Sur la Topologie des Singularités Réelles

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UNIVERSITÉ SAVOIE MONT BLANC

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ SAVOIE MONT BLANC

Spécialité : **Mathématiques**

Arrêté ministériel : 25 Mai 2016

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préparée au sein du **Laboratoire de Mathématiques**
dans **l'École Doctorale MSTII**

Sur la Topologie des Singularités Réelles

Thèse soutenue publiquement le **18/12 2020**,
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Résumé

Le sujet de cette thèse est la topologie des fibres de Milnor des singularités réelles. Il y est établie une formule pour le calcul des groupes d'homologie des fibres de Milnor d'un germe de singularité d'hypersurface $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$; cette formule est valable sous certaines conditions portant sur les indices de Morse des points critiques d'une petite perturbation de la singularité. Supposons que $X \subset \mathbb{R}^n$ est un sous-ensemble algébrique réel équidimensionnel ayant un point singulier isolé à l'origine et que $f : X \to \mathbb{R}$ est la restriction d'une application polynomiale $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$. Dans cette situation une formule est établie pour la caractéristique d'Euler-Poincaré des fibres de Milnor de f à l'origine, valide sous l'hypothèse que f ait une morsification.

Keywords. *Singularités Réelles, Fibre de Milnor, Fibre de Milnor Motivique Réelle, Equivalence Blow-Nash, Equivalence Analytique Par Arcs, Cycles Évanescents Réels, Morsification, Groupes d'Homologie Singulier, Caractéristique d'Euler-Poincaré*

Abstract

The subject of study in this thesis is the topology of the Milnor fibres of real singularities. A formula for the homology groups of the real Milnor fibres of a germ of hypersurface singularity $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ is established; the formula in question is valid under certain constraints pertaining to the Morse indices of the critical points of a small perturbation of the singularity. Suppose that $X \subset \mathbb{R}^n$ is an equidimensional real algebraic subset having an isolated singular point in the origin and that $f : X \to \mathbb{R}$ is the restriction of a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$. In this situation a formula is established for the Euler-Poincaré characteristic of the Milnor fibres of f at the origin, valid under the hypothesis that f has a morsification.

Keywords. *Real Singularities, Milnor Fibre, Real Motivic Milnor Fibres, Blow-Nash Equivalence, Arc-Analytic Equivalence, Real Vanishing Cycles, Morsification, Singular Homology Groups, Euler-Poincaré Characteristic*

Acknowledgements

In writing this thesis I have profited from the wisdom, patience and goodwill of my supervisors Georges Comte and Michel Raibaut. I wish to sincerely thank them for their many hours spent correcting my work, and the many words of advice that they have bestowed upon me as to make the exposition of this project as clear as possible. I wish to thank the other PhD students at LAMA, past and present, in particular Clovis, Mickael, Pierre and Rémy for their company and for many interesting discussions. I also wish to thank my very dear friends Coline and Yvain with whom I have had many happy opportunities to discuss literature and such (and with Yvain, sharing many late conversations oftentimes by a lakeside view).

However if it had not been for my father Arne, my mother Lilianne and my sister Ida this project would not have been carried through. If it had not been for the words of encouragements of my father I would likely not have continued working in the many times of complete dispair. I wish to thank him for being there, always, to talk with. Those many moments when life seem dark he has always a good book or a masterpiece in one form or another in reserve, to guide me to better thoughts, to future days. My sister's care has carried our family through difficult times. She has been strong when none of us could be and if it had not been for her both me and my father could not have managed. And if it had not been for my late mother, passed away suddenly November 23 2017, I would not have been able to come to France and pursue my studies here. Throughout my life she has been a light for me, in her many ways, and she still is. She was the one who would tell me so that I could believe it to be true, that everything will work out in the end.
This work is dedicated to her memory.

Contents

CHAPTER 1

Introduction en Français

"$'O\varphi\vartheta\alpha\lambda\mu o\tilde{\upsilon} \quad \kappa\varrho\acute{\alpha}\tau\epsilon\iota$"

Maxime Delphique

1. Introduction

Si l'un des buts principaux de la théorie de singularités est d'obtenir une classification des singularités, une connaissance d'invariants calculables de nature topologique ou algébrique des singularités est en premier lieu souhaitable, et en second lieu une relation d'équivalence convenable qui préserve l'un de ces invariants. Pour les singularités définies sur le corps des nombres complexes de tels invariants sont depuis longtemps utilisés pour ce problème de classification[1], et plusieurs choix sont à faire entre différentes relations d'équivalence de germes, comme l'équivalence contact ou l'équivalence à μ-constant, qui préservent un ou plusieurs de ces invariants. Le but principal de cette thèse est d'étudier les invariants algébro-topologiques des fibres de Milnor des singularités réelles, et en premier lieu leurs nombres de Betti, mais aussi leur caractéristique d'Euler-Poincaré.

C'est en général un problème délicat de calculer efficacement des invariants pour les fibres de Milnor de singularités réelles, et par conséquent les différents problèmes de classification restent en général complètement ouverts. Un progrès notable a été fait récemment en considérant des invariants de nature motiviques, comme différentes *fonctions zêta motiviques*, à l'aide desquels G. Fichou [**22**] et J.-B. Campesato [**9**] ont classifié respectivement les singularités ADE et les polynômes de Brieskorn, pour *l'équivalence Blow-Nash* des germes de Nash. Un autre but de cette thèse est l'étude des fonctions zêta motiviques introduites par G. Comte et G. Fichou dans [**11**] où est montré que la valeur du *polynôme de Poincaré virtuel* des fibres de Milnor motiviques d'une famille de fonctions polynomiales $f_t : \mathbb{R}^n \to \mathbb{R}$

[1] Comme les nombres de Milnor, et de Tjurina pour ne citer que ces exemples.

paramétrée semi-algébriquement par un ensemble T, est génériquement constant, i.e ne dépend pas du paramètre $t \in T$ en-dehors d'un ensemble semi-algébrique de codimension positive dans T.

2. Singularités Réelles

2.1. Les Fibres de Milnor Réelles. Dans [**39**] John Milnor a étudié la topologie des ensembles analytiques réels et complexes dans un voisinage d'un point singulier isolé d'une application analytique, établissant l'existence d'un fibré lisse localement trivial au-dessus d'une boule épointée de rayon suffisamment petit, centrée en la valeur du point critique. En d'autres termes soit $f : (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^k, 0)$ un germe de fonction analytique réelle. Choisissons-en un représentant $f : M \to \mathbb{R}^k$ sur un voisinage ouvert contenant l'origine. Pour $k \geq 2$, Milnor montre l'énoncé suivant.

THEOREM 1.1 ([**46**, Theorem 4.2]). *Si la matrice jacobienne $Jac(f)$ est de rang maximal en dehors de l'origine, alors il existe $\delta_0 > 0$ tel que, si $\delta \in (0, \delta_0]$, il existe $\epsilon_0 > 0$ tel que pour tout $\epsilon \in (0, \epsilon_0]$ la restriction*

$$f : f^{-1}(\mathbb{B}^k_\epsilon \setminus \{0\}) \cap \mathbb{B}^{n+k}_\delta \to \mathbb{B}^k_\epsilon \setminus \{0\}$$

est une fibration lisse localement triviale.

Les fibres de cette fibration sont alors des variétés lisses et s'appellent les *fibres de Milnor* de f à l'origine. Pour $k \geq 2$ la boule épointée est connexe et donc les fibres de Milnor

$$\mathcal{F}_\eta = f^{-1}(\eta) \cap \mathbb{B}^{n+k}_\delta, \qquad \eta \in \mathbb{B}^k_\epsilon \setminus \{0\}$$

sont difféomorphes. En revanche, pour $k = 1$ il existe généralement, à difféomorphisme près, deux fibres de Milnor ; la positive $f^{-1}(\eta) \cap \mathbb{B}^{n+k}_\delta$ et la négative $f^{-1}(-\eta) \cap \mathbb{B}^{n+k}_\delta$, $\eta \in (0, \epsilon_0)$. On les désigne respectivement par

$$\mathcal{F}^+_\eta \text{ et } \mathcal{F}^-_\eta.$$

Très peu de propriétés topologiques générales de ces fibres réelles sont connues. Pour les singularités isolées d'hypersurfaces complexes et pour les singularités isolées d'intersections complètes on sait depuis J. Milnor [**39**, Theorem 6.5] et H. Hamm [**34**, Corollary 5.10] que les fibres de Milnor complexes ont le type d'homotopie d'un bouquet de sphères et que le nombre de ces sphères peut être calculé algébriquement [**34**, Formula 5.11.c] comme somme alternée de rang de certains modules, sous l'hypothèse que ces modules sont de type fini. Aucun résultat comparable n'est connu pour les fibres de Milnor réelles.

En particulier il n'existe pas de méthode systématique pour calculer leurs nombres de Betti, bien que des formules aient été établies pour leur caractéristique d'Euler-Poincaré, par G. M. Khimshiashvili [**29**] (voire [**16**, Theorem 2.3]) en 1977 quand $k = 1$ pour les fibres de Milnor positives et négatives $\mathcal{F}_\eta^+$ et $\mathcal{F}_\eta^-$, et plus récemment par N. Dutertre, D. Dreibelbis et R. Araújo dos Santos [**44**, Corollary 3.4]) pour $\mathcal{F}_\eta$ quand $k \geq 2$.

3. Fibres de Milnor Motiviques

À une variété algébrique non-singulière irréductible X définie sur un corps k de caractéristique zero et un morphisme non-constant $f : X \to \mathbb{A}_k^1$, J. Denef et F. Loeser [**14**] ont associé une fonction zêta motivique $Z(f)$, qui est une série formelle à coefficients dans l'anneau de Grothendieck $K_0(\mathrm{Var}_k)$ des variétés sur k. La fibre de Milnor motivique de f est alors définie comme une limite formelle de $Z(f)$. Quand $k = \mathbb{C}$ on retrouve à partir de cette fibre motivique la caractéristique d'Euler de la fibre de Milnor topologique complexe.

Si $k = \mathbb{R}$ ces méthodes ne mènent pas à une formule analogue pour la caractéristique d'Euler de $\mathcal{F}_\eta^+$ et $\mathcal{F}_\eta^-$. En revanche les fonctions zêta motiviques introduites par G. Comte et G. Fichou permettent de retrouver, au travers de la réalisation par la caractéristique d'Euler-Poincaré, $\chi_c(\mathcal{F}_\eta^+)$ et $\chi_c(\mathcal{F}_\eta^-)$. Ces fonctions zêta ont des coefficients dans l'anneau de Grothendieck des formules semi-algébriques basiques que nous décrivons maintenant.

3.1. Formules Semi-algébriques Basiques. Dans [**11**] G. Comte et G. Fichou définissent l'anneau de Grothendieck $K_0(\mathrm{BSA}_\mathbb{R})$ de formules semi-algébriques basiques comme un substitut, portant plus de structures géométriques, de l'anneau de Grothendieck $K_0(\mathrm{SA}_\mathbb{R}) \cong \mathbb{Z}$ des ensembles semi-algébriques. L'idée est de considérer à la place des ensembles semi-algébriques les formules-mêmes qui les définissent.

DEFINITION 1.2 ([**11**, § 1.3]). *Une formule semi-algébrique basique A en n variables est un nombre fini d'égalités, inégalités et inéquations de polynômes en n variables. Une telle formule est donc la donnée de $I, J, L \subset \mathbb{N}$ et $p_i, q_j, r_l \in \mathbb{R}[x_1, \ldots, x_n]$ pour $i \in I, j \in J, l \in L$ tels que*

$$A := \{p_i = 0, \quad q_j \neq 0, \quad r_l > 0, \quad \forall i \in I, \forall j \in J, \forall l \in L\}.$$

Une formule semi-algébrique basique sans inégalité et sans inéquation est dite algébrique (et constructible si la formule ne comporte pas d'inégalité). L'ensemble des formules semi-algébriques basiques en n variables est désigné par BSA_n et on écrit $BSA = \bigcup_{n\in\mathbb{N}} BSA_n$.

Si A est une formule de la forme donnée dans la définition 1.2 elle donne lieu à un ensemble semi-algébrique réel

$$A(\mathbb{R}) = \{\bar{x} \in \mathbb{R}^n \quad | \quad p_i(\bar{x}) = 0, \quad q_j(\bar{x}) \neq 0, \quad r_l(\bar{x}) > 0, \quad i \in I, j \in J, l \in L\}$$

qui s'appelle l'ensemble de points réels de la formule A.

DEFINITION 1.3 ([**11**, § 1.3]). *Soit G le groupe abélien libre $\langle [A] | A \in BSA \rangle$ des classes d'equivalences de formules semi-algébriques basiques modulo la relation*

$$\textit{si } A, B \textit{ sont algébriques et } A \cong B$$

$$\textit{par un isomorphisme algébrique alors } [A] = [B].$$

Soit $K_0(BSA_{\mathbb{R}})$ le quotient de G par les relations

(1) $[A, q = 0] + [A, q \neq 0] = [A], \quad A \in BSA_{n,\mathbb{R}}, \quad q \in \mathbb{R}[x_1, \ldots, x_n]$,
(2) $[A, q > 0] + [A, q < 0] + [A, q = 0] = [A], \quad A \in BSA_{n,\mathbb{R}}, q \in \mathbb{R}[x_1, \ldots, x_n]$,
(3) *Si A, B sont des formules semi-algébriques basiques avec des ensembles disjoints de variables alors*

$$[A][B] = [A, B],$$

où A, B est la conjonction de A et B.

Rappelons qu'étant donné un corps k le groupe Grothendieck $K_0(\mathrm{Var}_k)$ des variétés algébriques sur k est construit de façon à y obtenir la propriété d'excision pour les classes des sous-variétés fermées. Plus précisément :

DEFINITION 1.4 ([**37**, Definition 1.1]). *Soit G le groupe abélien libre des classes d'isomorphismes de variétés sur k. L'anneau Grothendieck $K_0(Var_k)$ est le quotient de G par les relations*

(1) *Si $Y \subset X$ est une sous-variété fermée alors $[X] = [X \setminus Y] + [Y]$,*
(2) $[X \times_k Y] = [X][Y]$.

La classe $[\mathbb{A}_k^1]$ de la droite affine est désignée par $\mathbb{L}_k$ et s'appelle le motif de Lefschetz.

DEFINITION 1.5 ([**37**, Definition 2.3]). *Le polynôme de Poincaré virtuel est l'unique morphisme d'anneau $\beta_{Var} : K_0(Var_{\mathbb{R}}) \to \mathbb{Z}[u]$ tel que si X est compacte, non-singulière et avec des nombres de Betti $b_i(X) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(X; \mathbb{Z}/2\mathbb{Z})$, alors*

$$\beta_{Var}(X)(u) = \sum_{i \geq 0} (-1)^i b_i(X) u^i.$$

Un lien entre les deux anneaux Grothendieck $K_0(\mathrm{Var}_{\mathbb{R}})$ et $K_0(\mathrm{BSA}_{\mathbb{R}})$ est donné par la proposition suivante.

PROPOSITION 1.6 ([**11**, Proposition 1.3]). *Le morphisme*

$$i : K_0(Var_{\mathbb{R}}) \to K_0(BSA_{\mathbb{R}}), \quad [A(\mathbb{R})] \to [A]$$

qui envoie la classe d'une variété algébrique sur la classe de la formule qui la définit, est injectif.

Grâce à cette proposition on peut étendre $\beta_{\mathrm{Var}} : K_0(\mathrm{Var}_{\mathbb{R}}) \to \mathbb{Z}[u]$ à un morphisme d'anneaux $K_0(\mathrm{BSA}) \to \mathbb{Z}[u][1/2]$. La définition de ce morphisme est donnée par récurrence sur le nombre d'inégalités dans les formules basiques.

DEFINITION 1.7 ([**11**, Proposition 3.1]). *Le polynôme de Poincaré des formules semi-algébiques basiques est l'unique morphisme d'anneaux $\beta_{BSA} : K_0(BSA) \to \mathbb{Z}[u][1/2]$ tel que*

(1) *Si $A \in BSA$ et $p \in \mathbb{R}[x_1, \ldots, x_n]$ alors*
$$\beta_{BSA}([A, p > 0]) = \frac{1}{4}\beta_{BSA}([A, p = z^2]) - \frac{1}{4}\beta_{BSA}([A, p = -z^2]) + \frac{1}{2}\beta_{BSA}([A, p \neq 0]),$$
(2) *Si $[A] \in K_0(Var_{\mathbb{R}})$ alors $\beta_{BSA}([A]) = \beta_{Var_{\mathbb{R}}}([A])$.*

Ce morphisme est alors tel que l'évaluation en $u = -1$ de $\beta_{\mathrm{BSA}}([A])(u)$ est la caractéristique d'Euler-Poincaré à supports compacts de l'ensemble de points réels de la formule A.

PROPOSITION 1.8 ([**11**, Proposition 3.4]). *Si $A \in BSA$ alors*

$$\beta_{BSA}([A])(-1) = \chi_c(A(\mathbb{R}))$$

où χ_c est la caractéristique d'Euler-Poincaré à supports compacts.

3.2. Fibres de Milnor Motiviques Réelles. Nous rappelons dans ce qui suit la construction des fonctions zêta motiviques de [**11**].

Considérons le *schéma des arcs* $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}})$ sur le spectre de $\mathbb{R}$. Son ensemble de points réels $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}})(\mathbb{R})$ est l'ensemble des points $\mathbb{R}[[t]]$-rationnels de $\mathbb{A}^{n+1}_{\mathbb{R}}$. Il existe un sous-schéma $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)$ ayant pour ensemble de points réels les arcs formels aboutissant à l'origine

$$\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)(\mathbb{R}) = \{\gamma \in (\mathbb{R}[[t]])^{n+1} \quad | \quad \gamma(0) = 0\}.$$

Alors pour chaque $k \in \mathbb{N}$, il existe un $\mathbb{R}$-schéma de type fini $\mathcal{L}_k(\mathbb{A}^{n+1}_{\mathbb{R}})$, le *schéma des arcs tronqués* à ordre $k+1$, ayant comme ensemble de points réels l'ensemble de points $\mathbb{R}[[t]]/t^{k+1}$-rationnels de $\mathbb{A}^{n+1}_{\mathbb{R}}$.

Dans ce qui suit nous considérons seulement les ensembles de points réels de ces schémas, et nous écrivons $\mathcal{L}(\mathbb{R}^{n+1})$, $\mathcal{L}(\mathbb{R}^{n+1}, 0)$ et $\mathcal{L}_k(\mathbb{R}^{n+1}, 0)$ au lieu de $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}})(\mathbb{R})$, $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)(\mathbb{R})$ et $\mathcal{L}_k(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)(\mathbb{R})$ respectivement.

Pour chaque $k \in \mathbb{N}$, il existe un morphisme de troncation

$$\pi_k : \mathcal{L}(\mathbb{R}^{n+1}, 0) \to \mathcal{L}_k(\mathbb{R}^{n+1}, 0)$$

qui associe à un arc sa $(k+1)$-ième troncation. Pour chaque arc tronqué

$$\gamma_k \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0), \qquad \gamma_k(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0$$

on définit une fonction

$$\mathrm{ord} : \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \to \mathbb{N} \cup \{\infty\},$$

associant son ordre $\operatorname{ord}(\gamma_k)$ à γ_k, et associant $\operatorname{ord}(0) = \infty$ à l'arc $\gamma_k \equiv 0$. Il existe également une application composante angulaire

$$\operatorname{ac} : \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \to \mathbb{R}, \qquad \operatorname{ac}(\gamma_k) = a_{\operatorname{ord}(\gamma_k)}.$$

On considère alors les formules semi-algébriques basiques suivantes

$$X^1_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \operatorname{ord}(f \circ \gamma) = k,\ \operatorname{ac}(f \circ \gamma) = 1\},$$

$$X^{-1}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \operatorname{ord}(f \circ \gamma) = k,\ \operatorname{ac}(f \circ \gamma) = -1\},$$

$$X^{>}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \operatorname{ord}(f \circ \gamma) = k,\ \operatorname{ac}(f \circ \gamma) > 0\},$$

$$X^{<}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \operatorname{ord}(f \circ \gamma) = k,\ \operatorname{ac}(f \circ \gamma) < 0\}.$$

DEFINITION 1.9 ([**11**, § 4.1]). *Pour chaque symbole $\epsilon \in \{\pm 1, <, >\}$ posons*

$$Z^\epsilon(f) := \sum_{k \geq 0} [X^\epsilon_{k,f}] \mathbb{L}^{-(n+1)k} T^k \in K_0(BSA_\mathbb{R})[\mathbb{L}^{-1}][T].$$

REMARK 1.10. $Z^1(f), Z^{-1}(f) \in K_0(Var_\mathbb{R})[\mathbb{L}^{-1}][T]$.

REMARK 1.11. *Par* [**11**, Proposition 1.3] *en appliquant $\beta_{BSA} : K_0(BSA) \to \mathbb{Z}[1/2]$ à $Z^{\pm 1}(f)$ on obtient les fonctions zêta motiviques définies dans* [**21**].

Soit $\epsilon \in \{\pm 1, <, >\}$. Pour démontrer la rationalité des $Z^\epsilon(f)$, dans [**11**] est établie une formule calculant les fonctions zêta motiviques via une résolution des singularités. Pour la décrire on commence avec la donnée d'un morphisme birationnel propre $\sigma : M \to \mathbb{R}^{n+1}$ d'une variété lisse M tel que

(1) La restriction $\sigma : M \setminus (f \circ \sigma)^{-1}(0) \to \mathbb{C}^{n+1} \setminus f^{-1}(0)$ est un isomorphisme algébrique,
(2) Les diviseurs réduits de $f \circ \sigma$ et $\det \operatorname{Jac} \sigma$ sont à croisements normaux,
(3) Le diviseur exceptionnel $E = (\sigma^{-1}(0))_{red}$ est une réunion de composantes irréductibles de $Y := ((f \circ \sigma)^{-1}(0))_{red}$.

Il existe ainsi des sous-ensembles $\mathcal{J} \subset \mathbb{N}$ et $\mathcal{K} \subset \mathcal{J}$ tels que

$$Y = \bigcup_{j \in \mathcal{J}} E_j,\ E = \bigcup_{k \in \mathcal{K}} E_k.$$

DEFINITION 1.12. *Pour chaque $i \in \mathcal{J}$ soit*

$$N_i = \operatorname{mult}_{E_i}(f \circ \sigma),\ \nu_i - 1 := \operatorname{mult}_{E_i} \det \operatorname{Jac}(f \circ \sigma).$$

Comme le diviseur de $f \circ \sigma$ est à croisements normaux dans M il existe pour chaque point $p \in Y$ un ouvert affine $U \subset M$ contenant p, une suite regulière de fonctions regulières $x_i : U \to \mathbb{R}$ et un $u \in \mathcal{O}_U^{\times}$ tels que

$$f \circ \sigma(x) = u(x) \prod_{i \in \mathcal{J}} x_i^{N_i}, \qquad \forall x \in U. \tag{1}$$

On stratifie Y afin que E soit réunion de strates, de la façon suivante.

DEFINITION 1.13. *Pour chaque sous-ensemble non-vide $I \subset \mathcal{J}$ posons*

$$E_I^o := \bigcap_{i \in I} E_i \setminus \bigcup_{j \in \mathcal{J} \setminus I} E_j$$

et soit $N_I = \gcd_{i \in I}(N_i)$.

Si $\epsilon \in \{\pm 1, <, >\}$ on construit une formule semi-algébrique basique $E_I^{o,\epsilon} \in \mathrm{BSA}$ telle que $E_I^{o,1}(\mathbb{R})$ est l'ensemble de points réels d'un revêtement étale de degré N_I de l'extension des scalaires de E_I^o à $\mathbb{C}$. Afin que cela définisse correctement une formule on peut utiliser (1) et argumenter localement sur des ouverts affines.

DEFINITION 1.14 ([**11**, § 4.1]). *Pour chaque symbole $\epsilon \in \{\pm 1, <, >\}$, si $U \subset M$ est l'ouvert affine donné par (1) posons*

$$R_{I,U}^1 := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) = 1\},$$

$$R_{I,U}^{-1} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) = -1\},$$

$$R_{I,U}^{>} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) > 0\},$$

$$R_{I,U}^{<} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) < 0\}.$$

où N_I est comme dans la Définition 1.13 et u comme dans (1). Si $M = (U_l)_{l \in L}$ est un recouvrement fini, posons

$$[E_I^{0,\epsilon}] = \sum_{S \subset L} (-1)^{|S|+1} [R^{\epsilon}_{I, \cap_{s \in S} U_s}] \in K_0(BSA).$$

La rationalité des fonctions zêta motiviques $Z^{\epsilon}(f)(T)$ est alors conséquence du théorème suivant.

THEOREM 1.15 ([**11**, Theorem 4.2], [**14**]). *Pour chaque symbole $\epsilon \in \{\pm 1, <, >\}$ les fonctions $Z^{\epsilon}(f)(T)$ satisfont*

$$Z^{\epsilon}(f)(T) = \sum_{I \cap \mathcal{K} \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{0,\epsilon}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

DEFINITION 1.16 ([**11**, Definition 4.5]). *Pour chaque symbole* $\epsilon \in \{\pm 1, <, >\}$, *posons*

$$S^{\epsilon}(f) := \sum_{I \cap K \neq 0} (1-\mathbb{L})^{|I|-1} [\tilde{E}_I^{0,\epsilon}] \in K_0(BSA_{\mathbb{R}}).$$

En accord avec la terminologie établie dans [**11**] on appelle $S^1(f)$ la *fibre de Milnor motivique positive* et $S^{-1}(f)$ la *fibre de Milnor motivique negative* de f en l'origine. En introduisant de nouvelle terminologie, nous allons appeler $S^{>}(f)$ le *tube de Milnor motivique positif*, et $S^{<}(f)$ le *tube de Milnor motivique négatif*. Ils ne dépendent pas du choix de la résolution de singularités σ, d'après [**11**, Remark 4.3].

Supposons maintenant que $f : \mathbb{R}^{n+1} \to \mathbb{R}$ ait un point critique isolé en l'origine. Soit (ϵ_0, δ_0) des données de Milnor comme dans l'énoncé du Théorème 1.1 et pour chaque $\delta \in (0, \delta_0)$ et $\eta \in (0, \epsilon(\delta))$ écrivons

$$\mathcal{F}_\eta^{>} := f^{-1}((0,\eta)) \cap \mathbb{B}_\delta,$$

$$\mathcal{F}_\eta^{<} := f^{-1}((-\eta,0)) \cap \mathbb{B}_\delta,$$

$$\mathcal{F}_\eta^{1} = f^{-1}(\eta) \cap \mathbb{B}_\delta,$$

$$\mathcal{F}_\eta^{-1} = f^{-1}(-\eta) \cap \mathbb{B}_\delta.$$

THEOREM 1.17 ([**11**, Theorem 4.12]). *Si*

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad f(0) = 0$$

est une fonction polynomiale avec un point critique isolé en l'origine, alors pour chaque $\delta \in (0, \delta_0)$ *et* $\eta \in (0, \epsilon(\delta))$,

$$\beta_{BSA}(S^{\epsilon}(f))(-1) = (-1)^n \chi_c(\mathcal{F}_\eta^{\epsilon}), \quad \epsilon \in \{\pm 1, <, >\}.$$

CHAPTER 2

Introduction

Between the idea
And the reality
Between the motion
And the act
Falls the Shadow

"The Hollow Men"
T.S Elliot

1. Introduction

If one of the principal objectives in the study of the singularities of maps of algebraic varieties is to obtain reasonable classifications thereof, then an adequate understanding of calculable invariants of topological or algebraic nature is desirable first of all, and secondly, suitable equivalence relations preserving a given such invariant. For singularities defined over the field of complex numbers several such invariants has been used for the classification problem, amongst others the Milnor number and the Tjurina number, and several equivalence relations, amongst others the contact equivalence and the μ-constant equivalence relations, preserving one or more of these invariants. The principal objective of this thesis is to study the invariants of algebro-topological nature of the Milnor fibres of real singularities, and in particular their Betti numbers, but also their Euler-Poincaré characteristic.

It is a delicate problem to effectively calculate topological invariants for the Milnor fibres of real [1] singularities and as a consequence, the different classification problems at hand, are mostly completely open at the present date. There

[1] More generally, in the words of Réné Thom as quoted in the introduction (p. 5) to the book [**6**], real algebraic geometry, in Thom's view an example of mathematics pertaining to phenomena where "reality plays an essential role", has historically been neglected in lieu of the "even too beautiful" complex geometry. This might arguably explain at least in part why relatively little is known of the topology of real singularities.

has however recently been made a significant progress by considering instead of invariants of topological nature, "motivic" invariants in the form of different *motivic zeta functions*. Using these G. Fichou [**22**] and J.-B. Campesato [**9**] have classified respectively ADE-singularities and Brieskorn polynomials, up to the *Blow-Nash equivalence* relation on germs of Nash functions. In light of these advances, the secondary objective of this thesis is to study of the motivic zeta functions introduced by G. Comte and G. Fichou in [**11**] and to establish that the value of the *virtual Poincaré polynomial* of the motivic Milnor fibres of a semialgebraic family of polynomial functions $f_t : \mathbb{R}^n \to \mathbb{R}$ parametrised by T, is generically constant, that is does not depend on the parameter $t \in T$ outside of a semialgebraic set of positive codimension in T.

2. Real Singularities

2.1. Introductory Remarks. In his article [**40**] of 1952 John Nash adressed the question of when a compact C^∞-manifold can be given the structure of a real algebraic variety. He defined therein that which are now called Nash manifolds and Nash morphisms, which have since then become important objects of research in real algebraic geometry and singularity theory[2] and he proved that any compact smooth manifold is diffeomorphic (in fact isotopic) to a connected component of a real algebraic set. The latter result was then improved upon by Tognoli, who in 1973 [**49**] proved that any such manifold is diffeomorphic to some nonsingular real algebraic set. Further on, in 1981, Akbulut and King [**3**] proved that any nonsingular real algebraic set is diffeomorphic to the interior of a compact C^∞-manifold. As a consequence, one thus has a classification of *nonsingular* real algebraic sets, up to diffeomorphism.

2.2. A Milnor Fibration. The topological behaviour of real and complex algebraic (or more generally analytic) sets near an isolated singular point was investigated by John Milnor in his now classical treatise [**39**] from 1968. In it he showed, as a means of constructing differentiable structures on topological manifolds and in particular to produce examples of exotic spheres, that to any singularity of a complex hypersurface one can associate [**39**, Theorem 4.8] a locally trivial fibration over the pointed disc. Furthermore he proved the following real analogue

[2] In another article, of 1969, Nash defined and studied arc spaces, used by Denef and Loeser [**14**] to define motivic integration and motivic zeta functions of singularities.

(for $k > 1$).

Let $f : (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^k, 0)$ be a germ of real analytic function and let us choose a representative $f : M \to \mathbb{R}^k$ on a small neighborhood $M \subset \mathbb{R}^{n+k}$ containing the origin and let $\mathrm{Jac}(f)$ denote the Jacobian matrix of this representative.

THEOREM 2.1 ([**46**, Theorem 4.2]). *If the Jacobian matrix $Jac(f)$ has rank k in any $x \neq 0$, then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the restriction*

$$f : f^{-1}(\mathbb{B}^k_\epsilon \setminus \{0\}) \cap \mathbb{B}^{n+k}_\delta \to \mathbb{B}^k_\epsilon \setminus \{0\} \tag{2}$$

is a locally trivial topological fibration where $\mathbb{B}^{n+k}_\delta \subset \mathbb{R}^{n+k}$ denotes the open $(n+k)$-dimensional ball centered at the origin and of radius δ and where $\mathbb{B}^k_\epsilon \subset \mathbb{R}^k$ denotes the open k-dimensional ball centered at the origin and of radius ϵ.

When $k = 1$ there are thus up to diffeomorphism two fibres

$$\mathcal{F}^{\pm}_\eta := f^{-1}(\pm\eta) \cap \mathbb{B}^{n+1}_\delta, \qquad 0 < \eta \leq \epsilon$$

of the locally trivial fibration 2 in the statement of the Theorem 2.1. These are referred to as the *positive* and *negative* open *Milnor fibres* of f at the origin. The closures $\bar{\mathcal{F}}^{\pm}_\eta$ for the euclidean topology of the positive and negative open Milnor fibres are then manifolds with boundaries and are referred to as the positive and negative *closed Milnor fibres* of f at the origin.

When $k \geq 2$ there is up to diffeomorphism a unique fibre

$$\mathcal{F}_\eta = f^{-1}(\eta) \cap \mathbb{B}^{n+k}_\delta, \qquad \eta \in \mathbb{B}^k_\epsilon \setminus \{0\}$$

of (2) because then the codomain is connected. The manifold $\mathcal{F}_\eta$ is then referred to as the open Milnor fiber at the origin. The closure $\bar{\mathcal{F}}_\eta$ for the euclidean topology of the open Milnor fibre is a smooth manifold with boundary and is referred to as the closed real Milnor fibre of f at the origin.

2.3. An Outline of Known Results. As for the topology of the real Milnor fibres, relatively little is known, at least if one were to compare with the situation over the complex numbers. We now outline what has been established, beginning with a startling congruence which is somewhat typical in real algebraic geometry. Here $p \in \mathbb{R}^{n+1}$ is an isolated critical point of a germ of polynomial function $f : (\mathbb{R}^{n+1}, p) \to (\mathbb{R}, p)$ if p is an isolated critical point of any representative $f : M \to \mathbb{R}$ of this germ.

PROPOSITION 2.2 ([**36**, Proposition 1.2]). *If the origin is an isolated critical point of the germ of a real analytic function $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ and if (ϵ_0, δ_0) is Milnor data at the origin as in (Theorem 2.1) then for any $\eta \in (0, \epsilon(\delta)]$ the Euler-Poincaré characteristic of the closed Milnor fibres satisfy the congruence relation*

$$\chi(\bar{\mathcal{F}}_\eta^+) \equiv \chi(\bar{\mathcal{F}}_\eta^-) \qquad (mod\ 2).$$

This congruence relation has been proven by C. McCrory and A. Parusiński [**36**]) as follows[3].
Consider a representative $f : M \to \mathbb{R}$ of the germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ where $M \subset \mathbb{R}^{n+1}$ is a small open neighborhood of the origin and let

$$f_\mathbb{C} : M_\mathbb{C} \to \mathbb{C}$$

denote a complexification of this representative where $M_\mathbb{C} \subset \mathbb{C}^{n+1}$ is an open neighborhood of the origin such that $M_\mathbb{C} \cap \mathbb{R}^{n+1} = M$. Let $c : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ denote complex conjugation and let $\mathcal{F}_{\mathbb{C},\eta}$ denote a Milnor fibre (see [**39**, Theorem 4.8]) at the origin of the holomorphic function $f_\mathbb{C}$. Using the fact that the geometric monodromy

$$h : \mathcal{F}_{\mathbb{C},\eta} \to \mathcal{F}_{\mathbb{C},\eta}$$

satisfies, up to isotopy, the relation $chch = Id$. McCrory and Parusiński then obtain the congruence above and moreover obtain another proof of the Theorem [**13**, Theorem 1] of M. Coste and K. Kurdyka on the genericity modulo four of links of irreducible real algebraic sets. Another proof of Proposition 2.2, using real motivic zeta functions, is given by G.Fichou in [**23**]. Yet another proof follows easily in the case of polynomial map germs from the following formula, obtained by G.M. Khimshiashvili [**29**] in 1977, which we now recall.

THEOREM 2.3 ([**16**, Theorem 2.3]). *If the origin is an isolated critical point of the polynomial map germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ and if (ϵ_0, δ_0) is a Milnor data at*

[3] As remarked to me by A. Parusiński the result also follows from the theory of algebraically constructible functions [**41**, Theorem 6.6]

the origin, as in Theorem 2.1, then for any $\eta \in (0, \epsilon(\delta)]$,

$$\chi(\bar{\mathcal{F}}_\eta^+) = 1 - (-1)^{n+1} \deg_0(\nabla f), \qquad \chi(\bar{\mathcal{F}}_\eta^-) = 1 - \deg_0(\nabla f)$$

where $\deg_0(\nabla f)$ *is the topological degree of the map*

$$\frac{\nabla f}{\|\nabla f\|} : \mathbb{S}_\delta^n \to \mathbb{S}_\delta^n$$

where $\mathbb{S}_\delta^n = \partial \bar{\mathbb{B}}_\delta^{n+1}$ *denotes the* n*-dimensional sphere of radius* δ *centered at the origin.*

One can note here that the following interesting corollary results.

Corollary 2.4. *If* n *is odd then*

$$\chi(\bar{\mathcal{F}}_\eta^+) = \chi(\bar{\mathcal{F}}_\eta^-).$$

In the case of higher codimensional ($k \geq 2$) isolated real singularities N. Dutertre, D. Dreibelbis and R. Araújo dos Santos have quite recently found the following analogue of Khimshiashvilis formula.

Corollary 2.5 ([**44**, Corollary 3.4]). *Let* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ *with* $n > k \geq 2$ *be a polynomial map germ with an isolated singularity at the origin.*
Let $f = (f_1, \ldots, f_k)$ *be an arbitrary representative of the germ. Denote by* $\deg_0(\nabla f_i)$ *the topological degree of*

$$\frac{\nabla f_i}{\|\nabla f_i\|} : \mathbb{S}_\delta^{n-1} \to \mathbb{S}_\delta^{n-1}$$

and let $\bar{\mathcal{F}}_\eta$ *denote a closed Milnor fibre as in Theorem 2.1. Then the following holds.*

(1) *If* n *is even then* $\chi(\bar{\mathcal{F}}_\eta) = 1 - \deg_0(\nabla f_1)$ *and*

$$\deg_0(\nabla f_1) = \cdots = \deg_0(\nabla f_k).$$

(2) *If* n *is odd then* $\chi(\bar{\mathcal{F}}_\eta) = 1$ *and*

$$\deg_0(\nabla f_1) = \cdots = \deg_0(\nabla f_k) = 0.$$

In the complex realm, as already stated, Milnor proved the existence of a fibration corresponding to holomorphic germs and he moreover proved the following classical result

THEOREM 2.6 ([**39**, Theorem 6.5], [**46**, Theorem 7.2]). *If the origin is an isolated critical point of the germ of holomorphic function* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *then the Milnor fibre*

$$\mathcal{F}_{\mathbb{C}} = f^{-1}(\eta) \cap \bar{\mathbb{B}}_{\delta}^{n+1}(\mathbb{C})$$

has the homotopy type of a bouquet of spheres $\bigvee_{i=1}^{\mu(f,0)} \mathbb{S}^n$. *The number of spheres* $\mu = \mu(f, 0)$ *is called the* Milnor number *of* f *at the origin. It is nonnegative and* $\mu = 0$ *if and only if* $0 \in \mathbb{C}^{n+1}$ *is a regular point of* f.

REMARK 2.7. *That the analoguous Milnor fibre exists and has the homotopy type of such a bouquet of spheres have since been established for complete intersections* $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$ *by H. Hamm (see e.g* [**34**, Corollary 5.10]*)*[4].

No analogue of Theorem 2.6 has to this date been established for the real Milnor fibers of germs of isolated real singularities $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^k, 0)$ and as for the homotopy type of the real Milnor fibers, not much can be said. Even if one were to restrict ones attention to the singular homology groups of the real Milnor fibers, again, not much can be said except for two results which were established in 2014 by R. Araújo dos Santos, M. Hohlenwerger, O. Saeki and T. Souza. We state their results here for the sake of completeness.

PROPOSITION 2.8 ([**45**, Proposition 5.4]). *Let* $f : (\mathbb{R}^{2k}, 0) \to (\mathbb{R}^k, 0)$ *be a polynomial map germ with an isolated singularity at the origin,* $k \geq 2$. *Given a representative* $f = (f_1, \dots, f_k)$, *if* $\bar{\mathcal{F}}_\eta$ *denotes a Milnor fibre at the origin (Theorem 2.1) then*

(1) $\beta_{k-1} := \dim H_{k-1}(\bar{\mathcal{F}}_\eta, \mathbb{Z}) = (-1)^k \deg_0(\nabla f_1)$.
(2) *The Milnor fibre* $\bar{\mathcal{F}}_\eta$ *has the homotopy type of a bouquet*

$$\bar{\mathcal{F}}_\eta \sim \bigvee_{i=1}^{\beta_{k-1}} \mathbb{S}^{k-1}$$

where it is meant a point if $\beta_{k-1} = 0$.

They also showed

[4] Such bouquet decomposition theorems exist also more generally for holomorphic map germs $f : (X, 0) \to (\mathbb{C}, 0)$ such that both f and X have an isolated singular point in the origin and $\dim X \neq 3$. D. Siersma ([**47**, Theorem 0.1]) proved namely in this case that the Milnor fiber has the homotopy type of a bouquet $lk_0(X) \vee \mathbb{S}^n \vee \cdots \vee \mathbb{S}^n$. A generalisation of Siersma's result was later established by M. Tibar [**48**, "Bouquet Theorem"].

PROPOSITION 2.9 ([**45**, Proposition 5.6]). *Let* $f : (\mathbb{R}^{2k+1}, 0) \to (\mathbb{R}^k, 0)$ *with* $k \geq 3$ *be a polynomial map germ with an isolated singularity at the origin. Then the* $(k-1)$*-th homology group* $H_{k-1}(\bar{\mathcal{F}}_\eta, \mathbb{Z})$ *of the Milnor fibre is torsion free if and only if there is a homotopy equivalence*

$$\bar{\mathcal{F}}_\eta \sim \bigvee_{i=1}^{\beta_{k-1}} \mathbb{S}^{k-1} \vee \mathbb{S}^k.$$

3. Real Motivic Milnor Fibers

In this section we shall discuss motivic zeta functions in the case of real isolated singularities. To put this into perspective we first recall a formula of Norbert A'Campo and briefly discuss the classical motivic zeta functions of J. Denef and F. Loeser.

3.1. A'Campo's Formula. The classical formula [**2**, Théorème 1] of A'Campo implies that in the case of hypersurface singularities the Lefschetz numbers of the iterates of the monodromy acting on the complex Milnor fibre can be computed from a resolution of singularities and in particular this yields a formula for the Euler characteristics of the Milnor fibre.

The situation is as follows. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map with $f(0) = 0$ and suppose that the origin is a critical point. Write $\mathcal{F}_\eta = f^{-1}(\eta) \cap \bar{\mathbb{B}}_\delta$ for a representative of the Milnor fibre at the origin. By the resolution of singularities theorem of Hironaka [**27**] there exists a proper birational map $\sigma : M \to \mathbb{C}^{n+1}$ from a smooth connected complex algebraic variety M such that the following hold.

(1) The restriction $\sigma : M \setminus (f \circ \sigma)^{-1}(0) \to \mathbb{C}^{n+1} \setminus f^{-1}(0)$ is an algebraic isomorphism.
(2) The reduced divisor $Y := (f \circ \sigma)^{-1}_{red}(0) \subset M$ has normal crossings.
(3) The reduced divisor $E := (\sigma^{-1}(0))_{red}$ is a union of irreducible components of Y.

