem 6.1.1 (Lara Rodriguez-Thakur [LRT14]). *Let* $1 \leq i \leq q$ *and* $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$. *Then* $\zeta_A(i, j(q-1))$ *is zeta-like.*

They conjectured that the converse also holds. A weak form of this conjecture was stated in [LRT14, Conjecture 4.4]. Later, Thakur [Tha17] gave a slightly stronger form which is given below (see [Tha17, the discussion after Conjecture 7.3, p. 1010]).

Conjecture 6.1.2 (Lara Rodriguez-Thakur [LRT14, Tha17]). *All zeta-like tuples of weight at most* q^2 *and depth 2 are exactly* $(i, j(q-1))$ *such that* $1 \leq i \leq q$ *and* $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$.

The proof of Theorem 6.1.1 is of algebraic nature and based on explicit formulas of power sums (see [LRT14, §5]). We mention that Lara Rodriguez and Thakur have extended their result for a more general setting (see [LRT20, Tha92]). On the other hand, the statement that there are no other zeta-like MZV's is of a different nature, which may need some elaborated transcendental tools.

6.1.4 Statement of main results

We are ready to state the main results of our chapter. First we present an affirmative answer to Conjecture 6.1.2.

Theorem 6.1.3. *All zeta-like tuples of weight at most* q^2 *and depth 2 are exactly* $(i, j(q-1))$ *such that* $1 \leq i \leq q$ *and* $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$.

Next we extend our method and prove a similar result for zeta-like MZV's of weight at most q^2 and depth 3.

Theorem 6.1.4. *All zeta-like tuples of weight at most* q^2 *and depth 3 are exactly* $(1, q-1, q(q-1))$. *In particular, there are no Eulerian MZV's of weight at most* q^2 *and depth 3.*

Finally, we obtain a complete list of all zeta-like MZV's of weight at most q^2.

Theorem 6.1.5. *All zeta-like tuples of weight at most* q^2 *are exactly*

- *the tuples of depth 2:* $(i, j(q-1))$ *such that* $1 \leq i \leq q$ *and* $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$,
- *one tuple of depth 3:* $(1, q-1, q(q-1))$.

Let us briefly outline the main ideas of the proofs of Theorems 6.1.3, 6.1.4 and 6.1.5.

1. First, by using a motivic interpretation of MZV's due to Anderson and Thakur in [AT09] and the Anderson-Brownawell-Papanikolas criterion for linear independence in positive characteristic in [ABP04], Chang, Papanikolas and Yu [CPY19] succeeded in devising a criterion called *the CPY criterion* deciding whether a MZV is zeta-like (resp. Eulerian). As a consequence we are led to find non-trivial solutions of a system of difference equations having Anderson-Thakur polynomials as parameters.

2. Second, we apply the previous CPY criterion to determine all zeta-like MZV's of weight at most q^2 and depth 2. In order to do so we manage to completely solve the corresponding system of difference equations. We use explicit formulas for Anderson-Thakur polynomials of weight at most q^2 and carefully investigate both Eulerian and non-Eulerian cases. It settles Theorem 6.1.3.

3. Third, we apply the CPY criterion to determine the full list of all zeta-like MZV's $\zeta_A(s_1, s_2, s_3)$ of weight at most q^2 and depth 3. We observe that the CPY criterion implies that $\zeta_A(s_2, s_3)$ is Eulerian. Thus by Theorem 6.1.3 above we obtain a very short list of (s_2, s_3). We then repeat the first two steps to determine s_1 and prove Theorem 6.1.4.

4. Finally, we deduce from the CPY criterion and Theorem 6.1.4 that there are no zeta-like MZV's of weight at most q^2 and depth at least 4. Theorem 6.1.5 is shown, and we are all done.

6.2 A criterion for zeta-like and Eulerian MZV's

We continue with the notation given in the Introduction. Further, letting t be another independent variable, we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ equipped with the Gauss norm $\|.\|_\infty$, and by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

6.2.1 Dual t-motives

We recall the notion of dual t-motives due to to Anderson (see [BP20, §4] and [HJ20, §5] for more details). We refer the reader to [And86] for the related notion of t-motives.

For $i \in \mathbb{N}$ we consider the i-fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t)) \\ f = \sum_j a_j t^j &\mapsto f^{(i)} := \sum_j a_j^{q^i} t^j. \end{aligned}$$

We extend i-fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise.

Definition 6.2.1. An effective dual t-motive is a $\overline{K}[t, \sigma]$-module $\mathcal{M}'$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t - \theta)^\ell (\mathcal{M}'/\sigma\mathcal{M}') = \{0\}.$$

We mention that effective dual t-motives are called Frobenius modules in [CPY19, §2.2]. Note that Hartl and Juschka [HJ20, §4] introduced a more general notion of dual t-motives. In particular, effective dual t-motives are always dual t-motives.

Throughout this chapter we will always work with effective dual t-motives. Therefore, we will sometimes drop the word "effective" where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t-motives. Then a morphism of effective dual t-motives $\mathcal{M} \to \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t, \sigma]$-modules. We denote by $\mathcal{F}$ the category of effective dual t-motives equipped with the trivial object $\mathbf{1}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$-module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$-basis for $\mathcal{M}$. We recall that $\mathbb{L}$ denotes the fraction field of the Tate algebra $\mathbb{T}$. We say that an object $\mathcal{M}$ of $\mathcal{F}$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$. By [Pap08, Proposition 3.3.9] there exists a rigid analytic trivialization Ψ_0 of $\mathcal{M}$ with $\Psi_0 \in \mathrm{GL}_r(\mathbb{T})$. Further, if Ψ is a rigid analytic trivialization of $\mathcal{M}$, then $\Psi = \Psi_0 M$ with $M \in \mathrm{Mat}_r(\mathbb{F}_q(t))$.

6.2.2 Ext^1-modules

Let $\mathcal{M}'$ be an effective dual t-motive of rank r over $\overline{K}[t]$. We denote by $\Phi' \in \mathrm{Mat}_r(\overline{K}[t])$ the matrix defining the σ-action on $\mathcal{M}'$ with respect to some $\overline{K}[t]$-basis of $\mathcal{M}'$. Let $\mathcal{M}$ be the dual t-motive given by the matrix

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix}, \quad \text{with } \mathbf{v} = (v_1, \dots, v_r) \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

We note that $\mathcal{M}$ fits into an exact sequence of the form

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathbf{1} \to 0,$$

and so is an extension of the trivial dual t-motive $\mathbf{1}$ by $\mathcal{M}'$, i.e., $\mathcal{M}$ represents a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$.

Note that $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ has a natural $\mathbb{F}_q[t]$-module structure defined as follows. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two objects of $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ defined by the matrices

$$\Phi_1 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_1 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_1 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]),$$

and

$$\Phi_2 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_2 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

Then for any $a_1, a_2 \in \mathbb{F}_q[t]$, $a_1 * \mathcal{M}_1 + a_2 * \mathcal{M}_2$ is defined to be the class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ represented by

$$\begin{pmatrix} \Phi' & 0 \\ a_1\mathbf{v}_1 + a_2\mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

6.2.3 Dual t-motives connected to MZV's

Following Anderson and Thakur [AT09] we introduce dual t-motives connected to MZV's. We briefly review Anderson-Thakur polynomials introduced in [AT90]. For $k \geq 0$, we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^{k} [\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$ we write $n - 1 = \sum_{j\geq 0} n_j q^j$ with $0 \leq n_j \leq q-1$ and define

$$\Gamma_n := \prod_{j\geq 0} D_j^{n_j}.$$

We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^{j} (\theta^{q^j} - t^{q^\ell})$ for $j \geq 1$. Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n\geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j\geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j}\right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t

$$H_n(t) = \alpha_n(t)\big|_{t=\theta,\, \theta=t}. \tag{6.2.1}$$

By [AT90, Eq. (3.7.3)] we get $\|H_n\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$.

