# Questions of approximation and compactness for geometric variational problems

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# Questions d'approximation et de compacité pour des problèmes variationnels géométriques

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# Questions d'approximation et de compacité pour des problèmes variationnels géométriques

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# Introduction

Deux énergies géométriques de surface sont au cœur de ce manuscrit de thèse : l'aire et l'énergie de Willmore. L'intérêt pour la notion d'aire est ancien puisque les premiers travaux sur le problème isopérimétrique datent de l'antiquité et qu'il existe toujours aujourd'hui une riche activité de recherche autour des surfaces minimales en analyse, en géométrie, en théorie géométrique de la mesure mais aussi en probabilités et en statistique. Le très vaste champ d'applications des surfaces minimales motive également de nombreux travaux portant sur leur approximation discrète. L'énergie de Willmore est peut-être un peu moins connue, ainsi que ses points critiques, les surfaces de Willmore. On rappelle que l'énergie de Willmore désigne l'intégrale sur la surface du carré de la courbure moyenne, qu'elle doit son nom aux travaux du géomètre T.J. Willmore dans les années 60 [137, 136] mais que son existence est bien plus ancienne. En dimension 1, Bernoulli et Euler l'avaient déjà proposée au 18e siècle comme énergie de torsion d'une tige élastique. Son étude pour les 2-surfaces dans $\mathbb{R}^3$ a commencé dès le début du 19e siècle avec Poisson [117] puis Germain [65] qui l'ont utilisée pour caractériser les plaques élastiques fines. Plusieurs raisons expliquent l'engouement fort dont l'énergie de Willmore(–Poisson–Germain...) bénéficie aujourd'hui :

- C'est un invariant pour les transformations conformes de $\mathbb{R}^3$ et, à topologie fixée, elle offre par minimisation une bonne définition d'immersion optimale. Mieux encore, l'énergie de Willmore vérifie une propriété de quantification dépendant de la topologie : si $\Sigma$ est une surface orientable et fermée (i.e., compacte et sans bord) de genre $g$ alors pour toute immersion $f$ de $\Sigma$ dans $\mathbb{R}^3$ on a $W(f) \geq \beta_g$ où $\beta_g$ est une constante positive. En outre, il existe une surface de Willmore (i.e. un point critique $f$ de l'énergie) orientée, fermée et de genre $g$ d'énergie $\beta_g$ [128, 14]. L'inégalité $W(f) \geq \beta_g$ reliant géométrie et topologie de la surface est remarquable. Elle constitue le pendant pour la courbure moyenne (et sous forme d'inégalité) du lien établi par le théorème de Gauss-Bonnet entre l'intégrale de la courbure gaussienne et la caractéristique d'Euler. L'identification précise des $\beta_g$ et des surfaces dont c'est l'énergie a fait l'objet de nombreux travaux depuis Willmore. Celui-ci avait démontré que $\beta_0 = 4\pi$, qui coïncide avec l'énergie des sphères plongées. Le cas $g = 1$, objet de la fameuse conjecture de Willmore, a été résolu par Marques et Neves [94] qui ont confirmé que $\beta_1 = 2\pi^2$, énergie qu'on savait déjà atteinte par le tore de Clifford. Le problème est toujours ouvert pour $g \geq 2$. Une conjecture due à Kusner affirme que l'énergie optimale serait atteinte par les projections stéréographiques des surfaces de Lawson, une classe de surfaces minimales dans $\mathbb{S}^3$ [79, 86]. Il est d'ailleurs important de souligner, pour mettre en avant le lien étroit qui lie surfaces minimales et surfaces de Willmore, que la projection stéréographique sur $\mathbb{R}^3$ d'une surface minimale de $\mathbb{S}^3$ est une surface de Willmore.

- L'énergie de Willmore n'est pas seulement utile comme outil de caractérisation géométrique et topologique, elle joue aussi un rôle fondamental dans des applications variées : en biologie, elle est contenue dans l'énergie de Canham-Helfrich qui régit la forme des membranes bilipidiques, par exemple celle des globules rouges, voir [17, 98, 99] et les références incluses. Comme on l'a vu en évoquant Bernoulli, Euler, Poisson et Germain, elle est aussi utilisée en mécanique comme énergie de déformation des tiges ou des plaques élastiques. Elle apparaît également en relativité générale dans la masse d'Hawking [68]. Elle intervient par ailleurs dans des modèles

de reconstruction ou de segmentation d'images numériques [95, 9, 39, 13, 36, 96] et elle sert d'énergie de régularisation pour des problèmes de complétion ou de régularisation de surfaces discrètes ou de nuages de points [55, 29, 45].

Toutes ces applications sont à l'origine d'une très grande variété de méthodes pour l'approximation numérique de minimiseurs de l'énergie de Willmore. On se focalisera dans cette thèse sur les méthodes dites de champ de phase qui permettent d'approcher l'aire ou l'énergie de Willmore du bord d'un domaine ouvert (donc des énergies singulières concentrées) par des fonctionnelles très régulières définies pour des fonctions régulières [30]. L'approche par champ de phase permet à la fois d'obtenir des bonnes garanties de convergence (au sens de la $\Gamma$-convergence) des énergies approchées vers les énergies limites et se prête très bien à l'approximation numérique.

Décrivons maintenant plus en détail le contenu de ce manuscrit. Il comporte deux parties indépendantes rédigées en anglais. Bien que l'énergie de Willmore soit commune à ces deux parties, les questions abordées ainsi que les techniques et outils utilisés sont très différents.

La partie I, intitulée *De l'approximation champ de phase du flot de Willmore au flot de courbure moyenne approché avec auto-évitement*, a été l'objet d'une collaboration avec Elie Bretin. Elle comporte quatre chapitres. Au chapitre 1, on rappelle les principes de l'approximation champ de phase, en particulier pour l'approximation de l'aire ou de l'énergie de Willmore. On rappelle également quelques résultats connus sur les propriétés des flots-gradients associés à certaines approximations champ de phase et leur convergence vers les flots gradients limites que sont le flot de courbure moyenne et le flot de Willmore (i.e. le flot gradient $L^2$ de l'énergie de Willmore). Le chapitre 2 est consacré à l'analyse théorique et à l'approximation numérique du flot de De Giorgi-Bellettini-Paolini, c'est-à-dire du flot gradient $L^2$ associé au modèle classique d'approximation de l'énergie de Willmore dû à De Giorgi, Bellettini et Paolini [18]. Ce flot converge vers le flot de Willmore lorsque tout est régulier, mais les singularités qu'il laisse apparaître ne sont pas compatibles avec les propriétés de la relaxée de l'énergie de Willmore. Au contraire, un autre modèle d'approximation dû à Mugnai [108] empêche l'apparition de ces singularités indésirables. C'est dû à la présence d'un terme de réaction que nous qualifions de terme d'"auto-évitement" car il empêche, lors du flot, le contact entre deux points différents de l'interface approchée en générant une force répulsive au voisinage des singularités de la fonction distance. Le chapitre 3 est consacré à une brève étude de ce terme avec une illustration numérique de ses propriétés. Dans le chapitre 4, nous proposons un nouveau terme d'auto-évitement qui offre davantage de robustesse. Nous montrons en particulier que ce terme, dit de saut, converge vers une mesure portée par l'ensemble singulier d'un champ normal associé aux formes étudiées. On déduit d'une analyse asymptotique et de simulations numériques une conjecture faisant apparaître à des ordres variés les sous-ensembles de dimensions variées de cet ensemble singulier. Puis nous étudions l'apport de ce terme de saut lorsqu'il est couplé au flot de courbure moyenne approché. Le flot perturbé obtenu est particulièrement adapté pour obtenir des approximations numériques de solutions du problème de Steiner en dimension 3, qui est notoirement difficile. Il est également bien adapté pour résoudre numériquement le problème de Plateau et permet d'obtenir une approximation de surfaces minimales aussi bien orientables que non orientables.

La partie II du manuscrit est intitulée *Analyse asymptotique pour les sphères de Willmore dans une variété tri-dimensionnelle*, elle est composée du chapitre 5 et d'une annexe. L'étude des surfaces de Willmore dans les espaces courbes est relativement récente, et plusieurs résultats d'existence ou de non existence ont été prouvés depuis peu. En particulier, des petites sphères de Willmore ont été construites comme étant des sphères géodésiques perturbées, cf. les travaux de Lamm & Metzger [81], Mondino [104, 105], Lamm, Metzger & Schulze [82] et Chen & Li [40]. Toutes ces constructions reposent sur le théorème des fonctions implicites et certaines non-dégénérescences de la variété ambiante sont requises, notamment dans un voisinage d'un point critique de la courbure scalaire. Réciproquement, on est ramené naturellement à la question suivante :

*Soit $p$ un point dans une variété tri-dimensionnelle $(M, h)$. Supposons que, pour $r > 0$ suffisamment petit, la boule géodésique $B^h_r(p)$ contienne une surface de Willmore (ou plus généralement une surface de Willmore*

*sous* contrainte d'aire*), alors $p$ est-il essentiellement un point critique de la courbure scalaire de $(M, h)$ ?*

Lamm et Metzger ont montré dans leur travail pionnier [80] que ce phénomène de concentration apparaît si on ne considère que les surfaces de Willmore de courbure moyenne positive. D'une manière équivalente, un tel phénomène de concentration a lieu pour les surfaces de Willmore d'énergie inférieure à $4\pi + \varepsilon$ pour $\varepsilon > 0$ suffisamment petit. Laurain et Mondino ont généralisé ce résultat dans [84] lorsque la borne d'énergie est inférieure à $8\pi - \delta$ pour tout $\delta > 0$. Plus précisément, ils montrent que si $(M, h)$ est une variété riemannienne de dimension 3 et $(\Phi^k(\mathbb{S}^2))_{k\in\mathbb{N}}$ une suite de sphères de Willmore dans $M$ (éventuellement sous contrainte d'aire) dont l'énergie de Willmore est uniformément bornée par $8\pi - \delta$ et telles que $\mathrm{diam}(\Phi^k(\mathbb{S}^2)) \to 0$ lorsque $k \to +\infty$ alors, à une extraction près, la suite $(\Phi^k(\mathbb{S}^2))_{k\in\mathbb{N}}$ converge vers un point critique $p$ de la courbure scalaire de $(M, h)$. Notons que, grâce à une inégalité de Li et Yau [89], on ne peut avoir sous ces hypothèses que des sphères de Willmore plongées.

On s'intéresse dans la partie II du manuscrit à ce qu'on peut dire quand on relâche la contrainte d'énergie. Quitte à faire un changement d'échelle approprié autour du point $p$, nous étudions le cas particulier où les surfaces limites peuvent être vues asymptotiquement comme deux sphères euclidiennes qui se collent l'une à l'autre. Nous montrons comment obtenir des estimées ponctuelles fortes pour la suite $(\Phi^k)_{k\in\mathbb{N}}$ en appliquant la technique développée par Druet, Hebey & Robert [57] au système triple d'équations vérifiées par les surfaces de Willmore dont l'une est la reformulation de Rivière sous forme divergence et les deux autres sont celles données par le paramétrage conforme des surfaces. Ces trois équations sont des EDPs elliptiques avec des termes non-linéaires dominants sous forme de Wente. En étudiant soigneusement le recollement entre les deux sphères et en adaptant la méthode utilisée par Laurain [83] pour les surfaces à courbure moyenne constante, nous montrons à l'aide des estimées fortes que ces sphères de Willmore doivent se concentrer en un point critique de la courbure scalaire de $(M, h)$. Nous ne savons pas pour l'instant comme étendre ces résultats pour traiter le cas plus général d'un nombre quelconque de sphères avec apparition de caténoïdes. Nous pensons néanmoins que la méthode utilisée pour obtenir des estimées fortes doit pouvoir être généralisée.

# Part I: From the approximation of the Willmore flow to a self-avoiding approximate mean curvature flow

# Chapter 1

# Phase-field approximations of the perimeter and Willmore energies

## 1.1 Van der Waals-Cahn–Hilliard energy

An historical reference for the phase-field approach is van der Waals's work on the thermodynamic theory of capillary, where the free energy of a liquid-gas interface is studied (see [123] for an English translation with insightful comments). Van der Waal advocates that there is no sharp transition between the liquid and gas phase states, but rather a smooth transition. The local concentration of liquid with respect to gas can therefore be represented by a smooth scalar field $u$ which is not binary, but instead varies smoothly from 0 to 1. Van der Waals proposes a two-term model for the free energy of such a diffuse liquid-gas interface:

$$\int_V \left(\lambda|\nabla u|^2 + f_0(u)\right)\, dx, \tag{1.1.1}$$

where $V$ denotes the volume enclosing the liquid-gas interface, $\lambda$ is the capillarity coefficient, $\lambda|\nabla u|^2$ is the local free inhomogeneity energy density, and $f_0(u)$ denotes the local bulk free energy density associated with $u$.

Cahn and Hilliard [35] derived in 1958 a similar expression for the free energy of binary alloys but emphasized the energy tradeoff between the two terms: as the width of the transition zone increases, the decrease of the gradient term $|\nabla u|^2$ is counterbalanced by an increase of the bulk free energy. This leads to the *van der Waals–Cahn–Hilliard energy* (usually simply called the Cahn–Hilliard energy):

$$F_\varepsilon(u) = \int_V \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\, dx, \tag{1.1.2}$$

where $\varepsilon$ denotes the width of the transition layer and $W$ is usually taken to be a smooth double-well function of the form:

$$W(s) = \frac{(1-s^2)^2}{4}$$

in the case where the two pure phase states are represented by $\{u = -1\}$ and $\{u = 1\}$. Obviously, the energy promotes the smoothness of gas and liquid densities while penalizing the non pure states.

It is natural to study the asymptotic behavior of $F_\varepsilon$ as $\varepsilon \to 0$. A formal argument shows that, when $u$ is close to the characteristic function of a smooth set $\Omega$, $F_\varepsilon(u)$ is close to the area of $\partial\Omega$ (up to a multiplicative constant depending only on $W$). This has been proved rigorously by Modica and Mortola [103, 102] in the more general context of sets of finite perimeter and using the notion of $\Gamma$-convergence that we recall now.

## 1.2 $\Gamma$-convergence

One way to describe and understand the asymptotic behavior of a family of functionals in a variational context is the notion of $\Gamma$-*convergence* introduced by De Giorgi and Franzoni [52]. We recall below its definition and some of its basic properties, see [47, 28] for more detailed presentations on the topic.

**Definition 1.2.1.** *Let $X$ be a topological space and let $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functionals on $X$. We say that the sequence $(F_\varepsilon)_{\varepsilon>0}$* $\Gamma$-converges *to the* $\Gamma$-limit *$F : X \to \mathbb{R} \cup \{+\infty\}$ on $X$ as $\varepsilon \to 0$ if the following assertions hold for every $u \in X$:*

1. *Lower bound inequality: For every sequence $(u^\varepsilon)_{\varepsilon>0}$ of $X$ converging to $u$ in $X$ as $\varepsilon \to 0$, we have that:*
$$F(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u^\varepsilon).$$
2. *Upper bound inequality: There exists a sequence $(v^\varepsilon)_{\varepsilon>0}$ of $X$ converging to $u$ in $X$ as $\varepsilon \to 0$ such that*
$$\limsup_{\varepsilon\to 0} F_\varepsilon(v^\varepsilon) \leq F(u).$$

*In this case, we write that*
$$\Gamma(X) - \lim_{\varepsilon\to 0} F_\varepsilon = F \text{ or } \Gamma - \lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u) \text{ for every } u \in X.$$

*Remark* 1.2.2. In practice, combining both assertions in Definition 1.2.1, it is sometimes also convenient to replace the upper bound inequality by the existence of a so-called *recovery sequence* $(v^\varepsilon)_{\varepsilon>0}$ of $X$ converging to $u$ in $X$ as $\varepsilon \to 0$, and such that
$$\lim_{\varepsilon\to 0} F_\varepsilon(v^\varepsilon) = F(u).$$

Notice that $\Gamma$-convergence remains stable under continuous perturbations [47]:

**Proposition 1.2.3.** *Let $X$ be a topological space and $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functionals such that $\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon = F$. Then for every continuous function $G : X \to \mathbb{R}$, we have that*
$$\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon + G = F + G.$$

One of the most significant consequences of the $\Gamma$-convergence in a context of optimization is the link between (global) minimizers of a $\Gamma$-convergent sequence $(F_\varepsilon)$ and (global) minimizers of the $\Gamma$-limit:

**Theorem 1.2.4** ( [47]). *Assume that $(F_\varepsilon)_{\varepsilon>0}$ $\Gamma$-converges to a function $F$ in $X$ as $\varepsilon \to 0^+$. For every $\varepsilon > 0$, let $x_\varepsilon$ be a minimizer of $F_\varepsilon$ in $X$. The following properties hold:*

- *If $x$ is a cluster point of $(x_\varepsilon)$ then $x$ is a minimizer of $F$ in $X$ and $F(x) = \limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.*
- *If $(x_\varepsilon)$ converges to $x$ in $X$ then $x$ is a minimizer of $F$ in $X$ and $F(x) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.*
- *In addition, if $(F_\varepsilon)$ is equi-coercive (i.e., for every $t$ there exists a closed countably compact subset $K_t$ of $X$ such that $\{F_\varepsilon \leq t\} \subset K_t$ for every $\varepsilon$) then :*
  - $\min_{y\in X} F(y) = \lim_{\varepsilon\to 0} \inf_{y\in X} F_\varepsilon(y) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.
  - *If $x$ is a cluster point of $(x_\varepsilon)$ then $x$ is a minimizer of $F$ in $X$ and $F(x) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.*

*Remark* 1.2.5. It is important to observe that, unfortunately, $\Gamma$-convergence does not imply the convergence of local minimizers, see [28].

## 1.3 Phase-field approximation of perimeter

To get a proper Γ-convergent approximation of perimeter, it is useful to extend the definition of the Cahn–Hilliard energy to general $L^1$-functions in the following way.

**Definition 1.3.1.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$. We define $\mathcal{P}_\varepsilon : L^1(\Omega) \to \mathbb{R}_+$ by*

$$\mathcal{P}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \left( \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.3.1}$$

*where* $W(s) = \dfrac{(1-s^2)^2}{4}$.

Following a conjecture of De Giorgi, Modica and Mortola [103, 102] proved that, up to a multiplicative constant, the sequence $(\mathcal{P}_\varepsilon)_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0$ to the perimeter functional in $L^1(\Omega)$. The result holds in the context of functions of bounded variation and sets of finite perimeter, see [6].

**Theorem 1.3.2** (Modica & Mortola [103, 102])**.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$. Then* $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{P}_\varepsilon = c_0\mathcal{P}$ *on* $L^1(\Omega)$, *with* $c_0 := \displaystyle\int_{-1}^{1} \sqrt{2W(s)}\, ds$ *and* $\mathcal{P}$ *the* perimeter functional *defined by*

$$\mathcal{P}(u) = \begin{cases} |Du|(\Omega) & \text{if } u \in \mathrm{BV}(\Omega, \{-1, 1\}), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.3.2}$$

*with $|Du|(\Omega)$ the total variation of $u$ defined by*

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u(x)\ \mathrm{div}\, \varphi(x)\, dx,\ \ \varphi \in C_c^1(\Omega, \mathbb{R}^N),\ \|\varphi\|_{L^\infty(\Omega)} \le 1 \right\}.$$

*Remark* 1.3.3. $\mathcal{P}$ is called the perimeter functional for the following reason: when $E$ is a subset of $\Omega$ such that $\chi_E := 1 - 2\mathbb{1}_E \in \mathrm{BV}(\Omega)$ with $\mathbb{1}_E$ the characteristic function of $E$, $\mathcal{P}(\chi_E)$ coincides with twice the perimeter of $E$.

*Remark* 1.3.4. Assuming that $E$ has smooth boundary, there exists a so-called *recovery sequence* $(u^\varepsilon)$ such that $u^\varepsilon \to \chi_E$ in $L^1$ and

$$c_0\mathcal{P}(\chi_E) = \liminf_{\epsilon\to 0} \mathcal{P}_\varepsilon(u^\varepsilon)$$

from which follows the upper bound inequality needed to prove the Γ-convergence. The recovery sequence involves the function

$$q\left(\frac{d_E(\cdot)}{\varepsilon}\right),$$

where $d_E$ is the signed distance function to $E$ (negative in $E$, positive outside) and $q$ is a so-called *optimal profile*, i.e. a suitable minimizer of the 1D Cahn-Hilliard energy. More precisely, $q : \mathbb{R} \to \mathbb{R}$ is of class $C^1$, $q(0) = 0$, $q' = \sqrt{2W(q)}$, $\lim\limits_{s\to\pm\infty} q(t) = \pm 1$ and $q$ is a minimizer of $\displaystyle\int_{\mathbb{R}} \left( \frac{|\varphi'(t)|^2}{2} + W(\varphi(t)) \right) dt$. Recalling that $W(s) = \dfrac{(1-s^2)^2}{4}$, an explicit expression for $q$ is $q(t) = \tanh\left(\dfrac{t}{\sqrt{2}}\right)$ and

$$c_0 = \int_{-1}^{1} \sqrt{2W(s)} ds = \int_{\mathbb{R}} \left( \frac{|q'(t)|^2}{2} + W(q(t)) \right) dt = \frac{2\sqrt{2}}{3}.$$

## 1.4 De Giorgi-Bellettini-Paolini's phase-field approximation of the Willmore energy

The second part of the manuscript is devoted to the Willmore energy of smooth immersions into a Riemannian manifold, and more precisely to Willmore surfaces, i.e. images of critical points of the energy. A brief historical introduction to the Willmore energy is provided at the beginning of Section 5.1. In this first part of the manuscript, we use the following simpler definition: given an open set $E \subset \mathbb{R}^N$ with smooth boundary $\partial E$, the Willmore energy of $E$ is

$$\mathcal{W}(E) = \int_{\partial E} H_{\partial E}^2 d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure and $H$ is the (scalar) mean curvature of $\partial E$, i.e. the trace of its second fundamental form, see Definition 1.8.7.

*Remark* 1.4.1. We use two different definitions of the scalar mean curvature in the first and second parts of this manuscript to stick with the conventions used in the respective literatures. The mean curvature is the trace of the second fundamental form in the first part, i.e. the sum of the principal curvatures, but it is the half-sum in the second part! Up to these two different conventions, everything is consistent.

It is well known that the $L^2$-gradient flow of the perimeter of smooth sets is the mean curvature flow of their boundaries, see for instance [127]. It seems therefore natural to use the $L^2$-gradient of the van der Waals–Cahn–Hilliard functional to approximate the mean curvature. This is exactly what motivates the phase field model proposed by Bellettini and Paolini [18], following a conjecture of De Giorgi [51], to approximate the Willmore functional. A few other models have been proposed by various authors, see [30] for a survey, but we shall focus here on two of them which are convenient for our purpose of identifying a reaction term which promotes self-avoidance. De Giorgi–Bellettini–Paolini's model is defined as follows:

**Definition 1.4.2.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$.* De Giorgi-Bellettini-Paolini's phase field model *is $\mathcal{W}_\varepsilon : L^1(\Omega) \to \mathbb{R}_+$ defined by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \dfrac{1}{2\varepsilon} \displaystyle\int_\Omega \left( \varepsilon \Delta u - \frac{W'(u)}{\varepsilon} \right)^2 dx & \text{if } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.4.1}$$

The $\Gamma$–convergence as $\varepsilon \to 0$ of $\mathcal{W}_\varepsilon$ to the Willmore energy $\mathcal{W}$ (for the convergence in $L^1$ and in arbitrary dimension) has been first studied by Bellettini and Paolini in [18], where they proved for smooth sets (and by extension for any set with finite relaxed Willmore energy, see the next section) the upper bound inequality defined in Definition 1.2.1. In the case where $N = 2, 3$, Röger and Schätzle [125] completed the proof of $\Gamma$–convergence by showing the lower bound inequality for the Willmore energy of smooth sets. Independently, Nagasa and Tonegawa [111] also proved the $\Gamma$–convergence in the case $N = 2$.

**Theorem 1.4.3** ([125], [111])**.** *Let $N = 2, 3$. The $L^1 - \Gamma$ limit of $\mathcal{W}_\varepsilon$ coincides with $\mathcal{W}$ on smooth sets, i.e., for any $\chi_E = 1 - 2\mathbb{1}_E$ with $E \subset \Omega$ and $\partial E \in C^2$, we have that:*

$$(\Gamma_{\mathrm{L}^1(\Omega)}\text{-}\lim_{\varepsilon \to 0} (\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon))(\chi_E) = c_0 \left(\mathcal{P} + \mathcal{W}\right)(E). \tag{1.4.2}$$

*with* $c_0 := \displaystyle\int_{-1}^{1} \sqrt{2W(s)}\, ds$.

## 1.5 $\Gamma$-limit of the phase field approximation and relaxed Willmore energy

The Willmore energy is properly defined for subsets of $\mathbb{R}^N$ whose boundary is sufficiently smooth to admit an $L^2$-integrable mean curvature. In many situations, however, it is necessary to extend the definition to less regular sets. This is possible by relaxation using the notion of lower-semicontinuous envelope. We recall that the $L^1(\Omega)$-lower-semicontinuous envelope of the Willmore energy $\mathcal{W}$ is defined for any set $E$ of finite perimeter in $\Omega$ by

$$\overline{\mathcal{W}}(E) := \inf\{\liminf_{n\to+\infty} \mathcal{W}(E_n),\ E_n \subset \Omega,\ \partial E_n \in C^2,\ \chi_{E_n} \to \chi_E \text{ in } L^1(\Omega) \text{ as } n \to +\infty\}. \tag{1.5.1}$$

By locality of the mean curvature [9, 88, 97], $\overline{\mathcal{W}}$ coincides with $\mathcal{W}$ on smooth sets. Since the $\Gamma$-convergence of $\mathcal{W}_\varepsilon$ to the Willmore energy $\mathcal{W}$ is true for smooth sets, is it true that the $\Gamma$-limit of $\mathcal{W}_\varepsilon$ coincides with the *relaxed Willmore energy* $\overline{\mathcal{W}}$ on sets with finite relaxed Willmore energy? Unfortunately, it is not true: one can exhibit unsmooth sets for which the $\Gamma$-limit of $\mathcal{W}_\varepsilon$ does not coincide with the relaxed Willmore energy.

**Proposition 1.5.1** (Bellettini-Dal Maso-Paolini [16], Dang-Fife-Peletier [50])**.**

1. *There exists a bounded set $E_1 \subset \mathbb{R}^2$ of finite perimeter such that*

$$\Gamma_{\mathrm{L}^1(\Omega)}\text{-}\lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(E_1) < +\infty \ \text{ and } \ \overline{\mathcal{W}}(E_1) = +\infty; \tag{1.5.2}$$

2. *There exists a bounded set $E_2 \subset \mathbb{R}^2$ of finite perimeter such that*

$$\Gamma_{\mathrm{L}^1(\Omega)}\text{-}\lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(E_2) < \overline{\mathcal{W}}(E_2) < +\infty, \tag{1.5.3}$$

*Both sets $E_1$ and $E_2$ are illustrated in Figure 1.1.*

*Proof.* On the one hand, we have $\overline{\mathcal{W}}(E_1) = +\infty$, since, thanks to Theorem 4.1 in Bellettini-Dal Maso-Paolini [16], a domain with finite relaxed Willmore energy must admit a continuous unoriented tangent everywhere on its boundary. The fact that $\overline{\mathcal{W}}(E_2) < +\infty$ comes from Theorem 6.5 in [16], which shows that a planar set which admit a continuous unoriented tangent everywhere on its boundary has finite relaxed Willmore energy if and only if its boundary has an even number of cusp points. On the other hand, the reason why $\Gamma - \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon$ is finite on both $E_1$ and $E_2$ stems from the existence of smooth solutions with singular nodal sets for the Allen-Cahn equation:

$$\Delta u - W'(u) = 0.$$

Indeed, thanks to a result of Dang, Fife, and Peletier [50], there exists a unique solution $u \in C^\infty(\mathbb{R}^2)$ with values in $(-1, 1)$ such that

$$\begin{cases} u(x,y) < 0 & \text{if } xy > 0, \\ u(x,y) > 0 & \text{if } xy < 0 \text{ and} \\ u(x,y) = 0 & \text{if } xy = 0. \end{cases}$$

□

According to the above proposition, De Giorgi-Bellettini-Paolini's approximation $\mathcal{W}_\varepsilon$ does not $\Gamma$-converge to the relaxed Willmore energy $\overline{\mathcal{W}}$, or more precisely they coincide on smooth sets but not always on general sets of finite relaxed Willmore energy. It is then natural to ask if there exists a diffuse approximation for which the $\Gamma$-convergence holds also for unsmooth sets. As we will see in the next section, the answer is positive in $2D$.

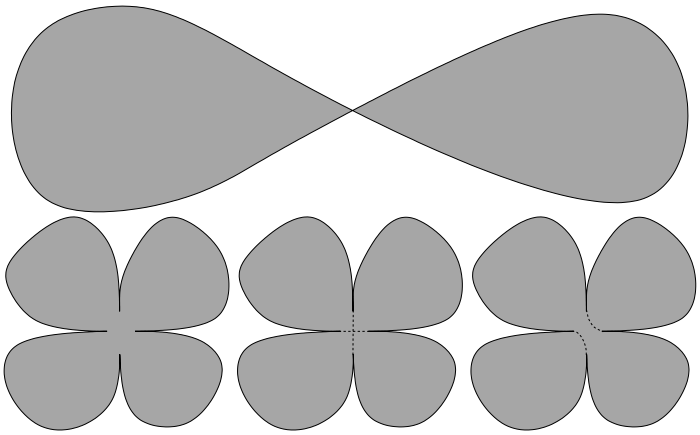

Figure 1.1: First line: the set $E_1$ satisfying (1.5.2). Its boundary does not have everywhere a continuous unoriented tangent, thus its related Willmore energy is infinite [16]. On the other hand, the characteristic function of $E_1$ is the limit of Dang-Fife-Peletier–type solutions whose De Giorgi–Bellettini–Paolini's energy is uniformly bounded. Second line: from left to right, the set $E_2$ satisfying (1.5.3), a limit configuration of Dang–Fife–Peletier–type solutions whose De Giorgi–Bellettini–Paolini's energy is uniformly bounded but whose relaxed Willmore energy is infinite, and a configuration showing that the relaxed Willmore energy $\overline{\mathcal{W}}(E_2)$ is finite.

## 1.6 Bellettini-Mugnai's phase field approximation of the second fundamental form energy

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and define

$$\mathcal{K}(E) = \int_{\Omega \cap \partial E} K_{\partial E} \, d\mathcal{H}^{N-1} \tag{1.6.1}$$

where $E \subset \Omega$ is an open set with $C^2$-boundary in $\Omega$ and $K_{\partial E}$ is the product of the principal curvatures of $\partial E$ (i.e. the Gauss curvature when $N = 3$). Let also

$$\mathcal{A}(E) := \int_{\Omega \cap \partial E} |A_{\partial E}|^2 \, d\mathcal{H}^{N-1} \tag{1.6.2}$$

with $A_{\partial E}$ the second fundamental form of $\partial E$. Notice that if $N = 2$, $|A_{\partial E}|^2 = H^2_{\partial E}$ thus $\mathcal{A}(E) = \mathcal{W}(E)$. And if $N = 3$, $\mathcal{A}(E) = \mathcal{W}(E) - 2\mathcal{K}(E)$ thus, thanks to Gauss-Bonnet theorem, minimizing in $\mathbb{R}^3$ the Willmore energy $\mathcal{W}$ with constrained genus is equivalent to minimizing $\mathcal{A}$ since $\mathcal{K}(E)$ depends only on the topology of $E$.

Bellettini and Mugnai introduced in [17] a phase-field approximation of $\mathcal{A}$ and used it to prove the $\Gamma$-convergence of a phase-field approximation of the Helfrich energy under certain conditions.

**Definition 1.6.1** (Bellettini-Mugnai [17])**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. The* Bellettini-Mugnai's *phase field model $\mathcal{W}_\varepsilon^{Mu}$ is defined by*

$$\mathcal{W}_\varepsilon^{Mu}(u) = \frac{1}{2\varepsilon} \int_{\Omega \setminus \{|\nabla u| = 0\}} \left| \varepsilon \nabla^2 u - \frac{W'(u)}{\varepsilon} \vec{n}_u \otimes \vec{n}_u \right|^2 dx. \tag{1.6.3}$$

Bellettini and Mugnai showed that, in dimensions $2, 3$, the $\Gamma$-limit of $\mathcal{W}_\varepsilon$ coincides with $\mathcal{A}$ on subsets of $\Omega$ with $C^2$-boundary:

**Theorem 1.6.2** (Corollary 4.3 in [17]). *Let $\Omega$ be a smooth domain in $\mathbb{R}^N$ with $N = 2, 3$. For any $\chi_E = 1 - 2\mathbb{1}_E$ with $E \subset \Omega$ and $\partial E \in C^2$, we have:*

$$\Gamma_{\mathrm{L}^1(\Omega)}\text{-}\lim_{\varepsilon\to 0}(\mathcal{P}^\varepsilon + \mathcal{W}_\varepsilon^{Mu})(\chi_E) = c_0(\mathcal{P}(E) + \mathcal{A}(E)). \tag{1.6.4}$$

*with $c_0 := \int_{-1}^{1} \sqrt{2W(s)}\, ds$.*

In contrast with Proposition 1.5.1, which shows that the $\Gamma$-limit of De Giorgi-Bellettini-Paolini's model does not coincide in general with the relaxed Willmore energy on unsmooth sets, it was proved by Mugnai in [109] in the two-dimensional case that such coincidence holds for the $\Gamma$-limit of Bellettini-Mugnai's model.

**Theorem 1.6.3** (Theorem 4.1 in [109]). *Let $N = 2$. For any set $E$ of finite perimeter and $\chi_E = 1 - 2\mathbb{1}_E$, we have that*

$$\Gamma_{\mathrm{L}^1(\Omega)}\text{-}\lim_{\varepsilon\to 0}(\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon^{Mu})(\chi_E) = c_0(\mathcal{P} + \overline{\mathcal{W}})(E). \tag{1.6.5}$$

Proposition 1.5.1 and Theorem 1.6.3 show that, although De Giorgi-Bellettini-Paolini's model $\mathcal{W}_\varepsilon$ and Bellettini-Mugnai's model $\mathcal{W}_\varepsilon^{Mu}$ have the same $\Gamma$-limit on smooth sets, this is not true for unsmooth sets. What can be said about the associated $L^2$-gradient flows? More precisely, do the gradient flows of $\mathcal{W}_\varepsilon$ and $\mathcal{W}_\varepsilon^{Mu}$ converge to the Willmore flow in a smooth setting? How do the flows behave just before the appearance of singularities? These questions will be addressed in Chapters 2 and 3.

## 1.7 Signed distance function to a set and medial axis

If $E \subset \mathbb{R}^N$ is nonempty we denote by $d(x, E) := \inf_{y\in E} \|x - y\|$ the Euclidean distance to $E$ of a point $x \in \mathbb{R}^N$. We also let $d(x, \emptyset) = +\infty$.

**Definition 1.7.1.** *Let $E \subset \mathbb{R}^N$. The* signed distance function *to $E$ is defined by:*

$$\forall x \in \mathbb{R}^N, \quad d_E(x) := d(x, E) - d(x, \mathbb{R}^N \setminus E).$$

We collect below some basic regularity properties of the signed distance function, see [5, 54, 66].

**Proposition 1.7.2.** *Let $E \subset \mathbb{R}^N$ be such that $\Gamma = \partial E$ is nonempty. The following properties hold:*

- *$d_E$ is 1-Lipschitz, thus differentiable a.e. in $\mathbb{R}^N$.*
- *$d_E$ is differentiable at $x \in \mathbb{R}^N \setminus \Gamma$ if and only if there exists a unique point $y \in \Gamma$ such that $d_E(x) = |y - x|$. In this case,*

$$\nabla d_E(x) = \begin{cases} \dfrac{x-y}{|x-y|} \text{ if } x \in \mathbb{R}^N \setminus E, \\ -\dfrac{x-y}{|x-y|} \text{ if } x \in E, \end{cases}$$

  *In particular, $|\nabla d_E| = 1$ at any differentiability point.*
- *If $\Gamma \in C^{1,1}$ and $n_\Gamma$ denotes the outer unit normal to $E$ on $\Gamma$, there exists a neighborhood $U$ of $\Gamma$ such that for any $x \in U$ [54]:*
  - *$x$ has a unique orthogonal projection on $\Gamma$ given by $\pi_\Gamma(x) = x - d_E(x)\nabla d_E(x)$.*
  - *$\nabla d_E(x) = n_\Gamma(\pi_\Gamma(x)) = \nabla d_E(z)$, $\forall z \in [x, \pi_\Gamma(x)]$ the line segment joining $x$ to $\pi_\Gamma(x)$.*

– *In addition, if* $\Gamma \in C^2$ *and* $s = \pi_\Gamma(x)$, $\Delta d_E(x) = \sum_{i=1}^{N-1} \frac{\kappa_i(s)}{1+\kappa_i(s)d_E(x)}$ *where* $\kappa_i(s)$ *are the principal curvatures of* $\Gamma$ *at* $s$ *along* $\nu = \nabla d_E(s)$.
*In particular, for every* $z \in \Gamma$, $\Delta d_E(z) = \sum_{i=1}^{N-1} \kappa_i(z) =$ *mean curvature at* $z$.

- *If* $k \in [2, +\infty]$ *is an integer, then*

$$\partial E \text{ is a } (N-1)\text{-manifold of class } C^k \iff \exists\, U \text{ open}, U \supset \partial E,\ d_E \in C^k(U)$$

*In such a case, we write* $\partial E \in C^k$.

There is a rich literature on *medial axes*, *skeletons* and *cut locus* of sets with various definitions. For it is more suitable for what we need later, we use the definition of medial axis based on the signed distance function.

**Definition 1.7.3** ([4]). *The* medial axis *of* $E \subset \mathbb{R}^N$ *is the singular set* $\mathrm{Sing}_E$ *of* $d_E$ *defined by:*

$$\mathrm{Sing}_E = \{x \in \mathbb{R}^N,\ d_E \text{ is not differentiable at } x\}$$

*Remark* 1.7.4. It follows that $\nabla d_E$ is well defined on $\mathbb{R}^N \setminus \mathrm{Sing}_E$ and takes values in $\mathbb{S}^{N-1}$. In addition, $x \in \mathrm{Sing}_E \setminus \partial E$ if and only if there exist at least two distinct points $y_1, y_2 \in \partial E$ such that $d(x, y_1) = d(x, y_2)$.

The following regularity properties of the medial axis are proved in [93] for the distance function to a closed, nonempty set, and easily extend to the signed distance function $d_E$ associated with a general set $E \subset \mathbb{R}^N$ with nonempty boundary.

**Proposition 1.7.5.** *If* $E \subset \mathbb{R}^N$ *has nonempty boundary then*

- $\mathrm{Sing}_E \setminus \partial E$ *is* $C^2 - (N-1)$*–rectifiable.*
- *If, in addition,* $\partial E$ *is of class* $C^r$ *with* $r \geq 3$ *then the closure* $\overline{\mathrm{Sing}_E}$ *of* $\mathrm{Sing}_E$ *is* $C^{r-2} - (N-1)$*–rectifiable. In particular,*
  - *the Hausdorff dimension of* $\overline{\mathrm{Sing}_E}$ *is at most* $(N-1)$.
  - *the vector field* $\nabla d_E$ *belongs to the space* $\mathrm{SBV}_{loc}(\mathbb{R}^N)$.

*Remark* 1.7.6. $\overline{\mathrm{Sing}_E}$ is not $(N-1)$-rectifiable in general. There is an example in [93] of a convex open set in $\mathbb{R}^2$ with $C^{1,1}$ boundary such that the closure of its medial axis has positive Lebesgue measure.

The boundary of $E$ is the set of points where $d_E$ vanishes. Is there a similar way to link the medial axis of $E$ with the (complementary of the) support of a map defined with $d_E$ or $\nabla d_E$? There is no easy way to do it directly, but the medial axis can be obtained as (a subset of) the support of the distributional limit of a sequence of well defined approximate maps, as we shall see in Chapter 4 together with various applications. There are essentially two difficulties with the medial axis:

- A smooth shape may have a non-smooth, and possibly very non smooth, medial axis [93].
- There is no continuous dependence of the medial axis of a set $E$ with respect to perturbations of $E$. A disk in $\mathbb{R}^2$ has its center as medial axis but it is easy to design arbitrarily small perturbations of the disk's boundary to get a collection of medial axes with Hausdorff dimension 1 and whose Hausdorff distance to the disk's center is bounded from below by a positive number.

## 1.8 Mean curvature flow and Willmore flow

We take exactly the definitions and characterizations of smooth flows, and in particular the mean curvature flow, given by Bellettini in [15].

We first define the parametric smooth flow of codimension 1 immersed manifolds. Recall that, given a $k$-dimensional oriented connected $C^\infty$ submanifold $S \subset \mathbb{R}^N$ without boundary, an immersion of $S$ in $\mathbb{R}^N$ is a map $\varphi \in C^\infty(S, \mathbb{R}^N)$ such that $\varphi$ is proper and, for any $s \in S$, $d\varphi(s)$ is injective. We denote $\varphi \in \mathrm{Imm}(S, \mathbb{R}^N)$. This immersion is an embedding if, in addition, $\varphi$ is an homeomorphism between $S$ and $\varphi(S)$. In such a case, we denote $\varphi \in \mathrm{Emb}(S, \mathbb{R}^N)$.

**Definition 1.8.1** (Parametric smooth flow [15])**.** *Let $S \subset \mathbb{R}^N$ be a smooth $(N-1)$-dimensional embedded oriented connected manifold without boundary, let $a, b \in \mathbb{R}$ be with $a < b$ and let $\varphi \in C^\infty([a,b] \times S; \mathbb{R}^N)$. We say that $\varphi$ is a parametric smooth flow, and we write $\varphi \in \mathcal{X}([a,b]; \mathrm{Imm}(S, \mathbb{R}^N))$, if for any $t \in [a,b]$, $\varphi(t, \cdot) \in \mathrm{Imm}(S, \mathbb{R}^N)$.*

*If, in addition, $\varphi(t, \cdot) \in \mathrm{Emb}(S, \mathbb{R}^N)$ for any $t \in [a,b]$, then we write $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(S, \mathbb{R}^N))$ and we say that $\varphi$ is an embedded smooth flow.*

*If $\varphi \in \mathcal{X}([a,b]; \mathrm{Imm}(S, \mathbb{R}^N))$, one can always choose locally a smooth unit vector field $\nu(t, \cdot)$ normal to $\varphi(t, S)$. The normal velocity vector associated with the parametric smooth flow is then defined as the map $\mathbf{V} : [a,b] \times S \to \mathbb{R}^n$ given by*

$$\mathbf{V}(t,s) := \langle \nu(t,s), \frac{\partial \varphi}{\partial t}(t,s) \rangle \nu(t,s), \quad (t,s) \in [a,b] \times S.$$

*Remark* 1.8.2. Remark that the velocity vector does not depend on the choice of orientation for $\nu$, but the velocity $V(t,s) = \langle \nu(t,s), \frac{\partial \varphi}{\partial t}(t,s) \rangle$ does

We now define smooth flows implicitly using the signed distance function. We denote as $\mathcal{P}(\mathbb{R}^N)$ the collection of subsets of $\mathbb{R}^N$.

**Definition 1.8.3** (Implicit smooth flow [15])**.** *We say that $f$ is an implicit smooth flow if:*

- *there exist $a, b \in \mathbb{R}$, $a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^N)$,*
- *for any $t \in [a,b]$ the set $f(t)$ is closed,*
- *if we denote by $d(t,z) := d_{f(t)}(z)$, $(t,z) \in [a,b] \times \mathbb{R}^N$, the time-dependent signed distance function to $f(t)$ then for any $t \in [a,b]$ there exists an open set $A_t \subset \mathbb{R}^N$ containting $\partial f(t)$ such that, setting $Q := \bigcup_{t \in [a,b]} (\{t\} \times A_t)$, we have*

$$d \in C^\infty(Q).$$

*The outer normal velocity vector of the flow at $x \in \partial f(t)$ is defined as $-\frac{\partial d}{\partial t}(t,x) \nabla d(t,x)$, where $\nabla$ denotes the gradient with respect to the space variables.*

*The outer normal velocity of the flow at $x \in \partial f(t)$ is defined as $-\frac{\partial d}{\partial t}(t,x)$.*

*Remark* 1.8.4. Notice in particular that, by Proposition 1.7.2, $\partial f(t) \in C^\infty$ for any $t \in [a,b]$.

**Definition 1.8.5** ([15])**.** *Let $E \subset \mathbb{R}^N$ be a closed set with $\partial E \in C^\infty$, and let $f : [a,b] \to \mathcal{P}(\mathbb{R}^N)$ be an implicit smooth flow. If $f(a) = E$ we say that $f$ starts from $E$ at time $a$.*

*Remark* 1.8.6 (Implicit vs parametric flows [15]).
Assume that $S \subset \mathbb{R}^N$ is a smooth $(N-1)$-dimensional embedded, oriented, connected, and compact manifold without boundary and let $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(S, \mathbb{R}^N))$. If we denote by $f(t)$ the closure of one connected component of $\mathbb{R}^N \setminus \varphi(t, S)$, then [4] $f : [a,b] \to \mathcal{P}(\mathbb{R}^N)$ is an implicit smooth flow. Conversely, if $f : [a,b] \to \mathcal{P}(\mathbb{R}^N)$ is an implicit smooth flow and $\partial f(t)$ is connected, then there exists [4] a smooth $(N-1)$-dimensional embedded oriented connected manifold $S$ without boundary

(for instance $S = \partial f(a)$) and a map $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(S, \mathbb{R}^N))$ such that $\partial f(t) = \varphi(t, S)$ for any $t \in [a,b]$.

If $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(S, \mathbb{R}^N))$ and $\partial f(t) = \varphi(t, S)$, a reasonable choice of an inner normal unit vector field is

$$\nu(t,s) = -\nabla d(t,x), \quad x = \varphi(t,s), \quad \forall (t,s) \in [a,b] \times S,$$

and it follows from $d(t, \varphi(t,x)) = 0$, $t \in [a,b]$, $s \in S$, that the inner velocity of the flow is $V(t,s) = \frac{\partial d}{\partial t}(t, \varphi(t,s))$ and the velocity vector is:

$$\mathbf{V}(t,s) = -\frac{\partial d}{\partial t}(t,x) \nabla d(t,x), \quad x := \varphi(t,s) \in \partial f(t), \quad (t,s) \in [a,b] \times S.$$

**Definition 1.8.7** (Second fundamental form and mean curvature of a codimension 1 immersion)**.** *Let $S \subset \mathbb{R}^N$ be a smooth $(N-1)$-dimensional embedded oriented connected manifold without boundary and $\varphi \in \mathrm{Imm}(S, \mathbb{R}^N)$. Let $s_1, \cdots, s_{N-1}$ be local coordinates on $S$ around $s \in S$ given by a parameterization $c$ taking some open neighborhood $O$ of an $(N-1)$-dimensional Euclidean vector space into $S$ and such that $\widehat{\varphi} := \varphi \circ c : O \to \mathbb{R}^N$ is the expression of $\varphi$ in the local system of coordinates $s_1, \cdots, s_{N-1}$. We denote by $g = (g_{\alpha\beta})_{\alpha,\beta}$ the Riemannian metric tensor induced on $S$ by the scalar product of $\mathbb{R}^N$, i.e.:*

$$g(s) = (g_{\alpha\beta}(s))_{\alpha,\beta} := (\langle \frac{\partial \widehat{\varphi}}{\partial s_\alpha}(s_1, \ldots, s_{N-1}), \frac{\partial \widehat{\varphi}}{\partial s_\beta}(s_1, \ldots, s_{N-1}) \rangle)_{\alpha,\beta}, \quad s = c(s_1, \ldots, s_{N-1}).$$

*By $g^{\alpha\beta}(s)$ we denote the $\alpha,\beta$-component of the inverse matrix $g^{-1}$ of $g$. Since $\varphi$ is locally an embedding, we can choose for $\sigma \in S$ belonging to a suitable neighborhood of a point $s \in S$ a smooth unit normal vector field to the corresponding image point $\varphi(\sigma)$, that will be denoted by $\nu(\sigma)$. If $\alpha, \beta \in [\![1, \ldots, N-1]\!]$ we define for $s = c(s_1, \ldots, s_{N-1})$:*

$$h(s) = (h_{\alpha\beta}(s))_{\alpha,\beta} := (\langle \nu(s), \frac{\partial^2 \widehat{\varphi}}{\partial s_\alpha, \partial s_\beta} \rangle)_{\alpha,\beta},$$

*and*

- *the second fundamental form at $s$, $A(s) = (h_{\alpha\beta}(s)\nu(s))_{\alpha,\beta}$, of the immersion $\varphi$;*
- *its scalar mean curvature $H(s) := \sum_{\alpha,\beta} g^{\alpha\beta}(s) h_{\alpha\beta}(s) = \mathrm{tr}(g^{-1}h)$;*
- *its mean curvature vector $\mathbf{H}(s) := H(s)\nu(s)$.*
- *when $N = 3$, its Gaussian curvature $K(s) = \det(g^{-1}h)$.*

*Remark* 1.8.8. Remark that $A$, $\mathbf{H}$ (as well as $K$ in dimension 3) are invariant under a change of sign of $\nu$, while $H$ changes sign.

*Remark* 1.8.9. As mentioned earlier, we use two different definitions of the mean curvature in the first and second parts of this manuscript to stick with the conventions used in the respective literatures. The mean curvature is the trace of the second fundamental form in the first part, i.e. the sum of the principal curvatures, but it is the half-sum in the second part.

*Remark* 1.8.10 (Mean curvature of an embedding). If $\varphi \in \mathrm{Emb}(S, \mathbb{R}^N)$, we consider the open set $E$ such that $\partial E = \varphi(S)$ and we let $d_E$ be the associated signed distance function. We choose in the arguments above the unit normal vector $\nu$ so that $\nu(s) = -\nabla d_E(\varphi(s))$, i.e. $\nu$ is an inner unit normal with respect to $E$. Then it can be proved [8] that, $\varphi$ being an embedding:

$$\left\{ \begin{array}{rcl} \mathbf{H}(s) & = & -\Delta d(x) \nabla d(x) \\ H(s) & = & \Delta d(x) \end{array} \right. , \quad x = \varphi(s).$$

**Definition 1.8.11** (Implicit smooth mean curvature flow [15])**.** *We say that $f$ is an implicit smooth mean curvature flow if:*

- *there exist $a, b \in \mathbb{R}$, $a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^N)$ is a smooth flow;*
- *the following equations hold for the signed distance function $d$ of Definition 1.8.3:*

$$\left\{ \begin{array}{rcl} \dfrac{\partial d}{\partial t}(t,x) & = & \Delta d(t,x) \\ d(t,x) & = & 0 \end{array} \right. , \quad t \in [a,b],\ x \in \partial f(t)$$

*Remark* 1.8.12. Since $|\nabla d(t,x)|^2 = 1$ for any $(t,z) \in Q$ (see Definition 1.8.3), the mean curvature flow can be equivalently defined using the velocity vector, i.e. $f$ is an implicit smooth mean curvature flow if:

$$-\frac{\partial d}{\partial t}(t,x)\nabla d(t,x) = -\Delta d(t,x)\nabla d(t,x), \quad t \in [a,b],\ x \in \partial f(t)$$

**Definition 1.8.13** (Parametric smooth mean curvature flow [15])**.** *Let $S \subset \mathbb{R}^N$ be a smooth $(N-1)$-dimensional embedded oriented connected manifold without boundary. We say that $\varphi \in \mathcal{X}([a,b]; \mathrm{Imm}(S, \mathbb{R}^N))$ is a parametric smooth mean curvature flow if, with the notations of Definition 1.8.1,*

$$\mathbf{V}(t,s) = H(t,s)\nu(t,s), \quad t \in [a,b],\ s \in S$$

*where $H(t,s)$ denotes the scalar mean curvature at $s \in S$ of the immersion $\varphi(t,\cdot) : S \to \mathbb{R}^N$.*

*Remark* 1.8.14 (Parametric vs implicit mean curvature flows). If $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(S, \mathbb{R}^N))$ is a parametric smooth mean curvature flow, it follows from the definitions and results above that the implicit and the parametric definitions of the flow are equivalent, the normal velocity vector satisfies

$$\mathbf{V}(t,s) = \mathbf{H}(t,s) = -\Delta d(t,x)\nabla d(t,x), \quad t \in [a,b],\ s \in S,\ x = \varphi(s)$$

and the inner normal velocity is:

$$V(t,s) = H(t,s) = \Delta d(t,x), \quad t \in [a,b],\ s \in S,\ x = \varphi(s)$$

**Definition 1.8.15** (Parametric smooth Willmore flow)**.** *Let $S \subset \mathbb{R}^3$ be a smooth 2-dimensional embedded oriented connected manifold without boundary. We say that $\varphi \in \mathcal{X}([a,b]; \mathrm{Imm}(S, \mathbb{R}^3))$ is a parametric smooth Willmore flow if, with the notations of Definition 1.8.1,*

$$\mathbf{V}(t,s) = -[\Delta_\varphi H + H(\frac{H^2}{2} - 2K)](t,s)\nu(t,s), \quad t \in [a,b],\ s \in S, \tag{1.8.1}$$

*where $\nu(t,\cdot)$ is the unit inner normal and $\Delta_\varphi(t,\cdot)$, $H(t,\cdot)$, and $K(t,\cdot)$ denote, respectively, the Laplace-Beltrami operator, the scalar mean curvature and the Gaussian curvature associated with the immersion $\varphi(t,\cdot)$, and with the assumption that $\nu(t,\cdot)$ is a unit inner normal to $\varphi(t,S)$.*

*Remark* 1.8.16. Recall that we use in this first part the definition of $H$ as the sum of the principal curvatures whereas the half-sum is used in the second part. If $H$ is replaced with $2H$, the right-hand side in the above definition of the Willmore flow is consistent with equation (5.3.1) in the second part.

**Definition 1.8.17** (Parametric smooth Euler-Bernoulli flow)**.** *Let $S \subset \mathbb{R}^2$ be a smooth 1-dimensional embedded connected manifold without boundary. We say that $\varphi \in \mathcal{X}([a,b]; \mathrm{Imm}(S, \mathbb{R}^2))$ is a parametric smooth Euler-Bernoulli flow (or, in short, an elastica flow) if, with the notations of Definition 1.8.1,*

$$\mathbf{V}(t,s) = -[\Delta_\varphi \kappa + \frac{\kappa^3}{2}](t,s)\nu(t,s), \quad t \in [a,b],\ s \in S,$$

*where $\kappa(t,\cdot)$ denotes the curvature associated with the immersion $\varphi(t,\cdot)$ and $\nu(t,\cdot)$ is a unit inner normal to $\varphi(t,S)$.*

## 1.9 Gradient flows and phase field approximation

### 1.9.1 Gradient flow associated with a functional

We first recall the definition of the gradient flow associated with a functional defined on a Hilbert space.