It follows that if $Y = \bigcup_{j \in \mathcal{J}} E_j$ denotes the decomposition into irreducible components of Y then there exists a subset of indices $\mathcal{K} \subset \mathcal{J}$ such that $E = \bigcup_{i \in \mathcal{K}} E_i$ is a union of irreducible components.

DEFINITION 2.10. *For any* $i \in \mathcal{J}$*, let* $N_i = mult_{E_i}(f \circ \sigma)$ *be the multiplicity of* $(f \circ \sigma)^{-1}(0)$ *along* E_i*. For any* $I \subset \mathcal{J}$ *let* $N_I = \gcd_{i \in I} N_i$*.*

Let us stratify Y in such a way that E is a union of strata.

DEFINITION 2.11. *For any nonempty subset* $\emptyset \neq I \subset \mathcal{J}$ *let*

$$E_I^o = \bigcap_{i \in I} E_i \setminus \bigcup_{j \in \mathcal{J} \setminus I} E_j.$$

The theorem of A'Campo now reads

THEOREM 2.12 ([**2**, Théorème 1]). *For any* $k \in \mathbb{N}$*, the Lefschetz number* $\Lambda(h_k)$ *of the* k*-th iterate of the monodromy action* $h : \mathcal{F}_\eta \to \mathcal{F}_\eta$ *of the Milnor fibre satisfies*

$$\Lambda(h^k) = \sum_{I \subset K, N_I | k} N_I \chi(E_I^0).$$

In particular the Euler characteristic of the Milnor fibre satisfies

$$\chi(\mathcal{F}_\eta) = \Lambda(h^0) = \sum_{I \subset K} N_I \chi(E_I^0).$$

A'Campo proved this theorem in 1973, by using the spectral sequence of the sheaf of vanishing cycles as to reduce the calculation to the case of a normal crossings singularity, for which the Lefschetz numbers had been computed by him in a previous article.

Whilst constructing their theory of motivic integration, J. Denef and F. Loeser associated in their article [**14**] to any nonconstant morphism

$$f : X \to \mathbb{A}_k^1$$

from an irreducible nonsingular algebraic variety over a field k of characteristic zero, motivic zeta functions. These are formal power series with coefficients in a ring of motives, which is to say in this setting, a localisation of the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$. The coefficients are then the motives corresponding to certain elements of the arc spaces introduced by Nash. A certain formal limit of the motivic zeta functions is then called the *motivic Milnor fibre* of f. We will not delve further into this matter, lest to say that when $k = \mathbb{C}$ one recovers most of the usual invariants of the complex Milnor fibre (such as the Euler characterstic) from the motivic Milnor fibre, by applying a certain realisation morphism.

We will however discuss a version for real polynomial maps, due to G.Comte and G.Fichou, from which one obtains the Euler characteristic of the real Milnor fibres. As their construction is based on “motives” as given by a localisation of the ring of *basic semialgebraic formulas*, we first need to discuss the latter.

3.2. Basic Semialgebraic Formulas. In the article [**11**] G. Comte and G. Fichou introduced the Grothendieck ring $K_0(\mathrm{BSA}_{\mathbb{R}})$ of basic semialgebraic formulas. They were brought to do so because of the lack of algebraic structure of the Grothendieck ring $K_0(\mathrm{SA}_{\mathbb{R}})$ of real semialgebraic sets; indeed the latter ring is isomorphic to the ring of integers generated by the class of a point. The idea then was to consider not semialgebraic sets themselves but rather the formulas defining them and thus produce a ring with richer structure. In what follows we will denote $\mathrm{Var}_{\mathbb{R}}$ the category of varieties over $\mathrm{Spec}(\mathbb{R})$ and if $X \in \mathrm{Var}_{\mathbb{R}}$ is a real variety we shall denote by $X(\mathbb{R})$ its set of real points.

Definition 2.13 ([**11**, § 1.3]). *A basic semialgebraic formula in n variables A is a finite number of equalities, inequalities and inequations of polynomials in n variables. That is, there exists $I, J, L \subset \mathbb{N}$ and polynomials $p_i, q_j, r_l \in \mathbb{R}[x_1, \dots, x_n]$ for $i \in I, j \in J$ and $l \in L$ such that*

$$A = \{p_i = 0, \quad q_j \neq 0, \quad r_l > 0, \quad \forall i \in I, \forall j \in J, \forall l \in L\}.$$

A basic semialgebraic formula without inequalities and inequations is called algebraic. The set of semialgebraic formulas in n variables is denoted by BSA_n. The union of the sets BSA_n over $\mathbb{N}$ is denoted by BSA.

Remark 2.14. *A basic semialgebraic formula is not determined by its set of real points*

$$A(\mathbb{R}) = \{\bar{x} \in \mathbb{R}^n \mid p_i(\bar{x}) = 0, \quad q_j(\bar{x}) \neq 0, \quad r_l(\bar{x}) > 0, \quad i \in I, j \in J, l \in L\}.$$

For instance $A = \{x^2 + a > 0\}$ and $B = \{x^2 + b > 0\}$ with a, b positive real numbers, define different formulas although their sets of real points are the same.

An isomorphism of formulas is defined as follows

Definition 2.15 ([**11**, Definition 2.6]). *Let*

$$A = \{A', p_i > 0, i = 1, \dots, l\}, \qquad B = \{B', q_i > 0, i = 1, \dots, l\}$$

be formulas such that $A', B' \subset \mathbb{R}^n$ *are Zariski constructible subsets and* $p_i, q_i \in \mathbb{R}[x_1, \dots, x_n]$*. We say that* A *and* B *are isomorphic if there exists an isomorphism of real algebraic varieties*

$$\phi : \tilde{A} \to \tilde{B}$$

where

$$\tilde{A} = \{A', p_1 = z_1^2, \dots, p_l = z_l^2\},$$

$$\tilde{B} = \{B', q_1 = z_1^2, \dots, q_l = z_l^2\}$$

such that ϕ *is equivariant with respect to the action of* $\{\pm 1\}^l$ *on* $\tilde{A}$ *and* $\tilde{B}$ *and such that the morphism induced by extension of scalars* $\phi_{\mathbb{C}} : \tilde{A}_{\mathbb{C}} \to \tilde{B}_{\mathbb{C}}$ *is a complex algebraic isomorphism.*

DEFINITION 2.16 ([**11**, § 1.3]). *Let* G *be the free abelian group* $\langle [A] | A \in BSA \rangle$ *of equivalence classes of basic semialgebraic formulas where if* A *and* B *are algebraic, that is have no inequalities, and if* $A \cong B$ *by an algebraic isomorphism then* $[A] = [B]$*. Let* $K_0(BSA_{\mathbb{R}})$ *be the quotient of* G *by the relations*

(1) $[A, q = 0] + [A, q \neq 0] = [A], \quad A \in BSA_{n,\mathbb{R}}, \quad q \in \mathbb{R}[x_1, \dots, x_n],$

(2) $[A, q > 0] + [A, q < 0] + [A, q = 0] = [A], \quad A \in BSA_{n,\mathbb{R}}, q \in \mathbb{R}[x_1, \dots, x_n].$

(3) *If* A, B *are basic semialgebraic formulas with disjoints sets of variables then*

$$[A][B] = [A, B]$$

where A, B *is the conjunction of the formulas* A *and* B*.*

Recall that given a field k the Grothendieck group[5] $K_0(\mathrm{Var}_k)$ of algebraic varieties over the spectrum of k is formed by isomorphism classes of varieties in such a way that one has the excision property for closed subvarieties.

DEFINITION 2.17 ([**37**, Definition 1.1]). *Let* G *be the free abelian group of isomorphism classes* $[X]$ *of varieties* X *over the spectrum of* k*. The Grothendieck group* $K_0(\mathit{Var}_k)$ *is the quotient of* G *modulo the relation*

(1) *If* $Y \subset X$ *is a closed subvariety then* $[X] = [X \setminus Y] + [Y]$*.*

The quotient of G *by the above relation and by the supplementary relation*

$$[X \times_k Y] = [X][Y]$$

[5]More generally, in any exact category there is a notion of Grothendieck group

gives $K_0(Var_k)$ the structure of a commutative ring. The class $[\mathbb{A}^1_k]$ of the affine line is denoted by $\mathbb{L}_k$ and is called the Lefschetz motive.

We have that $K_0(\mathrm{Var}_{\mathbb{R}})$ is isomorphic to a subring of $K_0(\mathrm{BSA})$ by the following proposition:

Proposition 2.18 ([**11**, Proposition 1.3]). *The morphism*

$$i : K_0(Var_{\mathbb{R}}) \to K_0(BSA_{\mathbb{R}})$$

defined by sending the class of an algebraic variety to the formula defining it, is an injection of rings.

By the universal property of the Grothendieck ring $K_0(\mathrm{Var}_k)$ if $(R, \cdot, +)$ is a commutative ring and if $e : \mathrm{Var}_k \to R$ is any application such that

(1) $e(X) = e(Y)$ if X and Y are isomorphic k-varieties,
(2) $e(X) = e(Y) + e(X \setminus Y)$ if $Y \subset X$ is a closed subvariety,
(3) $e(X \times_k Y) = e(X) \cdot e(Y)$

then there exists a unique ring homomorphism $\tilde{e} : K_0(\mathrm{Var}_k) \to R$ such that $\tilde{e}([X]) = e(X)$ for any $X \in \mathrm{Var}_k$. Applications $e : \mathrm{Var}_k \to R$ as above are called (additive and multiplicative) invariants in the literature and ring homomorphisms $\tilde{e} : K_0(\mathrm{Var}_k) \to R$ are called realisations of $K_0(\mathrm{Var}_k)$. When $k = \mathbb{C}, \mathbb{R}$ the Euler characteristics with compact supports gives an example of a realisation of $K_0(\mathrm{Var}_k)$; another is given by the virtual Poincaré polynomial[6] the existence of which in the real case was first proven by C. McCrory and A. Parusiński.

Definition 2.19 ([**37**, Definition 2.3]). *The virtual Poincaré polynomial is the unique ring morphism $\beta_{Var} : K_0(Var_{\mathbb{R}}) \to \mathbb{Z}[u]$ such that*

$$\beta_{Var}(X)(u) = \sum_{i \geq 0} (-1)^i b_i(X) u^i$$

whenever X is a compact nonsingular real variety with Betti numbers

$$b_i(X) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(X; \mathbb{Z}/2\mathbb{Z}), \qquad i \geq 0$$

in homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

[6]In the complex case another example is given by the Hodge-Deligne polynomial

This morphism can be extended to a morphism on the Grothedieck ring of basic semialgebraic formulas $K_0(\mathrm{BSA})$, giving a realisation of this ring. The definition is by induction on the number of inequalities appearing in a representative of a class $[A] \in K_0(\mathrm{BSA})$. More precisely

DEFINITION 2.20 ([**11**, Proposition 3.1]). *The virtual Poincaré polynomial of basic semialgebraic formulas is the unique morphism of rings $\beta_{BSA} : K_0(BSA) \to \mathbb{Z}[u][1/2]$ such that*

(1) *If $A \in BSA$ and $p \in \mathbb{R}[x_1, \dots, x_n]$ then*

$$\beta_{BSA}([A, p > 0]) = \frac{1}{4}\beta_{BSA}([A, p = z^2]) - \frac{1}{4}\beta_{BSA}([A, p = -z^2]) + \frac{1}{2}\beta_{BSA}([A, p \neq 0]).$$

(2) *If $[A] \in K_0(Var_{\mathbb{R}})$ then $\beta_{BSA}([A]) = \beta_{Var_{\mathbb{R}}}([A])$.*

The above morphism has then the property that evaluated at -1 one recovers the Euler characteristic with compact support of the set of real points of the formula:

PROPOSITION 2.21 ([**11**, Proposition 3.4]). *For any basic semialgebraic formula A,*

$$\beta_{BSA}([A])(-1) = \chi_c(A(\mathbb{R}))$$

where χ_c denotes Euler characteristic with compact support.

One thus gets an extension[7] of $\chi_c : K_0(\mathrm{Var}_{\mathbb{R}}) \to \mathbb{Z}$ to a morphism

$$\tilde{\chi}_c : K_0(\mathrm{BSA}) \to \mathbb{Z}[1/2].$$

The following proposition implies that isomorphic formulas have the same images by the virtual Poincaré polynomial.

PROPOSITION 2.22. *Suppose given formulas $A, B \in BSA_{\mathbb{R}}$. If there exists an isomorphism of formulas $\phi : A \to B$ then the classes of A and B have the same image $\beta_{BSA}([A]) = \beta_{BSA}([B])$ under $\beta_{BSA} : K_0(BSA_{\mathbb{R}}) \to \mathbb{Z}[u][1/2]$*

[7]As a matter of fact, the virtual Poincaré polynomial was defined by Comte and Fichou exactly as to impose this property.

PROOF. This follows from [**11**, Proposition 2.2] and from the fact that isomorphic real algebraic varieties define the same class in $K_0(\mathrm{Var}_{\mathbb{R}})$ hence have the same images under β_{Var}. $\square$

REMARK 2.23 ("Topology of Formulas"). *Little is known on the exact relationship between on the one hand the topology of the set* $A(\mathbb{R})$ *of real points of a formula and on the other hand the value* $\beta_{BSA}([A])$ *of the virtual Poincaré polynomial of the class of the formula in question. The problem is quite interesting but indeed very complicated, for, in order to find* $\beta_{BSA}([A])$ *one is required to caculate* 3^k *terms where* k *is the number of inequalities appearing in* A*. To calculate just one term one is then required to find the Betti numbers of a real Zariski-constructible set. By additivity one is eventually reduced to the case of a compact nonsingular algebraic variety by compactifying and resolving singularities. But finding the Betti numbers, or even the zeroeth one, of a compact, nonsingular real algebraic set is a notably difficult problem, albeit that one knows by the results of Akbulut and King* [**3**] *that it is diffeomorphic to some smooth manifold.*

3.3. Real Motivic Milnor Fibres. In analogy with the motivic zeta functions of Denef and Loeser [**14**] for nonconstant complex algebraic morphisms, and the motivic zeta functions of S. Koike and A. Parusiński [**30**] and of G. Fichou [**20**] for real analytic maps, Comte and Fichou associated in the article [**11**] motivic zeta functions to any real polynomial map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $f(0) = 0$. As already stated, these motivic zeta functions are formal power series with coefficients given by motives corresponding to certain subsets of truncated arc-spaces. By a motive is meant here an element of the localisation $K_0(\mathrm{BSA})[\mathbb{L}_{\mathbb{R}}^{-1}]$ of the Grothendieck ring of basic semialgebraic formulas with respect to the Lefschetz motive. We recall in what follows the construction of said motivic zeta functions and their known properties.

Consider the *arc scheme* $\mathcal{L}(\mathbb{A}_{\mathbb{R}}^{n+1})$ over the spectrum of $\mathbb{R}$. Its set of real points $\mathcal{L}(\mathbb{A}_{\mathbb{R}}^{n+1})(\mathbb{R})$ is the set of $\mathbb{R}[[t]]$-rational points of $\mathbb{A}_{\mathbb{R}}^{n+1}$. There exists a subscheme $\mathcal{L}(\mathbb{A}_{\mathbb{R}}^{n+1}, 0)$ having as set of real points

$$\mathcal{L}(\mathbb{A}_{\mathbb{R}}^{n+1}, 0)(\mathbb{R}) = \{\gamma \in (\mathbb{R}[[t]])^{n+1} \mid \gamma(0) = 0\}.$$

For each $k \in \mathbb{N}$ there exists a $\mathbb{R}$-scheme of finite type $\mathcal{L}_k(\mathbb{A}^{n+1}_{\mathbb{R}})$, called the *scheme of truncated arcs* of order $k+1$, having as set of real points the set of $\mathbb{R}[[t]]/t^{k+1}$-rationals points of $\mathbb{A}^{n+1}_{\mathbb{R}}$.

In what follows we shall only consider the set of real points of these schemes, and we shall write $\mathcal{L}(\mathbb{R}^{n+1}), \mathcal{L}(\mathbb{R}^{n+1}, 0)$ and $\mathcal{L}_k(\mathbb{R}^{n+1}, 0)$ instead of $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}})(\mathbb{R})$, $\mathcal{L}(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)(\mathbb{R})$ and $\mathcal{L}_k(\mathbb{A}^{n+1}_{\mathbb{R}}, 0)(\mathbb{R})$, respectively.

For each $k \in \mathbb{N}$ there exists a truncation map

$$\pi_k : \mathcal{L}(\mathbb{R}^{n+1}, 0) \to \mathcal{L}_k(\mathbb{R}^{n+1}, 0)$$

sending an arc to its $(k+1)$-th truncation. For a truncated arc

$$\gamma_k \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0), \qquad \gamma_k(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_0$$

one defines a function

$$\mathrm{ord} : \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \to \mathbb{N} \cup \{\infty\}$$

sending γ_k to $\mathrm{ord}(\gamma_k)$ and sending $\gamma_k \equiv 0$ to $\mathrm{ord}(0) = \infty$. There also exists a map, the angular component map

$$\mathrm{ac} : \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \to \mathbb{R}, \qquad \mathrm{ac}(\gamma_k) = a_{\mathrm{ord}\,\gamma_k}$$

which sends a truncated arc to its leading coefficient.

One then defines basic semialgebraic formulas

$$X^1_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \mathrm{ord}(f \circ \gamma) = k, \quad \mathrm{ac}(f \circ \gamma) = 1\},$$
$$X^{-1}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \mathrm{ord}(f \circ \gamma) = k, \quad \mathrm{ac}(f \circ \gamma) = -1\},$$
$$X^{>}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \mathrm{ord}(f \circ \gamma) = k, \quad \mathrm{ac}(f \circ \gamma) > 0\},$$
$$X^{<}_{k,f} := \{\gamma \in \mathcal{L}_k(\mathbb{R}^{n+1}, 0) \mid \mathrm{ord}(f \circ \gamma) = k, \quad \mathrm{ac}(f \circ \gamma) < 0\}.$$

The motivic zeta functions corresponding to f are then defined as follows

DEFINITION 2.24 ([**11**, § 4.1]). *For any symbol $\epsilon \in \{\pm 1, <, >\}$ define*

$$Z^{\epsilon}(f) := \sum_{k \geq 0} [X^{\epsilon}_{k,f}] \mathbb{L}^{-(n+1)k} T^k \in K_0(BSA_{\mathbb{R}})[\mathbb{L}^{-1}][T].$$

Note that $Z^{\pm 1}(f)$ are elements of $\mathcal{M}_{\mathbb{R}}[T] = K_0(\mathrm{Var}_{\mathbb{R}})[\mathbb{L}^{-1}][T]$, that is, their coefficients are motives in the sense of Denef and Loeser.

REMARK 2.25. *By applying $\beta_{BSA} : K_0(BSA) \to \mathbb{Z}[1/2]$ to $Z^{\pm 1}(f)$ one recovers the motivic zeta functions of G. Fichou (see* [**21**]*). This follows from* [**11**, Proposition 1.3].

In order to prove the rationality of the above motivic zeta functions, Comte and Fichou established a certain formula for $Z^{\epsilon}(f)$, analoguous to the formula [**14**, Theorem 2.4] established by Denef and Loeser for their motivic zeta functions. The content of this formula is that one can compute the zeta functions via a resolution of singularities. To describe it one starts with a proper birational morphism $\sigma : M \to \mathbb{R}^{n+1}$ from a smooth manifold M such that

(1) The restriction $\sigma : M \setminus (f \circ \sigma)^{-1}(0) \to \mathbb{C}^{n+1} \setminus f^{-1}(0)$ is an algebraic isomorphism,
(2) The divisors of $f \circ \sigma$ and $\det \mathrm{Jac}\, \sigma$ are normal crossings,
(3) The reduced divisor $E = (\sigma^{-1}(0))_{red}$ is a union of irreducible components of $Y := ((f \circ \sigma)^{-1}(0))_{red}$.

Under these hypotheses one can write

$$Y = \bigcup_{j \in \mathcal{J}} E_j, \qquad E = \bigcup_{k \in \mathcal{K}} E_k,$$

for some subsets $\mathcal{J} \subset \mathbb{N}$ and $\mathcal{K} \subset \mathcal{J}$.

DEFINITION 2.26. *For any $i \in \mathcal{J}$ let*

$$N_i = mult_{E_i}(f \circ \sigma), \qquad \nu_i - 1 := mult_{E_i} \det Jac(f \circ \sigma)$$

As the divisor of $f \circ \sigma$ is normal crossings in M there exists for any point $p \in Y$ an affine open $U \subset M$ containing p a regular sequence of regular functions $x_i : U \to \mathbb{R}$ and a unit $u \in \mathcal{O}_U^{\times}$ such that

$$f \circ \sigma(x) = u(x) \prod_{i \in \mathcal{J}} x_i^{N_i}, \qquad \forall x \in U. \tag{3}$$

One now stratifies Y in such a way that E is a union of strata.

DEFINITION 2.27. *For any nonempty subset $I \subset \mathcal{J}$ let*

$$E_I^o := \bigcap_{i\in I} E_i \setminus \bigcup_{j\in\mathcal{J}\setminus I} E_j$$

and let $N_I = \gcd_{i\in I}(N_i)$.

For any symbol $\epsilon \in \{\pm 1, <, >\}$ and for any stratum E_I^o one constructs a basic semialgebraic formula $E_I^{o,\epsilon}$ such that $E_I^{o,1}(\mathbb{R})$ is the set of real points of an étale covering of degree N_I of the extension of scalars of E_I^o to $\mathbb{C}$. For this to indeed yield a basic semialgebraic formula one first has to define it locally on affine open sets, using (3).

DEFINITION 2.28 ([**11**, § 4.1]). *For any symbol $\epsilon \in \{\pm 1, <, >\}$, if $U \subset M$ is the affine open set in (3) above then*

$$R_{I,U}^1 := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) = 1\},$$

$$R_{I,U}^{-1} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) = -1\},$$

$$R_{I,U}^{>} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) > 0\},$$

$$R_{I,U}^{<} := \{(x,t) \in (E_I^o \cap U) \times \mathbb{R} \mid t^{N_I} u(x) < 0\}$$

where N_I is an in Definition 2.26 and u as in (3) above. For a covering $M = (U_l)_{l\in L}$, set

$$[E_I^{0,\epsilon}] = \sum_{S\subset L} (-1)^{|S|+1} [R_{I,\cap_{s\in S} U_s}^{\epsilon}] \in K_0(BSA).$$

The rationality of the zeta functions $Z^\epsilon(f)(T)$ follows from the following theorem.

THEOREM 2.29 ([**11**, Theorem 4.2], [**14**]). *For any symbol $\epsilon \in \{\pm 1, <, >\}$ the zeta functions $Z^\epsilon(f)(T)$ satisfy*

$$Z^\epsilon(f)(T) = \sum_{I\cap\mathcal{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1} [\tilde{E}_I^{0,\epsilon}] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

One considers the following formal limits of the zeta functions above:

DEFINITION 2.30 ([**11**, Definition 4.5]). *For any symbol $\epsilon \in \{\pm 1, <, >\}$, let*

$$S^{\epsilon}(f) := \sum_{I \cap K \neq \emptyset} (1-\mathbb{L})^{|I|-1}[\tilde{E}_I^{0,\epsilon}] \in K_0(BSA_{\mathbb{R}}).$$

In accordance with the terminology used in the article [**11**] (see also [**14**]) one calls $S^{\pm 1}(f)$ the *motivic positive and negative Milnor fibres* of f at the origin. Introducing new terminology we will refer to $S^{>}(f)$ as the *motivic positive Milnor tube*, and to $S^{<}(f)$ as the *motivic negative Milnor tube*. They do not depend on the choice of a strong desingularisation, by [**11**, Remark 4.3].

In the case of isolated singularities one recovers the Euler characteristics of the classical Milnor fibres by applying the realisation

$$\tilde{\chi}_c : K_0(\mathrm{BSA}_{\mathbb{R}}) \to \mathbb{Z}[1/2], \qquad \tilde{\chi}_c([X]) := \beta_{\mathrm{BSA}}([X])(-1)$$

as according to the following real (partial) analogue of the A'Campo formula. So suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a polynomial function having an isolated critical point in the origin. Let (ϵ_0, δ_0) be Milnor data of f at the origin as in Theorem 2.1 and for any $\delta \in (0, \delta_0)$ and any $\eta \in (0, \epsilon(\delta))$ write

$$\mathcal{F}_{\eta}^{>} := f^{-1}((0,\eta)) \cap \mathbb{B}_{\delta},$$

$$\mathcal{F}_{\eta}^{<} := f^{-1}((-\eta,0)) \cap \mathbb{B}_{\delta},$$

and write

$$\mathcal{F}_{\eta}^{1} = f^{-1}(\eta) \cap \mathbb{B}_{\delta},$$

$$\mathcal{F}_{\eta}^{-1} = f^{-1}(-\eta) \cap \mathbb{B}_{\delta}$$

for the positive and negative Milnor fibres at the origin.

THEOREM 2.31 ([**11**, Theorem 4.12]). *Let*

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad f(0) = 0$$

be a polynomial function having the origin as an isolated critical point. Then for any $\delta \in (0, \delta_0)$ and any $\eta \in (0, \epsilon(\delta))$,

$$\tilde{\chi}_c(S^{\epsilon}(f)) = (-1)^n \chi_c(\mathcal{F}_{\eta}^{\epsilon}), \qquad \epsilon \in \{\pm 1, <, >\}$$

where χ_c denotes Euler characteristic with compact support.

4. A Brief Outline of Arc-Analytic Geometry

Real algebraic varieties present themselves with an immediate difference with complex analytic varieties in that a connected component (for the euclidean topology) of an irreducible real algebraic set X need not be itself algebraic. However, as it turns out, certain connected components are "rigid" in the sense that after blowing up the set of their preimages in the strict transform $\tilde{X}$ constitutes exactly the set of connected components of $\tilde{X}$.

In order to have a category containing all such rigid connected components K. Kurdyka introduced in his article [**32**] from 1988, the notion of *arc-analytic sets.* A slight modification of this notion, namely the $\mathcal{AS}$-sets, was shortly thereafter presented by A. Parusiński. These sets are then precisely the closed arc-analytic sets, for the euclidean topology. For posterior reference we now briefly discuss these notions.

Definition 2.32 ([**33**, § 1]). *A semialgebraic map $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be* arc-analytic *if for any real analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ the composition $f \circ \gamma$ is real analytic.*

A semialgebraic set is $\mathcal{AS}$ if any real analytic arc meeting it can be prolonged so as to be entirely included in it, except possibly in isolated points. More precicely:

Definition 2.33 ([**33**, Proposition 3.2]). *A semialgebraic set $X \subset \mathbb{P}^n$ is called an $\mathcal{AS}$-set if for any real analytic arc $\gamma : (-1, 1) \to \mathbb{P}^n$ such that $\gamma((-1, 0)) \subset X$ is entirely included in X, there exists an $\epsilon > 0$ such that $\gamma((0, \epsilon) \subset X$. A semialgebraic map is called an $\mathcal{AS}$-map if its graph is an $\mathcal{AS}$-set.*

One then forms the category with objects $\mathcal{AS}$-sets and arrows $\mathcal{AS}$-maps. The following then says that this category is constructible.

Theorem 2.34 ([**33**, Theorem 4.4]). *The $\mathcal{AS}$-sets forms a* constructible *category. That is, the following holds:*

(1) *$\mathcal{AS}$ contains the real algebraic sets.*
(2) *$\mathcal{AS}$ is stable by boolean operations. That is, arbitrary intersections and unions of $\mathcal{AS}$-sets are $\mathcal{AS}$ and the difference of two $\mathcal{AS}$-sets is an $\mathcal{AS}$-set.*
(3) *The inverse image of an $\mathcal{AS}$-set by an $\mathcal{AS}$-map is an $\mathcal{AS}$-set. The image of an $\mathcal{AS}$-set under an injective $\mathcal{AS}$-map is an $\mathcal{AS}$-set.*

(4) *For each locally compact $\mathcal{AS}$-set X there exists a semialgebraic subset $Y \subset X$ with $\dim Y \leq \dim X - 2$ such that $X \setminus Y$ is* Euler, *that is, for any $x \in X \setminus Y$ the Euler characteristics of the link of x in $X \setminus Y$ is an even integer.*

In particular the category $\mathcal{AS}$ contains the Zariski-constructible sets.

THEOREM 2.35 ([**33**, Theorem 4.5]). *The category of $\mathcal{AS}$-sets is strictly included in the category of real analytic sets and is the smallest constructible category containing the connected components of compact real algebraic sets.*

The relationship between the category $\mathcal{AS}$ and the category $\mathcal{N}$ of Nash-constructible sets is described by the following result, which was first proved by A. Parusiński [**42**, Theorem 4.3].

THEOREM 2.36 ([**33**, Theorem 3.9]). *For a semialgebraic set $X \subset \mathbb{P}^n$ the following are equivalent:*

(1) $X \in \mathcal{AS}$.
(2) *The indicator function of X is Nash-constructible on $\mathbb{P}^n$ (see* [**33**, Definition 3.7]*).*
(3) *There exists a proper, regular morphism of real algebraic varieties $f : Z \to \mathbb{P}^n$ and a union Z' of connected components of Z such that*

$$x \in X \iff \chi(f^{-1}(x) \cap Z') \equiv 1 \quad (mod\ 2).$$

$$x \notin X \iff \chi(f^{-1}(x) \cap Z') \equiv 0 \quad (mod\ 2).$$

5. Summary of Results

5.1. Motivic Milnor Fibres of Families of Singularities.

5.1.1. In chapter 3 we consider the arc-analytic equivalence of germs of Nash functions due to J.-B. Campesato [**10**].

DEFINITION (Definition 3.5). *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of Nash functions. If there exists a germ of semialgebraic homeomorphism*

$$h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

with $f = g \circ h$ such that h is arc-analytic and if there exists a constant $c > 0$ such that

$$|\det Jac(h)| > c$$

there where the Jacobian determinant is defined, then f is said to be arc-analytically equivalent to g via h and one writes $f \sim_a g$ or $f \sim g$ if no ambiguity is possible.

The result [**20**, Theorem 4.9] of G. Fichou implies that if $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are arc-analytically equivalent germs of Nash functions then the virtual Poincaré polynomial of the positive motivic Milnor fibres of f and g are equal, $\beta_{\mathrm{Var}}(S^1(f)) = \beta_{\mathrm{Var}}(S^1(g))$, and likewise the virtual Poincaré polynomials of the negative motivic Milnor fibres of f and g are equal, $\beta_{\mathrm{Var}}(S^{-1}(f)) = \beta_{\mathrm{Var}}(S^{-1}(g))$.

5.1.2. The first result in chapter 3 is an extension of [**20**, Theorem 4.9] to the positive and negative Milnor tubes.

Theorem (Theorem 3.10). *If $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are arc-analytically equivalent germs of Nash functions then for any symbol $\epsilon \in \{\pm 1, <, >\}$ the virtual Poincaré polynomial of the corresponding motivic Milnor fibres and motivic Milnor tubes are equal:*

$$\beta_{BSA}(S^\epsilon(f)) = \beta_{BSA}(S^\epsilon(g)).$$

5.1.3. The next result in chapter 3 concerns families

$$f_t : \mathbb{R}^n \to \mathbb{R}, \qquad f_t(0) = 0, \quad \forall t \in T$$

of polynomial maps semialgebraically parametrised by a real algebraic set $T \subset \mathbb{R}^m$. The methods of A. Parusiński and L. Paunescu leading to their proof of the Whitney Fibering Conjecture [**43**] enable us to prove that there exists a locally finite stratification of T having real analytic strata, such that above each strata there exists only one equivalence class for the arc-analytic equivalence, and that if T is compact then the stratification can be taken to have at most finitely many strata.

Theorem (Theorem 3.13). *Let $T \subset \mathbb{R}^m$ be a real algebraic set and $f : T \times (\mathbb{R}^n, 0) \to \mathbb{R}$ a polynomial map of polynomial function germs such that $f(t, 0) = 0$ for any $t \in T$. There exists a locally finite real analytic stratification $\mathcal{S}$ of T such that if $S \in \mathcal{S}$ is a stratum then*

$$f_t = f(t, \cdot) \sim_a f_{t'} = f(t', \cdot), \qquad \forall t, t' \in S$$

in the notation of Definition 3.5. In particular

$$\beta_{BSA}(S^\epsilon(f_t)) = \beta_{BSA}(S^\epsilon(f_{t'})), \qquad \forall t, t' \in S \in \mathcal{S}, \qquad \forall \epsilon \in \{\pm 1, <, >\}.$$

Moreover if T is compact then $\mathcal{S}$ can be taken to be a finite stratification.

5.2. Isolated Real Hypersurface Singularities.

5.2.1. In chapter 4 we study the homotopy type and the singular homology groups of the Milnor fibres of real polynomial map germs $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ such that $f^{-1}(0)$ has an isolated singular point in the origin. Since the idea is to perturb a representative of the germ f so as to have only nondegenerate critical points we are lead to consider $\mathbb{R}$-*morsifications*[8] of polynomial map germs.

Definition (Definition 4.28). *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of polynomial maps. A $\mathbb{R}$-morsification of f is a representative*

$$F : M \times U \to \mathbb{R}$$

of a polynomial map germ $F : (\mathbb{R}^{n+1} \times \bar{\mathcal{I}}, 0) \to (\mathbb{R}, 0)$ such that

(1) *$\bar{\mathcal{I}} \cong [-1, 1]^k$ for some $k \in \mathbb{N}$,*
(2) *$F(x, 0) = f(x)$ as germs of polynomial maps,*
(3) *There exists a dense subset $V \subset U \setminus \{0\}$ such that for all $t \in V$, the function $f_t : M \to \mathbb{R}$ is a Morse function with distinct critical values.*

A $\mathbb{R}$-morsification F is strong *if for any $t \in V$ the function $f_t : M \to \mathbb{R}$ has $\mu(f_{\mathbb{C}}, 0)$ critical points and it is* weak *if for any $t \in V$ it has strictly less than $\mu(f_{\mathbb{C}}, 0)$ critical points.*

In the case where $f^{-1}(0)$ has an isolated singular point in the origin an $\mathbb{R}$-morsification of f always exist.

Proposition (Proposition 4.29). *If $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ is a germ of real polynomial map and if $f^{-1}(0)$ has an isolated singular point in the origin then f has an $\mathbb{R}$-morsification.*

5.2.2. The following result is the first of the two main theorems in chapter 4. It says that after succesively attaching handles to a space homotopy equivalent to the real Milnor fibers of f one obtains a contractible space, where each handle corresponds to a nondegenerate critical point of $f_t = F(\cdot, t)$ for $t \in V$.

Theorem (Theorem 4.47). *Suppose that $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ is a germ of real polynomial map such that $f^{-1}(0)$ has an isolated singular point in the origin*

[8]"Real morsifications" have been considered by V.I Arnold and his school (see e.g [**19**]). These are however holomorphic maps and since we shall in the end apply Stratified Morse Theory it is arguably more natural for our purposes to consider real polynomial maps.

and let (δ_0, ϵ_0) be Milnor data at the origin. Let

$$F : M \times U \to \mathbb{R}$$

be a $\mathbb{R}$-morsification of f with $V \subset U \setminus \{0\}$ a dense subset such that for any $t \in V$ the function $f_t : M \to \mathbb{R}$ is Morse with distinct critical values. For any $\eta \in (0, \epsilon(\delta))$ there exists $t_0 = t_0(\eta) \in \mathbb{R}$ such that the following holds. Let $t \in V$ such that $|t| \leq t_0$ be fixed and let $p_1, \ldots, p_m \in \bar{\mathbb{B}}_\delta$ denote the critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ centered at the origin and let $\lambda(p_i)$ denote their indices. Then $\mathcal{X}_{t,\eta}$ is contractible and there exist embeddings

$$h_i^+ : \begin{cases} \mathbb{D}^{\lambda(p_i)} \times \partial\mathbb{D}^{n+1-\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1 \leq j \leq i-1 \\ h_j^+}} \left(\mathbb{D}^{\lambda(p_j)} \times \mathbb{D}^{n+1-\lambda(p_j)}\right), & i \geq 2 \\ \mathbb{D}^{\lambda(p_1)} \times \partial\mathbb{D}^{n+1-\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^+ \times (0,1), \quad i = 1, \end{cases}$$

$$h_i^- : \begin{cases} \mathbb{D}^{n+1-\lambda(p_i)} \times \partial\mathbb{D}^{\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^- \times (0,1) \cup \bigcup_{\substack{1 \leq j \leq i-1 \\ h_j^-}} \left(\mathbb{D}^{n+1-\lambda(p_j)} \times \mathbb{D}^{\lambda(p_j)}\right), & i \geq 2 \\ \mathbb{D}^{n+1-\lambda(p_1)} \times \partial\mathbb{D}^{\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^- \times (0,1), \quad i = 1 \end{cases}$$

and homeomorphisms

$$\mathcal{X}_{t,\eta} \cong \bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1 \leq i \leq m \\ h_i^+}} \left(\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}\right)$$

$$\mathcal{X}_{t,\eta} \cong \bar{\mathcal{F}}_\eta^- \times (0,1) \cup \bigcup_{\substack{1 \leq i \leq m \\ h_i^-}} \left(\mathbb{D}^{n+1-\lambda(p_i)} \times \mathbb{D}^{\lambda(p_i)}\right)$$

where each handle $\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}$ (respectively $\mathbb{D}^{n+1-\lambda(p_i)} \times \mathbb{D}^{\lambda(p_i)}$) is attached along $\mathbb{D}^{\lambda(p_i)} \times \partial\mathbb{D}^{n+1-\lambda(p_i)}$ (respectively along $\mathbb{D}^{n+1-\lambda(p_i)} \times \partial\mathbb{D}^{\lambda(p_i)}$) via h_i^+ (respectively via h_i^-).

Corollary (Corollary 4.60). *Keep the assumptions of Theorem 4.47. Suppose that $f_t : M \to \mathbb{R}$ has a unique critical point p of index $\lambda(p)$. If $\bar{\mathcal{F}}_\eta^+$ is nonempty then*

$$\beta(\bar{\mathcal{F}}_\eta^+) = \begin{cases} 1 + u^{n-\lambda(p)}, & \text{if } \lambda(p) \leq n, \\ 1, & \text{if } \lambda(p) = n+1. \end{cases}$$

If $\bar{\mathcal{F}}_\eta^-$ is nonempty then

$$\beta(\bar{\mathcal{F}}_\eta^-) = \begin{cases} 1 + u^{\lambda(p)-1}, & \text{if } \lambda(p) \geq 1, \\ 1, & \text{if } \lambda(p) = 0. \end{cases}$$

5.2.3. The analogous statement for holomorphic map germs $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $g^{-1}(0) \subset \mathbb{C}^{n+1}$ having an isolated singular point in the origin says that after attaching the Milnor number $\mu = \mu(g, 0)$ of discs of dimension n the complex Milnor fibre $\mathcal{F}_{\mathbb{C},\eta}(g)$ becomes contractible (see [**18**, Proposition 5.6]). A classical proof of this fact is due to E. Brieskorn [**7**]. There one takes a small perturbation of g having the Milnor number $\mu = \mu(g, 0)$ of nondegenerate critical points with pairwise distinct critical values. In this case each of the discs can be proven by a geometrical argument [**18**, Lemma 5.3] to be attached directly to $\mathcal{F}_{\mathbb{C},\eta}(g)$.

Furthermore, these arguments enable one to use the long exact sequence of a pair together with the Excision Theorem [**26**, Theorem 2.20] to deduce that the reduced singular homology groups of $\mathcal{F}_{\mathbb{C},\eta}(g)$ are

$$\tilde{H}_k(\mathcal{F}_{\mathbb{C},\eta}(g)) = \begin{cases} \mathbb{Z}^\mu, & \text{if } k = n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

There are two major *obstacles* for this reasoning to be carried through successfully in the case envisioned in Theorem 4.47 for real isolated hypersurface singularities. On the one hand if given an arbitrary $\mathbb{R}$-morsification F of f and a $t \in V$ such that f_t has critical points $p_1, \ldots, p_m \in \bar{\mathbb{B}}_\delta$ there might very well be the case that there exists $j, j' \in \{1, \ldots, m\}$ such that

$$\bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1 \le i \le m \\ h_i^+}} \left(\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}\right)$$

has the same homotopy type as

$$\bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1 \le i \ne j, j' \le m \\ h_i^+}} \left(\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}\right).$$

On the other hand, there is no *a priori* reason to believe that the discs in the statement of Theorem 4.47 can always be assumed to be attached directly to $\bar{\mathcal{F}_\eta}^+$ (respectively $\bar{\mathcal{F}}_\eta^-$). This seems in our view to be the most difficult obstacle. We shall now state conditions under the validity of which these two obstacles can be circumvented and the homology groups of the Milnor fibres can indeed be obtained.

5.2.4. We start by showing in Lemma 4.54 the following auxiliary result. If $F' : M \times U' \to \mathbb{R}$ is a $\mathbb{R}$-morsification of $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ then there exists another $\mathbb{R}$-morsification $F : M \times U \to \mathbb{R}$ of f such that there exists a dense open subset $V \subset U$ such that the number $m = m(t)$ of critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ is independent of $t \in V$ and each critical point is Morse with pairwise distinct critical values. The following result is the second main theorem in chapter 4.

Theorem (Theorem 4.58). *Let $n \in \mathbb{N}$ be odd and suppose that $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ is a germ of polynomial map such that $f^{-1}(0)$ has an isolated singular point in the origin. Let (δ_0, ϵ_0) be Milnor data at the origin. Suppose that*

$$F : M \times U \to \mathbb{R}, \qquad f_t = F(\cdot, t)$$

is a $\mathbb{R}$-morsification of f and let $V \subset U$ be a dense open subset such that the number $m = m(t)$ of critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ is independent of $t \in V$ and each critical point is Morse with pairwise distinct critical values. Let $\delta \in (0, \delta_0]$ and $\eta \in (0, \epsilon_0]$ be fixed. Then there exists $t_0 \in \mathbb{R}$ such that if $t \in V \cap \{|t| \leq t_0\}$, if $p_1, \dots, p_m \in \bar{\mathbb{B}}_\delta$ denotes the critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ and if $\lambda(p_i) = (n+1)/2, i = 1, \dots, m$ then

$$H_k(\bar{\mathcal{F}}_\eta^+) = H_k(\bar{\mathcal{F}}_\eta^-) \cong H_k(\bigvee_{i=1}^m \mathbb{S}^{(n-1)/2}), \qquad k \geq 0.$$

If given a smooth manifold or manifold with boundary X let

$$\beta(X) = \sum_{n \geq 0} \text{rank } H_n(X; \mathbb{Z})$$

denote the Poincaré polynomial of X in singular homology with integral coefficients.