We consider the dual t-motives $\mathcal{M}_{\mathfrak{s}}$ and $\mathcal{M}'_{\mathfrak{s}}$ attached to $\mathfrak{s}$ given by the matrices

$$\Phi_{\mathfrak{s}} = \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ H_{s_1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ 0 & H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & H_{s_r}^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s}} \in \mathrm{Mat}_r(\overline{K}[t])$ is the upper left $r \times r$ sub-matrix of $\Phi_{\mathfrak{s}}$. Then $\mathcal{M}_{\mathfrak{s}}$ represents a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.

Throughout this chapter, we work with the Carlitz period $\widetilde{\pi}$ which is a fundamental period of the Carlitz module (see [Gos96, Tha04]). We fix a choice of $(q-1)$st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i\geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t-\theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \widetilde{\pi}.$$

Given $\mathfrak{s}$ as above, Chang introduced the following series (see [Cha14, Lemma 5.3.1] and also [CPY19, Eq. (2.3.2)])

$$\mathfrak{L}(\mathfrak{s}) = \mathfrak{L}(s_1, \dots, s_r) := \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_r} H_{s_r})^{(i_r)} \dots (\Omega^{s_1} H_{s_1})^{(i_1)}. \tag{6.2.2}$$

Letting $\Gamma(\mathfrak{s}) = \Gamma_{s_1} \dots \Gamma_{s_r}$, by [Cha14, Eq. (5.5.3)] we have

$$\mathfrak{L}(\mathfrak{s})(\theta) = \Gamma(\mathfrak{s})\zeta_A(\mathfrak{s})/\widetilde{\pi}^{w(\mathfrak{s})}. \tag{6.2.3}$$

In particular, $\mathfrak{L}(\mathfrak{s})(\theta)$ is non-zero since $\zeta_A(\mathfrak{s})$ is known to be non-zero by Thakur [Tha09a].

If we denote $\mathcal{E}$ the ring of series $\sum_{n\geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n\to+\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$, then any $f \in \mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}(\mathfrak{s}) \in \mathcal{E}$ (see [Cha14, Lemma 5.3.1]).

Then the matrix given by

$$\Psi_{\mathfrak{s}} = \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1,\dots,s_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2,\dots,s_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1,\dots,s_r) & \mathfrak{L}(s_2,\dots,s_r) & \dots & \mathfrak{L}(s_r) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s}}^{(-1)} = \Phi_{\mathfrak{s}} \Psi_{\mathfrak{s}}.$$

Thus $\Psi_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}_{\mathfrak{s}}$.

We also denote by $\Psi'_{\mathfrak{s}}$ the upper $r \times r$ sub-matrix of $\Psi_{\mathfrak{s}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}'_{\mathfrak{s}}$.

To end this section, for $r \geq 2$ we let $\mathcal{N}_{\mathfrak{s}} \in \mathcal{F}$ be the dual t-motive defined by the matrix

$$\begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ H_{s_1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ 0 & H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ H_{w(\mathfrak{s})}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

Then $\mathcal{N}_{\mathfrak{s}}$ represents also a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.

6.2.4 A criterion for zeta-like and Eulerian MZV's in positive characteristic

We recall the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [ABP04, Theorem 3.1.1]).

Theorem 6.2.2 (Anderson-Brownawell-Papanikolas). *Let $\Phi \in \mathrm{Mat}_\ell(\overline{K}[t])$ be a matrix such that $\det \Phi = c(t-\theta)^s$ for some $c \in \overline{K}$ and $s \in \mathbb{Z}^{\geq 0}$. Let $\psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ be a vector satisfying $\psi^{(-1)} = \Phi\psi$ and $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{K})$ such that $\rho\psi(\theta) = 0$. Then there exists a vector $P \in \mathrm{Mat}_{1\times\ell}(\overline{K}[t])$ such that*

$$P\psi = 0 \quad \textit{and} \quad P(\theta) = \rho.$$

We now state the following result for zeta-like (resp. Eulerian) MZV's proved in [CPY19].

Theorem 6.2.3 ([CPY19], Theorems 4.4.2 and 4.2.2)**.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. Then $\zeta_A(\mathfrak{s})$ is zeta-like (resp. Eulerian) if and only if there exist $c, d \in \mathbb{F}_q[t]$ (resp. $c \in \mathbb{F}_q[t]$) with $c \neq 0$ such that $c * \mathcal{M}_{\mathfrak{s}} + d * \mathcal{N}_{\mathfrak{s}}$ (resp. $c * \mathcal{M}_{\mathfrak{s}}$) represents a trivial class in $Ext^1_{\mathcal{T}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.*

We stress that since $\zeta_A(\mathfrak{s})$ is non-zero, this result is an immediate consequence of [CPY19, Theorem 2.5.2] whose key tool is the Anderson-Brownawell-Papanikolas (ABP) criterion as stated in Theorem 6.2.2. Roughly speaking, using rigid analytic trivializations $\Psi_{\mathfrak{s}}$ defined as above, one applies the ABP criterion to lift a K-linear relation among MZV's to a $\overline{K}[t]$-linear relation among corresponding series defined in (6.2.2), which gives enough information to conclude.

We also recall the following corollary of Theorem 6.2.3 which was also conjectured by Lara Rodriguez and Thakur (see [LRT14, Conjecture 4.1, Part 2]).

Corollary 6.2.4 ([CPY19], Corollary 4.4.1)**.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. Suppose that $\zeta_A(\mathfrak{s})$ is zeta-like. Then each of*

$$\zeta_A(s_2, \dots, s_r), \dots, \zeta_1(s_r)$$

is Eulerian.

In particular, each s_i is divisible by $q-1$ for all $2 \leq i \leq r$.

By [CPY19, Remark 3.1] it implies the following criterion which will be used in the sequel.

Theorem 6.2.5 ([CPY19])**.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ as above. Then $\zeta_A(\mathfrak{s})$ is zeta-like (resp. Eulerian) if and only if there exist $c, d \in \mathbb{F}_q[t]$ (resp. $c \in \mathbb{F}_q[t]$ and $d = 0$) with $c \neq 0$ and polynomials $\delta_1, \dots, \delta_r \in \overline{K}[t]$ such that*

$$\begin{aligned}
\delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w, \\
\delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} + \delta_3^{(-1)} H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r}, \\
&\dots \\
\delta_{r-1} &= \delta_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} + \delta_r^{(-1)} H_{s_{r-1}}^{(-1)}(t-\theta)^{s_{r-1}+s_r}, \\
\delta_r &= \delta_r^{(-1)}(t-\theta)^{s_r} + cH_{s_r}^{(-1)}(t-\theta)^{s_r}.
\end{aligned} \tag{6.2.4}$$

Remark 6.2.6. 1) By [KL16, Theorem 2] we know that $\delta_1, \dots, \delta_r$ belong to $K[\theta] = \mathbb{F}_q[t, \theta]$ and

$$\deg_\theta \delta_i \leq \frac{q(s_i + \cdots + s_r)}{q-1}. \tag{6.2.5}$$

2) Note that if $(\delta_1, \dots, \delta_r, c, d) \in \mathbb{F}_q[\theta, t]^r \times \mathbb{F}_q[t]^2$ is a solution of the above system (6.2.4), then $(f\delta_1, \dots, f\delta_r, fc, fd) \in \mathbb{F}_q[\theta, t]^r \times \mathbb{F}_q[t]^2$ is also a solution of (6.2.4) for all $f \in \mathbb{F}_q[t]$.

6.3 Proof of Theorem 6.1.3

This section aims to present a proof of Theorem 6.1.3.

6.3.1 Setup

Let $\mathfrak{s} = (s_1, s_2) \in \mathbb{N}^2$ with $s_1 + s_2 \leq q^2$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. By Corollary 6.2.4 we can write $s_2 = \ell_2(q-1)$ for some $\ell_2 \in \mathbb{N}$. It suffices to show that we cannot have $\ell_2 < s_1$ and $s_1 + \ell_2(q-1) \leq q^2$.

Suppose that we do have $\ell_2 < s_1$ and $s_1 + \ell_2(q-1) \leq q^2$. In particular,

$$\ell_2 < q \tag{6.3.1}$$

since $\ell_2 q < s_1 + \ell_2(q-1) \leq q^2$. We will deduce a contradiction.

