



HAL
open science

Geometry of the holomorphic symplectic manifolds and the characteristic foliation

Renat Abugaliev

► **To cite this version:**

Renat Abugaliev. Geometry of the holomorphic symplectic manifolds and the characteristic foliation. Algebraic Geometry [math.AG]. Université Paris-Saclay, 2021. English. NNT : 2021UPASM048 . tel-03555776

HAL Id: tel-03555776

<https://theses.hal.science/tel-03555776>

Submitted on 3 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Géométrie des variétés
hyper-kählériennes et le
feuilletage caractéristique

Geometry of the holomorphic
symplectic manifolds and the
characteristic foliation

Thèse de doctorat de l'Université Paris-Saclay

Ecole Doctorale de Mathématique Hadamard (EDMH) n° 574
Spécialité de doctorat : Mathématique fondamentale
Unité de recherche : Université Paris-Saclay, CNRS, Laboratoire de
mathématiques d'Orsay, 91405, Orsay, France
Référent : Faculté des sciences d'Orsay

Thèse présentée et soutenue à Paris-Saclay, le 10 décembre 2021, par

Renat Abugaliev

Composition du jury

Olivier DEBARRE Professeur, Institut de Mathématiques de Jussieu-Paris Rive Gauche	Président
Dimitri MARKOUCHEVITCH Professeur, Université de Lille	Rapporteur & Examineur
Jorge PEREIRA Professeur, IMPA	Rapporteur & Examineur
Frédéric CAMPANA Professeur émérite, Institut Élie Cartan de Lorraine	Examineur
Emanuele MACRI Professeur, Université Paris-Saclay	Examineur

Direction de la thèse

Ekaterina AMERIK Professeure, Université Paris-Saclay	Directrice de thèse
---	---------------------

université
PARIS-SACLAY

FACULTÉ
DES SCIENCES
D'ORSAY



Fondation mathématique
FMJH
Jacques Hadamard

Remerciements

Je souhaiterais tout d'abord remercier ma directrice de thèse, Ekaterina Amerik, pour la sagesse de ses conseils et sa patience. Le contenu de ce manuscrit est entièrement issu des conseils, remarques et questions qui ponctuèrent les discussions que nous avons pu avoir. Je souhaiterais ensuite remercier les deux rapporteurs de cette thèse, Dimitri Markouchevitch et Jorge Pereira, dont la relecture minutieuse a sans conteste contribué à améliorer la lisibilité et la clarté du manuscrit. J'exprime également toute ma reconnaissance à Frédéric Campana, Olivier Debarre et Emanuele Macrì pour avoir accepté de faire partie de mon jury de thèse.

Je voudrais mentionner les mathématiciens sans lesquels je n'aurais pas les résultats de ma thèse. En premier ma directrice de thèse qui a fait bon choix du problème et son aide. Je voudrais également remercier Jorge Pereira qui a proposé la solution élégante du cas d'une hypersurface non nef, Emanuele Macrì qui a fait un commentaire très utile après mon rapport à la fin de ma deuxième année de doctorat, Misha Verbitsky qui m'a aidé à mieux comprendre les fibrations Lagrangiennes, Alexey Gorinov qui m'a montré le livre "Stratified Morse theory" de Goresky et MacPherson, qui contient les théorèmes importants pour ma thèse.

Contents

1	Introduction	3
2	Preliminaries	7
2.1	Hodge structures	7
2.2	Holomorphic symplectic manifolds	9
2.3	Cohomology of holomorphic symplectic manifolds	15
2.4	Lefschetz's theorems	17
2.5	The Torelli theorem and the Period Map	18
2.6	Foliations and invariant subvarieties	20
2.7	Characteristic foliation	21
3	The case of a hypersurface with zero square	23
3.1	Lagrangian fibrations on irreducible holomorphic symplectic manifolds	23
3.2	Monodromy of a vertical hypersurface	25
3.3	Proof of Theorem 2.56	27
3.4	Smoothness of vertical hypersurfaces	28
4	The case of a hypersurface with positive square	30
4.1	The case of an ample hypersurface	30
4.2	The LHT for a nef and big hypersurface in IHS	32
4.3	The case of a nef and big hypersurface	34
4.4	The case of a non numerically effective hypersurface	35
5	Singular counterexamples to the conjecture of Campana	36
5.1	Vertical hypersurfaces	36
5.2	The variety of lines in a cubic fourfold	37
6	A special divisor on the Debarre-Voisin manifold	39
6.1	Introduction	40
6.2	Peskine variety	43
6.3	Special lines in Peskine variety	45
6.4	A foliation on the threefold Z	49
6.5	Proof of theorem 6.26	53
6.6	Appendix	55
	References	55

1 Introduction

Dans cette thèse nous étudions le feuilletage caractéristique sur une hypersurface lisse dans une variété hyper-kählérienne. D'abord nous définissons une variété hyper-kählérienne irréductible.

Définition Une variété lisse projective X est hyper-kählérienne irréductible si elle satisfait les propriétés suivantes:

- L'espace $H^0(X, \Omega_X^2)$ est engendré par une forme σ non-dégénérée (symplectique);
- $\pi_1(X) = 0$.

Nous donnons les exemples de variétés hyper-kählériennes irréductibles au paragraphe 2.2. Aux paragraphes 2.5 et 2.3 nous rappelons les résultats importants sur les variétés hyper-kählériennes irréductibles.

Soit Y une hypersurface lisse dans une variété hyper-kählérienne irréductible projective X de dimension $2n$ et σ une forme holomorphiquement symplectique sur X . Pour chaque point $x \in Y$ la forme σ est une forme non-dégénérée sur $T_{X,x}$. Donc la forme restreinte à $T_{Y,x}$ est de corang 1 (c'est à dire $\sigma|_{T_{Y,x}}$ a le noyau de dimension une). Le feuilletage caractéristique F sur une hypersurface Y est le noyau de la forme symplectique σ restreinte à Y . Nous discutons les feuilletages et le feuilletage caractéristique aux paragraphes 2.6 et 2.7. Si les feuilles (c'est-à-dire, les courbes intégrales) de ce feuilletage sont quasi-projectives, on appelle ce feuilletage algébriquement intégrable. Jun-Muk Hwang et Eckart Viehweg ont démontré dans [40] que si Y est de type général, alors F n'est pas algébriquement intégrable. Dans le papier [2] Ekaterina Amerik et Frédéric Campana ont complété ce résultat comme suit.

Théorème[2] Soit Y une hypersurface lisse sur une variété hyper-kählérienne lisse projective irréductible X de dimension supérieure à 2. Alors le feuilletage caractéristique sur Y est algébriquement intégrable si et seulement si Y est unireglée, c'est à dire recouverte par des courbes rationnelles.

Après, on peut poser la question quelle est la dimension de la fermeture de Zariski de la feuille générale de F . Si $n = 2$ la situation est bien comprise dans le théorème exposé ci-dessous.

Théorème[3] Soit X une variété hyper-kählérienne irréductible projective de dimension 4 et soit Y une hypersurface lisse dans X . Supposons qu'une feuille générale du feuilletage caractéristique sur Y n'est pas algébrique, mais il y a une fibration méromorphe $p : Y \dashrightarrow C$ par les surfaces F -invariantes. Alors il existe une fibration lagrangienne rationnelle $\pi : X \dashrightarrow B$ completant le diagramme commutatif suivant.

$$\begin{array}{ccc} Y & \dashrightarrow^p & C \\ \downarrow & & \downarrow \\ X & \dashrightarrow^\pi & B \end{array}$$

En particulier, la fermeture de Zariski d'une feuille générale du feuilletage caractéristique est une surface abélienne .

Frédéric Campana a fait la conjecture suivante.

Conjecture de Campana Soit Y une hypersurface lisse dans une variété hyper-kählérienne projective irréductible X et soit q la forme de Beauville-Bogomolov sur $H^2(X, \mathbb{Q})$. Alors exactement une des trois affirmations suivantes est vraie.

1. $q(Y, Y) < 0$. Le feuilletage F est algébriquement intégrable et Y unireglée;
2. $q(Y, Y) = 0$. La dimension de la fermeture de Zariski d'une feuille générale de F est n et il existe une fibration lagrangienne $f : X \dashrightarrow B$ comme ci-dessous;
3. $q(Y, Y) > 0$. Une feuille générale de F est Zariski dense en Y .

Le premier cas est facile. S. Boucksom dans [14, Théorème 4.2 et Proposition 4.7] a démontré que une hypersurface Y est unireglée si $q(Y, Y) < 0$. Il y a donc une fibration rationnelle $Y \dashrightarrow W$ tel que les fibres sont rationnellement connexes (voir [16] et [46, Chapitre IV.5]). Alors, $\sigma|_Y$ est l'image réciproque d'une forme sur W et les fibres de cette fibration sont les feuilles du feuilletage caractéristique sur Y .

L'auteur de cette thèse a essayé de démontrer la conjecture de Campana et il l'a démontré pour une hypersurface nef ¹. Dans **Chapter 2** nous rappelons la définition d'une variété hyper-kählérienne et sa propriétés. Après nous rappelons la définition d'un feuilletage et du feuilletage caractéristique. **Chapitre 3** nous considérons le cas de $q(Y, Y) = 0$ et demotrons le résultat suivant.

Théorème 2.56 Soit X une variété hyper-kählérienne irréductible projective de dimension $2n$ et $\pi : X \rightarrow \mathbb{P}^n$ une fibration lagrangienne. On considère une hypersurface D dans \mathbb{P}^n tel que son image réciproque Y est une hypersurface lisse en X . Alors la fermeture de Zariski d'une feuille générale du feuilletage caractéristique sur Y est une fibre de π .

Pour les hypersurfaces Y tel que $q(Y, Y) = 0$, la conjecture lagrangienne nous dit que l'hypersurface Y est l'image réciproque d'une hypersurface dans la base d'une fibration rationnelle lagrangienne Dans le **théorème 2.56** nous prenons la fibration Lagrangienne au-dessus de l'espace projectif. Jun-Muk Hwang a prouvé dans [38] que si la base de la fibration lagrangienne est lisse, alors c'est l'espace projectif. Hwang aussi conjecture que la base est toujours lisse. Pour $n = 2$ cette conjecture a été prouvé dans [12] et [37]. Donc, le deuxième cas de la **conjecture de Campana** pour une hypersurface nef est une conséquence du **théorème 2.56**, de la **conjecture lagrangienne** et de la **conjecture de Hwang**.

Au **Chapitre 4** nous considérons le cas de $q(Y, Y) > 0$. Une hypersurface avec le carré de Beauville-Bogomolov positif si et seulement si cette hypersurface est big. Nous avons

¹Admettant deux conjectures bien connues dans le deuxième cas; voir ci-dessous.

démontré le théorème suivante.

Théorème 2.57 Soient X une variété hyper-kählérienne irréductible projective de dimension $2n$ et Y dans X une hypersurface lisse nef et big. Alors la feuille générale du feuilletage caractéristique est Zariski dense dans Y .

Dans une conversation Jorge Vitorio Pereira a esquissé la preuve du résultat suivant:

Théorème 2.58 Soient X une variété hyper-kählérienne irréductible projective de dimension $2n$ et Y dans X une hypersurface lisse non nef. Alors Y est unireglée.

Ce résultat nous dit que si $q(Y, Y) \geq 0$, alors Y est singulière (corollary 4.23). Donc, on ne doit pas démontrer la conjecture 2.55 pour une hypersurface non-nef.

On peut étudier le feuilletage caractéristique d'une hypersurface singulière comme un feuilletage défini sur le lieu lisse de l'hypersurface. Au **Chapitre 5** nous donnons les exemples connus des hypersurfaces singulières tel que la dimension de la fermeture de Zariski d'une feuille générale du feuilletage caractéristique est plus petite que la conjecture de Campana prédit pour une hypersurface lisse. En particulier, nous exhibons les exemples de la hypersurfaces verticales tel que la fermeture d'une feuille générale est une sous-variété propre de la fibre de la fibration lagrangienne. Après, nous étudions la variété des droites sur une hypersurface cubique de \mathbb{P}^5 . Dans [8] Beauville et Donagi ont montré que cette variété des droites est un variété hyper-kählérienne de dimension 4. Pour une hypersurface cubique très générale le groupe de Picard de la variété des droites est engendré par un diviseur ample. Nous décrivons deux exemples d'hypersurfaces singulières dont la classe de cohomologie est proportionnelle à ce diviseur. Dans le premier exemple la fermeture de Zariski d'une feuille générale du feuilletage caractéristique est une surface lagrangienne. Dans le deuxième exemple la fermeture de Zariski d'une feuille générale du feuilletage caractéristique est une courbe rationnelle.

Au **Chapitre 6** nous étudions la variété hyperkählerienne de dimension 4 construite par O. Debarre et C. Voisin dans [22]. Cette variété est similaire à la variété des droites dans une hypersurface cubique dans \mathbb{P}^5 . En général le groupe de Picard de cette variété est engendré par un diviseur ample. Il y a une variété de dimension 6 liée au la variété de Debarre et Voisin qu'on appelle la variété de Peskine. Le variété des droites dans la variété de Peskine est de dimension 6 et elle est fibrée par des surfaces cubiques sur la variété de Debarre et Voisin ([9, théorème 2.20]). Nous trouvons une sous-variété de dimension 3 (probablement singulière) des droites spéciales dans la variété de Peskine qui nous donne une hypersurface dans un variété de Debarre et Voisin très générale. Après, nous construisons un feuilletage naturel de rang un sur cette hypersurface. Nous conjecturons que ce feuilletage est le feuilletage caractéristique.

Notation

In the current work we study projective varieties over the field of complex numbers \mathbb{C} . Let X be such a variety, then we denote by:

- \mathcal{O}_X the structure sheaf of X ;
- Ω_X the sheaf of the Kähler differentials on X and by Ω_X^n we denote its n -th exterior power $\Lambda^n \Omega_X$;
- T_X the tangent bundle of X ;
- ω_X the canonical sheaf on X , i.e. $\omega_X := \Omega_X^{\dim X}$;
- K_X the canonical class of X ;
- $X^{(n)}$ the n -th symmetric power of X , i.e. $X^{(n)} := X^n / \mathfrak{S}_n$. Additionally, let (x_1, x_2, \dots, x_n) be a point in X^n , we denote its image in $X^{(n)}$ by $x_1 + x_2 + \dots + x_n$;
- $X^{[n]}$ the Hilbert scheme of n points on X ;
- $[Z]$ the point of $Hilb X$ corresponding to a closed subscheme Z of X ;
- $HC : X^{[n]} \rightarrow X^{(n)}$ the Hilbert-Chow morphism;
- Assume X is an abelian variety we denote by $X[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2 \dim X}$ the n -torsion subgroup of X .

Let V be a finite-dimensional vector space of dimension n , then we denote by:

- $\mathbb{P}(V)$ the variety parametrizing lines in V (i.e. we use the "classical notation");
- $Gr(k, V)$ the variety parametrizing k -dimensional subspaces of V .

We also use the "classical notation" for the projectivization of a vector bundle. Let X be a variety and \mathcal{E} be a vector bundle on X . Then we denote by $\mathbb{P}_X(\mathcal{E})$ the projective variety $\text{Proj}(\text{Sym}^* \mathcal{E}^*)$, where $\text{Sym}^*(\mathcal{E}^*)$ is the sheaf of the symmetric algebras of \mathcal{E}^* .

2 Preliminaries

The preliminaries chapter introduces chapters 3 and 4, which we would like to be essentially self-contained. The chapters 5 and 6 are more technical. We don't give all the necessary preliminaries to them, but provide the reader with the references.

2.1 Hodge structures

In this section we briefly recall some definitions and results of the Hodge theory. Let R be a subring of \mathbb{R} (usually \mathbb{Z} , \mathbb{Q} or \mathbb{R}). Let A_R be a finitely generated R -module. Denote by $A_{\mathbb{C}}$ the tensor product $A_R \otimes \mathbb{C}$. For $a \in A_{\mathbb{C}}$ we denote by $a \mapsto \bar{a}$ the complex conjugation on $A_{\mathbb{C}}$.

Definition 2.1. One can define pure R -Hodge structure of the weight k on A_R (one says integral, rational or real instead of "R-") in two equivalent ways.

The first is to take a decreasing filtration $0 = F^k A \subset F^{k-1} A \subset \dots \subset F^0 A = A_{\mathbb{C}}$, satisfying the following properties:

- For any $p + q = k + 1$, $F^p A_{\mathbb{C}} \cap \overline{F^q A_{\mathbb{C}}} = 0$, where $\overline{F^q A_{\mathbb{C}}}$ is the image of $F^q A_{\mathbb{C}}$ under the complex conjugation.
- For any $p + q = k + 1$, $F^p A_{\mathbb{C}} + \overline{F^q A_{\mathbb{C}}} = A_{\mathbb{C}}$.

The second is to define a decomposition $A_{\mathbb{C}} = A^{k,0} \oplus A^{k-1,1} \oplus \dots \oplus A^{0,k}$, such that $A^{p,q} = \overline{A^{q,p}}$ for any $p + q = k$.

These definitions are equivalent. In order to obtain the decomposition from the filtration we take $A^{p,q} := F^p A_{\mathbb{C}} \cap \overline{F^q A_{\mathbb{C}}}$ for $p + q = k$. Conversely, to construct the filtration from the decomposition we take $F^p A = \bigoplus_{i \geq p} A^{i,k-i}$.

Definition 2.2. Let A be a pure Hodge structure of an even weight $k = 2m$. The vector $\alpha \in A_R$ is called **of the Hodge type** if $\alpha \in A^{m,m}$. In other words, the space of the Hodge vectors $Hdg(A) := A_R \cap A^{m,m}$.

Example 2.3. For a real Hodge structure A , we have an equality $Hdg(A) = A^{m,m}$.

Example 2.4. Take $k = 2m$ and $A_R = R$. A Hodge structure with $A^{p,q} = 0$ for $p \neq q$ and $A^{m,m} = R$, denoted by $R(-m)$ and called **the Hodge-Tate**.

Definition 2.5. Let A and B be pure R -Hodge structures of weight k . Then a morphism $f : A \rightarrow B$ of the Hodge structures is a R -linear map $A_R \rightarrow B_R$ such that its complexification $f_{\mathbb{C}}$ maps $A^{p,q}$ to $B^{p,q}$ for any $p, q = k$.

Definition 2.6. The tensor product of two R -Hodge structures A and B of weights k and m respectively is $A_R \otimes B_R$ with Hodge structure $(A \otimes B)^{p,k+m-p} = \bigoplus_{p_1+p_2=p} A^{p_1,k-p_1} \otimes B^{p_2,m-p_2}$ of weight $k + m$.

Definition 2.7. Let A be a R -Hodge structure A of weight k and let $\phi : A \otimes A \rightarrow R(-k)$ be a morphism of the Hodge structures. Then $\phi(A^{p,q} \otimes A^{p',q'}) = 0$ if $p \neq q'$ and $q \neq p'$. The form ϕ is a polarization of the Hodge structure A if it satisfies the following properties.

- For any $a, b \in A_{\mathbb{C}}$, $\phi(a, b) = (-1)^k \phi(b, a)$. Which means that ϕ is symmetric for an even k and alternating for an odd k .
- For any non-zero vector a from a component $A^{p,q}$, $i^{p-q} \phi(a, \bar{a}) > 0$.

Theorem 2.8. *Let (X, ω) be a compact Kähler manifold of dimension n , then:*

- *There is an integral, rational and real Hodge structure $H^{p,q}(X)$ on $H^k(X, \mathbb{Z})$, $H^k(X, \mathbb{Q})$ and $H^k(X, \mathbb{R})$ respectively;*
- *$H^{p,q}(X) \cong H^p(X, \Omega^q)$;*
- *$\cup \omega^{n-k} : H^k(X, \mathbb{R}) \otimes H^k(X, \mathbb{R}) \rightarrow H^{2n}(X, \mathbb{R}) \cong \mathbb{R}$ is a polarization of \mathbb{R} -Hodge structure on $H^k(X, \mathbb{R})$. Moreover, if X is projective and ω is an ample class, then $\cup \omega^{n-k}$ is a polarization of the integral Hodge structure on $H^k(X, \mathbb{Z})$.*

Proof. See for example [66]. □

Theorem 2.9. *Let (X, ω) be a compact Kähler manifold of dimension n . We define the group $NS(X)$ as the lattice of the Hodge vectors in the second integral cohomologies of X . A class of α is contained in $NS(X)$ if and only if there is a line bundle L on X such that $c_1(L) = \alpha$.*

We end this section introducing the most important type of the Hodge structures for this thesis.

Definition 2.10. A Hodge structure A with a polarization q is of **K3-type** if

$$A^{2,0} = A^{0,2} = \mathbb{C}.$$

The space $A^{1,1}$ is the orthogonal complement under the form q to $A^{2,0}$ and $A^{0,2}$ in $A_{\mathbb{C}}$. Hence, $Hdg(A)$ is the orthogonal complement of $A^{2,0}$ in A_R . If $R = K$ is a field we define the **transcendental Hodge substructure** $T(A)$ of A as the minimal Hodge substructure of A containing $A^{2,0}$. If R is not a field we can't choose a minimal Hodge substructure, because we always can multiply it by a non-invertible element of R . So, we define $T(A)$ as the minimal Hodge substructure of A containing $A^{2,0}$ such that $T(A)_R$ is not divisible in A_R by a non-invertible element of R .

Lemma 2.11. *The Hodge structures $T(A)$ and $Hdg(A)$ are related in the following way:*

1. $T(A) \cap Hdg(A) = 0$;
2. $T(A)_{\mathbb{C}} + Hdg(A)_{\mathbb{C}} = A_{\mathbb{C}}$;
3. $T(A)$ and $Hdg(A)$ are the orthogonal complements of each other.

Proof. To prove this lemma we can assume that R is a field. The first statement follows from the minimality of $T(A)$. The quotient of A by $T(A) + Hdg(A)$ is a Hodge structure Q with $Q^{2,0} = 0$. Since R is a field, $A = T(A) \oplus Hdg(A) \oplus Q$. Hence Q contains only the Hodge vectors, but they must lie in $Hdg(A)$. Hence, $Q = 0$.

For any Hodge class a of A , the morphism $q(a, *) : A \rightarrow R(-2)$ defined by

$$\alpha \mapsto q(a, \alpha)$$

is a morphism of the Hodge structures. Thus its kernel is a Hodge substructure of A . The intersection of these kernels with respect to elements of $Hdg(A)$ is a Hodge structure containing $A^{2,0}$. Hence, the orthogonal complement to $Hdg(A)$ contains $T(A)$. But if it contains a non-zero Hodge element, the polarization q is degenerate. Hence $Hdg(A)^\perp = T(A)$.

So, orthogonal complement of $T(A)$ contains $Hdg(A)$. But if it contains an element of $T(A)$ the polarization q is again degenerate. \square

2.2 Holomorphic symplectic manifolds

We start with the definition of our main object of study, that is irreducible holomorphic symplectic manifolds.

Definition 2.12. Let X be a smooth projective variety over \mathbb{C} . It is an irreducible holomorphic symplectic (IHS) if it satisfies the following properties:

- $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$, where σ is a holomorphic symplectic form (at any point of X);
- $H^1(X, \mathcal{O}_X) = 0$;
- $\pi_1(X) = 0$.

Since σ is symplectic, X is even-dimensional. From now on we assume that $\dim X = 2n$.

Remark 2.13. Since σ is symplectic at every point, its determinant $\wedge^n \sigma$ is a nowhere vanishing section of $\omega_X = \Omega_X^{2n}$. Thus $\omega_X \cong \mathcal{O}_X$.

The first example of a holomorphic symplectic manifold is a $K3$ surface.

Definition 2.14. Let S be a smooth projective surface. If S satisfies the two following properties we call it $K3$:

1. $\omega_S \cong \mathcal{O}_S$;
2. $H^1(S, \mathcal{O}_S) = 0$ (In classical algebraic geometry such surfaces are called *regular*).

Since $n = 1$, $\omega_S = \Omega_S^2$. The only section of ω_S up to proportionality is a holomorphic symplectic form. Such surfaces are always simply connected but in higher dimension this statement is not always true.

Proposition 2.15. *Let S be a K3–surface. The topological Euler characteristic of S is equal to 24. In particular, the Hodge diamond of a K3–surface is the following:*

$$\begin{array}{ccccc} & & & & 1 \\ & & & & 0 & 0 \\ & & & 1 & 20 & 1 \\ & & & 0 & 0 & \\ & & & & & 1 \end{array}$$

Proof. The Noether formula [4, Chapter I, Theorem 5.5] yields:

$$12 \cdot \chi(S, \mathcal{O}_S) = K_S^2 + \chi_{top}(S).$$

Since the canonical bundle of S is trivial and $H^1(S, \mathcal{O}_S) = 0$, we obtain $\chi_{top}(S) = 24$. Using Hodge’s and Serre’s dualities we find the Hodge diamond. \square

Theorem 2.16 (Kodaira-Enriques classification). *A smooth projective surface with trivial canonical bundle is either a K3–surface or an abelian surface.*

Proof. See [4, Chapter VI, Theorem 1.1]. \square

Let us give some examples of K3–surfaces.

