Singularities of linear families of symmetric matrices

Mohadeseh Vakili

UNIVERSITÉ SAVOIE MONT BLANC

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ SAVOIE MONT BLANC

Spécialité : **Mathématiques**

Arrêté ministériel : 25 Mai 2016

Présentée par

« Mohadeseh / VAKILI»

Thèse dirigée par **« Krzysztof / KURDYKA»** et
Codirigée par **« Olivier / Le Gal »**

Préparée au sein du **Laboratoire de Mathématiques (LAMA)**
Dans **l'École Doctorale MSTII**

Singularités de familles linéaires de matrices symétriques

Thèse soutenue publiquement le **« 17/12/2020 »**,
devant le jury composé de :

M. Krzysztof KURDYKA
Professeur, LAMA, Université Savoie Mont Blanc, Directeur de thèse
M. Laurentiu PAUNESCU
Professeur, School of Mathematics and Statistics, The University of Sydney, Rapporteur
M. Antonio LERARIO
Associate professor, SISSA, International School for Advanced Studies, Rapporteur
M. Georges COMTE
Professeur, LAMA, Université Savoie Mont Blanc, Examinateur
Mme Camille PLÉNAT
Maître de conférences, Institut de Mathématiques de Marseille, Aix-Marseille Université, Examinatrice
M. Olivier LE GAL
Maître de conférences, LAMA, université savoie mont blanc, Invité


UNIVERSITÉ
SAVOIE
MONT BLANC

«Il faut être toujours ivre. Tout est là : c'est l'unique question. Pour ne pas sentir l'horrible fardeau du Temps qui brise vos épaules et vous penche vers la terre, il faut vous enivrer sans trêve.

Mais de quoi ? De vin, de poésie ou de vertu, à votre guise. Mais enivrez-vous.

Et si quelquefois, sur les marches d'un palais, sur l'herbe verte d'un fossé, dans la solitude morne de votre chambre, vous vous réveillez, l'ivresse déjà diminuée ou disparue, demandez au vent, à la vague, à l'étoile, à l'oiseau, à l'horloge, à tout ce qui fuit, à tout ce qui gémit, à tout ce qui roule, à tout ce qui chante, à tout ce qui parle, demandez quelle heure il est ; et le vent, la vague, l'étoile, l'oiseau, l'horloge, vous répondront : « Il est l'heure de s'enivrer ! Pour n'être pas les esclaves martyrisés du Temps, enivrez-vous ; enivrez-vous sans cesse ! De vin, de poésie, de mathématiques ou de vertu, à votre guise. »

-Baudelaire, Le Spleen de Paris, XXXIII

Dediée à Paul

Remerciements

Je ne doute pas que cette page sera certainement la plus lue de tout mon manuscrit, donc je vais essayer de n'oublier personne...

Tout d'abord, je souhaite remercier Olivier Le Gal, pour sa grande implication, sa patience infinie et ses bonnes intuitions quand je ne voyais plus de chemins. Merci pour tout ce temps passé, de l'apprentissage des mathématiques en français, en passant par la difficile ouverture d'un compte bancaire (pour une iranienne, c'est un chemin de croix !), jusqu'à la correction du manuscrit très en détail ! Merci d'avoir cru en moi, même quand je ne croyais plus en moi.

Merci Krzysztof pour avoir choisi cet intéressant sujet qui s'est révélé porteur de résultats et de nombreuses autres questions. Merci pour tes précieuses remarques pendant cette thèse et nos discussions sur des sujets aussi divers que variés! Merci également pour tes attentions lors des périodes plus difficiles de cette thèse.

Je tiens à remercier Laurentiu PAUNESCU et Antonio LERARIO d'avoir accepté d'être mes rapporteurs, merci pour votre temps et vos précieux commentaires. Merci notamment à Laurentiu pour avoir été présent à ma soutenance « de nuit » en direct d'Australie !

Merci à Camille PLENAT et Georges COMTE d'avoir été examinateurs dans mon jury. George, merci de m'avoir également accueilli dans ton laboratoire, et de finir ma thèse en la présidant !

Dans ce laboratoire, je voudrais remercier également Michel, Frédéric, Sébastien, Céline, Rémi, Marion, pour votre bonne humeur, votre aide (informatique ou autre !).

Je voudrais remercier Bernard TEISSIER, Hussein MURTADA et Marc CHARDIN, pour leur gentillesse et leurs intéressantes discussions mathématiques à Paris. C'est grâce à vous que j'ai pu revenir en France pour commencer cette thèse au Bourget-du-Lac.

Kheili merci az Siamak YASSEMI, pour m'avoir donné une chance de passer 3 mois à Paris pendant mon Master. Merci de m'avoir proposé un sujet en topologie combinatoire et ouvert vers des nouveaux horizons professionnels.

Merci à François Dahmani et Eric Dumas de m'avoir permis de finir cette thèse sereinement (ou presque...) avec cet ATER à Grenoble dans de bonnes conditions.

Stefana, Octave et Kévin, merci pour ces bons moments passés ensemble en séminaire à Istanbul, Lille, Marseille, Chambéry, et j'en oublie...

Michèle et Yves, merci infiniment pour votre accueil, votre gentillesse, et pour tous les cours de français qui m'ont permis de vivre, de voyager, et d'enseigner en France !

A Gizem, Jamie, Suelen, Julien, Faezeh, Wilfried, Mona, Rezvan, Ahmad Reza, Surena, Hadis, Priscilla, Seb, Johann et Daniel: merci, thank you, tesekkür ederim, mamnun, obrigado, pour votre présence dans ma vie, je ne saurais vous exprimer ma profonde amitié en quelques lignes, je le ferai donc « en présentiel »... Je vous aime tous.

A Bella, merci de m'avoir réchauffé les pieds quand je rédigeais mon manuscrit, d'avoir rajouté des « qsjdklmfksldjklsf » dedans, cachés mes stylos pour que je joue avec toi, merci mon petit chat.

A ma belle-famille, Aleth, Laure, Sophie, Joseph, merci pour m'avoir accueilli avec affection, pour le temps passé ensemble et pour ces si bons Cheetoz... merci infiniment !

A ma très chère famille, Farhang, Nasrin, Ali, Ehsan, Adeleh, Paria, et très très chère Mehrsa et Ariana. Je vous aime avec tout mon cœur, merci de m'avoir fait confiance et de croire en moi... Vous ne pouvez pas imaginer comment vous m'avez manqué durant ces 4 années, et j'ai besoin quotidiennement de vous entendre pour fonctionner...

Enfin, merci à mon Paul, pour tes attentions quotidiennes, ton soutien sans failles, et tes petits plats. Seul on va plus vite, ensemble on va plus loin...

Contents

List of Figures

Résumé

Dans cette thèse, on étudie les régularités des valeurs propres et vecteurs propres de familles linéaires de matrices symétriques réelles $A(t) \in Sym_d(\mathbb{R})$, $t \in \Omega \subset \mathbb{R}^k$. Les éléments propres de $A(t)$ sont singuliers en général, et Kurdyka et Paunescu (2008) en donnent une résolution par éclatements dans l'espace Ω. Le long d'une courbe, Rellich (1937) montre que les éléments propres de $A(t)$ ont un prolongement analytique. Mais le prolongement le long d'un lacet peut s'avérer différent du point de départ: le premier groupe d'homotopie de l'ensemble des paramètres réguliers agit par permutation sur le spectre de $A(t)$, une action que l'on appelle monodromie de $A(t)$.

Nous relions cette monodromie avec un autre invariant, la monodromie antipodale, et caractérisons complètement les permutations apparaissant comme monodromies de familles à deux paramètres. En montrant que les points singuliers d'une famille diagonale générique à un paramètre admettent des perturbations indépendantes, nous réalisons toute permutation donnée comme la monodromie antipodale d'une famille à deux paramètres. Pour les familles affines à deux paramètres, nous décrivons le comportement des valeurs propres au voisinage des points où $A(t)$ a une valeur propre double.

On s'intéresse ensuite à l'existence d'une diagonalisation, ou d'une réduction par bloc, analytique, pour $A(t) \in Sym_d(\mathbb{R})$. On remarque que l'absence de monodromie antipodale n'est pas suffisante pour factoriser une valeur propre dans le polynôme caractéristique P_A, ni l'existence d'un facteur dans P_A n'assure l'existence d'une réduction par bloc. On obtient cependant des résultats positifs pour les valeurs propres extrémales. Ceux-ci permettent de montrer que $A(t)$ admet une diagonalisation analytique complète si et seulement si sa monodromie antipodale est triviale: la monodromie antipodale est l'unique obstruction à l'existence d'éléments propres analytiques.

Enfin, nous nous concentrons sur les familles à deux paramètres dans $Sym_3(\mathbb{R})$. Ici, les vecteurs propres et valeurs propres forment chacun une cubique de $\mathbb{P}^2$. Nous classifions entièrement les couples de telles cubiques et montrons qu'il n'y existe que 9 types de telles paires. Inspiré par cette étude, on s'intéresse à la question de savoir si les vecteurs propres fournissent une résolution des singularités des valeurs propres, et on donne une réponse positive dans le cas des courbes, et lorsque la famille est l'espace complet $Sym_d(\mathbb{R})$.

ABSTRACT

In this thesis, we study the regularities of eigenvalues and eigenvectors of linear families of real symmetric matrices $A(t) \in Sym_d(\mathbb{R})$, $t \in \Omega \subset \mathbb{R}^k$. The eigenelements of $A(t)$ are singular in general, and Kurdyka and Paunescu (2008) give a resolution by blowing-ups in the space Ω. Along a curve, Rellich (1937) shows that the eigenelements of $A(t)$ have an analytic continuation. But the prolongation along a loop might be different from where one started: the first homotopy group of the set of regular parameters acts by permutation on the spectrum of $A(t)$, an action we call monodromy of $A(t)$.

We rely this monodromy with another invariant, the antipodal monodromy, and fully characterize those permutations that appear as monodromies of two-parameter families. Showing that singular points of one-parameter generic diagonal families have independent perturbations, we realize any given permutation as the antipodal monodromy of a two-parameter family. Then, for two-parameter affine families, we describe the behavior of the eigenvalues near the points where $A(t)$ has a double eigenvalue.

We then study whether $A(t) \in Sym_d(\mathbb{R})$ is analytically diagonalizable, or splits into blocks. We remark that neither the lack of antipodal monodromy is sufficient for an eigenvalue to be factored out of the characteristic polynomial P_A, nor a factor in P_A necessarily corresponds to a splitting block. However, we get positive results for extremal eigenvalues. It leads us to prove that $A(t)$ has full analytic diagonalization if and only if it has trivial antipodal monodromy: the antipodal monodromy is the only obstruction to get analytic eigenelements.

Finally, we focus on two-parameter families in $Sym_3(\mathbb{R})$. Here both eigenvectors and eigenvalues form a cubic curve in $\mathbb{P}^2$. We fully classify the couples of those cubics and show that there are only 9 types of such pairs. Inspired by this study, we consider the question whether eigenvectors form a resolution of singularities for the eigenvalues, and prove that it has a positive answer in the curve situation, and when the family is the full space $Sym_d(\mathbb{R})$.

Introduction

Consider a family of monic polynomials which their coefficients belong to a certain class of functions, the perturbation problem is whether it is possible to choose their roots as a function of this class. The origin of this problem is in the work of Lord Rayleigh (1894) in acoustics, and of Schrödinger (1926) in quantum mechanics. But, the mathematical foundations of this problem were only laid by Franz Rellich (1937) [29, 30, 31] for hyperbolic polynomials whose coefficients are analytic functions depending on one variable, and then it has been developed in the work of Tosio Kato [17] for linear operators.

The classical result of Rellich, Rellich Theorem 1.2.1, is connected with investigation of the behavior of the eigenvalues of symmetric matrices under one-parameter analytic perturbation. This result may be stated as follows. Rellich considered an analytic family $A(t)$ of $d\times d$ symmetric matrices, where $t \in I$ and $I \subset \mathbb{R}$ is an interval. He proved that the eigenvalues of $A(t)$ can be chosen analytically on I. Consequently, we can choose analytically eigenvectors of $A(t)$, so we can diagonalize uniformly and analytically the family of $A(t)$. But, if we consider a multi-parameter analytic family $A(t)$, $t \in \Omega$ of $d \times d$ symmetric matrices, where $\Omega \subset \mathbb{R}^k$ is open, then for $k \geq 2$ Rellich's theorem fails, even for linear families, Example 0.0.1.

Example 0.0.1. Let

$$A(t_1, t_2) = \begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix} = t_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, we get

$$\chi_{A(t_1,t_2)} = X^2 - (t_1^2 + t_2^2),$$
$$\Delta_{A(t_1,t_2)} = 4(t_1^2 + t_2^2),$$

where χ_A and Δ_A are the characteristic polynomial and the discriminant of $A(t_1, t_2)$, respectively. The eigenvalues of the matrix $A(t_1, t_2)$ are:

$$X_1 = \sqrt{t_1^2 + t_2^2}$$
$$X_2 = -\sqrt{t_1^2 + t_2^2}.$$

Note that the eigenvalues of $A(t_1, t_2)$ are not analytic functions at the point $(t_1, t_2) =$

$(0,0)$. These eigenvalues satisfy the equation $X^2 - t_1^2 - t_2^2 = 0$ and the surface of the eigenvalues is a circular double cone in the space (t_1, t_2, X), Figure 1.

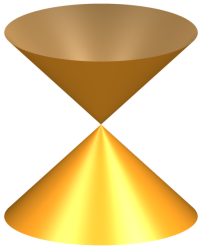


Figure 1: The eigenvalues of $A(t_1, t_2)$

In [23], Von Neumann and Wigner proved that generically the multiple eigenvalues of a two-parameter family of real symmetric matrices happen at isolated points. Indeed, Arnold [3] showed that the family of matrices with multiple eigenvalues has codimension two in the manifold of real symmetric matrices. Let us consider the family of 2×2 symmetric matrices that is given as follows.

$$\begin{array}{rccc} A & : & \mathbb{R}^3 & \to & Sym_2(\mathbb{R}) \\ & & (t_1, t_2, t_3) & \mapsto & \begin{pmatrix} t_1 & t_3 \\ t_3 & t_2 \end{pmatrix} \end{array}.$$

Then, the discriminant of A can be written as a sum of squares, $\Delta_A = (t_1 - t_2)^2 + 4t_3^2$. Note that $\Delta_A = 0$ if and only if the two eigenvalues of A coincide. Thus, we have $\Delta_A = 0$ if and only if

$$\begin{cases} t_1 - t_2 = 0 \\ t_3 = 0. \end{cases} . \tag{1}$$

Hence, in dimension two, the problem of locating the parameters such that eigenvalues coincide has a straightforward solution. But, as the dimension of the family grows, the discriminant becomes complicated and it cannot be used in practical computations. However, it can be written as a sum of squares [8], [14, 15, 16], [19], [25] and the number of summands grows quickly as we increase the dimension of the family. Recently, the multi-parameter version of Rellich's theorem has been studied. First, in 2008, by Kurdyka and Paunescu [18] for real symmetric and anti-symmetric matrices and then by Rainer (2009) [27], (2011) [28], Parusinski and Rond (2018) [26] and Grandjean (2019) [12] for normal matrices.

In [18], Kurdyka and Paunescu proved the following:

There exists a mapping

$$\sigma : \tilde{\Omega} \to \Omega$$

which is locally finite composition of blowing-ups with smooth centers, such that the family $A \circ \sigma$ locally satisfies the conclusion of the theorem of Rellich. In this way, the eigenvalues (and even eigenvectors) of the original family $A(t)$, $t \in \Omega \subset \mathbb{R}^k$ can be seen as multi-valued (arc-analytic) functions on Ω. Hence, there is a natural monodromy action of the first fundamental group of the complement of the centers on eigenvalues and eigenvectors. In 2009, Rainer generalized the result of Kurdyka and Paunescu for normal operators, [27]. In [26], Parusinsky and Rond showed that a family of normal matrices depending on a multi-parameter ring of formal power series can be diagonalized if we assume that the discriminant of its characteristic polynomial is a normal crossing. In 2019, Grandjean [12] presented a new proof of the results of Kurdyka and Paunescu, and of Rainer, for normal matrices. For further results on the perturbation problem of hyperbolic polynomials see [7], [9], [11], [22].

We consider $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^k$. Then, by the Implicit Function Theorem, $A(t)$ is diagonalizable, in an analytic way, in a neighborhood of a point $t_0 \in \mathbb{R}^k$ such that $A(t_0) \notin \Sigma$ where

$$\Sigma = \{A \in Sym_d(\mathbb{R}) \ \mid \ \Delta_A = 0\}$$

and Δ_A is the discriminant of the characteristic polynomial of A. Therefore, we are interested in studying the behavior of the eigenvalues in a neighborhood of the points $t_0 \in \mathbb{R}^k$ such that $A(t_0)$ has at least one pair of repeated eigenvalues.

The results in brief

Let us now explain our results in brief.

In chapter 2, for a multi-parameter analytic and linear family of real symmetric matrices, we define different types of monodromies, monodromy σ, and antipodal monodromy τ', respectively. The main result of this chapter is that the monodromy of a linear family depending on two parameters is a square. We state it as follows.

Theorem 0.0.2. *(see Theorem 2.5.1). A permutation $\sigma \in \mathfrak{S}_d$ is a square if and only if there are $A_1, A_2 \in Sym_d(\mathbb{R})$ such that σ is the monodromy of the eigenvalues of the matrix $A(t) = t_1A_1 + t_2A_2$ along a loop $\gamma : [0,1] \to \mathbb{R}^2\backslash(0,0)$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \Sigma$.*

In chapter 3, the main results are obtained by focusing on affine families depending on one and two parameters. Indeed, by focusing on one-parameter affine families, we

prove that given d distinct values $a_1, \ldots, a_d$ in $\mathbb{R}$, and σ a permutation of $\{a_1, \ldots, a_d\}$, there exists a linear family $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^2$, whose eigenvalues at $(1,0)$ are $a_1, \ldots, a_d$, and whose antipodal monodromy from $(1,0)$ in the direction $(0,1)$ is precisely σ. This result is stated as the following.

Theorem 0.0.3. *(see Theorem 3.3.5). Let $(b_1, \ldots, b_d) \in \mathbb{R}^d$ such that $b_i \neq b_j$ for $i \neq j$. Let τ be a permutation of $\{b_1, b_2, \ldots, b_d\}$. Then, there exists a 2-linear family $A(t_1, t_2)$ whose antipodal monodromy from $(0,1)$ in the direction of $(1,0)$ is τ.*

Then, by focusing on two-parameter affine families, we give a description of the behavior of the eigenvalues in a neighborhood of a matrix $A(t_0)$ which has a double eigenvalue, Subsection 3.4.

In chapter 4, we study possible reductions of linear families of symmetric matrices. In Section 4.2, we prove that if the monodromy of a linear family $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^k$, has an extremal orbit of length one or two, then there exists a homogeneous polynomial of degree one or two that divides the characteristic polynomial of $A(t)$. After that, we manage to find an invariant subspace $E(t)$ of dimension ℓ such that the characteristic polynomial of the restriction of $A(t)$ to $E(t)$ is the known divisor $P(t)$ of degree ℓ of $\chi_{A(t)}$. Finally, in Section 4.3, we prove that $A(t)$ has full analytic diagonalization if and only if it has trivial antipodal monodromy: the existence of non-trivial antipodal monodromy is the only obstruction for having eigenvalues as analytic functions. We state this result as the following.

Theorem 0.0.4. *(see Theorem 4.3.1). Let $A(t) \in Sym_d(\mathbb{R}), t \in \mathbb{R}^k$ be a k-linear family, $A \not\subset \Sigma$. Then, the following statements are equivalent:*

1. *There exists a basis $(e_1, e_2, \ldots, e_k)$ of $\mathbb{R}^k$, with $A(e_1) \notin \Sigma$ and $\tau'_{Span(e_1,e_i)} = \mathbb{I}_d$ for all $i = 2, \ldots, k$.*
2. *For any linear 2-plane $\mathcal{P} \subset \mathbb{R}^k$, $\tau'_{\mathcal{P}} = \mathbb{I}_d$.*
3. *0 is a quasi-regular point of $A(t)$.*
4. *There exists a family $P(t) \in O_d(\mathbb{R})$, analytic in t, such that $P^{-1}(t)A(t)P(t)$ is diagonal.*
5. *There exists an orthogonal matrix $P \in O_d(\mathbb{R})$ such that $P^{-1}A(t)P$ is diagonal.*
6. *There exists a basis $(e_1, e_2, \ldots, e_k)$ of $\mathbb{R}^k$, such that $A(e_1) \notin \Sigma$ and $A(e_1)A(e_i) = A(e_i)A(e_1)$ for $i = 2, \ldots, k$.*
7. *For all $(t, t') \in (\mathbb{R}^k)^2$, $A(t)A(t') = A(t')A(t)$.*

In chapter 5, we consider A, $A \not\subset \Sigma$, a linear subspace of dimension k of the $d \times d$ symmetric matrices and we introduce different objects. The **eigenelements set** of A which is a variety made of the triple matrix and the associated eigenvalues and eigenspaces. The **eigenvalue set** which is the image of the eigenelements set by the projection that forgets the eigenvector entry. The **eigenvectors set** which is the image of the eigenelements set by the projection that forgets the eigenvalue entry. We also define **eigenvector portrait** of A which is the image of the eigenvectors set by the projection that forgets the matrix entry. Since all of these objects have homogeneous behavior, it is natural to work with their counterpart in projective space. We note that the projective eigenvector set $\Gamma(A)$ is not smooth in general, so we define the set of the **strict eigenvectors** $G(A)$ which is constructed by first removing from $\Gamma(A)$ all eigenvectors that appears for the singular matrices of A, and then we take the topological closure. In Claim 5.1.1, we prove that $\Phi : G(A) \subseteq \Gamma(A) \to \Lambda(A)$ is a proper birational map. In Section 5.2, the main result is obtained by focusing on the eigenelements of the two-dimensional subspaces of 3×3 matrices. We consider two cubic curves: the cubic curve $V(A)$ which is associated to eigenvectors and the cubic curve $\Lambda(A)$ that is associated to the eigenvalues of A. We give a classification of the pair $(V(A), \Lambda(A))$. For this aim, we follow a projective classification of cubic curves of $\mathbb{P}^2$ containing 16 configurations where each configuration is the union of at most 3 analytic components. Then, we introduce a notion of marked cubic type for eigenvalues (resp. eigenvectors) by associating to each component C_i of $\Lambda(A)$ (resp. of $V(A)$), the number of eigenvalues (resp. eigenvectors) that belongs to C_i, for a regular member of the family. Thus, we fully classify the couples of these cubics and show that there are only 9 types of pairs $(V(A), \Lambda(A))$. This result is given as the following.

Theorem 0.0.5. *(see Theorem 5.2.4). Let A be a 2-family of 3×3 symmetric matrices, $A \not\subset \Sigma$. Then the eigenvectors type of A determines the eigenvalues type of A. More precisely, the following 9 couples of marked cubics types, and only those, appear as eigenvectors and eigenvalues types.*

Table 1: Table of cubics types.

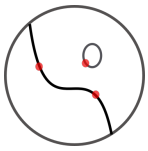
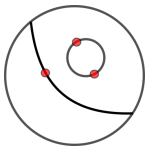
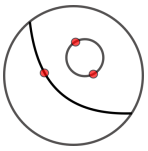
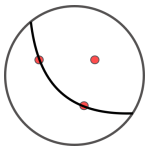
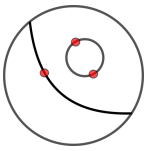
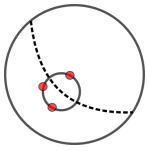
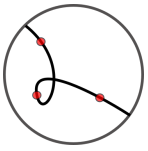
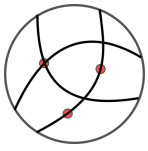
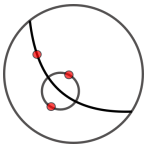
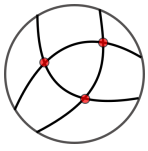
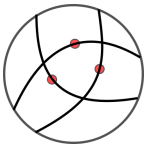
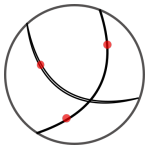
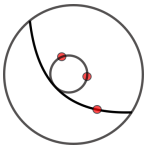
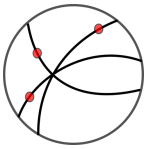
Marked cubic of eigenvectors	***Marked cubic of eigenvalues***
	
	
	
	
	
	
	
Continued on next page	

Table 1 – Continued from previous page

Marked cubic of eigenvectors	***Marked cubic of eigenvalues***
	
	

Finally, in the last section, we prove that $G(A) \to \Lambda(A)$ is a resolution of singularities of algebraic varieties in two special case: when the family is the full space $Sym_d(\mathbb{R})$, Proposition 5.3.2, and when the family depends on two parameters:

Theorem 0.0.6. *(see Theorem 5.3.1). For any 2-linear family $A \not\subset \Sigma$, the set $G(A)$ of strict eigenvectors of A is a non-singular algebraic curve.*

Chapter 1

Preliminaries

In this chapter, we present some notions which will be used throughout the thesis and we also give an extension of the Rellich Theorem to higher dimensions.

Notation and conventions 1.0.1. All matrices that we discuss are over the real numbers. We denote by $M_{m,n}(R)$ the set of matrices with entries in R with m rows and n columns. When $m = n = d$, we denote this set by $M_d(R)$. A matrix $S \in M_d(\mathbb{R})$ is called symmetric if $S = S^t$. We denote by $Sym_d(\mathbb{R})$ the set of all real symmetric matrices of dimension d.

Theorem 1.0.2. *(Spectral theorem) If $A \in Sym_d(\mathbb{R})$, then there exists an orthonormal matrix P such that $A = P^t D P$, with $D = diag(\lambda_1, \ldots, \lambda_d)$ where $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ are the eiegnvalues of A.*

Corollary 1.0.3. *Let $A \in Sym_d(\mathbb{R})$ be a symmetric matrix and let us denote by λ_{max} its largest eigenvalue and by λ_{min} its smallest eigenvalue, then we have:*

$$\max_{||x||=1} x^t A x = \lambda_{max}, \qquad x \in \mathbb{R}^d$$
$$\min_{||x||=1} x^t A x = \lambda_{min}, \qquad x \in \mathbb{R}^d.$$

We deduce from this corollary that the largest eigenvalue of a symmetric matrix is convex and the smallest eigenvalue is concave.

Lemma 1.0.4. *Let $A \in Sym_d(\mathbb{R})$. The following properties hold true:*

1. *The eigenvalues of A are real.*
2. *Eigenvectors of A corresponding to different eigenvalues are two-by-two orthog-*

onal.

Definition 1.0.5. *The set of all eigenvalues of $A \in Sym_d(\mathbb{R})$, denoted by $Spec(A)$, is called the spectrum of A.*

Definition 1.0.6. *We consider $A \in M_d(\mathbb{R})$. The characteristic polynomial of A, denoted by $\chi_A(x)$, is the polynomial defined by*

$$\begin{array}{rccc} \chi_A(x) & : & Sym_d(\mathbb{R}) \times \mathbb{R} & \to & \mathbb{R} \\ & & (A, x) & \mapsto & det(A - x\mathbb{I}_d). \end{array}$$

where $\mathbb{I}_d$ denotes the identity matrix of dimension d.

Definition 1.0.7 (Hyperbolic Polynomials)**.** *A monic polynomial*

$$p(x) = x^d + \sum_{i=1}^{d} a_i x^{d-i}$$

with real coefficients is called a hyperbolic polynomial, if all of the roots of $p(x)$ are real.

Since the eigenvalues of symmetric matrices are real, the characteristic polynomial of a real symmetric matrix is a hyperbolic polynomial.

1.1 Resultant and discriminant of polynomials

In this section, following Benedetti and Risler [5], we present the definitions and some properties of the notions of the resultant and the discriminant of polynomials.

1.1.1 Resultant

Definition 1.1.1. *Let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ and $Q(x) = b_0 + b_1 x + \cdots + b_q x^q$ be two polynomials of degrees p and q, respectively, with coefficients in an arbitrary field F. Their **resultant** $R(P, Q) = R_{p,q}(P, Q)$ is the element of F given by the determinant of the $(p + q) \times (p + q)$ **Sylvester matrix** $Syl_{p,q}(P, Q)$ given by*

$$\begin{pmatrix} a_0 & \dots & a_{q-1} & \dots & a_{p-1} & a_p & \\ & \ddots & & & & & \\ & & a_0 & \dots & \dots & & a_p \\ b_0 & \dots & \dots & b_q & & & \\ & b_0 & \dots & \dots & b_q & & \\ & & \ddots & & & & \\ & & & b_0 & \dots & \dots & b_q \end{pmatrix}.$$

Proposition 1.1.2. *$R_{p,q}(P,Q)$ is a homogeneous polynomial with coefficients a_i, b_i in an arbitrary field F.*

1. *$R_{p,q}(P,Q)$ is homogeneous of degree p in $a_0, \dots, a_p$ and degree q in $b_0, \dots, b_q$.*
2. *If a_i and b_i are regarded as having degree i, then $R_{p,q}(P,Q)$ is homogeneous of degree pq.*

The main importance of the resultant lies in the following proposition, which often is taken as definition.

Proposition 1.1.3. *Let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ and $Q(x) = b_0 + b_1 x + \cdots + b_q x^q$ be two polynomials of degrees p and q, respectively, with coefficients in an arbitrary field F. Suppose that in some extension of F, P has p roots $\alpha_1, \dots, \alpha_p$ and Q has q roots $\beta_1, \dots, \beta_q$. Then,*

1. *$R(P,Q) = a_p^q b_q^p \prod(\alpha_i - \beta_j) \qquad (1 \le i \le p, 1 \le j \le q)$*
2. *$R(P,Q) = a_p^q \prod_{i=1}^{p} Q(\alpha_i) = (-1)^{pq} b_q^p \prod_{j=1}^{q} P(\beta_j)$.*

Proof. See for instance [5], page 29. □

1.1.2 Discriminant

Definition 1.1.4. *Let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ be a polynomial of degree $p \ge 1$ with coefficients in an arbitrary field F. Suppose $P'(x)$ is the derivative of P. The **discriminant** Δ_P is given by*

$$\Delta_P = \begin{cases} 0 & \text{if } a_p = 0 \\ (\frac{1}{a_p}) R(P,P') & \text{if } a_p \neq 0 \end{cases}. \tag{1.1}$$

In general, it is often more convenient to use the following version.

Definition 1.1.5. *Let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ be a polynomial of degree $p \geq 1$ with coefficients in an arbitrary field F. The discriminant of P is*

$$\Delta_P = a_p^{2p-2} \prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j)^2,$$

where $\alpha_1, \ldots, \alpha_p$ are the roots of P in some extension of F.

Proposition 1.1.6. *Let P be a polynomial of degree $p \geq 1$ with coefficients in a field F. Then, $\Delta_P = 0$ if and only if P has a double root in some extension of F.*

We denote by Δ_A the discriminant of the characteristic polynomial of $A \in Sym_d(\mathbb{R})$. The discriminant is a fundamental object and it appears in several areas of mathematics, from mathematical physics to real algebraic geometry, see for instance [4, 2].

Corollary 1.1.7. *Let $A \in Sym_d(\mathbb{R})$. Then, the discriminant of the characteristic polynomial of A is*

$$\Delta_A = \prod_{1 \leq i < j \leq d} (\lambda_i(A) - \lambda_j(A))^2,$$

where $\lambda_1(A), ..., \lambda_d(A)$ are the eigenvalues of A.

Proposition 1.1.8. *Let $A \in Sym_d(\mathbb{R})$. Then Δ_A is a non-negative homogeneous polynomial of degree $d(d-1)$.*

Proof. See [5], page 27. □

1.2 Rellich's theorem

Theorem 1.2.1 (Rellich)**.** *Let $p(x,t) = x^d + \sum_{i=1}^d a_i(t) x^{d-i}$ be a monic hyperbolic polynomial; where $a_i(t)$ are real analytic functions on an open set $I \subset \mathbb{R}$. Then, there are analytic functions $f_i : I \mapsto \mathbb{R}$, $i = 1, ..., d$ such that $P(x,t) = \prod_{i=1}^d (x - f_i(t))$. In other words, we can choose analytically the roots of P.*

Proof. Due to [29]. A proof can be found in [18] by Kurdyka and Paunescu. In this paper, they gave a short proof of the Rellich Theorem by using the classical theorem of Newton-Puiseux which we will mention in the following theorem. □

Theorem 1.2.2 (Newton-Puiseux)**.** *Let $p(x,t) = x^d + \sum_{i=1}^d a_i(t) x^{d-i}$ be a monic polynomial such that a_i are real analytic functions in a neighborhood of $0 \in \mathbb{R}$. Then*

there are holomorphic functions h_i , $i = 1, ..., d$ *and* $r \in \mathbb{Z}$ *such that*

$$P(x,t) = \prod_{i=1}^{d}(x - h_i(t^{\frac{1}{r}}))$$

for $t \geq 0$ *close enough to* 0 *and for any* $x \in \mathbb{C}$. *We call* $f_i(t) = h_i(t^{\frac{1}{r}})$ *a Newton-Puiseux root of* P.

Proof. A proof can be found in [21]. □

Proposition 1.2.3. *Let* $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}$ *be an analytic family. Then, the eigenvalues of* $A(t)$ *are analytic functions on* t. *Moreover, eigenvectors of* $A(t)$ *can be chosen as analytic functions on* t.

Proof. See for instance [17] page 121. □

Here, we give an extension of the Rellich Theorem to higher dimensions.

Proposition 1.2.4. *Let* $P_{(t,x)}(\lambda) = \lambda^d + \sum_{i=1}^{d} a_i(t,x)\lambda^{d-i}$, $(t,x) \in U \subset \mathbb{R} \times \mathbb{R}^{k-1}$ *be an analytic family of hyperbolic polynomials. Suppose that:*

1. $P_{(0,0)}$ *has a root of order* ℓ *and* $d - \ell$ *simple roots;*
2. $P_{(0,x)}$ *has an analytic root* $\lambda_0(x)$ *of multiplicity* ℓ*;*
3. $P_{(t,0)}$ *has* ℓ *analytic roots* $\lambda_1(t), \ldots, \lambda_\ell(t)$ *with* $\lambda_1(0) = \cdots = \lambda_\ell(0) = \lambda_0(0)$ *and* $\lambda_i'(0) \neq \lambda_j'(0)$ *for* $i \neq j$.

Then, there exists a neighborhood V *of* $(0,0)$ *and analytic functions* $\lambda_1, \ldots, \lambda_\ell$ *on* V *such that* $P_{(t,x)}(\lambda_i(t,x)) = 0$, $\lambda_i(t,0) = \lambda_i(t)$ *and* $\lambda_i(0,x) = \lambda_0(x)$.

Proof. Up to replace $P_{(t,x)}(\lambda)$ by $P_{(t,x)}(\lambda + \lambda_0(x))$, we suppose that $\lambda_0(x) = 0$. The $d - \ell$ simple roots of $P_{(0,0)}$ admit analytic continuations $z_i(t,x)$, $i = \ell + 1, \ldots, d$ for (t,x) small. Set $S_{(t,x)}(\lambda) = \prod_{i=\ell+1}^{d}(\lambda - z_i(t,x))$ and note that $S_{(t,x)}(\lambda)$ divides $P_{(t,x)}(\lambda)$. Let

$$Q_{(t,x)}(\lambda) = \lambda^\ell + a_1(t,x)\lambda^{\ell-1} + \cdots + a_{\ell-1}(t,x)\lambda + a_\ell(t,x)$$

be the quotient of $P_{(t,x)}(\lambda)$ by $S_{(t,x)}(\lambda)$.

For a fixed $x \in \mathbb{R}^{k-1}$, $Q_{(t,x)}(\lambda)$ is an analytic one-parameter homogeneous polynomial which admits ℓ analytic roots. Since $Q_{(0,x)}(\lambda) = \lambda^\ell$, these roots vanish at $t = 0$. Then, Vieta's formulas show that for all $i = 1, \ldots, \ell$, $a_i(t,x)$ is divisible by t^i.

Now, for $x = 0$, write $a_i(t,0) = b_i t^i + o(t^i)$ and $\lambda_i(t) = m_i t + o(t)$. From $Q_{(t,0)}(\lambda_i(t)) = 0$, one gets

$$t^\ell(m_i^\ell + b_1 m_i^{\ell-1} + b_2 m_i^{\ell-2} + \cdots + b_{\ell-1} m_i + b_\ell) + o(t^\ell) = 0.$$

Hence, for each $i = 1, \ldots, \ell$, $R(m_i) = 0$ where

$$R(X) = X^\ell + b_1 X^{\ell-1} + \cdots + b_{\ell-1} X + b_\ell.$$

Furthermore, for all $i, j \in \{1, \ldots, \ell\}$, $i \neq j$, we have $m_i(x) \neq m_j(x)$. So, the polynomial $R(X)$ has ℓ simple roots. In particular, $R'(m_i) \neq 0$. For given $i \in \{1, \ldots, d\}$, set

$$W_{(t,x)}(\mu_i) := \frac{P_{(t,x)}((m_i + \mu_i)t)}{t^\ell}.$$

Notice that, since $a_i(t,x)$ is divisible by t^i, so $P_{(t,x)}((m_i+\mu_i)t)$ is divisible by t^ℓ. Thus, $W_{(t,x)}(\mu_i)$ is analytic in terms of (t,x).

We will apply the Implicit Function Theorem to express analytically a root $\mu_i(t,x)$ of $W_{(t,x)}(\mu_i)$ that vanish at $(0,0)$. Indeed, this gives the desired extension of $\lambda_i(t)$, by setting $\lambda_i(t,x) = m_i t + \mu_i(t,x)t$. For this, notice that

$$W_{(t,0)}(0) = \frac{P_{(t,0)}(m_i t)}{t^\ell} = R(m_i) = 0.$$

In particular, 0 is a root of $W_{(t,0)}(\mu)$. Moreover,

$$\begin{aligned}
\frac{\partial W_{(t,0)}(0)}{\partial \mu} &= \frac{1}{t^\ell} \times \frac{\partial P_{(t,0)}}{\partial \lambda}(m_i t) \times t \\
&= \frac{1}{t^{\ell-1}} \times \frac{\partial P_{(t,0)}}{\partial \lambda}(m_i t) \\
&= \frac{1}{t^{\ell-1}} \left[\frac{\partial S_{(t,0)}}{\partial \lambda}(m_i t) Q_{(t,0)}(m_i t) \right] \\
&+ \frac{1}{t^{\ell-1}} \left[S_{(t,0)}(m_i t) \frac{\partial Q_{(t,0)}}{\partial \lambda}(m_i t) \right].
\end{aligned}$$

But $Q_{(t,0)}(m_i t)$ is divisible by t^ℓ,

$$\frac{\partial Q_{(t,0)}}{\partial \lambda}(m_i t) = t^{\ell-1} R'(m_i) + o(t^{\ell-1}),$$

and $S_{(0,0)}(0) \neq 0$. Thus,

$$\frac{\partial W_{(t,0)}(0)}{\partial \mu}|_{t=0} = S_{(0,0)}(0)R'(m_i) \neq 0.$$

To conclude, the Implicit Function Theorem applies to $W_{(t,x)}(\mu)$:

There exists a neighborhood of $(0,0)$ and an analytic function $\mu_i(t,x)$ such that $\mu_i(0,0) = 0$ and $W_{(t,x)}(\mu_i(t,x)) = 0$. This means that $\lambda_i(t,x) = m_i t + \mu_i(t,x)t$ is an analytic root of $P_{(t,x)}(\lambda)$ that satisfies $\lambda_i(0,x) = 0$ and $\frac{\partial \lambda_i}{\partial t}(0,0) = m_i$. Notice that since the slopes of $\lambda_i(t)$, $i = 1, \ldots \ell$ are different at $t = 0$, $P_{(t,0)}$ has a unique analytic root with the slope m_i at $t = 0$. □

Chapter 2

Monodromy of eigenelements of a linear family of $Sym_d(\mathbb{R})$

Given $A(t)$, an analytic family of real symmetric matrices

$$\begin{array}{rccc} A & : & U & \to & Sym_d(\mathbb{R}) \\ & & t & \mapsto & A(t), \end{array}$$

where U is an open subset of $\mathbb{R}^k$, we study the behavior of the eigenvalues of $A(t)$ above the loops which turn around singular points of $A(t)$. Along a curve, a theorem of Rellich states that the eigenvalues extend in a unique way. But, the image of an eigenvalue by its extension along a loop can turn out to be different from the eigenvalue from which we started: therefore, the first homotopy group of the complement of the singular points acts by permutation on the spectrum of the family. This action is called monodromy of the family. In this chapter, we introduce various definitions of monodromies associated with analytic and linear families of real symmetric matrices.

In the first section, we introduce a set that we call **super-singular set** which we denote by S_A. A point $t \in \mathbb{R}^k$ is a super-singular point if there is no analytic continuation of the eigenvalues around this point.

In the second section, we construct the analytic continuation of the eigenvalues along continuous curves. In order to deal with paths that contain curves in Σ (the set of symmetric matrices with multiple eigenvalues), the values of the analytic continuation that we consider are in the set of germs of analytic functions. We show that if a continuous path γ in the space of parameters does not meet S_A, then there exists a

unique analytic continuation of the germs of the eigenvalues along γ. We also show that the end-point of the analytic continuation of a germ of an eigenvalue along γ, only depends on the homotomy class of $\gamma \in \mathbb{R}^k \backslash S_A$. As a corollary, the eigenvalues are analytic over any open simply-connected subset of $\mathbb{R}^k$ that does not intersect S_A.

In the third section, we define the monodromy of the eigenvalues along a loop γ, based at p, that does not meet the super-singular set. As a corollary of Section 2, we see that the monodromy a long γ only depends on the homotomy class of γ in $\mathbb{R}^k \backslash S_A$.

In the fourth section, we introduce another type of monodromy associated to k-linear families $A : \mathbb{R}^k \to Sym_d(R)$, $t \mapsto t_1 A_1 + \cdots + t_k A_k$ with $A_1, \ldots A_k \in Sym_d(\mathbb{R})$. Indeed, we see in Example 2.5.11 that the lack of the monodromy is not sufficient to have eigenvalues as analytic functions while the existence of non-trivial antipodal monodromy is the only obstruction for having eigenvalues as analytic functions, Theorem 4.3.1.

In section 5, we prove that the monodromy of a 2-linear family is a square. Indeed, we show that it is the square of the antipodal monodromy. Then, for a given permutation σ, we construct a family $t_1 A_1 + t_2 A_2$ such that the antipodal monodromy of $t_1 A_1 + t_2 A_2$ and σ have the same cycle structure. These results may be summarized in Theorem 2.5.1.

Finally, in the last section, we give some general properties of the eigenvalues of a one-parameter affine family that we use later.

2.1 Super-singular set

Definition 2.1.1. *Let $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^k$ be an analytic family. A point $t_0 \in \mathbb{R}^k$ is called a **singular point** of $A(t)$, if $A(t_0) \in \Sigma$.*

Definition 2.1.2. *A point $t_0 \in \mathbb{R}^k$ is called a **quasi-regular** point of an analytic family $A(t) \in Sym_d(\mathbb{R})$, if there exists an open neighborhood V of t_0 and analytic functions $\lambda_1, ..., \lambda_d : V \to R$ such that*

$$\chi_{A(t)}(x) = \prod_{i=1}^{d} (x - \lambda_i(t))$$

for any $t \in V$ and $x \in \mathbb{R}$.

Remark 2.1.3. From isolated zero principle, if t_0 is a quasi-regular point, the (an-

alytic) maps λ_i are uniquely determined up to reindexing. In particular, the germs $[\lambda_i]_{t_0}$ of the maps λ_i at t_0 are chosen in a finite set.

Definition 2.1.4. *A point $t_0 \in \mathbb{R}^k$ is called a **super-singular** point of an analytic family $A(t) \in Sym_d(\mathbb{R})$ if t_0 is not a quasi-regular point. The set of super-singular points of $A(t)$ is denoted by S_A*

$$S_A := \{t_0 \in \mathbb{R}^k \mid t_0 \text{ is a super-singular point of } A\}.$$

Remark 2.1.5. The set S_A is a closed subset of $\mathbb{R}^k$, since the set of quasi-regular points is open.

Definition 2.1.6. *A point $t_0 \in \mathbb{R}^k$ is called a **quasi-ordinary** point of an analytic family $A(t) \in Sym_d(\mathbb{R})$ if $\Delta_A(t)$ is a normal crossing at t_0.*

Theorem 2.1.7. *Let $P(t,x) = x^d + \sum_{i=1}^{d} a_i(t)x^{d-i}$ be a monic hyperbolic polynomial; where $a_i(t)$ are real analytic functions on an open set $\Omega \subset \mathbb{R}^k$. If $t_0 \in \Omega$ is a quasi-ordinary point, then it is a quasi-regular point.*

Proof. A proof is given in [18]. □

In the following example, we show that the converse of Theorem 2.1.7, is not true.

Example 2.1.8. Consider the linear 2-family $A(t_1,t_2) \in Sym_3(\mathbb{R})$ that is given as follows.

$$A(t_1,t_2) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_1+t_2 & 0 \\ 0 & 0 & t_2 \end{pmatrix}.$$

The eigenvalues of $A(t_1,t_2)$, $\lambda_1 = t_1$, $\lambda_2 = t_1 + t_2$ $\lambda_3 = t_2$, are analytic functions on $\mathbb{R}^2$. However, the discriminant of $A(t_1,t_2)$

$$\Delta_A(t) = t_1^2 t_2^2 (t_1 - t_2)^2$$

is not a normal crossing at the origin.

Proposition 2.1.9. *There exists a semi-analytic closed subset $X \subset \mathbb{R}^k$ of codimension at least 2 such that $S_A \subset X$.*

Proof. See [18]. □

2.2 Analytic continuation of eigenvalues

Let Ω be an open subset of $\mathbb{R}^k$ and

$$\Xi_\Omega = \{(t, f);\ t \in \Omega,\ f \in \mathcal{O}_t\},$$

where $\mathcal{O}_t$ stands for the ring of germs of analytic functions at $t \in \Omega \subset \mathbb{R}^k$.

Definition 2.2.1. *Consider a continuous path $\gamma : [0,1] \to \Omega$. We say that Γ is an **analytic continuation (prolongation) of the germ** $[\lambda]_{\gamma(0)}$ along γ if $(\gamma, \Gamma) : [0,1] \to \Xi_\Omega$ satisfies that for each $x \in [0,1]$, there are $\varepsilon > 0$, a neighborhood $V_x \subset \Omega$ of $\gamma(x)$ and an analytic function $f : V_x \to \mathbb{R}$ such that $\gamma([x-\varepsilon, x+\varepsilon]) \subset V_x$ and $\forall x' \in [x-\varepsilon, x+\varepsilon], \Gamma(x') = [f]_{\gamma(x')}$.*

Proposition 2.2.2. *Let $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$ be a continuous path. Then, there exists a **unique** analytic continuation Γ of the analytic germs $[\lambda_1]_{\gamma(0)}, ..., [\lambda_d]_{\gamma(0)}$ of the eigenvalues of an analytic family $A(t)$ along γ.*

Proof. Fix t_0 in $\mathbb{R}^k \backslash S_A$. Since t_0 is a quasi-regular point, an analytic choice λ_i, $i = 1, \ldots, d$ can be made on an open neighborhood V of t_0. Choose an index i. Denote by λ the corresponding analytic eigenvalue on V and set $[\lambda]_{t_0}$ for the germ of λ at t_0. We prove that we can extend analytically $[\lambda]_{t_0}$ along a (continuous) path $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, with $\gamma(0) = t_0$. For this, we define

$$E = \{x \in [0,1];\ \exists \Gamma : [0,x] \to \mathbb{R}^k \backslash S_A,\ \text{analytic continuation of } [\lambda]_{t_0} \text{ along } \gamma_{|[0,x]}\}$$

and we set $x_{\max} = \sup E$. Then $x_{max} > 0$, since there is an $a > 0$ such that $\gamma([0,a]) \subset V$ and for $x \in [0,a]$, $\Gamma(x) = [\lambda]_{\gamma(x)}$ is an analytic continuation of $[\lambda]_{t_0}$ along $\gamma_{|[0,x]}$. Moreover, since $\gamma(x_{max}) \notin S_A$, there exists a neighborhood W of $\gamma(x_{max})$ over which the eigenvalues of $A(t)$ are analytic functions $\mu_1, \ldots, \mu_d$. Suppose $x_{max} \neq 1$, and set $\varepsilon > 0$ and $y = x_{max} - \varepsilon$ such that $\gamma([y, x_{max} + \varepsilon]) \subset W$. Since $y \in E$, the germ $[\lambda]_{t_0}$ has an analytic continuation Γ along $\gamma_{|[0,y]}$, and since the germs of the eigenvalues are chosen in a finite set, there is a j such that $\Gamma(y) = [\mu_j]_{\gamma(y)}$. Extending Γ to $[y, x_{max} + \varepsilon]$ by setting $\Gamma(x) = [\mu_j]_{\gamma(x)}$ for $x \in [y, x_{max} + \varepsilon]$ constructs an analytic continuation of $[\lambda]_{t_0}$ along $\gamma_{|[0,x_{max}+\varepsilon]}$, which contradicts that x_{max} is the supremum of E. So $x_{max} = 1$, and the preceding construction (but on $[1-\varepsilon, 1]$) shows that $1 \in E$, which was to be proven.

We now prove the uniqueness of the analytic continuation $[\lambda]_{t_0}$ along γ. Suppose

that Γ_1, Γ_2 are analytic continuations of $[\lambda]_{t_0}$ along γ. We show that for all $t \in [0,1]$, $\Gamma_1(t) = \Gamma_2(t)$. Let

$$T = \{t \in [0,1] \mid \Gamma_1(t) = \Gamma_2(t)\}.$$

Then, T is closed in $[0,1]$ and it is non-empty since $0 \in T$. We show that T is open in $[0,1]$. Take $t \in T$, then $\Gamma_1(t) = \Gamma_2(t)$. By the definition of analytic continuation, for $i = 1, 2$, there is $\varepsilon_i > 0$, open connected subsets V_i of $\gamma(t)$, analytic functions $\alpha_i : V_i \to \mathbb{R}$ such that $\gamma([t-\varepsilon_i, t+\varepsilon_i]) \subset V_i$ and for all $t' \in [t-\varepsilon_i, t+\varepsilon_i]$, $\Gamma_i(t') = [\alpha_i]_{\gamma(t')}$. Since $\Gamma_1(t) = \Gamma_2(t)$, then $[\alpha_1]_{\gamma(t)} = [\alpha_2]_{\gamma(t)}$ and so $\alpha_1 = \alpha_2$ on $V_1 \cap V_2$. Let $\varepsilon := min(\varepsilon_1, \varepsilon_2)$, then $\Gamma_1(t') = \Gamma_2(t')$, for all $t' \in [t-\varepsilon, t+\varepsilon]$. By connectivity of $[0,1]$, we get $T = [0,1]$ which finishes the proof. □

Proposition 2.2.3. *Let Γ be an analytic continuation of $[\lambda]_{t_0}$ along $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = t_0, \gamma(1) = t$. Then, $\Gamma(1)$ only depends on the homotopy class (with fixed extremities) of γ in $\mathbb{R}^k \backslash S_A$.*

Proof. Let $F : [0,1]^2 \to \mathbb{R}^k \backslash S_A$ be an homotopy with $\gamma = F(0, \cdot)$. For each $y \in [0,1]$, $[\lambda]_{t_0}$ has an analytic continuation $\Gamma(y, \cdot)$ along $F(y, \cdot)$. It is sufficient to prove that $y \mapsto \Gamma(y, 1)$ is locally constant. Fix $y_0 \in [0,1]$. The path $F(y_0, [0,1])$ is compact, and covered by open neighborhood $V_{F(y_0,x)}$ of $F(y_0, x)$ where $\Gamma(y_0, x')$ is the germ at x' of some analytic eigenvalue λ_x on $V_{F(y_0,x)}$. We extract from it a finite covering V_j, $j = 1 \ldots, k$ and choose a sequence $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $F(y_0, [a_{j-1}, a_j]) \subset V_j$ and $\forall x \in [a_{j-1}, a_j], \Gamma(y_0, x) = [\lambda_j]_{F(y_0,x)}$. For y sufficiently close to y_0, the inclusion $F(y, [a_{j-1}, a_j]) \subset V_j$ remains, and the map $\widetilde{\Gamma}(y, \cdot)$ given by $\widetilde{\Gamma}(y, x) = [\lambda_j]_{F(y,x)}$ for $x \in [a_{j-1}, a_j]$ defines an analytic continuation of $[\lambda_j]_{t_0}$ along $F(y, \cdot)$. Unicity of analytic continuation shows that $\widetilde{\Gamma}(y, \cdot) = \Gamma(y, \cdot)$. In particular, $\Gamma(y, 1) = \widetilde{\Gamma}(y, 1) = [\lambda_k]_{\gamma(1)} = \Gamma(y_0, 1)$. So $\Gamma(\cdot, 1)$ is locally constant, as announced, then it is constant. □

Definition 2.2.4. *Let $t_0 \in \Omega$, where Ω is an open subset of $\mathbb{R}^k \backslash S_A$. We call $\lambda(t)$ the **end-point** of the analytic continuation Γ of $[\lambda]_{t_0}$ along $\gamma : [0,1] \to \Omega$, $\gamma(0) = t_0, \gamma(1) = t$. The end-point $\lambda(t)$ is defined by $\lambda(t) := \Gamma(1)(t)$.*

Corollary 2.2.5. *Let $\Omega \subset \mathbb{R}^k$ be open, simply-connected such that $\Omega \cap S_A = \emptyset$. Then, there are analytic functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, d$ such that*

$$\chi_{A(t)}(x) = \prod_{i=1}^{d} (x - \lambda_i(t)),$$

for any $t \in \Omega$, $x \in \mathbb{R}$.

Proof. Given $t_0 \in \Omega$. Since t_0 is a quasi-regular point, there exists an open neighborhood V of t_0 and analytic functions $\lambda_1, \dots \lambda_d : V \to \mathbb{R}$. Moreover, for any $t \in \Omega$, $t \neq t_0$, there exists a path $\gamma : [0,1] \to \Omega$, $\gamma(0) = t_0$, $\gamma(1) = t$. Since Ω is path-connected, the path γ that joins t_0 and t exists. We now fix an index $i \in \{1, \dots d\}$ and we call λ the associated eigenvalue. Let $\lambda(t) := \Gamma(1)(t)$. Since $\Gamma(1)(t)$ is homotopy-invariant and Ω is simply-connected, $\lambda(t)$ is a well-defined function. By construction, λ is analytic (it locally coincides with analytic functions). Therefore, any analytic germ at t_0 of an eigenvalue of $A(t)$ can be extended analytically on Ω. The equality $\chi_{A(t)}(x) = \prod_{i=1}^{d}(x - \lambda_i(t))$ is an equality between analytic functions, that holds in a neighborhood of $\{t_0\} \times \mathbb{R}$; from isolated zero principle, the equality holds everywhere on the connected set $\Omega \times \mathbb{R}$. □

2.3 Monodromy around super-singular set

Definition 2.3.1. *Let $A(t) \in Sym_d(\mathbb{R})$ be an analytic family. Let $t_0 \in \mathbb{R}^k \backslash S_A$. The **germ spectrum** of the eigenvalues of $A(t)$ at t_0, denoted by $Spec([A]_{t_0})$, is defined to be*

$$Spec([A]_{t_0}) = \{[\lambda_1]_{t_0}, \dots, [\lambda_d]_{t_0}\},$$

where $\lambda_1(t), \dots, \lambda_d(t)$ are analytic functions in $Spec(A(t))$ in a neighborhood of t_0.