We start proving some preliminary results in §6.3.2 and then obtain a contradiction by distinguishing two cases for the zeta-like MZV $\zeta_A(\mathfrak{s})$: the non-Eulerian case in §6.3.3 and the Eulerian

case in §6.3.4. To do so we use two key ingredients: the bound given in (6.2.5) and the explicit formulas for Anderson-Thakur polynomials of weight at most q^2. Roughly speaking, we consider δ_1 as a polynomial in θ with coefficients in $\mathbb{F}_q[t]$. The explicit formulas for Anderson-Thakur polynomials of weight at most q^2 forces that δ_1 is divisible by a product of certain factors. Then we manage to prove that the latter product has degree strictly bigger than the bound given in (6.2.5) so that $\delta_1 = \delta_2 = 0$, which is a trivial solution, and we are done.

From now on, we will use capital characters or Greek characters (e.g. F, G, δ) for polynomials in $\mathbb{F}_q[t, \theta]$ and usual characters (e.g. f, g) for polynomials in $\mathbb{F}_q[t]$.

6.3.2 Preliminaries

We first recall Lucas' theorem and refer the reader to [Gra97] for more details.

Lemma 6.3.1. *For $m, n \in \mathbb{N}$, we express m and n in base q*

$$m = m_k q^k + \cdots + m_1 q + m_0,$$
$$n = n_k q^k + \cdots + n_1 q + n_0,$$

with $0 \leq m_j, n_j \leq q-1$. Then we have the following equality in $\mathbb{F}_p$

$$\binom{m}{n} = \prod_{j=0}^{k} \binom{m_j}{n_j}.$$

We now prove some preliminary results which will be used later.

Lemma 6.3.2. *Let $F \in \mathbb{F}_q[t, \theta]$ be a polynomial. Suppose that $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$. Then $F = (t-\theta)^{q-1} G$ for some $G \in \mathbb{F}_q[t, \theta^q]$.*

Proof. We first suppose that $\deg_\theta F < q$. We write $F = a_0 + a_1\theta + \cdots + a_{q-1}\theta^{q-1}$ with $a_0, \ldots, a_{q-1} \in \mathbb{F}_q[t]$. Then

$$F(t-\theta) = a_0 t + \sum_{j=1}^{q-1} (a_j t - a_{j-1})\theta^j + a_{q-1}\theta^q.$$

Since $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$, it follows that for all $1 \leq j \leq q-1$ we have $a_j t - a_{j-1} = 0$, that means $a_j = a_{q-1} t^{q-1-j}$. Therefore,

$$F = a_{q-1}(t^{q-1} + t^{q-1}\theta + \cdots + \theta^{q-1}) = a_{q-1}(t-\theta)^{q-1}.$$

Here the last equality follows from the fact that for all $0 \leq j \leq q-1$, we have $(-1)^{q-1-j}\binom{q-1}{j} = 1$ in $\mathbb{F}_p$. We put $G = a_{q-1} \in \mathbb{F}_q[t]$ and we are done in this case.

We now move to the general case. We can always write $F = F_0 + \theta^q F_1 + \cdots + \theta^{kq} F_k$ for some $k \in \mathbb{N}$ and some polynomials $F_0, \ldots, F_k \in \mathbb{F}_q[t, \theta]$ with $\deg_\theta F_j < q$ for all $0 \leq j \leq k$. The hypothesis $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$ implies that $F_j(t-\theta) \in \mathbb{F}_q[t, \theta^q]$ for all $0 \leq j \leq k$. Thus by the previous discussion we deduce that there exist $G_0, \ldots, G_k \in \mathbb{F}_q[t]$ such that $F_j = G_j(t-\theta)^{q-1}$ for all $0 \leq j \leq k$. Therefore,

$$\begin{aligned} F &= F_0 + \theta^q F_1 + \cdots + \theta^{kq} F_k \\ &= G_0(t-\theta)^{q-1} + \theta^q G_1(t-\theta)^{q-1} + \cdots + \theta^{kq} G_k (t-\theta)^{q-1} \\ &= (G_0 + \theta^q G_1 + \cdots + \theta^{kq} G_k)(t-\theta)^{q-1}. \end{aligned}$$

We put $G = G_0 + \theta^q G_1 + \cdots + \theta^{kq} G_k \in \mathbb{F}_q[t, \theta^q]$, and we are also done. □

As an immediate consequence we obtain the following result.

Lemma 6.3.3. *Let $k \in \mathbb{N}$ and $F \in \mathbb{F}_q[t, \theta]$ be a polynomial. Suppose that $F(t-\theta)^k \in \mathbb{F}_q[t, \theta^q]$. We denote by ℓ the unique integer such that $0 \leq \ell \leq q-1$ and $k + \ell \equiv 0 \pmod{q}$. Then $F = (t-\theta)^\ell G$ for some $G \in \mathbb{F}_q[t, \theta^q]$.*

Proof. The hypothesis $F(t-\theta)^k \in \mathbb{F}_q[t,\theta^q]$ implies that $F(t-\theta)^{q-\ell} \in \mathbb{F}_q[t,\theta^q]$. Thus $F(t-\theta)^{q-\ell-1}(t-\theta) \in \mathbb{F}_q[t,\theta^q]$. By Lemma 6.3.2 there exists $G \in \mathbb{F}_q[t,\theta^q]$ such that $F(t-\theta)^{q-\ell-1} = (t-\theta)^{q-1}G$, i.e., $F = (t-\theta)^\ell G$ as required. □

We now give explicit formulas for Anderson-Thakur polynomials H_n with $n \leq q^2$. By direct calculations we deduce from (6.2.1)

- for $1 \leq n \leq q$, we have $H_n(t) = 1$,
- for $q+1 \leq n \leq q^2$, we put $k = \lfloor \frac{n-1}{q} \rfloor$ and get

$$H_n(t) = \sum_{j=0}^{k} \binom{(n-1)-jq+j}{j} (t^q - t)^{k-j}(t^q - \theta^q)^j. \tag{6.3.2}$$

For example, if $q+1 \leq n \leq 2q$, then

$$H_n(t) = (t^q - t) + n(t^q - \theta^q).$$

Furthermore, we prove the following results (see also [CK18, Proposition 4.10]).

Lemma 6.3.4. *Let $n \in \mathbb{N}$ such that $n = \ell(q-1)$ with $1 \leq \ell \leq q-1$. Then*

$$H_n = H_{\ell(q-1)} = (-1)^\ell \frac{(t-\theta^q)^\ell - (t^q - \theta^q)^\ell}{t^q - t}.$$

Proof. Since $1 \leq \ell \leq q-1$, we get $n = \ell(q-1) \leq q^2$, and

$$\lfloor \frac{n-1}{q} \rfloor = \lfloor \frac{\ell(q-1)-1}{q} \rfloor = \lfloor \ell - \frac{\ell+1}{q} \rfloor = \ell - 1.$$

We use (6.3.2) to obtain

$$\begin{aligned}
H_n(t) &= \sum_{j=0}^{\ell-1} \binom{(n-1)-jq+j}{j} (t^q - t)^{k-j}(t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{(\ell(q-1)-1)-jq+j}{j} (t^q - t)^{\ell-j-1}(t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{(\ell-1-j)q+q-\ell-1+j}{j} (t^q - t)^{\ell-j-1}(t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{q-\ell-1+j}{j} (t^q - t)^{\ell-j-1}(t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell}{j} (t^q - t)^{\ell-j-1}(t^q - \theta^q)^j \\
&= (-1)^\ell \frac{(t-\theta^q)^\ell - (t^q - \theta^q)^\ell}{t^q - t}.
\end{aligned}$$

Here the fourth equality holds by Lucas' theorem (see Lemma 6.3.1), and the last equality follows from the binomial expansion of $(t-\theta^q)^\ell = ((t^q - \theta^q) - (t^q - t))^\ell$. The proof is finished. □