Example 2.17 (Quartic surface).

Consider a smooth surface S of degree 4 in the projective space \mathbb{P}^3 . By the adjunction formula $\omega_S \cong \mathcal{O}_S(4) \otimes \omega_{\mathbb{P}^3} \cong \mathcal{O}_S$. Notice that this surface is not an abelian variety. Indeed, considering the exact sequence of the sheaves

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_S \rightarrow 0,$$

one can see that $H^1(S, \mathcal{O}_S) = 0$ (for an abelian surface this space has dimension 2). Here is a similar example of a K3–surface which we will use later.

Example 2.18 (Double cover of the plane ramified in a sextic curve).

Let $R \subset \mathbb{P}^2$ be a smooth sextic curve in the projective plane. There exists a double cover $p : S \rightarrow \mathbb{P}^2$ of the plane with a branch locus equal to R .

Proof. Let $\mathbb{P}(y : x_1 : x_2 : x_3)$ be the weighted projective space, where y has weight 3 and x_1, x_2 and x_3 have weight 1. Let $p(x_1, x_2, x_3)$ be a homogeneous polynomial of degree 6. Assume its zero locus C is smooth. The surface S is zero set of the polynomial $y^2 = p(x_1, x_2, x_3)$. Let π be the projection of S from the point $x_1 = x_2 = x_3 = 0$. Clearly it is double covering of projective plane ramified at a smooth sextic curve. One can check that S is smooth if C is smooth. Riemann-Hurwitz formula says:

1. $K_S = \pi^*K_{\mathbb{P}^2} + R$, where R is the set theoretical preimage of C .

$$2. \chi_{top}(S) = 2 \cdot (\chi_{top} \mathbb{P}^n - \chi_{top}(C)) + \chi_{top}(C).$$

Consider the first one. Applying π_* , we obtain that $K_S = 0$:

$$K_S = \pi.\pi^*K_{\mathbb{P}^2} + C = -6H + 6H = 0.$$

The second formula gives that $\chi_{top}(S) = 24$. Indeed, $\chi_{top}(\mathbb{P}^2) = 3$ and $\chi_{top}(C) = -18$ (by the adjunction formula $g(C) = \frac{(6-1)(6-2)}{2} = 10$).

$$\chi_{top}(S) = 2 \cdot (3 + 18) - 18 = 24.$$

Again S is either $K3$ or abelian. Euler characteristic of an abelian surface is zero. Thus, S is a $K3$ surface. \square

Let us recall another important construction of a $K3$ -surface introduced by Ernst Kummer.

Example 2.19 (Kummer surface).

We start with an abelian surface A . Consider the quotient surface S of A by the multiplication by -1 . The involution $-Id$ acts freely on the open subset $A \setminus A[2]$ (where $A[2]$ is the group of points of A of order 2). The images of sixteen two-torsion points of A are singular. Fortunately, such singularities are not complicated. Clearly, the involution $-Id$ acts on the tangent space to A at a point of $A[2]$ as the matrix

$$-E := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

such quotient singularities are called A_1 (for details look for example at [41][Chapter 7]). Analytically locally they look like the point $(0,0)$ in $\mathbb{C}^2 / \langle -E \rangle = Spec \mathbb{C}[x, y]^{\langle -E \rangle}$. We can generate the invariant algebra $\mathbb{C}[x, y]^{\langle -E \rangle}$ by the monomials x^2 , y^2 and xy with the relation $x^2y^2 = (xy)^2$. Thus, our quotient singularity is isomorphic to the singular point $(0,0,0)$ at the quadratic cone $Spec \mathbb{C}[u, t, w]/u^2 - tw$. Let us study these singularities.

Lemma 2.20. *Let S be a projective surface with a singular point x of A_1 -type. Then the minimal resolution $\pi : \tilde{S} \rightarrow S$ of the singularity x is the blow-up of S at the point x . Moreover,*

- $\pi^*K_S = K_{\tilde{S}}$ (such a singularity is called *crepant* or *du Val*)
- *The exceptional locus of π is a smooth, rational curve with square -2 .*

Proof. Since the question is local, we may assume that S is a quadratic cone in \mathbb{P}^3 and x is the vertex. Then \tilde{S} is the Hirzebruch surface F_2 [4][Chapter V.4]. In order to construct the morphism $\pi : \tilde{S} \rightarrow S$, we study the divisors on \tilde{S} . The Picard group of \tilde{S} is generated by the class of the fiber f and by the class of the exceptional section Σ . The canonical class of \tilde{S} is $-2\Sigma - 4f$. The intersection numbers are the following:

$$f^2 = 0, \quad \Sigma^2 = -2, \quad f \cdot \Sigma = 1.$$

The divisor $H := 2f + \Sigma$ provides us with a morphism to \mathbb{P}^3 . Since $H^2 = 2$, the image of this morphism is a quadric. One can see, that $H \cdot \Sigma = 0$. Hence, this morphism contracts the section Σ and the image is an irreducible singular quadric i.e. quadric cone. So, we constructed the morphism π . Now we prove, that it is the blow up. By the universal property of blow up, there is a birational morphism from \tilde{S} to $\text{Bl}_x S$. Because of the equality of the Picard numbers it is an isomorphism. It remains to see the last two statements. They follow easily from the equalities $\Sigma^2 = -2$ and $\Sigma \cdot K_{\tilde{S}} = 0$. \square

$K3$ -surfaces have been studied by many authors, see for example [36]. To construct some irreducible holomorphic symplectic varieties in a higher dimension we use Hilbert schemes of points on surfaces. So, in the current section we briefly recall the definition and some properties of the Hilbert scheme of points on surfaces. For a detailed study we refer to [56]. Start with the definition of the Hilbert scheme for an arbitrary polynomial.

Theorem 2.21 (Grothendieck). *Let X be a projective scheme of finite type over \mathbb{C} with a very ample line bundle $\mathcal{O}_X(1)$ and let P be a polynomial. There is a contravariant functor \mathfrak{Hilb}_X^P from the category of all locally noetherian schemes over \mathbb{C} to the category of sets, which associates to a scheme S the set of all such families:*

$$\mathfrak{Hilb}_X^P(S) = \{Y \subset X \times S \mid Y \text{ is flat with the Hilbert polynomial } P \text{ over } S\}.$$

This functor is representable by a projective scheme of finite type Hilb_X^P .

In this section we are interested in the case of a constant polynomial. One can remark that for a constant polynomial n , the Hilbert scheme does not depend on the choice of a very ample divisor on X . We call this Hilbert scheme $X^{[n]} := \text{Hilb}_X^n$ the Hilbert scheme of n points on X , because a general point of $X^{[n]}$ corresponds to a n -tuple of different points on X . Strictly speaking, there is the so-called Hilbert-Chow map from $X^{[n]}$ to the n -th symmetric power $X^{(n)} := X^n / \mathfrak{S}_n$, which maps a point corresponding to a finite subscheme Z to a cycle of dimension zero in the following way:

$$HC : Z \mapsto \sum_{x \in X} \text{length } \mathcal{O}_{Z,x} \cdot [x],$$

where $\mathcal{O}_{Z,x}$ is the local ring of \mathcal{O}_Z at a point x . Next, we recall some more specific results on $X^{[n]}$.

Theorem 2.22. *Let X be a connected variety, then:*

1. $X^{[n]}$ is also a connected variety;
2. The Hilbert-Chow map is an isomorphism on the open subset corresponding to reduced finite subschemes;
3. The Zariski tangent space to $X^{[n]}$ at the point corresponding to a subscheme Z is $\text{Hom}_{\mathcal{O}_X}(\mathcal{J}_Z, \mathcal{O}_Z)$, where \mathcal{J}_Z is the sheaf of ideals of Z .

Now we are ready to study the Hilbert schemes of points on surfaces. First, we prove that they are smooth.

Theorem 2.23 ([25]). *Let S be a smooth surface, then $S^{[n]}$ is a smooth variety. In particular, $S^{[n]}$ is a resolution of singularities of $S^{(n)}$.*

Proof. Let $[Z] \in S^{[n]}$ be a point corresponding to a closed subscheme Z of S , defined by the sheaf of ideals \mathcal{J}_Z . The Zariski tangent to $S^{[n]}$ at $[Z]$ is $\text{Hom}_{\mathcal{O}_S}(\mathcal{J}_Z, \mathcal{O}_Z)$. Furthermore, we can represent it as $\prod_i \text{Hom}_{\mathcal{O}_S}(\mathcal{J}_{Z_i}, \mathcal{O}_{Z_i})$, where Z_i is a local Artinian scheme of length n_i with support s_i . Thus, it is enough to check that $\dim \text{Hom}_{\mathcal{O}_S}(\mathcal{J}_{Z_i}, \mathcal{O}_{Z_i}) = 2n_i$. By the exact sequence $0 \rightarrow \mathcal{J}_{Z_i} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{Z_i} \rightarrow 0$, we obtain an isomorphism $\text{Hom}_{\mathcal{O}_S}(\mathcal{J}_{Z_i}, \mathcal{O}_{Z_i}) \cong \text{Ext}^1(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})$. By the Serre duality $\text{Ext}^2(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})^* \cong \text{Hom}(\mathcal{O}_S, \mathcal{O}_{Z_i} \otimes \omega_S) = \text{Hom}(\mathcal{O}_S, \mathcal{O}_{Z_i}) \cong \mathbb{C}^n$. The sheaf $\text{Ext}^k(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})$ for any integer k has support at s_i . Thus by the local-to-global Ext spectral sequence, $\text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) = H^0(S, \text{Ext}^k(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})) = \text{Ext}_{\mathcal{O}_{S, s_i}}^k(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})$, where \mathcal{O}_{S, s_i} is the local ring of S at s_i . It is easy to see that $\text{Hom}_{\mathcal{O}_{S, s_i}}(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) \cong \mathbb{C}^n$. Since \mathcal{O}_{S, s_i} is a local regular ring of dimension 2, the ring \mathcal{O}_{Z_i} possesses a free resolution of the length 2: $0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{O}_{Z_i} \rightarrow 0$. Hence, the Euler characteristic $\dim \text{Hom}_{\mathcal{O}_{S, s_i}}(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) - \dim \text{Ext}_{\mathcal{O}_{S, s_i}}^1(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) + \dim \text{Ext}_{\mathcal{O}_{S, s_i}}^2(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) = 0$. Thus, $\text{Ext}_{\mathcal{O}_{S, s_i}}^1(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) \cong \mathbb{C}^{2n}$. \square

Below we will need the following technical theorem.

Theorem 2.24 ([6]). *Let S be a projective smooth surface, then:*

1. $(HC)^* K_{S^{(n)}} = K_{S^{[n]}}$;
2. $H^1(S^{[n]}, \mathcal{O}_{S^{[n]}}) \cong H^1(S^{(n)}, \mathcal{O}_{S^{(n)}}) \cong H^1(S^n, \mathcal{O}_{S^n})^{\mathfrak{S}_n}$;
3. $H^2(S^{[n]}, \mathbb{Q})$ is isomorphic to $H^2(S^n, \mathbb{Q})^{\mathfrak{S}_n} \oplus \mathbb{Q}(-1)$ as a rational Hodge structure. In particular $b_2(S^{[n]}) = b_2(S) + 1$.
4. Assume S has a holomorphic symplectic form σ , then the form $(HC)^*(\sigma, \sigma, \dots, \sigma)$ on $S^{[n]}$ is also holomorphic symplectic.

Proof. As we know the Hilbert-Chow map resolves the singularities of $S^{(n)}$. We state that this resolution of singularities has only one prime exceptional divisor. In order to prove it, we observe that $S^{(n)}$ is stratified by subsets $\Delta_{(n_1, n_2, \dots, n_k)}$ corresponding to the splittings $n_1 + n_2 + \dots + n_k = n$ of the number n (for example the diagonal is the closure of $\Delta_{(2, 1, 1, \dots, 1)}$) and the Hilbert-Chow map has the fibers over the points of this stratification. In particular, the fiber over a point of the stratification $\Delta_{(n_1, n_2, \dots, n_k)}$ has dimension $\sum_i n_i - 1 = n - k$ and $\Delta_{(n_1, n_2, \dots, n_k)}$ has dimension $2k$. Thus dimension of the preimage of $\Delta_{(n_1, n_2, \dots, n_k)}$ is equal to $n + k$. It is easy to see that $n + k = 2n - 1$ if and only if $k = n - 1$. Thus, we obtain that the only exceptional divisor of the Hilbert-Chow is the closure of the preimage of $\Delta_{(2, 1, 1, \dots, 1)}$, we denote it by Σ . We have an isomorphism $H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S^{(n)}, \mathbb{Z}) \oplus \mathbb{C}\Sigma$. Moreover, $H^2(S^{(n)}, \mathbb{Z}) \cong H^2(S^n, \mathbb{Z})^{\mathfrak{S}_n}$. So, we prove the third statement of theorem. Moreover, the singularities of $S^{(n)}$ along $\Delta_{(2, 1, 1, \dots, 1)}$ have the type A_1 . It is a crepant singularity. So, we have proved the first statement. To prove the second statement we observe $S^{(n)}$ is a normal variety. Hence, the fibers of the Hilbert-Chow map are connected. In other words, $(HC)_* \mathcal{O}_{S^{[n]}} \cong \mathcal{O}_{S^{(n)}}$.

Now we prove the last part of the theorem. Since S has a symplectic form, S^n has trivial canonical bundle. Note, that the quotient map $S^n \rightarrow S^{(n)}$ is unramified in codimension one. Hence, the canonical class of $S^{(n)}$ is torsion. Note that S^n possess a \mathfrak{S}_n -invariant symplectic form. Hence, the canonical bundle of $S^{(n)}$ has a non-zero section. The line bundle corresponding to a torsion class has a non-zero section if and only if it is trivial. By statement 1 of this theorem $\omega_{S^{[n]}} \cong \mathcal{O}_{S^{[n]}}$. Let σ be a symplectic form on S . The sum of the pull-backs of σ under all projections is a \mathfrak{S}_n -invariant symplectic form on the n -th Cartesian power of S . Hence, we have a symplectic form on $S^{(n)} \setminus \Delta$. This form extends to a rational two-form on $S^{[n]}$. Denote this form by ω . Its determinant is a rational volume form on $S^{[n]}$. Since ω is symplectic on $S^{(n)} \setminus \Delta$, the image of $\det(\omega)$ under the isomorphism of sheaves $\omega_{S^{[n]}} \cong \mathcal{O}_{S^{[n]}}$ is a rational function on $S^{[n]}$ with zeros and poles along Σ , but Σ is an irreducible divisor. Hence, it is a regular function, and $\det(\omega)$ is a regular non-degenerate volume form. Thus, ω is symplectic. \square

Proposition 2.25 ([6]). *Let S be a K3-surface, then $S^{[n]}$ is an irreducible holomorphic symplectic manifold.*

Proof. By the theorem 2.23, it is a smooth variety and by the theorem 2.24, it has a symplectic form. It remains to find, that $H^1(S^n, \mathcal{O}_{S^n}) = 0$ and $H^0(S^n, \Omega_{S^n}^2) = \mathbb{C}$. The n -th Cartesian power S is a smooth variety of dimension $2n$ with the natural action of the symmetric group \mathfrak{S}_n . The space of two-forms $H^0(S^n, \Omega_{S^n}^2)$ is isomorphic to $H^0(S, \Omega_S^2)^{\oplus n}$. It is easy to see, that this space is generated by the differential forms $p_i^* \sigma$, where σ is a symplectic form on S and p_i is the i -th projection of S^n to S . The only (up-to proportionality) \mathfrak{S}_n -invariant holomorphic 2-form is $p_1^* \sigma + p_2^* \sigma + \dots + p_n^* \sigma$. By the third statement of theorem 2.24 this form is the pull-back of the generator $\sigma_{S^{[n]}}$ of $H^0(S^{[n]}, \Omega_{S^{[n]}}^2)$. The n -th power $\sigma_{S^{[n]}}$ is a section of the canonical bundle of $S^{[n]}$. It follows from the first statement of theorem 2.24 canonical bundle of $S^{[n]}$ is trivial. Hence, the form $\sigma_{S^{[n]}}$ is non-degenerate. By the second statement of theorem 2.24, the $H^1(S^n, \mathcal{O}_{S^n})$ is zero, since $H^1(S, \mathcal{O}_S) = 0$. \square

Example 2.26 (Generalized Kummer variety). Let A be an abelian surface and let n be an integer greater than 2. The morphism of the abelian varieties

$$A^n \rightarrow A; (a_1, a_2, \dots, a_n) \mapsto [a_1 + a_2 + \dots + a_n]$$

clearly factors through the symmetric power of A . Denote the resulting morphism from $A^{(n)}$ to A by *sum*. Define morphism $\pi : A^{[n]} \rightarrow A$ as the composition of *sum* and the Hilbert-Chow map. Finally, we define the generalized Kummer variety $K_n(A)$ as the fiber of π over zero. One can see, that if $n = 2$ applying this construction we obtain the Kummer surface of A .

Theorem 2.27 (Beauville). *The generalized Kummer variety K_n is an irreducible holomorphic symplectic manifold.*

Proof. Applying theorem 2.24 we obtain the following isomorphisms of the rational Hodge structures

$$H^2(A^{[n]}, \mathbb{C}) \cong H^2(A^n, \mathbb{C})^{\mathfrak{S}_n} \oplus \mathbb{Q}(-1) \cdot [\Sigma] \cong H^2(A, \mathbb{C}) \oplus \mathbb{Q}(-1) \cdot \Sigma$$

and

$$H^1(A^{[n]}, \mathbb{C}) \cong H^1(A^n, \mathbb{C})^{\mathfrak{S}_n} \cong H^1(A, \mathbb{C}).$$

The fibration π is isotrivial. More precisely, there is the following commutative square:

$$\begin{array}{ccc} K_n(A) \times A & \longrightarrow & A^{[n]} \\ \downarrow & & \downarrow \pi \\ A & \xrightarrow{\cdot n} & A \end{array}$$

where the upper horizontal arrow is the translation of a subscheme of A by an element of A and the left vertical map is the projection. Thus, we obtain another isomorphism of the rational Hodge structures $H^0(A, R^i \pi_* \mathbb{C}) \cong H^i(K_n, \mathbb{C})$. Applying the Leray spectral sequence to π we obtain an exact sequence of the rational Hodge structures

$$0 \rightarrow H^1(A, \mathbb{C}) \rightarrow H^1(A^{[n]}, \mathbb{C}) \rightarrow H^1(K_n(A), \mathbb{C}) \rightarrow \ker(H^2(A, \mathbb{C}) \rightarrow H^2(A^{[n]}, \mathbb{C})) \rightarrow H^1(K_n(A), \mathbb{C})$$

Because of the different weights the morphism $H^1(K_n(A), \mathbb{C}) \rightarrow \ker(H^2(A, \mathbb{C}) \rightarrow H^2(A^{[n]}, \mathbb{C}))$ is zero. Since $H^1(A, \mathbb{C}) \cong H^1(A^{[n]}, \mathbb{C})$, the group of the first cohomologies of $K_n(A)$ is zero. Thus, we obtain the following short exact sequence:

$$0 \rightarrow H^2(A, \mathbb{C}) \rightarrow H^2(A^{[n]}, \mathbb{C}) \rightarrow H^2(K_n(A), \mathbb{C}) \rightarrow 0.$$

Hence $H^2(K_n(A), \mathbb{C}) \cong H^2(A, \mathbb{C}) \oplus \mathbb{C}\Sigma$. In particular the restriction of the symplectic form (σ, \dots, σ) to $K_n(A)$ generates the space $H^0(K_n(A), \Omega_{K_n(A)})$. Since the normal bundle of $K_n(A)$ to $A^{[n]}$ and the canonical bundle of $A^{[n]}$ are trivial, the canonical bundle of $K_n(A)$ is also trivial. Hence, $K_n(A)$ is an irreducible holomorphic symplectic manifold. \square

There are two other families of *IHS* manifolds discovered by Kieran O'Grady. The varieties in the first family have dimension 6 [58]. In order to construct this family K. O'Grady has considered the desingularisation of moduli space of sheaves with $rk = 2$, $c_1 = 0$ and $c_2 = 2$ on a Jacobian of a genus-2 curve. This variety has a morphism to the square of the Jacobian of this curve. The fiber over $(0, 0)$ is a holomorphic symplectic manifold of dimension 6. The varieties in the second family have dimension 10 [59]. These manifolds are obtained as desingularisation of moduli space of sheaves on a *K3* with $rk = 2$, $c_1 = 0$ and $c_2 = 4$. All known examples of the holomorphic symplectic manifolds are deformations of examples above. Let us write the list of them: a Hilbert scheme of a *K3*-surface, a generalized Kummer manifold, O'Grady 6 and O'Grady 10. We call these deformations $K^{[n]}$ -type, $Kum^{[n]}$ -type, O'Grady 6-type and O'Grady 10-type respectively. In the article [42] the authors find some evidence to expect that there exist no other *IHS* manifolds.

2.3 Cohomology of holomorphic symplectic manifolds

Let S be a smooth compact complex surface. There is a natural polarization of the Hodge structures $H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \rightarrow \mathbb{Z}(-2)$. This pairing is dual to the intersection map of homologies. For an arbitrary variety X of bigger dimension we have only Lefschetz pairing

$q(\alpha, \beta) = h^{\dim X - 2} \cup \alpha \cup \beta$, where h is an ample class in $H^2(X, \mathbb{Z})$. We will consider it in the next section. However, it has a drawback that it depends on the choice of h . So it is not natural. One of the wonderful properties of holomorphic symplectic manifolds is that they carry a natural quadratic form on the second cohomologies like a surface. Let us define it.

Definition 2.28 (Beauville-Bogomolov-Fujiki form. See [7] and [26]). Let X be a holomorphic symplectic manifold. There is a quadratic form q on $H^2(X, \mathbb{Z})$ defined in the following way for an element α of $H^2(X, \mathbb{Z})$:

$$q(\alpha) = \frac{n}{2} \int_X \alpha \wedge \alpha \wedge \sigma^{n-1} \wedge \bar{\sigma}^{n-1} - (1-n) \int_X \alpha \wedge \sigma^n \wedge \bar{\sigma}^{n-1} \cdot \int_X \alpha \wedge \sigma^{n-1} \wedge \bar{\sigma}^n.$$

We will abuse the notation and denote the corresponding bilinear form

$$q(\alpha, \beta) = \frac{1}{2}(q(\alpha + \beta) - q(\alpha) - q(\beta))$$

by the same letter q . This form has the following nice property.

Theorem 2.29 (Fujiki formula). *There is a constant c_X depending only on topology of X such that for any $\alpha \in H^2(X, \mathbb{C})$*

$$\alpha^{2n} = 1 \cdot 3 \cdot \dots \cdot (2n-1) \cdot c_X \cdot q(\alpha)^n.$$

If X is of $S^{[n]}$ -type or an O'Grady 10, $c_x = 1$. For $X = Kum^{[n]}$, $c_x = n+1$ and for O'Grady 6 $c_x = 4$. Ten-dimensional O'Grady's example is constructed from a $K3$ -surface and the six-dimensional one is constructed from an abelian surface. One can remember the invariants above by a heuristic rule: "If the deformation class of manifolds arises from a $K3$ -surface, then $c_X = 1$ and $c_X = n+1$ if it arises from an abelian surface."

A more general version of the Fujiki formula is as follows:

$$\prod_{i=1}^{2n} \alpha_i = c_X \sum_{C \in \text{Couples}_{2n}} q(C),$$

where Couples_{2n} is the set of partitions of $1, 2, \dots, 2n$ in pairs². For a partition C , $q(C)$ is the product of the values BBF pairing in all the pairs in C . For example, if $n = 2$ and $C = \{\{1, 3\}, \{2, 4\}\}$, $q(C) = q(\alpha_1, \alpha_3) \cdot q(\alpha_2, \alpha_4)$.

Theorem 2.30. *The BBF form:*

$$S^2 H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}(-2)$$

is a polarization of the integral Hodge structure $H^2(X, \mathbb{Z})$. So, $H^2(X, \mathbb{Z})$ is of $K3$ -type.

There is another important theorem about cohomologies of holomorphic symplectic manifolds proved in the Ph.D. thesis of Misha Verbitsky.

² $|C_{2n}| = \alpha^n = 1 \cdot 3 \cdot \dots \cdot (2n-1) = \frac{2n!}{2^n \cdot n!}$

Theorem 2.31. [63, Theorem 1.7] *Let X be a holomorphic symplectic manifold of dimension $2n$, then the cup-product map*

$$S^k H^2(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{Z})$$

is injective for all $k \leq n$.