Corollary (Corollary 4.59). *Keep the assumptions of Theorem 4.58. The Poincaré polynomial of the real Milnor fibres are*

$$\beta(\bar{\mathcal{F}}_\eta^+) = 1 + mu^{n-\lambda}$$

and

$$\beta(\bar{\mathcal{F}}_\eta^-) = 1 + mu^{\lambda-1}$$

5.3. Applications to Real ADE-Singularities. As an application we can obtain the Poincaré polynomials in singular integral homology of the Milnor fibres of real ADE-singularities, except for one special case. Let us consider real polynomial map germs $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ of the form

(1) $A_k^{\pm,s} : f = x^{k+1} \pm y^2 + \sum_{i=1}^t x_i^2 - \sum_{j=t+1}^{s+t} x_j^2, \quad k \geq 2.$

(2) $D_k^{\pm,s} : f = x^2y \pm y^{k-1} + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2, \quad k \geq 4$
(3) $E_6^{\pm s} : f = x^3 \pm y^4 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$
(4) $E_7^{s} : f = x^3 + xy^3 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$
(5) $E_8^{s} : f = x^3 + y^5 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$

where $t+s = n-1, s, t \geq 0$ and where $(x, y, x_1, \ldots, x_{n-1})$ are coordinates on $\mathbb{R}^{n+1}$.

We first treat the case of curves (namely $n = 1$). When the map f is clear from the context we shall let $\mathcal{F}^+$ and $\mathcal{F}^-$ denote a positive, respectively negative, Milnor fibre of f at the origin in $\mathbb{R}^{n+1}$.

THEOREM (Theorem 5.1). *The Poincaré polynomials in singular homology of the Milnor fibres of ADE-singularities of curves are given as follows.*

(1) $A_k^+ : f(x,y) = x^{k+1} + y^2$:

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1, \qquad \text{if } k \equiv 0 \pmod 2$$

$$\beta(\bar{\mathcal{F}}^+) = 1 + u, \quad \beta(\bar{\mathcal{F}}^-) = 0 \qquad \text{if } k \equiv 1 \pmod 2.$$

(2) $A_k^- : f(x,y) = x^{k+1} - y^2$:

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(3) $D_k^+ : f(x,y) = x^2y + y^{k-1}, \quad k \geq 4$,

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(4) $D_k^- : f(x,y) = x^2y - y^{k-1}, \quad k \geq 4$,

$$\beta(\bar{\mathcal{F}}^\pm) = \begin{cases} 3, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(5) $E_6^\pm : f(x,y) = x^3 \pm y^4$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1$.
(6) $E_7 : f(x,y) = x^3 + xy^3$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 2$.
(7) $E_8 : f(x,y) = x^3 + y^5$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1$.

COROLLARY (Corollary 5.2). *The following holds.*

(1) A_k^{-s}, D_k^{+s}:

$$\beta(\bar{\mathcal{F}}^+) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 1 + u^{n-s-1}, & k \equiv 1 \pmod 2 \end{cases},$$

$$\beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod{2} \\ 1+u^s, & k \equiv 1 \pmod{2} \end{cases}$$

(2) A_k^{+s} :

$$\begin{cases} \beta(\bar{\mathcal{F}}^+) = 1+u^{n-s}, & k \equiv 1 \pmod{2} \\ \beta(\bar{\mathcal{F}}^-) = 1+u^{s-1}, & s \neq 0, k \equiv 1 \pmod{2} \\ \beta(\bar{\mathcal{F}}^-) = 0, & s = 0, k \equiv 1 \pmod{2} \\ \beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1, & k \equiv 0 \pmod{2}. \end{cases}$$

(3) E_7^s :

$$\beta(\bar{\mathcal{F}}^+) = 1+u^{n-s-1},$$

$$\beta(\bar{\mathcal{F}}^-) = 1+u^s.$$

(4) $E_6^{\pm s}, E_8^s : \beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1.$

The Poincaré polynomials of the Milnor fibres of D_k^{-s}-singularities for k odd and $n > 1$ are given as follows:

$$D_k^{-s} : \begin{cases} \beta(\bar{\mathcal{F}}^+) = 1+u^{n-s-1}, & k \equiv 1 \pmod{2} \\ \beta(\bar{\mathcal{F}}^-) = 1+u^s, & k \equiv 1 \pmod{2} \end{cases}$$

5.4. Morsifiable Real Singularities $f : X \to \mathbb{R}$. In the last chapter 6 we give a generalisation of the theorem 4.47. We shall consider the following situation.

5.4.1. Assume that $X \subset \mathbb{R}^n$ is a real algebraic subset containing the origin in $\mathbb{R}^n$ and that

$$f : X \to \mathbb{R}, \qquad f(0) = 0$$

is the restriction of a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$. Assume that there exist a Whitney stratification $\tilde{\mathcal{S}} = (\tilde{\mathscr{S}_i})_{i \in I}$ of X such that $f^{-1}(0) \subset X$ is a union of strata such that $\tilde{\mathcal{S}}$ satisfies the a_f-condition of Thom [**50**].

5.4.2. Suppose that $f : X \to \mathbb{R}$ is as in situation 5.4.1. Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the restriction

$$f : f^{-1}([-\epsilon, \epsilon] \setminus \{0\}) \cap X \cap \bar{\mathbb{B}}_\delta^n \to [-\epsilon, \epsilon] \setminus \{0\} \tag{4}$$

is a locally trivial topological fibration where $\mathbb{B}_\delta^n \subset \mathbb{R}^n$ denotes the open n-dimensional ball centered at the origin of radius δ. The bars designates closures for the euclidean topology. The fibres of (4) are the (closed) Milnor fibers of f at the origin and are denoted by

$$\bar{\mathcal{F}}_\eta^+ := f^{-1}(\eta) \cap X \cap \bar{\mathbb{B}}_\delta, \quad \bar{\mathcal{F}}_\eta^- := f^{-1}(-\eta) \cap X \cap \bar{\mathbb{B}}_\delta \quad \eta \in (0, \epsilon].$$

The set $\bar{\mathcal{F}}_\eta^+$ is the positive closed Milnor fibre and the set $\bar{\mathcal{F}}_\eta^-$ is the negative closed Milnor fibre.

5.4.3. To obtain the wanted generalisation of Theorem 4.47 one shall perturb $\tilde{f}$ so that restricted to each strata of X one obtains functions having only nondegenerate critical points. It turns out that the following perturbations suffices for this purpose.

Definition (Definition 6.3). *Let $Y \subset \mathbb{R}^n$ be a Whitney stratified real subanalytic set and let $g : Y \to \mathbb{R}$ the restriction of a polynomial map $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$. A* stratified $\mathbb{R}$-morsification *of g is the restriction $G : Y \times \bar{U} \to \mathbb{R}$ of a polynomial map*

$$\tilde{G} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R},$$

such that

(1) *$\bar{U} \subset \mathbb{R}^k$ is a smooth k-dimensional manifold with boundary having boundary $\partial\bar{U}$ and interior U,*
(2) *$\bar{U} \cong [-1, 1]^k$ and $\partial\bar{U}$ contains the origin in $\mathbb{R}^k$,*
(3) *$\tilde{G}(x, 0) = \tilde{g}$,*
(4) *The map*

$$\Phi : \mathbb{R}^n \times U \to T^*\mathbb{R}^n, \qquad \Phi(x, t) = d(\tilde{g}_t)(x),$$

where $T^*\mathbb{R}^n$ is the cotangent bundle of $\mathbb{R}^n$, is a submersion.

Stratified $\mathbb{R}$-morsifications give indeed rise to functions having only nondegenerate critical points on each strata, according to the following result.

Proposition (Proposition 6.4). *Let $Y \subset \mathbb{R}^n$ be a Whitney stratified compact real subanalytic set and let $G : Y \times \bar{U} \to \mathbb{R}$ be a stratified $\mathbb{R}$-morsification of $g : Y \to \mathbb{R}$ where g is the restriction of a polynomial function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$. Then there exists a dense open subset $V \subset U$ such that $g_t : Y \to \mathbb{R}$ is a stratified Morse function for any $t \in V$, in the sense of* [**25**, Definition 2.1].

5.4.4. To state the first of the main theorems as succinctly as possible we introduce the following notion.

DEFINITION (Definition 6.8). *A map* $f : X \to \mathbb{R}$ *is* morsifiable *if condition 5.4.1 is satisfied and if there exists a stratified* $\mathbb{R}$*-morsification (Definition 6.3)*

$$F : X \times \bar{U} \subset \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \qquad f_t(x) = F(x,t).$$

of f.

5.4.5. One also needs the following lemma and definition.

LEMMA (Lemma 6.9). *Suppose that* $f : X \to \mathbb{R}$ *is morsifiable and suppose that*

$$F : X \times \bar{U} \subset \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \qquad f_t(x) = F(x,t)$$

is a stratified $\mathbb{R}$*-morsification of* f. *There exists an open dense set* $V \subset U$ *such that for each* $t \in V$,

$$f_t : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

is a stratified Morse function.

DEFINITION (Definition 6.10). *For any* $t \in V$ *put*

$$\mathcal{C}(F,t) = \{p(t) \in X \cap \bar{\mathbb{B}}_\delta \mid p(t) \textit{ is a stratified critical point of } f_t : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}\}$$

5.4.6. The following abbreviations will be used in the statement of the theorem. Let $t \in V$ be fixed.

- If $p_j = p_j(t) \in \mathcal{C}(F,t)$ and p_j belongs to a stratum $\mathscr{S} \in \mathcal{S}$ denote by $\lambda_{\mathscr{S}}(p_j)$ the Morse index of $f_{t|\mathscr{S}_l}$ at p_j.
- If $p_j \in \mathcal{C}(F,t)$ belongs to a stratum $\mathscr{S} \in \mathcal{S}$ then because $\mathcal{S}$ satisfies the Whitney conditions there exists $\rho_j > 0$ such that if $\bar{\mathbb{B}}_{\rho_j}(p_j) \subset \mathbb{R}^n$ is the closed ball centered at p_j and of radius ρ_j then $\partial\bar{\mathbb{B}}_{\rho_j}(p)$ intersects the strata of $\mathcal{S}$ transversally, see [**25**, 3.5]. Denote by

$$l^+(p_j) = N(p_j) \cap \mathbb{B}_{\rho_j} \cap f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(s_j + \gamma_j),$$

$$l^-(p_j) = N(p_j) \cap \mathbb{B}_{\rho_j} \cap f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(s_j - \gamma_j),$$

  for $0 < \gamma_j << \rho_j$, the upper and lower half-links [**25**, Definition 3.9.1] of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ at the point p_j. Here $N(p_j)$ is the normal slice through the stratum $\mathscr{S} \in \mathcal{S}$ at p_j [**25**, § 1.4].
- Let $\dim(\mathscr{S})(p_j)$ denote the dimension of the stratum containing p_j.

5.4.7. The main theorem of chapter 6 says that after succesively attaching products of discs and cones of upper half-links to the positive Milnor fiber $\bar{\mathcal{F}}^+_\eta$

one obtains a contractible space. Similarly after succesively attaching products of discs and cones over lower half-links to the negative Milnor fiber $\bar{\mathcal{F}}_\eta^-$ one obtains a contractible space.

THEOREM (Theorem 6.13). *Suppose that $X \subset \mathbb{R}^n$ is a real algebraic set and that $f : X \to \mathbb{R}$ is a morsifiable map (see Definition 6.8) having Milnor data (ϵ_0, δ_0) at the origin. Suppose that*

$$F : X \times \bar{U} \subset \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \quad f_t = F(\cdot, t)$$

is a stratified $\mathbb{R}$-morsification of f (see Definition 6.3)), with V as in Lemma 6.9. For any $\delta \in (0, \delta_0]$ there exists $\epsilon \in (0, \epsilon_0]$ such that for any $\eta \in (0, \epsilon(\delta)]$ there exists $t_0(\eta) \in \mathbb{R}$ such that if $t \in V \cap \{|t| \leq t_0\}$ then there exist semialgebraic sets $\mathcal{N}(\bar{\mathcal{F}}_\eta^+)$ and $\mathcal{N}(\bar{\mathcal{F}}_\eta^-)$ having the homotopy type of $\bar{\mathcal{F}}_\eta^+$ respectively $\bar{\mathcal{F}}_\eta^-$ and attaching maps

$$h_j^+ : \partial \left(\mathbb{D}^{\dim \mathscr{S}(p_j) - \lambda_{\mathscr{S}}(p_j)} \times cone(l^+(p_j)) \right) \to$$

$$\begin{cases} \mathcal{N}(\bar{\mathcal{F}}_\eta^+) \cup \bigcup_{l=1}^{j-1} \mathbb{D}^{\dim \mathscr{S}(p_j) - \lambda_{\mathscr{S}}(p_j)} \times cone(l^+(p_j)), & 2 \leq j \leq m \\ \mathcal{N}(\bar{\mathcal{F}}_\eta^+), & j = 1 \end{cases}$$

$$h_j^- : \partial \left(\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)} \times cone(l^-(p_j)) \right) \to$$

$$\begin{cases} \mathcal{N}(\bar{\mathcal{F}}_\eta^-) \cup \bigcup_{l=1}^{j-1} \mathbb{D}^{\lambda_{\mathscr{S}}(p_j)} \times cone(l^-(p_j)), & 2 \leq j \leq m \\ \mathcal{N}(\bar{\mathcal{F}}_\eta^-), & j = 1 \end{cases}$$

with $p_j \in \mathcal{C}(F, t)$ and $m = |\mathcal{C}(F, t)|$ as in Definition 6.10, such that

$$\mathcal{N}(\bar{\mathcal{F}}_\eta^+) \cup \bigcup_{\substack{1 \leq i \leq m \\ h_i^+}} \left(\mathbb{D}^{\dim(\mathscr{S}) - \lambda_{\mathscr{S}}(p_j)} \times cone(l^+(p_j)) \right),$$

$$\mathcal{N}(\bar{\mathcal{F}}_\eta^-) \cup \bigcup_{\substack{1 \leq i \leq m \\ h_i^-}} \left(\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)} \times cone(l^-(p_j)) \right)$$

are contractible.

5.5. Applications.

5.5.1. As an application of the theorem 6.13 we can obtain the following generalisation of Khimshiashvili's formula [**16**, Theorem 2.3].

COROLLARY (Corollary 6.15). *Keep the assumptions of the Theorem 6.13 and suppose furthermore that X is equidimensional. The Euler-Poincaré characteristic* *of the negative Milnor fibre with $\mathbb{Q}$-coefficients is given by*

$$\chi(\bar{\mathcal{F}_\eta}^-) = 1 + \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S}(p_j))=\dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j)-1} + \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S}((p_j))<\dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j)}\tilde{\chi}(l^-(p_j))))$$

and the Euler-Poincaré characteristic with $\mathbb{Q}$-coefficients of the positive Milnor fibre is given by

$$\chi(\bar{\mathcal{F}}_\eta^+) = 1 + \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S})(p_j))=\dim(X)}} (-1)^{\dim(\mathscr{S})(p_j)+1-\lambda_{\mathscr{S}}(p_j)}$$

$$+ \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S})(p_j))<\dim(X)}} (-1)^{\dim(\mathscr{S})(p_j)-\lambda_{\mathscr{S}}(p_j)}\tilde{\chi}(l^+(p_j)))).$$

Moreover if $\dim(\mathscr{S}(p)) = \dim X$ *for any* $p \in \mathcal{C}(F,t)$ *then these relations hold for the Euler-Poincaré characteristics with $\mathbb{Z}$-coefficients.*

CHAPTER 3

Motivic Milnor Fibres of Families of Singularities

1. Introduction

This chapter is devoted to an analysis of the real motivic Milnor fibres of a semialgebraic family of polynomial functions parametrised by a real algebraic set, and to showing the existence of a stratification of the parameter space such that the virtual Poincaré polynomials of the motivic Milnor fibres is invariant along the strata.

2. The Arc-Analytic Equivalence

2.1. Nash Functions and Manifolds. We start by first recalling the arc-analytic equivalence for Nash germs, introduced by J-B Campesato in his thesis [**8**].

Definition 3.1 ([**40**] [**10**, Definition 6.1]). *A function $f : U \to \mathbb{R}$ defined on an open semialgebraic set $U \subset \mathbb{R}^n$ is said to be* Nash *if it is semialgebraic and of class C^∞. A map $f : U \to \mathbb{R}^m$ is said to be Nash if each of its components functions are Nash.*

By [**6**, Proposition 8.1.8] one has that a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is Nash if and only if it is real analytic and satisfies a nontrivial polynomial equation. In particular, by [**6**, Corollary 8.1.6], the ring of germs of Nash functions is isomorphic to the ring of algebraic power series. This yields a category whose objects are semialgebraic sets and whose arrows are Nash functions. The isomorphisms in this category are then:

Definition 3.2. *A Nash diffeomorphism is a semialgebraic map $\phi : U \to V$ such that ϕ and its inverse are Nash functions.*

The notion of manifold in this category is given by the following.

Definition 3.3 ([**10**, Definition 6.3]). *A Nash manifold of dimension $d \in \mathbb{N}$ is a semialgebraic subset $M \subset \mathbb{R}^n$ such that for any $x \in M$ there exists an open* *semialgebraic neighborhood $U \subset \mathbb{R}^n$ of x, an open semialgebraic neighborhood $V \subset \mathbb{R}^n$ of the origin and a Nash diffeomorphism $\phi : U \to V$ with $\phi(x) = 0$ such that $\phi(M \cap U) = \mathbb{R}^d \times \{0\}$.*

2.2. Nash-Isomorphic $\mathcal{AS}$-Sets. The virtual Poincaré polynomial on compact real algebraic varieties is invariant under Nash diffeomorphisms, as follows from the result [**20**, Theorem 3.3] (compare [**24**, Proposition 1.2]).

For noncompact varieties this is no longer true; one gets a counter-example thereof by taking $X \subset \mathbb{R}^2$ to be a hyperbola. Then X is Nash diffeomorphic to the union Y of two disjoint affine lines $\mathbb{A}^1$. So on the one hand $\beta(X) = u - 1$ because if $\bar{X} \subset \mathbb{P}^2$ is a projective compactification then $\bar{X} \setminus X$ consists of two points, whereas on the other hand $\beta(Y) = 2\beta(\mathbb{A}^1) = 2u$ by additivity. In the case of $\mathcal{AS}$-sets one defines

Definition 3.4 ([**20**, Definition 3.2]). *If $A, B \in \mathcal{AS}$ then A and B are said to be* Nash-isomorphic *if there exists compact Nash manifolds $A \subset X$ and $B \subset Y$ and a Nash diffeomorphism $\phi : X \to Y$ such that $\phi(A) = B$.*

Then Fichou proved [**20**, Theorem 3.3] that the virtual Poincaré polynomial as defined on the $\mathcal{AS}$-collection (which contains the real algebraic sets) is invariant under Nash isomorphisms.

2.3. The Arc-Analytic Equivalence. We now turn to the arc-analytic equivalence.

Definition 3.5 ([**10**, Definition 7.1]). *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of Nash functions. If there exists a germ of semialgebraic homeomorphism*

$$h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

with $f = g \circ h$ such that h is arc-analytic and if there exists a constant $c > 0$ such that

$$|\det Jac(h)| > c$$

there where the Jacobian determinant is defined, then f is said to be arc-analytically equivalent to g via h and one writes $f \sim_a g$ or $f \sim g$ if no ambiguity is possible.

This notion is due to J.-B. Campesato. He introduced it in order to prove that the *Blow-Nash equivalence* on germs of Nash functions [**20**], [**10**] is indeed an equivalence relation. To describe the latter we need the notion of Nash modification. To this aim, let us first recall the notion of complexification of a real analytic manifold.

DEFINITION 3.6 ([**10**, Definition 1.4]). *A complexification of a real analytic manifold M of dimension n is a complex analytic manifold $M_{\mathbb{C}}$ endowed with a real analytic isomorphism*

$$\phi : M \to \phi(M) \subset M_{\mathbb{C}}$$

with $\phi(M)$ a real analytic subvariety of $M_{\mathbb{C}}$ such that for any $x \in M_{\mathbb{C}}$ there exists a neighborhood $U_{\mathbb{C}} \subset M_{\mathbb{C}}$ of x and a complex analytic isomorphism

$$\psi : U_{\mathbb{C}} \to \psi(U_{\mathbb{C}}) \subset \mathbb{C}^n$$

such that

$$\psi(\phi(M) \cap U_{\mathbb{C}}) \subset \mathbb{R}^n \cap \psi(U_{\mathbb{C}}).$$

In other words M can be identified with a real analytic subvariety of its complexification via ϕ and this identification respects the charts. One can show that any real analytic manifold possesses a complexification. In the Nash category, the construction is functorial in the sense that if $g : M \to N$ is a proper *Nash* map between Nash manifolds and if $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$ are complexifications then g extends to a proper complex analytic map $g_{\mathbb{C}} : U \to V$ of open neighborhoods $U \subset M_{\mathbb{C}}$ and $V \subset N_{\mathbb{C}}$ of M and N, respectively.

DEFINITION 3.7 ([**10**, § 6.2]). *A Nash map $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is said to be a Nash modification if it is proper, surjective and if its complexification is proper and bimeromorphic.*

DEFINITION 3.8 ([**20**, Definition 4.2]). *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of Nash functions. One says that f and g are blow-Nash equivalent if the following holds.*

(1) *There exist two germs of Nash modifications*

$$\sigma_f : (M, \sigma_f^{-1}(0)) \to (\mathbb{R}^n, 0), \qquad \sigma_g : (M', \sigma_g^{-1}(0)) \to (\mathbb{R}^n, 0)$$

such that the associated divisors of $f \circ \sigma_f$ and $g \circ \sigma_g$ are normal crossings, and such that the associated divisors of $Jac(\sigma_f)$ and $Jac(\sigma_g)$ are normal crossings.

(2) *There exists a germ of Nash isomorphisms*

$$\Psi : (M, \sigma_f^{-1}(0)) \to (M', \sigma_g^{-1}(0))$$

preserving the multiplicities of the divisors divJac(σ_f) *and divJac*(σ_g) *along each of the irreducible components of the exceptional divisors* $\sigma_f^{-1}(0)$ *and* $\sigma_g^{-1}(0)$ *of the modifications.*

(3) *The map* Ψ *induces a germ of semialgebraic homeomorphism*

$$\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

making the following diagram commute

$$\begin{tikzcd}
(M, \sigma_f^{-1}(0)) \arrow[rr, "\Psi"] \arrow[d, "\sigma_f"] & & (M', \sigma_g^{-1}(0)) \arrow[d, "\sigma_g"] \\
(\mathbb{R}^n, 0) \arrow[rr, "\psi"] \arrow[dr, "f"] & & (\mathbb{R}^n, 0) \arrow[dl, "g"] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

We write $f \sim_{b-N} g$ *if* f *and* g *are blow-Nash equivalent.*

Campesato then proved the following theorem.

THEOREM 3.9 ([**10**, Theorem 7.9]). *Two germs of Nash functions*

$$f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

are arc-analytically equivalent if and only if they are blow-Nash equivalent.

In his thesis [**8**] Campesato showed that the arc-analytic equivalence is indeed an equivalence relation.

PROPOSITION 3.10 ([**8**, Proposition 3.7.5]). *The arc-analytic equivalence is an equivalence relation.*

As a consequence one obtains the corollary.

COROLLARY 3.11 ([**8**, Corollary 3.7.8]). *The Blow-Nash equivalence is an equivalence relation.*

3. Invariance Under the Arc-Analytic Equivalence

According to the result [**20**, Theorem 4.9] the virtual Poincaré polynomials of the motivic Milnor fibres are invariant under the arc-analytic equivalence. We shall begin by proving that this is also the case for the motivic Milnor tubes; the proof is in effect a minor modification of the proof of Fichou's result.

THEOREM 3.12. *If $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are arc-analytically equivalent germs of Nash functions then for any symbol $\epsilon \in \{\pm 1, <, >\}$ the virtual Poincaré polynomial of the respective motivic Milnor fibres and tubes $S^\epsilon(f)$ and $S^\epsilon(g)$, are equal.*

PROOF. (1) According to [**10**, Proposition 7.9] one has that $f \sim_a g$ if and only if $f \sim_{b-N} g$. Therefore there exist germs of Nash modifications

$$\sigma_f : (M, \sigma_f^{-1}(0)) \to (\mathbb{R}^n, 0), \qquad \sigma_g : (M', \sigma_g^{-1}(0)) \to (\mathbb{R}^n, 0)$$

having the properties that the associated divisors of $f \circ \sigma_f$ and of $g \circ \sigma_g$ are normal crossings and that the associated divisors $\mathrm{Jac}(\sigma_f)$ and $\mathrm{Jac}(\sigma_g)$ are normal crossings, and that there exists a germ of Nash isomorphisms ([**20**, Definition 3.2])

$$\Psi : (M, \sigma_f^{-1}(0)) \to (M', \sigma_g^{-1}(0))$$

preserving the multiplicities of the divisors div $\mathrm{Jac}(\sigma_f)$ and div $\mathrm{Jac}(\sigma_g)$ along the irreducible components of the corresponding exceptional divisors $\sigma_f^{-1}(0)$ and $\sigma_g^{-1}(0)$.

(2) Write

$$(g \circ \sigma_g)^{-1}(0) = \bigcup_{i=1}^{k} E_i'$$

for the decomposition into irreducible components of $(g \circ \sigma_g)^{-1}(0)$ and let $p \in M'$ be a point. Since $\mathrm{div}(g \circ \sigma_g)$ is a normal crossings divisor there exists an affine open set $U' \subset M'$ containing p and a system of (étale) local coordinates $x_1, \dots, x_k$ on $U' \subset M'$ such that

$$g \circ \sigma_g(x) = v(x) \prod_{i=1}^{k} x_i^{N_i}$$

for $v : U' \to \mathbb{R}$ an invertible Nash function. Since $\sigma_g^{-1}(0)$ is compact one can find a finite set $L \subset \mathbb{N}$ and open affine subsets $U_l', l \in L$ of a neighborhood of $\sigma_g^{-1}(0)$ such that

$$\sigma_g^{-1}(0) \subset \bigcup_{l \in L} U'_l.$$

For any subset $I \subset \{1, \ldots, k\}$ consider the stratification of $(g \circ \sigma_g)^{-1}(0)$ with strata

$$E_I'^{\,o} = \bigcap_{i \in I} E'_i \setminus \bigcup_{j \notin I} E'_j.$$

If $E_I'^{\,0} \cap \sigma_g^{-1}(0) \neq \emptyset$ then it follows that

$$E_I'^{o} = \bigcup_{l \in L} E_I'^{\,o} \cap U'_l.$$

(3) Then $E_I^{'o} \in \mathcal{AS}$ since g is a germ of Nash function, by [**10**, Remark 6.10], and $U' \in \mathcal{AS}$ because $\mathcal{AS}$ is a constructible category which contains the category of Zariski constructibles, by Theorem 2.34 ([**33**, Theorem 4.4]). Hence $E_I^{'o} \cap U' \in \mathcal{AS}$ again by Theorem 2.34 ([**33**, Theorem 4.4]) because $\mathcal{AS}$ is a constructible category.

(4) For each symbol $\epsilon \in \{\pm 1, <, >\}$ consider the classes in $K_0(\mathrm{BSA})$ corresponding to g and U' as in Definition 2.28 and in the statement of [**11**, Theorem 4.2], namely

$$[R^{\pm}_{I,U'}(g)] = [(x,t) \in (E_I'^{o} \cap U') \times \mathbb{R} \mid t^{N_I} v(x) = \pm 1],$$

$$[R^{>}_{I,U'}(g)] = [(x,t) \in (E_I'^{o} \cap U') \times \mathbb{R} \mid t^{N_I} v(x) > 0],$$

$$[R^{<}_{I,U'}(g)] = [(x,t) \in (E_I'^{o} \cap U') \times \mathbb{R} \mid t^{N_I} v(x) < 0].$$

Since Ψ is a Nash isomorphism the union of the preimages $E_I^o := \Psi^{-1}(E_I'^{o})$ gives a stratification of

$$(f \circ \sigma_f)^{-1}(0)$$

such that $\sigma_f^{-1}(0)$ is a union of strata. Moreover if $U' \subset M'$ is an affine open set then $U = \Psi^{-1}(U')$ is an affine open set because a Nash isomorphism is semialgebraic by [**20**, Definition 3.2]. Write

$$\Psi = (\Psi_1, \ldots, \Psi_n) : (M, \sigma_f^{-1}(0)) \to (M', \sigma_g^{-1}(0)).$$

Since Ψ is a germ of Nash isomorphism it preserves the multiplicities N_i of $\mathrm{div}(g \circ \sigma_g)$ along E_i for each $i = 1, \ldots, k$ so commutativity of the diagram in the Definition 3.8 of the Blow-Nash equivalence yields

$$f \circ \sigma_f(y) = g \circ \sigma_g \circ \Psi(y) = v(\Psi(y)) \prod_{i=1}^{k} \Psi_i(y)^{N_i}$$

where $v \circ \Psi : U \to \mathbb{R}$ is an invertible Nash function. All said,

$$[R^{\pm}_{I,U}(f)] = [(y,t) \in (E^o_I \cap U) \times \mathbb{R} \mid t^{N_I} v(\Psi(y)) = \pm 1],$$

$$[R^{>}_{I,U}(f)] = [(y,t) \in (E^o_I \cap U) \times \mathbb{R} \mid t^{N_I} v(\Psi(y)) > 0],$$

$$[R^{<}_{I,U}(f)] = [(y,t) \in (E^o_I \cap U) \times \mathbb{R} \mid t^{N_I} v(\Psi(y)) < 0]$$

are the classes in $K_0(\mathrm{BSA})$ corresponding to f and $U = \Psi^{-1}(U')$ as in the statement of [**11**, Theorem 4.2].

(5) Consider now the realisations of these sets under the virtual Poincaré polynomial (Definition 2.20). We claim that the underlying set $R^{\pm 1}_{I,U'}(g)$ is an intersection of $\mathcal{AS}$-sets. Indeed $E'^o_I \cap U'$ is $\mathcal{AS}$ by the above. Furthermore v is Nash hence semialgebraic and real analytic so its graph is $\mathcal{AS}$, as in the second step of the proof of ([**20**, Theorem 3.3]). The inverse image of an $\mathcal{AS}$-set under an $\mathcal{AS}$-map is $\mathcal{AS}$, by Theorem 2.34 ([**33**, Theorem 4.4]), hence $\{v(x) = \pm 1\}$ are $\mathcal{AS}$. Therefore $R^{\pm 1}_{I,U'}(g)$ is an intersection of $\mathcal{AS}$-sets hence is an $\mathcal{AS}$-set by Theorem 2.34 ([**33**, Theorem 4.4]).

(6) Again by the second step of the proof of [**20**, Theorem 3.3] the graph of Ψ is $\mathcal{AS}$ so Ψ is an $\mathcal{AS}$-map. Since it is an analytic isomorphism it is injective so the images of each of the sets $R^{\pm 1}_{I,U'}(g)$ under the injective $\mathcal{AS}$-map

$$(x,t) \mapsto (\Psi^{-1}(x), t)$$

are $\mathcal{AS}$-sets, Theorem 2.34 ([**33**, Theorem 4.4]). Hence $R^{\pm 1}_{I,U}(f)$ are $\mathcal{AS}$-sets. Since Ψ is a Nash isomorphism the application $(y,t) \mapsto (\Psi(y), t)$ gives a Nash isomorphism between $R^{\epsilon}_{I,U'}(g)$ and $R^{\epsilon}_{I,U}(f)$. Since β_{Var} is invariant under Nash isomorphisms by [**20**, Theorem 3.3] it follows that

$$\beta_{\mathrm{BSA}}(R^{\epsilon}_{I,U}(f)) = \beta_{\mathrm{Var}}(R^{\epsilon}_{I,U}(f)) = \beta_{\mathrm{Var}}(R^{\epsilon}_{I,\Psi(U)}(g)) = \beta_{\mathrm{BSA}}(R^{\epsilon}_{I,\Psi(U)}(g))$$

for any $I \subset \{1, \dots, k\}$ and any symbol $\epsilon \in \{\pm 1\}$. If $\bigcup_{l \in L} U'_l$ is a covering of a neighborhood of $\sigma_g^{-1}(0)$ by affine open sets as in the previous step 2 then

$$\bigcup_{l \in L} \Psi^{-1}(U_l)$$

is a covering of a neighborhood of $\sigma_f^{-1}(0)$ by affine open sets hence

$$\beta_{\mathrm{Var}}(E_I^{o,\epsilon}(f)) = \sum_{S\subset L}(-1)^{|S|+1}\beta_{\mathrm{Var}}(R_{I,\cap_{s\in S}\Psi^{-1}(U_s)}(f))$$

$$= \sum_{S\subset L}(-1)^{|S|+1}\beta_{\mathrm{Var}}(R_{I,\cap_{s\in S}U_s}(g)) = \beta_{\mathrm{Var}}(E_I^{o,\epsilon}(g))$$

so $\beta_{\mathrm{BSA}}(S_f^\epsilon) = \beta_{\mathrm{BSA}}(S_g^\epsilon)$ by [**11**, Theorem 4.2].

(7) We shall prove that $\beta_{\mathrm{BSA}}(S_f^>) = \beta_{\mathrm{BSA}}(S_g^>)$, the proof of $\beta_{\mathrm{BSA}}(S_f^<) = \beta_{\mathrm{BSA}}(S_g^<)$ is similar. It follows by the Definition 2.20 of the virtual Poincaré polynomial on basic semialgebraic formulas that

$$\beta_{\mathrm{BSA}}(R_{I,U}^>(f)) = \frac{1}{4}\beta_{Var}([(y,t)\in E_I^o\cap U\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) = z^2])$$

$$-\frac{1}{4}\beta_{Var}([(y,t)\in E_I^o\cap U\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) = -z^2])$$

$$+\frac{1}{2}\beta_{Var}([(y,t)\in E_I^o\cap U\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) \neq 0]).$$

To show $\beta_{\mathrm{BSA}}(R_{I,U}^>(f)) = \beta_{\mathrm{BSA}}(R_{I,\Psi(U)}^>(g))$ it therefore suffices to show that

$$\beta_{Var}([(x,t)\in E_I'^o\cap U'\times\mathbb{R} \mid t^{N_I}v(x) = \pm z^2]) =$$

$$\beta_{Var}([(y,t)\in E_I^o\cap U\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) = \pm z^2])$$

and that

$$\beta_{Var}([(x,t)\in E_I'^o\cap U'\times\mathbb{R} \mid t^{N_I}v(x) \neq 0]) =$$

$$\beta_{Var}([(y,t)\in E_I^o\cap U\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) \neq 0]).$$

The underlying sets of these classes are in each of the above cases intersections of $\mathcal{AS}$-sets and Zariski constructible sets hence are $\mathcal{AS}$-sets by Theorem 2.34 ([**33**, Theorem 4.4]). Since Ψ is a Nash isomorphism it follows that $(y,t,z)\mapsto(\Psi(y),t,z) = (x,t,z)$ gives a Nash isomorphism between

$$\{(y,t,z)\in E_I^o\cap U\times\mathbb{R}\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) = z^2\}$$

and

$$\{(x,t,z)\in E_I'^o\cap U'\times\mathbb{R}\times\mathbb{R} \mid t^{N_I}v(x) = z^2\}$$

and between

$$\{(y,t,z)\in E_I^o\cap U\times\mathbb{R}\times\mathbb{R} \mid t^{N_I}v(\Psi(y)) \neq 0\}$$

and

$$\{(x,t,z) \in E_I'^o \cap U' \times \mathbb{R} \times \mathbb{R} \mid t^{N_I} v(x) \neq 0\}.$$

Thus $\beta_{\mathrm{BSA}}(R^>_{I,U}(f)) = \beta_{\mathrm{BSA}}(R^>_{I,\Psi(U)}(g))$ for all $I \subset \{1,\ldots,k\}$ by [**20**, Theorem 3.3] so the same argument as in the previous step 6 gives

$$\beta_{\mathrm{BSA}}(S^>(f)) = \beta_{\mathrm{BSA}}(S^>(g)).$$

□

4. Arc-Wise Analytic Triviality

4.1. Arc-Wise Analytic Triviality of Function Germs. Before continuing we need to recall the notion of arc-wise analytic triviality for germs of families of analytic functions; we refer to the article [**43**] for further details. In the following we shall denote by $\mathbb{K}$ either the field of complex, or real numbers.

DEFINITION 3.13 ([**43**, § 8.3]). *Let $T \subset \mathbb{K}^m$ be an open analytic set. A $\mathbb{K}$-analytic family $f : T \times (\mathbb{K}^{n-1}, 0) \to \mathbb{K}$ of germs of $\mathbb{K}$-analytic functions is* arc-wise analytically trivial *over T if there exists a neighborhood $\Lambda \subset \mathbb{K}^m \times \mathbb{K}^{n-1}$ of $T \times \{0\}$ and a neighborhood $\Lambda_0 \subset \mathbb{K}^{n-1}$ of $\{0\}$ and an arc-wise analytic trivialisation* [**43**, Definition 1.2]

$$\sigma : T \times \Lambda_0 \to \Lambda$$

such that $f \circ \sigma(t,x) = f(0,x)$.

The key result which is going to be used later on is the following theorem. Here we only assume that $T \subset \mathbb{K}^m$ is analytic.

THEOREM 3.14 ([**43**, Theorem 8.5]). *Let $f : T \times (\mathbb{K}^{n-1}, 0) \to \mathbb{K}$ be a $\mathbb{K}$-analytic family of $\mathbb{K}$-analytic function germs and let $t_0 \in T$. Then there exists a neighborhood $U \subset T$ of t_0 and a $\mathbb{K}$-analytic subset $Z \subset U$ with $\dim Z < \dim U$ such that f is arc-wise analytically trivial along $U \setminus Z$.*

4.2. The Main Theorem. The main result of this chapter is the following Theorem 3.15. The content of the proof is to show that in the proof of the Theorem 3.14 above, the constructed arc-analytic trivialisation of σ has the property that the jacobian determinant of $\sigma_t := \sigma(t,\cdot)$ is bounded where it is defined and thus provides the wanted arc-analytic equivalence.

THEOREM 3.15. *Let $T \subset \mathbb{R}^m$ be a real algebraic set and let $f : T \times (\mathbb{R}^n, 0) \to \mathbb{R}$ be a polynomial function of polynomial map germs such that $f(t,0) = 0$ for any*

$t \in T$. There exists a locally finite real analytic stratification $\mathcal{S}$ of T such that if $S \in \mathcal{S}$ is a stratum then

$$f_t = f(t, \cdot) \sim_a f_{t'} = f(t, \cdot), \qquad \forall t, t' \in S$$

in the notation of Definition 3.5. In particular

$$\beta_{BSA}(S^\epsilon(f_t)) = \beta_{BSA}(S^\epsilon(f_{t'})), \qquad \forall t, t' \in S \in \mathcal{S}, \qquad \forall \epsilon \in \{\pm 1, <, >\}.$$

Moreover if T is compact then $\mathcal{S}$ is a finite stratification.

Proof. (1) Recalling the proof of the Theorem 3.14 ([**43**, Theorem 8.5]) let us write

$$F : T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \qquad F(t, z, x) = z - f(t, x)$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define

$$Y := \{F = 0\} \subset T \times \mathbb{R} \times \mathbb{R}^n.$$

Fix $t_0 \in T$. Then according to [**43**, Lemma 6.3] there exists a neighborhood $U(t_0)$ containing t_0, a proper analytic subset

$$Z(t_0) \subset U(t_0), \qquad \dim Z(t_0) < \dim U(t_0)$$

and a transverse local system $(F_i)_{i=0}^{n+1}$ of Zariski equisingular pseudo-polynomials (see [**43**, Definition 4.1, 3.1]) such that F_{n+1} is the Weierstrass polynomial of F in $(t, 0)$ for all $t \in U(t_0) \setminus Z(t_0)$. By [**43**, Theorem 3.3] there exist for any such local system a real number $\epsilon > 0$ and neighborhoods

$$\mathbb{B}_\epsilon = \{t \in U(t_0) \setminus Z(t_0) \mid \|t - t_0\| \le \epsilon\} \subset \mathbb{R}^m,$$

$$\Omega_0 \subset \mathbb{R}^{n+1}, \qquad \Omega \subset \mathbb{R}^{n+m+1}$$

of t_0 and of the respective origins and a homeomorphism

$$\Phi : \mathbb{B}_\epsilon \times \Omega_0 \to \Omega$$

preserving $\{z_0 = f(t, x)\}$ for each $z_0 \in \mathbb{R}$ (see [**43**, Definition 1.2]) such that $(z_0, x) \in \Omega_0$, and such that the conditions $(Z2 - Z3)$ of [**43**, Theorem 3.3] are satisfied, that is for all $t \in \mathbb{B}_\epsilon$ and all $(z, x_1, \ldots, x_n) \in \Omega_0$ the following holds,

(a) Φ is an arc-analytic trivialisation of the standard projection $\Omega \to \mathbb{B}_\epsilon$. In particular Φ is real analytic in t and arc-analytic in $(z, x_1, \ldots, x_n)$ and Φ^{-1} is arc-analytic.

(b) $\Phi(t, 0) = (t, 0)$ and

$$\Phi(t_0, z, x_1, \ldots, x_n) = (t_0, z, x_1, \ldots, x_n).$$

(c) $\Phi(t, z, x_1, \ldots, x_n) = (t, \Psi_1(t, z), \ldots, \Psi_{n+1}(t, z, x_1, \ldots, x_n))$.

(d) For each $i = 2, \ldots, n+1$,

$$\Psi_i(t, z, x_1, \ldots, x_{i-1}, \bullet) : \mathbb{R} \to \mathbb{R}$$

are bi-Lipschitz with the Lipschitz constants of Ψ_i and of Ψ_i^{-1} independent on $(t, z, x_1, \ldots, x_{i-1})$ and

$$\Psi_1(t, \cdot) : \mathbb{R} \to \mathbb{R}$$

is bi-Lipschitz with Lipschitz constants of Ψ_1 and of Ψ_1^{-1} independent of t.

Write $\Phi(t, z, x) = (t, \Psi(t, z, x))$ (so that $\Psi : \mathbb{B}_\epsilon \times \Omega_0 \to \Omega_0$) and consider the functions

$$\Psi_t = \Psi(t, \cdot, \cdot) : \Omega_0 \to \Omega_0, \qquad t \in U(t_0) \setminus Z(t_0)$$

$$\Phi_t = (t, \Psi_t) : \Omega_0 \to \mathbb{B}_\epsilon \times \Omega_0.$$

The above conditions (1c) and (1d) imply, as in the proof of [**43**, Proposition 3.6], that there exist constants $\tilde{C}, \tilde{c} > 0$, which are by condition (1d) independent of t, such that the Jacobian determinants are bounded

$$\tilde{c} \leq |\det \operatorname{Jac} \Psi_t| \leq \tilde{C} \tag{5}$$

there where they are defined. In particular Ψ_t is a diffeomorphism for $t \in U(t_0) \setminus Z(t_0)$ (see also [**43**, Proposition 3.5]). Denote by

$$\pi : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n, \qquad \pi(t, z, x) = (t, x)$$

the standard projection and put

$$\tilde{\pi} : \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad \tilde{\pi}(z, x) = x.$$

Let $\Lambda_0 = \tilde{\pi}(\Omega_0)$ and $\Lambda = \pi(\Omega)$. Then

$$\sigma : U(t_0) \setminus Z(t_0) \times \Lambda_0 \to \Lambda$$

given by

$$\sigma(t, x) = \pi \circ \Phi(t, f_0(x), x)$$

is an arc-wise analytic trivialisation of f along $U(t_0) \setminus Z(t_0)$, by 3.14 ([**43**, Theorem 8.5]).