**Definition 1.9.1.** *Let $\Omega \subset \mathbb{R}^N$ be open and let $H = H(\Omega, \mathbb{R})$ be a Hilbert space of functions from $\Omega$ to $\mathbb{R}$. Given a functional $F : H \to \mathbb{R} \cup \{+\infty\}$, its $H$-gradient flow is the equation*

$$\frac{\partial u}{\partial t}(t,x) = -\frac{\delta F}{\delta u}(t,x), \tag{1.9.1}$$

*where $\frac{\delta F}{\delta u}(t,\cdot)$ is the first variation of $F$ with respect to $u$ defined as the function of $H$ which satisfies:*

$$\forall \varphi \in X, \quad \left\langle \frac{\delta F}{\delta u}(t,\cdot), \varphi \right\rangle_H = \frac{dF(u(t,\cdot)+\epsilon\varphi)}{d\epsilon}\Big|_{\epsilon=0},$$

*with $\langle \cdot,\cdot \rangle_H$ the inner product in $H$. We shall also denote $\frac{\delta F}{\delta u} = \nabla_H F(u)$.*

We recalled earlier that $\Gamma$-convergence is a suitable notion of convergence for functionals for it links global minimizers of a sequence of functionals with global minimizers of its $\Gamma$-limit. There is no such link in general for local minimizers, see [28] for examples with no convergence of local minimizers. Since local minimizers are stationary points of gradient flows, it is reasonable to expect that, in general, the $\Gamma$-convergence of a sequence of functionals does not imply the convergence of the associated gradient flows to the gradient flow of the $\Gamma$-limit. There are some exceptions though, in particular for the mean curvature flow and the Willmore flow.

### 1.9.2 Gradient flows associated with phase-field approximations of the perimeter and Willmore energies

**Definition 1.9.2** ($L^2$-gradient flow of the phase field approximate perimeter)**.**
*The $L^2$-gradient flow of the phase field approximate perimeter $\mathcal{P}^\varepsilon$ coincides with the Allen-Cahn equation up to time rescaling $t \leftarrow \varepsilon t$, i.e. $\varepsilon \partial_t u^\varepsilon = -\nabla_{L^2}\mathcal{P}^\varepsilon(u^\varepsilon)$ gives :*

$$\begin{cases} \partial_t u^\varepsilon = v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{1.9.2}$$

**Definition 1.9.3** ($H^{-1}$-gradient flow of the phase field approximate perimeter)**.**
*The $H^{-1}$-gradient flow of the phase field approximate perimeter $\mathcal{P}^\varepsilon$ coincides with the Cahn–Hilliard equation, i.e. $\partial_t u^\varepsilon = -\nabla_{H^{-1}}\mathcal{P}^\varepsilon(u^\varepsilon)$ gives :*

$$\begin{cases} \partial_t u^\varepsilon = -\varepsilon \Delta v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{1.9.3}$$

The computations of the gradient flow of De Giorgi-Bellettini-Paolini's model (1.4.1) and Bellettini-Mugnai's model (1.6.3), and their connections with the Willmore flow have been discussed in [59, 60, 133, 30]. In particular, the following result holds:

**Definition 1.9.4** ($L^2$-gradient flow of De Giorgi-Bellettini-Paolini's phase field model)**.**
*The $L^2$-gradient flow of De Giorgi-Bellettini-Paolini's model $\partial_t u^\varepsilon = -\nabla_{L^2}\mathcal{W}^\varepsilon(u^\varepsilon)$ is given, up to time rescaling, by*

$$\begin{cases} \partial_t u^\varepsilon = -\varepsilon^2 \Delta v^\varepsilon + W''(u^\varepsilon) v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{1.9.4}$$

In the sequel, we shall refer to this flow as *De Giorgi-Bellettini-Paolini's flow*.

### 1.9.3 Gradient flows of phase field approximations: existence of solutions and convergence

We mentioned earlier that the gradient flows of a family of $\Gamma$-convergent functionals need not converge to the gradient flow of the $\Gamma$-limit. There are some exceptions, though, as we shall see now with a brief summary on known results on the Allen-Cahn equation, the Cahn–Hilliard equation, and the $L^2$-gradient flow of De Giorgi-Bellettini-Paolini's phase field approximation of Willmore energy.

1. Existence of solutions to the gradient flow equations

   **Allen-Cahn equation:** there is a rich literature on the existence and uniqueness of solutions to this equation, see [15, 4] and the many references therein.

   **Cahn–Hilliard equation:** Similarly, many contributions have addressed the existence and uniqueness of solutions to this model, see [113, 100] and the references therein.

   **De Giorgi-Bellettini-Paolini's approximation of the Willmore flow:** Using a time-discretization scheme, Colli and Laurençot [43] first proved the existence and uniqueness of (weak) solutions, along with well-posedness properties, for the following De Giorgi-Bellettini-Paolini's approximation of the Willmore flow *with a volume constraint*:

   $$\begin{cases} \partial_t u^\varepsilon = -\varepsilon^2 \Delta v^\varepsilon + W''(u^\varepsilon) v^\varepsilon + \overline{W''(u^\varepsilon) v^\varepsilon} \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{1.9.5}$$

   where $\overline{W''(u^\varepsilon) v^\varepsilon}$ denotes the spatial average of $W''(u^\varepsilon) v^\varepsilon$, namely,

   $$\overline{W''(u^\varepsilon) v^\varepsilon} = \frac{1}{|\Omega|} \int_\Omega W''(u^\varepsilon) v^\varepsilon \, dx.$$

   Similar results have been proved in [44] for solutions to the flow *with volume and area constraints*:

   $$\begin{cases} \partial_t u^\varepsilon = -\varepsilon^2 \Delta v^\varepsilon + W''(u^\varepsilon) v^\varepsilon + A^\varepsilon + B^\varepsilon v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{1.9.6}$$

   where $A^\varepsilon$ and $B^\varepsilon$ are time-depending functions in $L^2$ such that

   $$A^\varepsilon + B^\varepsilon \overline{v^\varepsilon} = \overline{W''(u^\varepsilon) v^\varepsilon}.$$

   Based on the ideas developed in [43, 44], the results have been generalized for *periodic solutions* of the flow (1.9.4) by Zwilling [139].

   The three flows above are either $L^2$ or $H^{-1}$ gradient flows. More general gradient flows can actually be defined and studied in metric spaces, see the thorough study in [7].

2. Strong convergence to sharp geometric flows

- The convergence of the Allen-Cahn equation to the mean curvature flow of smooth interfaces before the appearance of singularities has been proved by various authors with various techniques and in various forms, see [15, 53, 32, 41, 78, 124, 61]. For example, Evans, Soner, and Souganidis proved in [61] that the 0-level set $\Gamma^\varepsilon$ of solutions $u^\varepsilon$ to the Allen-Cahn equation (1.9.2) with well-prepared initial conditions converges to a smooth front $\Gamma$ evolving under the mean curvature flow. More precisely,

$$\operatorname{dist}(\Gamma_t^\varepsilon, \Gamma_t) \to 0, \text{ as } \varepsilon \to 0$$

for every $0 < t < T^*$ where $T^* > 0$ is the first time where $\Gamma$ encounters topological changes under the mean curvature flow.

- Pego [116] first formally showed that, with well-prepared initial conditions, the 0-level set of the solutions $u^\varepsilon$ to the Cahn–Hilliard equation (1.9.3) converges to a front $\Gamma$ evolving under the *Hele-Shaw* motion. Namely, we have, for every $0 < t < T^*$,

$$\begin{cases} \operatorname{dist}(\Gamma_t^\varepsilon, \Gamma_t) \to 0 \text{ as } \varepsilon \to 0, \\ V_{\Gamma_t} = \dfrac{1}{2}\left[\dfrac{\partial v}{\partial n}\right]_{\Gamma_t} \\ \Delta v = 0 \text{ on } \Omega \backslash \Gamma_t \\ v = cH_{\Gamma_t} \text{ on } \Gamma_t \\ \dfrac{\partial v}{\partial N} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.9.7}$$

where $c > 0$ depends only on $W$, $n$ and $N$ are the outer normals of $\Gamma_t$ and $\partial\Omega$, respectively, $\left[\frac{\partial v}{\partial n}\right]_{\Gamma_t}$ is the difference of approximate limits of $v$ with respect to $n$ on $\Gamma_t$ (see Chapter 4 for more details) and $T^*$ is the time before the appearance of singularities of $\Gamma$. A rigorous proof of the convergence of the Cahn–Hilliard equation (1.9.3) has been later given in [3]. Then it was proved in [34] that under time rescaling $t \leftarrow \varepsilon t$, and with well-prepared initial conditions, the 0-level set of the solutions converges to a smooth front $\Gamma$ with normal velocity $V_\Gamma = -\Delta_\Gamma H_\Gamma$.

- Under time rescaling $t \leftarrow \varepsilon^2 t$, the convergence of the rescaled De Giorgi-Bellettini-Paolini flow (1.9.4) to the Willmore flow has been shown formally in [92, 133]. It was proved rigorously only recently in [63], the main arguments of the proof will be presented in Chapter 2.

3. Weak convergence to sharp geometric flows.
Alongside with the results on the strong convergence of phase-field flows, which are limited to smooth flows without singularities, more general results of weak convergence in the sense of varifolds have been proved, for instance:

   - the convergence of the Allen-Cahn equation to Brakke's mean curvature flow [74, 101].
   - the convergence of the Cahn–Hilliard equation to Hele-Shaw's motion [42].

   The reader may refer to [15] for many more references regarding strong and weak approximations of the mean curvature flow. An interesting overview of phase field approximation of geometric flows, with both theoretical and numerical aspects, is [58].

# Chapter 2

# Analysis of De Giorgi-Bellettini-Paolini's flow

This chapter is devoted to the analysis of the approximation of the Willmore flow by the $L^2-$gradient flow associated with De Giorgi-Bellettini-Paolini's model (1.4.1). We give a brief presentation of the paper of Fei and Liu [63] which addresses the question of whether the gradient flow approximates the Willmore flow. It had been shown formally to be true using the method of matched asymptotic expansion, see [30] and the references therein, Fei and Liu provide a rigorous proof.

In the end of this chapter, we address the numerical approximation of De Giorgi-Bellettini-Paolini's flow. We propose a scheme based on the splitting of the associated operator into a convex part and a (semi-)concave part. The convex part is processed implicitly and the concave part explicitly, which guarantees that the scheme is *energetically stable*.

## 2.1 Convergence of De Giorgi-Bellettini-Paolini's flow to the Willmore flow for smooth surfaces

The $L^2-$gradient flow of De Giorgi-Bellettini-Paolini's model (1.4.1) defined in Chapter 1 is $\partial_t u^\varepsilon = -\nabla_{L^2}\mathcal{W}^\varepsilon$. By direct computations, we have that

$$\begin{cases} \partial_t u^\varepsilon = -\varepsilon\Delta v^\varepsilon + \dfrac{1}{\varepsilon}W''(u^\varepsilon)v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{2.1.1}$$

We recall our choice for the double well potential $W(s) = \frac{(1-s^2)^2}{4}$. Since we expect that a solution $u^\varepsilon$ to the equation can be characterized as a minimizer of the functional $\mathcal{W}^\varepsilon$ which converges to $\pm 1$ almost everywhere as $\varepsilon \to 0$, we are interested by the behavior of $u^\varepsilon$ in a slower timescale. Up to time rescaling $t \leftarrow \varepsilon t$, the $L^2-$gradient flow can be rewritten as:

(De Giorgi-Bellettini-Paolini's flow)
$$\begin{cases} \partial_t u^\varepsilon = -\Delta v^\varepsilon + \dfrac{1}{\varepsilon^2}W''(u^\varepsilon)v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} \end{cases} \tag{2.1.2}$$

The well-posedness of De Giorgi-Bellettini-Paolini's flow (2.1.2) has been proved in the previous section. The associated energy, De Giorgi-Bellettini-Paolini's phase field model, is known to $\Gamma$-converge to the Willmore energy of smooth sets in space dimensions $2, 3$, see [18, 125, 30]. It is

thus natural to ask the following question: *Does De Giorgi-Bellettini-Paolini's flow* (2.1.2) *converge to the Willmore flow* (1.8.1)*?*

Loreti and March first provide some elements of response in [92]. By using the formal method of matched asymptotic expansion, they show that given a smooth front $\Gamma := \partial E$ evolving by Willmore flow (1.8.1), then an approximate solution $(u_A^\varepsilon, v_A^\varepsilon)$ of (2.1.2) (that is, satisfying equation (2.1.2) up to the order $\varepsilon^k$ for some $k \in \mathbb{N}^*$) such that the normal velocity of the front $\Gamma_\varepsilon := (u_A^\varepsilon)^{-1}(0)$ converges to the normal velocity of $\Gamma$, is expected to take the following form:

$$\begin{cases} u_A^\varepsilon(x,t) = q\left(\dfrac{d_E(x)}{\varepsilon}\right) + \varepsilon^2\left(\|A\|^2 - \dfrac{H^2}{2}\right)\eta_1\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^3), \\ v_A^\varepsilon(x,t) = Hq'\left(\dfrac{d_E(x)}{\varepsilon}\right) - \varepsilon H^2\eta_2\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^2), \end{cases} \tag{2.1.3}$$

where $H$ and $\|A\|^2$ are respectively the mean curvature and the second fundamental form of $\Gamma$, $\eta_1$ and $\eta_2$ are respectively solutions on $\mathbb{R}$ of

$$\begin{cases} \eta_1''(s) - W''(q(s))\eta_1(s) = sq'(s) \text{ with} & \lim\limits_{s\to\pm\infty}\eta_1(s) = 0 \\ \eta_2''(s) - W''(q(s))\eta_2(s) = q''(s) \text{ with} & \lim\limits_{s\to\pm\infty}\eta_2(s) = 0. \end{cases} \tag{2.1.4}$$

Notice that $\eta_1$ and $\eta_2$ only depend on the double well potential $W$.

It is only recently that Fei and Liu provided in [63] a rigorous justification of this result. They showed that, for well-prepared initial data, as $\varepsilon$ goes to zero, the 0-level set of the solution to De Giorgi-Bellettini-Paolini's flow converges to a surface evolving by the Willmore flow before any formation of singularities occurs.

**Theorem 2.1.1** (Fei & Liu [63])**.** *Given a smooth front $\Gamma(t) = \partial E(t)$ evolving by Willmore flow for $0 \le t \le T$. Then there exists a family of functions $(u_0^\varepsilon)_{\varepsilon>0}$ such that the solution $(u^\varepsilon, v^\varepsilon)$ of* (2.1.2)

$$\begin{cases} \partial_t u^\varepsilon = -\Delta v^\varepsilon + \dfrac{1}{\varepsilon^2}W''(u^\varepsilon)v^\varepsilon \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2}, \end{cases}$$

*supplemented with initial and boundary conditions*

$$\begin{cases} u^\varepsilon(x,0) & = u_0^\varepsilon(x) & \text{in } \Omega \\ \partial_\nu u^\varepsilon(x,t) & = u^\varepsilon(x,t) - 1 = 0 & \text{in } \partial\Omega\times(0,T), \end{cases} \tag{2.1.5}$$

*satisfies*

$$\begin{array}{ll} u^\varepsilon & \to \pm 1 \text{ in } \Omega^\pm(\delta) \text{ for every } \delta > 0 \text{ and} \\ v^\varepsilon(\Gamma^\varepsilon, t) & \to H_\Gamma(t) \text{ for every } 0 < t < T \end{array} \tag{2.1.6}$$

*where $\Gamma^\varepsilon = \{u^\varepsilon = 0\}$, $H_\Gamma(t)$ is the mean curvature of $\Gamma$, $\Omega^+(\delta) = \{x \in \Omega |\, d_E(x) \ge \delta\}$, and $\Omega^-(\delta) = \{x \in \Omega |\, d_E(x) \le -\delta\}$.*

For readers' convenience, we give a sketch of the proof which divides in two parts: the first part consists in constructing an approximate solution to the flow (2.1.2) whose level set converges to motion by Willmore flow; in the second part, the stability of the phase-field system (2.1.2) is proved, which implies that the solution to (2.1.2) is close to the approximate solution if they are close enough at initial time.

- Construction of an approximate solution:

  **Theorem 2.1.2** (Theorem 1.1 in [63])**.** *Given a smooth front $\Gamma(t)$ evolving by Willmore flow for $0 \le t \le T$. Then, for an integer $k$ large enough, there exist two functions $u_A^\varepsilon$ and $v_A^\varepsilon$ such that*

1. *$u^\varepsilon$ converges to $\pm 1$ outside of $\Gamma$, that is,*

$$u_A^\varepsilon \to \pm 1 \text{ in } \Omega^\pm(\delta) \text{ for every } \delta > 0,$$

2. *$(u^\varepsilon, v^\varepsilon)$ satisfies the following system*

$$\begin{cases} \partial_t u_A^\varepsilon = \Delta v_A^\varepsilon - \dfrac{W''(u_A^\varepsilon)}{\varepsilon^2} v_A^\varepsilon + \varepsilon^k R_1 & \text{in } \Omega \times (0,T) \\ v_A^\varepsilon = \Delta u_A^\varepsilon - \dfrac{W'(u_A^\varepsilon)}{\varepsilon} + \varepsilon^{k+2} R_2 & \text{in } \Omega \times (0,T), \end{cases} \tag{2.1.7}$$

*where $R_1(x,t)$ and $R_2(x,t)$ are uniformly bounded functions in $\varepsilon$.*

3. *$(u_A^\varepsilon, v_A^\varepsilon)$ have the following Taylor expansions (with a control of the error terms):*

$$\begin{cases} u_A^\varepsilon(x,t) = q\left(\dfrac{d_E(x)}{\varepsilon}\right) + \varepsilon^2 \left(\|A\|^2 - \dfrac{H^2}{2}\right) \eta_1\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^3), \\ v_A^\varepsilon(x,t) = Hq'\left(\dfrac{d_E(x)}{\varepsilon}\right) - \varepsilon H^2 \eta_2\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^2), \end{cases}$$

The construction of such an approximate solution is based on the approach via matched asymptotic expansions of Alikakos, Bates and Chen in [3], where a general approximate solution to the Cahn–Hilliard equation is designed to prove that the Cahn–Hilliard equation converges to the Hele-Shaw model.

- Stability of the flow (2.1.2):

**Theorem 2.1.3** (Theorem 1.2. [63]). *Let $u^\varepsilon$ be a solution of (2.1.2) such that*

$$\int_\Omega |\nabla^l (u^\varepsilon - u_A^\varepsilon)(x,0)|^2 \, dx \le c\varepsilon^{48-10l} \text{ for every } 0 \le l \le 2,$$

*for some $c > 0$, where $u_A^\varepsilon$ is defined as in Theorem 2.1.2. Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$,*

$$\sup_{0 \le t < T} \int_\Omega |\nabla^l (u^\varepsilon - u_A^\varepsilon)(x,t)|^2 \, dx \le C\varepsilon^{48-10l} \text{ for every } 0 \le l \le 2.$$

In order to prove Theorem 2.1.3, as in De Mottoni-Schatzman [53] for the convergence of the Allen-Cahn equation and in Alikakos-Bates-Chen [3] for the convergence of the Cahn–Hilliard equation, the crucial step in [63] is to show a *spectrum condition* of the linearized operator associated with the system (2.1.2) at the approximate solution. More precisely, let $R^\varepsilon = u^\varepsilon - u_A^\varepsilon$, then direct computations give

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (R^\varepsilon)^2 \, dx + Q_A^\varepsilon(R^\varepsilon) = \int_\Omega H_A^\varepsilon(R^\varepsilon) R^\varepsilon \, dx, \tag{2.1.8}$$

where $Q_A^\varepsilon$ is a functional defined by

$$\begin{aligned} Q^\varepsilon(\varphi) &:= \int_\Omega (L_A^\varepsilon(\varphi))^2 \, dx - \frac{1}{\varepsilon^2}\int_\Omega W^{(3)}(u_A^\varepsilon) v_A^\varepsilon(\varphi)^2 \, dx \\ \text{with} \quad L_A^\varepsilon(\varphi) &:= \Delta\varphi - \frac{W''(u_A^\varepsilon)}{\varepsilon^2}\varphi \qquad \forall \varphi \in H^2(\Omega) \end{aligned} \tag{2.1.9}$$

and $H_A^\varepsilon(R^\varepsilon)$ is the non-linear perturbation term with respect to $R^\varepsilon$ satisfying

$$\|H_A^\varepsilon(R^\varepsilon)\|^2_{L^2(\Omega)} = o\left(\|R\|^2_{H^2(\Omega)}\right).$$

In particular, $Q^\varepsilon$ is the quadric form of the fourth-order operator:

$$\mathcal{L}_A^\varepsilon(\varphi) := (L_A^\varepsilon)^2(\varphi) - \frac{1}{\varepsilon^2} W^{(3)}(u_A^\varepsilon) v_A^\varepsilon \varphi. \tag{2.1.10}$$

By studying the lower bound of the spectral of $\mathcal{L}_A^\varepsilon$, Fei and Liu show the following estimate on $Q_A^\varepsilon$:

**Lemma 2.1.4** (Fei-Liu [63]). *There exist $C > 0$ and $\varepsilon_0 > 0$ depending only on $k$ such that for every $0 < \varepsilon \leq \varepsilon_0$ and for every $\varphi \in H^2(\Omega)$,*

$$Q^\varepsilon(\varphi) \geq \frac{\varepsilon^4}{4} \int_\Omega |\Delta\varphi|^2 \, dx - C \int_\Omega \varphi^2 \, dx. \tag{2.1.11}$$

Using Lemma 2.1.4 in (2.1.8), together with some bootstrap arguments, gives an estimate of $\|R^\varepsilon\|^2_{H^2(\Omega)}$ which proves Theorem 2.1.3 of stability. Finally, combining Theorems 2.1.2 and 2.1.3, we have the convergence of De Giorgi-Bellettini-Paolini's flow (2.1.2) to the Willmore flow (1.8.1) before singularities occur.

The study of the quadric form $Q_A^\varepsilon$ plays an important role to prove the convergence to the Willmore flow. Heuristically, we can write that

$$\mathcal{W}^\varepsilon(u + t\varphi) = \mathcal{W}^\varepsilon(u) + \varepsilon t \int_\Omega \nabla_{L^2} \mathcal{W}^\varepsilon(u)\varphi \, dx + \frac{(\varepsilon t)^2}{2} Q^\varepsilon(\varphi) + o(\varepsilon^2 t^2). \tag{2.1.12}$$

In the following section, by reformulating the quadric term $Q_A^\varepsilon$, we show that De Giorgi-Bellettini-Paolini's model $\mathcal{W}^\varepsilon$ can be written as the sum of a convex functional and an (almost)-concave functional which provides a numerical scheme with unconditional stability.

## 2.2 Splitting numerical schemes

Let us first present the principle of splitting methods for gradient flow equations. We consider the $L^2-$gradient flow of a given functional $F$:

$$\begin{cases} \partial_t u = -\nabla_{L^2} F(u) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{2.2.1}$$

In practice, numerical methods based on explicit time discretization are not convenient due to numerical instabilities.

The splitting method has originally been introduced by Eyre in [62] in the context of implicit and semi-implicit numerical methods for approximating the solutions to the Cahn–Hilliard equation. The method has been implemented later in different contexts, and numerous variants and generalizations have been actively developed since then.

The main idea of the method in its basic form is pretty straightforward: Assume that the functional $F$ can be decomposed into a convex part and a non linear part:

$$F(u) = F_+(u) + F_-(u),$$

where $F_+$ is convex quadratic and $F_-$ is non linear. The method consists in treating implicitely the convex part and explicitly the non linear part. More precisely, the splitting scheme is implemented at each time step as follows:

$$\frac{u_{n+1}-u_n}{h} = -\nabla_{L^2} F_+(u_{n+1}) - \nabla_{L^2} F_-(u_n). \tag{2.2.2}$$

The numerical advantage of the above scheme is that usually $\nabla_{L^2} F_+$ can be chosen to be linear, which makes the computation of $u_{n+1}$ relatively easy.

One can verify that the splitting scheme defined in (2.2.2) corresponds actually to the following minimization problem:

$$\begin{cases} R_n(u_{n+1}) = \min\limits_{v-u_n \in V} R_n(v), \\ R_n(v) := \dfrac{1}{2h}\langle v-u_n, v-u_n\rangle_V + F_+(v) + \langle \nabla F_-(u_n), v-u_n\rangle_V + F_-(u_n), \end{cases} \tag{2.2.3}$$

where $V$ is a Hilbert space and $\langle\cdot,\cdot\rangle_V$ its inner product.

In the case where $F_-$ is concave, the scheme (2.2.2) is *unconditionally $F$-stable*. More precisely, the scheme (2.2.2) ensures the decreasing of $F$, i.e.,

$$F(u_{n+1}) \le F(u_n) \text{ for every } n \in \mathbb{N}. \tag{2.2.4}$$

Indeed, by the definition of $(u_n)$ in (2.2.3), we have that

$$R_n(u_{n+1}) \le R_n(u_n) \text{ for every } n \in \mathbb{N},$$

which gives

$$\frac{1}{2h}\|u_{n+1}-u_n\|_V^2 + F_+(u_{n+1}) + \langle \nabla F_-(u_n), u_{n+1}-u_n\rangle_V + F_-(u_n) \le F(u_n). \tag{2.2.5}$$

Since $F_-$ is concave, we have that

$$\langle \nabla F_-(u_n), u_{n+1}-u_n\rangle_V + F_-(u_n) \ge F_-(u_{n+1})$$

which, together with (2.2.5), gives (2.2.4).

In the following, we first introduce the splitting scheme for the Cahn–Hilliard phase field approximation to the perimeter functional which has been widely studied in numerous works such as [138, 91, 87]. Next, we turn our attention to De Giorgi-Bellettini-Paolini's model for the approximation of the Willmore energy. We show that, despite that it is unlikely to write the functional $\mathcal{W}^\varepsilon$ as the sum of a convex part and a concave part, it is possible to write $\mathcal{W}^\varepsilon$ as a sum of a convex part and a *locally concave* part in a neighborhood of each $(u_n)$. The proof is based on the reformulation of the quadratic term $Q_A^\varepsilon$ defined in (2.1.9).

### 2.2.1 Splitting scheme for the Allen-Cahn approximate mean curvature flow

We consider the Allen-Cahn equation:

$$\begin{cases} \partial_t u = \Delta u - \dfrac{W'(u)}{\varepsilon^2} \\ u(x,0) = u_0^\varepsilon(x) \end{cases} \tag{2.2.6}$$

where the optimal profile associated with an initial shape $S$ is chosen in practice as initial condition, i.e., $u_0^\varepsilon = q(\frac{d(\cdot,S)}{\varepsilon})$.

Recall that (2.2.6) is the gradient flow associated with the (rescaled) Cahn–Hilliard approximation of the perimeter:

$$\mathcal{P}_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon^2}\, dx. \tag{2.2.7}$$

We can write that

$$\mathcal{P}_\varepsilon(u) = \mathcal{P}_+^\alpha(u) + \mathcal{P}_-^\alpha(u), \tag{2.2.8}$$

where $\mathcal{P}_+^\alpha$ and $\mathcal{P}_-^\alpha(u)$ are functionals defined by

$$\mathcal{P}_+^\alpha(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \alpha u^2\right)\, dx \text{ and } \mathcal{P}_-^\alpha(u) = -\int_\Omega \left(\frac{\alpha}{2}u^2 + \frac{W(u)}{\varepsilon^2}\right)\, dx \tag{2.2.9}$$

for some $\alpha > 0$. It is clear that for every $\alpha > 0$, the functional $\mathcal{P}_+^\alpha$ is always convex.

We claim that $\mathcal{P}_-^\alpha$ is concave as soon as we take

$$\alpha \geq \frac{2}{\varepsilon^2}. \tag{2.2.10}$$

Indeed, we easily calculate that

$$D^2(\mathcal{P}_-^\alpha)(u)(\varphi,\varphi) = -\int_\Omega \left(\alpha - \frac{1}{\varepsilon^2}W''(u)\right)\varphi^2\, dx. \tag{2.2.11}$$

We get that

$$D^2(\mathcal{P}_-^\alpha)(u) \leq 0 \text{ if and only if } \alpha \geq \frac{M(u)}{\varepsilon^2}, \tag{2.2.12}$$

where $M(u)$ is defined by

$$M(u) := \sup_{|s|\leq|u|} |W''(s)|.$$

Moreover, thanks to maximum principle, we have that

$$\|u\|_\infty \leq \|u_0^\varepsilon\|_\infty \leq 1, \tag{2.2.13}$$

which gives $M(u) \leq \sup_{|s|\leq 1}|W''(s)| = 2$, and we can conclude that $\mathcal{P}_-^\alpha$ is concave whenever (2.2.10) holds. With such choices of $\mathcal{P}_+^\alpha$, $\mathcal{P}_-^\alpha$, the splitting scheme associated with the gradient flow of the approximate perimeter is:

$$\frac{u_{n+1}-u_n}{h} = -\nabla_{L^2}\mathcal{P}_+^\alpha(u_{n+1}) - \nabla_{L^2}\mathcal{P}_-^\alpha(u_n)$$

### 2.2.2 Splitting scheme for De Giorgi-Bellettini-Paolini's flow

Up to space and time rescalings, we can assume that $\varepsilon = 1$. Let $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) be an open bounded subset with smooth boundary. Consider

$$\begin{cases} \partial_t u = \Delta v - W''(u)v & \text{in } \Omega\times(0,T), \\ v = -\Delta u + W'(u) & \text{in } \Omega\times(0,T), \\ \partial_\nu u = u - 1 = 0 & \text{in } \partial\Omega\times(0,T), \\ u(0) = u_0 & \text{for } x \in \Omega, \end{cases} \tag{2.2.14}$$

for $u \in H_1^2(\Omega) := 1 + H_0^2(\Omega)$, where $H_0^2(\Omega) = \{f \in H^2(\Omega)\,|\, f = \partial_\nu f = 0 \text{ on } \partial\Omega\}$ and $W(s) = \frac{(1-s^2)^2}{4}$. Recall that (2.2.14) corresponds to the $L^2-$gradient flow of the functional

$$E(u) := \frac{1}{2}\int_\Omega \left(\Delta u - W'(u)\right)^2 dx = \frac{1}{2}\|v\|_2^2.$$

The following theorem shows that one can decompose $E$ as the sum of a convex part and a locally concave part under proper settings:

**Theorem 2.2.1.** *Let $u_0 \in H_1^2(\Omega)$, $T > 0$ and $N \in \mathbb{N}^*$. Then there exist $\alpha,\ \beta > 0$ and $h > 0$ such that the following decomposition holds:*

$$E = E_+ + E_-$$

*where $E_+$ is a convex functional defined on $H_1^2(\Omega)$ by*

$$E_+(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \alpha\int_\Omega u^2\,dx + \beta\int_\Omega |\nabla u|^2\,dx$$

*and $E_-$ defined on $H_1^2(\Omega)$ by*

$$E_-(u) := \frac{1}{2}\int_\Omega W'(u)^2 + \int_\Omega W''(u)|\nabla u|^2 - \alpha\int_\Omega u^2 - \beta\int_\Omega |\nabla u|^2$$

*is concave in a neighborhood of each $u_n \in H_1^2(\Omega)$, for all $n \in \mathbb{N}$, defined by the iterative splitting scheme*

$$\frac{u_{n+1} - u_n}{h} = -\nabla_{L^2} E_+(u_{n+1}) - \nabla_{L^2} E_-(u_n)$$

*Proof.* First, we write that $E(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 + E_1(u) + E_2(u)$ where $\forall u \in H_1^2(\Omega)$,

$$E_1(u) := \frac{1}{2}\int_\Omega W'(u)^2 \quad \text{and} \quad E_2(u) := -\int_\Omega \Delta u W'(u) = \int_\Omega W''(u)|\nabla u|^2.$$

Notice that the last identity derives from integrations by parts. Direct calculations show that, $\forall R \in H_0^2(\Omega)$,

$$D^2E_1(u)(R,R) + D^2E_2(u)(R,R) = \int_\Omega (W''(u))^2R^2 + 2\int_\Omega W''(u)|\nabla R|^2 + \int_\Omega W^{(3)}(u)(W'(u) - \Delta u)R^2.$$

Thanks to Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\left|\int_\Omega W^{(3)}(u)(W'(u) - \Delta u)R^2\right| &\le \|W'(u) - \Delta u\|_2 \|R^2 W^{(3)}(u)\|_2 \\ &\le CF(u)\|u\|_\infty \|R^2\|_2 \\ &\le CF(u)\|u\|_\infty \|R\|_4^2,\end{aligned}$$

for some $C > 0$. Remark that thanks to the Sobolev embedding theorem, $\|u\|_\infty$ is finite for all $u \in H_1^2(\Omega)$.

Moreover, thanks again to the Sobolev embedding theorem, we have that $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ for $n = 2, 3$, which implies that there exists $C(\Omega) > 0$ such that $\forall R \in H_0^2(\Omega)$,

$$\|R\|_4 \le C(\Omega)\|\nabla R\|_2.$$

Combining the above inequalities, we obtain that

$$\left|\int_\Omega W^{(3)}(u)(W'(u) - \Delta u)R^2\right| \le C(\Omega)F(u)\|u\|_\infty \int_\Omega |\nabla R|^2.$$

Finally, for $u_0 \in H_1^2(\Omega)$ given, by choosing $\alpha, \beta \in \mathbb{R}^+$ such that

$$\begin{aligned}\alpha &> \sup_{|s| \le 2\|u_0\|_\infty} W''(s)^2 \text{ and} \\ \beta &> \sup_{|s| \le 2\|u_0\|_\infty} 2W''(s) + C(\Omega)F(u_0)\|u_0\|_\infty,\end{aligned}$$

we define

$$E_+(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 + \alpha \int_\Omega u^2 + \beta \int_\Omega |\nabla u|^2 \text{ and}$$

$$E_-(u) := E_1(u) + E_2(u) - \alpha \int_\Omega u^2 - \beta \int_\Omega |\nabla u|^2.$$

It is clear by the choice of $\alpha$ and $\beta$ that $E_+$ is convex and for every $R \in H_0^2(\Omega)$, we have that

$$D^2 E_-(u_0)\,(R,R) < 0. \tag{2.2.15}$$

For $h > 0$ small enough, we can prove by induction that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by the splitting scheme

$$\frac{u_{n+1}-u_n}{h} = -\nabla_{L^2} E_+(u_{n+1}) - \nabla_{L^2} E_-(u_n)$$

is energetically stable. Indeed, as in (2.2.3), $u_{n+1}$ is characterized by $R_n(u_{n+1}) = \min_{v-u_n\in H_0^2(\Omega)} R_n(v)$ where $R_n(v) := \frac{1}{2h}\langle v-u_n, v-u_n\rangle_{L^2} + E_+(v) + \langle \nabla_{L^2} E_-(u_n), v-u_n\rangle_{L^2} + E_-(u_n)$. In particular, $R_n(u_{n+1}) \le R_n(u_n)$, which leads to

$$\begin{aligned} E(u_n) = R_n(u_n) &\ge R_n(u_{n+1}) \ge E_+(u_{n+1}) + E_-(u_n) + \langle \nabla_{L^2} E_-(u_n), u_{n+1}-u_n\rangle \\ &= E_+(u_{n+1}) + E_-(u_{n+1}) - \frac{1}{2} D^2 E_-(w_n)\langle u_{n+1}-u_n, u_{n+1}-u_n\rangle \\ &\ge E_+(u_{n+1}) + E_-(u_{n+1}) = E(u_{n+1}) \end{aligned}$$

with $w_n$ on the vector segment $[u_n, u_{n+1}]$ and using the fact that (2.2.15) holds also for $w_n$ close enough to $u_0$. □

## 2.3 De Giorgi-Bellettini-Paolini's flow and formation of singularities not consistent with the relaxed Willmore energy

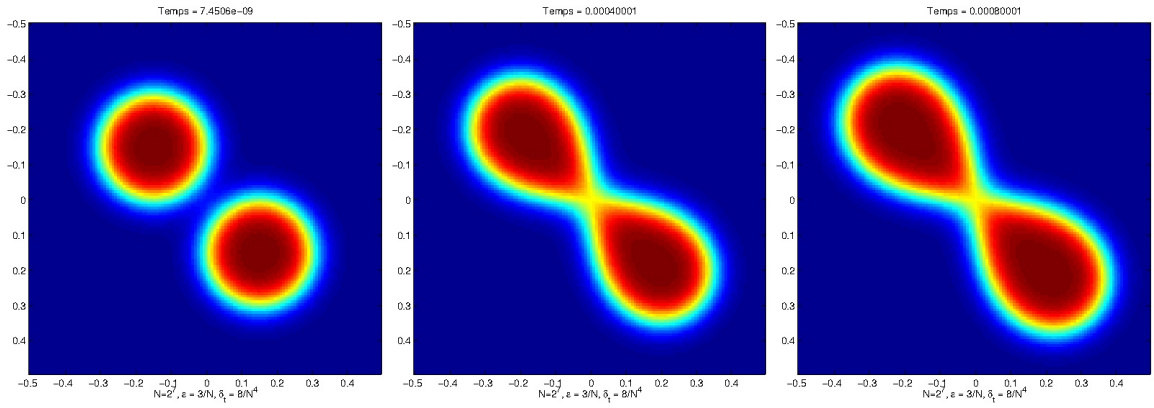


Figure 2.1: Numerical illustrations from [30]: two circles in 2D evolving by De Giorgi-Bellettini-Paolini's flow. Singularities appear in finite time, in particular the two disks merge to form a shape which has infinite relaxed Willmore energy.

We recalled in a previous section that De Giorgi-Bellettini-Paolini's phase field flow converges to the Willmore flow as long as everything remains smooth. We also recalled in Section 1.5 that De Giorgi-Bellettini-Paolini's flow may allow the creation in finite time of singularities which correspond

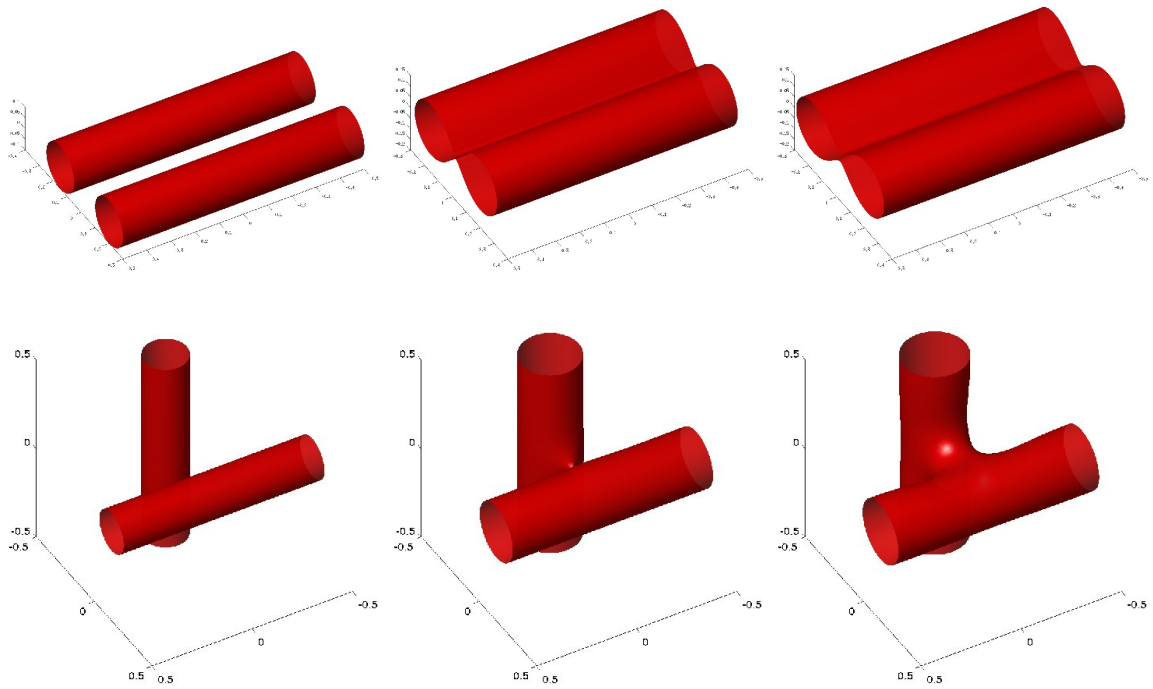

Figure 2.2: Numerical illustrations from [30]: cylinders in 3D evolving by De Giorgi-Bellettini-Paolini's flow. Singularities appear in finite time and shapes appear which have infinite relaxed Willmore energy.

to unsmooth nodal sets of smooth solutions to the Allen-Cahn equation. This can be illustrated numerically as in Figures 2.1 and 2.2 taken from [30]. As shown formally in [30], the $L^2-$gradient flow of Mugnai's model is equivalent to De Giorgi-Bellettini-Paolini's flow until the latter is close to let a singularity appear. But in contrast, with Mugnai's flow, level sets of the phase field solution deform themselves rather than colliding, see Figures 2.3 and 2.4 taken from [30]. It was actually emphasized in [30] that Mugnai's flow contains an additional reaction term with respect to De Giorgi-Bellettini-Paolini's flow. The next chapter is devoted to a more detailed analysis of this additional term.

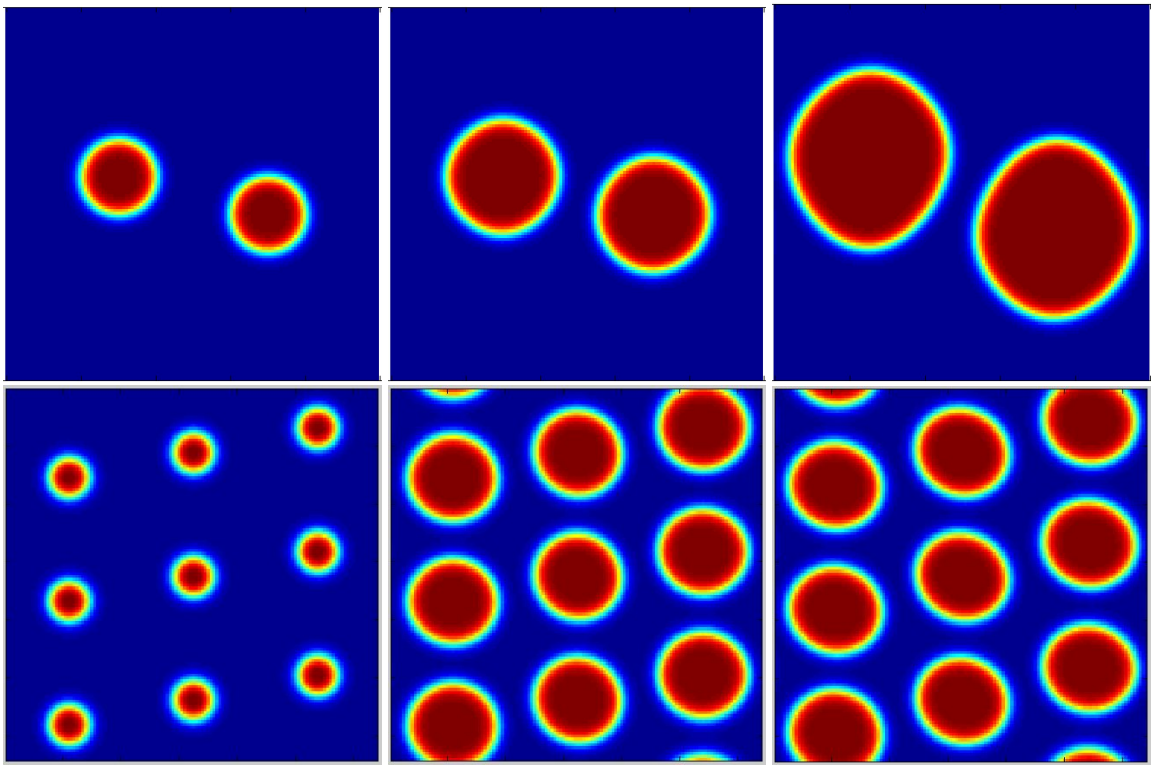

Figure 2.3: Illustrations in 2D taken from [30]. Mugnai's flow prevents from colliding, the interfaces deform themselves rather than merging.

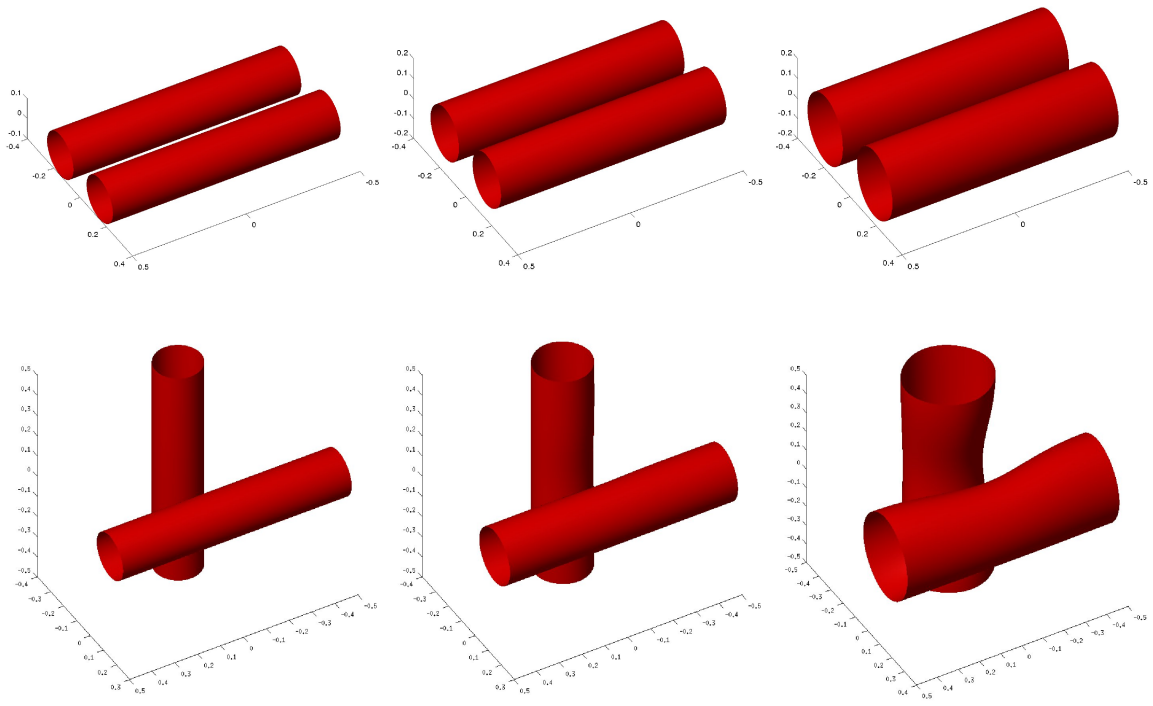

Figure 2.4: Illustrations in 3D taken from [30]. Mugnai's flow prevents from colliding, the interfaces deform themselves rather than merging.

# Chapter 3

# The self-avoidance term in Mugnai's flow

This chapter is devoted to the analysis and interpretation of a perturbation term carried by Mugnai's flow, i.e. the $L^2$-gradient flow of Mugnai's phase field approximation of the Willmore energy. This perturbation term prevents the interfaces moved by the flow from colliding, see the previous section.

## 3.1 Approximation of the Willmore flow with Mugnai's model

Let us recall that Mugnai's approximation of the Willmore energy is given by

$$\mathcal{W}_\varepsilon^{Mu}(u) := \begin{cases} \dfrac{1}{2\varepsilon}\displaystyle\int_{\Omega\setminus\{|\nabla u|\neq 0\}} \left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\vec{n}_u\otimes\vec{n}_u\right|^2 dx & \text{if } u\in L^1(\Omega)\cap W^{2,2}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}, \tag{3.1.1}$$

where $\vec{n}_u := \frac{\nabla u}{|\nabla u|}$ on $\{|\nabla u|\neq 0\}$, and $\vec{n}_u$ is set to an arbitrary unit vector on $\{|\nabla u| = 0\}$.

The associated $L^2$-gradient flow has been derived first in [30]. Up to time rescaling $t \leftarrow \varepsilon t$ and on the set $\{|\nabla u^\varepsilon|\neq 0\}$, the flow

$$\partial_t u^\varepsilon = -\nabla\mathcal{W}_\varepsilon^{Mu}(u^\varepsilon)$$

is actually equivalent to De Giorgi-Bellettini-Paolini's flow (2.1.2) plus a *perturbation term* (marked in red):

$$\text{(Mugnai's flow)} \qquad \begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon + \dfrac{1}{\varepsilon^2}W''(u^\varepsilon)v^\varepsilon + \dfrac{1}{\varepsilon^2}W'(u^\varepsilon)\mathcal{B}(\vec{n}^\varepsilon) \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2}, \end{cases} \tag{3.1.2}$$

where

$$\mathcal{B}(\vec{n}^\varepsilon) = \operatorname{div}\left(\operatorname{div}(\vec{n}^\varepsilon)\vec{n}^\varepsilon\right) - \operatorname{div}\left((\nabla\vec{n}^\varepsilon)\,\vec{n}^\varepsilon\right) \tag{3.1.3}$$

and $\nabla\vec{n}^\varepsilon = \nabla\left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\right)$ is a square $N\times N$ matrix such that $(\nabla\vec{n}^\varepsilon)_{i,j} = \frac{\partial\vec{n}_i^\varepsilon}{\partial x_j}$ for every $1\leq i,j\leq N$. Notice that $\mathcal{B}(\vec{n}^\varepsilon)$ is apparently of third order and highly non-linear with respect to its profile function $u^\varepsilon$.

A formal asymptotic expansion of a solution to this flow is provided in [30] and shows that, at least in dimensions 2 and 3, Mugnai's flow approximates the Willmore flow as long as everything remains smooth. As illustrated by the numerical experiments of Section 2.3, the flows start to diverge significantly when two parts of the 0-level set of a solution to Mugnai's flow become too close. This is due to the reaction term $\mathcal{B}(\vec{n}^\varepsilon)$ that we will now analyze further.

In the following lemma, we first show that, surprisingly, $\mathcal{B}(\vec{n}^\varepsilon)$ can be written in a form *à la Wente*, which plays an important role as a compensated compactness tool in the second part of this manuscript. Under this form, $\mathcal{B}(\vec{n}^\varepsilon)$ is of second order only.

**Lemma 3.1.1.** *If $u^\varepsilon \in C^2(\Omega)$ has non vanishing gradient $\nabla u^\varepsilon$ and $\vec{n}^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$ then*

$$\mathcal{B}(\vec{n}^\varepsilon) = 2 \sum_{1\leq i<j\leq N} \left( \frac{\partial \vec{n}_i^\varepsilon}{\partial x_i} \frac{\partial \vec{n}_j^\varepsilon}{\partial x_j} - \frac{\partial \vec{n}_j^\varepsilon}{\partial x_i} \frac{\partial \vec{n}_i^\varepsilon}{\partial x_j} \right). \tag{3.1.4}$$

*In the particular case where $N = 2$, the expression reduces to*

$$\mathcal{B}(\vec{n}^\varepsilon) = 2 \det(\nabla \vec{n}^\varepsilon) = 0.$$

*Proof.* First, we write that

$$\begin{aligned}
\operatorname{div}(\operatorname{div}(\vec{n}^\varepsilon)\vec{n}^\varepsilon) &= \sum_{1\leq j\leq N} \partial_j \left[ \left( \sum_{1\leq i\leq N} \partial_i \vec{n}_i^\varepsilon \right) \vec{n}_j^\varepsilon \right] \\
&= \sum_{1\leq i,j\leq N} \left( (\partial_{ji}^2 \vec{n}_i^\varepsilon)\vec{n}_j^\varepsilon + \partial_i \vec{n}_i^\varepsilon \partial_j \vec{n}_j^\varepsilon \right) \text{ and} \\
\operatorname{div}\left( (\nabla \vec{n}^\varepsilon)\, \vec{n}^\varepsilon \right) &= \sum_{1\leq j\leq N} \partial_j \left[ (\nabla \vec{n}^\varepsilon)\vec{n}^\varepsilon \right]_j = \sum_{1\leq j\leq N} \partial_j \left( \sum_{1\leq i\leq N} (\partial_i \vec{n}_j^\varepsilon)\vec{n}_i^\varepsilon \right) \\
&= \sum_{1\leq i,j\leq N} \left( (\partial_{ij}^2 \vec{n}_j^\varepsilon)\vec{n}_i^\varepsilon + \partial_i \vec{n}_j^\varepsilon \partial_j \vec{n}_i^\varepsilon \right).
\end{aligned}$$

Since

$$\sum_{1\leq i,j\leq N} (\partial_{ji}^2 \vec{n}_i^\varepsilon)\vec{n}_j^\varepsilon = \sum_{1\leq i,j\leq N} (\partial_{ij}^2 \vec{n}_j^\varepsilon)\vec{n}_i^\varepsilon,$$

we get that

$$\mathcal{B}(\vec{n}^\varepsilon) = \sum_{1\leq i,j\leq N} \left( \partial_i \vec{n}_i^\varepsilon \partial_j \vec{n}_j^\varepsilon - \partial_i \vec{n}_j^\varepsilon \partial_j \vec{n}_i^\varepsilon \right) = 2 \sum_{1\leq i<j\leq N} \left( \partial_i \vec{n}_i^\varepsilon \partial_j \vec{n}_j^\varepsilon - \partial_i \vec{n}_j^\varepsilon \partial_j \vec{n}_i^\varepsilon \right),$$

which gives (3.1.4). In particular, we have $\mathcal{B}(\vec{n}^\varepsilon) = 2\left( \partial_x \vec{n}_1^\varepsilon \partial_y \vec{n}_2^\varepsilon - \partial_x \vec{n}_2^\varepsilon \partial_y \vec{n}_1^\varepsilon \right) = 2 \det(\nabla \vec{n}^\varepsilon)$ if $N = 2$.