Lemma 6.3.5. *We put $n = q(q-1)$. Then*

$$H_n = H_{q(q-1)} = (t^q - t)^{q-2}.$$

Proof. We have

$$\lfloor \frac{n-1}{q} \rfloor = \lfloor \frac{q(q-1)-1}{q} \rfloor = q-2.$$

We use (6.3.2) to obtain

$$\begin{aligned} H_n(t) &= \sum_{j=0}^{q-2} \binom{(n-1)-jq+j}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\ &= \sum_{j=0}^{q-2} \binom{(q(q-1)-1)-jq+j}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\ &= \sum_{j=0}^{q-2} \binom{(q-1-j)q+j-1}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\ &= (t^q-t)^{q-2}. \end{aligned}$$

Here the last equality follows from the fact that for all $1 \leq j \leq q-2$, by Lucas' theorem (see Lemma 6.3.1),

$$\binom{(q-1-j)q+j-1}{j} = \binom{j-1}{j} = 0.$$

The proof is finished. □

6.3.3 The non-Eulerian case: $(q-1) \nmid w$

The proof in this case is divided into three steps. By Theorem 6.2.5 and Remark 6.2.6 there exist $c, d \in \mathbb{F}_q[t]$ with $c \neq 0$ such that there exist $\delta_1, \delta_2 \in K[t]$ verifying

$$\begin{aligned} \delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w, \\ \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}. \end{aligned}$$

Step 1. We first compute the Anderson-Thakur polynomials.

- Since $1 < s_1 < w \leq q^2$, by (6.3.2) we get explicit formulas for $H_{s_1}, H_w \in \mathbb{F}_q[t,\theta]$. Further,

$$\begin{aligned} \deg_\theta H_{s_1} &\leq q\lfloor \frac{s_1-1}{q} \rfloor, \\ \deg_\theta H_w &\leq q\lfloor \frac{w-1}{q} \rfloor. \end{aligned} \tag{6.3.3}$$

- We know that $1 \leq \ell_2 < q$ by (6.3.1). By Lemma 6.3.4 we have

$$H_{s_2} = H_{\ell_2(q-1)} = (-1)^{\ell_2} \frac{(t-\theta^q)^{\ell_2} - (t^q-\theta^q)^{\ell_2}}{t^q-t}.$$

Step 2. We solve the second equation

$$\delta_2 = \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}$$

for suitable $c \in \mathbb{F}_q[t]$. By the above explicit formula for H_{s_2} it is clear that we can take

$$\begin{aligned} c &= t^q - t, \\ \delta_2 &= (-1)^{\ell_2}(t-\theta)^{\ell_2 q} = (\theta-t)^{\ell_2 q}. \end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor$. We now solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + f\delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w \tag{6.3.4}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \leq n_1$ by (6.2.5),
- $f, d \in \mathbb{F}_q[t]$ with $f \neq 0$.

Note that by Remark 6.2.6 we need an extra polynomial $f \in \mathbb{F}_q[t]$.

Since the right-hand side of (6.3.4) is divisible by $(t-\theta)^w$, it implies that δ_1 is also divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we write

$$\delta_1 = F(t-\theta)^w, \quad F \in \mathbb{F}_q[t, \theta], \ \deg_\theta(F) \leq n_1 - w.$$

Replacing this expression in (6.3.4) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w + f(\theta - t)^{\ell_2 q} H_{s_1} + dH_w. \tag{6.3.5}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$.

We claim that

$$\deg_\theta F(t-\theta)^w \leq \deg_\theta(f(\theta - t)^{\ell_2 q} H_{s_1} + dH_w).$$

Otherwise $\deg_\theta F(t-\theta)^w > \deg_\theta(f(\theta-t)^{\ell_2 q} H_{s_1} + dH_w)$. It follows that $\deg_\theta F^{(1)} = \deg_\theta(F(t-\theta)^w)$. We deduce $\deg_\theta F = w/(q-1)$, which implies that $q-1$ divides w. We obtain then a contradiction.

We write

$$s_1 = \ell_1 q + k_1 + 1$$

with $0 \leq k_1 \leq q-1$. Then

$$w = s_1 + \ell_2(q-1) = (\ell_1 + \ell_2)q + k_1 + 1 - \ell_2.$$

By (6.3.3) $\deg_\theta H_{s_1} \leq \ell_1 q$ and $\deg_\theta H_w \leq (\ell_1 + \ell_2)q$. It follows that

$$\deg_\theta F(t-\theta)^w \leq \deg_\theta(f(\theta - t)^{\ell_2 q} H_{s_1} + dH_w) \leq (\ell_1 + \ell_2)q.$$

In particular, $w \leq (\ell_1 + \ell_2)q$. Thus

$$k_1 + 1 \leq \ell_2 \tag{6.3.6}$$

Therefore, $\ell_1 \geq 1$ since $s_1 = \ell_1 q + k_1 + 1 > \ell_2$.

On the other hand, $\deg_\theta F(t-\theta)^w \geq w = (\ell_1 + \ell_2)q + k_1 + 1 - \ell_2 > (\ell_1 + \ell_2)q + 1 - q$. Then the polynomial $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$ satisfies

$$(\ell_1 + \ell_2)q + 1 - q < \deg_\theta F(t-\theta)^w \leq (\ell_1 + \ell_2)q.$$

By Lemma 6.3.3 we conclude that

$$\delta_1 = F(t-\theta)^w = g(t-\theta)^{(\ell_1+\ell_2)q}$$

for some $g \in \mathbb{F}_q[t]$.

Replacing this equality in (6.3.5) and using explicit formulas for H_{s_1} and H_w given in (6.3.2) yields

$$\begin{aligned}
&g(t-\theta^q)^{\ell_2-(k_1+1)} \\
&= g(t-\theta)^{(\ell_1+\ell_2)q} + f(\theta^q - t^q)^{\ell_2} \sum_{j=0}^{\ell_1} \binom{(\ell_1 q + k_1) - jq + j}{j} (t^q - t)^{\ell_1 - j} (t^q - \theta^q)^j \\
&\quad + d \sum_{j=0}^{\ell_1+\ell_2-1} \binom{((\ell_1 + \ell_2)q + k_1 - \ell_2) - jq + j}{j} (t^q - t)^{\ell_1+\ell_2-1-j} (t^q - \theta^q)^j
\end{aligned}$$

We set

$$X := (t-\theta)^q = t^q - \theta^q$$

and rewrite the above equality as

$$\begin{aligned}
&g(X - (t^q - t))^{\ell_2-(k_1+1)} \\
&= gX^{\ell_1+\ell_2} + (-1)^{\ell_2} f X^{\ell_2} \sum_{j=0}^{\ell_1} \binom{\ell_1 q + k_1 - jq + j}{j} (t^q - t)^{\ell_1 - j} X^j \\
&\quad + d \sum_{j=0}^{\ell_1+\ell_2-1} \binom{(\ell_1 + \ell_2)q + k_1 - \ell_2 - jq + j}{j} (t^q - t)^{\ell_1+\ell_2-1-j} X^j.
\end{aligned}$$

Comparing the coefficients of X^{ℓ_2} yields

$$\begin{aligned} 0 &= (-1)^{\ell_2} f(t^q - t)^{\ell_1} + d\binom{(\ell_1+\ell_2)q + k_1 - \ell_2 - \ell_2 q + \ell_2}{\ell_2}(t^q - t)^{\ell_1 - 1} \\ &= (-1)^{\ell_2} f(t^q - t)^{\ell_1} + d\binom{\ell_1 q + k_1}{\ell_2}(t^q - t)^{\ell_1 - 1} \\ &= (-1)^{\ell_2} f(t^q - t)^{\ell_1}. \end{aligned}$$

Here the last equality holds by Lucas' theorem (see Lemma 6.3.1) since $\ell_2 < q$ by (6.3.1) and the fact that $k_1 + 1 \leq \ell_2$ by (6.3.6). Thus $f = 0$.

Next comparing the coefficients of $X^{\ell_1+\ell_2}$ yields

$$0 = g + (-1)^{\ell_2}\binom{\ell_1 q + k_1 - \ell_1 q + \ell_1}{\ell_1} f.$$

Since $f = 0$, it follows that $g = 0$, which is a contradiction.