(2) Consider

$$\sigma_t = \sigma(t, \cdot) : \Lambda_0 \to \Lambda.$$

Then by construction,

$$\sigma_t = \pi \circ \Phi(t, f_0(\cdot), \cdot) =$$

$$= \pi \circ (t, \Psi(t, f_0(\cdot), \cdot) = \tilde{\pi} \circ \Psi_t(f_0(\cdot), \cdot)$$

so that one can write

$$\sigma_t = \tilde{\pi} \circ \Psi_{t|H} = \tilde{\pi} \circ \Psi_t \circ g$$

where $H \subset \Omega_0$ is the smooth hypersurface $H = \Gamma(f_0) \cap \Omega_0$, where

$$\Gamma(f_0) = \{(z, x) \in \mathbb{R}^{n+1} \mid z = f_0(x)\}$$

and where

$$g : \Lambda_0 \to \Omega_0, \qquad g(x) = (f_0(x), x).$$

(3) Let $t \in U(t_0) \setminus Z(t_0)$. Then

$$\mathrm{Jac}(\sigma_t)(x) = \mathrm{Jac}(\tilde{\pi})(\Psi_t(f_0(x), x))\mathrm{Jac}(\Psi_t)(f_0(x), x)\mathrm{Jac}(g)(x)$$

by the chain rule. Taking local coordinates one finds that the Jacobian matrix of g has the form

$$\mathrm{Jac}(g)(x) = \begin{pmatrix} (\partial f_0/\partial x_i)(x) & I_n \end{pmatrix}$$

with $I_n = \mathrm{Id}_{\mathbb{R}^n}$ the identity matrix, so $\mathrm{Jac}(g)$ has full rank. By 5 one has that $\mathrm{Jac}(\Psi_t)$ likewise has full rank, since its determinant is nonzero.

(4) We claim that, after replacing if necessary $U(t_0)$ by a smaller neighborhood of the origin, for all $t \in U(t_0) \setminus Z(t_0)$ the restriction $\tilde{\pi}_{|\Psi_t(H)}$ of $\tilde{\pi}$ to $\Psi_t(H)$ is a submersion. To show this, remark that

$$\tilde{\pi}_{|H} : H \to \mathbb{R}^n$$

is a submersion, by the previous step (3). This implies that there exists a real number $\delta > 0$ such that if $\mathbb{B}_\delta \subset \mathbb{R}^{n+1}$ denotes the open ball of radius δ centered at the origin then $\bar{\mathbb{B}}_\delta \subset \Omega_0$ and

$$\forall s \in \tilde{\pi}(H \cap \partial\bar{\mathbb{B}}_\delta), \qquad \tilde{\pi}^{-1}(s) \pitchfork (H \cap \partial\bar{\mathbb{B}}_\delta).$$

In particular

$$\tilde{\pi}_{|H\cap\partial\bar{\mathbb{B}}_\delta} : H \cap \partial\bar{\mathbb{B}}_\delta \to \mathbb{R}^n$$

is a submersion. Replacing Ω_0 by $\mathbb{B}_\delta$ we can therefore assume that

$$\tilde{\pi}_{|H\cap\bar{\Omega}_0} : H \cap \bar{\Omega}_0 \to \mathbb{R}^n$$

is a submersion. Remark that $t \mapsto \Psi_t$ is continuous by (1a) and that Ψ_{t_0} is the identity map on Ω_0 by (1b). Since being a submersion is an open condition and since $H \cap \bar{\Omega}_0$ is compact it follows that there exists an open neighborhood $U'(t_0) \subset \mathbb{R}^m$ of t_0 such that $U'(t_0) \subset U(t_0)$ and such that

$$\tilde{\pi}_{|\Psi_t(H)} : \Psi_t(H) \to \mathbb{R}^n$$

is a submersion. This proves the claim.

(5) Fix $t \in U(t_0) \setminus Z(t_0)$. By the previous step (4) $\det \mathrm{Jac}(\sigma_t)$ is nonzero at any $x \in \Lambda_0$. Recall Hadamard's determinant inequality: if $A \in M(k, k, R)$ is a $k \times k$ matrix over a commutative ring R then

$$|\det(A)| \leq \|A \cdot e_1\| \dots \|A \cdot e_k\|$$

where $A \cdot e_i$ are the column vectors of A. Again by (1d),

$$\|\mathrm{Jac}\Psi_t \cdot e_i\| \leq K, \qquad i = 1, \dots, n+1.$$

for a constant K not depending on t. Let

$$M := \max_{1\leq i\leq n} \sup_{x\in\bar{\Lambda}_0} \partial f_0/\partial x_i(x)$$

where $\bar{\Lambda}_0 \subset \mathbb{R}^n$ is the closure of Λ_0 for the euclidean topology. Since $\mathrm{Jac}(\sigma_t)$ is invertible, the chain rule and Hadamard's inequality yield

$$|\det \mathrm{Jac}(\sigma_t^{-1})(\sigma_t(x))| = \frac{1}{|\det \mathrm{Jac}(\tilde{\pi} \circ \Psi_{t|H})(x)|}$$
$$\geq \frac{1}{\prod_{i=1}^{n} \|\mathrm{Jac}(\tilde{\pi} \circ \Psi_{t|H}) \cdot e_i\|}$$
$$\geq \frac{1}{lM \prod_{i=1}^{n+1} \|\mathrm{Jac}(\Psi_t) \cdot e_i\|} \geq \frac{1}{lMK^{n+1}}$$

for some nonzero $l \in \mathbb{N}$. Therefore there exists a positive constant $c = 1/(lMK^{n+1})$ not depending on the parameter t (because K and M are independent of t), such that

$$|\det \mathrm{Jac}(\sigma_t^{-1})| \geq c.$$

By the definition [**43**, Definition 1.2] of an arc-wise trivialisation Φ_t is arc-analytic and its inverse is arc-analytic hence σ_t is arc-analytic with arc-analytic inverse. And as its determinant is bounded where it is defined it follows by Definition 3.5 of the arc-analytic equivalence that for each $t \in U(t_0) \setminus Z(t_0)$ the function σ_t^{-1} yields an arc-analytic equivalence $f_t \sim_a f_0$. Therefore,

$$\beta_{\mathrm{BSA}}(S^\epsilon(f_t)) = \beta_{\mathrm{BSA}}(S^\epsilon(f_{t_0})), \qquad \forall t \in U(t_0) \setminus Z(t_0)$$

by Theorem 3.12.

(6) For each $t_0 \in T$ one finds a finite analytic stratification of a neighborhood of t_0 as follows. Let $U_0 = U(t_0)$ be the neighborhood of t_0 constructed in the previous step and let $Z_0 = Z(t_0)$ be the corresponding proper analytic subset. Repeating the previous step we find for each $t_1 \in Z_0$ a neighborhood $U_1 \subset Z_0$ of t_1, a proper analytic subset $Z_1 \subset U_1$ with $\dim Z_1 < \dim U_1 \leq \dim Z_0$ and an arc-wise analytic trivialisation

$$\sigma_1 : U_1 \setminus Z_1 \times \Lambda_0^1 \to \Lambda^1, \qquad f \circ \sigma_1(t,x) = f(0,x).$$

By induction on the dimension one finds a finite list of subsets

$$S_0(t_0), \ldots, S_n(t_0) \subset U_0$$

giving a stratification $\mathcal{S}(t_0)$ of the neighborhood U_0 of $t_0 \in T$ such that

$$\beta_{\mathrm{BSA}}(S^\epsilon(f_t)) = \beta_{\mathrm{BSA}}(S^\epsilon(f_{t'})), \qquad \forall t, t' \in S_i(t_0).$$

(7) One takes a covering of T by neighborhoods of the form $U = U(t)$ and one considers for each $t \in T$ the corresponding finite stratifications $\mathcal{S}(t)$. One defines $\mathcal{S}$ as a common refinement of the stratifications $\mathcal{S}(t)$. If T is compact then there exists a finite subcovering

$$T \subset \bigcup_{i=1}^{l} U(t_i)$$

with finite stratifications $\mathcal{S}(t_i)$ of $U(t_i)$, for each $i = 1, \ldots, l$ so in this case the common refinement $\mathcal{S}$ is finite as well.

$\square$

CHAPTER 4

Isolated Real Hypersurface Singularities

1. Introduction

This chapter concerns the topology of real isolated hypersurface singularities. We prove in Theorem 4.47 that after attaching a certain number of handles the real Milnor fibres become contractible, with each handle corresponding to a critical point of a $\mathbb{R}$-morsification (Definition 4.28); in particular one recovers the formula of Khimshiashvili ([**16**, Theorem 2.3]) for the Euler characteristic of the Milnor fibres.

We then give sufficient conditions, in Theorem 4.58 for having that the integer homology groups of the real Milnor fibres are isomorphic to the homology groups of a bouquet of spheres, where each sphere corresponds to a critical point of the given $\mathbb{R}$-morsification. In particular it follows in this case that the homology groups of the real Milnor fibres are uniquely determined by the real vanishing cycles (Definition 4.61). Our approach is by and large to adapt the proof of E. Brieskorn [**7**, Appendix] of Milnors theorem [**39**, Theorem 7.2] by replacing arguments which fails in the real case by arguments coming from Stratified Morse Theory [**25**] as constructed by Goresky and MacPherson.

Throughout, Stratified Morse Theory will serve as an important aide, which we therefore recall to facilitate the lecture.

2. Isolated Real Hypersurface Singularities

2.1. On Stratified Morse Theory. In their 1974 book [**25**] G. Goresky and R. MacPherson presented an extension of classical Morse theory on compact manifolds, to Whitney stratified spaces. We here present some of their main results. Let $X \subset M$ be a subset of a smooth manifold M and let $\mathcal{S}$ be a Whitney stratification of X.

DEFINITION 4.1 ([**25**, Definition 2.1]). *If $f_{|X} : X \to \mathbb{R}$ is the restriction of a smooth function $f : M \to \mathbb{R}$ then $p \in X$ is said to be a critical point of $f_{|X}$ if it is a critical point of the restriction $f_{|S}$ where $S \in \mathcal{S}$ is the stratum containing p, that is, $df(p)(T_pS) = 0$. A critical value $s = f(p) \in \mathbb{R}$ is said to be an* isolated critical value *if there exist $\gamma > 0$ such that p is the unique critical point in $f^{-1}([s-\gamma, s+\gamma])$.*

The notion of stratified Morse functions is as follows.

DEFINITION 4.2 ([**25**, Definition 2.1]). *A Morse function on X is the restriction $f_{|X} : X \to \mathbb{R}$ of a smooth function $f : M \to \mathbb{R}$ such that*

(1) *$f_{|X} : X \to \mathbb{R}$ is proper and if $p \neq q$ are distinct critical points of $f_{|X}$, then $f(p) \neq f(q)$.*
(2) *For any stratum $S \in \mathcal{S}$ the critical points of $f_{|S} : S \to \mathbb{R}$ are nondegenerate.*
(3) *For any critical point p belonging to a stratum $S \in \mathcal{S}$, if $S' \in \mathcal{S}$ is a stratum such that $S \subset \overline{S'}$ and if $(p_i) \in S'$ is a sequence of points converging to p, if*
$$Q := \lim_{p_i \to p} T_{p_i} S'$$
then
$$df(p)(Q) \neq 0 \qquad \text{whenever } Q \neq T_pS.$$

One can prove [**25**, Theorem 2.2.1] that if X is a closed subanalytic subset of an analytic manifold M then the set of smooth real-valued functions on M which restrict to a stratified Morse function on X forms an open dense subset of the space $C_p^\infty(M, \mathbb{R})$ of proper smooth maps $M \to \mathbb{R}$, with respect to the Whitney topology. Many of the results established by Goresky and MacPherson using stratified Morse functions, as defined above, hold as well for a larger class of functions; namely those functions having at most nondepraved critical points.

DEFINITION 4.3 ([**25**, Definition 2.3]). *Let $f : M \to \mathbb{R}$ be a smooth function and suppose that $X \subset M$ is a Whitney stratified subset with Whitney stratification $\mathcal{S}$. A critical point $p \in X$ of $f_{|X} : X \to \mathbb{R}$ is* nondepraved *if the following holds:*

(1) *p is an isolated critical point.*

(2) *The restriction* $f_{|S} : S \to \mathbb{R}$ *to the stratum containing* p *is nondepraved, that is, for any sequence of points* $(p_i) \in S$ *with* $p_i \to p$ *if*

$$v_i := \frac{p_i - p}{|p_i - p|} \to v, \qquad \ker df_{|S}(p_i) \to \tau$$

and $v \notin \tau$ *then for all* i *sufficiently large*

$$df(p_i)(v_i) \cdot (f(p_i) - f(p)) > 0.$$

(3) *If* $S \in \mathcal{S}$ *is the stratum containing* p *then for any stratum* $S' \in \mathcal{S}$ *such that* $S \subset \overline{S'}$ *if* $(p_i) \in S'$ *is a sequence of points converging to* p *and*

$$Q = \lim_{p_i \to p} T_{p_i} S'$$

then

$$df(p)(Q) \neq 0 \qquad \textit{whenever } Q \neq T_p S.$$

Let us now recall the Main Theorem of Stratified Morse theory in the proper nonrelative case. Let $X \subset M$ be a Whitney stratified subset of a smooth manifold as above and let $f : X \to \mathbb{R}$ be a proper smooth function with a nondepraved critical point $p \in X$ with value $s = f(p)$. We endow M with a Riemannian metric and we let

$$\bar{\mathbb{B}}^M_\eta(p) \subset M$$

denote the closed ball of radius $\eta > 0$ and centre p, with respect to this metric. Since X is Whitney stratified there exists $\eta > 0$ such that the boundary of the ball intersects any stratum $S' \in \mathcal{S}$ transversally

$$\partial \bar{\mathbb{B}}^M_\eta(p) \pitchfork S', \qquad S' \in \mathcal{S}.$$

Using [**25**, Lemma 3.5.1] it follows, by replacing η by a smaller value if necessary, that one can assume that

$$f_{|\bar{\mathbb{B}}^M_\eta(p) \cap X} : \bar{\mathbb{B}}^M_\eta(p) \cap X \to \mathbb{R}$$

has no other critical point of value s other than p. As p is nondepraved and nondepraved critical points are isolated there exists $\gamma > 0$ such that $f^{-1}([s - \gamma, s + \gamma])$ contains no other critical points than p.

Let S be the stratum containing p. Let $N' \subset M$ be a smooth submanifold of M such that

$$N' \pitchfork S', \qquad \forall S' \in \mathcal{S}$$

$$N' \cap S = \{p\}, \qquad \dim(N') + \dim(S) = \dim(M).$$

By the Whitney conditions there exists $\eta > 0$ be such that

$$\partial \bar{\mathbb{B}}^M_\eta(p) \pitchfork S', \qquad \partial \bar{B}^M_\eta(p) \pitchfork S' \cap N', \qquad \forall S' \in \mathcal{S}.$$

DEFINITION 4.4 ([**25**, Definition 1.4]). *The normal slice through the stratum S at p is*

$$N(p) = X \cap N' \cap \bar{\mathbb{B}}^M_\eta(p).$$

The link though the stratum S at the point p is

$$L(p) = X \cap N' \cap \partial \bar{\mathbb{B}}^M_\eta(p).$$

REMARK 4.5. *If the stratum S containing p is of maximal dimension* $\dim(S) = \dim(M)$ *then $N(p) = p$. If* $\dim(S) = 0$ *then $N(p)$ is a regular neighbourhood*[1] *of p in X.*

We fix from now on such quantities (η, γ).

DEFINITION 4.6 ([**25**, Definition 3.5.2]). *The Local Morse datum of f at the point p is the pair*

$$(\mathscr{A}, \mathscr{B}) := (\bar{\mathbb{B}}^M_\eta(p) \cap X \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap X \cap f^{-1}(s-\gamma)).$$

Given $a, b \in \mathbb{R}$ with $a \leq b$ if one writes

$$X_{\leq a} = f^{-1}((-\infty, a]) \cap X$$

then the following theorem says that the local Morse datum determine the topology of the above space when a cross over a critical value. First recall

DEFINITION 4.7 ([**25**, Notational Definition 3.0]). *If $B \subset A$ and $Y \subset X$ are topological spaces such that there exists a continuous map*

$$h : B \to Y$$

[1] This means that $N(p)$ is a neighborhood of p which is a locally flat compact codimension zero topological submanifold of X. Moreover the inclusion $\{p\} \subset N$ is a simple homotopy equivalence and $\{p\}$ is a strong deformation retract of $N(p)$.

such that the identity map $Y \to Y$ has an extension to a homeomorphism

$$\phi : X \to Y \cup_h A$$

then one says that X is obtained from Y by attaching A along B via the attaching map h.

THEOREM 4.8 ([**25**, Theorem 3.5.3]). *If p is a nondepraved critical point of the proper function $f : X \to \mathbb{R}$ then there exists a topological embedding $h : \mathscr{B} \to X_{\leq s-\gamma}$ such that the identity map $X_{\leq s-\gamma} \to X_{\leq s-\gamma}$ extends to a homeomorphism*

$$X_{\leq s+\gamma} \to X_{\leq s-\gamma} \cup_h \mathscr{A}$$

where $(\mathscr{A}, \mathscr{B})$ is the local Morse datum at p.

The conclusion of the theorem can be written more succinctly by the above Definition 4.7 as

$$X_{\leq s+\gamma} \cong X_{\leq s-\gamma} \cup_{\mathscr{B}} \mathscr{A} \qquad (*)$$

that is, that $X_{\leq s+\gamma}$ is obtained from $X_{\leq s-\gamma}$ by attaching $\mathscr{A}$ along $\mathscr{B}$ via the embedding h[2].

REMARK 4.9. *In the case where $X \subset \mathbb{R}^N$ is a smooth submanifold without boundary of dimension $\dim X = n$, Classical Morse theory implies that local Morse data are given by the Morse index and the dimension: there is a homeomorphism of pairs*

$$(\mathscr{A}, \mathscr{B}) = (\mathbb{D}^{\lambda(p)} \times \mathbb{D}^{n-\lambda(p)}, \partial\mathbb{D}^{\lambda(p)} \times \mathbb{D}^{n-\lambda(p)}).$$

In the more general situation considerered here with $X \subset M$ a Whitney stratified subset a smooth manifold, in order to determine local Morse data one needs not only to know the Morse index and the dimension but also Normal Morse data, which depend on the nature of the singularities of X.

The fundamental theorem of Stratified Morse theory in the proper, nonrelative case says that local Morse datum at p is the product of the tangential and the normal Morse data at p. We now define these terms

DEFINITION 4.10 ([**25**, Definition 3.6.1]). *The tangential Morse datum of f at the nondepraved critical point p is the pair*

$$(\mathscr{T}, \mathscr{T}') = (\bar{\mathbb{B}}^M_\eta(p) \cap S \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap S \cap f^{-1}(s-\gamma))$$

[2]any pair $(\mathscr{A}, \mathscr{B})$ such that $(*)$ holds for some embedding $h : \mathscr{B} \to X_{\leq s-\gamma}$ is called *Morse datum* of f at the point p

where $S \in \mathcal{S}$ is the stratum containing p. The normal Morse datum of f at p is the pair

$$(\mathscr{N}, \mathscr{N}') = (\bar{\mathbb{B}}^M_\eta(p) \cap N(p) \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap N(p) \cap f^{-1}(s-\gamma)).$$

In other words the tangential Morse datum of f at p is the local Morse datum of the restriction $f_{|S} : S \to \mathbb{R}$ to the stratum S containing p, whereas the normal Morse datum is the local Morse datum of the restriction $f_{|N(p)} : N(p) \to \mathbb{R}$ to the normal slice of the stratum S through the point p.

PROPOSITION 4.11 ([**25**, Proposition 3.5.3]). *The tangential and normal Morse data of f at the nondepraved point p are independent of the choice of Riemannian metric on M and on the choice of η, γ as long as these are chosen in accordance with the above procedure.*

The fundamental Theorem of Morse theory in this situation reads

THEOREM 4.12 ("Fundamental Theorem of Stratified Morse Theory"[**25**, Theorem 3.7]). *If $X \subset M$ is a Whitney stratified subset of a smooth manifold M and $f : X \to \mathbb{R}$ is a proper smooth function with a nondepraved critical point $p \in X$ then there is a $\mathscr{S}$-decomposition ([**25**, Definition 1.1]) preserving homeomorphism of pairs*

$$(\mathscr{A}, \mathscr{B}) \cong (\mathscr{T} \times \mathscr{N}, \mathscr{T} \times \mathscr{N}' \cup \mathscr{N} \times \mathscr{T}')$$

where $(\mathscr{A}, \mathscr{B})$ denotes the local Morse datum of f at p, $(\mathscr{T}, \mathscr{T}'))$ denotes the tangential Morse datum of f at p and $(\mathscr{N}, \mathscr{N}')$ denotes the normal Morse datum of f at p.

The following Proposition describes the tangential Morse data:

PROPOSITION 4.13 ([**25**, Proposition 4.5]). *Let $f : X \to \mathbb{R}$ be a proper smooth function with a nondepraved critical point $p \in X$ and let $S \in \mathcal{S}$ be the stratum containing p. There exists a homeomorphism of pairs*

$$(\mathscr{T}, \mathscr{T}') \cong (\mathbb{D}^{\dim(S)-\lambda_S(p)} \times \mathbb{D}^{\lambda_S}, \mathbb{D}^{\dim(S)-\lambda_S(p)} \times \partial\mathbb{D}^{\lambda_S(p)})$$

where $\lambda_S(p)$ is the Morse index of the restriction $f_{|S} : S \to \mathbb{R}$.

In order to find the homotopy type of the normal Morse datum at a point it suffices to know the lower halflink of the stratum containing the given point.

DEFINITION 4.14 ([**25**, Definition 3.9.1]). *The upper and lower half-links of X at the point $p \in X$ with respect to f are*

$$l^+(p) := N(p) \cap \bar{\mathbb{B}}_\eta^M \cap f^{-1}(s+\gamma),$$

$$l^-(p) := N(p) \cap \bar{\mathbb{B}}_\eta^M \cap f^{-1}(s-\gamma).$$

DEFINITION 4.15 ([**25**, § 1.4]). *The cone over a nonempty semialgebraic set L is the topological space*

$$cone(L) = L \times [0,1]/(x,0) \sim (y,0), \qquad \forall x, y \in L.$$

To make this definition complete we define the cone over the empty set to be a point. Indeed this situation might very well occur when one considers the cone over an upper or lower half link if for instance $N(p) = p$ and $p \in f^{-1}(s)$.

COROLLARY 4.16 ([**25**, Corollary 3.11.2]). *If $f : X \to \mathbb{R}$ is a smooth proper function with a nondepraved critical point in $p \in X$ then the normal Morse datum at p has the homotopy type of the pair*

$$(cone(l^-(p)), l^-(p))$$

where $l^-(p)$ is the lower half-link.

This gives

THEOREM 4.17 ([**25**, Theorem 3.12]). *If $f : X \to \mathbb{R}$ is a stratified proper Morse function on a Whitney stratified subset $X \subset M$ of a smooth manifold with a critical point $p \in X$ with critical value $s = f(p)$ and if $[a,b] \subset \mathbb{R}$ is an interval containing no other critical values except $s \in (a,b)$ then there exists a homotopy equivalence*

$$X_{\leq s+\gamma} \sim X_{\leq s-\gamma} \cup_{(\partial(\mathbb{D}^{\lambda_S(p)}, cone(l^-(p)))} (\mathbb{D}^{\lambda_S(p)}, cone(l^-(p))).$$

REMARK 4.18. *In the book of Goresky and MacPherson this theorem is given with $f : X \to \mathbb{R}$ a stratified proper Morse function, but as stated in the introduction of their book (see [**25**] page x.) it also holds in the case where $f : X \to \mathbb{R}$ is a proper smooth function with a nondepraved critical point in $p \in X$.*

2.2. Nonproper Stratified Morse Theory. There is a variant of the above results in the case of nonproper functions. Let M be a smooth manifold and let $X \subset M$ be a closed subset endowed with a Whitney stratification $\mathcal{S} = \{S_i | i \in \mathscr{I}\}$

with strata indexed by a partially ordered set $\mathscr{I}$. Let $\tilde{X} \subset X$ be a subset which is a union of strata: there is a partially ordered subset $\mathscr{I}' \subset \mathscr{I}$ such that

$$\tilde{X} = \bigcup_{i \in \mathscr{I}'} S_i.$$

Let $f : X \to \mathbb{R}$ be the restriction of a smooth function on M such that f is proper with a nondepraved critical point $p \in X$ and suppose $f(p) = 0$. Let us fix a Riemannian metric and let data (η, γ) be chosen for $f_{|X}$ in accordance with the procedure given before the definition 4.6 above.

Definition 4.19 ([**25**, Definition 10.1, 10.3]). *Let $p \in X$ and let $S \in \mathcal{S}$ be the stratum containing p. The normal slice through S in $\tilde{X}$ at the point p is*

$$N_{\tilde{X}}(p) = N(p) \cap \tilde{X}$$

where $N(p)$ is the normal slice at p with respect to $f : X \to \mathbb{R}$. The link of S in $\tilde{X}$ at p is

$$L_{\tilde{X}}(p) = L(p) \cap \tilde{X}$$

where $L(p)$ is the link at p with respect to $f : X \to \mathbb{R}$.

One defines the stratified sets

$$\tilde{X}_{\leq a} = \tilde{X} \cap f^{-1}(-\infty, a], \qquad \tilde{X}_{[a,b]} = \tilde{X} \cap f^{-1}[a, b].$$

Definition 4.20 ([**25**, Definition 10.3]). *The Local Morse datum of $f_{|\tilde{X}}$ at the point p is the pair*

$$(\mathscr{A}_{\tilde{X}}, \mathscr{B}_{\tilde{X}}) := (\bar{\mathbb{B}}^M_\eta(p) \cap \tilde{X} \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap \tilde{X} \cap f^{-1}(s-\gamma)).$$

The Tangential Morse datum of $f_{|\tilde{X}}$ at p is the pair

$$(\mathscr{T}_{\tilde{X}}, \mathscr{T}'_{\tilde{X}}) := (\bar{\mathbb{B}}^M_\eta(p) \cap S \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap S \cap f^{-1}(s-\gamma)).$$

The Normal Morse datum of $f_{|\tilde{X}}$ at p is the pair

$$(\mathscr{N}_{\tilde{X}}, \mathscr{N}'_{\tilde{X}}) = (\bar{\mathbb{B}}^M_\eta(p) \cap N(p) \cap \tilde{X} \cap f^{-1}([s-\gamma, s+\gamma]), \bar{\mathbb{B}}^M_\eta(p) \cap N(p) \cap \tilde{X} \cap f^{-1}(s-\gamma)).$$

In other words the local respectively normal Morse data for $f_{|\tilde{X}} : \tilde{X} \to \mathbb{R}$ are the local respectively normal Morse data for $f : X \to \mathbb{R}$ intersected with $\tilde{X}$, whereas tangential Morse data for $f_{|\tilde{X}}$ is the tangential Morse data for f.

THEOREM 4.21 ([**25**, Theorem 10.4]). *For $\gamma > 0$ sufficiently small there is a $\mathscr{S}'$-decomposition preserving homeomorphism*

$$\tilde{X}_{\leq\gamma} \cong \tilde{X}_{\leq -\gamma} \cup_{\mathscr{B}_{\tilde{X}}} \mathscr{A}_{\tilde{X}}.$$

One can then prove the following Fundamental Theorem of Stratified Morse Theory in the nonproper case.

THEOREM 4.22 ([**25**, Theorem 10.5]). *There is a $\mathscr{S}$-decomposition preserving homeomorphism*

$$\begin{aligned}(\mathscr{A}_{\tilde{X}}, \mathscr{B}_{\tilde{X}}) &\cong (\mathscr{T}_{\tilde{X}}, \mathscr{T}'_{\tilde{X}}) \times (\mathscr{N}_{\tilde{X}}, \mathscr{N}'_{\tilde{X}}) \\ &:= (\mathscr{T}_{\tilde{X}} \times \mathscr{N}_{\tilde{X}}, \mathscr{T}_{\tilde{X}} \times \mathscr{N}'_{\tilde{X}} \cup \mathscr{T}'_{\tilde{X}} \times \mathscr{N}_{\tilde{X}}).\end{aligned}$$

Define

DEFINITION 4.23. *The upper and lower half-links of X at the point p with respect to $f_{|\tilde{X}}$ are*

$$(l^+_{\tilde{X}}(p), \partial l^+_{\tilde{X}}(p)) = (l^+(p) \cap \tilde{X}, \partial l^+(p) \cap \tilde{X}),$$

$$(l^-_{\tilde{X}}(p), \partial l^-_{\tilde{X}}(p)) = (l^-(p) \cap \tilde{X}, \partial l^-(p) \cap \tilde{X})$$

where $l^+(p)$ and $l^-(p)$ are the upper and lower half-links of X at p with respect to f.

As in the proper case, albeit that the homeomorphism type of the tangential Morse data readily can be found, this is not the case for the normal Morse data. However its homotopy type is known; namely one can prove (see [**25**, Proposition 10.7.1]) the following. If the nondepraved critical point $p \in X$ lies in a stratum of $\tilde{X}$ then the normal Morse datum at p has the homotopy type

$$(\mathscr{N}_{\tilde{X}}, \mathscr{N}_{\tilde{X}}) \sim (\mathrm{cone}(l^-_{\tilde{X}}(p)), l^-(p)).$$

If p lies in a stratum of $X \setminus \tilde{X}$ (that is, it is a critical point at infinity) then one can prove (see [**25**, Proposition 10.7.2]) that the normal Morse datum at p has the homotopy type

$$(\mathscr{N}_{\tilde{X}}, \mathscr{N}_{\tilde{X}}) \sim (l^+_{\tilde{X}}(p), \partial l^+_{\tilde{X}}(p)).$$

This implies the following theorem

THEOREM 4.24 ([**25**, Theorem 10.8]). *Suppose that $[a,b] \subset \mathbb{R}$ contains no critical value of $f : X \to \mathbb{R}$ except for an isolated critical value $s \in (a,b)$ corresponding* *to a nondepraved critical point p belonging to a stratum S. If $p \in \tilde{X}$ then there is a homotopy equivalence*

$$\tilde{X}_{\leq b} \cong \tilde{X}_{\leq a} \cup_{\partial(\mathbb{D}^{\lambda_S(p)} \times cone(l^-_{\tilde{X}}(p)))} (\mathbb{D}^{\lambda_S(p)} \times cone(l^-_{\tilde{X}}(p))).$$

If $p \in X \setminus \tilde{X}$ then there is a homotopy equivalence

$$\tilde{X}_{\leq b} \cong \tilde{X}_{\leq a} \cup_{\partial(\mathbb{D}^{\lambda_S(p)} \times l^+_{\tilde{X}}(p))} (\mathbb{D}^{\lambda_S(p)} \times l^+_{\tilde{X}}(p))$$

where $\lambda_S(p)$ is the Morse index of $f_{|S}$ at p.

2.3. $\mathbb{R}$-Morsifications. A principal idea which presents itself in geometry is that of deforming a given object as to eliminate nongeneric properties of the object in question whilst conserving a given geometrical invariant. One example is the moving lemma in intersection theory whereby one can perturb two given cycles of algebraic varieties as to intersect transversally, whilst preserving their intersection number. Another example presents itself in the theory of singularities, where one can deform a germ of holomorphic functions, defining an isolated hypersurface singularity, as to split the singularity into isolated quadratic singularities[3].

Definition 4.25 ([**18**, Definition 3.8]). *A morsification of a germ*

$$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$$

of holomorphic functions is a representative $F : M \times U \to \mathbb{C}$ of a germ of holomorphic functions

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0), \qquad (z,t) \mapsto f_t(z)$$

such that

1. *$F(\cdot, 0) = f(\cdot)$ as germs of holomorphic functions.*
2. *There exists a subset $V \subset U \setminus \{0\}$ of complement of Lebesque measure zero such that for all $t \in V$ the function $f_t : M \to \mathbb{C}$ is a Morse function.*

Moreover, $F : M \times U \to \mathbb{C}$ a called a strict morsification if in addition, for all $t \in V$ if $p \neq q$ are critical points of $f_t : M \to \mathbb{C}$ then $f_t(p) \neq f_t(q)$.

Suppose now that the germ of holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ has an isolated singular point in the origin, that is, if $f : M \to \mathbb{C}$ is any representative

[3] This preserves the *total* Milnor number

then $0 \in f^{-1}(0)$ is an isolated singular point. One can show (see e.g [**18**, Proposition 3.18]) that in this case a strict morsification exists. Recalling that the Milnor fibre at the origin has the homotopy type of a bouquet of real n-dimensional spheres [**46**, Theorem 7.2] and that the Milnor number $\mu(f, 0)$ is defined as the number of such spheres, one has moreover

PROPOSITION 4.26. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic functions with an isolated singular point in the origin. Let $F : M \times U \to \mathbb{C}$ be a strict morsification of f. The Milnor number $\mu(f, 0)$ is equal to the number of critical points of $f_t : M \to \mathbb{C}$, for any $t \in V$.*

PROOF. This follows from [**18**, Theorem 5.2]. □

We give an example

EXAMPLE 4.27. *Consider the polynomial map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^3 - y^2$ and let*

$$F : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}, \qquad F(z, t) = z^3 - 3tz - y^2.$$

Then $F(z, 0) = f(z)$ for all $z \in \mathbb{C}$ and

$$Jac(f_t) = [3(z^2 - t), -2y].$$

Consequently, for any $t \in \mathbb{C}$ the critical points of $f_t : \mathbb{C} \to \mathbb{C}$ are

$$p_1(t) = (|t|^{1/2} e^{i \arg(t)/2}, 0), \qquad p_2(t) = (|t|^{1/2} e^{(\arg(t)/2 + \pi)i}, 0).$$

The Hessian matrix is

$$Hess(f_t) = \begin{pmatrix} 6z & 0 \\ 0 & -2 \end{pmatrix}$$

. Consequently

$$\det Hess(f_t)(p_i(t)) = -12 p_i(t) \neq 0, \qquad \forall t \in \mathbb{C}^*$$

so that the critical points $p_i(t)$ are nondegenerate whenever $t \neq 0$. Their values are distinct hence F is a strict morsification (with $V = \mathbb{C}^*$) and therefore by Proposition 4.26 above the Milnor number is

$$\mu(f, 0) = 2.$$

In particular the complex (closed) Milnor fibre

$$\bar{\mathcal{F}}_{\mathbb{C},\eta} = \{z^3 - y^2 = \eta, \quad |z|^2 + |y|^2 \leq \delta\}, \qquad |\eta| << \delta$$

is homotopy equivalent to a bouquet of spheres $\mathbb{S}^1 \vee \mathbb{S}^1$. One can remark here that if we consider the restriction $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of F to the real numbers then $f_t : \mathbb{R} \to \mathbb{R}$ has two nondegenerate critical point when $t > 0$ and it has no critical points when $t < 0$.

This serves as the inspiration for the following definition.

DEFINITION 4.28. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of polynomial maps. A $\mathbb{R}$-morsification of f is a representative*

$$F : M \times U \to \mathbb{R}$$

of a polynomial map germ $F : (\mathbb{R}^{n+1} \times \bar{\mathcal{I}}, 0) \to (\mathbb{R}, 0)$ such that

(1) *$\bar{\mathcal{I}} \cong [-1, 1]^k$ for some $k \in \mathbb{N}$,*
(2) *$F(x, 0) = f(x)$ as germs of polynomial maps,*
(3) *There exists a dense subset $V \subset U \setminus \{0\}$ such that for all $t \in V$, the function $f_t : M \to \mathbb{R}$ is a Morse function with distinct critical values.*

An $\mathbb{R}$-morsification F is strong *if for any $t \in V$ the function $f_t : M \to \mathbb{R}$ has $\mu(f_{\mathbb{C}}, 0)$ critical points and it is* weak *if for any $t \in V$ it has strictly less than $\mu(f_{\mathbb{C}}, 0)$ critical points.*

In the case where the polynomial map germ has an isolated critical point in the origin[4] an $\mathbb{R}$-morsification of f always exist:

PROPOSITION 4.29. *If $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ has an isolated critical point in the origin then f has an $\mathbb{R}$-morsification.*

PROOF. The proof of [**18**, Proposition 3.18] applies almost verbatim, but since it is constructive we recall the argument. Namely let $f : M \to \mathbb{R}$ with $f(0) = 0$ be a representative of the germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$, for some neighborhood $M \subset \mathbb{R}^{n+1}$ of the origin. Consider the real algebraic subset $\mathcal{D} \subset \mathbb{R}^{n+1}$ of critical values of the gradient

$$\operatorname{grad}(f) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

and let $T\mathcal{D} \subset \mathbb{R}^{n+1}$ denote the tangent cone of $\mathcal{D}$ at the origin. By the Brown-Sard Theorem [**31**, Theorem 3.1], $\dim \mathcal{D} \leq n$. But then $\dim T\mathcal{D} \leq n$ as well. In particular we can choose a union

$$l = \{(ta_1, \ldots, ta_{n+1}) \mid t \in (-a, 0) \cup (0, a)\}, \qquad a \in \mathbb{R}$$

[4]that is: for any representative $f : M \to \mathbb{R}$ the origin is an isolated critical point

of two line segments such that each line segment does not meet any critical value. Letting

$$f_t = f + t\sum_{i=1}^{n+1} a_i x_i$$

then the critical points of f_t are exactly the points $p \in M$ for which

$$\mathrm{grad}(f)(p) = (ta_1, \ldots, ta_{n+1}).$$

This is a noncritical value of the gradient $\mathrm{grad}(f)$ hence the Hessian matrix $\mathrm{Hess}(f_t)(p)$ at p is invertible. But this means that p is a nondegenerate critical point of f_t. Therefore if one defines the subset $V \subset \mathbb{R}$ by $V = (-a, 0) \cup (0, a)$ then for any $t \in V$ one has that f_t has only nondegenerate critical points in M. Up to changing the vector $(a_1, \ldots, a_{n+1})$ by a small perturbation one can assume that f_t has only nondegenerate critical points with distinct critical values for any $t \in V$. Then the representative

$$F : M \times (-a, a) \to \mathbb{R}, \qquad F(x, t) = f_t(x)$$

of the germ

$$F : (\mathbb{R}^{n+1} \times [-a, a], 0) \to (\mathbb{R}, 0)$$

satisfies the conditions of the Definition 4.28 of a $\mathbb{R}$-morsification, with $U = (-a, a)$ and $V = U \setminus \{0\}$. □

Remark 4.30. *As the proof shows one can always obtain a weak $\mathbb{R}$-morsification of a germ of isolated singularity $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ having the property that $V = U \setminus \{0\}$. Note however that the* number *of critical points of $f_t : M \to \mathbb{R}$ might be different for different choices of $t \in V$.*

We give a few examples.

Example 4.31. *In the example 4.27 the restriction of F to $F : \mathbb{R} \times \bar{\mathcal{I}} \to \mathbb{R}$ with $\bar{\mathcal{I}} = [0, 1]$ is a strong $\mathbb{R}$-morsification, whereas if $\bar{\mathcal{I}} = [-1, 0]$ it is a weak $\mathbb{R}$-morsification.*

In the example above one found a strong $\mathbb{R}$-morsification by a change of coordinates in the parameter space of a weak morsification. This is not always possible, as the following example illustrates

EXAMPLE 4.32. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^4$ and let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be $F(x, t) = x^4 - tx$. Then*

$$f_t'(x) = 4x^3 - t = 0 \qquad x^3 = t/4$$

and as there are only one real third root of unity there is only one critical point of $f_t : \mathbb{R} \to \mathbb{R}$ for any $t \in \mathbb{R}$. But the Milnor number of f in the origin is $\mu(f, 0) = 3$. Therefore there is no closed interval $\bar{\mathcal{I}}$ containing the origin such that $F_{|\mathbb{R}\times\bar{\mathcal{I}}}$ is a strong $\mathbb{R}$-morsification.

However, in this example one can find another one-parameter family which is a strong $\mathbb{R}$-morsification.

EXAMPLE 4.33. *Let $F : \mathbb{R} \times I \to \mathbb{R}$ with $I = [0, 32/27]$ be given by $F(x,t) = x^4 - 4tx^2 + 4t^2x$. Then for any $t \in [0, 32/27)$, the derivative $f_t' = 4(x^3 - 2tx + t^2)$ has discriminant*

$$\Delta(f_t') = t^3(32 - 27t)$$

which is positive when $0 < t < 32/27$ whence it follows that f_t have exactly $\mu(f, 0) = 3$ real roots. The set of $t \in (0, 32/27]$ such that the critical points of f_t are not Morse is the image $\pi(X)$ of

$$X = \{(x, t) \mid 3x^2 - 2t = 0, \quad x^3 - 2tx + t^2 = 0\} \subset \mathbb{R} \times (0, 32/27]$$

under the projection $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. As X is a proper intersection of two curves it follows that $\dim X = 0$ whence $\pi(X)$ has finite cardinality. Therefore $V = (0, 32/27] \setminus \pi(X)$ is dense in I and for any $t \in V$ the functions f_t are Morse. Hence F is a strong $\mathbb{R}$-morsification.

As for the existence of strong $\mathbb{R}$-morsifications let us recall the notion of M-morsification, due to Vladimir I. Arnold.

DEFINITION 4.34 ([**28**]). *A M-morsification of a germ of holomorphic functions $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity in the origin and of Milnor number $\mu = \mu(f, 0)$, is a representative*

$$F : M \times U \to \mathbb{C}$$

of a germ of holomorphic functions $F : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu-1}, 0) \to (\mathbb{C}, 0)$ such that

(1) *$F(\cdot, 0) = f(\cdot)$ as germs of holomorphic functions,*

(2) *$U \subset \mathbb{R}^{\mu-1}$ and there exists a subset $V \subset U \backslash \{0\}$ of complement of Lebesgue measure zero such that for all $t \in U$ the function $f_t = F(\cdot, t)$ is Morse with exactly μ real critical points.*

A M-morsification is said to be very nice *if moreover the critical values of f_t are distinct, for all $t \in U$.*

Stated differently, an M-morsification is defined from a representative of the truncated miniversal deformation[5] of f by restricting the base to the (open) set of real parameters for which all nondegenerate critical points of f_t are real.

Arnold established in his article [**5**] from 1991 the existence of very nice M-morsifications for simple singularities of class A_k; more precisely he calculated therein the number of connected components of the space of such morsifications. In 1993 M.R. Entov established the existence of very nice M-morsifications for D_k-singularities. And in his article [**28**] V.V. Guryunov proved their existence for simple singularities of class E_6, E_7 and E_8, showing that the number of connected components of the space of very nice M-morsifications are $82, 768$ and 4056, respectively.