The fact that $\det(\nabla \vec{n}^\varepsilon) = 0$ comes from the identity $|\vec{n}^\varepsilon|^2 = 1$ even if $N \neq 2$. Indeed, the derivative of the identity with respect to $x_j$ gives

$$\sum_{1\leq i\leq N} (\partial_j \vec{n}_i^\varepsilon)\vec{n}_i^\varepsilon = 0, \text{ for every } 1 \leq j \leq N$$

which can be written as

$$(\nabla \vec{n}^\varepsilon)^T \vec{n}^\varepsilon = \vec{0} \in \mathbb{R}^N, \tag{3.1.5}$$

where $(\nabla \vec{n}^\varepsilon)^T$ is the transpose of the matrix $\nabla \vec{n}^\varepsilon$. Since $\vec{n}^\varepsilon$ is not a null vector, (3.1.5) implies that the matrix $\nabla \vec{n}^\varepsilon$ is not invertible. Therefore, $\det(\nabla \vec{n}^\varepsilon) = 0$. □

## 3.2 Numerical illustration of the self-avoidance term

The lemma of the previous section shows that, in dimension 2, $\mathcal{B}(\vec{n}^\varepsilon)$ vanishes at smooth points where $\nabla u^\varepsilon \neq 0$. Numerical experiments show that it seems to be essentially true, up to negligible terms,

in dimension 3 as well. Yet $\mathcal{B}(\vec{n}^\varepsilon)$ charges some singular points. We will study in the next chapter another term which vanishes at smooth points but charges singular points, the so-called jump term $S\vec{n}^\varepsilon$ defined by

$$\begin{aligned} S\vec{n}^\varepsilon &= \langle \vec{n}^\varepsilon, (\nabla \vec{n}^\varepsilon)^T \vec{n}^\varepsilon \rangle \\ &= \sum_{1 \leq i,j \leq N} \vec{n}_i^\varepsilon \frac{\partial \vec{n}_i^\varepsilon}{\partial x_j} \vec{n}_j^\varepsilon. \end{aligned}$$

We compare in the next numerical experiments the values taken by $\mathcal{B}(\vec{n}^\varepsilon)$ and $S\vec{n}^\varepsilon$ on various configurations. The computation is done as follows: if $\Omega$ denotes the shape under study, $u^\epsilon$ is defined as $q(\frac{d(\cdot,\Omega)}{\varepsilon})$ with $q$ the usual optimal profile, and $\vec{n}^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$.

Figure 3.1 shows, in the first column, the values of $u^\epsilon = q(\frac{d(\cdot,\Omega)}{\varepsilon})$ with $\Omega$ defined successively as two different annuli formed of two concentric circles. The middle column shows the values of $\mathcal{B}(\vec{n}^\varepsilon)$ and the third column the values of $S\vec{n}^\varepsilon$. Remark that none of the terms charges smooth points. $\mathcal{B}(\vec{n}^\varepsilon)$ shows a little hump at the center of the concentric circles, whereas $S\vec{n}^\varepsilon$ charges the center of the circles and the "median" circle between them. In other words, $S\vec{n}^\varepsilon$ charges the whole singular set of the signed distance function to the annuli, whereas $\mathcal{B}(\vec{n}^\varepsilon)$ charges only the center point.

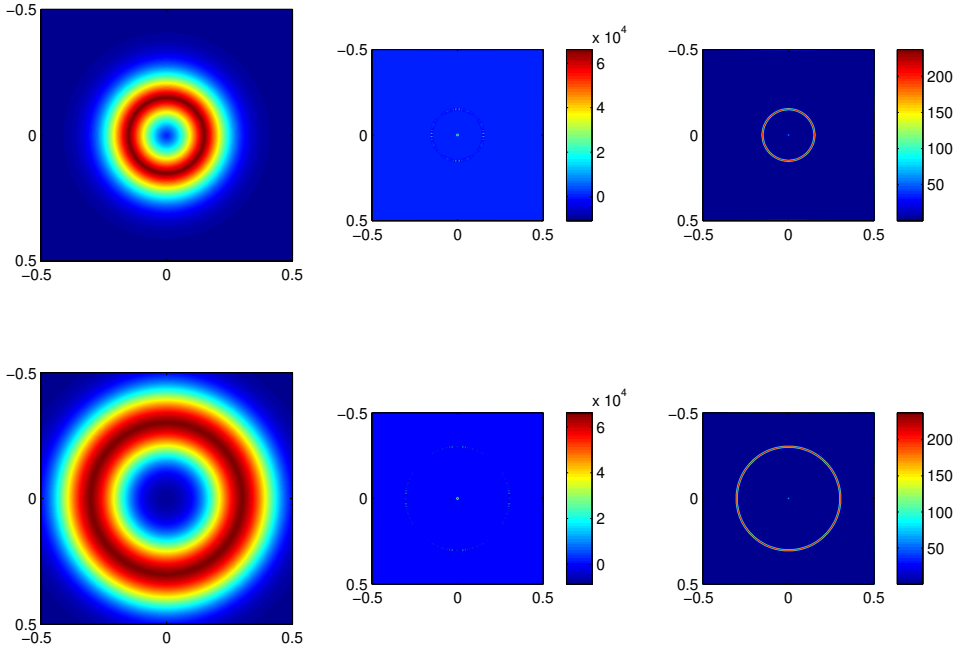


Figure 3.1: First column: values of $u^\epsilon = q(\frac{d(\cdot,\Omega)}{\varepsilon})$ with $\Omega =$ two different annuli; Second column: associated values of $\mathcal{B}(\vec{n}^\varepsilon)$ where $\vec{n}^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$; Third column: associated values of the jump term $S\vec{n}^\varepsilon$.

In the case of the two spheres of Figure 3.2, both $\mathcal{B}(\vec{n}^\varepsilon)$ and $S\vec{n}^\varepsilon$ charge their centers. In addition $S\vec{n}^\varepsilon$ charges also partially the median plane between the spheres, which is almost not charged by $\mathcal{B}(\vec{n}^\varepsilon)$. The two terms behave essentially the same way with both tori shown in Figure 3.2

Finally, Figure 3.3 shows some values taken by $\mathcal{B}(\vec{n}^\varepsilon)$ for two 3D examples. $\mathcal{B}(\vec{n}^\varepsilon)$ does not charge anything between two parallel planes but does charge the median axis between the two cylinders.

What do these numerical experiments suggest? That both terms $\mathcal{B}(\vec{n}^\varepsilon)$ and $S\vec{n}^\varepsilon$ charge only some singular points of the signed distance function to the shape under study, and not the smooth points. However, $\mathcal{B}(\vec{n}^\varepsilon)$ does not seem to charge consistently all parts of the singular set of the signed distance function.

## 3.3 Formal asymptotic expansion of Mugnai's flow

There are no known results on the existence and well-posedness of solutions to Mugnai's flow (3.1.2). The high non-linearity of $\mathcal{B}(\vec{n}^\varepsilon)$ with respect to $u^\varepsilon$ makes the problem difficult. However, a formal asymptotic expansion can be performed to identify the sharp limit. This is done in [30] where it is shown that, given a smooth front $\Gamma := \partial E$ evolving by Willmore flow (1.8.1) before the singularities occur, an approximate solution $(u_A^\varepsilon, v^\varepsilon)$ of Mugnai's flow (3.1.2) such that the normal velocity of the front $\Gamma_\varepsilon := (u_A^\varepsilon)^{-1}(0)$ converges to the normal velocity of $\Gamma$, is expected to take the following form:

$$\begin{cases} u_A^\varepsilon = q\left(\dfrac{d_E(x)}{\varepsilon}\right) + \varepsilon^2 \dfrac{\|A\|^2}{2}\eta_1\left(\dfrac{d_E(x)}{\varepsilon}\right) + O\left(\varepsilon^3\right), \\ v_A^\varepsilon = Hq'\left(\dfrac{d_E(x)}{\varepsilon}\right) - \varepsilon\|A\|^2\eta_2\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^2), \end{cases} \tag{3.3.1}$$

where $H$ and $\|A\|^2 = H^2 - K$ are respectively the mean curvature and the second fundamental form of $\Gamma$, and $\eta_1$ and $\eta_2$ are respectively solutions on $\mathbb{R}$ of

$$\begin{cases} \eta_1''(s) - W''(q(s))\eta_1(s) = sq'(s), & \text{with } \lim\limits_{s\to\pm\infty} \eta_1(s) = 0, \\ \eta_2''(s) - W''(q(s))\eta_2(s) = q''(s), & \text{with } \lim\limits_{s\to\pm\infty} \eta_2(s) = 0. \end{cases} \tag{3.3.2}$$

*Remark* 3.3.1. Recall that the formal analysis of Loreti and March [92] shows that the approxime solution to De Giorgi-Bellettini-Paolini's flow (2.1.2) admits the following Taylor:

$$\begin{cases} u_A^\varepsilon(x,t) = q\left(\dfrac{d_E(x)}{\varepsilon}\right) + \varepsilon^2\left(\|A\|^2 - \dfrac{H^2}{2}\right)\eta_1\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^3), \\ v_A^\varepsilon(x,t) = Hq'\left(\dfrac{d_E(x)}{\varepsilon}\right) - \varepsilon H^2\eta_2\left(\dfrac{d_E(x)}{\varepsilon}\right) + O(\varepsilon^2). \end{cases}$$

In particular, when $N = 2$, the two approximate solutions coincide up to order $\varepsilon^3$. And in dimension 3, also the expansions are different, they actually are both good approximations of the Willmore flow as long as there is no topology change and no appearance of singularity, see [30].

## 3.4 Flow associated with a regularized Mugnai's model

In the numerical scheme presented in [30] to approximate solutions of Mugnai's flow, the normal vector $\vec{n}^\varepsilon$ is regularized with a small parameter $\sigma > 0$, that is

$$\vec{n}_\sigma^\varepsilon := \frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \sigma^2}}.$$

In order to preserve the variational aspect of the gradient flow, one can also regularize the normal vector in the expression of Mugnai's model (1.6.3) before computing the flow. For instance, one can consider a regularized Mugnai's model with *regularized normal vector*:

$$\mathcal{W}_{\varepsilon,\sigma}^{Mu}(u^\varepsilon) = \frac{1}{2\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon}\vec{n}_\sigma^\varepsilon \otimes \vec{n}_\sigma^\varepsilon\right|^2 dx, \tag{3.4.1}$$

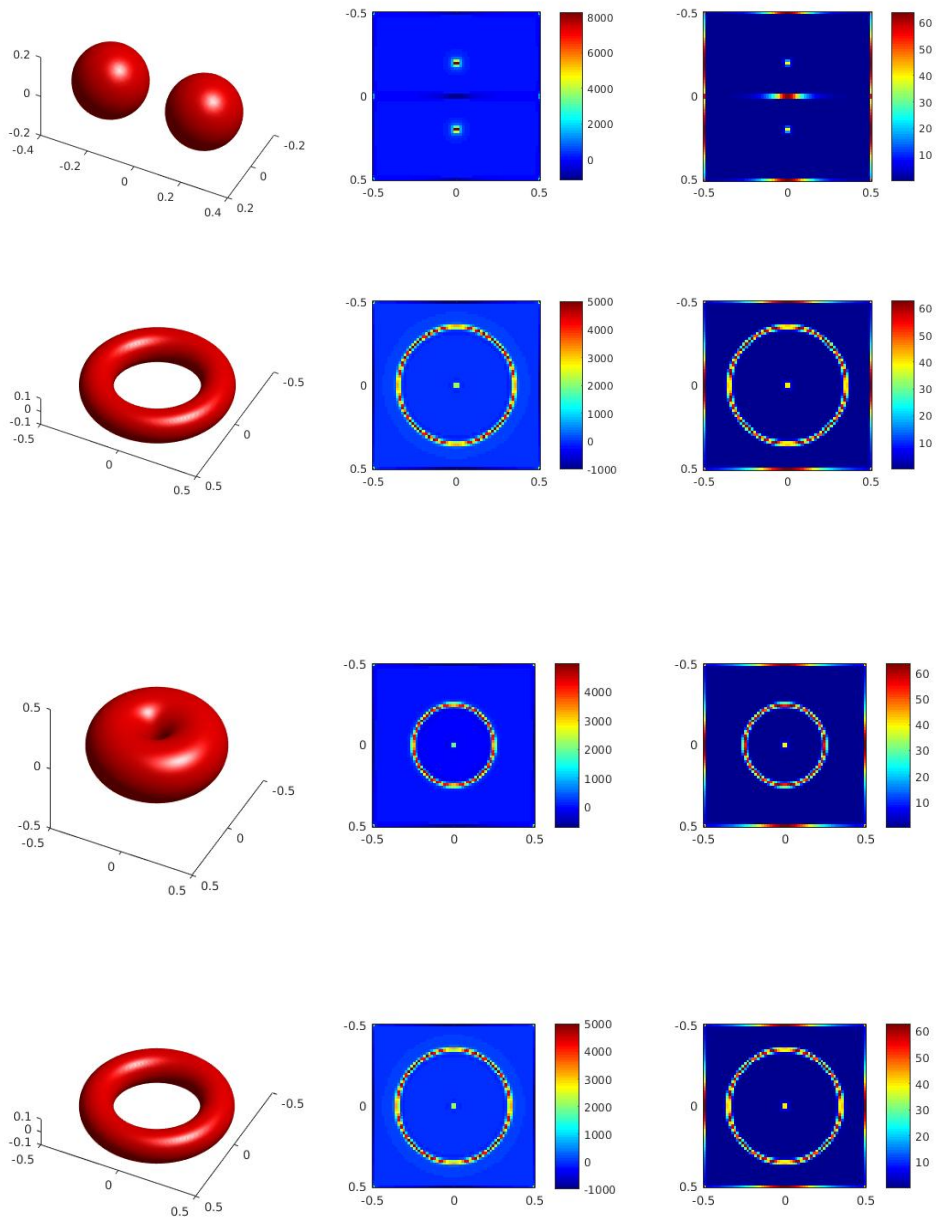

Figure 3.2: First column: Two round spheres and two tori. Middle and right columns: values of $\mathcal{B}(\vec{n}^\varepsilon)$ and $S\vec{n}^\varepsilon$ on the plane $\{x_3 = 0\}$. The following notations are used: if $\Omega$ denotes a shape, then $u^\epsilon = q(\frac{d(\cdot,\Omega)}{\varepsilon})$ and $\vec{n}^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$.

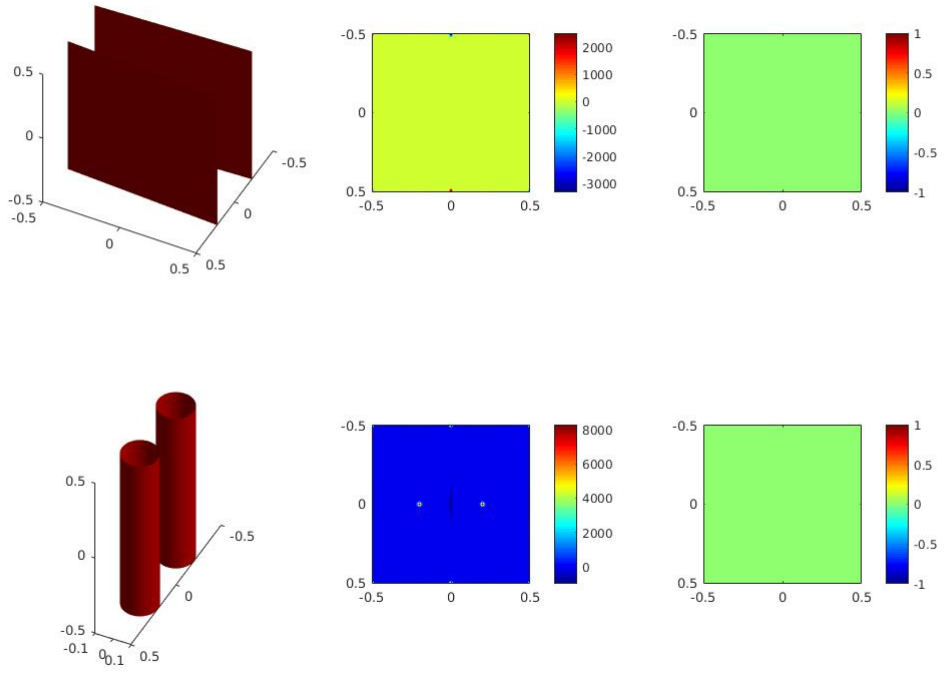


Figure 3.3: First column: two parallel planes and two parallel cylinders. Middle column: values of $\mathcal{B}(\vec{n}^{\varepsilon})$ on the plane $\{x_3 = 0\}$. Right column: values of $\mathcal{B}(\vec{n}^{\varepsilon})$ on the plane $\{x_1 = 0\}$.

In this case, thanks to Lemma 3.1.1 and by direct computations, the gradient flow of the regularized model (3.4.1) can be written as

$$\begin{cases} \partial_t u^\varepsilon = -\Delta v^\varepsilon + \dfrac{1}{\varepsilon^2} W''(u^\varepsilon) v^\varepsilon + \dfrac{1}{\varepsilon^2} W'(u^\varepsilon) \mathcal{B}(\vec{n}_\sigma^\varepsilon) + \mathcal{A}_\sigma^\varepsilon(u^\varepsilon) \\ v^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2}, \end{cases} \tag{3.4.2}$$

where

$$\begin{aligned} \mathcal{A}_\sigma^\varepsilon(u^\varepsilon) = \frac{\sigma^2}{\varepsilon^2} \Bigg[ & \frac{1}{|\nabla u^\varepsilon|^2 + \sigma^2} \left( \Delta W'(u^\varepsilon) + W''(u^\varepsilon) \Delta u^\varepsilon \right) \\ & - \frac{2}{(|\nabla u^\varepsilon|^2 + \sigma^2)} W''(u^\varepsilon) \langle (\nabla^2 u^\varepsilon) \vec{n}_\sigma^\varepsilon, \vec{n}_\sigma^\varepsilon \rangle \\ & - \frac{2}{\varepsilon^2} W'(u^\varepsilon) \operatorname{div} \left( W'(u^\varepsilon) |\vec{n}_\sigma^\varepsilon|^2 \vec{n}_\sigma^\varepsilon \right) \\ & - \frac{1}{\varepsilon^2} W''(u^\varepsilon) W'(u^\varepsilon) \left( \frac{4|\nabla u^\varepsilon|^4 + 3\sigma^2 |\nabla u^\varepsilon|^2 + \sigma^4}{(|\nabla u^\varepsilon|^2 + \sigma^2)^3} \right) \Bigg]. \end{aligned} \tag{3.4.3}$$

Notice that for $\sigma > 0$ small enough, we have $\mathcal{A}_\sigma^\varepsilon(u^\varepsilon) = \circ\left(\frac{1}{\varepsilon^2} W'(u^\varepsilon) \mathcal{B}(\vec{n}^\varepsilon)\right)$.

## 3.5 Conclusion

In comparison with De Giorgi-Bellettini-Paolini's flow, Mugnai's flow contains an additional second-order reaction term which charges essentially only singular points of the gradient of the solution to the flow, and which seems to prevent, according to numerical simulations, topology changes and appearance of singularities. However, we also observed that the parts of the singular set of the solution's gradient are not all charged consistently. The purpose of the next chapter is threefold:

- introduce and study a new, more consistent reaction term;
- associate it with an approximate mean curvature flow to get a new *self-avoiding approximate mean curvature flow*;
- study applications of this new flow to the Steiner problem and the Plateau problem.

# Chapter 4

# A self-avoiding approximate mean curvature flow

## 4.1 The discontinuity set of singular maps

In the previous chapter, we showed that, according to the numerical simulations of [30] together with Lemma 3.1.1, the additional term $\mathcal{B}(\vec{n}^\varepsilon)$ in Mugnai's flow (3.1.2) charges singular points of the profile function $u^\varepsilon$. More precisely, $\mathcal{B}(\vec{n}^\varepsilon)$ behaves as a repulsive force from the set where the gradient of $u^\varepsilon$ vanishes, thus prevents the front from topological changes in certain cases.

In this section, we first recall some well-known results on the *distributional Jacobian*, which is widely considered as an optimal indicator describing the singular set of Sobolev maps. However, by providing some examples, we show that the distributional Jacobian is not relevant to our framework where the normal vector $\vec{n}^\varepsilon$ needs not be a Sobolev map in general. This leads us to introduce a new *jump term*. We show that this jump term is able to charge the higher dimensional part of the skeleton, we also illustrate on simple configurations that other dimensional parts seem to be represented in the asymptotic expansion of the jump term.

### 4.1.1 Distributional Jacobian of Sobolev maps

Let $f = (f_1, \dots, f_N) \in W^{1,N-1}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ with $\Omega$ an open set of $\mathbb{R}^M$. The *distributional Jacobian* of $f$ can be defined as

$$Jf := \frac{1}{N} d \left( \sum_{i=1}^{N} (-1)^{i-1} f_i \bigwedge_{j \neq i} df_j \right), \tag{4.1.1}$$

in the distributional sense. Notice that in case where $f \in W^{1,N}(\Omega, \mathbb{R}^N)$, the Jacobian of $f$ is basically the determinant of the matrix $(\nabla f)_{i,j} = (\partial_j f^i)_{i,j}$:

$$Jf = \det(\nabla f) dx, \tag{4.1.2}$$

in the classical sense.

The distributional Jacobian has been actively studied in the last two decades. In particular, the distributional Jacobian for Sobolev maps taking values in the unit sphere $\mathbb{S}^{N-1}$ has been widely addressed. In the case where $M = N$, Brézis-Coron-Lieb proved that the distributional Jacobian can be written as a sum of Dirac masses.

**Theorem 4.1.1** (Brézis, Coron & Lieb [31]). *Let $u \in W^{1,N-1}(\Omega, \mathbb{S}^{N-1})$ with $\Omega \subset \mathbb{R}^N$. There exists an at most countable collection of points $\{a_i\}_{i\in J} \subset \Omega$ and integers $\{d_i\}_{i\in J} \subset \mathbb{Z}$ such that*

$$Ju = \alpha_N \sum_{i\in J} d_i \delta_{a_i},$$

*where $\alpha_N = \frac{1}{N}\mathcal{H}^{N-1}(\mathbb{S}^{N-1})$, $\delta_{a_i}$ is the Dirac mass centered at the point $a_i$ for $i \in J$ and $d_i$ is the Brouwer degree of $u$ at $a_i$.*

The above theorem implies that a Sobolev map in $W^{N-1}(\Omega, \mathbb{S}^{N-1})$ may have a gradient whose determinant vanishes almost everywhere, but the Jacobian is not necessarily zero. For instance, consider $E$ a ball centered at origin $0 \in \mathbb{R}^2$ and $d_E$ the signed distance function to $E$. One can verify that $d_E \in W^{1,1}(\Omega, \mathbb{S}^1)$ and its gradient

$$f_E := \nabla d_E(x) = \frac{x}{|x|} \text{ if } x \neq 0$$

is well-defined everywhere except at $0$. Moreover, we have that

$$Jf_E = \pi\delta_0 \text{ and } \det(\nabla f_E) = 0 \text{ in } \mathbb{R}^2 \setminus \{0\}.$$

Indeed, by differentiating the identity $|f_E|^2 = 1$ in $\mathbb{R}^2 \setminus \{0\}$, we get that

$$(\nabla f_E)^T f_E = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \mathbb{R}^2 \setminus \{0\}, \tag{4.1.3}$$

where $(\nabla f_E)^T$ is the transpose of the matrix $(\nabla f_E)$. Thanks to (4.1.3), the matrix $(\nabla f_E)$ is not invertible and thus $\det(\nabla f_E) = 0$. For the computation of the distributional Jacobian $Jf_E$, we refer the interested reader to Examples 3.1 in Jerrard-Soner [75]. Finally, several generalizations of Theorem (4.1.1) to general Sobolev maps have been studied as well, see [75, 1].

While the distributional Jacobian is well-suited for characterizing singular points of Sobolev maps in $W^{1,N}(\Omega, \mathbb{S}^{N-1})$ as shown in Theorem 4.1.1, it does not necessarily fit to our context where the signed distance function can easily fail to be a Sobolev map, which, to the author's knowledge, causes difficulties to define its distributional Jacobian. For example, consider two parallel ellipsoids in $\mathbb{R}^3$. The associated skeleton consists of two parallel segments and the signed distance function is not regular. Moreover, even if one may regularize the distance function in a symmetric way (for example, by convoluting it with a radial mollifier) in order to approximate the Jacobian, the approximate Jacobian (which can be computed by the determinant) is always zero since the third coordinate of the gradient vanishes.

In fact, since the skeleton of a domain is meagre, as shown in the next section, it is more convenient to work with functions of *bounded variation*.

### 4.1.2 BV functions and the jump term

In the following, we first introduce the notion of function of bounded variation and its basic properties, see [6] for more detailed information on the topic.

Given $u \in L^1(\Omega)$, we say that $u$ is a *function of bounded variation* if its distributional gradient $Du$ is a vector-valued Radon measure with finite total variation in $\Omega$. More precisely, we have that

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}(\vec{\varphi})\, dx \,|\, \vec{\varphi} \in C_c^1(\Omega, \mathbb{R}^N),\ \|\vec{\varphi}\|_\infty \leq 1 \right\} < +\infty.$$

The class of such functions is denoted as $BV(\Omega)$. Endowed with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, it is a Banach space. The notion of bounded variation can naturally be extended for vector-valued functions.

For every $u \in BV(\Omega)$, $Du$ is a Radon measure which can be written as the sum of an absolutely continuous part and a singular part with respect to the Lebesgue measure, i.e.:

$$Du = D^A u + D^S u.,$$

where $D^A u = \nabla^A u dx$ with $\nabla^A u$ the Radon-Nikodym derivative of $Du$ with respect to the Lebesgue measure in $\mathbb{R}^N$, and $D^S u \perp dx$. Also notice that $u$ is approximately differentiable a.e. in $\Omega$ and its approximate differential coincides with $\nabla^A u$ a.e. in $\Omega$, see[6].

The singular part $D^S u$ decomposes into the *jump part* $D^J u$ and the *Cantor part* $D^C u$. More precisely, we can write that

$$D^J u = ([u] \otimes \nu)\mathcal{H}^{N-1} \llcorner \Sigma(u), \tag{4.1.4}$$

where $\Sigma(u)$ is the *jump set* of $u$ oriented by a unit vector $\nu$, and $[u] := u^+ - u^-$ is the difference of the approximate limits of $u$ with respect to $\nu$ on $\Sigma(u)$. Namely, we have

$$u^{\pm}(x) := \lim_{r \to 0} \fint_{B^{\pm}(x,r)} u(y)\, dy$$

where $B^+(x,r) = \{y \in \mathbb{R}^N \mid \langle y, \nu(x)\rangle_{\mathbb{R}^N} > 0\}$ and $B^-(x,r) = \{y \in \mathbb{R}^N \mid \langle y, \nu(x)\rangle_{\mathbb{R}^N} < 0\}$.

The Cantor part $D^C u$ is a signed measure which does *not* charge any $\mathcal{H}^{N-1}-$finite set, that is

$$D^C u(O) = 0,\ \forall O \subset \Omega \text{ s.t. } \mathcal{H}^{N-1}(O) < +\infty.$$

Finally, we say that $u \in BV(\Omega)$ is a function of *special bounded variation*, denoted by $u \in SBV(\Omega)$, if its Cantor part vanishes, i.e. $D^C u = 0$.

Now we are at the point to introduce the notion of *jump term*.

**Definition 4.1.2.** *For every $\vec{n} \in SBV(\Omega, \mathbb{S}^{N-1})$, we define the (regularized)* jump term *$S\vec{n}^\sigma$ of $\vec{n}$ by*

$$S\vec{n}^\sigma = \sum_{1 \le i,j \le N} \vec{n}_i^\sigma \partial_j \vec{n}_i^\sigma \vec{n}_j^\sigma = \langle n^\sigma, (\nabla n^\sigma)^T n^\sigma \rangle,$$

*where $\sigma > 0$ and $\vec{n}^\sigma = f^\sigma * \vec{n}$ is the mollification of $\vec{n}$ by a smooth kernel $f^\sigma = \frac{1}{\sigma^N} f\left(\frac{\cdot}{\sigma}\right)$ with $f \in C_c^\infty(\Omega)$ and $\int_\Omega f = 1$.*

The intention of the jump term is pretty straightforward: by formally deriving the identity

$$|\vec{n}|^2 = 1$$

and then by considering its inner product with $\vec{n}$, in the case where $\vec{n}$ is regular, we get that

$$\sum_{1 \le i,j \le N} \vec{n}_i \partial_j \vec{n}_i \vec{n}_j \equiv 0.$$

Therefore, we expect that the expression of the jump term introduced in the above definition preserves only the singular part of the map $\vec{n}$.

In the following, we show that the regularized jump term of a $\vec{n} \in SBV(\Omega, \mathbb{S}^{N-1})$ converges asymptotically to a measure supported on the jump set $\Sigma$ of $\vec{n}$.

**Theorem 4.1.3.** *Let $\vec{n} \in SBV(\Omega, \mathbb{S}^{N-1})$ with $C^1-$ jump set $\Sigma$ oriented by the unit vector $\nu$. Then*

$$S\vec{n}^\sigma \to \frac{1}{12} |[\vec{n}]|^2 \langle [\vec{n}], \nu \rangle\, ds_\Sigma \quad \text{in } \mathcal{D}'(\Omega) \quad \text{as } \sigma \to 0, \tag{4.1.5}$$

*where $ds_\Sigma$ is the measure volume on $\Sigma$ and $\vec{n}^\sigma = f^\sigma * \vec{n}$ is the mollification of $\vec{n}$ by a smooth kernel $f^\sigma = \frac{1}{\sigma^N} f\left(\frac{\cdot}{\sigma}\right)$ with $f \in C_c^\infty(\Omega)$ and $\int_\Omega f = 1$.*

*Proof.* Since $\vec{n}^\sigma = f^\sigma * \vec{n}$, we have that

$$\partial_j \vec{n}_i^\sigma = f^\sigma * \partial_j \vec{n}_i = f^\sigma * \partial_j^A \vec{n}_i + f^\sigma * \partial_j^J \vec{n}_i = f^\sigma * \partial_j^A \vec{n}_i + f^\sigma * [\vec{n}_i]\nu_j \, ds_\Sigma.$$

Therefore, for every function with compact support $\varphi \in C_c^\infty(\Omega)$, we can write that

$$\sum_{i,j} \int_\Omega \vec{n}_i^\sigma \partial_j \vec{n}_i^\sigma \vec{n}_j^\sigma \varphi dx = I_1^\sigma + I_2^\sigma,$$

where

$$I_1^\sigma = \sum_{i,j} \int_\Omega \vec{n}_i^\sigma \partial_j^A \vec{n}_i^\sigma \vec{n}_j^\sigma \varphi \, dx \quad \text{and} \quad I_2^\sigma = \sum_{i,j} \int_\Omega \vec{n}_i^\sigma g_{ij}^\sigma \vec{n}_j^\sigma \varphi \, dx \tag{4.1.6}$$

with

$$g_{ij}^\sigma = f^\sigma * [\vec{n}_i]\nu_j \, ds_\Sigma.$$

Without loss of generality, we can assume that $\Omega = \Sigma(1) := \{x \in \Omega \,|\, d(x, \Sigma) \leq 1\}$ and that the map

$$\begin{aligned} \tau : \Sigma(1) &\to [-1, 1] \times \Sigma \\ x &\mapsto (r, s) \end{aligned}$$

is a diffeomorphism, with $s = \pi_\Sigma(x)$ the normal projection of $x$ on $\Sigma$ and $r = \langle x - s, \nu(s) \rangle$, so that $x = s + r\nu(s)$, . Thanks to the fact that

$$\vec{n}_i^\sigma = f^\sigma * \vec{n}_i \to \vec{n}_i \quad \text{a.e.,} \quad f^\sigma * \partial_j^A \vec{n}_i^\sigma \to \partial_j^A \vec{n}_i \ \text{ a.e. as } \sigma \to 0,$$

and $|\vec{n}|^2 = \sum_i \vec{n}_i^2 = 1$, we get that, for every $\sigma > 0$, $|\vec{n}_i^\sigma (f^\sigma \star \partial_j^A \vec{n}_i^\sigma)| \leq C |\partial_j^A \vec{n}_i|$ for some $C > 0$ and

$$\sum_i \vec{n}_i^\sigma \left( f^\sigma * \partial_j^A \vec{n}_i^\sigma \right) \to \sum_i \vec{n}_i \partial_j^A \vec{n}_i = \frac{1}{2} \partial_j \left( |\vec{n}|^2 \right) = 0 \quad \text{a.e. as } \sigma \to 0, \tag{4.1.7}$$

which, thanks to Lebesgue's theorem, leads to

$$\lim_{\sigma \to 0} I_1^\sigma = \sum_{i,j} \int_\Omega \vec{n}_i \partial_j^A \vec{n}_i \vec{n}_j \varphi \, dx = 0. \tag{4.1.8}$$

In order to estimate $I_2^\sigma$ defined in (4.1.6), by identifying $x = (r, s)_\tau$ and taking $z = r/\sigma$, we can write that

$$\begin{aligned} I_2^\sigma &= \sum_{i,j} \int_{-1}^1 \int_\Sigma \vec{n}_i^\sigma(r, s) g_{ij}^\sigma(r, s) \vec{n}_j^\sigma(r, s) \varphi(r, s) J_\Sigma(r, s) \, ds \, dr \\ &= \sum_{i,j} \int_{-\infty}^\infty \int_\Sigma \vec{n}_i^\sigma(\sigma z, s) g_{ij}^\sigma(\sigma z, s) \vec{n}_j^\sigma(\sigma z, s) \varphi(\sigma z, s) \chi_{[-1/\sigma, 1/\sigma]}(z) \sigma J_\Sigma(\sigma z, s) \, ds \, dz, \end{aligned} \tag{4.1.9}$$

where $\chi_{[-1/\sigma, 1/\sigma]}$ is the characteristic function of $[-1/\sigma, 1/\sigma]$ and $J_\Sigma = \det \left( \partial_{(r,s)} \tau^{-1}(r, s)_\tau \right)$ is the Jacobian of $\tau^{-1}$. We are now at the point of studying the pointwise limit of the integrand function as $\sigma$ goes to 0.

Let $(\sigma z_0, s_0)_\tau \in [-1, 1] \times \Sigma$, we have that

$$\sigma g_{ij}^\sigma(\sigma z_0, s_0)_\tau = \int_\Sigma \sigma f^\sigma((\sigma z_0, s_0)_\tau - (0, s)_\tau)[\vec{n}_i](s)\nu_j(s) ds. \tag{4.1.10}$$

As $\Sigma$ is $C^1$, we can parametrize locally $\Sigma$ as a graph around $s_0$. More precisely, up to rotation, we can assume that $\nu(s_0) = (0_{\mathbb{R}^{N-1}}, 1)$ and then there exist a neighbourhood $V(s_0)$ of $s_0$ and $\delta > 0$ such that $\forall s \in V(s_0)$, we can write that

$$s = s(u) = s_0 + (u, \Psi(u)), \tag{4.1.11}$$

where $\Psi : B^{N-1}(0,\delta) \to \mathbb{R}$ is a $C^1$-map such that $\Psi(0) = |\nabla\Psi(0)| = 0$. For $\sigma > 0$ small enough, with help of (4.1.11), (4.1.10) can thus be written as

$$\begin{aligned} \sigma g_{ij}^{\sigma}(\sigma z_0, s_0) &= \int_{B^{N-1}(0,\delta)} \sigma f^{\sigma}\left((\sigma z_0, s_0)_\tau - (0, s(u))_\tau\right)[\vec{n}_i]\nu_j\left(s(u)\right)\sqrt{1+|\nabla\Psi(u)|^2}\,du \\ &= \int_{B^{N-1}(0,\delta/\sigma)} f\left(\frac{(\sigma z_0, s_0)_\tau - (0, s(\sigma u))_\tau}{\sigma}\right)[\vec{n}_i]\nu_j(s(\sigma u))\sqrt{1+|\nabla\Psi(\sigma u)|^2}\,du, \end{aligned} \tag{4.1.12}$$

where the second equality is obtained by replacing $u$ by $\sigma u$. Moreover we have that

$$\begin{aligned} \frac{(\sigma z_0, s_0)_\tau - (0, s(\sigma u))_\tau}{\sigma} &= \frac{s_0 + (0_{\mathbb{R}^{N-1}}, \sigma z_0) - s_0 - (\sigma u, \Psi(\sigma u))}{\sigma} \\ &= \left(-u, z_0 - \frac{\Psi(\sigma u)}{\sigma}\right) \to (-u, z_0) \text{ as } \sigma \to 0, \end{aligned} \tag{4.1.13}$$

thanks to (4.1.11). Together with (4.1.12) and (4.1.13), we get that

$$\sigma g_{ij}^{\sigma}(\sigma z_0, s_0) \to g(z_0)[\vec{n}_i](s_0)\nu_j(s_0) \text{ as } \sigma \to 0, \tag{4.1.14}$$

where $g(z_0) = \int_{\mathbb{R}^{N-1}} f(-u, z_0)du = \int_{\mathbb{R}^{N-1}} f(u, z_0)\,du$.

By analogy, for $\sigma$ small enough, we also have

$$\begin{aligned} \vec{n}_i^{\sigma}(\sigma z_0, s_0) &= \int_{-1/\sigma}^{1/\sigma}\int_{\Sigma} f^{\sigma}\left((\sigma z_0, s_0)_\tau - (\sigma z, s)_\tau\right)\vec{n}_i(\sigma z, s)_\tau \sigma J_\Sigma(\sigma z, s)_\tau\,ds\,dz \\ &= \int_{-1/\sigma}^{1/\sigma}\int_{B^{N-1}(0,\delta/\sigma)} f\left(\frac{(\sigma z_0, s_0)_\tau - (\sigma z, s(\sigma u))_\tau}{\sigma}\right) \\ &\times \vec{n}_i(\sigma z, s(\sigma u))_\tau J_\Sigma(\sigma z, s(\sigma u)_\tau)\sqrt{1+|\nabla\Psi(\sigma u)|^2}\,du\,dz \end{aligned} \tag{4.1.15}$$

Moreover, since we have that

$$\begin{aligned} \vec{n}_i(\sigma z, s(\sigma u))_\tau &\to \begin{cases} \vec{n}_i^+(s_0) \text{ if } z > 0, \\ \vec{n}_i^-(s_0) \text{ if } z < 0, \end{cases} \\ J_\Sigma(\sigma z, s(\sigma u))_\tau &\to 1 \text{ and} \\ \frac{(\sigma z_0, s_0)_\tau - (\sigma z, s(\sigma u))_\tau}{\sigma} &\to (-u, z_0 - z)\,, \end{aligned} \tag{4.1.16}$$

as $\sigma \to 0$. Notice that the second limit is due to the fact that $J_\Sigma(r,s)_\tau = 1 + H_\Sigma(s)r + o(r)$, see [127].

Thanks to (4.1.15) and (4.1.16), we get that

$$\begin{aligned} \vec{n}_i(\sigma z_0, s_0) &\to \int_0^{+\infty}\int_{\mathbb{R}^{N-1}} f(-u, z_0 - z)\vec{n}_i^+(s_0)\,du\,dz + \int_{-\infty}^{0}\int_{\mathbb{R}^{N-1}} f(-u, z_0 - z)\vec{n}_i^-(s_0)\,du\,dz \\ &= \left(\int_{-\infty}^{z_0} g(z)\,dz\right)\vec{n}_i^+(s_0) + \left(\int_{z_0}^{+\infty} g(z)\,dz\right)\vec{n}_i^-(s_0) \\ &= G(z_0)\vec{n}_i^+(s_0) + (1 - G(z_0))\,\vec{n}_i^-(s_0), \end{aligned} \tag{4.1.17}$$

as $\sigma \to 0$, where $G(z_0) = \int_{-\infty}^{z_0} g(z)dz$.

Finally, together with (4.1.9), (4.1.14), (4.1.16) and (4.1.17), we obtain that

$$\begin{aligned}\lim_{\sigma\to 0} I_2^\sigma =& \sum_{i,j}\int_\Sigma\int_{\mathbb{R}} \left(G(z)\vec{n}_i^+(s)+(1-G(z))\,\vec{n}_i^-(s)\right)(g(z)[\vec{n}_i]\nu_j)\\ &\times\left(G(z)\vec{n}_j^+(s)+(1-G(z))\,\vec{n}_j^-(s)\right)\,dz\,\varphi(0,s)_\tau\,ds\\ =&\sum_i\int_\Sigma\int_{\mathbb{R}} G(z)g(z)[\vec{n}_i](G(z)\vec{n}_i^+ + (1-G(z))n_i^-)\langle\vec{n}^+,\nu\rangle(s)\varphi(0,s)_\tau\,ds\\ &+\sum_i\int_\Sigma\int_{\mathbb{R}}(1-G(z))g(z)[\vec{n}_i](G(z)\vec{n}_i^+ + (1-G(z))n_i^-)\langle\vec{n}^-,\nu\rangle(s)\varphi(0,s)_\tau\,ds,\end{aligned} \tag{4.1.18}$$

which, thanks to the fact that $\sum_i\left(1-\vec{n}_i^+\vec{n}_i^-\right)=\frac{|[\vec{n}]|^2}{2}$, leads to

$$\begin{aligned}\lim_{\sigma\to 0} I_2^\sigma =&\sum_i\int_\Sigma\int_{\mathbb{R}} G(z)g(z)(2G(z)-1)(1-\vec{n}_i^+\vec{n}_i^i)\langle\vec{n}^+,\nu\rangle(s)\varphi(0,s)_\tau\,ds\\ &+\sum_i\int_\Sigma\int_{\mathbb{R}}(1-G(z))g(z)(2G(z)-1)(1-\vec{n}_i^+\vec{n}_i^i)\langle\vec{n}^-,\nu\rangle(s)\varphi(0,s)_\tau\,ds\\ =&\int_\Sigma\left(\int_{\mathbb{R}} G(z)g(z)(2G(z)-1)\,dz\right)\frac{|[\vec{n}]|^2}{2}\langle\vec{n}^+,\nu\rangle(s)\varphi(0,s)_\tau\,ds\\ &+\int_\Sigma\left(\int_{\mathbb{R}}(1-G(z))g(z)(2G(z)-1)\,dz\right)\frac{|[\vec{n}]|^2}{2}\langle\vec{n}^-,\nu\rangle(s)\varphi(0,s)_\tau\,ds\\ =&\frac{1}{12}\int_\Sigma|[\vec{n}]|^2\langle[\vec{n}],\nu\rangle\varphi(0,s)\,ds=\left\langle\frac{1}{12}|[\vec{n}]|^2\langle[\vec{n}],\nu\rangle\,ds_\Sigma,\varphi\right\rangle.\end{aligned} \tag{4.1.19}$$

Hence, together with (4.1.8) and (4.1.19), we prove Theorem 4.1.3. □

*Remark* 4.1.4. In the case where $N=2$ and $\vec{n}=\nabla d_E$ is the gradient of a signed distance function on a certain domain $E$, $\vec{n}^\perp:=\begin{pmatrix}-\partial_y d_E\\ \partial_x d_E\end{pmatrix}$ satisfies

$$\nabla\cdot\vec{n}^\perp=0 \text{ in } D'(\Omega),$$

(or equivalently, $\nabla\times\vec{n}=0$ in $D'(\Omega)$). Thanks to the divergence-free hypothesis and the trace theory of functions of bounded variation, we can write that

$$\vec{n}^+=\cos\theta\nu^\perp+\sin\theta\nu \text{ and } \vec{n}^-=\cos\theta\nu^\perp-\sin\theta\nu$$

on the jump set $\Sigma(\vec{n})$, where $\theta$ is the angle between $\vec{n}^+$ and $\nu^\perp$. Therefore, we have that $[\vec{n}]=2\sin\theta=\langle[\vec{n}],\nu\rangle$, which, combined with Theorem 4.1.3, implies

$$\lim_{\sigma\to 0} S\vec{n}^\sigma=\frac{1}{12}\langle[\vec{n}],\nu\rangle^3 ds_\Sigma \text{ in } D'(\Omega).$$

Notice that, up to a multiplicative constant, the quantity $\int_\Sigma\frac{1}{12}\langle[\vec{n}],\nu\rangle^3 ds_\Sigma$ coïncides with the line-energy functional $I_{t\to t^3}$ which appears as the asymptotic energy of the *Aviles-Giga model* $(G_\varepsilon)_\varepsilon$ defined by

$$G_\varepsilon(m^\varepsilon)=\int_\Omega\left(\varepsilon|\nabla m^\varepsilon|^2+\frac{1}{\varepsilon}g\left(|1-|m^\varepsilon|^2|\right)\right)dx \tag{4.1.20}$$

for $m^\varepsilon\in H^1(\Omega,\mathbb{R}^2)$ satisfying the divergence-free hypothesis $\nabla\cdot m^\varepsilon=0$ in $\Omega$ and with $g:t\mapsto t^2$. The variational model (4.1.20) has been originally introduced by Aviles and Giga [10] in 1987 and its asymptotic behavior has been studied in their seminal works [11]-[12]. The Aviles-Giga model is related to numerous physical applications such as theory of liquid crystals in smectic states, Bloch walls in micromagnetics, thin-film blisters, etc., see e.g. [10, 72, 114]. For more recent results on the link between line-energy functions and $BV$ functions with vanishing divergence, we refer to [70, 71] and[73].

## 4.2 Asymptotic behavior of the jump term on simple configurations

Thanks to Theorem 4.1.3, the jump term $S\vec{n}^\sigma$ approximates the 1-codimensional jump set of $\vec{n}$ weighted by the weight of the jump and the angle with the normal to the set. However, $S\vec{n}^\sigma$ does not necessarily charge at the limit as $\sigma$ goes to 0 singular parts of higher codimension. For instance, consider $\vec{n}(x) = x/|x| \in W^{1,1}(\Omega, \mathbb{S}^1)$. As shown in the previous section, the distributional Jacobian satisfies

$$J\vec{n}^\sigma \to \pi\delta_0 \text{ in } \mathcal{D}'(\Omega) \text{ as } \sigma \to 0,$$

whereas the jump term satisfies

$$S\vec{n}^\sigma \to 0 \text{ in } \mathcal{D}'(\Omega) \text{ as } \sigma \to 0.$$

Indeed, since the singular part of $D\vec{n}$ vanishes, thanks to (4.1.6) and (4.1.8), we have that, for every $\varphi \in C_c^\infty(\Omega)$,

$$\langle S\vec{n}^\sigma, \varphi\rangle = I_1^\sigma \to 0 \text{ as } \sigma \to 0.$$

However, in the following we study the case where the map $\vec{n} = \nabla d_E$ is chosen to be the gradient of a signed distance function to a given domain $E$. It is clear that the discontinuity set of $\vec{n}$ corresponds to the skeleton of $E$, we illustrate with some basic examples that higher codimensional parts of the jump set $\Sigma(\vec{n})$ actually appear in the Taylor expansion of $S\vec{n}^\sigma$ at higher orders of $\sigma$.

### 4.2.1 Case of a round sphere

In this case, the associated skeleton $\Sigma$ is a singleton. Without loss of generality, we can assume that $\Sigma = \{0_{\mathbb{R}^N}\}$ and therefore we have

$$\vec{n}(x) = \frac{x}{|x|}, \ \forall x \neq 0.$$

Thanks to the definition of $\vec{n}^\sigma$ and its symmetric properties, we get that

$$\vec{n}^\sigma(x) = h\left(\frac{|x|}{\sigma}\right)\vec{n}(x), \ \forall x \neq 0, \tag{4.2.1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function such that $h(0) = 0$, $\lim_{x\to+\infty} h(x) = 1$ and

$$0 \leq h(x) \leq 1 \text{ for every } x \geq 0.$$

By injecting (4.2.1), since $\sum_{i,j} n_i \partial_j n_i n_j = 0$ almost everywhere, we get that

$$\begin{aligned} S\vec{n}^\sigma(x) &= \sum_{i,j} h\left(\frac{|x|}{\sigma}\right)\vec{n}_i\left(h\left(\frac{|x|}{\sigma}\right)\partial_j\vec{n}_j\right)h\left(\frac{|x|}{\sigma}\vec{n}_j\right) \\ &= \sum_j \left(h\left(\frac{|x|}{\sigma}\right)\right)^2 h'\left(\frac{|x|}{\sigma}\right)\vec{n}_j\frac{x_j}{\sigma|x|} \\ &= \frac{1}{\sigma}\left(h\left(\frac{|x|}{\sigma}\right)\right)^2 h'\left(\frac{|x|}{\sigma}\right) \end{aligned}$$

Therefore, for every $\varphi \in C_c^\infty(\Omega)$, we get that

$$\begin{aligned}
\langle S\vec{n}^\sigma, \varphi\rangle &= \frac{1}{\sigma}\int_{\mathbb{R}^N} \left(h\left(\frac{|x|}{\sigma}\right)\right)^2 h'\left(\frac{|x|}{\sigma}\right)\varphi(x)\,dx \\
&= \frac{1}{\sigma}\int_0^\infty \int_{\mathbb{S}_r} \left(h\left(\frac{r}{\sigma}\right)\right)^2 h'\left(\frac{r}{\sigma}\right)\varphi(r,s) ds_{\mathbb{S}_r} dr \\
&= \frac{1}{\sigma}\int_0^\infty \left(h\left(\frac{r}{\sigma}\right)\right)^2 h'\left(\frac{r}{\sigma}\right) r^{N-1}\left(\int_{\mathbb{S}^{N-1}} \varphi(r,s)\,ds\right) dr \\
&= \sigma^{N-1}\int_0^\infty h(u)^2 h'(u)\left(\int_{\mathbb{S}^{N-1}} \varphi(\sigma u, s)\,ds\right) u^{N-1}\,du \\
&\sim \sigma^{N-1}\int_{\mathbb{R}^N} h(|x|)^2 h'(|x|) dx \varphi(0),
\end{aligned}$$

which gives

$$\frac{1}{\sigma^{N-1}}\langle S\vec{n}^\sigma, \varphi\rangle \to \left(\int_{\mathbb{R}^N} h(|x|)^2 h'(|x|) dx\right)\varphi(0), \tag{4.2.2}$$

as $\sigma \to 0$.

### 4.2.2 Case of a circular tube in $\mathbb{R}^3$

We now discuss the case where the skeleton $\Sigma$ is a regular curve in $\mathbb{R}^3$ and $\Gamma := \Sigma_\sigma = \{x \in \mathbb{R}^3 \,|\, d(x,\Sigma) \le \sigma\}$ is a tube obtained by thickening the curve.

For every $\varphi \in C_c^\infty(\Gamma)$, we claim that

$$\langle S\vec{n}^\sigma, \varphi\rangle \sim \sigma \int_\Sigma \left(\int_{\mathbb{D}} h(u,s)^2 \partial_u h(u,s)\,du\right)\varphi(0,s)\,ds, \tag{4.2.3}$$

as $\sigma \to 0$.

*Proof.* For $\sigma > 0$ small enough, we can write that for every $x \in \Sigma_\sigma$, there exist $s \in \Sigma$ and $u \in \mathbb{D}$ such that $x = s + \sigma u$.

Therefore, by analogy with (4.1.15), we have that

$$\begin{aligned}
\vec{n}^\sigma(x_0) &= \int_{\Sigma_\sigma} \vec{n}(y) f^\sigma(x_0^\sigma - y)\,dy \\
&= \int_{\mathbb{D}_\sigma}\int_\Sigma \vec{n}(y) f^\sigma(x_0^\sigma - y) J(z,s)\,ds\,dz \\
&= \sigma^2 \int_{\mathbb{D}}\int_\Sigma \vec{n}(s,\sigma u) f^\sigma(x_0^\sigma - y) J(\sigma u, s)\,ds\,du \\
&\sim \int_{\mathbb{D}}\int_{\mathbb{R}} f(-z, u_0 - u)\vec{n}_{s_0}^\infty(u)\,dz\,du \\
&= h^\sigma \vec{n}_{s_0}^\infty(u_0)
\end{aligned} \tag{4.2.4}$$

where $\vec{n}_{s_0}^\infty(u) = \lim_{t\to 0^+} \vec{n}_{s_0}(tu)$ and $h^\sigma(\cdot) = h(\frac{\cdot}{\sigma})$. Combining (4.2.4) with polar coordinates, we have that

$$\begin{aligned}
\langle \vec{S\vec{n}}^\sigma, \varphi\rangle &= \sum_{i,j}\int_{\Sigma_\sigma} \vec{n}_i^\sigma \partial_j \vec{n}_i^\sigma \vec{n}_j^\sigma\,\varphi\,dx \\
&= \int_\Sigma \int_0^\sigma \int_0^{2\pi} \frac{1}{\sigma} h\left(\frac{r}{\sigma},\theta\right)^2 h_r\left(\frac{r}{\sigma},\theta\right) r\varphi(r,\theta,s) J(r,\theta,s) dr d\theta\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \sigma \int_\Sigma \int_0^1 \int_0^{2\pi} h(u,\theta) h_u(u,\theta) u \varphi(\sigma u, \theta, s) J(\sigma u, \theta, s)\, du\, d\theta\, ds \\
&= \sigma \int_\Sigma \int_0^1 \int_0^{2\pi} h(u,\theta) h_u(u,\theta) \varphi(0, \theta, s) J(0, \theta, s) u\, du\, d\theta\, ds \\
&\sim \sigma \int_\Sigma \left( \int_{\mathbb{D}} h(u,s)^2 \partial_u h(u,s) \right) \varphi(0,s)\, ds,
\end{aligned}
$$

as $\sigma \to 0$. □

### 4.2.3 Asymptotic comparisons between $\mathcal{B}\vec{n}$ and $S\vec{n}$

We now compare the asymptotics of $\mathcal{B}\vec{n}$ and of the jump term $S\vec{n}$ on simple examples.

#### 2D examples

In 2D, thanks to Lemma 3.1.1, $\mathcal{B}\vec{n}$ is proportional to the distributional Jacobian, i.e. $\mathcal{B}\vec{n} = 2J\vec{n}$.