Definition 2.3.2. *Let $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^k$ be an analytic family. Let $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = \gamma(1) = t_0$ be a continuous loop in the space of parameters. The permutation*

$$\begin{array}{ccccc} \overline{\sigma_\gamma} & : & Spec([A]_{t_0}) & \to & Spec([A]_{t_0}) \\ & & [\lambda_i]_{t_0} & \mapsto & \Gamma_i(1) \end{array},$$

*where Γ_i is the analytic continuation of the germ $[\lambda_i]_{t_0}$ along γ, is called **the monodromy** of $Spec([A]_{t_0})$ along γ.*

Definition 2.3.3. *Let $A(t) \in Sym_d(\mathbb{R})$, $t \in \mathbb{R}^k$ be an analytic family. Let $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = \gamma(1) = t_0$ be a continuous loop in the space of parameters, and suppose $A(t_0) \notin \Sigma$. Then, the permutation*

$$\begin{array}{ccccccccc} \sigma_\gamma & : & Spec(A(t_0)) & \to & Spec([A]_{t_0}) & \to & Spec([A]_{t_0}) & \to & Spec(A(t_0)) \\ & & \lambda & \mapsto & [\lambda]_{t_0} & \mapsto & [\mu]_{t_0} = \overline{\sigma_\gamma}([\lambda]_{t_0}) & \mapsto & [\mu]_{t_0}(t_0) \end{array},$$

where $\overline{\sigma_\gamma}([\lambda]_{t_0})$ is the analytic continuation of the germ $[\lambda]_{t_0}$ along γ, with $[\mu]_{t_0}(t_0) = \mu$, is called ***the monodromy*** *of $Spec(A(t_0))$ along γ.*

Corollary 2.3.4. *From Proposition 2.2.3, the monodromies $\overline{\sigma_\gamma}$ and σ_γ of $A(t) \in Sym_d(\mathbb{R})$ along $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = \gamma(1) = t_0$ (with $A(t_0) \notin \Sigma$ for σ_γ) only depend on the homotopy class of γ in $\mathbb{R}^k \backslash S_A$.*

It is natural to ask about the dependence of the monodromies on the base point t_0 of $\pi_1(\mathbb{R}^k \backslash S_A, t_0)$. For this, let $t_1 \neq t_0 \in \mathbb{R}^k \backslash S_A$, and denote by

$$\begin{array}{rccc} \mathfrak{S}^{t_0} \; : & \pi_1(\mathbb{R}^k \backslash S_A, t_0) & \to & \mathfrak{S}(Spec([A]_{t_0})) \\ & \bar{\gamma} & \mapsto & \overline{\sigma_\gamma} \end{array},$$

$$\begin{array}{rccc} \mathfrak{S}^{t_1} \; : & \pi_1(\mathbb{R}^k \backslash S_A, t_1) & \to & \mathfrak{S}(Spec([A]_{t_1})) \\ & \bar{\gamma} & \mapsto & \overline{\sigma_\gamma} \end{array}.$$

Then, by Proposition 2.1.9, $\mathbb{R}^k \backslash S_A$ is path connected, so there exists $\delta : [0,1] \to \mathbb{R}^k \backslash S_A$, $\delta(0) = t_0, \delta(0) = t_1$. The analytic prolongation of the germs of the eigenvalues at t_0 along δ defines a bijection ϕ from $Spec([A]_{t_0})$ to $Spec([A]_{t_1})$. We get

$$\mathfrak{S}^{t_1} \circ \phi = \phi \circ \mathfrak{S}^{t_0},$$

and the following diagram commute.

$$\begin{array}{ccc} Spec([A]_{t_0}) & \xrightarrow{\mathfrak{S}^{t_0}} & Spec([A]_{t_0}) \\ \downarrow \phi & & \downarrow \phi \\ Spec([A]_{t_1}) & \xrightarrow{\mathfrak{S}^{t_1}} & Spec([A]_{t_1}) \end{array}$$

By the following lemma, the cycle structure of $\mathfrak{S}^{t_1}$ is the same as the cycle structure of $\mathfrak{S}^{t_0}$.

For any permutation $\alpha \in \mathfrak{S}_n$, we can write α as the product of disjoint cycles. Suppose that in the cycle decomposition of α, there are cycles of length $k_1, k_2, \ldots, k_\ell$, where $k_1 \geq k_2 \geq \cdots \geq k_\ell$. We call $(k_1, k_2, \ldots, k_\ell)$ the **cycle type** of α.

Let $\alpha, \beta \in \mathfrak{S}_n$, if there exists $\tau \in \mathfrak{S}_n$ such that $\beta = \tau\alpha\tau^{-1}$, we say that α and β are conjugate in $\mathfrak{S}_n$.

Lemma 2.3.5. *Let $\sigma, \rho \in \mathfrak{S}_n$ be two permutations. Then, ρ and σ are* ***conjugate*** *if and only if they have the same cycle type.*

Proof. Suppose that σ has the cycle type $(k_1, k_2, \ldots, k_\ell)$, so σ can be written as $\sigma = c_1 c_2 \ldots c_\ell$ where each c_i is a cycle of length k_i. Let $\tau \in \mathfrak{S}_n$ such that $\rho = \tau\sigma\tau^{-1}$, then we have

$$\tau\sigma\tau^{-1} = \tau c_1 \ldots c_\ell \tau^{-1} = (\tau c_1 \tau^{-1})(\tau c_2 \tau^{-1}) \ldots (\tau c_\ell \tau^{-1}). \tag{2.1}$$

Since for each $i = 1, \ldots, \ell$, c_i is a cycle of length k_i, $\tau c_\ell \tau^{-1}$ is also a cycle of length k_i. Moreover, since for all $i \neq j$, c_i and c_j are distinct and $\tau \in \mathfrak{S}_n$ is a bijection, it follows that $\tau c_i \tau^{-1}$ and $\tau c_j \tau^{-1}$ are also distinct. Thus, $\tau\sigma\tau^{-1}$ is a product of disjoint cycles $\tau c_i \tau^{-1}$ of length k_i. Therefore, any conjugate of σ, $\rho = \tau\sigma\tau^{-1}$, has the same cycle type of σ.

Now, suppose that σ and ρ have the cycle type $(k_1, k_2, \ldots, k_\ell)$, we show that σ and ρ are conjugate. Suppose that $\sigma = \alpha_1 \ldots \alpha_\ell$ and $\rho = \beta_1 \ldots \beta_\ell$ be the cycle decomposition of σ and ρ into the product of the disjoint cycles α_i and β_i of length k_i. For each i, let us write $\alpha_i = (a_{i_1} a_{i_2} \ldots a_{i_{k_i}})$ and $\beta_i = (b_{i_1} b_{i_2} \ldots b_{i_{k_i}})$ and we define the permutation $\tau \in \mathfrak{S}_n$ by $\tau(a_{ij}) = b_{ij}$, $i = 1, \ldots, \ell$, $j = 1, \ldots, k_i$. Then

$$\tau\alpha_i\tau^{-1} = \tau a_{i_1} a_{i_2} \ldots a_{i_{k_i}} \tau^{-1} = \tau a_{i_1}\tau^{-1}\tau a_{i_2}\tau^{-1} \ldots \tau a_{i_{k_i}}\tau^{-1} = b_{i_1} b_{i_2} \ldots b_{i_{k_i}}.$$

From this, we have

$$\tau\alpha\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_1\tau^{-1}) \ldots (\tau\alpha_\ell\tau^{-1}) = \beta_1\beta_2 \ldots \beta_\ell = \rho.$$

Hence, any two elements of $\mathfrak{S}_n$ with the same cycle type are conjugate. □

Definition 2.3.6. *Let $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \sum$. Consider the bijection*

$$\begin{array}{ccccc} \mathcal{D} & : & Spec(A(t_0)) & \to & \mathcal{E}(A(t_0)) \\ & & \lambda_i & \mapsto & E_{\lambda_i} \end{array},$$

where $\mathcal{E}(A(t_0)) = \cup_{i=1}^{d} E_{\lambda_i}$ and E_{λ_i} is the one-dimensional eigenspace associated to λ_i. The permutation $\sigma_{E_{\lambda_i}} := \mathcal{D} \circ \sigma_\gamma \circ \mathcal{D}^{-1}$ is called the ***monodromy of eigenspace*** *of $A(t_0)$ along γ.*

Remark 2.3.7. If Γ_i is the analytic prolongation of the eigenvalue λ_i along a continuous loop $\gamma : [0,1] \to \mathbb{R}^k \backslash S_A$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \sum$. Then $\mathcal{D} \circ \Gamma_i$ is the analytic prolongation of $\mathcal{D} \circ \lambda_i$ along γ.

2.4 Antipodal monodromy

The absence of monodromy around super-singular set is not sufficient to have analytic eigenvalues. Thus, we introduce another type of monodromy.

In this section, we consider a linear application $A : \mathbb{R}^k \to Sym_d(\mathbb{R})$ such that for any $t = (t_1, ..., t_k) \in \mathbb{R}^k$,

$$A(t) := t_1 A_1 + ... + t_k A_k$$

where $A_1, ..., A_k \in Sym_d(\mathbb{R})$. We call it shortly by a k-linear family.

Definition 2.4.1 (Half-turn path)**.** *Take $t_0 \in \mathbb{R}^k \backslash \{0\}$, and choose $u \in \mathbb{R}^k \backslash \{0\}$ such that $u \perp t_0$ and $\|u\| = \|t_0\|$. The **half-turn path** from t_0 in the direction u is the path $\gamma : [0, 1] \to \mathbb{R}^k \backslash \{0\}$ given by*

$$\gamma(s) = \cos(\pi s) t_0 + \sin(\pi s) u.$$

If the half-turn from t_0 in the direction u avoids S_A, the analytic prolongation of the germs of the eigenvalues at t_0 defines a bijection $\overline{\tau_{(t_0,u)}}$ from $Spec([A]_{t_0})$ to $Spec([-A]_{t_0})$. While 0 might be super-singular, we also can define a continuation of the eigenvalues of $A(-t_0)$ along the segment $[-t_0, t_0]$. Indeed, since the family is linear, $A(kt_0) = kA(t_0)$ for any k, so if $\lambda \in Spec(A(t_0))$, $[-t_0, t_0] \ni kt_0 \mapsto k\lambda$ is the unique analytic prolongation of the eigenvalue $-\lambda \in Spec(A(-t_0))$ over $[-t_0, t_0]$. (Note that if $A(t_0) \notin \Sigma$, $[-t_0, t_0]$ being an analytic arc, the existence of an analytic prolongation of a given eigenvalue along $[-t_0, t_0]$ exists by Rellich's Theorem 1.2.1; here, the linearity of the family makes this prolongation simple). We define the antipodal monodromy of the germs from t_0 in the direction u as the composition of the analytic prolongation along the half-turn path with this prolongation along $[-t_0, t_0]$.

Definition 2.4.2 (Antipodal monodromy of the germs)**.** *Let $t_0 \in \mathbb{R}^k \backslash \{0\}$, $u \perp t_0$, $\|u\| = \|t_0\|$. If the half-turn path from t_0 in the direction u avoids S_A, we define the **antipodal monodromy** from t_0 in the direction u to be the permutation $\overline{\tau'_{(t_0,u)}}$ of $Spec([A]_{t_0})$ given by*

$$\begin{array}{rccc} \overline{\tau'_{(t_0,u)}} : & Spec([A]_{t_0}) & \to & Spec([A]_{t_0}) \\ & [\lambda]_{t_0} & \mapsto & -\overline{\tau_{(t_0,u)}}([\lambda]_{t_0}) \end{array}$$

where $\overline{\tau_{(t_0,u)}}$ is the analytic prolongation of the germs along the half-turn path from t_0

in the direction u.

Definition 2.4.3 (Antipodal monodromy of the eigenvalues)**.** *Let $t_0 \in \mathbb{R}^k \backslash \{0\}$, $u \perp t_0$, $\|u\| = \|t_0\|$. If the half-turn from t_0 in the direction u avoids S_A and $A(t_0) \notin \Sigma$, we define the* ***antipodal monodromy of the eigenvalues*** *from t_0 in the direction u to be the permutation $\tau'_{(t_0,u)}$ of $Spec(A(t_0))$ given by*

$$\begin{array}{ccccccccc} \tau'_{(t_0,u)} & : & Spec(A(t_0)) & \to & Spec([A]_{t_0}) & \to & Spec([-A]_{t_0}) & \to & Spec(A(t_0)) \\ & & \lambda & \mapsto & [\lambda]_{t_0} & \mapsto & [\mu]_{t_0} = \overline{\tau_{(t_0,u)}}([\lambda]_{t_0}) & \mapsto & -[\mu]_{t_0}(t_0) \end{array},$$

where $\overline{\tau_{(t_0,u)}}$ is the analytic prolongation of the germs along the half-turn path from t_0 in the direction of u with $[\mu]_{t_0}(t_0) = \mu$.

Consider an analytic family $A(t)$, $t \in \mathbb{R}^k$. Let $t_0 \in \mathbb{R}^k$ such that the eigenvalues of $A(t_0)$ and $A(-t_0)$ are all simple. We define the antipodal monodromy from t_0 in the direction u, $u \perp t_0$ and $\|u\| = \|t_0\|$ as the composition of the analytic prolongation of the eigenvalues of $A(t_0)$ along the half-turn path from t_0 in the direction u with the analytic prolongation of the eigenvalues of $A(-t_0)$ along $[-t_0, t_0]$. If there exists the antipodal monodromy, then the eigenvalues of $A(t)$ are not analytic functions. But, we will see in an example that the antipodal mondromy of an affine family depends on the base point t_0. So, the antipodal monodromy of an analytic family is less interesting than the antipodal monodromy of a k-linear family.

2.5 Combinatorics of monodromies

In this section, we study on monodromy around a super-singular set, antipodal monodromy and the relation between them for a 2-linear family $A(t)$, and we will prove Theorem 2.5.1.

We say that a permutation $\sigma \in \mathfrak{S}_d$ is a square if there exists a permutation $p \in \mathfrak{S}_d$ such that $\sigma = p^2$.

Theorem 2.5.1. *A permutation $\sigma \in \mathfrak{S}_d$ is a square if and only if there are $A_1, A_2 \in Sym_d(\mathbb{R})$ such that σ is the monodromy of the eigenvalues of the matrix $A(t) = t_1A_1 + t_2A_2$ along a loop $\gamma : [0,1] \to \mathbb{R}^2 \backslash (0,0)$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \Sigma$.*

The proof of Theorem 2.5.1 is divided into two steps. First, we show that the monodromy of a 2-linear family is a square, Proposition 2.5.3. Then, in Proposition 2.5.10, for a given permutation $p \in \mathfrak{S}_d$, we construct two symmetric matrices A_1

and A_2 such that $\sigma(A(t_1,t_2)) = p^2$, where σ is the monodromy of the eigenvalues of $A(t_1,t_2) = t_1A_1 + t_2A_2$. The following lemma shows that the cycle type of a permutation determines whether a given permutation is a square.

Lemma 2.5.2. *A permutation $\sigma \in \mathfrak{S}_d$ is a square if and only if its unique decomposition into the product of distinct cycles contains an even number of cycles of even length.*

Proof. Let $\sigma = C_1^2 \dots C_{i_2}^2 \dots C_1^l \dots C_{i_l}^l$ be the decomposition of σ into distinct cycles C_j^k of length k where $2 \leqslant k \leqslant l$, $1 \leqslant j \leqslant i_k$. We suppose that the decomposition of σ contains an even number of cycles of length even and we prove that σ has a square root. For $j = 0, \dots, \frac{i_k}{2}$, if $C_{2j+1}^k = (a_1 \dots a_k)$ and $C_{2j+2}^k = (b_1 \dots b_k)$ be two cycles of even length k in the decomposition of σ, then $C_{2j+1}^k C_{2j+2}^k = (\tilde{C}_j^{\,2k})^2$ with $\tilde{C}_j^{\,2k} = (a_1 b_1 \dots a_k b_k)$. For any odd number $k' \in \{2, \dots, l\}$ and for all $j \in \{1, \dots, i'_k\}$, if $C_j^{k'} = (d_1 d_2 \dots d_{k'})$, then $C_j^{k'} = (\tilde{C}_j^{\,k'})^2$ with $\tilde{C}_j^{\,k'} = (d_1 d_3 \dots d_{k'} d_2 \dots d_{k'-1})$. Finally,

$$\sigma = \prod_{k' \text{ odd}, j=1}^{i_{k'}} C_j^{k'} \prod_{k \text{ even}, j=1}^{i_k} C_j^k \tag{2.2}$$

$$= \prod_{k' \text{ odd}, j=1}^{i_{k'}} (\tilde{C}_j^{\,k'})^2 \prod_{k \text{ even}, j=0}^{\frac{i_k}{2}} (\tilde{C}_j^{\,k})^2 \tag{2.3}$$

$$= \Big(\prod_{k' \text{ odd}, j=1}^{i_{k'}} \tilde{C}_j^{\,k'} \prod_{k \text{ even}, j=0}^{\frac{i_k}{2}} \tilde{C}_j^{\,k} \Big)^2 = p^2. \tag{2.4}$$

Now assume that $\sigma = p^2$ and let $p = d_1 \dots d_{l'} c_1 \dots c_l$ be the decomposition of p into odd cycles d_i of length k'_i, $i \in \{1, \dots, l'\}$ and even cycles c_j of length k_j, $j \in \{1, \dots, l\}$. Then

$$p^2 = d_1^2 d_2^2 \dots d_{l'}^2 c_1^2 \dots c_l^2 = (d'_1 d'_2 \dots d'_{l'})(e_1 f_1 \dots e_l f_l) = \sigma,$$

where for $i \in \{1, \dots, l'\}$ and $j \in \{1, \dots, l\}$, $d'_i = d_i^2$ is an odd cycle of length k'_i and if $c_j = (a_1 b_1 a_2 b_2 \dots a_k b_k)$, then $e_j = (a_1 \dots a_k)$, $f_j = (b_1 \dots b_k)$ are even cycles of length $\frac{k_j}{2}$. Therefore, the cycles of even length in σ come in pairs. Thus, in the decomposition of σ into distinct cycles, the number of cycles of even length is even. □

2.5.1 The monodromy of a 2-linear family is a square

The following proposition proves that the monodromy of a 2-linear family is a square, which finishes one part of the proof of Theorem 2.5.1.

Proposition 2.5.3. *Let $A(t) \in Sym_d(\mathbb{R})$ be a 2-linear family, $t_0 \in \mathbb{R}^2\backslash\{0\}$ and $u \perp t_0$, $\|u\| = \|t_0\|$ with $A(t_0) \notin \Sigma$. Let $\gamma : [0,1] \to \mathbb{R}^2\backslash\{0\}$ be the loop given by $\gamma(s) = \cos(2\pi s)(t_0) + \sin(2\pi s)(u)$. Then, the monodromy along γ is*

$$\sigma_\gamma = (\tau'_{(t_0,u)})^2,$$

where $\tau'_{(t_0,u)}$ is the antipodal monodromy from t_0 in the direction of u.

The main idea of the proof is to decompose γ as the combination of two loops along which the monodromy coincide. Namely, γ is the concatenation of the half-turn path from t_0 in the direction of u followed by the straight path from $-t_0$ to t_0, and of the straight path from t_0 to $-t_0$ followed by the half-turn path from $-t_0$ in the direction of $-u$.

Proof. Let $\delta : [0,1] \to \mathbb{R}^2\backslash(0,0)$ be the half-turn path from t_0 in the direction u and $\eta = -\delta$.

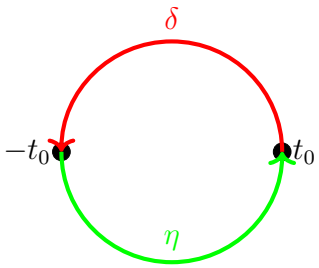


Figure 2.1: Monodromy along the unit circle centered at the origin

We define

$$\begin{array}{rccc} \Lambda_\delta & : & [0,1] \times Spec(A(t_0)) & \to & Spec(A(\delta(t))) \\ & & (t,\lambda) & \mapsto & \Lambda_{\delta(t)}(\lambda) \end{array},$$

where $\Lambda_{\delta(t)}(\lambda)$ is the value at $\delta(t)$ of the analytic continuation of λ along δ, and similarly, we let $\Lambda_{\eta(t)}(\lambda)$ be the value at $\eta(t)$ of the analytic continuation of $\lambda \in Spec(A(-t_0))$ along η, so that $\sigma_\gamma(\lambda) = \Lambda_{\eta(1)}(\Lambda_{\delta(1)}(\lambda))$.

For a given $\lambda \in Spec(A(t_0))$, remark that $t \mapsto -\Lambda_{\delta(t)}(\lambda)$ is analytic in terms of

$t \in [0,1]$ and belongs to $-Spec(A(\delta(t))$. But, since A is linear,

$$-Spec(A(\delta(t)) = Spec(A(-\delta(t)) = Spec(A(\eta(t)).$$

By uniqueness of analytic continuation, we then get that $\Lambda_{\eta(t)}(-\lambda) = -\Lambda_{\delta(t)}(\lambda)$. In particular, for any $\lambda \in Spec(A(t_0))$,

$$\begin{aligned} \sigma_\gamma(\lambda) &= \Lambda_{\eta(1)}(\Lambda_{\delta(1)}(\lambda)) \\ &= -\Lambda_{\delta(1)}(-\Lambda_{\delta(1)}(\lambda)) \\ &= \tau'_{(t_0,u)} \circ \tau'_{(t_0,u)}(\lambda), \end{aligned}$$

where $\tau'_{(t_0,u)}(\lambda) = -\Lambda_{\delta(1)}(\lambda)$ is the antipodal monodromy from t_0 in the direction of u. □

In Proposition 2.5.3, we proved that the monodromy along the loop γ which turns one time around the origin is a square. Since the fundamental group of $\mathbb{R}^2\backslash(0,0)$ is isomorphic to the integers, with the path γ acting as a generator, the proposition is true for any homotopy class of a loop γ in $\mathbb{R}^2\backslash(0,0)$ which turns k times around the origin.

Corollary 2.5.4. *(Corollary of Proposition 2.5.3) If σ is the monodromy of the eigenvalues of a 2-linear family $A(t)$ along a loop $\gamma : [0,1] \to \mathbb{R}^2\backslash(0,0)$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \Sigma$, then σ is a square.*

2.5.2 Construction of a linear family with prescribed monodromy

We will construct the two matrices of Theorem 2.5.1 by blocks that are related to the cycle decomposition of a square root of σ. These matrices are build from the following two matrices:

$$A_1 := \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & (d-1) & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

and $A_2 = diag(0, \dots, 0, 1)$. We first study the monodromy of the family $A(t_1, t_2) = t_1 A_1 + t_2 A_2$. For this, we blow-up the origin in the space of parameters. Namely, we let $\pi_0 : (u, v) \to (u, uv)$ and $\pi_1 : (u, v) \to (uv, v)$ be the expressions of the blow-up π of $(0, 0) \in \mathbb{R}^2$ in the two classical system of coordinates. The pull-back of A by π is given in coordinates by

$$A \circ \pi_0 = uA_1 + uvA_2 = u(A_1 + vA_2) = uB(v),$$
$$A \circ \pi_1 = uvA_1 + vA_2 = v(uA_1 + A_2) = vB'(u).$$

Hence, for $t_1 \neq 0$ we get $Spec(A(t_1, t_2)) = t_1 Spec(B(\frac{t_2}{t_1}))$ and for $t_2 \neq 0$, we get $Spec(A(t_1, t_2)) = t_2 Spec(B'(\frac{t_1}{t_2}))$.

The study of the eigenvalues of $A \circ \pi$ is made in the following lemmas. We set $a_0 = -\infty$, $a_k = k$ for $k = 1, \dots, d-1$, and $a_d = +\infty$.

Lemma 2.5.5. *For any $v \in \mathbb{R}$, the matrix $B(v) = A_1 + vA_2$ admits d distinct analytic eigenvalues $\lambda_1(v), \dots, \lambda_d(v)$ such that for each $k \in \{1, \dots, d\}$, $\lambda_k(v) \in (a_{k-1}, a_k)$. In particular, $A_1 \notin \Sigma$.*

Proof. Note that the super-singular set of the one-parameter family $B(v)$ is empty. So, all the points are quasi-regular and all the eigenvalues of $B(v)$ are analytic functions. We show that for each $v \in \mathbb{R}$, $\chi_{B(v)}(k) = (-1)^k (k-1)!(d-k-1)!$ for a fixed $k \in \{1, \dots, d-1\}$. From $\chi_{B(v)}(x) = det(A_1 + vA_2 - x\mathbb{I}_d)$, we get

$$\chi_{B(v)}(k) = \begin{vmatrix} 1-k & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & 2-k & \ddots & & & & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\ \vdots & & \ddots & -1 & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & 0 & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & 1 & \ddots & \vdots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & d-1-k & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & v-k \end{vmatrix}.$$

Expanding this determinant with respect to the k-th row gives

$$\chi_{B(v)}(k) := (-1)^{k+d} \begin{vmatrix} 1-k & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \\ 0 & 2-k & \ddots & \ddots & & & & & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \vdots & \\ \vdots & & \ddots & -1 & 0 & \ddots & & & \vdots & \\ \vdots & & & \ddots & 0 & 1 & \ddots & & \vdots & \\ \vdots & & & & \ddots & \ddots & 2 & \ddots & \vdots & \\ \vdots & & & & & \ddots & \ddots & \ddots & 0 & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & d-1-k & \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \end{vmatrix}.$$

Now expanding with respect to the k-th column, we get

$$\chi_{B(v)}(k) = (-1)^{d+k} \times (-1)^{d-k-1}(d-1-k)! \times (-1)^{k-1}(k-1)! \tag{2.5}$$
$$= (-1)^k (d-1-k)!(k-1)! \tag{2.6}$$

Therefore, for all $k \in \{1, ..., d-1\}$, $\chi_B(k)$ is a non-zero constant, and for $k < d-1$, $\chi_B(k)\chi_B(k+1) < 0$. In particular, by the Mean Value Theorem, $\chi_{B(v)}$ admits a root in the interval $(k, k+1)$, for $k = 1, \ldots, d-2$. Moreover, as $x \to \pm\infty$, $\chi_{B(v)}(x) \sim_{\pm\infty} (-1)^d x^d$ while $\chi_B(d-1) = (-1)^{d-1}(d-2)!$ and $\chi_B(1) = -(d-2)!$, so in a similar way, $\chi_{B(v)}$ admits a root in $(-\infty, 1)$ and another one in $(d-1, +\infty)$. Then, for any v, $\chi_B(v)$ has d distinct eigenvalues $\lambda_1(v), \ldots, \lambda_d(v)$ such that for each $k \in \{1, \ldots, d\}$, $\lambda_k(v) \in (a_{k-1}, a_k)$ with $a_k = k$ if $k \neq d$, $a_0 = -\infty$ and $a_d = +\infty$. □

Lemma 2.5.6. *For any $v \in \mathbb{R}$, $B(v) = A_1 + vA_2$ has analytic eigenvalues $\lambda_1(v) < \cdots < \lambda_d(v)$ such that for each $k = 1, \ldots, d$,*

$$\lim_{v \to -\infty} \lambda_k(v) = a_{k-1} \text{ and } \lim_{v \to +\infty} \lambda_k(v) = a_k.$$

Proof. By Lemma 2.5.5, for any $v \in \mathbb{R}$, $B(v)$ has d distinct analytic eigenvalues $\lambda_1(v), \ldots, \lambda_d(v)$. Without loss of generality, we can suppose that $\lambda_1(v) < \cdots < \lambda_d(v)$.

Let us first show that for any fixed parameter x,

$$deg(v \mapsto \chi_{B(v)}(x)) \leqslant 1. \tag{2.7}$$

$$\chi_{B(v)}(x) = \begin{vmatrix} 1-x & 0 & \cdots & 0 & 1 \\ 0 & 2-x & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & d-1-x & 1 \\ 1 & \cdots & \cdots & 1 & v-x \end{vmatrix}.$$

We expand this determinant with respect to the d-th column. We get

$$\chi_{B(v)}(x) = (v-x) \begin{vmatrix} 1-x & 0 & \cdots & 0 \\ 0 & 2-x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d-1-x \end{vmatrix} + R,$$

where R does not depend on v. Hence,

$$\chi_{B(v)}(x) = (1-x)(2-x)\dots(d-1-x)v + g(x) \tag{2.8}$$

where $g(x)$ is a polynomial of degree d in terms of x which does not depend on v.

Now, we claim that for all $k \in \{1, \dots, d\}$, λ_k is strictly monotone. Indeed, if there are $v_1, v_2 \in \mathbb{R}$ such that $\lambda_k(v_1) = \lambda_k(v_2) = \alpha$, then $\chi_{B(v_1)}(\alpha) = \chi_{B(v_2)}(\alpha) = 0$. Notice that $\alpha \notin \{1, \dots, d-1\}$ (because if $\alpha \in \{1, \dots, d-1\}$, (2.6) shows that $\chi_{B(v)}(\alpha) \neq 0$ for all v), so (2.8) gives $v_1 = v_2$. Since λ_k is injective and continuous, λ_k is strictly monotone.

Our next claim is that $\lambda_k'(0) > 0$. Since $\chi_{B(0)}(x)$ has d distinct roots, for any $i \in \{1, \dots, d-1\}$,

$$\frac{\partial \chi_{B(0)}}{\partial x}(\lambda_i(0)) \, \frac{\partial \chi_{B(0)}}{\partial x}(\lambda_{i+1}(0)) < 0.$$

Moreover, $\chi_{B(0)}(x) \sim_{x \to -\infty} (-1)^d x^d$, so $\lim_{x \to -\infty} \chi_{B(0)}(x) = +\infty$, then $\frac{\partial \chi_{B(0)}}{\partial x}(\lambda_1(0)) < 0$, and finally,

$$sgn \, \frac{\partial \chi_{B(0)}}{\partial x}(\lambda_k) = sgn \, (-1)^k. \tag{2.9}$$

Now, we fix $x \in \mathbb{R}$. By (2.8), we get $\dfrac{\partial \chi_{B(0)}}{\partial v}(x) = \displaystyle\prod_{i=1}^{d-1}(i-x)$. Since, from Lemma (2.5.5), $\lambda_k(0) \in (a_{k-1}, a_k)$,

$$sgn \left(\frac{\partial \chi_{B(0)}}{\partial v}(\lambda_k(0)) \right) = sgn \left(\prod_{i=1}^{d-1}(i - \lambda_k(0)) \right) = sgn \, (-1)^{k-1}. \tag{2.10}$$

On the other hand, $\chi_{B(v)}(\lambda_k(v)) \equiv 0$. Thus,

$$\frac{\partial \chi_{B(v)}}{\partial x}(\lambda_k(v)) + \frac{\partial \chi_{B(v)}}{\partial v}(\lambda_k(v))\lambda_k'(v) = 0. \tag{2.11}$$

Replacing v by 0 in (2.11) gives

$$\lambda_k'(0) = \frac{-\frac{\partial \chi_{B(0)}}{\partial x}(\lambda_k(0))}{\frac{\partial \chi_{B(0)}}{\partial v}(\lambda_k(0))}. \tag{2.12}$$

Then by (2.9) and (2.10), we get

$$sgn\,(\lambda_k'(0)) = sgn\left((-1)^{k+1} \times (-1)^{-k+1}\right) = sgn\,(-1)^2 = +$$

Finally, λ_k is monotone and has positive derivative at 0, so λ_k is strictly increasing.

Now, from (2.8), for given $x \in (a_{k-1}, a_k)$, $\chi_{B(v)}(x)$ has degree exactly one in v, then has a root. So x belongs to the image of an eigenvalue. But only λ_k has values in (a_{k-1}, a_k), so x is in the image of λ_k. In other terms, λ_k is surjective onto (a_{k-1}, a_k).

Finally, λ_k is an increasing bijection from $\mathbb{R}$ to (a_{k-1}, a_k), so $\lim_{v\to-\infty} \lambda_k(v) = a_{k-1}$ and $\lim_{v\to+\infty} \lambda_k(v) = a_k$. □

Lemma 2.5.7. *Let $B'(u) = uA_1 + A_2$, $u \in \mathbb{R}$. Then, $d-1$ analytic eigenvalues of B', $\mu_1, \ldots, \mu_{d-1}$ vanish at $u = 0$ and their slopes $\mu_1'(0), \ldots, \mu_{d-1}'(0)$ are $1, 2, \ldots, d-1$.*

We prove the lemma by relating the slopes of the eigenvalues of the matrix B' that vanish at $u = 0$, with the finite limits of the eigenvalues of $B(v) = A_1 + vA_2$ at $v = +\infty$.

Proof. Let μ_i, $i = 1, \ldots, d$ be the analytic eigenvalues of B', and denote by $\lambda_1 < \cdots < \lambda_d$ the analytic eigenvalues of B, with $B(v) = A_1 + vA_2$ (see Lemma (2.5.5)). Notice that

$$\begin{aligned} B'(u) &= uA_1 + A_2 \\ &= u\left(A_1 + \frac{1}{u}A_2\right) = uB\left(\frac{1}{u}\right). \end{aligned}$$

So for $u \neq 0$, each eigenvalue μ_i is obtained as $\mu_i(u) = u\lambda_k\left(\frac{1}{u}\right)$, for some k. Since $\lambda_1 < \cdots < \lambda_d$, up to reindexing, we get for $u > 0$, $\mu_k(u) = u\lambda_k\left(\frac{1}{u}\right)$. Now for

$k = 1, \ldots, d-1$,

$$\lim_{u \to 0^+} \mu_k(u) = \lim_{u \to 0^+} u\lambda_k\left(\frac{1}{u}\right) = 0,$$

since $\lim_{u\to 0^+} \lambda_k\left(\frac{1}{u}\right) = \lim_{v\to+\infty} \lambda_k(v) = k$, by Lemma 2.5.6.

Moreover, for $k = 1, \ldots, d-1$,

$$\mu_k'(0) = \lim_{u \to 0^+} \frac{\mu_k(u)}{u} = \lim_{u \to 0^+} \lambda_k\left(\frac{1}{u}\right) = k.$$

□

Proposition 2.5.8. *Let $A(t_1, t_2) = t_1A_1 + t_2A_2$, $(t_1, t_2) \in \mathbb{R}^2$. Then, the antipodal monodromy τ' from $t_0 = (1, 0)$ in the direction $u = (0, 1)$ is the permutation of the ordered set $Spec(A(1, 0))$ defined by*

$$\tau' = \begin{pmatrix} 1 & 2 & \ldots & d-1 & d \\ 2 & 3 & \ldots & d & 1 \end{pmatrix}.$$

Proof. Set $A(\theta) = \cos(\theta)A_1 + \sin(\theta)A_2$. Notice that $A(0) = A_1$, $A_1 \notin \Sigma$ and denote by $\lambda_1, \ldots, \lambda_d : [0, \pi] \to \mathbb{R}$ the analytic eigenvalues of $A(\theta)$ with $\lambda_1(0) < \cdots < \lambda_d(0)$.

Let us show that $\lambda_d(\pi) = -\lambda_1(0)$. Since $Spec(A(\frac{\pi}{2})) = \{0, 1\}$, $\lambda_d(\frac{\pi}{2}) \in \{0, 1\}$. We claim that $\lambda_d(\frac{\pi}{2}) = 1$. Otherwise, $\lambda_d(\frac{\pi}{2}) = 0$, and there exists $k \in \{1, \ldots, d-1\}$ such that $\lambda_k(\frac{\pi}{2}) = +1$ so $\lambda_d(\frac{\pi}{2}) < \lambda_k(\frac{\pi}{2})$ while $\lambda_d(0) > \lambda_k(0)$. This leads to a contradiction, since from Lemma 2.5.5, $A(\theta) = \cos(\theta)\ B(\tan(\theta))$ has distinct eigenvalues over $(0, \frac{\pi}{2})$. The same argument on $(\frac{\pi}{2}, \pi)$ shows that $\lambda_d(\pi) > \lambda_k(\pi)$ for $k = 1, \ldots, d-1$ so $\lambda_d(\pi) = \max\left(Spec(A(\pi))\right) = -\lambda_1(0)$.

We next show that for all $k \neq d$, $\lambda_k(\pi) = -\lambda_{k+1}(0)$. Notice that for all $k \neq d$, $\lambda_k(\frac{\pi}{2}) = 0$, since otherwise $\lambda_d(\frac{\pi}{2}) = \lambda_k(\frac{\pi}{2}) = 1$, contrary to the fact that the multiplicity of the eigenvalue 1 is 1 at $\theta = \frac{\pi}{2}$. Moreover, since $A(\theta) = \sin(\theta)\left(B'(\text{cotan}\,(\theta))\right)$ with $B'(u) = uA_1 + A_2$, so $\lambda_k(\theta) = \sin(\theta)\ \mu_k(\text{cotan}\,(\theta))$, where μ_k is the analytic eigenvalue of $B'(u)$ that vanishes at $u = 0$. Thus,

$$\lambda_k'(0) = \Big(\sin(\theta)\left(\mu_k(\text{cotan}\,(\theta))\right)\Big)'_{|\theta=0} = -\mu_k'(0).$$

Then by Lemma 2.5.7, the slopes $\lambda_1'(0), \ldots, \lambda_d'(0)$ are all different. Hence, the eigenvalues $\lambda_1, \ldots, \lambda_{d-1}$ coincide at $\frac{\pi}{2}$ with different slopes, so their ordering reverse at $\frac{\pi}{2}$. Since again by Lemma 2.5.5, these eigenvalues have constant ordering on $(0, \frac{\pi}{2})$ and

$(\frac{\pi}{2}, \pi)$ we get $\lambda_1(\pi) > \cdots > \lambda_{d-1}(\pi)$. However,

$$\{\lambda_1(\pi), \ldots, \lambda_{d-1}(\pi)\} = Spec(-A_1)\backslash\{\lambda_d(\pi)\} = \{-\lambda_2(0), \ldots, -\lambda_d(0)\},$$

yet $-\lambda_d(0) < \cdots < -\lambda_2(0)$, so we get finally $\lambda_k(\pi) = -\lambda_{k+1}(0)$ for $k \in \{1, \ldots, d-1\}$.

Thus, the antipodal monodromy from $(1,0)$ in the direction of $(0,1)$ is given by

$$\begin{aligned}\tau'(\lambda_k(0)) &= -\lambda_k(\pi) \\ &= -(-\lambda_{k+1}(0)) = \lambda_{k+1}(0),\end{aligned}$$

which finishes the proof. □

Proposition 2.5.9. *Let $A_i(t_1, t_2) \in Sym_{d_i}(\mathbb{R})$, $i = 1, \ldots, k$, $d_1 + \cdots + d_k = d$ be 2-linear families and $A(t_1, t_2) = diag(A_1(t_1, t_2), \ldots, A_k(t_1, t_2))$. Suppose that $A(1,0) \notin \Sigma$, then the antipodal monodromy τ' of $A(t_1, t_2)$ from $(1,0)$ in the direction of $(0,1)$ is given by $\tau' = \tau_1 \circ \cdots \circ \tau_k$, where*

$$\tau_i(\lambda) = \begin{cases} \tau_i'(\lambda) & \text{if } \lambda \in Spec(A_i(0)) \\ \lambda & Otherwise \end{cases},$$

where τ_i' is the antipodal monodromy of A_i from $(1,0)$ in the direction of $u = (0,1)$.

Proof. Note that $\chi_A = \prod_{i=1}^{k} \chi_{A_i}$. Moreover, $\mathbb{R}^2\backslash(0,0)$ is made of the quasi-regular points of $A(t_1, t_2)$ and each A_i. Thus, in a neighborhood V of the point $(1,0)$, we have $\chi_A = \prod_{\ell=1}^{d}(x - \lambda_\ell)$ and $\chi_{A_i} = \prod_{j=1}^{d_i}(x - \lambda_{ij})$ where the λ_ℓ and λ_{ij} are analytic and uniquely determined. Yet, $\chi_A = \prod_{\ell=1}^{d}(x - \lambda_\ell) = \prod_{i=1}^{k}\prod_{j=1}^{d_i}(x - \lambda_{ij})$. Then,

$$\{\lambda_\ell \ ; \ l = 1, \ldots, d\} = \{\lambda_{ij} \ ; \ i = 1 \ldots, k, j = 1, \ldots, d_i\},$$

over V. For a given $\lambda \in Spec(A)$, let $\ell \in \{1, \ldots, d\}$ be the unique index such that $\lambda = \lambda_\ell(1,0)$ and and i, j such that $\lambda_\ell = \lambda_{ij}$ over V. Then, the analytic continuation of λ_{ij} along the half-turn path from $(1,0)$ in the direction of $(0,1)$ is an analytic continuation of λ_ℓ, so $\tau'(\lambda_\ell) = \tau_i'(\lambda_{ij}(1,0)) = \tau_1 \circ \cdots \circ \tau_k(\lambda)$. □

Proposition 2.5.10. *Suppose that $p \in \mathfrak{S}_d$ has the cycle decomposition $p = C_1C_2...C_l$, where C_i is a cycle of length k_i. Let*

$$A_i(t_1,t_2) := \begin{pmatrix} t_1 & 0 & \cdots & 0 & t_1 \\ 0 & 2t_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & (k_i-1)t_1 & t_1 \\ t_1 & \cdots & \cdots & t_1 & t_2 \end{pmatrix}.$$

Then, there are $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$ such that $\sigma = p^2$ is the monodromy of the eigenvalues of the matrix $A(t_1,t_2) = diag(A_1(t_1,t_2) + \alpha_1\mathbb{I}_{k_1}, \ldots, A_\ell(t_1,t_2) + \alpha_\ell\mathbb{I}_{k_\ell})$ along a loop $\gamma : [0,1] \to \mathbb{R}^2\backslash(0,0)$, $\gamma(0) = \gamma(1) = t_0$ with $A(t_0) \notin \Sigma$.

Proof. Let $t_0 = (1,0)$ and $u = (0,1)$. Then, Proposition 2.5.8 proves that the antipodal monodromy from t_0 in the direction u of the matrix A_i is a cycle of length k_i. We choose real numbers $\alpha_1, \ldots, \alpha_\ell$ such that for all $i \neq j \in \{1, \ldots, \ell\}$, $Spec(A_i(t_0) + \alpha_i\mathbb{I}_{k_i}) \cap Spec(A_j(t_0) + \alpha_j\mathbb{I}_{k_j}) = \emptyset$. Then

$$A(t_1,t_2) := \begin{pmatrix} A_1(t_1,t_2) + \alpha_1\mathbb{I}_{k_1} & & & \\ & A_2(t_1,t_2) + \alpha_2\mathbb{I}_{k_2} & & \\ & & \ddots & \\ & & & A_\ell(t_1,t_2) + \alpha_\ell\mathbb{I}_{k_\ell} \end{pmatrix}$$

is a block diagonal matrix associated with the permutation p such that $A(t_0) \notin \Sigma$. By Proposition 2.5.9, the antipodal monodromy $\tau'_{(t_0,u)}$ from t_0 in the direction u of $A(t_1,t_2)$ is given by $\tau'_{(t_0,u)} = p$. We conclude from Proposition 2.5.3 that the monodromy of eigenvalues of $A(t_1,t_2)$ along γ is $\sigma_\gamma = (\tau'_{(t_0,u)})^2 = p^2$. □

Example 2.5.11. In this example, we see that the lack of monodromy is not sufficient to have eigenvalues as analytic functions. Let

$$A(t_1,t_2) = \begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix} = t_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In Example 0.0.1 We saw that the eigenvalues of $A(t_1,t_2)$ cannot be chosen as the analytic functions on $\mathbb{R}^2$. Since as we move on the circle $t_1^2+t_2^2 = 1$ we get $\lambda_1 = +1$ and $\lambda_2 = -1$, the monodromy of eigenvalues of $A(t_1,t_2)$ is the identity permutation, Figure 2.2. This shows that the lack of the monodromy is not sufficient to get eigenvalues as analytic functions.

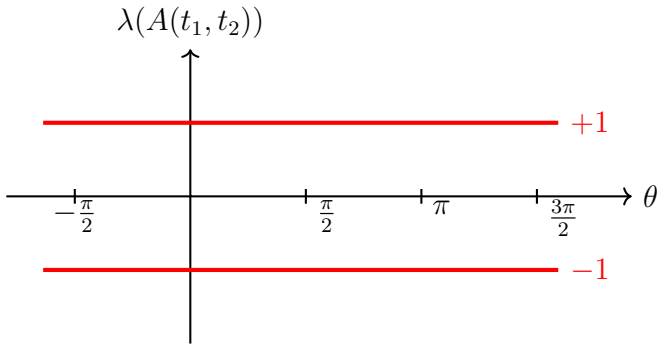


Figure 2.2: The eigenvalues of $A(\theta) = \cos\theta A_1 + \sin\theta A_2$

However, the antipodal monodromy from $t_0 = (1,0)$ in the direction of $u = (0,1)$ is a cycle of length two $\tau'_{(t_0,u)} = (\lambda_1\lambda_2)$.

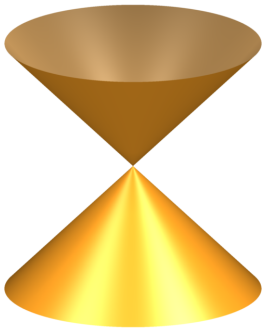


Figure 2.3: The eigenvalues of $A(t_1,t_2)$

In Theorem 4.3.1, we will see that the existence of non-trivial antipodal monodromy is the only obstruction for getting the eigenvalues as analytic functions.

Example 2.5.12. Let $A(t) = t_1A_1 + t_2A_2$ be the linear family of symmetric matrices dimension three, where A_1 and A_2 are in the form of the matrices of Subsection 2.5.2. We see that the monodromy of the eigenvalues along the unit circle centered at the origin is non-trivial. Consider

$$A(t_1,t_2) = \begin{pmatrix} t_1 & 0 & t_1 \\ 0 & 2t_1 & t_1 \\ t_1 & t_1 & t_2 \end{pmatrix} = t_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $Spec(A_1) = \{-0.8, 1.3, 2.5\}$ and $Spec(A_2) = \{0,0,1\}$. By Lemmas 2.5.5 and 2.5.6, the eigenvalues of the $B(v) = A_1 + vA_2$ are distinct and strictly increasing such

that for $k = 1, 2, 3$, $\lim_{v\to-\infty} \lambda_k(v) = a_{k-1}$ and $\lim_{v\to+\infty} \lambda_k(v) = a_k$, with $a_k = k$ for $k = 1, 2$, $a_0 = -\infty$, $a_3 = +\infty$ In the following figure, we see the graph of the eigenvalues of $B(v)$.

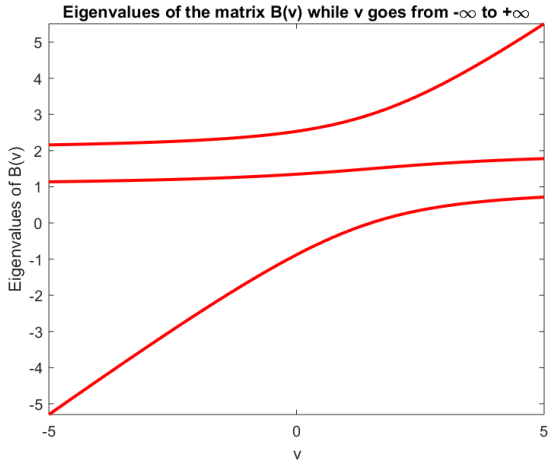


Figure 2.4: The eigenvalues of $B(v)$

By Lemma 2.5.7, the slopes of the eigenvalues of $B'(u) = uA_1 + A_2$ at $u = 0$ are $\{1, 2\}$. By Proposition 2.5.8, the antipodal monodromy $\tau'_{(t_0,u)}$ is a cycle of length 3. Finally, by Proposition 2.5.10, the monodromy of the eigenvalues of $A(t_1, t_2)$ is given by $\sigma = (\tau'_{(t_0,u)})^2$. Hence, $\sigma = (\lambda_1\lambda_3\lambda_2)$ is a cycle of length three, Figure 2.5.

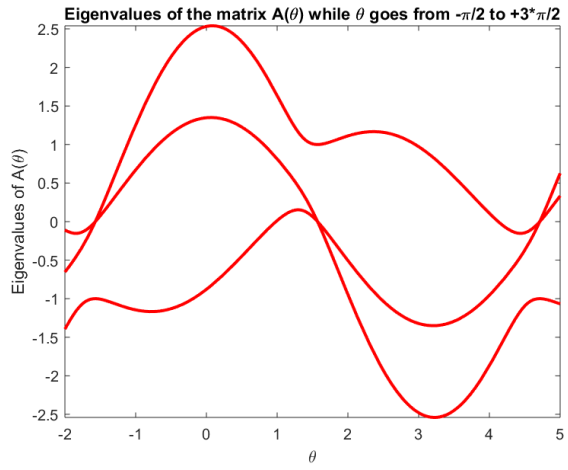


Figure 2.5: Monodromy of the eigenvalues of $A(\theta) = \cos\theta A_1 + \sin\theta A_2$

2.6 Slopes and asymptotes of eigenvalues

In Subsection 2.5.2, we were driven to prove some properties of eigenvalues associated to a special family. Some of them are consequences of general facts we state below, as we use them later.

Lemma 2.6.1. *Let $B : v \to A_1 + vA_2$. If $\lambda : \mathbb{R} \to \mathbb{R}$ is a continuous eigenvalue of $B(v)$, then $\lim_{v\to\pm\infty} \frac{\lambda(v)}{v}$ exists and it belongs to the spectrum of A_2.*

Proof. Since $v \mapsto \lambda(v)$ is a semi-algebraic function, $\lim_{v\to+\infty} \frac{\lambda(v)}{v}$ exists in $\mathbb{R}\cup\{\pm\infty\}$. Let us denote by $\lambda_k(A)$ the k-th largest (with repetition) eigenvalue of A. Note that the k-th eigenvalue function

$$\begin{array}{rccc} \phi_k & : & Sym_d(\mathbb{R}) & \to & \mathbb{R} \\ & & A & \mapsto & \lambda_k(A) \end{array}$$

is continuous and we take $\lambda(v) = \phi_k(B(v))$ in a neighborhood of $+\infty$.

Consider the matrix

$$\frac{B(v)}{v} = \frac{A_1 + vA_2}{v}$$

whose eigenvalues are of the form $\frac{\lambda_i(v)}{v}$, $\lambda_i(v) \in Spec(B(v))$. Since the eigenvalue function is continuous, the eigenvalues of $\lim_{v\to+\infty} \frac{B(v)}{v} = A_2$ are $\lim_{v\to+\infty} \frac{\lambda_i(v)}{v} = S$. Same argument works for $v = -\infty$, thus, the slopes of the eigenvalues of the matrix $B(v)$ at $\pm\infty$ are the eigenvalues of A_2. □

Lemma 2.6.2. *Let*

$$A(\theta) = \cos(\theta)A_1 + \sin(\theta)A_2,$$

where

$$A_2 = diag(0, ..., 0, 1).$$

Then, the finite limits of the eigenvalues of the matrix $B(v) = A_1 + vA_2$ at $v = \pm\infty$ are the slopes of the eigenvalues of the matrix $A(\theta)$ at the point $\theta = -\frac{\pi}{2}$.

Proof. Consider an eigenvalue $\lambda_i(v)$ of the matrix $B(v)$ with finite limit at $v = -\infty$. Suppose that

$$\lim_{v\to-\infty} \lambda_i(v) = C_i,$$

where $C_i \in \mathbb{R}$. Suppose that $\mu_i(\theta)$ is the corresponding eigenvalue of the matrix $A(\theta)$.

Since $u = \cos\theta$ and $v = \frac{\sin\theta}{\cos\theta}$, we have

$$\mu_i(\theta) = \lambda_i(\tan\theta)\cos\theta, \tag{2.13}$$

Since $\cos\theta \sim_{-\frac{\pi}{2}} (\theta + \frac{\pi}{2})$, $\lambda_i(\tan\theta) \sim_{-\frac{\pi}{2}} C_i$, we get

$$\mu_i(\theta) \sim_{-\frac{\pi}{2}} C_i \times (\theta + \frac{\pi}{2}) \tag{2.14}$$

We now calculate the slope of the eigenvalue μ_i of $A(\theta)$ at the point $\theta = -\frac{\pi}{2}$,

$$\mu_i'(-\frac{\pi}{2}) := \lim_{\theta \to -\frac{\pi}{2}} \frac{\mu_i(\theta)}{\theta + \frac{\pi}{2}}.$$

Since $\mu_i(\theta)$ depending on one parameter is analytic and $\mu_i(-\frac{\pi}{2}) = 0$, then by Hopital's rule and 2.14 $\mu_i'(-\frac{\pi}{2}) = C_i$.

Now consider a constant eigenvalue $\lambda_j(v) \neq \lambda_i(v)$ of $B(v)$ at $v = +\infty$ and suppose that

$$\lim_{v \to +\infty} \lambda_j(v) = C_j.$$

Let $\mu_j(\theta)$ be the corresponding eigenvalue of matrix $A(\theta)$. We have

$$\mu_j(\theta) = \lambda_j(\tan\theta)\cos\theta, \tag{2.15}$$

Since $\cos\theta \sim_{+\frac{\pi}{2}} (-\theta + \frac{\pi}{2})$, $\lambda_j(\tan\theta) \sim_{+\frac{\pi}{2}} C_j$, we get

$$\mu_j(\theta) \sim_{+\frac{\pi}{2}} C_j \times (-\theta + \frac{\pi}{2}) \tag{2.16}$$

Hence, the slope of the eigenvalue μ_j of $A(\theta)$ at the point $\theta = +\frac{\pi}{2}$,

$$\mu_j'(+\frac{\pi}{2}) := \lim_{\theta \to +\frac{\pi}{2}} \frac{\mu_j(\theta)}{-\theta + \frac{\pi}{2}}.$$

By Hopital's rule and 2.16, we get $\mu_j'(+\frac{\pi}{2}) = -C_j$.

Since $A(\theta) = -A(\theta + \pi)$, the slopes of eigenvalues of the matrix $A(\theta)$ at the points at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are opposite reciprocals which finishes the proof. □

Now, it is of interest to look at the slopes of the eigenvalues of the matrix $A(\theta)$ at $\theta = -\frac{\pi}{2}$, Lemma 2.6.3.

Lemma 2.6.3. *If $A_2 \in Sym_d(\mathbb{R})$ is diagonal with $\#Spec(A_2) = d$. Then, the slopes of the eigenvalues $A(\theta) = \cos(\theta)A_1 + \sin(\theta)A_2$ at $\theta = -\frac{\pi}{2}$ are the diagonal elements of the matrix A_1.*

Proof. Let $t_1 = \cos\theta = uv$, $t_2 = \sin\theta = v$. We consider $B' : u \mapsto A_2 + uA_1$, where

$$A_2 = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_d \end{pmatrix},$$

such that for all $i, j \in \{1, \ldots, d\}$, $\alpha_j \neq \alpha_j$ and

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{12} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{dd} \end{pmatrix}.$$

Since $B'(u)$ is a one parameter family, the eigenvalues of $B'(u)$, $\lambda_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, d$ are analytic functions. Choose an index i and denote by $\lambda(u)$ the corresponding eigenvalue such that $\lambda(u) = \alpha_1 + m_1 u + o(u)$ in a neighborhood of zero. So we have

$$\det(B'(u) - \lambda(u)\mathbb{I}_d) = 0.$$

Therefore,

$$\det \left(\begin{array}{c|ccc} \alpha_1 + ua_{11} - \lambda(u) & ua_{12} & ... & ua_{1d} \\ \hline ua_{12} & \alpha_2 + ua_{22} - \lambda(u) & ... & ua_{2d} \\ \vdots & \vdots & & \\ ua_{1d} & ua_{2d} & ... & \alpha_d + ua_{dd} - \lambda(u) \end{array} \right) = 0. \quad (2.17)$$

Let

$$a_1' := \begin{pmatrix} a_{12} \\ \vdots \\ a_{1d} \end{pmatrix}, A_1' := \begin{pmatrix} a_{22} & \cdots & a_{2d} \\ \vdots & \ddots & \vdots \\ a_{2d} & \cdots & a_{dd} \end{pmatrix}, A_2' := \begin{pmatrix} \alpha_2 & & \\ & \ddots & \\ & & \alpha_d \end{pmatrix}.$$

Hence, we can rewrite 2.17 as

$$\det\left(\begin{array}{c|c} \alpha_1 + ua_{11} - \lambda(u) & ua_1'^t \\ \hline ua_1' & A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1} \end{array}\right) = 0.$$

We can write

$$A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1} = A_2' - \alpha_1\mathbb{I}_{d-1} + uA_1' - (\lambda(u) - \alpha_1)\mathbb{I}_{d-1}.$$

Since for all $i \in \{2, \ldots, d\}$, $\alpha_1 \neq \alpha_i$ and $\lambda(0) = \alpha_1$, then at $u = 0$, the matrix $(A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1})$ is invertible. We can choose u sufficiently small such that $(A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1})$ stays invertible. Define $x(u) \in \mathbb{R}^{d-1}$ by

$$x(u) = -u(A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1})^{-1}a_1'. \tag{2.18}$$

so that

$$(A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1})x(u) = -ua_1'.$$

Denote by

$$C' = \begin{pmatrix} ua_1'^t \\ A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1} \end{pmatrix},$$

the $d \times (d-1)$ sub-matrix of the determinant of the matrix $B'(u) - \lambda(u)\mathbb{I}_d$. Let us replace the first column C_1 of the determinant of the matrix $B'(u) - \lambda(u)\mathbb{I}_d$ by $C_1 + C'x(t)$:

$$C_1 + C'x(t) \to C_1.$$

Note that under these procedures the determinant of the matrix $B'(u) - \lambda(u)\mathbb{I}_d$ does not change. We get,

$$det(B'(u) - \lambda(u)\mathbb{I}_d) = det\left(\begin{array}{c|c} \Lambda(u) & ua_1'^t \\ \hline 0 & A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1} \end{array}\right) = 0,$$

where

$$\Lambda(u) = \alpha_1 + ua_{11} - \lambda(u) + ua_1'^t x(u). \tag{2.19}$$

Moreover, by replacing 2.18 in 2.19 we get,

$$\Lambda(u) = \alpha_1 + ua_{11} - \lambda(u) - u^2 a_1'^t (A_2' + uA_1' - \lambda(u)\mathbb{I}_{d-1})^{-1}(a_1'). \tag{2.20}$$

Thus,

$$\begin{aligned}\chi_{B'}(\lambda(u)) &= \Lambda(u) \times det(A'_2 + uA'_1 - \lambda(u)\mathbb{I}_{d-1}) \\ &= P_1(u, \lambda(u)) \times P_2(u, \lambda(u)),\end{aligned}$$

where

$$P_1(u, \lambda(u)) = \Lambda(u),$$

and

$$P_2(u, \lambda(u)) = det(A'_2 + uA'_1 - \lambda(u)\mathbb{I}_{d-1}).$$

Since at $u = 0$, $P_2(u, \lambda(u)) \neq 0$, then $\chi_{B'(u)} = 0$ if and only if $P_1(u, \lambda(u)) = 0$. If $\Lambda(u) = 0$, we get

$$\lambda(u) = \alpha_1 + ua_{11} - u^2 a_1'^t (A'_2 + uA'_1 - \lambda(u)\mathbb{I}_{d-1})^{-1}(a'_1).$$

From this, the slope of the eigenvalue $\lambda(u)$ at the point $u = 0$ is a_{11}, the corresponding diagonal element of A_1. Since we can do the same procedure respect to any diagonal element of $(B'(u) - \lambda_i(u)\mathbb{I}_d)$, it can be concluded that slopes of the eigenvalues of $B'(u)$ at the point $u = 0$ are the diagonal elements of A_1. By linearity of the family we have the same for the eigenvalues of the matrix $A(\theta)$. □

Chapter 3

Intersection of linear families with Σ_2

Given a linear family $A(t)$ in $Sym_d(\mathbb{R})$, we showed in the previous chapter obstructions – the monodromy around S_A and the antipodal monodromy – for the existence of analytic eigenvalues. The monodromy is fully supported by the super-singular set. Thus, it is of prime importance to identify in Σ the points that are super-singular for $A(t)$. We dedicate this chapter to study of the intersections of linear families with Σ_2, this is, matrices that have exactly one double eigenvalue. Since eigenvalues are homogeneous, it is sometimes convenient to restrict the study to affine subspaces of $A(t)$ of codimension one, (that can be seen as charts of the corresponding projective family). We state most of our results in this framework.