Remark 2.32. This morphism is generally not surjective. For example, if X is a generalized Kummer variety, $b_4 = 199$ and b_2 is only 8. So dimension count shows that cup-product can't be surjective. However, sometimes this injective morphism is also surjective. For a Hilbert scheme of two points on a K3-surface, $b_2 = 23$ and $b_4 = 276$. Thus we have an isomorphism

$$S^2 H^2(X, \mathbb{Q}) \cong H^4(X, \mathbb{Q}).$$

2.4 Lefschetz's theorems

In our results below we will use some specific versions of the Lefschetz's theorem about hyperplane section and hard Lefschetz's theorem. Let us first recall the hard Lefschetz theorem.

Theorem 2.33. *Let X be a compact Kähler manifold of dimension m (for example projective) and let $\omega \in H^2(X, \mathbb{C})$ be a Kähler class (an ample class). Define the Lefschetz operator*

$$L : H^k(X, \mathbb{C}) \rightarrow H^{k+2}(X, \mathbb{C})$$

as the cup-product with the Kähler class ω . Then the operator

$$L^{m-k} : H^k(X, \mathbb{C}) \xrightarrow{\sim} H^{2m-k}(X, \mathbb{C})$$

is an isomorphism for every $k \leq m$.

Proof. See [65, Theorem 6.25] □

Next we recall the classical version of the Lefschetz hyperplane section theorem.

Theorem 2.34 (LHT). *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n . Take a smooth hyperplane section $Y := X \cap \mathbb{P}^{N-1}$. For any $0 \leq k < n - 1$ the restriction map*

$$H^k(X, \mathbb{Z}) \rightarrow H^k(Y, \mathbb{Z}).$$

is an isomorphism, and for $k = n - 1$ is injection.

The Lefschetz theorem about hyperplane section also gives some isomorphisms of homotopy groups.

Theorem 2.35 (LHT*). [28, Part II, Sect 1.1] *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n . Take a smooth hyperplane section $Y := X \cap \mathbb{P}^{N-1}$. For any $0 \leq k < n - 1$ the pushforward map induces isomorphism of the homotopy groups:*

$$\pi_k(Y, \mathbb{Z}) \xrightarrow{\sim} \pi_k(X, \mathbb{Z}).$$

In this thesis we shall use a consequence of the following general version of the LHT.

Theorem 2.36. [28, Part II Chapter 5.1] *Let M be a purely n -dimensional nonsingular connected algebraic variety. Let $f : M \rightarrow \mathbb{C}\mathbb{P}^N$ be a morphism and let $H \subset \mathbb{C}\mathbb{P}^N$ be a linear subspace of codimension c . Let H_δ be the δ -neighborhood of H with respect to some real analytic Riemannian metric. Define $\phi(k)$ to be dimension of the set of points $z \in \mathbb{C}\mathbb{P}^N \setminus H$ such that the fiber $f^{-1}(z)$ has dimension k . (If this set is empty define $\phi(k) = -\infty$.) If δ is sufficiently small, then the homomorphism induced by inclusion, $\pi_i(f^{-1}(H_\delta)) \rightarrow \pi_i(X)$ is an isomorphism for all $i < \hat{n}$ and is a surjection for $i = \hat{n}$, where*

$$\hat{n} = n - \sup_k (2k - n + \phi(k) + \min\langle \phi(k), c - 1 \rangle) - 1.$$

We will not be interested in the higher homotopy groups. What we shall need is the following corollary of the above theorem for π_1 .

Theorem 2.37. *Let $\Delta \subset \mathbb{P}^n$ ($n \geq 2$) be a divisor in the projective space \mathbb{P}^n and $U = \mathbb{P}^n \setminus \Delta$ its complement. Let D be a hypersurface in \mathbb{P}^n . Assume the following:*

1. D is smooth;
2. D intersects the smooth locus Δ^{sm} of the hypersurface Δ transversely;
3. The intersection of D and of the singular locus Δ^{sing} of the hypersurface has dimension less than $n - 2$.

Then the pushforward map of the fundamental groups $\pi_1(D \cap U) \rightarrow \pi_1(U)$ is surjective.

Proof. We put M equal to $\mathbb{P}^n \setminus \Delta$ and f equal to the composition of its embedding to \mathbb{P}^n and of the degree- d Veronese map, where d is degree of the hypersurface D . One can check that $\hat{n} = n - 1$. Since $\dim D \cap \Delta_{sing} < n - 2$, $D \cap \Delta_{sing}$ has complex codimension at least 2 in D and we can neglect it when computing the fundamental group of D . There is a homeomorphism between D_δ and the total space of the line bundle $\mathcal{N}_{D/\mathbb{P}^n}$. Since Δ is transversal to D , $D_\delta \cap \Delta$ is homeomorphic to $\mathcal{N}_{Y/\mathbb{P}^n}|_{Y \cap \Delta}$. The zero-section induces an isomorphism on the fundamental groups. \square

2.5 The Torelli theorem and the Period Map

The crucial fact of the theory of holomorphic symplectic manifolds is the Torelli theorem. To state it for an irreducible holomorphic symplectic manifold of an arbitrary dimension we need some preliminaries. Let us first formulate it for a $K3$ -surface.

Theorem 2.38 ([61] and [15]). *Let S and S' be $K3$ -surfaces such that $H^2(S, \mathbb{Z}) \cong H^2(S', \mathbb{Z})$ as polarized Hodge structures with respect to the intersection pairing. Then $S \cong S'$.*

Let us consider a holomorphic symplectic manifold of dimension at least four. We can replace the intersection pairing with BBF quadratic form. However, the analogous statement is wrong.

Proposition 2.39 ([21]). *There exist two non-isomorphic holomorphic symplectic manifolds X and X' of dimension four such that there is an isomorphism of polarized Hodge structures $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$, where the polarizations are given by the BBF forms.*

The global Torelli theorem was proved by M. Verbitsky in [64]. We also refer to [35] for another exposition of this result. For this thesis the local Torelli theorem is enough. Let X be an IHS manifold and $Def(X)$ the space of its local deformations. The manifold $Def(X)$ is smooth of dimension $b_2(X) - 2$ by [11]. Locally we can avoid the monodromy action. We mean, if we take a complex ball B in $Def(X)$ containing $[X]$. Let X' be a deformation of X from the ball B . The local system $H^2(X', \mathbb{Z})$ with the BBF-form is constant over B . Let $\sigma' \in H^2(X', \mathbb{C}) \cong H^2(X, \mathbb{C})$ be the symplectic form of this manifold X' , using this cohomology class we define the local period map $Per : Def(X) \rightarrow \mathbb{P}(H^2(X, \mathbb{C}))$.

$$Per : X' \mapsto \mathbb{C} \cdot \sigma'$$

It is easy to see from the properties of the polarization of a Hodge structure 2.7 that, $q(\sigma', \bar{\sigma}') > 0$ and $q(\sigma', \sigma') = 0$. Thus the image of Per is contained in the quadric $Q \subset \mathbb{P}(H^2(X, \mathbb{C}))$ defined by the BBF quadratic form. The Local Torelli theorem states that locally it is an isomorphism.

Theorem 2.40 (The Local Torelli theorem). *The period map*

$$Per : B \rightarrow Q \subset \mathbb{P}(H^2(X, \mathbb{C}))$$

is surjective and biholomorphic on a small open subset of Q .

Next, we describe a relation between a deformation X' and its image under the period map.

Example 2.41. The space of deformations of the Hilbert scheme of a $K3$ -surface has dimension 21, because its second Betti number is 23 (see Theorem 2.24).

Lemma 2.42. *Let α be a Hodge class in $H^2(X, \mathbb{Z})$. Then it remains Hodge class on a deformation $X' \in B$ if and only if the class σ' as above is orthogonal to α . We denote this subspace of B by B^α .*

Proof. It is an easy corollary of lemma 2.11. □

So the set of deformations of X preserving a divisor D is a codimension one submanifold of B . We give some examples of families of the IHS manifolds which are deformations of the second Hilbert scheme of a $K3$ -surface preserving a divisor H .

Example 2.43 ([8]).

Let $C \subset \mathbb{P}(V_6)$ be a smooth cubic fourfold corresponding to a cubic polynomial $p \in S^3V_6^*$. Define its Fano variety of lines $F(C) := \text{Hilb}_X^{1+t}$ as the Hilbert scheme of lines on C . The variety $F(C)$ is naturally embedded in $Gr(2, V)$. Moreover, $F(C)$ is the zero set of the section of the vector bundle $S^3\mathcal{U}^*$ (where, \mathcal{U} is the tautological subbundle on $Gr(2, V)$)

corresponding to the polynomial p . Using this fact one can show that the variety $F(C)$ is a holomorphic symplectic manifold. Moreover, for some special cubic polynomials p there exist a $K3$ -surface S and an isomorphism $F(Y) \cong S^{[2]}$. Hence, for any polynomial p the variety $F(C)$ is diffeomorphic to the Hilbert scheme of two point of a $K3$ -surface. Each member of this family of holomorphic symplectic manifolds $F(C)$ is equipped with a polarization H . This divisor H is the restriction of the Plücker hyperplane section σ_1 . The Schubert calculus gives that $H^4 = 108$ and hence $q(H) = 6$. This family has dimension 20 and for a very general member of this family $\text{NS}(F(C)) \cong \mathbb{Z}H$.

Example 2.44. [22]

Let V be a ten-dimensional vector space and let $\sigma \in \wedge^3 V^*$ be a general alternating three-form on V . Consider the variety $X \subset Gr(6, V)$ of the six-dimensional subspaces $U_6 \subset V$ such that σ restricts to U_6 as zero. Let \mathcal{U} be the tautological subbundle on $Gr(6, V)$. The variety X is the zero locus of the section of the vector bundle $\wedge^3 \mathcal{U}^*$ corresponding to the three-form σ . The variety X turns out to be an irreducible holomorphic symplectic manifold of dimension 4 diffeomorphic to the Hilbert scheme of two points on a $K3$ -surface. Let H be the restriction of the Plücker hyperplane section from $Gr(6, V)$. By the calculations in Macaulay2 the authors of [22] have obtained that $H^4 = 1452$ and hence $q(H) = 22$. Similarly to the previous example this family of irreducible holomorphic symplectic manifolds has dimension 20 and for a very general variety X we have $\text{NS}(X) \cong \mathbb{Z}H$.

2.6 Foliations and invariant subvarieties

In this section we recall some definitions related to foliations and preliminary results about the Zariski closure of the leaves. First, we define the foliations and leaves and also mention the Frobenius theorem.

Definition 2.45. Let X be a smooth variety. A (singular) foliation is a saturated subsheaf $F \subset T_X$ which is closed under the Lie bracket, i.e. $[F, F] \subset F$. The singularity locus $Sing(F)$ of F is the subset of X on which T_X/F is not locally free, and it has codimension at least 2 in X . A leaf of F is the maximal connected injectively immersed complex analytic submanifold $L \subset X \setminus Sing(F)$ such that $T_L = F|_L$.

A saturated subsheaf $F \subset T_X$ which does not necessarily satisfy the property $[F, F] \subset F$ is called a distribution. The property $[F, F] \subset F$ is needed for the existence of leaves.

Theorem 2.46 (Frobenius). *Let X be a smooth variety and $F \subset T_X$ a distribution on X . We say that $F \subset T_X$ is integrable if there exists a leaf (i.e. locally closed submanifold $L \subset X \setminus Sing(F)$ such that $T_L = F|_L$) through every point of $X \setminus Sing(F)$. The distribution F is integrable if and only if F is closed under the Lie bracket, i.e. $[F, F] \subset F$.*

Proof. See for example [66, Book I, Theorem 2.20]. □

Definition 2.47. If every leaf of a foliation is algebraic we call this foliation algebraically integrable.

In this paper we study only foliations of rank one. It follows from theorem 2.46 that all distributions of rank one are foliations (i.e. integrable).

Definition 2.48. Let Y be a closed smooth subvariety of X . One says it is invariant under the foliation F or F -invariant if T_Y contains $F|_Y$.

The Zariski closure of a leaf through a point x is the smallest invariant under F subvariety containing this point. We denote it by $\overline{Leaf}^{Zar}(x, F)$. We recall some results of Philippe Bonnet from the work [14]. He states them for an affine variety X . Nevertheless, these statements for an affine X obviously lead to the analogous statements for a projective variety. Thus, let us reformulate them for a projective variety X .

Theorem 2.49. [13, Theorem 1.3] *Let X be a projective variety with a foliation F . There is an integer m such that m is equal to the dimension of the Zariski closure of the leaf of F through a very general³ point $x \in X$. We call this integer m the dimension of the Zariski closure of a generic leaf. Moreover, the dimension of the Zariski closure of the leaf through every point $x \in X$ is not greater than m .*

Proposition 2.50. [13, Theorem 1.4] *Let X be a smooth projective variety of dimension n with a foliation F . Assume that the Zariski closure of a very general leaf of F has dimension $m < n$. Then there exists a rational map $X \dashrightarrow W$ with F -invariant fibers of dimension m and a very general fiber of this map is the Zariski closure of a leaf of F .*

In the work [44] the authors proved the following consequence of the Bogomolov-McQuillan theorem ([5]) and of the Reeb stability theorem. We formulate it for a foliation of rank one.

Theorem 2.51. [44, Theorem 2] *Let X be a smooth variety with a regular foliation $F \subset T_X$ of rank one. Assume that there is a curve $C \subset X$, such that $\deg F|_C > 0$. Then all leaves of F are rational curves.*

2.7 Characteristic foliation

Note that a holomorphic symplectic form σ on a smooth variety X induces an isomorphism between the vector bundles T_X and Ω_X . Indeed, one can map a vector field v to the differential form $\sigma(v, *)$.

Definition 2.52. Let Y be a hypersurface in X and Y^{sm} be smooth locus of Y . Consider the restriction of T_X to Y . The restriction of a symplectic form to any codimension one subspace has rank $2n - 2$ i.e. has one-dimensional kernel. The orthogonal complement of the bundle $T_{Y^{sm}}$ in $T_X|_{Y^{sm}}$ is a line subbundle F of $T_{Y^{sm}} \subset T_X|_{Y^{sm}}$. We call the rank one subbundle $F \subset T_{Y^{sm}}$ the **characteristic foliation**.

Assume Y is smooth. Since $Y = Y^{sm}$, F is a subbundle of T_Y . Furthermore, F is isomorphic to the conormal bundle $\mathcal{N}_{Y/X}$ (which is isomorphic to $\mathcal{O}_Y(-Y)$ by the adjunction formula). Indeed, consider the following short exact sequence:

$$0 \rightarrow T_Y \rightarrow T_X|_Y \rightarrow \mathcal{O}_Y(Y) \rightarrow 0.$$

³Outside of a countable union of proper closed subvarieties.

Applying the isomorphism $T_X \cong \Omega_X$, we obtain that $F \cong \mathcal{O}_Y(-Y)$.

We are going to apply the techniques of the previous section to the characteristic foliation. Our first question is whether it is algebraically integrable. Jun-Muk Hwang and Eckart Viehweg showed in [40] that if Y is of general type, then F is not algebraically integrable. In paper [2] Ekaterina Amerik and Frédéric Campana completed this result to the following.

Theorem 2.53. [2, Theorem 1.3] *Let Y be a smooth hypersurface in an irreducible holomorphic symplectic manifold X of dimension at least 4. Then the characteristic foliation on Y is algebraically integrable if and only if Y is uniruled, i.e. covered by rational curves.*

The next step is to ask what could be the dimension of the Zariski closure of a generic leaf of F . In dimension 4 the situation is understood thanks to Theorem 2.54.

Theorem 2.54 ([3]). *Let X be an irreducible holomorphic symplectic fourfold and let Y be an irreducible smooth hypersurface in X . Suppose that the characteristic foliation F on Y is not algebraically integrable, but there exists a meromorphic fibration on $p : Y \dashrightarrow C$ by surfaces invariant under F (see Definition 2.48). Then there exists a rational Lagrangian fibration $X \dashrightarrow B$ extending p . In particular, the Zariski closure of a generic leaf is an abelian surface.*

This leads to the following conjecture.

Conjecture 2.55. *Let Y be a smooth hypersurface in an irreducible holomorphic symplectic manifold X and let q be the Beauville-Bogomolov form on $H^2(X, \mathbb{Q})$. Then:*

1. *If $q(Y, Y) > 0$, a generic leaf of F is Zariski dense in Y ;*
2. *If $q(Y, Y) = 0$, the Zariski closure of a generic leaf of F is an abelian variety of dimension n ;*
3. *If $q(Y, Y) < 0$, F is algebraically integrable and Y is uniruled.*

Let us explain why this conjecture is plausible and formulate the main results of chapter 3 (theorem 2.56) and of chapter 4 (theorem 2.57).

Case of $q(Y, Y) < 0$. In this case the conjecture is easy to prove. By [14, Theorem 4.2 and Proposition 4.7] Y is uniruled, if $q(Y, Y) < 0$. There is a dominant rational map $f : Y \dashrightarrow W$, such that the fibers of f are rationally connected (see [16] and [46, Chapter IV.5]). Rationally connected varieties do not have non-zero holomorphic differential forms. Thus, the form $\sigma|_Y$ is the pull-back of some form $\omega \in H^0(W, \Omega_W^2)$ and for any point x in Y the relative tangent space $T_{Y/W, x}$ is the kernel of the form $\sigma|_Y$. The kernel of $\sigma|_Y$ has dimension one at every point. So, the rational map $f : Y \dashrightarrow W$ is a fibration in rational curves and these rational curves are the leaves of the foliation F .

Case of $q(Y, Y) = 0$. Conjecturally, X admits a rational Lagrangian fibration, and the hypersurface Y is the inverse image of a hypersurface of its base (this conjecture was proved for manifolds of K3 type in [5] and for manifolds of Kummer type [69], for O'Grady 6 type

in [53] and for O’Grady 10 type in [52]). Moreover a rational Lagrangian fibration can be replaced with a regular Lagrangian fibration, if we assume that the divisor Y is numerically effective. In chapter 3 we consider an irreducible holomorphic symplectic manifold X equipped with a regular Lagrangian fibration $\pi : X \rightarrow B$ and assume that the base B is smooth. By [38], this means $B \cong \mathbb{P}^n$. Hwang and Oguiso in [39] prove that the characteristic foliation on the discriminant hypersurface is algebraically integrable, but this hypersurface is very singular. Our result is as follows.

Theorem 2.56. *Let X be a projective irreducible holomorphic symplectic manifold and $\pi : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration. Consider a hypersurface D in \mathbb{P}^n such that its preimage Y is a smooth irreducible hypersurface in X . Then the closure of a generic leaf of the characteristic foliation on Y is a fiber of π (hence an abelian variety of dimension n).*

Case of $q(Y, Y) > 0$. Here we have the similar problems as in the previous case. Unfortunately, we did not manage to handle the case of the non-numerically effective hypersurfaces. For a non-nef divisor with positive BBF -square there exists a birational transformation of X making it nef. The main problem that the strict transform of this divisor can become singular (see section 4.4). However, we proved the conjecture 2.55 for nef hypersurface.

Theorem 2.57. *Let Y be a smooth nef and big hypersurface in an irreducible holomorphic symplectic manifold X . Then a generic leaf of the characteristic foliation of Y is Zariski dense in Y .*

In a conversation with Jorge Vitorio Pereira he proposed a solution of the case of a not nef hypersurface. This result is a nice application of the characteristic foliation.

Theorem 2.58. *Let X be an irreducible holomorphic symplectic manifold and Y a non numerically effective hypersurface in X . Then Y is either uniruled or singular.*

We retell this prove in the section 4.4.

3 The case of a hypersurface with zero square

The goal of this chapter is to prove conjecture 2.55 for a vertical divisor (i.e. theorem 2.56).

3.1 Lagrangian fibrations on irreducible holomorphic symplectic manifolds

In this section we define Lagrangian fibrations and recall some results on them.

Theorem 3.1 ([49]). *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and $\pi : X \rightarrow B$ be a regular morphism with connected fibers. Assume that B is a normal variety and $0 < \dim B < 2n$. Then:*

- B has dimension n ;
- Every fiber of π is a **Lagrangian** subvariety i.e. the restriction of σ is zero;

- Moreover, if a fiber is smooth, it is an abelian variety.

Definition 3.2. The morphism π as in the previous theorem is called a *Lagrangian fibration*.

Theorem 3.3 ([38]). *If B is smooth, then $B \cong \mathbb{P}^n$.*

The base B is conjectured to be always smooth. For $n = 2$ this conjecture was proved in [37]. The authors of [12] proved this conjecture for all n but for some specific type of singularities of B . In the present paper we will assume that this conjecture is true.

A Lagrangian subvariety is not always a fiber of a Lagrangian fibration.

Example 3.4. Let S be a $K3$ -surface and C be a smooth curve in S . Clearly, C is Lagrangian in S , but let us construct a more interesting example. Let Z be the strict transform of $C^{(n)}$ in the Hilbert scheme $S^{[n]}$ under the Hilbert-Chow map. The subvariety Z is Lagrangian. Indeed, the preimage of $\sigma_{S^{[n]}}$ in S^n is $p_1^*\sigma_S + \dots + p_n^*\sigma_S$, where p_i is the i -th projection of S^n to S . Since $\sigma_S|_C = 0$, the restriction of $p_1^*\sigma_S + \dots + p_n^*\sigma_S$ to $C^{(n)}$ is zero.

There is an interesting result about deformation of IHS manifolds with a Lagrangian subvariety.

Theorem 3.5 ([65]). *Let X be an IHS manifold and $Z \subset X$ be a Lagrangian subvariety of X . Let X' be a deformation of X and σ' be the corresponding class in $H^2(X, \mathbb{C})$ (we view the second cohomologies as a constant vector bundle, as in section 2.5). This deformation X' "preserves the Lagrangian subvariety" (i.e. X' contains a Lagrangian subvariety Z' which is a deformation of Z) if and only if $\sigma'|_Z = 0$.*

In example 3.4 the Lagrangian subvariety deforms in codimension 2. Keiji Oguiso in [60] observed the following consequence of this theorem.

Theorem 3.6 ([60]). *Let $\pi : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of a projective irreducible holomorphic symplectic manifold X and let X_b be a smooth fiber of π . Then $\text{rank im}(H^2(X, \mathbb{Q}) \rightarrow H^2(X_b, \mathbb{Q}))$ is one.*

Proof. In the case of a manifold X with a Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^n$ over the base \mathbb{P}^n , the deformation X' preserves Lagrangian fibration if and only if it preserves the class $c_1(\pi^* \mathcal{O}_{\mathbb{P}^n}(1))$ [50]. As we have seen in section 2.5 the class of a divisor deforms in codimension one. In the end, applying theorem 3.5 for X and a fiber of the Lagrangian fibration π we obtain that $\text{rank im}(H^2(X, \mathbb{Q}) \rightarrow H^2(X_b, \mathbb{Q})) = 1$. \square

It is well-known that a smooth fiber of a Lagrangian fibration is an abelian variety. Concerning singular fibers there is the following result.

Proposition 3.7 (Proposition 2.2,[39]). *Let $\Delta \subset \mathbb{P}^n$ be the set of points $b \in \mathbb{P}^n$ such that the fiber X_b is singular. Then:*

1. *The set Δ is a hypersurface in \mathbb{P}^n . We call it the **discriminant hypersurface**.*
2. *The normalization of a component of a general fiber of π over Δ is smooth*

3. The singular locus of a component of a general singular fiber is a disjoint union of $(n - 1)$ -dimensional complex tori.

Remark 3.8. In particular, there is a subvariety $X_1 \subset X$ of codimension 1 or 2, such that the morphism of sheaves $\pi^* \Omega_{\mathbb{P}^n} \rightarrow \Omega_X$ restricted to X_1 is not injective and $\pi(X_1)$ is the discriminant hypersurface $\Delta \subset \mathbb{P}^n$. Moreover, a fiber X_b over a point $b \in \Delta$ is singular along its intersection with X_1 and $\pi^{-1}\Delta$ is singular along X_1 .

Definition 3.9. Let Y be a hypersurface in X . If there exists a hypersurface D in the base of a Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^n$ such that $Y = \pi^{-1}(D)$, Y is called a **vertical** hypersurface.

Let us give the simplest example of a Lagrangian fibration.