1. Case $\Gamma = \{(-1,0)\} \cup \{(1,0)\} \cup \{x = 0\}$, which corresponds to the skeleton of 2 circles of same radius inferior to 1. We calculate that

$$
\vec{n}(x) = \begin{cases} \dfrac{(x+1, y)}{\sqrt{(x+1)^2 + y^2}} \text{ if } x < 0 \text{ and } (x,y) \neq (-1,0). \\ \dfrac{(x-1, y)}{\sqrt{(x-1)^2 + y^2}} \text{ if } x > 0 \text{ and } (x,y) \neq (1,0). \end{cases}
$$

   Moreover, for $(x,y) \notin \Gamma$,

$$
\begin{aligned}
\partial_x n_1 &= \frac{y^2}{\left((x \pm 1)^2 + y^2\right)^{3/2}}. \\
\partial_y n_2 &= \frac{(x \pm 1)^2}{\left((x \pm 1)^2 + y^2\right)^{3/2}}. \\
\partial_x n_2 &= -\frac{(x \pm 1) y}{\left((x \pm 1)^2 + y^2\right)^{3/2}} = \partial_y n_1.
\end{aligned}
$$

   Here we also set $\nu = (1,0)$ which is the normal vector on the median line $\{x = 0\}$. Therefore we have

$$
[\vec{n}]_{|\{x=0\}} = \frac{(-2,0)}{\sqrt{1+y^2}} = \langle [\vec{n}], \nu \rangle \nu.
$$

   (a) Since $\vec{n}$ is locally $W^{1,1}$ around $(1,0)$ and $(-1,0)$, for every $\varphi \in C_c^\infty(\mathbb{R}^2 \backslash \{x=0\})$, one has

$$
\langle \mathcal{B}\vec{n}^\sigma, \varphi \rangle \to 2\pi \left( \varphi(1,0) + \varphi(-1,0) \right).
$$

   For every $\varphi \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$, by integration by parts, we have that

$$
\begin{aligned}
\langle B\vec{n}^\sigma, \varphi \rangle &= 2 \left( -\langle n_1^\sigma \partial_y n_2^\sigma, \partial_x \varphi \rangle + \langle n_1^\sigma \partial_x n_2^\sigma, \partial_y \varphi \rangle \right) \\
&= -2 \left( \int_{\{x<0\}} n_1^\sigma \partial_y n_2^\sigma \varphi\, dxdy + \int_{\{x>0\}} n_1^\sigma \partial_y n_2^\sigma \varphi\, dxdy + o(1) \right) \\
&\to \int_{\{x=0\}} \varphi(0,y) \left[ n_1 \partial_y n_2 \right]_{\{x=0\}} dy,
\end{aligned}
$$

where

$$[n_1\partial_y n_2]_{\{x=0\}} = n_1\partial_y n_2(0^+, y) - n_1\partial_y n_2(0^-, y) = -\frac{2}{(1+y^2)^2}.$$

Therefore, $\mathcal{B}\vec{n}$ charges both circles centers as well as the median plan.

(b) Since $\vec{n}$ is $W^{1,1}$ in neighbourhoods of $(1,0)$ and $(-1,0)$, $S\vec{n}$ does not charge these points, namely, we have

$$\langle S\vec{n}^\sigma, \varphi\rangle \to 0 \text{ as } \sigma \to 0,$$

for every compactly supported function $\varphi$ in $\Omega\backslash\{x = 0\}$. Thanks to Theorem 4.1.3, we deduce that, for every $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\langle S\vec{n}^\sigma, \varphi\rangle &\to \frac{1}{12}\int_{\{x=0\}} \varphi|[\vec{n}]|^2\langle[\vec{n}], \nu\rangle\, ds_{\{x=0\}} \\ &= -\frac{2}{3}\int_{-\infty}^{+\infty} \varphi(0,y)\frac{1}{(1+y^2)^{3/2}}\, dy\end{aligned}$$

as $\sigma \to 0$. Notice that if we consider the configuration contained in a domain $\Omega \subset \{(x,y)\,|\,a \le y \le b\}$, then $S\vec{n}$ is "equivalent" to the uniform measure on $\{x = 0\}$ in the sense that there exist $C_1, C_2 > 0$ such that

$$C_1\int_a^b \varphi(0,y)\, dy \le \int_a^b \varphi(0,y)\frac{1}{(1+y^2)^{3/2}} dy \le C_2\int_a^b \varphi(0,y)\, dy,$$

for any $\varphi \ge 0$.

The previous example shows that $S\vec{n}$ charges only the median line at first order.

2. Case $\Gamma = \{(0,0)\} \cup \partial B(0,1)$ which corresponds to the skeleton of an annulus. We have

$$\vec{n}(x) = \begin{cases} \dfrac{x}{|x|} \text{ if } |x| > 1. \\ -\dfrac{x}{|x|} \text{ if } |x| < 1. \end{cases}$$

By setting $\nu = x$ for $|x| = 1$, we get $\langle[\vec{n}], \nu\rangle = 2$

(a) In a local neighbourhood of the origin point, it is clear that $\mathcal{B}\vec{n}^\sigma \to 2J\vec{n} = 2\pi\delta_0$.

In order to study the asymptotic behavior of $\mathcal{B}\vec{n}^\sigma$ around the circle $\partial B(0,1)$, we choose a radial regularized normal vector $\vec{n}^\sigma$. Namely, we consider $\vec{n}^\sigma = (n_1^\sigma, n_2^\sigma)$ where $n_1^\sigma = f^\sigma\cos\theta$ and $n_2^\sigma = f^\sigma\sin\theta$ for some smooth radial functions $f^\sigma : \mathbb{R}_+ \to \mathbb{R}$ such that, as $\sigma \to 0$, $f^\sigma(r) \to 1$ if $r > 1$ and $f^\sigma(r) \to -1$ if $r < 1$. Using Lemma 3.1.1 in polar coordinates, we can write

$$\begin{aligned}\mathcal{B}\vec{n}^\sigma &= 2\left(\frac{\partial n_1^\sigma}{\partial x}\frac{\partial n_2^\sigma}{\partial y} - \frac{\partial n_1^\sigma}{\partial y}\frac{\partial n_2^\sigma}{\partial x}\right) \\ &= \frac{2}{r}f^\sigma\frac{\partial f^\sigma}{\partial r}\left(\cos^2\theta + \sin^2\theta\right)^2 \\ &= \frac{2}{r}f^\sigma\frac{\partial f^\sigma}{\partial r}.\end{aligned}$$

For every $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{0\})$, we get that

$$\begin{aligned}\langle \mathcal{B}\vec{n}^\sigma, \varphi\rangle &= \int_{\mathbb{R}^2} \mathcal{B}\vec{n}^\sigma \varphi \\ &= \int_0^{2\pi}\int_0^{+\infty} \mathcal{B}\vec{n}^\sigma r\varphi(r,\theta)\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^{+\infty} f^\sigma \partial_r(f^\sigma)\varphi(r,\theta)\,dr\,d\theta \\ &\to C ds_{\partial B(0,1)} \text{ in } \mathcal{D}'(\mathbb{R}^2\backslash\{0\})\end{aligned}$$

as $\sigma \to 0$.

(b) Thanks to Theorem 4.1.3, we have

$$S\vec{n}^\sigma \to \frac{2}{3} ds_{\partial B(0,1)} \text{ in } \mathcal{D}'(\mathbb{R}^2)$$

as $\sigma \to 0$.

The example of the annuli shows that $S\vec{n}$ charges only the circle whereas $J\vec{n}$ charges both the origin and the circle.

**3D examples**

1. Case $\Gamma = \{x = 1, y = 0\} \cup \{x = -1, y = 0\} \cup \{x = 0\}$ which corresponds to the skeleton of two parallel vertical cylinders.

   (a) It is clear that the third coordinate of $\vec{n}$ remains constant outside of $\Gamma$. Therefore, thanks to 3.1.1, we have

   $$\mathcal{B}\vec{n} = 2\left(\frac{\partial \vec{n}_1}{\partial x}\frac{\partial \vec{n}_2}{\partial y} - \frac{\partial \vec{n}_2}{\partial x}\frac{\partial \vec{n}_1}{\partial y}\right)$$

   in the distributional sense, which, by analogy with 1.(a) in the 2D case, gives

   $$\begin{aligned}\langle \mathcal{B}\vec{n}, \varphi\rangle = \pi\int_{\{x=1,y=0\}} \varphi(1,0,z)\,dz - \pi\int_{\{x=-1,y=0\}} \varphi(-1,0,z)\,dz \\ + \int_{\{x=0,y=0\}} \varphi(0,0,z)[n_1\partial_y n_2]_{\{x=0,y=0\}}\,dz.\end{aligned}$$

   (b) Similarly as in the case of two circles in 2D, we have

   $$\langle S\vec{n}^\sigma, \varphi\rangle \to -\frac{2}{3}\int_{\mathbb{R}^2} \varphi(0,y,z)\frac{1}{(1+y^2+z^2)^{3/2}}\,dy$$

   as $\sigma \to 0$.

2. Case $\Gamma = \{x = 0\}$ which corresponds to the skeleton of two parallel planes. We have $\vec{n}(x,y,z) = (1,0,0)$ if $x < 0$ and $\vec{n}(x,y,z) = (-1,0,0)$ if $x > 0$.

   (a) Here we can take for instance the regularized normal vector $\vec{n}^\sigma$ of the form $(n_1^\sigma, 0, 0)$ with $n_1^\sigma$ a smooth function accross the median plane $\Gamma$. Thanks to Lemma 3.1.1, we have $\mathcal{B}\vec{n}^\sigma = 0$, therefore $\mathcal{B}\vec{n}^\sigma \to 0$ as $\sigma \to 0$.

   (b) By Theorem 4.1.3, we have $S\vec{n}^\sigma \to \frac{2}{3} ds_{\{x=0\}}$ as $\sigma \to 0$.

   In this example, we observe that $\mathcal{B}\vec{n}^\sigma$ vanishes with some symmetric regularizations while the limit $S\vec{n}^\sigma$ is independent on the choice of the regularization.

**Summary on the consistency of $\mathcal{B}\vec{n}$ and the jump term $S\vec{n}$ for $\mathrm{BV}$ maps:** With the above 2D or 3D examples, one can conclude that no matter the given configuration, the jump term always charges the 1-codimensional part of the corresponding skeleton at first order, whereas the reaction term $\mathcal{B}\vec{n}$ from Mugnai's model charges systematically the 0-dimensional parts of the skeleton and charges the rest of the skeleton rather inconsistently depending on the given configurations.

Theoretically, to the author's knowledge, it seems so far not clear if we can canonically define either $\mathcal{B}\vec{n}$ or the Jacobian for BV maps. The above examples also indicate that even by regularizing the normal vector to properly compute $\mathcal{B}\vec{n}^\sigma$ in the classical sense, the asymptotic limits depend on the choice of the regularization. This does not meet the criteria in our study, since the main interest of this chapter is to be able to capture the skeleton of any given configuration up to suitably negligible sets.

### 4.2.4 Numerical illustration of the asymptotic behavior of $S\vec{n}^\sigma$

We now compute some approximations of $S\vec{n}^\sigma$ for various numerical examples on a Cartesian grid either in $2D$ or in $3D$. More precisely, $\vec{n}^\sigma = f^\sigma * \vec{n}$ is computed using a Gaussian kernel $f^\sigma = \frac{1}{\sigma^N}\exp^{-\pi\frac{|x|^2}{\sigma^2}}$ with $N = 2, 3$.

#### 2D numerical examples

In the $2D$ case, we provide some illustrations of the jump term in Figure 4.1 obtained with a resolution of $2^7$ nodes for each dimension of the grid, where we consider three different sets $E$ whose skeletons consist of, respectively:

- two concentric circles,
- two parallel lines,
- and a rectangle.

We compute the jump term $S\vec{n}^\sigma$ using different values of the regularization parameter $\sigma > 0$ (see the caption for the details).

#### 3D numerical examples

We also try with the same $3D$ test presented in Figure (4.2), with a numerical resolution of $2^7$ nodes for each dimension of grid. Examples in Figure (4.2) are:

- two concentric spheres,
- two parallel planes,
- and a circular torus.

The values taken by the jump term are captured on the three hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$ and $\{x_3 = 0\}$.

In both figures 4.1 and 4.2, one can clearly observe that the skeleton of the set $E$ is well localized in each configuration. Thanks to the observations together with Theorem 4.1.3 and the heuristic results (4.2.1) and (4.2.3) in the case of round spheres and circular tubes in $\mathbb{R}^3$, one could conjecture that if $\vec{n}$ is the gradient of the signed distance function associated with $E$, its jump term $S\vec{n}^\sigma$ characterizes asymptotically the jump set $\Sigma(\vec{n})$, which corresponds to the skeleton of $E$. More precisely, it is attempting to propose the following conjecture:

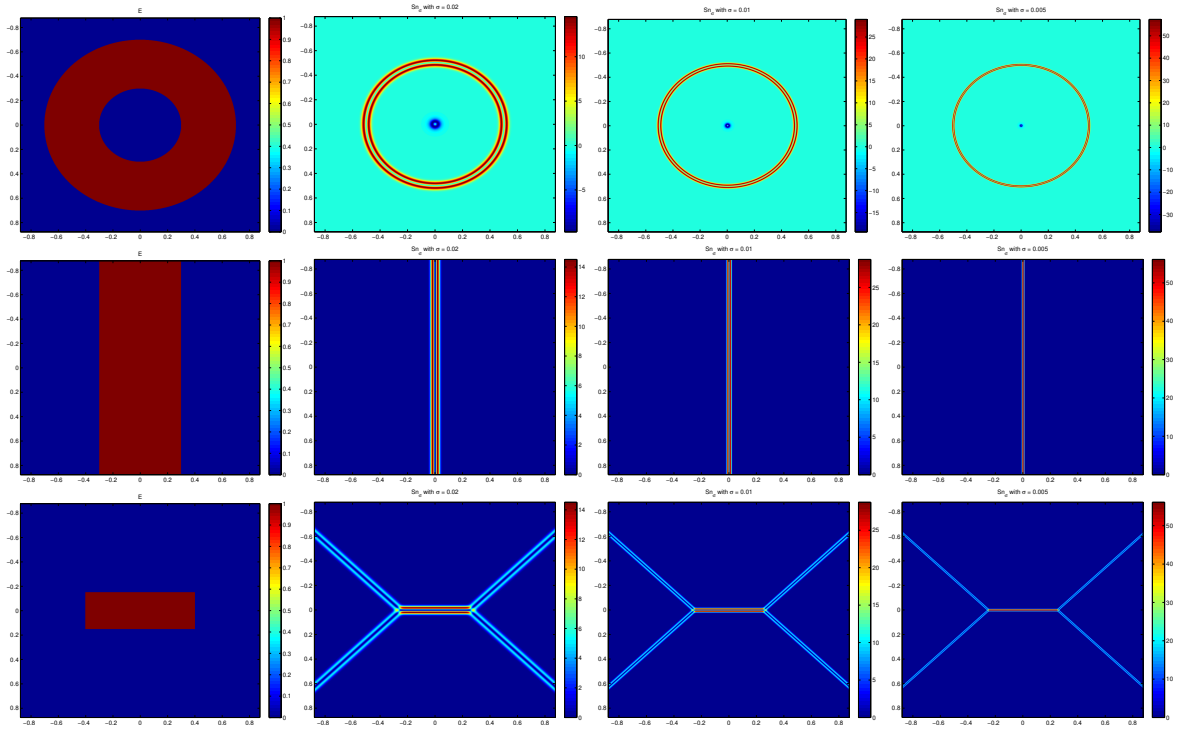

Figure 4.1: Numerical values of $S\vec{n}^{\sigma}$ in $2D$ for various configurations and various values of $\sigma$; The first column of each line corresponds to the considered set $E$; On each line, the columns $2$, $3$, and $4$ correspond to $S\vec{n}^{\sigma}$ with, respectively, $\sigma = 0.02$, $0.01$ and $0.005$.

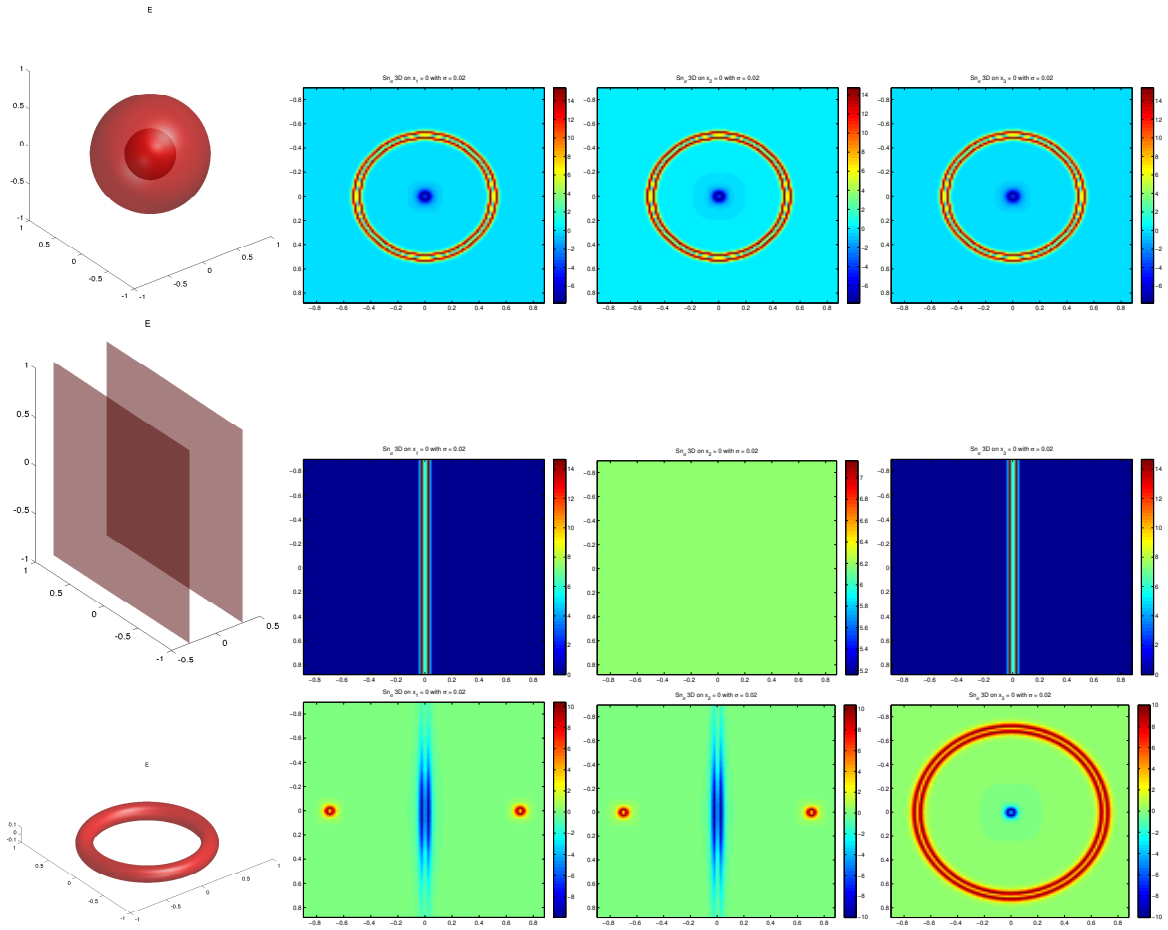

Figure 4.2: Numerical values of $S\vec{n}^{\sigma}$ for $3D$ examples. Each line shows the set $E$, then the values taken by the jump term on the planes $\{x_1 = 0\}$, $\{x_2 = 0\}$, and $\{x_3 = 0\}$, respectively.

**Conjecture 4.2.1.**

$$Sn^\sigma \sim \sum_{j=0}^{N-1} \sigma^j \, \alpha_j \, \mathcal{H}^{N-1-j}_{\llcorner \Sigma_j} \tag{4.2.5}$$

*where*

- $\alpha_j$ *are* $\mathbb{R}$*-valued density functions;*
- $\Sigma_j$ *are* $(N-1-j)$*-dimensional sets;*
- *the discontinuity set of* $n$ *satisfies* $\Sigma(n) = \bigcup_j \Sigma_j$

Actually, the numerical computations of Figures 4.1 and 4.2 show that the magnitude on the jump term appears to be linear with respect to $\sigma$ and does not depend on the dimension of the skeleton. For instance, in the case of two concentric circles as in Figure 4.1, one can notice that the jump term computed at the center has the same order of magnitude (up to a sign difference) as it has on the circle. This is not in contradiction with (4.2.5) for the following reason: on the one hand, the Dirac mass $\delta_0$ at the center can be numerically represented by the inverse of the pixel's area, which gives in 2D

$$\mu_0(\Sigma) \simeq \frac{1}{\sigma^2}.$$

On the other hand, the uniform measure on the discrete circle satisfies $\mu_1(\sigma) \simeq \frac{1}{L\sigma}$ since the circle of length $L$ is approximated numerically by a band of area $L\sigma$ in $2D$. Therefore, thanks to (4.2.5), we have that $\sigma\mu_0(\Sigma) \simeq L\mu_1(\sigma) \simeq \frac{1}{\sigma}$, which justifies the fact that the jump term has approximately the same magnitude at the center and on the circle in Figure 4.1.

## 4.3 Approximate mean curvature flow with a forcing term

In this section, we address perturbed Allen-Cahn equations of the form:

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} + g^\varepsilon & \text{for } (x,t) \in \Omega \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{for } x \in \Omega, \\ \partial_\nu u^\varepsilon_{|\partial\Omega} \equiv 0 & \text{for } t > 0 \end{cases}, \tag{4.3.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is an open bounded set with smooth boundary and the forcing term $g^\varepsilon$ satisfies

$$\sup_{\varepsilon>0} \int_0^T \int_\Omega \varepsilon g^\varepsilon(x,t) dx dt < +\infty. \tag{4.3.2}$$

In [110], Mugnai and Röger showed that equation (4.3.1) is a phase-field approximation of the forced mean curvature flow

$$V = H + g,$$

in the varifold sense, where $g$ is an approximate limit of $g^\varepsilon \nabla u^\varepsilon$ defined by

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \eta \cdot \nabla u^\varepsilon g^\varepsilon \, dx \, dt = \int_0^T \int_\Omega \eta \cdot g \, d\mu,$$

for every $\eta \in C^0_c((0,T) \times \Omega, \mathbb{R}^N)$, with $\mu$ is a Radon measure on $(0,T) \times \Omega$ obtained as the limit of *diffuse surface area measures* $\mu^\varepsilon$ defined as

$$\mu^\varepsilon := \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) \mathcal{L}^{N+1}.$$

One can also consider the case where $g^\varepsilon$ does not depend on $\varepsilon$:

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2}(1+f^\sigma) & \text{for } (x,t) \in \Omega \times (0,T) \\ u^\varepsilon(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{4.3.3}$$

where $u \in C^\infty(\Omega\times]0,T]) \cap C^0([0,T], C^\infty(\Omega))$, $f^\sigma \in C^\infty(\overline{\Omega})$ and $W(s) = \frac{1}{4}(1-s^2)^2$ is the bistable potential. In [118], Qi and Zheng showed that, if $1 + f^\sigma \geq a$ for some $a > 0$, then the equation (4.3.3) converges to the forced mean curvature flow as

$$V = H + \frac{\nabla^\perp f^\sigma}{2(1+f^\sigma)}, \tag{4.3.4}$$

in the varifold sense, where $\nabla^\perp f^\sigma := \nabla f^\sigma \cdot \nu$ is the normal component of $\nabla f^\sigma$ with $\nu$ the outer normal of the front $\Gamma$.

In the following sections, we first show the convergence result (4.3.4) by using the formal method of matched asymptotic expansion. Next, we discuss the behavior of the forcing term in (4.3.4) and we prove that, under certain conditions on $f^\sigma$, (4.3.4) provides a flow with obstacle.

*Remark* 4.3.1. One can verify that equation (4.3.3) is the $L^2-$gradient flow (up to time rescaling) of the following functional:

$$E(u^\varepsilon) = \int_\Omega \left( \frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} dx + \int_\Omega \frac{W(u^\varepsilon)}{\varepsilon} f^\sigma \right) dx.$$

The first integral is well-known as the phase-field approximation of perimeter of the interface $\Gamma$, whereas the second integral can be regarded as a "penalized weight" on the obstacle.

*Remark* 4.3.2. In practice, as shown in the next section where we are interested in numerical applications of the perturbed phase-field approximation, we usually consider *periodic* solutions of (4.3.3)

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2}(1+f^\sigma) & \text{for } (x,t) \in \Omega \times (0,T) \\ u^\varepsilon(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{4.3.5}$$

where $\Omega = [0,L]^N$ with $N = 2, 3$.

### 4.3.1 Formal asymptotic expansion: first order

Similarly as for the asymptotic analysis of Mugnai's flow in [30], see Chapter 3, we find the inner solution of equation (4.3.3) up to order $\varepsilon^{-2}$ and we show that formally that the level set of an approximate solution of (4.3.3) with well prepared initial data approximates a front $\Gamma$ moved by the forced mean curvature flow (4.3.4). We assume that the zero level set of a solution $u^\varepsilon$ to equation (4.3.3) is a smooth $n-1$ dimensional front $\Gamma$ which is the boundary of a domain $E \subset \Omega$ evolving in time. For convenience, we assume that $\Gamma$ does not intersect $\partial\Omega$ and that as $\varepsilon \to 0$, $u^\varepsilon$ converges to $-1$ in $E$ and to $1$ outside of $E$ almost everywhere, that is, $\forall t \in ]0,T[$, for every compact sets $K_1 \subset\subset E$ and $K_2 \subset\subset \Omega \setminus E$, we have that

$$u^\varepsilon(x) \to \begin{cases} -1, \text{ if } x \in K_1 \\ 1, \text{ if } x \in K_2 \end{cases}, \text{ as } \varepsilon \to 0.$$

Following the method of matched asymptotic expansion used in [53, 3, 63], we assume that the solution $u^\varepsilon$ of (4.3.3) has the following Taylor expansion in a neighbourhood of $\Gamma$:

$$u(x,t) = U(z,x,t) = \sum_{i=0}^{\infty} \varepsilon^i U^i(z,s,t), \tag{4.3.6}$$

where for every $x$ close to $\Gamma$, we write $x = (z, s) \in \mathbb{R} \times \Gamma$ with $s := \pi(x) \in \Gamma$ the normal projection of $x$ on $\Gamma$, $U^i$ are smooth bounded fonctions such that

$$U^i(0, s, t) = 0 \text{ and } \lim_{z \to \pm\infty} U^i(z, s, t) = 0, \ \forall i \in \mathbb{N}, \forall (s, t) \in \Gamma \times ]0, T] \tag{4.3.7}$$

and the parameter $z$ is the *stretched* signed distance in a neighbourhood of $\Gamma$:

$$z = \frac{d(x, t)}{\varepsilon},$$

where $d(x, t)$ is the signed distance defined by

$$d(x, t) = \begin{cases} \operatorname{dist}(x, \Gamma) & \text{if } x \in \Omega \setminus E \\ -\operatorname{dist}(x, \Gamma) & \text{if } x \in E \end{cases}.$$

Then it follows that

$$\begin{cases} \nabla u^\varepsilon = \nabla_\Gamma U + \dfrac{1}{\varepsilon} U_z \nabla d, \\ \Delta u = \Delta_\Gamma U + \dfrac{1}{\varepsilon} U_z \Delta d + \dfrac{1}{\varepsilon^2} U_{zz}, \\ \partial_t u = \partial_t U + \dfrac{1}{\varepsilon} V U_z \end{cases}$$

where $V = \partial_t d(x, t)$ is the normal velocity of $\Gamma$, which is positive when the front moves toward $E$ (i.e when $E$ "shrinks"). Moreover, we can write that

$$f^\sigma(x) = \sum_{i=0}^{\infty} \frac{\varepsilon^i z^i}{i!} f^i(s), \tag{4.3.8}$$

where

$$f^i(s) = \left(\partial_\nu^i f^\sigma\right)(0, s),$$

with $\left(\partial_\nu^i f^\sigma\right)$ the ith normal derivative of $f^\sigma$. In particular, we have

$$\begin{aligned} f^\sigma &= f_0 \\ \nabla^\perp f^\sigma &= f_1 \end{aligned} \quad \text{on } \Gamma. \tag{4.3.9}$$

### 4.3.2 Formal asymptotic expansion: second order

At order $\varepsilon^{-2}$, equation (4.3.3) gives

$$U^0_{zz} - W'(U^0)(1 + f^0(s)) = 0, \text{ for } z \in \mathbb{R}. \tag{4.3.10}$$

It is clear that

$$U^0(z) := q\left(\sqrt{1 + f^0(s)} z\right),$$

where

$$q(s) := \tanh\left(\frac{s}{\sqrt{2}}\right),$$

is a solution of (4.3.10) and satisfies the matching condition (4.3.7). Notice that by differentiating (4.3.10), we also get that

$$U^0_{zzz} - W''(U^0)(1 + f^0(s)) U^0_z = 0. \tag{4.3.11}$$

Next at order $\varepsilon^{-1}$, we get that

$$V U^0_z = U^1_{zz} - W''(U^0)(1 + f^0(s)) + H U^0_z - W'(U^0) z f^1(s).$$

By multiplying by $U_z^0$ and integrating, we get that

$$\begin{aligned} V\int_{-\infty}^{+\infty}(U_z^0)^2 &= \int_{-\infty}^{+\infty}U_z^0(U_{zz}^1 - W''(U^0)(1+f^0(s))) + \int_{-\infty}^{+\infty}H(U_z^0)^2 - \int_{-\infty}^{+\infty}W'(U^0)zf^1(s)U_z^0 \\ &= \int_{-\infty}^{+\infty}U^1\left(U_{zzz}^0 - W''(U^0)(1+f^0(s))U_z^0\right) + H\int_{-\infty}^{+\infty}(U_z^0)^2 + \\ &= H\int_{-\infty}^{+\infty}(U_z^0)^2 + \frac{1}{2}f^1\int_{-\infty}^{+\infty}W'(U^0)zU_z^0, \end{aligned}$$

thanks to (4.3.11) and integration by parts. Moreover, since $W'(U^0) = \frac{U_{zz}^0}{1+f^0(s)}$, we get that

$$\begin{aligned} V\int_{-\infty}^{+\infty}(U_z^0)^2 &= H\int_{-\infty}^{+\infty}(U_z^0)^2 - \frac{f^1(s)}{1+f^0(s)}\int_{-\infty}^{+\infty}U_{zz}^0U_z^0z \\ &= H\int_{-\infty}^{+\infty}(U_z^0)^2 + \frac{f^1(s)}{2(1+f^0(s))}\int_{-\infty}^{+\infty}(U_z^0)^2, \end{aligned}$$

which implies

$$\begin{aligned} V &= H + \frac{f^1(s)}{2(1+f^0(s))} \\ &= H + \frac{\nabla^\perp f^\sigma}{2(1+f^\sigma)} \text{ on } \Gamma. \end{aligned}$$

### 4.3.3 Perturbed mean curvature flow with static obstacles

Here, we show that if the perturbation term $f^\sigma$ satisfies a growth condition around a given point then the front $\Gamma$ moved by the flow (4.3.4) will never intersect with the point if the initial distance at $t=0$ between the interface $\Gamma$ and the point $0$ is far enough. In this case, we say that the flow (4.3.4) is a perturbed mean curvature flow with a static obstacle point.

Without loss of generality, we can assume that the obstacle point is the origin point $0$. We denote $\delta := \text{dist}(\Gamma, 0) > 0$ and take $s_0 \in \Gamma$ such that $\delta = |s_0 - 0|$. Additionally, we assume in the following that the obstacle point $0$ is at the interior of $\Gamma$, i.e $0 \in \Omega^-$. The study of the opposite case can be proved in a similar way.

First of all, thanks to the definition of $\delta$, we have the following lemma giving an upper-bound control on the mean curvature.

**Lemma 4.3.3.** *If $0 \in \Omega^-$, $H(s_0) \leq \frac{N-1}{\delta}$.*

*Proof.* Thanks to the definition of $s_0$, up to some linear transformation on the plane, we can assume that the obstacle point has coordinates $(0,\delta)$, $s_0$ has coordinates $(0,0)$ and the curve can be locally considered as the graph of a regular fonction $\psi$ defined on a neighborhood of $0$ such that $\psi(0) = \psi'(0) = 0$. Therefore, it is clear that the (signed) curvature at $(0,0)$ is

$$k(s_0) = \frac{\psi''(0)}{(1+\psi'(0))^{\frac{3}{2}}} = \psi''(0),$$

and we have that

$$\psi(x) = \psi''(0)\frac{x^2}{2} + o(x^2), \text{ as } x \to 0.$$

Thanks to the definition of $\delta$, we have that, for all $x$ in a neighborhood of $0$,

$$(x-0)^2 + (\psi(x) - \delta)^2 \geq \delta^2,$$

which gives

$$x^2 - 2\delta\psi(x) + \psi(x)^2 \geq 0.$$

Hence, by dividing the above inequality by $x^2$, as $x$ goes to $0$, we obtain that

$$\psi''(0)\delta \leq 1,$$

which leads to

$$k(s_0) \leq \frac{1}{\delta}.$$

□

We are now in position to prove the following theorem thanks to Lemma 4.3.3.

**Theorem 4.3.4.** *Let $f^\sigma(x) = \sum_{i=0}^{+\infty} \frac{\varepsilon^i z^i}{i!} f^i(s)$ defined as in (4.3.8). If there exists $\delta_N > 0$ such that*

$$\frac{\delta_N f^1(s_0)}{2(1+f^0(s_0))} < 1 - N, \tag{4.3.12}$$

*for every $s_0 \in \Gamma(0)$ satisfying $\operatorname{dist}(\Gamma(0), 0) = |s_0 - 0| = \delta_N$, and if $\operatorname{dist}(\Gamma(0), 0) > \delta_N$, then*

$$\operatorname{dist}(\Gamma(t), 0) \geq \delta_N,$$

*for every $0 \leq t < T^*$ before the singularities occur.*

*Proof.* Assume that there exists $t > 0$ such that $\operatorname{dist}(\Gamma(t), 0) = \delta_N$. Thanks to Lemma 4.3.3 and the definition of forced mean curvature flow (4.3.4), there exists $s \in \Gamma(t)$ such that $\nabla^\perp f^\sigma(s) = f^1(s)$ and that

$$V(s) = H(s) + \frac{\nabla^\perp f^\sigma(s)}{2(1+f^\sigma(s))} \leq \frac{N-1}{\delta_N} + \frac{f^1(s)}{2(1+f^0(s))} < 0,$$

thanks to (4.3.12). Therefore, the distance between $\Gamma$ and $0$ is bounded below by $\delta_N$, which completes the proof. □

**A Gaussian kernel as perturbative term:** Consider

$$f^\sigma(x) = \frac{1}{\sigma^N} e^{-\frac{|x|^2}{\sigma^2}}.$$

It is clear that

$$f^\sigma \to \delta_0 \text{ as } \sigma \to 0,$$

in the distribution sense, where $\delta_0$ is the Dirac function centered at $0$.

**Claim 4.3.5.** *There exists $\sigma_n > 0$ such that, for every $\sigma \leq \sigma_n$, $V(s_0) < 0$ if $c_{\sigma,n} \leq \delta \leq d_{\sigma,n}$, for some $0 < c_{\sigma,n} < d_{\sigma,n}$.*

*Proof.* By the definitions of $\delta$ and $s_0$, we first notice that

$$\nabla^\perp f^\sigma(s_0) = \partial_{\nu(s_0)} f^\sigma$$

where

$$\nu(s_0) = \frac{s_0 - 0}{|s_0 - 0|} = \frac{s_0}{\delta}.$$

Therefore, we have that

$$f^0(s_0) = \frac{1}{\sigma^N} e^{-\frac{\delta^2}{\sigma^2}} \text{ and } f^1(s_0) = -\frac{2\delta}{\sigma^{N+2}} e^{-\frac{\delta^2}{\sigma^2}}.$$

Thanks to (4.3.4) and Lemma 4.3.3, we have that

$$V(s_0) \leq \frac{n-1}{\delta} + \frac{1}{2}\frac{-\frac{2\delta}{\sigma^{(n+2)}}e^{-\frac{\delta^2}{\sigma^2}}}{1+\frac{1}{\sigma^n}e^{-\frac{\delta^2}{\sigma^2}}}.$$

In order to have $V < 0$, it is sufficient to require that,

$$g(\delta) := \frac{(n-1)\frac{\delta^2}{\sigma^{(n+2)}}e^{-\frac{\delta^2}{\sigma^2}}}{1+\frac{1}{\sigma^n}e^{-\frac{\delta^2}{\sigma^2}}} > n-1.$$

Since $g$ is continuous on $[0,+\infty[$ such that $\lim_{\delta\to+\infty} g(\delta) = g(0) = 0$ and that, for $\sigma < 1$,

$$\max g \geq g(-\sigma^2 \ln \sigma^n) = \frac{-\ln \sigma^n}{2}.$$

Hence, the theorem is proved by taking $\sigma$ small enough and the existences of $c_{\sigma,n}$ and $d_{\sigma,n}$ derive from the intermediate value theorem. □

*Remark* 4.3.6. The above example indicates that, in the case $0 \in \Omega^-$, the Gaussian kernel $f^\sigma$ in (4.3.3) acts as a "repulsive source" avoiding the interface from getting in collision with the point $0$. Moreover, the repulsive zone contains an annulus whose smaller radius is at most of order $\sigma$ and whose bigger one is at least of order $-\sigma \ln \sigma$. These observations can actually be extended to the case where the obstacle is a general $C^2$-subset.

*Remark* 4.3.7. One of the interests of using the flow (4.3.3) is that when the interface is far from the obstacle, it behaves exactly as a mean curvature flow. When the interface starts approaching the obstacle, the perturbed term will prevent the interface from self colliding by forcing the interface and the obstacle to remain beyond a minimal distance, denoted by $\tilde{\sigma}$ depending on $\sigma$ when the interface become stationary. It is natural to extend this idea to dynamic obstacles, and even obstacles defined by the interface itself, actually its skeleton. This is the purpose of the next section.

## 4.4 Self-avoiding approximate mean curvature flow

Inspired by Theorem 4.1.3 on the approximation of the skeleton and by Theorem 4.3.4 on the mean curvature flow with static obstacles, we propose a phase-field approximation of the mean curvature flow with a *self-avoidance* term. The motion is similar to mean curvature flow as long as the interface is far enough from its skeleton, but a reaction term (actually the jump term associated with the skeleton) starts to be active when the interface gets closer to its skeleton, and prevents them to be too close. We will provide arguments that this new self-avoiding mean curvature flow applied to filaments in $\mathbb{R}^3$ approximates the codimension 2 mean curvature flow. Instead of defining directly the motion of a filament, we consider a tubular fattening of it and let is evolve according to the new flow. Thanks to Theorem 4.3.4, the fattening does not degenerate. These results could be generalized to higher dimensions.

### 4.4.1 Definition of the flow and numerical approximation scheme

In this section, we introduce a mean curvature flow perturbed with a term depending on a skeleton-based obstacle. Thanks to the results on the characterization of singular sets by the jump term defined in Definition 4.1.2, together with our analysis of the perturbed Allen-Cahn equation (4.3.1), a natural phase field candidate is the Allen-Cahn equation coupled with the *jump term*, i.e.,

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}(1 + f^\sigma_{u^\varepsilon}) \tag{4.4.1}$$

where

$$f^{\sigma}_{u^{\varepsilon}} = c(f^{\sigma} * |S\vec{n}^{\sigma}_{u^{\varepsilon}}|) \quad \text{and} \quad n^{\sigma}_{u^{\varepsilon}} = f^{\sigma} * \frac{\nabla u^{\varepsilon}}{|\nabla u^{\varepsilon}|}.$$

Here, $f^{\sigma}$ is the Gaussian kernel of size $\sigma > 0$ seen at the end of the previous section and $c$ is a constant to be chosen in order to enforce or relax the topological constraint.

To approximate numerically this flow, we propose a quasi-static approach, i.e. the sequence of approximate solutions $(u^n)_n$ is defined recursively as $u^{n+1}(x) = v(x,h)$ where $v$ is solution of the following PDE

$$\begin{cases} \partial_t v(x,t) & = \Delta v(x,t) - \frac{W'(v(x,t))}{\varepsilon^2}(1 + f^{\sigma}_{u^n}(x)) \\ v(\cdot,0) & = u^n \end{cases} \tag{4.4.2}$$

and $h$ represents the time step of the quasi-static formulation. Notice that the above equation is a $L^2$-gradient flow of the perturbed Cahn–Hilliard energy

$$J^{\varepsilon}_{u^n}(v) = \int \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{\varepsilon}W(v)\,(1 + f^{\sigma}_{u^n}))\,dx.$$

From a numerical point of view, equation (4.4.2) is solved in a box $Q$ with periodic boundary conditions. Moreover, we use a semi-implicit approach based on a convex-concave splitting of the perturbed Cahn–Hilliard energy. More precisely, we define the solution $v^p$ as an approximation of $v$ at the time $p\delta_t$ given by the following scheme

$$\frac{v^{p+1} - v^p}{\delta_t} = (\Delta v^{p+1} - \frac{\alpha}{\epsilon^2}v^{p+1}) - \left((1 + f^{\sigma}_{u^n})\frac{W'(v^p)}{\varepsilon^2} - \frac{\alpha}{\epsilon^2}v^p\right).$$

Here, $\delta_t$ represents the time step of the scheme. The stabilized coefficient $\alpha$ is assumed to be sufficiently large to ensure the concavity of the the functional

$$v \mapsto \int_Q (1 + f^{\sigma}_{u^n})\frac{W(v)}{\epsilon^2} - \alpha\frac{v^2}{2\varepsilon^2}dx.$$

Indeed, in that case, we can prove the decreasing of the perturbed Cahn–Hilliard energy

$$J^{\varepsilon}_{u^n}(v^{p+1}) \leq J^{\varepsilon}_{u^n}(v^p),$$

and obtain an energy-unconditionally stable scheme. More precisely, this scheme can be written as

$$v^{p+1} = (I_d - \delta_t(\Delta - \frac{\alpha}{\epsilon^2}I_d))^{-1}\left(v^p - \delta_t\left((1 + f^{\sigma}_{u^n}(x))\frac{W'(v^p)}{\varepsilon^2} - \frac{\alpha}{\epsilon^2}v^p\right)\right)$$

where the operator $(I_d - \delta_t(\Delta - \frac{\alpha}{\epsilon^2}I_d))^{-1}$ is computed in Fourier space using the Fast Fourier Transform thanks to the periodic boundary conditions on the box $Q$.
In practice, we consider the box $Q = [-0.5, 0.5]^3$, discretized with $N$ nodes on each axis. All the $3D$ numerical experiments presented in this section are obtained using a resolution of $N = 2^7$, a phase-field parameter $\varepsilon = 2/N$, and $h = \delta_t = \varepsilon^2$. As for the jump term, we set $\sigma^2 = 0.1\varepsilon^2$ and $c = 0.35\varepsilon N^3$. In order to ensure the stability of the splitting scheme as shown in Chapter 2, section 2.2, the regularized parameter $\alpha$ is defined as

$$\alpha = \frac{1}{\varepsilon^2}\sup_{x\in Q}|W''(u(x))|.$$

### 4.4.2 Application to a dumbbell

The first numerical experiment illustrates the influence of the jump term in our new self-avoiding approximate mean curvature flow. We consider a dumbbell as the initial set, and it is well known

that the classical mean curvature flow yields a topology change in finite time. It is illustrated on the first line of Figure 4.3 where we plot the $0-$level set of an approximate solution $u^n$ to the Allen-Cahn equation computed at different times $t^n = nh$ without using the jump term (which corresponds to using $c = 0$). On the second line, we plot the $0-$level set of the approximate solution $u^n$ to the self-avoiding approximate mean curvature flow. We clearly observe here that the jump term forces the topological conservation.

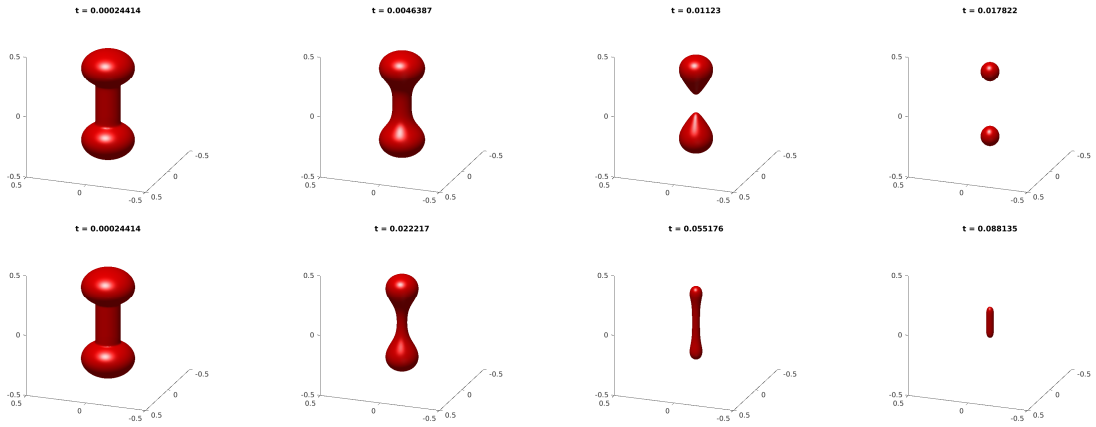


Figure 4.3: Evolution of a dumbbell: the solution $u^n$ plotted at different times $t$. The first and second lines correspond to the approximate mean curvature flow with or without, respectively, the additional jump term.

### 4.4.3 Application to a circle in 3D and codimension 2 mean curvature flow

We will now advocate, both with a numerical example and an asymptotic analysis, that the self-avoiding approximate mean curvature flow is a good candidate for approximating a codimension 2 mean curvature flow.

#### Numerical illustration

The second numerical experiment concerns the evolution of a circle in $\mathbb{R}^3$. We plot in Figure 4.4 the solution at different times $t^n = nh$. We can clearly observe a fattened circle which decreases along the iterations. We also illustrate in the last plot the evolution of the squared mass of $u$,

$$t \mapsto \left( \int_Q u(x,t)\, dx \right)^2$$

along the iterations. Notice that in the case of a fattened circle, this term is proportional to the square of the radius of the circle which is expected to decrease linearly in the case of the mean curvature flow, see Remark 4.4.1 below. We can clearly observe numerically such a decreasing (except at the every beginning of the simulation) which means that the circle evolves consistently with the mean curvature flow. One reason for the "bad" behavior at the very beginning is presumably an inconsistent initialization which is quickly regularized by the flow.

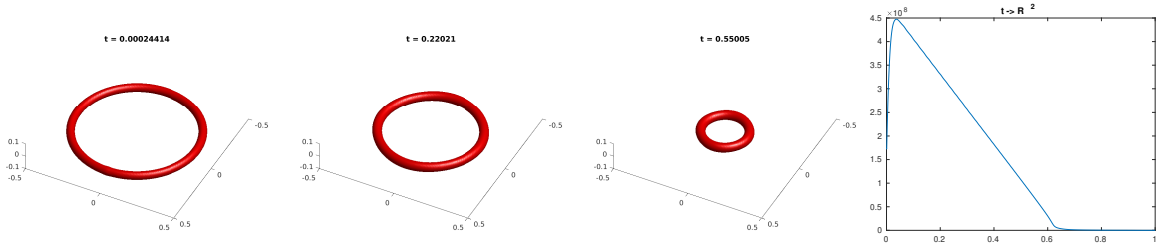


Figure 4.4: Evolution of a circle: the solution $u^n$ plotted at different times $t$ and the squared mass $t \mapsto (\int_Q u(x,t)dx)^2$ along the iterations.

**Formal analysis of the velocity**

Consider, for all $u \in \mathbb{S}$, $\Psi(u) = \begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}$, which parametrizes a circle $\Gamma$ of radius $R > 0$ lying on the horizontal plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^3$. The curvature of $\Gamma$ is $1/R$.

We start by fattening the curve with an offset $\delta > 0$, hence we obtain a circular torus. Let $\Phi$ be a parameterization of the torus defined by

$$\Phi(u,v) = \begin{pmatrix} (R+\delta\cos v)\cos u \\ (R+\delta\cos v)\sin u \\ \delta\sin v \end{pmatrix}.$$

By direct computations, the outer normal $\vec{n}$ and the scalar mean curvature $H$ can be written as

$$\vec{n}(u,v) = \begin{pmatrix} \cos u\cos v \\ \sin u\cos v \\ \sin v \end{pmatrix}$$

and

$$H(u,v) = -\left(\frac{1}{\delta} + \frac{\cos v}{R+\delta\cos v}\right).$$

The velocity of the torus on $\Phi(u,v)$ under the flow (4.3.4) is

$$V_{\Phi(u,v)} = H(u,v) + g_\sigma(\delta). \tag{4.4.3}$$

We denote the mean curvature vector as $\vec{H} := H\vec{n}$. For every $u \in \mathbb{S}$, the motion of the point $\Psi(u)$ is the average velocity of the torus on the associated sectional points, that is the mean value of $V_{\Phi(u,v)}$ for all $v \in \mathbb{S}$.

$$\begin{aligned} \vec{V}_{\Psi(u)} &= \frac{1}{2\pi}\int_0^{2\pi} V_{\Psi(u,v)}\vec{n}(u,v)dv \\ &= \frac{1}{2\pi}\int_0^{2\pi} \vec{H}(u,v)dv + \frac{1}{2\pi}\int_0^{2\pi} g_\sigma(\delta)\vec{n}(u,v)dv \\ &= \frac{1}{2\pi}\begin{pmatrix} \cos u\int_0^{2\pi}\frac{\cos^2 v}{R+\delta\cos v}dv \\ \sin u\int_0^{2\pi}\frac{\cos^2 v}{R+\delta\cos v}dv \\ \int_0^{2\pi}\frac{\cos v\sin v}{R+\delta\cos v}dv \end{pmatrix}. \end{aligned} \tag{4.4.4}$$

Notice that the forcing term $g_\sigma(\delta)$ in (4.4.3) does not have in average any impact on the velocity in (4.4.4).

Since we have that

$$\int_0^{2\pi} \frac{\cos v \sin v}{R + \delta \cos v} dv = 0$$

and that

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 v}{R + \delta \cos v} dv &= \int_0^{\frac{\pi}{2}} \left( \frac{\cos^2 v}{R + \delta \cos v} + \frac{\cos^2 v}{R - \delta \cos v} \right) dv + \int_{-\frac{\pi}{2}}^{0} \left( \frac{\cos^2 v}{R + \delta \cos v} + \frac{\cos^2 v}{R - \delta \cos v} \right) dv \\
&= 4R \int_0^{\frac{\pi}{2}} \frac{\cos^2 v}{R^2 - \delta^2 \cos^2 v} dv \\
&= \frac{4R}{\delta^2} \left( R^2 \int_0^{\frac{\pi}{2}} \frac{1}{R^2 - \delta^2 \cos^2 v} dv - \frac{\pi}{2} \right).
\end{aligned} \tag{4.4.5}$$

Moreover, by the change of variables $x = \tan v$, we get that

$$\int_0^{\frac{\pi}{2}} \left( \frac{1}{R^2 - \delta^2 \cos^2 v} \right) dv = \int_0^{\infty} \left( \frac{1}{R^2(1 + x^2) - \delta^2} \right) dx = \frac{\pi}{2} \frac{1}{R\sqrt{R^2 - \delta^2}}. \tag{4.4.6}$$

Therefore, together with (4.4.4), (4.4.5) and (4.4.6), we get that

$$\vec{V}_{\Psi(u)} = \frac{R}{\delta^2} \left( 1 - \frac{1}{\sqrt{1 - \frac{\delta^2}{R^2}}} \right) \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix} \sim -\frac{1}{2R} \left( 1 + O\left( \frac{\delta^2}{R^2} \right) \right) \vec{n}_{\Psi(u)}, \tag{4.4.7}$$

where $\vec{n}_{\Psi(u)}$ is the normal vector of $\Psi$. Consequently, the circle of radius $R > 0$ moves with a normal velocity $V \sim -\frac{1}{2R}$ at first order, i.e. consistently with the codimension 2 mean curvature flow.

*Remark* 4.4.1. Thanks to (4.4.7), one can show that the square of the radius $R$ of the circle decreases linearly as illustrated in Figure 4.4. Indeed, we have that

$$V = \frac{d}{dt} R \sim -\frac{1}{2R},$$

which gives

$$\frac{1}{2} \frac{d}{dt} (R^2) = R \frac{d}{dt} R \sim -\frac{1}{2}.$$

Hence, we get that

$$R^2 \sim -t.$$

### 4.4.4 Applications to smooth curves (and filaments) in 3D

The study of the previous section can be extended to more general smooth curves in 3D.

#### Numerical experiments

We present two evolutions of filaments (i.e., fattened smooth curves) in Figure 4.5. The first column shows the evolution of a simple filament which converges, using periodic boundary conditions, to a simple line. The second column shows the evolution of a more complex filament with the presence of a triple junction.

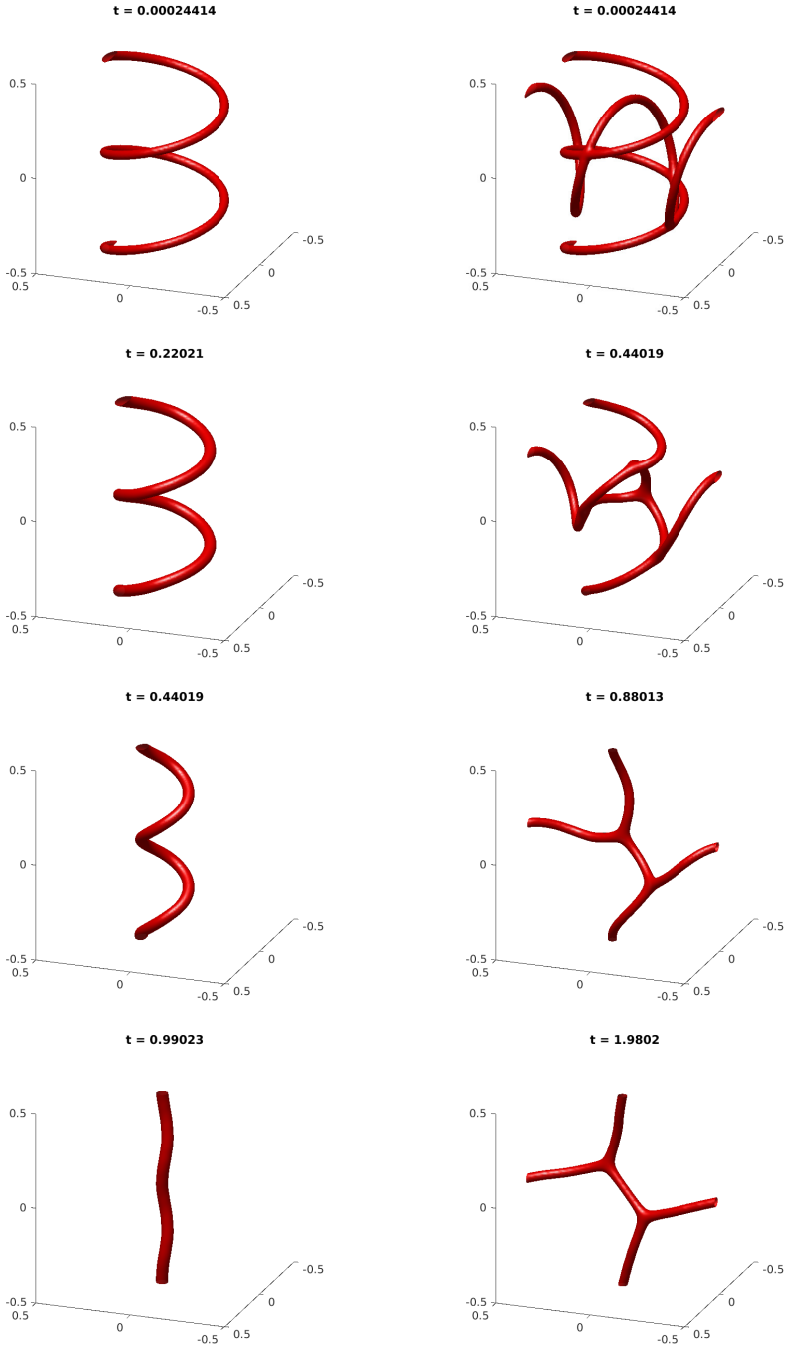


Figure 4.5: Each column illustrates the numerical evolution along time of a filament flowed by the self-avoiding mean curvature flow.