In the first section, we describe some underlying geometric structure of $Sym_d(\mathbb{R})$. We recall that $Sym_d(\mathbb{R})$ is stratified according to the multiplicities of eigenvalues. The action of the orthogonal group $O(d)$ respects the stratification, and the geometry of $O(d)$ shows in particular through the local geometry of the smooth part of Σ, that is Σ_2. We give a local decomposition of (the tangent space to) $Sym_d(\mathbb{R})$ in a neighborhood of $A_0 \in \Sigma_2$ that respects $T_{A_0}\Sigma_2$, and distinguish the tangent to the orbit of A_0 by $O(d)$ and the centralizer of A_0.

In the second section, we recall some results of Lax [20] and of S. Friedland, J. Robbin and J. Sylvester [10]. Asking for the nature of the intersection of $A(t)$ with Σ, one could hope for some lower bound for $A(t) \cap \Sigma$ in terms of $\dim(A)$. Naively, Σ being codimension one, linear 3-families (i.e., affine 2-families) could always intersect $\Sigma \setminus \{0\}$ (0 always belongs to $A \cap \Sigma$). Lax Theorem shows that the later is true in $Sym_d(\mathbb{R})$ for $d \equiv 2 \bmod 4$. But, Friedland, Robbin and Sylvester refines the later and gives the minimal $k(d)$ such that any $k(d)$-linear subspace in $Sym_d(\mathbb{R})$ intersects

$\Sigma \setminus \{0\}$. Surprisingly, there are subsequences of $k(d)$ that grow to infinity. We remark that for those large families with no non-zero intersection points with Σ, the origin is always a super-singular point.

In another section, we focus on affine 1-families. There, thanks to Rellich, eigenvalues are analytic in terms of the parameter. We relate the slopes of collapsing eigenvalues to the transversality of the family with Σ_2, which will be of great use later. The maximal number of intersection points of $A(t)$ with Σ_2 that is allowed by the degree of $\chi_{A(t)}$, is reached for simultaneously diagonalizable families. We then prove that the singular points of generic diagonal families admit independent perturbations. In other words, given a generic diagonal affine family $D(t)$ (this means, such that $D(t) \cap \Sigma \subset \Sigma_2$), and choosing a subset I of $D(t) \cap \Sigma_2$, there exists an affine family $A(t)$ close to $D(t)$ such that $A(t) \cap \Sigma_2$ is exactly made of $\mathrm{Card}(I)$ points close to the points of I, Proposition 3.3.4. This result can be regarded as a version of Brusotti Theorem (singularities of plane curves with only simple singular points can be perturbed independently one to the other) for determinential curves, where matrices are perturbed instead of curves. This result leads us to construct families with prescribed "set" antipodal monodromy. Precisely, given d distinct values $a_1, \ldots, a_d$ in $\mathbb{R}$, and σ a permutation of $\{a_1, \ldots, a_d\}$, there exists a 2-linear family $A(t) \in Sym_d(\mathbb{R})$, whose eigenvalues at $(1,0)$ are $a_1, \ldots, a_d$, and whose antipodal monodromy from $(1,0)$ in the direction $(0,1)$ is precisely σ, Theorem 3.3.5.

In the last section of this chapter, we focus on affine two-dimensional families. We give a description of the behavior of the eigenvalues in a neighborhood of a point $A(t_0)$ of $A(t) \cap \Sigma_2$. Here we use the description of $T_{A_0}\Sigma_2$ that is given in the first section of the chapter, and break down the analysis in terms of the dimension of the intersection of $T_{A_0}A(t)$ with $T_{A_0}\Sigma_2$ and then of the intersection of $A(t)$ with Σ_2. The diversity of the encountered cases dissuades to push the analysis further (increasing the dimension of $A(t)$) with no new theoretical result. Among the interesting behaviors that we enlight, we should mention that we construct non-trivial plane curves included in Σ_2, and cases where the antipodal monodromy of the affine family depends on the base point. We also produce lines along which the contact order with Σ_2 is 3, that refute the naive local representation of Σ_2 as a sort of bended cylindrical surface. (as it is locally the orbit of a "flat" linear space made of diagonal matrices by the action of the "round" group $O(d)$).

3.1 Generalities on $Sym_d(\mathbb{R})$, Σ_2, $O(d)$.

3.1.1 Stratifying $Sym_d(\mathbb{R})$

We fix an integer $d \in \mathbb{N}^*$, and consider the space $Sym_d(\mathbb{R})$ of symmetric $d \times d$ matrices. We first introduce some notations. Let $\Delta : Sym_d(\mathbb{R}) \to \mathbb{R}$ be the discriminant function, so that Σ, the set of symmetric matrices with multiple eigenvalues, is the zero set of Δ:

$$\Sigma = \{X \in Sym_d(\mathbb{R});\ \Delta(X) = 0\}.$$

The space $Sym_d(\mathbb{R})$ can be decomposed according to the multiplicities and the order of the eigenvalues. Given $(i_1, \ldots, i_\ell) \in (\mathbb{N}^*)^\ell$ with $i_1 + \cdots + i_\ell = d$, we denote by $\Sigma_{i_1,\ldots,i_\ell}$ the matrices whose eigenvalues are of the form $\lambda_1 < \cdots < \lambda_\ell$, λ_k having multiplicity i_k:

$$\Sigma_{i_1,\ldots,i_\ell} = \left\{ X \in Sym_d(\mathbb{R});\ \exists \lambda_1 < \cdots < \lambda_\ell,\ \chi_X(\lambda) = \prod_{k=1}^{\ell} (\lambda_k - \lambda)^{i_k} \right\}$$

Forgetting the order of the eigenvalues drives to consider the sets $\mathcal{S}_J$, indexed by the finite sequences $J = (j_1, \ldots, j_s) \in (\mathbb{N}^*)^s$ with $\sum_{k=1}^{s} k j_k = d$, and made of the matrices that have exactly j_k eigenvalues of order k. The set $\mathcal{S}_{j_1,\ldots,j_s}$ is the disjoint union of the $\Sigma_{i_1,\ldots,i_\ell}$ where j_k is the occurrence of k in the sequence $i_1, \ldots, i_\ell$:

$$\mathcal{S}_{j_1,\ldots,j_s} = \bigsqcup_{J(i_1,\ldots,i_\ell)=(j_1,\ldots,j_s)} \Sigma_{i_1,\ldots,i_\ell},$$

where $J(i_1, \ldots, i_\ell) = (j_1, \ldots, j_s)$ if $j_k = Card\{n;\ i_n = k\}$.

For short, and since we manly focus on it, we write Σ_2 for the set $\mathcal{S}_{d-2,1}$, made of those matrices that have exactly 1 double eigenvalue (and $d-2$ simple eigenvalues), and which is the disjoint union of all $\Sigma_{i_1,\ldots,i_{d-1}}$ where all i_k's are 1 but one that is 2.

Note that symmetric matrices also can be considered as quadratic forms, assuming that a quadratic form can be written in coordinates as a symmetric matrix. In ([3], Appendix 10) Arnold represent the open area $Sym_d^+(\mathbb{R})$ of the positive definite forms in $Sym_d(\mathbb{R})$ as the area of ellipsoid. For the first time, the problem of the stratification of symmetric matrices was asked by Arnold. In [4], he discusses the properties of this stratification. Then, in ([32], Lemma 1) Shapiro and Vainshtein give a stratification of $Sym_d(\mathbb{R})$ according to the multiplicity of the eigenvalues. In our notations, it is

shown that :

Theorem 3.1.1. *[Arnold [3], Shapiro and Vainshtein [32]]*

1. *The decomposition*

$$Sym_d(\mathbb{R}) = \bigsqcup_{\sum_{k=1}^{s} k j_k = d} \mathcal{S}_{j_1,\ldots,j_s}$$

is a stratification of $Sym_d(\mathbb{R})$ into smooth semi-algebraic manifolds. In particular, $\Sigma = Sym_d(\mathbb{R}) \setminus \mathcal{S}_d$ is a finite disjoint union of smooth semi-algebraic manifolds.

2. *The decomposition*

$$\mathcal{S}_{j_1,\ldots,j_s} = \bigsqcup_{J(i_1,\ldots,i_\ell)=(j_1,\ldots,j_s)} \Sigma_{i_1,\ldots,i_\ell},$$

is a decomposition into smooth connected components of equal dimension, and the codimension of $\mathcal{S}_{j_1,\ldots,j_s}$ is given by

$$Codim(\mathcal{S}_{j_1,\ldots,j_s}) = \sum_{k=1}^{s} \frac{(k-1)(k+2)}{2} j_k.$$

In particular, $\Sigma_2 = \mathcal{S}_{d-2,1}$ is a smooth manifold of codimension 2 and has $d-1$ connected components.

3.1.2 Action of the orthogonal group

We denote by $O(d)$ the **orthogonal group** of $d \times d$ matrices,

$$O(d) := \{Q \in M_d(\mathbb{R}) \mid Q^T Q = \mathbb{I}_d\}.$$

It acts on $Sym_d(\mathbb{R})$ by conjugation, this is, with the action α:

$$\begin{array}{rccc} \alpha & : & O(d) \times Sym_d(\mathbb{R}) & \to & Sym_d(\mathbb{R}) \\ & & (Q, A_0) & \mapsto & Q^T A_0 Q. \end{array}$$

The orbit $\mathcal{O}_{A_0}$ of an $A_0 \in Sym_d(\mathbb{R})$ under this action

$$\mathcal{O}_{A_0} = \{Q^T A_0 Q;\ \ Q \in O(d)\},$$

is the image of the map

$$\begin{array}{rccc} \overline{\alpha}_{A_0}: & O(d) & \to & Sym_d(\mathbb{R}) \\ & Q & \mapsto & Q^T A_0 Q, \end{array}$$

and consists of the symmetric matrices which have the same eigenvalues (with multiplicities) as A_0.

We fix $A_0 \in Sym_d(\mathbb{R})$ and describe the tangent space $T_{A_0}\mathcal{O}_{A_0}$. For this, we compute the differential $D_{\mathbb{I}_d}\overline{\alpha}_{A_0} \in L(T_{\mathbb{I}_d}O(d), T_{A_0}Sym_d(\mathbb{R}))$. From the definition of $O(d)$, we have

$$T_{\mathbb{I}_d}O(d) = ker\, D_{\mathbb{I}_d}(M \mapsto M^T M - \mathbb{I}_d) = \{K \in M_d(\mathbb{R});\ K^T = -K\}.$$

So $T_{\mathbb{I}_d}O(d)$ is the space of $d \times d$ skew-symmetric, that we denote $Skew_d(\mathbb{R})$. Notice $Sym_d(\mathbb{R})$ is linear so $T_{A_0}Sym_d(\mathbb{R}) = Sym_d(\mathbb{R})$. Direct calculation gives:

$$\begin{array}{rccc} D_{\mathbb{I}_d}(\overline{\alpha}_{A_0}): & Skew_d(\mathbb{R}) & \to & Sym_d(\mathbb{R}) \\ & K & \mapsto & KA_0 - A_0K. \end{array} \tag{3.1}$$

Now, $T_{A_0}\mathcal{O}_{A_0} = Im\, D_{\mathbb{I}_d}(\overline{\alpha}_{A_0})$, so

$$T_{A_0}\mathcal{O}_{A_0} := \{KA_0 - A_0K \mid K \in Skew_d(\mathbb{R})\}.$$

In particular,

$$\begin{aligned} \dim \mathcal{O}_{A_0} &= \dim T_{A_0}\mathcal{O}_{A_0} \\ &= \dim\ T_{\mathbb{I}_d}O(d) - \dim\ \ker(D_{\mathbb{I}_d}(\overline{\alpha}_{A_0})) \\ &= \tfrac{d(d-1)}{2} - \dim\{K \in Skew_d(\mathbb{R}) \mid KA_0 - A_0K = 0\}. \end{aligned}$$

3.1.3 Construction of tangent planes

Proposition 3.1.2. *Suppose that $A_0 \in \Sigma_2$. Then,*

$$Sym_d(\mathbb{R}) = T_{A_0}\mathcal{O}_{A_0} \oplus C(A_0) \cap T_{A_0}\Sigma_2 \oplus C(A_0) \cap T_{A_0}\Sigma_2{}^{\perp}$$

and

$$T_{A_0}\Sigma_2 = T_{A_0}\mathcal{O}_{A_0} \oplus C(A_0) \cap T_{A_0}\Sigma_2$$

where $C(A_0) = \{M \in Sym_d(\mathbb{R}); MA_0 = A_0M\}$ *and* ${T_{A_0}\Sigma_2}^{\perp}$ *is the complement of* $T_{A_0}\Sigma_2$ *, orthogonal to* $T_{A_0}\Sigma_2$ *for the scalar product* $< A|B >= tr(AB)$*,* $A, B \in Sym_d(\mathbb{R})$*.*

Proof. Let $P \in O(d)$ such that $P^TA_0P = B_0$ with $B_0 = diag(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_d)$, $\alpha_1 = \alpha_2 = \alpha$ and for all $i, j \in \{3, \ldots, d\}$, $i \neq j$, $\alpha_i \neq \alpha_j \neq \alpha$. For all $i, j \in \{1, \ldots, d\}$, let δ_{ij} be the matrix whose only non-zero entry is the (i, j)-th that is 1 and define $D_i = \delta_{ii}$. Let $Diag \cap T_{B_0}\Sigma_2$ be the diagonal direction in $T_{B_0}\Sigma_2$ that contains B_0, this is,

$$Diag \cap T_{B_0}\Sigma_2 = span(D, D_3, \ldots, D_d)$$

where $D = \delta_{11} + \delta_{22}$. Let $K_{ij} \in Skew_d(\mathbb{R})$ be $K_{ij} = \delta_{ij} - \delta_{ji}$, then K_{ij}, $i < j$, form a basis for $Skew_d(\mathbb{R})$. Note that $\{K \in Skew_d(\mathbb{R}); KB_0 - B_0K = 0\} = span(K_{12})$ and for all $i < j$, $K_{ij}B_0 - B_0K_{ij} = (\alpha_j - \alpha_i)\delta_{ij} + (\alpha_j - \alpha_i)\delta_{ji}$ with $\alpha_j - \alpha_i \neq 0$ for all $i, j \in \{3, ..., d\}$. Define $O_{ij} = \delta_{ij} + \delta_{ji}$. Then, $((O_{i,j}); i < j, (i, j) \neq (1, 2))$ is a family of $\frac{d(d-1)}{2} - 1$ independent matrices which form a basis for $T_{B_0}\mathcal{O}_{B_0}$,

$$T_{B_0}\mathcal{O}_{B_0} = Span((O_{i,j}); i < j, (i, j) \neq (1, 2)).$$

Note that $Diag \cap T_{B_0}\Sigma_2$ is a subspace of $T_{B_0}\Sigma_2$. Since $B_0 \in \Sigma_2$ and Σ_2 is smooth, $\mathcal{O}_{B_0}$ is a sub-manifold of Σ_2. So, $T_{B_0}\mathcal{O}_{B_0}$ also is a subspace of $T_{B_0}\Sigma_2$ and we have

$$\dim Diag \cap T_{B_0}\Sigma_2 + \dim T_{B_0}\mathcal{O}_{B_0} = \dim T_{B_0}\Sigma_2.$$

Since any matrix of $T_{B_0}\mathcal{O}_{B_0}$ has zero diagonal, for any matrix $M_1 \in Diag \cap T_{B_0}\Sigma_2$ and $M_2 \in T_{B_0}\mathcal{O}_{B_0}$, $tr(M_1M_2) = 0$. So, $Diag \cap T_{B_0}\Sigma_2$ is the orthogonal complement of $T_{B_0}\mathcal{O}_{B_0}$ in $T_{B_0}\Sigma_2$ for the scalar product $< A|B >= tr(AB)$. Thus,

$$T_{B_0}\Sigma_2 = T_{B_0}\mathcal{O}_{B_0} \oplus Diag \cap T_{B_0}\Sigma_2.$$

Since $T_{B_0}\Sigma_2^{\perp}$ is the complement of $T_{B_0}\Sigma_2$, it has dimension two and $Sym_d(\mathbb{R}) = T_{B_0}\Sigma_2 \oplus {T_{B_0}\Sigma_2}^{\perp}$. Since $T_{B_0}\Sigma_2 = T_{B_0}\mathcal{O}_{B_0} \oplus Diag \cap T_{B_0}\Sigma_2$, we have

$$Sym_d(\mathbb{R}) = T_{B_0}\mathcal{O}_{B_0} \oplus Diag \cap T_{B_0}\Sigma_2 \oplus {T_{B_0}\Sigma_2}^{\perp}.$$

Now, consider the invertible linear map $\mathcal{C} : Sym_d(\mathbb{R}) \to Sym_d(\mathbb{R})$, $X \mapsto PXP^T$. Since $\mathcal{C}$ is an invertible linear map, it preserves the direct sums of subspaces of

$Sym_d(\mathbb{R})$, we have

$$Sym_d(\mathbb{R}) = \mathcal{C}(T_{B_0}\mathcal{O}_{B_0}) \oplus \mathcal{C}(Diag \cap T_{B_0}\Sigma_2) \oplus \mathcal{C}(T_{B_0}{\Sigma_2}^{\perp}).$$

Notice that $\mathcal{C}(\mathcal{O}_{B_0}) = \mathcal{O}_{B_0}$, $\mathcal{C}(T_{B_0}\mathcal{O}_{B_0}) = T_{A_0}\mathcal{O}_{A_0}$ and $\mathcal{C}(T_{B_0}\Sigma_2) = T_{A_0}\Sigma_2$. We show that

- $\mathcal{C}(Diag \cap T_{B_0}\Sigma_2) = C(A_0) \cap T_{A_0}\Sigma_2$:

$\mathcal{C}(Diag \cap T_{B_0}\Sigma_2) \subseteq C(A_0) \cap T_{A_0}\Sigma_2$: let $B_1 \in Diag \cap T_{B_0}\Sigma_2$ so that $\mathcal{C}(B_1) = A_1$. Since $B_1 \in T_{B_0}\Sigma_2$, we get $A_1 \in T_{A_0}\Sigma_2$. Since $P^T A_1 P = B_1$, $P^T A_0 P = B_0$ and B_0 and B_1 are diagonal, A_0 and A_1 are simultaneously diagonalizable, so $A_1 \in C(A_0)$.

$C(A_0) \cap T_{A_0}\Sigma_2 \subseteq \mathcal{C}(Diag \cap T_{B_0}\Sigma_2)$: let $A_1 \in C(A_0) \cap T_{A_0}\Sigma_2$, so that $\mathcal{C}^{-1}(A_1) = B_1$. We show that $B_1 \in Diag \cap T_{B_0}\Sigma_2$. Since $A_1 \in T_{A_0}\Sigma_2$, so $B_1 \in T_{B_0}\Sigma_2$. Since $A_1 A_0 = A_0 A_1$, $\mathcal{C}^{-1}(A_1)\mathcal{C}^{-1}(A_0) = \mathcal{C}^{-1}(A_0)\mathcal{C}^{-1}(A_1)$. From this, $B_1 B_0 = B_0 B_1$. So, $B_1 \in C(B_0)$. But $B_1 \in T_{B_0}\Sigma_2$, so B_1 is diagonal.

- $\mathcal{C}(T_{B_0}\Sigma_2^{\perp}) = C(A_0) \cap T_{A_0}\Sigma_2^{\perp}$:

$\mathcal{C}(T_{B_0}\Sigma_2^{\perp})) \subseteq C(A_0) \cap T_{A_0}\Sigma_2^{\perp}$: since for any $A, B \in Sym_d(\mathbb{R})$, $\mathcal{C}$ preserves the inner product $< A|B >= tr(AB)$, for any $S \in T_{B_0}\Sigma_2^{\perp}$, $\mathcal{C}(S) \in T_{A_0}\Sigma_2^{\perp}$. Since

$$S \in span\left(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}\right),$$

so $SB_0 = B_0 S$. Thus, $\mathcal{C}(S)\mathcal{C}(B_0) = \mathcal{C}(B_0)\mathcal{C}(S)$. So, $\mathcal{C}(S) \in C(A_0)$.

$C(A_0) \cap T_{A_0}\Sigma_2^{\perp} \subseteq \mathcal{C}(T_{B_0}\Sigma_2^{\perp})$: let $A_1 \in C(A_0) \cap T_{A_0}\Sigma_2^{\perp}$. Since orthogonality is preserved by $\mathcal{C}^{-1}$, $\mathcal{C}^{-1}(A_1) \in T_{B_0}\Sigma_2^{\perp}$. So, $A_1 = \mathcal{C}(B_1)$ with $B_1 \in T_{B_0}\Sigma_2^{\perp}$.

Therefore,

$$Sym_d(\mathbb{R}) = T_{A_0}\mathcal{O}_{A_0} \oplus C(A_0) \cap T_{B_0}\Sigma_2 \oplus C(A_0) \cap T_{A_0}{\Sigma_2}^{\perp}$$

and $T_{A_0}\Sigma_2 = T_{A_0}\mathcal{O}_{A_0} \oplus C(A_0) \cap T_{B_0}\Sigma_2$ which finishes the proof. □

3.2 Empty intersection

We recall a theorem of Lax (see [20]) from 1982, which shows that if $d \equiv 2 \pmod 4$, any 3-linear plane in $Sym_d(\mathbb{R})$ intersects Σ_2 somewhere else than the origin. We let $\mathcal{M}$ be the set of real $d \times d$ matrices with real eigenvalues, and say that $M \in \mathcal{M}$ is singular if it does not have d distinct eigenvalues.

Theorem 3.2.1 (Lax in [20])**.** *Let $A_0, A_1, A_2 \in M_d(\mathbb{R})$, and for $t = (t_0, t_1, t_2) \in \mathbb{R}^3$, denote $A(t) = t_0A_0 + t_1A_1 + t_2A_2$. Suppose $\forall t \in \mathbb{R}^3, A(t) \in \mathcal{M}$. If $d \equiv 2 \pmod 4$ then there exists $t \neq 0$ such that $A(t)$ is singular.*

The theorem of Lax applies in particular to families of real symmetric matrices. As a corollary, we state the symmetric version.

Corollary 3.2.2. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family. Suppose $d \equiv 2 \pmod 4$ and $k \geq 3$. Then $A(t) \cap \Sigma \neq \{0\}$.*

For $d \not\equiv 2 \pmod 4$ S. Friedland, J. Robbin and J. Sylvester in [10], extended Lax's theorem and give the minimal $k(d)$ such that any $k(d)$-linear family has a non-trivial intersection with Σ. Their proof relies on the construction of vector fields on high dimensional spheres and uses Adams result [1]. We must also mention that, for $d = 3$ and $d = 4$, a proof can be found in [24, Remark 4.3]. To state the theorem, we first need the following definition (and notation):

Definition 3.2.3. *Given $m \in \mathbb{N}$, we denote by $\rho(m)$ the Radon-Hurwitz number of m:*

$$\rho(m) = 2^c + 8d$$

where $m = (2a + 1)2^{c+4d}$ with $a, d \in \mathbb{Z}$ and $c = 0, 1, 2$ or 3.

Theorem 3.2.4. *(S. Friedland, J. Robbin and J. Sylvester) For positive integers d and k the following are equivalent:*

1. *$k < \sigma(d)$, where $\sigma(d)$ is given by*

$$\sigma(d) = 2 \text{ for } d \neq 0, \pm 1 \pmod 8,$$
$$\sigma(d) = \rho(4b) \text{ for } d = 8b, 8b \pm 1;$$

2. *Either $k = 1$ or else there is an integer m with $k < \rho(m)$ and d is one of the $2m - 1$, $2m$, $2m + 1$;*

3. *There is a linear map*

$$\Psi : \mathbb{R}^{k+1} \to M_d(\mathbb{R})$$

such that each matrix $\Psi(\alpha)$, $\alpha \in \mathbb{R}^{k+1}\backslash(0, ..., 0)$ *has* d *distinct real eigenvalues.*

4. *There exists an odd continuous map*

$$\psi : \mathbb{S}^k \to M_d(\mathbb{R})$$

such that each $\psi(\alpha)$, $\alpha \in \mathbb{S}^k$, *has* d *distinct real eigenvalues.*

It can be deduced from the proof of Theorem 3.2.4 that this theorem applies to symmetric matrices. Indeed, the proof gives an explicit construction of the family $\Psi(\alpha)$ that appears in the third item, and it happens that Ψ is made of symmetric matrices. Below, we follow the proof and we construct an explicit 4-linear family of 8×8 symmetric matrices that has no non-trivial intersection with Σ.

Example 3.2.5 (Constructed from the proof of Theorem 3.2.4, [10])**.** By using the implication (2) $\Rightarrow$ (3) of Theorem 3.2.4, we give the minimal number of parameters k such that a k-linear family of 8×8 matrices has a non-trivial intersection with Σ. According to the part (2) of Theorem 3.2.4, we search for $m \in \mathbb{Z}$ such that $8 = 2m-1$ or $8 = 2m$ or $8 = 2m+1$. From this, either $m = 4$, $m = \frac{7}{2}$ or $m = \frac{9}{2}$. Since $m \in \mathbb{Z}$, then $m = 4$. Since $4 = (2 \times 0 + 1)2^{2+4\times 0}$, we get $\rho(4) = 2^2 + 8 \times 0 = 4$. We follow the proof of Theorem 3.2.4, and for $m = 4$ with $k < \rho(4) = 4$, we construct

$$\Psi : \mathbb{R}^{k+1} \to M_d(\mathbb{R})$$

such that $\Psi(\alpha)$ is non-singular. Choose $k = 3$ and define

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For all $i, j = 1, 2, 3, i \neq j$ we have

$$\begin{aligned} A_i + A_i^T &= 0 \\ A_i A_i &= -\mathbb{I}_m \\ A_i A_j + A_j A_i &= 0. \end{aligned}$$

Consider the map

$$\begin{array}{rccc} \Phi & : & \mathbb{R}^4 & \to & M_4(\mathbb{R}) \\ & & \alpha & \mapsto & \alpha_0\mathbb{I}_4 + \sum_{i=1}^3 \alpha_i A_i \end{array}.$$

We have

$$\Phi(\alpha) = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix}$$

and we note that

$$\Phi(\alpha)\Phi(\alpha)^T = |\alpha|^2\mathbb{I}_4,$$

for $\alpha \in \mathbb{R}^4$. We define $\Gamma : \mathbb{R}^4 \to M_8(\mathbb{R})$ by

$$\Gamma(\alpha) = \begin{pmatrix} \mathbb{I}_4 & -\mathbb{I}_4 \\ \Phi(\alpha) & \Phi(\alpha) \end{pmatrix}.$$

We can verify that for $|\alpha| = 1$, we get $\Gamma(\alpha)\Gamma(\alpha)^T = 2\mathbb{I}_8$. We choose,

$$D = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

with $A = diag(a_1, a_2, a_3, a_4)$ and $a_1 > a_2 > a_3 > a_4 > 0$ are the eigenvalues of A and we define $\Psi(\alpha) = \Gamma(\alpha)D\Gamma(\alpha)^T$. Then, we get

$$\Psi(\alpha) = \begin{pmatrix} 0 & 2A\Phi(\alpha)^T \\ 2\Phi(\alpha)A & 0 \end{pmatrix} \tag{3.2}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 2a_1\alpha_0 & -2a_1\alpha_1 & 2a_1\alpha_2 & 2a_1\alpha_3 \\ 0 & 0 & 0 & 0 & -2a_2\alpha_1 & 0 & 2a_2\alpha_3 & -2a_2\alpha_2 \\ 0 & 0 & 0 & 0 & -2a_3\alpha_2 & -2a_3\alpha_3 & 0 & -2a_3\alpha_1 \\ 0 & 0 & 0 & 0 & -2a_4\alpha_3 & 2a_4\alpha_2 & -2a_4\alpha_1 & 0 \\ 2a_1\alpha_0 & -2a_2\alpha_1 & -2a_3\alpha_2 & -2a_4\alpha_3 & 0 & 0 & 0 & 0 \\ -2a_1\alpha_1 & 0 & -2a_3\alpha_3 & 2a_4\alpha_2 & 0 & 0 & 0 & 0 \\ 2a_1\alpha_2 & 2a_2\alpha_3 & 0 & -2a_4\alpha_1 & 0 & 0 & 0 & 0 \\ 2a_1\alpha_3 & -2a_2\alpha_2 & -2a_3\alpha_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

We note that the eigenvalues of the symmetric matrix of $\Psi(\alpha)$ are $\pm 2|\alpha|^2 a_i$, $i = 1, \ldots, 4$. The map $\Psi : \mathbb{R}^4 \to Sym_8(\mathbb{R})$ satisfies the hypothesis of the part 3 of

Theorem 3.2.4.

From Theorem 3.2.4 and Definition 3.2.3,

- If $d \neq 0, \pm 1 \pmod 8$, then $k(d) = 3$.
- If $d = 0, \pm 1 \pmod 8$: We write

$$d = 2 \times 2^{a+4b}(2c+1) + x$$

with $x \in \{-1, 0, 1\}$, $a \in \{0, 1, 2, 3\}$ and $b, c \in \mathbb{N}$ and $a + 4b \geq 2$. Then, the minimum number of parameters $k(d)$ for intersecting Σ is $k(d) = 2^a + 8b + 1$. Now since $a \in \{0, 1, 2, 3\}$, $b \sim \frac{1}{4}\log_2 d$. This gives $k(d) \sim \frac{8}{4}\log_2 d + 9$. We illustrate the behavior of $k(d)$ for $d < 1000$ in a picture bellow.

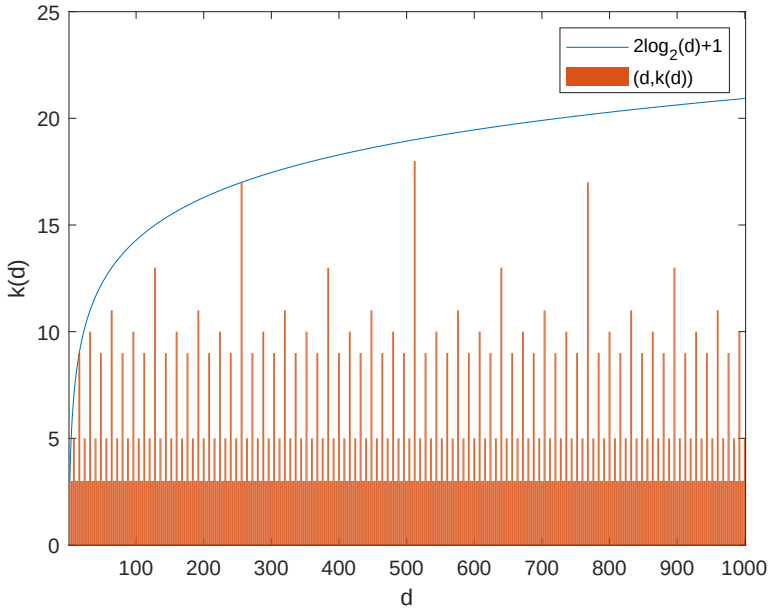


Figure 3.1: The minimum number of parameters for intersecting Σ

To conclude this section, we state the following straightforward proposition. It shows in particular that for the families that are constructed in the former (that do not intersect Σ but at 0), 0 is always a super-singular point.

Proposition 3.2.6. *Let $A(t) \in Sym_d(\mathbb{R})$, $d \geq 2$ be a k-linear family, and suppose that 0 is not super-singular for $A(t)$. Then, for all non zero t_0, u, $t_0 \perp u$, there exists $0 \neq t_1 \in Span(t_0, u)$, $A(t_1) \in \Sigma$.*

Proof. Suppose $A(t_0) \notin \Sigma$ (otherwise $t_0 = t_1$ works) and denote $\lambda_1 < \cdots < \lambda_d$ the eigenvalues of $A(t_0)$. The analytic prolongation of each λ_i, $i = 1, \ldots, d$ over the half turn $[0,1] \ni s \mapsto \cos(\pi s)t_0 + \sin(\pi s)u$ defines an analytic function $\lambda_i(s)$ on $[0,1]$. Since 0 is not super-singular, the antipodal monodromy from t_0 in the direction u is trivial: $\tau'_{t_0,u} = \mathbb{I}_d$. In particular, for all i, $\lambda_i(1) = -\lambda_i(0)$. So $\lambda_d(s) - \lambda_1(s)$ is a continuous function on $[0,1]$, positive for $s = 0$ and negative for $s = 1$. Then there exists $s_1 \in (0,1)$ such that $\lambda_1(s_1) = \lambda_d(s_1)$. The announced t_1 is $t_1 = \cos(\pi s_1)t_0 + \sin(\pi s_1)u$. □

3.3 Affine lines and Σ_2

In this section, we study the intersections of affine lines with Σ_2. We first state a useful lemma: along a transverse intersection with Σ_2, the collapsing eigenvalues intersect transversally (Lemma 3.3.1). We then notice that the maximal number of the intersection points of an affine family with Σ_2 is realized for diagonal families. Constructing perturbations of these diagonal families allows us to show that any permutation of d distinct values can be produced as the antipodal monodromy of a 2-linear family in $Sym_d(\mathbb{R})$ (Theorem 3.3.5).

Lemma 3.3.1. *Let $A(t) = A_0 + tA_1 \in Sym_d(\mathbb{R})$ with $A_0 \in \Sigma_2$ and $A_1 \notin T_{A_0}\Sigma_2$. Then, there exist $\varepsilon > 0$ such that $A(t) \notin \Sigma$ for $t \in (-\varepsilon, \varepsilon)$, $t \neq 0$ and $\lambda_1, \lambda_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}$ two analytic functions such that $\lambda_1(t), \lambda_2(t) \in Spec(A(t))$, $t \in (-\varepsilon, \varepsilon)$, $\lambda_1(0) = \lambda_2(0)$ and $\lambda_1'(0) \neq \lambda_2'(0)$.*

Proof. Since $A(t)$ is a one parameter family, by Rellich Theorem the eigenvalues of $A(t)$ are defined as analytic functions. Denote these analytic eigenvalues by $\lambda_i(t), i = 1, \ldots, d$.

Let $\ell \in \mathbb{R}$ be the multiple eigenvalue of A_0. Then, up to diagonalization of A_0, subtracting $\ell\mathbb{I}_d$ from $A(t)$, we can assume that $A_0 = diag(0, 0, \alpha_3, \ldots, \alpha_d)$, where for all $i \neq j$, $\alpha_i \neq \alpha_j$ are non-zero real numbers since $A_0 \in \Sigma_2$. Up to reindexing, $\lambda_1(0) = \lambda_2(0) = 0$.

Since $A_1 \notin T_{A_0}\Sigma_2$, after suitable change of basis in $\ker(A_0)$, we can assume that

A_1 has the following form

$$A_1 = \begin{pmatrix} x & 0 & a_{1,3} & \cdots & a_{1,d} \\ 0 & y & a_{2,3} & & \vdots \\ a_{1,3} & a_{2,3} & a_{3,3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ a_{1,d} & \cdots & \cdots & \cdots & a_{d,d} \end{pmatrix}, \tag{3.4}$$

with $x \neq y$ (See proposition 3.1.2).

We write the Taylor expansions of $\lambda_1(t)$ and $\lambda_2(t)$ at 0:

$$\begin{aligned} \lambda_1(t) &= 0 + m_1 t + o(t) \\ \lambda_2(t) &= 0 + m_2 t + o(t), \end{aligned}$$

and we let $e_1(t)$ and $e_2(t)$ be analytic unit eigenvectors associated with $\lambda_1(t)$ and $\lambda_2(t)$, respectively. We write

$$\begin{aligned} e_1(t) &= e_0^1 + e_1^1 t + o(t) \\ e_2(t) &= e_0^2 + e_1^2 t + o(t). \end{aligned}$$

Since e_0^1 is eigenvector associated with the eigenvalue 0 of A_0,

$$e_1(0) = e_0^1 = (a, b, 0, \ldots, 0) \text{ with } a^2 + b^2 = 1.$$

We will show that either $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$. We have $A(t)e_1(t) = \lambda_1(t)e_1(t)$. So,

$$\begin{aligned} &(A_0 + tA_1)(e_0^1 + e_1^1 t + o(t)) = (m_1 t + o(t))(e_0^1 + e_1^1 t + o(t)) \\ &A_0 e_0^1 + t(A_1 e_0^1 + A_0 e_1^1) + o(t) = t m_1 e_0^1 + o(t). \end{aligned}$$

Considering the terms in t, we get

$$A_1 e_0^1 + A_0 e_1^1 = m_1 e_0^1. \tag{3.5}$$

Denote $e_1^1 = (v_1, v_2, \dots, v_d)$. Then, from (3.5) we get

$$\begin{pmatrix} ax \\ by \\ \alpha_3 v_3 + a_{1,3}a + a_{2,3}b \\ \vdots \\ \alpha_d v_d + a_{1,d}a + a_{2,d}b \end{pmatrix} = \begin{pmatrix} m_1 a \\ m_1 b \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

From this $ax = m_1 a$ and $by = m_1 b$. If none among a, b is 0, it gives $m_1 = x$ and $m_1 = y$, which contradicts $x \neq y$. Therefore, either $a = 0$ or $b = 0$. Assuming $e_0^1 = (1, 0, \dots, 0)$ (the other choice exchange the roles of e_1 and e_2) gives $m_1 = x$, and by the same argument, $m_2 = y$. Consequently, $m_1 = \lambda_1'(0) \neq \lambda_2'(0) = m_2$, which finishes the proof. Notice that, λ_1 and λ_2 having different slopes, they are distinct for $t \neq 0$ on some $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, and ε might be reduced if needed to avoid other equality between eigenvalues. □

From Rellich Theorem, if $A(t) = A_0 + tA_1$ is an affine line in $Sym_d(\mathbb{R})$, its eigenvalues are analytic functions $\lambda_1(t), \dots, \lambda_d(t)$. A point $A(t) \in \Sigma$ corresponds to a parameter t_0 for which two different λ_i, λ_j coincide: $\lambda_i(t_0) = \lambda_j(t_0)$. In other terms, since the graphs of the λ_i form the zero set of the characteristic polynomial $\chi_{A(t)}$, a point in Σ is a double point of the curve $\chi_{A(t)}^{-1}(0)$. Now, $\chi_{A(t)}(\lambda)$ is a polynomial of degree d in (t, λ). So, the number of its double points, if it is finite, is bounded by $\frac{d(d-1)}{2}$. In particular, if $A(t) \not\subset \Sigma$, $\mathrm{Card}(A(t) \cap \Sigma) \leq \frac{d(d-1)}{2}$. It happens that this value is reached by diagonal families, when $\chi_{A(t)}(\lambda)$ factors in d lines.

Claim 3.3.2. *Let $D(t) = Diag(a_1, \dots, a_d) + tDiag(b_1, \dots, b_d)$, and suppose*

$$\big(i \neq j\big) \Rightarrow b_i \neq b_j, \text{ and } \Big(i < j,\, i' < j',\, (i,j) \neq (i',j')\Big) \Rightarrow \frac{a_i - a_j}{b_i - b_j} \neq \frac{a_{i'} - a_{j'}}{b_{i'} - b_{j'}}.$$

Then $D(t) \cap \Sigma_2$ is made of $\frac{d(d-1)}{2}$ points.

Notice that the condition above on the a_i, b_i is a generic condition in $(\mathbb{R}^d)^2$. We say that an affine family $D(t) \in Sym_d(\mathbb{R})$ of diagonal matrices is **in general position** when $D(t) = A + tB$ with $A = diag(a_1, \dots, a_d)$, $B = diag(b_1, \dots, b_d)$ and the $a_i, b_i, i = 1, \dots, d$ satisfy the hypothesis of claim 3.3.2. Because in this case, the eigenvalues form d lines in general position.

Proof of claim 3.3.2. Here the d eigenvalues are linear, given by $\lambda_i(t) = a_i + tb_i$. For

all $i \neq j$, the equation $\lambda_i(t) = \lambda_j(t)$ has a unique solution $t_{i,j} = -\frac{a_i - a_j}{b_i - b_j}$, and for different couples (i,j) with $i < j$, the $t_{i,j}$ differs.

Then, for each $i < j$, exactly two eigenvalues coincide at $t_{i,j}$, so $D(t_{i,j}) \in \Sigma_2$, and for other parameters t, the d eigenvalues are different and $D(t) \notin \Sigma$. Since there are $\frac{d(d-1)}{2}$ couples (i,j) with $i < j$, the claim is proven. □

Starting from an affine family of diagonal matrices $D(t)$ in general position, we will show that there exists a small perturbation $A(t) = D(t) + E$, $E \in Sym_d(\mathbb{R})$ of $D(t)$ which exactly preserves any given intersection points with Σ_2. The existence of an affine curve having prescribed intersection with Σ_2 could also be proven as an application of the Brusotti Theorem, that we recall below [6], and of [13] that state that in dimension 2, hyperbolic algebraic curves are the determinential curves of affine families. Here we state the Brusotti Theorem, which shows that we can smooth ordinary double points of a plane curve independently.

Theorem 3.3.3 (Brusotti)**.** *Let C be a real algebraic curve of degree d in $\mathbb{RP}^2$ with ordinary double points as its only singularities. For any of these singularities, choose a local deformation. Then it is possible to vary the curve C in the space of real algebraic curves of degree d in $\mathbb{RP}^2$ in such a way that all previously chosen deformations are realized.*

Our construction is however more explicit, and our perturbation appears as a continuous deformation of the family $D(t)$. The idea is to consider a diagonal family $A + tB$ in general position, and to find a perturbation of the lines made by the eigenvalues in a way that preserves some intersection points of the lines while ignoring the other intersection.

Proposition 3.3.4. *Let $D(t) = diag(a_1, \dots, a_d) + tdiag(b_1, \dots, b_d) \in Sym_d(\mathbb{R})$ be a diagonal family in general position. For $1 \leq i < j \leq d$, set $t_{i,j} = -\frac{a_i - a_j}{b_i - b_j}$, and choose a subset $I \subset \{(i,j);\ 1 \leq i < j \leq d\}$. Then, there exist $E \in Sym_d(\mathbb{R})$, arbitrary small, and $T = \{t'_{i,j};\ (i,j) \in I\}$ with $t'_{i,j}$ close to $t_{i,j}$ such that*

$$A + tB + E \in \Sigma \Leftrightarrow A + tB + E \in \Sigma_2 \Leftrightarrow t \in T.$$

Proof. For notation purpose, we let δ_{ij} be the matrix whose only non-zero entry is the (i,j)-th that is 1. Note that $D(t) \in \Sigma \Leftrightarrow D(t) \in \Sigma_2 \Leftrightarrow t \in \{t_{i,j}, 1 \leq i < j \leq d\}$. We let $E \in Sym_d(\mathbb{R})$ be a matrix with zero diagonal and indeterminate other coefficients $\varepsilon_{i,j} = \varepsilon_{j,i}$, $i \neq j$, and we investigate for conditions on the entries of E that determines

whether $A+tB+E$ has a point or not in Σ for t close to $t_{i,j}$.

Fix (i,j), $1 \leq i < j \leq d$.

By definition of $t_{i,j}$, the (i,i)-th and (j,j)-th entries of the diagonal matrix $D_{i,j} := A+t_{i,j}B$ coincide ($D_{i,j} \in \Sigma_2$) and are $\lambda_{ij} := a_i + t_{i,j}b_i = a_j + t_{i,j}b_j$. We let $t' = t - t_{i,j}$, and set:

$$B_1 = diag(b_1, \ldots, b_{i-1}, \frac{b_i+b_j}{2}, b_{i+1}, \ldots, b_{j-1}, \frac{b_i+b_j}{2}, b_{j+1}, \ldots, b_d),$$

$B_2 = \delta_{ii} - \delta_{jj}$, $\Delta_{i,j} = \delta_{i,j} + \delta_{j,i}$ and $\overline{E}_{i,j} = E - \varepsilon_{i,j}\Delta_{i,j}$.

This way, we get

$$A + tB + E = D_{i,j} + t'B_1 + \overline{E}_{i,j} + \frac{b_i - b_j}{2}\, t'\, B_2 + \varepsilon_{i,j}\Delta_{i,j}.$$

In this decomposition, remark that, from Proposition 3.1.2, B_1 and $\overline{E}_{i,j}$ are independent in $T_{D_{i,j}}\Sigma_2$ (as soon as $\overline{E}_{i,j} \neq 0$), while B_2 and $\Delta_{i,j}$ form a basis of a complementary space of $T_{D_{i,j}}\Sigma_2$ in $Sym_d(\mathbb{R})$. Since Σ_2 is locally the graph of an analytic function over its tangent space, there exists a neighborhood $W_{i,j}$ of $0 \in \mathbb{R} \times \mathbb{R}^{\frac{d(d-1)}{2}-1}$, a neighborhood $V_{i,j}$ of $D_{i,j}$ in $Sym_d(\mathbb{R})$, and an analytic function

$$\begin{array}{rccccc} \varphi_{i,j} : W_{i,j} \subset & \mathbb{R} & \times & \mathbb{R}^{\frac{d(d-1)}{2}-1} & \to & \mathbb{R}^2 \\ & (t' & , & \overline{E}_{i,j}) & \mapsto & (\Phi_{i,j}(t', \overline{E}_{i,j}), \Psi_{i,j}(t', \overline{E}_{i,j})) \end{array}$$

such that

$$A + tB + E \in V_{i,j} \cap \Sigma_2 \Leftrightarrow \left\{ \begin{array}{l} (t', \overline{E}_{i,j}) \in W_{i,j}, \\ \frac{b_i - b_j}{2}\, t' = \Phi_{i,j}(t', \overline{E}_{i,j}), \\ \varepsilon_{i,j} = \Psi_{i,j}(t', \overline{E}_{i,j}). \end{array} \right. \tag{3.6}$$

Notice that $d\varphi_{i,j}(0) = 0$, as $T_{D_{i,j}}\Sigma_2$ is obviously tangent to Σ_2 at $D_{i,j}$. As a first consequence, the second item in the right hand side of (3.6) can be inverted. Indeed,

$$\frac{\partial}{\partial t'}\left[\frac{b_i - b_j}{2}\, t' - \Phi_{i,j}(t', \overline{E}_{i,j})\right]_{(t', \overline{E}_{i,j}) = (0,0)} = \frac{b_i - b_j}{2} \neq 0,$$

so from the Implicit Functions Theorem,

$$\frac{b_i - b_j}{2}\, t' = \Phi_{i,j}(t', \overline{E}_{i,j}) \Leftrightarrow t' = \alpha_{i,j}(\overline{E}_{i,j}), \tag{3.7}$$

for some analytic function $\alpha_{i,j}$ in a neighborhood of 0. We shrink $W_{i,j}$ and $V_{i,j}$ so that

(3.7) holds for $(t', \overline{E}_{i,j}) \in W_{i,j}$, on the left hand side, and for $\overline{E}_{i,j}$ in a neighborhood of 0, say $U_{i,j}$, on the right hand side. From this, the equivalence (3.6) becomes:

$$A + tB + E \in V_{i,j} \cap \Sigma_2 \Leftrightarrow \begin{cases} \overline{E}_{i,j} \in U_{i,j}, \\ t' = \alpha_{i,j}(\overline{E}_{i,j}), \\ \varepsilon_{i,j} - \Psi'_{i,j}(\overline{E}_{i,j}) = 0, \end{cases} \tag{3.8}$$

where $\Psi'_{i,j}(\overline{E}_{i,j}) := \Psi_{i,j}(\alpha_{i,j}(\overline{E}_{i,j}), \overline{E}_{i,j})$. This will be the condition near $t_{i,j}$ we were looking for.

Now, let $\Psi : \mathbb{R}^{\frac{d(d-1)}{2}} \ni E = (\varepsilon_{i,j})_{1 \leq i < j \leq d} \mapsto (\varepsilon_{i,j} - \Psi'_{i,j}(\overline{E}_{i,j}))_{1 \leq i < j \leq d} \in \mathbb{R}^{\frac{d(d-1)}{2}}$. We claim that Ψ is invertible at 0. Indeed, recall that the functions $\varphi_{i,j}$ have all zero differential at 0, which implies $d\Psi'_{i,j}(0) = 0$, then $d\Psi(0) = Id$. We need to adjust some neighborhoods to conclude (shrinking $V_{i,j}$, $W_{i,j}$ and $U_{i,j}$ might be needed). We let $U \subset \mathbb{R}^{\frac{d(d-1)}{2}}$ be a neighborhood of 0, and $\delta_t > 0$, $\delta_E > 0$ be such that:

1. $|t_{i,j} - t_{i',j'}| > 2\delta_t$ if $(i,j) \neq (i',j')$;
2. $A + (t_{ij} - \delta_t, t_{i,j} + \delta_t)B + \mathcal{B}(0, \delta_E) \subset V_{i,j}$;
3. Ψ has an inverse defined on U, with $\Psi^{-1}(U) = U' \subset \mathcal{B}(0, \delta_E)$,
4. $\forall t,\ \forall (i,j),\ 1 \leq i < j \leq d,\ |t - t_{i,j}| \geq \delta_t \Rightarrow d(A + tB, \Sigma) > \delta_E$.

We can now conclude. Recall that I is a set of indices, and we look for E such that $A + tB + E$ has only points in Σ_2 near the $t_{i,j}$, for $(i,j) \in I$. Choose $(\zeta_{i,j})_{1 \leq i < j \leq d} \in U$ such that $\zeta_{i,j} = 0 \Leftrightarrow (i,j) \in I$, and set $E = \Psi^{-1}((\zeta_{i,j})_{1 \leq i < j \leq d})$. We claim that E works.

First fix $(i,j) \in I$. We show that there exists a unique $t'_{i,j} \in (t_{i,j} - \delta_t, t_{i,j} + \delta_t)$ such that $A + t'_{i,j}B + E \in \Sigma$. Indeed, for $t \in (t_{i,j} - \delta_t, t_{i,j} + \delta_t)$, $A + tB + E \in V_{i,j}$ and since $\zeta_{i,j} = 0$, we have $\varepsilon_{i,j} - \Psi'_{i,j}(\overline{E}_{i,j}) = 0$. So (3.8) gives

$$A + tB + E \in \Sigma_2 \Leftrightarrow t' = \alpha_{i,j}(\overline{E}_{i,j}),$$

so $t'_{i,j} = t_{i,j} + \alpha_{i,j}(\overline{E}_{i,j})$ works and is unique.

Now fix $(i,j) \notin I$. We show that for all $t \in (t_{i,j} - \delta_t, t_{i,j} + \delta_t)$, $A + tB + E \notin \Sigma$. Again, for $t \in (t_{i,j} - \delta_t, t_{i,j} + \delta_t)$, $A + tB + E \in V_{i,j}$. But $\zeta_{i,j} \neq 0$, so $\varepsilon_{i,j} - \Psi'_{i,j}(\overline{E}_{i,j}) \neq 0$. Thus (3.8) shows that $A + tB + E \in \Sigma$ has no solution.

Finally, if $\forall (i,j),\ |t - t_{i,j}| \geq \delta_t$, then $d(A + tB, \Sigma) > \delta_E$. But $E \in \mathcal{B}(0, \delta_E)$, so

$A + tB + E \notin \Sigma$. Therefore, there are no other point in $A + tB + E \cap \Sigma$ than the $t'_{i,j}$, $(i,j) \in I$, as announced. □

The previous proposition allows us to construct a 2-linear family with prescribed monodromy.

Theorem 3.3.5. *Let $(b_1, \ldots, b_d) \in \mathbb{R}^d$ such that $b_i \neq b_j$ for $i \neq j$. Let τ be a permutation of $\{b_1, b_2, \ldots, b_d\}$. Then, there exists a 2-linear family $A(t_1, t_2)$ whose antipodal monodromy from $(0,1)$ in the direction of $(1,0)$ is τ.*

Proof. Up to reindexing, we suppose $b_1 > \cdots > b_d$, and let $B = diag(b_1, \ldots, b_d)$. Given $A = diag(a_1, \ldots, a_d)$, such that $A + tB$ is in general position, we denote by

$$t_{i,j} = -\frac{a_i - a_j}{b_i - b_j}$$

the abscissa of the intersection of the lines $y = a_i + tb_i$ and $y = a_j + tb_j$, for $1 \leq i \leq d$, $1 \leq j \leq d$, ($i \neq j$ and $t_{i,j} = t_{j,i}$), and given $I \subset \{(i,j); 1 \leq i < j \leq d\}$, we set $E(I)$ for the perturbation matrix given by Proposition 3.3.4.

We prove by induction on d the following:

Claim 3.3.6 (Inductive hypothesis)**.** *Given B and τ as above, there exists $A = diag(a_1, \ldots, a_d)$, there exists $I \subset \{(i,j); 1 \leq i < j \leq d\}$ such that:*

1. *The family $A + tB$ is in general position;*
2. *$\forall (i,j), (i',j')$, $1 \leq i < j \leq d$, $1 \leq i' < j' \leq d$*

$$\big((j > j') \text{ or } (j = j' \text{ and } i > i')\big) \Rightarrow t_{i,j} < t_{i',j'};$$

3. *The antipodal monodromy of $A(t_1, t_2) = t_1(A + E(I)) + t_2 B$ from $(0,1)$ in the direction of $(1,0)$ is τ.*

Whenever $d = 1$, $A = 0, I = \emptyset$ works.

Suppose the claim holds for $d-1$ and prove it for d. Let $b_k = \tau(b_d)$, and let τ' be the permutation of $\{1, \ldots, d-1\}$ given, if $b_\ell = \tau(b_i)$, by $\tau'(i) = \ell$ if $\ell < k$, and $\tau'(i) = \ell - 1$ if $\ell \geq k$.

We apply the inductive hypothesis to $B' = (b_1, \ldots, b_{d-1})$ and τ', and get $A' = diag(a_1, \ldots, a_{d-1})$, $I' \subset \{(i,j); 1 \leq i < j \leq d-1\}$. Choose $a_d \in \mathbb{R}$ such that

$a_d + b_d(t_{d-2,d-1} - 2) < a_1 + b_1(t_{d-2,d-1} - 2)$. Since $a_{d-1} + b_{d-1}(t_{d-2,d-1} - 2) > \cdots > a_1 + b_1(t_{d-2,d-1} - 2)$, we get $t_{i,d} < t_{d-2,d-1} - 2$, $\forall i = 1, \ldots, d$. Then, since $b_d < b_i$ for $i = 1, \ldots, d-1$, the points (1) and (2) are satisfied for $A = diag(a_1, \ldots, a_d)$.