Example 3.10. Let S be a $K3$ surface with an elliptic fibration $\pi : S \rightarrow \mathbb{P}^1$. This fibration induces a morphism

$$\pi^{(n)} : S^{(n)} \rightarrow \mathbb{P}^n; s_1 + s_2 + \dots + s_n \mapsto \pi(s_1) + \pi(s_2) + \dots + \pi(s_n),$$

where \mathbb{P}^n is considered as n -th symmetric power of \mathbb{P}^1 . Composing this morphism with the Hilbert-Chow map, we obtain a morphism with connected fibers from $S^{[n]}$ to \mathbb{P}^n . Thus, by Theorem 3.1 it is a Lagrangian fibration. Let $b_1, b_2, \dots, b_m \in \mathbb{P}^1$ be the points such that the fiber $\pi^{-1}(b_i)$ is singular. The discriminant locus of the fibration π is the union of the hyperplanes $H_i := b_i + x_1 + x_2 + \dots + x_{n-1}$ and of the hypersurface $\Delta_0 := 2x_1 + x_2 + \dots + x_{n-1}$, which is tangent to each H_i . In particular, for $n = 2$ the discriminant hypersurface is the union of the diagonal conic and of the lines tangent to this conic.

3.2 Monodromy of a vertical hypersurface

To prove theorem 2.56 we will need to know some information about the monodromy action on a smooth fiber of a smooth vertical hypersurface. Thus we generalize theorem 3.6 to a vertical hypersurface.

Theorem 3.11. *Let X be a projective irreducible holomorphic symplectic manifold and let $\pi : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration. Consider a hypersurface D in \mathbb{P}^n such that its preimage Y is a smooth irreducible hypersurface in X . Let $X_b \subset Y$ be a smooth fiber of π , then the morphism $H^2(Y, \mathbb{Q}) \rightarrow H^2(X_b, \mathbb{Q})$ has rank one.*

We prove this theorem after proving some lemmas. To study the monodromy action on the fibers one should throw away the singular fibers. Proposition 3.6 says that $\pi_1(\mathbb{P}^n \setminus \Delta)$ fixes only one-dimensional vector space in $H^2(X_b, \mathbb{Z})$. Consider a hypersurface $D \subset \mathbb{P}^n$ in the base of the Lagrangian fibration. If the pushforward of the fundamental group $\pi_1(\mathbb{P}^n \setminus \Delta) \rightarrow \pi_1(D \setminus \Delta \cap D)$ is surjective, then theorem 3.11 is clear. This morphism is surjective for a general D or more concretely D which is in a good position with respect to Δ . Applying theorem 2.37 to \mathbb{P}^N and Δ we obtain the following lemma.

Lemma 3.12. *Let D be a hypersurface in projective space \mathbb{P}^N . Assume the following:*

1. D is smooth;
2. D intersects the smooth locus Δ^{sm} of the discriminant hypersurface Δ transversely;
3. The intersection of D and of the singular locus Δ^{sing} of the discriminant hypersurface has dimension less than $n - 2$.

Then the homomorphism of the fundamental groups $\pi_1(D \setminus \Delta \cap D) \rightarrow \pi_1(\mathbb{P}^n \setminus \Delta)$ is surjective.

To apply Lemma 3.12 to our problem, we recall the following results.

Theorem 3.13. (Deligne's invariant cycle theorem, see e.g [66, Theorem 3.2]) *Let $X \rightarrow Y$ be a projective morphism of quasi-projective varieties. Then for any point $y \in Y$ and for any integer k the space of the invariants under the monodromy action $\pi_1(Y, y) \rightarrow \text{Aut } H^k(X_y, \mathbb{Q})$ is equal to the image of the restriction map $H^k(\bar{X}, \mathbb{Q}) \rightarrow H^k(X_y, \mathbb{Q})$, for any smooth projective compactification \bar{X} of X .*

Combining this theorem with theorem 3.6 and lemma 3.12, we obtain an immediate corollary.

Corollary 3.14. *The statement of theorem 3.11 becomes true if we add the assumption that D is smooth, dimension of $D \cap \Delta_{sing}$ is less than $n - 2$ and D intersects Δ^{sm} transversely.*

Remark 3.15. In Section 3.4 we show that the smoothness of Y implies that D is smooth and D intersects Δ^{sm} transversely. However, the condition that $\dim D \cap \Delta_{sing} < n - 2$ is not always satisfied even when Y is smooth.

Finally, the following Lemma makes the proof work for an arbitrary smooth vertical hypersurface.

Lemma 3.16. *Let X be a smooth projective variety and D_0, D_1 the linearly equivalent smooth hypersurfaces in X . Let Z be a smooth subvariety of X contained in $D_0 \cap D_1$, then $H^i(D_0, \mathbb{Q})$ and $H^i(D_1, \mathbb{Q})$ have the same images in $H^i(Z, \mathbb{Q})$, for any $i \in \mathbb{Z}$.*

Proof. Let $I := [0, 1] \hookrightarrow |D_0 - Z| := \mathbb{P}(H^0(X, \mathcal{O}_X(D_0) \otimes \mathcal{J}_Z))$ be a path between the points $[D_0]$ and $[D_1]$, avoiding the points corresponding to singular hypersurfaces in the linear system $|D_0 - Z|$. Let $\mathcal{D}_I \subset X \times I = \{(x, t) | x \in D_t\}$ and $Z_I = \{(x, t) | x \in Z\}$. Since all hypersurfaces D_t are smooth, \mathcal{D}_I is diffeomorphic to $D_0 \times I$ and Z_I is diffeomorphic to $Z \times I$. We have the following commutative squares for all $t \in I$:

$$\begin{array}{ccc} Z & \longrightarrow & Z_I \\ \downarrow & & \downarrow \\ D_t & \longrightarrow & D_I \end{array}$$

Since the horizontal maps induce an isomorphism on the cohomology groups, the images of the vertical maps are the same. \square

⁴Here X_y is the fiber $f^{-1}(y)$.

Lemma 3.16 implies Theorem 3.11: take $D_0 = Y$ an arbitrary smooth vertical hypersurface, $Z = X_b$ a smooth fiber of the Lagrangian fibration π in Y and $D_1 = Y'$ a smooth vertical hypersurface through X_b linearly equivalent to Y , such that Y' satisfies the conditions of Lemma 3.12. Note that a generic hypersurface from the linear system $|Y|$ is smooth by Kleiman's version of the Bertini theorem ([45]).

3.3 Proof of Theorem 2.56

In the last section we apply theorem 3.11 to prove theorem 2.56. Let us recall the second.

Theorem 2.56 Let X be a projective irreducible holomorphic symplectic manifold and $\pi : X \rightarrow \mathbb{P}^n$ a Lagrangian fibration. Consider a hypersurface D in \mathbb{P}^n such that its preimage Y is a smooth irreducible hypersurface in X . Then the closure of a generic leaf of the characteristic foliation on Y is a fiber of π (hence an abelian variety of dimension n).

Lemma 3.17. *Let $\pi : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration, let $Y := \pi^{-1}(D)$ be a smooth irreducible vertical hypersurface in X , where D is an irreducible hypersurface in B . Then every fiber of the fibration $\pi : Y \rightarrow D$ is invariant under the characteristic foliation F on Y .*

Proof. Consider a smooth fiber X_b of the fibration $\pi : Y \rightarrow D$ over a point $b \in D$. Let x be a point in X_b . The tangent space to X_b at the point x is the orthogonal complement of itself in $T_{X,x}$. Since $T_{D,x}$ contains $T_{X_b,x}$, the space $T_{X_b,x}$ contains the orthogonal complement of $T_{D,x}$ i.e. F_x . The singular fibers are invariant as well because of the closedness of this property. \square

Hence, the Zariski closure of every leaf of the characteristic foliation on a vertical hypersurface is of dimension at most n . Our purpose is to show that this closure is X_b (for very general b) and not a proper subvariety of X_b .

Proposition 3.18. *In the assumptions of the Lemma 3.17, let Z be an irreducible subvariety of a smooth fiber X_b , invariant under F . Fix a group law on X_b , such that Z contains the zero point. For any $a \in X_b$, the translate of Z by a point a is an invariant subvariety. In particular, if Z is a minimal invariant subvariety (i.e. the Zariski closure of a leaf), then it is an abelian variety.*

Proof. Since the tangent bundle to X_b is trivial, we may view the restriction of the characteristic foliation to X_b as a one-dimensional subspace of $H^0(X_b, T_{X_b})$. A translation acts trivially on $H^0(X_b, T_{X_b})$. Thus, we obtain the first statement. Let $a \in Z$ be a point of Z . The translation $Z + a$ is an invariant subvariety. The intersection of $Z + a$ and Z is a non-empty invariant subvariety. Because of the minimality of Z , $Z \cap (Z + a) = Z$. In other words, $Z = Z + a$. Hence, Z is an abelian subvariety of X_b . \square

Proposition 3.19. *Assume that the Zariski closure of a generic leaf of the characteristic foliation on Y has dimension less than n . Then for a general fiber X_b of π the image of the restriction map $H^2(X, \mathbb{Q}) \rightarrow H^2(X_b, \mathbb{Q})$ has rank at least 2. This contradicts Theorem 3.11.*

Proof. Consider a rational fibration $p : Y \dashrightarrow \Sigma_{gen}$, which was constructed in Proposition 2.50. By Proposition 3.18 its fibers are abelian subvarieties of fibers of π . Thus, $\Sigma_{gen} \dashrightarrow D$ is a fibration in abelian varieties (quotients of fibers of π). Let G be a relatively ample divisor on the fibration $\Sigma_{gen} \dashrightarrow D$ and let $p^\bullet G$ be the closure of its preimage in Y . The restriction of $p^\bullet G$ to a very general fiber is X_b is a pull-back of an ample divisor from its quotient. Hence, $p^\bullet G|_{X_b}$ is effective but not ample. Let H be an ample divisor on Y . The restrictions of G and H are not proportional in $H^2(X_b, \mathbb{Q})$. \square

3.4 Smoothness of vertical hypersurfaces

To conclude this chapter, we would like to discuss when a vertical hypersurface $Y = \pi^{-1}D$ is smooth. We show that the first and the second conditions on D of lemma 3.12 are necessary for Y to be smooth (see corollaries 3.21 and 3.22). However, the situation with the third condition is more complicated. It is natural to guess that the fibers over the singular points of the discriminant hypersurface are "more singular" (that is the fibers with the greater dimension of the Zariski tangent space at the singular points) than others. Unfortunately, this turns out to be false for some types of singularities of Δ . For simplicity, consider the case of $\dim X = 4$, then the base is the projective plane and the discriminant divisor Δ is a plane curve. We show that the fiber over a double point of Δ is "more singular" than the others (i.e. contains a point with Zariski tangent space of dimension 4). Thus any $D \subset \mathbb{P}^2$ passing through a double point of Δ has singular preimage Y in X (see lemma 3.23). However, as we show in example 3.24, the analogous statement is not true for a cusp of Δ . Most of the preimages of curves passing through a cusp point of Δ are smooth.

Lemma 3.20. *In the notations as above, let $\mathcal{O}_{\mathbb{P}^n, b}$ be the local ring of a point $b \in \mathbb{P}^n$ and D be the zero set of a regular function $f \in \mathcal{O}_{\mathbb{P}^n, b}$. The hypersurface $Y = \pi^{-1}D$ is singular at a point x in the fiber X_b if and only if the differential form $\pi^*(df)$ is zero at x .*

Proof. Write down the natural exact sequence:

$$\begin{array}{ccccccc} \mathbb{C} \cdot \pi^*(df) = \mathcal{O}_Y(-Y) & \longrightarrow & \Omega_X|_Y & \longrightarrow & \Omega_Y & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ \mathbb{C} \cdot \pi^*(df) = \pi^* \mathcal{O}_D(-D) & \longrightarrow & \pi^* \Omega_{\mathbb{P}^n}|_D & \longrightarrow & \pi^* \Omega_D & \longrightarrow & 0 \end{array}$$

Restricting this sequence to the point x , we see that the form $\pi^*(df)$ generates the vector space $\mathcal{N}_{Y/X}$. The hypersurface Y is singular at x if and only if its conormal bundle is not locally free at x . This condition is equivalent non-vanishing of $\pi^*(df)$ at x . That finishes the proof. \square

Corollary 3.21. *If D is singular, Y is also singular.*

Corollary 3.22. *Consider a regular point b of Δ and assume that a hypersurface D is tangent to Δ at b . Then Y is singular along $\text{sing}(X_b)$.*

Proof. Let $f \in \mathcal{O}_{\mathbb{P}^n, b}$ be the function defining Δ . By Lemma 3.20 $\pi^*(df)_x = 0$ for any $x \in \text{sing}(X_b)$. Since Δ and D are tangent at point b , D is also singular at these points. \square

Lemma 3.23. *Assume $Z \subset \Delta$ is a subvariety of Δ of codimension 1, such that one can choose the coordinates x_1, x_2, \dots, x_n at a local analytic neighborhood of a general point of Z in \mathbb{P}^n , such that $Z = \{x_1 = x_2 = 0\}$ and the hyperplanes $x_1 = 0$ and $x_2 = 0$ are contained in Δ . In the case $n = 2$, Z is a double point of Δ . Assume the hypersurface $D := \{f = 0\}$ contains Z , then $Y := \pi^{-1}D$ is singular.*

Proof. Let us continue to work in these analytic coordinates. Let X^i be the space of points where $\pi : X \rightarrow B$ is singular at $\{x_i = 0\}$, $i = 1, 2$. By remark 3.8, X^i has codimension not greater than one at any irreducible component of $\pi^{-1}Z$. Hence the scheme $X^1 \cap X^2 \cap \pi^{-1}Z$ is not empty. So, take some point P in this scheme. The differential forms $\pi^*(dx_1)$ and $\pi^*(dx_2)$ are zero at P . Since D contains Z , the differential form df is a linear combination of dx_1 and dx_2 . Thus, the form $\pi^*(df)$ is zero at the point P . Applying Lemma 3.20, we conclude that Y is singular at P . \square

Example 3.24. Let S be a $K3$ surface with an ample line bundle L . Assume $\text{Pic } S = \mathbb{Z}L$ and $c_1(L)^2 = 2$ (see example 2.18). Thus, the surface S is equipped with a finite morphism p of degree 2 to the projective plane $\mathbb{P}(V)$ ramified at a smooth sextic curve R , where V is the space dual to $H^0(S, L)$ ($\dim H^0(S, L) = 3$ by the Riemann-Roch theorem and the Kodaira vanishing). The projective plane $\mathbb{P}(V^*)$ is the linear system of the preimages of the lines on $\mathbb{P}(V)$. Consider the Beauville-Mukai system introduced in [7] and [54]. Let X be the moduli space of stable sheaves G on S such that $rk G = 0$, $c_1(G) = c_1(L)$ and $c_2(G) = 0$. It is an irreducible holomorphic symplectic manifold. A point of X is the parameter point of a sheaf $i_{C*}F$, where $C \subset S$ is a curve from the linear system $\mathbb{P}(V)$, i_C is the embedding of C to S and F is a reflexive rank one sheaf of degree zero at C . The morphism mapping a sheaf to its supporting curve is the Lagrangian fibration $\pi : X \rightarrow \mathbb{P}(V^*)$ in Jacobians of the curves from the linear system $\mathbb{P}(V^*)$. The fiber X_C of π is smooth iff C is smooth. Hence, the discriminant hypersurface of π is the dual curve R^* to R . One can compute that for a general sextic plane curve R , the dual curve R^* is a curve of the geometric genus 10 (the same as R), of degree $30 = \deg R(\deg R - 1)$ by the Plücker formula [23, Section 1.2.3]). Moreover, the dual curve has 324 double points and 72 cusps. Indeed, dimensional count shows that a general sextic curve does not have tritangent lines and tangent lines of order 4. Which means that the singularities of R^* are the double points and the cusps (a bitangent line to R corresponds to a double point of R and a line tangent to R at an inflection point corresponds to a cusp of R^*). Both of these singularities increase the arithmetic genus by one. The arithmetic genus of R^* is $\frac{(\deg(R^*)-1)(\deg(R^*)-2)}{2} = 406$. Thus R^* has 396 singular points. The number of the inflection points is the intersection of R with its Hessian curve $He(R)$ [23, Theorem 1.1.10]. The Hessian curve of R has degree 12. Hence, R has 72 inflections points and 324 bitangent lines.

There is a birational map:

$$f : S^{[2]} \dashrightarrow X; [Z] \mapsto i_{C*}(\mathcal{I}_Z) \otimes L$$

undefined along the subvariety $\{[Z] \in S^{[2]} \mid Z = p^{-1}P \text{ for } P \in \mathbb{P}(V)\}$ which is naturally isomorphic to the projective plane $\mathbb{P}(V)$. The indeterminacy locus of the inverse map is the zero section of the Jacobian fibration π . Moreover, $X \setminus \mathbb{P}(V^*) \cong S^{[2]} \setminus \mathbb{P}(V)$.

Since a fibration is smooth along a section, to find the singularities of π it is enough to study the rational map $\pi' := \pi \circ f$. The closure of the fiber of π' over a point of $\mathbb{P}(V^*)$ corresponding to a curve C from the linear system $\mathbb{P}(V)^*$ is $C^{[2]}$. A double point $[l]$ of R^* corresponds to a bitangent line l to R . Its preimage in S is a rational curve B with two double points P_1 and P_2 . The Zariski tangent space to $B^{[2]}$ at the point $P_1 + P_2$ has dimension 4. Thus, $\pi^* \Omega_{\mathbb{P}^n} \rightarrow \Omega_X$ is zero at the point $P_1 + P_2$. By Lemma 3.20, the preimage $Y := \pi^{-1}D$ in X of any hypersurface $D \subset \mathbb{P}(V^*)$ containing $[l]$ is singular. That is an illustration of Lemma 3.23.

Let $[l] \in \mathbb{P}(V^*)$ be a cusp of R^* . This point corresponds to the line l tangent to R at an inflection point $Q \in \mathbb{P}(V)$ of R . The preimage of the line l in S is a curve C of geometric genus 1 with a cusp $Q_S = p^{-1}Q$. By [48, Lemma 2.6], $C^{[2]}$ is singular at the points $Q_S + P$ for any point $P \in C$ different from Q_S and at the point $[Z]$ corresponding to the scheme Z "tangent" to the cusp. In the local ring \mathcal{O}_{C, Q_S} all the ideals I with $\dim \mathcal{O}_{C, Q_S} / I = 2$ are principal, except the ideal corresponding to the scheme Z . Moreover, the Zariski tangent space to $C^{[2]}$ at its singular points has dimension 3. Hence, the linear map $\Omega_{\mathbb{P}(V), [l]} \rightarrow \Omega_{X, x}$ is of rank one for any singular point x of $C^{[2]}$. One can compute that the kernel of this map is the conormal space $\mathcal{N}_{l_Q / \mathbb{P}(V^*), [l]}^*$ of the line $l_Q \subset \mathbb{P}(V^*)$ corresponding to the inflection point Q of R . Thus, by lemma 3.20 if a hypersurface $D \subset \mathbb{P}(V^*)$ through the point $[l]$ is not tangent to l_Q , then the hypersurface $Y := \pi^{-1}D$ is smooth along the fiber over $[l]$.

Conclusion: A smooth vertical hypersurface Y may be smooth even if D does not satisfy property 2 of lemma 3.12.

4 The case of a hypersurface with positive square

4.1 The case of an ample hypersurface

In this section we prove conjecture 2.55 for a smooth and ample hypersurface Y .

Theorem 4.1. *Let Y be a smooth and ample hypersurface in an irreducible holomorphic symplectic manifold X . Then a generic leaf of the characteristic foliation is Zariski dense in Y .*

In order to prove theorem 4.1 we assume the contrary. Let $Y \dashrightarrow B$ be a rational fibration such that its general fiber is invariant under the characteristic foliation F . Without loss of generality one can assume that B is the projective line. We replace $Y \dashrightarrow B$ by the composition $Y \dashrightarrow B \dashrightarrow \mathbb{P}^1$, where $B \dashrightarrow \mathbb{P}^1$ is a pencil of hypersurfaces in B . Let Z be a fiber of this rational fibration.

Definition 4.2. A subvariety Z of codimension k is called **coisotropic** if the restriction of the symplectic form σ to the tangent space to Z at a general point has the smallest possible rank $n - k$ (if the rank of the restriction is smaller, then the form σ is degenerate on T_X).

Lemma 4.3. *Let (X, σ) be an irreducible holomorphic symplectic manifold and Y a smooth hypersurface in X . Consider a (possibly singular) subvariety Z of codimension 2 in X contained in Y (i.e. a hypersurface in Y). The following statements are equivalent*⁵

1. *The variety Z is invariant under the characteristic foliation;*
2. *Z is coisotropic in (X, σ) (the restriction of σ to the smooth locus of Z has the least possible rank $n - 2$).*

Proof. Let z be a smooth point of Z . Consider the vector spaces $T_{Z,z} \subset T_{Y,z} \subset T_{X,z}$.
 \implies Since Z is F_Y -invariant, $T_{Z,z}$ contains $T_{Y,z}^\perp$. The line $T_{Y,z}^\perp$ is orthogonal to any vector of $T_{Z,z}$. Thus $\sigma|_{T_{Z,z}}$ is degenerate. Since $T_{Z,z}$ has codimension 2, it is coisotropic.
 \impliedby Since Z is coisotropic, $T_{Z,z}$ contains $T_{Y,z}^\perp$ and hence it contains $T_{Y,z}^\perp = F_{Y,z}$ ⁶. \square

Theorem 4.1 obviously follows from the next result.

Proposition 4.4. *Let Y be an ample smooth hypersurface in an irreducible holomorphic symplectic manifold X . Then Y contains no coisotropic subvariety of codimension 2 in X .*

In the rest of this section, we prove proposition 4.4. First we remark that being coisotropic is a cohomological property. In other words, a subvariety Z of codimension k (possibly singular) is coisotropic if and only if $[Z] \cup [\sigma^{n-k+1}] = 0 \in H^{2n+2}(X, \mathbb{C})$ (see [68, lemma 1.4] for the details). Now we use the ampleness of Y to apply the Lefschetz hyperplane theorem (LHT). It yields that there is a (not necessarily effective) divisor D , such that $[Z] = [D] \cdot [Y]$.

Lemma 4.5. *Let $\alpha, \beta \in \text{NS}(X)$. The class $\alpha \cup \beta$ is coisotropic if and only if $q(\alpha, \beta) = 0$.*

Proof. If Z is coisotropic $[Z] \cup [\sigma^{n-1}] = 0$ and hence $[Z] \cup [\sigma^{n-1}] \cup [\bar{\sigma}^{n-1}] = \alpha \cup \beta \cup [\sigma]^{n-1} \cup [\bar{\sigma}]^{n-1} = 0$. Let us look at definition 2.28. Since α and β of type $(1, 1)$, $q(\alpha, \beta)$ is proportional to $\alpha \cup \beta \cup [\sigma]^{n-1} \cup [\bar{\sigma}]^{n-1}$. Hence, if Z is coisotropic, then $q(\alpha, \beta) = 0$. \square

So, $[D]$ is *BBF*-orthogonal to $[Y]$. We show that it is impossible.

Lemma 4.6. *Let $\alpha, \beta \in \text{NS}(X)$ and $q(\beta, \beta) > 0$. Then the signs of $q(\alpha, \beta)q(\beta, \beta)^{n-1}$ and of $\alpha\beta^{2n-1}$ are the same.*

Proof. The Fujiki formula (see Theorem 2.29) says that there is a positive constant c such that for every $\gamma \in H^2(X, \mathbb{Z})$

$$q(\gamma, \gamma)^n = c\gamma^{2n}.$$

Let k be an integer. Applying the Fujiki formula for $k\beta + \alpha$ we obtain the equality of polynomials in k :

$$c(k\beta + \alpha)^{2n} = (k^2q(\beta) + 2q(\alpha, \beta)k + q(\alpha))^n. \quad (1)$$

This equality of polynomials gives us the equality of the coefficients of the term of degree $2n - 1$:

⁵It is easy to see that the implication $2 \Rightarrow 1$ is true for any codimension of Z .

⁶As one may notice, for Z of higher codimension the first implication is wrong but the second implication holds.

$$2nc(\alpha\beta^{2n-1}) = 2nq(\alpha, \beta)q(\beta, \beta)^{n-1}.$$

□

Corollary 4.7. *Let Y be an ample divisor and D a divisor such that $[Z] = [Y] \cup [D]$ is an effective divisor. Then $q(Y, D) > 0$. In particular, Z is not coisotropic.*

Proof. Take $\alpha = [D]$ and $\beta = [Y]$. Since Z is effective and Y is ample we have

$$[Y]^{2n-1} \cup [D] = [Y]^{2n-2} \cup [Z] > 0$$

and hence $q(D, Y) > 0$.

□

In the end of this section we make a conjecture generalizing proposition 4.4.