PROPOSITION 4.35. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of polynomial maps defining an isolated singularity in the origin. Suppose that for any representative $f : M \to \mathbb{R}$, where M is an open neighbourhood of the origin, the extension of scalars $f_{\mathbb{C}} : M_{\mathbb{C}} \to \mathbb{C}$ is a simple isolated singularity of class ADE. Then there exists a strong $\mathbb{R}$-morsification of f.*

PROOF. (1) Suppose $n = 1$. As $f_{\mathbb{C}} : M_{\mathbb{C}} \to \mathbb{C}$ is a simple ADE-singularity, by the results [**5**], [**19**] and [**28**], there exists a very nice M-morsification

$$F_{\mathbb{C}} : M_{\mathbb{C}} \times U \to \mathbb{C}$$

of $f_{\mathbb{C}}$ where $U \subset \mathbb{R}^{\mu-1}$ is an open neighbourhood of the origin. By definition there exists a subset $V \subset U \setminus \{0\}$ of complement of measure zero such that the functions

$$f_{\mathbb{C},t} : M_{\mathbb{C}} \to \mathbb{C}, \qquad t \in V$$

[5] often called the truncated universal unfolding

are Morse with μ real critical points $p_1, \dots, p_\mu$ and distinct critical values. There might be other critical points of $f_{\mathbb{C},t}$ which are not real, but as we show now this is not the case. Indeed the complement of a set $A \subset \mathbb{R}^k$ of measure zero is dense, so V is dense. And as $F_{\mathbb{C}}$ is a representative of a miniversal deformation it follows by [**18**, Lemma 3.9] that it is a strict morsification. By [**18**, Proposition 3.19] as $F_{\mathbb{C}}$ is a morsification of $f_{\mathbb{C}}$ it follows that $f_{\mathbb{C},t}$ has all in all μ nondegenerate critical points. Therefore all the critical points are real. Consequently if one considers the restriction

$$F : M \times U \to \mathbb{R}$$

to the open neighborhood M of the origin in $\mathbb{R}^2$ then $F(x, 0) = f(x)$ for all $x \in M$ and moreover, for any $t \in V$, the function $f_t = F(\cdot, t)$ is Morse with distinct critical values. Hence F is a strong $\mathbb{R}$-morsification.

(2) If $n \geq 1$ and if $f : M \to \mathbb{R}$ is a representative of the germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ then one can by assumption write

$$f(x_1, \dots, x_{n+1}) = \tilde{f}(x_1, x_2) + \sum_{i=3}^{n+1} \epsilon_i x_i^2, \qquad \epsilon_i \in \{\pm 1\}$$

where the extension of scalars of $\tilde{f} : M \cap \mathbb{R}^2 \to \mathbb{R}$ is a simple curve singularity of class ADE. Notice that

$$\mu = \mu(f_{\mathbb{C}}, 0) = \mu(\tilde{f}_{\mathbb{C}}, 0).$$

By the step above there exists a strong $\mathbb{R}$-morsification

$$\tilde{F} : M \cap \mathbb{R}^2 \times U \to \mathbb{R}$$

of $\tilde{f}$, where $U \subset [-\delta, \delta]^{\mu - 1}$ for some $\delta > 0$. By definition there exists therefore a dense subset $V \subset U \setminus \{0\}$ such that $\tilde{f}_t = \tilde{F}(\cdot, t)$ has $\mu(\tilde{f}_{\mathbb{C}}, 0)$ critical points $\tilde{p}_1, \dots, \tilde{p}_\mu$ with distinct critical values, for any $t \in V$. Let

$$F : M \times U \to \mathbb{R},$$

$$F(x_1, \dots, x_{n+1}) = \tilde{F}(x_1, x_2) + \sum_{i=3}^{n+1} \epsilon_i x_i^2.$$

Then for each $t \in V$ the critical points of $f_t = F(\cdot, t)$ are

$$p_i = \tilde{p}_i \times (0, \dots, 0) \subset M,$$

where $i = 1, \dots, \mu$. Therefore F is a strong $\mathbb{R}$-morsification.

$\square$

3. Preparatory Lemmas

Consider a germ of polynomial maps $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ with an isolated singular point in the origin. Let $f : M \to \mathbb{R}$ be a representative of this germ in a neighborhood $M \subset \mathbb{R}^{n+1}$ of the origin.

THEOREM 4.36 (see e.g [**46**, Theorem 4.2]). *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,*

$$f : f^{-1}((0, \epsilon]) \cap \bar{\mathbb{B}}_\delta \to (0, \epsilon],$$

$$f : f^{-1}((0, -\epsilon]) \cap \bar{\mathbb{B}}_\delta \to (0, -\epsilon].$$

are the projections of trivial topological fibrations, where $\bar{\mathbb{B}}_\delta \subset \mathbb{R}^{n+1}$ denotes the closed ball of radius δ centered at the origin. Moreover each fibre of these fibrations are smooth manifolds with boundary.

Let us fix Milnor data (ϵ_0, δ_0) as in the Theorem 4.36 and let $\epsilon \in (0, \epsilon_0)$. For any $\eta \in (0, \epsilon]$ let us write

$$\bar{\mathcal{F}}_\eta^+ = f^{-1}(\eta) \cap \bar{\mathbb{B}}_\delta, \qquad \bar{\mathcal{F}}_\eta^- = f^{-1}(-\eta) \cap \bar{\mathbb{B}}_\delta$$

for the positive and negative Milnor fibres. It follows from the triviality of the fibrations above that if $\tilde{\eta}$ is another real number such that $\tilde{\eta} \in (0, \epsilon]$ then there are homeomorphisms

$$\bar{\mathcal{F}}_\eta^+ \cong \bar{\mathcal{F}}_{\tilde{\eta}}^+, \qquad \bar{\mathcal{F}}_\eta^- \cong \bar{\mathcal{F}}_{\tilde{\eta}}^-.$$

Recall the first of the Isotopy lemmas of Réné Thom:

LEMMA 4.37 ("Thom's First Isotopy Lemma" [**35**, Proposition 11.1]). *Let $g : \mathcal{M} \to \mathcal{N}$ be a smooth function of smooth manifolds and let $X \subset \mathcal{M}$ be a closed Whitney stratified subset. If $g_{|X} : X \to \mathcal{N}$ is proper and if for any stratum $S \subset X$ the restriction $g_{|S} : S \to \mathcal{N}$ is a submersion then $f_{|X} : X \to f(X)$ is a locally trivial topological fibration.*

We now can prove

LEMMA 4.38. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of polynomial maps defining an isolated singular point in the origin. Let $f : M \to \mathbb{R}$ be a representative on a* *neighborhood of the origin and let* (ϵ_0, δ_0) *be Milnor data. Suppose that*

$$F : M \times [-1,1]^k \to \mathbb{R}, \qquad F(\cdot, t) = f_t$$

is a polynomial map such that $F(x,0) = f(x)$ *for all* $x \in M$. *Then for any* $\delta \in (0, \delta_0)$ *and any* $\eta \in (0, \epsilon(\delta))$ *there exists* $t_0' = t_0'(\eta) \in \mathbb{R}$ *such that for any* $t \in [-1,1]^k$ *such that* $|t| \le t_0'$ *there exists a homeomorphism*

$$\phi : \bar{\mathcal{F}}_\eta^\pm \to f_t^{-1}(\pm\eta) \cap \bar{\mathbb{B}}_\delta.$$

Proof. We will prove the assertion for the positive Milnor fibre, the other case is analogous. By the Curve Selection Lemma [**39**, Lemma 3.1] one can choose $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ the sphere $\mathbb{S}_\delta \subset \mathbb{R}^{n+1}$ intersects $f^{-1}(0)$ transversally. Moreover one can choose $\epsilon_0(\delta) > 0$ such that any $\eta \in (0, \epsilon_0)$ is a regular value of $f : \mathbb{B}_\delta \to \mathbb{R}$. Let us fix $\delta \in (0, \delta_0)$. Then by continuity, $f^{-1}(\eta)$ and $\mathbb{S}_\delta$ intersect transversally for all $\eta \in (0, \epsilon_0)$. Let

$$\tilde{F} : M \times [-1,1]^k \to \mathbb{R} \times [-1,1]^k, \qquad \tilde{F}(x,t) = (f_t(x), t).$$

and denote by

$$\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k, \qquad \pi(x,t) = t$$

the standard projection on the parameter t. We will show that there exists $t_0'(\eta)$ such that the restricted map

$$\pi : \tilde{F}^{-1}(\{\eta\} \times (-t_0', t_0')^k) \cap (\bar{\mathbb{B}}_\delta \times \mathbb{R}^k) \to \mathbb{R}^k$$

is a proper stratified submersion and then use the Isotopy Lemma to obtain the result. To achieve this we will first find a Whitney stratification of the domain.

(1) First note that since any $\eta \in (0, \epsilon_0)$ is a regular value of $f : \mathbb{B}_\delta \to \mathbb{R}$ the differential $d\tilde{F}_{|M \times (-1,1)^k}(x, 0)$ has maximal rank $k+1$ for any $x \in f^{-1}(\eta) \cap \mathbb{B}_\delta$. Therefore, we can assume by continuity that if η is fixed then

$$\text{rank } (d\tilde{F}_{|M \times (-1,1)^k})(x,t) = k+1$$

for any $x \in f_t^{-1}(\eta) \cap \mathbb{B}_\delta$ and any $t \in (-t_0, t_0)^k$. This means that

$$\tilde{F} : \mathbb{B}_\delta \times (-t_0, t_0)^k \to \mathbb{R} \times \mathbb{R}^k$$

has no critical values in $\{\eta\} \times (-t_0, t_0)^k$.

(2) In particular,

$$\mathscr{T}_1 := \tilde{F}^{-1}(\{\eta\} \times (-t_0, t_0)^k) \cap \left(\mathbb{B}_\delta \times (-t_0, t_0)^k\right)$$

is a smooth manifold.

(3) We claim that there exists $\tilde{t}_0 = \tilde{t}_0(\eta) \in (0, 1]$ such that

$$\mathscr{T}_2 := \tilde{F}^{-1}(\{\eta\} \times (-\tilde{t}_0, \tilde{t}_0)^k) \cap \left(\mathbb{S}_\delta \times (-\tilde{t}_0, \tilde{t}_0)^k\right)$$

is a smooth manifold. For this it suffices to show the following claim: the intersection

$$\tilde{F}^{-1}(\{\eta\} \times (-\tilde{t}_0, \tilde{t}_0)^k) \cap \mathbb{S}_\delta \times (-\tilde{t}_0, \tilde{t}_0)^k$$

is transverse. This will be shown using the fact that transversality is an open condition. We have that the set of (x, t) with $x \in f_t^{-1}(\eta)$ such that $f_t^{-1}(\eta) \cap \mathbb{S}_\delta$ is not transverse at x is precisely the set of (x, t) such that

$$T_{(x,t)} f_t^{-1}(\eta) \perp v(x)$$

where $v(x) = (x, 0)$. Since this is a closed condition, the set where $T_{(x,t)} f_t^{-1}(\eta)$ is not orthogonal to $v(x)$ is an open condition. Since the intersection

$$f^{-1}(\eta) \cap \mathbb{S}_\delta$$

is transverse, $T_{(x,0)} f_t^{-1}(\eta)$ is not orthogonal to $v(x)$. Therefore, by openness, there exists a neighborhood of $t = 0$ in $\mathbb{R}^k$ such that for any t in that neighborhood and any $x \in f_t^{-1}(\eta)$, the intersection $f_t^{-1}(\eta) \cap \mathbb{S}_\delta$ is transverse. Thus there exists a $\tilde{t}_0 \in \mathbb{R}$ as claimed.

(4) Let $t_0' = \min(t_0, \tilde{t}_0)$ and consider

$$\pi_{|\mathscr{T}_i} : \mathscr{T}_i \to (-t_0', t_0')^k, \qquad i = 1, 2$$

We claim that up to replacing t_0' by a smaller value one has that $\pi_{|\mathscr{T}_i}$ is a submersion, for $i = 1, 2$. Suppose otherwise and that for any t_0' there exists a point $(x, t) \in \mathscr{T}_1$ in which $\pi_{|\mathscr{T}_1}$ has a critical point. By definition this means that

$$T_{(x,t)} \mathscr{T}_1 \perp T_t \mathbb{R}^k \qquad (*)$$

and that $d\tilde{F}(x, t)$ is parallel to $T_t \mathbb{R}^k$. Since $(*)$ is a closed condition, if $(x_t, t) \in \mathscr{T}_1$ is a sequence such that $|t| \to 0$ then there would exist a limit point $(\tilde{x}, 0)$ with $\tilde{x} \in f^{-1}(-\eta) \cap \bar{\mathbb{B}}_\delta$ such that $d\tilde{F}(\tilde{x}, 0)$ is parallel to $T_0\mathbb{R}^k$. But then $df(\tilde{x})$ would be zero which is impossible since $\tilde{x} \in f^{-1}(\eta)$. Therefore $\pi_{|\mathscr{T}_1}$ is a submersion. In the same vein suppose that $\pi_{|\mathscr{T}_2}$ has a critical point (x, t) which is to say that

$$T_{(x,t)}\mathscr{T}_2 \perp T_t\mathbb{R}^k.$$

Letting $(x_t, t) \in \mathscr{T}_2$ be a sequence such that $|t| \to 0$ there would exist a limit point $(\tilde{x}, 0)$ with $\tilde{x} \in f^{-1}(\eta) \cap \mathbb{S}_\delta$ such that $d\tilde{F}(\tilde{x}, 0)$ is parallel to $T_0\mathbb{R}^k$ hence $df(\tilde{x}) = 0$ which is impossible since $f^{-1}(\eta) \cap \mathbb{S}_\delta$ is transverse at $\tilde{x}$.

(5) Therefore if

$$\mathscr{W} := \mathscr{T}_1 \cup \mathscr{T}_2$$

then by the the steps 2 and 3 above $\mathscr{W}$ is a smooth manifold with boundary hence $\mathcal{T} = \{\mathscr{T}_1, \mathscr{T}_2\}$ is a Whitney stratification. Moreover, by the step 4

$$\pi_{|\mathscr{W}} : \mathscr{W} \to (-t_0', t_0')^k$$

is a proper stratified submersion.

(6) By Thom's first Isotopy Lemma 4.37 (see [**35**, Proposition 11.1]) $\pi_{|\mathscr{W}}$ is the projection of a locally trivial fibration. As the base is contractible it is a trivial fibration. The fibers of a trivial fibration are homeomorphic whence it follows that for any $t \in (-t_0', t_0')^k$ there exists a homeomorphism

$$\phi_t : \pi_{|\mathscr{W}}^{-1}(0) \to \pi_{|\mathscr{W}}^{-1}(t),$$

or in other words,

$$\bar{\mathcal{F}}_\eta^+ \cong f_t^{-1}(\eta) \cap \bar{\mathbb{B}}_\delta$$

which is what was to be proven.

□

REMARK 4.39. *Restricted to each strata of the fibers the homeomorphism ϕ_t is in fact a diffeomorphism.*

REMARK 4.40. *The set*

$$f_t^{-1}(\eta) \cap \bar{\mathbb{B}}_\delta$$

is not *a Milnor fibre of $f_t : \mathbb{R}^{n+1} \to \mathbb{R}$ since the function f_t has in general critical points in $\bar{\mathbb{B}}_\delta \setminus \{0\}$.*

Applying this lemma in the case where $F : M \times U \to \mathbb{R}$ is a $\mathbb{R}$-morsification we get the following corollary:

Corollary 4.41. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of polynomial maps defining an isolated singular point in the origin and let $F : M \times U \to \mathbb{R}$ be a $\mathbb{R}$-morsification. Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 > 0$ such that for any $\eta \in (0, \epsilon_0)$ there exists $t_0' = t_0'(\eta) \in \mathbb{R}$ such that for any $t \in U \cap \{|t| \leq t_0'\}$ there exists a homeomorphism*

$$\phi : \bar{\mathcal{F}}_\eta^\pm \to f_t^{-1}(\pm\eta) \cap \bar{\mathbb{B}}_\delta.$$

Let us denote

$$\mathcal{X}_{t,\eta} = f_t^{-1}(-\eta, \eta) \cap \bar{\mathbb{B}}_\delta,$$
$$\bar{\mathcal{F}}_{t,\eta}^+ = f_t^{-1}(\eta) \cap \bar{\mathbb{B}}_\delta,$$
$$\bar{\mathcal{F}}_{t,\eta}^- = f_t^{-1}(-\eta) \cap \bar{\mathbb{B}}_\delta.$$

Remark 4.42. *We have that $\bar{\mathcal{F}}_{t,\eta}^+$ and $\bar{\mathcal{F}}_{t,\eta}^-$ belong to the boundary of $\bar{\mathcal{X}}_{t,\eta}$ and that the map*

$$f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$$

has the property that its restriction to $\bar{\mathcal{F}}_{t,\eta}^+$ and to $\bar{\mathcal{F}}_{t,\eta}^-$ respectively has no critical points, by the proof of Lemma 4.38.

Lemma 4.43. *For any $\eta \in (0, \epsilon_0(\delta))$ there exists a real number $t_0'' > 0$ such that for any $t \in U$ with $|t| \leq t_0''$, the subset $\mathcal{X}_{t,\eta} \subset M$ is a smooth manifold with boundary.*

Proof. Let $\eta \in (0, \epsilon_0(\delta))$ be fixed. By the Lemma 4.38 there exists $t_0' = t_0'(\eta)$ such that for any $t \in U$ such that $|t| \leq t_0'$ one has that η is not a critical value of $f_t : \mathbb{B}_\delta \to \mathbb{R}$. Now the open sets

$$f_t^{-1}((-\infty, \eta) \cap \mathbb{B}_\delta, \qquad f_t^{-1}(-\eta, \infty) \cap \mathbb{B}_\delta$$

are smooth manifolds for any $t \in U$ such that $|t| \leq t_0'$. Transversality is an open condition so there exists $t_0'' \in \mathbb{R}$ such that the intersection

$$f_t^{-1}(-\eta, \eta) \cap \mathbb{B}_\delta$$

is transverse, for $t \in U \cap \{|t| \leq t_0''\}$ hence is a smooth manifold. The same arguments applied to $f_t : \mathbb{S}_\delta \to \mathbb{R}$ shows that

$$f_t^{-1}(-\eta, \eta) \cap \mathbb{S}_\delta$$

is a smooth manifold. This proves the assertion. □

We now show that $\bar{\mathcal{X}}_{t,\eta}$ is contractible.

LEMMA 4.44. *Suppose that*

$$F : M \times U \to \mathbb{R}$$

is a $\mathbb{R}$-morsification of f where $U \subset [-1, 1]^k$ is contractible. Then for any $\eta \in (0, \epsilon_0(\delta))$ there exists $t_0 \in \mathbb{R}$ such that for any $t \in U$ with $|t| \leq t_0$,

$$\bar{\mathcal{X}}_{t,\eta} = f_t^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta$$

is contractible.

PROOF. We put $t_0 = \min(t_0'(\eta), t_0''(\eta))$ in the notation of Lemma 4.38 and 4.43. Let

$$\tilde{F} : M \times U \to \mathbb{R} \times U, \qquad \tilde{F}(x, t) = (f_t(x), t)$$

and let

$$\pi : \mathbb{R}^{n+1} \times \mathbb{R}^k \to \mathbb{R}^k, \qquad \pi(x, t) = t$$

be the standard projection. Consider

$$\mathscr{T}_1 = \tilde{F}^{-1}((-\eta, \eta) \times U \cap \{|t| < t_0\})) \cap \mathbb{B}_\delta \times U,$$

$$\mathscr{T}_2 = \tilde{F}^{-1}((-\eta, \eta) \times U \cap \{|t| < t_0\})) \cap \mathbb{S}_\delta \times U,$$

$$\mathscr{T}_3^\pm = \tilde{F}^{-1}(\{\pm\eta\} \times U \cap \{|t| < t_0\})) \cap \mathbb{B}_\delta \times U,$$

$$\mathscr{T}_4^\pm = \tilde{F}^{-1}(\{\pm\eta\} \times U \cap \{|t| < t_0\})) \cap \mathbb{S}_\delta \times U.$$

Then the $\mathscr{T}_1$ and $\mathscr{T}_2$ are smooth manifolds and by the proof of the Lemma 4.38 $\mathscr{T}_3^\pm$ and $\mathscr{T}_4^\pm$ are smooth manifolds. Therefore if

$$\mathscr{Y} = \tilde{F}^{-1}([-\eta, \eta] \times U \cap \{|t| < t_0\}) \cap (\bar{\mathbb{B}}_\delta \times U)$$

then

$$\mathcal{T} = \{\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3^\pm, \mathscr{T}_4^\pm\}$$

is a Whitney stratification of $\mathscr{Y}$ by transversality of their intersections. We will prove that the restriction of the projection

$$\pi : \mathscr{Y} \to U \cap \{|t| < t_0\}$$

is a proper stratified submersion. First of all it is a proper map since any fiber is compact, of the form

$$\pi_{|\mathscr{Y}}^{-1}(s) = f_t^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta.$$

By the proof of Lemma 4.38

$$\pi_{|\mathscr{T}_i^\pm} : \mathscr{T}_i^\pm \to U \cap \{|t| < t_0\}, \qquad i = 3, 4$$

are submersions. Now since

$$\dim \mathscr{T}_1 = \dim(M \times U) = n + k + 1$$

is of maximal dimension and since the stratum $\mathscr{T}_1$ is a smooth manifold and in particular open, $\pi_{|\mathscr{T}_1}$ is a submersion and since the projection $\pi_{|\mathbb{B}_\delta \times U}$ is a submersion. It thus remains to show that $\pi_{|\mathscr{T}_2}$ is a submersion. Again this follows since

$$\dim \mathscr{T}_2 = \dim(M \times U) - 1 = n + k,$$

since $\mathscr{T}_2$ is smooth manifold and in particular open and since the projection $\pi_{|\mathbb{S}_\delta \times U}$ is a submersion. Therefore one has that

$$\pi_{|\mathscr{Y}} : \mathscr{Y} \to U \cap \{|t| < t_0\}$$

is a stratified proper submersion. By the First Isotopy Lemma of Thom 4.37 (see [**35**, Proposition 11.1]) it is therefore the projection of a locally trivial fibration, and as the codomain is contractible by assumption, it is a trivial fibration. In particular its fibres are homeomorphic thus

$$\pi_{|\mathscr{Y}}^{-1}(t) = \bar{\mathcal{X}}_{t,\eta} \cong f^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta = \pi_{|\mathscr{Y}}^{-1}(0).$$

Therefore, to finish the proof it suffices to show that $f^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta$ is contractible. But it follows from ([**15**, Proposition 1.6]) that the inclusion $f^{-1}(0) \cap \bar{\mathbb{B}}_\delta \subset f^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta$ is a homotopy equivalence. By the Local Conic Structure of algebraic sets $f^{-1}(0) \cap \bar{\mathbb{B}}_\delta$ is a cone over its boundary hence contractible. Hence $f^{-1}([-\eta, \eta]) \cap \bar{\mathbb{B}}_\delta$ and therefore also $\bar{\mathcal{X}}_{t,\eta}$ is contractible. □

4. On the Homotopy Type of The Real Milnor Fibres

4.1. Introduction. The idea now is to Whitney stratify $\bar{\mathcal{X}}_{t,\eta}$ and to use the nonproper version of Stratified Morse Theory to obtain the homotopy type of the real Milnor fibres. More precisely we will in Theorem 4.47 show that a tubular neighborhood of a real Milnor fibre becomes homeomorphic to a contractible space after succesively adjoining handles attached via embeddings, with each handle corresponding to a critical point of an $\mathbb{R}$-morsification.

4.2. The Situation. Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of isolated singularity and let

$$F : M \times U \to \mathbb{R}, \qquad f_t = F(\cdot, t)$$

be a $\mathbb{R}$-morsification of this germ, where $M \subset \mathbb{R}^{n+1}$ is a neighborhood of the origin. Fix Milnor data (ϵ_0, δ_0) at the origin for $f : M \to \mathbb{R}$, as in Theorem 4.36.

We shall only consider those critical points which are bounded in the following sense.

NOTATION 4.45. *Let $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \delta)$ be fixed. For any $t \in V$ with $V \subset U \setminus \{0\}$ as in Definition 4.28 let $p_1, \ldots, p_{m(t)}$ denote the critical points of $f_t : M \to \mathbb{R}$ such that $p_i \in \bar{\mathbb{B}}_\delta$ for all $i = 1, \ldots, m(t)$ and let $s_i = f_t(p_i)$ denote their values.*

REMARK 4.46. *We claim that there exists $t_0 \in \mathbb{R}$ such that $s_i(t) \in (-\epsilon, \epsilon)$ for all $t \in V \cap \{|t| \le t_0\}$. Write $U' = U \cap \{|t| \le t_0\}$. We first claim that if $\mathscr{C}_t$ denotes the set of $t \in U'$ such that $p_i(t) \in \bar{\mathbb{B}}_\delta$ is a critical point of $f_t : M \to \mathbb{R}$ then $\mathscr{C}_t$ is closed in U. Consider*

$$\mathscr{C} = \{(x, t) \in \bar{\mathbb{B}}_\delta \times U' \mid \mathit{rank}\ \mathit{Jac}(f_t)(x) = 0\}.$$

Then $\mathscr{C}$ is closed in the compact set $\bar{\mathbb{B}}_\delta \times U' \subset M \times \mathbb{R}^k$ with respect to the Euclidean topology and so if

$$\pi : \mathbb{R}^{n+1} \times \mathbb{R}^k \to \mathbb{R}^k, \qquad \pi(x, t) = t$$

denotes the standard projection onto the second factor then the image $\mathscr{C}_t = \pi(\mathscr{C})$ is closed in U'. Therefore, by continuity of $t \mapsto df_t$, if

$$(p_{t_n})_{n \in \mathbb{N}}, \qquad |p_{t_n}| \le \delta$$

is a bounded sequence of critical points such that $t_n \in U'$ *and* $\lim_{n\to\infty}|t_n| = 0$ *then the limit* $p = p_0$ *exists, is bounded* $|p_0| \leq \delta$*, and is a critical point of* f*. As the origin is the unique critical point of* f *in the ball* $\bar{\mathbb{B}}_\delta \subset M$ *it follows that*

$$\forall p_i(t) \in \bar{\mathbb{B}}_\delta, \qquad \lim_{|t|\to 0} p_i(t) = 0.$$

Since $f_t : M \to \mathbb{R}$ *is a representative of a germ* $f_t : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ *one has* $f_t(0) = 0$ *hence the critical values satisfy* $s_i(t) \in (-\eta, \eta)$ *for all* $i = 1, \ldots, m(t)$ *and for all* $t \in V \cap \{|t| \leq t_0\}$ *for* t_0 *sufficiently small.*

4.3. The Theorem. The following theorem concerns the homotopy type of the real Milnor fibres of f at the origin.

THEOREM 4.47. *Suppose that* $F : M \times U \to \mathbb{R}$ *is a* $\mathbb{R}$*-morsification. For any* $\eta \in (0, \epsilon(\delta))$ *there exists* $t_0 = t_0(\eta) \in \mathbb{R}$ *such that the following holds. Let* $t \in V$ *(where* V *is as in Definition 4.28) such that* $|t| \leq t_0$ *be fixed and let* $p_1, \ldots, p_m \in \bar{\mathbb{B}}_\delta$ *denote the critical points of* $f_t : M \to \mathbb{R}$ *lying inside the ball of radius* δ *centered at the origin and let* $\lambda(p_i)$ *denote their indices. Then* $\mathcal{X}_{t,\eta}$ *is contractible and there exist embeddings*

$$h_i^+ : \begin{cases} \mathbb{D}^{\lambda(p_i)} \times \partial\mathbb{D}^{n+1-\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1\leq j\leq i-1 \\ h_j^+}} \left(\mathbb{D}^{\lambda(p_j)} \times \mathbb{D}^{n+1-\lambda(p_j)}\right), & i \geq 2 \\ \mathbb{D}^{\lambda(p_1)} \times \partial\mathbb{D}^{n+1-\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^+ \times (0,1), \quad i = 1. \end{cases}$$

$$h_i^- : \begin{cases} \mathbb{D}^{n+1-\lambda(p_i)} \times \partial\mathbb{D}^{\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^- \times (0,1) \cup \bigcup_{\substack{1\leq j\leq i-1 \\ h_j^-}} \left(\mathbb{D}^{n+1-\lambda(p_j)} \times \mathbb{D}^{\lambda(p_j)}\right), & i \geq 2 \\ \mathbb{D}^{n+1-\lambda(p_1)} \times \partial\mathbb{D}^{\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^- \times (0,1), \quad i = 1. \end{cases}$$

and homeomorphism

$$\mathcal{X}_{t,\eta} \cong \bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \bigcup_{\substack{1\leq i\leq m \\ h_i^+}} \left(\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}\right),$$

$$\mathcal{X}_{t,\eta} \cong \bar{\mathcal{F}}_\eta^- \times (0,1) \cup \bigcup_{\substack{1\leq i\leq m \\ h_i^-}} \left(\mathbb{D}^{n+1-\lambda(p_i)} \times \mathbb{D}^{\lambda(p_i)}\right)$$

where each handle $\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}$ *(respectively* $\mathbb{D}^{n+1-\lambda(p_i)} \times \mathbb{D}^{\lambda(p_i)}$*) is attached along* $\mathbb{D}^{\lambda(p_i)} \times \partial\mathbb{D}^{n+1-\lambda(p_i)}$ *(respectively along* $\mathbb{D}^{n+1-\lambda(p_i)} \times \partial\mathbb{D}^{\lambda(p_i)}$*) via* h_i^+ *(respectively via* h_i^-*).*

PROOF. Let $t_0(\eta) = \min(t_0'(\eta), t_0''(\eta))$ in the notation of Lemma 4.44. Then for any $t \in U \cap \{|t| \leq t_0\}$ we have that $\pm\eta$ are regular values of $f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$, and consequently no point of $\bar{\mathcal{F}}^{\pm}_{t,\eta}$ are critical points. We now fix $t \in V$ such that $|t| < t_0$.

(1) Let

$$\mathscr{S}_1 = \mathrm{int}\mathcal{X}_{t,\eta}, \quad \mathscr{S}_2 = \mathcal{X}_{t,\eta} \cap \mathbb{S}_\delta$$

$$\mathscr{S}_3 = \mathcal{F}^+_{t,\eta}, \quad \mathscr{S}_4 = \mathcal{F}^-_{t,\eta}, \quad \mathscr{S}_5 = \partial\bar{\mathcal{F}}^+_{t,\eta}, \quad \mathscr{S}_6 = \partial\bar{\mathcal{F}}^-_{t,\eta}$$

and let $\mathcal{S} = \{\mathscr{S}_1, \ldots \mathscr{S}_6\}$. As $|t| < t_0''$ it follows from the Lemma 4.43 that $\mathscr{S}_i$ are smooth manifolds, for $i = 1, 2$. As $|t| < t_0'$ it follows from the steps 1 and 3 of the proof of 4.38 that $\mathscr{S}_i$ are smooth manifolds, for $i = 3, \ldots, 6$. Moreover as

$$f_t^{-1}([-\eta, \eta]) \cap \mathbb{B}_\delta = \mathscr{S}_1 \cup \mathscr{S}_3 \cup \mathscr{S}_4$$

is a manifold with boundary its strata satisfy the Whitney conditions (see e.g [**25**, Definition 1.2]). As the same holds for

$$f_t^{-1}([-\eta, \eta]) \cap \mathbb{S}_\delta = \mathscr{S}_2 \cup \mathscr{S}_5 \cup \mathscr{S}_6$$

its strata satisfy the Whitney conditions. But then by tranversality (step 3 in the proof of 4.38) the strata of $\mathcal{S}$ satisfy also the Whitney conditions. Therefore it is a Whitney stratification of $\bar{\mathcal{X}}_{t,\eta}$ such that $\mathcal{X}_{t,\eta}$ is a union of strata.

(2) Let us consider the function $f_t : M \to \mathbb{R}$. It restricts to a proper function

$$f_t : \bar{\mathcal{X}}_{t,\eta} \to [-\eta, \eta] \subset \mathbb{R}.$$

At the points $p \in \bar{\mathcal{F}}^{\pm}_{t,\eta}$ this function have depraved stratified critical points since f_t is constant there. However any critical point

$$p \in \bar{\mathcal{X}}_{t,\eta} \setminus \bar{\mathcal{F}}^+_{t,\eta} \cup \bar{\mathcal{F}}^-_{t,\eta}$$

is necessarily nondepraved, as we now prove. So we need to show that if $p \in \mathscr{S}_i$ is a critical point of

$$f_{t|\mathscr{S}_i} : \mathscr{S}_i \to \mathbb{R}, \qquad i = 1, 2$$

then it is nondepraved. For this, we first show $f_{t|\mathscr{S}_2}$ has no critical point. Indeed suppose not. Then

$$f_{t|\mathbb{S}_\delta} : \mathbb{S}_\delta \to \mathbb{R}$$

would have a critical point $p \in \mathbb{S}_\delta$ with critical value $s \in (-\eta, \eta)$ hence $df_{t|\mathbb{S}_\delta}(p) = 0$ and

$$T_p\mathbb{S}_\delta \subset \ker df_{t|\mathbb{S}_\delta}(p). \qquad (*)$$

However the intersections $f_t^{-1}(\pm\eta) \cap \mathbb{S}_\delta$ are transverse, by Lemma 4.38 for all t sufficiently small (that is, up to replacing t_0 be a smaller value). Therefore by continuity if t is fixed, then $f_t^{-1}(s) \cap \mathbb{B}_\delta$ is transverse for all $s \in (-\eta, \eta)$. Then $(*)$ implies that $\ker df_{t|\mathbb{S}_\delta}(p) = T_p f_t^{-1}(s)$ and we conclude that $T_p\mathbb{S}_\delta \subset T_p f_t^{-1}(s)$, which is impossible since we had a transverse intersection. Therefore $\mathscr{S}_2$ has no critical points and it remains to show that any critical point p in $\mathscr{S}_1$ is nondepraved. Since $\mathscr{S}_1$ is a smooth manifold one can apply [**25**, Proposition 2.4] to

$$f_{t|\mathscr{S}_1} : \mathscr{S}_1 \to \mathbb{R}$$

which gives that any critical point is nondepraved. All said, any stratified critical point of $f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$ which belongs to a stratum $\mathscr{S}_i$ for $i = 1, 2$ necessarily belong to the first, and is a nondepraved point.

(3) We now apply the nonproper version of Stratified Morse Theory with the stratified subset $\mathcal{X}_{t,\eta}$ and the proper function $f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$. We will use Theorem 4.21 ([**25**, Theorem 10.4]) in conjunction with the Fundamental Theorem 4.22 ([**25**, Theorem 10.5]).

Let $p \in \mathcal{X}_{t,\eta}$ be a critical point with value $s = f_t(p)$. By the previous step, $p \in \mathscr{S}_1$ is an interior point. As $\mathscr{S}_1$ has maximal dimension $\dim \mathscr{S}_1 = \dim M$ it follows that its normal slice is $N_{\mathcal{X}_{t,\eta}}(p) = p$ whence it follows by its very definition that the normal Morse data is

$$(\mathscr{N}_{\mathcal{X}_{t,\eta}}, \mathscr{N}_{\mathcal{X}_{t,\eta}}) = (p, \emptyset).$$

Therefore by the Fundamental Theorem 4.22 ([**25**, Theorem 10.5]) the local Morse data for $f_{t|\mathcal{X}_{t,\eta}}$ at the point $p \in \mathscr{S}_1$ is homeomorphic to the tangential Morse data

$$(\mathscr{A}_{\mathcal{X}_{t,\eta}}, \mathscr{B}_{\mathcal{X}_{t,\eta}}) \cong (\mathscr{T}_{\mathcal{X}_{t,\eta}}, \mathscr{T}'_{\mathcal{X}_{t,\eta}})$$

for $f_{t|\mathcal{X}_{t,\eta}}$ at the point, which is by definition the tangential Morse data of f_t at p. By 4.13 ([**25**, Proposition 4.5]) the tangential Morse data at p

is homeomorphic by a decomposition-preserving homeomorphism to

$$(\mathscr{T}_{\mathcal{X}_{t,\eta}}, \mathscr{T}'_{\mathcal{X}_{t,\eta}}) = (\mathbb{D}^{n+1-\lambda_S(p)} \times \mathbb{D}^{\lambda_S(p)}, \mathbb{D}^{n+1-\lambda_S(p)} \times \partial\mathbb{D}^{\lambda_S(p)}).$$

Thus by Theorem 4.21 ([**25**, Theorem 10.4]) one has that if $[s-\gamma, s+\gamma]$ contains no critical values except $s = f_t(p)$ then

$$f_t^{-1}(-\eta, s+\gamma) \cap \bar{\mathbb{B}}_\delta \cong f^{-1}(-\eta, s-\gamma) \cap \bar{\mathbb{B}}_\delta \cup_{\mathscr{T}'_{\mathcal{X}_{t,\eta}}} \mathscr{T}_{\mathcal{X}_{t,\eta}}$$

which is to say that there exists an embedding

$$h_s : \mathbb{D}^{n+1-\lambda_S(p)} \times \partial\mathbb{D}^{\lambda_S(p)} \to f_t^{-1}(-\eta, s-\gamma) \cap \bar{\mathbb{B}}_\delta$$

such that the identity map extends to a homeomorphism

$$f_t^{-1}(-\eta, s+\gamma) \cap \bar{\mathbb{B}}_\delta \cong$$
$$f_t^{-1}(-\eta, s-\gamma) \cap \bar{\mathbb{B}}_\delta \cup_h \left(\mathbb{D}^{n+1-\lambda_S(p)} \times \mathbb{D}^{\lambda_S(p)}\right).$$

(4) By the Remark 4.46 if $p_1, \ldots, p_m \in \mathbb{B}_\delta$ denote the critical values of $f_{t|\mathscr{S}_1}$ we can order the corresponding critical values $s_i = f_t(p_i)$ as follows

$$-\eta < s_1 < \cdots < s_m < \eta.$$

For each $i = 1, \ldots, m$ choose a real number $\gamma_i > 0$ such that s_i is the only critical value contained in $[s_i - \gamma_i, s_i + \gamma_i]$ and such that

$$s_i + \gamma_i = s_{i+1} - \gamma_{i+1}, \qquad i = 1, \ldots, m-1.$$

The previous step 3 gives that there exist an embedding

$$h_m^- : \mathbb{D}^{n+1-\lambda_S(p_m)} \times \partial\mathbb{D}^{\lambda_S(p_m)} \to f_t^{-1}(-\eta, s_m - \gamma_m) \cap \bar{\mathbb{B}}_\delta$$

such that

$$\mathcal{X}_{t,\eta} \cong f_t^{-1}(-\eta, s_m - \gamma_m) \cap \bar{\mathbb{B}}_\delta \cup_{h_m^-} \left(\mathbb{D}^{n+1-\lambda_{\mathscr{S}_1}(p_m)} \times \mathbb{D}^{\lambda_{\mathscr{S}_1}(p_m)}\right).$$

Similarly there exists an embedding

$$h_{m-1}^- : \mathbb{D}^{n+1-\lambda_S(p_{m-1})} \times \partial\mathbb{D}^{\lambda_S(p_{m-1})} \to f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}) \cap \bar{\mathbb{B}}_\delta$$

such that

$$f_t^{-1}(-\eta, s_m - \gamma_m) \cap \bar{\mathbb{B}}_\delta = f_t^{-1}(-\eta, s_{m-1} + \gamma_{m-1}) \cap \bar{\mathbb{B}}_\delta$$

$$\cong f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}) \cap \bar{\mathbb{B}}_\delta \cup_{h_{m-1}^-} \left(\mathbb{D}^{n+1-\lambda_{\mathscr{S}_1}(p_{m-1})} \times \mathbb{D}^{\lambda_{\mathscr{S}_1}(p_{m-1})}\right).$$

Therefore $\mathcal{X}_{t,\eta}$ is homeomorphic to

$$\left(f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}) \cap \bar{\mathbb{B}}_\delta \cup_{h_{m-1}^-} \left(\mathbb{D}^{n+1-\lambda_{\mathscr{S}_1}(p_{m-1})} \times \mathbb{D}^{\lambda_{\mathscr{S}_1}(p_{m-1})}\right)\right)$$
$$\cup_{h_m^-} \left(\mathbb{D}^{n+1-\lambda_{\mathscr{S}_1}(p_m)} \times \mathbb{D}^{\lambda_{\mathscr{S}_1}(p_m)}\right).$$

Continuing inductively this yields

$$\mathcal{X}_{t,\eta} \cong f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cup \bigcup_{\substack{1 \le i \le m \\ h_i^-}} \left(\mathbb{D}^{n+1-\lambda_{\mathscr{S}_1}(p_i)} \times \mathbb{D}^{\lambda_{\mathscr{S}_1}(p_i)}\right).$$

(5) Since the dimension of the stratum $\mathscr{S}_1$ is maximal, the index of $f_{t|\mathscr{S}_1}$ at a critical point is equal to the index of $f_t : M \to \mathbb{R}$ so

$$\lambda_{\mathscr{S}_1}(p_i) = \lambda(p_i).$$

It therefore remains to show that

$$f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong \bar{\mathcal{F}}_\eta^- \times (0, 1).$$

Now,

$$f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong f_t^{-1}(-\eta + \gamma_1/2) \cap \bar{\mathbb{B}}_\delta \times (-\eta, s_1 - \gamma_1)$$

and

$$f_t^{-1}(-\eta - \gamma_1, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong f_t^{-1}(-\eta) \cap \bar{\mathbb{B}}_\delta \times (-\eta - \gamma_1, s_1 - \gamma_1)$$

by Thom's Isotopy Lemma 4.37 ([**35**, Proposition 11.1]) because by construction $f_t : M \to \mathbb{R}$ has no critical values in $(-\eta - \gamma_1, s_1 - \gamma_1)$ and it moreover follows that

$$f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong f_t^{-1}(-\eta - \gamma_1, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta.$$

But then

$$\bar{\mathcal{F}}_{t,\eta}^- = f_t^{-1}(-\eta) \cap \bar{\mathbb{B}}_\delta \cong f_t^{-1}(-\eta + \gamma_1/2) \cap \bar{\mathbb{B}}_\delta$$

and consequently $f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong \bar{\mathcal{F}}_\eta^- \times (0, 1)$ because $\bar{\mathcal{F}}_{t,\eta}^- \cong \bar{\mathcal{F}}_\eta$ by Lemma 4.38. Finally that $\mathcal{X}_{t,\eta}$ is contractible follows from Lemma 4.44.

$\square$

Figures (1–2) illustrate Theorem 4.47 in the case of a quadratic singularity.

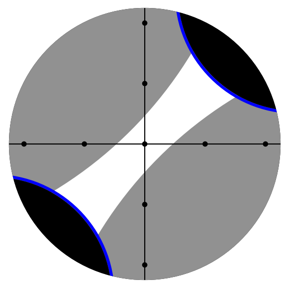

FIGURE 1. The positive Milnor fibre, in blue, of $f = xy$ with the handle $\mathbb{D}^1 \times \mathbb{D}^1$, represented by the white strip, attached along the two disjoint segments $\mathbb{D}^1 \times \mathbb{S}^0$.

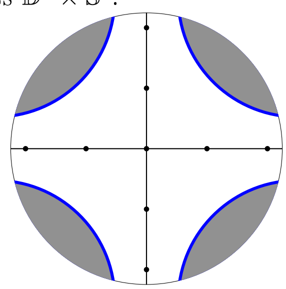

FIGURE 2. The region in white, representing $\mathcal{X}_{t,\eta}$ is homeomorphic to the space obtained from the positive Milnor with a handle attached as in Figure 1.