6.3.4 The Eulerian case: $(q-1) \mid w$

Since $(q-1) \mid w$ and $s_2 = \ell_2(q-1)$ for some $\ell_2 \in \mathbb{N}$, it follows that $s_1 = \ell_1(q-1)$ for some $\ell_1 \in \mathbb{N}$. Since $s_1 + s_2 \leq q^2$, we get $\ell_1 + \ell_2 \leq q + 1$.

As before, the proof in this case is also divided into three steps. By Theorem 6.2.5 and Remark 6.2.6 there exist $c \in \mathbb{F}_q[t]$ with $c \neq 0$ such that there exist $\delta_1, \delta_2 \in K[t]$ verifying

$$\begin{aligned} \delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w, \\ \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2} + c H_{s_2}^{(-1)}(t-\theta)^{s_2}. \end{aligned}$$

Step 1. We compute the Anderson-Thakur polynomials. By Lemma 6.3.4 and the fact that $\ell_2 < q$ by (6.3.1) we have

$$H_{s_2} = H_{\ell_2(q-1)} = (-1)^{\ell_2}\frac{(t-\theta^q)^{\ell_2} - (t^q - \theta^q)^{\ell_2 q}}{t^q - t}.$$

Step 2. We solve the equation

$$\delta_2 = \delta_2^{(-1)}(t-\theta)^{s_2} + c H_{s_2}^{(-1)}(t-\theta)^{s_2}$$

for suitable $c \in \mathbb{F}_q[t]$. As before, by the above explicit formula for H_{s_2} it is clear that we can take

$$\begin{aligned} c &= t^q - t, \\ \delta_2 &= (-1)^{\ell_2}(t-\theta)^{\ell_2 q} = (\theta - t)^{\ell_2 q}. \end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor = (\ell_1 + \ell_2)q$. We have to solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} f H_{s_1}^{(-1)}(t-\theta)^w \tag{6.3.7}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \leq n_1$ by (6.2.5),
- $f \in \mathbb{F}_q[t]$ with $f \neq 0$.

We see that δ_1 is divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we can write

$$\delta_1 = F(t-\theta)^w, \quad F \in \mathbb{F}_q[t, \theta],\ \deg_\theta(F) \leq n_1 - w.$$

Replacing this expression in (6.3.7) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w + f(\theta - t)^{\ell_2 q} H_{s_1}(t^q - t). \tag{6.3.8}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$.

We distinguish four subcases.

Subcase 1: $\ell_1 + \ell_2 < q$

Since $\ell_1 < \ell_1 + \ell_2 < q$, then by Lemma 6.3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q - \theta^q)^{\ell_1}}{t^q - t}.$$

We know $w = (q-1)(\ell_1+\ell_2) \leq \deg_\theta \delta_1 \leq n_1 = q(\ell_1+\ell_2)$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. By Lemma 6.3.3 we deduce that $\delta_1 = g(t-\theta)^{q(\ell_1+\ell_2)}$ with $g \in \mathbb{F}_q[t]$. Thus $F = g(t-\theta)^{\ell_1+\ell_2}$. Replacing it in (6.3.8) we get

$$g(t-\theta^q)^{\ell_1+\ell_2} = g(t-\theta)^{q(\ell_1+\ell_2)} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

If $f \neq 0$, then we obtain a contradiction since the right-hand side is divisible by $\theta - t$ but not the left-hand side.

Subcase 2: $\ell_1 + \ell_2 = q$ **with** $1 \leq \ell_2 < q$

We have supposed that $\ell_2 < s_1 = \ell_1(q-1)$ (see §6.3.1). Thus $1 \leq \ell_2 \leq q-2$.

Since $\ell_1 q - \ell_2 < q$, then by Lemma 6.3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q - \theta^q)^{\ell_1}}{t^q - t}.$$

Since $w = (q-1)(\ell_1+\ell_2) = (q-1)q$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$, it follows that $F \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta F \leq q$. By (6.3.8),

$$F^{(1)} = F(t-\theta)^{q(q-1)} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

The right-hand side is divisible by $(\theta-t)^{\ell_2 q}$. Thus F is divisible by $(t^q-\theta)^{\ell_2}$. Since $F \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta F \leq q$, we get $F = g(t^q-\theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q-\theta^q)^q = g(t^q-\theta)^q(t-\theta)^{q(q-1)} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Thus

$$g(t^q - t^{q^2})(t-\theta)^{q(q-1)} = f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Since $1 \leq \ell_2 \leq q-2$, we get $g = f = 0$, which is a contradiction.

Subcase 3: $\ell_1 + \ell_2 = q+1$ **with** $1 < \ell_2 < q$

Since $\ell_1 = (q+1) - \ell_2 < q$, then by Lemma 6.3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q - \theta^q)^{\ell_1}}{t^q - t}.$$

Since $\ell_1 + \ell_2 = q+1$, it follows that $w = (q-1)(\ell_1+\ell_2) = q^2-1$. We know that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. By Lemma 6.3.3 we get $\delta_1 = G(t-\theta)^{q^2}$ with $G \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta G \leq q$. Thus $F = G(t-\theta)$.

By (6.3.8),

$$G^{(1)}(t-\theta^q) = G(t-\theta)^{q^2} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

The right-hand side is divisible by $(t-\theta)^{\ell_2 q}$. It implies that G is divisible by $(t^q-\theta)^{\ell_2}$. Since $G \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta G \leq q$, then $G = g(t^q-\theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q-\theta^q)^q(t-\theta^q) = g(t^q-\theta)^q(t-\theta)^{q^2} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

We get

$$g(t^q-\theta^q)^q(t-t^{q^2}) = f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Since $1 < \ell_2 < q$, comparing the power of $(t-\theta)$ yields $g = f = 0$, which is a contradiction.

Subcase 4: $\ell_1 = q$ **and** $\ell_2 = 1$

The arguments are similar to those of Case 3 except the explicit formula for H_{s_1}. By Lemma 6.3.5 we have

$$H_{s_1} = H_{q(q-1)} = (t^q - t)^{q-2}.$$

Since $w = (q-1)(\ell_1 + \ell_2) = q^2 - 1$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$, it follows that $\delta_1 = G(t-\theta)^{q^2}$ with $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$. Thus $F = G(t-\theta)$. By (6.3.8),

$$G^{(1)}(t-\theta^q) = G(t-\theta)^{q^2} + f(\theta - t)^q (t^q - t)^{q-1}.$$

The right-hand side is divisible by $(t-\theta)^q$. It follows that G is divisible by $(t^q - \theta)$. Since $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$, $G = g(t^q - \theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q - \theta^q)^q (t - \theta^q) = g(t^q - \theta)^q (t-\theta)^{q^2} + f(\theta - t)^q (t^q - t)^{q-1}.$$

We get

$$g(t^q - \theta^q)^q (t - t^{q^2}) = f(\theta - t)^q (t^q - t)^{q-1}.$$

Comparing the power of $(t-\theta)$ yields $g = f = 0$, which is a contradiction.

To summarize, in all cases we obtain a contradiction. Then the proof of Theorem 6.1.3 is finished.

6.4 Proof of Theorem 6.1.4

In this section we prove Theorem 6.1.4.

Let $\mathfrak{s} = (s_1, s_2, s_3) \in \mathbb{N}^3$ with $s_1 + s_2 + s_3 \leq q^2$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. Corollary 6.2.4 implies that $\zeta_A(s_2, s_3)$ is Eulerian. By Theorem 6.1.3 either $(s_2, s_3) = (q-1, (q-1)^2)$ or $(s_2, s_3) = (q-1, q(q-1))$.

If $(s_2, s_3) = (q-1, q(q-1))$, then $s_1 \leq q^2 - s_2 - s_3 = 1$. Thus $s_1 = 1$. It turns out that $\zeta_A(1, q-1, q(q-1)$ is zeta-like (see [LRT14, Theorem 3.2]), and we are done.