Conjecture 4.8. *Let X be an IHS manifold of dimension $2n$ and Y a smooth ample hypersurface in X . Then the hypersurface Y contains no coisotropic subvariety except itself.*

If we apply the arguments we used to prove proposition 4.4, we can decompose the class of a coisotropic subvariety Z as $[Z] = [Y] \cup \alpha$, where $\alpha \in H^{2\dim Z-2}(X, \mathbb{Z})$. For $\text{codim} Z > 2$ we meet the problem that the class α is not necessarily polynomial (i.e. is not contained in the image of the morphism $S^{\dim Z-1} H^2(X, \mathbb{Z}) \rightarrow H^{2\dim Z-2}(X, \mathbb{Z})$ defined in theorem 2.31). And we can not use the estimations with BBF form to this class α .

4.2 The LHT for a nef and big hypersurface in IHS

In the previous section we used the LHT for an ample hypersurface. In order to adapt this proof for a nef and big hypersurface we show that the LHT holds for such hypersurfaces. The main result of this section is the following.

Theorem 4.9. *Let Y be a smooth, nef and big hypersurface in an irreducible holomorphic symplectic manifold X . Then for $i < \dim X$ the restriction induces an isomorphism on the cohomology groups $H^i(X, \mathbb{Q}) \cong H^i(Y, \mathbb{Q})$.*

First we recall that the LHT (with rational coefficients) follows from the Kodaira–Akizuki–Nakano vanishing.

Lemma 4.10. *Let X be a smooth variety and \mathcal{L} be an effective line bundle on X . Consider a smooth hypersurface $Y \subset X$ such that $\mathcal{O}_X(Y) \cong \mathcal{L}$. Assume that Kodaira–Akizuki–Nakano vanishing holds for \mathcal{L} . This means the following:*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^*) = 0 \text{ for } p + q < \dim X.$$

Then $H^i(X, \mathbb{Q}) \cong H^i(Y, \mathbb{Q})$ for $i < \dim X$.

Proof. See [66, Chapter 13.3].

□

Thus, to prove the LHT it is enough to prove the Kodaira–Akizuki–Nakano vanishing. Let us formulate it.

Proposition 4.11. *Let \mathcal{L} be a nef and big line bundle on an irreducible holomorphic symplectic manifold X . Then the Kodaira–Akizuki–Nakano vanishing (we will write it as the KAN for shortness) holds for \mathcal{L} :*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^*) = 0 \text{ for } p + q < \dim X.$$

Remark that the KAN and the LHT are not true for a nef and big line bundle.

Remark 4.12. In [47, Remark 4.3.3] R. Lazarsfeld gives an example, where the KAN (or the LHT for a generic section) is not true for a nef and big line bundle. This line bundle is a pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$ on the projective space \mathbb{P}^3 to the blowing-up $\text{Bl}_P \mathbb{P}^3$ of this space at a point $P \in \mathbb{P}^3$.

The clue in the holomorphic symplectic case is that the morphism induced by a nef and big line bundle is of a special kind. Namely, these morphisms are semismall.

Definition 4.13. Let $f : X \rightarrow Y$ be a morphism of algebraic varieties. We call f **semismall** if for every subvariety Z of X , the following inequality holds

$$2 \dim Z \leq \dim X + \dim f(Z).$$

Let us recall the definition of a *lef* line bundle (l.e.f. = Lefschetz effettivamente funziona) from [19].

Definition 4.14. A line bundle \mathcal{L} is called *lef* if some its power $\mathcal{L}^{\otimes k}$ is generated by global section and induces a semismall morphism to projective space.

E. Esnault and E. Viehweg proved the KAN vanishing for *lef* line bundles.

Theorem 4.15. [24, Theorem 2.4] *Let L be a lef line bundle on a smooth variety X . Then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^*) = 0 \text{ for } p + q < \dim X.$$

We are going to prove that a nef and big line bundle \mathcal{L} on a holomorphic symplectic manifold X is *lef*. By the Kawamata–Shokurov base-point-free theorem (see [18, Chapter 10]) the line bundle $\mathcal{L}^{\otimes k}$ for $k \gg 0$ is generated by global sections.

It remains to prove that the morphism $\phi : X \xrightarrow{|kY|} \mathbb{P}^N$ is semismall. For this purpose we show that despite the fact that Y is not ample, the hard Lefschetz theorem remains true for $[Y] \in H^2(X, \mathbb{Z})$ because Y deforms to an ample divisor.

Lemma 4.16. *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and Y be a hypersurface with $q(Y, Y) > 0$. Then for every $0 < r < 2n$ we have an isomorphism*

$$\cup[Y]^{2n-r} : H^r(X, \mathbb{Q}) \xrightarrow{\sim} H^{4n-r}(X, \mathbb{Q}).$$

Proof. As in section 2.5 we consider the local deformations of X . Let $B_M \subset Def(X)$ be a small analytic ball around the point corresponding to X and take the subspace B^Y defined in lemma 2.42. The ball B^Y parametrizes local deformations of X such that the class $[Y]$ remains of type $(1, 1)$. Moreover, a very general Hodge structure A in $Per(B^Y)$, does not have a Hodge class not proportional to $[Y]$. Hence, for X' the holomorphic symplectic manifold corresponding to a very general point of B^Y , $[Y]$ generates $NS(X')$. Since X is projective, the Beauville-Bogomolov square of the class $[Y]$ in $H^2(X, \mathbb{Z})$ is positive. Hence, the Beauville-Bogomolov square of the class $[Y]$ in $H^2(X', \mathbb{Z})$ is also positive. By [34, Theorem 2] if X' has a Hodge class (in our case this class is $[Y]$) with the positive BBF-square (see [34, Theorem 2]), X' is projective. Which means that $NS(X')$ contains an ample class. The Neron-Severi lattice of X' is generated by $[Y]$ and $[Y]$ is positive. Hence, $[Y]$ is an ample class in $NS(X')$. Thus by the hard Lefschetz theorem for $[Y]$ and X' and the isomorphism of $H^*(X, \mathbb{Z}) \cong H^*(X', \mathbb{Z})$, the hard Lefschetz theorem is true for X and $[Y]$. \square

Now we are ready to prove that $\phi : X \rightarrow \mathbb{P}^N$ is semismall. Let Z be a closed subvariety of X of codimension r . By lemma 4.16, $[Z] \cup [Y]^{2n-2r} \neq 0 \in H^{4n-2r}(X, \mathbb{Z})$. Hence the section of $\phi(Z)$ by a linear space of codimension $2n - 2r$ in \mathbb{P}^N is not empty. Finally we obtain,

$$\dim \phi(Z) \geq 2n - 2r = 2(2n - r) - 2n = 2 \dim Z - \dim X.$$

Thus, the line bundle \mathcal{L} is nef. By theorem 4.15 the KAN holds for \mathcal{L} . By lemma 4.10 the LHT with rational coefficients is true for Y .

4.3 The case of a nef and big hypersurface

Now after proving theorem 4.9 we are ready to prove theorem 2.57. Let us recall it.

Theorem 2.57 Let Y be a smooth nef and big hypersurface in an irreducible holomorphic symplectic manifold X . Then a generic leaf of the characteristic foliation of Y is Zariski dense in Y .

As it was shown in section 4.1 it is enough to prove the following.

Proposition 4.17. *Let Y be a smooth nef and big hypersurface in an irreducible holomorphic symplectic manifold X . Then Y can not be covered by a family of coisotropic subvarieties of codimension 2 in X .*

The problem is that even though the LHT holds, it is not enough to apply the arguments of section 4.1 to a nef and big hypersurface Y . Indeed when Y is not ample one may have $Y^{2n-1} \cdot D = 0$ (cf. proof of Corollary 4.7). But using the fact that the family of coisotropic subvarieties covers Y we can show that $D^2 \cdot Y^{2n-2} \geq 0$. Indeed, let Z_1, Z_2 be two distinct members of this family. Then $Z_1 \cap Z_2$ is an effective cycle of codimension 4 in X . Thus the intersection number $Z_1 \cdot Z_2 \cdot Y^{2n-4} = D^2 \cdot Y^{2n-2}$ is not negative. This observation contradicts the following lemma.

Lemma 4.18. *In the assumption of lemma 4.6 if $q(\beta, \beta) > 0$, then $\alpha^2 \cup \beta^{2n-2} < 0$.*

Proof. The signature of the restriction of the BBF form to $H^{1,1}(X)$ is $(1, h^{1,1}(X) - 1)$. Hence, the BBF square of α is negative. Considering the equality of polynomials (1) at the terms of degree $2n - 2$ we obtain:

$$c \cdot \frac{2n(2n-1)}{2} \cdot \alpha^2 \cup \beta^{2n-2} = \frac{n(n-1)}{2} q(\alpha, \alpha) q(\beta, \beta)^{n-1}.$$

Since $q(\alpha, \alpha) < 0$ and $q(\beta, \beta) > 0$, the intersection number $\alpha^2 \beta^{2n-2}$ is negative. \square

4.4 The case of a non numerically effective hypersurface

In this section we prove that a generic leaf of the characteristic foliation of Y is also dense in Y if Y is not nef under some additional condition on Y . A divisor with positive Beauville-Bogomolov-Fujiki square is not necessarily nef. But it can be transformed to a nef divisor by a birational modification of X . The following proposition is a consequence of theorem 1.2 and lemma 2.2 in [51].

Proposition 4.19. [51] *Let X be an irreducible holomorphic symplectic manifold and Y an irreducible hypersurface with $q(Y, Y) > 0$. Then there is an irreducible holomorphic symplectic manifold X' with birational map $\psi : X' \dashrightarrow X$, such that the divisor $Y' = \psi^*Y$ is nef.*

Let us recall an important fact on the birational transformation of IHS manifolds.

Lemma 4.20. [43, page 420] *Let $\psi : X \dashrightarrow X'$ be a birational transformation of IHS manifolds, then ψ is an isomorphism in codimension 1.*

It is easy to see that if a generic leaf of the characteristic foliation is dense in Y' , then the same is true for Y . Theorem 2.57 (Campana's conjecture for a nef and big hypersurface) has the following corollary.

Corollary 4.21. *In the assumptions of proposition 4.19, if Y' is smooth, then a generic leaf of the characteristic foliation is dense in Y .*

In a recent conversation Jorge Vitorio Pereira proposed the following solution of the non-nef case. This result is also very interesting out of the context of conjecture 2.55.

Theorem 2.58 Let X be a holomorphic symplectic manifold and Y a smooth non-nef hypersurface in X . Then Y is uniruled.

Proof. First, we notice that the line bundle $\mathcal{O}_Y(Y)$ is not nef. Indeed, $\mathcal{O}_X(Y)$ is not nef. Hence, there exists a curve $C \subset X$, such that the line bundle $\mathcal{O}_C(Y)$ has negative degree. Since Y intersects C negatively, C is contained in Y .

Recall that the characteristic foliation F is isomorphic to $\mathcal{O}_Y(-Y)$. Thus the restriction of F to the curve C has positive degree. That allows us apply theorem 2.51 to the variety Y and the foliation F . We obtain that all leaves of the characteristic foliation on Y are rational curves. Hence, Y is uniruled. \square

Lemma 4.22. *If Y is smooth and uniruled then $q(Y, Y) < 0$.*

Proof. The proof of this statement is contained in the work [14]. Let us highlight it. By [14, theorem 4.5] an "exceptional" hypersurface has negative BBF square. We need to show that a smooth uniruled divisor is "exceptional" according to definition of [14]. S. Boucksom defines a "modified nef divisor" (the definition is not important for us). For an irreducible divisor being exceptional is equivalent to being not "modified nef" [14, Definition 3.10]. What is important for us is that the restriction of a modified nef divisor to any prime divisor is pseff [14, Proposition 2.4].

Note that $\mathcal{O}_Y(Y)$ is not pseff. Indeed, $\omega_Y \cong \mathcal{O}_Y(Y)$. Canonical bundle of a uniruled variety is not pseff. Hence, Y is exceptional and we can apply [14, theorem 4.5] to a smooth uniruled hypersurface Y . \square

Corollary 4.23. *A non nef hypersurface Y with non-negative Beauville-Bogomolov square is singular.*

That completes the proof of conjecture 2.55.

5 Singular counterexamples to the conjecture of Campana

In this chapter we give a few known examples of singular hypersurfaces in an irreducible holomorphic symplectic manifold, such that the dimension of a generic leaf of the characteristic foliation is smaller than predicted by the conjecture 2.55 for a smooth hypersurface.

5.1 Vertical hypersurfaces

As we proved in theorem 2.56, if X is an IHS manifold Lagrangian fibered over \mathbb{P}^n , then the Zariski closure of a general of the characteristic foliation on a smooth vertical hypersurface is a fiber of the Lagrangian fibration. Let us give few examples of singular vertical hypersurfaces such that the Zariski closure of a generic leaf of the characteristic foliation is a proper subvariety in a fiber.

The most natural singular divisor on a holomorphic symplectic manifold is the preimage of the discriminant divisor. Hwang and Oguiso proved that its characteristic foliation is algebraically integrable.

Proposition 5.1. [39] *Let X be a holomorphic symplectic manifold and $f : X \rightarrow B$ a holomorphic Lagrangian fibration over a complex manifold B . Assume that each fiber of f is of class C , i.e. bimeromorphic to a compact Kähler manifold. Let D be a component of the discriminant hypersurface of f . Then the characteristic foliation on every irreducible component Y of $f^{-1}D$ has algebraic leaves and the closures of the leaves are either rational curves or elliptic curves.*

Example 5.2. Let $p : S \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface and $\pi : X \rightarrow \mathbb{P}^2$ the induced Lagrangian fibration on the Hilbert scheme of the subschemes of length 2 in S (see example 3.10). The preimage of the diagonal conic has two components: the exceptional divisor of the blow-up $S^{[2]} \rightarrow S^{(2)}$ and the relative symmetric square $S_{\mathbb{P}^1}^{(2)}$ of our elliptic fibration. It is easy to see, that both of them are uniruled. Thus, the characteristic foliations are **algebraically integrable**.

Example 5.3. For the same X as above, consider a line l in \mathbb{P}^2 tangent to the conic $\Delta_{\mathbb{P}^1}$. It can be defined as

$$\{b_1 + b_2 \in (\mathbb{P}^1)^{(2)} \mid b_1 = b \text{ or } b_2 = b\}$$

for some fixed point b in \mathbb{P}^1 . Let Y_l be the preimage of the line l . Considering the strict transform of Y_l in the Cartesian square $S \times S$, one can show that the leaves of the characteristic foliation on Y_l are isomorphic to the elliptic curve $\pi^{-1}(b)$. By Lemma 3.22 Y_l is singular. This example shows, that smoothness of Y is a necessary condition in Theorems 2.56 and 2.53.

5.2 The variety of lines in a cubic fourfold

In this section we consider a "complete" (of dimension 20) family of holomorphic symplectic manifolds introduced by A. Beauville and R. Donagi in [8]. A general variety of this family possesses only ample hypersurfaces. Theorem 2.57 says that if an ample hypersurface Y is smooth, then a generic leaf of the characteristic foliation is dense in Y . We will give two examples of a singular hypersurface Y such a generic leaf of the characteristic foliation on Y is not dense in Y .

Let $C \subset \mathbb{P}(V)$ ($\dim V = 6$) be a smooth cubic fourfold. The variety of lines in C is a holomorphic symplectic fourfold X . By definition, X is embedded in $Gr(2, V)$. Let \mathcal{U} be the universal sub-bundle and \mathcal{Q} the universal quotient bundle. One can define X as the zero locus of a section of $S^3\mathcal{U}^*$ in $Gr(2, V)$. For a very general C , the variety X has Picard number equal to 1. Indeed, by [8, Proposition 4] $H^2(X, \mathbb{Z})$ and $H^4(C, \mathbb{Z})$ are isomorphic as Hodge structures and by the Noether-Lefschetz theorem (see e.g. [66, Book II, Section 3.3.2]) $Hdg^4(C, \mathbb{Z}) \cong \mathbb{Z}$. The Picard group of X is generated by the restriction of the Plücker hyperplane section H of $Gr(2, V)$. Clearly, it is ample. The embedding into the Grassmannian also induces the morphism on the fourth rational cohomologies: $H^4(Gr(2, V), \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})$. This vector space $H^4(Gr(2, V), \mathbb{Q})$ is generated by the Schubert cycles σ_{11}^G and σ_2^G . Here σ_{11}^G can be obtained as the set of lines contained in a hyperplane and σ_2^G as the set of lines intersecting a projective plane in $\mathbb{P}(V)$. We denote their restrictions to X by σ_{11} and σ_2 respectively. Let us describe the subring of $H^*(X, \mathbb{Q})$ generated by these algebraic classes. We have the following relations between them [1, Lemma 4]:

$$H^4 = 108, H^2 = \sigma_2 + \sigma_{11}, \sigma_2\sigma_{11} = 18, \sigma_2^2 = 27 \text{ and } \sigma_{11}^2 = 45.$$

Applying the formulas from section 2.3 we obtain that $q(H) = 6$. We know 2 examples of a singular hypersurface Y in X such that the leaves of the characteristic foliation are not dense in Y . The closure of a generic leaf of the characteristic foliation is a Lagrangian surface in

the first example and a rational curve in the second.

Let us construct the first one. Let $V_4 \subset V$ be a vector subspace of V of dimension 4. The subspace V_4 defines a Schubert cell in $Gr(2, V)$ as a variety of lines intersecting the projective space $\mathbb{P}(V_4)$. This is the hyperplane section in the Plücker embedding corresponding to $\det V_4$ via the isomorphism $\wedge^2 V^* \cong \wedge^4 V$ given by an element of $\wedge^6 V$. Intersecting it with X we obtain the hypersurface

$$Y_{V_4} = \{l \in C \mid l \cap \mathbb{P}(V_4) \neq \emptyset\}. \quad (2)$$

Clearly, $Y_{V_4} \in |H|$. For a general subspace V_4 the intersection of $\mathbb{P}(V_4)$ and C is a smooth cubic surface $C(V_4)$. The classical fact of algebraic geometry says that a smooth cubic surface contains exactly 27 lines l_1, l_2, \dots, l_{27} . They give us 27 points in Y_{V_4} . Denote the set of these points by L . The hypersurface Y_{V_4} is singular at L . If a line l is not one of these 27, it intersects with $\mathbb{P}(V_4)$ by a point of the cubic surface $C(V_4)$. It defines a rational map $Y_{V_4} \dashrightarrow C(V_4)$. But we are interested in another map. A line from $Y_{V_4} \setminus L$ lies in a unique hyperplane with V_4 . This defines a rational map $\phi : Y_{V_4} \dashrightarrow \mathbb{P}(V/V_4)$, $l \mapsto l + \mathbb{P}(V_4)$. A general fiber of ϕ over a hyperplane V_5 containing V_4 is a surface of lines in the cubic threefold $C(V_5) := \mathbb{P}(V_5) \cap C$. This surface of lines was studied in the famous work [17] of P. Griffiths and H. Clemens. Let us denote this surface by $F(V_5)$.

$$F(V_5) = \{l \in \mathbb{P}(V_5) \cap C\} \quad (3)$$

It is known that $F(V_5)$ is a Lagrangian surface in X and hence ϕ is invariant under the characteristic foliation on the divisor Y_{V_4} (lemma 4.3). To show this we need to recall the following properties of the surface $F(V_5)$.

Lemma 5.4. [17] *Let F be a surface of lines of a smooth cubic threefold $C_3 \subset \mathbb{P}(V_5)$. It has the Hodge numbers $h^{1,0} = 5$, $h^{2,0} = 10$, $h^{1,1} = 25$. Moreover, the cotangent bundle Ω_F is isomorphic to the restriction of $\mathcal{U}^*|_F$ (where \mathcal{U} is the universal subbundle on $Gr(2, V_5)$). Moreover, $H^0(F, \Omega_F) = V^*$.*

Lemma 5.5. [65, Section 3, Example 7] *Let V_5 be a hyperplane in V . Then $F(V_5)$ is a Lagrangian surface in X .*

These classical results give the following proposition about the characteristic foliation on the hypersurface Y_{V_4} :

Proposition 5.6. *Let X be the Fano variety of lines in a cubic fourfold $C \subset \mathbb{P}(V)$ and H a class of a hyperplane with respect to the embedding of X to $\mathbb{P}(\wedge^2 V)$ induced by the Plücker embedding of $Gr(2, V)$. Take a four-dimensional subspace V_4 of V . Let $Y_{V_4} \in |H|$ be a hypersurface in X defined by formula (2). Then for any five-dimensional subspace V_5 of V containing V_4 , the surface $F(V_5)$ (see formula (3)) is invariant under the characteristic foliation of the singular hypersurface Y_{V_4} .*

In order to construct the second hypersurface which is unirational we need to construct a two-dimensional family of rational curves in X . A general line in C has the normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and a special line has the normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Geometrically this means that there is a unique plane tangent

to C at a general line l and a pencil of planes tangent to X at a special one. Each of these planes intersects C by $2l$ and another line l' (this property was studied in [67] in order to construct a rational endomorphism of X). The set of special lines is the surface S of class $5\sigma_2$ [1]. This surface was well described in the recent work [29].

$$S := \{l \in X \mid \mathcal{N}_{l/C} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\}$$

The surface S can also be defined as the set of lines such that there exists a $\mathbb{P}_l^3 \subset \mathbb{P}(V)$ tangent to C at l . A line from S defines a rational curve R_l in X , the set of lines l' intersecting l and such that the plane $l + l'$ is tangent to C at l . This curve is isomorphic to the variety of planes tangent to C at l (i.e. the planes in \mathbb{P}_l^3 containing l). Thus this curve is rational. The curve R_l has degree 3 in the Plucker embedding of X . Indeed, a general $\mathbb{P}^3 \subset \mathbb{P}(V)$ intersects \mathbb{P}_l^3 in a line intersecting the cubic hypersurface C at three points. Every point in $C \cap \mathbb{P}_l^3$ gives us a unique plane tangent to C at l . We define the hypersurface Y as the union of all curves R_l .

$$Y := \{l' \in X \mid \exists l \in S \text{ such that } \mathbb{P}(l' + l) \cap C = 2l + l'\}$$

The hypersurface Y is clearly uniruled and thus its characteristic foliation is algebraically integrable. In the article [57, Corollary 0.3] the authors prove that the Fano variety of a very general cubic fourfold does not contain a uniruled hypersurface except the hypersurface Y constructed above.

Let X be an IHS manifold of dimension 4. It follows from lemma 4.3 that if a generic leaf of the characteristic foliation on a (may be singular) hypersurface Y is not dense in Y , then Y is covered by Lagrangian varieties. There are very few known examples of Lagrangian varieties in an IHS manifolds X with $NS(X)$ generated by an ample divisor. There is the following open question.

Question. *Let X be a projective IHS manifold of dimension 4 with $NS(X) \cong \mathbb{Z}$. Does X contain a Lagrangian surface?*

Proposition 5.6 motivates us to make the following conjecture which might help to answer this question.

Conjecture 5.7. *Let X be an IHS manifold embedded in the projective space \mathbb{P}^N . Then there exists a hyperplane $H \subset \mathbb{P}^N$, such that a generic leaf of a characteristic foliation of the hyperplane section $Y := H \cap X$ is not dense in Y .*

6 A special divisor on the Debarre-Voisin manifold

In this chapter we study a certain divisor in the holomorphic symplectic manifold constructed by O. Debarre and C. Voisin in [22]. In the end of this chapter we construct a foliation on this divisor and conjecture that this foliation is characteristic.

6.1 Introduction

Let V be a complex vector space of dimension 10 and let $\sigma \in \wedge^3 V^*$ be a very general alternating form of rank 3. There are several types of varieties associated to σ . We shall recall their definitions. For a very general form σ all these varieties are smooth (see for example [10] and [9]).

- The variety $Y \subset Gr(3, V)$ defined by the hyperplane section corresponding to $\sigma \in \wedge^3 V^*$ in the Plücker embedding;
- The Peskine variety $P \subset \mathbb{P}(V)$ defined as $P = \{V_1 \subset V, \dim V_1 = 1 \mid rk\sigma(V_1, *, *) \leq 6\}$ (for a very general σ we can replace " \leq " by the " $=$ ");
- The congruence variety of lines $\Sigma \subset Gr(2, V)$ defined as $\Sigma = \{V_2 \subset V, \dim V_2 = 2 \mid \sigma(V_2, V_2, V) = 0\}$;
- The Debarre-Voisin variety $X \subset Gr(6, V)$ defined as $X = \{V_6 \subset V \mid \dim V_6 = 6 \mid \sigma(V_6, V_6, V_6) = 0\}$;
- The Fano variety F of lines in the Peskine variety P .