**Formal analysis of the velocity**

In the following, we will describe how to generalize the result of the circular case to any $C^2$-curve in $\mathbb{R}^3$. Given a closed curve $\Gamma$ in $\mathbb{R}^3$, as shown in the previous section, we consider the tube with thickness $\delta > 0$ obtained by fattening the curve. For every $s \in \Gamma$, one can construct a circular torus $\tau(s)$ as follows:

*The torus $\tau(s)$ is generated by the circle consisting of the orthogonal sectional points at $s$ (which is a circle of radius $\delta$) and the osculating circle of $\Gamma$ on $s$ of radius $|R(s)|$, where $R(s)$ is the inverse of the curvature of $\Gamma$ at $s$.*

Thanks to the previous result, since $s$ is the barycenter of its orthogonal sectional points, the velocity at $s$ is therefore the average of the velocities on the sectional set, which leads to

$$\vec{V}_s = \frac{R(s)}{\delta^2}\left(1 - \frac{1}{\sqrt{1 - \frac{\delta^2}{R(s)^2}}}\right)\vec{n}(s),$$

where $\vec{n}(s)$ is the normal vector of $\Gamma$ on $s$.

**Open space curves**

For open curves in $\mathbb{R}^3$, the process is the same as for closed curves except at the end points. The velocity on each end point of the curve is indeed the mean value of the velocities on the semi-spherical cap of radius $\delta$ centered on the end point.

Up to some linear transformation, we can assume that the end point is $(0, 0, 0)$ and the semi-spherical cap is parametrized by $\Phi : [0, 1] \times \mathbb{S}$ with

$$\Phi(r, \theta) := \delta \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{1 - r^2} \end{pmatrix}.$$

It is clear that $H(r, \theta) = -\frac{2}{\delta}$ and the normal vector is

$$\vec{n}(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{1 - r^2} \end{pmatrix}.$$

Direct computations shows that the mean velocity vector on the end point is

$$\vec{V} = \left(-\frac{2}{\delta} + g_\sigma(\delta)\right)\frac{1}{2\pi}\int_0^{2\pi} \vec{n}(r, \theta)|\Phi_r \wedge \Phi_\theta|(r, \theta)drd\theta = \left(-\frac{2}{\delta} + g_\sigma(\delta)\right)\begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}.$$

Notice that in absence of the forcing term $g_\sigma(\delta)$, i.e under the classical mean curvature flow, the end point moves towards the curve, therefore shortens the curve. However with the presence of $g_\sigma$, the end point becomes stationary in finite time. Since the regular part of the curves keeps evolving under the mean curvature flow at first order, the flow actually provides a natural approach to the Steiner problem in $\mathbb{R}^3$.

**Videos**

To get a better idea of numerical flows computed with our model, videos (in .avi format) can be downloaded at the following addresses.

- Video of the flow shown in Figure 4.5, left: Filament 1.
- Video of the flow shown in Figure 4.5, right: Filament 2.
- In the previous video, the filament remains connected but the inner cycle disappears. It is actually possible to play with that by simply changing the weight of the jump term: a higher weight yields a higher sensitivity. This is the choice made for the numerical simulation shown in this third video where the inner cycle is conserved: Filament 3.

**Conclusion**

The above numerical experiments together with the formal analysis indicate that our self-avoiding approximate mean curvature flow:

- manages to preserve the topology of general (smooth) sets thanks to the jump term;
- gives an approximation of the mean curvature flow in codimension 2 in the case where we have filaments as initial sets.

## 4.5 Application to the Steiner problem in 3D

The numerical experiments and observations of the previous sections on the Allen-Cahn equation coupled with the jump term (4.4.1) lead us naturally to address the application to Steiner's problem in dimension 3. Recall that Steiner's problem consists in finding, for a given collection of points $a_0, \dots, a_N \in Q$, a compact connected set $K \subset Q$ containing all the $a_i$'s and having minimal length. In other words, it amounts to solving the following minimization problem

$$\min\{\mathcal{H}^1(K),\ K \subset Q,\ K \text{ connected},\ a_i \in K, \forall i\}, \tag{4.5.1}$$

where $\mathcal{H}^1(K)$ stands for the one-dimensional Hausdorff measure of $K$. The numerical approximation of solutions to this problem is notoriously difficult, especially in dimension $\geq 3$, see for instance [77, 76, 49, 27, 24, 25, 37, 38, 26]. The model we propose provides an effective and natural way to approximate these solutions in 3D (and actually even in higher dimension for the extension is straightforward).

**A fattened problem**

We consider an approximation of Steiner's problem in dimension 3 for the $\tilde{\sigma}$-tubular set $K_{\tilde{\sigma}} := \{x \in \mathbb{R}^3 \mid \operatorname{dist}(x, K) \leq \tilde{\sigma}\}$ which is the $\tilde{\sigma}$-fattening of $K$. We choose the fattening parameter $\tilde{\sigma}$ to be small enough as in Claim 4.3.5 so that the minimal distance between the boundary of $K_{\tilde{\sigma}}$ and its skeleton $K$ is ensured.
It is clear that the length of $K$ is approximated by the perimeter of $K_{\tilde{\sigma}}$ in the sense that

$$\mathcal{P}(K_{\tilde{\sigma}}) \simeq 2\pi\tilde{\sigma}\mathcal{H}^1(K).$$

Moreover, the property that $K$ contains all the $a_i$ points can be replaced by the *inclusion constraint*:

$$\cup_{i=1}^N B(a_i, \tilde{\sigma}) \subset K_{\tilde{\sigma}}, \tag{4.5.2}$$

which means that $K_{\tilde{\sigma}}$ contains all the balls $B(a_i, \tilde{\sigma})$ of radius $\tilde{\sigma}$.

Therefore, we are now interested in a phase-field approximation of the following approximate minimization problem:

$$\min\{\mathcal{H}^2(K_{\tilde{\sigma}}) \mid \cup_{i=1}^N B(a_i, \tilde{\sigma}) \subset K_{\tilde{\sigma}} \text{ and } \operatorname{dist}(\partial K_{\tilde{\sigma}}, \Sigma(K_{\tilde{\sigma}})) \geq \tilde{\sigma}\},$$

where $\Sigma(K_{\tilde{\sigma}})$ is the skeleton of $\partial K_{\tilde{\sigma}}$.

**Phase-field approximation of the fattened Steiner problem**

There is a natural way to approximate the inclusion constraint (4.5.2) using phase fields. This can be done by first introducing the function $u_{\text{in}}^{\varepsilon}$ defined by

$$u_{\text{in}}^{\varepsilon}(x) := q\left(\frac{\text{dist}(x, \cup_{i=1}^{N}\{a_i\}) - \tilde{\sigma}}{\varepsilon}\right),$$

where $\text{dist}(x, \cup_{i=1}^{N}\{a_i\})$ denotes the distance function to the points $a_i$ and $q$ is the usual optimal profile associated with the double-well function. The inclusion constraint (4.5.2) can thus be implemented by considering the inequality constraint:

$$u_{\text{in}}^{\varepsilon} \leq u^{\varepsilon}.$$

Indeed, it is not difficult to observe that

$$u_{\text{in}}^{\varepsilon} \leq u^{\varepsilon} \quad \Longrightarrow \quad \cup_{i=1}^{N} B(a_i, \tilde{\sigma}) \subset \{x; u^{\varepsilon}(x) \geq 0\}$$

Therefore, we define recursively the sequence $\{u^n\}_{n\in\mathbb{N}}$ by

$$u^{n+1}(x) = \max(v(x,h), u_{\text{in}}^{\varepsilon}(x)), \tag{4.5.3}$$

where $v$ is solution to the equation:

$$\begin{cases} \partial_t v(x,t) = \Delta v(x,t) - \dfrac{W'(v(x,t))}{\varepsilon^2}(1 + f_{u^n}^{\sigma}(x)), \\ v(\cdot, 0) = u^n. \end{cases} \tag{4.5.4}$$

**Numerical experiments**

We define $Q = [-0.5, 0.5]^3$ and use the following settings: $N = 2^7$, $\varepsilon = 2/N$, $h = \delta_t = \varepsilon^2$, $\sigma^2 = 0.1\varepsilon^2$, $c = 0.35\varepsilon N^3$, and $\tilde{\sigma} = 0.02$.
Remark that in $3D$, according to conjecture (4.2.5):

$$f_{u^n}^{\sigma} \simeq \mu(\Sigma_2^n) + \sigma\mu(\Sigma_1^n) + \sigma^2\mu(\Sigma_0^n)$$

where $\Sigma_i^n$ is the $i$-dimensional component of the skeleton $\Sigma^n$ of $u^n$. Therefore, in order to make sure the entire skeleton is an obstacle for the flow, we set

$$c \simeq \varepsilon N^3 \simeq \frac{1}{\sigma^2}$$

which is of order $\sigma^{-2}$ so that even in the worst case where the skeleton is a union of discrete points, the perturbation term $cf_{u^n}^{\sigma}$ charges effectively the skeleton $\Sigma^n$ at time $nh$.

The first example in Figure 4.6 represents the case where the $a_i$'s are the vertices of a cube. The initial set is also a cube containing all nodes $a_i$. We show in Figure 4.6 approximate solutions at different times of the phase-field flow (4.5.3)-(4.5.4).

Notice that even if the initial set is of dimension 3 (the initial cube), the stationary solution is close to a tubular thickening of a 1-dimensional set $K$ containing all the nodes $a_i$. We observe that this solution approximates well the Steiner tree associated with the vertices of the cube, in particular at each triple junction all angles equal $2\pi/3$.

As a comparison, we consider in Figure 4.7 an example with 10 points randomly distributed in $Q$ which leads exactly to the same conclusion.

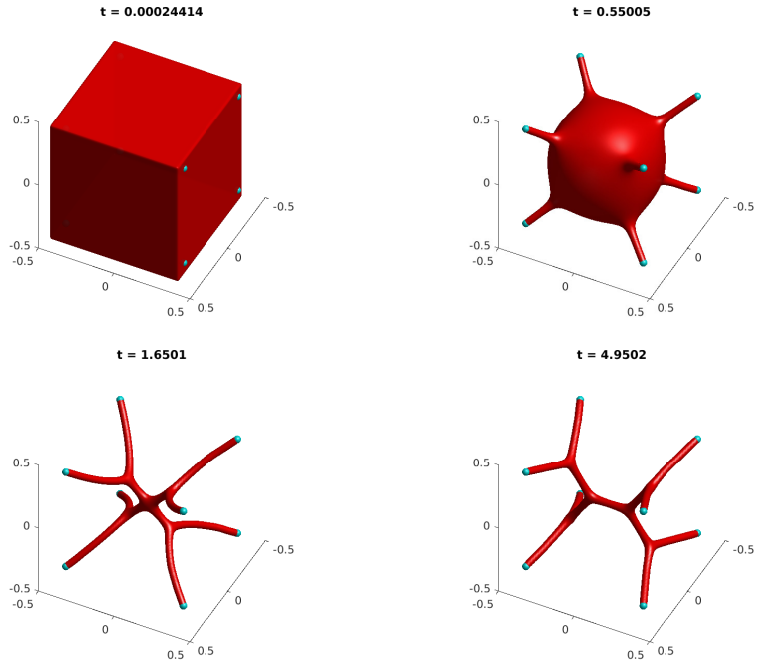


Figure 4.6: A Steiner tree associated with the vertices of a cube: illustration of the approximate solutions at different times along the numerical flow. The red and blue interfaces are, respectively, the 0-level set of $u^n$ and the set $\cup_{i=1}^{N} B(a_i, \tilde{\sigma})$.

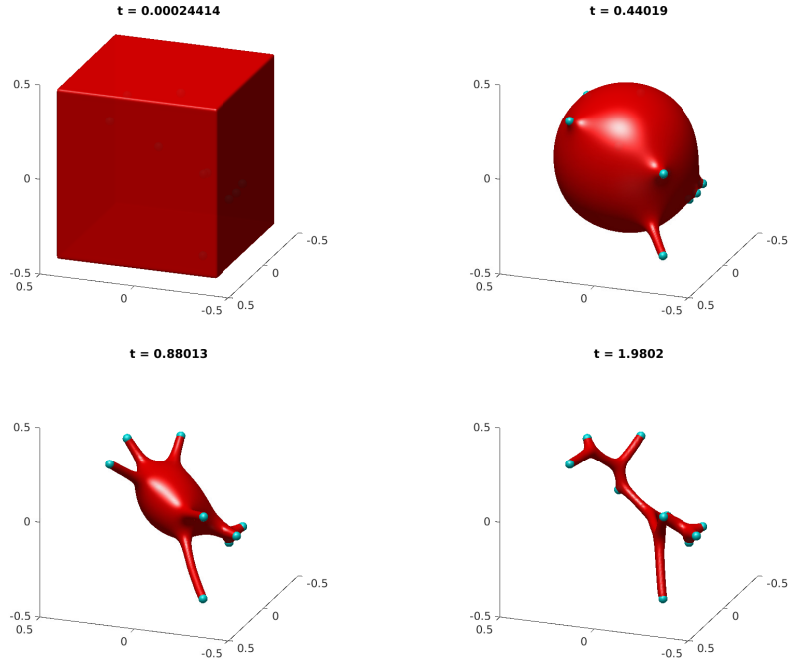


Figure 4.7: A Steiner tree associated 10 points randomly chosen in $Q$: illustration of the approximate solutions at different times along the numerical flow. The red and blue interfaces are, respectively, the 0-level set of $u^n$ and the set $\cup_{i=1}^{N} B(a_i, \tilde{\sigma})$.

**Videos**

To get a better idea of numerical flows computed with our model, videos (in .avi format) can be downloaded at the following addresses.

- Video of the flow illustrated in Figure 4.6: Approximation of the Steiner set of cube's vertices.
- Approximation of the Steiner tree associated with 50 randomly chosen points: Approximation of the Steiner set of 50 random points.

## 4.6 Application to the Plateau problem

Our last numerical application is devoted to the celebrated Plateau problem. Recall that the Plateau problem in dimension 3 consists in finding, for a given closed Jordan curve $\gamma$, a surface $E$ in $\mathbb{R}^3$ with (locally) minimal area such that the boundary of $E$ coincides with $\gamma$. In other words, it amounts to solving the following minimization problem:

$$\min\{\mathcal{H}^2(E),\ E \subset \Omega,\ \text{connected and such that } \partial E = \gamma\}, \tag{4.6.1}$$

where $\mathcal{H}^2(E)$ stands for the 2-dimensional Hausdorff measure of $E$.

**A fattened Plateau problem**

We consider a $\tilde{\sigma}$-fattened minimal surface problem for the $\tilde{\sigma}$-tubular thickening of a given Jordan curve $\gamma$

$$\gamma_{\tilde{\sigma}} := \{x; \operatorname{dist}(x, \gamma) < \tilde{\sigma}\},$$

by considering the following minimization problem:

$$\min\left\{\mathcal{P}(E_{\tilde{\sigma}}) + \frac{c}{\tilde{\sigma}}\mathcal{H}^3(E_{\tilde{\sigma}}); E_{\tilde{\sigma}} \subset \Omega, \text{connected and } \gamma_{\tilde{\sigma}} \subset E_{\tilde{\sigma}}\right\}. \tag{4.6.2}$$

where $\mathcal{H}^3(E_{\tilde{\sigma}})$ stands for the volume of $E_{\tilde{\sigma}}$. Here, $\tilde{\sigma}$ being chosen sufficiently small, the volume term is present to ensure that $E_{\tilde{\sigma}}$ has a thickness of size $\tilde{\sigma}$ which requires the existence of a connected set $E$ such that

$$E_{\tilde{\sigma}} \simeq \{x \in Q; \operatorname{dist}(x, E) \leq \tilde{\sigma}\}, \quad \text{and} \quad \mathcal{P}(E_{\tilde{\sigma}}) \simeq 2\mathcal{H}^2(E).$$

Figure 4.8 shows what may happen if this volume term is not used: it it not possible to get in the limit a "thin" volume which approximates a surface.

As previously, starting from an initial connected set, the connectedness property is conserved thanks to the jump term in the self-avoiding mean curvature flow.

*Remark* 4.6.1. In contrast with the fattened Plateau problem, it is not necessary to penalize the volume in the fattened Steiner problem. The reason is straightforward: in the Steiner problem the vertices constraints coupled with an area-minimizing property of the flow yield a natural decreasing of the volume to reach a final tubular set.

**Phase-field approximation of the fattened Plateau Problem**

Similarly to the fattened Steiner problem, we build recursively a sequence $(u^n)_n$ by combining a perturbed Allen Cahn equation with an additional inclusion constraint, i.e. we compute:

$$u^{n+1}(x) = \max(v(x, h), u^{\varepsilon}_{\text{in}}), \tag{4.6.3}$$

where $v$ is solution of the following flow

$$\begin{cases} \partial_t v(x,t) = \Delta v(x,t) - \dfrac{W'(v(x,t))}{\varepsilon^2}(1 + f^{\sigma}_{u^n}(x)) + \dfrac{c_{\text{volume}}}{\varepsilon\sigma}\sqrt{2W(u^n)}, \\ v(\cdot,0) = u^n. \end{cases} \tag{4.6.4}$$

with $u^{\varepsilon}_{\text{in}}$ defined as

$$u^{\varepsilon}_{\text{in}}(x) = q\left(\frac{\text{dist}(x,\gamma_{\tilde{\sigma}})}{\varepsilon}\right),$$

and the last term in the first equation of (4.6.4) corresponds to the approximation of the volume term.

**Numerical experiments**

As previously, we define $Q = [-0.5, 0.5]^3$ and we set $N = 2^7$, $\varepsilon = 2/N$, $h = \delta_t = \varepsilon^2$, $\sigma^2 = 0.1\varepsilon^2$, $c = 0.35\varepsilon N^3$, and $\tilde{\sigma} = 0.02$.
In Figure 4.8, we plot the solution obtained at different times using the parameter $c_{\text{volume}} = 0$. In this case, the volume is not penalized during the iteration. In particular, we can observe that the stationary set $E_\sigma$ has not the right form in the sense that its thickness is not of size $\tilde{\sigma}$ and cannot be associated with a minimal 2-dimensional surface. This experiment shows all the interest to penalize the volume in our computations. All following numerical experiments in this section are done with the setting $c_{\text{volume}} = 1$.

The main motivation of Figure 4.9 is to illustrate the influence of the initial set and its topology on the result. Each column corresponds to a particular choice of an initial configuration, the images show the numerical solution of the flow at different times. We observe that our model is able to compute a minimal surface associated with the given boundary but the topology of this minimal surface depends logically on the choice of the initial set.

With Figure 4.10, we illustrate the ability of our method to approximate non orientable minimal surfaces and minimal surfaces with triple line singularities. The first column shows an example of a non smooth (approximate) minimal surface whereas the second one gives an approximation of a Möbius strip. Notice that in the latter case, we start from an initial connected set given by a cube with a cylinder hole.

**Videos**

To get a better idea of numerical flows computed with our model, videos (in .avi format) can be downloaded at the following addresses.

- Video of the flow illustrated in Figure 4.9, right: An (approximate) Plateau solution.
- Approximation of a Möbius strip with a singularity line: A singular (approximate) Möbius strip.
- An hybrid solution combining a Steiner set and a minimal surface: A Steiner-Plateau hybrid solution.

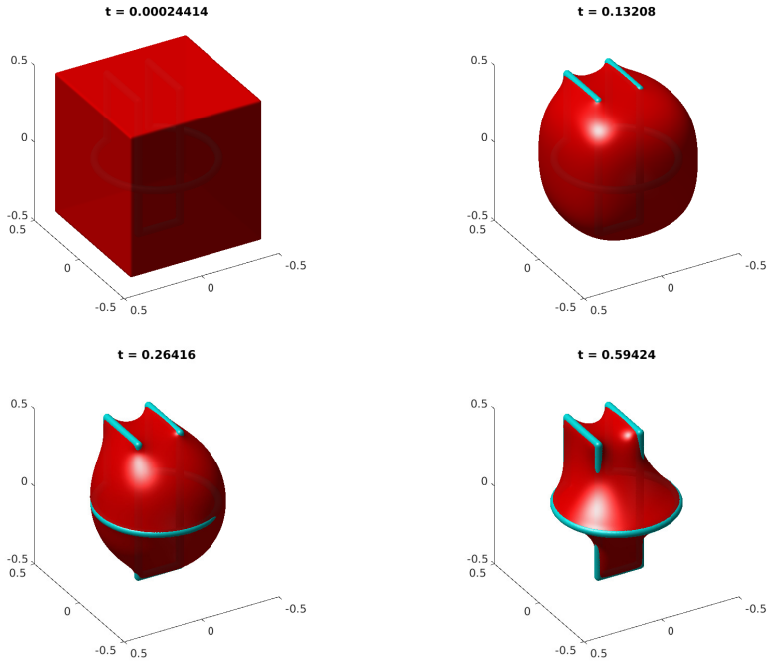


Figure 4.8: Using the self-avoiding mean curvature flow without volume penalization: the stationary shape is not at all close to a minimal surface

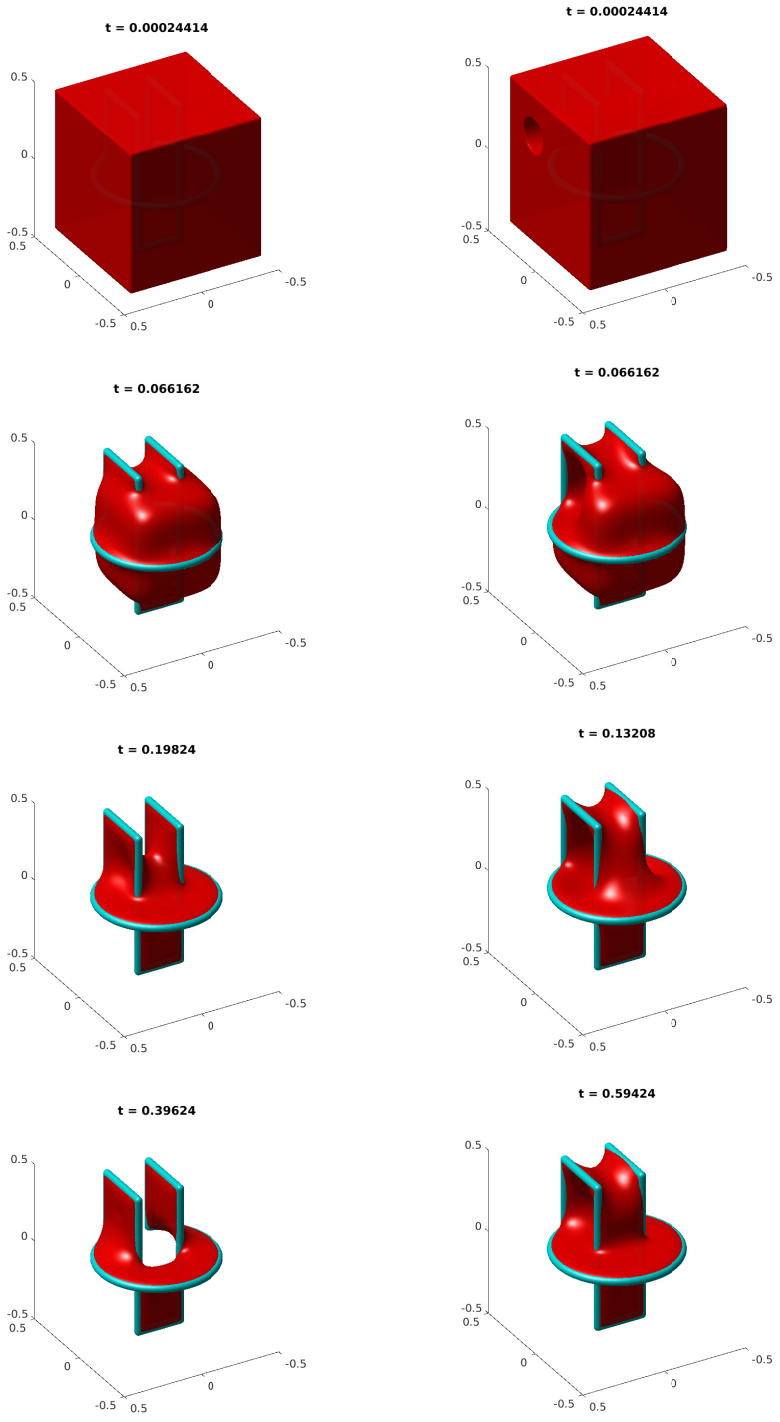


Figure 4.9: Numerical approximation of solutions to the Plateau problem using the proposed flow. The experiment illustrates the influence on the result of the initial set's topology. Each column shows the numerical solution at different times starting from two different initial configurations.

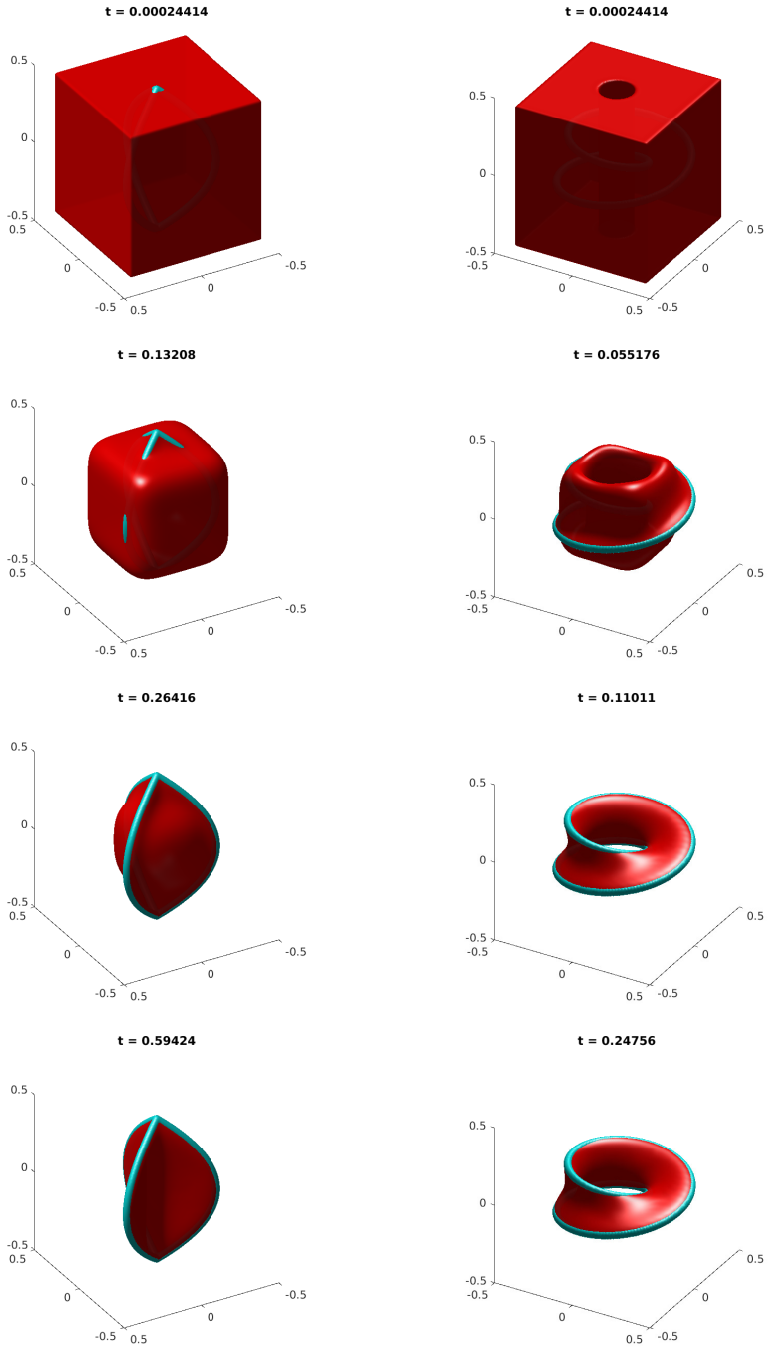


Figure 4.10: Numerical approximation of solutions to the Plateau problem using the proposed flow. First column: convergence to a minimal surface with a triple line singularity. Second column: convergence to a Möbius strip, an example of non-orientable surface that our model can compute.

# Asymptotic analysis of Willmore spheres in Riemannian manifolds

# Chapter 5

# Asymptotic analysis of embedded Willmore spheres in Riemannian manifolds of dimension 3

In this chapter, we present some partial results in the direction of proving that small embedded Willmore spheres concentrating in a Riemannian manifold of dimension 3 must concentrate at a critical point of the scalar curvature. This type of results were proven for constant mean curvature spheres by Laurain [83] using the Wente structure of the associated equation. It has also been proven by Laurain-Mondino [84] with a small energy condition.

We present here a way to get strong pointwise estimates for such a sequence, in the spirit of [83], showing how to use the Rivière formulation of Willmore equation in terms of a triple system of elliptic PDE's with nonlinear leading terms in Wente form.

However, this remains programmatic since, in the decomposition of the sequence of Willmore surfaces into standard "bubbles", spheres and catenoids should appear and we are not, at now, able to deal with the appearance of catenoids. But we are convinced that the way we find strong estimates should be very useful for the general case. So we present them here. Since it has no meaning to deal with multiple spheres without considering catenoids, we just restrict ourself to show how it works for two spheres. And we shall explain from place to place what is lacking to deal with the general situation.

## 5.1 Willmore energy and Willmore immersions

Let $(M,h)$ be a Riemannian manifold and $\Phi : \Sigma \to (M,h)$ be a smooth immersion of a closed surface $\Sigma$ into $(M,h)$. The Willmore energy of $\Phi$ is given by

$$\mathcal{W}_h(\Phi) := \int_\Sigma H^2 d\sigma_g, \tag{5.1.1}$$

where $d\sigma_g$ is the volume form associated to the pullback metric $g := \Phi^*(h_{|\Sigma})$ on $\Sigma$, and the mean curvature $H$ is half of the trace of the second fundamental form $A$. We can write simply $\mathcal{W}(\Phi)$ instead of $\mathcal{W}_h(\Phi)$ if there is no risk of confusion. In addition, we say that $\Phi$ is a *Willmore immersion* if $\Phi$ is a critical point of the Willmore functional $\mathcal{W}$ and $\Phi(\Sigma)$ is then said to be a *Willmore surface* in $(M,h)$.

The investigation of Willmore energy in 3-dimensional Euclidean space has originally been introduced by Poisson [117] and Sophie Germain [65], inspired by the works of Bernoulli and Euler on the theory of elasticity and the study of Chladni figures in the early XIX century (interested readers can refer to [46], for example). In the 20's, Blaschke [23] and his student Thomsen [132] started the

study of the conformal invariant theory of critical points of this functional. This functional was later referred as Willmore energy, after the name of the english mathematician Thomas Willmore who rediscovered the functional in the 60's, see [137] and [136]. Since then, the subject has become one active branch in geometric analysis. We also refer to Nitsche [112], Palmer [115], Dall'Acqua [48], Grunau [67] and recently Alessandroni-Kuwert [2] ...etc. for topics on Willmore surfaces with fixed and/or free boundaries.

## 5.2 Existence of Willmore surfaces in manifolds and concentration phenomena

Concerning Willmore surfaces in manifolds, several existence and non-existence results have been recently revealed: using a pertubative method, Willmore spheres (with possibly *small* area constraint or not) have been constructed as perturbations of small geodesic spheres by Lamm-Metzger [81], Mondino [104]-[105], Lamm-Metzger-Schulze [82] and Chen-Li [40]. Mondino-Rivière [107] generalize the existence result for any value of area constraint. All these constructions are based on the implicit function theorem, where certain non-degeneracies of the ambient manifold are required, notably in a neighborhood of a critical point of the scalar curvature.

Hence, conversely, we are naturally lead to the following question: let $p$ be a point of a given 3-dimensional Riemannian manifold $(M, h)$. Assume that, for $r$ small enough, the geodesic ball $B_r^h(p)$ contains a Willmore surface (or more generally Willmore surface under area constraint), what can we say about $p$?

In several different contexts, it appears that $p$ has to be a critical point of the scalar curvature. Druet [56] first gave the response in the case of isoperimetric domains, when he showed that isoperimetric domains of small volume necessarily concentrate at a critical point of the scalar curvature (more precisely a maximum point of scalar curvature). In the case of constant mean curvature surfaces, the answer was given by Laurain [83] for embedded CMC spheres with small diameter while the mean curvature goes to infinity. In the case of Willmore surfaces, Lamm and Metzger first showed that the concentration phenomena appears if we restrict ourself to Willmore surfaces with positive mean curvature in their pioneer work [80]. In an equivalent way, such concentration phenomena holds for Willmore surfaces with its energy below $4\pi + \varepsilon$ for $\varepsilon > 0$ small enough. Laurain and Mondino in [84] next generalized the result to any energy threshold $8\pi - \delta$ for any $\delta > 0$. More precisely, they prove the following theorem.

**Theorem 5.2.1** (Laurain-Mondino [84])**.** *Let $(M, h)$ be a 3-dimensional Riemannian manifold. Let $p \in M$ and let $\delta > 0$ be given. If, for any $r$ small enough, there exists an embedded Willmore sphere with area constraint contained in the geodesic ball $B_r^h(p)$ such that its Willmore energy is bounded above by $8\pi - \delta$, then $p$ is a critical point of the scalar curvature of $(M, h)$.*

Notice that, thanks to the Li-Yau inequality in [89], only embedded Willmore spheres occur under these assumptions. Instead of considering the Willmore energy, one can also consider the energy functional given by the second fundamental form:

**Definition 5.2.2.** *Let $(M, h)$ be a Riemannian manifold and $\Phi : \Sigma \to (M, h)$ be a smooth immersion of a closed surface $\Sigma$ into $(M, h)$. We define the energy $\mathcal{A}_h(\Phi)$ by*

$$\mathcal{A}_h(\Phi) := \frac{1}{2}\int_\Sigma |A|^2 \, d\sigma_g = 2\int_\Sigma H^2 - \int_\Sigma \left(K - \overline{K}_h\left(T\Phi\right)\right) \, d\sigma_g \tag{5.2.1}$$

*where $K$ is the Gauss curvature and $\overline{K}_h\left(T\Phi\right)$ is the sectional curvature of the ambient manifold $(M, h)$ on the tangent space of $\Phi(\Sigma)$*

Thanks to the definition 5.2.2 of $\mathcal{A}_h$, Theorem 5.2.1 can be reformulated as follows:

**Theorem 5.2.3.** *Let $(M,h)$ be a smooth 3-dimensional Riemannian manifold and let $(\Phi^k(\mathbb{S}^2))_{k\in\mathbb{N}}$ be a sequence of area-constrained Willmore spheres in $M$ satisfying the following assumptions:*

1. *$diam(\Phi^k(\mathbb{S}^2)) = \varepsilon_k$ with $lim_{k\to+\infty}\varepsilon_k = 0$,*
2. *the energies of $\Phi^k(\mathbb{S}^2)$ are uniformly bounded above by $12\pi - \delta$ i.e,*

$$\sup_{k\in\mathbb{N}} \mathcal{A}_h(\Phi^k) < 12\pi - \delta.$$

*Then, up to a subsequence of $(\Phi^k(\mathbb{S}^2))_{k\in\mathbb{N}}$, $(\Phi^k(\mathbb{S}^2))_{k\in\mathbb{N}}$ converge to $p \in M$ which is a critical point of the scalar curvature of $(M,h)$.*

Indeed, since we always have that

$$\int_\Sigma K\, d\sigma_g = 4\pi \text{ for } \Sigma = \mathbb{S}^2,$$

we get that

$$\mathcal{A}_h(\Phi^k) = 2\mathcal{W}_h(\Phi^k) - 4\pi + \int_{\mathbb{S}^2} \overline{K}_h(T\Phi^k)\, d\sigma_{g^k}, \tag{5.2.2}$$

which implies

$$\sup_{k\in\mathbb{N}} \mathcal{A}_h(\Phi^k) < 12\pi - \delta \iff \sup_{k\in\mathbb{N}} \mathcal{W}_h(\Phi^k) < 8\pi - \delta. \tag{5.2.3}$$

since under the hypothesis of Theorem 5.2.1, we have that

$$\int_{\mathbb{S}^2} \overline{K}_h\left(T\Phi^k\right)\, d\sigma_{g^k} \lesssim C_p diam\left(\Phi^k(\mathbb{S}^2)\right) \to 0 \text{ as } k\to\infty.$$

In this part of the thesis, we will focus on the analysis aspect of the nature of the problem, and provide key elements which seems promising to generalize the result to the most general case for any energy upper bound threshold $C > 0$.

In Section 5.3, we will first introduce the conservative form given by Mondino-Rivière [106] for Willmore immersions in manifolds. Under proper rescaling of the metric, we show that the Lagrange multiplier due to the area constraint can be treated as a perturbative error. Hence, we are lead to consider Willmore spheres in $(\mathbb{R}^3, h_\varepsilon)$ where $h_\varepsilon$ is the rescaled metric, close to the usual Euclidean metric $h_0$.

In Section 5.5, working with relevant parametrization for our Willmore spheres, we perform a local blow-up analysis at the points where concentration phenomena appear: we describe asymptotically the Willmore spheres as a sum of bubbles and remainder, by using the pointwise technique developed in Druet-Hebey-Robert [57], which allows to obtain first pointwise estimates on the remainder. In Section 5.7, exploiting compensated compactness results in Lorentz spaces, together with help of Green's representation formula, we are able to improve the pointwise estimates by re-injecting them in the Willmore system. At the end of the section, under the assumption that the remainder and bubble interactions can be absorbed by the Riemannian error term due to $h_\varepsilon$, we prove Theorem 5.2.3.

Finally in Section 5.7.2, using the fact that, with relevant initial conditions, the linearized Willmore operator contains only trivial solutions, we study carefully the interaction between bubbles and prove the assumption made in Section 5.7.

## 5.3 Preliminaries

Throughout the rest of the thesis, $(M,h)$ will be a 3-dimensional Riemannian manifold. By abuse of notations, we denote by $D$ both the Levi-Civita connection of $(M,h)$ and the associated covariant

exterior derivative. $\star_h$ designs the Hodge operator associated to $h$ on multi-vectors of $M$ from $\Lambda^p M$ into $\Lambda^{3-p} M$. The Riemannian curvature tensor is given by the following formula:

$$Riem_h(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where $[X,Y]$ is the Lie bracket of $X$ and $Y$.

Since here we deal particularly with immersions $\Phi$ of $\mathbb{S}^2$ onto $(M,h)$, with help of stereographic projection and local conformal coordinates, it would be convenient to consider $\Phi$ as a conformal immersion of a disc or a plane into $(M,h)$. Recall that an immersion $\Phi$ of the 2-dimensional unit disc $\mathbb{D}$ into $(M,h)$ is said to be a *conformal immersion* if there exists $\lambda \in C^\infty(\mathbb{D})$ such that the pullback metric $g := \Phi^* h = e^{2\lambda}(dx^2 + dy^2)$. We call $\lambda$ the *conformal factor* of $\Phi$. We also denote by $(\vec{e}_1, \vec{e}_2)$ the orthonormal basis of $\Phi_*(T\mathbb{D})$ given by $\vec{e}_1 := e^{-\lambda}\partial_x \Phi$ and $\vec{e}_2 := e^{-\lambda}\partial_y \Phi$ where $e^\lambda = |\partial_x \Phi| = |\partial_y \Phi|$. Finally, the unit normal vector $\vec{n}$ to $\Phi(\mathbb{D})$ is given by

$$\vec{n} = \star_h(\vec{e}_1 \wedge \vec{e}_2).$$

Discovered in the Ph.D work of Thomsen [132] and attributed to the work of Schadow, an Euler-Lagrange equation of the Willmore functional has been established for surfaces immersed in $\mathbb{R}^3$. Weiner [134] next derived its form for surfaces immersed in higher dimensions and in curved spaces. An immersion $\Phi : \Sigma \to (M,h)$ is then said to be a *Willmore immersion* if it satisfies

$$\Delta_g H + 2H(H^2 - (K_g - K_h)) + H Ric_h(\vec{n},\vec{n}) = 0 \text{ on } \Sigma, \tag{5.3.1}$$

where $\Delta_g = div_g \nabla$ is the negative Laplace-Beltrami operator corresponding to the induced metric $g$, $K_g$ is the Gauss curvature on $(\Sigma, g)$, $K_h$ is the sectional curvature of $(M,h)$ on $T\Phi(\Sigma)$ and $Ric_h$ is the Ricci tensor of $(M,h)$. More generally, Willmore surfaces *under area-constraint* are characterized by

$$\Delta_g H + 2H(H^2 - (K_g - K_h)) + H Ric_h(\vec{n},\vec{n}) = \alpha H \text{ on } \Sigma, \tag{5.3.2}$$

where $\alpha \in \mathbb{R}$ is the *Lagrange multiplier*.

Remark that the above equation (5.3.2) is a non-linear elliptic PDE of fourth order with respect to its parametrization. Despite of its simplicity, the equation is not quite convenient for variational approaches. In fact, under weak formulation, $H$ is required to be at least $L^3_{loc}(\Sigma)$, which does not fit with the Willmore functional, giving only $L^2$-integrability of $H$.

In consecutive works [120]-[121]-[122], Rivière develops powerful analytical tools for the study of Willmore immersions into $\mathbb{R}^3$. In particular, by describing the conservative laws of the conformally invariant functional, Rivière manages to give a local reformulation of the Willmore equation (5.3.6) in a divergence form. Moreover, derived from the divergence form, a coupled system of *Wente-type* equations permits to use compensated compactness developped by Wente [135] and Tartar [130] and leads to numerous outcomings: in [21], Bernard and Rivière proved an energy quantization result for any sequence of Willmore surfaces with fixed genus with uniformly bounded energy and non-degenerating conformal type; in [85], Laurain and Rivière recently generalized this result in degenerating case, and showed strong compactness of Willmore immersions with energy below $12\pi$.

In this section, we focus on the following divergence form generalized by Mondino and Rivière in [106] for Willmore immersions into manifolds, or more generally, for conformal immersion satisfying the Euler-Lagrange equation (5.3.2) with area constraint. We have:

**Proposition 5.3.1** (see Mondino-Rivière [106], Corollary II.1.). *A conformal immersion $\Phi : \mathbb{D} \to (M,h)$ is an area-constrained Willmore immersion if and only if*

$$2e^{2\lambda}\left[R^\perp_\Phi(T\Phi) + \vec{H} Ric_h(\vec{n},\vec{n}) - \alpha \vec{H}\right] = D^*\left[D\vec{H} - 3\pi_{\vec{n}}(D\vec{H}) - \star_h\left(\vec{H} \wedge D^\perp \vec{n}\right)\right], \tag{5.3.3}$$

*for some $\alpha \in \mathbb{R}$, where we denote by $\pi_{\vec{n}} : T_\Phi M \to (\Phi_*(T\mathbb{D}))^\perp$ the orthogonal projection from the tangent bundle to $(M,h)$ onto the normal bundle to $\Phi(\mathbb{D})$, $\vec{H} := H\vec{n}$ the mean curvature vector, $D\cdot := \left(D_{\partial_x \Phi}\cdot\,,\, D_{\partial_y \Phi}\cdot\right)$,*

*$D^\perp \cdot := \left(-D_{\partial_y \Phi} \cdot,\, D_{\partial_x \Phi} \cdot\right)$ and $D^*$ represents an operator acting on couples of vector fields $(\vec{V}_1, \vec{V}_2)$ along $\Phi_*(T\Sigma)$ defined by*

$$D^*(\vec{V}_1, \vec{V}_2) := D_{\partial_x \Phi} \vec{V}_1 + D_{\partial_y \Phi} \vec{V}_2,$$

*and finally we have $e^{2\lambda} R^\perp_\Phi(T\Phi) := *_h \left(\vec{n} \wedge Riem_h(\partial_y \Phi, \partial_x \Phi)\vec{H}\right)$.*

Given any conformal immersion $\Phi : \mathbb{R}^2 \to (M, h)$, it is well known that

$$D^* D\Phi = 2e^{2\lambda}\vec{H} = \star_h D\vec{n} \wedge \nabla^\perp \Phi,$$

where $\nabla \cdot = \left(\frac{\partial}{\partial x} \cdot,\, \frac{\partial}{\partial y} \cdot\right)$ and $\nabla^\perp \cdot = \left(-\frac{\partial}{\partial y} \cdot,\, \frac{\partial}{\partial x} \cdot\right)$.

To be supplemented with the conformal equation and the usual perturbed harmonic map equation for the Gauss map (which can be obtained from Lemma II.1. in [107], for example) , it is thus clear that a conformal area-constrained Willmore immersion from $\mathbb{R}^2$ into $(M, h)$ satisfies the following system of equations:

$$\begin{cases} D^* D\Phi = \star_h D\vec{n} \wedge \nabla^\perp \Phi \\ D^* D\vec{n} = -\star_h \left(D\vec{n} \wedge D^\perp \vec{n}\right) \\ \qquad\qquad - \star_h \left(\vec{n} \wedge Riem_h(\partial_y \Phi, \partial_x \Phi)\vec{n}\right) - 2 \star_h \left(D\vec{H} \wedge \nabla^\perp \Phi\right), \\ \quad D^* \left[D\vec{H} - 3\pi_{\vec{n}}(D\vec{H}) - \star_h \left(\vec{H} \wedge D^\perp \vec{n}\right)\right] = \\ \qquad 2e^{2\lambda} \left[R^\perp_\Phi(T\Phi) + \vec{H} Ric_h(\vec{n}, \vec{n})\right] - \alpha \star_h D\vec{n} \wedge \nabla^\perp \Phi \end{cases} \tag{5.3.4}$$

In geodesic normal coordinates, the above system can be rewritten as

$$
\begin{cases}
\Delta(\Phi)^k = -\left(\Gamma^k_{ij}\right)(\Phi)\left(\nabla(\Phi)^i\cdot\nabla(\Phi)^j\right)+\sqrt{|h|}h^{ik}\left(\nabla\vec{n}\wedge\nabla^\perp\Phi\right)_i \\
\qquad\qquad +\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla\Phi)^n\wedge\nabla^\perp\Phi\right)_i \\
\Delta(\vec{n})^k = -2\left(\Gamma^k_{ij}\right)(\Phi)\left(\nabla(\vec{n})^i\cdot\nabla(\Phi)^j\right) \\
\qquad\qquad -(\Gamma^k_{ij})(\Phi)(\vec{n})^i(\Delta\Phi)^j \\
\qquad\qquad -(\Gamma^k_{ij,\,l})(\Phi)(\vec{n})^i(\nabla(\Phi)^j\cdot\nabla(\Phi)^l) \\
\qquad\qquad -\sqrt{|h|}h^{ik}\left(\nabla\vec{n}\wedge\nabla^\perp\vec{n}\right)_i \\
\qquad\qquad -2\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla\Phi)^n\wedge\nabla^\perp\vec{n}\right)_i \\
\qquad\qquad -\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla\Phi)^n\wedge\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla^\perp\Phi)^n\right)_i \\
\qquad\qquad -2\sqrt{|h|}h^{ik}\left(\nabla\vec{H}\wedge\nabla^\perp\Phi\right)_i \\
\qquad\qquad -2\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{H})^m(\nabla\Phi)^n\wedge\nabla^\perp\Phi\right)_i \\
\qquad\qquad -e^{-2\lambda}\mathcal{R}(\partial_x\Phi,\partial_y\Phi,\vec{n},\partial_y\Phi)\partial_x\Phi \\
\qquad\qquad -e^{-2\lambda}\mathcal{R}(\partial_y\Phi,\partial_x\Phi,\vec{n},\partial_x\Phi)\partial_y\Phi \\
div\left(\nabla(\vec{H})^k-3\left(\nabla\vec{H}\cdot_{h^\varepsilon}\vec{n}\right)(\vec{n})^k\right) = -2\left(\Gamma^k_{ij}\right)(\Phi)\left(\nabla(\vec{H})^i\cdot\nabla(\Phi)^j\right) \\
\qquad\qquad -(\Gamma^k_{ij})(\Phi)(\vec{H})^i(\Delta\Phi)^j \\
\qquad\qquad -(\Gamma^k_{ij,\,l})(\Phi)(\vec{H})^i(\nabla(\Phi)^j\cdot\nabla(\Phi)^l) \\
\qquad\qquad +3div\left(\left(h_{\alpha\beta}(\Phi)\Gamma^\alpha_{ij}(\Phi)(\vec{H})^i(\nabla\Phi)^j(\vec{n})^\beta\right)\vec{n}\right) \\
\qquad\qquad +3\partial_x H\left(\Gamma^k_{ij}(\Phi)(\vec{n})^i\partial_x\Phi^j\right) \\
\qquad\qquad +3\partial_y H\left(\Gamma^k_{ij}(\Phi)(\vec{n})^i\partial_y\Phi^j\right) \\
\qquad\qquad +\sqrt{|h|}h^{ik}\left(\nabla\vec{H}\wedge\nabla^\perp\vec{n}\right)_i \\
\qquad\qquad +\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{H})^m(\nabla\Phi)^n\wedge\nabla^\perp\vec{n}\right)_i \\
\qquad\qquad +\sqrt{|h|}h^{ik}\left(\nabla\vec{H}\wedge\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla^\perp\Phi)^n\right)_i \\
\qquad\qquad +\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{H})^m(\nabla\Phi)^n\wedge\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla^\perp\Phi)^n\right)_i \\
\qquad\qquad +3e^{-2\lambda}\mathcal{R}\left(\partial_x\Phi,\partial_y\Phi,\vec{H},\partial_y\Phi\right)\partial_x\Phi \\
\qquad\qquad +3e^{-2\lambda}\mathcal{R}\left(\partial_y\Phi,\partial_x\Phi,\vec{H},\partial_x\Phi\right)\partial_y\Phi \\
\qquad\qquad +2e^{2\lambda}\vec{H}Ric_h(\vec{n},\vec{n}) \\
\qquad\qquad -\alpha\sqrt{|h|}h^{ik}\left(\nabla\vec{n}\wedge\nabla^\perp\Phi\right)_i \\
\qquad\qquad -\alpha\sqrt{|h|}h^{ik}\left(\vec{\Gamma}_{mn}(\Phi)(\vec{n})^m(\nabla\Phi)^n\wedge\nabla^\perp\Phi\right)_i
\end{cases}, \tag{5.3.5}
$$

where

$$\nabla\vec{H}\cdot_{h^\varepsilon}\vec{n} := h^\varepsilon(\Phi)(\nabla\vec{H},\vec{n}) = h^\varepsilon_{ij}(\Phi)(\nabla\vec{H})^i\,(\vec{n})^j$$

the $(\Gamma^k_{ij})$'s are the Christoffel symbols of the Levi-Civita connection $D$, $\mathcal{R}(X,Y,Z,W)$ is given by

$$\mathcal{R}(X,Y,Z,W) := h(Riem_h(X,Y)Z,W) = R_{ijmk}X^jY^jZ^mW^k,$$

and $\vec{\Gamma}_{mn}X^mY^n$ is a vector in $\mathbb{R}^3$ with coordinates $(\Gamma^k_{mn}X^mY^n)$. We also have

$$\vec{n} = \frac{\star_h \Phi_x \wedge \Phi_y}{|\Phi_x \wedge \Phi_y|_h} \text{ and } \vec{H} = \frac{\Delta_g \Phi}{|\nabla \Phi|^2_h}$$

with

$$(\star_h \Phi_x \wedge \Phi_y)_k = \sqrt{|h|} h^{ik} (\Phi_x \wedge \Phi_y)_i \text{ and } \Delta_g \Phi^k = \Delta \Phi^k + \Gamma^k_{ij}(\Phi)(\nabla \Phi^i \cdot \nabla \Phi^j).$$

Later we will exploit the above system (5.3.5) in order to study the asymptotic behavior of area-constrained Willmore spheres.

In comparison, we also describe the system verified by conformal Willmore immersions into $\mathbb{R}^3$ with area constraint:

$$\begin{cases} \Delta \Phi = \nabla \vec{n} \wedge \nabla^\perp \Phi \\ \Delta \vec{n} = \nabla^\perp \vec{n} \wedge \nabla \vec{n} - 2 \nabla \vec{H} \wedge \nabla^\perp \Phi \\ \mathcal{L}_{\vec{n}}(\nabla \vec{H}) = \alpha \nabla \vec{n} \wedge \nabla^\perp \Phi \end{cases} \tag{5.3.6}$$

where, for every $\vec{N} \in W^{1,2}(\mathbb{R}^2, \mathbb{S}^2)$, $\mathcal{L}_{\vec{N}}$ denotes

$$\mathcal{L}_{\vec{N}}(\nabla \vec{f}) := div \left( \nabla \vec{f} - 3 \pi_{\vec{N}}(\nabla \vec{f}) \right) + \nabla^\perp \vec{N} \wedge \nabla \vec{f}.$$

Notice that $\mathcal{L}_{\vec{N}}$ is a self-adjoint operator in $W^{1,2}(\mathbb{R}^2; \mathbb{R}^3)$.

In order to simplify our notation and to enhance the clarity of the lecture for readers, from now on we will denote by $\varepsilon$ the index of sequences instead of $\varepsilon_k$ and $(\Sigma^\varepsilon)$ instead of $(\Sigma_k)_{k \in \mathbb{N}}$.