To conclude, we have to show that for all $1 \leq s_1 \leq q$, $\zeta_A(s_1, s_2, s_3)$ where $s_2 = q-1$ and $s_3 = (q-1)^2$ is not zeta-like. Suppose that it is not the case, i.e., $\zeta_A(s_1, s_2, s_3)$ is zeta-like where $1 \leq s_1 \leq q$, $s_2 = q-1$ and $s_3 = (q-1)^2$. Thus

$$w = s_1 + s_2 + s_3 = s_1 + q(q-1).$$

Lemma 6.4.1. *With the above notation, we have* $H_{s_1} = 1$ *and*

$$H_w(t) = \sum_{j=0}^{q-s_1} \binom{s_1 - 1 + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j. \tag{6.4.1}$$

Proof. Since $1 \leq s_1 \leq q$, $H_{s_1} = 1$, and it is clear that $\lfloor \frac{w-1}{q} \rfloor = \lfloor \frac{s_1 + q(q-1) - 1}{q} \rfloor = q-1$. Thus by (6.3.2) we get

$$\begin{aligned} H_w(t) &= \sum_{j=0}^{q-1} \binom{(w-1) - jq + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j \\ &= \sum_{j=0}^{q-1} \binom{(s_1 + q(q-1) - 1) - jq + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j. \end{aligned}$$

By Lucas' theorem (see Lemma 6.3.1), for all $0 \leq j \leq q-1$,

$$\binom{(s_1 + q(q-1) - 1) - jq + j}{j} = \binom{s_1 - 1 + j}{j}.$$

Since $1 \leq s_1 \leq q$, this term is equal to 0 if $q - s_1 + 1 \leq j \leq q - 1$. Thus

$$\begin{aligned} H_w(t) &= \sum_{j=0}^{q-1} \binom{(s_1 + q(q-1) - 1) - jq + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j \\ &= \sum_{j=0}^{q-s_1} \binom{s_1 - 1 + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j. \end{aligned}$$

The proof is finished. □

By Theorem 6.2.5 and Remark 6.2.6 there exist $c, d \in \mathbb{F}_q[t]$ with $c \neq 0$ such that there exist $\delta_1, \delta_2, \delta_3 \in K[t]$ verifying

$$\begin{aligned} \delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)} (t-\theta)^w + d H_w^{(-1)} (t-\theta)^w, \\ \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+s_3} + \delta_3^{(-1)} H_{s_2}^{(-1)} (t-\theta)^{s_2+s_3}, \\ \delta_3 &= \delta_3^{(-1)}(t-\theta)^{s_3} + c H_{s_3}^{(-1)} (t-\theta)^{s_3}. \end{aligned}$$

As before, if $q - 1 \mid w$, then we can suppose that $d = 0$ (see Theorem 6.2.5) and divide the proof into three steps.

Step 1. We first compute the Anderson-Thakur polynomials. By Lemma 6.3.4,

$$\begin{aligned} H_{s_2} &= H_{q-1} = 1, \\ H_{s_3} &= H_{(q-1)^2} = \frac{(t-\theta^q)^{q-1} - (t^q - \theta^q)^{q-1}}{t^q - t}. \end{aligned}$$

Step 2. We solve the equations

$$\begin{aligned} \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+s_3} + \delta_3^{(-1)} H_{s_2}^{(-1)} (t-\theta)^{s_2+s_3}, \\ \delta_3 &= \delta_3^{(-1)}(t-\theta)^{s_3} + c H_{s_3}^{(-1)} (t-\theta)^{s_3}. \end{aligned}$$

for suitable $c \in \mathbb{F}_q[t]$. By the above explicit formula for H_{s_2} and H_{s_3} it is clear that we can take

$$\begin{aligned} c &= (t^q - t)^{q+1}, \\ \delta_3 &= (t^q - t)^q (t-\theta)^{(q-1)q}, \\ \delta_2 &= -(t^q - \theta)^q (t-\theta)^{(q-1)q}. \end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor = \lfloor \frac{qs_1}{q-1} \rfloor + q^2$ and recall that $1 \leq s_1 \leq q$. We have to solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + f\delta_2^{(-1)} H_{s_1}^{(-1)} (t-\theta)^w + d H_w^{(-1)} (t-\theta)^w. \tag{6.4.2}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \leq n_1$ by (6.2.5),
- $f, d \in \mathbb{F}_q[t]$ with $f \neq 0$.

We see that δ_1 is divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we write

$$\delta_1 = F(t-\theta)^w$$

for some $F \in \mathbb{F}_q[t, \theta]$ with $\deg_\theta(F) \leq n_1 - w = \lfloor \frac{s_1}{q-1} \rfloor + q$.

Replacing this expression in (6.4.2) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w - f(t^q - \theta)^q (t-\theta)^{(q-1)q} H_{s_1} + d H_w. \tag{6.4.3}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. By Lemma 6.3.3 we get $F = G(t-\theta)^{q-s_1}$ with $G \in \mathbb{F}_q[t,\theta^q]$. In particular,

$$\deg_\theta G \leq \deg_\theta F - (q-s_1) \leq \lfloor \frac{s_1}{q-1} \rfloor + q - (q-s_1) = s_1 + \lfloor \frac{s_1}{q-1} \rfloor.$$

We distinguish three subcases.

Subcase 1: $1 \leq s_1 < q-1$**.**

In this case, since $\deg_\theta G \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor < q$ and $G \in \mathbb{F}_q[t,\theta^q]$, it follows that $G = g \in \mathbb{F}_q[t]$. Further, H_w is given as in (6.4.1). Putting all together into (6.4.3) we obtain

$$\begin{aligned} g(t-\theta^q)^{q-s_1} &= g(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q} \\ &\quad + d\sum_{j=0}^{q-s_1} \binom{s_1-1+j}{j} (t^q-t)^{q-1-j}(t^q-\theta^q)^j. \end{aligned}$$

We set

$$X := (t-\theta)^q = t^q - \theta^q$$

and rewrite the above equality as

$$\begin{aligned} &g(X-(t^q-t))^{q-s_1} \\ &= gX^q - f(t^{q^2}-t^q+X)X^{q-1} + d\sum_{j=0}^{q-s_1} \binom{s_1-1+j}{j} (t^q-t)^{q-1-j}X^j. \end{aligned}$$

We compare the coefficients of X^q yields $g = f$.

- If $1 < s_1 \leq q-1$, then comparing the coefficients of X^{q-1} yields $f = 0$, which is a contradiction.
- Otherwise $s_1 = 1$, then by replacing $g = f$ in the above equation we obtain

$$f(X-(t^q-t))^{q-1} = -f(t^{q^2}-t^q)X^{q-1} + d\sum_{j=0}^{q-1}(t^q-t)^{q-1-j}X^j.$$

We compare the constant coefficients and get $d = f$. Then using $d = f$ and looking at the coefficients of X^{q-1} yields $f = -f(t^{q^2}-t^q) + d = -f(t^{q^2}-t^q) + f$. Thus $f = 0$, and we also get a contradiction.

Subcase 2: $s_1 = q-1$**.**

In this case, $F = G(t-\theta)$ with $\deg_\theta G \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor = q$ and $G \in \mathbb{F}_q[t,\theta^q]$. Further, we know that $q-1 \mid w$, then we can suppose that $d = 0$ in Eq. (6.4.3) (see Theorem 6.2.5). Putting all together into (6.4.3) yields

$$G^{(1)}(t-\theta^q) = G(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

The right-hand side is divisible by $(t-\theta)^{(q-1)q}$. It implies that G is divisible by $(t^q-\theta)^{q-1}$. Since $G \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta G \leq q$, then $G = g(t^q-\theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q-\theta^q)^q(t-\theta^q) = g(t^q-\theta)^q(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

We get

$$g(t^q-\theta^q)^q(t-t^{q^2}) = -f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

Comparing the power of $(t-\theta)$ yields $f = g = 0$, which is a contradiction.