Lemma 6.1. [22, Proposition 3.1] *Let $U_3 \subset V$ be a three-dimensional space from Y (i.e. $\sigma(U_3, U_3, U_3) = 0$), then Y is singular at U_3 if $\sigma(U_3, U_3, V) = 0$. For a general σ there is no such V_3 .*

Proposition 6.2. [22] *Debarre-Voisin variety X is an irreducible holomorphic symplectic manifold of dimension four of $K3^{[2]}$ type. The Plücker hyperplane section gives us the polarization on X with BBF-square equal to 22.*

The authors of [22] present Y as an analogue of the cubic fourfold and X as an analogue of the Fano variety of lines in it. As analogue of the incidence variety in the product of the cubic fourfold and its Fano variety of lines they consider the incidence variety $I_{3,6}$ in $Y \times X$ defined as

$$I_{3,6} = \{(U_3, U_6) \mid U_3 \subset U_6 \text{ and } \sigma(U_6, U_6, U_6) = 0\}.$$

That gives some Hodge-theoretical results (see for example [22, Corollary 2.7]). The projection of $I_{3,6}$ to Y is not surjective due to the dimensional reasons. Indeed, $\dim Y = \dim Gr(3, 10) - 1 = 20$ and $\dim I_{3,6} = \dim Gr(3, 6) + \dim X = 13$. In this chapter we consider another analogy. We take the Peskine variety P instead of a cubic fourfold and its Fano variety of lines F instead of the Fano variety of lines in a cubic fourfold. The variety P has only dimension 6. The variety F is not an *IHS* manifold, but Benedetti and Song proved that it is fibered in cubic surfaces over X (see proposition 6.6).

Proposition 6.3. *The Peskine variety P has degree 15 and dimension 6. Its canonical bundle is equal to $\mathcal{O}_P(-3)$ with respect to the embedding into $\mathbb{P}(V)$. It is equipped with a vector bundle $\mathcal{K} \subset V \otimes \mathcal{O}_P$ of rank 4 with the fiber K_{V_1} over the point corresponding to a line V_1 . Here the space K_{V_1} is defined by the equation $\sigma(V_1, K_{V_1}, V) = 0$. There is the following exact sequence of vector bundles on P :*

$$0 \rightarrow \mathcal{O}_P(-1) \rightarrow \mathcal{K} \rightarrow \mathcal{N}_{P/\mathbb{P}(V)}(-3) \rightarrow 0.$$

Proof. See e.g. [20, Section 4.5]. □

Proposition 6.4. [20, Theorem 3.8] *The congruence variety Σ is the zero set of a section of the vector bundle $Q_8^* \otimes \mathcal{O}_{Gr(2,V)}(1)$ in $Gr(2, V)$ (where Q_8 is the universal quotient bundle and $\mathcal{O}_{Gr(2,V)}(1)$ is the ample line bundle associated to the Plücker embedding). Hence it has dimension 8.*

Proposition 6.5. [20, Corollary 4.18 and Table 1 on page 44] *For a point $[V_1] \in \mathbb{P}(V) \setminus P$ there is exactly one projective line from Σ passing through it. For a point $[V_1] \in P$ the set of lines from Σ passing through the point $[V_1]$ is the projective plane $\mathbb{P}(K_{V_1}/V_1)$. It gives us the following commutative diagram:*

$$\begin{array}{ccc} Bl_P \mathbb{P}(V) & \xrightarrow{\sim} & \mathbb{P}_\Sigma(\mathcal{U}_2) \\ \downarrow p & & \downarrow q \\ \mathbb{P}(V) & & \Sigma \end{array}$$

Moreover, a general line from Σ intersects P by four points. In other words, let E be the exceptional divisor of the blow-up p . The restriction of q to E is the morphism of degree 4.

See [20] for more details on the congruence varieties of lines.

Proposition 6.6. [9, Theorem 2.20] *The variety F has dimension 6. There exists a fibration in cubic surfaces $\pi : F \rightarrow X$. We denote the fiber $\pi^{-1}[U_6]$ for $[U_6] \in X$ by C_{U_6} .*

We end this section with a sketch of the construction of this fibration in cubic surfaces from [9, Section 2.7]. First of all, the authors prove the following lemma.

Lemma 6.7. *For a very general form $\sigma \in \wedge^3 V^*$ the variety F is isomorphic to the variety of the seven-dimensional spaces U_7 in V such that σ restricts to U_7 as a three-form of rank 5 (i.e. rank of the morphism $U_7 \rightarrow \wedge^2 U_7^*$ induced by σ is 5).*

To obtain such a space U_7 from the plane $[U_2] \in F$, we take U_7 as the sum $\sum_{U_1 \subset U_2} K_{U_1}$. Since $\mathbb{P}(U_2) \subset P$ the form $\sigma(u, *, *)$ on V has rank 6 for every $u \in U_2$. Hence, the form $\sigma(u, *, *)$ has the kernel $K_{\langle u \rangle}$ ⁷ of dimension 4. The authors of [9] show that the sum of all $K_{\langle u \rangle}$ for all $u \in U_2$ has dimension 7 for a very general three-form σ . This is the required space U_7 . Moreover, they show that $\sigma(U_2, U_2, U_7) = 0$ (and hence $\sigma|_{U_7}$ has rank not greater than 5).

For a very general σ there is no seven-dimensional space U_7 such that the restriction of σ to U_7 has rank less than 5. Hence, a seven-dimensional space U_7 with $rk \sigma|_{U_7} \leq 5$ contains a unique two-dimensional space U_2 such that $\sigma(U_2, U_7, U_7) = 0$. For every $u \in U_2$ the two-form $\sigma(u, *, *)$ has a seven-dimensional coisotropic space U_7 and hence it has rank not greater than 6. Which means that $\mathbb{P}(U_2)$ is a line in the Peskine variety.

Finally, we construct the fibration π from F to X .

⁷Here we mean K_{U_1} for $U_1 = \langle u \rangle$.

Lemma 6.8. *There exists a unique six-dimensional space U_6 such that $U_2 \subset U_6 \subset U_7$ and $\sigma(U_6, U_6, U_6) = 0$.*

Proof. Any alternating three-form of rank 5 on U_7 with the kernel U_2 can be represented as $x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$ for some basis of the space of the linear forms $(U_7/U_2)^*$. Then the hyperplane U_6 in U_7 corresponding to the linear form x_1 satisfies the property $\sigma|_{U_6} = 0$. \square

Thus we constructed the morphism π from F to X .

Remark that if $\sigma|_U = 0$ for some subspace of V , then every vector $q \in V/U$ defines the two-form $\sigma(q, *, *)$ on U . By lemma 6.7 we can think of F as the variety of seven-dimensional spaces U_7 such that $\sigma|_{U_7}$ has rank 5. The fiber of the morphism $\pi : F \rightarrow X$ over the point of X corresponding to a space U_6 is the set of the seven-dimensional spaces U_7 containing U_6 such that the form $\sigma(U_7/U_6, *, *)$ has two-dimensional kernel (i.e. is degenerate).

Definition 6.9. Let U be a vector space of dimension $2n$ and let W be a subspace of $\wedge^2 U^*$. Consider the morphism

$$S^n W \rightarrow S^n(\wedge^2 U^*) \rightarrow \det U^* \cong \mathbb{C}.$$

If this morphism is not zero it defines a hypersurface of degree n in $\mathbb{P}(W)$. We call this hypersurface **Pfaffian**.

Lemma 6.10. *The set of seven-dimensional subspaces U_7 of V , such that U_7 contains U_6 and the form $\sigma(U_7/U_6, *, *)$ on U_6 is degenerate, gives the Pfaffian cubic in $\mathbb{P}(V/U_6)$ which we denote by C_{U_6} .*

As we said σ gives the morphism $V/U_6 \rightarrow \wedge^2 U_6^*$. It induces the morphism $S^3(V/U_6) \rightarrow \wedge^6 U_6^* \cong \mathbb{C}$. This morphism defines the cubic surface C_{U_6} in $\mathbb{P}(V/U_6)$. This fibration in cubic surfaces provides the following correspondence between P and X .

$$\begin{array}{ccccc}
 & & \mathbb{P}_F(\mathcal{U}_2) & & \\
 & \swarrow & & \searrow^{\mathbb{P}(U_2)} & \\
 P & & & & F \longleftrightarrow \mathbb{P}_X(\mathcal{Q}_4) \\
 & & & & \downarrow C_{U_6} \\
 & & & & X \longleftarrow \mathbb{P}(V/U_6)
 \end{array}$$

Here \mathcal{Q}_4 is the restriction of the tautological quotient bundle on $Gr(6, V)$ to X and \mathcal{U}_2 is the restriction of the tautological subbundle of $Gr(2, V)$ to F .

For each vector $q \in C_{U_6}$, the form $\sigma(q, *, *)$ on U_6 has two-dimensional kernel U_2 . That gives a vector bundle $\mathcal{U}_2|_{C_{U_6}} \subset U_6 \otimes \mathcal{O}_{C_{U_6}}$ on C_{U_6} . For a general $U_6 \in X$ the intersection $\mathbb{P}(U_6) \cap P$ is isomorphic to $\mathbb{P}_{C_{U_6}}(\mathcal{U}_2|_{C_{U_6}})$. This scroll is called the Palatini threefold. See [31] and [32] for the details about the Palatini threefold and its connection with the Peskine variety.

The goal of this chapter is to describe a special threefold $Z \subset F$ (namely, $Z = F \cap \Sigma$) and map it to a divisor in X via the fibration π . In the end of this chapter we construct a foliation of rank one on $\pi(Z)$ which we conjecture to be the characteristic foliation.

6.2 Peskine variety

Before constructing a threefold Z in the Fano variety F of lines in the Peskine variety P , we need to study the geometry of the Peskine variety itself.

Let the line $V_1 = \langle v_1 \rangle$ correspond to a point of the Peskine variety P . We denote by K_{V_1} the kernel of the alternating form $\sigma(v_1, *, *)$ on V . For a general σ , for every $V_1 \in P$ the space K_{V_1} has dimension 4 (see [9, Section 2.5]). Choose a basis: $K_{V_1} = \langle v_1, k_1, k_2, k_3 \rangle$, by the definition $\sigma(v_1, k_i, v) = 0$ for every $v \in V$. Consider an arbitrary bi-vector in $\wedge^2(K_{V_1}/V_1)$. Since dimension of K_{V_1}/V_1 is three, any bi-vector from $\wedge^2(K_{V_1}/V_1)$ is the skew-product of two vectors. Without loss of generality we can assume that this bi-vector is $k_1 \wedge k_2$. The linear form $\sigma(k_1, k_2, *)$ is not zero on V , otherwise we obtain the equation $\sigma(U_3, U_3, V) = 0$ for the space $U_3 = \langle v_1, k_1, k_2 \rangle$, which contradicts lemma 6.1. Thus the bi-vectors $k_i \wedge k_j \in \wedge^2(K_{V_1}/V_1)$ define three distinct non-zero linear forms $\sigma(k_i, k_j, *)$ on V . Hence, there is a unique seven-dimensional space \overline{T}_{V_1} such that $\sigma(K_{V_1}, K_{V_1}, \overline{T}_{V_1}) = 0$. Thus we obtain two vector bundles \mathcal{K} and \mathcal{T} on P . Since both K_{V_1} and \overline{T}_{V_1} contain V_1 , one can take the quotients $\mathcal{K}/\mathcal{O}_P(-1)$ and $\mathcal{T}/\mathcal{O}_P(-1)$.

Lemma 6.11. *There are the isomorphisms $\mathcal{K}/\mathcal{O}_P(-1) \cong \mathcal{N}_{P/\mathbb{P}(V)}(-3)$ and $\mathcal{T}/\mathcal{O}_P(-1) \cong T_P(-1)$. The second isomorphism means that the projective space $\mathbb{P}(\overline{T}_{V_1}) \cong \mathbb{P}^6$ is tangent to P at $[V_1]$.*

Proof. There is an exact sequence (see [32, Corollary 2.4] and [20, Section 4.5]):

$$0 \rightarrow \mathcal{N}_{P/\mathbb{P}(V)}(-3) \rightarrow T_{\mathbb{P}(V)}(-1)|_P \rightarrow \Omega(2)_{\mathbb{P}(V)}|_P \rightarrow \mathcal{N}_{P/\mathbb{P}(V)}^*(4) \rightarrow 0,$$

where the middle map is defined by the following operator:

$$V \otimes \mathcal{O}_{\mathbb{P}(V)} / \mathcal{O}_{pr(V)}(-1) \rightarrow (V/V_1)^* \otimes \mathcal{O}_{\mathbb{P}(V)}; \quad u \in V/V_1 \mapsto \sigma(u, v, *)$$

for a chosen $v \in V_1$. Clearly, the kernel of this map is $\mathcal{K}/\mathcal{O}_P(-1)$. That gives us the first isomorphism.

To prove the second we will show that $\mathbb{P}(\overline{T}_{V_1})$ is tangent P at V_1 . Let $R = \mathbb{C}[\varepsilon]/\varepsilon^2$ be the ring of the dual numbers. The point corresponding to $V_1 = \langle v \rangle$ lies in the Peskine variety if $\wedge^4(\sigma(v, *, *)) = 0$ (the fourth power of the two-form $\sigma(v, *, *)$ on V). The line $\langle v, t \rangle$ is tangent to the Peskine variety at the point corresponding to the line $\langle v \rangle$ if and only if in addition $\wedge^4(\sigma(v + \varepsilon t, *, *)) = 0$. We prove that this condition is equivalent to $\sigma(t, K_{V_1}, K_{V_1}) = 0$ (i.e. $t \in \overline{T}_{V_1}$). Indeed, using the trilinear property of the form σ we obtain that:

$$\wedge^4(\sigma(v + \varepsilon t, *, *)) = \varepsilon \sigma(v, *, *) \wedge \sigma(v, *, *) \wedge \sigma(v, *, *) \wedge \sigma(t, *, *).$$

$t \in \mathbf{T}_{V_1} \Rightarrow \wedge^3(\sigma(v, *, *)) \wedge \sigma(t, *, *) = \mathbf{0}$. It is an alternating eight-form on V . Choose a basis $\langle v, k_1, k_2, k_3, u_1, u_2, \dots, u_6 \rangle$ on V such that $k_1, k_2, k_3 \in K_{V_1}$. If we choose eight vectors of this basis at least two of them should be from K_{V_1} . Without loss of generality we may assume that they are k_1 and k_2 . If one of them plugs into $\sigma(v, *, *)$ the whole eight-form is zero. The only possibility is to plug both of k_1 and k_2 to $\sigma(t, *, *)$, but it is also zero because of the equality $\sigma(t, K_{V_1}, K_{V_1}) = 0$.

$t \in \mathbf{T}_{V_1} \Leftarrow \wedge^3(\sigma(v, *, *)) \wedge \sigma(t, *, *) = \mathbf{0}$. If $\sigma(t, k_i, k_j) \neq 0$ for some $1 \leq i \leq j \leq 3$,

the eight-form above is not zero. Indeed, the rank of $\sigma(v, *, *)$ is exactly six and it is non-degenerate on $V/K_{V_1} = \langle u_1, u_2, \dots, u_6 \rangle$. Thus we may order the vectors u_1, u_2, \dots, u_6 such that $\sigma(v, u_1, u_2) \wedge \sigma(v, u_3, u_4) \wedge \sigma(v, u_5, u_6) \neq 0$. Since $\sigma(t, k_i, k_j)$ is also not zero, we obtain that

$$\sigma(v, u_1, u_2) \wedge \sigma(v, u_3, u_4) \wedge \sigma(v, u_5, u_6) \wedge \sigma(t, k_i, k_j) \neq 0.$$

□

Corollary 6.12. *We the following isomorphism of the line bundles on P :*

- $\det \mathcal{K} \cong \mathcal{O}_P(-3)$;
- $\det \mathcal{T} \cong \mathcal{O}_P(-4)$.

Where $\mathcal{O}_P(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to P .

Proof. The canonical bundle of the Peskine variety is $\mathcal{O}_P(-3)$ (see proposition 6.3). Thus $\det(\mathcal{N}_{P/\mathbb{P}(V)}) \cong \mathcal{O}_P(7)$ and $\det(T_P) \cong \mathcal{O}_P(3)$. For any vector bundle \mathcal{E} of rank r the determinant of a twist $\mathcal{E}(-m)$ is $\det(\mathcal{E}) \otimes \mathcal{O}(-rm)$. Using this fact, we obtain that $\det(\mathcal{N}_{P/\mathbb{P}(V)}(-3)) \cong \mathcal{O}_P(-2)$ and $\det(T_P(-1)) \cong \mathcal{O}_P(-3)$. We apply the isomorphisms from lemma 6.11 to finish the proof. □

Lemma 6.13. *Let $[V_1]$ be a point of the Peskine variety P . The following statements are equivalent:*

1. $K_{V_1} \cap \bar{T}_{V_1} \neq V_1$;
2. $\sigma(K_{V_1}, K_{V_1}, K_{V_1}) = 0$;
3. $K_{V_1} \subset \bar{T}_{V_1}$.

Proof. **1** \Rightarrow **2** Choose a basis: $K_{V_1}/V_1 = \langle u_1, u_2, u_3 \rangle$. Let u_1 lie in the intersection $K_{V_1} \cap \bar{T}_{V_1}$. Since $u_1 \in \bar{T}_{V_1}$, $\sigma(u_1, u_2, u_3) = 0$.

2 \Rightarrow **3** Take a vector $u \in K_{V_1}$. By condition 2, $\sigma(u, K_{V_1}, K_{V_1}) = 0$. Hence, u lies in \bar{T}_{V_1} .

The last implication **3** \Rightarrow **1** is obvious. □

Lemma 6.14. *The set of points in P as above is a divisor on P which we denote by Q . There is an isomorphism of the line bundles $\mathcal{O}_P(Q) \cong \mathcal{O}_P(2)$ with respect to embedding of P to $\mathbb{P}(V)$.*

Proof. The form σ defines a section of the line bundle $\wedge^3(\mathcal{K} / \mathcal{O}_P(-1))^*$. Let us calculate its Chern class. We are going to use the isomorphism $\mathcal{K} / \mathcal{O}_P(-1) \cong \mathcal{N}_{P/\mathbb{P}(V)}(-3)$ (see lemma 6.11). Since $\omega_P \cong \mathcal{O}_P(3)$ and the rank of $\mathcal{N}_{P/\mathbb{P}(V)}$ is 3, $\det(\mathcal{N}_{P/\mathbb{P}(V)}) \cong \mathcal{O}_P(7)$. We obtain $\wedge^3(\mathcal{K} / \mathcal{O}_P(-1)) \cong \mathcal{O}_P(2)$. □

Lemma 6.15. *The intersection of the projective space $\mathbb{P}(K_{V_1})$ and the Peskine variety P contains a cubic surface which we denote by S_{V_1} . Moreover, the point V_1 is contained in S_{V_1} if and only if $V_1 \in Q$.*

Proof. Denote by B the blow-up of $\mathbb{P}(K_{V_1})$ at the point V_1 , by $p : B \rightarrow \mathbb{P}(K_{V_1})$ the morphism of the blow-up and by $q : B \rightarrow \mathbb{P}(K_{V_1}/V_1)$ the projection from the point V_1 . A point b of B represents a flag $U_1 \subset U_2 \subset K_{V_1}$ ($\dim U_1 = 1$ and $\dim U_2 = 2$) such that U_2 contains V_1 . The vector spaces U_1 are the fibers of the line bundle $p^* \mathcal{O}_{\mathbb{P}(K_{V_1})}(-1)$ and the quotient spaces U_2/U_1 are the fibers of the line bundle $q^* \mathcal{O}_{\mathbb{P}(K_{V_1}/V_1)}(-1)$. We denote by \mathcal{U}_2^B the vector bundle with the fibers U_2 . Since $\sigma(U_1, U_2, V) = 0$, the vector $u \in U_1$ defines an alternating two-form $\sigma(u_1, *, *)$ on V/U_2 . So we have a twisted skew morphism:

$$\sigma_{U_2} : ((V \otimes \mathcal{O}_B)/\mathcal{U}_2^B) \otimes p^* \mathcal{O}_{\mathbb{P}(K_{V_1})}(-1) \rightarrow ((V \otimes \mathcal{O}_B)/\mathcal{U}_2^B).$$

It follows from [33, Theorem 10] that this map is degenerate at the zero locus of a section of the line bundle $p^* \mathcal{O}_{\mathbb{P}(K_{V_1})}(4) \otimes q^* \mathcal{O}_{\mathbb{P}(K_{V_1}/V_1)}(-1)$. Moreover, this map is degenerate if and only if U_1 is a point of the Peskine variety P . Let E be the exceptional divisor of the blow-up $p : B \rightarrow \mathbb{P}(K_{V_1})$. We have an isomorphism $\mathcal{O}_B(E) \cong p^* \mathcal{O}_{\mathbb{P}(K_{V_1})}(1) \otimes q^* \mathcal{O}_{\mathbb{P}(K_{V_1}/V_1)}(-1)$. The points of E parametrize the flags $V_1 \subset U_2$. Hence, σ_{U_2} is degenerate at E . The rest of the degeneration locus of σ_{U_2} is the preimage of a cubic surface in $\mathbb{P}(K_{V_1})$.

To prove the last statement of the lemma consider a three-dimensional space $K' \subset K_{V_1}$ containing V_1 , there exists a unique hyperplane $V_9 \subset V$ such that $\sigma(K', K', V_9) = 0$. The form σ defines a map $K' \rightarrow \wedge^2(V_9/K')^*$. For a vector $u \in K'$ the corresponding form is degenerate at the cubic curve which we denote by $E_{K'}$. This curve $E_{K'}$ is the intersection of S and $\mathbb{P}(K')$. The form on V_9/K' corresponding to $v \in V_1$ is degenerate if and only if $K_{V_1} \subset V_9$ or equivalently $V_1 \in Q$. \square

We suppose that the cubic S_{V_1} for a general $V_1 \in P$ is smooth. Unfortunately we do not know how to show this at the moment.

6.3 Special lines in Peskine variety

In this section we obtain the threefold $Z = \Sigma \cap F$ first as a degeneracy locus in Σ and afterwards in F . We prove that the fibration π defined in proposition 6.6 maps Z birationally onto a hypersurface in the Debarre-Voisin variety X .

Proposition 6.16. *The variety $Z = \Sigma \cap F$ is the zero locus of the vector bundle $S^4 \mathcal{U}_2^*(-1)$ on Σ , where \mathcal{U}_2^8 is the restriction of the tautological vector bundle on $Gr(2, V)$ and $\mathcal{O}_\Sigma(-1)$ is its determinant.*

Proof. Let $l = \mathbb{P}(V_2)$ be a line from Σ , there is a quartic on l defined by the morphism

$$S^4 V_2 \rightarrow \wedge^8(V/V_2)^* \cong \mathbb{C},$$

induced by the following map:

$$V_2 \rightarrow \wedge^2(V/V_2)^*; \quad u \mapsto \sigma(u, *, *).$$

⁸We abuse the notation and use the symbol for the tautological bundle on $Gr(2, V)$ and its restriction to its subvarieties.

The zero locus of this quartic is the intersection of l and P . These quartics give a morphism of vector bundles on Σ :

$$S^4\mathcal{U}_2 \rightarrow \mathcal{O}_\Sigma(-1).$$

Let Z' be the zero locus of this morphism. Its expected dimension is 3. If the quartic vanishes at l then the line l lies in the Peskine variety. Thus, $Z' = F \cap \Sigma = Z$. \square

Corollary 6.17. *The expected dimension of Z is 3 and the expected class of Z in $Gr(2, V)$ is $480\sigma_{85} + 468\sigma_{76}$ ⁹. In particular, dimension of Z is at least 3 for any alternating three-form σ .*

Proof. Recall that Σ is the zero locus of a section of the bundle \mathcal{Q}_8^* on $Gr(2, V)$. To calculate the expected class of Z we calculate the intersection of the top Chern classes of the vector bundles $\mathcal{Q}_8^*(1)$ and $S^4\mathcal{U}_2^*(-1)$ on $Gr(2, V)$ in calculation 6.41. Since it is not zero Z can not have dimension less than 3. \square

Lemma 6.18. *A line $l = \mathbb{P}(V_2) \in Z$ is contained in Q (see lemma 6.14).*

Proof. A line $l \subset P$ is tangent to P at every point of l . Since $l \in \Sigma$, for any $V_1 \subset V_2$ the space K_{V_1} contains V_2 . Thus $V_2 \subset \overline{T}_{V_1} \cap K_{V_1}$. By lemma 6.13, V_1 is contained in Q . \square

Remark 6.19. The expected dimension of the image of $\mathbb{P}_Z(\mathcal{U}_2|_Z)$ in Q is four. A general point V_1 of Q is contained in a cubic surface S_{V_1} (see lemma 6.15). A point V_1 from the image of $\mathbb{P}_Z(\mathcal{U}_2|_Z)$ in Q is contained in a line of the cubic surface S_{V_1} .

In proposition 6.16 we defined Z as the zero locus of a section of the vector bundle $S^4\mathcal{U}_2(-1)$ on Σ . Let us describe Z as a subvariety of F . For this we need to define the following rational map from F to P , which might be of independent interest.