Now, we let $I = I' \cup \{(j, d); j < k\}$. The antipodal monodromy σ of $A(t_1, t_2) = t_1(A + E(I)) + t_2 B$ from $(0, 1)$ in the direction $(1, 0)$ is given by $\sigma(b_i) = b_j$ if the analytic prolongation of the eigenvalue of $A + E(I) + vB$ that is asymptotic to $b_i v$ at $+\infty$ is asymptotic to $b_j v$ at $-\infty$. We denote by $\lambda_1(v), \ldots, \lambda_d(v)$ the analytic eigenvalues of $A + E(I) + vB$ indexed in such a way that $\lambda_i(v) \sim_{+\infty} b_i v$.

Let $i \in \{1, \ldots, d-1\}$. Then $\lambda_i(t_{d-2,d-1} - 1)$ is the $j+1$-th larger eigenvalue of $A + E(I) + (t_{d-2,d-1} - 1)B$ with $j = \tau'(i)$ because $\lambda_d(t_{d-2,d-1} - 1) < \lambda_i(t_{d-2,d-1} - 1)$.

If $j < k$, then $(j, d) \in I$, so the analytic prolongation of λ_i intersects the perturbation of the line $y = a_d + v b_d$. Then the rank of λ_i among the λ's is raised by 1, i.e, becomes the j-th. So $\lambda_i(v) \sim_{-\infty} b_j v$, and $\sigma(b_i) = b_{\tau'(i)} = \tau(b_i)$.

If $j \geq k$, then for all $j' > j$, $(j, j') \notin I$, so the rank of λ_i is conserved for $t < t_{d-2,d-1} - 1$, and $\lambda_i(v) \sim_{-\infty} b_{j+1} v$. But $b_{j+1} = b_{\tau'(i)+1} = \tau(b_i)$. So $\sigma(b_i) = \tau(b_i)$ again.

Finally, for $i = d$, then λ_d is asymptotic to $b_d v$ at $+\infty$, intersects $k-1$ eigenvalues since $(1, d), \ldots, (k-1, d)$ belong to I, and no other, so its rank among the λ's at $-\infty$ is k: $\lambda_d(v) \sim_{-\infty} b_k v$. Hence $\sigma(b_d) = b_k = \tau(b_d)$, which concludes the proof of the claim, and the theorem follows. □

We illustrate the previous proof with the following picture, where we show the construction of the set I in the case of a particular permutation of 5 eigenvalues. Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

Since $\tau(5) = 3 = k_1$, we get $I_1 = \{(1, 5), (2, 5)\}$ and

$$\tau_1' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

Since $\tau_1'(4) = 2 = k_2$, $I_2 = \{(1, 4)\}$ and

$$\tau_2' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Since $\tau_2'(3) = 1 = k_3$, $I_3 = \emptyset$ and $\tau_3' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. We define $I = I_1 \cup I_2 \cup I_3 = \{(1,5),(2,5),((1,4))\}$, Figure 3.2.

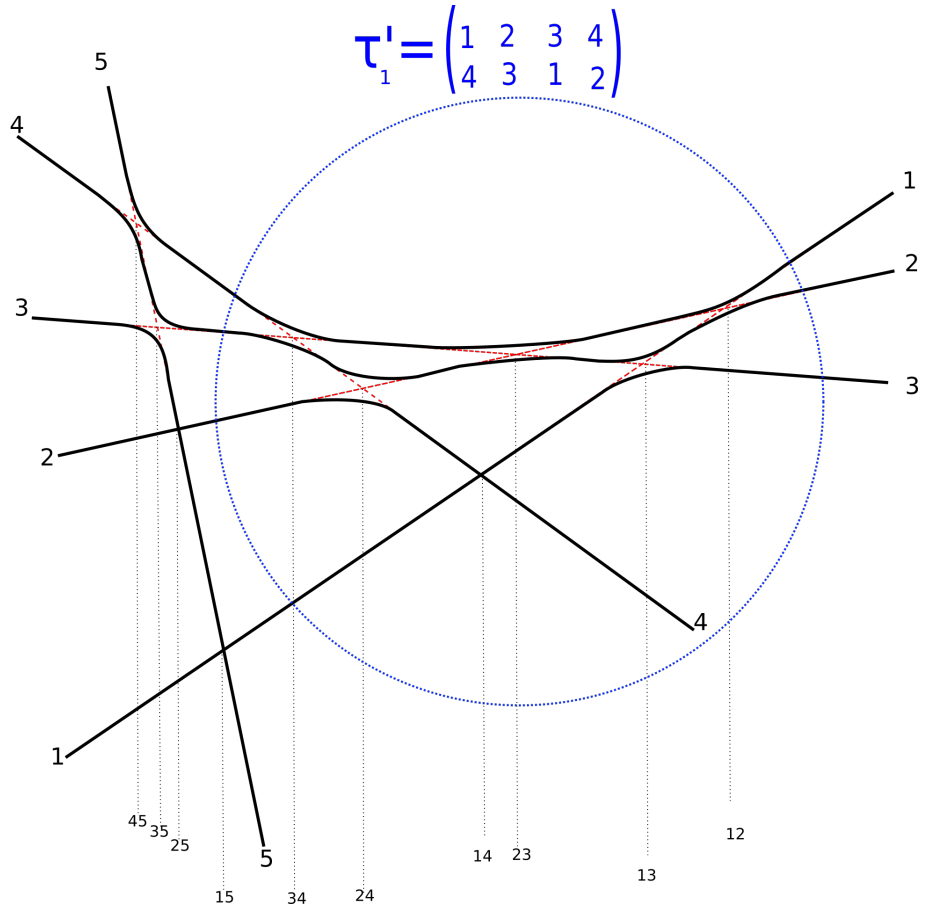


Figure 3.2: Perturbation of lines

3.4 Local study of intersections of Σ_2 with affine planes

In this section, we study the local behavior of the eigenvalues of an affine plane family $\mathcal{P}$ in a neighborhood of a point A_0 of $\mathcal{P} \cap \Sigma_2$, in order to decide whether 0 is a super-singular point of the family. We will frequently compute antipodal monodromies, and for this we always imply that $\mathcal{P}$ is parameterized as $A_0 + L(t), t \in \mathbb{R}^2$, where L is linear. We separate the cases according to the dimension of the intersection of the tangent planes to $\mathcal{P}$ and Σ_2 at the point A_0, and we consider the different possibilities in turn.

3.4.1 $\dim T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = 0$.

We first suppose that there is no intersection between $T_{A_0}\mathcal{P}$ and $T_{A_0}\Sigma_2$ but 0, so no direction in $T_{A_0}\mathcal{P}$ belongs to $T_{A_0}\Sigma_2$. We have the following proposition.

Proposition 3.4.1. *Let $A(t), t \in \mathbb{R}^2$ be an affine family with $A(0) = A_0 \in \Sigma_2$. If $\dim T_{A_0}A(t) \cap T_{A_0}\Sigma_2 = 0$, then for all small $t_0 \neq 0$, the antipodal monodromy of the eigenvalues of $A(t)$ from $t_0 \in \mathbb{R}^2$ in the direction of u, $u \perp t_0$, $\|t_0\| = \|u\|$ is the transposition of the two eigenvalues whose analytic continuations coincide at the origin.*

Proof. Since $T_{A_0}A(t) \cap T_{A_0}\Sigma_2 = \{A_0\}$, there exists a neighborhood V of the origin such that for all $t \in V$, $t \neq (0,0)$, $A(t) \notin \Sigma_2$. Let us take $t_0 \in V$, $t_0 \neq (0,0)$ such that $\gamma \subset V$, where γ is the concatenation of the half-turn path from t_0 in the direction of u, $h_{(t_0,u)}$, followed by the straight path from $-t_0$ to t_0, $[-t_0, t_0]$. Since $A(t_0) \notin \Sigma$, we have $\#Spec(A(t_0)) = d$.

Note that $h_{(t_0,u)} \subset V$, so the order of the eigenvalues of $A(t_0)$ is preserved along $h_{(t_0,u)}$. The same argument shows that this order is also preserved along the segment $(-t_0, 0)$. Let

$$C(s) := A_0 + s(A_0 - A(-t_0)) = A_0 + sB,$$

with $B = A_0 - A(-t_0)$ and $s \in [-1, 1]$, so that $C(-1) = A(-t_0)$, $C(0) = A_0$. Note $A_0 \in \Sigma_2$, $B \notin T_{A_0}\Sigma_2$ and for all $s \in [-1, 0)$, $s \neq 0$, we have $C(s) \notin \Sigma_2$. So, by Lemma 3.3.1 there exists $\lambda_1, \lambda_2 \in Spec(C(-1))$ whose analytic prolongations along the segment $[-1, 0]$ coincide at 0 and have two different slopes at 0. Hence, the order of the analytic prolongations of λ_1 and λ_2 reverse at the origin. But $A_0 \in \Sigma_2$, so the order between the other eigenvalues is preserved along the segment $[-t_0, 0]$. Finally, the order of the eigenvalues of $A(t)$ is preserved along the segment $(0, t_0)$.

From this, the antipodal monodromy from t_0 in the direction of u, is the transposition that reverse λ_1 and λ_2. This finishes the proof. □

Corollary 3.4.2. Under the hypotheses of Proposition 3.4.1, there exists a non-trivial antipodal monodromy for the eigenvalues of $A(t)$ which forbids the eigenvalues to be analytic. Hence, $(0,0) \in S_A$.

Example 3.4.3. Let $A_0 = diag(0,0,1)$, $A_1, A_2 \notin T_{A_0}\Sigma_2$ be given as follows.

$$A_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Consider the family

$$A(t) = A_0 + t_1A_1 + t_2A_2 = \begin{pmatrix} t_1 & t_2 & t_2 - t_1 \\ t_2 & -t_1 & t_1 + t_2 \\ t_2 - t_1 & t_1 + t_2 & 2t_2 + 1 \end{pmatrix},$$

The characteristic polynomial of $A(t)$ is

$$\chi_{A(t_1,t_2)}(x) = -x^3 + x^2(1 + 2t_2) + x(3t_2^2 + 3t_1^2) - 8t_1^2t_2 - t_1^2 - t_2^2.$$

The family $A(t)$ satisfies the hypothesis of Proposition 3.4.1. So, $A(t) \cap \Sigma = \{A_0\}$ in a neighborhood of the origin, and there is a non-trivial antipodal monodromy which forbids the eigenvalues to be analytic. We see the eigenvalues of $A(t)$ in a neighborhood of the origin, Figure 3.3.

Figure 3.3: Eigenvalues of $A(t)$ near the origin

3.4.2 $\dim T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = 1$

We now suppose the intersection of $T_{A_0}\mathcal{P}$ with the tangent plane to Σ_2 at A_0 is a line. Since Σ_2 has codimension two, the intersection $\Sigma_2 \cap \mathcal{P}$ in a neighborhood of A_0 should generically contains nothing but A_0. However, it happens that Σ_2 contains many

plane curves, and this case admits different subcases. In a neighborhood of A_0, either $\mathcal{P} \cap \Sigma_2 = \{A_0\}$ or $\dim \mathcal{P} \cap \Sigma_2 = 1$. Again, we consider these different possibilities in turn.

3.4.2.1 $\mathcal{P} \cap \Sigma_2 = \{A_0\}$:

Proposition 3.4.4. *Let $A(t) = A_0 + t_1A_1 + t_2A_2 \in \mathcal{P}$ with $A_1 \in T_{A_0}\Sigma_2$ and $A_2 \notin T_{A_0}\Sigma_2$. Suppose that $A(t) \cap \Sigma_2 = \{A_0\}$ in a neighborhood of the origin. Then, there exists $\varepsilon > 0$ such that the antipodal monodromy of the eigenvalues of $A(t)$ from $(0, \varepsilon)$ in the direction of $(\varepsilon, 0)$ is the transposition of the two eigenvalues of $A(0, \varepsilon)$ whose analytic continuations coincide at 0.*

Proof. Since $A(t) \cap \Sigma_2 = \{A_0\}$, there exists a neighborhood V of the origin such that for all $t \in V$, $t \neq (0, 0)$, $A(t) \notin \Sigma_2$. There exists $\varepsilon > 0$ such that the concatenation of the half-turn path from $(0, \varepsilon)$ in the direction of $(\varepsilon, 0)$, $h_{(t_0,u)}$, followed by the straight path from $-(0, \varepsilon)$ to $(0, \varepsilon)$ belongs to V. Since $A(0, \varepsilon) \notin \Sigma$, we have $\#Spec(A(0, \varepsilon)) = d$. Moreover, $h_{(t_0,u)} \subset V$, so the order of the eigenvalues of $A(0, \varepsilon)$ is preserved along $h_{(t_0,u)}$. Now, $A_2 \notin T_{A_0}\Sigma_2$ and $A_0 \in \Sigma_2$. So, by Lemma 3.3.1 there exists $\lambda_1, \lambda_2 \in Spec(A(0, \varepsilon))$ such that the analytic continuations of λ_1 and λ_2 along the segment $(-\varepsilon, \varepsilon)$ coincide at 0 and have two different slopes at 0.

Hence, the order of λ_1 and λ_2 reverse at the origin. But $A_0 \in \Sigma_2$, so the order between λ_i and λ_j is preserved along the segment $(-\varepsilon, \varepsilon)$ for all $\{i, j\} \neq \{1, 2\}$. This finishes the proof. □

Corollary 3.4.5. *Under the hypotheses of Proposition 3.4.4, $A_0 \in S_A$.*

In the following example, which illustrate the situation, we see that in a non-linear case, the antipodal monodromy is not a satisfying invariant in the affine situation. Indeed, it shows that this antipodal monodromy might depend on base point t_0.

Example 3.4.6. Let $A(t) = A_0 + t_1A_1 + t_2A_2 \in \mathcal{P}$ be given as follows.

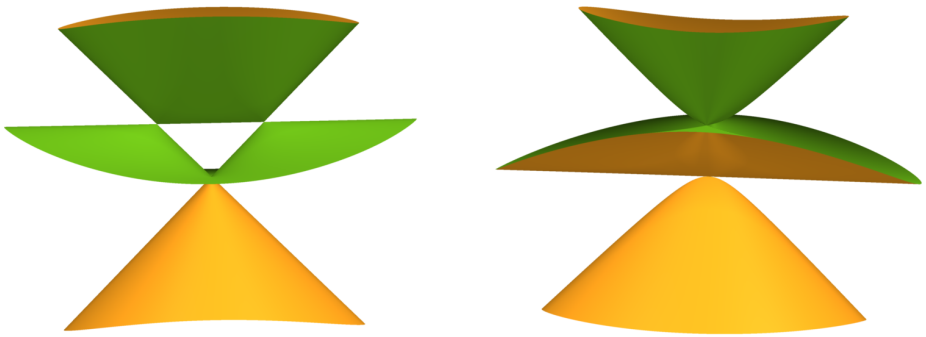
$$A(t) = \begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & 0 & 0 \\ t_1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $A_0 \in \Sigma_2$, $A_1 \in T_{A_0}\Sigma_2$ and $A_2 \notin T_{A_0}\Sigma_2$. We have

$$\chi_{A(t_1,t_2)(\lambda)} = -\lambda^3 + \lambda^2 + \lambda(t_1^2 + t_2^2) - t_2^2,$$
$$\Delta_{A(t_1,t_2)} = (2t_1^3)^2 + (\sqrt{12}t_1^2t_2)^2 + (t_1^2)^2 + (\sqrt{12}t_1t_2^2)^2 + (\sqrt{20}t_1t_2)^2 + (2t_2(t_2^2-1))^2,$$

where $\chi_{A(t_1,t_2)(\lambda)}$ and $\Delta_{A(t_1,t_2)}$ are the characteristic polynomial and the discriminant of the matrix $A(t_1,t_2)$, respectively. We have $Z(\Delta_{A(t_1,t_2)}) = \{(0,0),(0,\pm 1)\}$. So $A(t) \cap \Sigma_2 = \{A_0\}$ in a small neighborhood of the origin.

The calculation gives $Spec(A(t_1,0)) = \{0, \frac{1}{2}(1 \pm \sqrt{4t_1^2+1})\}$ and $Spec(A(0,t_2)) = \{-t_2, 1, t_2\}$. Note that for $t_0 = (0,t_2)$, $t_2 \neq 0$ small, the antipodal monodromy from t_0 in the direction of $u = (-t_2, 0)$ is a transposition of the two eigenvalues whose analytic continuation coincide at 0. But, for small $t_1 \neq 0$, the antipodal monodromy from $(t_1,0)$ in the direction of $(0,t_1)$ is trivial. Indeed, $t \mapsto 1 - \sqrt{4t_1^2+1}$ is a non-positive analytic function on $(-t_1,t_1)$ so that two eigenvalues 0 and $\frac{1}{2}(1-\sqrt{4t_1^2+1})$ do not exchange along the segment $(-t_1,t_1)$. In Figure 3.4, we see the eigenvalues of $A(t_1,t_2)$.



(a) non-trivial monodromy (b) trivial monodromy

Figure 3.4: Antipodal monodromies of the eigenvalues

3.4.2.2 $\dim \mathcal{P} \cap \Sigma_2 = 1$:

We first prove that the intersection $\mathcal{P} \cap \Sigma_2$ is an analytic curve .

Proposition 3.4.7. *Let $\mathcal{P}$ be the image of $A(t) = A_0 + t_1A_1 + t_2A_2$, $A_0 \in \Sigma_2$. Suppose that:*

1. $T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = Span(A_1)$

2. $\dim \mathcal{P} \cap \Sigma_2 = 1$.

Then, the intersection $\mathcal{P} \cap \Sigma_2$ *is a smooth analytic curve.*

Proof. We choose $A' \in Sym_d(\mathbb{R})$ such that $Span(A_2, A')$ is a complementary subspace of $T_{A_0}\Sigma_2$ in $Sym_d(\mathbb{R})$:

$$Sym_d(\mathbb{R}) = T_{A_0}\Sigma_2 \oplus Span(A_2, A').$$

Denote by π the associated projection $\pi : Sym_d(\mathbb{R}) \to T_{A_0}\Sigma_2$. We note that locally Σ_2 is the graph of an analytic function over $T_{A_0}\Sigma_2$. This means that there exists an analytic function

$$\begin{array}{rccc} \phi & : & T_{A_0}\Sigma_2 & \to & \mathbb{R}^2 \\ & & B & \mapsto & (\phi_1(B), \phi_2(B)) \end{array},$$

a neighborhood $\mathcal{V} \in Sym_d(\mathbb{R})$ of A_0 and a neighborhood $\mathcal{W} \in T_{A_0}\Sigma_2$ of 0 such that

$$A \in \mathcal{V} \cap \Sigma_2 \Leftrightarrow \begin{cases} \pi(A - A_0) \in \mathcal{W}, \\ A = A_0 + \pi(A - A_0) + \phi_1(\pi(A - A_0))A_2 + \phi_2(\pi(A - A_0))A'. \end{cases}$$

In particular, for (t_1, t_2) is a small neighborhood of the origin,

$$A_0 + t_1A_1 + t_2A_2 \in \mathcal{V} \cap \Sigma_2 \Leftrightarrow \begin{cases} t_1A_1 \in \mathcal{W}, \\ t_2 = \phi_1(t_1A_1), \\ \phi_2(t_1A_1) = 0. \end{cases}$$

Since, $\dim A(t) \cap \Sigma_2 = 1$ and $A_1 \in T_{A_0}\Sigma_2$, for t_1 close to 0, we have $t_1A_1 \in \mathcal{W}$ and $\phi_2(t_1A_1) = 0$. So, $A_0 + t_1A_1 + t_2A_2 \in \mathcal{V} \cap \Sigma_2$ if and only if $t_2 = \phi_1(t_1A_1)$. Hence, the intersection $\mathcal{P} \cap \Sigma_2$ is given by the parameterization $A_0 + t_1A_1 + \phi_1(t_1A_1)A_2$ which defines an analytic curve as announced. □

We now study the eigenvalues of the family.

Proposition 3.4.8. *Let* $A(t) = A_0 + t_1A_1 + t_2A_2 \in \mathcal{P}$ *with* $A_0 \in \Sigma_2$ *such that* $\dim T_{A_0}A(t) \cap T_{A_0}\Sigma_2 = 1$. *If* $\dim A(t) \cap \Sigma_2 = 1$, *then* $A(t)$ *admits analytic eigenvalues* $\mu_1(t_1, t_2)$ *and* $\mu_2(t_1, t_2)$ *in a neighborhood of the origin which coincide at* $(0, 0)$.

Proof. Without loss of generality, we suppose that $A_1 \in T_{A_0}A(t) \cap T_{A_0}\Sigma_2$ and $A_2 \notin T_{A_0}\Sigma_2$. By Proposition 3.4.7, the intersection $A(t) \cap \Sigma_2$ is an analytic curve. Let

$A_0 + xA_1 + \phi(x)A_2$ be an analytic parameterization of $A(t) \cap \Sigma_2$, where ϕ is analytic. Let $B(t,x) = A_0 + xA_1 + \phi(x)A_2 + tA_2$. Then, $P_{(t,x)}(\lambda) = \chi_{B(t,x)}(\lambda)$ is an analytic family of hyperbolic polynomials such that:

1. $P_{(0,0)}(\lambda) = \chi_{B(0,0)} = \chi_{A_0}(\lambda)$ has $d-2$ simple roots and one root of multiplicity two,
2. $P_{(0,x)}(\lambda) = \chi_{B(0,x)}(\lambda) = \chi_{A_0+xA_1+\phi(x)A_2}(\lambda)$. Note that $A_0 + xA_1 + \phi(x)A_2$ is a one-parameter analytic family in Σ_2. So, by Rellich's Theorem, the eigenvalues of $A_0 + xA_1 + \phi(x)A_2$ are analytic functions. Since $A_0 + xA_1 + \phi(x)A_2 \in \Sigma_2$, it has an analytic multiple root. We denote it by $\lambda_0(x)$.
3. Since $A_0 \in \Sigma_2$ and $A_2 \notin T_{A_0}\Sigma_2$, then by Lemma 3.3.1, $P_{(t,0)}(\lambda) = \chi_{A_0+tA_2}(\lambda)$ has two analytic roots $\lambda_1(t)$, $\lambda_2(t)$ with $\lambda_1(0) = \lambda_2(0)$, $\lambda_1'(0) = \lambda_2'(0)$.

So, $\chi_{B(t,x)}(\lambda)$ satisfies the hypotheses of Proposition of 1.2.4. So, it has two analytic roots $\lambda_1(t,x)$ and $\lambda_2(t,x)$ in a neighborhood of $(0,0)$. Now, $A_0 + tA_1 + t_2A_2 = B(t_2 - \phi(t_1), t_1)$. So,

$$\mu_1(t_1,t_2) = \lambda_1(t_2 - \phi(t_1), t_1)$$
$$\mu_2(t_1,t_2) = \lambda_2(t_2 - \phi(t_1), t_1).$$

are analytic eigenvalues of $A(t_1,t_2)$. □

Examples:

Recall Σ_2 has codimension two in $Sym_d(\mathbb{R})$. However, it happens that Σ_2 contains many plane curves. We give several examples of this fact. A trivial case happens when $A_0 + t_1A_1$ is block diagonal, with a constant eigenspace of dimension two (case 1). There are also non-trivial examples, where $\mathcal{P} \cap \Sigma_2$ is not a line and in particular, $\mathcal{P} \cap \Sigma_2 \neq T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2$ (case 2).

• **Case 1:** The intersection $\mathcal{P} \cap \Sigma_2$ is a line:

Example 3.4.9. Let $A(t) = A_0 + t_1A_1 + t_2A_2$ where $A_0 = diag(0,0,1)$ and $A_1 = diag(1,1,2)$ and

$$A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that $A_0, A_1 \in \Sigma_2$ are diagonal and $A_2 \notin T_{A_0}\Sigma_2$. So, the intersection of $A(t)$ and

Σ_2 is the line $A_0 + t_1 A_1$. Here, $A(t)$ satisfies the hypothesis of Proposition 3.4.8. So, the eigenvalues of $A(t)$ are analytic in a neighborhood of the origin, Figure 3.5.

Figure 3.5: Intersection along a line

Example 3.4.10. We give another example in dimension four, to emphasize the fact that $\mathcal{P}$ might intersect Σ_2 along a line that is not made of diagonal matrices, but only block diagonal. Indeed, suppose that $A_0 = diag(0, 0, \alpha_3, \ldots, \alpha_d) \in \Sigma_2$, $A_2 \notin T_{A_0}\Sigma_2$ and A_1 has the following form.

$$A_1 = \left(\begin{array}{c|c} k\mathbb{I}_2 & 0_{2,d-2} \\ \hline 0_{d-2,2} & B \end{array}\right), \tag{3.9}$$

where B is a $d-2 \times d-2$ symmetric matrix. Then, for all $(t_1, 0)$ in a neighborhood of the origin $A_0 + t_1 A_1 + t_2 A_2 \in \mathcal{P}$ has a double eigenvalue $t_1 k$. For instance, let $A_0 = diag(0, 0, 1, 2)$,

$$A_1 = 0.1 \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

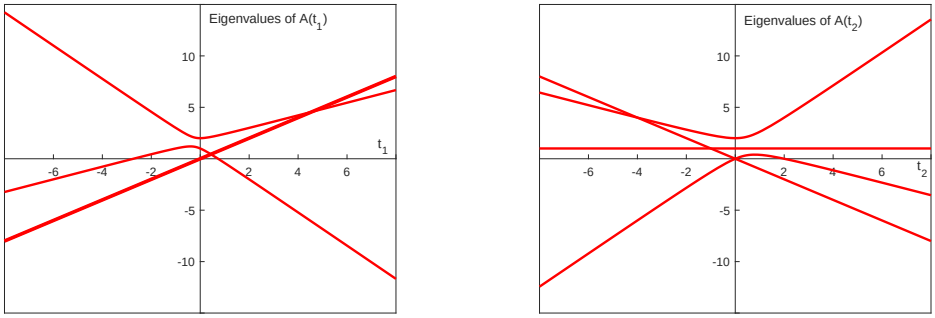
Note that $A_0 \in \Sigma_2$, $A_2 \notin T_{A_0}\Sigma_2$ and A_1 has the shape given in (3.9). We have

$$\chi_{A(t_1,t_2)}(x) = (0.1t_1 - t_2 - x)[(0.1t_1 + t_2 - x)((2-x)(-0.1t_1 + 1 - x) - 0.01t_1^2) + t_2^2(-0.1t_1 + 1 - x)]. \tag{3.10}$$

We have $\chi_{A(t_1,0)}(x) = (0.1t_1 - x)^2(x^2 + x(0.1t_1 - 3) + 2 - 0.2t_1)$, whose roots are

$$\begin{aligned} x_1 &= 0.1t_1 \\ x_2 &= 0.1t_1 \\ x_3 &= \frac{1}{2}(3 - 0.1t_1 + \sqrt{0.05(t_1^2 + 4t_1 + 20)}) \\ x_4 &= \frac{1}{2}(3 - 0.1t_1 - \sqrt{0.05(t_1^2 + 4t_1 + 20)}) \end{aligned}$$

which has a double eigenvalue t_1, Figure 3.6a, and $Spec(A(0,t_2)) = \{1, -t_2, \frac{1}{2}(t_2 + 2 \pm \sqrt{5t_2^2 - 4t_2 + 4})\}$, Figure 3.6b.



(a) (b)

Figure 3.6: The eigenvalues of the one parameter families

Note that $A(0,0) \in \Sigma_2$ and for all $(t_1, 0)$ in a neighborhood of the origin, $A(t_1, t_2)$ has the double eigenvalue $0.1t_1$. In Figure 3.7, we see the eigenvalues of $A(t)$ in a neighborhood of the origin.

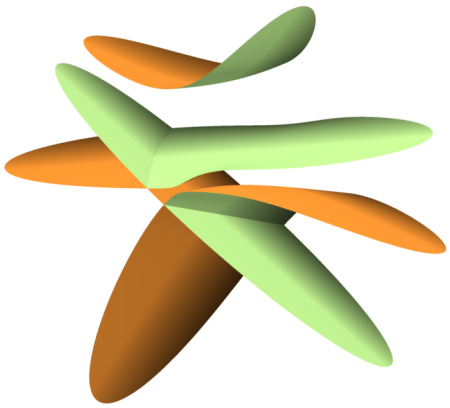


Figure 3.7: Intersection along a line

• **Case 2:** The intersection $\mathcal{P} \cap \Sigma_2$ is not a line:

There are also planes curves in Σ_2 that are not lines. Our examples is based on the observation that the orbits of A_0 under of the action of certain one-parameter subgroups of $O(d)$ which are plane curves are plane curves themselves. For instance, if $A_0 = diag(0, 0, \alpha_3, \ldots, \alpha_d) \in \Sigma_2$ and

$$G(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & & & \vdots \\ \sin\theta & 0 & \cos\theta & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix},$$

then, the orbit of A_0 by the subgroup $G = \{G(\theta);\ \theta \in \mathbb{R}\} \subset O(d)$ is the image of

$$\gamma(\theta) = G(\theta)A_0G(\theta)^T = \begin{pmatrix} r(\theta) & 0 & q(\theta) & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & & & \vdots \\ q(\theta) & 0 & z(\theta) & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \alpha_4 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_d \end{pmatrix},$$

with $r(\theta) = \alpha_3 \sin^2\theta$, $q(\theta) = -\alpha_3 \sin\theta\cos\theta$,$z(\theta) = \alpha_3\cos^2\theta$. So, for all $\theta \in \mathbb{R}$, $\gamma(\theta)$ belongs to the affine plane $\mathcal{P} = A_0 + Span(A_1, A_2)$, where

$$A_1 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & -1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & & \vdots \\ 1 & 0 & 0 & 0 & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}.$$

In particular, the plane $\mathcal{P}$ contains a curve included in Σ_2. Here, since $A_1 \notin T_{A_0}\Sigma_2$, this curve is the intersection $\mathcal{P} \cap \Sigma_2$.

Example 3.4.11. Let $A(t_1, t_2) = A_0 + t_1A_1 + t_2A_2$ be the affine family given by

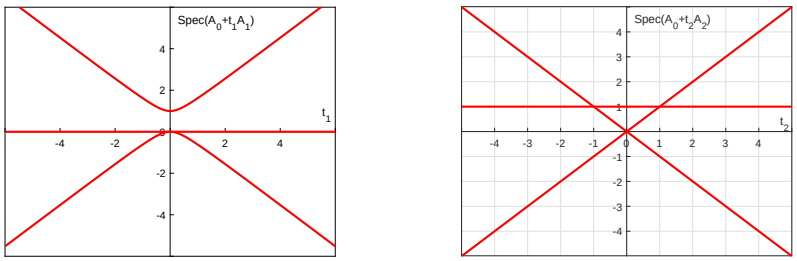
$$A(t_1,t_2) = \begin{pmatrix} t_2 & 0 & t_1 \\ 0 & -t_2 & 0 \\ t_1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $A(t_1, t_2)$ is tangent to Σ_2 in the direction of A_1 and it is transverse to Σ_2 in the direction of A_2. We have

$$\begin{aligned} \chi_{A(t_1,t_2)}(x) &= t_1^2t_2 + t_1^2x + t_2^2x - t_2^2 - x^3 + x^2 \\ &= (t_2 + x)(t_1^2 + t_2x - t_2 - x^2 + x) \end{aligned}$$

Thus, $Spec(A(t_1, t_2)) = \{\frac{1}{2}(t_2 + 1 \pm \sqrt{4t_1^2 + t_2^2 - 2t_2 + 1}), -t_2\}$.

If $t_2 = 0$ then $Spec(A_0 + t_1A_1) = \{0, \frac{1}{2}(1 \pm \sqrt{4t_1^2 + 1})\}$, Figure 3.8a. Now, if $t_1 = 0$ then $Spec(A_0 + t_2A_2) = \{-t_2, 1, t_2\}$, Figure 3.8b.



(a) (b)

Figure 3.8: The eigenvalues of the one parameter families

We calculate the intersection of Σ with $A(t_1, t_2)$. Notice that, $A(t_1, t_2) \in \Sigma_2$, then **either** $-t_2$ is a multiple eigenvalue of $A(t_1, t_2)$, **or** the sub-matrix

$$A_{22} := \begin{pmatrix} t_2 & t_1 \\ t_1 & 1 \end{pmatrix} \in \Sigma_2.$$

But the discriminant of A_{22} is $\Delta_{A_{22}} = (t_2 - 1)^2 + 4t_1^2$, which vanishes only at $(0, 1)$. Now, if $-t_2$ is a multiple eigenvalue of $A(t_1, t_2)$, we have $\det \begin{pmatrix} t_2 + t_2 & t_1 \\ t_1 & 1 + t_2 \end{pmatrix} = 0$. From this, we get $(1 + t_2)(2t_2) - t_1^2 = 0$. So, near $(0, 0)$, $A(t_1, t_2) \in \Sigma$ along this curve (hyperbola).

From proposition 3.4.7, the eigenvalues of $A(t)$ are analytic functions in a neighborhood of the origin: $(0, 0) \notin S_A$. In Figure 3.9, we see the eigenvalues of A(t) in a neighborhood of the origin.

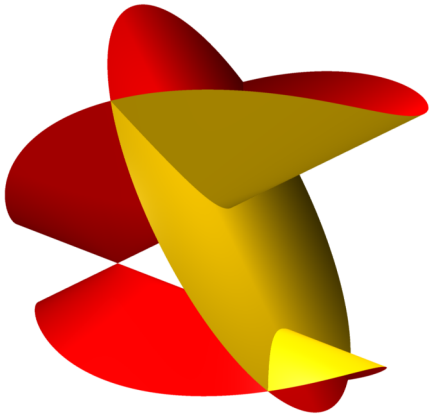


Figure 3.9: Intersection along a hyperbola

3.4.3 $\dim T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = 2$

The plane tangent to Σ_2 at A_0 and the tangent to $\mathcal{P}$ coincide. We note $\mathcal{P} = A_0 + t_1A_1 + t_2A_2$, $\mathcal{P} \subset T_{A_0}\Sigma_2$. There are again subcases:

1. $\mathcal{P} \cap \Sigma_2 = \{A_0\}$;
2. $\dim \mathcal{P} \cap \Sigma_2 = 1$;
3. $\dim \mathcal{P} \cap \Sigma_2 = 2$.

Except for the subcase $\dim \mathcal{P} \cap \Sigma_2 = 2$, i.e., $\mathcal{P} \subset \Sigma_2$, we don't have a description simply based on $\dim \mathcal{P} \cap \Sigma_2$. Indeed, the behavior of the eigenvalues of $\mathcal{P}$ strongly depends on orders of contact between $\mathcal{P}$ and Σ_2, that can be high as the dimension of the ambient space grows. We illustrate this fact with various examples that includes the complete classification for $Sym_3(\mathbb{R})$. The following proposition shows that in dimension three, in this case, $(0,0) \notin S_A$.

Proposition 3.4.12. *Let $\mathcal{P} = A_0 + tA_1 + t_2A_2 \in Sym_3(\mathbb{R})$ with $A_0 \in \Sigma_2$. Suppose that $\dim T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = 2$. Then, the eigenvalues of $\mathcal{P}$ are analytic functions near the origin.*

Proof. Since $\dim T_{A_0}\mathcal{P} \cap T_{A_0}\Sigma_2 = 2$, we have $\mathcal{P} \subset T_{A_0}\Sigma_2$. Without loss of generality, we suppose that $A_0 = diag(0,0,a)$ with $a \in \mathbb{R}$, $a \neq 0$. Up to subtract $k(t_1,t_2)\mathbb{I}_3$ with some linear function k to the family, we suppose that

$$A(t_1,t_2) = A_0 + B(t_1,t_2) = A_0 + \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha & \beta & \gamma \end{pmatrix},$$

with α, β, γ linear in (t_1,t_2). We show that the eigenvalues of $A_0 + B(t_1,t_2)$ are analytic functions. The characteristic polynomial is

$$\begin{aligned} \chi_{A_0+B(t_1,t_2)}(x) &= \det \begin{pmatrix} -x & 0 & \alpha \\ 0 & -x & \beta \\ \alpha & \beta & \gamma + a - x \end{pmatrix} \\ &= -x\left(-x(\gamma + a - x) - \beta^2\right) + \alpha^2 x \\ &= -x\left(x^2 - x(a+\gamma) - \beta^2 - \alpha^2)\right). \end{aligned}$$

Then $Spec(A_0 + B(t_1,t_2)) = \left\{0, \frac{1}{2}\left(\gamma + a \pm \sqrt{(a+\gamma)^2 + 4(\alpha^2+\beta^2)}\right)\right\}$. Since $a \neq 0$

and $\gamma(0,0) = 0$, this are analytic functions near $(0,0)$. □

Remark 3.4.13. In the previous proof, we have seen that, given a direction $B \in T_{A_0}\mathcal{P}$, the contact order of the two eigenvalues that coincide at 0 along $A_0 + tB$ is always 2, except when the linear forms α and β have a common zero line that contains B (in which case, $A+tB$ lies in Σ_2). We will see that this order might vary in higher dimensional spaces, and this sometimes create non-trivial antipodal monodromy.

3.4.3.1 $\mathcal{P} \cap \Sigma_2 = \{A_0\}$:

Example 3.4.14. We first give an example in dimension 3. In view of the proof of the preceding proposition, this case happens for the family $A(t_1,t_2) = A_0 + t_1A_1 + t_2A_2 \in Sym_3(\mathbb{R})$ given as follows.

$$A(t_1,t_2) = \begin{pmatrix} 0 & 0 & t_1 \\ 0 & 0 & t_2 \\ t_1 & t_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have $\chi_{A(t_1,t_2)} = t_1^2x + t_2^2x - x^3 + x^2 = x(t_1^2 + t_2^2 - x^2 + x)$ and so

$$Spec(A(t_1,t_2)) = \left\{ \frac{1}{2}(1 - \sqrt{4t_1^2 + 4t_2^2 + 1}), 0, \frac{1}{2}(1 + \sqrt{4t_1^2 + 4t_2^2 + 1}) \right\}.$$

In Figure 3.10, we see the analytic eigenvalues of $A(t_1,t_2)$.

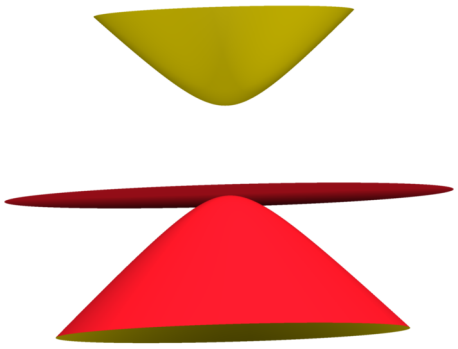


Figure 3.10: The intersection is a point

Example 3.4.15. We give another example for $d = 4$. For $\mathcal{P} \in Sym_d(\mathbb{R})$, $d > 3$, there are examples where $\mathcal{P} \cap \Sigma_2 = \{A_0\}$ and there exists a direction along which

the order of contact between the eigenvalues is odd, and this produces non-trivial antipodal monodromy. Indeed, if this happens, the analytic continuation along a segment $[-t, t]$ exchange the order of two eigenvalues, while the order is preserved along the half turn (since only $A_0 \in \Sigma$). So the antipodal monodromy in this direction is non-trivial, and $(0,0) \in S_A$.

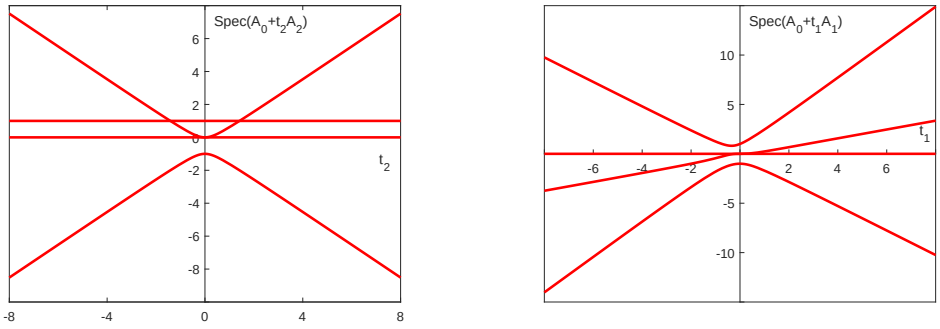
For instance, let $A(t_1, t_2) = A_0 + t_1 A_1 + t_2 A_2 \in Sym_4(\mathbb{R})$ be given as follows.

$$A(t_1,t_2) = \begin{pmatrix} 0 & 0 & 0 & t_2 \\ 0 & 0 & t_1 & t_1 \\ 0 & t_1 & 1+t_1 & 0 \\ t_2 & t_1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of $A(t_1, t_2)$ is

$$\chi_{A(t_1,t_2)} = t_1^3 x + t_1^2 t_2^2 - 2t_1^2 x^2 + t_1 t_2^2 x - t_1 x^3 - t_1 x^2 - t_2^2 x^2 + t_2^2 x + x^4 - x^2.$$

In Figures 3.11a and 3.11b, we see the eigenvalues of $A(0, t_2)$ and $A(t_1, 0)$, respectively.



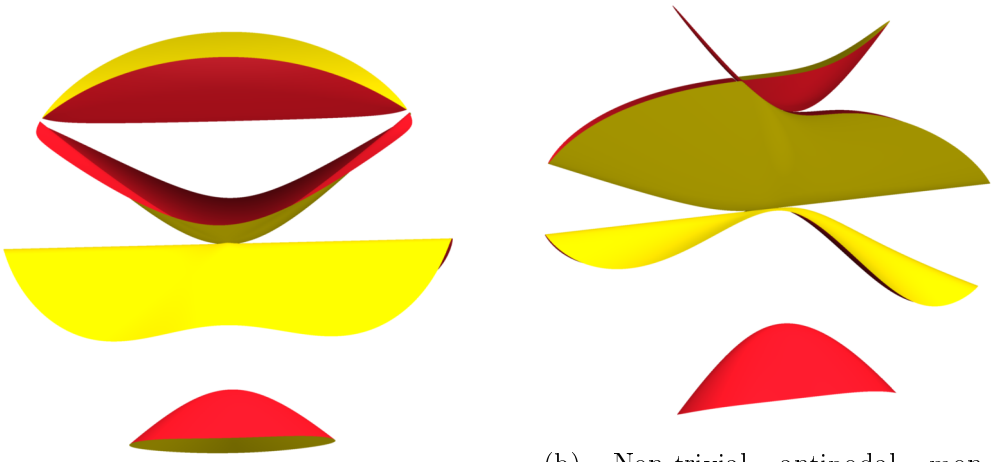
(a) (b)

Figure 3.11: The eigenvalues of the one parameter families

In the direction of A_2, the order of contact between the eigenvalues is even (order 2) while the order of contact in the direction of A_1 is odd (order 3), Figure 3.12. So the antipodal monodromy from $(t_1, 0)$ in the direction of $(0, t_1)$ exchange two eigenvalues. In particular, $(0,0) \in S_A$.

Once again, this example shows that the antipodal monodromy might depend on the base point in the case of affine families. Since in the direction of A_2, the order of contact between the eigenvalues is 2, the antipodal monodromy from $(0, t_2)$ in the

direction of $(t_2, 0)$ is the identity.



(a) Trivial antipodal monodromy

(b) Non-trivial antipodal monodromy

Figure 3.12: Antipodal monodromies of eigenvalues

Since the antipodal monodromy from $(t_1, 0)$ in the direction of $(0, t_1)$ is non-trivial, so there is no analytic orthogonal matrix $Q(t) \in O(d)$ which diagonalize $A(t_1, t_2)$.

3.4.3.2 $\dim \mathcal{P} \cap \Sigma_2 = 1$

We give two examples, in dimension 3 and 5. In dimension three, Example 3.4.16, $(0, 0)$ is not a super-singular point as expected from Proposition 3.4.12. In this example, the intersection $\mathcal{P} \cap \Sigma_2$ is a smooth curve. In dimension five, Example 3.4.17, $(0, 0)$ also is not a super-singular point while the intersection $\mathcal{P} \cap \Sigma_2$ is a singular curve, a cusp.

Example 3.4.16. In **dimension three**, the order of contact between the eigenvalues is two. So, the antipodal monodromy from t_0, in the direction of u, $u \perp t_0$, $\|t_0\| = \|u\|$ is trivial. Thus, there exists an analytic orthogonal matrix $Q(t) \in O(d)$ which diagonalize $\mathcal{P}$. For instance, let $A(t_1, t_2) = A_0 + t_1 A_1 + t_2 A_2 \in Sym_3(\mathbb{R})$ be given as follows.

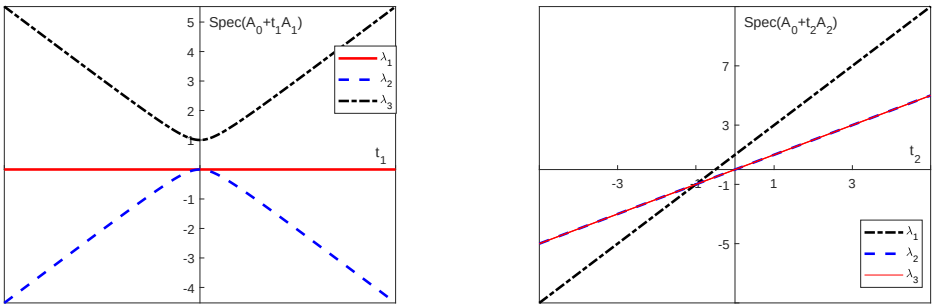
$$A(t_1, t_2) = \begin{pmatrix} t_2 & 0 & t_1 \\ 0 & t_2 & 0 \\ t_1 & 0 & 1 + 2t_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then,

$$\chi_{A(t_1,t_2)} = -t_1^2 t_2 + t_1^2 x + 2t_2^3 - 5t_2^2 x + t_2^2 + 4t_2 x^2 - 2t_2 x - x^3 + x^2 \tag{3.11}$$

$$= -(t_2 - x)(t_1^2 - 2t_2^2 + 3t_2 x - t_2 - x^2 + x) = 0. \tag{3.12}$$

Note that $Spec(A(t_1, 0)) = \{0, \frac{1}{2}(1 \pm \sqrt{4t_1^2 + 1}\}$, Figure 3.13a and $Spec(A(0, t_2)) = \{t_2, t_2, 2t_2 + 1\}$, Figure 3.13b.



(a) (b)

Figure 3.13: The eigenvalues of the one parameter families

By (3.12), we get $Spec(A(t_1, t_2)) = \{t_2, \frac{1}{2}(1 + 3t_2 \pm \sqrt{4t_1^2 + (t_2 + 1)^2})\}$, where for $t_1 = 0, t_2 = -1$, $A(t)$ has a triple eigenvalue -1 and $A(0, 0) \in \Sigma_2$. Note that for small $t_1 \neq 0$, the antipodal monodromy from $(t_1, 0)$ in the direction of $(0, t_1)$ is trivial. Indeed, $t \mapsto 1 - \sqrt{4t_1^2 + 1}$ is a non positive analytic function on $(-t_1, t_1)$ so that two eigenvalues 0 and $\frac{1}{2}(1 - \sqrt{4t_1^2 + 1})$ do not exchange along the segment $(-t_1, t_1)$. Therefore, the eigenvalues of $A(t_1, t_2)$ are analytic functions in a neighborhood of the origin, Figure 3.14.

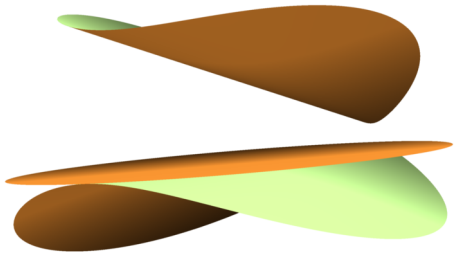


Figure 3.14: Eigenvalues of $A(t_1, t_2)$ in a neighborhood of the origin

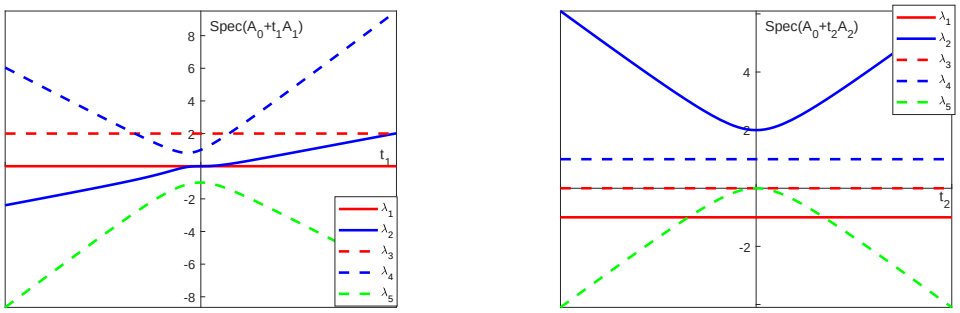
Example 3.4.17. This example prove that the intersection of $\mathcal{P} \cap \Sigma_2$ is not necessary a smooth curve when $\mathcal{P} \subset T_{A_0}\Sigma_2$. Let $A(t_1, t_2) = A_0 + t_1A_1 + t_2A_2 \in Sym_5(\mathbb{R})$ be given as follows.

$$A(t_1, t_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & t_2 \\ 0 & 0 & t_1 & t_1 & 0 \\ 0 & t_1 & 1+t_1 & 0 & 0 \\ 0 & t_1 & 0 & -1 & 0 \\ t_2 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of $A(t_1, t_2)$ is

$$\begin{aligned} \chi_{A(t_1,t_2)}(x) &= (-t_2^2 + x^2 - 2x)(-t_1^3 + 2t_1^2x + t_1x^2 + t_1x - x^3 + x) \\ &= P_1(t_2, x)P_2(t_1, x) \end{aligned}$$

In Figure 3.15 we see the eigenvalues of $A(t_1, 0)$ and $A(0, t_2)$.



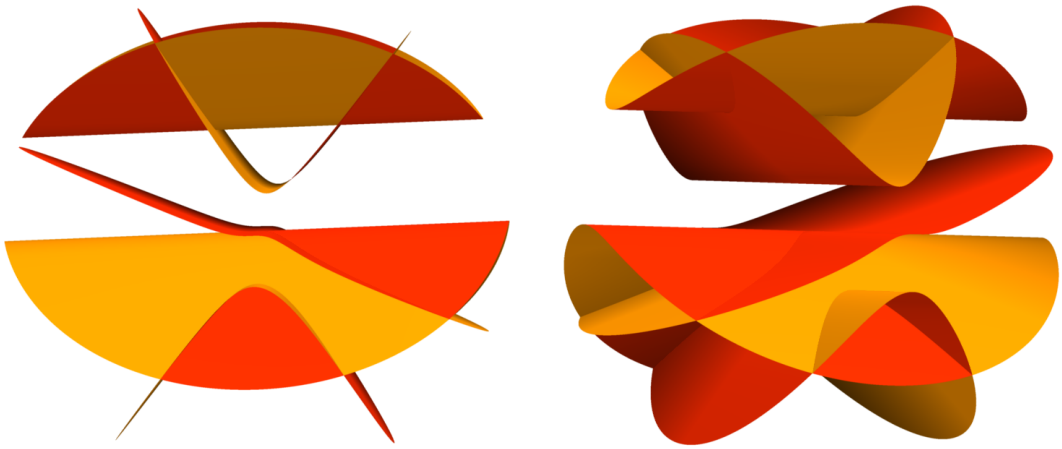
(a) (b)

Figure 3.15: The eigenvalues of the one parameter families

Let $x(t_2)$ be a root of $P_1(t_2, x)$ with $x(0) = 0$. We write the Taylor expansion of $x(t_2)$ at $t_2 = 0$:

$$x(t_2) = \sum_{i>0} a_i t_2^i. \tag{3.13}$$

By replacing (3.13) in $P_1(t_2, x) = 0$, we get $-t_2^2 + (a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 + o(t_2^3))^2 - 2(a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 + o(t_2^3)) = 0$. Now, by considering the terms in t_2, t_2^2 and t_2^3, we get $a_1 = 0$, $a_2 = 0$ and $a_3 = \frac{-1}{2}$, respectively. So, a root of $P_1(t_2, x)$ with the value zero at the origin, is of the form $x(t_2) = \frac{-1}{2} t_2^2 + o(t_2^3)$. Similarly, it can be shown that $x(t_1) = t_1^3 + o(t_1^3)$ is a root of $P_2(t_1, x)$ which has the value zero at the origin. These two roots intersect along $t_1^3 + \frac{1}{2} t_2^2 = o(|t|^3)$ which is singular at $(0, 0)$. In Figure 3.16, we see the eigenvalues of $A(t_1, t_2)$.



(a) (b)

Figure 3.16: The eigenvalues of $A(t_1, t_2)$

Notice that for any t_0 and $u \perp t_0$, the antipodal monodromy from t_0 in the direction of u is trivial. So $(0, 0)$ is not a super-singular point.

3.4.3.3 $\dim \mathcal{P} \cap \Sigma_2 = 2$

In this case, $\mathcal{P} \subset \Sigma_2$. Since Σ_2 is smooth, so the eigenvalues of $\mathcal{P}$ are analytic functions. Consequently, there is an orthogonal matrix $Q(t) \in O(d)$ which diagonalize $\mathcal{P}$.

Chapter 4

Reductions

In this chapter, we study possible reductions of linear families of symmetric matrices. In a first attempt, if $A(t)$ is a family of $d \times d$ symmetric matrices, one can investigate whether there exists a regular (analytic, linear ...) family $O(t)$ of orthogonal matrices such that $A'(t) := O(t)^t A(t) O(t)$ is reduced, this is $A'(t) = Diag(A_1(t), \ldots, A_n(t))$ is diagonal by blocs. Recently, the question was studied by Kurdyka and Paunescu in the case where $A(0)$ admits n different multiple eigenvalues for analytic families [18] and then by Parusinski and Rond, Lemma 2.1 of [26], for the families of normal matrices depending on a formal multi-parameter. We say that a family $A(t)$ splits into n blocks if n regular invariant linear subspaces $E_i(t)$ exist. In full generality, to get true splitting, one also needs to find "regular" basis of each $E_i(t)$, but it won't be problematic in the linear case.

Obtaining such reduction aggregates two difficulties. Indeed, if a family $A(t)$ splits, then it has a reducible characteristic polynomial $\chi_{A(t)}$, since it is the product of the characteristic polynomials $\chi_{A_i(t)},\ i = 1, \ldots, n$, where $A_i(t)$ is the restriction of $A(t)$ to $E_i(t)$. Now, when a polynomial family $P(t) \in (\mathbb{R}[t])[\lambda]$ is the product of n factors $P_1(t), \ldots, P_n(t)$, the monodromy σ associated to the roots of $P(t)$ also factorizes. A cycle in the decomposition of σ acts on the roots of one $P_i(t)$, so σ can be seen as the product of n permutations with disjoint support. We identify this situation by the classes of the quotient of the roots of $P(t)$ by the action of σ (that only aggregate roots of the same factor $P_i(t)$).

In this chapter, we try to find sufficient conditions to reverse the implications (a) and (b):

$$A(t) \text{ splits } \overset{(a)}{\Longrightarrow} \chi_{A(t)} \text{ factorizes } \overset{(b)}{\Longrightarrow} \sigma \text{ factorizes.}$$

We call **factorization problem** the reverse of (b), this is, finding a factor $P_1(t)$ in $\chi_A(t)$ with $P_1(t_0) = \prod_{i=1}^{\ell}(\lambda - \lambda_i(t_0))$, when the subset $\{\lambda_i(t_0);\ i = 1, \dots, \ell\}$ of the roots of $\chi_{A(t_0)}$ is known to be invariant by σ. We call **splitting problem** the reverse of (a), this means, finding an invariant subspace $E(t)$ of dimension ℓ such that the characteristic polynomial of the restriction of $A(t)$ to $E(t)$ is the known divisor $P(t)$ of degree ℓ of $\chi_{A(t)}$.

In the first section, we notice some rigidities that proceed from the linearity of the family. To be differentiable (and in particular analytic), an eigenvalue $\lambda(t)$ of $A(t)$ needs to be linear, and in the same vein, if a linear family $A(t)$ splits, the associated invariant spaces needs to be constant. It happens that (b) cannot be reversed in full generality. Indeed, there is an example of a 2-linear family in $Sym_3(\mathbb{R})$ that admits an eigenvalue that is invariant by the antipodal monodromy but is not linear, Proposition 4.1.4. Then we study eigenvectors associated to linear eigenvalues, and we show that they have constraints, but not enough to make them constant, except in the case where the eigenvalue is extremal. It happens indeed (a) is not an equivalence either, that we illustrate by Example 4.1.10.