Let $(\Phi^\varepsilon)$ be a sequence satisfying the assumptions in Theorem 5.2.3. Take a system of coordinates $(y^1, y^2, y^3)$ around $p$. For every $\varepsilon > 0$, we consider the exponential chart centered at a point $p^\varepsilon \in (M, h)$ such that $\Sigma^\varepsilon \subset B^{h_\varepsilon}_{2\varepsilon}(p^\varepsilon)$. Notice that, up to a subsequence, $p^\varepsilon \to p$ as $\varepsilon \to 0$. Then, by rescaling the exponential chart by a factor $\frac{1}{\varepsilon}$ with respect to 0, we obtain a sequence of immersions, still denoted by $\Phi^\varepsilon$, from $\mathbb{S}^2$ into $(\mathbb{R}^3, h_\varepsilon)$, where $h_\varepsilon$ is the rescaled metric defined by

$$h_\varepsilon(y)(\cdot\,, \cdot) := h(\varepsilon y)(\varepsilon^{-1} \cdot\,, \varepsilon^{-1} \cdot).$$

Thanks to the invariance of the Willmore functional under rescaling, it is clear that $(\Phi^\varepsilon)$ remains a sequence of area-constrained Willmore immersions in $(\mathbb{R}^3, h_\varepsilon)$ such that

$$diam_{h_\varepsilon}(\Phi^\varepsilon(\mathbb{S}^2)) = 1,\ \Phi^\varepsilon(\mathbb{S}^2) \subset B^{h_\varepsilon}_2(0) \text{ and } \sup_{\varepsilon > 0} \mathcal{W}_{h_\varepsilon}(\Phi^\varepsilon) < +\infty. \tag{5.3.7}$$

**Lemma 5.3.2.** *Let $\Phi^\varepsilon : \mathbb{S}^2 \to (\mathbb{R}^3, h_\varepsilon)$ be a sequence of smooth immersions such that $diam_{h_\varepsilon}(\Phi^\varepsilon(\mathbb{S}^2)) = 1$, $\Phi^\varepsilon(\mathbb{S}^2) \subset B^{h_\varepsilon}_2(0)$ and $\sup_{\varepsilon>0} \mathcal{W}_{h_\varepsilon}(\Phi^\varepsilon) < +\infty$. Then, we have*

$$\frac{1}{2} \leq diam_{h_0}(\Phi^\varepsilon(\mathbb{S}^2)) \leq 2,\ \Phi^\varepsilon(\mathbb{S}^2) \subset B^{h_0}_3(0) \text{ and } \sup_{\varepsilon > 0} \mathcal{W}_{h_0}(\Phi^\varepsilon) < +\infty, \tag{5.3.8}$$

*where $h_0$ is the usual Euclidean metric in $\mathbb{R}^3$. Moreover, there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq Area_{h_0}(\Phi^\varepsilon) \leq C. \tag{5.3.9}$$

*Proof.* (5.3.8) is a consequence of the expansion of the metric in normal coordinates, see Lemma 6.5.1 in Appendix 6. The inequality (5.3.9) directly stems from the fact that, by Lemma 1.1 in Simon [129], there exists $C > 0$ such that, for all compact connected surface $\Sigma$ in $\mathbb{R}^3$,

$$\sqrt{\frac{Area_{h_0}(\Sigma)}{\mathcal{W}_{h_0}(\Sigma)}} \leq diam(\Sigma) \leq C \sqrt{Area_{h_0}(\Sigma) \mathcal{W}_{h_0}(\Sigma)}.$$

□

Next, we show that the associated Lagrange multiplier $\alpha^\varepsilon$ can be treated as a small perturbation term. The idea is based on the observation that, as shown in Lamm-Metzger [81] and in Laurain-Mondino [84], area-constrained Willmore surfaces embedded in $\mathbb{R}^3$ are actually "true" Willmore surfaces. Indeed, let $\Sigma$ be some area-constrained Willmore surface embedded in $\mathbb{R}^3$ and let $\alpha$ be the corresponding Lagrange multiplier. By considering the first variations of the Willmore and the area functionals with respect to the position vector field of $\Sigma$, one get that

$$\delta_{\vec{x}}\mathcal{W}(\Sigma) = \alpha\delta_{\vec{x}}Area(\Sigma).$$

Since the Willmore energy is invariant by homothetic transformations, $\delta_{\vec{x}}\mathcal{W}(\Sigma) = 0$, while $\delta_{\vec{x}}Area(\Sigma) = 2Area(\Sigma)$, so that it follows immediately that $\alpha = 0$.

In case of surfaces embedded in curved space, the Lagrange multiplier does not *a priori* vanish, but more the curved space is flat, less the Lagrange multiplier becomes.

**Lemma 5.3.3** (see Lamm-Metzger [81] and Laurain-Mondino [84])**.** *Let $(\Phi^\varepsilon)_{\varepsilon>0}$ be a sequence of smooth area-constrained Willmore immersions of $\mathbb{S}^2$ into $(\mathbb{R}^3, h_\varepsilon)$ and $\Phi^\varepsilon(\mathbb{S}^2) \subset B_{h_0}(0,2)$. Then the Lagrange multipliers $\alpha^\varepsilon$ satisfy:*

$$\alpha^\varepsilon = O(\varepsilon^2). \tag{5.3.10}$$

Next, we show that, under area and energy controls, for any conformal immersion of the standard sphere $\mathbb{S}^2$, we get Lorentz estimates on its associated conformal factor.

**Lemma 5.3.4.** *Let $\Phi^\varepsilon : (\mathbb{S}^2, g_0) \to (M, h_\varepsilon)$ be a conformal immersion such that*

$$\sup_{\varepsilon>0}\left(Area_{h_\varepsilon}(\Phi^\varepsilon(\mathbb{S}^2)) + \mathcal{W}_{h_\varepsilon}(\Phi^\varepsilon)\right) < +\infty,$$

*where $g_0$ is the usual metric on $\mathbb{S}^2$. We denote by $\lambda^\varepsilon$ the associated conformal factor, that is, $e^{\lambda^\varepsilon}g_0 = g^\varepsilon := (\Phi^\varepsilon)^*(h_\varepsilon)$, where $g_\varepsilon = (\Phi^\varepsilon)^*(h_\varepsilon)$. Then, there holds*

$$\sup_{\varepsilon>0}\|\nabla\lambda^\varepsilon\|_{L^{2,+\infty}_{g_0}(\mathbb{S}^2)} < +\infty, \tag{5.3.11}$$

*where $\|f\|_{L^{2,+\infty}_{g_0}(\mathbb{S}^2)} := \sup_{t>0} t^2\mu_{g_0}(|f| \geq t)$ with $\mu_{g_0}$ the induced measure defined on $(\mathbb{S}^2, g_0)$, for all measurable functions $f$ on $(\mathbb{S}^2, g_0)$.*

*Proof.* Since $K_{g_0} \equiv 1$, the well-known Liouville's equation gives the following identity:

$$\Delta_{g_0}\lambda^\varepsilon = -e^{2\lambda^\varepsilon}K_{g_\varepsilon} + 1 \text{ on } \mathbb{S}^2 \tag{5.3.12}$$

where $\Delta_{g_0} = div(\nabla_{g_0}\cdot)$. Since

$$\frac{1}{2}|A^\varepsilon|^2 - 1 = 2H^2 - K_{g_\varepsilon}$$

where $A^\varepsilon$ is the second fundamental form of $\Phi^\varepsilon$, we get that

$$|K_{g_\varepsilon}| \leq 4H^2 + 2 - K_{g_\varepsilon}.$$

Integrating the Liouville's equation (5.3.12) over $(\mathbb{S}^2, g_0)$ gives that

$$\begin{aligned}\int_{\mathbb{S}^2}|\Delta_{g_0}\lambda^\varepsilon|d\sigma_{g_0} &\leq \int_{\mathbb{S}^2}e^{2\lambda^\varepsilon}|K_{g_\varepsilon}|d\sigma_{g_0} + \int_{\mathbb{S}^2}1d\sigma_{g_0}\\ &\leq 4\int_{\mathbb{S}^2}H^2d\sigma_{g_\varepsilon} + 2\int_{\mathbb{S}^2}1d\sigma_{g_\varepsilon} - \int_{\mathbb{S}^2}K_{g_\varepsilon}d\sigma_{g_\varepsilon} + 4\pi\\ &\leq 4\mathcal{W}_{h_\varepsilon}(\Phi^\varepsilon(\mathbb{S}^2)) + 2Area_{h_\varepsilon}(\Phi^\varepsilon(\mathbb{S}^2)),\end{aligned} \tag{5.3.13}$$

thanks to Gauss-Bonnet theorem. Hence we obtain with Lemma 5.3.2 that

$$\sup_{\varepsilon>0} \|\Delta_{g_0}\lambda^\varepsilon\|_{L^1(\mathbb{S}^2)} < +\infty.$$

Finally, thanks to standard elliptic estimates (see Theorem 3.3.6. in Hélein [69], for example), we deduce (5.3.11). □

**Lemma 5.3.5.** *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any conformal immersion $\Phi : \mathbb{D} \to \mathbb{R}^3$, if*

$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx \leq \varepsilon_0,$$

*then we have*

$$\|\lambda - \overline{\lambda}\|_{L^\infty(\mathbb{D}_{1/2})} \leq C \left( \int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx + \|\nabla \lambda\|_{L^{2,+\infty}(\mathbb{D})} \right) \tag{5.3.14}$$

*where $\lambda$ is the conformal factor of $\Phi$ defined as $e^\lambda = |\partial_x \Phi| = |\partial_y \Phi|$ and*

$$\overline{\lambda} := \frac{1}{\pi} \int_{\mathbb{D}} \lambda \, dx.$$

*Proof.* Assume $\|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D})} < +\infty$ and $\varepsilon_0 < \frac{8\pi}{3}$. Thanks to Hélein's moving frames theorem (see, e.g, Hélein [69]), there exists $(\vec{e}_1, \vec{e}_2) \in \left(W^{1,2}(\mathbb{D}, \mathbb{R}^3) \cap C^\infty(\mathbb{D}, \mathbb{R}^3)\right)^2$ such that

$$\begin{cases} \|\vec{e}_1\|^2 = \|\vec{e}_2\|^2 = 1, \ \vec{e}_1 \cdot \vec{e}_2 = 0 \text{ on } \mathbb{D}, \\ \Delta\lambda = \nabla^\perp \vec{e}_1 \cdot \nabla \vec{e}_2 \text{ on } \mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} |\nabla \vec{e}_i|^2 \, dx \leq \int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx, \quad \forall i = 1, 2. \end{cases} \tag{5.3.15}$$

Write $\lambda = \mu + \nu$, where $\mu$ and $\nu$ are solutions of

$$\begin{cases} \Delta\mu = \nabla^\perp \vec{e}_1 \cdot \nabla \vec{e}_2 \text{ on } \mathbb{D} \\ \mu = 0 \text{ on } \partial\mathbb{D} \end{cases} \text{ and } \begin{cases} \Delta\nu = 0 \text{ on } \mathbb{D} \\ \nu = \lambda \text{ on } \partial\mathbb{D} \end{cases}. \tag{5.3.16}$$

Thanks to Wente's inequality 6.3.4, we have

$$\|\mu\|_{L^\infty(\mathbb{D})} + \|\nabla\mu\|_{L^2(\mathbb{D})} \leq C \|\nabla \vec{e}_1\|_{L^2(\mathbb{D})} \|\nabla \vec{e}_2\|_{L^2(\mathbb{D})} \leq C \int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx \tag{5.3.17}$$

for some $C > 0$. Thus, we obtain that

$$\begin{aligned} \|\nabla\nu\|_{L^{2,+\infty}(\mathbb{D})} &\leq \|\nabla\mu\|_{L^{2,+\infty}(\mathbb{D})} + \|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D})} \\ &\leq C \left( \int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx + \|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D})} \right). \end{aligned} \tag{5.3.18}$$

for some $C > 0$.

Next, thanks to Poincaré's inequality, we have that

$$\|\nu - \overline{\nu}\|_{L^1(\mathbb{D})} \leq C \|\nabla\nu\|_{L^1(\mathbb{D})} \leq C \|\nabla\nu\|_{L^{2,+\infty}(\mathbb{D})} \tag{5.3.19}$$

for some $C > 0$, where $\overline{\nu} := \frac{1}{\pi} \int_{\mathbb{D}} \nu \, dx$.

Since $W^{1,1}(\mathbb{D}) \hookrightarrow L^1(\partial\mathbb{D})$, we deduce that

$$\|\nu - \overline{\nu}\|_{L^1(\partial\mathbb{D})} \leq C \|\nabla\nu\|_{L^{2,+\infty}(\mathbb{D})} \tag{5.3.20}$$

for some $C > 0$.

Moreover, since $\nu$ is harmonic, we get that

$$\|\nu - \overline{\nu}\|_{L^\infty\left(\mathbb{D}_{1/2}\right)} \leq C\|\nu - \overline{\nu}\|_{L^1(\partial\mathbb{D})} \leq C\|\nabla\nu\|_{L^{2,+\infty}(\mathbb{D})}. \tag{5.3.21}$$

Combining (5.3.17) and (5.3.18) with (5.3.21), we get that

$$\begin{aligned}\|\lambda - \overline{\lambda}\|_{L^\infty\left(\mathbb{D}_{1/2}\right)} &\leq \|\mu - \overline{\mu}\|_{L^\infty\left(\mathbb{D}_{1/2}\right)} + \|\nu - \overline{\nu}\|_{L^\infty\left(\mathbb{D}_{1/2}\right)} \\ &\leq \|\nabla\mu\|_{L^{2,+\infty}(\mathbb{D})} + \|\nabla\nu\|_{L^{2,+\infty}(\mathbb{D})} \\ &\leq C\left(\int_{\mathbb{D}} |\nabla\vec{n}|^2\, dx + \|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D})}\right)\end{aligned} \tag{5.3.22}$$

for some $C > 0$ with $\overline{\mu} = \frac{1}{\pi}\int_{\mathbb{D}} \mu\, dx$. □

**Corollary 5.3.6.** *Let $\Phi^\varepsilon : (\mathbb{S}, g_0) \to (M, h)$ be a sequence of conformal immersions of $\mathbb{S}$ into $(M, h)$. For any connected compact set $K \subset \mathbb{S}^2\backslash\{S\}$ containing the north pole $N$, we have that*

$$\sup_{\varepsilon>0} \|\lambda^\varepsilon - \lambda^\varepsilon(N)\|_{L^\infty(K)} \leq C_K,$$

*where $C_K$ only depends on $\sup_{\varepsilon>0} \|\nabla_{g_0}\vec{n}^\varepsilon\|_{L^\infty(K)}$.*

*Proof.* For any $x \in K$ and $\rho > 0$ such that $B^{g_0}_\rho(x) \subset K$, we have that

$$\sup_{\varepsilon>0} \int_{B^{g_0}_\rho(x)} |\nabla_{g_0}\vec{n}^\varepsilon|^2\, d\sigma_{g_0} \leq C'_K 2\pi(1 - \cos\frac{\rho}{2}) \leq C_K\rho^2$$

where $C'_K = \sup_{\varepsilon>0}\sup_K |\nabla_{g_0}\vec{n}^\varepsilon|^2$.

We choose $\rho > 0$ small enough such that $C_K\rho^2 \leq \varepsilon_0$ where $\varepsilon_0$ is defined as in Lemma 5.3.5. Therefore, thanks to Lemma 5.3.5, for $\varepsilon > 0$, there exists $c^\varepsilon_x \in \mathbb{R}$ such that

$$\sup_{\varepsilon>0} \sup_{y\in B^{g_0}_{\rho/2}(x)} |\lambda^\varepsilon(y) - c^\varepsilon_x| \leq C_\rho \tag{5.3.23}$$

for some $C_\rho > 0$ only depending on $\rho > 0$.

Finally, thanks to the connectedness and the compactness of $K$ and the Vitali's covering theorem, one can cover $K$ by a finite number of balls of radius $\frac{\rho}{4}$, such that every two distinct points of $K$ can be joined by a chain of balls of the cover, which gives the desired result. □

### $\varepsilon-$regularity and Strong convergence

**Theorem 5.3.7** ($\varepsilon-$regularity theorem for conformal Willmore immersions, Rivière [121])**.** *There exists $\varepsilon_0 > 0$ such that, for every conformal Willmore immersion $\Phi : \mathbb{D} \to \mathbb{R}^3$, if we have*

$$\int_{\mathbb{D}} |\nabla\vec{n}|^2\, dx \leq \varepsilon_0,$$

*then, for every $l \in \mathbb{N}^*$, there exists $C_l\left(\|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D}_{1/2})}\right) > 0$ such that*

$$\|e^{-\lambda}\nabla^l\Phi\|_{L^\infty(\mathbb{D}_{1/2})} \leq C_l\left(\|\nabla\lambda\|_{L^{2,+\infty}(\mathbb{D}_{1/2})}\right)\left[\int_{\mathbb{D}} |\nabla\vec{n}|^2\, dx + 1\right]^{\frac{1}{2}}$$

**Corollary 5.3.8.** *Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ be a sequence of conformal Willmore immersion such that*

$$\sup_{\varepsilon>0} \int_{\mathbb{D}} |\nabla \vec{n}^\varepsilon|^2 \, dx \leq \varepsilon_0$$

*and* $\limsup_{\varepsilon>0} \|\nabla \lambda^\varepsilon\|_{2,+\infty} < +\infty$. *Then, up to a subsequence of $(\Phi^\varepsilon)_{\varepsilon>0}$, we have, for every $l \in \mathbb{N}$,*

$$\frac{\Phi^\varepsilon - c^\varepsilon}{r^\varepsilon} \to \Phi^\infty \text{ in } C^l_{loc}(\mathbb{D})$$

*where $r^\varepsilon > 0$, $c^\varepsilon \in \mathbb{R}^3$ and $\Phi^\infty$ is a conformal Willmore immersion on $\mathbb{D}$.*

*Proof.* Set $\tilde{\Phi}^\varepsilon = e^{-\lambda^\varepsilon(0)} \left(\Phi^\varepsilon - \Phi^\varepsilon(0)\right)$. Thanks to Corollary 5.3.6 and the defintion of $\tilde{\Phi}^\varepsilon$, we have that

$$\begin{aligned} \tilde{\Phi}^\varepsilon(0) &= 0, \quad \forall \varepsilon > 0, \\ \sup_{\varepsilon>0} \|\tilde{\lambda}^\varepsilon\|_{L^\varepsilon(\overline{\mathbb{D}}_\rho)} &= \sup_{\varepsilon>0} \|\lambda^\varepsilon - \lambda^\varepsilon(0)\|_{L^\infty(\overline{\mathbb{D}}_\rho)} \leq C_\rho \text{ and} \\ \sup_{\varepsilon>0} \int_{\mathbb{D}} |\nabla \tilde{n}^\varepsilon|^2 \, dx &= \sup_{\varepsilon>0} \int_{\mathbb{D}} |\nabla \vec{n}^\varepsilon|^2 \, dx \leq \varepsilon_0 \end{aligned} \tag{5.3.24}$$

for every $0 < \rho < 1$ and for some $C_\rho > 0$, where $\tilde{n}^\varepsilon$ is the normal vector of $\tilde{\Phi}^\varepsilon$. Thanks to the $\varepsilon-$regularty theorem 5.3.7, together with (5.3.24), we get that, for every $l \in \mathbb{N}$,

$$\sup_{\varepsilon>0} \|\tilde{\Phi}^\varepsilon\|_{L^\infty(\overline{\mathbb{D}}_\rho)} + \|\nabla^l \tilde{\Phi}^\varepsilon\|_{L^\infty(\overline{\mathbb{D}}_\rho)} \leq C_{l,\rho} \tag{5.3.25}$$

for some $C_{l,\rho} > 0$. Therefore, thanks to Arzelà-Ascoli theorem, there exists a conformal immersion $\Phi^\infty$ such that, up to a subsequence of $(\Phi^\varepsilon)_{\varepsilon>0}$,

$$\tilde{\Phi}^\varepsilon = \frac{\Phi^\varepsilon - c^\varepsilon}{r^\varepsilon} \to \Phi^\infty \text{ in } C^l_{loc}(\mathbb{D}) \tag{5.3.26}$$

with $c^\varepsilon = \Phi^\varepsilon(0)$ and $r^\varepsilon = e^{-\lambda^\varepsilon(0)} > 0$.

Finally, $\Phi^\infty$ is a Willmore immersion since, by strong convergence, $\Phi^\infty$ also verifies the Euler-Lagrange equation (5.3.1). □

## 5.4 Willmore spheres in $\mathbb{R}^3$

As in many works on compactness problem and bubbling phenomena, we need to know global solutions of the Willmore equation in the entire domain. In [33], Bryant showed that the only embedded Willmore spheres in $\mathbb{R}^3$ are naturally Euclidean spheres:

**Lemma 5.4.1** (Bryant, [33]). *If $\Phi : \mathbb{S}^2 \to \mathbb{R}^3$ is a Willmore immersion without transversal self-intersection, then $\Phi$ is embedded and $\Phi(\mathbb{S})$ is a round sphere.*

*Remark* 5.4.2. Moreover, up to compose $\Phi$ with $\pi_N$, the stereographic projection with respect to the North pole $N$, if $\Phi : \mathbb{R}^2 \to \mathbb{C}$ is conformal then there exist $r > 0$, $C \in \mathbb{R}^3$ and $P, Q \in \mathbb{C}[X]$ irreducible polynomials with $\max\{deg(P), deg(Q)\} = 1$ such that

$$\Phi = r\omega \text{ with } \omega(z) = \pi_N^{-1}\left(\frac{P(z)}{Q(z)}\right) + C. \tag{5.4.1}$$

In particular, setting $\omega^\varepsilon = \omega\left(\frac{\cdot - a^\varepsilon}{\mu^\varepsilon}\right)$ for some $\mu^\varepsilon > 0$ and $a^\varepsilon \in \mathbb{R}^2$, we get that

$$|\nabla \omega^\varepsilon(x)| = O\left(\frac{\mu^\varepsilon}{|x - a^\varepsilon|^2 + (\mu^\varepsilon)^2}\right). \tag{5.4.2}$$

For the purpose of convenience, we set $\hat{\omega} := \omega - p$ where $p$ is the barycenter of $\omega$. It is clear that $|\nabla \omega| = |\nabla \hat{\omega}|$, $\vec{n}_{r\omega} = -\hat{\omega}$ and $\vec{H}_{r\omega} = -\frac{1}{r}\hat{\omega}$. Notice that $(\hat{\omega}_x, \hat{\omega}_y, \hat{\omega})$ forms an orthogonal basis of $\mathbb{R}^3$.

## 5.5 Energy tracking and bubble extraction

In this section, we will describe the bubbling phenomenon for $\Phi^\varepsilon$ and show that one can write asymptotically $\Phi^\varepsilon$ as the sum of round spheres. Precisely, we have, up to a subsequence of $(\Phi^\varepsilon)$,

$$\Phi^\varepsilon = r_0^\varepsilon \mathcal{B}_0^\varepsilon + r_1^\varepsilon \mathcal{B}_1^\varepsilon + \varphi^\varepsilon$$

where $r_i \mathcal{B}_i^\varepsilon$ are so called "bubbles" which corresponds to Willmore spheres in $\mathbb{R}^3$ and $\varphi^\varepsilon$ is the remainder converging to zero in some sense which we will precise later.

In the more general situation of arbitrary energy, it is believed that $\Phi^\varepsilon$ can be written as a sum of Willmore surfaces in $\mathbb{R}^3$, these surfaces being round spheres and catenoids. Below, we show how to extract successively these surfaces in the decomposition. However, we need to stop at the second sphere since the third "bubble" could be a catenoid and we were not able to classify the surfaces that could appear in this process (even if we conjecture that only spheres and catenoids should appear).

### 5.5.1 Bubble extraction and weak estimates

In [21], Bernard and Rivière have proved the quantization of energy while the conformal factor stay in a compact of moduli space. Here we establish a similar result for our embedded spheres with pointwise estimates. To do so, our technique is based on a systematic method developed by Druet, Hebey and Robert in [57].

The following construction of bubbles can originally be found in Druet [56] for Yamabe-type equations and Laurain [83] for CMC surfaces. However, some new ingredients are needed here since we need to control three quantities, $\nabla\Phi^\varepsilon$, $\nabla\vec{n}^\varepsilon$ and $\nabla\vec{H}^\varepsilon$.

The claims in what follows are up to a subsequence of $(\Phi^\varepsilon)$.

**First bubble extraction** Up to compose $\Phi^\varepsilon$ with a rotation of $\mathbb{S}^2$, we can assume that, for every $\varepsilon > 0$, the North pole $N \in \mathbb{S}^2$ is the maximum point of the quantity $|\nabla_{g_0}\Phi^\varepsilon| + |\nabla_{g_0}\vec{n}^\varepsilon|$:

$$\frac{1}{\mu^\varepsilon} := \left(|\nabla_{g_0}\Phi^\varepsilon| + |\nabla_{g_0}\vec{n}^\varepsilon|\right)(N) = \max_{\mathbb{S}^2}\left(|\nabla_{g_0}\Phi^\varepsilon| + |\nabla_{g_0}\vec{n}^\varepsilon|\right).$$

Replacing $\Phi^\varepsilon$ by $\Phi^\varepsilon(x) := \Phi^\varepsilon\left(\pi_N^{-1}\left(\mu^\varepsilon \pi_N(x)\right)\right)$ for all $x \in \mathbb{S}^2$, where $\pi_N : \mathbb{S}^2 \to \mathbb{R}^2$ is the stereographic projection with respect to the North pole $N$, one can verify that the quantity $|\nabla_{g_0}\Phi^\varepsilon| + |\nabla_{g_0}\vec{n}^\varepsilon|$ is bounded on every compact subset of $\mathbb{S}^2\backslash\{S\}$.

- The case where $\lambda^\varepsilon(N) \to +\infty$ is excluded, otherwise we would have

  $$\text{Area}(\Phi^\varepsilon(K)) \to +\infty$$

  for any compact set $K \subset \mathbb{S}^2\backslash\{S\}$.

- If $\sup_{\varepsilon>0}|\lambda^\varepsilon(N)| < +\infty$, then for any compact set $K \subset \mathbb{S}^2\backslash\{S\}$ containing the North pole, $x \in K$ and for every $\rho > 0$, we have that

  $$\int_{B_\rho^{g_0}(x)\cap K} |\nabla_{g_0}\vec{n}^\varepsilon|^2 \, d\sigma_{g_0} \leq C_K \rho^2.$$

  By taking $\rho > 0$ such that $C_K\rho^2 \leq \varepsilon_0$, we get that, up to a subsequence of $(\Phi^\varepsilon)$,

  $$\frac{\Phi^\varepsilon - c^\varepsilon}{r^\varepsilon} \to \Phi^0 \text{ in } C^l_{loc}(\mathbb{S}^2\backslash S),$$

  where $\Phi^0$ is a conformal Willmore immersion on $\mathbb{S}^2\backslash\{S\}$ and $\limsup r^\varepsilon < +\infty$. Moreover, since $\Phi^0$ has finite energy and finite area, $\Phi^0$ can be extended to a branched conformal Willmore immersion on $\mathbb{S}^2$. Indeed, up to compose $\Phi_0, \vec{n}_0, \vec{H}_0$ with $\pi_N^{-1}$, it is clear that $\Phi_0, \vec{n}_0, \vec{H}_0 \in$

$C^\infty(\mathbb{R}^2\backslash\{0\})$ satisfy the Willmore equation (5.3.6) with $\alpha = 0$ on $\mathbb{R}^2\backslash\{0\}$. Moreover for all $r > 0$, we have

$$\int_{B_r(0)} \mathcal{L}_{\vec{n}_{\Phi^\varepsilon\circ\pi_N^{-1}}}\left(\nabla\vec{H}_{\Phi^\varepsilon\circ\pi_N^{-1}}\right)dx = \alpha^\varepsilon\int_{B_r(0)}\Delta(\Phi^\varepsilon\circ\pi_N^{-1})dx + o(1). \tag{5.5.1}$$

By divergence theorem, we get

$$\begin{aligned}&\int_{\partial B_r(0)}\left(\partial_\nu\vec{H}_{\Phi^\varepsilon\circ\pi_N^{-1}} - 3\pi_{\vec{n}_{\Phi^\varepsilon\circ\pi_N^{-1}}}(\partial_\nu\vec{H}_{\Phi^\varepsilon\circ\pi_N^{-1}}) + \vec{H}_{\Phi^\varepsilon\circ\pi_N^{-1}}\wedge\partial_\tau\vec{n}_{\Phi^\varepsilon\circ\pi_N^{-1}}\right)rd\theta\\ &= \alpha^\varepsilon\int_{\partial B_r(0)}\partial_\nu(\Phi^\varepsilon\circ\pi_N^{-1})rd\theta + o(1),\end{aligned} \tag{5.5.2}$$

where $\nu$ is the unit normal vector and $\tau$ the unit tangent vector to the boundary of $B_r(0)$. The strong convergence of $\Phi^\varepsilon\circ\pi_N^{-1}$ to $\Phi_0$ and the fact that $\alpha^\varepsilon = o(1)$ in lemma 5.3.3 allow us to pass to the limit in the last identity. As $\varepsilon\to 0$, we obtain that

$$\int_{\partial B_r(0)}\left(\partial_\nu\vec{H}_{\Phi_0} - 3\pi_{\vec{n}_{\Phi_0}}(\partial_\nu\vec{H}_{\Phi_0}) + \vec{H}_{\Phi_0}\wedge\partial_\tau\vec{n}_{\Phi_0}\right)rd\theta = 0. \tag{5.5.3}$$

Thanks to the above identity for all $r > 0$, we can thus apply the singularity removability theorem established in Theorem I.2. of [20] which shows that $\Phi_0$ extends to a branched smooth Willmore immersion in $\mathbb{R}^2$. Moreover, since $\Phi_0$ is obtained as the limit of smooth embeddings, $\Phi_0$ has in fact no branched points in $\mathbb{R}^2$, see Lemma A.1. in Laurain [83] or Li [90]. Hence, it turns out that we have

$$\vec{n}_0 = \vec{n}_{\Phi_0} := \frac{(\Phi_0)_x\wedge(\Phi_0)_y}{|(\Phi_0)_x\wedge(\Phi_0)_y|} \text{ and } \vec{H}_0 = \vec{H}_{\Phi_0} := \frac{\Delta\Phi_0}{|(\Phi_0)_x\wedge(\Phi_0)_y|}.$$

Similarly, we can also show that $\Phi_0$ has no branched points at infinity by considering $\Phi_0\circ\mathcal{I}$ where $\mathcal{I} : x\mapsto\frac{x}{|x|^2}$ is the inversion centered at $0$, and thus $\Phi_0$ can be extended to a Willmore immersion in $\mathbb{S}^2$. Finally, by the classification result of Bryant [33], $\Phi_0$ parametrizes a round sphere and we can write $\Phi_0 = r_0\omega_0$ for some $r_0 > 0$ and a simple embedded unit sphere $\omega_0$.

- If $\lambda^\varepsilon(N)\to-\infty$, then set $\hat{\Phi}^\varepsilon = e^{-\lambda^\varepsilon(N)}(\Phi^\varepsilon - \Phi^\varepsilon(0))$. Thanks to Corollary 5.3.6, we have that

$$\sup_{\varepsilon>0}\|\hat{\lambda}^\varepsilon\|_{L^\infty(K)} \le C_K$$

for every connected compact set $K$ containing the North pole $N$ and for some $C_K > 0$. Therefore, similarly to the previous case, up to a subsequence, we have that

$$\hat{\Phi}^\varepsilon\to\hat{\Phi}^\infty \text{ in } C^l_{loc}(\mathbb{S}^2\backslash\{S\}),\quad \forall l\in\mathbb{N}$$

where $\hat{\Phi}^\infty$ is a conformal Wilmore immersion in $\mathbb{S}^2\backslash\{S\}$.

1. If $\hat{\Phi}^\infty$ has finite area, similarly to the previous case, we have that $\hat{\Phi}^\infty$ can be extended to a embedded Willmore sphere and thus is a round sphere.
2. If $\hat{\Phi}^\infty$ has infinite area, noting that $\hat{\Phi}^\infty$ has integer density 1 at the South pole $S$ (obtained as a limit of embeddings), by Li [90], $\hat{\Phi}^\infty$ is a complete conformal Willmore immersion with finite energy and thus $\hat{\Phi}^\infty$ is a plane, which contradicts the fact that

$$1 = |\nabla_{g_0}\Phi^\varepsilon|(N) + |\nabla_{g_0}\vec{n}^\varepsilon|(N) = |\nabla_{g_0}\Phi^\varepsilon|(N) + |\nabla_{g_0}\hat{\vec{n}}^\varepsilon|(N)\to 0 \text{ as } \varepsilon\to 0.$$

Finally, let $a_0\in\mathbb{R}^2$ be a maximum point of $|\nabla\omega_0|$, by replacing $\omega_0$ by $\omega_0(\cdot + a_0)$, we can assume that $|\nabla\omega_0|$ attains its maximum at $0$.

**Reparametrization of Willmore spheres** Up to compose $\Phi^\varepsilon$ by $\Phi^\varepsilon \circ \pi_N^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$, we have that

1. $\Phi^\varepsilon$ satisfies the Willmore system (5.3.6) on the entire plane $\mathbb{R}^2$ such that $\Phi^\varepsilon$ is conformal and
$$\sup_{\varepsilon>0} \|\nabla\lambda^\varepsilon\|_{2,+\infty} < +\infty.$$
2. for all $l \in \mathbb{N}$,
$$\frac{\Phi_0^\varepsilon - c_0^\varepsilon}{r_0^\varepsilon} \to \omega_0 \text{ in } C^l_{loc}(\mathbb{R}^2\backslash\{0\}) \text{ as } \varepsilon \to 0. \tag{5.5.4}$$
where $\Phi_0^\varepsilon = \Phi^\varepsilon(\cdot + a_0)$, $c_0^\varepsilon \in \mathbb{R}^3$ and $\limsup_{\varepsilon>0} r_0^\varepsilon < +\infty$,
3. $\omega_0$ is a simple embedded unit sphere and $|\nabla\omega_0|$ attains its maximum at $0$.
4. We also have, for $R > 0$
$$\lim_{\varepsilon\to 0} \sup_{|x|\geq R} (1+|x|^2)\left(\left|\nabla\left(\frac{\Phi^\varepsilon}{r_0^\varepsilon} - \omega_0\right)\right| + |\nabla(\vec{n}^\varepsilon - \vec{n}_0)|\right) = 0,$$
$$\sup_{\varepsilon>0} \|\Phi^\varepsilon\|_\infty < +\infty, \sup_{\varepsilon>0} \|\nabla\Phi^\varepsilon\|_2 < +\infty \text{ and } \sup_{\varepsilon>0} \frac{1}{2}\|\nabla\vec{n}^\varepsilon\|_2^2 < 12\pi - \delta.$$

## Construction of bubble tree

After choosing proper reparametrizations, we are now at the point to extract the second bubble. We prove the following theorem:

**Theorem 5.5.1.** *We have either*
$$\frac{\Phi_0^\varepsilon - c_0^\varepsilon}{r_0^\varepsilon} \to \omega_0, \text{ in } C^l_{loc}(\mathbb{R}^2), \text{ for every } l \in \mathbb{N} \tag{5.5.5}$$
*or there exist*

1. *$\omega_1$ a embedded unit sphere for which $|\nabla\omega_1|$ attains its maximum at $0$,*
2. *$a_1^\varepsilon$ a sequences of $\mathbb{R}^2$ and $r_1^\varepsilon, \mu_1^\varepsilon$ sequences of positive real numbers such that*
$$\lim_{\varepsilon\to 0} a_1^\varepsilon = 0, \limsup_{\varepsilon>0} r_1^\varepsilon < +\infty \text{ and } \lim_{\varepsilon\to 0} \mu_1^\varepsilon = 0,$$

*such that, up to a subsequence of $\Phi^\varepsilon$, we have the following assertions:*
$$\frac{\Phi_1^\varepsilon - c_1^\varepsilon}{r_1^\varepsilon} \to \omega_1 \text{ in } C^l_{loc}(\mathbb{R}^2) \text{ as } \varepsilon \to 0 \tag{5.5.6}$$
*for all $l \in \mathbb{N}$, where $\Phi_1^\varepsilon = \Phi^\varepsilon(\mu_1^\varepsilon \cdot + a_1^\varepsilon)$ and*
$$\sup_{x\in\mathbb{R}^2} (1+|x|^2) d_1^\varepsilon(x) \left(|\nabla\varphi^\varepsilon(x)| + |\nabla\nu^\varepsilon(x)|\right) = o(1) \text{ as } \varepsilon \to 0 \tag{5.5.7}$$
*with*
$$d_1^\varepsilon(x) := \frac{|x - a_1^\varepsilon| + \mu_1^\varepsilon}{\sqrt{1+|x|^2}},$$
$$\varphi^\varepsilon := \Phi^\varepsilon - r_0^\varepsilon\omega_0^\varepsilon - r_1^\varepsilon\omega_1^\varepsilon \text{ and}$$
$$\nu^\varepsilon := \vec{n}^\varepsilon - \hat{\vec{n}}_0^\varepsilon - \hat{\vec{n}}_1^\varepsilon$$
*where $\hat{\vec{n}}_0^\varepsilon$ and $\hat{\vec{n}}_1^\varepsilon$ are respectively the normal vector of $\omega_0^\varepsilon$ and $\omega_1^\varepsilon$ with $(a_0^\varepsilon, \mu_0^\varepsilon) = (a_0, 1)$ and $\omega_i^\varepsilon = \omega_i(\frac{\cdot - a_i^\varepsilon}{\mu_i^\varepsilon})$. Notice that we have $0 \leq d_1^\varepsilon(x) \leq 1$.*

*Proof of Theorem 5.5.1.* If we have

$$\sup_{x\in\mathbb{R}^2}\left(|x-a_0^\varepsilon|+\mu_0^\varepsilon\right)\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(x)\to 0$$

as $\varepsilon\to 0$, then we get automatically (5.5.5) by $\varepsilon$-regularity theorem.

Now assume that

$$\liminf_{\varepsilon>0}\sup_{x\in\mathbb{R}^2}\left(|x-a_0^\varepsilon|+\mu_0^\varepsilon\right)\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(x)>0. \tag{5.5.8}$$

Let $a_1^\varepsilon\in\mathbb{R}^2$ be such that

$$\left(|a_1^\varepsilon-a_0^\varepsilon|+\mu_0^\varepsilon\right)\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(a_1^\varepsilon)=\sup_{x\in\mathbb{R}^2}\left(|x-a_0^\varepsilon|+\mu_0^\varepsilon\right)\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(x). \tag{5.5.9}$$

The supremum is achieved in $\mathbb{R}^2$ since, thanks to the choice of parametrization, we have that

$$\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(x)=O\left(\frac{1}{1+|x|^2}\right)\text{ as }x\to+\infty.$$

Moreover, we get that

$$a_1^\varepsilon\to 0\text{ as }\varepsilon\to 0,$$

since, by construction,

$$\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(x)\to 0$$

as $\varepsilon\to 0$ on $\mathbb{R}^2\backslash\{0\}$.

Setting

$$\frac{1}{\mu_1^\varepsilon}:=\left(|a_1^\varepsilon-a_0^\varepsilon|+\mu_0^\varepsilon\right)\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(a_1^\varepsilon)\,,$$

we have that

$$\mu_1^\varepsilon\to 0\text{ as }\varepsilon\to 0,$$

otherwise the quantity

$$(\Phi^\varepsilon-c_0^\varepsilon)-r_0^\varepsilon\omega_0^\varepsilon$$

would be uniformly bounded in $C^1(\mathbb{R}^2)$ and thus would converge uniformly to $0$ on $\mathbb{R}^2$ which would contradict (5.5.8).

Next, we set

$$\begin{aligned}\Phi_1^\varepsilon&=\Phi^\varepsilon(\mu_1^\varepsilon\cdot+a_1^\varepsilon)\,,\\ \vec{n}_1^\varepsilon&=\vec{n}^\varepsilon(\mu_1^\varepsilon\cdot+a_1^\varepsilon)\text{ and}\\ \vec{H}_1^\varepsilon&=\vec{H}^\varepsilon(\mu_1^\varepsilon\cdot+a_1^\varepsilon)\,.\end{aligned}$$

Then we get that

$$\lambda_1^\varepsilon=\log|\partial_x\Phi_1^\varepsilon|=\log\mu_1^\varepsilon+\lambda^\varepsilon\text{ and} \tag{5.5.10}$$

$$\sup_{\varepsilon>0}\|\nabla\lambda_1^\varepsilon\|_{2,+\infty}=\sup_{\varepsilon>0}\|\nabla\lambda^\varepsilon\|_{2,+\infty}<+\infty. \tag{5.5.11}$$

Moreover, for all $x\in\mathbb{R}^2$, we have that

$$\begin{aligned}&\left(|\nabla\Phi_1^\varepsilon|+|\nabla\vec{n}_1^\varepsilon|\right)(x)\\ &=\mu_1^\varepsilon\left(|\nabla\Phi^\varepsilon|+|\nabla\vec{n}^\varepsilon|\right)(\mu_1^\varepsilon x+a_1^\varepsilon)\\ &\leq\mu_1^\varepsilon\left(|\nabla\Phi^\varepsilon-r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\vec{n}^\varepsilon-\nabla\hat{\vec{n}}_0^\varepsilon|\right)(\mu_1^\varepsilon x+a_1^\varepsilon)+\mu_1^\varepsilon\left(|r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\hat{\vec{n}}_0^\varepsilon|\right)(\mu_1^\varepsilon x+a_1^\varepsilon)\\ &\leq 1+o(1)+\mu_1^\varepsilon\left(|r_0^\varepsilon\nabla\omega_0^\varepsilon|+|\nabla\hat{\vec{n}}_0^\varepsilon|\right)(\mu_1^\varepsilon x+a_1^\varepsilon)\end{aligned} \tag{5.5.12}$$

since

$$\mu_1^\varepsilon \left( |r_0^\varepsilon \nabla \omega_0^\varepsilon| + |\nabla \hat{\vec{n}}_0^\varepsilon| \right) (\mu_1^\varepsilon x + a_1^\varepsilon) = O(\mu_1^\varepsilon) = o(1). \tag{5.5.13}$$

Then, using the definition of $a_1^\varepsilon$, (5.5.12) and (5.5.13), we obtain that, for all $x \in \mathbb{R}^2$,

$$\left( |\nabla \Phi_1^\varepsilon| + |\nabla \vec{n}_1^\varepsilon| \right)(x) \leq 1 + o(1). \tag{5.5.14}$$

Hence $|\nabla \Phi_1^\varepsilon| + |\nabla \vec{n}_1^\varepsilon|$ is bounded on every compact subset of $\mathbb{R}^2$. Moreover, $\Phi_1^\varepsilon$ still verifies the Willmore system (5.3.5), since the system is conformally invariant. Thanks to (5.5.11), (5.5.14) and Corollary 5.3.8, up to a subsequence, there exist $c_1^\varepsilon \in \mathbb{R}^3$ and $r_1^\varepsilon > 0$ and a unit embedded sphere $\omega_1$ such that we have, for all $l \in \mathbb{N}$,

$$\frac{\Phi_1^\varepsilon - c_1^\varepsilon}{r_1^\varepsilon} \to \omega_1 \text{ in } C^l_{loc}(\mathbb{R}^2) \text{ as } \varepsilon \to 0. \tag{5.5.15}$$

It remains to show (5.5.7). Assume by contradiction that there exists $\delta_0 > 0$ such that, for all $\varepsilon > 0$,

$$\sup_{x \in \mathbb{R}^2} (1 + |x|^2) d_1^\varepsilon(x) \left( |\nabla \varphi^\varepsilon(x)| + |\nabla \nu^\varepsilon(x)| \right) \geq \delta_0. \tag{5.5.16}$$

Let $a_2^\varepsilon \in \mathbb{R}^2$ be such that

$$(1 + |a_2^\varepsilon|^2) d_1^\varepsilon(a_2^\varepsilon) \left( |\nabla \varphi^\varepsilon(a_2^\varepsilon)| + |\nabla \nu^\varepsilon(a_2^\varepsilon)| \right) = \sup_{x \in \mathbb{R}^2} (1 + |x|^2) d_1^\varepsilon(x^\varepsilon) \left( |\nabla \varphi^\varepsilon(x)| + |\nabla \nu^\varepsilon(x)| \right).$$

For every $\varepsilon > 0$, the supremum is achieved in $\mathbb{R}^2$ since, thanks to the choice of parametrization and Remark 5.4.2, we have that

$$|\nabla \varphi^\varepsilon(x)| + |\nabla \nu^\varepsilon(x)| = O\left( \frac{1}{1 + |x|^2} \right) \text{ and } d_1^\varepsilon(x) \to 1 \text{ as } |x| \to +\infty.$$

Moreover, we get that

$$a_2^\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

since $|\nabla \varphi^\varepsilon| + |\nabla \nu^\varepsilon| \to 0$ on $\mathbb{R}^2 \backslash \{0\}$.

Next, we set

$$\frac{1}{\mu_2^\varepsilon} := |\nabla \varphi^\varepsilon(a_2^\varepsilon)| + |\nabla \nu^\varepsilon(a_2^\varepsilon)|.$$

Since we have $(1 + |a_2^\varepsilon|^2) d_1^\varepsilon(a_2^\varepsilon) \to 0$ as $\varepsilon \to 0$, thanks to (5.5.16), we get that

$$\mu_2^\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Now, we distinguish between two cases according to

$$\liminf_{\varepsilon \to 0} \frac{\min\{d_0^\varepsilon(a_2^\varepsilon), d_1^\varepsilon(a_2^\varepsilon)\}}{\mu_2^\varepsilon} > 0.$$

<u>First case:</u>

$$\liminf_{\varepsilon \to 0} \frac{\min\{d_0^\varepsilon(a_2^\varepsilon), d_1^\varepsilon(a_2^\varepsilon)\}}{\mu_2^\varepsilon} = +\infty. \tag{5.5.17}$$

In this case, we set

$$\Phi_2^\varepsilon = \Phi^\varepsilon(\mu_2^\varepsilon \cdot + a_2^\varepsilon) \text{ and}$$
$$\vec{n}_2^\varepsilon = \vec{n}^\varepsilon(\mu_2^\varepsilon \cdot + a_2^\varepsilon).$$

Then, for all $x \in \mathbb{R}^2$, we get that

$$\begin{aligned}
&(|\nabla\Phi_2^\varepsilon| + |\nabla\vec{n}_2^\varepsilon|)(x)\\
&= \mu_2^\varepsilon \left(|\nabla\Phi^\varepsilon| + |\nabla\vec{n}^\varepsilon|\right)(\mu_2^\varepsilon x + a_2^\varepsilon)\\
&\leq \mu_2^\varepsilon(|\nabla\varphi^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon)| + |\nabla\nu^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon)| + O\left(\mu_2^\varepsilon \left|\nabla\left(r_0^\varepsilon\omega_0^\varepsilon + r_1^\varepsilon\omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon)\right)\\
&\quad + O\left(\mu_2^\varepsilon \left|\nabla\left(\hat{n}_0^\varepsilon + \hat{n}_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon)\right)\\
&\leq 1 + o(1) + O\left(\mu_2^\varepsilon \left|\nabla\left(\omega_0^\varepsilon + \omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon)\right).
\end{aligned} \tag{5.5.18}$$

Thanks to (5.4.2) and (5.5.17), we get that

$$\mu_2^\varepsilon \left|\nabla\left(\omega_0^\varepsilon + \omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon) = o(1). \tag{5.5.19}$$

Then, using the definition of $a_2^\varepsilon$, (5.5.17), (5.5.18) and (5.5.19), we obtain that, for all $x \in \mathbb{R}^2$,

$$(|\nabla\Phi_2^\varepsilon| + |\nabla\vec{n}_2^\varepsilon|)(x) \leq 1 + o(1). \tag{5.5.20}$$

Hence $|\nabla\Phi_2^\varepsilon|$ and $|\nabla\vec{n}_2^\varepsilon|$ are bounded on every compact subset of $\mathbb{R}^2$. Moreover, $\Phi_2^\varepsilon$, $\vec{n}_2^\varepsilon$ and $\vec{H}_2^\varepsilon$ still verify the Willmore system (5.3.5), since the system is conformally invariant. Standard elliptic theory and simple bootstrap arguments confirm that there exists some $\Phi_2 \in C^\infty(\mathbb{R}^2)$ satisfying (5.3.6) with $\alpha = 0$ such that, up to a subsequence, we have, for all $l \in \mathbb{N}$,

$$\frac{\Phi_2^\varepsilon - c_2^\varepsilon}{r_2^\varepsilon} \to \Phi_2 \text{ in } C^l_{loc}(\mathbb{R}^2) \text{ as } \varepsilon \to 0. \tag{5.5.21}$$

Thanks to Laurain [83] and Li [90], we can show that $\nabla\Phi_2$ does not vanish on $\mathbb{R}^2$ and that $\Phi_2$ extends as a smooth immersion in $\mathbb{S}^2$. Hence, by the classification result of Bryant [33], $\Phi_2$ parametrizes a round sphere: $\Phi_2 = r_2\omega_2$ for some $r_2 > 0$ and $\omega_2$ embedded unit sphere.

Let $R > 0$. It appears that, thanks to (5.5.17), for $\varepsilon$ small enough, we can write

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{n}^\varepsilon|^2 dx \geq \sum_{i=0}^{2} \frac{1}{2}\int_{\mathbb{R}^2 \cap B_{R\mu_i^\varepsilon}(a_i^\varepsilon)\setminus\mathcal{S}_i^\varepsilon(R)} |\nabla\vec{n}^\varepsilon|^2\, dx \tag{5.5.22}$$

where $\mathcal{S}_i^\varepsilon(R) = \left\{x \in B_{R^{-1}\mu_j^\varepsilon}(a_j^\varepsilon) \,|\, j \text{ s.t. } \lim_{\varepsilon\to 0} \frac{\mu_j^\varepsilon}{\mu_i^\varepsilon} = 0\right\}$.

Moreover, thanks to (5.5.4), (5.5.15) and (5.5.21), we obtain that

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{n}^\varepsilon|^2\, dx &\geq \sum_{i=0}^{2} \frac{1}{2}\int_{\left(B_{(\mu_i^\varepsilon)^{-1}}\left(-\frac{a_i^\varepsilon}{\mu_i^\varepsilon}\right)\cap B_R(0)\right)\setminus\mathcal{S}_i(R)} |\nabla\hat{n}_i|^2\, dx + \delta_R^\varepsilon\\
&\geq 3 \times 4\pi + \delta_R^\varepsilon
\end{aligned} \tag{5.5.23}$$

where $\mathcal{S}_i(R) = \bigcup_{x\in\mathcal{S}_i} B_{\frac{1}{R}}(x)$ and $\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\delta_R^\varepsilon = 0$. Therefore, we have that

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\|\nabla\vec{n}^\varepsilon\|_2^2 \geq 12\pi,$$

which contradicts with the energy threshold assumption.

Second case:

$$0 < \lim_{\varepsilon\to 0} \frac{\min\{d_0^\varepsilon(a_2^\varepsilon), d_1^\varepsilon(a_2^\varepsilon)\}}{\mu_2^\varepsilon} < +\infty. \tag{5.5.24}$$

First we have that

$$\frac{\mu_1^\varepsilon}{\mu_2^\varepsilon} + \frac{\mu_2^\varepsilon}{\mu_1^\varepsilon} + \frac{a_1^\varepsilon - a_2^\varepsilon}{\mu_1^\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{5.5.25}$$

Indeed, assuming by contradiction this is not the case, then, up to a subsequence,

$$\mu_1^\varepsilon = O(\mu_2^\varepsilon),\ \mu_2^\varepsilon = O(\mu_1^\varepsilon) \text{ and } \left|\frac{a_2^\varepsilon - a_1^\varepsilon}{\mu_1^\varepsilon}\right| = O(1) \text{ as } \varepsilon \to 0, \tag{5.5.26}$$

which contradicts with the fact that

$$|\nabla\varphi^\varepsilon(\mu_1^\varepsilon \cdot + a_1^\varepsilon)| + |\nabla\nu^\varepsilon(\mu_1^\varepsilon \cdot + a_1^\varepsilon)| \to 0 \text{ on } \mathbb{R}^2 \text{ as } \varepsilon \to 0, \tag{5.5.27}$$

thanks to (5.5.7).

Next, we set

$$\begin{aligned}
\Phi_2^\varepsilon &= \Phi^\varepsilon(\mu_2^\varepsilon \cdot + a_2^\varepsilon),\\
\vec{n}_2^\varepsilon &= \vec{n}^\varepsilon(\mu_2^\varepsilon \cdot + a_2^\varepsilon),\\
b_{12} &:= \lim_{\varepsilon\to 0} \frac{a_1^\varepsilon - a_2^\varepsilon}{\mu_2^\varepsilon}\,.
\end{aligned}$$

For all $x \in \mathbb{R}^2\backslash\{b_{12}\}$, we have that

$$\begin{aligned}
&\left(|\nabla\Phi_2^\varepsilon| + |\nabla\vec{n}_2^\varepsilon|\right)(x)\\
&= \mu_2^\varepsilon \left(|\nabla\Phi^\varepsilon| + |\nabla\vec{n}^\varepsilon|\right)(\mu_2^\varepsilon x + a_2^\varepsilon)\\
&\le \mu_2^\varepsilon \left(|\nabla\varphi^\varepsilon| + |\nabla\nu^\varepsilon|\right)(\mu_2^\varepsilon x + a_2^\varepsilon) + O\left(\mu_2^\varepsilon \left|\nabla\left(\omega_0^\varepsilon + \omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon)\right)\\
&\le \frac{\min\{d_0^\varepsilon(a_2^\varepsilon), d_1^\varepsilon(a_2^\varepsilon)\}}{\min\{d_0^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon), d_1^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon)\}} + O\left(\mu_2^\varepsilon \left|\nabla\left(\omega_0^\varepsilon + \omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon)\right).
\end{aligned} \tag{5.5.28}$$

Thanks to (5.4.2) and (5.5.24), we get that

$$\frac{\min\{d_0^\varepsilon(a_2^\varepsilon), d_1^\varepsilon(a_2^\varepsilon)\}}{\min\{d_0^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon), d_1^\varepsilon(\mu_2^\varepsilon x + a_2^\varepsilon)\}} + \mu_2^\varepsilon \left|\nabla\left(\omega_0^\varepsilon + \omega_1^\varepsilon\right)\right|(\mu_2^\varepsilon x + a_2^\varepsilon) = O\left(\frac{1}{|x - b_{12}|}\right). \tag{5.5.29}$$

Then, combining (5.5.24), (5.5.28) and (5.5.29), we obtain that, for all $x \in \mathbb{R}^2\backslash\{b_{12}\}$,

$$\left(|\nabla\Phi_2^\varepsilon| + |\nabla\vec{n}_2^\varepsilon|\right)(x) \le O\left(\frac{1}{|x - b_{12}|}\right). \tag{5.5.30}$$

Hence, $(|\nabla\Phi_2^\varepsilon| + |\nabla\vec{n}_2^\varepsilon|)$ is bounded on every compact set of $\mathbb{R}^2\backslash\{b_{12}\}$. Up to subsequence, there exists a $\Phi_2 \in C^\infty(\mathbb{R}^2\backslash\{b_{12}\})$ such that, for every $l \in \mathbb{N}$,

$$\frac{\Phi_2^\varepsilon - c_2^\varepsilon}{r_2^\varepsilon} \to \Phi_2 \text{ in } C^l_{loc}(\mathbb{R}^2\backslash\{b_{12}\}). \tag{5.5.31}$$

As previously shown in Reparametrization of Willmore spheres, we can apply the singularity removability theorem at $b_{12}$ and at infinity, so that $\Phi_2$ can be extended to a branched immersion in $\mathbb{S}^2$ and satisfies the Willmore equation (5.3.1). Again, since $\Phi_2$ is obtained as the limit of smooth embeddings, by Laurain [83] and Li [90], $\Phi_2$ has no branched points in $\mathbb{S}^2$. Therefore, by the classification result of Bryant [33], there exist $r_2 > 0$ and a embedded unit sphere $\omega_2$ given by (5.4.1) such that $\Phi_2 = r_2\omega_2$.