COROLLARY 4.48. *With the same assumptions as in Theorem 4.47 there exist attaching maps*

$$H_i^+ : \begin{cases} \partial \mathbb{D}^{n+1-\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^+ \cup \bigcup_{1\le j \le i-1} {}_{H_j^+} \mathbb{D}^{n+1-\lambda(p_j)}, & i \geq 2 \\ \partial \mathbb{D}^{n+1-\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^+, & i = 1. \end{cases}$$

$$H_i^- : \begin{cases} \partial \mathbb{D}^{\lambda(p_i)} \to \bar{\mathcal{F}}_\eta^- \cup \bigcup_{1\le j \le i-1} {}_{H_j^-} \mathbb{D}^{\lambda(p_j)}, & i \geq 2 \\ \partial \mathbb{D}^{\lambda(p_1)} \to \bar{\mathcal{F}}_\eta^-, & i = 1. \end{cases}$$

and homotopy equivalences

$$\mathcal{X}_{t,\eta} \sim \bar{\mathcal{F}}^+ \cup_{H_1^+} \mathbb{D}^{n+1-\lambda(p_1)} \cup \cdots \cup_{H_m^+} \mathbb{D}^{n+1-\lambda(p_m)},$$

$$\mathcal{X}_{t,\eta} \sim \bar{\mathcal{F}}^+ \cup_{H_1^-} \mathbb{D}^{\lambda(p_1)} \cup \cdots \cup_{H_m^-} \mathbb{D}^{\lambda(p_m)}$$

where each disc $\mathbb{D}^{n+1-\lambda(p_i)}$ (respectively $\mathbb{D}^{\lambda(p_i)}$) is attached along its boundary via H_i^+ (respectively via H_i^-).

PROOF. The proof consists of repeating steps (3–5) of the previous Theorem 4.47 but using homotopy Morse data [**25**, Definition 3.3] instead of Morse data.

Let $t \in V \cap \{|t| \leq t_0\}$ be fixed, with t_0 is as Lemma 4.44. In particular $\bar{\mathcal{F}}_{t,\eta}^{-} \cong \bar{\mathcal{F}}_{\eta}$ and $\mathcal{X}_{t,\eta}$ is contractible. Let again $p \in \mathcal{X}_{t,\eta}$ be a critical point of $f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$ and let $s = f_t(p)$. In step 3 of the proof of Theorem 4.47 was established that the local Morse data of $f_{t|\mathcal{X}_{t,\eta}}$ at p is homeomorphic to the tangential Morse data of f_t at p. Furthermore it was established that the tangential Morse data of f_t at p is homeomorphic to

$$(\mathbb{D}^{n+1-\lambda_S(p)} \times \mathbb{D}^{\lambda_S(p)}, \mathbb{D}^{n+1-\lambda_S(p)} \times \partial\mathbb{D}^{\lambda_S(p)}).$$

By [**25**, Remark 3.5.4] there exists a homotopy equivalence

$$(\mathbb{D}^{n+1-\lambda_S(p)} \times \mathbb{D}^{\lambda_S(p)}, \mathbb{D}^{n+1-\lambda_S(p)} \times \partial\mathbb{D}^{\lambda_S(p)}) \sim (\mathbb{D}^{\lambda_S(p)}, \partial\mathbb{D}^{\lambda_S(p)}).$$

By [**25**, Remark 3.3] this implies that if $\gamma = \gamma(p) > 0$ is chosen as in step 3 of the proof of Theorem 4.47 then there exists an attaching map

$$H_s : \partial\mathbb{D}^{\lambda(p)} \to f_t^{-1}(-\eta, s-\gamma) \cap \bar{\mathbb{B}}_\delta$$

and a homotopy equivalence

$$f_t^{-1}(-\eta, s+\gamma) \cap \bar{\mathbb{B}}_\delta \sim f_t^{-1}(-\eta, s-\gamma) \cap \bar{\mathbb{B}}_\delta \cup_{H_s} \mathbb{D}^{\lambda(p)}.$$

(1) In general if real numbers $\gamma_1, \dots, \gamma_m > 0$ are chosen as in step 4 of Theorem 4.47 then it follows that there exist for each $i = 1, \dots, m$ attaching maps

$$\tilde{H}_i^- : \partial\mathbb{D}^{\lambda(p_i)} \to f_t^{-1}(-\eta, s_i - \gamma_i) \cap \bar{\mathbb{B}}_\delta$$

and homotopy equivalences

$$f_t^{-1}(-\eta, s_i + \gamma_i) \cap \bar{\mathbb{B}}_\delta \sim f_t^{-1}(-\eta, s_i - \gamma_i) \cap \bar{\mathbb{B}}_\delta \cup_{\tilde{H}_i^-} \mathbb{D}^{\lambda(p_i)}.$$

This gives

$$\mathcal{X}_{t,\eta} \sim f_t^{-1}(-\eta, s_m - \gamma_m) \cap \bar{\mathbb{B}}_\delta \cup_{\tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}$$

$$\sim f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}) \cap \bar{\mathbb{B}}_\delta \cup_{\tilde{H}_{m-1}^-} \mathbb{D}^{\lambda(p_{m-1})} \cup_{\tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}.$$

Continuing in this manner one obtains

$$\mathcal{X}_{t,\eta} \sim f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cup_{\tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup \cdots \cup_{\tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}.$$

Since by step 5 of the proof of Theorem 4.47,

$$f_t^{-1}(-\eta, s_1 - \gamma_1) \cap \bar{\mathbb{B}}_\delta \cong \bar{\mathcal{F}}_\eta^- \times (0,1)$$

one gets that

$$\mathcal{X}_{t,\eta} \sim \bar{\mathcal{F}}^- \times (0,1) \cup_{\tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup \cdots \cup_{\tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}.$$

(2) Let

$$\phi_1 : \bar{\mathcal{F}}^- \times (0,1) \to \bar{\mathcal{F}}^-$$

be a deformation retraction. Then ϕ_1 extends by [**38**, Lemma 3.7] to a homotopy equivalence

$$\phi_2 : \bar{\mathcal{F}}^- \times (0,1) \cup_{\tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \to \bar{\mathcal{F}}^- \cup_{\phi_1 \circ \tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)}.$$

In turn applying [**38**, Lemma 3.7] to ϕ_2 gives a homotopy equivalence

$$\begin{aligned} \phi_3 : \bar{\mathcal{F}}^- \times (0,1) \cup_{\tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup_{\tilde{H}_2^-} \mathbb{D}^{\lambda(p_2)} \\ \to \bar{\mathcal{F}}^- \cup_{\phi_1 \circ \tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup_{\phi_2 \circ \tilde{H}_2^-} \mathbb{D}^{\lambda(p_2)}. \end{aligned}$$

Continuing in this manner one eventually obtains a homotopy equivalence between

$$\mathcal{X}_{t,\eta} \sim \bar{\mathcal{F}}^- \times (0,1) \cup_{\tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup \cdots \cup_{\tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}$$

and

$$\bar{\mathcal{F}}^- \cup_{\phi_1 \circ \tilde{H}_1^-} \mathbb{D}^{\lambda(p_1)} \cup \cdots \cup_{\phi_m \circ \tilde{H}_m^-} \mathbb{D}^{\lambda(p_m)}.$$

Then $H_i^- := \phi_i \circ \tilde{H}_i^-, i = 1, \ldots, m$ are the wanted attaching maps.

□

One recovers the formula of Khimshiashvili [**16**, Theorem 2.3] from this Corollary by taking the Euler-Poincaré characteristic.

Corollary 4.49. *With the same assumptions as in Theorem 4.47 one has*

$$\chi(\bar{\mathcal{F}}_\eta^+) = 1 - \sum_{i=1}^m (-1)^{n+1-\lambda(p_i)}, \qquad \chi(\bar{\mathcal{F}}_\eta^-) = 1 - \sum_{i=1}^m (-1)^{\lambda(p_i)}.$$

Proof. It was established in the Corollary 4.48 that

$$\bar{\mathcal{F}}_{t,\eta}^+ \cup_{H_1^+} \mathbb{D}^{n+1-\lambda(p_1)} \cup \cdots \cup_{H_m^+} \mathbb{D}^{n+1-\lambda(p_m)}$$

is contractible. For any $k = 1, \dots, m$ let

$$\mathcal{M}_k := \bar{\mathcal{F}}^+_{t,\eta} \cup_{H_1^+} \mathbb{D}^{n+1-\lambda(p_1)} \cup \cdots \cup_{H_k^+} \mathbb{D}^{n+1-\lambda(p_k)},$$

$$\mathcal{M}_0 := \bar{\mathcal{F}}^+_{t,\eta}.$$

For any pair (X, A) of topological spaces consider the Euler characteristic

$$\chi(X, A) = \sum_{n \geq 0} (-1)^n \text{rank } H_n(X, A; \mathbb{Z})$$

in relative singular homology with $\mathbb{Z}$-coefficients. From the long exact sequence of a triple (X, A, B) with $B \subset A \subset X$ one has that χ is additive in the sense that

$$\chi(X, B) = \chi(X, A) + \chi(A, B).$$

Since

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_m$$

is an ascending filtration of topological spaces and since $\mathcal{M}_i \sim \mathcal{M}_{i-1} \cup_{H_i^+} \mathbb{D}^{n+1-\lambda(p_i)}$ for $i = 1, \dots, m$ one obtains

$$\chi(\mathcal{M}_m, \mathcal{M}_0) = \sum_{i=1}^{m} \chi(\mathcal{M}_i, \mathcal{M}_{i-1}).$$

Using the Excision Theorem [**26**, Theorem 2.20] applied to the pair $(\mathcal{M}_i, \mathcal{M}_{i-1})$ and the subspace $\mathcal{M}_{i-1} \setminus H^+_{i-1}(\partial \mathbb{D}^{n+1-\lambda(p_i)})$ gives

$$\chi(\mathcal{M}_i, \mathcal{M}_{i-1}) = \chi(\mathbb{D}^{n+1-\lambda(p_i)}, \partial \mathbb{D}^{n+1-\lambda(p_i)}) = (-1)^{n+1-\lambda(p_i)}$$

(see also [**38**, I § 5]) hence

$$\chi(\mathcal{M}_m, \mathcal{M}_0) = \sum_{i=1}^{m} (-1)^{n+1-\lambda(p_i)}.$$

One the other hand, applying additivity of χ to the ascending filtration

$$\emptyset \subset \mathcal{M}_0 = \bar{\mathcal{F}}^+_{t,\eta} \subset \mathcal{M}_m$$

of topological spaces one obtains

$$\chi(\bar{\mathcal{F}}^+_{t,\eta}) = \chi(\mathcal{M}_m) - \chi(\mathcal{M}_m; \mathcal{M}_0).$$

Since $\bar{\mathcal{X}}_{t,\eta} \sim \mathcal{M}_m$ is contractible and since $\bar{\mathcal{F}}^+_{t,\eta} \cong \bar{\mathcal{F}}^+_{\eta}$ this yields

$$\chi(\bar{\mathcal{F}}^+_{\eta}) = 1 - \sum_{i=1}^{m} (-1)^{n+1-\lambda(p_i)}$$

whence the first assertion. The proof of the second assertion is analoguous. □

REMARK 4.50. *The proof of Khimshiashvili's formula which is known to us*[6] *is the one given in* [**16**, Theorem 2.3]. *It uses Morse theory for manifolds-with-boundaries. Theorem 4.47 is not stated in the literature and it does not follow from of the proof* [**16**, Theorem 2.3].

Another corollary is the following.

COROLLARY 4.51. *In the notation of Theorem 4.47 if there exists $t \in V$ with $|t| < t_0$ such that $f_t = F(x,t)$ has no critical points inside the ball of radius δ centered at the origin then the positive and negative Milnor fibres are contractible.*

REMARK 4.52. *If $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a germ of complex hypersurface singularities then it is never the case that a Milnor fibre (see e.g* [**39**, Theorem 4.8]*) of f is contractible. Indeed the Milnor fibre is contractible if and only if the origin is a nonsingular point of $(f^{-1}(0), 0)$, according to a theorem* [**1**, Théorème 3] *of Norbert A'Campo, generalising a result of Milnor for isolated singularities.*

We give another example of Theorem 4.47 .

EXAMPLE 4.53. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = y^2 - x^3$. Then*

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F(x,y,t) = y^2 - x^3 - tx$$

is a $\mathbb{R}$-morsification. Indeed

$$Jac(f_t)(x,y) = [-(3x^2 + t), 2y]$$

has full rank, for any $t > 0$. Therefore $f_t : \mathbb{R}^2 \to \mathbb{R}$ has no critical points and hence the Milnor fibres $\bar{\mathcal{F}}^{\pm}_{\eta}$ are both contractible by Corollary 4.51 . Remark that if we instead consider the $\mathbb{R}$-morsification

$$G : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad G(x,y,t) = y^2 - x^3 + tx$$

then

$$Jac(g_t)(x,y) = [t - 3x^2, 2y]$$

has for $t > 0$ nonmaximal rank in $p_1(t) = (\sqrt{t/3}, 0)$ and in $p_2(t) = (-\sqrt{t/3}, 0)$. Since

$$Hess(g_t) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$$

[6] The original article [**29**] is not avaible online

one obtains the Morse indices $\lambda(p_1(t)) = 1$ *and* $\lambda(p_2(t)) = 0$. *As a consequence* $\bar{\mathcal{F}}_\eta^+ \times (0,1)$ *remains contractible after attaching a handle* $\mathbb{D}^1 \times \mathbb{D}^1$ *along* $\mathbb{D}^1 \times \partial\mathbb{D}^1$ *and then attaching a handle* $\mathbb{D}^0 \times \mathbb{D}^2$ *along* $\mathbb{D}^0 \times \partial\mathbb{D}^2$.

5. On the Homology of The Real Milnor Fibres

According to the previous Theorem 4.47 one obtains a contractible space by succesively attaching handles (Definition 4.7) to (a space having the homotopy type of) the positive (respectively negative) real Milnor fibres. However this gives no information about the manner in which the handles are attached. Moreover it might be the case that there exist $t \in V$ and $1 \leq i, j \leq m$ and a pair of handles

$$(\mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n+1-\lambda(p_i)}, \mathbb{D}^{\lambda(p_j)} \times \mathbb{D}^{n+1-\lambda(p_j)}),$$

such that $\bar{\mathcal{X}}_{t,\eta}$ has the homotopy type of

$$\bar{\mathcal{F}}_\eta^+ \times (0,1) \cup \left(\bigcup_{1 \leq l \neq i,j \leq m} \mathbb{D}^{\lambda(p_l)} \times \mathbb{D}^{n+1-\lambda(p_l)} \right).$$

An example of this phenomenon is provided by Example 4.53 where one knows by using the $\mathbb{R}$-morsification F that the positive Milnor fibre is contractible so that the pair of handles provided by the $\mathbb{R}$-morsification G does not contribute to the homotopy type of $\bar{X}_{t,\eta}$.

One therefore concludes that the homology groups of the Milnor fibres cannot be obtained directly from Theorem 4.47. However we shall now state conditions under the validity of which, all of the handles are up to homotopy attached to a real Milnor fibre.

Throughout this section we fix Milnor data (δ_0, ϵ_0) for $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$.

5.1. A Lemma. By restricting the parameter space U one can always find a $\mathbb{R}$-morsification such that the number of nondegenerate critical points lying inside the closed ball of radius δ is independent of the parameter.

LEMMA 4.54. *Let* $F : M \times U \to \mathbb{R}$ *be a* $\mathbb{R}$*-morsification of* $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$. *There exists a finite set of nonempty connected semialgebraic subsets*

$$U_1, \dots, U_N \subset U \subset \mathbb{R}^k$$

each containing the origin such that if $t, t' \in U_i$ then $m(t) = m(t')$ (Definition 4.45) and such that $V \cap U_i = U_i \setminus \{0\}$. That is, the number of critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ is independent of $t \in V \cap U_i$ and each critical point is Morse with pairwise distinct critical values.

PROOF. Since $F : M \times U \to \mathbb{R}$ is a representative of a polynomial map, M and U are semialgebraic. Since

$$\mathcal{P} : t \in U \quad p \in M : \quad \text{rank Jac}(f_t)(p) = 0, \quad \det \text{Hess}(f_t)(p) \neq 0$$

$$\mathcal{Q} : t \in U \quad p, q \in M : \text{rank Jac}(f_t)(p) = \text{rank Jac}(f_t)(q) = 0,$$
$$p \neq q \implies f_t(p) \neq f_t(q)$$

are semialgebraic conditions one can assume (Definition 4.28) that $V \subset U \setminus \{0\}$ is semialgebraic. Let

$$\mathscr{C}_V = \{(p, t) \in M \times V \mid \text{rank Jac}(f_t)(p) = 0\}.$$

Then $\mathscr{C}_V$ is a semialgebraic set. Let

$$\pi : \mathbb{R}^{n+1} \times \mathbb{R}^k \to \mathbb{R}^k, \qquad \pi(x, t) = t$$

denote the standard projection onto the second factor. Then $\pi_{|\mathscr{C}_V}$ is a semialgebraic, continuous map and its fiber dimension is the number of critical points of $f_{t|V}$. By Hardt's Theorem [**12**, Theorem 4.1] one can decompose V into a finite union of semialgebraic subsets $V_1, \ldots, V_L \subset V$ such that $\pi_{|\mathscr{C}_V}$ is semialgebraically trivial (see e.g [**12**, § 4.1.1]) over each V_i. The fiber dimension $\dim \pi^{-1}_{|\mathscr{C}}(t)$ is therefore independent of $t \in V_i$. In other words the number of critical points of f_t is independent of $t \in V_i$, each critical point is Morse and the critical values are pairwise distinct. Since $(0, \ldots, 0) \in \bar{V}$ by the Morse Lemma [**38**, Lemma 2.2] there exists a subcollection $\{V_1, \ldots, V_N\} \subset \{V_1, \ldots, V_L\}$ such that $(0, \ldots, 0) \in \bar{V}_i, i = 1, \ldots, N$. Every semialgebraic set has a decomposition into a finite number of connected semialgebraic components by [**6**, Theorem 2.4.5] so we can assume that $V_1, \ldots, V_N$ are connected. Then $U_i = V_i \cup \{0\}, i = 1, \ldots, N$ satisfy the conditions of the Lemma. □

REMARK 4.55. *One can thus always obtain a $\mathbb{R}$-morsification F such that that $f_t : M \to \mathbb{R}$ is a Morse function with exactly $m = m(t)$ critical points with distinct critical values for* all *$t \in U \setminus \{0\}$. Indeed this follows from the previous Lemma*

4.54 since one can replace a given $\mathbb{R}$-morsification $F : M \times U \to \mathbb{R}$ by a new $\mathbb{R}$-morsification $F' : M \times U_i \to \mathbb{R}$ where U_i is one of the connected semialgebraic sets given by Lemma 4.54 and where the dense set $V \subset U$ is replaced by by $U_i \setminus \{0\}$.

5.2. The Local Milnor Fibres. We shall now introduce the local Milnor fibres. In this subsection $F : M \times U \to \mathbb{R}$ is a $\mathbb{R}$-morsification of f such that $m(t)$ is independent of t, denoted $m = m(t)$, and such that $V = U \setminus \{0\}$.

Let $t \in V$. By the Morse Lemma [**38**, Lemma 2.2] each of the critical points $p_i = p_i(t) \in \bar{\mathbb{B}}_\delta$ of $f_t : M \to \mathbb{R}$ defines a quadratic singularity in the fibre $f_t^{-1}(s_i)$, where $s_i = f_t(p_i)$. One can therefore apply the Fibration Theorem of Milnor 4.36 [**46**, Theorem 4.2] to deduce the following.

For each $i = 1, \ldots, m$ there exists a $\delta_{i,0} = \delta_{i,0}(t) > 0$ such that for any $\delta_i \in (0, \delta_{i,0}]$ there exist an $\epsilon_{i,0} > 0$ such that for any $\epsilon_i \in (0, \epsilon_{i,0})$ the restrictions

$$f_t : f_t^{-1}((s_i, s_i + \epsilon_i]) \cap \mathbb{B}_{\delta_i}(p_i) \to (s_i, s_i + \epsilon_i] \tag{6}$$

and

$$f_t : f_t^{-1}([s_i - \epsilon_i, s_i)) \cap \mathbb{B}_{\delta_i}(p_i) \to [s_i - \epsilon_i, s_i) \tag{7}$$

are projections of trivial (topological) fibrations where $\mathbb{B}_{\delta_i}(p_i)$ is the open ball centered at p_i and of radius δ_i. The pair $(\delta_{i,0}, \epsilon_{i,0}) = (\delta_{i,0}(t), \epsilon_{i,0}(t))$ will be called *local Milnor data* at the point p_i.

REMARK 4.56. *We shall write $\mathbb{B}_{\delta_i}$ instead of $\mathbb{B}_{\delta_i}(p_i)$ when no confusion is possible.*

DEFINITION 4.57. *Let $t \in V$ be fixed and let $\delta_i \in (0, \delta_{i,0}]$ and $\eta_i \in (0, \epsilon_{i,0}]$. The fibers*

$$\mathcal{F}^+_{i,loc} := f_t^{-1}(s_i + \eta_i) \cap \mathbb{B}_{\delta_i}, \qquad i = 1, \ldots, m$$
$$\mathcal{F}^-_{i,loc} := f_t^{-1}(s_i - \eta_i) \cap \mathbb{B}_{\delta_i}, \qquad i = 1, \ldots, m$$

of the fibrations 6 and 7 are called the local positive (respectively negative) open Milnor fibres at the point $p_i = p_i(t)$ with respect to the $\mathbb{R}$-morsification F. Their closures

$$\bar{\mathcal{F}}^\pm_{i,loc} := f_t^{-1}(s_i + \eta_i) \cap \bar{\mathbb{B}}_{\delta_i}, \qquad i = 1, \ldots, m$$

$$\bar{\mathcal{F}}^{\pm}_{i,loc} := f_t^{-1}(s_i - \eta_i) \cap \bar{\mathbb{B}}_{\delta_i}, \qquad i = 1, \dots, m$$

are called the local positive (respectively negative) closed Milnor fibres at the point p_i with respect to F.

The following schematic Figure 3 serves to illustrate the local Milnor fibres and also the proof of the next Theorem 4.58.

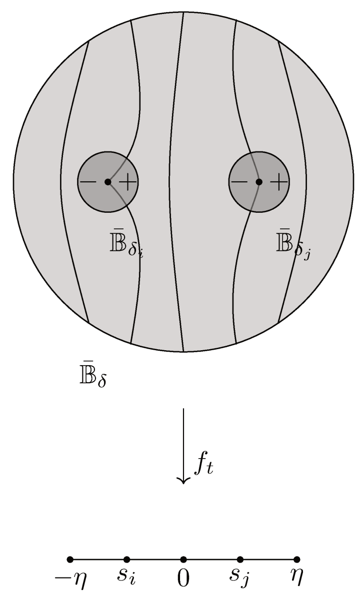

FIGURE 3. A set of two local Milnor fibres

5.3. The Main Theorem. The main result of this section is the following theorem. We assume throughout that $\bar{\mathcal{F}}^{+}_{\eta}$ and $\bar{\mathcal{F}}^{-}_{\eta}$ are nonempty

THEOREM 4.58. *Let $F : M \times U \to \mathbb{R}$ be a $\mathbb{R}$-morsification of a germ of isolated singularity $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ with $n \equiv 1 \pmod 2$ and $n > 1$. Let $\delta \in (0, \delta_0]$ and $\eta \in (0, \epsilon_0]$ and recall that $m = m(t)$ for all $t \in V$ (where $m(t)$ is as in Definition 4.45 and V is as in Definition 4.28). Let t_0 be as in Lemma 4.44 and let $t \in V \cap \{|t| \le t_0\}$ be fixed. Denote by $p_1, \dots, p_m \in \bar{\mathbb{B}}_\delta$ the critical points of $f_t : M \to \mathbb{R}$ lying inside the ball of radius δ and let $\lambda(p_i)$ denote their Morse indices. If $\lambda(p_i) = (n+1)/2, i = 1, \dots, m$ then*

$$H_k(\bar{\mathcal{F}}^{+}_{\eta}) = H_k(\bar{\mathcal{F}}^{-}_{\eta}) \cong H_k\Big(\bigvee_{i=1}^{m} \mathbb{S}^{(n-1)/2}\Big), \qquad k \ge 0.$$

PROOF. Using the Isotopy Lemma 4.37 ([**35**, Proposition 11.1]) together with Corollary 4.48 we first show that the contractible space $\bar{\mathcal{X}}_{t,\eta}$ is obtained by attaching discs $\mathbb{D}^{\lambda(p_i)}$ along their boundaries $\partial\mathbb{D}^{\lambda(p_i)}$ to certain spaces which by using [**25**, Proposition 3.2] are shown to be homotopy equivalent to different copies of the negative Milnor fibre of f. Using the Excision Theorem [**26**, Theorem 2.20] one is the able to obtain the integral homology groups of the negative Milnor fibre. The case with the positive Milnor fibres is then obtained by considering $-f_t$ instead of f_t.

(1) Fix $t \in V$. Let $\delta_i = \delta_i(t) \in (0, \delta_{i,0}(t))$ and $\epsilon_i = \epsilon_i(t) \in (0, \epsilon_{i,0}(\delta_i))$ be local Milnor data as in Definition 4.57. For each $i = 1, \ldots, m$ one can assume by the Morse Lemma [**38**, Lemma 2.2] that δ_i is chosen such that there exists local coordinates $x_1, \ldots, x_{n+1} : \mathbb{B}_{\delta i} \to \mathbb{R}^{n+1}$ with $x_j(p_i) = (0, \ldots, 0), j = 1, \ldots, n+1$ such that

$$f_{|\mathbb{B}_{\delta_i}} = s_i - \sum_{j=1}^{\lambda} x_j^2 + \sum_{l=\lambda+1}^{n+1} x_i^2,$$

up to precomposing f with a diffeomorphism, where $\lambda := \lambda(p_i), i = 1, \ldots, m$. Put

$$\mathcal{Z}_i = f_t^{-1}(s_i - \epsilon_i, s_i + \epsilon_i) \cap \bar{\mathbb{B}}_\delta,$$

$$\bar{\mathcal{Z}}_i = f_t^{-1}([s_i - \epsilon_i, s_i + \epsilon_i]) \cap \bar{\mathbb{B}}_\delta,$$

$$\mathcal{Y}_i = \mathcal{Z}_i \cap \mathbb{B}_{\delta_i} = f_t^{-1}(s_i - \epsilon_i, s_i + \epsilon_i) \cap \mathbb{B}_{\delta_i}.$$

For each $i = 1, \ldots, m$ choose real positive numbers $\gamma_i > 0$ such that the interval $[s_i - \gamma_i, s_i + \gamma_i]$ contains no critical value of $f_{t|\bar{\mathbb{B}}_\delta}$ except s_i and such that

$$s_i + \gamma_i = s_{i+1} - \gamma_{i+1}, \qquad i = 1, \ldots, m-1,$$

$$-\eta = s_1 - \gamma_1, \quad \eta = s_m + \gamma_m.$$

Put

$$\mathcal{X}_i = f_t^{-1}(s_i - \gamma_i, s_i + \gamma_i) \cap \bar{\mathbb{B}}_\delta,$$

$$\bar{\mathcal{X}}_i = f_t^{-1}([s_i - \gamma_i, s_i + \gamma_i]) \cap \bar{\mathbb{B}}_\delta.$$

Then the union of the sets $\bar{\mathcal{X}}_i, i = 1, \ldots, m$ is the manifold with corners $\bar{\mathcal{X}}_\eta := \bar{\mathcal{X}}_{t,\eta}$ of Lemma 4.44. In particular it is contractible. By Lemma 4.43 one has that each $\bar{\mathcal{X}}_i$ and $\bar{\mathcal{Z}}_i$ are smooth manifolds with corners. By smoothening their corners one deduces that $\bar{\mathcal{X}}_i \sim \mathcal{X}_i$ and $\bar{\mathcal{Z}}_i \sim \mathcal{Z}_i$. Let $i \in \{1, \dots, m\}$. We shall now use Goresky and MacPherson's method of "moving the wall" to deduce that $\bar{\mathcal{X}}_i$ is homeomorphic to $\bar{\mathcal{Z}}_i$. For this consider

$$\mathcal{W} = \{(x, r) \in \mathbb{R} \times [0,1] \mid x \in [s_i - r\gamma_i - (1-r)\epsilon_i, s_i + r\gamma_i + (1-r)\epsilon_i]\}$$

and endow $\mathcal{W}$ with its natural Whitney stratification. If $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the standard projection $\pi(x, r) = r$ then $\pi_{|\mathcal{W}}$ is a proper submersion by construction. Now, by the choice of $t_0 \in \mathbb{R}$ one has that $f_t : \bar{\mathbb{B}}_\delta \to \mathbb{R}$ is transverse to $[-\epsilon, \epsilon]$ hence $f_{t|\bar{\mathbb{B}}_\delta}$ is transverse to

$$\mathcal{W}_r = [s_i - r\gamma_i - (1-r)\epsilon_i, s_i + r\gamma_i + (1-r)\epsilon_i] \subset [-\epsilon, \epsilon]$$

for all $r \in [0, 1]$. One can therefore apply [**25**, Theorem 4.8]. This gives a stratum-preserving homeomorphism

$$f_t^{-1}([s_i - \epsilon_i, s_i + \epsilon_i]) \cap \bar{\mathbb{B}}_\delta \cong f_t^{-1}([s_i - \gamma_i, s_i + \gamma_i]) \cap \bar{\mathbb{B}}_\delta.$$

As a consequence $\bar{\mathcal{X}}_i \cong \bar{\mathcal{Z}}_i$ and therefore also $\mathcal{X}_i \sim \mathcal{Z}_i$.

(2) For any $i = 1, \dots, m$ consider the map

$$f_t : \mathcal{Z}_i \setminus \mathcal{Y}_i \to (s_i - \epsilon_i, s_i + \epsilon_i).$$

It is proper since its fibers are of the form

$$f^{-1}_{t|\mathcal{Z}_i \setminus \mathcal{Y}_i}(s) = f_t^{-1}(s) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_i})$$

and hence are compact. From the assumption

$$\lambda(p_1) = \cdots = \lambda(p_m) = \frac{n+1}{2}$$

follows that if we choose $\eta_i \in (0, \epsilon_i], i = 1, \dots, m$ then local Milnor fibres (Definition 4.57) are of the form

$$\bar{\mathcal{F}}^+_{i,loc} \cong \{x \in \bar{\mathbb{B}}_{\delta_i} \mid -\sum_{j=1}^{(n+1)/2} x_j^2 + \sum_{j=(n+3)/2}^{n+1} x_j^2 = \eta_i\} \cong \bar{\mathcal{F}}^-_{i,loc}.$$

In particular the boundaries of the local Milnor fibres $\bar{\mathcal{F}}^{\pm}_{t,i,loc}$ are nonempty so the fibers of $f_{t|\mathcal{Z}_i \setminus \mathcal{Y}_i}$ are nonempty. Note that the only stratified critical point of $f_{t|\mathcal{Z}_i}$ is by construction $p_i \in \mathcal{Y}_i$. Therefore, in order to show that

$f_{t|\mathcal{Z}_i \setminus \mathcal{Y}_i}$ is a proper stratified submersion it suffices to show that

$$f_t : f_t^{-1}(s_i - \epsilon_i, s_i + \epsilon_i) \cap \mathbb{S}_{\delta_i} \to \mathbb{R}$$

is a submersion, that is, that the map

$$f_t : \mathbb{S}_{\delta_i} \to \mathbb{R}$$

has no critical values in $(s_i - \epsilon_i, s_i + \epsilon_i)$. It has no critical values $s \neq s_i$ because

$$f_t : f_t^{-1}((s_i, s_i + \epsilon_i]) \cap \bar{\mathbb{B}}_{\delta_i} \to (s_i, s_i + \epsilon_i],$$

$$f_t : f_t^{-1}((s_i - \epsilon_i, s_i]) \cap \bar{\mathbb{B}}_{\delta_i} \to [s_i - \epsilon_i, s_i)$$

are stratified submersions and because $f_t^{-1}(s_i) \cap \mathbb{S}_{\delta_i}$ is transverse by construction (Definition 4.57) and so $f_t : \mathbb{S}_{\delta_i(t)} \to \mathbb{R}$ has no critical point above s_i. All said $f_{t|\mathcal{Z}_i \setminus \mathcal{Y}_i}$ is a proper stratified submersion so by Thom's First Isotopy Lemma 4.37 ([**35**, Proposition 11.1]),

$$\mathcal{Z}_i \setminus \mathcal{Y}_i \cong f_t^{-1}(s) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_i}) \times (s_i - \epsilon_i, s_i + \epsilon_i),$$

for all $s \in (s_i - \epsilon_i, s_i + \epsilon_i)$. This yields in particular a deformation retraction

$$\mathcal{Z}_i \sim f_t^{-1}(s) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_i}) \cup \mathcal{Y}_i =$$

$$= f_t^{-1}(s) \cap \bar{\mathbb{B}}_\delta \cup \mathcal{Y}_i, \qquad s \in (s_i - \epsilon_i, s_i + \epsilon_i).$$

(3) We claim that $\mathcal{Z}_i \sim \mathcal{Z}_{i-1}$ for all $i = 1, \dots, m$. To prove this we shall show that both $\mathcal{Z}_i$ and $\mathcal{Z}_{i-1}$ deformation retracts to homeomorphic spaces. From the Morse Lemma [**38**, Lemma 2.2] follows that there exists a diffeomorphism $\mathcal{Y}_{i-1} \cong \mathcal{Y}_i$. By the previous step 2 it suffices therefore to show that

$$f_t^{-1}(s_i - \eta_i) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_i}) \cong f_t^{-1}(s_{i-1} + \eta_{i-1}) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_{i-1}}).$$

The interval $[s_i - \gamma_i, s_i)$ contains no critical values of the proper map $f_{t|\bar{\mathbb{B}}_\delta}$. There exists therefore by [**25**, Proposition 3.2] a stratum-preserving[7] homeomorphism

$$\phi_i : f_t^{-1}((-\infty, s_{i-1} + \eta_{i-1}]) \cap \bar{\mathbb{B}}_\delta \to f_t^{-1}((-\infty, s_i - \eta_i]) \cap \bar{\mathbb{B}}_\delta$$

[7] By the proof of [**25**, Proposition 3.2], which is a simple consequence of [**25**, Theorem 4.4], it follows that ϕ_i is in fact smooth on strata. This follows also from [**25**, Remark 7.2] where one uses the First Isotopy Lemma to lift a certain smooth vector field to construct the homeomorphism ϕ_i.

which hence restricts to a stratum-preserving homeomorphism

$$\phi_{i|A_i} : A_i := f_t^{-1}(s_{i-1} + \eta_{i-1}) \cap \bar{\mathbb{B}}_\delta \to B_i = f_t^{-1}(s_i - \eta_i) \cap \bar{\mathbb{B}}_\delta.$$

Since $\mathcal{F}^+_{i-1,loc} \subset \mathrm{int}(A_i)$ it follows that $\phi_{i|\mathcal{F}^+_{i-1,loc}}$ is a homeomorphism onto its image. Since $\mathcal{F}^+_{i-1,loc}$ and $\mathcal{F}^-_{i,loc}$ are diffeomorphic by [**38**, Lemma 2.2] it follows that the image of $\phi_{i|\mathcal{F}^+_{i-1,loc}}$ is homeomorphic to $\mathcal{F}^-_{i,loc}$. Then

$$\tilde{\phi}_{i|A_i \setminus \mathcal{F}^+_{i-1,loc}} : A_i \setminus \mathcal{F}^+_{i-1,loc} \to B_i \setminus \phi_i(\mathcal{F}^+_{i-1,loc}) \cong B_i \setminus \mathcal{F}^-_{i,loc}$$

gives the wanted homeomorphism. This yields

$$\begin{aligned} \mathcal{Z}_i &\sim f_t^{-1}(s_i - \eta_i) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_i}) \cup \mathcal{Y}_i \\ &\cong f_t^{-1}(s_{i-1} + \eta_{i-1}) \cap (\bar{\mathbb{B}}_\delta \setminus \mathbb{B}_{\delta_{i-1}}) \cup \mathcal{Y}_{i-1} \sim \mathcal{Z}_{i-1}. \end{aligned}$$

(4) By the proof of Theorem 4.47 and Corollary 4.48 applied to

$$f_t : f_t^{-1}([s_i - \epsilon_i, s_i + \epsilon_i]) \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

there exists an attaching map

$$h_i^- : \partial\mathbb{D}^\lambda \to f_t^{-1}(s_i - \eta_i) \cap \bar{\mathbb{B}}_{\delta_i}$$

and a homotopy equivalence

$$\mathcal{Y}_i \sim f_t^{-1}(s_i - \eta_i) \cap \bar{\mathbb{B}}_{\delta_i} \cup_{h_i^-} \mathbb{D}^\lambda$$

where one uses the fact that $\mathcal{Y}_i \sim f_t^{-1}(s_i - \epsilon_i, s_i + \epsilon_i) \cap \bar{\mathbb{B}}_\delta$ which follows from the Tubular Neighborhood Theorem [**26**, Proposition 3.42] for topological manifolds with boundaries. Applying instead Theorem 4.47 and Corollary 4.48 to

$$-f_t : f_t^{-1}([s_i - \epsilon_i, s_i + \epsilon_i]) \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

one obtains an attaching map

$$h_i^+ : \partial\mathbb{D}^\lambda \to f_t^{-1}(s_i + \eta_i) \cap \bar{\mathbb{B}}_{\delta_i}$$

and a homotopy equivalence

$$\mathcal{Y}_i \sim f_t^{-1}(s_i + \eta_i) \cap \bar{\mathbb{B}}_{\delta_i} \cup_{h_i^+} \mathbb{D}^\lambda.$$

Applying the previous step 2 yields deformation retractions

$$\mathcal{Z}_i \sim \begin{cases} f_t^{-1}(s_i+\eta_i)\cap\bar{\mathbb{B}}_\delta \cup_{h_i^+} \mathbb{D}^\lambda, \\ f_t^{-1}(s_i-\eta_i)\cap\bar{\mathbb{B}}_\delta \cup_{h_i^-} \mathbb{D}^\lambda. \end{cases}$$

(5) Using [**25**, Proposition 3.2] one obtains homeomorphisms

$$\phi_i^+ : f_t^{-1}(s_i+\eta_i)\cap\bar{\mathbb{B}}_\delta \to f_t^{-1}(s_i+\gamma_i)\cap\bar{\mathbb{B}}_\delta,$$

$$\phi_i^- : f_t^{-1}(s_i-\eta_i)\cap\bar{\mathbb{B}}_\delta \to f_t^{-1}(s_i-\gamma_i)\cap\bar{\mathbb{B}}_\delta$$

so that applying [**38**, Lemma 3.7] and step 4 one obtains homotopy equivalences

$$\mathcal{Z}_i \sim \begin{cases} f_t^{-1}(s_i+\gamma_i)\cap\bar{\mathbb{B}}_\delta \cup_{\phi_i^+\circ h_i^+} \mathbb{D}^\lambda, \\ f_t^{-1}(s_i-\gamma_i)\cap\bar{\mathbb{B}}_\delta \cup_{\phi_i^-\circ h_i^-} \mathbb{D}^\lambda. \end{cases}$$

(6) By the first step 1 one has $\bar{\mathcal{X}}_i \sim \mathcal{X}_i$ and $\mathcal{X}_i \sim \mathcal{Z}_i$ for all $i=1,\dots,m$. By step 3 one has $\mathcal{Z}_i \sim \mathcal{Z}_{i-1}$ for all $i=2,\dots,m$. Hence

$$\bar{\mathcal{X}}_i \sim f_t^{-1}(-\eta)\cap\bar{\mathbb{B}}_\delta \cup_{\tilde{H}_i} \mathbb{D}^\lambda$$

for certain attaching maps $\tilde{H}_i$, by step 5.

(7) Consider the long exact sequence of the pair $(\bar{\mathcal{X}}_{t,\eta}, \bar{\mathcal{F}}^-_{t,\eta})$ in integral homology:

$$\to \tilde{H}_n(\bar{\mathcal{F}}^-_{t,\eta}) \to \tilde{H}_n(\bar{\mathcal{X}}_{t,\eta}) \to H_n(\bar{\mathcal{X}}_{t,\eta}, \bar{\mathcal{F}}^-_{t,\eta})$$

Since by Lemma 4.44 $\bar{\mathcal{X}}_{t,\eta}$ is contractible this gives

$$\tilde{H}_k(\bar{\mathcal{F}}^-_{t,\eta}) = H_{k+1}(\bar{\mathcal{X}}_{t,\eta}, \bar{\mathcal{F}}^-_{t,\eta}), \qquad \forall k \geq 0.$$

By the previous step 6 and the fact that $\bar{\mathcal{X}}_i \sim \mathcal{X}_i$ gives

$$\bar{\mathcal{X}}_\eta \sim \bar{\mathcal{F}}^-_{t,\eta} \cup_{\tilde{H}_1} \mathbb{D}^\lambda \cup \cdots \cup_{\tilde{H}_m} \mathbb{D}^\lambda$$

where

$$\tilde{H}_i : \partial\mathbb{D}^\lambda \to \bar{\mathcal{F}}^-_{t,\eta}, \qquad i=1,\dots,m$$

are attaching maps. Therefore

$$\tilde{H}_k(\bar{\mathcal{F}}^-_{t,\eta}) \cong H_{k+1}(\bar{\mathcal{F}}^-_{t,\eta} \cup_{\tilde{H}_1} \mathbb{D}^\lambda \cup \cdots \cup_{\tilde{H}_m} \mathbb{D}^\lambda, \bar{\mathcal{F}}^-_{t,\eta}).$$

Since $\lambda > 0$ the closure of

$$\bar{\mathcal{F}}_\eta^- \setminus \bigcup_{i=1}^m \tilde{H}_i(\partial\mathbb{D}^\lambda)$$

is contained in the interior of $\bar{\mathcal{F}}_\eta^-$ so using the Excision Theorem [**26**, Theorem 2.20] yields

$$\begin{aligned} \tilde{H}_k(\bar{\mathcal{F}}_{t,\eta}^-) &= \bigcup_{i=1}^m H_{k+1}(\mathbb{D}^\lambda, \partial\mathbb{D}^\lambda) \\ &= \bigoplus_{i=1}^m \mathbb{Z}, \qquad k = \lambda - 1. \end{aligned}$$

Therefore

$$H_k(\bar{\mathcal{F}}_\eta^-) = H_k(\bigvee_{i=1}^m \mathbb{S}^{\lambda-1})$$

where we have used the fact that $\bar{\mathcal{F}}_{t,\eta}^- \cong \bar{\mathcal{F}}_\eta^-$ by Lemma 4.44.

□

5.4. Consequences. We now discuss some consequences.

COROLLARY 4.59. *Keep the assumptions of the Theorem 4.58. The Poincaré polynomial of the real Milnor fibres are*

$$\beta(\bar{\mathcal{F}}_\eta^+) = 1 + mu^{n-\lambda}$$

and

$$\beta(\bar{\mathcal{F}}_\eta^-) = 1 + mu^{\lambda-1}$$

The following result, which follows from 4.47, will be of considerable use in the next chapter when we consider ADE-singularities.