This invites us to consider extremal eigenvalues, section two. We achieve to solve the factorization problem when σ has an extremal orbit of length 1 or 2. The result might be generalized, but our method for an extremal orbit of length n requires a construction of particular hyperbolic curves of degree n that we have not reached in arbitrary degree. Then, still for extremal factors, we manage to solve the splitting problem in full generality.

The last section combines consequences of the formers. In particular, we state as a theorem (Theorem 4.3.1) the equivalence of many propositions, including the following:

A linear family admits an analytic diagonalization if and only if the associated antipodal monodromy is trivial if and only if any two matrices in the family commute.

4.1 Generalities

In this section, we study the splitting and factorization problems for linear families. We first state some generalities that confirm the rigidity of linear cases, and we give examples which show that neither the lack of monodromy is sufficient for an eigenvalue to be factored out of the characteristic polynomial, nor a factor in the characteristic

polynomial corresponds necessarily to a splitting block of the family. However, in the second subsection, we will obtain positive results under the assumption that the considered eigenvalues and factors are extremal.

4.1.1 Rigidity of eigenelements

The behavior of "regular" eigenelements of linear families is very rigid. Indeed, eigenvalues need to be linear, and eigenvectors to be constant, as shown by the following two propositions.

Proposition 4.1.1. *Let $A(t) \in Sym_d(\mathbb{R}), t \in \mathbb{R}^k$ be a linear family. Suppose $\lambda : \mathbb{R}^k \to \mathbb{R}$ is an eigenvalue of $A(t)$, continuous on $\mathbb{R}^k$, and differentiable at 0. Then λ is linear.*

Proof. For given $t \neq 0$ and $s \in \mathbb{R}$, $A(st) = sA(t)$, so $Spec(A(st)) = sSpec(A(t))$. Then, since $Spec(A(t))$ is discrete (finite) and λ is continuous, λ is positively homogeneous of degree 1: $\forall s > 0, \lambda(st) = s\lambda(t)$. For $s > 0$, we then get

$$\lambda(t) - d\lambda(0)(t) = \frac{\lambda(st) - d\lambda(0)(st)}{s} = \frac{o(st)}{s} \xrightarrow[s\to 0]{} 0.$$

So $\lambda(t) = d\lambda(0)(t)$ for all t (the equality is clear for $t = 0$), and λ is linear. □

Proposition 4.1.2. *Let $A(t) \in Sym_d(\mathbb{R}), t \in \mathbb{R}^k$ be a linear family, $A \not\subset \Sigma$. The following are equivalent:*

1. *There are k independent parameters $(t_1, \dots, t_k) \in \mathbb{R}^{k^2}$ such that $A(t_1), \dots, A(t_k)$ have a common eigenvector.*
2. *There exists a continuous $e : \mathbb{R}^k \to \mathbb{S}^{d-1}$ that is an eigenvector of $A(t)$.*
3. *There exists an analytic $e : \mathbb{R}^k \to \mathbb{S}^{d-1}$ that is an eigenvector of $A(t)$.*
4. *There exists $e \in \mathbb{S}^{d-1}$ that is an eigenvector of $A(t)$.*

If these propositions are satisfied, the associated eigenvalue is linear.

Proof. Let us prove (2) ⇒ (4). Since $A \not\subset \Sigma$, the set Ω of regular parameters (such that $A(t) \notin \Sigma$) is a dense open set in $\mathbb{R}^k$. Let $t \in \Omega$. Then $\mathbb{R}^* t \subset \Omega$, since A is linear. The unit eigenvectors of $A(st)$, for $s > 0$ are made of $2d$ distinct points that do not depend on s. The map e being continuous, $0 < s \mapsto e(st)$ is constant, and in

particular, $e(t) = \lim_{s\to 0} e(st) = e(0)$. Finally, $\forall t \in \Omega, e(t) = e(0)$, so e is constant on Ω. Then, by continuity of e and density of Ω, e is constant on $\mathbb{R}^k$.

Since (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are clear, and since a common eigenvector of different matrices is an eigenvector of any linear combination of them, we get (1) $\Rightarrow$ (4) which finishes the proof of the proposition.

Finally, if e is a constant eigenvector, the associated eigenvalue is $\lambda(t) =< A(t)e, e >$, which is linear. □

The following proposition generalizes the principle, working with invariant spaces instead of eigenvectors. The hypothesis $A(t) \not\subset \Sigma$ below is necessary. Any continuous $E : \mathbb{R}^2 \to \mathbb{G}_3^1$ whose image is included in the hyperplane $(0,0,1)^\perp$ is a continuous invariant space of the family $A(t_1, t_2) = t_1 diag(1,1,0) + t_2 \mathbb{I}_3$.

Proposition 4.1.3. *Let $A(t) \in Sym_d(\mathbb{R}), t \in \mathbb{R}^k$ be a linear family such that $A(t) \not\subset \Sigma$. Suppose $E(t)$ is a continuous invariant space of $A(t)$, then $E(t)$ is constant.*

Proof. Let Ω be the set of regular parameters t such that $A(t) \notin \Sigma$. Since $A(t) \not\subset \Sigma$, Ω is a dense open set in $\mathbb{R}^k$. Take $t_0 \in \Omega$. Since $A(t)$ is linear, for any $s \in \mathbb{R}^*$ the eigenspaces of $A(st_0)$ are d independent lines associated to simple eigenvalues of $A(st_0)$. Since the invariant spaces of $A(t)$ are all direct sums of subspaces of eigenspaces of $A(t)$, the set $E(st_0)$, $s \in \mathbb{R}^*$, is discrete. Thus, by continuity of E, $s \mapsto E(st_0)$ is constant and $E(t_0) = \lim_{s\to 0} E(st_0) = E(0)$. Finally, for all $t \in \Omega$, we have $E(t) = E(0)$. By continuity of $E(t)$ and density of Ω in $\mathbb{R}^k$, $E(t)$ is constant on $\mathbb{R}^k$. □

4.1.2 Factorization problem

In view of Proposition 4.1.1, the following example shows that the antipodal monodromy fails to predict the factorization of the characteristic polynomial.

Proposition 4.1.4. *There exists a 2-linear family $A(t) \in Sym_3(\mathbb{R})$ and $\lambda \in \mathbb{R}$ such that :*

- *λ is an eigenvalue of $A(1,0)$ and is invariant by the antipodal monodromy from $(1,0)$ in the direction of $(0,1)$: $\tau'_{(1,0),(0,1)}(\lambda) = \lambda$;*
- *λ has no analytic prolongation on $\mathbb{R}^2$. Indeed, λ has a unique analytic prolongation on $\mathbb{R}^2 \setminus \{0\}$ that is not a linear function.*

Proof. The proof is given by the following example. Let

$$A(t_1, t_2) = t_1 A_1 + t_2 A_2 = \begin{pmatrix} 0 & 0 & t_1 \\ 0 & t_2 & t_1 \\ t_1 & t_1 & t_1 - t_2 \end{pmatrix},$$

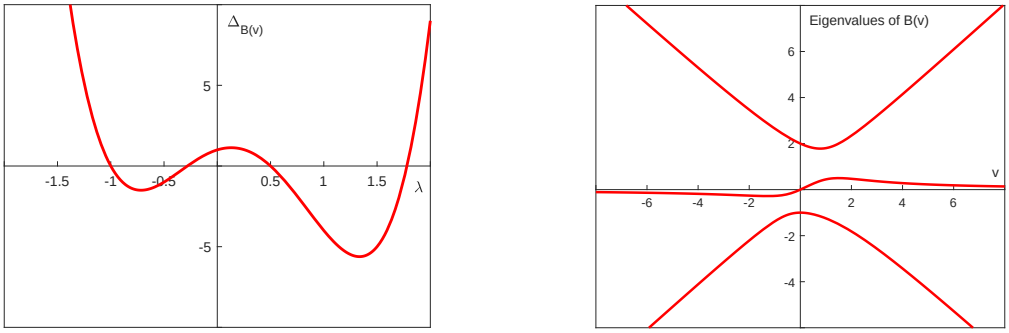
where $Spec(A_1) = \{-1, 0, 2\}$ and $Spec(A_2) = \{0, 0, 1\}$. Notice that 0 is an eigenvalue of $A(1, 0)$. We show that $\tau'_{((1,0),(0,1))}(0) = 0$. Let $t_1 = u$ and $t_2 = uv$ and

$$B(v) = A_1 + vA_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & v & 1 \\ 1 & 1 & 1 - v \end{pmatrix}.$$

The characteristic polynomial and the discriminant of $B(v)$ are

$$\chi_{B(v)}(\lambda) = -\lambda^3 + \lambda^2 + (2 + v^2 - v)\lambda - v = \lambda v^2 - (\lambda + 1)v - \lambda^3 + \lambda^2 + 2\lambda,$$
$$\Delta_{B(v)} = (\lambda + 1)^2 - 4\lambda(-\lambda^3 + \lambda^2 + 2\lambda) = (\lambda + 1)(2\lambda - 1)(\lambda(2\lambda - 3) - 1).$$

Therefore, $Z(\Delta_{B(v)}) = \{\lambda \in \mathbb{R} | \Delta_{B(v)}(\lambda) = 0\} = \{-1, \frac{1}{2}, \frac{3}{4} - \frac{\sqrt{17}}{4}, \frac{3}{4} + \frac{\sqrt{17}}{4}\}$. Since $\Delta_{B(v)} \sim_{\lambda \to \pm\infty} 4\lambda^4$, then $\Delta_{B(v)}$ is negative in the two intervals $I_1 = [-1, \frac{3}{4} - \frac{\sqrt{17}}{4}]$ and $I_2 = [\frac{1}{2}, \frac{3}{4} + \frac{\sqrt{17}}{4}]$, Figure 4.1a.



(a) The graph of $\lambda \mapsto \Delta_{B(v)}$ (b) The eigenvalues of $B(v)$

Figure 4.1: Non-analytic eigenvalue with trivial antipodal monodromy

From this, if we choose a $\lambda \in I_1 \cup I_2$, then there is no parameter v such that λ is an eigenvalue of the matrix $B(v)$. Thus, the eigenvalues of $B(v)$ are distinct, so for all v, $\lambda_1(v) < \lambda_2(v) < \lambda_3(v)$. Note that $\lambda_2(0) = 0$ and the slopes of the eigenvalues

of $B(v)$ at $\pm\infty$ are -1 , 0 and 1, Lemma 2.6.1. So, $\lambda_2(v)$ has a finite limit 0 at $\pm\infty$, Lemma 2.6.2 and 2.6.3. In Figure 4.1b, we see the graph of eigenvalues of $B(v)$. Now, let h be the half turn path from $(1,0)$ in the direction of $(0,1)$ and pr_h be the analytic prolongation along h. Then, for each $i = 1,2,3$, $pr_h(\lambda_i) = \lambda_i$. Since $\tau_{((1,0),(0,1))}(\lambda_i) = -pr_h(\lambda_i)$, the antipodal monodromy from $(1,0)$ in the direction of $(0,1)$ is the transposition of the two extremal eigenvalues -1 and 2 at $(1,0)$. So, $\tau_{((1,0),(0,1))}(0) = 0$.

Now, we show that the eigevalue 0 does not have any anlytic continuation on $\mathbb{R}^2$. We write the Taylor expansion of $\lambda_2(v)$ at $v = 0$:

$$\lambda_2(v) = \sum_{i>0} a_i v^i. \tag{4.1}$$

By replacing (4.1) in $\chi_{B(v)}$, we get $-(a_1v + a_2v^2 + o(v^2))^3 + (a_1v + a_2v^2 + o(v^2))^2 + (v^2 - v + 2)(a_1v + a_2v^2 + o(v^2)) - v = 0$. Now, by considering the terms in v, we get $(2a_1 - 1)v = 0$. From this, the slope of the eigenvalue of $\lambda_2(v)$ at $v = 0$ is $\frac{1}{2}$. We have

$$\lambda_2(v) = \frac{1}{2}v + o(v).$$

Note that $\lambda_2(v)$ is the unique analytic prolongation of the eigenvalue 0 of $\mathbb{R}^2\backslash(0,0)$. Since $\lambda_2(v)$ is not linear, the associated eigenvalue $\mu_2(t_1,t_2)$ of $A(t_1,t_2)$, $\mu_2(t_1,t_2) = t_1\lambda_2(v)$ is not linear in t_1 and t_2. In particular, by Proposition 4.1.1 $\mu_2(t_1,t_2)$ is not analytic on $\mathbb{R}^2$. So, 0 does not have an analytic continuation $\mathbb{R}^2$. In Figure 4.2, we see the eigenvalues of $A(t_1,t_2)$.

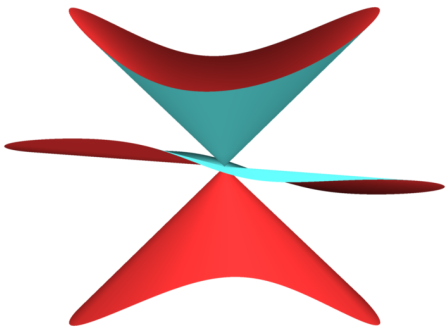


Figure 4.2: The eigenvalues of $A(t)$

□

4.1.3 Splitting problem

From the preceding, the factorization problem has no positive answer in full generality, even for orbits of length 1 of σ. We now consider the splitting problem. For this we focus on linear eigenvalues (factors of degree one in χ_A), and try to investigate the regularities of the associated eigenvectors. From Proposition 4.1.2, getting a splitting for the family equates to have a constant eigenvector. We will see that again, this cannot be reached in general. Indeed, examples below show that even the full given of the eigenvalue set – all (t, λ) with $\lambda \in Spec(A(t))$ –, is not sufficient to determine whether $A(t)$ splits. However, the journey reveals that extremal eigenvalues play a particular role, and we study them in a second phase.

The following is the **key lemma** that outlines the regularities of eigenvectors associated to linear eigenvalues.

Lemma 4.1.5. *Let $A(t) \in Sym_d(\mathbb{R})$, be a 2-linear family, and suppose $A(t)$ has a linear eigenvalue $\lambda(t)$ that is not always multiple : $\exists t \in \mathbb{R}^2, \chi'_{A(t)}(\lambda(t)) \neq 0$. Then, there exists an eigenvector $e(t) \in \mathbb{S}^{d-1}$ associated to $\lambda(t)$, analytic on $\mathbb{R}^2 \setminus \{0\}$, unique up to multiplication by -1, and we have:*

$$\forall (t, t_1, t_2) \in (\mathbb{R}^2 \setminus \{0\})^3,\ e(t) \perp \big(A(t_1) - \lambda(t_1)\mathbb{I}_d\big)e(t_2) + \big(A(t_2) - \lambda(t_2)\big)e(t_1). \quad (4.2)$$

Proof. Without loss of generalities, suppose $(1, 0)$ is a parameter such that $\lambda(1, 0)$ is a simple eigenvalue of $A(1, 0)$, and choose $e_0 \in \mathbb{S}^{d-1}$ to be one of the two unit eigenvector of $A(1, 0)$ associated with $\lambda(1, 0)$. For given $\theta \in \mathbb{R}$ we denote by Θ the parameter $\Theta = (\cos(\theta), \sin(\theta))$, and $\Theta + \Pi = (\cos(\theta + \pi), \sin(\theta + \pi))$.

From Rellich Theorem, there exists a unique analytic vector $e_1 : \mathbb{R} \to \mathbb{S}^{d-1}$ such that $A(\Theta)e_1(\theta) = \lambda(\Theta)e_1(\theta)$ and $e_1(0) = e_0$. We claim that e_1 is 2π periodic. Indeed, notice that the equation $A(\Theta)v = \lambda(\Theta)v$ for $v \in \mathbb{S}^{d-1}$ has only two continuous solutions (in terms of θ), that can be distinguished by their value e_0 or $-e_0$ at $\theta = 0$, then happens to be e_1 and $-e_1$. Now, since $A(\Theta + \Pi) = -A(\Theta)$ and $\lambda(\Theta + \Pi) = -\lambda(\Theta)$, the problem $A(\Theta + \Pi))v = \lambda(\Theta + \Pi)v, v \in \mathbb{S}^{d-1}$ has the same continuous solutions. From this we deduce that either $e_1(\theta + \pi) = e_1(\theta)$, and then $e_1(\theta + 2\pi) = e_1(\theta)$, or else, $e_1(\theta + \pi) = -e_1(\theta)$ and then $e_1(\theta + 2\pi) = -e_1(\theta + \pi) = -(-e_1(\theta)) = e_1(\theta)$. In both cases, e_1 is 2π periodic. The function e_1 being analytic and 2π periodic, the map $e : \mathbb{R}^2 \setminus \{0\} \to S^{d-1}$ given by $e(t) = e_1(t/||t||)$ is analytic on $\mathbb{R}^2 \setminus \{0\}$, and

since $A(t)$ and $\lambda(t)$ are linear, $e(t)$ is an eigenvector of $A(t)$ associated with $\lambda(t)$. The unicity up to central symmetry is straightforward.

We come to the relation (4.2). Let t_1, t_2 be two independent parameters (otherwise the relation is trivial), and denote $A_1 = A(t_1)$, $A_2 = A(t_2)$, $\lambda_1 = \lambda(t_1)$, $\lambda_2 = \lambda(t_2)$, $e_1 = e(t_1)$, $e_2 = e(t_2)$, so $(A_1 - \lambda_1 \mathbb{I}_d)e_1 = (A_2 - \lambda_2 \mathbb{I}_d)e_2 = 0$. For $t \in \mathbb{R}^2$, we set $t = \alpha t_1 + \beta t_2$ (α and β depends on t). Since $e(t)$ is an eigenvector of $A(t)$ for $\lambda(t)$, we have $(A(t) - \lambda(t)\mathbb{I}_d)e(t) = 0$. Then, in particular

$$\forall t \in \mathbb{R}^2 \setminus \{0\},\ < (A(t) - \lambda(t)\mathbb{I}_d)e(t)\ ;\ \alpha e_1 + \beta e_2 >= 0.$$

This gives, by linearity of A and λ, and by symmetry of $A - \lambda \mathbb{I}_d$:

$$\begin{aligned}
0 &= < (\alpha(A_1 - \lambda_1 \mathbb{I}_d) + \beta(A_2 - \lambda_2 \mathbb{I}_d))e(t)\ ;\ \alpha e_1 + \beta e_2 > \\
&= < e(t)\ ;\ (\alpha(A_1 - \lambda_1 \mathbb{I}_d) + \beta(A_2 - \lambda_2 \mathbb{I}_d))(\alpha e_1 + \beta e_2) > \\
&= \alpha^2 < e(t)\ ; (A_1 - \lambda_1 \mathbb{I}_d)e_1 > \dots \\
&\quad \dots + \beta^2 < e(t)\ ;\ (A_2 - \lambda_2 \mathbb{I}_d))e_2 > \dots \\
&\quad \dots + \alpha\beta < e(t)\ ;\ (A_1 - \lambda_1 \mathbb{I}_d)e_2 + (A_2 - \lambda_2 \mathbb{I}_d))e_1 > \\
&= \alpha\beta < e(t)\ ;\ (A_1 - \lambda_1 \mathbb{I}_d)e_2 + (A_2 - \lambda_2 \mathbb{I}_d))e_1 > .
\end{aligned}$$

Then, for $\alpha\beta \neq 0$, we get $< e(t)\ ;\ (A_1 - \lambda_1 \mathbb{I}_d)e_2 + (A_2 - \lambda_2 \mathbb{I}_d))e_1 >= 0$. By continuity and density, the equation is finally satisfied for all t, as announced.[1] □

The vector $\big(A(t_1) - \lambda(t_1)\mathbb{I}_d\big)e(t_2) + \big(A(t_2) - \lambda(t_2)\big)e(t_1)$ which appears in the key lemma is associated with the linearity of eigenvectors, as suggested by the proof. Indeed, we have the following:

Proposition 4.1.6. *Let $A(t_1, t_2) = t_1 A_1 + t_2 A_2$ be a 2 linear family, and for $i = 1, 2$, let $\lambda_i \in \mathbb{R}$ be an eigenvalue of A_i, with associated eigenvector $e_i \in \mathbb{R}^d \setminus \{0\}$. The following are equivalent:*

- $(A_1 - \lambda_1 \mathbb{I}_d)e_2 + (A_2 - \lambda_2 \mathbb{I}_d)e_1 = 0$,
- *$t_1 e_1 + t_2 e_2$ is an eigenvector of $A(t_1, t_2)$.*

If this case, the eigenvector $t_1 e_1 + t_2 e_2$ is associated with the linear eigenvalue $\lambda(t_1, t_2) = \lambda_1 t_1 + \lambda_2 t_2$.

[1] notice that the first part of the proof (the existence of an analytic eigenvector on $\mathbb{R}^2 \setminus \{0\}$) does not involve the linearity of λ but only its idempotency by the antipodal monodromy. In particular, if λ has no monodromy, λ has an associated eigenvector, analytic on $\mathbb{R}^2 \setminus \{0\}$.

Proof. First suppose $e(t_1, t_2) = t_1 e_1 + t_2 e_2$ is an eigenvector of $A(t_1, t_2)$. Denote by $\lambda(t_1, t_2)$ the associated eigenvalue. So $\lambda_1 = \lambda(1, 0), \lambda_2 = \lambda(0, 1)$. Notice that, if e_1 and e_2 are linearly dependent, they are both eigenvectors of the two matrices, and the result follows immediately. Thus, we suppose e_1 and e_2 independent.

We have $\big(A(t_1, t_2) - \lambda(t_1, t_2)\mathbb{I}_d\big)e(t_1, t_2) = 0$, which gives

$$t_1(t_1\lambda_1 - \lambda(t_1, t_2))e_1 + t_2(t_2\lambda_2 - \lambda(t_1, t_2))e_2 + t_1t_2(A_1e_2 + A_2e_1) = 0. \tag{4.3}$$

So, for $t_1t_2 \neq 0$,

$$\frac{(t_1\lambda_1 - \lambda(t_1, t_2))}{t_2}e_1 + \frac{t_2\lambda_2 - \lambda(t_1, t_2)}{t_1}e_2 = -A_1e_2 - A_2e_1.$$

Since the right hand side vector does not depends on (t_1, t_2) and e_1, e_2 are independent, this implies $\frac{t_1\lambda_1 - \lambda(t_1,t_2)}{t_2} = c_1$ and $\frac{t_2\lambda_2 - \lambda(t_1,t_2)}{t_1} = c_2$ are constant. In particular, $\lambda(t_1, t_2) = t_1\lambda_1 - c_1t_2 = -t_1c_2 + t_2\lambda_2$, so $\lambda(t_1, t_2) = \lambda_1t_1 + \lambda_2t_2$. Notice the equality also holds if $t_1t_2 = 0$, so λ is linear as announced. Finally, replacing $\lambda(t_1, t_2)$ by $\lambda_1t_1 + \lambda_2t_2$ in (4.3) for $t_1 = 1, t_2 = 1$ precisely gives $(A_1 - \lambda_1\mathbb{I}_d)e_2 + (A_2 - \lambda_2\mathbb{I}_d)e_1 = 0$.

We now suppose $(A_1 - \lambda_1\mathbb{I}_d)e_2 + (A_2 - \lambda_2\mathbb{I}_d)e_1 = 0$. Then

$$\begin{aligned}
&\big(A(t_1, t_2) - (t_1\lambda_1 + t_2\lambda_2)\mathbb{I}_d\big)(t_1e_1 + t_2e_2)\\
&= \big(t_1(A_1 - \lambda_1\mathbb{I}_d) + t_2(A_2 - \lambda_2\mathbb{I}_d)\big)(t_1e_1 + t_2e_2)\\
&= t_1t_2\big((A_1 - \lambda_1\mathbb{I}_d)e_2 + (A_2 - \lambda_2\mathbb{I}_d)e_1\big)\\
&= 0.
\end{aligned}$$

So $t_1e_1 + t_2e_2$ is an eigenvector of $A(t_1, t_2)$ for $t_1\lambda_1 + t_2\lambda_2$. □

This fact can be generalized to bigger families, as follows.

Corollary 4.1.7. *Let $A(t) = t_1A_1 + \cdots + t_kA_k$ be a k-linear family. Let λ_i be an eigenvalue of A_i and $e_i \in \mathbb{R}^d\backslash\{0\}$, $i = 1, \ldots, d$, be an associated eigenvector. Then, the following are equivalent:*

- *For all $i \neq j$, $(A_i - \lambda_i\mathbb{I}_d)e_j + (A_j - \lambda_j\mathbb{I}_d)e_i = 0$*

- *$t_1e_1 + \cdots + t_ke_k$ is an eigenvector of $A(t_1, \ldots, t_k)$.*

In this case, the associated eigenvalue is linear, $\lambda(t_1, \ldots, t_k) = t_1\lambda_1 + \cdots + t_k\lambda_k$.

Proof. First, we suppose that $e(t) = t_1e_1 + \cdots + t_ke_k$ is an eigenvector of $A(t)$. Then, on each plane $P_{ij} = t_iA_i + t_jA_j$, $i \neq j$, we have $(A_i - \lambda_i\mathbb{I}_d)e_j + (A_j - \lambda_j\mathbb{I}_d)e_i = 0$, Proposition 4.1.6.

Now, suppose that for all $i \neq j$, $(A_i - \lambda_i\mathbb{I}_d)e_j + (A_j - \lambda_j\mathbb{I}_d)e_i = 0$. We prove that $t_1e_1 + \cdots + t_ke_k$ is an eigenvector of A(t) associated to $t_1\lambda_1 + \cdots + t_k\lambda_k$. The proof is by induction on k, the number of parameters. If $k = 2$, then by Proposition 4.1.6 there is nothing to prove. We fix a $k > 2$ and we suppose that the corollary holds for $k - 1$. We have

$$
\begin{aligned}
&[t_1(A_1 - \lambda_1\mathbb{I}_d) + \cdots + t_{k-1}(A_{k-1} - \lambda_{k-1}\mathbb{I}_d) + t_k(A_k - \lambda_k\mathbb{I}_d)](t_1e_1 + \cdots + t_ke_k)\\
&= (t_1(A_1 - \lambda_1\mathbb{I}_d) + \cdots + t_{k-1}(A_{k-1} - \lambda_{k-1}\mathbb{I}_d))(t_ke_k) + t_k(A_k - \lambda_k\mathbb{I}_d)(t_1e_1 + \cdots + t_{k-1}e_{k-1})\\
&= \Sigma_{i=1}^{k-1}t_it_k(A_i - \lambda_i\mathbb{I}_d)e_k + \Sigma_{i=1}^{k-1}t_it_k(A_k - \lambda_k\mathbb{I}_d)e_i\\
&= \Sigma_{i=1}^{k-1}t_it_k((A_i - \lambda_i\mathbb{I}_d)e_k + (A_k - \lambda_k\mathbb{I}_d)e_i) = 0
\end{aligned}
$$

So, $t_1e_1 + \cdots + t_ke_k$ is an eigenvector of A(t) associated to the linear eigenvalue $t_1\lambda_1 + \cdots + t_k\lambda_k$ which finishes the proof. □

As a corollary, the key Lemma 4.1.5 together with Proposition 4.1.6, show that eigenvectors associated to linear eigenvalues are at least constrained in a hyperplane. This corollary is weak compared to the lemma, but we state it because its conclusion seems more eloquent, and is sufficient for some applications.

Corollary 4.1.8. *Let $A(t) \in Sym_d(\mathbb{R})$, be a 2-linear family, and suppose $A(t)$ has a linear eigenvalue $\lambda(t)$ that is not always multiple. Denote by $e(t)$ the associated eigenvector. Then there exists a hyperplane H such that $\forall t, e(t) \in H$.*

Proof. We fix two independent parameters t_1, t_2 and we define

$$v = \big(A(t_1) - \lambda(t_1)\mathbb{I}_d\big)e(t_2) + \big(A(t_2) - \lambda(t_2)\big)e(t_1).$$

Then, either $v \neq 0$, or $v = 0$. In the first case, the conclusion stands from Lemma 4.1.5, with $H = v^\perp$. In the latter case, $e(t)$ belongs to a two dimensional plane P, Proposition 4.1.6. Any hyperplane H that contains the plane P fits. □

Another corollary of the key lemma is obtained by specifying t_2 to be t in its conclusion. It gives the following corollary which is more powerful than the preceding.

The result is remarkable because it gives a constraint on the eigenvector that is associated with a single matrix of the family.

Corollary 4.1.9. *Let $A(t) \in Sym_d(\mathbb{R})$, be a 2-linear family, that admits a linear eigenvalue $\lambda(t)$, associated with the analytic eigenvector $e(t)$ for $t \neq 0$. Suppose $\lambda_0 := \lambda(t_0)$ is a single eigenvalue of $A_0 = A(t_0)$. Then, for all t, $e(t)$ belongs to the isotropic cone of $A_0 - \lambda_0 \mathbb{I}_d$, this means,*

$$\forall t \in \mathbb{R} \setminus \{0\}, e(t)^t (A_0 - \lambda_0 \mathbb{I}_d) e(t) = 0.$$

Proof. By Lemma 4.1.5, $\forall t \in \mathbb{R}\setminus\{0\}$, $e(t) \perp \big(A(t_0)-\lambda(t_0)\mathbb{I}_d\big)e(t)+\big(A(t)-\lambda(t)\big)e(t_0)$. Thus,

$$< e(t) \mid (A_0 - \lambda_0 \mathbb{I}_d) e(t) > + < e(t) \mid (A(t) - \lambda(t)\mathbb{I}_d) e(t_0) >= 0 \tag{4.4}$$

But, $< e(t) \mid (A(t) - \lambda(t)\mathbb{I}_d) e(t_0) >= 0$, since $\mathrm{Im}(A(t) - \lambda(t)\mathbb{I}_d) \subset e(t)^\perp$. Hence, for all $t \in \mathbb{R} \setminus \{0\}$, $< e(t) \mid (A_0 - \lambda_0 \mathbb{I}_d) e(t) >= 0$ which finishes the proof. □

If, in what precedes, λ_0 is an extremal eigenvalue of A_0, then the quadratic form associated to $A_0 - \lambda_0 \mathbb{I}_d$ is non-negative (or non-positive), and its isotropic cone coincides with its kernel, this is, the eigenspace of A_0 associated with λ_0. Since λ_0 is a single eigenvalue, $e(t)$ then range in a 1-dimensional subspace. So, $A(t)$ admits a constant eigenvector associated with $\lambda(t)$. Recall that having a constant eigenvector (or eigenspace) is the condition to get splitting of the family. Hence, if $\lambda(t)$ is a linear eigenvalue, that is extremal for some parameter, the former proposition allows to solve the splitting problem. This is a declination of a general fact studied in detail later.

Before focusing on extremal eigenvalues, let us illustrate that (a) is not an equivalence in general.

Example 4.1.10. In this example, we consider $A(t) \in Sym_3(\mathbb{R})$ a 2-linear family given by

$$A(t_1, t_2) = t_1 A_1 + t_2 A_2 = \begin{pmatrix} -t_1 & -t_1 & -t_1 \\ -t_1 & -t_1 + t_2 & t_1 \\ -t_1 & t_1 & t_1 - t_2 \end{pmatrix}.$$

We show that -1 is an eigenvalue of $A(1,0)$ so that $\tau'_{(1,0),(0,1)}(-1) = -1$ and it has an analytic continuation on $\mathbb{R}^2$ which is a linear function. But, we show that there is no analytic block diagonalization for $A(t)$ on $\mathbb{R}^2$.

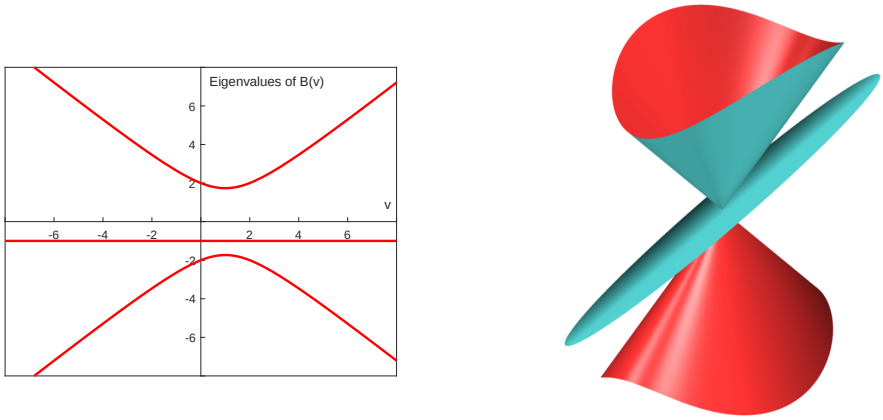
Let $t_1 = u$ and $t_2 = uv$, then we have

$$A(u,v) = uA_1 + uvA_2 = uB(v) = u\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1+v & 1 \\ -1 & 1 & 1-v \end{pmatrix}.$$

We have $Spec(A_1) = \{-2,-1,2\}$ and $Spec(A_1) = \{-1,0,1\}$. The characteristic polynomial of the matrix $B(v)$ is

$$\chi_{B(v)}(\lambda) = (1+\lambda)v^2 - (2\lambda+2)v - \lambda^3 - \lambda^2 + 4\lambda + 4 = (1+\lambda)(v^2 - 2v - \lambda^2 + 4).$$

Thus, $\lambda_1(v) = +\sqrt{v^2-2v+4}$, $\lambda_2(v) = -1$ and $\lambda_3(v) = -\sqrt{v^2-2v+4}$ are the eigenvalues of $B(v)$. In Figure 4.3a, we see the eigenvalues of $B(v)$ and $A(t)$.



(a) The eigenvalues of $B(v)$ (b) The eigenvalues of $A(t)$

Figure 4.3: Analytic eigenvalue with trivial antipodal monodromy

Note that the associated eigenvector $\mu_2(t_1,t_2) = t_1\lambda_2(v) = t_1$ of $A(t_1,t_2)$ is linear.

Now we study the behavior of an eigenvector $X(v)$ associated with $\lambda_2(v) = -1$. Let $X(v) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$, so we have $(B(v) - \lambda_2(v)\mathbb{I}_3)X(v) = 0$. From this we get

$$\begin{aligned} -x_2 - x_3 &= 0 \\ -x_1 + vx_2 + x_3 &= 0 \\ -x_1 + x_2 + (2-v)x_3 &= 0, \end{aligned}$$

Thus, $X(v) = \begin{pmatrix}(1-v)x_3 & -x_3 & x_3\end{pmatrix} = x_3\begin{pmatrix}1-v & -1 & 1\end{pmatrix}$. Let

$$\tilde{X}(v) := \frac{X(v)}{||X(v)||} = \pm(\frac{1-v}{\sqrt{v^2-2v+3}}, \frac{-1}{\sqrt{v^2-2v+3}}, \frac{+1}{\sqrt{v^2-2v+3}}). \tag{4.5}$$

Note that

$$\tilde{X}(v) = \begin{cases} \pm(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) & v = 0 \\ \pm(1,0,0) & v = -\infty \\ \pm(-1,0,0) & v = +\infty. \end{cases}$$

Since we have $v = \frac{t_2}{t_1}$, we get

$$\begin{aligned} V(t_1,t_2) &= \pm(\frac{\frac{t_1-t_2}{t_1}}{\sqrt{(\frac{t_2}{t_1})^2 - 2(\frac{t_2}{t_1})+3}}, \frac{-1}{\sqrt{(\frac{t_2}{t_1})^2 - 2(\frac{t_2}{t_1})+3}}, \frac{1}{\sqrt{(\frac{t_2}{t_1})^2 - 2(\frac{t_2}{t_1})+3}}) \\ &= \pm\begin{pmatrix}\frac{t_1-t_2}{\sqrt{(t_2-t_1)^2+2t_1^2}} & \frac{-t_1}{\sqrt{(t_2-t_1)^2+2t_1^2}} & \frac{t_1}{\sqrt{(t_2-t_1)^2+2t_1^2}}\end{pmatrix} \end{aligned}$$

which are eigenvectors of $A(t_1,t_2)$ associated to $\mu_2(t_1,t_2) = t_1\lambda_2(v)$. Note that these eigenvectors are not analytic on $\mathbb{R}^2$ and turn in the plane $v_2 + v_3 = 0$. So, there is no analytic block diagonalization of $A(t_1,t_2)$ in $\mathbb{R}^2$.

Proposition 4.1.11. *There exists a 2-linear family $t_1A_1 + t_2A_2$ of $Sym_3(\mathbb{R})$ that admits a linear eigenvalue, and $(A_1-\lambda_1)e_2 + (A_2-\lambda_2)e_1 = 0$ with $e_1 \neq e_2$.*

Proof. Let $A_1 = diag(-1,0,1)$,

$$A_2 = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 3 \end{pmatrix}$$

First, we note that $\lambda_1 = 0$ is an eigenvalue of A_1 associated to eigenvector $e_1 = (0,1,0)$ and $\lambda_2 = 0$ is an eigenvalue of A_2 associate to eigenvector $e_2 = (1,1,1)$. We have,

$$\chi_{A(t_1,t_2)}(\lambda) = \lambda(-\lambda^2 + t_1^2 + 2t_1t_2 + 3t_2^2 + 4\lambda t_2)$$

Thus, $\lambda(t) = 0$ is the linear eigenvalue of $A(t_1,t_2)$ and we have

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \tag{4.6}$$

with $e_1 \neq e_2$. □

The following example comes as a complement of the fact that (a) cannot be reversed. It shows it is hopeless to search for additional data concerning the eigenvalues to get a criterion that determines if a linear eigenvalue corresponds to a splitting.

Proposition 4.1.12. *There are two linear 2-families $A(t)$, $B(t)$ in $Sym_3(\mathbb{R})$ such that:*

- $\chi_{A(t)} = \chi_{B(t)}$. *So $A(t)$ and $B(t)$ share the same eigenvalues.*
- *$A(t)$ and $B(t)$ have a (common) linear eigenvalue.*
- *$A(t)$ has a constant eigenvector, and in particular, splits.*
- *$B(t)$ has no constant eigenvector, and in particular, doesn't split.*

Proof.

$$A(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & \sqrt{2}y \\ 0 & \sqrt{2}y & -x \end{pmatrix} \quad B(x,y) = \begin{pmatrix} 0 & y & y \\ y & x & 0 \\ y & 0 & -x \end{pmatrix}$$

The characteristic polynomial of both is $\chi(x,y,\lambda) = -\lambda(\lambda^2 - x^2 - 2y^2)$. So, 0 is a constant eigenvalue of $A(x,y)$ and $B(x,y)$ associated with $(1,0,0)$ for $A(t)$ (constant eigenvector) and with $(x,-y,y)$ for $B(t)$ (non constant eigenvector). Notice that $(x,-y,y)$ belongs to the plane $Y+Z=0$. □

4.2 Extremal eigenvalues

In this section we investigate if the factorization and splitting problems can be solved with the additional hypothesis that the considered eigenvalues (or block of eigenvalues) are extremal. We first consider the factorization problem.

4.2.1 Factorization problem for extremal eigenvalues

We prove that the factorization problem has a positive answer for extremal eigenvalues for orbits of length 1 and 2. In other terms, when an orbit of the antipodal monodromy at some t_0 is made of extremal eigenvalues, then the characteristic polynomial admits a factor whose roots at t_0 are these eigenvalues, if the orbit has length 1 or 2. We

only consider the maximal eigenvalues, but the results holds for the minimal ones, by replacing $A(t)$ by its opposite.

If $A(t), t \in \mathbb{R}^k$, is a k-linear family and $\mathcal{P}$ a linear subspace of dimension 2 in $\mathbb{R}^k$, notice the cycle structure of the antipodal monodromy $\tau'_{t_0,u}$ of $A(t)$ from t_0 in the direction u, where (t_0, u) is an orthonormal basis of $\mathcal{P}$, does not depends on (t_0, u). Indeed, super-singular sets have codimension 2, so the origin is the only potential super-singular point of the restriction of A to $\mathcal{P}$, and $\tau'_{t_0,u}$ and $\tau'_{t_0,-u}$ are inverse one to the other, then have the same cycle structure. It allows us to write $\tau'_{\mathcal{P}} = \mathbb{I}_d$ as a shorthand for $\tau'_{t_0,u} = \mathbb{I}_d$ for all orthonormal basis (t_0, u) of $\mathcal{P}$.

Proposition 4.2.1 (Factorization of extremal 1-orbit). *Let $A(t) \in Sym_d(\mathbb{R})$ be a 2-linear family, $t_0 \in \mathbb{R}^2 \setminus S_A$, and $\overline{\lambda_1} > \overline{\lambda_2} \geq \cdots \geq \overline{\lambda_d}$ be the eigenvalues of $A(t_0)$. Suppose $\overline{\lambda_1}$ is fixed by the antipodal monodromy: $\tau'_{(t_0,u)}(\overline{\lambda_1}) = \overline{\lambda_1}$. Then, $A(t)$ admits a linear eigenvalue $\lambda_1 : \mathbb{R}^2 \to \mathbb{R}$ with $\lambda_1(t_0) = \overline{\lambda_1}$. In other terms, $(\lambda - \lambda_1(t))$ divides $\chi_{A(t)}(\lambda)$.*

Proof. Without loss of generality, we suppose $t_0 = (0, 1)$, and set $A(t_1, t_2) = t_1 A_1 + t_2 A_2$. Since $(0, 1) \notin S_A$, the eigenvalues of $A(t)$ are uniquely determined as analytic functions in a neighborhood V of $(0, 1)$. We denote them by $\lambda_i(t_1, t_2)$, $i = 1, \ldots, d$, with $\lambda_i(0, 1) = \overline{\lambda_i}$. Since $\tau'_{(t_0,u)}(\overline{\lambda_1}) = \overline{\lambda_1}$, the analytic prolongation of the eigenvalue $\lambda_1(t)$ defines a function on $\mathbb{R}^2$ (analytic on $\mathbb{R}^2 \backslash (0, 0)$ and $\lambda_1(0, 0) = 0$).

Let us consider the one-parameter family $B(v) = A_1 + vA_2$. Then, for $t_1 \neq 0$ sufficiently close to 0, if $t_1 = \frac{1}{v}$, $\lambda_i(t_1, t_1 v)$ is an analytic function. Since $A(t_1, t_2)$ is linear, $\frac{\lambda_i(t_1,t_1 v)}{t_1}$ does not depend on t_1 and $\frac{\lambda_i(t_1,t_1 v)}{t_1} \in Spec(B(v))$. Moreover, by Rellich's Theorem the eigenvalues of $B(v)$ are analytic functions for all $v \in \mathbb{R}$. Denote them by $\mu_1(v), \ldots, \mu_d(v)$. Up to change the indexes, we have $\mu_i(v) = \frac{\lambda_i(t_1,t_1 v)}{t_1}$, for $t_1 > 0$ close to zero and $t_1 = \frac{1}{v}$. As $\tau'_{(t_0,u)}(\overline{\lambda_1}) = \overline{\lambda_1}$, we have

$$\begin{aligned} \lim_{v \to \pm\infty} \frac{\mu_1(v)}{v} &= \lim_{v \to \pm\infty} \frac{\lambda_1(t_1, t_1 v)}{t_1 v} \\ &= \lim_{t_1 \to 0} \lambda_1(t_1, 1) \\ &= \lambda_1(0, 1) = \overline{\lambda_1}. \end{aligned}$$

Hence, $\mu_1(v) \sim_{\pm\infty} \overline{\lambda_1} v$ so the slope of the eigenvalue $\mu_1(v)$ at $\pm\infty$ is $\overline{\lambda_1}$, the maximal eigenvalues of A_2. Yet if $i \neq 1$, $\lim_{v \to +\infty} \frac{\mu_i(v)}{v} = k_{i,1}$ and $\lim_{v \to -\infty} \frac{\mu_i(v)}{v} = k_{i,2}$ where for $i = 2, \ldots, d$ and $j = 1, 2$, $k_{i,j} \in Spec(A(0, 1))$ so $k_{i,j} < \overline{\lambda_1}$. Therefore, there

exists $C > 0$ such that for all $v \leq -C$, $\mu_1(v) < \mu_i(v)$ and for all $v \geq C$, $\mu_1(v) > \mu_i(v)$.

Now consider the line $\mathcal{L}$ passing trough the two points $p_1 = (-C, \mu_1(-C))$ and $p_2 = (C, \mu_1(C))$. Then, for any $i \neq 1$,

$$sgn\left(\mu_i(C) - \mu_1(C)\right) = -sgn\left(\mu_i(-C) - \mu_1(-C)\right).$$

Since $\mu_1(v), \ldots, \mu_d(v)$ are continuous, by Mean Value Theorem there are $d-1$ intersection points between the graphs of $\mu_i(v)$, $i \in \{2, \ldots, d\}$ and the line $\mathcal{L}$:

$$\#\big(\{(v, \mu_i(v)); v \in \mathbb{R}, i \neq 1\} \cap \mathcal{L}\big) = d - 1.$$

Moreover, $\#\left(\{(v, \mu_1(v)); v \in \mathbb{R}\} \cap \mathcal{L}\right) \geq 2$. From this, $\#\left(\chi_B^{-1}(0) \cap \mathcal{L}\right) \geq d + 1$. But $\deg(\chi_B) = d$, so by Bézout Theorem, if $\mathcal{L}_A \not\subset \{\chi_B = 0\}$, one gets $\#\left(\chi_B^{-1}(0) \cap \mathcal{L}\right) \leq d$. Thus $\mathcal{L} \subset \{\chi_B = 0\}$. So, $\mu_1(v)$ is an affine eigenvalue of $B(v)$,

$$\mu_1(v) = \overline{\lambda_1} v + \mu_1(0).$$

Finally, $\lambda_1(t_1, t_2) = t_1 \mu_1(\frac{t_2}{t_1}) = t_1(\overline{\lambda_1}\frac{t_2}{t_1} + \mu_1(0)) = \overline{\lambda_1} t_2 + \mu_1(0) t_1$ is linear in (t_1, t_2). □

This can be generalized to k-linear family as follows.

Proposition 4.2.2. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family and suppose that there exists $t_0 \in \mathbb{R}^k$, $r > 0$ and $B(t_0, r) \cap S_A = \emptyset$ such that for any $t \in B(t_0, r)$ and any $u \perp t$, $\tau'_{(t,u)}(\overline{\lambda_1(t)}) = \overline{\lambda_1(t)}$, where $\overline{\lambda_1(t)} > \overline{\lambda_2(t)} \geq \overline{\lambda_3(t)} \geq \cdots \geq \overline{\lambda_d(t)}$ are the eigenvalues of $A(t)$. Then, there exists a linear function $L : \mathbb{R}^k \to \mathbb{R}$ such that $(\lambda - L(t))$ divides $\chi_{A(t)}(\lambda)$ and $L(t_0) = max\ Spec(A(t_0))$.*

Proof. Let $(t_1, \ldots, t_k) \in B(t_0, r)$ be a basis of $\mathbb{R}^k$. We set, for $(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$:

$$L(\sum_{i=1}^{k} \alpha_i t_i) := \sum_{i=1}^{k} \alpha_i L(t_i). \tag{4.7}$$

We claim that

$$\forall y \in \mathrm{Conv}(t_1, \ldots, t_k), \ \ \lambda(y) = L(y) \tag{4.8}$$

where $\mathrm{Conv}(X)$ denotes the convex hull of X. We show this by induction on k. For $k = 1$ there is nothing to prove. Let $y \in \mathrm{Conv}(t_1, \ldots, t_k)$. Then, $y = \alpha y' + (1 - \alpha) t_k$

with $\alpha \in [0,1]$ and $y' \in \text{Conv}(t_1, \ldots, t_{k-1})$. Now, $\alpha y' \in B(t_0, r)$, and there exists $u \perp \alpha y'$ such that $t_k \in Span(\alpha y', u)$. From Proposition 4.2.1, λ is linear on this space. Hence, $\lambda(\alpha y' + (1-\alpha)t_k) = \alpha\lambda(y') + (1-\alpha)\lambda(t_k)$. From induction hypothesis, we conclude $\lambda(y') = L(y')$. Finally,

$$\begin{aligned}\lambda(y) &= \alpha L(y') + (1-\alpha)L(t_k) \\ &= L(\alpha y' + (1-\alpha)t_k) = L(y)\end{aligned}$$

which finishes the proof of the claim.

From this, we conclude that for any $t \in \text{Conv}(t_1, \ldots, t_k)$, $\chi_{A(t)}(L(t)) = 0$. But, $\text{Conv}(t_1, \ldots, t_k)$ is open. So, for any $t \in \mathbb{R}^k$, $\chi_{A(t)}(L(t)) = 0$. Then, $(\lambda - L(t))|\chi_{A(t)}(\lambda)$. □

Proposition 4.2.3 (Factorization of extremal 2-orbit)**.** *Let $A(t) \in Sym_d(\mathbb{R})$ be a 2-linear family, and let $\overline{\lambda_1} > \overline{\lambda_2} > \overline{\lambda_3} \geq \cdots \geq \overline{\lambda_d}$ be the eigenvalues of $A(t_0)$ for some t_0. Suppose that $\tau'_{(t_0,u)}(\overline{\lambda_1}) = \overline{\lambda_2}$ and $\tau'_{(t_0,u)}(\overline{\lambda_2}) = \overline{\lambda_1}$, where $u \perp t_0$, $||u|| = 1$. Then, the characteristic polynomial of $A(t)$ has a homogeneous divisor $p(t)$ of degree 2 with $p(t_0)(\lambda) = (\lambda - \overline{\lambda_1})(\lambda - \overline{\lambda_2})$.*

Proof. The principle follows the proof of Proposition 4.2.1. Our goal is to construct a hyperbola that has more intersection points with the characteristic polynomial than Bézout Theorem allows, so that the hyperbola is included in the eigenvalues, which provides a factor of degree 2 in χ_A.

We suppose $t_0 = (0,1)$, write $A(t_1, t_2) = t_1 A_1 + t_2 A_2$ and let $t_1 = u, t_2 = uv$. Then, $A(u, uv) = uB(v)$ with $B(v) = A_1 + vA_2$. Since $B(v)$ is a one parameter family, the eigenvalues of $B(v)$ are analytic functions $\mu_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, d$.

In a neighborhood V of t_0, the eigenvalues of A are analytic functions λ_i, $i = 1, \ldots, d$, with $\lambda_i(0,1) = \overline{\lambda_i}$. We write the order one Taylor expansion of λ_i at $(0,1)$:

$$\lambda_i(t_1, t_2) = \lambda_i(0,1) + \alpha_i t_1 + \beta_i(t_2 - 1) + o(t_1, (t_2 - 1)). \tag{4.9}$$

For $v > 0$ sufficiently large, $(\frac{1}{v}, 1) \in V$, so up to change the indexes, we have

$$\mu_i(v) = \frac{\lambda_i(\frac{1}{v}, 1)}{\frac{1}{v}}, \tag{4.10}$$

and for $v < 0$ sufficiently large also, $\mu_i(v)$ is one of the $\frac{\lambda_j(\frac{1}{v},1)}{\frac{1}{v}}$ for some j (j might differ from i).

From (4.9) and (4.10), we get the expansion of $\mu_i(v)$ at $+\infty$:

$$\mu_i(v) \underset{v\to+\infty}{=} \lambda_i(0,1)v + \alpha_i + (\beta_i \times 0) + o\left(\frac{1}{v}\right) \times v \underset{v\to+\infty}{=} \lambda_i(0,1)v + \alpha_i + o(1). \quad (4.11)$$

Since $\tau'_{(t_0,u)}(\overline{\lambda_1}) = \overline{\lambda_2}$ and $\tau'_{(t_0,u)}(\overline{\lambda_2}) = \overline{\lambda_1}$, it follows that

$$\lim_{v\to-\infty} \frac{\mu_1(v)}{v} = \overline{\lambda_2}, \qquad \lim_{v\to-\infty} \frac{\mu_2(v)}{v} = \overline{\lambda_1}, \quad (4.12)$$

so for $v < 0$ large, $\mu_1(v) = \frac{\lambda_2(\frac{1}{v},1)}{\frac{1}{v}}$ and $\mu_2(v) = \frac{\lambda_1(\frac{1}{v},1)}{\frac{1}{v}}$. This gives the expansions of $\mu_1(v)$ and $\mu_2(v)$ at $-\infty$:

$$\mu_1(v) \underset{v\to-\infty}{=} \overline{\lambda_2}v + \alpha_2 + o(1),$$
$$\mu_2(v) \underset{v\to-\infty}{=} \overline{\lambda_1}v + \alpha_1 + o(1).$$

Now, consider the family $(H_a)_{a\in\mathbb{R}}$ of hyperbolas with asymptotes $\mu = \overline{\lambda_1}v + \alpha_1$ and $\mu = \overline{\lambda_2}v + \alpha_2$ given by

$$H_a : (\mu - (\overline{\lambda_1}v + \alpha_1))(\mu - (\overline{\lambda_2}v + \alpha_2)) - a^2 = 0, \text{ for } a \in \mathbb{R}.$$

Notice that for any v, the equation of H_a has two solutions, so the branches of H_a are two graphs of analytic function defined all over the v-axis. We show that there exists $a_0 \in \mathbb{R}$, such that the hyperbola H_{a_0} is included in the roots of the characteristic polynomial of $B(v)$.

Recall that the larger eigenvalue of $B(v)$ is convex. For large $v > 0$, $\mu_1(v)$ is the maximal eigenvalue of $B(v)$, and since we have

$$\lim_{v\to+\infty} (\mu_1(v) - \overline{\lambda_1}v - \alpha_1) = 0,$$

we get, for large v, $\mu_1(v) \geq \overline{\lambda_1}v + \alpha_1$. Now, $\overline{\lambda_1} > \overline{\lambda_2}$, so there exists v_0 such that $\mu_1(v_0) \geq \overline{\lambda_1}v_0 + \alpha_1 > \overline{\lambda_2}v_0 + \alpha_2$. Let

$$a_0 = \sqrt{\big(\mu_1(v_0) - (\overline{\lambda_1}v_0 + \alpha_1)\big)\big(\mu_1(v_0) - (\overline{\lambda_2}v_0 + \alpha_2)\big)}.$$

(The inequality above shows that the term under the root is non-negative).

We count the number of intersection points of H_{a_0} with $\{\chi_B = 0\}$.

By construction of a_0, $(v_0, \mu_1(v_0)) \in H_{a_0}$, which gives the first point.

We claim that each branch of the hyperbola intersects all graphs of μ_i, for $i \geq 3$, which gives $2(d-2)$ other points. Indeed, note that, for $i \in \{1,2\}$ and $j \in \{3,\dots,d\}$, we got $\overline{\lambda_j}v + \alpha_j < \overline{\lambda_i}v + \alpha_i$ for large $v > 0$, and $\overline{\lambda_j}v + \alpha_j > \overline{\lambda_i}v + \alpha_i$ for large $v < 0$. Since the branches of the hyperbola are asymptotic to $\overline{\lambda_i}v + \alpha_i$ for $i = 1, 2$, and the asymptotic of the μ_j, $j > 2$ stand among the $\overline{\lambda_k}v + \alpha_k$, $k = 3, \dots, d$ we got, for $v > 0$ large enough:

$$\mu_j(v) < \min_{(v,\mu)\in H_{a_0}} \mu, \quad \text{and} \quad \mu_j(-v) > \max_{(-v,\mu)\in H_{a_0}} \mu, \text{ for } j = 3, \dots, d.$$

Since $\mu_3, \dots, \mu_d$ and the two branches of H_{a_0} are continuous, the previous inequalities implies, by Mean Value Theorem, that there are $2(d-2)$ intersection points between the graphs of the μ_j, $j \in \{3,\dots,d\}$ and the hyperbola H_{a_0}.