By analogy with the first case, thanks to (5.5.25), for $R > 0$ and $\varepsilon > 0$ small enough, we can write

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{n}^\varepsilon|^2 dx &\ge \sum_{i=0}^{2} \frac{1}{2}\int_{\mathbb{R}^2 \cap B_{R\mu_i^\varepsilon}(a_i^\varepsilon)\backslash\mathcal{S}_i^\varepsilon(R)} |\nabla\vec{n}^\varepsilon|^2\, dx\\
&\ge \sum_{i=0}^{2} \frac{1}{2}\int_{\left(B_{(\mu_i^\varepsilon)^{-1}}\left(-\frac{a_i^\varepsilon}{\mu_i^\varepsilon}\right)\cap B_R(0)\right)\backslash\mathcal{S}_i(R)} |\nabla\hat{n}_i|^2\, dx + \delta_R^\varepsilon\\
&\ge 3 \times 4\pi + \delta_R^\varepsilon
\end{aligned} \tag{5.5.32}$$

where

$$\mathcal{S}_i^\varepsilon(R) = \left\{ x \in B_{R^{-1}\mu_j^\varepsilon}(a_j^\varepsilon) | \, j \text{ s.t. } \lim_{\varepsilon\to 0} \frac{\mu_j^\varepsilon}{\mu_i^\varepsilon} = 0 \right\},$$

$\mathcal{S}_i(R) = \bigcup_{x\in\mathcal{S}_i} B_{\frac{1}{R}}(x)$ and $\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\delta_R^\varepsilon = 0$. (5.5.32) contradicts with the energy threshold assumption.

Thus in either case we have reached a contradiction, which shows that (5.5.7) is true and completes the proof of Theorem 5.5.1. □

*Remark* 5.5.2. In the general situation, as long as the weak estimate (5.5.7) of Proposition 5.5.1, one could continue the process as we did with this $a_2^\varepsilon$. And the aim would be to prove that the limiting Willmore surfaces, obtained after rescaling, are either spheres or catenoids (which all carry the same amount of energy). And the process has to stop once there is no energy left (as we did here).

Thanks to Theorem 5.5.1, we give in the following another pointwise estimates on $\nabla\varphi^\varepsilon$:

**Proposition 5.5.3.** *For any $0 < \alpha < 1$, we have*

$$(1+|x|^2)|\nabla\varphi^\varepsilon|(x) = o\left( d_1^\varepsilon(x)^{-\alpha} + \frac{r_1^\varepsilon + \mu_1^\varepsilon}{d_1^\varepsilon(x)} \right) \tag{5.5.33}$$

*Proof.* Assume by contradiction that

$$\liminf_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^2} \frac{(1+|x|^2)|\nabla\varphi^\varepsilon(x)|}{\left( d_1^\varepsilon(x)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x)} \right)} \geq \varepsilon_0 \tag{5.5.34}$$

for some $\varepsilon_0 > 0$.

For every $\varepsilon > 0$, $\sup_{x\in\mathbb{R}^2} \frac{(1+|x|^2)|\nabla\varphi^\varepsilon(x)|}{\left( d_1^\varepsilon(x)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x)} \right)}$ is attained by some $x^\varepsilon \in \mathbb{R}^2$ thanks to the fact that

$$|\nabla\varphi^\varepsilon(x)| = o\left( \frac{1}{1+|x|^2} \right) \text{ and } \left( d_1^\varepsilon(x)^\alpha + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x)} \right) \not\to 0 \text{ as } x \to +\infty.$$

**Claim 5.5.4.** *We have*

$$\frac{|x^\varepsilon - a_1^\varepsilon|}{\mu_1^\varepsilon} \to +\infty \text{ as } \varepsilon \to 0.$$

*Proof of Claim 5.5.4:* Otherwise, since $\frac{\Phi_1^\varepsilon - c_1^\varepsilon}{r_1^\varepsilon}$ converges to $\omega_1$, we have that

$$\frac{\mu_1^\varepsilon}{r_1^\varepsilon}|\nabla\varphi^\varepsilon(x^\varepsilon) + r_0^\varepsilon\nabla\omega_0^\varepsilon(x^\varepsilon)| = \frac{\mu_1^\varepsilon}{r_1^\varepsilon}|\nabla\Phi^\varepsilon(x^\varepsilon) - \nabla\omega_1^\varepsilon(x^\varepsilon)| = o(1)$$

which implies

$$|\nabla\varphi^\varepsilon(x^\varepsilon)| = o\left(\frac{r_1^\varepsilon}{\mu_1^\varepsilon}\right) + O(1).$$

Moreover, since $d_1^\varepsilon(x^\varepsilon) \sim C\mu_1^\varepsilon$ as $\varepsilon \to 0$ for some $C > 0$, we obtain that

$$|\nabla\varphi^\varepsilon(x^\varepsilon)| = O(1) + o\left(\frac{r_1^\varepsilon}{\mu_1^\varepsilon}\right) \gtrsim \varepsilon_0 \left( d_1^\varepsilon(x^\varepsilon)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)} \right) \gtrsim \left( \frac{1}{(\mu_1^\varepsilon)^\alpha} + \frac{1}{\mu_1^\varepsilon r_1^\varepsilon} \right),$$

which leads to a contradiction. □

Now, we set

$$\tilde{\varphi}^\varepsilon(x) := \frac{1}{d_1^\varepsilon(x^\varepsilon)|\nabla\varphi^\varepsilon|(x^\varepsilon)}\varphi^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)x)\ ,$$
$$\tilde{\Phi}^\varepsilon(x) := \frac{1}{d_1^\varepsilon(x^\varepsilon)|\nabla\varphi^\varepsilon|(x^\varepsilon)}\Phi^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)x) \text{ and}$$
$$\tilde{n}^\varepsilon(x) := n^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)x)\ .$$

Remark that, thanks to (5.5.34), on any compact set $K$ of $\mathbb{R}^2\backslash\{0\}$, we have that

$$\begin{aligned}\nabla\tilde{\Phi}^\varepsilon(y) &= \nabla\tilde{\varphi}^\varepsilon(y) + \frac{r_1^\varepsilon}{|\nabla\varphi^\varepsilon(x^\varepsilon)|}\nabla\omega_1^\varepsilon\,(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + \frac{r_0^\varepsilon}{|\nabla\varphi^\varepsilon(x^\varepsilon)|}\nabla\omega_0^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
&= \nabla\tilde{\varphi}^\varepsilon(y) + O\left(\frac{r_1^\varepsilon\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)^2|\nabla\varphi^\varepsilon(x^\varepsilon)|}\right) + O\left(\frac{r_0^\varepsilon}{|\nabla\varphi^\varepsilon(x^\varepsilon)}\right)\\
&= \nabla\tilde{\varphi}^\varepsilon(y) + O\left(\frac{r_1^\varepsilon\mu_1^\varepsilon}{\varepsilon_0 d_1^\varepsilon(x^\varepsilon)^2\left(d_1^\varepsilon(x^\varepsilon)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)}\right)}\right) + o(1)\\
&= \nabla\tilde{\varphi}^\varepsilon(y) + o(1)\end{aligned}$$

Write $x^\varepsilon = a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y^\varepsilon$ so that we have

$$y^\varepsilon \to y_0 \text{ as } \varepsilon \to 0$$

with $|y_0| = 1$ and

$$|\nabla\tilde{\varphi}^\varepsilon(y^\varepsilon)| = 1.$$

Thanks to (5.5.34), we have that, for every $y \in K$,

$$\begin{aligned}|\nabla\tilde{\varphi}^\varepsilon(y)| &\le \frac{d_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon\left(a_1^\varepsilon+d_1^\varepsilon(x^\varepsilon)y\right)}}{d_1^\varepsilon(x^\varepsilon)^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)}}\,(1+o(1))\\
&\le \frac{d_1(x^\varepsilon)^{1-\alpha}}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + r_1^\varepsilon + \mu_1^\varepsilon}|x|^{-\alpha} + \frac{r_1^\varepsilon+\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + r_1^\varepsilon + \mu_1^\varepsilon}|x|^{-1} + o(1)\end{aligned} \tag{5.5.35}$$

with equality at $y = y^\varepsilon$. Up to a subsequence, we set

$$C_1 := \lim_{\varepsilon\to 0}\frac{d_1^\varepsilon(x^\varepsilon)^{1-\alpha}}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + r_1^\varepsilon + \mu_1^\varepsilon} \in [0,1],$$

which gives

$$|\nabla\tilde{\Phi}^\varepsilon(y)| \le C_1|y|^{-\alpha} + (1-C_1)|y|^{-1} + o(1) \tag{5.5.36}$$

on every compact set $K$ of $\mathbb{R}^2\backslash\{0\}$.

Moreover, thanks to (5.5.7) and the fact that $\mu_1^\varepsilon = o\left(d_1^\varepsilon(x^\varepsilon)\right)$, for every $y \in K$, we get that

$$\begin{aligned}|\nabla\tilde{n}^\varepsilon(y)| &\le d_1^\varepsilon(x^\varepsilon)|\nabla\nu^\varepsilon|\,(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{N}_0^\varepsilon|\,(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{N}_1^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
&= o(1).\end{aligned} \tag{5.5.37}$$

Thanks to (5.5.36), (5.5.37) and the first equation of (5.3.5), by standard elliptic theory, up to a subsequence, $\tilde{\Phi}^\varepsilon - \tilde{\Phi}^\varepsilon(0)$ converges to some $\Phi_0$ in $C^\infty_{loc}(\mathbb{R}^2\backslash\{0\})$ where $\Phi_0$ satisfies

$$\Delta\Phi_0 = 0 \text{ on } \mathbb{R}^2\backslash\{0\} \tag{5.5.38}$$

and

$$|\nabla\Phi_0(y)| \le C_1|y|^{-\alpha} + (1-C_1)|y|^{-1} \text{ on } \mathbb{R}^2\backslash\{0\}. \tag{5.5.39}$$

Thanks to Bôchner's theorem (see, e.g. Raynor [119]) together with (5.5.38) and (5.5.39), we have that

$$\Phi_0 = \vec{\gamma}_0 \log r + \vec{C} \tag{5.5.40}$$

for some $\vec{\gamma}_0, \vec{C} \in \mathbb{R}^3$.

Finally, since $\Phi_0$ is conformal, we get that $\vec{\gamma}_0 = \vec{0}$ and thus $\nabla\Phi_0 \equiv 0$, which is in contradiction with the fact that $\nabla\tilde{\Phi}^\varepsilon(y^\varepsilon) \to 1$ as $\varepsilon \to 0$. □

Similarly, we can also obtain a pointwise estimate on $|\nabla\eta^\varepsilon|$ where

$$\eta^\varepsilon := \vec{H}^\varepsilon - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon - \frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon :$$

**Proposition 5.5.5.** *Set*

$$\eta^\varepsilon = \vec{H}^\varepsilon - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon - \frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon.$$

*Then we have, for any* $0 < \alpha < 1$,

$$(1+|x|^2)|\nabla\eta^\varepsilon(x)| = o\left(d_1^\varepsilon(x)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x)}\right) . \tag{5.5.41}$$

*Proof of Proposition 5.5.5:* Assume by contradiction that

$$\liminf_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^2} \frac{(1+|x|^2)|\nabla\eta^\varepsilon(x)|}{\left(d_1^\varepsilon(x)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x)}\right)} \geq \varepsilon_0 \tag{5.5.42}$$

for some $\varepsilon_0 > 0$.

For $\varepsilon > 0$, $\sup_{x\in\mathbb{R}^2} \frac{(1+|x|^2)|\nabla\eta^\varepsilon(x)|}{\left(d_1^\varepsilon(x)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x)}\right)}$ is attained by some $x^\varepsilon \in \mathbb{R}^2$ thanks to the fact that

$$|\nabla\eta^\varepsilon(x)| = o\left(\frac{1}{1+|x|^2}\right) \text{ and } \left(d_1^\varepsilon(x)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x)}\right) \not\to 0 \text{ as } x \to +\infty.$$

Moreover, since $|\nabla\eta^\varepsilon| \to 0$ in $\mathbb{R}^2\backslash\{0\}$ as $\varepsilon \to 0$, we also get that

$$\lim_{\varepsilon\to 0} x^\varepsilon = 0$$

and $\lim_{\varepsilon\to 0} r_1^\varepsilon|\nabla\eta^\varepsilon(x)| = +\infty$.

**Claim 5.5.6.** *We have*

$$\frac{|x^\varepsilon - a_1^\varepsilon|}{\mu_1^\varepsilon} \to +\infty \text{ as } \varepsilon \to 0.$$

*Proof of Claim 5.5.6:* Otherwise, since we have

$$\mu_1^\varepsilon r_1^\varepsilon|\nabla\eta^\varepsilon(x^\varepsilon) + \nabla\mathcal{H}_0^\varepsilon(x^\varepsilon)| = \mu_1^\varepsilon|r_1^\varepsilon\nabla\vec{H}^\varepsilon(x^\varepsilon) - \nabla\mathcal{H}_1^\varepsilon(x^\varepsilon)| = o(1)$$

we get that $\mu_1^\varepsilon r_1^\varepsilon|\nabla\eta^\varepsilon(x^\varepsilon)| = o(1) + O\left(\mu_1^\varepsilon r_1^\varepsilon\right)$ which implies

$$|\nabla\eta^\varepsilon(x^\varepsilon)| = o\left(\frac{1}{\mu_1^\varepsilon r_1^\varepsilon}\right) + O\left(1\right).$$

Moreover, since $d_1^\varepsilon(x^\varepsilon) \sim C\mu_1^\varepsilon$ as $\varepsilon \to 0$ for some $C > 0$, we obtain that

$$|\nabla\eta^\varepsilon(x^\varepsilon)| = o\left(\frac{1}{\mu_1^\varepsilon r_1^\varepsilon}\right) + O\left(1\right) \gtrsim \left(\frac{1}{(\mu_1^\varepsilon)^\alpha} + \frac{1}{\mu_1^\varepsilon r_1^\varepsilon}\right),$$

which gives contradiction. □

Next, we set

$$\begin{aligned}
\tilde{\Phi}^\varepsilon(y) &= \Phi^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
\tilde{n}^\varepsilon(y) &= \vec{n}^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
\tilde{H}^\varepsilon(y) &= \frac{1}{d_1^\varepsilon(x^\varepsilon)|\nabla\eta^\varepsilon(x^\varepsilon)|}\vec{H}^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
\tilde{\eta}^\varepsilon(y) &= \frac{1}{d_1^\varepsilon(x^\varepsilon)|\nabla\eta^\varepsilon(x^\varepsilon)|}\eta^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)
\end{aligned} \tag{5.5.43}$$

Notice that, thanks to (5.5.42), we get that

$$\begin{aligned}
\nabla\tilde{H}^\varepsilon(y) &= \nabla\tilde{\eta}^\varepsilon(y) + \frac{1}{r_0^\varepsilon|\nabla\eta^\varepsilon(x^\varepsilon)|}\nabla\mathcal{H}_0^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + \frac{1}{r_1^\varepsilon|\nabla\eta^\varepsilon(x^\varepsilon)|}\nabla\mathcal{H}_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
&= \nabla\tilde{\eta}^\varepsilon(y) + O\left(\frac{1}{|\nabla\eta^\varepsilon(x^\varepsilon)|}\right) + O\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon|\nabla\eta^\varepsilon(x^\varepsilon)d_1^\varepsilon(x^\varepsilon)^2}\right)\\
&= \nabla\tilde{\eta}^\varepsilon(y) + o(1) + O\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon\varepsilon_0\left(d_1^\varepsilon(x^\varepsilon)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x^\varepsilon)}\right)d_1^\varepsilon(x^\varepsilon)^2}\right)\\
&= \nabla\tilde{\eta}^\varepsilon(y) + o(1) + O\left(\frac{\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)}\right)\\
&= \nabla\tilde{\eta}^\varepsilon(y) + o(1)
\end{aligned} \tag{5.5.44}$$

on every compact set $K$ of $\mathbb{R}^2\backslash\{0\}$.

Write $x^\varepsilon = a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y^\varepsilon$ so that $y^\varepsilon \to y_0$ as $\varepsilon \to 0$ with $|y_0| = 1$ and $|\nabla\tilde{\eta}^\varepsilon(y^\varepsilon)| = 1$. Then we have, for every $y \in K$,

$$\begin{aligned}
|\nabla\tilde{\eta}^\varepsilon(y)| &\leq \frac{d_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)}}{d_1^\varepsilon(x^\varepsilon)^{-\alpha} + \frac{1}{r_1^\varepsilon d_1^\varepsilon(x^\varepsilon)}}(1 + o(1))\\
&\leq \frac{d_1^\varepsilon(x^\varepsilon)^{1-\alpha}}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + \frac{1}{r_1^\varepsilon}}|y|^{-\alpha} + \frac{\frac{1}{r_1^\varepsilon}}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + \frac{1}{r_1^\varepsilon}}|y|^{-1} + o(1)\\
&\lesssim \frac{1}{|y|} + o(1)
\end{aligned} \tag{5.5.45}$$

since

$$\lim_{\varepsilon\to 0}\frac{\frac{1}{r_1^\varepsilon}}{d_1^\varepsilon(x^\varepsilon)^{1-\alpha} + \frac{1}{r_1^\varepsilon}} = 1.$$

Moreover, thanks to estimates (5.5.7) and the fact that $\mu_1^\varepsilon = o(d_1^\varepsilon(x^\varepsilon))$, for every $y \in \mathbb{R}^2\backslash\{0\}$, we

have

$$\begin{aligned}
|\nabla\tilde{\Phi}^\varepsilon(y)| &\leq d_1^\varepsilon(x^\varepsilon)|\nabla\varphi^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + r_0^\varepsilon d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{B}_0^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + r_1^\varepsilon d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{B}_1^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
&= o\left(\frac{d_1^\varepsilon(x^\varepsilon)}{d_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)}\right) + o(1) + O\left(\frac{r_1^\varepsilon d_1^\varepsilon(x^\varepsilon)\mu_1^\varepsilon}{(\mu_1^\varepsilon)^2 + |d_1^\varepsilon(x^\varepsilon)y|^2}\right)\\
&= o\left(\frac{1}{\tilde{\mu}^\varepsilon + |y|}\right) + o\left(\frac{\tilde{\mu}^\varepsilon}{(\tilde{\mu}^\varepsilon)^2 + |y|^2}\right)\\
&= o\left(\frac{1}{\tilde{\mu}^\varepsilon + |y|}\right).\\
|\nabla\tilde{n}^\varepsilon(y)| &\leq d_1^\varepsilon(x^\varepsilon)|\nabla\nu^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{N}_0^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y) + d_1^\varepsilon(x^\varepsilon)|\nabla\mathcal{N}_1^\varepsilon|(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)\\
&= o\left(\frac{d_1^\varepsilon(x^\varepsilon)}{d_1^\varepsilon(a_1^\varepsilon + d_1^\varepsilon(x^\varepsilon)y)}\right) + o(1) + O\left(\frac{d_1^\varepsilon(x^\varepsilon)\mu_1^\varepsilon}{(\mu_1^\varepsilon)^2 + |d_1^\varepsilon(x^\varepsilon)y|^2}\right)\\
&= o\left(\frac{1}{\tilde{\mu}^\varepsilon + |y|}\right) + O\left(\frac{\tilde{\mu}^\varepsilon}{(\tilde{\mu}^\varepsilon)^2 + |y|^2}\right)
\end{aligned} \tag{5.5.46}$$

with $\tilde{\mu}_1^\varepsilon := \frac{\mu_1^\varepsilon}{d_1^\varepsilon(x^\varepsilon)} \to 0$ as $\varepsilon \to 0$.

For $R > 0$, using Hodge decomposition, we can write that

$$\nabla\tilde{H}^\varepsilon - 3\left(\nabla\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon\right)\tilde{n}^\varepsilon = \nabla C_R^\varepsilon + \nabla^\perp D_R^\varepsilon + \nabla E_R^\varepsilon \text{ on } \mathbb{D}_0(R)$$

where $\nabla\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon := h^\varepsilon(\tilde{\Phi}^\varepsilon)(\nabla\tilde{H}^\varepsilon, \tilde{n}^\varepsilon)$, $E_R^\varepsilon$ is a harmonic function on $\mathbb{D}_0(R)$, $C_R^\varepsilon$ and $D_R^\varepsilon$ verify

$$\begin{cases} \Delta C_R^\varepsilon = div\left(\nabla\tilde{H}^\varepsilon - 3\left(\nabla\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon\right)\tilde{n}^\varepsilon\right) \text{ on } \mathbb{D}_0(R) \\ C_R^\varepsilon = 0 \text{ on } \partial\mathbb{D}_0(R) \end{cases} \text{ and } \begin{cases} \Delta D_R^\varepsilon = 3div((\nabla^\perp\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon)\tilde{n}^\varepsilon) \text{ on } \mathbb{D}_0(R) \\ \dfrac{\partial D_R^\varepsilon}{\partial\nu} = 0 \text{ on } \partial\mathbb{D}_0(R) \end{cases}$$

Thanks to the third equation of the Willmore system (5.3.5), together with (5.5.45) and (5.5.46), for every $y \in \mathbb{R}^2\backslash\{0\}$, we can write that

$$\begin{aligned}
div\left(\nabla\tilde{H}^\varepsilon - 3(\nabla\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon)\tilde{n}^\varepsilon\right)(y) &= O(|\nabla\tilde{H}^\varepsilon(y)||\nabla^\perp\tilde{n}^\varepsilon(y)|) + o\left(|\nabla\tilde{H}^\varepsilon(y)||\nabla\tilde{\Phi}^\varepsilon(y)|\right)\\
&\quad + o\left(|\tilde{H}^\varepsilon(y)\nabla\tilde{\Phi}^\varepsilon(y)||\nabla\tilde{n}^\varepsilon(y)|\right) + o\left(|\nabla\tilde{n}^\varepsilon(y)||\nabla\tilde{\Phi}^\varepsilon(y)|\right)\\
&\leq o\left(\frac{1}{|y|(\tilde{\mu}_1^\varepsilon + |y|)}\right) + O\left(\frac{\tilde{\mu}_1^\varepsilon}{|y|\left((\tilde{\mu}_1^\varepsilon)^2 + |y|^2\right)}\right)\\
&\leq o\left(\frac{1}{|y|\left(\tilde{\mu}_1^\varepsilon + |y|\right)}\right) \text{ and}\\
3div((\nabla^\perp\tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon)\tilde{n}^\varepsilon)(y) &= O(|\nabla^\perp\tilde{H}^\varepsilon(y)||\nabla\tilde{n}^\varepsilon(y)|) + o\left(|\nabla\tilde{\Phi}^\varepsilon(y)||\nabla\tilde{H}^\varepsilon(y)|\right)\\
&\leq o\left(\frac{1}{|y|(\tilde{\mu}_1^\varepsilon + |y|)}\right)
\end{aligned} \tag{5.5.47}$$

Thanks to (5.5.47), there exist some $C_R, D_R \in C^\infty(\mathbb{D}_0(R)\backslash\{0\})$ and $d^\varepsilon \in \mathbb{R}^3$ such that, for every $l \in \mathbb{N}$,

$$C_R^\varepsilon \to C_R \text{ in } C^l_{loc}(\mathbb{D}_0(R)\backslash\{0\}) \text{ and } D_R^\varepsilon - d^\varepsilon \to D_R \text{ in } C^l_{loc}(\mathbb{D}_0(R)\backslash\{0\})$$

where $C_R$ and $D_R$ verify

$$\begin{cases} \Delta C_R(y) = 0 \text{ on } \mathbb{D}_0(R)\backslash\{0\} \\ C_R = 0 \text{ on } \partial\mathbb{D}_0(R) \end{cases} \text{ and } \begin{cases} \Delta D_R(y) = 0 \text{ on } \mathbb{D}_0(R)\backslash\{0\} \\ \dfrac{\partial D_R}{\partial\nu} = 0 \text{ on } \partial\mathbb{D}_0(R) \end{cases} \tag{5.5.48}$$

Using Green's representation formula and estimates on $\nabla \tilde{H}^\varepsilon$, we can also prove that $C_R$ and $D_R$ have at most logarithm singularities at 0. Therefore, thanks to Bôchner's theorem (see, e.g. Raynor [119]), we can write that

$$C_R(r) = \vec{\gamma} \ln(R/r) \text{ and } D_R(r) = \vec{\delta}_1 \ln(R/r) + \vec{\delta}_2 \tag{5.5.49}$$

for some $\vec{\gamma}, \vec{\delta}_1, \vec{\delta}_2 \in \mathbb{R}^3$.

Thanks to the divergence theorem and (5.5.47), we also have

$$\int_{\partial \mathbb{D}_0(R)} \frac{\partial C_R^\varepsilon}{\partial \nu_R} \, dl_R = - \int_{\partial \mathbb{D}_0(R)} \left( \frac{\partial \tilde{H}^\varepsilon}{\partial \nu_R} - 3 \left( \frac{\partial \tilde{H}^\varepsilon}{\partial \nu_R} \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon \right) \tilde{n}^\varepsilon \right) dl_R = o(1). \tag{5.5.50}$$

Therefore, we get that $\int_{\partial \mathbb{D}_0(R)} \partial_\nu C_R \, dl_R = 0$, which implies $\vec{\gamma} = \vec{0}$.

In addition, we also have $\vec{\delta}_1 = \vec{0}$, since $\frac{\partial D_R}{\partial \nu} = 0$ on $\partial \mathbb{D}_0(R)$. Thus we get

$$\nabla C_R \equiv \nabla D_R \equiv 0 \tag{5.5.51}$$

for every $R > 0$.

Finally, since $\lim_{|y| \to +\infty} \nabla \tilde{H}^\varepsilon(y) = 0$, we have that

$$E_R^\varepsilon \to 0 \text{ in } C^l\left(\mathbb{D}_0(R)\right) \text{ as } R \to +\infty. \tag{5.5.52}$$

Combining (5.5.51) and (5.5.52) with the diagonal extraction procedure, we obtain that

$$\nabla \tilde{H}^\varepsilon - 3(\nabla \tilde{H}^\varepsilon \cdot_{\tilde{h}^\varepsilon} \tilde{n}^\varepsilon)\tilde{n}^\varepsilon \to 0 \text{ in } C^l_{loc}(\mathbb{R}^2 \backslash \{0\})$$

and thus $\nabla \tilde{H}^\varepsilon \to 0$ in $C^l_{loc}(\mathbb{R}^2 \backslash \{0\})$, which is in contradiction with the fact that $\nabla \tilde{H}^\varepsilon(y^\varepsilon) = 1 + o(1)$. Hence (5.5.41) is proved. □

*Remark* 5.5.7. These improved weak estimates are necessary since the sizes of the various spheres and catenoids which could appear in the decomposition need not be comparable. Note also that, working only on the gradient of $\Phi^\varepsilon$, it should be possible to consider also bubbles modelled on catenoids. One would have to adapt these weak estimates.

### 5.5.2 Energy quantization

In this section, we show that, in addition to the weak estimates (5.5.7), we have

$$\|\nabla \nu^\varepsilon\|_{L^2} + \|\nabla \varphi^\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.5.53}$$

We first prove that

$$\|\nabla \nu^\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.5.54}$$

This is in fact a direct consequence of energy quantization established in Bernard-Rivière [21]. Indeed, it is well known that $(M, h)$ can be smoothly embedded in $\mathbb{R}^n$ for some $n \geq 3$ ($n = 120$ suffices), one can therefore consider $(\Phi^\varepsilon)$ as a sequence of conformal Willmore immersions with area constraint into $\mathbb{R}^n$.

We next apply the Energy tracking procedure (Lemma III.2 in [21]) around $a_i^\varepsilon$ for each $1 \leq i \leq p$, then, for $0 < \alpha < 1$, we can define the corresponding "$\alpha$-neck region" $\Omega^\varepsilon(\alpha)$ of $\Phi^\varepsilon$ such that the $L^2$-norm of $\nabla \vec{n}^\varepsilon$ converges to the sum of the bubble energy $\nabla \hat{\omega}_i$ on the complement of $\Omega^\varepsilon(\alpha)$, that is,

$$\forall\, 0 < \alpha < 1, \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \backslash \Omega^\varepsilon(\alpha)} |\nabla \nu^\varepsilon|^2 dx = 0. \tag{5.5.55}$$

This set $\Omega^\varepsilon(\alpha)$ characterizes the junction between the bubbles $r_0^\varepsilon \omega_0^\varepsilon$ and $r_1^\varepsilon \omega_1^\varepsilon$. It remains to show that asymptotically there is no energy left in neck regions.

Since $(\Phi^\varepsilon)_{\varepsilon>0}$ is a local Palais-Smale sequence for the Willmore functional (see [19] for more details), thanks to the control on conformal factor in Lemma 5.3.4, we are allowed to use Lemma V.1 and Lemma VI.1 in [21], where uniform controls of Willmore immersion in neck regions are established, on $\Omega^\varepsilon(\alpha)$. Therefore, we deduce the *no-neck energy property*

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon(\alpha)}|\nabla\vec{n}^\varepsilon|^2dx=0. \tag{5.5.56}$$

Combining (5.5.55) and (5.5.56), we get (5.5.54).

In order to prove $L^2$-convergence for $\nabla\varphi^\varepsilon$, one can check that $\varphi^\varepsilon$ verifies

$$\begin{aligned}\Delta\varphi^\varepsilon=&-\Big(\nabla\hat{\vec{n}}_0^\varepsilon\wedge\nabla^\perp(r_1^\varepsilon\omega_1^\varepsilon)+\nabla\hat{\vec{n}}_1^\varepsilon\wedge\nabla^\perp(r_0^\varepsilon\omega_0^\varepsilon)\Big)\\&-\nabla\hat{\vec{n}}_0^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon-\nabla\hat{\vec{n}}_1^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon\\&+\nabla\nu^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon+O(\varepsilon^2(|\nabla\Phi^\varepsilon|+|\nabla\vec{n}^\varepsilon|)^2).\end{aligned} \tag{5.5.57}$$

With the help of stereographic coordinates, we can consider $\varphi^\varepsilon$, $\omega_i^\varepsilon$ and $\nu^\varepsilon$ as maps defined on $\mathbb{S}^2$. Observe that we have

$$r_1^\varepsilon\nabla\hat{\vec{n}}_0^\varepsilon\wedge\nabla^\perp\omega_1^\varepsilon+r_0^\varepsilon\nabla\hat{\vec{n}}_1^\varepsilon\wedge\nabla^\perp\omega_0^\varepsilon\to 0\text{ in }L^1(\mathbb{S}^2).$$

For $i=0,1$, let $U_i^\varepsilon$ and $V^\varepsilon$ be solutions of

$$\Delta U_i^\varepsilon=\nabla\omega_i^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon\text{ and }\Delta V^\varepsilon=\nabla\nu^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon\text{ on }\mathbb{S}^2. \tag{5.5.58}$$

Thanks to the generalized Wente inequality proved by Ge [64] (see Lemma 6.3.6) applied to (5.5.58) together with (5.5.54) and the fact $\|\nabla\varphi^\varepsilon\|_{2,+\infty}\to 0$ as $\varepsilon\to 0$, we obtain that

$$\|\nabla U_i^\varepsilon\|_2=O(\|\nabla\hat{\omega}_i^\varepsilon\|_2\|\nabla\varphi^\varepsilon\|_{2,+\infty})=o(1)\text{ and }\|\nabla V^\varepsilon\|_2=O(\|\nabla\nu^\varepsilon\|_2\|\nabla\varphi^\varepsilon\|_{2,+\infty})=o(1). \tag{5.5.59}$$

Coming back to (5.5.57) with the fact that $\|\Phi^\varepsilon\|_\infty$, $\|\nabla\Phi^\varepsilon\|_2$ and $\|\nabla\vec{n}^\varepsilon\|_2$ are uniformly bounded, we get that

$$\|\nabla\varphi^\varepsilon\|_2^2\to 0\text{ as }\varepsilon\to 0. \tag{5.5.60}$$

Hence, we retrieve the $L^2$-convergence (5.5.53).

## 5.6 Adjustments of bubbles

We have previously shown that $\Phi^\varepsilon$ behaves asymptotically as a sum of two round spheres. Since Willmore spheres in manifolds are not exactly round and, thanks to the expansion of the metric in normal coordinates, one can verify that the derivatives of curvature terms appear at order $\varepsilon^3$ in the Willmore system (5.3.5). Therefore, we would like to modify the form of our bubble $r_\theta^\varepsilon\omega_\theta^\varepsilon$ such that it solves the Willmore system (5.3.5) at order $\varepsilon^2$. For each $\theta=0,1$, we set

$$\begin{aligned}r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon&=r_\theta^\varepsilon c_\theta^\varepsilon+r_\theta^\varepsilon(\hat{\omega}_\theta^\varepsilon+\varepsilon^2(r_\theta^\varepsilon)^2\varphi_\theta^\varepsilon),\\\mathcal{N}_\theta^\varepsilon&=-\hat{\omega}_\theta^\varepsilon+\varepsilon^2(r_\theta^\varepsilon)^2\nu_\theta^\varepsilon,\\\mathcal{H}_\theta^\varepsilon&=\frac{1}{r_\theta^\varepsilon}(-\hat{\omega}_\theta^\varepsilon+\varepsilon^2(r_\theta^\varepsilon)^2\eta_\theta^\varepsilon),\end{aligned} \tag{5.6.1}$$

where $\mathcal{N}_\theta^\varepsilon$ and $\mathcal{H}_\theta^\varepsilon$ represent respectively the linearized normal vector and linearized mean curvature vector of $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ at $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ with

$$\begin{aligned}\nu_\theta^\varepsilon &= \frac{2}{|\nabla\hat\omega_\theta^\varepsilon|^2}\left(((\varphi_\theta^\varepsilon)_x\cdot\hat\omega_\theta^\varepsilon)(\hat\omega_\theta^\varepsilon)_x+((\varphi_\theta^\varepsilon)_y\cdot\hat\omega_\theta^\varepsilon)(\hat\omega_\theta^\varepsilon)_y\right)\\ &\quad+\frac{1}{3}\mathcal{R}^\varepsilon(\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,\cdot)+\frac{1}{6}\mathcal{R}^\varepsilon(\hat\omega_\theta,c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon)\hat\omega_\theta^\varepsilon,\\ \eta_\theta^\varepsilon &= \frac{\Delta\varphi_\theta^\varepsilon}{|\nabla\hat\omega_\theta^\varepsilon|^2}+\frac{2}{|\nabla\hat\omega_\theta^\varepsilon|^2}\left((\varphi_\theta^\varepsilon)_x\cdot(\hat\omega_\theta^\varepsilon)_x+(\varphi_\theta^\varepsilon)_y\cdot(\hat\omega_\theta^\varepsilon)_y\right)\hat\omega_\theta^\varepsilon\\ &\quad+\frac{1}{3}Ric^\varepsilon(\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,\cdot)-\frac{1}{3}\mathcal{R}^\varepsilon(\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon,\cdot)+\frac{1}{6}\mathcal{R}^\varepsilon(\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon)\hat\omega_\theta^\varepsilon,\end{aligned}\tag{5.6.2}$$

where $c_\theta^\varepsilon=O(1)$ is a constant vector in $\mathbb{R}^3$ to be chosen later.

We also require that $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ is conformal up to order $\varepsilon^3$, that is, at $r_\theta^\varepsilon\mathcal{B}_i^\varepsilon$, we have

$$\begin{aligned}\langle(r_\theta\mathcal{B}_\theta^\varepsilon)_x,(r_\theta\mathcal{B}_\theta^\varepsilon)_y\rangle_{h_\varepsilon(r_\theta\mathcal{B}_\theta^\varepsilon)}&=O(\varepsilon^3(r_\theta^\varepsilon)^5|\nabla\mathcal{B}_\theta^\varepsilon|^2)\\ \langle(r_\theta\mathcal{B}_\theta^\varepsilon)_x,(r_\theta\mathcal{B}_\theta^\varepsilon)_x\rangle_{h_\varepsilon(r_\theta\mathcal{B}_\theta^\varepsilon)}-\langle(r_\theta\mathcal{B}_\theta^\varepsilon)_y,(r_\theta\mathcal{B}_\theta^\varepsilon)_y\rangle_{h_\varepsilon(r_\theta\mathcal{B}_\theta^\varepsilon)}&=O(\varepsilon^3(r_\theta^\varepsilon)^5|\nabla\mathcal{B}_\theta^\varepsilon|^2),\end{aligned}\tag{5.6.3}$$

which, thanks to expansion of the metric $h^\varepsilon$ in Lemma 6.5.1, gives

$$\begin{aligned}(\varphi_\theta^\varepsilon)_x\cdot(\hat\omega_\theta^\varepsilon)_y+(\varphi_\theta^\varepsilon)_y\cdot(\hat\omega_\theta^\varepsilon)_x&=-\frac{1}{3}\mathcal{R}^\varepsilon\left((\hat\omega_\theta^\varepsilon)_x,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,(\hat\omega_\theta^\varepsilon)_y\right)\\ 2\left((\varphi_\theta^\varepsilon)_x\cdot(\hat\omega_\theta^\varepsilon)_x-(\varphi_\theta^\varepsilon)_y\cdot(\hat\omega_\theta^\varepsilon)_y\right)&=-\frac{1}{3}\left(\mathcal{R}^\varepsilon\left((\hat\omega_\theta^\varepsilon)_x,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,(\hat\omega_\theta^\varepsilon)_x\right)-\mathcal{R}^\varepsilon\left((\hat\omega_\theta^\varepsilon)_y,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon+c_\theta^\varepsilon,(\hat\omega_\theta^\varepsilon)_y\right)\right)\end{aligned}\tag{5.6.4}$$

Finally, we take

$$\begin{aligned}(\varphi_\theta^\varepsilon)_k&:=\frac{1}{6}Ric^\varepsilon(\hat\omega_\theta^\varepsilon,\cdot)_k-\frac{1}{12}Scal(p^\varepsilon)(\hat\omega_\theta^\varepsilon)_k+\left(\left(\frac{\alpha^\varepsilon}{\varepsilon^2}-\frac{1}{3}Scal(p^\varepsilon)\right)f_\theta^\varepsilon+g_\theta^\varepsilon\right)(\hat\omega_\theta^\varepsilon)_k\\ &\quad-\frac{1}{6}\mathcal{R}^\varepsilon(\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon,c_\theta^\varepsilon,\cdot)_k-\frac{1}{3}\mathcal{R}^\varepsilon(\hat\omega_\theta^\varepsilon,c_\theta^\varepsilon,\hat\omega_\theta^\varepsilon,\cdot)_k\end{aligned}\tag{5.6.5}$$

where

$$\begin{aligned}f_\theta^\varepsilon(x)&=f\left(\frac{|x-a_\theta^\varepsilon|}{\mu_\theta^\varepsilon}\right)\text{ and}\\ g_\theta^\varepsilon&=\frac{1}{4}\left(R_{imnj}(p^\varepsilon)(c_\theta^\varepsilon)^m(c_\theta^\varepsilon)^n\right)\left((\hat\omega_\theta^\varepsilon)^i(\hat\omega_\theta^\varepsilon)^j+\frac{2\delta_{ij}}{(1+r^2)}\right)\\ &\quad+\frac{1}{12}\left(R_{imnj}(p^\varepsilon)(c_\theta^\varepsilon)^m(c_\theta^\varepsilon)^n-Ric_{ij}(p^\varepsilon)\right)\left((\hat\omega_\theta^\varepsilon)^i(\hat\omega_\theta^\varepsilon)^j-\frac{\delta_{ij}}{4(1+r^2)}\right),\end{aligned}$$

with

$$f(r):=\frac{2r^2}{1+r^2}\ln r-\ln(1+r^2)-\frac{1}{1+r^2}$$

and $r^2=x^2+y^2$. By Lemma 5.3.3, one can verify that

$$|\nabla\varphi_\theta^\varepsilon|^2+|\nabla\nu_\theta^\varepsilon|^2+|\nabla\eta_\theta^\varepsilon|^2=O(|\nabla\hat\omega_\theta^\varepsilon|^2).$$

Notice that the case where only one bubble appears is studied in [84] by Laurain and Mondino. In our case where we have (at least) two bubbles $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$, since these bubbles are not centered at the same point, we also need to take into account the presence of their barycenter in (5.6.5).

Thanks to the conformality (5.6.3) together with the fact that in dimension 3,

$$R_{imnj}=(h_{in}Ric_{mj}-h_{ij}Ric_{mn}+h_{mj}Ric_{in}-h_{mn}Ric_{ij})+\frac{Scal}{2}(h_{ij}h_{mn}-h_{in}h_{mj})\tag{5.6.6}$$

one can check that $(r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon, \mathcal{N}_\theta^\varepsilon, \mathcal{H}_\theta^\varepsilon)$ solves the first two equations in (5.3.5) up to order $\varepsilon^3$, since they are directly derived from the conformality of the immersion. The last equation in (5.3.5) is also satisfied up to order $\varepsilon^3$ thanks to the choice of $f$ we made. Computations are tedious but straightforward. Thus we have the following:

**Proposition 5.6.1.** *For $\theta \in \{0,1\}$, $r_\theta \mathcal{B}_\theta^\varepsilon$, $\mathcal{N}_\theta^\varepsilon$ and $\mathcal{H}_\theta^\varepsilon$ verify the following system of equations:*

$$
\begin{cases}
\begin{aligned}
\Delta(r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^k = &-(\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^i \cdot \nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j\right) \\
&+ \sqrt{|h_\varepsilon|} (h^\varepsilon)^{ik} (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla \mathcal{N}_\theta^\varepsilon \wedge \nabla^\perp r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon\right)_\theta \\
&+ \left(\vec{\Gamma}_{mn}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^m (\nabla r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^n \wedge \nabla^\perp r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon\right)_k \\
&+ O\left(\varepsilon^3 (r_\theta^\varepsilon)^4 |\nabla \mathcal{B}_\theta^\varepsilon|^2\right) \\
\Delta(\mathcal{N}_\theta^\varepsilon)^k = &-2(\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla (\mathcal{N}_\theta^\varepsilon)^i \cdot \nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j\right) \\
&- (\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^i (\Delta r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \\
&- (\Gamma^\varepsilon)_{ij,l}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^i (\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \cdot \nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^l) \\
&- \sqrt{|h_\varepsilon|} (h^\varepsilon)^{ik} (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla \mathcal{N}_\theta^\varepsilon \wedge \nabla^\perp \mathcal{N}_\theta^\varepsilon\right)_\theta \\
&- 2\left(\vec{\Gamma}_{mn}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^m (\nabla r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^n \wedge \nabla^\perp \mathcal{N}_\theta^\varepsilon\right)_k \\
&- 2\sqrt{|h_\varepsilon|} (h^\varepsilon)^{ik} (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla \mathcal{H}_\theta^\varepsilon \wedge \nabla^\perp r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon\right)_\theta \\
&- 2\left(\vec{\Gamma}_{mn}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^m (\nabla r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^n \wedge \nabla^\perp r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon\right)_k \\
&- \varepsilon^2 \frac{2}{|\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)|^2} \mathcal{R}^\varepsilon(\partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \mathcal{N}_\theta^\varepsilon, \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)) \partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \\
&- \varepsilon^2 \frac{2}{|\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)|^2} \mathcal{R}^\varepsilon(\partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \mathcal{N}_\theta^\varepsilon, \partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)) \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \\
&+ O\left(\varepsilon^3 (r_\theta^\varepsilon)^3 |\nabla \mathcal{B}_\theta^\varepsilon|^2\right)
\end{aligned} \\
\begin{aligned}
div\left(\nabla \mathcal{H}_\theta^\varepsilon - 3\left(\nabla \mathcal{H}_\theta^\varepsilon \cdot_{h_\theta^\varepsilon} \mathcal{N}_\theta^\varepsilon\right) \mathcal{N}_\theta^\varepsilon\right)_k = &-2(\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla (\mathcal{H}_\theta^\varepsilon)^i \cdot \nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j\right) \\
&- (\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^i (\Delta r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \\
&- (\Gamma^\varepsilon)_{ij,l}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^i (\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \cdot \nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^l) \\
&+ 3 div\left(\left((h^\varepsilon)_{\alpha\beta} (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \Gamma_{ij}^\alpha (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^i (\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j)(\mathcal{N}_\theta^\varepsilon)^\beta\right) \mathcal{N}_\theta^\varepsilon\right) \\
&+ 3\partial_x \left(h_{ij}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^i (\mathcal{N}_\theta^\varepsilon)^j\right) (\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^i (\partial_x r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \\
&+ 3\partial_y \left(h_{ij}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^i (\mathcal{N}_\theta^\varepsilon)^j\right) (\Gamma^\varepsilon)_{ij}^k (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^i (\partial_y r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^j \\
&+ \sqrt{|h_\varepsilon|} (h^\varepsilon)^{ik} (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \left(\nabla \mathcal{H}_\theta^\varepsilon \wedge \nabla^\perp \mathcal{N}_\theta^\varepsilon\right)_\theta \\
&+ \left(\vec{\Gamma}_{mn}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{H}_\theta^\varepsilon)^m (\nabla r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^n \wedge \nabla^\perp \mathcal{N}_\theta^\varepsilon\right)_k \\
&+ \left(\nabla \mathcal{H}_\theta^\varepsilon \wedge \vec{\Gamma}_{mn}^\varepsilon (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)(\mathcal{N}_\theta^\varepsilon)^m (\nabla^\perp r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^n\right)_k \\
&+ \varepsilon^2 \frac{6}{|\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)|^2} \mathcal{R}^\varepsilon(\partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \mathcal{H}_\theta^\varepsilon, \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)) \partial_x (r_\theta \mathcal{B}_\theta^\varepsilon) \\
&+ \varepsilon^2 \frac{6}{|\nabla (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)|^2} \mathcal{R}^\varepsilon(\partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon), \mathcal{H}_\theta^\varepsilon, \partial_x (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)) \partial_y (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon) \\
&+ \varepsilon^2 Ric(\mathcal{N}_\theta^\varepsilon, \mathcal{N}_\theta^\varepsilon) \Delta (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)^k \\
&- \alpha^\varepsilon \left(\nabla \mathcal{N}_\theta^\varepsilon \wedge \nabla^\perp (r_\theta^\varepsilon \mathcal{B}_\theta^\varepsilon)\right)^k \\
&+ O\left(\varepsilon^3 (r_\theta^\varepsilon)^3 |\nabla \mathcal{B}_\theta^\varepsilon|^2\right)
\end{aligned}
\end{cases}
\tag{5.6.7}
$$

*where $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon, \mathcal{N}_\theta^\varepsilon$ and $\mathcal{H}_\theta^\varepsilon$ are defined as in (5.6.1) and*

$$\nabla\mathcal{H}_\theta^\varepsilon \cdot_{h_\theta^\varepsilon} \mathcal{N}_\theta^\varepsilon := h^\varepsilon(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)\left(\nabla\mathcal{H}_\theta^\varepsilon, \mathcal{N}_\theta^\varepsilon\right) = h_{ij}^\varepsilon(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)(\nabla\mathcal{H}_\theta^\varepsilon)^i(\mathcal{N}_\theta^\varepsilon)^j.$$

*Moreover, $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ verifies the conformality up to order $\varepsilon^3$, that is, at $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$, we have that*

$$\begin{aligned}\langle(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_x, (r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_y\rangle_{h_\varepsilon(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)} &= O(\varepsilon^3(r_\theta^\varepsilon)^5|\nabla\mathcal{B}_\theta^\varepsilon|^2)\\ \langle(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_x, (r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_x\rangle_{h_\varepsilon(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)} - \langle(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_y, (r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)_y\rangle_{h_\varepsilon(r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon)} &= O(\varepsilon^3(r_\theta^\varepsilon)^5|\nabla\mathcal{B}_\theta^\varepsilon|^2)\end{aligned}. \tag{5.6.8}$$

Now, we try to rotate our modified bubbles $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ in such a way that each revised $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$ is tangent to the Willmore spheres $\Sigma^\varepsilon$ at extremal points. Set

$$\varphi^\varepsilon := \Phi^\varepsilon - r_0^\varepsilon\mathcal{B}_0^\varepsilon - r_1^\varepsilon\mathcal{B}_1^\varepsilon,\ \nu^\varepsilon := \vec{n}^\varepsilon - \mathcal{N}_0^\varepsilon - \mathcal{N}_1^\varepsilon,\ \text{ and } \eta^\varepsilon := \vec{H}^\varepsilon - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon - \frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon.$$

We have the following proposition:

**Proposition 5.6.2.** *For $\theta \in \{0,1\}$, up to change the $(a_\theta^\varepsilon)$'s and $(\mu_\theta^\varepsilon)$'s in Theorem 5.5.1, rotate and translate the bubbles $r_\theta^\varepsilon\mathcal{B}_\theta^\varepsilon$, we have:*

- *For all $l \in \mathbb{N}$, we have that*

$$\begin{aligned}&\frac{\Phi^\varepsilon(a_0^\varepsilon + \mu_0^\varepsilon\cdot)}{r_0^\varepsilon} - \mathcal{B}_0^\varepsilon(a_0^\varepsilon + \mu_0^\varepsilon\cdot) \to 0 \text{ in } C^l_{loc}(\mathbb{R}^2\backslash\{-a_0\}) \text{ as } \varepsilon \to 0 \text{ and}\\ &Span\{\Phi_x^\varepsilon(a_0^\varepsilon), \Phi_y^\varepsilon(a_0^\varepsilon)\} = Span\{(r_0^\varepsilon\mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon), (r_0^\varepsilon\mathcal{B}_0^\varepsilon)_y(a_0^\varepsilon)\},\\ &\Phi_x^\varepsilon(a_0^\varepsilon) = (r_0^\varepsilon\mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon).\\ &|\vec{n}^\varepsilon - \mathcal{N}_0^\varepsilon|(a_0^\varepsilon) = O\left(\varepsilon^3\right)\\ &\left|\vec{H}^\varepsilon - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon\right|(a_0^\varepsilon) = O(\varepsilon^2).\end{aligned} \tag{5.6.9}$$

  *where $c_0^\varepsilon$ is chosen so that $\Phi^\varepsilon(a_0^\varepsilon) = r_0^\varepsilon\mathcal{B}_0^\varepsilon(a_0^\varepsilon)$.*

- *For all $l \in \mathbb{N}$, we have that*

$$\begin{aligned}&\frac{\Phi^\varepsilon(a_1^\varepsilon + \mu_1^\varepsilon\cdot)}{r_1^\varepsilon} - \mathcal{B}_1^\varepsilon(a_1^\varepsilon + \mu_1^\varepsilon\cdot) \to 0 \text{ in } C^l_{loc}(\mathbb{R}^2) \text{ as } \varepsilon \to 0 \text{ and}\\ &Span\{\Phi_x^\varepsilon(a_1^\varepsilon), \Phi_y^\varepsilon(a_1^\varepsilon)\} = Span\{(r_1^\varepsilon\mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon), (r_1^\varepsilon\mathcal{B}_1^\varepsilon)_y(a_1^\varepsilon)\},\\ &\Phi_x^\varepsilon(a_1^\varepsilon) = (r_1^\varepsilon\mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon)\\ &|\vec{n}^\varepsilon - \mathcal{N}_1^\varepsilon|\,(a_1^\varepsilon) = O(\varepsilon^3(r_1^\varepsilon)^3)\\ &\left|\vec{H}^\varepsilon - \frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon\right|(a_1^\varepsilon) = O\left(\varepsilon^2 r_1^\varepsilon\right)\end{aligned} \tag{5.6.10}$$

  *where $c_1^\varepsilon$ is chosen so that $\Phi^\varepsilon(a_1^\varepsilon) - r_1^\varepsilon\mathcal{B}_1^\varepsilon(a_1^\varepsilon) = O(\varepsilon^2(r_1^\varepsilon)^3)$ and $c_1^\varepsilon = O(1)$. Moreover, we have that*

$$\begin{aligned}&\left|\nabla\left(\frac{\Phi^\varepsilon}{r_1^\varepsilon} - \mathcal{B}_1^\varepsilon\right)(a_1^\varepsilon)\right| = O\left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}\right),\\ &\left|\nabla^2\left(\frac{\Phi^\varepsilon}{r_1^\varepsilon} - \mathcal{B}_1^\varepsilon\right)(\nabla\mathcal{B}_1^\varepsilon)(a_1^\varepsilon)\right| = O\left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{(\mu_1^\varepsilon)^3}\right).\end{aligned} \tag{5.6.11}$$

- *The estimates (5.5.7) in Theorem 5.5.1 and the $L^2$-convergence (5.5.53) still remain true.*

*Proof.* We first prove (5.6.9). Thanks to Theorem 5.5.1 and the adjustment of $r_0^\varepsilon \mathcal{B}_0^\varepsilon$ in (5.6.1), there exists $\delta_0 > 0$ such that, for every $l \in \mathbb{N}$,

$$\frac{\nabla \Phi_0^\varepsilon}{r_0^\varepsilon} - \nabla \mathcal{B}_0^\varepsilon(\mu_0^\varepsilon \cdot + a_0^\varepsilon) \to 0 \text{ in } C^l(B(a_0^\varepsilon, \delta_0)) \text{ as } \varepsilon \to 0 \tag{5.6.12}$$

where $\Phi_0^\varepsilon = \Phi^\varepsilon(\mu_0^\varepsilon \cdot + a_0^\varepsilon)$. Thanks to (5.6.12), there exist $\tilde{\mu}_0^\varepsilon > 0, \tilde{\theta}_0^\varepsilon \in \mathbb{R}$ and $\tilde{R}_0^\varepsilon \in SO(3)$ satisfying

$$\begin{aligned} \tilde{\mu}_0^\varepsilon = \mu_0^\varepsilon(1 + o(1)), \tilde{R}_0^\varepsilon \to \mathrm{Id} \text{ as } \varepsilon \to 0, \\ Span\{(\tilde{\Phi}_0^\varepsilon)_x(0), (\tilde{\Phi}_0^\varepsilon)_y(0)\} = Span\{(\tilde{r}_0^\varepsilon \tilde{R}_0^\varepsilon \mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon), (\tilde{r}_0^\varepsilon \tilde{R}_0^\varepsilon \mathcal{B}_0^\varepsilon)_y(a_0^\varepsilon)\}, \\ (\tilde{\Phi}_0^\varepsilon)_x(0) = (\tilde{r}_0^\varepsilon \tilde{R}_0^\varepsilon \mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon). \end{aligned} \tag{5.6.13}$$

where $\tilde{f}_0^\varepsilon = f^\varepsilon(e^{i\tilde{\theta}_0^\varepsilon}\tilde{\mu}_0^\varepsilon \cdot + a_0^\varepsilon)$ and $\frac{1}{\tilde{r}_0^\varepsilon} := \tilde{H}_0^\varepsilon \cdot \tilde{n}_0^\varepsilon(0)$. Therefore, up to replace $\mathcal{B}_0^\varepsilon$ by $\tilde{R}_0^\varepsilon \mathcal{B}_0^\varepsilon \left(\frac{\mu_0^\varepsilon \cdot}{e^{i\tilde{\theta}_0^\varepsilon}\tilde{\mu}_0^\varepsilon}\right)$, $r_0^\varepsilon$ by $\tilde{r}_0^\varepsilon$ and $\mu_0^\varepsilon$ by $e^{i\tilde{\theta}_0^\varepsilon}\tilde{\mu}_0^\varepsilon$, we get that

$$\begin{aligned} Span\{\Phi_x^\varepsilon(a_0^\varepsilon), \Phi_y^\varepsilon(a_0^\varepsilon)\} = Span\{(r_0^\varepsilon \mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon), (r_0^\varepsilon \mathcal{B}_0^\varepsilon)_y(a_0^\varepsilon)\}, \\ \Phi_x^\varepsilon(a_0^\varepsilon) = (r_0^\varepsilon \mathcal{B}_0^\varepsilon)_x(a_0^\varepsilon). \end{aligned} \tag{5.6.14}$$

Moreover, we choose $c_0^\varepsilon \in \mathbb{R}^3$ such that $\Phi^\varepsilon(a_0^\varepsilon) = r_0^\varepsilon \mathcal{B}_0^\varepsilon(a_0^\varepsilon)$. Since $\vec{n}^\varepsilon = \star_{h^\varepsilon} \frac{\Phi^\varepsilon \wedge \Phi^\varepsilon}{|\Phi^\varepsilon \wedge \Phi^\varepsilon|}$, together with (5.6.1), (5.6.14) and (6.5.1), we get that

$$|\vec{n}^\varepsilon - \mathcal{N}_0^\varepsilon|(a_0^\varepsilon) = O(\varepsilon^3). \tag{5.6.15}$$

Thanks to the choice of $r_0^\varepsilon$, together with (5.6.15) and the fact that $\left|\frac{1}{r_0^\varepsilon}\mathcal{N}_0^\varepsilon(a_0^\varepsilon) - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon(a_0^\varepsilon)\right| = O\left(\varepsilon^2(r_0^\varepsilon)\right) = O(\varepsilon^2)$ given by (5.6.1), we also obtain that

$$\left|\vec{H}^\varepsilon - \frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon\right|(a_0^\varepsilon) = O(\varepsilon^2). \tag{5.6.16}$$

Next, we prove (5.6.10). Thanks to Theorem 5.5.1, there exists $\delta_1 > 0$ such that, for every $l \in \mathbb{N}$,

$$\frac{\Phi_1^\varepsilon}{r_1^\varepsilon} - \hat{\omega}_1 \to 0 \text{ in } C^l(B(0, \delta_1)) \tag{5.6.17}$$

as $\varepsilon \to 0$, where $\Phi_1^\varepsilon = \Phi^\varepsilon(\mu_1^\varepsilon \cdot + a_1^\varepsilon)$. Thanks to (5.6.17) and the fact that

$$|\nabla \hat{\omega}_1|(0) = \max_{\mathbb{R}^2} |\nabla \hat{\omega}_1| \tag{5.6.18}$$

there exists $\tilde{a}_1^\varepsilon \in \mathbb{R}^2$ such that

$$|\tilde{a}_1^\varepsilon - a_1^\varepsilon| = o(\mu_1^\varepsilon) \text{ and } \left|\nabla\left(\frac{\Phi_1^\varepsilon}{r_1^\varepsilon}\right)\right| \text{ attains its local maximum at } \tilde{a}_1^\varepsilon. \tag{5.6.19}$$

In addition, (5.6.17) also implies the existence of $\tilde{\mu}_1^\varepsilon > 0, \tilde{\theta}_1^\varepsilon \in \mathbb{R}$ and $\tilde{R}_1^\varepsilon \in SO(3)$ satisfying

$$\begin{aligned} \tilde{\mu}_1^\varepsilon = \mu_1^\varepsilon(1 + o(1)), \quad \tilde{R}_1^\varepsilon \to \mathrm{Id} \text{ as } \varepsilon \to 0, \\ Span\{(\tilde{\Phi}_1^\varepsilon)_x(0), (\tilde{\Phi}_1^\varepsilon)_y(0)\} = Span\{(r_1^\varepsilon \tilde{R}_1^\varepsilon \omega_1)_x(0), (r_1^\varepsilon \tilde{R}_1^\varepsilon \omega_1)_y(0)\}, \\ (\tilde{\Phi}_1^\varepsilon)_x(0) = (r_1^\varepsilon \tilde{R}_1^\varepsilon \omega_1)_x(0). \end{aligned} \tag{5.6.20}$$

where $\tilde{\Phi}_1^\varepsilon = \Phi^\varepsilon(e^{i\tilde{\theta}_1^\varepsilon}\tilde{\mu}_1^\varepsilon \cdot + \tilde{a}_1^\varepsilon)$.