COROLLARY 4.60. *Keep the notations of Theorem 4.47 and suppose $f_t : M \to \mathbb{R}$ has a unique critical point p of index $\lambda(p)$. If $\bar{\mathcal{F}}_\eta^+$ is nonempty then*

$$\beta(\bar{\mathcal{F}}_\eta^+) = \begin{cases} 1 + u^{n-\lambda(p)}, & \textit{if } \lambda(p) \le n, \\ 1, & \textit{if } \lambda(p) = n+1. \end{cases}$$

If $\bar{\mathcal{F}}_\eta^-$ is nonempty then

$$\beta(\bar{\mathcal{F}}_\eta^-) = \begin{cases} 1 + u^{\lambda(p)-1}, & \textit{if } \lambda(p) \ge 1, \\ 1, & \textit{if } \lambda(p) = 0. \end{cases}$$

PROOF. We shall treat the second assertion, the first assertion follows by considering $-f_t$ instead of f_t and repeating the arguments below. By the Corollary 4.48 there is an attaching map

$$h : \partial\mathbb{D}^{\lambda(p)} \to \bar{\mathcal{F}}^-_{t,\eta}$$

such that

$$\bar{\mathcal{X}}_{t,\eta} \sim \bar{\mathcal{F}}^-_{t,\eta} \cup_{\partial\mathbb{D}^{\lambda(p)}} \mathbb{D}^{\lambda(p)}$$

is contractible. If $\lambda(p) = 0$ then $\mathbb{D}^\lambda$ is a point attached to $\bar{\mathcal{F}}^-_{t,\eta}$ along the empty set. Then either $\bar{\mathcal{F}}^-_{t,\eta} = \emptyset$ in which case $\bar{\mathcal{F}}^-_\eta = \emptyset$, or $\bar{\mathcal{F}}^-_{t,\eta}$ is contractible in which case $\beta(\bar{\mathcal{F}}^-_\eta) = 1$. So suppose $\bar{\mathcal{F}}^-_\eta \neq \emptyset$ and that $\lambda(p) > 0$. Then $\bar{\mathcal{F}}^-_{t,\eta}$ is nonempty as well so one can apply the long exact sequence in homology [**18**, Proposition 4.12], yielding

$$\tilde{H}_k(\bar{\mathcal{F}}^-_\eta) \cong H_{k+1}(\bar{\mathcal{X}}_{t,\eta}, \bar{\mathcal{F}}^-_{t,\eta}).$$

Since $\lambda(p) > 0$ the closure of $\bar{\mathcal{F}}^-_{t,\eta} \setminus h(\partial\mathbb{D}^{\lambda(p)})$ is contained in the interior of $\bar{\mathcal{F}}^-_{t,\eta}$ so one can use the Excision Theorem [**26**, Theorem 2.20][8]. This gives

$$\begin{aligned}\tilde{H}_k(\bar{\mathcal{F}}^-_\eta) &= H_{k+1}(\bar{\mathcal{F}}^-_{t,\eta} \cup_{\partial\mathbb{D}^{\lambda(p)}} \mathbb{D}^{\lambda(p)}, \bar{\mathcal{F}}^-_{t,\eta}) \\ &\cong H_{k+1}(\mathbb{D}^{\lambda(p)}, \partial\mathbb{D}^{\lambda(p)}).\end{aligned}$$

The claim follows. □

5.5. Real Vanishing Cycles. In this subsection $F : M \times U \to \mathbb{R}$ is a $\mathbb{R}$-morsification such that $m = m(t)$ and such that $V = U \setminus \{0\}$, as in subsection 5.2. We assume furthermore that $\bar{\mathcal{F}}^+_\eta$ and $\bar{\mathcal{F}}^-$ are nonempty. Fix $t \in V$ and consider for each $i = 1, \ldots, m$ the local Milnor fibres (Definition 4.57)

$$\mathcal{F}^+_{i,loc} = f_t^{-1}(s_i + \eta_i) \cap \mathbb{B}_{\delta_i}$$

$$\mathcal{F}^-_{i,loc} = f_t^{-1}(s_i - \eta_i) \cap \mathbb{B}_{\delta_i}$$

at the critical point $p_i = p_i(t) \in \bar{\mathbb{B}}_\delta$ of $f_t = F(\cdot, t)$. Let $\lambda(p_i)$ be the Morse index of f_t at p_i. Then Corollary 4.60 gives

$$H_k(\mathcal{F}^+_{i,loc}) = \begin{cases} \mathbb{Z}, & k = 0, n - \lambda(p_i), \\ \{0\}, & \text{otherwise} \end{cases}$$

[8]One can also take a neighbourhood $A \supset \bar{\mathcal{F}}^-_{t,\eta}$ such that $\mathrm{int}(A) \cup \mathbb{D}^{\lambda(p)}$ deformation retracts to $\bar{\mathcal{F}}^-_{t,\eta} \cup_{\partial\mathbb{D}^{\lambda(p)}} \mathbb{D}^{\lambda(p)}$ and such that $A \cap \mathbb{D}^{\lambda(p)} = \partial\mathbb{D}^{\lambda(p)}$ and apply the Excision Theorem to A and $B = \mathbb{D}^{\lambda(p)}$.

$$H_k(\mathcal{F}^-_{i,loc}) = \begin{cases} \mathbb{Z}, & k = 0, \lambda(p_i) - 1, \\ \{0\}, & \text{otherwise} \end{cases}$$

for each $i = 1, \ldots, m$. Thus the local Milnor fibres have the homology groups of spheres.

Definition 4.61. *Suppose that $\lambda(p_i) \leq n$. A generator*

$$\gamma_i^+ \subset \tilde{H}_{n-\lambda(p_i)}(\mathcal{F}^+_{i,loc}) \cong \tilde{H}_{n-\lambda(p_i)}(\mathbb{S}^{n-\lambda(p_i)})$$

is called a positive vanishing cycle with respect to F at the point p_i.
Suppose that $\lambda(p_i) \geq 1$. A generator

$$\gamma_i^- \subset \tilde{H}_{\lambda(p_i)-1}(\mathcal{F}^-_{i,loc}) \cong \tilde{H}_{\lambda(p_i)-1}(\mathbb{S}^{\lambda(p_i)-1})$$

is called a negative vanishing cycle with respect to F at the point p_i.
If $\lambda(p_i) = n+1$ one says that F has no positive vanishing cycles at p_i. If $\lambda(p_i) = 0$ one says that F has no negative vanishing cycles at p_i.

In contrast to the situation for complex hypersurface singularities this notion of vanishing cycle depends of course on the choice of morsification.

If the conditions of the Theorem 4.58 are satisfied then a set of positive, or equivalently negative, real vanishing cycles is a generating set of the top-dimensional homology groups of the real positive and negative Milnor fibres.

Corollary 4.62. *Suppose that the conditions of Theorem 4.58 are satisfied. Then $\gamma_1^-, \ldots, \gamma_m^-$ is a set of generators of the homology group*

$$\tilde{H}_k(\mathcal{F}^+_\eta) \cong \tilde{H}_k(\mathcal{F}^-_\eta), \qquad k = (n-1)/2.$$

Proof. Consider the spaces

$$\mathcal{Y}_i = f_t^{-1}(s_i - \epsilon_i, s_i + \epsilon_i) \cap \mathbb{B}_i$$

as in the proof of Theorem 4.58. There are isomorphisms

$$\tilde{H}_k(\bar{\mathcal{F}}^-_\eta) \cong H_{k+1}(\bar{\mathcal{X}}_{t,\eta}, \bar{\mathcal{F}}^-_{t,\eta}) \cong \bigoplus_{i=1}^m H_{k+1}(Y_i, \mathcal{F}^-_{i,loc})$$

because $Y_i \sim \mathcal{F}_i \cup_{\partial\mathbb{D}^\lambda} \mathbb{D}^\lambda$ by the proof of Theorem 4.58. We claim that $\mathcal{Y}_i$ is contractible. The Morse Lemma [**38**, Lemma 2.2] gives that $\mathcal{Y}_i$ is homeomorphic to

$$Q_i^{-1}(-\epsilon, \epsilon) = \{-\epsilon_i < Q_i(x_1, \ldots, x_{n+1}) < \epsilon_i\}$$

where $Q_i : U \to \mathbb{R}$ is a quadratic form of index λ defined in a neighborhood of the origin $U \subset \mathbb{R}^{n+1}$ with $Q_i(0,\ldots,0) = 0$. Then $Q_i^{-1}(0) \subset Q_i^{-1}(-\epsilon,\epsilon)$ is by [**15**, Proposition 1.6] a deformation retract. By the Local Conic Structure of algebraic sets $Q_i^{-1}(0)$ is contractible. Hence $\mathcal{Y}_i$ is contractible. Since $\mathcal{F}^-_{i,loc} \neq \emptyset$ as $\lambda = (n+1)/2 > 0$ it follows from the long exact sequence of the pair $(\mathcal{Y}_i, \mathcal{F}^-_{i,loc})$ that

$$H_{k+1}(\mathcal{Y}_i, \mathcal{F}^-_{i,loc}) = \tilde{H}_k(\mathcal{F}^-_{i,loc}), \qquad k \geq 0.$$

hence

$$\tilde{H}_k(\bar{\mathcal{F}}^-_\eta) \cong \bigoplus_{i=1}^{m} \tilde{H}_k(\mathcal{F}^-_{i,loc}), \qquad k \geq 0.$$

The only nontrivial homology group is in dimension $k = (n-1)/2$ so the right-hand side is generated by the negative vanishing cycles $\gamma_1^-, \ldots, \gamma_m^-$. Hence these generate the left-hand side as well. $\square$

A set of vanishing cycles corresponding to an arbitrary $\mathbb{R}$-morsification does not generate the homology groups of the Milnor fibres (for an example see e.g Example 4.53). This is to compare with the situation over the complex numbers, where the vanishing cycles determine not only the homology groups of the complex Milnor fibre but also its homotopy type.

CHAPTER 5

Applications to Real ADE-Singularities

1. Introduction

In this brief chapter we gather some further consequences of the results of the previous chapters. We show in Corollary 5.2 how the Theorems 4.47 and 4.58, taken in tandem, enables us to find in the case of an ADE-singularity, the Poincaré polynomials of the Milnor fibres, except for the case with a $D_k^{-,s}$-singularity (see the list 2 below) where k is even.

2. On the Topology of Real ADE-Singularities

In this section we shall consider isolated singularities $f : \mathbb{R}^{n+1} \to \mathbb{R}$ of the following form:

(1) $A_k^{\pm,s} : f = x^{k+1} \pm y^2 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2, \quad k \geq 2$
(2) $D_k^{\pm,s} : f = x^2 y \pm y^{k-1} + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2, \quad k \geq 4$
(3) $E_6^{\pm s} : f = x^3 \pm y^4 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$
(4) $E_7^{s} : f = x^3 + xy^3 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$
(5) $E_8^{s} : f = x^3 + y^5 + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$

where $t+s = n-1, s, t \geq 0$ and where $(x, y, x_1, \ldots, x_{n-1})$ are coordinates on $\mathbb{R}^{n+1}$. To simplify the notation we shall write $\mathcal{F}^+ = \mathcal{F}_\eta^+(f)$ for a positive Milnor fibre and $\mathcal{F}^- = \mathcal{F}^-(f)$ for a negative Milnor fibre at the origin.

The idea will be to first treat the case of curves (namely $n = 1$) and find for each of the classes above a $\mathbb{R}$-morsification of a particularly simple character, with few critical points. It is then easy to find $\mathbb{R}$-morsifications in the general case (namely $n \geq 2$).

The following result gives the Poincaré polynomials of the Milnor fibres of ADE-singularities of curves.

THEOREM 5.1. *For a topological space X let $\beta(X)$ denote the Poincaré polynomial in singular homology with $\mathbb{Z}$-coefficients. Then*

(1) $A_k^+ : f(x,y) = x^{k+1} + y^2$:

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1, \qquad \text{if } k \equiv 0 \pmod 2$$

$$\beta(\bar{\mathcal{F}}^+) = 1 + u, \quad \beta(\bar{\mathcal{F}}^-) = 0 \qquad \text{if } k \equiv 1 \pmod 2.$$

(2) $A_k^- : f(x,y) = x^{k+1} - y^2$:

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(3) $D_k^+ : f(x,y) = x^2 y + y^{k-1}, \quad k \geq 4$,

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(4) $D_k^- : f(x,y) = x^2 y - y^{k-1}, \quad k \geq 4$,

$$\beta(\bar{\mathcal{F}}^\pm) = \begin{cases} 3, & k \equiv 0 \pmod 2 \\ 2, & k \equiv 1 \pmod 2 \end{cases}$$

(5) $E_6^\pm : f(x,y) = x^3 \pm y^4$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1$.

(6) $E_7 : f(x,y) = x^3 + xy^3$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 2$.

(7) $E_8 : f(x,y) = x^3 + y^5$, $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1$.

PROOF. We are going to find, in each of the cases above, a $\mathbb{R}$-morsification having either one unique critical point, or no critical points at all and in the case D_k^- with k odd, a $\mathbb{R}$-morsification with two critical points. To begin with we treat the case of an A_k-singularity.

(1) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^{k+1} + y^2$.
Suppose that $k \equiv 0 \pmod 2$ and let

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$$

$$F(x,y,t) = x^{k+1} + tx + y^2.$$

Then

$$\mathrm{Jac}(f_t) = [(k+1)x^k + t, 2y],$$

so f_t has no critical point for $t \neq 0$. Hence applying Theorem 4.47 we get

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1$$

since the Milnor fibres are contractible.
Suppose that $k \equiv 1 \pmod 2$. Then $\bar{\mathcal{F}}^- = \emptyset$ because f is nonnegative. Letting

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F(x,y,t) = x^{k+1} - (k+1)tx + y^2$$

then

$$\mathrm{Jac}(f_t) = [(k+1)(x^k - t), 2y],$$

so that there is a unique critical point $p = (t^{1/k}, 0)$ and then

$$\mathrm{Hess}(f_t)(p) = \begin{pmatrix} k(k+1)t^{(k-1)/k} & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $t > 0$ it follows that $\lambda(p) = 0$. Therefore we can apply Corollary 4.60 to deduce that

$$\beta(\bar{\mathcal{F}}^+) = 1 + u.$$

(2) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^{k+1} - y^2$.
Suppose that $k \equiv 0 \pmod 2$ and define

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F(x,y,t) = x^{k+1} + tx - y^2.$$

Then

$$\mathrm{Jac}(f_t) = [(k+1)x^k + t, -2y], \qquad t \in [0,1]$$

Since k is even, f_t has no critical points whenever t is nonzero. Therefore the positive and negative Milnor fibres are both contractible, by Theorem 4.47 so

$$\beta(\bar{\mathcal{F}}^\pm) = 1.$$

Suppose that $k \equiv 1 \pmod 2$. Then one considers $F : \mathbb{R}^2 \times [-1,0] \to \mathbb{R}$ given by the same formula as in the case k even and one gets instead that $f_t = F(x,t)$ has a unique critical point

$$p = ((-t/(k+1))^{1/k}, 0).$$

in which the Hessian matrix is

$$\mathrm{Hess}(f_t)(p) = \begin{pmatrix} \frac{k(k+1)}{(k+1)^{(k-1)/k}}(-t)^{(k-1)/k} & 0 \\ 0 & -2 \end{pmatrix}.$$

Since t is negative there is only one negative eigenvalue so the index is $\lambda(p) = 1$. One applies Corollary 4.60 and concludes that

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 2.$$

(3) Let $k \geq 4$ and consider a D_k^+-singularity $f(x, y) = x^2y + y^{k-1}$. Suppose first that $k \equiv 0 \pmod 2$ and consider

$$F : \mathbb{R}^2 \times [-1, 0] \to \mathbb{R},$$

$$F(x, y, t) = x^2y + 2tx^2 + y^{k-1} - ty.$$

Then

$$\mathrm{Jac}(f_t) = [2x(y + 2t), x^2 + (k-1)y^{k-2} - t]$$

has nonmaximal rank in a point $p = (x, y)$ if and only if either one of the following sets of equations is satisfied

$$\begin{cases} x = 0, \quad (k-1)y^{k-2} = t, \quad \text{or} \\ x^2 = t\left(1 - 2^{k-2}(k-1)t^{k-3}\right), \quad y = -2t. \end{cases}$$

Remark first of all that if $t \neq 0$ then $(k-1)y^{k-2} = t$ has no real solutions because k is even and $t < 0$. Furthermore, since $t < 0$ and since $(k-3)$ is odd,

$$\forall t < 0, \qquad t^{k-3} < 0 < \frac{1}{(k-1)2^{k-2}}$$

hence $1 - 2^{k-2}(k-1)t^{k-3} > 0$. Therefore the second set of equations has no real solutions for $t \neq 0$ either. As a consequence $f_t : \mathbb{R}^2 \to \mathbb{R}$ has no critical points whenever $t \neq 0$. Thus applying Theorem 4.47 one gets that the Milnor fibres are contractible and so $\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^+) = 1$ by homotopy invariance.

Suppose now that $k \equiv 1 \pmod 2$. Put

$$I = (-\frac{1}{((k-1)2^{k-2})^{1/(k-3)}}, 0]$$

and consider the $\mathbb{R}$-morsification

$$F : \mathbb{R}^2 \times I \to \mathbb{R},$$

$$F(x,y,t)=x^2y+2tx^2+y^{k-1}-ty.$$

Since $k-3\equiv 0 \pmod 2$ one has

$$\forall t\in I,\qquad t^{k-3}<\frac{1}{(k-1)2^{k-2}}$$

Therefore if $t\in I\setminus\{0\}$ then

$$x^2=t\left(1-2^{k-2}(k-1)t^{k-3}\right)$$

has no real solutions. As a consequence the only critical point of f_t is $p(t)=(0,(t/k-1)^{k-2})$. The Hessian $\mathrm{Hess}(f_t)(p(t))$ is

$$\begin{pmatrix} 2t(2+\frac{t^{k-3}}{(k-1)^{k-2}}) & 0 \\ 0 & (k-1)(k-2)(\frac{t}{k-1})^{(k-2)(k-3)} \end{pmatrix}.$$

Since

$$k-3\equiv(k-2)(k-3)\equiv 0 \pmod 2$$

it follows that if $t\in I\setminus\{0\}$ then $\lambda(p)=1$. Using Corollary 4.60 one deduces that

$$\beta(\bar{\mathcal{F}}^{+})=\beta(\bar{\mathcal{F}}^{-})=2.$$

(4) Let $k\geq 4$ and consider a D_k^--singularity $f(x,y)=x^2y-y^{k-1}$. Suppose that $k\equiv 1 \pmod 2$ and let

$$F:\mathbb{R}^2\times[0,1]\to\mathbb{R},\qquad F(x,y,t)=x^2y-y^{k-1}+(k-1)ty.$$

Then

$$\mathrm{Jac}(f_t)=[2xy,x^2+(k-1)(t-y^{k-2})].$$

If $t\neq 0$ then there is only one critical point

$$p(t)=(0,t^{1/(k-2)})$$

and one find the Hessian to be given by

$$\mathrm{Hess}(f_t)(p(t))=\begin{pmatrix} 2t^{1/(k-2)} & 0 \\ 0 & -(k-1)(k-2)t^{(k-3)/(k-2)} \end{pmatrix}.$$

Therefore $\lambda(p(t))=1$ whenever $t\neq 0$ hence

$$\beta(\bar{\mathcal{F}}^{+})=\beta(\bar{\mathcal{F}}^{-})=\beta(\mathbb{S}^0)=2$$

by Corollary 4.60.

Suppose that $k \equiv 0 \pmod 2$ and let

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F(x,y,t) = x^2y - y^{k-1} + (k-1)ty.$$

Then

$$\mathrm{Jac}(f_t) = [2xy, x^2 + (k-1)(t - y^{k-2})].$$

If $t \neq 0$ then there are two critical points

$$p_1(t) = (0, t^{1/(k-2)}), \qquad p_2(t) = (0, -t^{1/(k-2)})$$

and the Hessian matrices are

$$\mathrm{Hess}(f_t)(p(t)) = \begin{pmatrix} \pm 2t^{1/(k-2)} & 0 \\ 0 & \mp(k-1)(k-2)t^{(k-3)/(k-2)} \end{pmatrix}.$$

As a consequence, since $t > 0$, the Morse indices are

$$\lambda(p_1) = \lambda(p_2) = 1.$$

Since $n + 1 - \lambda(p_i) = \lambda(p_i)$ one can apply the Theorem 4.47 to deduce that

$$\beta(\bar{\mathcal{F}}^{\pm}_{\eta}) = \beta(\mathbb{S}^0 \vee \mathbb{S}^0) = 3.$$

(5) Consider a $E_6^{\pm}$-singularity $f^{\pm}(x,y) = x^3 \pm y^4$. If

$$F^{\pm} : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F^{\pm}(x,y,t) = x^3 + 3tx \pm y^4$$

and if $f_t^{\pm} = F^{\pm}(\cdot, t)$ then

$$\mathrm{Jac}(f_t^{+}) = [3x^2 + 3t, 4y^3],$$

$$\mathrm{Jac}(f_t^{-}) = [3x^2 + 3t, -4y^3]$$

so that neither f_t^{+} nor f_t^{-} has any critical points whenever $t \neq 0$ hence

$$\beta(\bar{\mathcal{F}}^{+}) = \beta(\bar{\mathcal{F}}^{-}) = 1$$

by Theorem 4.47. The same holds for E_8-singularities because if one defines

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \qquad F(x,y,t) = F(x,y,t) = x^3 + 3tx + y^5$$

then

$$\mathrm{Jac}(f_t) = [3x^2 + 3t, 5y^4]$$

so f_t has no critical points whenever t is nonzero hence the Milnor fibres are contractible.

(6) It remains the case of an E_7-singularity. In this case, put

$$F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}, \quad F(x,y,t) = x^3 + 3tx + xy^3 + ty^3.$$

The Jacobian matrix is

$$\mathrm{Jac}(f_t) = [3(x^2+t) + y^3, 3y^2(x+t)]$$

which has nonmaximal rank in $p = (x,y)$ if and only if either

$$\begin{cases} x^2 + t = 0, \quad y = 0, \quad \text{or} \\ x = -t, \quad y^3 = -3(t^2+t). \end{cases}$$

However if $t \neq 0$ then the first set of equations has no real solutions and one finds that f_t has a unique critical point in

$$p(t) = (-t, -(3)^{1/3}(t^2+t)^{1/3})$$

The Hessian matrix is given by

$$\begin{pmatrix} 6x & 3y^2 \\ 3y^2 & 6y(x+t) \end{pmatrix}$$

and one finds that evaluated in $p(t)$ its eigenvalues are

$$\lambda_+(t) = -3t + \sqrt{9t^2 + 3^{10/3}(t^2+t)^{4/3}},$$

$$\lambda_+(t) = -3t - \sqrt{9t^2 + 3^{10/3}(t^2+t)^{4/3}}.$$

For t sufficiently small, $\lambda_+(p(t)) > 0$ and $\lambda_-(p(t)) < 0$. Hence the Morse index at $p = p(t)$ is $\lambda(p) = 1$ and, applying the Corollary 4.60 one gets that

$$\beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 2.$$

□

From this we can easily deduce the corresponding result for higher dimensions, except for the case D_k^- with k even and $n \geq 2$.

COROLLARY 5.2. *The Poincaré polynomials in singular homology with integer coefficients of the Milnor fibres of the isolated singularities* $A_k^{\pm s}, D_k^{+s}$ *and* E_i^s *are given as follows.*

(1) A_k^{-s}, D_k^{+s}:

$$\beta(\bar{\mathcal{F}}^+) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 1 + u^{n-s-1}, & k \equiv 1 \pmod 2 \end{cases},$$

$$\beta(\bar{\mathcal{F}}^-) = \begin{cases} 1, & k \equiv 0 \pmod 2 \\ 1 + u^{s}, & k \equiv 1 \pmod 2 \end{cases}$$

(2) E_7^s :

$$\beta(\bar{\mathcal{F}}^+) = 1 + u^{n-s-1},$$

$$\beta(\bar{\mathcal{F}}^-) = 1 + u^{s}.$$

(3) $E_6^{\pm s}, E_8^s : \beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1.$

(4) A_k^{+s} :

$$\begin{cases} \beta(\bar{\mathcal{F}}^+) = 1 + u^{n-s}, & k \equiv 1 \pmod 2 \\ \beta(\bar{\mathcal{F}}^-) = 1 + u^{s-1}, & s \neq 0, k \equiv 1 \pmod 2 \\ \beta(\bar{\mathcal{F}}^-) = 0, & s = 0, k \equiv 1 \pmod 2 \\ \beta(\bar{\mathcal{F}}^+) = \beta(\bar{\mathcal{F}}^-) = 1, & k \equiv 0 \pmod 2). \end{cases}$$

The Poincaré polynomials of the Milnor fibres of D_k^{-s}*-singularities for* k *odd and* $n > 1$ *are*

$$D_k^{-s} : \begin{cases} \beta(\bar{\mathcal{F}}^+) = 1 + u^{n-s-1}, & k \equiv 1 \pmod 2 \\ \beta(\bar{\mathcal{F}}^-) = 1 + u^{s}, & k \equiv 1 \pmod 2 \end{cases}$$

PROOF. Consider an $\mathbb{R}$-morsification

$$\tilde{F} : \mathbb{R}^2 \times \bar{U} \to \mathbb{R}, \qquad f_t = \tilde{F}(\cdot, t)$$

of any of the ADE-singularities $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ given in the proof of Theorem 5.1 and let

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad f = \tilde{f}(x, y) + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2.$$

If

$$F : \mathbb{R}^{n+1} \times \bar{U} \to \mathbb{R}, \qquad F = \tilde{F} + \sum_{i=1}^{t} x_i^2 - \sum_{j=t+1}^{s+t} x_j^2$$

then the critical points of $f_t = F(\cdot, t)$ are the points $p = (\tilde{p}, 0, \ldots, 0)$ with $\tilde{p}$ a critical point of $\tilde{f}_t$. Moreover these are Morse whenever $\tilde{p}$ is Morse, with Hessian

$$\mathrm{Hess}(f_t)(p) = \begin{pmatrix} \mathrm{Hess}(\tilde{f}_t)(\tilde{p}) & 0 & 0 \\ 0 & 2I_t & 0 \\ 0 & 0 & -2I_s \end{pmatrix}$$

where I_t and I_s denotes the identity matrices of size $t \times t$ and $s \times s$, respectively. As a consequence, if $\lambda(\tilde{p})$ is the Morse index of $\tilde{f}_t$ at $\tilde{p}$ then $\lambda(p) = s + \lambda(\tilde{p})$ is the Morse index of f_t at p. Since in each of the cases there is either one critical point or none at all the result follows immediately from Corollary 4.60 and from Theorem 4.47 of the previous chapter.

□

We can remark here that G. Fichou (see [**22**]) has classified the real *ADE*-singularities above using the blow-Nash equivalence relation. The following definitions can be found on page 184 of [**4**].

Definition 5.3 ([**4**, II]). *Let X be a manifold and suppose that a Lie group G acts on X. Let $x \in X$. The* modality *of x under the action of G is the least number m such that there exists a neighborhood $x \in U \subset M$ and a covering of U by finitely many m-parameter families of orbits. If $m = 0$ then one says that x is* simple. *If $m = 1$ one says that x is* unimodal *and if $m = 2$ one says that x is* bimodal.

In other words, a point $x \in X$ is simple if it has a neighborhood U intersecting only finitely many orbits of G.

Definition 5.4 ([**4**, II]). *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a real analytic function germ. The* modality *of f is the modality of its jet $j^k f$, for $k \in \mathbb{N}$ sufficiently large, in the space of jets of functions $\mathbb{R}^{n+1} \to \mathbb{R}$ having a critical point $0 \in \mathbb{R}^{n+1}$ and a critical value $0 \in \mathbb{R}$, under the action of the Lie group Diff$(\mathbb{R}^{n+1})$ of diffeomorphisms of $\mathbb{R}^{n+1}$.*

Returning to Fichou's result, he showed that if

$$f, g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$$

are two germs of Nash functions with one of them simple (as a germ of real analytic function), say f, then f, g are blow-Nash equivalent if and only if they are analytically equivalent, and then g is simple as well. So for simple Nash function germs the blow-Nash equivalence coincides with the analytic equivalence. Consequently, the list 2 given in the beginning of this section gives a complete list of simple singularities up to the blow-Nash equivalence.

CHAPTER 6

Morsifiable Real Singularities

1. Introduction

The purpose of this chapter is to generalise the results of Chapter 4 to the case of real singularities $f : X \to \mathbb{R}$ defined on a real algebraic set $X \subset \mathbb{R}^n$ containing the origin such that f is *morsifiable*, see Definition 6.8. In Theorem 6.13 is obtained a formula for the Euler characteristics of the Milnor fibres of such singularities, generalising the formula of Khimshiashvili [**16**, Theorem 2.3]. In Theorem 6.15 a result on the generic value of the Euler-Poincaré characteristic of the link of an irreducible algebraic subset is obtained.

2. The Situation

2.1. Introduction. Assume that $X \subset \mathbb{R}^n$ is a real algebraic subset containing the origin. Assume that

$$f : X \to \mathbb{R}, \qquad f(0) = 0$$

is the restriction of a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ and that there exists a stratification $\tilde{\mathcal{S}} = (\tilde{\mathscr{S}_i})_{i \in I}$ of X such that $\tilde{\mathcal{S}}$ satisfies the Whitney conditions, Thom's a_f-condition (see e.g[**50**]) and that $f^{-1}(0) \subset X$ is a union of strata of $\tilde{\mathcal{S}}$.

2.2. A Milnor Fibration. Under the assumptions above one can show the existence of a locally trivial topological fibration over a small punctured interval.

PROPOSITION 6.1. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the restriction*

$$f : f^{-1}([-\epsilon, \epsilon] \setminus \{0\}) \cap X \cap \bar{\mathbb{B}}^n_\delta \to [-\epsilon, \epsilon] \setminus \{0\} \tag{8}$$

is a locally trivial fibration, where $\bar{\mathbb{B}}^n_\delta \subset \mathbb{R}^n$ denotes the closed n-dimensional ball centered at the origin of radius δ.

PROOF. One copies verbatim the proof of [**17**, Theorem 1.1] for f and X complex analytic. □

The data (ϵ_0, δ_0) will be referred to as *Milnor data* for f at the origin. The fibers

$$\bar{\mathcal{F}}_\eta^+ = f^{-1}(\eta) \cap X \cap \bar{\mathbb{B}}_\delta^n, \qquad \bar{\mathcal{F}}_\eta^- = f^{-1}(-\eta) \cap X \cap \bar{\mathbb{B}}_\delta^n, \quad \eta \in (0, \epsilon]$$

of (8) will be referred to as a positive, respectively negative, Milnor fiber of f at the origin.

Since $\tilde{\mathcal{S}}$ satisfies the Whitney conditions [**25**] it follows that, if δ_0 is chosen small enough then for any $\delta \in (0, \delta_0]$ and any stratum $\tilde{\mathscr{S}}$ having the origin in its closure, the intersection $\tilde{\mathscr{S}} \cap \partial\bar{\mathbb{B}}_\delta^n$ is transverse. Furthermore, since $\tilde{\mathcal{S}}$ satisfies the a_f-condition it follows that if $\delta \in (0, \delta_0]$ and $\eta \in (0, \epsilon_0]$ are fixed and if $x \in f^{-1}(\eta)$ then for any $i \in I$ the intersection

$$f^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \partial\bar{\mathbb{B}}_\delta^n$$

is transverse at x. As a consequence the positive Milnor fiber is stratified by strata

$$f^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta^n, \qquad i \in I,$$

$$f^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \partial\bar{\mathbb{B}}_\delta^n, \qquad i \in I.$$

and the negative Milnor fiber is stratified by strata

$$f^{-1}(-\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta^n, \qquad i \in I,$$

$$f^{-1}(-\eta) \cap \tilde{\mathscr{S}}_i \cap \partial\bar{\mathbb{B}}_\delta^n, \qquad i \in I.$$

REMARK 6.2. *When no confusion is possible we shall write $\mathbb{B}_\delta$ instead of $\mathbb{B}_\delta^n$ for the n-dimensional ball of radius δ centered at the origin in $\mathbb{R}^n$ and write $\mathbb{S}_\delta$ instead of $\partial\bar{\mathbb{B}}_\delta^n$ for its boundary sphere.*

3. Stratified $\mathbb{R}$-morsifications

3.1. Stratified $\mathbb{R}$-morsifications. To obtain the wanted generalisation of Theorem 4.47 in chapter 4 the guiding principle will be to perturb $f : X \to \mathbb{R}$ to obtain a stratified Morse function. To this aim we introduce the following notion.

DEFINITION 6.3. *Let $Y \subset \mathbb{R}^n$ be a Whitney stratified real subanalytic set and let $g : Y \to \mathbb{R}$ the restriction of a polynomial map $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$. A* stratified

$\mathbb{R}$-morsification *of g is the restriction $G : Y \times \bar{U} \to \mathbb{R}$ of a polynomial map*

$$\tilde{G} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R},$$

such that

(1) *$\bar{U} \subset \mathbb{R}^k$ is a smooth k-dimensional manifold with boundary having boundary $\partial\bar{U}$ and interior U,*
(2) *$\bar{U} \cong [-1, 1]^k$ and $\partial\bar{U}$ contains the origin in $\mathbb{R}^k$,*
(3) *$\tilde{G}(x, 0) = \tilde{g}$,*
(4) *The map*

$$\Phi : \mathbb{R}^n \times U \to T^*\mathbb{R}^n, \qquad \Phi(x, t) = d(\tilde{g}_t)(x),$$

where $T^*\mathbb{R}^n$ is the cotangent bundle of $\mathbb{R}^n$, is a submersion.

The following proposition shows that if $G : Y \times \bar{U} \to \mathbb{R}$ is a stratified $\mathbb{R}$-morsification of $g : Y \to \mathbb{R}$ as in Definition 6.3 and if $Y \subset \mathbb{R}^n$ is compact then there exists a dense open subset $V \subset U$ of such that for each $t \in V$ the function $G(\cdot, t) = g_t : Y \to \mathbb{R}$ is a stratified Morse function.

Proposition 6.4. *Let $Y \subset \mathbb{R}^n$ be a Whitney stratified compact real subanalytic set and let $G : Y \times \bar{U} \to \mathbb{R}$ be a stratified $\mathbb{R}$-morsification of $g : Y \to \mathbb{R}$ where g is the restriction of a polynomial function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$. Then there exists a dense open subset $V \subset U$ such that $g_t : Y \to \mathbb{R}$ is a stratified Morse function for any $t \in V$, in the sense of Definition 4.2* [**25**, Definition 2.1].

Proof. Since U is a smooth manifold, since Y is compact and since Φ is a submersion one can apply the Theorem [**25**, Theorem 2.2.3] to deduce that there exists a dense open subset $V \subset U$ such that $g_t : Y \to \mathbb{R}$ is a stratified Morse function, for any $t \in V$. □

4. Preparatory Lemmas

We start by proving the following lemma.

Lemma 6.5. *Suppose that $f : X \to \mathbb{R}$ satisfies the assumptions in subsection 2.1. Then Milnor data (ϵ_0, δ_0) for f at the origin can be chosen so that if $\delta \in (0, \delta_0]$ and $\epsilon \in (0, \epsilon_0]$ then*

$$f^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta$$

is contractible.

Before proving the lemma we shall need to invoke a result proved by A. H. Durfee in the article [**15**]. If $M \subset \mathbb{R}^n$ is a semialgebraic set and $Y \subset M$ is a compact semialgebraic subset recall [**15**] that a *semialgebraic rug function* of Y in M is a proper, semialgebraic function $\alpha : M \to \mathbb{R}$ such that

$$\forall x \in M \quad \alpha(x) \geq 0, \qquad \alpha^{-1}(0) = Y.$$

Recall furthermore that if α is a semialgebraic rug function for Y in M and if $\delta' > 0$ is small enough, then the inclusion $Y \subset \alpha^{-1}([0, \delta'])$ is a homotopy equivalence. Indeed, as remarked on page 525 of [**15**] this can be proved as [**15**, Proposition 1.6]. We shall now use this fact to prove the lemma.

Proof. Let $\delta \in (0, \delta_0]$ and consider

$$\alpha : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}, \qquad \alpha(x) = f(x)^2.$$

Then α is semialgebraic and since $X \cap \bar{\mathbb{B}}_\delta$ is compact α is proper. Furthermore,

$$\forall x \in X \cap \bar{\mathbb{B}}_\delta, \qquad \alpha(x) \geq 0,$$

$$\alpha^{-1}(0) = f^{-1}(0) \cap X \cap \bar{\mathbb{B}}_\delta$$

so α is a semialgebraic rug function for $f^{-1}(0) \cap X \cap \bar{\mathbb{B}}_\delta$ in $X \cap \bar{\mathbb{B}}_\delta$. If $y \in \mathbb{R}$ is such that $y \geq 0$ then

$$\alpha^{-1}([0, y]) = f^{-1}([-\sqrt{y}, \sqrt{y}]) \cap X \cap \bar{\mathbb{B}}_\delta.$$

By the proof of [**15**, Proposition 1.6] it follows that if ϵ_0 is sufficiently small then for all $\epsilon \in (0, \epsilon_0]$

$$f^{-1}(0) \cap X \cap \bar{\mathbb{B}}_\delta \subset f^{-1}([-\epsilon, \epsilon]) \cap X \cap \bar{\mathbb{B}}_\delta$$

is a homotopy equivalence. By the Local Conical Structure of algebraic sets,

$$f^{-1}(0) \cap X \cap \bar{\mathbb{B}}_\delta$$

is a cone over $f^{-1}(0) \cap X \cap \mathbb{S}_\delta$ hence is contractible. Therefore,

$$f^{-1}([-\epsilon, \epsilon]) \cap X \cap \bar{\mathbb{B}}_\delta$$

is contractible. Thus, up to replacing $\epsilon_0(\delta)$ by a smaller value $\epsilon'(\delta)$, one obtains Milnor data (δ_0, ϵ'_0) for which the statement of the lemma holds true. □

We now give a generalisation of Lemma 4.38 in the third chapter.

LEMMA 6.6. *Let $X \subset \mathbb{R}^n$ be a real algebraic set containing the origin in $\mathbb{R}^n$. Suppose that $f : X \to \mathbb{R}$ is the restriction of a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ satisfying the assumptions in subsection 2.1 and let (ϵ_0, δ_0) be Milnor data for f at the origin. Suppose that*

$$\tilde{F} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \quad l \in \mathbb{N}$$

is a polynomial map such that $\tilde{F}(\cdot, 0) = \tilde{f}$ and let $F = \tilde{F}_{|X \times \mathbb{R}^l}$. Then for any $\delta \in (0, \delta_0]$ and any $\eta \in (0, \epsilon(\delta)]$ there exists $t_0 = t_0(\eta) \in \mathbb{R}$ such that for any $t \in \mathbb{R}$ such that $|t| < t_0$ there exists a homeomorphism

$$\phi_t : f^{-1}(\eta) \cap X \cap \bar{\mathbb{B}}_\delta \to f_t^{-1}(\eta) \cap X \cap \bar{\mathbb{B}}_\delta.$$

PROOF. Fix $\delta \in (0, \delta_0]$. As the Whitney stratification $\tilde{\mathcal{S}}$ of X satisfies the Thom a_f-condition it follows that for any $\eta \in (0, \epsilon_0(\delta)]$ the sphere $\mathbb{S}_\delta$ intersects $f^{-1}(\eta) \cap \tilde{\mathscr{S}}_i, i \in I$ transversally at any $x \in f^{-1}(\eta)$. Fix $\eta \in (0, \epsilon_0]$. Then η is a regular value of the restriction

$$f : \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \to \mathbb{R}, \qquad \forall i \in I.$$

Consider

$$\tilde{G} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R} \times \mathbb{R}^l,$$
$$\tilde{G}(x, t) = (\tilde{F}(x, t), t)$$

and let

$$\pi : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l, \qquad \pi(x, t) = t$$

denote the standard projection onto the second factor. We will show that there exists $t_0 \in \mathbb{R}$ such that

$$\pi : \tilde{G}^{-1}(\{\eta\} \times (-t_0, t_0)^l) \cap X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l \to \mathbb{R}^l$$

is a proper stratified submersion and then use the First Isotopy Lemma 4.37 ([**35**, Proposition 11.1]) to obtain the result.

(1) For each $i \in I$ if $x \in f^{-1}(\eta)$ then

$$d\tilde{G}(x, 0) : T_x(\tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta) \times T_0\mathbb{R}^l \to T_{\tilde{G}(x,0)}(\mathbb{R} \times \mathbb{R}^l)$$

has full rank, because η is a regular value of $f_{|\tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta}$. It follows that there exists $t_0' \in \mathbb{R}$ such that $d\tilde{G}(x, t)$ has full rank for any $x \in f^{-1}(\eta)$ and any $t \in (-t_0', t_0')^l$. By continuity of the differential $d\tilde{G}$ this then holds for any $x \in f_t^{-1}(\eta)$.

(2) As a consequence if $i \in I$ then

$$\tilde{G}^{-1}(\{\eta\} \times (-t_0', t_0')^l) \cap \tilde{\mathscr{S}_i} \cap \mathbb{B}_\delta \times \mathbb{R}^l$$

is a smooth manifold.

(3) We will show that there exists $\tilde{t}_0 \in \mathbb{R}$ such that if $i \in I$ then

$$\tilde{G}^{-1}(\{\eta\} \times (-\tilde{t}_0, \tilde{t}_0)^l) \cap \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \times \mathbb{R}^l$$

is a smooth manifold. To this aim it suffices to show that

$$\tilde{G}^{-1}(\{\eta\} \times (-\tilde{t}_0, \tilde{t}_0)^l) \pitchfork \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \times (-\tilde{t}_0, \tilde{t}_0)^l \qquad (*)$$

is a transverse intersection. The set of points (x, t) where the intersection $(*)$ is not transverse is the set of (x, t) with $x \in X \cap \mathbb{S}_\delta$ and $t \in (-\tilde{t}_0, \tilde{t}_0)^l$ such that

$$T_{(x,t)} f^{-1}_{t|\tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta}(\eta) \perp (x, 0).$$

Notice that this is a closed condition and that

$$T_{(x,0)} f^{-1}_{|\tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta}(\eta) \not\perp (x, 0) \qquad \forall i \in I$$

because $f^{-1}(\eta)$ is transverse to $\tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta$ for any stratum $\tilde{\mathscr{S}_i}$ containing the origin in its closure. Therefore there exists a neighborhood U of the origin in $\mathbb{R}^l$ such that for any $t \in U$ and any $x \in X \cap \mathbb{S}_\delta$ the intersection $f_t^{-1}(\eta) \cap \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta$ is transverse for all $i \in I$. Hence there exists $\tilde{t}_0$ satisfying $(*)$.