Now, we count the intersection number at infinity. In homogeneous coordinates $[t_1 : t_2 : \lambda]$, the hyperbola H_{a_0} has equation

$$\big(\lambda - (\overline{\lambda_1}t_2 + \alpha_1 t_1)\big)\big(\lambda - (\overline{\lambda_2}t_2 + \alpha_2 t_1)\big) - a_0^2 t_1^2 = 0,$$

then has two points at infinity $[0 : 1 : \overline{\lambda_1}]$ and $[0 : 1 : \overline{\lambda_2}]$. The tangents of H_{a_0} at these points are given by $\lambda - \alpha_1 t_1 - \overline{\lambda_1}t_2 = 0$ for $[0 : 1 : \overline{\lambda_1}]$ and $\lambda - \alpha_2 t_1 - \overline{\lambda_2}t_2 = 0$ for $[0 : 1 : \overline{\lambda_2}]$. Because $\overline{\lambda_1}$ is an eigenvalue of $A(0,1)$, the point $[0 : 1 : \overline{\lambda_1}]$, belongs to the zero set of χ_A, and similarly, $\overline{\lambda_2}$ is an eigenvalue of $A(0,1)$ so $[0 : 1 : \overline{\lambda_2}] \in \chi_A^{-1}(0)$. Beside, the tangent to the curve $\chi_A^{-1}(0)$ at these points can be deduced from the expansions (4.9) of λ_1 and λ_2. For this notice that λ_i is homogeneous of degree one, which imposes $\beta_i = \lambda_i(0,1)$ in (4.9). We got:

$$\lambda_i(t_1, t_2) = \lambda_i(0,1) + \alpha_i t_1 + \lambda_i(0,1)(t_2 - 1) + o(t_1, (t_2 - 1)),$$

which shows that $\lambda - \alpha_i t_1 - \overline{\lambda_i}t_2 = 0$ is the tangent to $\chi_A^{-1}(0)$ at $[0 : 1 : \overline{\lambda_i}]$, for $i = 1, 2$. Finally, the two points $[0 : 1 : \overline{\lambda_i}]$, for $i = 1, 2$ belongs to both H_a and $\chi_A^{-1}(0)$, where the two curves H_a and $\chi_A^{-1}(0)$ have the same tangent. The intersection multiplicity of each of these two points is then at least 2. This provides an intersection number not smaller than 4 for the points at infinity in $H_a \cap \chi_A^{-1}(0)$.

From above, $\#(H_{a_0} \cap \chi_A^{-1}(0)) \geq 1 + 2(d-2) + 4 = 2d + 1$.

On another hand, $\deg(\chi_A) = d$ and $\deg(H_a) = 2$, so by Bézout Theorem, if the intersection if finite, $\#\left(H_{a_0} \cap \chi_A^{-1}(0)\right) \leq 2d$. So $H_{a_0} \subset \chi_A^{-1}(0)$ and the polynomial

$$P(t_1, t_2, \lambda) = \left(\lambda - (\overline{\lambda_1} t_2 + \alpha_1 t_1)\right)\left(\lambda - (\overline{\lambda_2} t_2 + \alpha_2 t_1)\right) - a_0^2 t_1^2$$

divides the characteristic polynomial of the family, with $P(0,1)(\lambda) = (\lambda - \overline{\lambda_1})(\lambda - \overline{\lambda_2})$ as claimed. □

This can be generalized to k-linear families:

Proposition 4.2.4. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family. Suppose that there exists $t_0 \in \mathbb{R}^k$, $r > 0$ and $B(t_0, r) \cap S_A = \emptyset$ such that $\forall t \in B(t_0, r)$, $\forall u \perp t$, $\tau'_{(t,u)}(\overline{\lambda_1(t)}) = \overline{\lambda_2(t)}$ and $\tau'_{(t,u)}(\overline{\lambda_2(t)}) = \overline{\lambda_1(t)}$, where $\overline{\lambda_1(t)} > \overline{\lambda_2(t)} > \overline{\lambda_3(t)} \geq \cdots \geq \overline{\lambda_d(t)}$ are the eigenvalues of $A(t)$. Then, the characteristic polynomial of $A(t)$ has a homogeneous divisor $p(t)$ of degree 2 with $p(t_0)(\lambda) = (\lambda - \overline{\lambda_1(t_0)})(\lambda - \overline{\lambda_2(t_0)})$.*

Proof. We need the following lemma:

Lemma 4.2.5. *Let $f : B(t_0, r) \to \mathbb{R}$ be such that for any affine line $L \in \mathbb{R}^k$, there exists a quadratic function q_L on L such that $f_{|_L} = q_L$. Then, there exists a quadratic function q such that $f = q_{|_{B(t_0,r)}}$.*

Proof. By induction on k, the dimension of the ambient space. For $k = 1$, there is nothing to prove. We suppose that the lemma holds for k and we prove it for $k+1$. Denote by (x', x_{k+1}) a system of affine coordinate centered at t_0. From induction hypothesis, for all x_{k+1} with $|x_{k+1}| < r$, $f_{x_{k+1}} : x' \mapsto f(x', x_{k+1})$ is the restriction of a quadratic function: Denoting by $(m_\alpha(x'))_{\alpha \in A}$, $(m_\beta(x'))_{\beta \in B}$ the collections of all monomials of respective degree 2 and 1 in therms of x',

$$f(x_1, \ldots, x_{k+1}) = \sum_{\alpha \in A} a_\alpha(x_{k+1}) m_\alpha(x') + \sum_{\beta \in B} b_\beta(x_{k+1}) m_\beta(x') + c(x_{k+1}). \tag{4.13}$$

On the other hand, for any x',

$$f_{x'} : x_{k+1} \mapsto f(x', x_{k+1}) \tag{4.14}$$

is quadratic. Choosing $(x'_i)_{i \in I}$ such that the vectors $((m_\alpha(x'_i))_{\alpha \in A}, (m_\beta(x'_i))_{\beta \in B}, 1)$ form a basis of $\mathbb{R}^{Card(A)+Card(B)+1}$, we obtain that $\forall \alpha \in A$, $\forall \beta \in B$, $a_\alpha(x_{k+1})$,$b_\beta(x_{k+1})$

and $c(x_{k+1})$ is quadratic. It remains to prove that the a_α are constant and the b_β have degree 1. For this, notice that given $\ell \in \mathbb{R}^k$, $s \in \mathbb{R} \mapsto f(s\ell, s)$ is quadratic, and

$$f(s\ell, s) = \sum_{\alpha \in A} a_\alpha(s) s^2 m_\alpha(\ell) + \sum_{\beta \in B} b_\beta(s) s m_\beta(\ell) + c(s); \tag{4.15}$$

so, choosing again $(\ell_i)_{i \in I}$ in such a way that the vectors $((m_\alpha(\ell_i))_{\alpha \in A}, (m_\beta(\ell_i))_{\beta \in B}, 1)$, for $i \in I$, form a basis of $\mathbb{R}^{Card(A)+Card(B)+1}$ gives rise to an invertible system with quadratic second member, which shows that $a_\alpha(s)s^2$ is quadratic, $b_\beta(s)s$ is quadratic and $c(s)$ is quadratic, so a_α is constant and b_α is linear. Finally,

$$f(x_1, \dots, x_{k+1}) = \sum_{\alpha \in A} a_\alpha(x_{k+1}) m_\alpha(x') + \sum_{\beta \in B} b_\beta(x_{k+1}) m_\beta(x') + c(x_{k+1})$$

is quadratic as announced. This finishes the proof of the lemma. □

Now we prove the proposition. Choose $t_1, \dots, t_k$ in such a way that $(t_0, t_1, \dots, t_{k-1})$ form a basis of $\mathbb{R}^k$. From Proposition 4.2.3, $L(t) = \overline{\lambda_1(t)} + \overline{\lambda_2(t)}$ is linear on each affine line of $B(t_0, r) \cap t_0 + Span(t_1, \dots, t_k)$ and $f(t) = \overline{\lambda_1(t)\lambda_2(t)}$ is quadratic on each affine line of $B(t_0, r) \cap t_0 + Span(t_1, \dots, t_k)$, and $\lambda^2 - L(t)\lambda + f(t)$ divides $\chi_{A(t)}$ for all $t \in B(t_0, r)$. Then, $L(t)$ is linear on $B(t_0, r)$ and by Lemma 4.2.5, $f(t)$ is quadratic on $B(t_0, r)$. Then, $p(t_0)(\lambda) := \lambda^2 - L(t)\lambda + f(t)$ is an homogeneous polynomial of degree two and $\chi_{A(t)}$ vanishes over an open subset of the connected zeros of p, so χ_A identically vanish over $p^{-1}(0)$, so p divides χ_A. This finises the proof of the proposition. □

We don't know if the procedure we develop in the former proofs can be adapted for an arbitrary large collection of extremal eigenvalues that is invariant by the antipodal monodromy, but the question deserves to be stated.

Question 4.2.6. *Let $A(t) \in Sym_d(\mathbb{R})$ be a 2-linear family, and let*

$$\overline{\lambda_1} > \cdots > \overline{\lambda_k} > \overline{\lambda_{k+1}} \geq \cdots \geq \overline{\lambda_d}$$

be the eigenvalues of $A(t_0)$ for some t_0. Suppose $\tau'_{(t_0,u)}(\{\overline{\lambda_1}, \dots, \overline{\lambda_k}\}) = \{\overline{\lambda_1}, \dots, \overline{\lambda_k}\}$. Does the characteristic polynomial of $A(t)$ have an homogeneous divisor $p(t)$ of degree k with $p(t_0)(\lambda) = \prod_{i=1}^k (\lambda - \overline{\lambda_i})$?

In an attempt to adapt the previous proofs, one constructs the same analytic eigenvalues $\mu_i(v)$ of the one-dimensional family $B(v) = A_1 + vA_2$, where $A(t_1, t_2) =$

$t_1A_1 + t_2A_2$, and $t_0 = (0, 1)$. The μ_i are asymptotic at $+\infty$ to d lines, say

$$\mu_i(v) \underset{v\to+\infty}{=} \overline{\lambda_i}v + \alpha_i + o(1).$$

At $-\infty$, each μ_i is also asymptotic to one of these lines, not necessary the same, but since the set $\{\lambda_i;\ i = 1, \ldots, k\}$ is invariant by the monodromy, if

$$\mu_j(v) \underset{v\to-\infty}{=} \overline{\lambda_i}v + \alpha_i + o(1),$$

then either i and j both belong to $\{1, \ldots, k\}$, or else i and j both belong to $\{k + 1, \ldots, d\}$.

The family of all hyperbolic polynomial of degree k, whose roots are k graphs (say, of $g_1, \ldots, g_k$), that are asymptotic at infinity to the lines $\mu = \overline{\lambda_i}v+\alpha_i$, for $i = 1, \ldots, k$ can be constructed. It is a $k(k-1)/2$ dimensional family, parameterized by all $k \times k$ symmetric matrices C with zero diagonal, to which one associates the polynomial

$$P_C(v, \lambda) = \det\Big(\mathrm{Diag}(\alpha_1, \ldots, \alpha_k) + C + v\mathrm{Diag}\left(\overline{\lambda_1}, \ldots, \overline{\lambda_1}\right) - \lambda\mathbb{I}_k\Big).$$

(We do not claim the parameterization is one to one).

There are intersections of the two curves $P_C^{-1}(0)$ and $\chi_A^{-1}(0)$ that exist independently from C. Indeed, the k points of $P_C^{-1}(0)$ at infinity belongs to $\chi_A^{-1}(0)$, and are double points in this intersection (same asymptotic line). This gives $2k$ points. There are also $k(d-k)$ points obtained with the extremal condition on the $\overline{\lambda_i}, i = 1, \ldots, k$. From Mean Value Theorem, each graph $\mu_i,\ i = k+1, \ldots, d$ intersects each graph $g_j,\ j = 1, \ldots, k$ since from the asymptotics, $\mu_i - g_i$ is positive at $-\infty$ and negative at $+\infty$. So we have $2k + k(d-k) = kd - k(k-2)$ intersection points for any C in the family, when Bézout allows kd points. The difficulty is then to choose C in such a way that the intersection number of P_C with the graphs of the $\mu_i,\ i = 1, \ldots, k$, realizes at least $k(k-2)+1$ other points. This would contradict Bézout, and then provide a factor in χ_A as wanted. This number, $k(k-2)+1$, must be compared with the dimension $k(k-1)/2$ of the family P_C. For $k = 1$, the family has dimension 0 and there are no missing points. The proof is direct for the line. For $k = 2$, the family has dimension 1 and there is one missing point. A choice must, and can be made (the value a_0 in the proof), to get the right hyperbola. For $k = 3$, the family has dimension 3, but 4 intersection points are missing. The situation seems to deteriorate when k grows, since $k(k-2)+1-k(k-1)/2$ increases rapidly. It let us think that, if

the process works, a good knowledge of the roots of P_C and some accurate properties of the graphs of the extremal eigenvalues are required to conclude.

4.2.2 Splitting problem for extremal eigenvalues

Here we show that the splitting problem always have a positive solution in the case of extremal eigenvalues.

Proposition 4.2.7. *Let $A(t) \in Sym_d(\mathbb{R})$ be a 2-linear family whose characteristic polynomial $\chi(t,\lambda)$ admits a homogeneous divisor $P(t,\lambda)$ of degree k: $\forall t,\ \chi(t,\lambda) = P(t,\lambda)Q(t,\lambda)$. Suppose that for some parameter t_0, the roots of $P(t_0)$ are strict maximal in $Spec(A(t_0))$, this is,*

$$P(t_0)(\lambda) = \prod_{i=1}^{k}(\lambda - \lambda_i),$$

where $\lambda_1 \geq \cdots \geq \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_d$ are the roots (enumerated with multiplicity) of $\chi(t_0)$. Then, there exists an orthogonal matrix O such that

$$O^t A(t) O = diag(A'(t), A''(t)),$$

where $A'(t) \in Sym_k(\mathbb{R})$, $A''(t) \in Sym_{d-k}(\mathbb{R})$ and $P(t)$ is the characteristic polynomial of $A'(t)$.

The proof requires to notice that the sum of the k values of a quadratic form q over an orthonormal family of k vectors is maximal if and only if these k vectors generate the invariant space associated with the k greater eigenvalues of q. It is the following lemma. For notation purpose, we set

$$V_k^d = \{(e_1, \ldots, e_k) \in (\mathbb{R}^d)^k;\ \forall (i,j),\ e_i^t e_j = \delta_i^j\},$$

where δ_i^j is 1 if $i = j$ and 0 otherwise. V_k^d is the set of all orthonormal families of k vectors in $\mathbb{R}^d$. Notice that V_k^d is a smooth compact sub-manifold of $(\mathbb{R}^d)^k$ (of codimension $\frac{k(k+1)}{2}$).

Lemma 4.2.8. *Let $A \in Sym_d(\mathbb{R})$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_d$, and let $f : V_k^d \to \mathbb{R}$ be given by $f(e_1, \ldots, e_k) = \sum_{i=1}^{k} e_i^t A e_i$. Then f admits its maximum $\lambda_1 + \cdots + \lambda_k$ at $e_1, \ldots, e_k$ if and only if $Vect(e_1, \ldots, e_k)$ is the invariant subspace of A associated with $\lambda_1, \ldots, \lambda_k$.*

Proof. It is an optimization with constraints problem on $(\mathbb{R}^d)^k$. The constraints are given by the

$$g_{ij}(e_1,\dots,e_k) := e_i^t e_j - \delta_i^j,\ i=1,\dots,d,\ j=i+1,\dots,d,$$

that have differentials $dg_{ij} = e_i^t de_j + e_j^t de_i$. The function to be optimized has differential

$$df = 2\sum_{i=1}^{k} e_i^t A de_i,$$

so, from Lagrange Multipliers Theorem, if f has a critical point under the constraints $g_{ij}=0$, then there are $\alpha_{ij}\in\mathbb{R}$ such that $df = \sum_{i,j}\alpha_{ij}dg_{ij}$. Since the linear forms de_i are independent, this gives, for all i:

$$e_i^t A = \sum_{j=1}^{k}(\alpha_{ij}+\alpha_{ji})e_j^t.$$

In particular, for all i, Ae_i belongs to $\mathrm{Vect}(e_1,\dots,e_k)$, so $\mathrm{Vect}(e_1,\dots,e_k)$ is an invariant space for A.

Recall that the invariant spaces of A are all direct sums of subspaces of eigenspaces of A, and also, that if H is a k-dimensional invariant subspace, $f(e_1,\dots,e_k)$ does not depend on the choice of an orthonormal basis $(e_1,\dots,e_k)$ of H since $f(e_1,\dots,e_k)$ is nothing but the trace of the restriction to H of the endomorphism associated to A. So, if $(e_1,\dots,e_k)$ is an orthonormal basis of a k-dimensional invariant subspace, $f(e_1,\dots,e_k)$ is the sum of k values among the eigenvalues of A (multiplicities taken into account). Among all possible choices, the sum $\lambda_1+\cdots+\lambda_k$ is maximal, so finally, $f(e_1,\dots,e_k)$ is maximal whenever $Vect(e_1,\dots,e_k)$ is the invariant subspace that is the direct sum of the eigenspaces of A associated to λ_i for $i=1,\dots,k$. □

We can now prove Proposition 4.2.7.

Proof of Proposition 4.2.7. Without loss of generality, we suppose that $t_0=(0,1)$ and set $A(1,0)=A_1, A(0,1)=A_2$ and $B(v)=A_1+vA_2$. Since $B(v)$ is a one parameter family, $B(v)$ admits analytic eigenvalues $\mu_1(v),\dots,\mu_d(v)$ associated with an analytic orthonormal basis of eigenvectors $(e_1(v),\dots,e_d(v))$. We set $\chi_B(v)(\mu) = \det(B(v)-\mu Id)$ and write $p(v)=P(1,v)$ and $q(v)=Q(1,v)$, so $\chi_B = pq$. The

polynomial P is homogeneous of degree k so p can be written as:

$$p(v)(\mu) = \sum_{i=0}^{k} p^i(v)\mu^{k-i}, \text{ with } \deg(p^i) = i.$$

We let $p^1(v) = p_0 + p_1 v$. Notice that $p^0 = 1$.

Recall that the limits $\lim_{v\to+\infty} \frac{\mu_i(v)}{v}$ are the eigenvalues of $A(0,1)$, taken with multiplicity, which are the λ_i for $i = 1, \ldots, d$. Up to reorder indices we suppose that for all i,

$$\mu_i(v) \underset{v\to+\infty}{=} \lambda_i v + o(v).$$

By unicity of analytic prolongation, $\forall v,\ p(v)(\mu) = \prod_{i=1}^{k}(\mu - \mu_i(v))$. In particular, for all v, $p_0 + p_1 v = \sum_{i=1}^{k} \mu_i(v)$, which leads to

$$p_1 = \sum_{i=1}^{k} \lambda_i \tag{4.16}$$

by identification of the dominant terms in the asymptotic at $+\infty$.

We now study the asymptotic near $v = 0$, and for this, we set

$$\forall i,\ e_i(v) \underset{v\to0}{=} e_i + ve_i' + o(v).$$

Since $\mu_i(v) = e_i(v)^t B(v) e_i(v)$, we get

$$\begin{aligned} p_0 + p_1 v &= \sum_{i=1}^{k} \mu_i(v) \\ &= \sum_{i=1}^{k} e_i(v)^t B(v) e_i(v) \\ &= \sum_{i=1}^{k} (e_i + ve_i' + o(v))^t (A_1 + vA_2)(e_i + ve_i' + o(v)) \\ &= \sum_{i=1}^{k} e_i^t A_1 e_i + v\left(2e_i^t A_1 e_i' + e_i^t A_2 e_i\right) + o(v) \end{aligned} \tag{4.17}$$

Notice that, since $e_i(v)^t e_i(v) \equiv 1$, we get $e_i^t e_i' = 0$. Since $e_i = e_i(0)$, $e_i^t A_1 = \mu_i(0) e_i^t$

so $e_i^t A_1 e_i' = 0$. Then the terms of order v in (4.17) reduce to:

$$p_1 = \sum_{i=1}^{k} e_i^t A_2 e_i,$$

which, according to (4.16) gives

$$\sum_{i=1}^{k} e_i^t A_2 e_i = \sum_{i=1}^{k} \lambda_i. \tag{4.18}$$

But A_2 has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_d$, so according to lemma 4.2.8, the equation (4.18) implies that $(e_1, \ldots, e_k)$ is an orthonormal basis of the invariant space H of A_2 associated to the eigenvalues $\lambda_1, \ldots, \lambda_k$.

We can now conclude. H is an invariant space of A_2, and since for $i = 1, \ldots, k$, $e_i = e_i(0)$ is an eigenvector of A_1, H is invariant for A_1 too. So H is invariant for all combinations $t_1 A_1 + t_2 A_2$, this means, for $A(t)$, and its orthogonal $H^\perp$ also. Then, if O is the orthogonal matrix of $(e_1, \ldots, e_d)$, the matrix $O^t A(t) O$ is diagonal by block, $O^t A(t) O = diag(A'(t), A''(t))$, and the identity $P(t) = \chi_{A'(t)}$ follows from the equality near $t = (0, 1)$ and unicity of analytic prolongation. □

This can be generalized to k-linear families.

Proposition 4.2.9. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family whose characteristic polynomial $\chi(t, \lambda)$ admits a homogeneous divisor $P(t, \lambda)$ of degree ℓ: $\forall t$, $\chi(t, \lambda) = P(t, \lambda) Q(t, \lambda)$. Suppose that for some parameter t_0, $P(t_0)(\lambda) = \prod_{i=1}^{\ell} (\lambda - \lambda_i)$, where $\lambda_1 \geq \cdots \geq \lambda_\ell > \lambda_{\ell+1} \geq \cdots \geq \lambda_d$ are the roots (enumerated with multiplicity) of $\chi(t_0)$. Then, there exists an orthogonal matrix O such that*

$$O^t A(t) O = diag(A'(t), A''(t)),$$

where $A'(t) \in Sym_\ell(\mathbb{R})$, $A''(t) \in Sym_{d-\ell}(\mathbb{R})$ and $P(t)$ is the characteristic polynomial of $A'(t)$.

Proof. Let $(e_1, \ldots, e_k)$ be a basis of $\mathbb{R}^k$. For all $t \in \mathbb{R}^k$, we have $t = t_1 e_1 + \cdots + t_k e_k$ and we set $A(e_i) = A_i$. Suppose that $t_0 = e_1$. There exists a neighborhood V of t_0 such that the eigenvalues of $A(t)$ are uniquely determined as analytic functions on V and for all $t \in V$, $P(t)(\lambda) = \prod_{i=1}^{\ell} (\lambda - \lambda_i(t))$ where $\lambda_1(t) \geq \cdots \geq \lambda_\ell(t) > \lambda_{\ell+1} \geq \cdots \geq \lambda_d(t)$ with $\lambda_i(e_1) = \lambda_i$, $i = 1, \ldots, d$. Let $t \in Span(e_1, e_i) \cap V$, $i \neq 1$, then there exists $t_1, t_i \in$

$\mathbb{R}$ such that $A(t_1, t_i) = t_1 A_1 + t_i A_i$. By Proposition 4.2.7, there exists an invariant subspace H of dimension ℓ for A_1 associated to $\lambda_1, \ldots, \lambda_\ell$ which is also invariant for each A_i, $i = 2, \ldots, k$. So, H is invariant for all combination of $t_1 A_1 + \cdots + t_k A_k$. Since for all $t \in V$, H is invariant by $A(t)$, $H^\perp$ is invariant by $A(t)$. Therefore, there exists an orthonormal matrix O, such that $O^t A(t) O = \mathrm{diag}(A'(t), A''(t))$, where $A'(t) \in Sym_\ell(\mathbb{R})$, $A''(t) \in Sym_{d-\ell}(\mathbb{R})$. This is an equality between analytic functions, that holds in a neighborhood of t_0; from isolated zero principle, the equality holds everywhere on R^k. Since $P(t) = \chi_{A'(t)}$ holds in a neighborhood of t_0, by unicity of analytic prolongation for all $t \in \mathbb{R}^k$ we get $P(t) = \chi_{A'(t)}$. □

4.3 Full reduction

In this section, we mix the previous results to study the problem of the full reduction of a linear family. Even through, in full generality, the antipodal monodromy cannot predict the factorization of the characteristic polynomial (Proposition 4.1.4), nor the factorization is sufficient to get splitting (Example 4.1.10). For a full reduction, one can use recursively the positive results obtained for extremal eigenvalues (Section 4.2) to get the following theorem. In particular, it shows that the existence of a non-trivial antipodal monodromy is the only obstruction for the existence of a diagonalization of the family.

We first state the result, and present a proof that relies on the postponed Corollary 4.3.3. Then we exhibit a sufficient condition to get a partial diagonalization (Proposition 4.3.2). The extremal case of this proposition is exactly Corollary 4.3.3 that is needed to complete the proof of Theorem 4.3.1. Finally, as an application, we give a complete description of the 2 linear families of $Sym_2(\mathbb{R})$.

Theorem 4.3.1. *Let $A(t) \in Sym_d(\mathbb{R}), t \in \mathbb{R}^k$ be a k-linear family, $A \not\subset \Sigma$. Then, the following statements are equivalent:*

1. *There exists a basis $(e_1, e_2, \ldots, e_k)$ of $\mathbb{R}^k$, with $A(e_1) \notin \Sigma$ and $\tau'_{Span(e_1, e_i)} = \mathbb{I}_d$ for all $i = 2, \ldots, k$.*
2. *For any linear 2-plane $\mathcal{P} \subset \mathbb{R}^k$, $\tau'_{\mathcal{P}} = \mathbb{I}_d$.*
3. *0 is a quasi-regular point of $A(t)$.*
4. *There exists a family $P(t) \in O_d(\mathbb{R})$, analytic in t, such that $P^{-1}(t) A(t) P(t)$ is diagonal.*

5. *There exists an orthogonal matrix $P \in O_d(\mathbb{R})$ such that $P^{-1}A(t)P$ is diagonal.*

6. *There exists a basis $(e_1, e_2, \dots, e_k)$ of $\mathbb{R}^k$, such that $A(e_1) \notin \Sigma$ and $A(e_1)A(e_i) = A(e_i)A(e_1)$ for $i = 2, \dots, k$.*

7. *For all $(t, t') \in (\mathbb{R}^k)^2$, $A(t)A(t') = A(t')A(t)$.*

Proof. Some implications between these statements are straightforward. We first show (7) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), and (5) $\Rightarrow$ (7). Remains (1) $\Rightarrow$ (5) that we deduce from the postponed Proposition 4.3.2 which contains in fact the core of the proof.

(7) $\Longrightarrow$ (6): For any $(t, t') \in (\mathbb{R}^k)^2$, $A(t)$ commutes with $A(t')$. So in particular, for any $(e_1, e_i) \in (\mathbb{R}^k)^2$, $i \neq 1$, $A(e_1)$ commutes with $A(e_i)$.

(6) $\Longrightarrow$ (5): Write $A(t) = \sum_{i=1}^{k} t_i A(e_i)$. For any $i \neq 1$, $A(e_i)$ commutes with $A(e_1)$, thus there exists an orthogonal basis of $\mathbb{R}^d$ where $A(e_i)$ and $A(e_1)$ are diagonal. But $A(e_1) \notin \Sigma$ so $A(e_1)$ is diagonal in only one basis (up to permutation). So all $A(e_i)$ are diagonal in this basis, and therefore $A(t)$ also.

(5) $\Longrightarrow$ (4): There is a $P \in O_d(\mathbb{R})$ which diagonalize $A(t)$, so in particular, there is an analytic diagonalization for $A(t)$.

(4) $\Longrightarrow$ (3): $\chi_A = \chi_{P^{-1}AP} = \prod_{i=1}^{d}(x - \lambda_i(t))$, where by assumption the λ_i are analytic, so 0 is a quasi-regular point.

(3) $\Longrightarrow$ (2): Since 0 is a quasi-regular point of $A(t)$, for each i, $\lambda_i(t)$ is analytic in a neighborhood of 0, so $\tau'_{\mathcal{P}} = Id$.

(2) $\Longrightarrow$ (1): For all linear 2-plane $\mathcal{P}$, $\tau'_{\mathcal{P}} = Id$, so in particular, $\tau'_{Span(e_1,e_i)} = Id$.

(5) $\Longrightarrow$ (7): Let $(t, t') \in (\mathbb{R}^k)^2$ be two independent parameters. By (5), there exists an orthogonal matrix P such that $P^{-1}A(t)P$ and $P^{-1}A(t')P$ are diagonal, so they commute.

Finally, we show (1) $\Rightarrow$ (5) which is the main part of the proof of the theorem. The key point of the proof of (1) $\Rightarrow$ (5) is Proposition 4.3.2.

Proposition 4.3.2. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family, and $(e_1, \dots, e_k)$ be a basis of $\mathbb{R}^k$. Suppose the eigenvalues (with multiplicity) of $A(e_1)$ satisfy the ordering $\overline{\lambda_1} > \cdots > \overline{\lambda_\ell} > \overline{\lambda_{\ell+1}} \geq \cdots \geq \overline{\lambda_d}$. Let $\Lambda = \{\overline{\lambda_1}, \cdots, \overline{\lambda_\ell}\}$ and assume that there exists a partition of Λ, $\Lambda = P_1 \cup \cdots \cup P_m$ formed by singletons or consecutive pairs $(\overline{\lambda_i}, \overline{\lambda_{i+1}})$,*

$i = 1 \ldots, \ell - 1$ such that for all $j = 1, \ldots, m$, $\tau'_{(e_1,e_i)}(P_j) = P_j$. Then, there exists an orthogonal matrix $P \in O(d)$ such that

$$P^{-1}A(t)P = \begin{pmatrix} \Lambda_1(t) & & & \\ & \ddots & & \\ & & \Lambda_m(t) & \\ & & & B(t) \end{pmatrix}$$

where $B(t) \in Sym_{d-\ell}(\mathbb{R})$ and for $j = 1, \ldots, m$, $\Lambda_j(t) \in Sym_{(\#P_j)}(\mathbb{R})$ and $\Lambda_j(e_1) = diag(P_j)$.

Proof. The proof is by induction on m, the number of extremal blocks fixed by the antipodal monodromies. For $t \in \mathbb{R}^k$, we write $t = \sum_{i=1}^k t_i e_i$ and $A(e_i) = A_i$.

For $m = 0$, there is nothing to prove.

We fix $m > 1$ and we suppose that the proposition holds for $m - 1$. Choose $i \in \{2, \ldots, k\}$, then the two linear family $t_1A_1 + t_iA_i$ satisfies the hypothesis of Proposition 4.2.1 or Proposition 4.2.3 for $(t_1, t_i) = (1, 0)$ and $u \perp (1, 0)$.

In the first case, from Proposition 4.2.1, for each $i \neq 1$, $t_1A_1 + t_iA_i$ admits a linear eigenvalue $\lambda_{1,i}(t_1, t_i) = t_1\overline{\lambda_1} + t_i\mu_i$ with $\lambda_{1,i}(1, 0) = \overline{\lambda_1}$. In particular, for any $i \neq 1$, $p(t_1, t_i) = (\lambda - \lambda_{1,i}(t_1, t_i))$ divides the characteristic polynomial of $t_1A_1 + t_iA_i$. Since for any $i \neq 1$, $t_1A_1 + t_iA_i$ satisfies the hypothesis of the Proposition 4.2.7, there exists an invariant subspace H of dimension one for A_1 associated to $\overline{\lambda_1}$ which is also invariant by each A_i, $i = 2, \ldots, k$. So, H is invariant for all combination $t_1A_1 + \cdots + t_kA_k$. Since $H^\perp$ is invariant by $A(t)$, there exists $P_1 \in O(d)$ such that

$$P_1^{-1}A(t)P_1 = \begin{pmatrix} \Lambda_1(t) & 0 \\ 0 & C(t) \end{pmatrix}, \tag{4.19}$$

where $\Lambda_1(t) = \lambda_1(t) = t_1\overline{\lambda_1} + t_2\mu_2 + \cdots + t_k\mu_k$ and $C(t)$ is a k-linear symmetric matrix of size $d - 1 \times d - 1$ whose eigenvalues for $t = e_1$, are $\overline{\lambda_2} > \cdots > \overline{\lambda_\ell} > \overline{\lambda_{\ell+1}} \geq \cdots \geq \overline{\lambda_d}$ with $\tau'_{(e_1,e_i)}(P_j) = P_j$ for $j = 2, \ldots, \ell$.

In the other case, from Proposition 4.2.3, the characteristic polynomial of $t_1A_1 + t_iA_i$ has a homogeneous divisor $p(t_1, t_i)$ of degree two such that $p(1, 0)(\lambda) = (\lambda - \overline{\lambda_1})(\lambda - \overline{\lambda_2})$. Since for any $i \neq 1$, $t_1A_1 + t_iA_i$ satisfies the hypothesis of Proposition 4.2.7, there exists an invariant subspace H of dimension two for A_1 associated to

$\overline{\lambda_1}, \overline{\lambda_2}$ which is also invariant by each A_i, $i = 2, \ldots, k$. So, H is invariant for all combination $t_1A_1 + \cdots + t_kA_k$. Since $H^\perp$ is invariant by $A(t)$, there exists $P_1 \in O(d)$ such that

$$P_1^{-1}(t)P_1 = \begin{pmatrix} \Lambda_1(t) & 0 \\ 0 & C(t) \end{pmatrix}, \tag{4.20}$$

where $\Lambda_1(t) \in Sym_2(\mathbb{R})$, $\Lambda_1(e_1) = diag(\overline{\lambda_1}, \overline{\lambda_2})$ and $C(t)$ is a k-linear symmetric matrix of size $d-2 \times d-2$ whose eigenvalues for $t = e_1$, are $\overline{\lambda_3} > \cdots > \overline{\lambda_\ell} > \overline{\lambda_{\ell+1}} \geq \cdots \geq \overline{\lambda_d}$ and satisfies $\tau'_{(e_1,e_i)}(P_j) = P_j$ for $j = 2, \ldots, \ell$.

From the inductive hypothesis, there exists a $d-1 \times d-1$ or $d-2 \times d-2$ orthogonal matrix P_2 such that

$$P_2^{-1}C(t)P_2 = \left(\begin{array}{cccc|c} \Lambda_2(t) & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \Lambda_m(t) & 0 \\ \hline 0 & \cdots & \cdots & 0 & B(t) \end{array}\right),$$

where the matrix $B(t)$ is a symmetric matrix of dimension $= d - \ell$.

Define the $d \times d$ matrix $R = \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix}$ if $P_2 \in O(d-1)$ or $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P_2 \end{pmatrix}$ if $P_2 \in O(d-2)$ and let $P = RP_1$. Then, P is an orthogonal matrix and we have

$$P^{-1}A(t)P = \begin{pmatrix} \Lambda_1(t) & 0 \\ 0 & P_2^{-1}C(t)P_2 \end{pmatrix}.$$

This finishes the proof. □

Corollary 4.3.3. *Let $A(t) \in Sym_d(\mathbb{R})$ be a k-linear family, and $(e_1, \ldots, e_k)$ be a basis of $\mathbb{R}^k$. Suppose the eigenvalues (with multiplicity) of $A(e_1)$ satisfy the ordering $\overline{\lambda_1} > \cdots > \overline{\lambda_d}$. Let $\Lambda = \{\overline{\lambda_1}, \cdots, \overline{\lambda_d}\}$ and assume that there exists a partition of Λ, $\Lambda = P_1 \cup \cdots \cup P_m$ formed by singletons or consecutive pairs $(\overline{\lambda_i}, \overline{\lambda_{i+1}})$, $i = 1, \ldots, d-1$ such that for all $j = 1, \ldots, m$, $\tau'_{(e_1,e_i)}(P_j) = P_j$. Then, there exists an orthogonal matrix $P \in O(d)$ such that $P^{-1}A(t)P$ is diagonal by 1×1 blocks and 2×2 blocks.*

Proof. This is exactly Proposition 4.3.2 with $\ell = d$. □

This corollary finishes the proof of (1) $\Rightarrow$ (5), and so the proof of the theorem is finished. $\square$

In the following proposition, we study on all 2-linear family $A \in Sym_2(\mathbb{R})$ and give a computational necessary and sufficient condition for having the eigenvalues as analytic functions.

Proposition 4.3.4. *Consider the 2-linear family*

$$A(t_1,t_2) = t_1A_1 + t_2A_2 = t_1\begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} + t_2\begin{pmatrix} a_1' & b' \\ b' & a_2' \end{pmatrix}; a_1,a_2,b,b',a_1',a_2' \in \mathbb{R},$$

and let

$$C = \begin{pmatrix} a_1 - a_2 & a_1' - a_2' \\ b & b' \end{pmatrix}.$$

Let $t_0 \in \mathbb{R}^2\backslash(0,0)$ and $u \perp t_0$. Then the following alternative holds.

1. *Either*
   - *(a) $det(C) \neq 0$,*
   - *(b) and $\tau'_{(t_0,u)}$ is a transposition,*
   - *(c) and the eigenvalues of $A(t_1,t_2)$ are not analytic functions near origin,*
2. *or else,*
   - *$(\tilde{a})$ $det(C) = 0$,*
   - *$(\tilde{b})$ and $\tau'_{(t_0,u)} = \mathbb{I}_d$,*
   - *$(\tilde{c})$ and there exists an orthogonal matrix P such that $P^tA(t_1,t_2)P$ is diagonal.*

Proof. It is to prove the equivalence of (a), (b), (c) and the negations $\neg(\tilde{a})$ of $(\tilde{a})$, $\neg(\tilde{b})$ of $(\tilde{b})$, and $\neg(\tilde{c})$ of $(\tilde{c})$. We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow \neg(\tilde{b}) \Rightarrow \neg(\tilde{c}) \Rightarrow \neg(\tilde{a}) \Rightarrow (a)$.

Let $\Delta_{A(t_1,t_2)}$ be the discriminant of $A(t_1,t_2)$. We remark that $\Delta_{A(t_1,t_2)}$ is the sum of the squares of the two linear forms $\mathcal{L}_1 = 2(bt_1+b't_2)$ and $\mathcal{L}_2 = (a_1-a_2)t_1+(a_1'-a_2')t_2$:

$$\begin{aligned}\Delta_{A(t_1,t_2)} &= 4(bt_1 + b't_2)^2 + [(a_1 - a_2)t_1 + (a_1' - a_2')t_2]^2 \\ &= \mathcal{L}_1^2(t_1,t_2) + \mathcal{L}_2^2(t_1,t_2).\end{aligned}$$

So $\Delta_{A(t_1,t_2)} = 0$ equates the linear system

$$\begin{cases} (a_1 - a_2)t_1 + (a_1' - a_2')t_2 = 0 \\ bt_1 + b't_2 = 0, \end{cases} \tag{4.21}$$

whose coefficient matrix is C.

If $det(C) \neq 0$, then $(0,0)$ is the unique parameter for $\Delta_{A(t_1,t_2)} = 0$. In this case, for given t_0, u, if $\lambda_1 < \lambda_2$ denotes the eigenvalues of $A(t_0)$, then the analytic continuations $\tau(\lambda_1)$, $\tau(\lambda_2)$ of λ_1 and λ_2 along the half-turn path from t_0 in the direction of u satisfy $\tau(\lambda_1) < \tau(\lambda_2)$. But $Spec(A(-t_0)) = \{-\lambda_1, -\lambda_2\}$. So, $\tau(\lambda_1) = -\lambda_2$ and $\tau(\lambda_2) = -\lambda_1$. Finally, $\tau'_{t_0,u}(\lambda_1) = -\tau(\lambda_1) = \lambda_2$ and $\tau'_{t_0,u}(\lambda_2) = -\tau(\lambda_2) = \lambda_1$, so $\tau'_{t_0,u}$ is a transposition: $(a) \Rightarrow (b)$. Then, $(b) \Rightarrow (c) \Rightarrow \neg(\tilde{b}) \Rightarrow \neg(\tilde{c})$ are given by Theorem 4.3.1. Now, the direct calculation of $A_1A_2 - A_2A_1$ shows that $det(C) = 0$ is equivalent to $A_1A_2 = A_2A_1$. So Theorem 4.3.1 again shows $\neg(\tilde{c}) \Rightarrow \neg(\tilde{a})$. Since $\neg(\tilde{a}) \Rightarrow (a)$ is clear, this finishes the proof. □

Chapter 5

Resolution of eigenvalues by eigenvectors

In this chapter, we study linear subspaces of $Sym_d(\mathbb{R})$ by means of their eigenspaces.

In the first section – Projectivization – we introduce different objects: the variety made of the triples matrix and the associated eigenvalues and eigenspaces, and those of its projections that are of interest to us. Since these objects have homogeneous behavior, it is convenient to systematically quotient by scalar multiplication and work in projective spaces. This quotients are described in the same section.

In the second section – Cubics of eigenelements –, we focus on the eigenelements of the two-dimensional subspaces of 3×3 matrices. This case is of particular interest as we will see that for these families, both the eigenspaces and the eigenvalues form a cubic curve in the projective plane $\mathbb{P}^2$. A 2-linear family of $Sym_3(\mathbb{R})$ hence realizes a matching between two cubic curves, that happens to be birationnal. We introduce a "marked cubic type" for those curves (the given of a cubic configuration together with a number of eigenelements for each component of the cubic). We prove that the marked cubic of the eigenspaces determines the marked cubic of the eigenvalues, Theorem 5.2.4. Indeed, we give a complete enumeration of all possible marked cubics of the pairs eigenvectors, eigenvalues. It happens that there exist only 9 such pairs, when the full combinatorics of pairs of marked cubic types is way heavier.

In the third section – Resolution of singularities of eigenvalues – we discuss a question that arises naturally. The preceding study enlightens the fact that eigenspaces sometimes contains artificial points, appearing as combination of eigenvectors associ-

ated to multiple eigenvalues but disappearing by small perturbation. Removing these artificial points from eigenvectors produces a set, that we call strict eigenvectors, and which is a smooth curve in all cases studied in section 2, even when the corresponding curve of eigenvalues is singular. It is then natural to ask whether the birationnal mapping that maps eigenvectors to eigenvalues is a resolution of singularities. We prove that this question has a positive answer in two situations : when the family is a 2-linear family and when the family is the full space $Sym_d(\mathbb{R})$, Section 5.3.

5.1 Projectivization

In this section, we introduce different sets made of symmetric matrices, eigenspaces and eigenvalues, and describe their elementary properties. Let A, $A \not\subset \Sigma$, be a k-linear family of $Sym_d(\mathbb{R})$, i.e., A is a linear subspace of dimension k of the $d \times d$ symmetric matrices. For short, we write $A(t)$ for the image of $\mathbb{R}^k \ni t \mapsto A(t) \in Sym_d(\mathbb{R})$ whenever A comes with a given linear parameterization.

5.1.1 Eigenelements and their projections

The **eigenelements** of A are the triples (A_0, λ, v) where A_0 range in A, λ is an eigenvalue of A_0 and v an eigenvector[1] of A_0 associated with λ. We denote the set of eigenelements of A by $\tilde{E}(A)$, or $\tilde{E}$ when A is clear from context. $\tilde{E}$ admits the following definition:

$$\tilde{E}(A) = \{(A_0, \lambda, v) \in Sym_d(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d;\ A_0 \in A,\ \det(A_0 - \lambda \mathbb{I}_d) = 0, (A_0 - \lambda \mathbb{I}_d)v = 0\},$$ [2]

which makes $\tilde{E}(A)$ an algebraic surface of dimension $k+1$ and degree d. [3]

Whenever $A = A(t)$ is parameterized always with the minimum number of parameters, we also refer to $\tilde{E}(A(t))$ for the similar set where the parameter of the family replaces the matrix entry:

$$\tilde{E}(A(t)) = \{(t, \lambda, v) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d;\ \det(A(t) - \lambda \mathbb{I}_d) = 0, (A(t) - \lambda)v = 0\}.$$

[1] Only in this subsection, we consider 0 as an eigenvector, even if it contradicts the usual definition, in order to get algebraic definitions below.

[2] Due to the fact that $A_0 v = \lambda v$ always holds when $v = 0$, $\tilde{E}$ cannot be simply described by the equation $(A - \lambda \mathbb{I}_d)v = 0$; this limitation disapears onces we consider projective sets.

[3] All the dimensions we give here and below are the generic ones. It might happen that the dimension behave badly. If A contains $\mathbb{I}_d$, the fiber of $E(A)$ over $(\mathbb{I}_d, 1)$ has dimension d, and d could be larger than k.

The sets we study in this chapter are various projections of $\tilde{E}$. Namely, we call **eigenvalues set** of the family A the image of $\tilde{E}(A)$ by the projection that forgets the eigenvectors entry, and **eigenvectors set** of the family A the image of $\tilde{E}(A)$ by the projection that forgets the eigenvalues entry. The eigenvalues set of A is denoted by $\tilde{\Lambda}(A)$ and given by

$$\tilde{\Lambda}(A) = \{(A_0, \lambda) \in Sym_d(\mathbb{R}) \times \mathbb{R};\ A_0 \in A,\ \det(A_0 - \lambda \mathbb{I}_d) = 0\},$$

and the eigenvector set is denoted by $\tilde{\Gamma}(A)$ and given by

$$\tilde{\Gamma}(A) = \{(A_0, v) \in Sym_d(\mathbb{R}) \times \mathbb{R}^d;\ A_0 \in A,\ A_0 v \wedge v = 0\}.$$

These two projections are algebraic sets, as it is shown by their equations. $\tilde{\Lambda}$ has dimension k and degree d, and $\tilde{\Gamma}$ dimension $k+1$ also and degree 3. For notation purpose, again we drop the dependence in A if the context allows, and, whenever A is parameterized, we make no distinction between the eigenvectors and eigenvalues sets of the family and the similar sets defined in terms of the parameters of the family instead of the matrix itself.

Finally, we call **eigenvectors portrait** of the family A, or **portrait** of A for short, the set denoted by $\tilde{V}(A)$ and made of the points in $\mathbb{R}^d$ that are eigenvectors of some member of the family, this is, the image of the eigenvectors set by the projection that forgets the matrix entry. Again, this projection happens to be an algebraic set. To see the later, fix a linear basis $A_1, \ldots, A_k$ of A. Then the vector v belongs to $\tilde{V}(A)$ if and only if there exists A_0 in A, and $\lambda \in \mathbb{R}$, such that $A_0 v = \lambda v$. So $v \in \tilde{V}(A)$ if and only if there are $(t_1, \ldots, t_k, \lambda) \in \mathbb{R}^{k+1}$, such that $t_1 A_1 v + t_2 A_2 v + \cdots + t_k A_k v - \lambda v = 0$. In other terms, $v \in \tilde{V}(A)$ iff the $k+1$ vectors $A_1 v, A_2 v, \ldots, A_k v, v$ are linearly dependent. Hence,

$$\tilde{V}(A) = \{v \in \mathbb{R}^d,\ rk(A_1 v, A_2 v, \ldots, A_k v, v) < k+1\}.$$

Since the inequality $rk(A_1 v, A_2 v, \ldots, A_k v, v) < k+1$ is equivalent to the nullity of all $(k+1) \times (k+1)$ minors of the matrix $(A_1 v, A_2 v, \ldots, A_k v, v)$ (it is always satisfied if $d < k+1$), the condition above shows the algebraicity of $\tilde{V}(A)$. The set $\tilde{V}(A)$ is then either $\mathbb{R}^d$ if $d < k+1$, or an algebraic set of degree $k+1$ and dimension less or equal than $k+1$.

To refer to the different projection mappings that links A and all the sets we have introduced, we use the symbol $\tilde{\pi}$ decorated with the source space as upscript and

target space as downscript. We then get $\tilde{\pi}_\Gamma^E : \tilde{E} \to \tilde{\Gamma}$, $\tilde{\pi}_V^E : \tilde{E} \to \tilde{V}$, $\tilde{\pi}_V^\Gamma : \tilde{\Gamma} \to \tilde{V}$, $\tilde{\pi}_\Lambda^E : \tilde{E} \to \tilde{\Lambda}, \tilde{\pi}_A^E : \tilde{E} \to A, \tilde{\pi}_A^\Lambda : \tilde{\Lambda} \to A, \tilde{\pi}_A^\Gamma : \tilde{\Gamma} \to A$.

It would be natural to define mappings between $\tilde{\Gamma}$ and $\tilde{\Lambda}$, since eigenvalues and eigenvectors are associated to each other. However, from eigenvectors to eigenvalues, the convention we adopt (the fact that we allow 0 to be an eigenvector), while needed to get algebraic sets, forbids us to construct such function, since 0 becomes an eigenvector associated to different eigenvalues (the mapping $\tilde{\pi}_\Gamma^E$ is not injective then can't be inverted). From eigenvalues to eigenvectors, the situation is even worse since full dimension of eigenvectors are associated to eigenvalues. These obstructions invite us to consider eigenspaces as subsets of projective instead of linear spaces, which will dispel unnecessary limitations.

5.1.2 Projective Eigenelements

Recall that, L being a linear space, the quotient $L \setminus \{0\}/\mathbb{R}^*$ of $L \setminus \{0\}$ by scalar invertible multiplication is called projective space associated to L. We denote it by $\mathbb{P}L$, except in the particular cases of projective spaces associated to $\mathbb{R}^{d+1}$, for which we keep the usual notation $\mathbb{P}^d$. If L is a linear space of dimension $d+1$, $\mathbb{P}L$ is an algebraic smooth compact variety of dimension d. When L is equipped with coordinates $x_1, \ldots, x_k$, $\mathbb{P}L$ has homogeneous coordinates denoted by $[x_1 : \cdots : x_k]$. We also denote by $[\cdot]$ the quotient operator. So $[x]$ denotes the class of $x \in L \setminus \{0\}$ in $\mathbb{P}L$, and $[X] \subset \mathbb{P}L$ is the image of $X \subset L$ by the quotient. We say that the quotient respects a subset $X \subset L$ if $X \setminus \{0\}$ is a union of fibers of the quotient, i.e., if X is invariant by invertible scalar multiplication.

The dual L^* of L, being isomorphic to L, is also associated to a projective space $\mathbb{P}L^*$ that is isomorphic to $\mathbb{P}L$. The choice of a non-degenerated bilinear form B determines all isomorphisms between L and L^*, via $x \mapsto B(x, .)$. Dealing with matrices of automorphisms that are self-adjoint for the canonical scalar product, we always identify $\mathbb{R}^{k+1}$ and its dual through the isomorphism defined by the canonical scalar product. Consequently, we equip $\mathbb{P}^k$ with an orthogonality relation ($[x] \perp [y]$ in $\mathbb{P}^k$ if and only if $< x, y >= 0$), and systematically identify $\mathbb{P}^k$ and $\mathbb{P}^{k*}$ through the isomorphism that is given by the canonical dot product. For instance, the orthogonal of a point p in $\mathbb{P}^2$ is, for us, the line $p^\perp$ in $\mathbb{P}^2$ made of the classes of all non-zero points of the plane of $\mathbb{R}^3$ that is orthogonal to the line in $\mathbb{R}^3$ whose non-zero points have class p in $\mathbb{P}^2$.

Let us now see how the former applies to our sets of eigenelements. As announced, eigenspaces are better understood as subsets of projective spaces. Indeed, if v is a non-zero eigenvector of A associated with λ, the whole class $[v]$ does. So the sets which we defined before have natural projective counterpart where the eigenvectors entries are replaced by projective coordinates. Similarly, since eigenvalues are homogeneous of degree one in terms of matrices, the couples $(A_0, \lambda) \in \tilde{\Lambda}$ admits coherent projective representative $[A_0 : \lambda] \in \mathbb{P}(Sym_d(\mathbb{R}) \times \mathbb{R}) \sim \mathbb{P}^{\frac{d(d+1)}{2}}$ when A_0 range in a linear space. Also, if the family $A = \{A(t);\ t \in \mathbb{R}^k\}$ has a linear parameterization, the couples $(A(t_0), \lambda)$ are homogeneous of degree one in terms of the parameterization, so the quotient $(A(t_0), \lambda) \in A \times \mathbb{R} \mapsto [t_0, \lambda] \in \mathbb{P}^k$ respects our set $\tilde{\Lambda}$. Finally, since eigenspaces are invariant by invertible scalar multiplication of the couple matrix, eigenvalue, the quotient $Sym_d(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{P}^{\frac{d(d+1)}{2}} \times \mathbb{P}^k$ (or $(\mathbb{R}^k \times \mathbb{R}) \times \mathbb{R}^d \to \mathbb{P}^k \times \mathbb{P}^{d-1}$ for parameterized families) respects the set $\tilde{E}$. The remaining projections, $\tilde{\pi}_A^\Gamma$ and $\tilde{\pi}_A^\Lambda$, also have projective counterparts, where the target space becomes $\mathbb{P}Sym_d(\mathbb{R})$ or $\mathbb{P}A = \mathbb{P}^{k-1}$.

To summarize, given a k-linear family of $d \times d$ symmetric matrices, we get a projective counterpart $E(A)$ of $\tilde{E}(A)$:

$$E(A) = \{([A_0 : \lambda], [v]) \in \mathbb{P}(Sym_d(\mathbb{R}) \times \mathbb{R}) \times \mathbb{P}^{d-1};\ [A_0] \in \mathbb{P}A,\ (A_0 - \lambda \mathbb{I}_d)v = 0\},^4$$

$$\text{or } E(A(t = (t_1, \dots, t_k))) = \{([t_1 : \cdots : t_k : \lambda], [v]) \in \mathbb{P}^k \times \mathbb{P}^{d-1};\ (A(t) - \lambda \mathbb{I}_d)v = 0\},$$

which is algebraic of degree 2, and has dimension $k - 1$. We have a projective counterpart $\Lambda(A)$ of $\tilde{\Lambda}(A)$:

$$\Lambda(A) = \{[A_0 : \lambda] \in \mathbb{P}(Sym_d(\mathbb{R}) \times \mathbb{R});\ A_0 \in A,\ \det(A_0 - \lambda \mathbb{I}_d) = 0\},$$

$$\text{or } \Lambda(A(t = (t_1, \dots, t_k))) = \{[t_1 : \cdots : t_k : \lambda] \in \mathbb{P}^k;\ \det(A(t) - \lambda \mathbb{I}_d) = 0\},$$

algebraic of degree d, with dimension $k - 1$, a projective counterpart $\Gamma(A)$ of $\tilde{\Gamma}(A)$:

$$\Gamma(A) = \{([A_0], [v]) \in \mathbb{P}Sym_d(\mathbb{R}) \times \mathbb{P}^{d-1};\ [A_0] \in \mathbb{P}A,\ A_0 v \wedge v = 0\},$$

$$\text{or } \Gamma(A(t)) = \{([t], [v]) \in \mathbb{P}^{k-1} \times \mathbb{P}^{d-1};\ A(t)v \wedge v = 0\},$$

[4] Notice that the unpleasant behavior of the non projective equation for $v = 0$ now disapears

algebraic of degree 3 and dimension $k-1$, a projective eigenvectors portrait $V(A)$:

$$V(A) = \{[v] \in \mathbb{P}^{d-1};\ rk(A_1 v, \ldots, A_k v, v) < k+1\}$$

(where $(A_1, \ldots, A_k)$ stands for a linear basis of A), algebraic of degree k with dimension $k-1$ if $d \geq k+1$ and $d-1$ otherwise. Except for V, all these sets map onto a projective family $[A]$ of dimension $k-1$. Notice that all equations that appear in our definitions are homogeneous in terms of their projective coordinates, and then, are well founded. We denote by the letter π with source upscript and target subscript the mapping that factors the corresponding $\tilde{\pi}$ through the quotient operators.

It is now possible to construct a mapping between eigenvectors set and eigenvalues set. Indeed, if $([A_0], [v])$ is given in $\Gamma(A)$, there is a unique point $([A_0' : \lambda], [v'])$ in $E(A)$ such that $[A_0'] = [A_0]$ and $[v] = [v']$. In fact, one has $[A_0 : \lambda] = [A_0 :< A_0 v, v >]$. In particular, there exists a unique map $\phi : \Gamma(A) \to \Lambda(A)$ such that the following diagram commute:

$$\begin{array}{ccc} E(A) & \xrightarrow{\pi_\Lambda^E} & \Lambda(A) \\ \downarrow{\scriptstyle \pi_\Gamma^E} & \nearrow{\scriptstyle \phi} & \\ \Gamma(A) & & \end{array}$$

The variety $\Gamma(A)$ is not smooth in general. For instance, if $A \subset Sym_3(\mathbb{R})$ is the 2-linear diagonal family given by $A(t_1, t_2) = t_1 \mathrm{diag}(1,1,0) + t_2 \mathrm{diag}(0,1,1)$, $\Gamma(A)$ is the union of the 6 lines $\ell_1, \ldots, \ell_6$ in $\mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\ell_1 = \mathbb{P}^1 \times \{[1:0:0]\}, \ell_2 = \mathbb{P}^1 \times \{[0:1:0]\},\ \ell_3 = \mathbb{P}^1 \times \{[0:0:1]\},$$

$$\ell_4 = \{[1:1]\} \times \{[v]; v_1 = 0\},\ \ell_5 = \{[0:1]\} \times \{[v]; v_2 = 0\},\ \ell_6 = \{[1:0]\} \times \{[v]; v_3 = 0\}.$$

Thus, $\Gamma(A)$ is singular at the 6 intersection points $\ell_i \cap \ell_j$ (that exist for $i = 1,2,3$ and $j = 4,5,6$ if $j \neq i+3$). In section 3, we will study a subset G of Γ, that we call **strict eigenvectors**, and we consider the question whether $G \xrightarrow{\Phi} \Lambda$ is a resolution of singularities for Λ. The set $G(A)$ is constructed by first removing from $\Gamma(A)$ all eigenvectors that appears for the singular matrices of A, and then take the topological closure. Precisely, suppose $A \subset Sym_d(\mathbb{R})$ is a k-linear family, and $A \not\subset \Sigma$. We set

$$G(A) = \mathrm{Clos}\left(\Gamma \setminus [\Sigma] \times \mathbb{P}^{d-1}\right). \tag{5.1}$$

In the former example, with $A(t_1, t_2) = t_1 \mathrm{diag}(1,1,0) + t_2 \mathrm{diag}(0,1,1)$, the set $G(A)$ is

the union of the three lines ℓ_1, ℓ_2, ℓ_3, which are the eigenvectors of the family A that matters, in the sense that all and only those appear as image of an analytic section of the mapping π_A^Γ.