Proposition 6.20. *Let $\mathbb{P}(U_2)$ be a line in the Peskine variety. Then the intersection $\bigcap_{u \in U_2} K_{\langle u \rangle}$ is not empty. Thus every line in the Peskine variety is contained in a cubic surface S_{V_1} (see lemma 6.15) for some point $[V_1] \in P$. For a general line in the Peskine variety this V_1 is unique. That defines a rational map of degree 27 from F to P .*

Proof. By lemma 6.7 there is a seven-dimensional space U_7 containing U_2 such that $\sigma(U_2, U_7, U_7) = 0$ and this space is the sum of all $K_{\langle u \rangle}$ for all $u \in U_2$. The form σ defines the linear map

$$U_7 \rightarrow U_2^* \otimes (V/U_7)^*. \quad (4)$$

By the dimensional reasons this map has a kernel of at least dimension one. Consider the following subvariety of $P \times F$:

$$I = \{(V_1, U_2) \mid \mathbb{P}(U_2) \subset P, V_1 \in P \text{ and } U_2 \subset K_{V_1}\}.$$

Since $\mathbb{P}(K_{V_1}) \cap P$ is the cubic surface S_{V_1} and every cubic surface contains a line, the projection of $I \rightarrow P$ is surjective. For a general $V_1 \in P$ the cubic surface S_{V_1} is smooth (thus contains 27 lines). Hence, a general fiber of the projection $I \rightarrow P$ is 27 points and $\dim I = 6$.

⁹See [27, Part 3] for the description of the cohomologies of the Grassmann varieties.

Since $\dim F = 6$, a general fiber of the projection $I \rightarrow F$ is a finite scheme. A fiber of this projection is a linear space. Indeed, assume the vectors v_1, v_2 satisfy the property $\sigma(v_1, U_2, V) = \sigma(v_2, U_2, V) = 0$, then their sum also satisfies this property. The projection $I \rightarrow F$ is of relative dimension 0 and has linear fibers. Hence, this projection is birational. \square

Remark 6.21. The subvariety of F where the morphism (4) has rank 5 has the expected codimension 2 if F . Let us denote it by F_1 . The fiber of the projection $I \rightarrow F$ (from the proof above) over a point of F_1 is a line in the Peskine variety (it cannot be a projective space of greater dimension by [9, page 19]). More concretely, let U_2 and V_2 satisfy the property $V_2 \subset \bigcap_{U_1 \subset U_2} K_{U_1}$ that means that $\sigma(U_2, V_2, V) = 0$. That leads to $U_2 \subset \bigcap_{V_1 \subset V_2} K_{V_1}$. In other words, there is an involution $F_1 \rightarrow F_1$. For a space U_2 from Z , we have $U_2 \subset \bigcap_{U_1 \subset U_2} K_{U_1}$. That means that Z is the set of fixed points of this involution.

The flags $U_2 \subset U_7$ for all $U_2 \in F$ (see lemma 6.7) form a flag of the vector bundles $\mathcal{U}_2 \subset \mathcal{U}_7$ on F . Since $\sigma(\mathcal{U}_2, \mathcal{U}_7, \mathcal{U}_7) = 0$, the subvariety Z of F can be defined as the zero locus of a section the bundle $\det \mathcal{U}_2 \otimes (V/\mathcal{U}_7)^*$ defined by the form σ . Hence, the expected dimension of Z is 3.

The composition of the embedding of Z to F and of the morphism $\pi : F \rightarrow X$ is a morphism $\pi|_Z : Z \rightarrow X$.

Let $U_2 \in F$ be line in the Peskine variety P . Let U_7 be the seven-dimensional space such that $\sigma(U_2, U_7, U_7) = 0$ (see lemma 6.7) and let U_6 be the six-dimensional such that $U_2 \subset U_7$ and $\sigma(U_6, U_6, U_6) = 0$ (see lemma 6.8). Let C_{U_6} be the fiber of $\pi : F \rightarrow X$. Recall that C_{U_6} is the cubic surface in $\mathbb{P}(V/U_6)$ (see lemma 6.10).

Lemma 6.22. *Let U_9 be the hyperplane in V such that $\sigma(U_2, U_2, V_9) = 0$. Then the projective plane $\mathbb{P}(U_9/U_6) \subset \mathbb{P}(V/U_6)$ is tangent to the cubic surface C_{U_6} at U_7/U_6 .*

Proof. Let U_6 be a six-dimensional space from X (i.e. $\sigma(U_6, U_6, U_6) = 0$). We recall that for any $q \in V/U_6$ the form $\sigma(q, *, *)$ is correctly defined on U_6 . We also recall that the seven-dimensional space U_7 corresponding to $\langle q \rangle \in \mathbb{P}(V/U_6)$ is contained in the cubic surface C_{U_6} (i.e. the fiber of π over U_6) if and only if $\wedge^3(\sigma(q, *, *)) = 0$ (see lemma 6.10).

We prove this lemma in the similar way as the second part of lemma 6.11. Take again the ring of the dual numbers $R = \mathbb{C}[\varepsilon]/\varepsilon^2$. Let $U_7/U_6 = \langle q_0 \rangle$ and q_1 be another vector from V/U_6 . We need to prove that $\wedge^3(\sigma(q_0 + \varepsilon q_1, *, *)) = 0$ if $\sigma(q_1, U_2, U_2) = 0$. Indeed, $\wedge^3(\sigma(q_0 + \varepsilon q_1, *, *)) = \varepsilon \sigma(q_0, *, *) \wedge \sigma(q_0, *, *) \wedge \sigma(q_1, *, *)$. This is the skew-form in six variables on U_6 . Plugging one vector from U_2 to $\sigma(q_0, *, *)$ we obtain zero. If we plug both of them to $\sigma(q_1, *, *)$ we also obtain zero. Thus, V_9/U_6 is tangent to C_{U_6} . \square

Corollary 6.23. *In the assumptions of lemma 6.22. If $U_2 \in Z$, then cubic surface C_{U_6} is singular at the point U_7/U_6 .*

Proof. If $U_2 \in Z = F \cap \Sigma$, any hyperplane V_9 satisfies the property $\sigma(U_2, U_2, V_9) = 0$. Hence, any V_9/U_6 containing U_7/U_6 is tangent to C_{U_6} at U_7/U_6 . That means that U_7/U_6 is a singular point of the cubic surface C_{U_6} . \square

Let us formulate a general statement about singularities of the Pfaffian cubics.

Lemma 6.24. *Let $Q_4 = \langle \omega_1, \omega_2, \omega_3, \omega_4 \rangle$ be a four-dimensional subspace of $\wedge^2 U_6^*$ (where $\dim U_6 = 6$). Denote the Pfaffian cubic in $\mathbb{P}(Q_4)$ by C_{U_6} . The cubic surface C_{U_6} is singular at the point ω_1 if and only if one of the following occurs:*

1. *The form ω_1 has rank less than 4;*
2. *The form ω_1 has rank 4. For all $i \in \{1, 2, 3, 4\}$ we have $\omega_i(K, K) = 0$, where K is the kernel of ω_1 .*

Proof. Again consider the ring $R = \mathbb{C}[\varepsilon]/\varepsilon^2$ of the dual numbers. The form ω_1 corresponds to a singular point of the Pfaffian cubic, if and only if :

$$\omega_1^3 = 0 \quad \text{and} \quad (\omega_1 + \varepsilon * \omega_i)^3 = 0$$

for all i . That is equivalent to:

$$\omega_1(*, *) \wedge \omega_1(*, *) \wedge \omega_i(*, *) = 0. \tag{5}$$

”If” part: We have already checked that the condition 2 leads to equation (5) in corollary 6.23. The first condition also leads to this equality. Indeed, the kernel of the form ω_1 has dimension 4. Hence, we should plug at least to vector of its kernel to ω_1 . That makes whole form $\omega_1^2 \wedge \omega_i$ equal to zero.

”Only if” part: Assume both of the conditions of lemma are false. Hence, ω_1 has the kernel K of dimension two and there is a form another form $\omega_i \in Q$ such that $\omega_i(K, K) \neq 0$. Choose a basis $\langle u_1, u_2, \dots, u_6 \rangle$ on U_6 such that $K = \langle u_1, u_2 \rangle$ and $\omega_1(u_3, u_4)$ and $\omega_1(u_5, u_6)$ are not zero. The form $\omega_1^2 \wedge \omega_i$ is not zero if we plug the vectors in the following order:

$$\omega_1(u_3, u_4) \wedge \omega_1(u_5, u_6) \wedge \omega_i(u_1, u_2) = 0.$$

□

Corollary 6.25. *For a general form σ the image of Z in X is the variety of the singular cubics of the family $\pi : F \rightarrow X$.*

Proof. As we said after lemma 6.7 for a general σ there is no seven-dimensional space such that σ restricts to it as a form of rank less than 5. Hence, the second case of this lemma does not happen. □

It is natural to expect that its codimension is one and a general singular cubic has only one singular point.

Theorem 6.26. *The image of Z in X is a divisor $\pi(Z) \subset X$ and the morphism $\pi|_Z : Z \rightarrow \pi(Z)$ is birational.*

Proof. We prove this theorem in the section 6.5. □

The Debarre-Voisin variety X parametrizes another family of varieties and $\pi(Z)$ is again the discriminant locus of this family.

Proposition 6.27. *Let U_6 be a six-dimensional vector space from X . Then $Gr(2, U_6) \cap \Sigma$ is a Fano fourfold $\Sigma(U_6)$ of degree 14 and index 2. If $\mathbb{P}(U_6)$ contains a line $\mathbb{P}(U_2)$ from Z and U_6 is contained in the corresponding U_7 (i.e. $[U_7/U_6] \in S_{U_6}$), then $\Sigma(U_6)$ is singular at $[U_2]$.*

Proof. Denote by Q_4 the quotient of V by U_6 . Since for every plane U_2 in U_6 $\sigma(U_2, U_6, U_6) = 0$, the variety $\Sigma \cap Gr(2, U_6)$ is the intersection of $Gr(2, U_6)$ and of the linear space of codimension 4 given by the following equation: $\sigma(Q_4, *, *) = 0$. By the Mukai list [55] it is a Fano fourfold of index 2 and degree 14.

Let $\mathbb{P}(U_2)$ be a line from $\Sigma(U_6)$. The tangent space $\bar{T}_{U_2} \subset \wedge^2 U_6$ to $\Sigma(U_6)$ at U_2 is the subspace of $U_2 \wedge U_6$ defined by $\sigma(\bar{T}_{U_2}, Q_4) = 0$. Now consider $\mathbb{P}(U_2)$ which lies in $\Sigma \cap F \cap \mathbb{P}(U_6)$, we obtain that $\sigma(U_2 \wedge U_6, U_7) = 0$. Hence, \bar{T}_{U_2} has codimension 3 in $U_2 \wedge U_6$ i.e. is of dimension 6. That means that $\Sigma(U_6)$ is singular at U_2 . \square

6.4 A foliation on the threefold Z

In this section we define a foliation of rank one on the threefold Z . Let V_2 be a two-dimensional subspace of V from $Z = \Sigma \cap F$. Denote the projective line $\mathbb{P}(V_2)$ by $[l]$. The tangent space of $Gr(2, V)$ at l is $V_2^* \otimes (V/V_2)$. The space $V_2^* \otimes (V/V_2)$ is the space of global section of the vector bundle $(V/V_2) \otimes \mathcal{O}_l(1)$ on l . The space $H^0(l, V/V_2 \otimes \mathcal{O}_l(1))$ is canonically isomorphic to $Hom_l(\mathcal{O}_l(-1), (V/V_2) \otimes \mathcal{O}_l)$. Thus a subbundle of the trivial bundle $(V/V_2) \otimes \mathcal{O}_l$ isomorphic to $\mathcal{O}_l(-1)$ gives a line tangent to $Gr(2, V)$ at $[l]$. We will define a subbundle $\mathcal{O}_l(-1) \subset (V/V_2) \otimes \mathcal{O}_l$ (namely $\mathcal{U}_3/(V_2 \otimes \mathcal{O}_l)$ from the proposition 6.28) for every $V_2 \in Z$. Afterwards we check that the obtained line tangent to $Gr(2, V)$ at l is also tangent to Z .

In order to construct a subbundle $\mathcal{O}_l(-1) \subset (V/V_2) \otimes \mathcal{O}_l$ we need to find some natural vector bundles on l with the particular properties (see proposition 6.28). The vector bundles from the bottom row of the diagram in proposition 6.28 are the trivial vector bundles $V_k \otimes \mathcal{O}_l$ depending only on V_2 , where k is dimension of V_k . To construct a vector bundle \mathcal{U}_k from the top row of this diagram, we need to construct its fiber (we denote by U_k^x the fiber of \mathcal{U}_k over a point x) over every point of the projective line l (here k is the rank of \mathcal{U}_k). The bundle \mathcal{U}_1 is just the tautological bundle on l . Thus, the space U_1^x is the line in V_2 corresponding to $x \in l$.

Besides the bundles \mathcal{U}_k we have already constructed the vector bundles \mathcal{T} (of rank 7) and \mathcal{K} (of rank 4) on the Peskine variety. For shortness we abuse the notation and denote their restrictions to l by the same signs. The fibers of the bundles \mathcal{T} and \mathcal{K} over a point $x \in l$ are $\bar{T}_{U_1^x}$ and $K_{U_1^x}$ respectively (see beginning of section 6.2). For convenience, in this section we change the notation. Namely, we denote $\bar{T}_{U_1^x}$ by T^x and $K_{U_1^x}$ by K^x .

Proposition 6.28. *For any $[V_2] \in Z$ and $U_1 \subset V_2$ there exist the following commutative diagram of the vector bundles on the line l :*

$$\begin{array}{ccccccccc}
 \mathcal{U}_1 & \longrightarrow & \mathcal{U}_3 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{U}_8 \\
 \downarrow & \nearrow & \downarrow & & \downarrow & & \nearrow & & \downarrow \\
 V_2 \otimes \mathcal{O}_l & \longrightarrow & V_4 \otimes \mathcal{O}_l & \longrightarrow & V_7 \otimes \mathcal{O}_l & \longrightarrow & V_9 \otimes \mathcal{O}_l & \longrightarrow & V \otimes \mathcal{O}_l
 \end{array}$$

Where all maps are the injections of vector bundles. Since all these bundles are the subbundles of $V \otimes \mathcal{O}_l$ we can apply the alternating form σ to them. The bundles satisfy the following equalities with the form σ :

1. $\sigma(V_2, V_2, V) = 0$;
2. $\sigma(V_2, V_7, V_7) = 0$;
3. $\sigma(V_2, V_4, V_9) = 0$;
4. $\sigma(\mathcal{U}_1, \mathcal{K}, V \otimes \mathcal{O}_l) = 0$;
5. $\sigma(\mathcal{K}, \mathcal{K}, \mathcal{T}) = 0$;
6. $\sigma(V_2 \otimes \mathcal{O}_l, \mathcal{K}, \mathcal{U}_8) = 0$;
7. $\sigma(\mathcal{U}_3, \mathcal{U}_3, V_9 \otimes \mathcal{O}_l) = 0$.

Proof. To prove this proposition we are going to construct the spaces V_k from the first row of the diagram and the fibers U_k^x of the bundles \mathcal{U}_k from the second row of the diagram.

We need to recall the main properties of $V_2 \in Z = F \cap \Sigma$.

- $\sigma(V_2, V_2, V) = 0$ (i.e. $V_2 \in \Sigma$);
- there exists the unique seven-dimensional subspace V_7 of V such that $\sigma(V_2, V_7, V_7) = 0$ (i.e. $\mathbb{P}(V_2) \subset P$);
- $V_2 \subset K^x$ for any $x \in l$;
- $K^x \subset T^x \cap V_7$ for any $x \in l$.

Lemma 6.29. *For every $x \in l$, the intersection $T^x \cap V_7$ has dimension at least six.*

Proof. Choose a basis $\langle u_0, u_1, u_2, u_3 \rangle$ on K^x such that $U_1^x = \langle u_0 \rangle$ and $V_2 = \langle u_0, u_1 \rangle$. Recall that $\sigma(u_0, u_i, V) = 0$ for all $0 \leq i \leq 3$ and the space T^x is defined as the orthogonal to three bi-vectors $u_1 \wedge u_2$, $u_1 \wedge u_3$ and $u_2 \wedge u_3$ with respect to the three-form σ . Since $\sigma(V_2, V_7, V_7) = 0$ the vector space V_7 is contained in the orthogonal to at least two of this triple of bi-vectors (namely, $u_1 \wedge u_2$ and $u_1 \wedge u_3$). Thus the intersection of U_7^x and T^x can be thought of as the orthogonal to $u_2 \wedge u_3$ in V_7 with respect to the form σ . Hence it has dimension at least 6. \square

We expect that for general $x \in l$ and $[l] \in Z$ this intersection is exactly six-dimensional (i.e. the spaces T^x and V_7 are not equal) and their sum is an eight-dimensional space. This is not necessary to prove for the goals of this chapter, because we can define an eight-dimensional space containing V_7 and T^x in another way.

Lemma 6.30. *There exists an eight-dimensional space $U_8^x \subset V$, such that $\sigma(V_2, K^x, U_8^x) = 0$. The space U_8^x contains V_7 and T^x .*

Proof. We take the same basis on K^x as in the proof of lemma 6.29. We have:

$$\sigma(u_0, u_1, V) = 0, \sigma(u_0, u_2, V) = 0 \text{ and } \sigma(u_0, u_3, V) = 0.$$

Thus we can define U_8^x as the orthogonal to two bivectors $u_1 \wedge u_2$ and $u_1 \wedge u_3$ with respect to σ . This orthogonal has dimension at least 8. It can not have bigger dimension, otherwise the intersection of $K^x \cap K^y$ (for another point $y \in l$) has dimension at least three which is impossible for a general σ (see lemma 6.1). Both T^x and V_7 are orthogonal to the bi-vectors $u_1 \wedge u_2$ and $u_1 \wedge u_3$. Hence, they are contained in U_8^x . \square

Next we define a hyperplane V_9 in V containing V_7 , which contains U_8^x for any choice of $x \in l$.

Lemma 6.31. *All the spaces U_8^x defined lemma 6.30 are contained in a nine-dimensional space. We denote it by V_9 .*

Proof. Apply the alternating form σ to $V_2 \wedge V_7 \wedge V$. Since $\sigma(V_2, V_7, V) = 0$. It defines the following morphism of the vector spaces:

$$V_2 \otimes V/V_7 \rightarrow (V_7/V_2)^*.$$

The left hand space has dimension 6 and the right hand space has dimension 5. Hence it has a non-zero bi-vector in the kernel. Choose a basis $\langle q_0, q_1, q_2 \rangle$ on V/V_7 such that a non-zero element of this kernel is

$$u_0 \otimes q_1 + u_1 \otimes q_0$$

(where again $K^x = \langle u_0, u_1, u_2, u_3 \rangle$ and $V_2 = \langle u_0, u_1 \rangle$).

One can see that $V_7 + \langle u_0 \rangle$ is the space U_8^x defined by the line $U_1^x = \langle u_0 \rangle$ (see lemma 6.30). Indeed, we have the equality :

$$\sigma(u_0 \otimes q_1 + u_1 \otimes q_0, V_7) = 0.$$

Since V_7 contains K^x , we can put K^x instead of V_7 in the equality above. Using the fact that $\sigma(u_0, K^x, V) = 0$ we obtain:

$$\sigma(u_1, K^x, V_7 + q_0) = 0.$$

If we choose other basis on $V_2 = \langle u'_0, u'_2 \rangle$ the resulting vectors q'_0 and q'_1 as above will generate the same two-dimensional subspace of V/V_7 and the space $U'_8 = V_7 + \langle q'_0 \rangle$ will satisfy $\sigma(V_2, K_{\langle u'_0 \rangle}, U'_8) = 0$. That means that the space $V_9 := V_7 + \langle q_0, q_1 \rangle$ is the required hyperplane. \square

Now we define a three-dimensional space U_3^x (depending on $x \in l$) using the hyperplane V_9 .

Lemma 6.32. *For every point $x \in l$ there exists a unique three-dimensional space U_3^x such that $V_2 \subset U_3^x \subset K^x$ and $\sigma(U_3^x, U_3^x, V_9) = 0$.*

Proof. By lemma 6.31 the space V_9 contains U_8^x and hence by lemma 6.30 it contains T^x . Thus V_9 is the orthogonal to some bi-vector λ from $\wedge^2 K^x$. Using again the fact that the hyperplane V_9 contains the space U_8^x (the orthogonal complement to $u_1 \wedge u_2$ and $u_1 \wedge u_3$, where as usual $K^x = \langle u_0, u_1, u_2, u_3 \rangle$ and $V_2 = \langle u_0, u_1 \rangle$), we obtain that this bi-vector λ is divided by u_1 . Without loss of generality we may assume that λ is $u_1 \wedge u_2$. Then the required space U_3^x is $\langle u_0, u_1, u_2 \rangle$. \square

The form $\sigma(U_1, *, *)$ on V_9 has bigger kernel than on V . There is a unique five-dimensional space U_5^x containing K_{U_1} such that $\sigma(U_1, U_5, V_9) = 0$.¹⁰ Finally, we define the last vector space V_4 of dimension 4 as the sum of all U_3^x from lemma 6.32.

Lemma 6.33. *The sum of all vector spaces U_3^x defined in lemma 6.32 has dimension 4. We denote it by V_4 . Moreover, it satisfies the property $\sigma(V_2, V_4, V_9) = 0$. In other words, V_4 is the intersection of all U_5^x .*

Proof. We prove that $V_4 := \sum_{x \in l} U_3^x$ can not have dimension smaller than 4. Indeed, if it has dimension 3, all U_3^x are equal. Thus the intersection $\bigcap_{x \in l} K^x$ has dimension 3 and this is impossible for a general σ (see lemma 6.1).

It is easy to see that the space $V_4 := \sum_{x \in l} U_3^x$ is contained in all U_5^x . Indeed, any for any $x \in l$ we know that $\sigma(V_2, U_3^x, V_9) = 0$. Taking the sum by all $x \in l$ we obtain that $\sigma(V_2, V_4, V_9) = 0$. Hence, $\sigma(U_1^x, V_4, V_9) = 0$ and $V_4 \subset U_5^x$. The space V_4 can have dimension bigger than 4 only if all U_5^x are equal, but that leads to a contradiction with lemma 6.1. \square

All these lemmas together give the proposition 6.28. \square

Corollary 6.34. *The hyperplane $\mathbb{P}(V_9) \subset \mathbb{P}(V)$ is tangent to P along l .*

Proof. By proposition 6.28 V_9 contains T^x for all $x \in l$. The projective space $\mathbb{P}(T^x)$ is the maximal tangent space to P at x (see lemma 6.11). Hence, $\mathbb{P}(V_9)$ is tangent to P at any point $x \in l$. \square

Next we find the isomorphism classes of some vector bundles from proposition 6.28.

Proposition 6.35. *The vector bundle $\mathcal{U}_3 / (V_2 \otimes \mathcal{O}_l)$ is isomorphic to $\mathcal{O}_l(-1)$ and the vector bundle $\mathcal{K} / (V_2 \otimes \mathcal{O}_l)$ is isomorphic to $\mathcal{O}_l(-1) \oplus \mathcal{O}_l(-2)$.*

Proof. The morphism $x \mapsto U_3^x / V_2$ induces an isomorphism of $\mathbb{P}(V_2)$ and $\mathbb{P}(V_4 / V_2)$. Indeed, if a point of $\mathbb{P}(V_4 / V_2)$ has at least two point x and y in the pre-image, then $U_3^x = U_3^y$ and $\sigma(U_3^x, U_3^x, V) = 0$, but it contradicts lemma 6.1. Thus $\mathcal{U}_3 / (V_2 \otimes \mathcal{O}_l) \cong \mathcal{U}_1 \cong \mathcal{O}_l(-1)$.