Subcase 3: $s_1 = q$**.**

In this case, we know that $F = G \in \mathbb{F}_q[t,\theta^q]$ with $\deg_\theta F \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor = q+1$. Thus we can write $F = a + \theta^q b$ with $a, b \in \mathbb{F}_q[t]$. Further, by (6.4.1) we get

$$H_w = H_{q^2} = (t^q-t)^{q-1}.$$

Putting all together into (6.4.3) yields

$$a+\theta^{q^2}b=(a+\theta^q b)(t-\theta)^{q^2}-f(t^q-\theta)^q(t-\theta)^{(q-1)q}+d(t^q-t)^{q-1}.$$

Comparing the coefficients of θ^{q^2+q} yields $b=0$. Then we use $b=0$ and compare the coefficients of $\theta^{(q-1)q}$ to get $0=-f(t^{q^2}-t^q)$. Thus $f=0$, which is a contradiction.

To summarize, in all cases we obtain a contradiction. Then the proof of Theorem 6.1.4 is finished.

6.5 Proof of Theorem 6.1.5

In this short section we present a proof of Theorem 6.1.5.

It suffices to prove that there is no zeta-like MZV's of weight at most q^2 and depth at least 4. Suppose that it is not the case. Then there exists $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$ with $s_1+\cdots+s_r\leq q^2$ and $r\geq 4$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. Corollary 6.2.4 implies that $\zeta_A(s_{r-2},s_{r-1},s_r)$ is Eulerian. By Theorem 6.1.4 this is impossible. Thus the proof of Theorem 6.1.5 is complete.

6.6 Final remarks

We end this chapter with some remarks.

Remark 6.6.1. We refer the reader to [CPY19, KL16, LRT14] for numerous numerical data concerning zeta-like and Eulerian MZV's in positive characteristic.

Remark 6.6.2. In this chapter we have succeeded in determining completely all zeta-like MZV's of weight at most q^2. Thus it is tempting to ask whether we could go further.

- Eulerian MZV's are at least conjecturally understood (see for example [CPY19, §6.2]).
- However, one should be aware that there are plenty of zeta-like MZV's of weight greater than q^2 (see for example [Che17, CK18, KL16]). At the moment, it seems very difficult to formulate a conjecture in a reasonable way to include all these examples. We hope to work on this question in a future work.

Bibliography

[ABP04] Greg W. Anderson, W. Dale Brownawell, and Matthew A. Papanikolas. Determination of the algebraic relations among special Γ-values in positive characteristic. *Ann. of Math. (2)*, 160(1):237–313, 2004. doi:10.4007/annals.2004.160.237.

[And86] Greg W. Anderson. t-motives. *Duke Math. J.*, 53(2):457–502, 1986. doi:10.1215/S0012-7094-86-05328-7.

[ANDTR17a] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. Special functions and twisted L-series. *J. Théor. Nombres Bordeaux*, 29(3):931–961, 2017. URL: http://jtnb.cedram.org/item?id=JTNB_2017__29_3_931_0.

[ANDTR17b] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. Stark units in positive characteristic. *Proc. Lond. Math. Soc. (3)*, 115(4):763–812, 2017. doi:10.1112/plms.12051.

[ANDTR19] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. Exceptional zeros of L-series and Bernoulli-Carlitz numbers. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):981–1024, 2019. URL: https://doi.org/10.2422/2036-2145.201706_006, doi:10.2422/2036-2145.201706_006.

[ANDTR20] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. On special L-values of t-modules. *Adv. Math.*, 372:107313, 33, 2020. doi:10.1016/j.aim.2020.107313.

[AP14] Bruno Anglès and Federico Pellarin. Functional identities for L-series values in positive characteristic. *J. Number Theory*, 142:223–251, 2014. doi:10.1016/j.jnt.2014.03.004.

[AP15] Bruno Anglès and Federico Pellarin. Universal Gauss-Thakur sums and L-series. *Invent. Math.*, 200(2):653–669, 2015. doi:10.1007/s00222-014-0546-8.

[APTR16] B. Anglès, F. Pellarin, and F. Tavares Ribeiro. Arithmetic of positive characteristic L-series values in Tate algebras. *Compos. Math.*, 152(1):1–61, 2016. With an appendix by F. Demeslay. doi:10.1112/S0010437X15007563.

[APTR18] Bruno Anglès, Federico Pellarin, and Floric Tavares Ribeiro. Anderson-Stark units for $\mathbb{F}_q[\theta]$. *Trans. Amer. Math. Soc.*, 370(3):1603–1627, 2018. doi:10.1090/tran/6994.

[AT90] Greg W. Anderson and Dinesh S. Thakur. Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)*, 132(1):159–191, 1990. doi:10.2307/1971503.

[AT09] Greg W. Anderson and Dinesh S. Thakur. Multizeta values for $\mathbb{F}_q[t]$, their period interpretation, and relations between them. *Int. Math. Res. Not. IMRN*, (11):2038–2055, 2009. doi:10.1093/imrp/rnp010.

[ATR17] Bruno Anglès and Floric Tavares Ribeiro. Arithmetic of function field units. *Math. Ann.*, 367(1-2):501–579, 2017. doi:10.1007/s00208-016-1405-2.

[BP20] W. Dale Brownawell and Matthew A. Papanikolas. A rapid introduction to Drinfeld modules, t-modules, and t-motives. *t-Motives: Hodge Structures, Transcendence and Other Motivic Aspects*, pages 3–30, may 2020. doi:10.4171/198.

[Bro12] Francis Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012. doi:10.4007/annals.2012.175.2.10.

[Car35] Leonard Carlitz. On certain functions connected with polynomials in a Galois field. *Duke Math. J.*, 1(2):137–168, 1935. doi:10.1215/S0012-7094-35-00114-4.

[Car42] L. Carlitz. Some topics in the arithmetic of polynomials. *Bull. Amer. Math. Soc.*, 48:679–691, 1942. doi:10.1090/S0002-9904-1942-07757-3.

[Cha14] Chieh-Yu Chang. Linear independence of monomials of multizeta values in positive characteristic. *Compos. Math.*, 150(11):1789–1808, 2014. doi:10.1112/S0010437X1400743X.

[Che17] Huei-Jeng Chen. Anderson-Thakur polynomials and multizeta values in positive characteristic. *Asian J. Math.*, 21(6):1135–1152, 2017. doi:10.4310/AJM.2017.v21.n6.a6.

[CK18] Huei-Jeng Chen and Yen-Liang Kuan. On depth 2 zeta-like families. *J. Number Theory*, 184:411–427, 2018. doi:10.1016/j.jnt.2017.08.027.

[CPY19] Chieh-Yu Chang, Matthew A. Papanikolas, and Jing Yu. An effective criterion for Eulerian multizeta values in positive characteristic. *J. Eur. Math. Soc. (JEMS)*, 21(2):405–440, 2019. doi:10.4171/JEMS/840.

[Dem15a] Florent Demeslay. A class formula for l-series in positive characteristic, 2015. arXiv:1412.3704.

[Dem15b] Florent Demeslay. *Formules de classes en caractéristique positive*. PhD thesis, 2015. Thèse de doctorat dirigée par Anglès, Bruno Mathématiques et leurs intéractions Caen 2015. URL: http://www.theses.fr/2015CAEN2044.

[FvdP04] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. doi:10.1007/978-1-4612-0041-3.

[GAD+19] David Goss, Bruno Anglès, Tuan Ngo Dac, Federico Pellarin, and Floric Tavares Ribeiro. The digit principle and derivatives of certain l-series. *Publications mathématiques de Besançon. Algèbre et théorie des nombres*, (1):81–102, 2019. URL: https://pmb.centre-mersenne.org/articles/10.5802/pmb.30/, doi:10.5802/pmb.30.

[GAND+19] David Goss, Bruno Anglès, Tuan Ngo Dac, Federico Pellarin, and Floric Tavares Ribeiro. The digit principle and derivatives of certain l-series. *Publications Mathématiques de Besançon - Algèbre et Théorie des Nombres*, (1):81–102, 2019. URL: http://www.numdam.org/articles/10.5802/pmb.30/, doi:10.5802/pmb.30.

[Gek88] Ernst-Ulrich Gekeler. On the coefficients of Drinfel′d modular forms. *Invent. Math.*, 93(3):667–700, 1988. doi:10.1007/BF01410204.