Claim 5.1.1. *Let A be a k-linear family in $Sym_d(\mathbb{R})$, and suppose that $A \not\subset \Sigma$. Then the mapping $\Phi : G(A) \to \Lambda(A)$ is a proper birationnal map.*

Proof. Let $([t_1 : \cdots : t_k], [v]) \in G(A)$, then $\Phi([t_1 : \cdots : t_k], [v]) = [t_1 : \cdots : t_k : \lambda]$ with $\lambda = \frac{<A(t)v,v>}{||v||^2}$ is a rational map that is defined everywhere which belongs to $\Lambda(A)$. We show that it has as a rational inverse $\Psi : \Lambda(A) \to G(A)$. Let us take $[t_1 : \cdots : t_k : \lambda] \in \Lambda(A)$. The eigenspace of $A(t)$ associated to λ is the set of solutions of the $d \times d$ system

$$(t_1 A_1 + \cdots + t_k A_k - \lambda \mathbb{I}_d)v = 0 \tag{5.2}$$

Since $A \not\subset \Sigma$, for a generic point $t \in \mathbb{R}^k$, this system has rank $d-1$, so we can delete one equation from (5.2) and we obtain an equivalent system

$$B(t)v = 0, \tag{5.3}$$

where $B(t)$ is a matrix of dimension $(d-1) \times d$. Let M_i be the determinant of the $(d-1) \times (d-1)$ matrix obtained from $B(t)$ by deleting the i-th column. By Cramer's rule we obtain that

$$v = [-M_1(t) : \cdots : (-1)^k M_k(t) : \cdots : (-1)^d M_d(t)] \tag{5.4}$$

is a non-zero solution of (5.2). So, $\Psi([t_1 : \cdots : t_k : \lambda]) = ([t_1, \ldots, t_k,], [v])$ is defined on a generic point $[t : \lambda] \in \Lambda(A)$. Finally, since $G(A)$ is compact and Φ is continuous, Φ is proper. □

The relations between the sets and functions we introduced until now are described in the following diagram, where the quotient operator [.] is written as a dashed arrow between two algebraic sets.

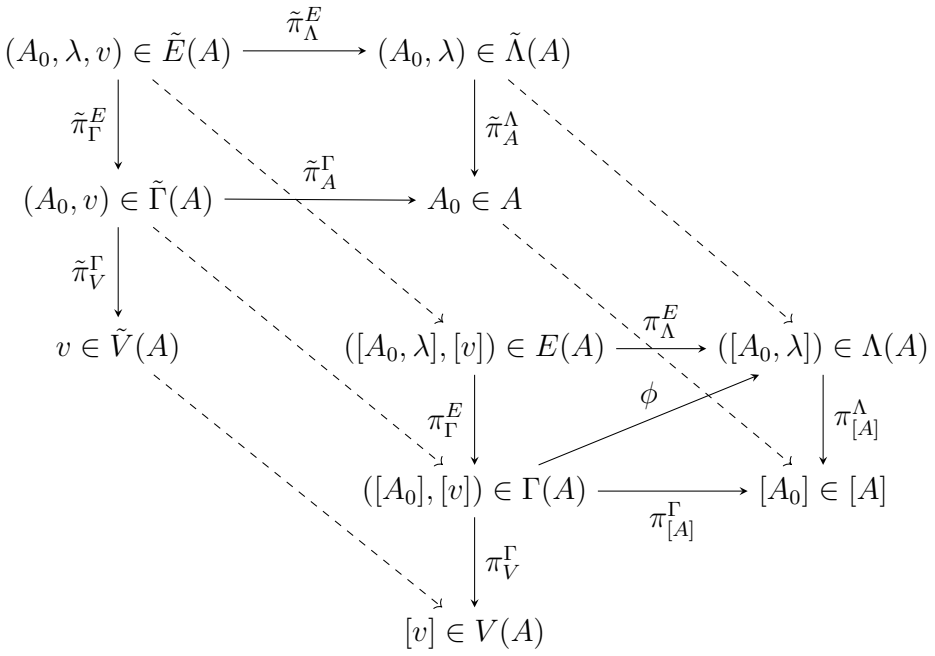


Figure 5.1: Projections

5.1.3 Rellich and monodromies

We finish this section by restating a version of Rellich Theorem in our framework, and use it to define the monodromies associated to the projective eigenelements.

Theorem 5.1.2 (Rellich). *Let $A \subset Sym_d(\mathbb{R})$ be a linear family, $A_0 \in A \setminus \Sigma$, and let $c : \mathbb{R} \to [A]$ be a parameterized analytic curve in $[A]$, with $c(0) = [A_0]$. Choose an eigenvalue $\lambda_0 \in Spec(A_0)$ and let v_0 be an associated eigenvector of A_0. Then c admits a unique analytic lift in $E(A)$ issued from $([A_0 : \lambda_0], [v_0])$ and $C(0) = ([A_0 : \lambda_0], [v_0])$.*

Proof. Let

$$E(A)|_c = \{(s, [B : \lambda], [v]) \in \mathbb{R} \times \mathbb{P}^k \times \mathbb{P}^{d-1} \ ; \ c(s) = [B], Bv = \lambda v\}.$$

Recall that we have a double analytic covering

$$Sym_d(\mathbb{R}) \supset \mathbb{S}^{\frac{d(d-1)}{2}} \to \mathbb{P}(Sym_d(\mathbb{R})),$$

so in particular, there exists a unique analytic $\tilde{c} : \mathbb{R} \to A \backslash \{0\}$ such that $[\tilde{c}] = c$ and $\tilde{c}(0) = A_0$. Since $\tilde{c}$ depends on one parameter, by Rellich Theorem, there exists

analytic functions $\lambda(s)$ and $v(s)$ such that $\lambda(0) = \lambda_0$ and $v(0) = v_0$. We define

$$\begin{array}{rcccc} C & : & \mathbb{R} & \to & E(A) \\ & & s & \mapsto & ([\tilde{c}(s) : \lambda(s)], [v(s)]) \end{array}$$

which is the analytic prolongation of λ_0 and v_0 along c in $E(A)$. We have $C(0) = ([A_0 : \lambda(0)], [v_0])$ and by unicity of analytic continuation this lift is unique. □

As in chapter 2, this version of Rellich Theorem allows to define a monodromy associated with the projective family and eigenvalues. Indeed, if $\ell : [0,1] \to [A] \setminus S_{[A]}$ (where $S_{[A]}$ is the image of the super-singular set of A by the projective quotient) and $[A_0 : \lambda] \in \Lambda(A)$ with $[A_0] = \ell(0)$, then ℓ has a unique lift ℓ_λ in $\Lambda(A)$ satisfying $\ell_\lambda(0) = [A_0 : \lambda]$. The lift ℓ_λ is not a loop in general. If $\ell_\lambda(1) = [A_0 : \mu]$, the function $\sigma_\ell : [A_0 : \lambda] \mapsto [A_0, \mu]$ defines a permutation of the fiber of Λ over $[A_0]$ that is called **projective monodromy of the eigenvalues** of A associated with ℓ. Replacing the given of an eigenvalue with an eigendirection $[v] \in \mathbb{P}^{d-1}$, one constructs a monodromy for eigenvectors. Denoting by $\ell_{[v]}$ the unique lift of ℓ in Γ with $\ell_{[v]}(0) = ([A_0], [v])$, and $\ell_{[v]}(1) = ([A_0], [w])$, the function $\delta_\ell : [v] \mapsto [w]$ defines a permutation of the fiber of Γ over $[A_0]$ that we call **projective monodromy of the eigenvectors** of A associated with ℓ.

It is clear that, if ℓ is the projectivization of a loop $\tilde{\ell} : [0,1] \to A \setminus S_A$, σ_ℓ is nothing but the projective image of the monodromy associated with $\tilde{\ell}$. However, not all loops in $[A] \setminus S_{[A]}$ are projectivization of loops in $A \setminus S_A$. For instance, if A_1, A_2 are independent symmetric matrices, $[0,1] \ni t \mapsto \cos(\pi t)A_1 + \sin(\pi t)A_2 \in Sym_d(\mathbb{R})$ is not a loop while $[0,1] \ni t \mapsto [\cos(\pi t)A_1 + \sin(\pi t)A_2] \in \mathbb{P}Sym_d(\mathbb{R})$ does. This explains that we use a different terminology, but we frequently drop the word "projective" for short when the context is clear. Actually, the projective monodromy truly differs from the linear monodromy since it recovers the antipodal monodromy. If $A(t)$ is a linear family and $A(t_0) \notin \Sigma$, $u \perp t_0$, then the projective image $[\gamma] : [0,1] \to [A]$ of the half turn γ from t_0 in the direction u is a loop in $[A]$, and is then associated to projective monodromies.

Claim 5.1.3. *Let $A(t)$ be a k-linear family. Suppose that $h : [0,1] \to A \setminus S_A$ is the half turn path from t_0 in the direction u, $u \perp t_0$ and $A(t_0) \notin \Sigma$. Then, the projective monodromy of the eigenvalues associated with $[h]$ is the projective image of the antipodal monodromy from t_0 in the direction u.*

Proof. Let $h : [0,1] \to A \backslash S_A$ be the half turn path from t_0 in the direction of u, σ be the antipodal monodromy and p the projective monodromy. Let $\lambda \in Spec(A_0)$ and denote by $pr_h(\lambda)$ the analytic prolongation of λ along h. Then, $\sigma(\lambda) = -pr_h(\lambda)$. So, the projective image of σ, $[\sigma]$, satisfy

$$[\sigma]([A(t_0) : \lambda]) = [A(t_0) : -pr_h(\lambda)].$$

On the other hand,

$$p([A(t_0) : \lambda]) = [-A(t_0) : pr_h(\lambda)] = [\sigma]([A(t_0) : \lambda]),$$

which finishes the proof. □

We therefore keep the terminology **antipodal monodromy** for the projective monodromy of eigenvalues associated to half turns, and extend the notion and define the **antipodal monodromy of eigenvectors**. Notice that, eigenvalues and eigenvectors being associated one to the other, the cycle structure of the antipodal monodromy is the same for eigenvalues and eigenvectors. There is a small glitch that must be revealed here: an antipodal monodromy for eigenvectors could take into account an orientation of the direction $[v]$, that we do not consider here. Our antipodal monodromy of eigenvectors is rather an antipodal monodromy of eigendirections. However, one occurrence of this oriented antipodal monodromy appears as a remark (Remark 5.2.24) in section 2 of this chapter.

5.2 Cubics of eigenelements

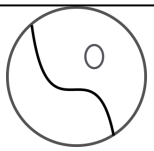
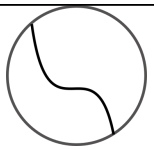
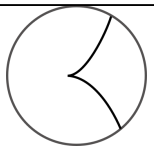
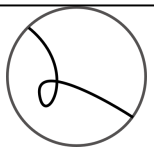
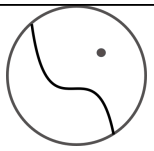
We have seen in the former section that for a given k-family $A(t)$ of $d \times d$ symmetric matrices, the set $\Lambda(A)$ is an algebraic set of degree d and dimension $k-1$ in $\mathbb{P}^k$, and the portrait $V(A)$ is an algebraic set of degree $k+1$ and dimension $k-1$ in $\mathbb{P}^{d-1}$ if $d \geq k+1$. Many reasons invite us to focus on the case $k = 2$, $d = 3$. Indeed, for $k = 1$, the projective family is one point, and for $k = 2$, $d < 3$, the portrait is the full space, so the case $k = 2, d = 3$ minimizes the values of k and d among the interesting ones, then probably, minimizes the complexity. Also, and more generally, $k = d - 1$ makes $V(A)$ an hypersurface, same as $\Lambda(A)$, which makes these cases of a prior interest. Finally, for $k = 2, d = 3$ the couple $(V(A), \Lambda(A))$ is made of two cubic curves of $\mathbb{P}^2$. The classification of cubics being sufficiently restraint, it is possible to go through entirely, but it is still large enough to allow interesting phenomenon to

occur.

5.2.1 Marked cubic types and results

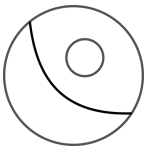
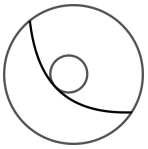
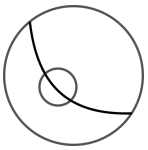
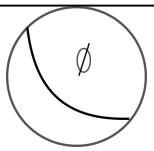
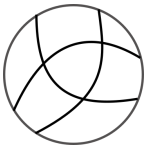
Our goal here is to give a complete classification of the pair $(V(A), \Lambda(A))$ that can be encountered where A is a 2-family of 3×3 symmetric matrices. For this we define a notion of marked cubic type. It is the invariant by which we classify. We follow the projective classification of cubic curves below of $\mathbb{P}^2$, which contains 16 configurations. Each configuration is the union of finitely (at most 3) analytic components (denoted by $(C_i)_{i \in I_C}$). To each component, we associate a number among $0, 1, 2, 3$, which will represent the number of eigenelements that belongs to a component for $A_0 \in A \setminus \Sigma$. We give in the following table the explicit list of all cubic types, together with their decomposition into components. We also associate a symbolic picture to each cubic and component, which will be used in our classification instead of the (unnecessary heavy) explicit description.

Table 5.1: Table of cubics types.

Cubic types	**Components**	**Example**	**Symbol**
Elliptic with an oval	Oval:  Pseudo-line: 	$x^3 - zy^2 - 3xz^2 = 0$	
Elliptic connected	Pseudo-line: 	$x^3 - y^2z - xz^2 - z^3 = 0$	
Rational cuspidal	Cusp: 	$x^3 - y^2z = 0$	
Rational crunodal	Crunodal: 	$x^3 - y^2z - 3xz^2 + 2z^3 = 0$	
Rational acnodal	Point: Pseudo-line: 	$zy^2 - x^2(x - z) = 0$	

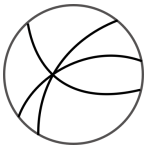
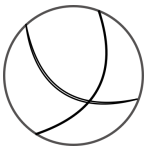
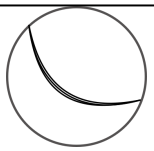
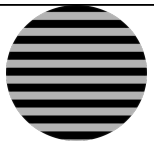
Continued on next page

Table 5.1 – *Continued from previous page*

Cubic types	**Components**	**Example**	**Symbol**
Disjoint union of a non degenerate conic and a line	Conic: ○ Line: ⌒	$(x - 2y - 4z)(x^2 + y^2 - z^2) = 0$	
Union of a non degenerate conic and a tangent line	Conic: ○ Line: ⌒	$(y - z)(x^2 + y^2 - z^2) = 0$	
Union of a non degenerate conic and a secant line	Conic: ○ Line: ⌒	$y(x^2 + y^2 - z^2) = 0$	
Product of a line and an empty conic	Line: ⌒	$x(x^2 + y^2 + z^2) = 0$	
Disjoint union of a conic point and a line	Point: • Line: ⌒	$(x - 2y - 4z)(x^2+y^2) = 0$	
Product of a line and a conic point on the line	Point: • Line: ⌒	$(x - 2y - 4z)((x^2+(y+2z)^2) = 0$	
Union of three lines in general position	Line: ⌒ Line: ⌒ Line: ⌒	$(x - 2y + 4z)(2x - y + 5z)(x + y - 3z) = 0$	

Continued on next page

Table 5.1 – *Continued from previous page*

Cubic types	**Components**	**Example**	**Symbol**
Three concurrent lines	Line:  Line:  Line: 	$(2x - 3y + 5z)(3x + 4y - 7z)(9x - 5y + 8z) = 0$	
Product of a double line and a single line	Double line:  Line: 	$(2x - 3y + 5)(3x + 4y - 7)^2 = 0$	
A triple line	Triple line: 	$x^3 = 0$	
$\mathbb{P}^2$	$\mathbb{P}^2$: 	$0 = 0$	

Definition 5.2.1. *A* **marked cubic type** *is the given of a cubic configuration C (an entry in the configuration list above) with components $(C_i)_{i\in I_C}$ together with a map $s : (C_i)_{i\in I_C} \to \{0, 1, 2, 3\}$.*

To associate a couple of marked cubics to A, we remark the following :

Lemma 5.2.2 (Eigenvalues type)**.** *Let A be a 2-family of 3×3 symmetric matrices, $A \not\subset \Sigma$. Let C be a component of $\Lambda(A)$. Then, the cardinality $Card(C \cap (\pi_A^\Lambda)^{-1}([A_0]))$ of the fiber of C over $[A_0]$ is at most 3 and does not depend on $A_0 \in A \setminus \Sigma$.*

Proof. Since $\chi_{A(t)}$ is a hyperbolic polynomial of degree 3, there are at most three points in the fiber of $\Lambda(A)$ over $[A_0]$.

Let $A_0 \neq B_0$ be two points in $[A]\backslash[\Sigma]$. Denote by ℓ, the analytic parameterization of $[A]$ given by

$$\begin{array}{rcccc} \ell & : & \mathbb{R} & \to & [A] \\ & & t & \mapsto & \cos(t)A_0 + \sin(t)B_0 \end{array}$$

Since $A_0 \notin \Sigma$, there exists $\lambda_1, \lambda_2, \lambda_3$ such that

$$\Lambda(A) \cap (\pi_A^\Lambda)^{-1}(A_0) = \{[A_0 : \lambda_1], [A_0 : \lambda_2], [A_0 : \lambda_3]\}.$$

Then, by Rellich Theorem, there are three unique lifts

$$\begin{array}{rcccc} \ell_i & : & \mathbb{R} & \to & \Lambda(A) \\ & & t & \mapsto & [A(t) : \lambda_i(t)] \end{array}$$

such that $\lambda_i(0) = \lambda_i$.

Suppose that $[A_0 : \lambda_i] \in C$. Then, there exists a neighborhood V of $[A_0 : \lambda_i]$ such that $V \cap C = \Lambda \cap C$, since $A_0 \notin \Sigma$. Then, λ_i being analytic, there exists $\varepsilon > 0$ such that for all $t \in (-\varepsilon, +\varepsilon)$, $[A(t) : \lambda_i(t)] \in C$. But, C is an analytic set and λ_i is analytic, so for all $t \in \mathbb{R}$, $[A(t) : \lambda_i(t)] \in C$. For $t = \frac{\pi}{2}$, this gives $[B_0 : \lambda_i(\frac{\pi}{2})] \in C$. Then,

$$Card(C \cap (\pi_A^\Lambda)^{-1}(B_0)) \geq Card(C \cap (\pi_A^\Lambda)^{-1}(A_0)).$$

By exchanging the role of A_0 and B_0, we get

$$Card(C \cap (\pi_A^\Lambda)^{-1}(B_0)) \leq Card(C \cap (\pi_A^\Lambda)^{-1}(A_0))$$

which shows that this number does not depend on A_0. This finishes the proof. □

Lemma 5.2.3 (Eigenvectors type,)**.** *Let A be a 2-family of 3×3 symmetric matrices, $A \not\subset \Sigma$. Let C be a component of $V(A)$. Then, the cardinality $Card(C \cap (\pi_A^\Gamma)^{-1}(A_0))$ is at most 3 and does not depend on $A_0 \in A \setminus \Sigma$.*

Proof. A regular symmetric matrix of dimension 3 has three eigendirection. The same principle of the proof of Lemma 5.2.2 shows that $Card(C' \cap (\pi_A^\Gamma)^{-1}(A_0))$ does not depend on $A_0 \notin A \setminus \Sigma$ where C' is a component of Γ. The result then follows from the fact that any component of $V(A)$ is the union of the images of certain components of Γ. □

From lemmas 5.2.2 and 5.2.3, if A is a 2-family of 3×3 symmetric matrices that is not included in Σ, one can associate a marked cubic type for eigenvalues (resp. eigenvectors), by defining a function s which associates to each components C_i of Λ (resp. of V), the number of eigenvalues (resp. eigenvectors) that belongs to C_i, for a regular member of the family. We respectively call those marked cubic types **eigenvalues type** and **eigenvectors type** of the family.

We can now state the main result of this section.

Theorem 5.2.4. *Let A be a 2-family of 3×3 symmetric matrices, $A \not\subset \Sigma$. Then the eigenvectors type of A determines the eigenvalues type of A. More precisely, the following 9 couples of marked cubics types, and only those, appear as eigenvectors and eigenvalues types.*

Table 5.2: Table of cubics types.

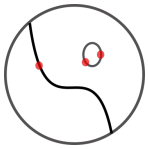
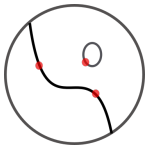
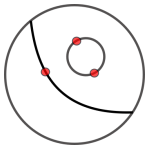
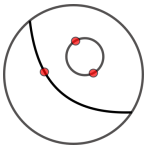
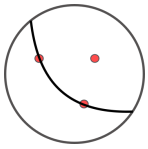
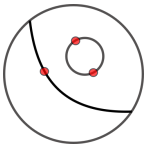
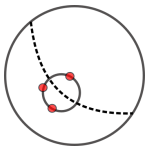
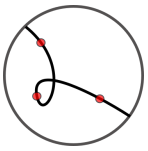
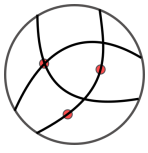
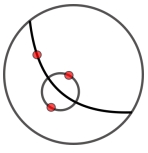
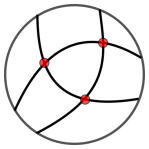
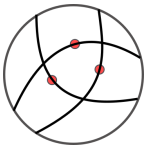
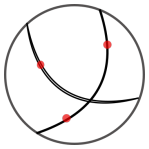
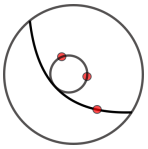
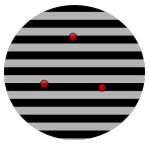
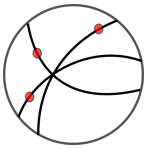
Marked cubic of eigenvectors	***Marked cubic of eigenvalues***
	
	
	
	
	
	
Continued on next page	

Table 5.2 – Continued from previous page

Marked cubic of eigenvectors	***Marked cubic of eigenvalues***
	
	
	

The proof goes as follows. In subsection 5.2.2, we give a collection of lemmas in which our proof is based on. In subsection 5.2.3, we enumerate all combinatoric candidates for the eigenvectors type, and either we eliminate the case from the lemmas, or we deduce the corresponding eigenvalues type. Finally, in Subsection 5.2.4, we construct an explicit family that have these prescribed eigenvectors and eigenvalues types.

5.2.2 Cubic types properties

In this subsection, we give some properties of eigenvectors types and eigenvalues types that will be sufficient for us to eliminate all marked cubic types that cannot be encountered as types of linear families. We first need to introduce the analogous of the set $G(A)$ in $V(A)$, this is, the set of strict eigenvectors. For this, we call **regular eigenvector** of A a point in $V(A)$ that is the eigenvector of a regular member $A_0 \in A \setminus \Sigma$ of A:

$$V_{reg}(A) = \{[v] \in \mathbb{P}^2;\ \exists A_0 \in A \setminus \Sigma,\ A_0 v \wedge v = 0\},$$

and a **virtual eigevector** is an eigenvector associated to multiple eigenvalues:

$$V_{vir}(A) = \{[v] \in \mathbb{P}^2;\ \exists([A_0, \lambda], [v]) \in E,\ \dim((\pi_\Lambda^E)^{-1}([A_0, \lambda])) > 0\}.$$

Notice that a regular eigenvector might also be virtual (if it is associated to both regular and singular elements of A), and that a non-virtual eigenvector might not be regular (if it is associated to the simple eigenvalue of an element in $A \cap \Sigma_2$). Finally, a **strict eigenvector** is an eigenvector that can be approached in $V_{reg}(A)$. So, the set of strict eigenvectors $V_{str}(A)$ is the topological closure of $V_{reg}(A)$. Notice that $V_{str}(A)$ is nothing but the image of $G(A)$ by π_V^Γ:

$$V_{str}(A) = \mathrm{Clos}(V_{reg}(A)) = \pi_V^\Gamma(G(A)). \tag{5.5}$$

Intersecting Σ has a deep influence in $V(A)$, since then, $V(A)$ has a linear factor, and in particular, V is reducible.

Fact 5.2.5. *Let $A \in Sym_3(\mathbb{R})$ be a 2-linear family, $A \not\subset \Sigma$. If $[A] \cap [\Sigma] \neq \emptyset$, $V(A)$ contains a line.*

Proof. If $[A_0] \in [A] \cap [\Sigma]$, A_0 has a multiple eigenvalue, whose associated eigenspace has dimension at least 2. The projective quotient of this eigenspace contains a projective line. □

Virtual eigenvectors come by packs of projective lines. There are at most 3 of them, if the family does not meet $[\mathbb{I}_d]$.

Fact 5.2.6. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \not\subset \Sigma$. Then, V_{vir} is a union of at most 3 lines, or $V_{vir} = \mathbb{P}^2$.*

Proof. In an affine chart, the discriminant of the characteristic polynomial has degree 6 and its roots are double, then it has at most three roots. The dimension of the associated eigenspaces is 2 or 3. Thus, either $V_{vir} = \mathbb{P}^2$, or it is the union of at most three lines. □

Lemma 5.2.7. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family such that $A \not\subset \Sigma$. Then, $V(A) = \mathbb{P}^2$ if and only if there exists a parameter t_0 such that $\lambda\mathbb{I}_3 \in A(t_0)$.*

Proof. If there exists a parameter $t_0 \in \mathbb{R}^2$ such that $\lambda\mathbb{I}_3 \in A(t_0)$, then the associated eigenspace is $\mathbb{R}^3$. So, $V(A) = \mathbb{P}^2$.

Note that the fiber of each element of $[A \backslash \Sigma]$, $[A \cap \Sigma_2]$ and $[A \cap \Sigma_3]$ in $\Gamma(A)$ has dimension 0, 1 and 2, respectively. Since $A \not\subset \Sigma$, the dimension of $A \cap \Sigma_2$ and $A \cap \Sigma_3$

is at most 0. We have $\dim(V(A)) \leq \dim(\Gamma(A))$ and

$$\dim(\Gamma(A)) = \max(\dim[A\backslash\Sigma] + 0, \dim[A \cap \Sigma_2] + 1, \dim[A \cap \Sigma_3] + 2).$$

Thus, $V(A) = \mathbb{P}^2$ gives $\dim[A \cap \Sigma_3] = 0$. This means that there exists a parameter t_0 such that $A(t_0) \in \Sigma_3$ which finishes the proof. □

Lemma 5.2.8. *Let $A(t) \in Sym_d(\mathbb{R})$ be a diagonal 2-linear family such that $A(t) \not\subset \Sigma$. Then, either $V(A)$ is the union of three lines in general position, or $V(A) = \mathbb{P}^2$.*

Proof. Let $A(t) = t_1A_1 + t_2A_2$ with $A_1 = diag(a, b, c)$, $A_2 = diag(a', b', c')$. Then, $V(A) = \{[x : y : z] \in \mathbb{P}^2 \mid kxyz = 0\}$ with

$$k = \begin{vmatrix} a & a' & 1 \\ b & b' & 1 \\ c & c' & 1 \end{vmatrix}.$$

If

$$rk\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 3,$$

then $\mathbb{I}_3 \notin A(t)$ and $V(A)$ is the union of three lines $x = 0$, $y = 0$, $z = 0$. Otherwise, $\mathbb{I}_3 \in A(t)$ which means $V(A) = \mathbb{P}^2$. This, finishes the proof. □

We notice that if for two different parameters, e is an eigenvector of $A(t)$, then it is an eigenvector of the family for all parameters.

Fact 5.2.9. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \not\subset \Sigma$. If e is an eigenvector of both $A(t_1), A(t_2)$ for independent t_1, t_2, then e is an eigenvector of $A(t)$ for all t. In this case, the associated eigenvalue is linear, and Λ contains a line.*

Proof. For a given $t \in \mathbb{R}^2$, set $t = \alpha(t)t_1 + \beta(t)t_2$, so α and β are linear in terms of t. Then, we have $A(t) = \alpha(t)A(t_1) + \beta(t)A(t_2)$. Thus,

$$A(t)e = (\alpha(t)\lambda_1 + \beta(t)\lambda_2)e,$$

where λ_1 and λ_2 are the respective eigenvalues associated with e for $A(t_1)$ and $A(t_2)$. So for all t, e is an eigenvector of $A(t)$ associated with the linear eigenvalue $\alpha(t)\lambda_1 +$

$\beta(t)\lambda_2$. The set

$$\{[t : \alpha(t)\lambda_1 + \beta(t)\lambda_2] \in \mathbb{P}^2;\ t \in \mathbb{R}^2 \setminus \{0\}\}$$

is included in Λ and is a projective line. □

Recall that the eigenspaces of a symmetric matrix are orthogonal.

Fact 5.2.10. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$. Then, for all parameters $t \in \mathbb{R}^2 \setminus \Sigma$, the three eigenvectors of $A(t)$ are two by two orthogonal.*

Lemma 5.2.11. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$, and let C be a component of $V(A)$. If $C \notin V_{vir}$, then $C \subset V_{str}$.*

Proof. By Fact 5.2.6, V_{vir} is the union of at most 3 lines or $V_{vir} = \mathbb{P}^2$. But, $C \notin V_{vir}$, so $V_{vir} \neq \mathbb{P}^2$. Let $\tilde{C} \in \Gamma(A)$ be the projective preimage of the component C of $V(A)$, $\tilde{C} = {\pi_V^\Gamma}^{-1}(C)$. Moreover, $\text{Clos}(\tilde{C} \backslash \Sigma) = \tilde{C}$, since Σ has codimension 1 in A, But, by the definition of strict eigenvectors of $\Gamma(A)$, Equation (5.1), we have $\text{Clos}(\tilde{C} \backslash \Sigma) \subset G(A)$. We have $V_{str}(A) = \pi_V^\Gamma(G(A))$, Equation (5.5). Finally, since $\tilde{C} \subset G(A)$, so $C \subset V_{str}$ which finishes the proof. □

Lemma 5.2.12. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$, and suppose $V(A) \neq \mathbb{P}^2$. If ℓ is a line included in $V(A)$, then $\ell \cap V_{str}$ contains at least two different points.*

Proof. Let $\mathbb{R} \ni s \mapsto [B(s)]$, be a parameterization of $[A]$. If $\ell \subset V_{str}$, then $\ell \cap V_{str}$ contains at least two different points. If $\ell \not\subset V_{str}$, then by Lemma 5.2.11, $\ell \subset V_{vir}$. Then, there exists a parameter s_0 and λ such that $\ell = \ker(B(s_0) - \lambda \mathbb{I}_d)$. Since $A \notin \Sigma$, for small $\varepsilon \neq 0$, $B(s_0 + \varepsilon) \notin \Sigma$. Let $\lambda_1(s_0 + \varepsilon)$, $\lambda_2(s_0 + \varepsilon)$ and $\lambda_3(s_0 + \varepsilon)$ be the analytic eigenvalues of $B(s_0 + \varepsilon)$. Up to change of indices, we suppose that $\lim_{\varepsilon \to 0} \lambda_i(s_0 + \varepsilon) = \lambda$, $i = 1, 2$. Let $e_1(s_0 + \varepsilon)$ and $e_2(s_0 + \varepsilon)$ be eigenvectors associated to the eigenvalues $\lambda_1(s_0 + \varepsilon)$ and $\lambda_2(s_0 + \varepsilon)$, respectively. Note that for all small ε, $e_i(s_0 + \varepsilon) \in V_{str}$, $i = 1, 2$. Since, $e_i(s_0)$, $i = 1, 2$, is an eigenvector of $B(s_0)$ associated to λ and $\ell = \ker(A(s_0) - \lambda \mathbb{I}_d)$, $e_i(s_0) \in \ell$. We have $\lim_{\varepsilon \to 0} e_i(s_0 + \varepsilon) = e_i(s_0)$, $i = 1, 2$, so $e_i(s_0) \in V_{str}$. Since for $\varepsilon \neq 0$, $< e_1(s_0 + \varepsilon), e_2(s_0 + \varepsilon) >= 0$ and the inner product function is continuous, we get $e_1(s_0) \perp e_2(s_0)$. This means $\ell \cap V_{str}$ contains at least two different points, which finishes the proof. □

We prove that the components of $V(A)$ that contains non-virtual eigenvectors are fully covered by strict eigenvectors. Indeed, these components are exactly the analytic

prolongation of them.

Lemma 5.2.13. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$. Let C be a component of $V(A)$ and $e \in C \setminus V_{vir}$. Then C is the image of the analytic prolongation of e.*

Proof. Let $s \mapsto [B(s)]$ be a 1-periodic parameterization $[A]$, and denote by $[e_1(s)]$, $[e_2(s)]$ and $[e_3(s)]$ the analytic eigenvectors of $[B(s)]$. Set

$$I_i = \{([B(s)], [e_i(s)]),\ s \in \mathbb{R}\},\ i = 1, 2, 3. \tag{5.6}$$

Since I_1, I_2 and I_3 are connected, and $G = \cup_{k=1}^{3} I_k$, each component of G is a union of some I_k. Moreover, if two I_k have non-empty intersection, they coincide, since each I_k is the analytic prolongation of each of its elements. So, the components of G are exactly the I_k (some I_k might coincide).

Recall that C is the image of a union of the analytic components of Γ by π_V^Γ. Since Since $e \in C \setminus V_{vir}$, $e \in C \setminus V_{str}$, Lemma 5.2.11. So, these components lies in G. Say $C = \bigcup_{j \in J} \pi_V^\Gamma(I_j)$. We claim that this union is disjoint. Indeed, if $v \in \pi_V^\Gamma(I_1) \cap \pi_V^\Gamma(I_2)$, $I_1 \neq I_2$. Then, there exists $s_1 \neq s_2$ such that $([B(s_1)], [v]) \in I_1$ and $([B(s_2)], [v]) \in I_2$, then $[v]$ is an eigenvector of $B(s)$ for all $s \in \mathbb{R}$, Lemma 5.2.9. So, $[A] \times \{[v]\}$ is a component of G, then $I_1 = [A] \times \{[v]\} = I_2$ which is forbidden. Thus,

$$C = \dot{\cup}_{j \in J}\ \pi_V^\Gamma(I_j). \tag{5.7}$$

But, the $\pi_V^\Gamma(I_j)$ are closed and C is connected, so $C = \pi_V^\Gamma(I_{j_1})$. In the other words, C is the image of the analytic prolongation of each of its points. This finishes the proof. □

Lemma 5.2.14. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$. If $e \in V_{reg} \cap V_{vir}$, then either $e \in V_{str}$, or for all t, $([A(t)], [e]) \in \Gamma$.*

Proof. Let $\mathbb{R} \ni s \mapsto [B(s)]$ be a parameterization of $[A]$ with $\ell(0) = [B(s_0)]$. Since e is a regular eigenvector, it has unique analytic prolongation $e(s)$, $e(s_0) = e$. If $e(s)$ is not constant, let C be the component of $V(A)$ which contains $e(s)$. For small $s \neq 0$, $B(s_0 + s) \notin \Sigma$, so $C \not\subset V_{vir}$. Then, $C \subset V_{str}$, Lemma 5.2.11, and $e \in V_{str}$. □

Proposition 5.2.15. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$. Suppose $V(A) \neq \mathbb{P}^2$. If $V(A)$ is irreducible, it has at least two components.*

Proof. Being irreducible, V contains no line, so from Fact 5.2.6, $V_{vir} = \emptyset$, then $A \cap \Sigma = \emptyset$.

Suppose V has only one component. Denote by $\lambda_1, \lambda_2, \lambda_3$ the three (different) eigenvalues of $A(1,0)$, and respectively by $e_1, e_2, e_3 \in \mathbb{P}^2$ the associated eigenvectors. The projectivized 1-periodic half-turn

$$p : \mathbb{R} \ni s \mapsto [\cos(\pi s)A(1,0) + \sin(\pi s)A(0,1)] \in [A]$$

is a parameterization of $[A]$, bijective when reduced to $[0,1)$, and has three different lifts in Γ, $(p(t), e_1(t))$, $(p(t), e_2(t))$, $(p(t), e_3(t))$, which satisfy $e_i(0) = e_i$ for $i = 1, 2, 3$.

We claim that for each i, $e_i(1) \neq e_i(0)$ and prove it for $i = 1$. If $e_1(1) = e_1$, then $e_1(t)$ is 1-periodic. But V is the image of $e_1(t)$ from Fact 5.2.13. So there exists $t \in (0,1)$, $e_1(t) = e_2$. So e_2 is an eigenvector of $A(1,0)$ and of $A(\cos(t), \sin(t))$, then of the whole family by Fact 5.2.9. In particular, the analytic continuation $e_2(t)$ of e_2 is constant, which contradicts that V is the image of $e_2(t)$ from Fact 5.2.13. Hence, $e_1(1) \neq e_1$.

Since all permutations of 3 elements that have no fixed points are cycles of length 3, the antipodal monodromy of eigenvectors of family is a cycle. From this, the antipodal monodromy of the eigenvalues of family is a cycle of length three. But, a cyclic antipodal monodromy needs a non empty intersection with Σ, which is a contradiction. □

Proposition 5.2.16. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \not\subset \Sigma$, and C_1, C_2 be two components of V_{str}. Then $C_1 \cap C_2 = \emptyset$.*

Proof. Recall that $G(A)$ is the disjoint union of its components, that have disjoint intersection by π_V^Γ. Since, C_1 and C_2 are components of V_{str}, they are the images of the components of $G(A)$ by π_V^Γ. So, $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$. □

Fact 5.2.17. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \not\subset \Sigma$. Then Λ belongs to the following list:*

1. *Elliptic with an oval: two eigenvalues on the oval and one on the pseudo-line.*
2. *Rational crunodal: three eigenvalues on the rational crunodal.*
3. *Union of a non-degenerated conic and a (disjoint, secant or tangent) line: Two eigenvalues on the conic and one on the line.*

4. *Union of three lines in general position or concurrent: each line contains one eigenvalue.*

Proof. Since $A \notin \Sigma$, in an affine chart, Λ is the union of three graphs of eigenvalues and there exists a parameter t_0 with $A(t_0) \notin \Sigma$. Now, since the characteristic polynomial of $A(t)$ is a hyperbolic polynomial of degree 3, there exists a point $p \notin \Lambda(A)$ such that all the lines passing through p intersect $\Lambda(A)$ in exactly three points. Thus, it remains only four possibilities for the cubic configuration of Λ: elliptic with an oval, rational crunodal, non degenerate conic with a line, three lines. Since in an affine chart, there are three eigenvalues lying in different graphs for non-singular parameters, it is sufficient to associate the number of eigenvalues to each component. □

Proposition 5.2.18. *Let $A(t) \in Sym_3(\mathbb{R})$ be a 2-linear family, $A(t) \notin \Sigma$. If $\Lambda(A)$ contains a line, V_{str} contains a line or an isolated point.*

Proof. By Corollary 4.1.8, an eigenvector associated to a linear eigenvalue is restrained to an hyperplane, this means a line here. If the line is not covered by the eigenvector, it is a constant isolated eigenvector. □

5.2.3 Proof of Theorem 5.2.4: forbidden types

In this subsection we prove the first part of Theorem 5.2.4. For this, we go through the full combinatoric of marked cubics, and discuss the compatibility of the given marked cubic with the properties of eigenvectors types that we saw in the former subsection. If the marked cubic is not found to be forbidden as an eigenvectors type, we show that it can be associated to only one eigenvalues type. An example of each type that is not proven to be forbidden is given in Subsection 5.2.4. For a fast journey through the enumeration, see Table 5.12 at the end of the section.

To organize the enumeration, we proceed as follows. We first pick up a cubic configuration to be the configuration of V. Then for each line $\ell \subset V$, we decide whether $\ell \subset V_{vir}$. In the pictures, a line in V_{vir} is symbolized by a doted line. Then, from Lemma 5.2.13, the remaining components contains at least 1 eigenvector, so we associate a number in $1 \dots 3$ to them, keeping in mind that the total number of regular eigenvectors is at most 3. If this total is smaller than 3, we associate the remaining ones to virtual lines. Below, we discuss all marked cubic types that satisfy these combinatoric constraints.

I) V **is elliptic with an oval.** There are two possibilities for the eigenvectors type, described in table 5.3.

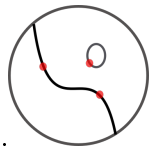
Cubic configuration	**Marked cubic type**
I) 	(a).  (b). 

Table 5.3: Elliptic cubic with an oval

It happens that both cases are realizable. We discuss the eigenvalues type. In both cases, the fact that V_{str} has two components determines an antipodal monodromy made of one cycle of length 2 and one fixed point. Beside, since V is irreducible, $[A] \cap \Sigma = \emptyset$. So, the corresponding eigenvalues type must have two components that do not intersect. From Fact 5.2.17, it is either elliptic with an oval or a disjoint union of a conic with a line. By Proposition 5.2.18, the later must be removed since V_{str} does not have a line or an isolated point. So the eigenvalue type is prescribed: it is elliptic with an oval.

(a) The oval has two non-virtual eigenvectors and the pseudo-line one, **Example 5.2.20**.

(b) The pseudo-line has two non-virtual eigenvectors and the oval one, **Example 5.2.21**.

II) V **is elliptic connected,** ...

III) ... **or** V **is rational crunodal,** ...

IV) ... **or** V **is rational cuspidal**, Table 5.4.

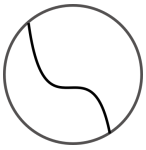
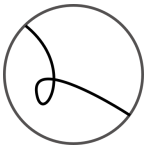
Cubic configurations
II)  III)  IV) 

Table 5.4: One component irreducible cubic types

In all of these cases, V is irreducible and has one component, which contradicts Proposition 5.2.15. The three cases are impossible.

V) V **is rational acnodal**, Figure 5.2.

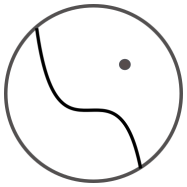


Figure 5.2: Cubic type V

The case is impossible. Indeed, V has no virtual eigenvectors, Fact 5.2.6. Then from 5.2.11, the isolated point p is an eigenvector for all parameters. By Fact 5.2.10, the other eigenvectors are restrained to $p^{\perp}$. This contradicts Fact 5.2.13, that the pseudo-line component is covered by regular eigenvectors, while it is not included in $p^{\perp}$.

VI) V **is the disjoint union of a non-degenerate conic and a line.** There are many subcases summarized in table 5.5 below.

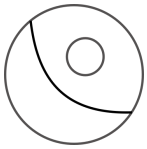
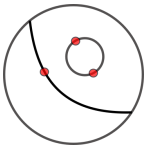
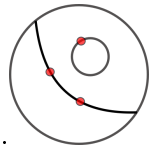
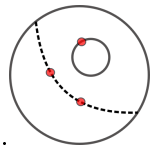
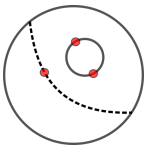
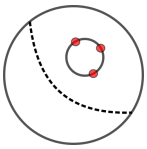
Cubic configuration	**Marked cubic type**
VI) 	(a).  (b).  (c).  (d).  (e). 

Table 5.5: Disjoint union of a non-degenerate conic and a line

(a) The line is not in V_{vir} and has one eigenvector, the cubic has two eigenvectors.

The case exists, we discuss the eigenvalues. The corresponding eigenvalues

type must have two components that do not intersect. By Fact 5.2.17, there are two possibilities. Either elliptic with an oval, or the disjoint union of a conic with a line. We show that the first case is impossible:

Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the orthonormal basis of $\mathbb{R}^3$ and suppose that the projective line ℓ of the cubic is generated by the two vectors e_2 and e_3. Thus, for each parameter, a vector of the form $[0 : y : z]$ belongs to the line ℓ. Since for all parameters, the line ℓ is covered by a strict eigenvector, there exists two different parameters t_1, t_2 such that e_3 and e_2 are regular eigenvectors of $A(t_1)$ and $A(t_2)$, respectively. We suppose $t_1 = (1,0)$ and $t_2 = (1,0)$ and we write

$$A(t_1,t_2) = t_1 \begin{pmatrix} \alpha & \gamma & 0 \\ \gamma & \beta & 0 \\ 0 & 0 & \lambda(1,0) \end{pmatrix} + t_2 \begin{pmatrix} a & 0 & b \\ 0 & \lambda(0,1) & 0 \\ b & 0 & c \end{pmatrix}.$$

$v = [0 : y : z]$ is an eigenvector of $A(t_1,t_2)$ if and only if $\det \begin{pmatrix} A_1 v & A_2 v & v \end{pmatrix} = 0$. We have

$$\begin{vmatrix} \gamma y & bz & 0 \\ \beta y & \lambda(0,1)y & y \\ \lambda(1,0)z & cz & z \end{vmatrix} = (b\lambda(1,0) - b\beta)yz^2 + (\gamma\lambda(0,1) - c\gamma)y^2 z = 0.$$

From this, we get $\lambda(1,0) = \beta$ and $\lambda(0,1) = c$. Now, we show that the associated eigenvalue $\lambda(t_1,t_2)$ is linear. We have $(t_1A_1 + t_2A_2)v = \lambda(t_1,t_2)v$, so we get

$$\begin{pmatrix} t_1\gamma y + t_2 bz \\ t_1\lambda(1,0)y + t_2\lambda(0,1)y \\ t_1\lambda(1,0)z + t_2\lambda(0,1)z \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda(t_1,t_2)y \\ \lambda(t_1,t_2)z \end{pmatrix}.$$

So, $\lambda(t_1,t_2) = t_1\lambda(1,0) + t_2\lambda(0,1)$ is linear.

Therefore, the marked cubic of eigenvalues is the disjoint union of a conic and a line. See **Example 5.2.22**.

(b) The line is not in V_{vir} and has two eigenvectors, the conic has one, ...

(c) ... or, the line is in V_{vir} and carry two distinct non-virtual eigenvectors,

the conic has one.

These cubics do not happen to be a cubic of eigenvectors. Indeed, by Fact 5.2.10, the two eigenvectors on the line ℓ are orthogonal to the third one, which is then assigned to be fixed at $\ell^\perp$ for all non singular parameters. This is a contradiction with Proposition 5.2.13, since the conic is not covered by non-virtual eigenvectors.

(d) The conic carries two eigenvectors ...

(e) ... or the conic carries three eigenvectors.

These cubics do not happen to be a cubic of eigenvectors. Indeed, in both of cases, the virtual line has at most one strict eigenvector, which contradicts Lemma 5.2.12.

VII) V **is the union of a non-degenerate conic and a line with two intersection points.** There are again subcases for the marked type, summarized in table 5.6.

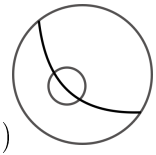
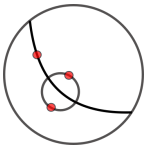
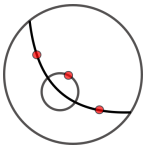
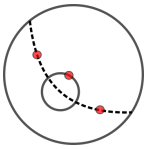
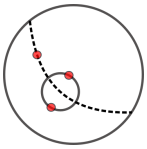
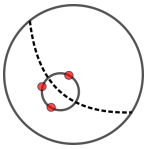
Cubic configuration	**Marked cubic type**
VII) 	(a).  (b).  (c).  (d).  (e). 

Table 5.6: Union of a line and a non-degenerate conic that intersect twice.

(a) The line is not virtual and has one ...

(b) ... or two eigenvectors, the conic has the other(s).

Both cases are impossible. Indeed, by Proposition 5.2.16, two components in V_{str} do not intersect.

(c) The line is virtual and carries two ...

(d) ... or one regular eigenvector(s).

By Lemma 5.2.14, each eigenvector on the virtual line is fixed. Now, from Fact 5.2.10, all other eigenvectors on the conic must be orthogonal to any fixed non-virtual eigenvector on the line. This is a contradiction with Lemma 5.2.13, since the conic curve is covered by strict eigenvectors and does not included in the orthogonal of a point. The cases are impossible.

(e) The line is virtual and carries no regular eigenvectors.

The case exists. Since V_{str} is connected, the antipodal monodromy is a cycle of length 3. So Λ has one component. By Fact 5.2.17, only rational crunodal cubic has one component in the configurations of the cubics, see **Example 5.2.25**.

VIII) V **is the union of a conic and a tangent line.** The subcases are in the following table 5.7.

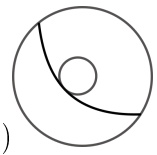
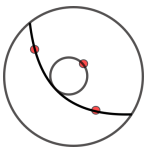
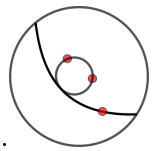
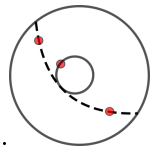
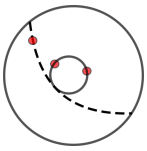
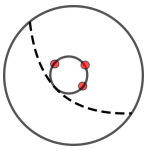
Cubic configuration	**Marked cubic type**
VIII) 	(a).  (b).  (c).  (d).  (e). 

Table 5.7: Union of a non-degenerate conic and a tangent line

(a) The line is not virtual and has two ...

(b) ... or one eigenvector(s), the conic has the other(s).

From Proposition 5.2.16, two components in V_{str} do not intersect. Impossible cases.

(c) The line is virtual and carries two ...

(d) ... or one regular eigenvector.

The cases are impossible. Indeed, by Lemma 5.2.14, the eigenvectors on a virtual line are constant. By Fact 5.2.10, all other eigenvectors must be orthogonal to any fixed non-virtual eigenvector on the line. But, the conic curve is covered but strict eigenvectors, Lemma 5.2.13, and not included in the orthogonal of a point.

(e) The line is virtual and carries no regular eigenvectors.

The only intersection of the line with V_{str} is the tangency point. But, by Lemma 5.2.12, there must be at least two different strict eigenvectors on the virtual line. This case is impossible.

IX) **V is the disjoint union of a conic point and a line.** There is only one type, but we discuss two possibilities, depending whether the line is regular. See table 5.8.

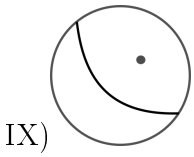
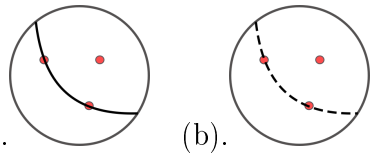
Cubic configuration	**Marked cubic type**
 IX)	 (a). (b).

Table 5.8: A conic point and a disjoint line

(a) The line is non-virtual, and has two eigenvectors, the conic point is the third one. This case happens. Since there is a constant eigenvector, the associated eigenvalue is linear and Λ contains a line. There is no virtual eigenvector, so $[A]$ does not meet Σ. Then, Λ contains only one line (any two lines intersect). So Λ is the union of a line with one eigenvalue, and a disjoint non-degenerate conic with the other two. See **Example 5.2.23**.

(b) The line is virtual and carries two regular eigenvectors. Then the three eigenvectors are constant, and the family is diagonal. But, the cubic type of eigenvectors of a diagonal family is either three lines in general position, or $\mathbb{P}^2$, Lemma 5.2.8. This case never happens.

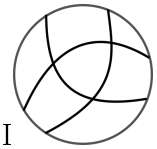
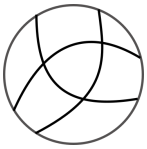
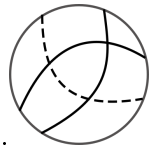
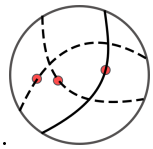
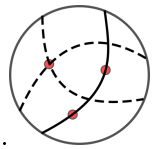
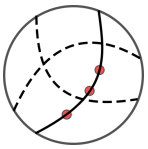
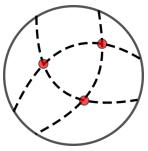
X) **V is the product of a line and a conic point on the line** ...

XI) ... **or V is the product of a line and an empty conic.**

There is now way to set three two-by-two orthogonal vectors on a line, so these cases are forbidden.

XII) V **is the union of three lines in general position.** There are many subcases. We do not enumerate all types when less than two lines are virtual, since we can eliminate all by the same argument. Table 5.9.

Table 5.9: Three lines in general position

Cubic configuration	**Marked cubic type**
XII 	(a).  (b).  (c).  (d).  (e).  (f). 

(a) No line ...

(b) ... or only one line is virtual.

Both cases are impossible. Indeed, non-virtual lines are fully covered by strict eigenvectors, Lemma 5.2.13, and cannot intersect by Proposition 5.2.16.

(c) One line is non-virtual, and carry one regular eigenvector.

By Lemma 5.2.14, there are two fixed regular eigenvectors on the virtual lines. So, the regular eigenvector on the non-virtual line is restrained at the point orthogonal to the eigenvectors on the virtual lines, then cannot cover the line, which is a contradiction with Lemma 5.2.13.

(d) One line ℓ is non-virtual, and carry two eigenvectors. The third one has

to be orthogonal to ℓ, and since by Lemma 5.2.12, any line must contain at least two strict eigenvectors, this point must be the intersection of the two virtual lines. This case exists. Since there is a constant eigenvector, the corresponding eigenvalue is linear and Λ contains a line. The rest of Λ must have a unique component since the antipodal monodromy contains a cycle of length 2. So, Λ is made of a line and a non-degenerate conic. There are two virtual lines, which correspond to two parameters such that the family intersect Σ_2. Thus, the conic and the line in Λ intersect in two points. See **Example 5.2.26**.

(e) A unique regular line carries three eigenvectors. This is impossible as the three vectors must be two-by-two orthogonal.

(f) The three lines are virtual.

By Lemma 5.2.12, each line must contain at least two strict eigenvectors. There is only one way to distribute three eigenvectors on the three lines in such a way that each line contains at least two vectors: the three eigenvectors lie at the three intersection points. This case exists. The three strict eigenvectors are fixed, so the family is diagonal. The associated eigenvalues are linear, so Λ is the union of three lines. These lines cannot have a common intersection point, otherwise $[\mathbb{I}_d]$ belongs to $[A]$, and $\mathbb{P}^2 \subset V$, Lemma 5.2.7. Hence, Λ is made of three lines in general position, **Example 5.2.27**.

XIII) V **is made of three concurrent lines**. Again we don't precise the type when there are more than one non-virtual line. Table 5.10

Table 5.10: Three distinct concurrent lines

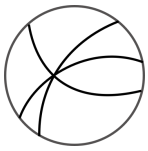
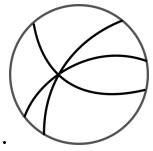
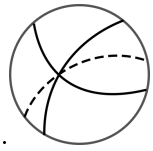
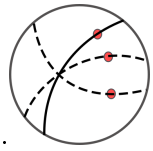
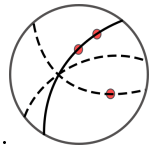
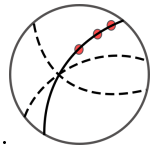
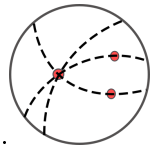
Cubic configuration	**Marked cubic type**
XIII) 	(a).  (b).  (c).  (d).  (e). 
Continued on next page	

Table 5.10 – *Continued from previous page*

Cubic configuration	**Marked cubic type**
	 (f).

(a) There are no virtual lines ...

(b) ... or one virtual line.

By Lemma 5.2.13, non-virtual lines are fully covered by strict eigenvectors. But, they cannot intersect, Proposition 5.2.16. Both cases are impossible.

(c) The only non-virtual line carries one eigenvector. Then, the two eigenvectors on the virtual lines are fixed, Lemma 5.2.14. From this, the eigenvector on the non-virtual line must be fixed. This is contrary to Lemma 5.2.13.

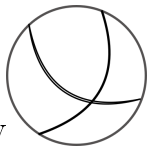
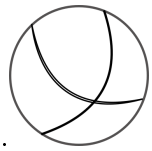
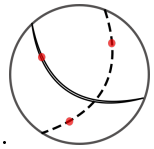
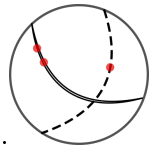
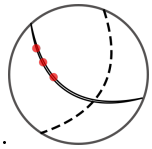
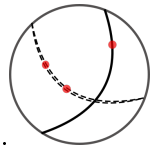
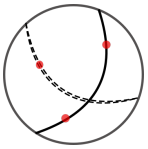
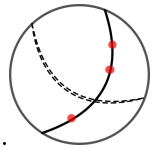
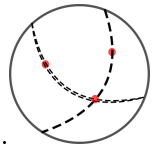
(d) The only non-virtual line carries two eigenvectors. One of the virtual lines gets only one intersection point with V_{vir}, which is a contradiction with Lemma 5.2.12.