Now we prove the second isomorphism. The determinant of \mathcal{K} is isomorphic to $\mathcal{O}_l(-3)$ (see corollary 6.12). Thus determinant of $\mathcal{K} / (V_2 \otimes \mathcal{O}_l)$ is also isomorphic to $\mathcal{O}_l(-3)$. The vector bundle \mathcal{K} is the subbundle of the trivial bundle $(V_4 / V_2) \otimes \mathcal{O}_l$. The only possibility for \mathcal{K} / V_2 except $\mathcal{O}_l(-1) \oplus \mathcal{O}_l(-2)$ is the vector bundle $\mathcal{O}_l \oplus \mathcal{O}_l(-3)$. In this case the quotient sheaf $\mathcal{K} / \mathcal{U}_3$ will not be a vector bundle, but the morphism $\mathcal{U}_3 \rightarrow \mathcal{K}$ is injective in any point of l . \square

¹⁰We do not mention U_5^x in the proposition 6.28, because U_5^x does not seem to be a subspace of T^x for a general $x \in l$

Next we are going to construct a foliation of rank one on Z . Chose a basis u_0, u_1, u_2, u_3 on V_4 such that $V_2 = \langle u_0, u_1 \rangle$. Recall that $\sigma(V_2, V_4, V_9) = 0$. Thus, the form σ defines the morphism

$$\phi' : V_2 \rightarrow (V_4/V_2)^* \otimes (V/V_9)^*; \quad u \in V_2 \mapsto \sigma(u, *, *).$$

Let x be a point of l and u be a vector from U_1^x , then $\phi'(u)$ is a bilinear form on $(V_4/V_2) \times (V/V_9)$. This form is zero at $(U_3^x/V_2) \times (V/V_9)$. Up to proportionality it gives a morphism $\phi \in \text{Hom}(V_2, V_4/V_2)$ (using the fact that $V_4/V_2 \cong (V_4/V_2)^*$ and $V/V_9 \cong \mathbb{C}$), such that:

$$\sigma(\phi(u_0) \wedge u_1 - \phi(u_1) \wedge u_0, V) = 0. \quad (6)$$

The morphism ϕ maps each U_1^x to $U_3^x/(V_2 \otimes \mathcal{O}_l)$ and gives an embedding of $\mathcal{U}_1 \rightarrow (\mathcal{K}/V_2) \otimes \mathcal{O}_l$. The tangent space to $Gr(2, V)$ at a point $[V_2]$ is $\text{Hom}(V_2, V/V_2)$. Thus the morphism ϕ defined above gives a vector tangent to $Gr(2, V)$ at V_2 . We prove that it also tangent to Z .

Proposition 6.36. *The morphism $\phi \in \text{Hom}(V_2, V_4/V_2)$ corresponds to a tangent vector to Z at V_2 .*

Proof. First we prove that it is tangent to F . By lemma 6.11 $T_P|_l$ is $\mathcal{T}(-1)/\mathcal{O}_l$. A morphism from $f \in \text{Hom}(V_2, V_4)$ corresponds to a vector tangent to F if and only if $f(U_1^x) \subset T^x/V_2$ for every $x \in l$. The morphism ϕ maps U_1^x to K^x/V_2 , which is contained in T^x/V_2 .

Next we prove that ϕ is tangent to Σ . Since Σ is the section of $Gr(2, V)$ by the hyperplanes $\sigma(v, *, *)$ (for all $v \in V$) in the Plücker embedding, a morphism $f \in \text{Hom}(V_2, V_4/V_4)$ corresponds to a vector tangent to Σ if and only if it satisfies

$$\sigma(f(u_0) \wedge u_1 - f(u_1) \wedge u_0, V) = 0.$$

The morphism ϕ satisfies this property by formula (6). □

Corollary 6.37. *The morphism ϕ generates a one-dimensional space tangent to Z at V_2 . That gives a foliation of rank one on Z , which we denote by \mathcal{F} .*

Conjecture 6.38. *Let $\pi : F \rightarrow X$ be the morphism defined in proposition 6.6. Restrict this fibration to Z and obtain a birational morphism $\pi|_Z : Z \rightarrow \pi(Z)$ of Z to a divisor $\pi(Z)$ in X . We expect that at a general point of Z the foliation \mathcal{F} is the pull-back of the characteristic foliation on $\pi(Z)$.*

6.5 Proof of theorem 6.26

In this section we prove theorem 6.26. Earlier in this chapter we have fixed a general alternating three-form $\sigma \in \wedge^3 V^*$. In the current section we will consider all $\sigma \in \wedge^3 V^*$. Consider the following incidence variety in $Gr(6, V) \times \mathbb{P}(\wedge^3 V^*)$:

$$I := \{(U_6, \sigma) \mid \sigma(U_6, U_6, U_6) = 0\}.$$

The fiber of I over a general σ has dimension 4 (the Debarre-Voisin variety X_σ). Let U_6 be a six-dimensional subspace of V and choose a section of Q_4 , then $V = U_6 \oplus Q_4$. The space of the alternating three-forms $\wedge^3 V^*$ decomposes as the following:

$$\wedge^3 V^* = \wedge^3 U_6^* \oplus \wedge^2 U_6^* \otimes Q_4^* \oplus U_6^* \otimes \wedge^2 Q_4^* \oplus \wedge^3 Q_4^*.$$

Hence, the kernel of $\wedge^3 V^* \rightarrow \wedge^3 U_6^*$ has dimension 100 and maps surjectively onto $\text{Hom}(Q_4, \wedge^2 U_6^*)$. That gives us the following diagram:

$$\begin{array}{ccc} I & \xrightarrow{\mathbb{P}(\ker(\wedge^3 V^* \rightarrow \wedge^3 U_6^*) \cong \mathbb{P}^{99})} & \text{Gr}(6, V) \\ \downarrow X_\sigma & & \\ \mathbb{P}(\wedge^3 V^*) & & \end{array}$$

For every couple $(U_6, \sigma) \in I$ one has the surjective morphism $\wedge^3 V^* \rightarrow \text{Hom}(Q_4, \wedge^2 U_6^*)$. Each morphism $f \in \wedge^3 V^* \rightarrow \text{Hom}(Q_4, \wedge^2 U_6^*)$ gives the morphism

$$f^{(3)} : S^3(V/U_6) \rightarrow \det U_6 \cong \mathbb{C}.$$

Let $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0$ be the variety of morphisms f such that $f^{(3)}$ is zero. We give an estimate of the dimension of the variety $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0$.

Lemma 6.39. *Codimension of $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0$ in $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))$ is at least 4.*

Proof. First note that the variety of degenerate morphisms (up-to proportionality) $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^d$ is of codimension $(\dim \wedge^2 U_6^* - \dim Q_4 + 1) = 12$ in $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))$. Hence, for this lemma we can consider only non-degenerate morphisms $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^{nd}$. Consider the following morphism

$$\xi : \mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^{nd} \rightarrow \text{Gr}(4, \wedge^2 U_6^*).$$

Clearly, all fibers of ξ are isomorphic. Let $Pf \subset \mathbb{P}(\wedge^2 U_6^*)$ be the cubic of all degenerate 2-forms. It is enough to estimate dimension of the variety of the linear projective spaces of dimension 3 contained in the cubic Pf . If the variety of these \mathbb{P}^3 has codimension less than 4 they cover whole projective space $\mathbb{P}(\wedge^2 U_6^*)$. \square

Denote the variety of the morphisms $f \in \mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))$ such that $f^{(3)} \neq 0$ by $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0$. We can define a morphism $\mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0 \rightarrow \mathbb{P}(S^3 Q_4^*)$, mapping f to $f^{(3)}$. F. Tanturri proved that this morphism is surjective.

Lemma 6.40. [62, Proposition 1.0.8] *Any cubic surface can be represented as the Pfaffian cubic for some morphism $f \in \text{Hom}(Q_4, \wedge^2 U_6^*)$.*

Let $I^0 \subset I$ be the an open subvariety of point of I such that the couple (σ, U_6) gives a morphism $f \in \mathbb{P}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0$. The complement $I \setminus I^0$ has codimension at least 4. Any couple $(\sigma, U_6) \in I^0$ gives the pfaffian cubic $C_{U_6, \sigma}$. Denote the total space of this family by \mathcal{C} . We have the following diagram:

$$\begin{array}{ccccc} \mathcal{C} & & & & \\ \downarrow C_{U_6, \sigma} & & & & \\ I^0 & \longrightarrow & \mathbb{P}_{\text{Gr}(6, V)}(\text{Hom}(Q_4, \wedge^2 U_6^*))^0 & \longrightarrow & \text{Gr}(6, V) \\ \downarrow X_\sigma & & & & \\ \mathbb{P}(\wedge^3 V^*) & & & & \end{array}$$

Where \mathcal{U}_6 is the tautological subbundle and \mathcal{Q}_4 it the tautological quotient bundle on $Gr(6, V)$. Since a general cubic surface is smooth and smoothness is an open property, for a general σ and U_6 the cubic surface $C_{U_6, \sigma}$ is smooth. This proves that for a general σ the morphism $\pi_Z : Z \rightarrow X$ is not surjective.

Next we prove that $\pi(Z)$ is a divisor. Let I^1 be the subvariety of I^0 such that the cubic C_{σ, U_6} is singular. Since any cubic surface can be represented as the Pfaffian cubic, I^1 has codimension one. A fiber of I^1 over σ is $\pi_\sigma(Z_\sigma)$. By dimension count we obtain that the dimension of the fiber of I^1 over a general σ is 3. That proves that $\pi(Z)$ is a divisor in X .

In the end we prove that $\pi_Z : Z \rightarrow \pi(Z)$ is birational. Let I^2 be the subvariety of I^1 such that the cubic $C_{U_6, \sigma}$ has two singular points. The fibers of I^1/I^2 over σ correspond to the cubics $C_{U_6, \sigma}$ with only one singular point. Any line in Z gives a singular point in one of these cubics. Since any cubic surface can be represented as the Pfaffian, I^2 has codimension one in I^1 . Hence, the dimension of the fiber of I^2 over a general σ is not greater than 2. Thus the dimension of I^2 over a general σ has dimension 2 and a general point the fibers of I^1 are not contained in I^2 . And hence a general singular cubic C_{U_6} has only one singular point. That means that $\pi_Z : Z \rightarrow \pi(Z)$ is birational.

6.6 Appendix

Calculation 6.41. *We use the following code in Macaulay2 [30] to calculate the intersection $c_8(S^2\mathcal{Q}_8^*(1)) \cdot c_5(S^4\mathcal{U}_2^*(-1))$ with σ_3 and σ_{21} .*

```
needsPackage "Schubert2";
G = flagBundle{2,8}; (U,Q) = bundles G;
O = det(Q);
h = chern_1 O;
E = O * (dual Q);
S = (symmetricPower_4 (dual U)) * (dual O) ;
c = chern_8 E;
d = chern_5 S;
s3 = chern_3 Q;
s21 = (chern_2 U) * h;
<< (s3*d*c, s21*d*c) / integral << endl; --(480, 468)
```

The table of the multiplication of $H^6(Gr(2, V), \mathbb{Z}) \times H^{10}(Gr(2, V), \mathbb{Z})$ is the following:

$$\begin{array}{cc} & \sigma_3 & \sigma_{21} \\ \sigma_{85} & 1 & 0 \\ \sigma_{76} & 0 & 1 \end{array}$$

That gives us that $c_8(S^2\mathcal{Q}_8^*(1)) \cdot c_5(S^4\mathcal{U}_2^*(-1)) = 480\sigma_{85} + 468\sigma_{76}$.

References

- [1] E. Amerik, *A computation of invariants of a rational self-map*, Ann. Fac. Sci. Toulouse **18** (2011).
- [2] E. Amerik and F. Campana, *Characteristic foliation on non-uniruled smooth divisors on projective hyperkaehler manifolds*, Journal of the London Mathematical Society **95** (2014), 115–127.

- [3] E. Amerik and L. Guseva, *On the characteristic foliation on a smooth hypersurface in a holomorphic symplectic fourfold*, Arxiv:1611.00416 (2016).
- [4] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, 2004.
- [5] A. Bayer and E. Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, *Inventiones mathematicae* **198** (2013), 505–590.
- [6] A. Beauville, *Varietes kahleriennes dont la premiere classe de chern est nulle*, *Journal of Differential Geometry* **18** (1984), 755–782.
- [7] ———, *Counting rational curves on K3 surfaces*, *Duke Math. J.* **97** (1999), 99–108.
- [8] A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, *C.R. acad. sc. Paris* **301** (1985), 703–706.
- [9] V. Benedetti and J. Song, *Divisors in the moduli space of Debarre-Voisin varieties*, arXiv:2106.06859 (2021).
- [10] M. Bernardara, E. Fatighenti, and L. Manivel, *Nested varieties of K3 type*, *Journal de l'École polytechnique — Mathématiques* **8** (2021), 733–778.
- [11] F. Bogomolov, *Hamiltonian kähler manifolds*, *Sov. Math. Dokl.* **19** (1978), 1462–1465.
- [12] F. Bogomolov and N. Kurnosov, *Lagrangian fibrations for ihs fourfolds*, arXiv:1810.11011 (2018).
- [13] P. Bonnet, *Minimal invariant varieties and first integrals for algebraic foliations*, *Bulletin Brazilian Mathematical Society* **37** (2006), 1–17.
- [14] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, *Annales Scientifiques de l'École Normale Supérieure* **37** (2004), 45–76.
- [15] D. Burns and M. Rapoport, *On the Torelli problem for kählerian K3 surfaces*, *Annales scientifiques de l'École Normale Supérieure Ser. 4*, **8** (1975), no. 2, 235–273.
- [16] F. Campana, *Connexité rationnelle des variétés de Fano*, *Annales Scientifiques De L Ecole Normale Superieure* **25** (1992), 539–545.
- [17] H. Clemens and P. Griffiths, *Annals of Mathematics*, **95** (1972), 281–356.
- [18] H. Clemens, J. Kollár, and M. Shigefumi, *Higher dimensional complex geometry*, *Astérisque*, Société mathématique de France, 1988.
- [19] M. A. A. de Cataldo and L. Migliorini, *The hard Lefschetz theorem and the topology of semismall maps*, *Annales scientifiques de l'École Normale Supérieure Ser. 4*, **35** (2002), no. 5, 759–772.
- [20] P. De Poi, D. Faenzi, K. Ranestad, and E. Mezzetti, *Fano congruences of index 3 and alternating 3-forms*, *Annales de l'Institut Fourier* **67** (2017), no. 5, 2099–2165.
- [21] O. Debarre, *Un contre-exemple au theoreme de torelli pour les varietes symplectiques irreductibles*, *C. R. Acad. Sc. Paris* **299** (1984), no. 14, 681–684.
- [22] O. Debarre and C. Voisin, *Hyper-kähler fourfolds and Grassmann geometry*, *J. Reine angew. Math.* **649** (2010), 63–87.
- [23] I. V. Dolgachev, *Classical algebraic geometry: A modern view*, Cambridge University Press, 2012.
- [24] H. Esnault and E. Viehweg, *Vanishing and non-vanishing theorems*, *Théorie de hodge - luminy*, juin 1987, 1989.
- [25] J. Fogarty, *Algebraic families on an algebraic surface*, *American Journal of Mathematics* **90** (1968), 511–521.
- [26] A. Fujiki, *On the de rham cohomology group of a compact kähler symplectic manifold*, *Adv. Studies in Pure Math. 1* **10** (1985), 105–165.

- [27] W. Fulton, *Young tableaux: With applications to representation theory and geometry*, Cambridge University Press, 1996.
- [28] M. Goresky and R. MacPherson, *Stratified Morse theory*, Springer-Verlag, 1988.
- [29] F. Gounelas and A. Kouvidakis, *Geometry of lines on a cubic fourfold*, arXiv: 2008.05162 (2020).
- [30] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*.
- [31] F. Han, *Pfaffian bundles on cubic surfaces and configurations of planes*, *Mathematische Zeitschrift* **278** (2014), 363–383.
- [32] ———, *Duality and quadratic normality*, *Rend. Istit. Mat. Univ. Trieste* **47** (2015).
- [33] J. Harris and T. W. Loring, *On symmetric and skew-symmetric determinantal varieties*, *Journal of Topology* **23** (1981), no. 1, 71–84.
- [34] D. Huybrechts, *Erratum to the paper: Compact hyperkaehler manifolds: basic results*, arXiv:math.0106014 (2001).
- [35] ———, *A global Torelli theorem for hyperkaehler manifolds (after Verbitsky)*, *Asterisque- Societe Mathematique de France* (2011).
- [36] ———, *Lectures on K3 surfaces*, Cambridge University Press, 2016.
- [37] D. Huybrechts and C. Xu, *Lagrangian fibrations of hyperkähler fourfolds*, arXiv:1902.10440 (2019).
- [38] J.-M. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, *Inventiones mathematicae* **174** (2007), 625–644.
- [39] J.-M. Hwang and K. Oguiso, *Characteristic foliation on the discriminant hypersurface of a holomorphic lagrangian fibration*, *American Journal of Mathematics* **131** (2007), 981–1007.
- [40] J.-M. Hwang and E. Viehweg, *Characteristic foliation on a hypersurface of general type in a projective symplectic manifold*, *Compositio Mathematica* **146** (2009), 497–506.
- [41] S. Ishii, *Introduction to singularities*, Springer-Verlag, 2014.
- [42] D. Kaledin, M. Lehn, and Ch. Sorger, *Singular symplectic moduli spaces*, *Inventiones mathematicae* **164** (2006), no. 3, 591–614.
- [43] Y. Kawamata, *Flops connect minimal models*, *Publications of the Research Institute for Mathematical Sciences* **44** (200705), 419–423.
- [44] S. Kebekus, L. Solá Conde, and M. Toma, *Rationally connected foliations after Bogomolov and McQuillan*, *Journal of Algebraic Geometry* **16** (2005).
- [45] S. Kleiman, *The transversality of a general translate*, *Compositio Mathematica* **28** (1974), no. 3, 287–297.
- [46] J. Kollár, *Rational curves on algebraic varieties*, Springer-Verlag, 1996.
- [47] R. Lazarsfeld, *Positivity in algebraic geometry*, Vol. 48, Springer, Berlin, Heidelberg, 2004.
- [48] H. Lee, *A note on the Hilbert scheme of points on a cusp curve*, *International Journal of Algebra and Computation* **22** (2012).
- [49] D. Matsushita, *Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds*, *Mathematical Research Letters* **7** (2000), 389–391.
- [50] ———, *Higher direct images of dualizing sheaves of Lagrangian fibrations*, *American Journal of Mathematics* **127** (2000), 243–259.
- [51] D. Matsushita and D.-Q. Zhang, *Zariski F-decomposition and Lagrangian fibration on hyperkähler manifolds*, *Mathematical Research Letters* **20** (2009).

- [52] G. Mongardi and C. Onorati, *Birational geometry of irreducible holomorphic symplectic tenfolds of O'Grady type*, arXiv:2010.12511 (2020).
- [53] G. Mongardi and A. Rapagnetta, *Monodromy and birational geometry of O'Grady's sixfolds*, ArXiv:1909.07173 (2019).
- [54] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, *Inventiones mathematicae* **77** (1984), no. 1, 101–116.
- [55] ———, *Biregular classification of Fano 3-folds of index 3*, *Proc. Natl. Acad. Sci.* **86** (1989), 3000–3002.
- [56] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Kyoto University, 1999.
- [57] G. Oberdieck, J. Shen, and Q. Yin, *Rational curves in holomorphic symplectic varieties and Gromov–Witten invariants*, *Advances in Mathematics* **357** (2019), 106829.
- [58] K. O'Grady, *Desingularized moduli spaces of sheaves on a K3 surface*, arXiv: 9805099 (1998).
- [59] ———, *A new six-dimensional irreducible symplectic variety.*, *Journal of Algebraic geometry* **12** (2003), 435–505.
- [60] K. Oguiso, *Picard number of the generic fiber of an abelian fibered hyperkähler manifold*, *Mathematische Annalen* **344** (2008), 929–937.
- [61] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *A Torelli theorem for algebraic surfaces of type K3*, *Mathematics of the USSR-Izvestiya* **5** (1971), 547–588.
- [62] F. Tanturri, *Pfaffian representations of cubic surfaces*, *Geometriae Dedicata* **168** (2014), no. 1, 69–86.
- [63] M. Verbitsky, *Cohomology of compact hyperkaehler manifolds*, arXiv:alg-geom.9501001 (1995).
- [64] ———, *A global Torelli theorem for hyperkähler manifolds*, arXiv:0908.4121 (2009).
- [65] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, *Complex projective geometry: Selected papers, 1992*, pp. 294–303.
- [66] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, 2003.
- [67] ———, *Intrinsic pseudovolume forms and K-correspondences*, *The Fano conference (2004)*, 761–792.
- [68] ———, *Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-kähler varieties*, Springer International Publishing, 2016.
- [69] K. Yoshioka, *Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface*, arXiv:1206.4838 (2012).

Titre : Géométrie des variétés hyper-kähleriennes et le feuilletage caractéristique.

Mots clés : Géométrie algébrique, Variétés hyper-kähleriennes, géométrie complexe, feuilletage.

Résumé : Dans cette thèse nous étudions le feuilletage caractéristique sur une hypersurface lisse dans une variété hyper-kählerienne. Voici une explication détaillée du problème. Soit Y une hypersurface lisse dans une variété hyper-kählerienne irréductible projective X de dimension $2n$ et σ une forme holomorphiquement symplectique sur X . Pour chaque point $x \in Y$ la forme σ est une forme non dégénérée sur $T_{X,x}$. Donc la forme restreinte à $T_{Y,x}$ est de corang 1 (c'est à dire le noyau de $\sigma|_{T_{Y,x}}$ est de dimension un). Le feuilletage caractéristique F sur une hypersurface Y est le noyau de la forme symplectique σ restreinte à Y .

On peut poser la question suivante : quelle est la dimension de la fermeture de Zariski de la feuille générale de F . Dans cette thèse nous avons trouvé la fermeture de Zariski d'une feuille générale de F dans certains cas.

Le premier cas est le suivant. Soit X une variété hyper-kählerienne irréductible projective de dimension $2n$. Soit X munie d'une fibration lagrangienne $\pi : X \rightarrow \mathbb{P}^n$. On appelle l'hypersurface Y dans X verticale s'il existe une hypersurface D dans \mathbb{P}^n telle que son image réciproque soit Y . Nous avons démontré que la fermeture de Zariski d'une feuille générale du feuilletage caractéristique sur Y est

une fibre de π , si Y est verticale et lisse.

Voici le deuxième cas. Soit Y une hypersurface lisse nef et big dans X . Nous avons démontré qu'une feuille générale du feuilletage caractéristique est Zariski dense dans Y . Récemment J. V. Pereira a montré qu'une hypersurface non-nef de carré de Beauville-Bogomolov non négatif ne peut pas être lisse.

Dans la suite de la thèse nous étudions le feuilletage caractéristique sur les hypersurfaces singulières. Nous présentons des exemples des hypersurfaces verticales telles que la fermeture d'une feuille générale est une sous-variété propre d'une fibre de la fibration lagrangienne. Après, nous étudions la variété X des droites sur une hypersurface cubique de \mathbb{P}^5 . Nous décrivons deux exemples d'hypersurfaces singulières Y dans X tel qu'une feuille générale du feuilletage caractéristique sur Y n'est pas Zariski dense dans Y . Vers la fin de la thèse nous étudions la variété hyperkählerienne de dimension 4 construite par O. Debarre et C. Voisin. Nous trouvons une hypersurface particulière dans cette variété et construisons un feuilletage naturel de rang un sur cette hypersurface. Nous conjecturons que ce feuilletage est le feuilletage caractéristique.

Title : Geometry of the holomorphic symplectic manifolds and the characteristic foliation.

Keywords : Algebraic geometry, holomorphic symplectic manifolds, complex geometry, foliations.

Abstract : In this thesis we study the characteristic foliation on a hypersurface in a smooth projective holomorphic symplectic manifold. Let us explain the problem in details. Let X be a smooth projective irreducible holomorphic symplectic manifold of dimension $2n$ and Y be a smooth hypersurface in X . Let σ be a holomorphic symplectic form on X . At every point x of Y the holomorphic symplectic form σ restricts to the tangent space $T_{Y,x}$ of Y at x as an alternating form of corang 1. Thus, it has one-dimensional kernel. The characteristic foliation F of Y is the kernel of symplectic form σ restricted to Y .

One can ask what could be the dimension of the Zariski closure of a generic leaf of F . In this thesis we find the answer to this question in certain cases.

In the first case the irreducible holomorphic symplectic manifold X is equipped with a Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^n$. One calls a hypersurface Y vertical if there exists a hypersurface D in \mathbb{P}^n such that Y is the pre-image of D . We proved that the Zariski closure of a generic leaf of the charac-

teristic foliation on Y is dense in a fiber of the Lagrangian fibration π .

In the second we consider a nef and big hypersurface Y in X . We prove that a generic leaf of the characteristic foliation on Y is Zariski dense in Y . Recently J. V. Pereira found that a non-nef hypersurface with non-negative Beaville-Bogomolov square can not be smooth.

Afterwards, we study the characteristic foliation on singular hypersurfaces. We give few examples of singular vertical hypersurfaces such that a generic leaf characteristic foliation is not Zariski dense in a fiber of the Lagrangian fibration. Next, we consider the variety X of lines in the cubic fourfold. We give two example of a singular hypersurface Y in X such that a generic leaf of the characteristic foliation on Y is not Zariski dense in Y . Towards the end of the thesis we study the holomorphic symplectic constructed by O. Debarre and C. Voisin. We find a special hypersurface in the manifold and construct a natural foliation of rang one on this hypersurface. We conjecture that this foliation is characteristic.