[Gez19] Oğuz Gezmiş. Taelman L-values for Drinfeld modules over Tate algebras. *Res. Math. Sci.*, 6(1):Paper No. 18, 25, 2019. doi:10.1007/s40687-019-0181-5.

[Gos80a] David Goss. The algebraist's upper half-plane. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):391–415, 1980. doi:10.1090/S0273-0979-1980-14751-5.

[Gos80b] David Goss. Modular forms for $\mathbf{F}_r[T]$. *J. Reine Angew. Math.*, 317:16–39, 1980. doi:10.1515/crll.1980.317.16.

[Gos80c] David Goss. π-adic Eisenstein series for function fields. *Compositio Math.*, 41(1):3–38, 1980. URL: http://www.numdam.org/item?id=CM_1980__41_1_3_0.

[Gos96] David Goss. *Basic structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1996. doi:10.1007/978-3-642-61480-4.

[GP18] Nathan Green and Matthew A. Papanikolas. Special L-values and shtuka functions for Drinfeld modules on elliptic curves. *Res. Math. Sci.*, 5(1):Paper No. 4, 47, 2018. doi:10.1007/s40687-018-0122-8.

[GP19] Oğuz Gezmiş and Matthew A. Papanikolas. The de Rham isomorphism for Drinfeld modules over Tate algebras. *J. Algebra*, 525:454–496, 2019. doi:10.1016/j.jalgebra.2019.02.006.

[Gra97] Andrew Granville. Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers. In *Organic mathematics (Burnaby, BC, 1995)*, volume 20 of *CMS Conf. Proc.*, pages 253–276. Amer. Math. Soc., Providence, RI, 1997.

[Gre17] Nathan Green. Tensor powers of rank 1 drinfeld modules and periods, 2017. arXiv:1706.03854.

[Gre19] Nathan Green. Special zeta values using tensor powers of Drinfeld modules. *Math. Res. Lett.*, 26(6):1629–1676, 2019. doi:10.4310/MRL.2019.v26.n6.a4.

[HJ20] Urs Hartl and Ann-Kristin Juschka. Pink's theory of Hodge structures and the Hodge conjecture over function fields. *t-Motives: Hodge Structures, Transcendence and Other Motivic Aspects*, pages 31–182, may 2020. doi:10.4171/198.

[IGF] José Ignacio, Burgos Gil, and Javier Fresán. Clay Mathematics Proceedings Multiple zeta values: from numbers to motives.

[KL16] Yen-Liang Kuan and Yi-Hsuan Lin. Criterion for deciding zeta-like multizeta values in positive characteristic. *Exp. Math.*, 25(3):246–256, 2016. doi:10.1080/10586458.2015.1069228.

[LND21a] Huy Hung Le and Tuan Ngo Dac. On identities for zeta values in Tate algebras. *Trans. Amer. Math. Soc.*, 374(8):5623–5650, 2021. doi:10.1090/tran/8357.

[LND21b] Huy Hung Le and Tuan Ngo Dac. Zeta-like multiple zeta values in positive characteristic. working paper or preprint, January 2021. URL: https://hal.archives-ouvertes.fr/hal-03093398.

[LRT14] José Alejandro Lara Rodríguez and Dinesh S. Thakur. Zeta-like multizeta values for $\mathbb{F}_q[t]$. *Indian J. Pure Appl. Math.*, 45(5):787–801, 2014. doi:10.1007/s13226-014-0089-0.

[LRT20] José Alejandro Lara Rodríguez and Dinesh S. Thakur. Zeta-like multizeta values for higher genus curves, 2020. arXiv:2003.12910.

[Pap08] Matthew A. Papanikolas. Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms. *Invent. Math.*, 171(1):123–174, 2008. doi:10.1007/s00222-007-0073-y.

[Pap15] Matthew A Papanikolas. Log-algebraicity on tensor powers of the carlitz module and special values of goss l-functions. *work in progress, 167 pages (last version: 28 April 2015)*, 2015.

[Pel12] Federico Pellarin. Values of certain L-series in positive characteristic. *Ann. of Math. (2)*, 176(3):2055–2093, 2012. doi:10.4007/annals.2012.176.3.13.

[Pel21] Federico Pellarin. The analytic theory of vectorial drinfeld modular forms, 2021. arXiv:1910.12743.

[PP18a] F. Pellarin and R. Perkins. On twisted a-harmonic sums and carlitz finite zeta values. *Journal of Number Theory*, 2018. URL: https://www.sciencedirect.com/science/article/pii/S0022314X18303299, doi:https://doi.org/10.1016/j.jnt.2018.10.018.

[PP18b] Federico Pellarin and Rudolph B. Perkins. Vectorial Drinfeld modular forms over Tate algebras. *Int. J. Number Theory*, 14(6):1729–1783, 2018. doi:10.1142/S1793042118501063.

[Spe17] David E. Speyer. Some sums over irreducible polynomials. *Algebra Number Theory*, 11(5):1231–1241, 2017. doi:10.2140/ant.2017.11.1231.

[Sti09] Henning Stichtenoth. *Algebraic function fields and codes*, volume 254 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, second edition, 2009.

[Tae10] Lenny Taelman. A Dirichlet unit theorem for Drinfeld modules. *Math. Ann.*, 348(4):899–907, 2010. doi:10.1007/s00208-010-0506-6.

[Tae12a] Lenny Taelman. A Herbrand-Ribet theorem for function fields. *Invent. Math.*, 188(2):253–275, 2012. doi:10.1007/s00222-011-0346-3.

[Tae12b] Lenny Taelman. Special L-values of Drinfeld modules. *Ann. of Math. (2)*, 175(1):369–391, 2012. doi:10.4007/annals.2012.175.1.10.

[Tha92] Dinesh S. Thakur. Drinfel′d modules and arithmetic in the function fields. *Internat. Math. Res. Notices*, (9):185–197, 1992. doi:10.1155/S1073792892000217.

[Tha93] Dinesh S. Thakur. Shtukas and Jacobi sums. *Invent. Math.*, 111(3):557–570, 1993. doi:10.1007/BF01231298.

[Tha04] Dinesh S. Thakur. *Function field arithmetic*. World Scientific Publishing Co., Inc., River Edge, NJ, 2004. doi:10.1142/9789812562388.

[Tha09a] Dinesh S. Thakur. Power sums with applications to multizeta and zeta zero distribution for $\mathbb{F}_q[t]$. *Finite Fields Appl.*, 15(4):534–552, 2009. doi:10.1016/j.ffa.2009.04.002.

[Tha09b] Dinesh S. Thakur. Relations between multizeta values for $\mathbb{F}_q[t]$. *Int. Math. Res. Not. IMRN*, (12):2318–2346, 2009. doi:10.1093/imrn/rnp018.

[Tha17] Dinesh S. Thakur. Multizeta values for function fields: a survey. *J. Théor. Nombres Bordeaux*, 29(3):997–1023, 2017. URL: http://jtnb.cedram.org/item?id=JTNB_2017__29_3_997_0.

[Tha20] Dinesh S. Thakur. Multizeta in function field arithmetic. *t-Motives: Hodge Structures, Transcendence and Other Motivic Aspects*, pages 441–452, may 2020. doi:10.4171/198.

[Tod18] George Todd. A conjectural characterization for $\mathbb{F}_q(t)$-linear relations between multizeta values. *J. Number Theory*, 187:264–287, 2018. doi:10.1016/j.jnt.2017.09.028.

[Yu91] Jing Yu. Transcendence and special zeta values in characteristic p. *Ann. of Math. (2)*, 134(1):1–23, 1991. doi:10.2307/2944331.

[Zag94] Don Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 497–512. Birkhäuser, Basel, 1994.

[Zei84] Doron Zeilberger. A combinatorial proof of newton's identities. *Discrete Mathematics*, 49(3):319, 1984. URL: https://www.sciencedirect.com/science/article/pii/0012365X84901717, doi:https://doi.org/10.1016/0012-365X(84)90171-7.