(e) The only virtual line carries three eigenvectors; but they must be two-by-two orthogonal, Fact 5.2.10. This case does not happen to be a cubic of eigenvectors.

(f) The three lines are virtual. By Lemma 5.2.14, there are three fixed eigenvectors. But, there is no way to distribute 3 regular eigenvectors in such a way that each line contains at least two of them. This is a contradiction with Lemma 5.2.12.

XIV) V **is the product of a double and a single line**. The types are described in table 5.11. The bold line stand for the double one.

Table 5.11: Product of a double and a single line.

Cubic configuration	**Marked cubic type**
XIV 	(a).  (b).  (c).  (d).  (e).  (f).  (g).  (h). 

(a) The three lines are non-singular. Thus, by Lemma 5.2.13, the lines are covered by strict eigenvectors. But, they cannot intersect, Proposition 5.2.16. This case is impossible.

(b) The single line is virtual and carries two fixed points. By orthogonality, the double line cannot be covered by a strict eigenvector which is a contradiction with Lemma 5.2.13.

(c) The single line is virtual and carries one point. By Lemma 5.2.14, there is one constant eigenvector on the virtual line which correspond to a linear eigenvalue λ of the family. Without loss of generality, we suppose that $\lambda = 0$ and $e = (1, 0, 0)$ is the associated eigenvector on the virtual line, and we let

$$A(t_1, t_2) = t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & c \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & a' & b' \\ 0 & b' & c' \end{pmatrix}.$$

Then, for all parameters two eigenvectors on the double line must be orthogonal to $e = (1, 0, 0)$. So, the double line is given by $x^2 = 0$.

Now, $v = (x, y, z)$ is an eigenvector of $A(t_1, t_2)$ if and only if $p(x, y, z) = \det\begin{pmatrix} A_1 v & A_2 v & v \end{pmatrix} = 0$, we have

$$p(x,y,z) = \begin{vmatrix} 0 & 0 & x \\ ay+bz & a'y+b'z & y \\ by+cz & b'y+c'z & z \end{vmatrix} = x \begin{vmatrix} ay+bz & a'y+b'z \\ by+cz & b'y+c'z \end{vmatrix}.$$

But, $\frac{p(x,y,z)}{x}$ does not depend on x. Thus, x^2 is not a divisor of $p(x, y, z)$. This case is impossible.

(d) The double line has 3 eigenvectors. But, then they cannot be two-by-two orthogonal. This case is impossible.

(e) The double line is virtual and has two regular eigenvectors. By orthogonality, the third one cannot cover the simple line which is a contradiction with Lemma 5.2.13.

(f) The double line is virtual and has a fixed regular eigenvector. This case exists. The family has a fixed eigenvector that correspond to a linear eigenvalue. The other eigenvectors have non-trivial monodromy, so Λ has only one other component, a non-degenerate conic. The double line correspond to one parameter such that the family intersect Σ_2. So, the line and the conic are tangent, see **Example 5.2.28.**

(g) The single line carries three eigenvectors. They cannot be two-by-two orthogonal. This case is impossible.

(h) The two lines are virtual. Thus, the eigenvectors are fixed, Lemma 5.2.14, and so the case is diagonal. But, by Lemma 5.2.8, the eigenvector type of the diagonal families is made of three general lines or $\mathbb{P}^2$. That's impossible.

XV) V **is a triple line**. The eigenvectors cannot be orthogonal two-by-two, which eliminate the case.

XVI) V **is** $\mathbb{P}^2$. Then, by Lemma 5.2.7, the family contains $[\mathbb{I}_d]$, and the eigenvalue type is made of three concurrent lines, **Example 5.2.29**.

The following table summarizes all discussed cases, together with a symbol standing for the main argument that eliminates the case when it is proven to be forbidden.

The symbols are as follows:

The symbol " < 2 comp." stands for Proposition 5.2.15, " $\perp$ " for Fact 5.2.10, "$\cap = \emptyset$" for Proposition 5.2.16, "Diag." for Lemma 5.2.8, " < 2 str." for Lemma 5.2.12 and "Impossible" for the argument given in XIV(c).

Table 5.12

Cubic configuration	**Marked cubic type**	Λ
I) 	 I(a). exists  I(b). exists	 
II)  < 2 comp.		
III)  < 2 comp.		
IV)  < 2 comp.		
V)  $\perp$		
VI) 	(a).  VI(a) exists (b).  $\perp$ (c).  $\perp$ (d).  < 2 str. (e).  < 2 str.	
VII) 	(a).  $\cap = \emptyset$ (b).  $\cap = \emptyset$	
Continued on next page		

Table 5.12 – *Continued from previous page*

Cubic configuration	**Marked cubic type**	Λ
	(c).  $\perp$ (d).  $\perp$ (e).  exists.	
VIII) 	(a).  $\cap = \emptyset$ (b).  $\cap = \emptyset$ (c).  $\perp$ (d).  $\perp$ (e).  $< 2\ str$	
IX) 	(a).  IX(a). exists (b).  Diag.	
X)  $\perp$		
XI)  $\perp$		
XII) 	(a).  $\cap = \emptyset$ (b).  $\cap = \emptyset$	

Continued on next page

Table 5.12 – *Continued from previous page*

Cubic configuration	**Marked cubic type**	Λ
	(c).  $\perp$ (d).  XII(d). exists (e).  $\perp$ (f).  XII(f). exists	 
XIII) 	(a).  $\cap = \emptyset$ (b).  $\cap = \emptyset$ (c).  $\perp$ (d).  < 2 str (e).  $\perp$ (f).  < 2 str	
XIV) 	(a).  $\cap = \emptyset$ (b).  $\perp$ (c).  Impossible (d).  $\perp$ (e).  $\perp$ (f).  XIV(f). exists (g).  $\perp$	

Continued on next page

Table 5.12 – *Continued from previous page*

Cubic configuration	**Marked cubic type**	Λ
	(h).  Diag.	
XV)  $\perp$		
XVI) 	 XVI. exists	

5.2.4 Examples of cubics

In this section, we give 9 examples for the marked cubics of the pairs of eigenvectors and eigenvalues of a 2-linear family.

Lemma 5.2.19. *Suppose that C is a real cubic curve of $\mathbb{P}^2$ whose cubic type is an elliptic with an oval and p is a point on C. Let L be a tangent line to C at the point p. Then, $L \cap C = \{p, q\}$ where $p \neq q$ and the point q belongs to the pseudo line.*

Proof. First, note that any line on the real projective plane intersect a cubic either at three real roots or at one point. Moreover, the number of intersection points of a line with an oval is even. Since p is a point of multiplicity two, q cannot be on the oval, so it belongs to the pseudo line. □

Example 5.2.20. Here, we give an example for the marked cubic I.(a) of eigenvectors.

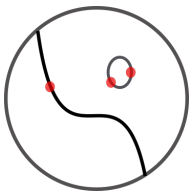


Figure 5.3: Marked cubic I.(a)

Let

$$A(t_1, t_2) = t_1 A_1 + t_2 A_2 = \begin{pmatrix} 0 & 2t_1 & t_1 \\ 2t_1 & t_2 & 0 \\ t_1 & 0 & -t_2 \end{pmatrix}. \tag{5.8}$$

A vector $v = [X : Y : Z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if $P(X, Y, Z) = \det \begin{pmatrix} A_1 v & A_2 v & v \end{pmatrix} = 0$,

$$\begin{aligned} \begin{vmatrix} 2Y+Z & 0 & X \\ 2X & Y & Y \\ X & -Z & Z \end{vmatrix} &= Y \begin{vmatrix} 2Y+Z & X \\ X & Z \end{vmatrix} + Z \begin{vmatrix} 2Y+Z & X \\ 2X & Y \end{vmatrix} \\ &= Y(2YZ + Z^2 - X^2) + Z(2Y^2 + ZY - 2X^2) \\ &= 4Y^2Z + 2YZ^2 - X^2Y - 2ZX^2 \end{aligned}$$

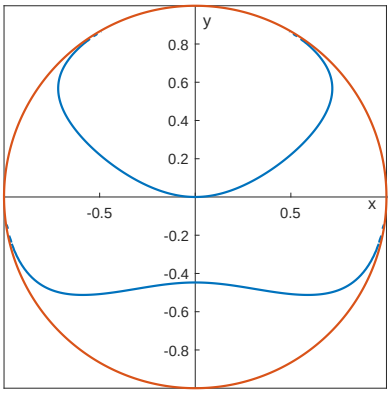
In the chart $Z = 1$, we get

$$\begin{aligned} p(x, y) &= 4y^2 + 2y - x^2y - 2x^2 \\ &= 4y^2 + y(2 - x^2) - 2x^2 \end{aligned}$$

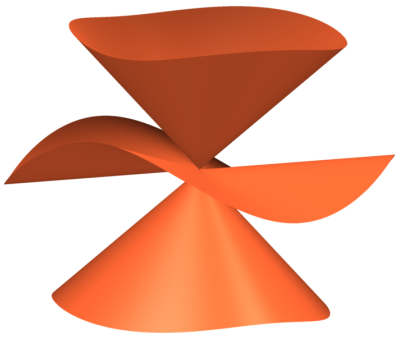
We have

$$\Delta_{p(x,y)}(x) = (2 - x^2)^2 + 32x^2.$$

So, for any fixed $x \in \mathbb{R}$, $p(x, y) = 0$ has two real roots $y = \frac{(x^2-2) \pm \sqrt{(2-x^2)^2+32x^2}}{8}$ such that $p(x, y) = 0$. In Figure 5.7, we see the graphs of $[X : Y : \sqrt{1 - X^2 - Y^2}] \mapsto P(X, Y, \sqrt{1 - X^2 - Y^2}) = 0$ and $v = (X, Y, Z) \mapsto P(X, Y, Z) = 0$.



(a) The graph of $P(X, Y, \sqrt{1 - X^2 - Y^2}) = 0$ in $\mathbb{P}^2$



(b) The graph of $P(X, Y, Z) = 0$ in $\mathbb{R}^3$

Figure 5.4: Cubic of eigenvectors and eigenvectors of $A(t)$

Note that the asymptotics of $p(x, y) = 0$ are $y = \frac{1}{4}x^2 - \frac{1}{2}$ and $y = -2$ and we have

$p(0,0) = p(0,-\frac{1}{2}) = 0$. So, any two lines $y = k$, $k \in (-\frac{1}{2}, 0)$ and $y = -2$ separates the projective plane $\mathbb{P}^2$ in two components such that each component of $\mathbb{P}^2$ contains a part of the cubic. So, the cubic contains 2 components. There are two possibilities. Either, the cubic type is an elliptic with an oval, or a line and a conic. The later is not possible. Indeed, for any x, $p(x,-2) \neq 0$, the cubic of eigenvectors does not contain a line. Hence, the cubic type of eigenvectors is an irreducible cubic with two components. The only possibility for the cubic type of eigenvectors is an elliptic with an oval.

Now, we show that there is only one non-virtual eigenvector on the pseudo line:

We have $\frac{\partial p}{\partial x}(x,y) = -2yx - 4x$ and $\frac{\partial p}{\partial x}(x,y) = 8y + 2 - x^2$. So, the tangent line to $p(x,y) = 0$ at the origin is $y = 0$. The intersection of $y = 0$ with $P(X,Y,Z) = 0$ gives $X^2Z = 0$. So, the intersection points are the double point $[0:0:1]$ and the simple point $[1:0:0]$. By Lemma 5.2.19, the point $[1:0:0]$ belongs to the pseudo line.

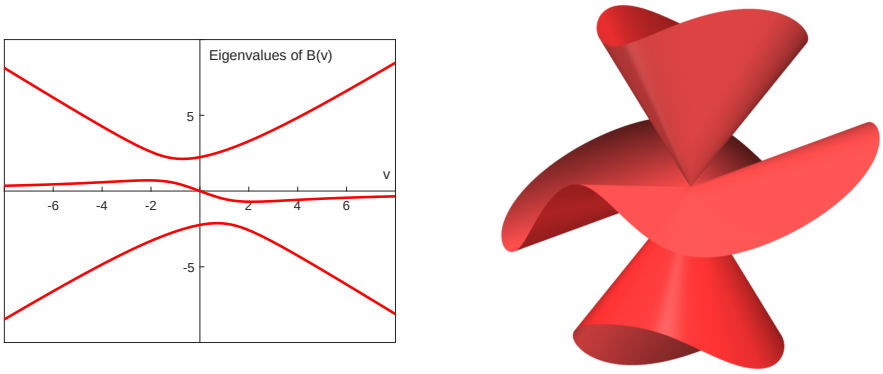
Since the cubic type of eigenvectors is an elliptic with an oval, $A(t_1,t_2) \cap \Sigma = \emptyset$. Now, $[1:0:0]$ is an eigenvector of $A(0,1)$ associated to the eigenvalue 0 and $Spec(A(0,1)) = \{-1,0,1\}$. Thus, there is a monodromy between two eigenvectors associated to the eigenvalues -1 and 1, so these eigenvectors are in the same components. So, there is one eigenvector on the pseudo line and two eigenvectors on the oval.

By Theorem 5.2.4, the cubic type of the eigenvalues is an elliptic with an oval. Let $B(v) = A_1 + vA_2$, then we have

$$\chi_{B(v)}(x) = -x^3 + (5+v^2)x + 3v \tag{5.9}$$

$$\Delta_{B(v)}(x) = 9 - 20x^2 + 4x^4 = 0 \tag{5.10}$$

We have $Z(\Delta_{B(v)})(x) = \{\pm\frac{1}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\}$. So, $\Delta_{B(v)}(x) < 0$ on the two intervals $I_1 = [-\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$ and $I_2 = [\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}]$. Then, if $x \in I_1 \cup I_2$, there is no $v \in \mathbb{R}$ such that $\chi_{B(v)}(x) = 0$. In Figure 5.5, we see the graph of the eigenvalues.



(a) The eigenvalues of B(v) (b) Eigenvalues of $A(t_1, t_2)$

Figure 5.5: Eigenvalues of $B(v)$ and $A(t_1, t_2)$

Example 5.2.21. Here, we give an example for the marked cubic I.(b) of eigenvectors, Figure 5.6.

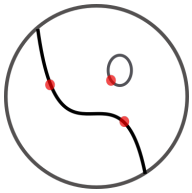


Figure 5.6: Marked cubic I.(b)

Let $A(t_1, t_2) = t_1 A_1 + t_2 A_2$ be given as follows.

$$A(t_1, t_2) = \begin{pmatrix} 0 & \sqrt{3}t_1 & 0 \\ \sqrt{3}t_1 & t_2 & \sqrt{2}t_1 \\ 0 & \sqrt{2}t_1 & -t_2 \end{pmatrix} = t_1 \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (5.11)$$

First, we show that the marked cubic of the eigenvalues is elliptic with an oval. For this, let $B(v) = A_1 + vA_2$, then we have

$$\chi_{B(v)}(x) = -x^3 + (5 + v^2)x + 3v \quad (5.12)$$

which is the characteristic polynomial of the matrix given in previous example. From Example 5.2.20, the cubic type of the eigenvalues is an elliptic with an oval. Thus, the cubic type of eigenvectors is elliptic with an oval.

We show that there is only one non-virtual eigenvector on the oval:

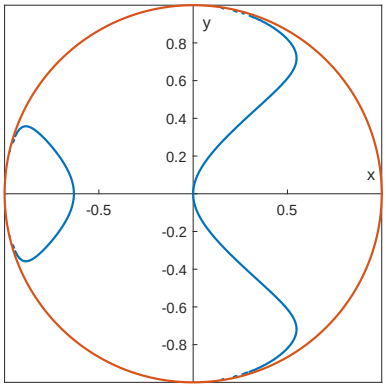
A vector $v = [X : Y : Z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if the three vectors $A_1 v, A_2 v$ and v are co-linear, i.e. $P(X, Y, Z) = \det \begin{pmatrix} A_1 v & A_2 v & v \end{pmatrix} = 0$,

$$\begin{aligned} \begin{vmatrix} \sqrt{3}Y & 0 & X \\ \sqrt{3}X + \sqrt{2}Z & Y & Y \\ \sqrt{2}Y & -Z & Z \end{vmatrix} &= Y \begin{vmatrix} \sqrt{3}Y & X \\ \sqrt{2}Y & Z \end{vmatrix} + Z \begin{vmatrix} \sqrt{3}Y & X \\ \sqrt{3}X + \sqrt{2}Z & Y \end{vmatrix} \\ &= -\sqrt{3}X^2Z - \sqrt{2}XY^2 - \sqrt{2}XZ^2 + 2\sqrt{3}ZY^2 \end{aligned}$$

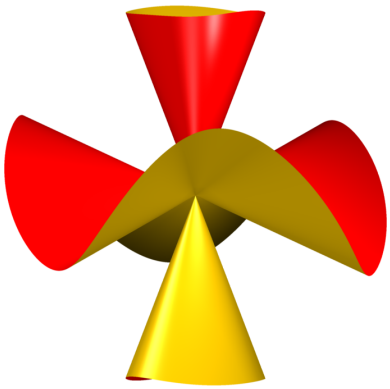
In the chart $Z = 1$, we get

$$\begin{aligned} p(x, y) &= -\sqrt{3}x^2 - \sqrt{2}xy^2 - \sqrt{2}x + 2\sqrt{3}y^2 \\ &= -\sqrt{3}x^2 - (\sqrt{2}y^2 + \sqrt{2})x + 2\sqrt{3}y^2 \end{aligned}$$

In Figure 5.7, we see the graphs of $[X : Y : \sqrt{1 - X^2 - Y^2}] \mapsto P(X, Y, \sqrt{1 - X^2 - Y^2}) = 0$ and $v = (X, Y, Z) \mapsto P(X, Y, Z) = 0$.



(a) The graph of $P(X, Y, \sqrt{1 - X^2 - Y^2}) = 0$ in $\mathbb{P}^2$



(b) The graph of $P(X, Y, Z) = 0$ in $\mathbb{R}^3$

Figure 5.7: Cubic of eigenvectors and eigenvectors of $A(t)$

Now, the tangent line to $p(x, y) = 0$ at the point $(0, 0)$ is

$$\frac{\partial p}{\partial x}(0, 0)(x - 0) + \frac{\partial p}{\partial y}(0, 0)(y - 0) = 0.$$

We have $\frac{\partial p}{\partial x}(x, y) = -2\sqrt{3}x - (\sqrt{2}y^2 + \sqrt{2})$ and $\frac{\partial p}{\partial x}(x, y) = 2(-\sqrt{2}x + 2\sqrt{3})y$. So, the tangent line to $p(x, y) = 0$ at the origin is $x = 0$. The intersection of $x = 0$

with $P(X,Y,Z)=0$ gives $Y^2Z=0$. So, the intersection points are the double point $[0:0:1]$ and the simple point $[0:1:0]$. By Lemma 5.2.19, the point $[0:1:0]$ belongs to the pseudo line.

The eigenvector $[0:1:0]$ is associated to the eigenvalue 1 of $A(0,1)$ and $Spec(A(0,1)) = \{-1,0,1\}$. By the same argument used in the previous example, two eigenvectors associated to the eigenvalues -1 and 1 are in the same component. Thus, two eigenvectors $[0:1:0]$ and $[0:0:1]$ belong to the pseudo line, while the only eigenvector on the oval is $[1:0:0]$.

Example 5.2.22. Here, we give an example for the marked cubic VI.(a) of eigenvectors, Figure 5.8.

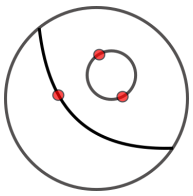


Figure 5.8: Marked cubic VI.(a)

Let $A(t_1,t_2)=t_1A_1+t_2A_2$ be given as follows.

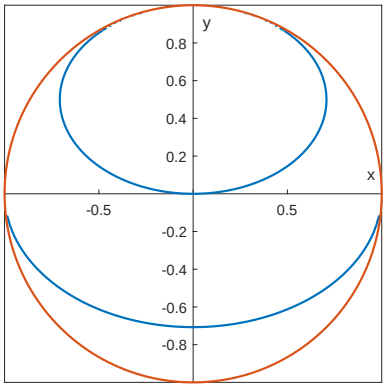
$$A(t_1,t_2)=\begin{pmatrix}0 & t_1 & t_1\\ t_1 & t_2 & 0\\ t_1 & 0 & -t_2\end{pmatrix}=t_1\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}+t_2\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{5.13}$$

A vector $v=[X:Y:Z]\in\mathbb{P}^2$ is an eigenvector of $A(t_1,t_2)$ if and only if

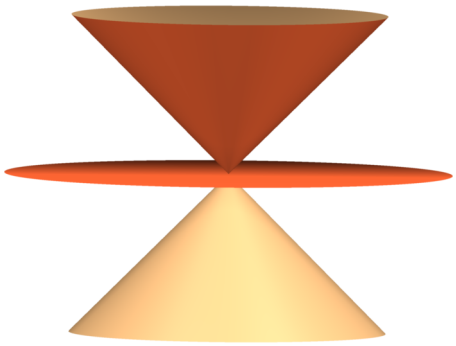
$$P(X,Y,Z)=\begin{vmatrix}Y+Z & 0 & X\\ X & Y & Y\\ X & -Z & Z\end{vmatrix}=Y\begin{vmatrix}Y+Z & X\\ X & Z\end{vmatrix}+Z\begin{vmatrix}Y+Z & X\\ X & Y\end{vmatrix} \tag{5.14}$$

$$=(Y+Z)(2YZ-X^2)=0 \tag{5.15}$$

Now, if $Z=1$, we get $p(x,y)=-(y+1)(2y-x^2)$ where $2y-x^2=0$ is the equation of a hyperbola which does not intersect the line $y=-1$. So, the cubic type of eigenvectors is the disjoint union of a line and a non degenerate conic. In Figure 5.9, we see the graphs of $[X:Y:\sqrt{1-X^2-Y^2}]\mapsto P(X,Y,\sqrt{1-X^2-Y^2})=0$ and $v=(X,Y,Z)\mapsto P(X,Y,Z)=0$.



(a) The graph of $P(X, Y, \sqrt{1 - X^2 - Y^2}) = 0$ in $\mathbb{P}^2$



(b) The graph of $P(X, Y, Z) = 0$ in $\mathbb{R}^3$

Figure 5.9: Cubic of eigenvectors and eigenvectors of $A(t)$

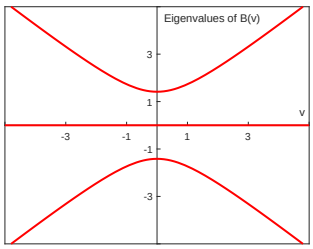
Since eigenvectors are two-by-two orthogonal, there is one eigenvector on the line while the others are on the conic.

By Theorem 5.2.4, the cubic type of eigenvalues is the disjoint union of a line and an irreducible conic. We have

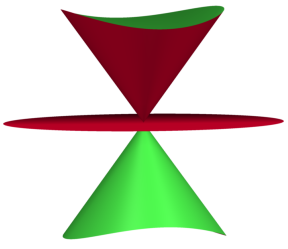
$$\Lambda = \{[t_1 : t_2 : \lambda] \mid \det(t_1 A_1 + t_2 A_2 - \lambda \mathbb{I}_3) = 0\} \tag{5.16}$$

$$= \{[t_1 : t_2 : \lambda] \mid \lambda(-\lambda^2 + 2t_1^2 + t_2^2) = 0\}. \tag{5.17}$$

In Figure 5.10, we see the graphs of the eigenvalues $B(v) = A_1 + vA_2$ and $A(t_1, t_2)$.



(a) The eigenvalues of B(v)



(b) Eigenvalues of $A(t_1, t_2)$

Figure 5.10: Eigenvalues of $B(v)$ and $A(t_1, t_2)$

Example 5.2.23. Here, we give an example for the cubic IX.(a) of eigenvectors,

Figure 5.11.

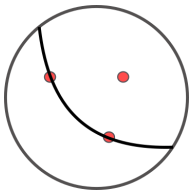


Figure 5.11: Marked cubic IX.(a)

$$A(t_1,t_2)=\begin{pmatrix}0&0&0\\0&t_2&\sqrt{2}t_1\\0&\sqrt{2}t_1&-t_2\end{pmatrix}=t_1\begin{pmatrix}0&0&0\\0&0&\sqrt{2}\\0&\sqrt{2}&0\end{pmatrix}+t_2\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}. \qquad (5.18)$$

Then, $\chi_{A(t_1,t_2)}(\lambda)=-\lambda^3+(2t_1^2+t_2^2)\lambda$ which is the characteristic polynomial of the matrix given in Example 5.2.22. So, the cubic type of the eigenvalues is the disjoint union of a line and a non-degenerate conic.

Now $v=[X:Y:Z]\in\mathbb{P}^2$ is an eigenvector of $A(t_1,t_2)$ if and only if $\det\begin{pmatrix}A_1v & A_2v & v\end{pmatrix}=0$,

$$\begin{vmatrix}0&0&X\\\sqrt{2}Z&Y&Y\\\sqrt{2}Y&-Z&Z\end{vmatrix}=X\begin{vmatrix}\sqrt{2}Z&Y\\\sqrt{2}Y&-Z\end{vmatrix}$$
$$=-\sqrt{2}X(Y^2+Z^2)$$

So, the marked cubic of the eigenvectors is the disjoint union of the conic point $Y^2+Z^2=0$ and the line $X=0$. There is one constant eigenvector $[1:0:0]$ associated to the analytic eigenvalue 0 on the conic point, and two non-virtual eigenvectors $[0:0:1]$ and $[0:1:0]$ are on the line $XZ^2=0$. The marked cubic of eigenvectors is the disjoint union of a conic point and a line.

Remark 5.2.24. The analytic prolongation of eigenvectors associated to the extremal eigenvalues 1 and -1 along the half turn path changes in Example 5.2.20 (resp. 5.2.22), but not in Example 5.2.21 (resp. 5.2.23). This orientation is a factor which cannot be seen by the antipodal monodromy of the eigenvectors, but this would have been manifested if we had constructed a monodromy in taking into account an orientation of eigenvectors.

Example 5.2.25. Here, we give an example for the cubic VII.(e) of eigenvectors, Figure 5.12.

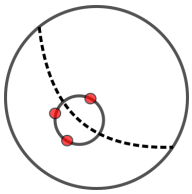


Figure 5.12: Marked cubic VII.(e)

Let $A(t_1, t_2) = t_1 A_1 + t_2 A_2$ be given as follows.

$$A(t_1, t_2) = \begin{pmatrix} t_1 & 0 & t_1 \\ 0 & 2t_1 & t_1 \\ t_1 & t_1 & t_2 \end{pmatrix} = t_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.19}$$

From Example 2.5.12, the monodromy of the eigenvalues is a cycle of length three. So, the cubic type of the eigenvalues is the rational crunodal. From this, the marked cubic of eigenvectors is the union of a non-degenerate conic and a line with two intersection points in which the conic has three regular eigenvectors.

Indeed, $v = [X : Y : Z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if

$$P(X, Y, Z) = \begin{vmatrix} X+Z & 0 & X \\ 2Y+Z & 0 & Y \\ X+Y & Z & Z \end{vmatrix} = -Z \begin{vmatrix} X+Z & X \\ 2Y+Z & Y \end{vmatrix} \tag{5.20}$$

$$= -Z(Y(X+Z) - X(2Y+Z)) \tag{5.21}$$

$$= -Z(-XY + YZ - XZ) = 0 \tag{5.22}$$

This is a reducible cubic and $Z = 0$ is the equation of the line of the cubic and $-XY + YZ - XZ$ is a non-degenerate conic. Note that the line $Z = 0$, intersect the conic in two points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. In Figure 5.13, we see the graph of eigenvectors.

Figure 5.13: The graph of $P(X, Y, Z) = 0$

Example 5.2.26. Here, we give an example for the cubic XII.(d) of eigenvectors, Figure 5.14.

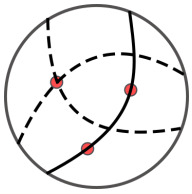


Figure 5.14: Marked cubic XII.(d)

$$A(t_1, t_2) = \begin{pmatrix} 0 & t_2 & 0 \\ t_2 & t_1 & 0 \\ 0 & 0 & -t_1 \end{pmatrix} = t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.23}$$

$v = [x : y : z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if $\det \begin{pmatrix} A_1 v & A_2 v & v \end{pmatrix} = 0$,

$$\begin{aligned} \begin{vmatrix} 0 & y & x \\ y & x & y \\ -z & 0 & z \end{vmatrix} &= -y \begin{vmatrix} y & y \\ -z & z \end{vmatrix} + x \begin{vmatrix} y & x \\ -z & 0 \end{vmatrix} \\ &= -y(2yz) + x^2 z \\ &= z(x^2 - 2y^2) = z(x - \sqrt{2}y)(x + \sqrt{2}y) \end{aligned}$$

So, the cubic type of eigenvectors is the union of three lines in general position. Note that $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are eigenvectors of the non-singular matrix $A(1, 0)$ associated to the eigenvalue 0 , 1 and -1, respectively. Two eigenvectors $[1 : 0 : 0]$ and $[0 : 1 : 0]$ belong only to the line $z = 0$, while $[0 : 0 : 1]$ belongs only to the intersection of the two lines $x - \sqrt{2}y = 0$ and $x + \sqrt{2}y = 0$.

Therefore, the cubic type of the eigenvalues is the union of a line and a non-

degenerate conic with two intersection points. We have

$$\Lambda = \{[t_1 : t_2 : \lambda] \mid \det(t_1 A_1 + t_2 A_2 - \lambda \mathbb{I}_3) = 0\} \tag{5.24}$$
$$= \{[t_1 : t_2 : \lambda] \mid (t_1 + \lambda)(t_1\lambda + t_2^2 - \lambda^2) = 0\} \tag{5.25}$$

In Figure 5.15, we see the eigenvalue of $A(t_1, t_2)$.

Figure 5.15: The eigenvalue of $A(t_1, t_2)$

Example 5.2.27. Here, we give an example for the cubic XII.(f) of eigenvectors, Figure 5.16.

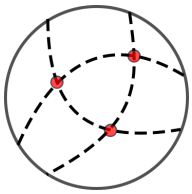


Figure 5.16: Marked cubic XII.(f)

$$A(t_1, t_2) = \begin{pmatrix} t_2 & 0 & 0 \\ 0 & t_1 + 2t_2 & 0 \\ 0 & 0 & -t_1 + 3t_2 \end{pmatrix} = t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + t_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{5.26}$$

We have $Spec(A(t_1, t_2)) = \{t_2, t_1 + 2t_2, -t_1 + 3t_2\}$ which two by two intersect along three different lines. So, the cubic type of the eigenvalues is union of three lines in general position.

Now, $v = [X : Y : Z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if

$$\begin{vmatrix} 0 & X & X \\ Y & 2Y & Y \\ -Z & 3Z & Z \end{vmatrix} = -X \begin{vmatrix} Y & Y \\ -Z & Z \end{vmatrix} + X \begin{vmatrix} Y & 2Y \\ -Z & 3Z \end{vmatrix}$$
$$= 3XYZ$$

which is the union of three lines $X = 0$, $Y = 0$, $Z = 0$ in general position. Note that eigenvectors of the non-singular parameter $(1, 0)$ are $v_1 = [1 : 0 : 0]$, $v_2 = [0 : 1 : 0]$ and $v_3 = [0 : 0 : 1]$. We have $v_1 \in \{Z = 0\} \cap \{Y = 0\}$, $v_2 \in \{Z = 0\} \cap \{X = 0\}$ and $v_3 \in \{X = 0\} \cap \{Y = 0\}$. Thus, the marked cubic of eigenvectors is the union of three lines in general position.

Example 5.2.28. Here, we give an example for the cubic XIV.(f) of eigenvectors, Figure 5.17.

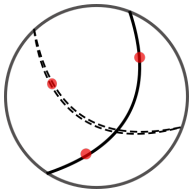


Figure 5.17: Marked cubic XIV.(f)

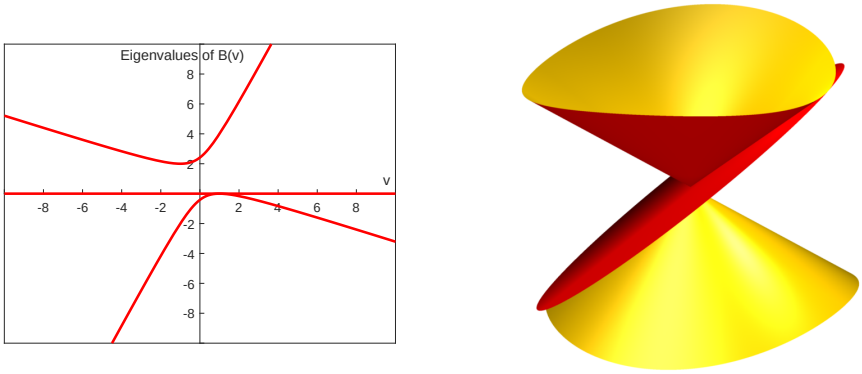
Let

$$A(t_1, t_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2t_1 & t_1 + t_2 \\ 0 & t_1 + t_2 & 2t_2 \end{pmatrix} = t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{5.27}$$

Let $B(v) = A_1 + vA_2$. We have

$$\chi_{B(v)}(x) = -x(x^2 - x(2 + 2v) - v^2 + 2v - 1) \tag{5.28}$$
$$Spec(B(v)) = \{0, (1 + v) \pm \sqrt{2(v^2 + 1)}\} \tag{5.29}$$

We have $Spec(B(1)) = \{0, 0, 4\}$, $\frac{\partial \chi_B}{\partial x}(v, x) = 2x - (2+2v)$ and $\frac{\partial \chi_B}{\partial v}(v, x) = -2x - 2v + 2$. So, the tangent line to the hyperbola $x^2 - x(2 + 2v) - v^2 + 2v - 1$ at $(v, x) = (1, 0)$ is $x = 0$. So, the marked cubic of the eigenvalues is the union of a non-degenerate conic and a tangent line, Figure 5.18.



(a) The eigenvalues of $B(v)$ (b) Eigenvalues of $A(t_1, t_2)$

Figure 5.18: Eigenvalues of $B(v)$ and $A(t_1, t_2)$

Now, $v = [X : Y : Z] \in \mathbb{P}^2$ is an eigenvector of $A(t_1, t_2)$ if and only if

$$P(X,Y,Z) = \begin{vmatrix} 0 & 0 & X \\ 2Y+Z & Z & Y \\ Y & y+2Z & Z \end{vmatrix} = x \begin{vmatrix} 2Y+Z & Z \\ Y & Y+2Z \end{vmatrix}$$
$$= 2X(Y+Z)^2$$

If $Z = 1$, we get $p(x, y) = 2x(y + 1)^2$. The cubic type of the eigenvector is the product of a double line and a single line such that the intersection point happens at $[0 : -1 : 1]$. For the non-singular parameter $(t_1, t_2) = (1, 0)$, $[0 : -1 : 1 + \sqrt{2}]$, $[1 : 0 : 0]$ and $[0 : 1, 1+\sqrt{2}]$ are eigenvectors of $A(t_1, t_2)$ associated with the eigenvalues $1 - \sqrt{2}$, 0 and $1 + \sqrt{2}$, respectively. Note that only $[1 : 0 : 0]$ belongs to the double line $(Y + Z)^2 = 0$ while two vectors $[0 : -1 : 1 + \sqrt{2}]$ and $[0 : 1 : 1 + \sqrt{2}]$ belong only to the simple line $X = 0$. So, the marked cubic of eigenvectors is the product of a double line with a simple line.

Example 5.2.29. Here, we give an example for the cubic XVI of eigenvectors, Figure 5.19.

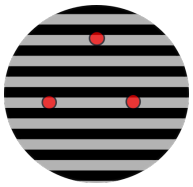


Figure 5.19: Marked cubic XVI

Let

$$A(t_1,t_2)=\begin{pmatrix} t_2+t_1 & 0 & 0\\ 0 & 2t_1+t_2 & 0\\ 0 & 0 & t_2+3t_1\end{pmatrix}=t_1\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}+t_2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \quad (5.30)$$

First, we note that the eigenvalues of $A(t_1,t_2)$ are $\lambda_1(t_1,t_2)=t_1+t_2$, $\lambda_2(t_1,t_2)=2t_1+t_2$, and $\lambda_2(t_1,t_2)=3t_1+t_2$ which intersect two-by-two over $t_1=0$ since $A(0,t_2)=t_2\mathbb{I}_3$. So, the cubic of the eigenvalues is made of three concurrent lines. From this, the cubic type of eigenvectors is the projective plane $\mathbb{P}^2$.

We have

$$\begin{aligned} V(A)&=\{v\in\mathbb{P}^2 \mid \det\begin{pmatrix}A_1v & A_2v & v\end{pmatrix}=0\}\\ &=\{v\in\mathbb{P}^2 \mid 0=0\}=\mathbb{P}^2\end{aligned}$$

and for a non-singular parameter $(t_1,0)$ there are three constant eigenvectors $v_1=[1:0:0]$, $v_2=[0:1:0]$, $v_3=[0:0:1]$ associated with three different eigenvalues t_1, $2t_1$ and $3t_1$ respectively.

5.3 Resolution of singularities of eigenvalues

In Subsection 5.1.2, we saw that there exists a map $\phi:\Gamma(A)\to\Lambda(A)$ where $\Gamma(A)$ is the set of eigenvectors and $\Lambda(A)$ is the set of the eigenvalues. Note that the variety $\Gamma(A)$ is not smooth in general. In Claim 5.1.1, we proved that $\Phi:G(A)\subseteq\Gamma(A)\to\Lambda(A)$ is a proper birational map, where for a k-linear family $A\in Sym_d(\mathbb{R})$, $A\not\subset\Sigma$,

$$G(A)=\mathrm{Clos}\left(\Gamma\setminus[\Sigma]\times\mathbb{P}^{d-1}\right). \quad (5.31)$$

Here, we show that the map Φ is a resolution of singularities of algebraic varieties when the family is a 2-linear family and when the family is the full space $Sym_d(\mathbb{R})$.

Recall that a subset of $\mathbb{R}^n$, respectively of $\mathbb{C}^n$, is called **constructible** if it is a finite union of differences of algebraic sets. The Euclidean closure of a complex constructible set is an algebraic set. However, in general, this not true in the real case. For instance, a real cubic with isolated point, when we delete this point we obtain a closed (for the Euclidean topology) constructible set which is not algebraic. So, a priori, our set $G(A)$ may not be algebraic. We conjecture that indeed $G(A)$ is

an algebraic set. However we are able to prove it only for curves, what is actually needed in this chapter.

5.3.1 Resolution of a 2-linear family

In can be concluded from the following theorem and Claim 5.1.1 that the map $\Phi : G(A) \to \Lambda(A)$ is a resolution of singularity for $\Lambda(A)$ if $A \in Sym_d(\mathbb{R})$ is a 2-linear family.

Theorem 5.3.1. *For any 2-linear family $A \not\subset \Sigma$ the set $G(A)$ of strict eigenvectors of A is a non-singular algebraic curve.*

Proof. We shall consider an affine chart of the projectivization of A, that is an affine family in one parameter of the form

$$A(t) = A_0 + tA_1, t \in \mathbb{R},$$

where A_0, A_1 are real symmetric matrices. We assume that $A_0 \in \Sigma$. In this chart

$$\Gamma_{\mathbb{R}} := \Gamma(A(t)) = \{(t, [v]) \in \mathbb{R} \times \mathbb{P}^{d-1}(\mathbb{R});\ A(t)v \wedge v = 0\}.$$

Now we consider its complexification

$$\Gamma_{\mathbb{C}} := \Gamma(A(z)) = \{(z, [v]) \in \mathbb{C} \times \mathbb{P}^{d-1}(\mathbb{C});\ A(z)v \wedge v = 0\}.$$

By the theorem of Rellich there are distinct real analytic functions $\lambda_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, d$ such that $SpecA(t) = \{\lambda_1(t), \ldots, \lambda_d(t)\}$ for all $t \in \mathbb{R}$. We consider their complexifications, denoted again by $\lambda_j(z)$, which are well defined and holomorphic in a small disk D around the origin in $\mathbb{C}$. Shrinking D, if necessary, we may assume that for any $z \in D^* := D \setminus \{0\}$ all $\lambda_1(z), \ldots, \lambda_d(z)$ are pairwise distinct. In other words, $\Delta(z) \neq 0$ for $z \in D^*$, where $\Delta(z)$ stands for the discriminant of the characteristic polynomial of the matrix $A(z)$. Again by the theorem of Rellich there are analytic functions $[v_j] : \mathbb{R} \to \mathbb{P}^{d-1}(\mathbb{R})$, $j = 1, \ldots, d$ which represent eigendirections associated to $\lambda_j(t)$, that is

$$A(t)v_j(t) = \lambda_j(t)v_j(t),\ t \in \mathbb{R}.$$

Again we can take the complexifications $[v_j] : D \to \mathbb{P}^{d-1}(\mathbb{C})$, $j = 1, \ldots, d$ which

satisfy clearly

$$A(z)v_j(t) = \lambda_j(z)v_j(z),\ z \in D. \tag{5.32}$$

We denote by V_j the graph of $[v_j]$ which is a smooth holomorphic submanifold of $\mathbb{C} \times \mathbb{P}^{d-1}(\mathbb{C})$. The following inclusion is important

$$(D^* \times \mathbb{P}^{d-1}(\mathbb{C})) \cap \Gamma_{\mathbb{C}} \subset \bigcup_{j=1}^{d} V_j. \tag{5.33}$$

Indeed, let $(z, [v]) \in D^* \times \mathbb{P}^{d-1}(\mathbb{C})) \cap \Gamma_{\mathbb{C}}$, then $[v]$ is an eigendirection of $A(z)$. Since $\Delta(z) \neq 0$, the matrix $A(z)$ has d distinct eigendirections. Therefore, $[v] = [v_j](z)$ for some $j \in \{1, \ldots, d\}$.

Let $\Sigma_{\mathbb{C}} := \{z \in \mathbb{C} : \Delta(z) = 0\}$ which is actually a finite set. The set

$$\Gamma_{\mathbb{C}} \setminus (\Sigma_{\mathbb{C}} \times \mathbb{P}^{d-1}(\mathbb{C}))$$

is a complex constructible set. So (see for instance Lojasiewicz [21], Proposition 2, VII.8.3, page 394) its Euclidean closure, denoted by $G_{\mathbb{C}}$, is a complex algebraic subset of $\mathbb{C} \times \mathbb{P}^{d-1}(\mathbb{C})$. Therefore

$$H = G_{\mathbb{C}} \cap (\mathbb{R} \times \mathbb{P}^{d-1}(\mathbb{R}))$$

is a real algebraic set. It is enough to prove that $G_{\mathbb{R}} = H$. The inclusion $G_{\mathbb{R}} \subset H$ is obvious. To prove the converse assume that $(0, [v]) \in H$, it means that there exists a sequence

$$(D^* \times \mathbb{P}^{d-1}(\mathbb{C})) \cap \Gamma_{\mathbb{C}}) \ni (z_n, [w(z_n)] \to (0, [v]),\ n \to \infty.$$

By the inclusion (5.33), there exists $j \in \{1, \ldots, d\}$ such that $[w(z_n)] = [v_j](z_n)$ for almost all $n \in \mathbb{N}$, because eigendirection maps $[v_i]$ are continuous with disjoint graphs. Therefore

$$(0, [v]) = \lim_{t \to 0}(t, [v_j](t)) \in G_{\mathbb{R}}.$$

To make the final conclusion that the set $G_{\mathbb{R}}$ is algebraic, we apply the above argument over each point $t \in \mathbb{R}$ such that $A(t) \in \Sigma$. Note that there are finitely many such points.

To prove that $G_{\mathbb{R}}$ is actually non-singular observe that its complexification $G_{\mathbb{C}}$ is non-singular (as analytic set) at each point $G_{\mathbb{R}}$, since it is locally a graph of a

holomorphic mapping. It follows from a lemma of Serre ([21], VII.16.1 Corollary, page 458) that each point of $G_{\mathbb{R}}$ is regular in the algebraic sense. In other words, it has a local description by a regular submersion. □

5.3.2 Resolution of a $\frac{d(d+1)}{2}$-linear family

From Claim 5.1.1 and the following proposition, the map $\Phi : G(A) = \Gamma(A) \to \Lambda(A)$ is a resolution of singularity for $\Lambda(A)$ if $A \in Sym_d(\mathbb{R})$ is a $\frac{d(d+1)}{2}$-linear family.

Proposition 5.3.2. *$\Gamma = \{([A],[V]) \in \mathbb{P}Sym_d(\mathbb{R}) \times \mathbb{P}^{d-1} \mid V$ is an eigenvector of $A\}$ is a smooth algebraic set of codimension $d-1$.*

Proof. Let ϕ be the map given by

$$\begin{array}{rccc} \phi & : & \mathbb{P}Sym_d(\mathbb{R}) \times \mathbb{P}^{d-1} & \to & \mathbb{P}\Lambda^2(\mathbb{R}^d) \\ & & ([A],[V]) & \mapsto & [A.V \wedge V] \end{array}.$$

We consider the map ψ given by

$$\begin{array}{rccc} \psi & : & Sym_d(\mathbb{R}) \times \mathbb{R}^d & \to & \Lambda^2(\mathbb{R}^d) \\ & & (A,V) & \mapsto & A.V \wedge V \end{array}$$

and we define $\tilde{\Gamma} = \{(A,V) \in Sym_d(\mathbb{R}) \times \mathbb{R}^d \mid V$ is a non-zero eigenvector of $A\}$. Note that $(A,V) \in \tilde{\Gamma}$ if and only if $\psi(A,V) = 0$, ψ being a polynomial, it follows that $\tilde{\Gamma} = \psi^{-1}(0)$ is algebraic. Now, we show that for any $(A,V) \in \tilde{\Gamma} \subset Sym_d(\mathbb{R}) \times \mathbb{R}^d$, the rank of the differential of ψ is $d-1$. For this, we show that the image of $d_{(A,V)}\psi$ has dimension $d-1$.

The differential of ψ is given by

$$d_{(A,V)}\psi(a,v) = a.V \wedge V + A.v \wedge V + A.V \wedge v$$

Now, if $(A,V) \in \tilde{\Gamma}$, then $V = e_1$ is an eigenvector of A. Let $\{e_2,\ldots,e_d\}$ such that $\{e_1,e_2,\ldots,e_d\}$ is an orthonormal basis of $\mathbb{R}^d$ which is formed by eigenvectors of A. We denote by $(\lambda_1,\ldots,\lambda_d)$ the associated eigenvalues of A. Let $a = (a_{ij})$ be a symmetric

matrix. Then,

$$\begin{aligned} d_{(A,e_1)}\psi(a,e_i) &= a.e_1 \wedge e_1 + A.e_i \wedge e_1 + A.e_1 \wedge e_i \\ &= a.e_1 \wedge e_1 + \lambda_i e_i \wedge e_1 + \lambda_1 e_1 \wedge e_i \\ &= a.e_1 \wedge e_1 + (\lambda_i - \lambda_1) e_i \wedge e_1 \\ &= (a.e_1 + (\lambda_i - \lambda_1) e_i) \wedge e_1 \end{aligned}$$

So, any member of the image of $d_{(A,e_1)}\psi(a,v)$ is included $\mathbb{R}^d \wedge e_1$. Conversely, let $w \in \mathbb{R}^d$, then there exists a symmetric matrix a such that $ae_1 = w$. So, $d_{(A,e_1)}\psi(a,0) = ae_1 \wedge e_1 = w \wedge e_1$. Therefore, $d_{(A,V)}\psi$ has rank $d-1$ on $\tilde{\Gamma}$.

Let $d_1 = \dim(Sym_d(\mathbb{R}) \times \mathbb{R}^d)$, $d_2 = \dim(\Lambda^2(\mathbb{R}^d))$. Since the rank of $d_{(A,V)}\psi$ is $d-1$, its matrix has some $d-1 \times d-1$ non-zero minor. By ordering the coordinates, we may assume that it is the upper-left minor. We label the coordinates $(x_1, \ldots, x_{d-1}, y_1, \ldots, y_{d_1-(d-1)})$ on $Sym_d(\mathbb{R}) \times \mathbb{R}^d$ and we label the coordinates $(u_1, \ldots, u_{d-1}, v_1, \ldots, v_{d_2-(d-1)})$ on $\Lambda^2(\mathbb{R}^d)$. Then, we may write

$$\psi(x,y) = (Q(x,y), R(x,y)),$$

where Q is the projection on $u = (u_1, \ldots, u_{d-1})$ and R is the projection on $v = (v_1, \ldots, v_{d_2-(d-1)})$ with $\frac{\partial Q}{\partial x}$ non-singular. Note that for $(A,V) \in \tilde{\Gamma}$, $d_{(A,V)}Q$ has the maximum rank $d-1$. Since determinant is continuous and the rank function is lower semi-continuous, there exists a neighborhood $U \subset Sym_d(\mathbb{R}) \times \mathbb{R}^d$ of (A,V) such that for any $u \in U$, $d_u Q$ has a constant rank $d-1$. So, $Q : U \to V$ has constant rank on U where V is an open set around $Q(A,V)$. Now, by constant rank theorem, for any point $p \in U$, we can find a smooth coordinate chart (U_0, g) centered at p and (V_0, h) centered at $Q(p)$ with $U_0 \subset U$ and $Q(U_0) \subset V_0 \subset V$ such that

$$h \circ Q \circ g^{-1}(x^1, \ldots, x^{d-1}, x^d, \ldots, x^{d_1}) = (x^1, \ldots, x^{d-1})$$

Note that $Q^{-1}(0) \cap U_0 = \{x^1 = \cdots = x^{d-1} = 0\}$. We deduce that $Q^{-1}(0)$ is a smooth algebraic set of codimension $d-1$. Up to restriction of U_0 on an open ball $B_r(a)$ centered at $a = (A,V)$, $Q^{-1}(0)$ is connected (path-connected). We show that $Q^{-1}(0) = \tilde{\Gamma}$ on this ball. We remark that $\tilde{\Gamma} \subset Q^{-1}(0)$ on $B_r(a)$. Now we show that $Q^{-1}(0) \subset \tilde{\Gamma}$ on $B_r(a)$. Let $m \in Q^{-1}(0) \cap B_r(a)$ and v be one of the functions which defines $\tilde{\Gamma}$. Then, we can write $v(m) = \int_\gamma dv$ where γ is a continuous path in $Q^{-1}(0)$ from 0 to m. Since the rank of ψ on all the points of $Q^{-1}(0) \cap B_r(a)$ is $p-1$ and

$du_1, \ldots, du_{d-1}$ are independent, then dv is a combination of $du_1, \ldots, du_{d-1}$ on all the points of $Q^{-1}(0) \cap B_r(a)$. Thus, we get $v(m) = 0$ which proves $Q^{-1}(0) \subset \tilde{\Gamma}$ on $B_r(a)$. Thus, we deduce that $\tilde{\Gamma}$ is a smooth algebraic set of codimension $d-1$. Therefore, Γ also is a smooth algebraic set of codimensoin $d-1$. This finishes the proof.

□

Corollary 5.3.3. *The algebraic variety*

$$E := \{([A : \lambda], [V]) \in \mathbb{P}(Sym_d(\mathbb{R}) \times \mathbb{R}) \times \mathbb{P}^{d-1} \mid (A - \lambda \mathbb{I}_d)V = 0\}$$

is smooth.

Index

References

[1] J. F. Adams, "Vector fields on spheres", *Annals of Mathematics*, pp. 603–632, 1962.

[2] V. I. Arnold, "Remarks on eigenvalues and eigenvectors of hermitian matrices, berry phase, adiabatic connections and quantum hall effect", *Selecta Mathematica*, vol. 1, no. 1, pp. 1–19, 1995.

[3] V. I. Arnold, *Mathematical methods of classical mechanics*. Springer Science & Business Media, 2013, vol. 60.

[4] ——, "Modes and quasimodes", *Funktsional'nyi Analiz i ego Prilozheniya*, vol. 6, no. 2, pp. 12–20, 1972.

[5] R. Benedetti and J.-J. Risler, *Real algebraic and semialgebraic sets*. Hermann, 1991.

[6] E. Brugallé, "Surfaces with many solitary points", *Geometriae Dedicata*, vol. 141, no. 1, p. 123, 2009.

[7] J. Chaumatet and A.-M. Chollet, "Division par un polynôme hyperbolique", *Canadian Journal of Mathematics*, vol. 56, no. 6, pp. 1121–1144, 2004.

[8] M. Dana and K. D. Ikramov, "On the codimension of the variety of symmetric matrices with multiple eigenvalues", *Zapiski Nauchnykh Seminarov POMI*, vol. 323, pp. 34–46, 2005.

[9] J. Dieudonné, "Sur un theoreme de glaeser", *Journal d'analyse Mathématique*, vol. 23, no. 1, pp. 85–88, 1970.

[10] S. Friedland, J. W. Robbin, and J. H. Sylvester, "On the crossing rule.", WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER, Tech. Rep., 1982.

[11] G. Glaeser, "Racine carrée d'une fonction différentiable", in *Annales de l'institut Fourier*, vol. 13, 1963, pp. 203–210.

[12] V. Grandjean, "Re-parameterizing and reducing families of normal operators", *Israel Journal of Mathematics*, vol. 230, no. 2, pp. 715–744, 2019.

[13] J. W. Helton and V. Vinnikov, "Linear matrix inequality representation of sets", *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, vol. 60, no. 5, pp. 654–674, 2007.

[14] N. V. Ilyushechkin, "Discriminant of the characteristic polynomial of a normal matrix", *Mathematical Notes*, vol. 51, no. 3, pp. 230–235, 1992.

[15] ——, "On relations between summands in the discriminant of a symmetric matrix", *Mathematical Notes*, vol. 93, no. 1-2, pp. 337–339, 2013.

[16] N. Ilyushechkin, "Some identities for elements of a symmetric matrix", *Journal of Mathematical Sciences*, vol. 129, no. 4, pp. 3994–4008, 2005.

[17] T. Kato, *Perturbation theory for linear operators*. Springer Science & Business Media, 2013, vol. 132.

[18] K. Kurdyka and L. Paunescu, "Hyperbolic polynomials and multiparameter real-analytic perturbation theory", *Duke Mathematical Journal*, vol. 141, no. 1, pp. 123–149, 2008.

[19] P. D. Lax, "On the discriminant of real symmetric matrices", *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, vol. 51, no. 11-12, pp. 1387–1396, 1998.

[20] ——, "The multiplicity of eigenvalues", *Bulletin of the American Mathematical Society*, vol. 6, no. 2, pp. 213–214, 1982.

[21] S. Lojasiewicz, *Introduction to complex analytic geometry*. Birkhäuser, 2013.

[22] M. Losik and A. Rainer, "Choosing roots of polynomials with symmetries smoothly", *arXiv preprint math/0603660*, 2006.

[23] J. von Neumann and E. P. Wigner, "Über merkwürdige diskrete eigenwerte", in *The Collected Works of Eugene Paul Wigner*, Springer, 1993, pp. 291–293.

[24] J. C. Ottem, K. Ranestad, B. Sturmfels, and C. Vinzant, "Quartic spectrahedra", *Mathematical programming*, vol. 151, no. 2, pp. 585–612, 2015.

[25] B. N. Parlett, "The (matrix) discriminant as a determinant", *Linear algebra and its applications*, vol. 355, no. 1-3, pp. 85–101, 2002.

[26] A. Parusinski and G. Rond, "Multiparameter perturbation theory of matrices and linear operators", *arXiv preprint arXiv:1807.04242*, 2018.

[27] A. Rainer, "Perturbation of complex polynomials and normal operators", *Mathematische Nachrichten*, vol. 282, no. 12, pp. 1623–1636, 2009.

[28] ——, "Quasianalytic multiparameter perturbation of polynomials and normal matrices", *Transactions of the American Mathematical Society*, vol. 363, no. 9, pp. 4945–4977, 2011.

[29] F. Rellich, "Störungstheorie der spektralzerlegung", *Mathematische Annalen*, vol. 113, no. 1, pp. 600–619, 1937.

[30] ——, "Störungstheorie der spektralzerlegung. v", *Mathematische Annalen*, vol. 118, no. 1, pp. 462–484, 1941.

[31] F. Rellich and J. Berkowitz, *Perturbation theory of eigenvalue problems*. CRC Press, 1969.

[32] M. Shapiro and A. Vainshtein, "Stratification of hermitian matrices and the alexander mapping", *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, vol. 321, no. 12, pp. 1599–1604, 1995.