(4) Letting $t_0 = \min(t_0', \tilde{t}_0)$ one gets a stratified map

$$\pi : \mathscr{W} = \tilde{G}^{-1}(\{\eta\} \times (-t_0, t_0)^l) \cap X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l \to \mathbb{R}^l$$

with compact fibers

$$\pi^{-1}_{|\mathscr{W}}(t) = f_t^{-1}(\eta) \cap X \cap \bar{\mathbb{B}}_\delta$$

hence $\pi_{|\mathscr{W}}$ is proper. We shall show that, up to replacing t_0 by a smaller value, it is is a submersion on each of the strata

$$\tilde{\mathscr{S}}_i' := \tilde{G}^{-1}(\{\eta\} \times (-t_0, t_0)^l) \cap \tilde{\mathscr{S}_i} \cap \mathbb{B}_\delta, \qquad \forall i \in I,$$

$$\tilde{\mathscr{S}}_i'' := \tilde{G}^{-1}(\{\eta\} \times (-t_0, t_0)^l) \cap \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \qquad \forall i \in I.$$

For this one argues as in the proof of the Lemma 4.38.

(a) Indeed suppose there exists $i \in I$ such that for any t_0 one has that

$$\pi_{|\tilde{\mathscr{S}}_i''} : \tilde{\mathscr{S}}_i'' \to \mathbb{R}^l$$

is not a submersion. This means that there exists a $x \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta$ such that for any t one has that

$$T_{(x,t)}\tilde{\mathscr{S}}_i'' \perp T_t\mathbb{R}^l \qquad (**).$$

which in turn implies that the differential $d\tilde{G}(x,t)$ is parallel to $T_t\mathbb{R}^l$. Since $(**)$ is a closed condition one can find a sequence (x_t, t) with $x_t \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta$ such that $d\tilde{G}(x_t,t)$ is parallel to $T_t\mathbb{R}^l$. Letting $|t| \to 0$ there would exist a limit point $y \in f^{-1}(\eta) \cap \bar{\tilde{\mathscr{S}}}_j \cap \mathbb{S}_\delta$ for some $j \in I$ such that $\tilde{\mathscr{S}}_i \subset \bar{\mathscr{S}}_j$. But then $d\tilde{G}(y,0)$ is parallel to $T_0\mathbb{R}$ so $df(y) = 0$. This contradicts the fact that the intersection $f^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta$ is transverse.

(b) In a similar way if

$$\pi_{|\tilde{\mathscr{S}}_i'} : \tilde{\mathscr{S}}_i' \to \mathbb{R}^l$$

were not a submersion then there would exist a $y \in f^{-1}(\eta) \cap \bar{\tilde{\mathscr{S}}}_j \cap \mathbb{S}_\delta$ such that $df(y) = 0$. Then $y \in f^{-1}(\eta)$ would be a critical point of $f : \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \to \mathbb{R}$ for some $i \in I$, contradiction.

(5) Therefore $\pi_{|\mathscr{X}}$ is a proper stratified submersion above $(-t_0,t_0)^l \subset \mathbb{R}^l$ so by the first Isotopy Lemma 4.37 ([**35**, Proposition 11.1]) it is the projection of a trivial topological fibration. In particular its fibers are homeomorphic which gives the homeomorphism of the statement.

□

The following lemma is an analogue of Lemma 4.44.

LEMMA 6.7. *Keep the assumptions of the Lemma 6.6. For any $\delta \in (0,\delta_0]$ there exists $\epsilon \in (0,\epsilon_0]$ such that for any $\eta \in (0,\epsilon]$ there exists $t_0' \in \mathbb{R}$ such that for any $t \in \mathbb{R}^l \cap \{|t| < t_0'\}$ the semialgebraic set*

$$f_t^{-1}([-\eta,\eta]) \cap X \cap \bar{\mathbb{B}}_\delta$$

is contractible.

PROOF. Keep the notation of the proof of Lemma 6.6. By the proof of Lemma 6.6 there exists $t_0 \in \mathbb{R}$ such that if for each $i \in I$ one defines

$$\mathscr{T}_{i1} = \tilde{G}^{-1}((-\eta,\eta) \times (-t_0,t_0)^l) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \times \mathbb{R}^l,$$

$$\mathscr{T}_{i2} = \tilde{G}^{-1}((-\eta,\eta) \times (-t_0,t_0)^l) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta \times \mathbb{R}^l,$$

$$\mathscr{T}_{i3}^{\pm} = \tilde{G}^{-1}(\{\pm\eta\} \times (-t_0,t_0)^l) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \times \mathbb{R}^l,$$

$$\mathscr{T}_{i4}^{\pm} = \tilde{G}^{-1}(\{\pm\eta\} \times (-t_0, t_0)^l) \cap \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \times \mathbb{R}^l$$

then $\mathscr{T}_{ij}^{\pm}$ are smooth manifolds for $j = 3, 4$. For each $i \in I$, $\dim \mathscr{T}_{i1} = \dim \tilde{\mathscr{S}_i} \cap \mathbb{B}_\delta \times \mathbb{R}^l$ and $\dim \mathscr{T}_{i2} = \dim \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \times \mathbb{R}^l$. Since

$$\mathscr{T}_{i1} \subset \tilde{\mathscr{S}_i} \cap \mathbb{B}_\delta \times \mathbb{R}^l$$

$$\mathscr{T}_{i2} \subset \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta \times \mathbb{R}^l$$

are open $\mathscr{T}_{i1}$ and $\mathscr{T}_{i1}$ are smooth manifolds. For each $i \in I$ put

$$\mathscr{T}_i = \{\mathscr{T}_{ij}, \mathscr{T}_{il}^{\pm}, \quad j = 1, 2, \quad l = 3, 4\}$$

and let $\mathcal{T} = (\mathscr{T}_i)_{i \in I}$. Then $\mathcal{T}$ is a stratification of

$$\mathscr{Y} := \tilde{G}^{-1}([-\eta, \eta] \times (-t_0, t_0)^l) \cap X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l.$$

Since $\tilde{\mathcal{S}}$ satisfies the a_f-condition $\tilde{f}_{|X \cap \bar{\mathbb{B}}_\delta} \pitchfork [-\eta, \eta]$ so, up to replacing t_0 by a smaller value,

$$\tilde{G}_{|X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l} \pitchfork \left([-\eta, \eta] \times (-t_0, t_0)^l\right).$$

But then

$$\tilde{G}^{-1}([-\eta, \eta] \times (-t_0, t_0)^l) \cap X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l$$

is Whitney stratified by the strata of $\mathcal{T}$ (see [**25**, § 1.3]). Consider the restriction

$$\pi_{|\mathscr{Y}} : \mathscr{Y} \to \mathbb{R}^l$$

of the standard projection $\pi : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l$ onto the second factor. We shall show that up to replacing t_0 by a smaller value one has that $\pi_{|\mathscr{Y}}$ is a proper stratified submersion. Its fibres are of the form

$$\pi_{|\mathscr{Y}}^{-1}(t) = f_t^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta$$

and if $K \subset \mathbb{R}^l$ is compact then

$$\pi_{|\mathscr{Y}}^{-1}(K) = \tilde{G}^{-1}([-\eta, \eta] \times K) \cap X \cap \bar{\mathbb{B}}_\delta \times \mathbb{R}^l$$

is compact so $\pi_{|\mathscr{Y}}$ is proper. For any $i \in I$ and any $j = 1, 2$ the maps

$$\pi_{|\mathscr{T}_{ij}} : \mathscr{T}_{ij} \to \mathbb{R}^l, \qquad j = 1, 2$$

are submersions. Indeed the strata $\mathscr{T}_{i1}$ and $\mathscr{T}_{i2}$ are smooth of dimension

$$\dim(\mathscr{T}_{i1}) = \dim(\mathscr{S}_i \cap \mathbb{B}_\delta) + l, \qquad \dim(\mathscr{T}_{i2}) = \dim(\mathscr{S}_i \cap \mathbb{S}_\delta) + l$$

respectively and the projections

$$\pi : \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \times \mathbb{R}^l \to \mathbb{R}^l,$$

$$\pi : \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta \times \mathbb{R}^l \to \mathbb{R}^l$$

are both submersions because each of the strata $\tilde{\mathscr{S}}_i$ intersects the ball $\mathbb{B}_\delta$ and the sphere $\mathbb{S}_\delta$ transversally, by the Whitney conditions. Hence these intersections are all smooth manifolds. It remains to show the claim for $i = 3, 4$. Suppose not. Then as in the proof of Lemma 6.6 there would exists $i \in I$ and a sequence (x_t, t) with $|t|$ tending to zero with

$$x_t \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta \cup f_t^{-1}(-\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta$$

in case $j = 3$ respectively

$$x_t \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta \cup f_t^{-1}(-\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta$$

in case $j = 4$, such that $d\tilde{G}(x, t)$ is parallel to $T_t\mathbb{R}^l$. Suppose that

$$x_t \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{B}_\delta$$

Then there would exists a limit $y \in f_t^{-1}(\eta) \cap \bar{\tilde{\mathscr{S}}}_i$ with $y \in \mathbb{B}_\delta$ such that $df(y) = 0$, for some $i \in I$. Suppose instead that

$$x_t \in f_t^{-1}(\eta) \cap \tilde{\mathscr{S}}_i \cap \mathbb{S}_\delta.$$

Then there would exists a limit $y \in f_t^{-1}(\eta) \cap \bar{\tilde{\mathscr{S}}}_i$ with $y \in \mathbb{S}_\delta$ such that $df(y) = 0$, for some $i \in I$. But this contradicts in both cases the fact that

$$f : f^{-1}(\mathbb{S}_{|\eta|}) \cap X \cap \bar{\mathbb{B}}_\delta \to \mathbb{S}_\eta$$

is a stratified submersion, by construction (see 2.1). Therefore $\pi_{|\mathscr{Y}}$ is a proper stratified submersion. By the First Isotopy Lemma 4.37 ([**35**, Proposition 11.1]) $\pi_{|\mathscr{Y}}$ is the projection of a trivial topological fibration. Therefore there exist for any $t \in (-t_0, t_0)^l$ a homeomorphism

$$\phi_t : \pi_{|\mathscr{Y}}^{-1}(t) \to \pi_{|\mathscr{Y}}^{-1}(0),$$

which is to say that

$$f_t^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta \cong f^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta.$$

Since by Lemma 6.5 the space to the right is contractible the result follows. $\square$

5. On the Homotopy Type of the Milnor Fibres

5.1. Morsifiable Maps. We shall now apply Stratified Morse Theory ([**25**]) to obtain a contractible space by adjoining cells to the real Milnor fibres. In the statement of the main theorem 6.13 one shall need to assume the existence of a stratified $\mathbb{R}$-morsification wherefore we introduce the following definition.

Definition 6.8. *Let $X \subset \mathbb{R}^n$ be a real algebraic set containing the origin. The restriction $f : X \to \mathbb{R}$ of a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ is called* morsifiable *if there exists a Whitney stratification $\tilde{\mathcal{S}}$ of X satisfying the Thom a_f-condition such that $f^{-1}(0) \subset X$ is a union of strata, and if there exists a stratified $\mathbb{R}$-morsification*

$$F : X \times \bar{U} \subset \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \qquad f_t(x) = F(x,t)$$

of $f : X \to \mathbb{R}$.

5.2. The Main Constructions. In the remainder of this section $f : X \to \mathbb{R}$ is morsifiable. The corresponding stratification is denoted by $\tilde{\mathscr{S}} = (\tilde{\mathscr{S}_i}), i \in I$ and the corresponding stratified $\mathbb{R}$-morsification of f is

$$F : X \times \bar{U} \subset \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \qquad f_t(x) = F(x,t).$$

Fix $\delta \in (0, \delta_0]$ and $\eta \in (0, \epsilon(\delta)]$. We now show that for an open dense subset of parameters the restriction of f_t to $X \cap \bar{\mathbb{B}}_\delta$ is a stratified Morse function

Lemma 6.9. *There exists an open dense set $V \subset U$ such that for each $t \in V$,*

$$f_t : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

is a stratified Morse function.

Proof. Since $\tilde{\mathcal{S}} = (\tilde{\mathscr{S}_i})_{i \in I}$ satisfies the Whitney conditions if $\mathcal{S} = (\mathscr{S}_i)_{i \in}$ is the stratification of $X \cap \bar{\mathbb{B}}_\delta \subset \mathbb{R}^n$ having strata

$$\mathscr{S}_i' = \tilde{\mathscr{S}_i} \cap \mathbb{B}_\delta, \quad \mathscr{S}_i'' = \tilde{\mathscr{S}_i} \cap \mathbb{S}_\delta, \qquad i \in I$$

then $\mathcal{S}$ is a Whitney stratification. Since $X \cap \bar{\mathbb{B}}_\delta$ is compact, by Proposition 6.4 it follows that there exist a dense subset $V \subset U$ such that

$$\forall t \in V, \qquad f_t : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

is a stratified Morse function with respect to $\mathcal{S}$. $\square$

DEFINITION 6.10. *If $t \in V$ put*

$$\mathcal{C}(F,t) = \{p(t) \in X \cap \bar{\mathbb{B}}_\delta \mid p(t) \text{ is a stratified critical point of } f_t : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}\}$$

As critical points of stratified Morse functions are isolated and since the critical values are pairwise distinct by [**25**, Definition 2.1], one can write

$$\mathcal{C}(F,t) = \{p_1(t), \dots, p_m(t)\}, \qquad f_t(\mathcal{C}(F,t)) = \{s_1(t), \dots, s_m(t)\},$$
$$m(t) = |\mathcal{C}(F,t)|.$$

REMARK 6.11. *Since $X \cap \bar{\mathbb{B}}_\delta$ is compact for any $i = 1, \dots, k$ the critical values of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ are bounded. By continuity of the differentials of the map*

$$(f_t, t) : X \cap \bar{\mathbb{B}}_\delta \times \bar{U} \to \mathbb{R} \times \bar{U},$$

$$(x,t) \mapsto (f_t(x), t)$$

the critical values of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ tend to the critical values of $f_{|X\cap\mathbb{B}_\delta}$ when $|t| \to 0$. But the origin is the only critical value of the latter by Proposition 6.1 so one can therefore assume that if $t_0 = min(t_0, t_0')$ in the notation of the Lemmas 6.6 and 6.7 then the critical values of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ lie in $(-\eta, \eta)$ for all $t \in V$ such that $|t| < t_0$.

Before stating the main theorem we shall introduce some further notation.

- If $p_j = p_j(t) \in \mathcal{C}(F,t)$ and p_j belongs to a stratum $\mathscr{S} \in \mathcal{S}$ denote by $\lambda_{\mathscr{S}}(p_j)$ the Morse index of $f_{t|\mathscr{S}_l}$ at p_j.
- If $p_j \in \mathcal{C}(F,t)$ belongs to a stratum $\mathscr{S} \in \mathcal{S}$ then because $\mathcal{S}$ satisfies the Whitney conditions there exists $\rho_j > 0$ such that if $\bar{\mathbb{B}}_{\rho_j}(p_j) \subset \mathbb{R}^n$ is the closed ball centered at p_j and of radius ρ_j then $\partial\bar{\mathbb{B}}_{\rho_j}(p)$ intersects the strata of $\mathcal{S}$ transversally, see [**25**, 3.5]. Let

  $$l^+(p_j) = N(p_j) \cap \mathbb{B}_{\rho_j} \cap f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(s_j + \gamma_j),$$

  $$l^-(p_j) = N(p_j) \cap \mathbb{B}_{\rho_j} \cap f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(s_j - \gamma_j)$$

  for $0 < \gamma_j << \rho_j$ denote the upper and lower half-links [**25**, Definition 3.9.1] of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ at the point p_j. Let $\dim(\mathscr{S})(p_j)$ denote the dimension of the stratum containing p_j.

REMARK 6.12. *Remark that*

$$l^{\pm}(p_j) = l^{\pm}(p_j) \cap f_t^{-1}(-\eta, \eta)$$

by definition. In other words

$$l^{\pm}(p_j) = l^{\pm}_{X\cap\bar{\mathbb{B}}_\delta\cap f_t^{-1}(-\eta,\eta)}(p_j).$$

In particular the upper (lower) half-links of the proper function $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ are the upper (lower) half-links of the restriction of f_t to $X\cap\bar{\mathbb{B}}_\delta\cap f_t^{-1}(-\eta,\eta)$.

5.3. The Main Theorem. The main result of this chapter is the following theorem.

THEOREM 6.13. *Suppose that $X\subset\mathbb{R}^n$ is a real algebraic set and that $f: X\to\mathbb{R}$ is a morsifiable map (see Definition 6.8) having Milnor data (ϵ_0,δ_0) at the origin. Suppose that*

$$F: X\times\bar{U}\subset\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R},\quad f_t=F(\cdot,t)$$

is a stratified $\mathbb{R}$-morsification of f (see Definition 6.3) with V as in Lemma 6.9. For any $\delta\in(0,\delta_0]$ there exists $\epsilon\in(0,\epsilon_0]$ such that for any $\eta\in(0,\epsilon(\delta)]$ there exists $t_0(\eta)\in\mathbb{R}$ such that if $t\in V\cap\{|t|\le t_0\}$ then there exist semialgebraic sets $\mathcal{N}(\bar{\mathcal{F}}^+_\eta)$ and $\mathcal{N}(\bar{\mathcal{F}}^-_\eta)$ having the homotopy type of $\bar{\mathcal{F}}^+_\eta$ and $\bar{\mathcal{F}}^-_\eta$ respectively, and attaching maps

$$h_j^+ : \partial\left(\mathbb{D}^{\dim\mathscr{S}(p_j)-\lambda_{\mathscr{S}}(p_j)}\times cone(l^+(p_j))\right)\to$$

$$\begin{cases}\mathcal{N}(\bar{\mathcal{F}}^+_\eta)\cup\bigcup_{l=1}^{j-1}\mathbb{D}^{\dim\mathscr{S}(p_j)-\lambda_{\mathscr{S}}(p_j)}\times cone(l^+(p_j)), & 2\le j\le m\\ \mathcal{N}(\bar{\mathcal{F}}^+_\eta), & j=1\end{cases}$$

$$h_j^- : \partial\left(\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)}\times cone(l^-(p_j))\right)\to$$

$$\begin{cases}\mathcal{N}(\bar{\mathcal{F}}^-_\eta)\cup\bigcup_{l=1}^{j-1}\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)}\times cone(l^-(p_j)), & 2\le j\le m\\ \mathcal{N}(\bar{\mathcal{F}}^-_\eta), & j=1\end{cases}$$

with $p_j\in\mathcal{C}(F,t)$ and $m=|\mathcal{C}(F,t)|$ as in Definition 6.10, such that

$$\mathcal{N}(\bar{\mathcal{F}}^+_\eta)\cup\bigcup_{\substack{1\le i\le m\\ h_i^+}}\left(\mathbb{D}^{\dim(\mathscr{S})-\lambda_{\mathscr{S}}(p_j)}\times cone(l^+(p_j))\right),$$

$$\mathcal{N}(\bar{\mathcal{F}}^-_\eta)\cup\bigcup_{\substack{1\le i\le m\\ h_i^-}}\left(\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)}\times cone(l^-(p_j))\right)$$

are contractible.

PROOF. Fix $\delta\in(0,\delta_0]$ and $\eta\in(0,\epsilon(\delta)]$ and let $t_0\in\mathbb{R}$ be as in Lemma 6.6 and Lemma 6.7. Using Lemma 6.9 one can find a dense open set $V\subset U$ such that if $t\in V$ then $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ is a stratified Morse function. Fix $t\in V\cap\{|t|\le t_0\}$. By

Remark 6.11 one can order the critical values as

$$-\eta < s_1 < \cdots < s_m < \eta$$

Since $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ is a stratified Morse function one can choose for each $j = 1, \dots, m$ a $\gamma_j > 0$ such that $[s_j - \gamma_j, s_j + \gamma_j]$ contains no other critical value except s_j. By [**25**, Remark 3.4] $\gamma_j > 0$ can be chosen so that

$$s_j + \gamma_j = s_{j+1} - \gamma_{j+1}, \qquad j = 1, \dots, m-1$$

because the topological type of

$$f^{-1}([s_j - \gamma_j, s_j + \gamma_j]) \cap X \cap \bar{\mathbb{B}}_\delta, \qquad j = 1, \dots, m$$

is independent of the choice of $\gamma_j > 0$ as long as $[s_j - \gamma_j, s_j + \gamma_j]$ contains no critical value except s_j. Let us consider the following semialgebraic sets

$$\bar{\mathcal{X}}_{t,\eta} = f_t^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta,$$

$$\mathcal{X}_{t,\eta} = f_t^{-1}(-\eta, \eta) \cap X \cap \bar{\mathbb{B}}_\delta,$$
$$\mathcal{X}'_{t,\eta} = f_t^{-1}(-\eta, \eta) \cap X \cap \mathbb{B}_\delta,$$
$$\mathcal{X}''_{t,\eta} = f_t^{-1}(-\eta, \eta) \cap X \cap \mathbb{S}_\delta.$$

Notice that $\bar{\mathcal{X}}_{t,\eta}$ is Whitney stratified by strata

$$f_t^{-1}(-\eta, \eta) \cap \mathscr{S}, \quad f_t^{-1}(\pm\eta) \cap \mathscr{S}, \qquad \mathscr{S} \in \mathcal{S}$$

and that if $\mathscr{S} \in \mathcal{S}$ then $\dim f_t^{-1}(-\eta, \eta) \cap \mathscr{S} = \dim \mathscr{S}$. We shall apply stratified Morse theory for the proper map $f_t : \bar{\mathcal{X}}_{t,\eta} \to \mathbb{R}$ and the subspace $\mathcal{X}_{t,\eta}$. We claim that f_t restricted to $\mathcal{X}_{t,\eta}$ has only nondepraved critical points. However since $f_{t|X\cap\mathbb{B}_\delta}$ is Morse, its critical points are isolated so if there was a critical point $p \in f_t^{-1}(-\eta, \eta) \cap X \cap \bar{\mathbb{B}}_\delta$ then this point must be isolated, as well. But then, by [**25**, Proposition 2.4] p is necessarily nondepraved.

(1) By [**25**, Theorem 10.8] there exists an attaching map

$$h_m^- : \partial(\mathbb{D}^{\lambda_{\mathscr{S}}(p_m)} \times \text{cone } l^-(p_m)) \to f_t^{-1}(-\eta, s_m - \gamma_m) \cap X \cap \bar{\mathbb{B}}_\delta$$

and a homotopy equivalence

$$f_t^{-1}(-\eta, s_m + \gamma_m) \cap X \cap \bar{\mathbb{B}}_\delta \sim$$
$$\big(f_t^{-1}(-\eta, s_m - \gamma_m]) \cap X \cap \bar{\mathbb{B}}_\delta\big) \cup_{h_m^-} (\mathbb{D}^{\lambda_{\mathscr{S}}(p_m)} \times \text{cone } l^-(p_m)).$$

In a similar way, using [**25**, Theorem 10.8] there exists an attaching map

$$h^-_{m-1} : \partial(\mathbb{D}^{\lambda_{\mathscr{S}}(p_{m-1})} \times \text{cone } l^-(p_{m-1})) \to f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}) \cap X \cap \bar{\mathbb{B}}_\delta$$

and a homotopy equivalence

$$f_t^{-1}(-\eta, s_m - \gamma_m) \cap X \cap \bar{\mathbb{B}}_\delta = f_t^{-1}(-\eta, s_{m-1} + \gamma_{m-1}) \cap X \cap \bar{\mathbb{B}}_\delta \sim$$

$$\big(f_t^{-1}(-\eta, s_{m-1} - \gamma_{m-1}]) \cap X \cap \bar{\mathbb{B}}_\delta\big) \cup_{h^-_{m-1}} (\mathbb{D}^{\lambda_{\mathscr{S}}(p_{m-1})} \times \text{cone } l^-(p_{m-1})).$$

Continuing in this way one obtains that $f_t^{-1}(-\eta, s_m + \gamma_m)$ has the homotopy type of

$$\mathcal{N}(\bar{\mathcal{F}}^-_\eta) \cup \bigcup_{\substack{1 \le i \le m \\ h_i^-}} \Big(\mathbb{D}^{\lambda_{\mathscr{S}}(p_j)} \times \text{cône}(l^-(p_j))\Big)$$

where

$$\mathcal{N}(\bar{\mathcal{F}}^-_\eta) := f_t^{-1}(-\eta, s_1 - \gamma_1) \cap X \cap \bar{\mathbb{B}}_\delta.$$

(2) The same argument as in the last part of the proof of Theorem 4.47, using the First Isotopy Lemma 4.37 ([**35**, Proposition 11.1]), gives that there exists[1] a homeomorphism

$$\mathcal{N}(\bar{\mathcal{F}}^-_\eta) \cong f_t^{-1}(-\eta) \cap X \cap \bar{\mathbb{B}}_\delta \times (-\eta - \gamma_1, s_1 - \gamma_1).$$

(3) By Lemma 6.7

$$f_t^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta \cong f^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta$$

by a stratum-preserving homeomorphism so

$$\mathcal{X}_{t,\eta} \cong f^{-1}(-\eta, \eta) \cap X \cap \bar{\mathbb{B}}_\delta.$$

We have that $f^{-1}([-\eta, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta$ is contractible and we claim that $f^{-1}(-\eta, \eta) \cap X \cap \bar{\mathbb{B}}_\delta$ is contractible as well. For this we consider the trivial fibrations

$$f : f^{-1}((0, \eta]) \cap X \cap \bar{\mathbb{B}}_\delta \to (0, \eta] \tag{9}$$

$$f : f^{-1}[(-\eta, 0)) \cap X \cap \bar{\mathbb{B}}_\delta \to (-\eta, 0] \tag{10}$$

By the Tubular Neighborhood Theorem (see e.g [**26**, Proposition 3.42]) there exists a homotopy equivalence h^+ between $(0, \eta]$ and $(0, \eta)$ such that

$$h^+_{(0,\tilde{\rho}]} = \mathrm{Id}_{(0,\tilde{\rho}]}, \qquad 0 < \tilde{\rho} << \eta.$$

[1] This also follows from the fact that the coarse Morse data [**25**, Definition 3.4] of f_t at p_1 is independent of the choice of $\gamma_1 > 0$, by [**25**, Remark 3.4]

By the homotopy lifting property there exists a lift of h^+ by (9) to a homotopy equivalence

$$H^+ : f^{-1}((0,\eta]) \cap X \cap \bar{\mathbb{B}}_\delta \to f^{-1}(0,\eta) \cap X \cap \bar{\mathbb{B}}_\delta$$

which is the identity on $f^{-1}((0,\tilde{\rho}]) \cap X \cap \bar{\mathbb{B}}_\delta$. In a similar way using (10) there exists a homotopy equivalence

$$H^- : f^{-1}([-\eta,0)) \cap X \cap \bar{\mathbb{B}}_\delta \to f^{-1}((-\eta,0)) \cap X \cap \bar{\mathbb{B}}_\delta$$

which is the identity on $f^{-1}((0,\tilde{\rho}]) \cap X \cap \bar{\mathbb{B}}_\delta$. Therefore

$$f^{-1}([-\eta,\eta]) \cap X \cap \bar{\mathbb{B}}_\delta \sim f^{-1}(-\eta,\eta) \cap X \cap \bar{\mathbb{B}}_\delta.$$

As a consequence $\mathcal{X}_{t,\eta}$ is contractible.

(4) Let

$$\mathcal{W} = \{(x,r) \in \mathbb{R} \times [0,1] \mid x \in [-\eta,(1-r)\eta + r(s_m+\gamma_m)]\} \subset \mathbb{R} \times \mathbb{R}$$

and let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the standard projection $\pi(x,r) = r$. For each $r \in [0,1]$ let

$$\mathcal{W}_r = \pi^{-1}(r) \cap \mathcal{W}.$$

Since by construction $[s_m+\gamma_m,\eta]$ contains no critical values of $f_{t|X\cap\bar{\mathbb{B}}_\delta}$ there exists by [**25**, Theorem 4.3] (see also [**25**, Remark 7.2]) a stratum-preserving homeomorphism

$$\bar{\mathcal{X}}_{t,\eta} = f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(\mathcal{W}_0) \cong f^{-1}_{t|X\cap\bar{\mathbb{B}}_\delta}(\mathcal{W}_1) = f_t^{-1}([-\eta,s_m+\gamma_m]) \cap X \cap \bar{\mathbb{B}}_\delta$$

hence

$$\mathcal{X}_{t,\eta} \cong f_t^{-1}(-\eta,s_m+\gamma_m) \cap X \cap \bar{\mathbb{B}}_\delta$$

Since

$$\bar{\mathcal{F}}^-_\eta \cong f_t^{-1}(-\eta) \cap X \cap \bar{\mathbb{B}}_\delta$$

by Lemma 6.6 the result for the negative Milnor fibre follows. The result for the positive Milnor fibre is obtained by considering $-f_t$ instead of f_t. □

6. Consequences

6.1. The Euler Characteristics of The Milnor Fibres. We shall now draw several consequences from Theorem 6.13. To begin with we can obtain the

Euler-Poincaré characteristic of the Milnor fibres. We denote by

$$\tilde{\chi}(\cdot) = \sum_{j \geq 0} (-1)^j \dim \tilde{H}_j(\cdot, R)$$

the Euler-Poincaré characteristic in reduced singular homology with coefficients in a given principal ideal domain R.

Corollary 6.14. *Keep the assumptions of the Theorem 6.13 and suppose furthermore that X is equidimensional. The Euler-Poincaré characteristic of the negative Milnor fibre with $\mathbb{Q}$-coefficients is given by*

$$\chi(\bar{\mathcal{F}_\eta}^-) = 1 + \sum_{\substack{1 \leq j \leq m \\ \dim(\mathscr{S}(p_j)) = \dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j) - 1} + \sum_{\substack{1 \leq j \leq m \\ \dim(\mathscr{S}((p_j)) < \dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j)} \tilde{\chi}(l^-(p_j))))$$

and the Euler-Poincaré characteristic with $\mathbb{Q}$-coefficients of the positive Milnor fibre is given by

$$\chi(\bar{\mathcal{F}}_\eta^+) = 1 + \sum_{\substack{1 \leq j \leq m \\ \dim(\mathscr{S})(p_j)) = \dim(X)}} (-1)^{\dim(\mathscr{S})(p_j) + 1 - \lambda_{\mathscr{S}}(p_j)}$$

$$+ \sum_{\substack{1 \leq j \leq m \\ \dim(\mathscr{S})(p_j)) < \dim(X)}} (-1)^{\dim(\mathscr{S})(p_j) - \lambda_{\mathscr{S}}(p_j)} \tilde{\chi}(l^+(p_j)))).$$

Moreover if $\dim(\mathscr{S}(p)) = \dim X$ for any $p \in \mathcal{C}(F, t)$ then these relations hold for the Euler-Poincaré characteristics with $\mathbb{Z}$-coefficients.

Proof. We shall prove the first formula in the statement; the second formula is analoguous. By e.g [**39**, I. § 5] if $A \subset X$ is a subspace of a topological space X and $R = \mathbb{Q}, \mathbb{Z}$ then the Euler characteristic

$$\chi(X, A) = \sum_{n \geq 0} (-1)^n \text{rank } H_n(X, A; R)$$

is additive; namely if $A \subset B \subset X$ then

$$\chi(X, A) = \chi(X, B) + \chi(B, A). \tag{11}$$

Let $\mathcal{N}(\bar{\mathcal{F}}_\eta^-) \subset \mathcal{X}_{t,\eta}$ be as in the statement of the Theorem 6.13 and let us write $\mathcal{N} := \mathcal{N}(\bar{\mathcal{F}}_\eta^-)$ to simplify the notation. Applying (11) to the filtration

$$\emptyset \subset \mathcal{N} \subset \mathcal{X}_{t,\eta}$$

gives

$$\chi(\bar{\mathcal{F}}_\eta^-) = \chi(\mathcal{N}, \emptyset) = \chi(\mathcal{X}_{t,\eta}, \emptyset) - \chi(\mathcal{X}_{t,\eta}, \mathcal{N})$$

$$= 1 - \chi(\mathcal{X}_{t,\eta}, \mathcal{N}).$$

On the other hand there exists by Theorem 6.13 a filtration

$$\mathcal{N} \subset \cdots \subset B_p \subset A_p \subset \cdots \subset X_{t,\eta} \tag{12}$$

where p goes through the set $\mathcal{C}(F,t)$ and where each consecutive term is obtained from the previous as

$$A_p = B_p \cup_{h_p} (\mathbb{D}^{\lambda_{\mathscr{S}}(p)} \times \text{cone } l^-(p))$$

for an attaching map

$$h_p : \partial(\mathbb{D}^{\lambda_{\mathscr{S}}(p)} \times \text{cone } l^-(p)) \to B_p.$$

Recall that $m = |\mathcal{C}(F,t)|$. Applying (11) to the filtration (12) one obtains

$$\chi(\mathcal{X}_{t,\eta}, \mathcal{N}) = \sum_{j=1}^{m} \chi(A_p, B_p)$$

$$= \sum_{j=1}^{m} \sum_{n \geq 0} (-1)^n \text{rank } H_n(A_p, B_p; R)$$

so that

$$\chi(\bar{\mathcal{F}}_\eta^-) = 1 - \sum_{j=1}^{m} \sum_{n \geq 0} (-1)^n \text{rank } H_n(A_p, B_p; R) \tag{13}$$

and by the Excision Theorem [**26**, Theorem 2.20] applied to

$$B_p \setminus h_p \left(\partial(\mathbb{D}^{\lambda_{\mathscr{S}}(p)} \times \text{cone } l^-(p)) \right)$$

one obtains

$$H_n(A_p, B_p) = H_n \left(\mathbb{D}^{\lambda_{\mathscr{S}}(p)} \times \text{cone}(l^-(p), \quad \partial(\mathbb{D}^{\lambda_{\mathscr{S}}(p)} \times \hat{\text{cone}}(l^-(p)) \right) \tag{14}$$

where the homology groups have R-coefficients. For any $i = 1, \ldots, k$ and any $j = 1, \ldots, m_i$ if $p = p_{ji} \in \mathcal{C}_i(F,t)$ is such that $\dim(\mathscr{S}(p_{ji}))$ is maximal, hence equal to $\dim X$, then the normal slice is $N(p) = p$ by [**25**, p.8 § 1.2]. Then $l^-(p) = \emptyset$ by Definition 4.14 [**25**, Definition 3.9.1] so the only nontrivial homology group is

$$H_n(A_p, B_p) = H_n \left(\mathbb{D}^{\lambda_{\mathscr{S}}(p)}, \quad \partial \mathbb{D}^{\lambda_{\mathscr{S}}(p)} \right) = \mathbb{Z}, \qquad n = \lambda_{\mathscr{S}}(p).$$

The equations (14) and (13) then give

$$\begin{aligned}\chi(\bar{\mathcal{F}}_\eta^-) &= 1 - \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S}(p_j))=\dim X}} (-1)^{\lambda_{\mathscr{S}}(p_j)}\\ &= 1 + \sum_{\substack{1\leq j\leq m\\ \dim(\mathscr{S}(p_j))=\dim X}} (-1)^{\lambda_{\mathscr{S}}(p_j)-1}\end{aligned}$$

where we can take the coefficients to be in $R = \mathbb{Z}$. So suppose that not all critical points lie in maximal dimensional strata and let us take $R = \mathbb{Q}$. In this case one can apply the Künneth Theorem for relative homology [**26**, Theorem 3.18]. This yields an isomorphism

$$H_n(A_p, B_p) \cong \bigoplus_{l\geq 0} H_l(\mathbb{D}^{\lambda_{\mathscr{S}}(p)}, \partial\mathbb{D}^{\lambda_{\mathscr{S}}(p)}) \otimes H_{n-l}(\text{cone } l^-(p), l^-(p))$$

which is trivial unless $l = \lambda_{\mathscr{S}}(p)$, because the relative homology of the pair $(\mathbb{D}^{\lambda_{\mathscr{S}}(p)}, \partial\mathbb{D}^{\lambda_{\mathscr{S}}(p)})$ is trivial in all degrees except $l = \lambda_{\mathscr{S}}(p)$, in which case one obtains

$$H_n(A_p, B_p) \cong H_{n-\lambda_{\mathscr{S}}(p)}(\text{cone } l^-(p), l^-(p)) \tag{15}$$

hence

$$\chi(\bar{\mathcal{F}}_\eta) = 1 - \sum_{j=1}^{m}\sum_{n\geq 0}(-1)^n \text{rank } H_{n-\lambda_{\mathscr{S}}(p)}(\text{cone } l^-(p), l^-(p)).$$

One must distinguish between two cases.

(1) If p does not belong to a stratum of maximal dimension then the normal slice is of positive dimension so $l^-(p)$ is nonempty and then the long exact sequence of the pair $(\text{cone}(l^-(p)), l^-(p))$ gives

$$\begin{aligned}H_{n-\lambda_{\mathscr{S}}(p)}(\text{cone } l^-(p), l^-(p)) &= \tilde{H}_{n-\lambda_{\mathscr{S}}(p)-1}(l^-(p))\\ &= \tilde{H}_n(S^{\lambda_{\mathscr{S}}(p)+1}(l^-)(p))\end{aligned}$$

where $S^{\lambda_{\mathscr{S}}(p)+1}(l^{-1}(p))$ denotes the $(\lambda_{\mathscr{S}}(p)+1)$-th iterated (unreduced) suspension [**26**] of the lower half-link. In particular the suspension of the lower half-link satisfies

$$\forall N \in \mathbb{N}, \qquad \tilde{\chi}(S^N(l^-)(p)) = (-1)^N \tilde{\chi}(l^-(p)).$$

(2) If p belong to a maximal dimensional stratum then $l^-(p) = \emptyset$. The cone over the empty set is a point (see the convention 4.14) so

$$H_{n-\lambda_{\mathscr{S}}(p)}(\text{cone } l^-(p), l^-(p)) = H_{n-\lambda_{\mathscr{S}}(p)}(pt) = \tilde{H}_n(\mathbb{S}^{\lambda_{\mathscr{S}}(p)}).$$

Combining 1 and 2 and applying equations (15) and (13) then give the result. □

6.2. On the Link of Algebraic Subsets. In this subsection illustrate one potential use of the Theorem 6.13 and in particular the general Khimshiashvili formula established in 6.14, to prove a result on the generic value of the Euler characteristic of links of irreducible real algebraic subsets.

For convenience we recall, following the article [**36**] of C. McCrory and A. Parusiński, that the *link* of a compact real algebraic subset Y of a real algebraic set $X \subset \mathbb{R}^N$ in a nonsingular point $x \in Y$ is defined as

$$X \cap N_x \cap \mathbb{S}_\epsilon^N(x)$$

where N_x is the normal tangent space of $Y \subset \mathbb{R}^N$ at x and $\mathbb{S}^N(x) \subset \mathbb{R}^N$ is a sphere of radius $\epsilon > 0$ centered at x. The link is thus undefined in singular points of Y which motivates the definition [**36**, § 1.3] of the *localised link*. As Y is per assumption compact there exists a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ restricting to a proper, nonnegative map $f : X \to \mathbb{R}$ such that $f^{-1}(0) = Y$. Then the localised link of Y in X at $x \in X$ is defined as

$$\tilde{lk}_x(Y; X) = f^{-1}(\epsilon) \cap \mathbb{B}_\delta(x), \qquad 0 < \epsilon << \delta.$$

Stated in other terms it is the positive Milnor fibre of $f : X \to \mathbb{R}$ at x. We can state the following theorem:

THEOREM 6.15. *Let $X \subset \mathbb{R}^N$ be an equidimensional real algebraic set endowed with a Whitney stratification $\mathcal{S} = (\mathscr{S}_i)_{i\in I}$ and let $Y \subset X$ be an irreducible real algebraic subset. Suppose that there exists a polynomial map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{f}$ restricts to a morsifiable proper, nonnegative map $f : X \to \mathbb{R}$ with $f^{-1}(0) = Y$. Then there exists a real algebraic subset $Z \subset Y$ with $\dim Z < \dim Y$ such that for any $x \in Y \setminus Z$ there exists a stratified Morse function*

$$g = g_x : X \cap \bar{\mathbb{B}}_\delta \to \mathbb{R}$$

such that

$$\chi(\tilde{lk}_x(Y; X)) = 2 \sum_{\substack{1 \le j \le m \\ \dim(\mathscr{S}(p_j)) = \dim(X), \\ \dim(\mathscr{S}) \equiv 1}} (-1)^{\lambda_{\mathscr{S}}(p_j)}$$

$$+ \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))<\dim(X)\\ \dim(\mathscr{S})\equiv 0}} (-1)^{\lambda_{\mathscr{S}}(p_j)}[\tilde{\chi}(l^+(p_j)) - \tilde{\chi}(l^-)(p_j)]$$

$$- \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))<\dim(X)\\ \dim(\mathscr{S})\equiv 1}} (-1)^{\lambda_{\mathscr{S}}(p_j)}[\tilde{\chi}(l^+(p_j)) + \tilde{\chi}(l^-)(p_j)]$$

where $p_j, j = 1, \ldots, m$ are the critical points of g and where $\lambda_{\mathscr{S}}(p_j)$ is the Morse index of p_j at the stratum $\mathscr{S} \in \mathcal{S}$.

Proof. Let $Z = \mathrm{Sing}(Y)$ denote the singular locus of Y. Then $\dim Z < \dim Y$ and since Y is irreducible the complement $Y \setminus Z$ is a dense open set for the Zariski topology. For any $x \in Y \setminus Z$ the localised link at x is the positive Milnor fibre

$$\tilde{lk}_x(Y;X) = f^{-1}(\epsilon) \cap \mathbb{B}_\delta(x), \qquad 0 < \epsilon << \delta$$

of $f : X \to \mathbb{R}$ and since f is nonnegative,

$$f^{-1}(-\epsilon) \cap \mathbb{B}_\delta(x) = \emptyset.$$

Since f is morsifiable it follows that there exists a stratified $\mathbb{R}$-morsification $F : Y \cap \bar{B}_\delta \times \bar{\mathcal{U}} \to \mathbb{R}$. Let $t \in V$ so that $g := f_t$ is a stratified Morse function and let its critical points be denoted $p_1, \ldots, p_m$. Remark that

$$\forall k \in \mathbb{N}, \qquad \tilde{\chi}(S^k(l^-(p))) = (-1)^k \tilde{\chi}(l^-(p))$$

where $S^k(l^-(p))$ denotes the k-th (unreduced) suspension [**26**] of $l^-(p)$, so by Corollary 6.14,

$$\chi(f^{-1}(\epsilon) \cap \mathbb{B}_\delta(x)) = 1 + \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))=\dim(X)}} (-1)^{\dim(\mathscr{S})(p_j)-\lambda_{\mathscr{S}}(p_j)-1}$$

$$+ \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))<\dim(X)}} (-1)^{\dim(\mathscr{S})(p_j)-\lambda_{\mathscr{S}}(p_j)}\tilde{\chi}(l^+(p_j))$$

and

$$\chi(f^{-1}(-\epsilon) \cap \mathbb{B}_\delta(x)) = 1 + \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))=\dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j)-1}$$

$$+ \sum_{\substack{1\le j\le m\\ \dim(\mathscr{S}(p_j))<\dim(X)}} (-1)^{\lambda_{\mathscr{S}}(p_j)}\tilde{\chi}(l^-(p_j)).$$

Substracting these quantities yields the result. $\square$

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