Thanks to (5.6.18) and (5.6.1), there exists $b_1^\varepsilon = o(1)$ such that $|\nabla(\tilde{R}_1^\varepsilon \tilde{\mathcal{B}}_1^\varepsilon)|$ attains its local maximum at $b_1^\varepsilon$. Moreover, thanks to the adjustment of bubble (5.6.1), we also have that that $\left|\nabla \tilde{R}_1^\varepsilon \left(\tilde{\omega}_1^\varepsilon - \tilde{\mathcal{B}}_1^\varepsilon\right)\right| = O(\varepsilon^2 (r_1^\varepsilon)^2)$ in a neighborhood of $0$, where $\tilde{\mathcal{B}}_1^\varepsilon = \mathcal{B}^\varepsilon(e^{i\tilde{\theta}_1^\varepsilon}\tilde{\mu}_1^\varepsilon \cdot + \tilde{a}_1^\varepsilon)$. Therefore, we get that

$$\begin{aligned} |b_1^\varepsilon| &= O(\varepsilon r_1^\varepsilon) \\ \left|\nabla\left(\frac{\tilde{\Phi}_1^\varepsilon}{r_1^\varepsilon}\right)(0) - \nabla \tilde{R}_1^\varepsilon \tilde{\mathcal{B}}_1^\varepsilon(b_1^\varepsilon)\right| &= O(\varepsilon r_1^\varepsilon), \end{aligned} \tag{5.6.21}$$

which implies that there exist $\hat{\mu}_1^\varepsilon > 0$, $\hat{\theta}_1^\varepsilon \in \mathbb{R}$ and $\hat{R}_1^\varepsilon \in SO(3)$ satisfying

$$\begin{aligned} \hat{\mu}_1^\varepsilon &= \tilde{\mu}_1^\varepsilon(1 + O(\varepsilon r_1^\varepsilon)), \\ |\hat{R}_1^\varepsilon - \mathrm{Id}| &= O(\varepsilon r_1^\varepsilon) \end{aligned} \tag{5.6.22}$$

such that

$$\begin{aligned} Span\{(\hat{\Phi}_1^\varepsilon)_x(0), (\hat{\Phi}_1^\varepsilon)_y(0)\} = Span\{(\hat{r}_1^\varepsilon \hat{R}_1^\varepsilon \tilde{R}_1^\varepsilon \tilde{\mathcal{B}}_1^\varepsilon)_x(b_1^\varepsilon), (\hat{r}_1^\varepsilon \hat{R}_1^\varepsilon \tilde{R}_1^\varepsilon \tilde{\mathcal{B}}_1^\varepsilon)_y(b_1^\varepsilon)\} \text{ and} \\ (\hat{\Phi}_1^\varepsilon)_x(0) = (\hat{r}_1^\varepsilon \hat{R}_1^\varepsilon \tilde{R}_1^\varepsilon \tilde{\mathcal{B}}_1^\varepsilon)_x(b_1^\varepsilon). \end{aligned} \tag{5.6.23}$$

where $\hat{f}_1^\varepsilon = \tilde{f}_1^\varepsilon(e^{i\hat{\theta}_1^\varepsilon}\hat{\mu}_1^\varepsilon \cdot)$ and $\frac{1}{\hat{r}_1^\varepsilon} := \hat{H}_1^\varepsilon \cdot \hat{n}_1^\varepsilon(0)$.

Up to replace $\mathcal{B}_1^\varepsilon$ by $\hat{R}_1^\varepsilon \tilde{R}_1^\varepsilon \tilde{\mathcal{B}}^\varepsilon\left(\frac{\cdot - \tilde{a}_1^\varepsilon}{e^{i\tilde{\theta}_1^\varepsilon}\mu_1^\varepsilon} + b_1^\varepsilon\right)$,$r_1^\varepsilon$ by $\hat{r}_1^\varepsilon$, $\mu_1^\varepsilon$ by $e^{i(\tilde{\theta}_1^\varepsilon + \hat{\theta}_1^\varepsilon)}\tilde{\mu}_1^\varepsilon\hat{\mu}_1^\varepsilon$ and $a_1^\varepsilon$ by $\tilde{a}_1^\varepsilon$, we get that

$$\begin{aligned} Span\left\{(\Phi_1^\varepsilon)_x(a_1^\varepsilon), (\Phi_1^\varepsilon)_y(a_1^\varepsilon)\right\} = Span\left\{(r_1^\varepsilon \mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon), (r_1^\varepsilon \mathcal{B}_1^\varepsilon)_y(a_1^\varepsilon)\right\} \\ (\Phi_1^\varepsilon)_x(a_1^\varepsilon) = (r_1^\varepsilon \mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon) \end{aligned} \tag{5.6.24}$$

Now, we take

$$r_1^\varepsilon c_1^\varepsilon := \Phi^\varepsilon(a_1^\varepsilon) - r_1^\varepsilon \hat{\omega}_1^\varepsilon(a_1^\varepsilon) = O(r_1^\varepsilon). \tag{5.6.25}$$

Thanks to the above adjustments (5.6.22), (5.6.24) and the choice (5.6.25) of $c_1^\varepsilon$ we obtain that

$$|\Phi^\varepsilon(a_1^\varepsilon) - r_1^\varepsilon \mathcal{B}_1^\varepsilon(a_1^\varepsilon)| = O(\varepsilon^2 (r_1^\varepsilon)^3) \tag{5.6.26}$$

and

$$\left|\left(\frac{\Phi^\varepsilon}{r_1^\varepsilon}\right)_y(a_1^\varepsilon) - (\mathcal{B}_1^\varepsilon)_y(a_1^\varepsilon)\right| = O(\varepsilon r_1^\varepsilon). \tag{5.6.27}$$

Using the quasi-conformality (5.6.8) of $r_1^\varepsilon \mathcal{B}_1^\varepsilon$ and the conformality of $\Phi^\varepsilon$ with respect to the metric $h^\varepsilon$, together with (5.6.26) and (5.6.27), we get that

$$\begin{aligned} \left|\nabla\left(\frac{\Phi^\varepsilon}{r_1^\varepsilon} - \mathcal{B}_1^\varepsilon\right)(a_1^\varepsilon)\right| &= O\left(\frac{\varepsilon^3 (r_1^\varepsilon)^3}{\mu_1^\varepsilon}\right), \\ \left|\nabla^2\left(\frac{\Phi^\varepsilon}{r_1^\varepsilon} - \mathcal{B}_1^\varepsilon\right)(\nabla \mathcal{B}_1^\varepsilon)(a_1^\varepsilon)\right| &= O\left(\frac{\varepsilon^3 (r_1^\varepsilon)^3}{(\mu_1^\varepsilon)^3}\right). \end{aligned} \tag{5.6.28}$$

Moreover, thanks to the fact that

$$(\vec{n}^\varepsilon)^k = \frac{(h^\varepsilon)^{ik}\nu_i}{|(h^\varepsilon)^{ij}\nu_i\nu_j|} \text{ at } \Phi^\varepsilon(a_1^\varepsilon) \text{ and } (\mathcal{N}_1^\varepsilon)^k = \frac{(h^\varepsilon)^{ik}\nu_i}{|(h^\varepsilon)^{ij}\nu_i\nu_j|} + O\left(\varepsilon^3 (r_1^\varepsilon)^3\right) \text{ at } r_1^\varepsilon \mathcal{B}_1^\varepsilon(a_1^\varepsilon)$$

with $\nu(a_1^\varepsilon) = \frac{\Phi_x^\varepsilon(a_1^\varepsilon) \wedge \Phi_y^\varepsilon(a_1^\varepsilon)}{|\Phi_x^\varepsilon(a_1^\varepsilon) \wedge \Phi_y^\varepsilon(a_1^\varepsilon)|} = \frac{(r_1^\varepsilon \mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon) \wedge (r_1^\varepsilon \mathcal{B}_1^\varepsilon)_y(a_1^\varepsilon)}{|(r_1^\varepsilon \mathcal{B}_1^\varepsilon)_x(a_1^\varepsilon) \wedge (r_1^\varepsilon \mathcal{B}_1^\varepsilon)_y(a_1^\varepsilon)|}$ thanks to (5.6.24), we get that

$$|\vec{n}^\varepsilon - \mathcal{N}_1^\varepsilon|(a_1^\varepsilon) = O\left(\varepsilon^2 |\Phi^\varepsilon - r_1^\varepsilon \mathcal{B}_1^\varepsilon|(a_1^\varepsilon) + \varepsilon^3 (r_1^\varepsilon)^3\right) = O(\varepsilon^3 (r_1^\varepsilon)^3). \tag{5.6.29}$$

Thanks to the choice of $r_1^\varepsilon$ in (5.6.23), together with (5.6.29) and the fact that $\left|\frac{1}{r_1^\varepsilon}\mathcal{N}_1^\varepsilon - \mathcal{H}_1^\varepsilon\right|(a_1^\varepsilon) = O\left(\varepsilon^2(r_1^\varepsilon)\right)$ given by (5.6.1), we also obtain that

$$\left|\vec{H}^\varepsilon - \frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon\right|(a_1^\varepsilon) = O\left(\varepsilon^2 r_1^\varepsilon\right) \tag{5.6.30}$$

Finally, one can also verify that the estimates (5.5.7) established in Theorem 5.5.1 and the $L^2$-convergence (5.5.53) still hold. □

Thanks to Proposition 5.6.1 and the expansion of metrics, we check that $\varphi^\varepsilon, \nu^\varepsilon$ and $\eta^\varepsilon$ satisfy the following linearized system of (5.3.5):

**Proposition 5.6.3.** *Under the preceding notations, we have*

$$\begin{cases}
\Delta\varphi^\varepsilon = \nabla\nu^\varepsilon \wedge \nabla^\perp\varphi^\varepsilon + \nabla\mathcal{N}_0^\varepsilon \wedge \nabla^\perp\varphi^\varepsilon + \nabla\mathcal{N}_1^\varepsilon \wedge \nabla^\perp\varphi^\varepsilon \\
\qquad + r_0^\varepsilon\left(\nabla\mathcal{N}_1^\varepsilon \wedge \nabla^\perp\mathcal{B}_0^\varepsilon + \nabla\nu^\varepsilon \wedge \nabla^\perp\mathcal{B}_0^\varepsilon\right) \\
\qquad + r_1^\varepsilon\left(\nabla\mathcal{N}_0^\varepsilon \wedge \nabla^\perp\mathcal{B}_1^\varepsilon + \nabla\nu^\varepsilon \wedge \nabla^\perp\mathcal{B}_1^\varepsilon\right) \\
\qquad + R^\varepsilon(\varphi) \\
\Delta\nu^\varepsilon = -\nabla\nu^\varepsilon \wedge \nabla^\perp\nu^\varepsilon - 2\left(\nabla^\perp\mathcal{N}_0^\varepsilon \wedge \nabla\nu^\varepsilon + \nabla^\perp\mathcal{N}_1^\varepsilon \wedge \nabla\nu^\varepsilon + \nabla^\perp\mathcal{N}_0^\varepsilon \wedge \nabla\mathcal{N}_1^\varepsilon\right) \\
\qquad - 2\nabla\eta^\varepsilon \wedge \nabla^\perp\varphi^\varepsilon \\
\qquad - 2\nabla\eta^\varepsilon \wedge \left(r_0^\varepsilon\nabla^\perp\mathcal{B}_0^\varepsilon + r_1^\varepsilon\nabla^\perp\mathcal{B}_1^\varepsilon\right) \\
\qquad - 2\left(\frac{1}{r_0^\varepsilon}\nabla\mathcal{H}_0^\varepsilon + \frac{1}{r_1^\varepsilon}\nabla\mathcal{H}_1^\varepsilon\right) \wedge \nabla^\perp\varphi^\varepsilon \\
\qquad - 2\frac{r_1^\varepsilon}{r_0^\varepsilon}\nabla\mathcal{H}_0^\varepsilon \wedge \nabla^\perp\mathcal{B}_1^\varepsilon - 2\frac{r_0^\varepsilon}{r_1^\varepsilon}\nabla\mathcal{H}_1^\varepsilon \wedge \nabla^\perp\mathcal{B}_0^\varepsilon \\
\qquad + R^\varepsilon(\nu), \\
div\left(\nabla\eta^\varepsilon - 3\left(\nabla\eta^\varepsilon \cdot_{h^\varepsilon} \vec{n}^\varepsilon\right)\vec{n}\right) = -\nabla^\perp\nu^\varepsilon \wedge \nabla\eta^\varepsilon - \nabla^\perp\mathcal{N}_0^\varepsilon \wedge \nabla\eta^\varepsilon - \nabla^\perp\mathcal{N}_1^\varepsilon \wedge \nabla\eta^\varepsilon \\
\qquad - \frac{1}{r_0^\varepsilon}\nabla^\perp(\nu^\varepsilon + \mathcal{N}_1^\varepsilon) \wedge \nabla\mathcal{H}_0^\varepsilon - \frac{1}{r_1^\varepsilon}\nabla^\perp(\nu^\varepsilon + \mathcal{N}_0^\varepsilon) \wedge \nabla\mathcal{H}_1^\varepsilon \\
\qquad + 3\left(div\left(\left(\nabla\frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon \cdot_{h_0^\varepsilon}(\vec{n} - \mathcal{N}_0^\varepsilon)\right)\vec{n}\right) + div\left(\left(\nabla\frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon \cdot_{h_0^\varepsilon}\mathcal{N}_0^\varepsilon\right)(\vec{n} - \mathcal{N}_0^\varepsilon)\right)\right) \\
\qquad + 3\left(div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon \cdot_{h_1^\varepsilon}(\vec{n} - \mathcal{N}_1^\varepsilon)\right)\vec{n}\right) + div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon \cdot_{h_1^\varepsilon}\mathcal{N}_1^\varepsilon\right)(\vec{n} - \mathcal{N}_1^\varepsilon)\right)\right) \\
\qquad + R^\varepsilon(\eta),
\end{cases} \tag{5.6.31}$$

*where, thanks to Lemma 6.5.1, the remainders $R^\varepsilon(\varphi), R^\varepsilon(\nu), R^\varepsilon(\eta)$ verify:*

$$
\begin{aligned}
R^\varepsilon(\varphi)_k :=& -(\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)\left((\nabla\Phi^\varepsilon)^i(\nabla\Phi^\varepsilon)^j - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^i(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j\right)\\
&+ B^\varepsilon_{ik}(\Phi^\varepsilon)\left((\nabla\vec{n}^\varepsilon)\wedge(\nabla^\perp\Phi^\varepsilon) - \sum_{\theta=0,1}(\nabla\mathcal{N}^\varepsilon_\theta)\wedge(\nabla^\perp r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)\right)_i\\
&+ \vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^m\left((\nabla\Phi^\varepsilon)^n\wedge\nabla^\perp\Phi^\varepsilon - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^n\wedge\nabla^\perp r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta\right)\\
&+ O\left(\varepsilon^2\sum_{\theta=0,1} r^\varepsilon_\theta|\Phi^\varepsilon - r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
&+ O\left(\varepsilon^3(|\Phi^\varepsilon - r^\varepsilon_1\mathcal{B}^\varepsilon_1|^2|\nabla\Phi^\varepsilon|^2 + |\Phi^\varepsilon - r^\varepsilon_1\mathcal{B}^\varepsilon_1|^3|\nabla\vec{n}^\varepsilon||\nabla\Phi^\varepsilon|)\right) + O\left(\varepsilon^3(r^\varepsilon_1)^4|\nabla\hat{\omega}^\varepsilon_1|^2\right)
\end{aligned}
\tag{5.6.32}
$$

$$
\begin{aligned}
R^\varepsilon(\nu)_k :=& -2(\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)\left((\nabla\vec{n}^\varepsilon)^i(\nabla\Phi^\varepsilon)^j - \sum_{\theta=0,1}(\nabla\mathcal{N}^\varepsilon_\theta)^i(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j\right)\\
&- (\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)(\vec{n}^\varepsilon)^i\left((\Delta\Phi^\varepsilon)^j - \sum_{\theta=0,1}(\Delta r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j\right)\\
&- \partial_l(\Gamma^\varepsilon)^k_{ij}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^i\left((\nabla\Phi^\varepsilon)^j(\nabla\Phi^\varepsilon)^l - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^l\right)\\
&- B^\varepsilon_{ik}(\Phi^\varepsilon)\left((\nabla\vec{n}^\varepsilon)\wedge(\nabla^\perp\vec{n}^\varepsilon) - \sum_{\theta=0,1}(\nabla\mathcal{N}^\varepsilon_\theta)\wedge(\nabla^\perp\mathcal{N}^\varepsilon_\theta)\right)_i\\
&- 2B^\varepsilon_{ik}(\Phi^\varepsilon)\left((\nabla\vec{H}^\varepsilon)\wedge(\nabla^\perp\Phi^\varepsilon) - \sum_{\theta=0,1}(\nabla\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta)\wedge(\nabla^\perp r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)\right)_i\\
&- 2\vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^m\left((\nabla\vec{n}^\varepsilon)^n\wedge\nabla^\perp\Phi^\varepsilon - \sum_{\theta=0,1}(\nabla\mathcal{N}^\varepsilon_\theta)^n\wedge\nabla^\perp r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta\right)\\
&- 2\vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{H}^\varepsilon)^m\left((\nabla\Phi^\varepsilon)^n\wedge\nabla^\perp\Phi^\varepsilon - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^n\wedge\nabla^\perp r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta\right)\\
&+ O\left(\varepsilon^2\sum_{\theta=0,1}|\Phi^\varepsilon - r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
&+ O\left(\varepsilon^2\sum_{\theta=0,1} r^\varepsilon_\theta|\vec{n}^\varepsilon - \mathcal{N}^\varepsilon_\theta||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
&+ O\left(\varepsilon^2\sum_{\theta=0,1}\left|\vec{H}^\varepsilon - \frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\right||\nabla(r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)|^2\right)\\
&+ O\left(\varepsilon^3|\Phi^\varepsilon - r^\varepsilon_1\mathcal{B}^\varepsilon_1|^3(|\nabla\vec{n}^\varepsilon|^2 + |\nabla\vec{H}^\varepsilon||\nabla\Phi^\varepsilon|)\right) + O\left(\varepsilon^3(r^\varepsilon_1)^3|\nabla\hat{\omega}^\varepsilon_1|^2\right)\ \textit{and}
\end{aligned}
\tag{5.6.33}
$$

$$
\begin{aligned}
R^\varepsilon(\eta)_k := & -2(\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)\left((\nabla\vec{H}^\varepsilon)^i(\nabla\Phi^\varepsilon)^j - \sum_{\theta=0,1}(\nabla\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta)^i(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j\right)\\
& - (\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)\left((\vec{H}^\varepsilon)^i(\Delta\Phi^\varepsilon)^j - \sum_{\theta=0,1}(\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta)^i(\Delta r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j\right)\\
& - \partial_l(\Gamma^k_{ij})^\varepsilon(\Phi^\varepsilon)(\vec{H}^\varepsilon)^i\left((\nabla\Phi^\varepsilon)^j(\nabla\Phi^\varepsilon)^l - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^l\right)\\
& + B^\varepsilon_{ik}(\Phi^\varepsilon)\left((\nabla\vec{H}^\varepsilon)\wedge(\nabla^\perp\vec{n}^\varepsilon) - \sum_{\theta=0,1}(\nabla\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta)\wedge(\nabla^\perp\mathcal{N}^\varepsilon_\theta)\right)_i\\
& + \vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{H}^\varepsilon)^m\left((\nabla\Phi^\varepsilon)^n\wedge\nabla^\perp\vec{n}^\varepsilon - \sum_{\theta=0,1}(\nabla r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^n\wedge\nabla^\perp\mathcal{N}^\varepsilon_\theta\right)\\
& + \nabla\vec{H}^\varepsilon\wedge\vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^m(\nabla^\perp\vec{n}^\varepsilon)^n - \sum_{\theta=0,1}\nabla\left(\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\right)\wedge\vec{\Gamma}^\varepsilon_{mn}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^m(\nabla^\perp\mathcal{N}^\varepsilon_\theta)^n\\
& + \sum_{\theta=0,1}3div\left((h^\varepsilon(\Phi^\varepsilon)-h^\varepsilon(r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta))_{ij}\left(\nabla\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\right)^i(\vec{n}^\varepsilon)^j(\vec{n}^\varepsilon)\right)\\
& + 3div\left((h^\varepsilon)_{\alpha\beta}(\Phi^\varepsilon)\vec{\Gamma}^\alpha_{ij}(\Phi^\varepsilon)\left((\vec{H})^i(\nabla\Phi^\varepsilon)^j(\vec{n})^\beta\vec{n} - \left(\sum_{\theta=0,1}\left(\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\right)^i\nabla(r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta)^j(\mathcal{N}^\varepsilon_\theta)^\beta\mathcal{N}^\varepsilon_\theta\right)\right)\right)\\
& + 3\nabla(\vec{H}^\varepsilon\cdot\vec{n}^\varepsilon)\bullet(\Gamma^\varepsilon)^k_{ij}(\Phi^\varepsilon)(\vec{n}^\varepsilon)^i(\nabla\Phi^\varepsilon)^j - 3\sum_{\theta=0,1}\nabla(\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\cdot\mathcal{N}^\varepsilon_\theta)\bullet(\Gamma^\varepsilon)^k_{ij}(\Phi^\varepsilon)(\mathcal{N}^\varepsilon_\theta)^i(\nabla(r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta))^j\\
& + O\left(\varepsilon^2\sum_{\theta=0,1}\frac{1}{r^\varepsilon_\theta}|\Phi^\varepsilon - r^\varepsilon_\theta\mathcal{B}^\varepsilon_\theta||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
& + O\left(\varepsilon^2\sum_{\theta=0,1}|\vec{n}^\varepsilon-\mathcal{N}^\varepsilon_\theta||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
& + O\left(\varepsilon^2\sum_{\theta=0,1}r^\varepsilon_\theta\left|\vec{H}^\varepsilon-\frac{1}{r^\varepsilon_\theta}\mathcal{H}^\varepsilon_\theta\right||\nabla\hat{\omega}^\varepsilon_\theta|^2\right)\\
& + O\left(\varepsilon^3|\Phi^\varepsilon - r^\varepsilon_1\mathcal{B}^\varepsilon_1|^3\left(|\nabla\vec{H}^\varepsilon||\nabla\vec{n}^\varepsilon|\right) + \varepsilon^3|\Phi^\varepsilon - r^\varepsilon_1\mathcal{B}^\varepsilon_1|^2|\nabla\vec{H}^\varepsilon||\nabla\Phi^\varepsilon|\right) + O\left(\varepsilon^3(r^\varepsilon_1)^2|\nabla\hat{\omega}^\varepsilon_1|^2\right).
\end{aligned}
\tag{5.6.34}
$$

*where* $B^\varepsilon_{ik}(\Phi^\varepsilon) = \left(\sqrt{|h^\varepsilon|}h^\varepsilon - \delta\right)^{ik}$.

In addition, combining (5.6.8) and the conformality of $\Phi^\varepsilon$, we have that

**Proposition 5.6.4.** *Under the preceding notations, $\varphi^\varepsilon$ satisfies the relation of quasi-conformality: for $\theta = 0, 1$,*

*at $r_\theta \mathcal{B}^\varepsilon_\theta$, we have that*

$$\begin{aligned}
\langle \varphi^\varepsilon_x, \varphi^\varepsilon_y\rangle_+ &\left(\langle (r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta)_x, \varphi^\varepsilon_y\rangle + \langle \varphi^\varepsilon_x, (r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta)_y\rangle\right)\\
&= O\left(r^\varepsilon_\theta r^\varepsilon_{1-\theta}\langle (\mathcal{B}^\varepsilon_\theta)_x, (\mathcal{B}^\varepsilon_{1-\theta})_y\rangle\right) + O\left(\varepsilon^2 (r^\varepsilon_\theta)^2 |\Phi^\varepsilon - r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta||\nabla \mathcal{B}^\varepsilon_\theta|^2\right)\\
&\quad + O\left(\varepsilon^2 r^\varepsilon_{1-\theta}|\nabla \mathcal{B}^\varepsilon_{1-\theta}|^2\right) + O\left(\varepsilon^3(|\Phi^\varepsilon - r^\varepsilon_\theta c^\varepsilon_\theta|^2|\nabla\Phi^\varepsilon|^2 + |\Phi^\varepsilon - r^\varepsilon_\theta c^\varepsilon_\theta|^3(|\nabla \vec{n}^\varepsilon||\nabla\Phi^\varepsilon|)\right)\\
&\quad + O\left(\varepsilon^3 (r^\varepsilon_\theta)^5 |\nabla \hat{\omega}^\varepsilon_\theta|^2\right) \text{ and}\\
\langle \varphi^\varepsilon_x, \varphi^\varepsilon_x\rangle - \langle \varphi^\varepsilon_y, \varphi^\varepsilon_y\rangle + &\left(\langle (r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta)_x, \varphi^\varepsilon_x\rangle - \langle (r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta)_y, \varphi^\varepsilon_y\rangle\right)\\
&= O\left(r^\varepsilon_\theta r^\varepsilon_{1-\theta}\langle (\mathcal{B}^\varepsilon_\theta)_x, (\mathcal{B}^\varepsilon_{1-\theta})_x\rangle_{h_\varepsilon} - \langle (\mathcal{B}^\varepsilon_\theta)_y, (\mathcal{B}^\varepsilon_{1-\theta})_y\rangle_{h_\varepsilon}\right)\\
&\quad + O\left(\varepsilon^2 (r^\varepsilon_\theta)^2 |\Phi^\varepsilon - r^\varepsilon_\theta \mathcal{B}^\varepsilon_\theta||\nabla \mathcal{B}^\varepsilon_\theta|^2\right)\\
&\quad + O\left(\varepsilon^2 r^\varepsilon_{1-\theta}|\nabla \mathcal{B}^\varepsilon_{1-\theta}|^2\right) + O\left(\varepsilon^3(|\Phi^\varepsilon - r^\varepsilon_\theta c^\varepsilon_\theta|^2|\nabla\Phi^\varepsilon|^2 + |\Phi^\varepsilon - r^\varepsilon_\theta c^\varepsilon_\theta|^3(|\nabla \vec{n}^\varepsilon||\nabla\Phi^\varepsilon|)\right)\\
&\quad + O\left(\varepsilon^3 (r^\varepsilon_\theta)^5 |\nabla \hat{\omega}^\varepsilon_\theta|^2\right)
\end{aligned} \tag{5.6.35}$$

## 5.7 Strong pointwise estimates

Throughout the rest of this section, we show how to use the system of equations with nonlinearities of Wente type to obtain strong pointwise estimates on our sequence, starting from the weak ones obtained previously. We shall also show how to deduce that the Willmore surfaces concentrate at a critical point of the scalar curvature once these estimates are obtained.

The aim would be to obtain these estimates with a sum of an arbitrary number of bubbles (spheres and catenoids) and one would have to adapt the arguments below to catenoids. However, even if we know that "two spheres imply a catenoid", we present the estimates with two spheres to show how to deal with interaction terms in the estimate. At last, we shall assume that the two bubbles $r^\varepsilon_0 \mathcal{B}^\varepsilon_0$ and $r^\varepsilon_1 \mathcal{B}^\varepsilon_1$ do not degenerate, that is,

$$\liminf_{\varepsilon\to 0} r^\varepsilon_0 > 0 \text{ and } \liminf_{\varepsilon\to 0} r^\varepsilon_1 > 0.$$

This last assumption is useful only for the conclusion, not for the estimates. If this is not the case, we believe (and heuristics tell so) that the interaction with the catenoid would be the leading term.

### 5.7.1 Strong pointwise estimates

Set

$$\begin{aligned}
\rho^\varepsilon_\varphi &:= \sup_{x\in\mathbb{R}^2} \left(1+|x|^2\right)|\nabla\varphi^\varepsilon(x)|,\\
\rho^\varepsilon_\nu &:= \sup_{x\in\mathbb{R}^2} \left(1+|x|^2\right)|\nabla\nu^\varepsilon(x)|,\\
\rho^\varepsilon_\eta &:= \sup_{x\in\mathbb{R}^2} \left(1+|x|^2\right)|\nabla\eta^\varepsilon(x)|,\\
\rho^\varepsilon &:= \max\{\rho^\varepsilon_\varphi, \rho^\varepsilon_\nu, \rho^\varepsilon_\eta\} \text{ and}
\end{aligned} \tag{5.7.1}$$

$$\begin{aligned}
\rho^\varepsilon_{\varphi,i,R} &:= \sup_{x\in\Omega^\varepsilon_i(R)} \left(1+|x|^2\right)|\nabla\varphi^\varepsilon(x)|,\\
\rho^\varepsilon_{\nu,i,R} &:= \sup_{x\in\Omega^\varepsilon_i(R)} \left(1+|x|^2\right)|\nabla\nu^\varepsilon(x)|,\\
\rho^\varepsilon_{\eta,i,R} &:= \sup_{x\in\Omega^\varepsilon_i(R)} \left(1+|x|^2\right)|\nabla\eta^\varepsilon(x)|\\
\rho^\varepsilon_{i,R} &:= \max\{\rho^\varepsilon_{\varphi,i,R}, \rho^\varepsilon_{\nu,i,R}, \rho^\varepsilon_{\eta,i,R}\}
\end{aligned} \tag{5.7.2}$$

for $R>0$ and $i=0,1$, where

$$\Omega_0^\varepsilon(R):=\mathbb{D}_{a_0^\varepsilon}(R\mu_0^\varepsilon)\backslash\mathbb{D}_{a_1^\varepsilon}\left(\frac{\mu_0^\varepsilon}{R}\right)\text{ and }\Omega_1^\varepsilon(R):=\mathbb{D}_{a_1^\varepsilon}(R\mu_1^\varepsilon)\backslash\mathbb{D}_{a_0^\varepsilon}\left(\frac{\mu_1^\varepsilon}{R}\right).$$

With the help of (5.6.31) in Proposition 5.6.3, we are now able to enhance the pointwise estimates (5.5.7) established in Theorem 5.5.1.

In order to obtain precise pointwise estimates, we will adapt a "bootstrap" approach which consists of studying successively the equations of the system (5.6.31) and getting estimates with the help of previous ones.

**Claim 5.7.1.** *For every $R>0$ and $x\in\mathbb{D}_0(R)$, we have that*

$$\begin{aligned}(1+|x|)|\nabla\varphi^\varepsilon|(x)\lesssim&\left(\rho^\varepsilon_{\varphi,0,R}+r_0^\varepsilon\rho^\varepsilon_{\nu,0,R}+\varepsilon^3+R^{-1}\left(\rho^\varepsilon_\varphi+r_0^\varepsilon\rho^\varepsilon_\nu\right)\right)(1+|x|)^{-1}\\&+\left(\left(\rho^\varepsilon_{\varphi,1,R}+r_1^\varepsilon\rho^\varepsilon_{\nu,1,R}\right)\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\\&+\left(\left(\rho^\varepsilon_\varphi+r_1^\varepsilon\rho^\varepsilon_\nu\right)\right)\left(R+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\\&+\left(\frac{\varepsilon^3(r_1^\varepsilon)^4}{\mu_1^\varepsilon}+1\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\\&+\sqrt{\|\nabla\nu^\varepsilon\|_2\|\nabla\varphi^\varepsilon\|_2\rho^\varepsilon_\varphi\rho^\varepsilon_\nu}\,(1+|x|)^{-1}\end{aligned}\tag{5.7.3}$$

*Proof of Claim 5.7.1:* Let us write with Proposition 5.6.3 and the Green's representation formula in Lemma 6.2.2 that

$$\begin{aligned}(1+|x|)\,|\nabla\varphi^\varepsilon(x)|\lesssim&(1+|x|)\,|\nabla\psi^\varepsilon_{\varphi,\nu}(x)|\\&+\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_0^\varepsilon(y)|\left(|\nabla\varphi^\varepsilon|+r_0^\varepsilon|\nabla\nu^\varepsilon|\right)(y)\,dy\\&+\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_1^\varepsilon(y)|\left(|\nabla\varphi^\varepsilon|+r_1^\varepsilon|\nabla\nu^\varepsilon|\right)(y)\,dy\\&+\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_0^\varepsilon(y)||\nabla\mathcal{B}_1^\varepsilon(y)|\,dy\\&+\varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(|\nabla\varphi^\varepsilon(y)|^2+|\nabla\nu^\varepsilon(y)||\nabla\varphi^\varepsilon(y)|\right)\,dy\\&+\varepsilon^2\sum_{i=0,1}r_i^\varepsilon\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_i^\varepsilon(y)|^2|\Phi^\varepsilon-r_i^\varepsilon\mathcal{B}_i^\varepsilon|(y)dy\\&+\varepsilon^3\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(|\nabla\mathcal{B}_0^\varepsilon|^2+(r_1^\varepsilon)^4|\nabla\mathcal{B}_1^\varepsilon|^2\right)(y)\,dy.\end{aligned}\tag{5.7.4}$$

**Claim 5.7.2.** *For every $x\in\mathbb{R}^2$, we have that*

$$(1+|x|^2)^2|\nabla\psi^\varepsilon_{\varphi,\nu}(x)|^2\lesssim\|\nabla\nu^\varepsilon\|_{L^2(\mathbb{R}^2)}\|\nabla\varphi^\varepsilon\|_{L^2(\mathbb{R}^2)}\rho^\varepsilon_\varphi\rho^\varepsilon_\nu.\tag{5.7.5}$$

*In particular, thanks to (5.5.53), we get that*

$$(1+|x|)\,|\nabla\psi^\varepsilon_{\varphi,\nu}|=o\left(\left(\rho^\varepsilon_\varphi+\rho^\varepsilon_\nu\right)(1+|x|)^{-1}\right).\tag{5.7.6}$$

*Proof of Claim 5.7.2:* With the help of stereographic coordinates $\pi : \mathbb{R}^2 \to \mathbb{S}^2$ and thanks to conformal invariance of the equation and the energy, $\psi^\varepsilon_{\varphi,\nu} \circ \pi^{-1}$, $\varphi^\varepsilon \circ \pi^{-1}$ and $\nu^\varepsilon \circ \pi^{-1}$ are maps defined on $(\mathbb{S}^2, g_0)$ satisfying

$$\Delta_{g_0}(\psi^\varepsilon_{\varphi,\nu} \circ \pi^{-1}) = \nabla_{g_0}(\varphi^\varepsilon \circ \pi^{-1}) \wedge \nabla_{g_0}(\nu^\varepsilon \circ \pi^{-1}) \text{ on } \mathbb{S}^2.$$

Thanks to Lemma 6.3.8 and , we have

$$\begin{aligned}\|\nabla_{g_0}(\psi^\varepsilon_{\varphi,\nu} \circ \pi^{-1})\|^2_\infty &\lesssim \|\nabla_{g_0}(\varphi^\varepsilon \circ \pi^{-1})\|_\infty \|\nabla_{g_0}(\nu^\varepsilon \circ \pi^{-1})\|_\infty \|osc(\psi^\varepsilon_{\varphi,\nu} \circ \pi^{-1})\|_\infty \\ &\lesssim \|\nabla_{g_0}(\varphi^\varepsilon \circ \pi^{-1})\|_\infty \|\nabla_{g_0}(\nu^\varepsilon \circ \pi^{-1})\|_\infty \|\nabla_{g_0}(\varphi^\varepsilon \circ \pi^{-1})\|_2 \|\nabla_{g_0}(\nu^\varepsilon \circ \pi^{-1})\|_2 \\ &\lesssim \|\nabla_{g_0}(\varphi^\varepsilon \circ \pi^{-1})\|_\infty \|\nabla_{g_0}(\nu^\varepsilon \circ \pi^{-1})\|_\infty \|\nabla\varphi^\varepsilon\|_2 \|\nabla\nu^\varepsilon\|_2.\end{aligned} \tag{5.7.7}$$

Moreover, since for every $f \in C^\infty(\mathbb{S}^2)$ and $\forall x \in \mathbb{R}^2$, we have

$$\|\nabla_{g_0} f(\pi(x))\|_{g_0(\pi(x))} = e^{-\lambda(x)} \|\nabla(f \circ \pi)(x)\|_{euc}$$

where $(\pi^*)(g_0) = e^{2\lambda} h_{euc}$ with $e^\lambda = \frac{2}{1+|x|^2}$. We get that, for every $x \in \mathbb{R}^2$

$$(1+|x|^2)^2 |\nabla\psi^\varepsilon_{\varphi,\nu}(x)|^2 \lesssim \|\nabla\varphi^\varepsilon\|_2 \|\nabla\nu^\varepsilon\|_2 \rho^\varepsilon_\varphi \rho^\varepsilon_\nu.$$

In particular, thanks to (5.5.53), $\nabla\varphi^\varepsilon \to 0$ and $\nabla\nu^\varepsilon \to 0$ in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0$, we obtain that

$$(1+|x|)^{-1} |\nabla\psi^\varepsilon_{\varphi,\nu}(x)| \lesssim \sqrt{\|\nabla\varphi^\varepsilon\|_2 \|\nabla\nu^\varepsilon\|_2 \rho^\varepsilon_\varphi \rho^\varepsilon_\nu}\, (1+|x|)^{-1} = o\left((\rho^\varepsilon_\varphi + \rho^\varepsilon_\nu)(1+|x|)^{-1}\right). \tag{5.7.8}$$

□

Thanks to the definitions (5.7.1) of $\rho^\varepsilon$ and (5.7.2) of $\rho^\varepsilon_{i,R}$, we can also write that, for $i = 0, 1$

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\mathcal{B}^\varepsilon_i(y)| \left(|\nabla\varphi^\varepsilon| + r^\varepsilon_i |\nabla\nu^\varepsilon|\right)(y)\, dy \\ &\lesssim (\rho^\varepsilon_{\varphi,i,R} + r^\varepsilon_i \rho_{\nu,i,R}) \int_{\Omega^\varepsilon_{i,R}} \frac{1}{|x-y|(1+|y|)} |\nabla\mathcal{B}^\varepsilon_i(y)|\, dy + (\rho^\varepsilon_\varphi + r^\varepsilon_i \rho^\varepsilon_\nu) \int_{\mathbb{R}^2 \setminus \Omega^\varepsilon_{i,R}} \frac{1}{|x-y|(1+|y|)} |\nabla\mathcal{B}^\varepsilon_i(y)| dy \\ &\lesssim \rho^\varepsilon_{i,R} \int_{\mathbb{R}^2} \frac{1}{|x-y|(1+|y|)} \frac{\mu^\varepsilon_i}{(\mu^\varepsilon_i)^2 + |y-a^\varepsilon_i|^2}\, dy + \rho^\varepsilon \int_{\mathbb{R}^2 \setminus \Omega^\varepsilon_{i,R}} \frac{1}{|x-y|(1+|y|)} \frac{\mu^\varepsilon_i}{(\mu^\varepsilon_i)^2 + |y-a^\varepsilon_i|^2}\, dy.\end{aligned}$$

This leads by direct computations to

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\mathcal{B}^\varepsilon_0(y)| \left(|\nabla\varphi^\varepsilon| + r^\varepsilon_0 |\nabla\nu^\varepsilon|\right)(y) dy \lesssim \left(\rho^\varepsilon_{0,R} + R^{-1}\rho^\varepsilon\right)(1+|x|)^{-1} \text{ and} \\ &\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\mathcal{B}^\varepsilon_1(y)| \left(|\nabla\varphi^\varepsilon| + r^\varepsilon_1 |\nabla\nu^\varepsilon|\right)(y) dy \\ &\lesssim \rho^\varepsilon_{1,R} \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right) + \rho^\varepsilon \left(R + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right).\end{aligned} \tag{5.7.9}$$

In order to estimate the term in $|\nabla\mathcal{B}^\varepsilon_0||\nabla\mathcal{B}^\varepsilon_1|$, we can write that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\mathcal{B}^\varepsilon_0(y)||\nabla\mathcal{B}^\varepsilon_1(y)| dy &\lesssim \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} \frac{\mu^\varepsilon_0}{(\mu^\varepsilon_0)^2 + |y-a^\varepsilon_0|^2} \frac{\mu^\varepsilon_1}{(\mu^\varepsilon_1)^2 + |y-a^\varepsilon_1|^2} dy \\ &\lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{1+|y|} \frac{\mu^\varepsilon_1}{(\mu^\varepsilon_1)^2 + |y-a^\varepsilon_1|^2} dy \\ &\lesssim \int_{\mathbb{R}^2} \frac{1}{\left|\frac{x-a^\varepsilon_1}{\mu^\varepsilon_1} - y\right|} \frac{1}{1+|\mu^\varepsilon_1 y + a^\varepsilon_1|} \frac{1}{1+|y|^2} dy \\ &\lesssim \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right).\end{aligned} \tag{5.7.10}$$

**Claim 5.7.3.** *For every $1 < \beta < 2$, we have*

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y)\, dy &= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{2-\beta} (1+|x|)^{-1}\right) \\ &\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{2-\beta} (r_1^\varepsilon + \mu_1^\varepsilon)^\beta (\mu_1^\varepsilon + |x - a_1^\varepsilon|)^{1-\beta} (1+|x|)^{\beta-2}\right).\end{aligned} \tag{5.7.11}$$

*In particular, by taking $\beta = \frac{4}{3}$, we get that*

$$\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y)\, dy = o\left((\varepsilon^3 + \rho_\varphi^\varepsilon)(1+|x|)^{-1}\right) + o\left(\frac{\varepsilon^3 (r_1^\varepsilon)^4}{\mu_1^\varepsilon} \left(1 + \frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right). \tag{5.7.12}$$

*Proof of Claim 5.7.3:* For $1 < \beta < 2$, we can write that

$$\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y)\, dy \le (\rho_\varphi^\varepsilon)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|\,(1+|y|)^{3-2\beta}} |\nabla \varphi^\varepsilon|^\beta(y)\, dy.$$

Thanks to the pointwise estimates (5.5.33) in Proposition 5.5.3, we have, for every $0 < \alpha < 1$,

$$|\nabla \varphi^\varepsilon|(y) = o\left(\frac{1}{(\mu_1^\varepsilon + |y - a_1^\varepsilon|)^\alpha (1+|y|)^{2-\alpha}} + \frac{r_1^\varepsilon + \mu_1^\varepsilon}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)(1+|y|)}\right),$$

so that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y) dy &= o\left((\rho_\varphi^\varepsilon)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{\alpha\beta} (1+|y|)^{3-\alpha\beta}}\, dy\right) \\ &\quad + o\left((\rho_\varphi^\varepsilon)^{2-\beta} (r_1^\varepsilon + \mu_1^\varepsilon)^\beta \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^\beta (1+|y|)^{3-\beta}}\, dy\right)\end{aligned} \tag{5.7.13}$$

Taking $0 < \alpha < 1$ such that $\alpha\beta < 1$, we can write that

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y)\, dy &= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{2-\beta} (1+|x|)^{-1}\right) \\ &\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{2-\beta} (r_1^\varepsilon + \mu_1^\varepsilon)^\beta (\mu_1^\varepsilon + |x - a_1^\varepsilon|)^{1-\beta} (1+|x|)^{\beta-2}\right).\end{aligned} \tag{5.7.14}$$

Take $\beta = \frac{4}{3}$. Thanks to Young's inequality, we have that

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon|^2(y)\, dy &= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{\frac{2}{3}} (1+|x|)^{-1}\right) \\ &\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{\frac{2}{3}} (r_1^\varepsilon + \mu_1^\varepsilon)^{\frac{4}{3}} (\mu_1^\varepsilon + |x-a_1^\varepsilon|)^{-\frac{1}{3}} (1+|x|)^{-\frac{2}{3}}\right). \\ &= o\left(\varepsilon^3 (1+|x|)^{-1}\right) + o\left(\varepsilon^{\frac{3}{2}} \rho_\varphi^\varepsilon (1+|x|)^{-1}\right) \\ &\quad + o\left(\varepsilon^3 (r_1^\varepsilon + \mu_1^\varepsilon)^4 (\mu_1^\varepsilon + |x-a_1^\varepsilon|)^{-1}\right) \\ &= o\left((\varepsilon^3 + \rho_\varphi^\varepsilon)(1+|x|)^{-1}\right) + o\left(\frac{\varepsilon^3 (r_1^\varepsilon)^4}{\mu_1^\varepsilon} \left(1 + \frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right).\end{aligned} \tag{5.7.15}$$

□

**Claim 5.7.4.** *For every $0 \leq \beta < 1$ and $0 \leq \gamma < 1$ such that $\beta + \gamma > 1$, we have that*

$$\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon || \nabla \nu^\varepsilon(y)| \, dy = o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} \left(1 + |x|^{-1}\right)\right) + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} \left(r_1^\varepsilon + \mu_1^\varepsilon\right)^\gamma \left(\mu_1^\varepsilon + |x - a_1^\varepsilon|\right)^{1-\beta-\gamma} (1+|x|)^{\beta+\gamma-2}\right). \tag{5.7.16}$$

*In particular, by taking $\beta = \frac{1}{2}$ and $\gamma = \frac{2}{3}$, we get that*

$$\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon || \nabla \nu^\varepsilon(y)| \, dy = o\left(\left(\varepsilon^3 + \rho_\varphi^\varepsilon + \rho_\nu^\varepsilon\right)(1+|x|)^{-1}\right) + o\left(\frac{\varepsilon^3 (r_1^\varepsilon)^4}{\mu_1^\varepsilon} \left(1 + \frac{|x - a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right). \tag{5.7.17}$$

*Proof of Claim 5.7.4:* Let $0 \leq \beta < 1$ and $0 \leq \gamma < 1$ such that $\beta + \gamma > 1$. Thanks to (5.5.7) in Theorem 5.5.1 and (5.5.33) in Proposition 5.5.3, proceeding as in Claim 5.7.3, we have that, for every $0 < \alpha < 1$ such that $\beta + \alpha\gamma < 1$,

$$\begin{aligned}
\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon || \nabla \nu^\varepsilon(y)| \, dy &= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{\left(\mu_1^\varepsilon + |y - a_1^\varepsilon|\right)^{\alpha\gamma+\beta} (1+|y|)^{3-(\alpha\gamma+\beta)}} \, dy\right) \\
&\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} (r_1^\varepsilon + \mu_1^\varepsilon)^\gamma \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{\left(\mu_1^\varepsilon + |y - a_1^\varepsilon|\right)^{\beta+\gamma} (1+|y|)^{3-(\beta+\gamma)}} \, dy\right) \\
&= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} (1+|x|)^{-1}\right) \\
&\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{1-\gamma} (\rho_\nu^\varepsilon)^{1-\beta} (r_1^\varepsilon + \mu_1^\varepsilon)^\gamma \left(\mu_1^\varepsilon + |x - a_1^\varepsilon|\right)^{1-(\beta+\gamma)} (1+|x|)^{(\beta+\gamma)-2}\right).
\end{aligned} \tag{5.7.18}$$

Take $\beta = \frac{1}{2}, \gamma = \frac{2}{3}$. Thanks to Young's inequality, we have that

$$\begin{aligned}
\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \varphi^\varepsilon || \nabla \nu^\varepsilon(y)| \, dy &= o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{\frac{1}{3}} (\rho_\nu^\varepsilon)^{\frac{1}{2}} (1+|x|)^{-1}\right) \\
&\quad + o\left(\varepsilon^2 (\rho_\varphi^\varepsilon)^{\frac{1}{3}} (\rho_\nu^\varepsilon)^{\frac{1}{2}} (r_1^\varepsilon + \mu_1^\varepsilon)^{\frac{2}{3}} \left(\mu_1^\varepsilon + |x - a_1^\varepsilon|\right)^{-\frac{1}{6}} (1+|x|)^{-\frac{5}{6}}\right) \\
&= o\left(\left(\varepsilon^{12} + \left((\rho_\varphi^\varepsilon)^{\frac{1}{3}} (\rho_\nu^\varepsilon)^{\frac{1}{2}}\right)^{\frac{6}{5}}\right)(1+|x|)^{-1}\right) \\
&\quad + o\left(\left(\varepsilon^{\frac{3}{2}} (\rho_\varphi^\varepsilon)^{\frac{1}{3}} (\rho_\nu^\varepsilon)^{\frac{1}{2}}\right)^{\frac{6}{5}} (1+|x|)^{-1}\right) \\
&\quad + o\left(\varepsilon^3 \left(r_1^\varepsilon + \mu_1^\varepsilon\right)^4 \left(\mu_1^\varepsilon + |x - a_1^\varepsilon|\right)^{-1}\right) \\
&= o\left(\left(\varepsilon^3 + \left((\rho_\varphi^\varepsilon)^{\frac{5}{6}} + (\rho_\nu^\varepsilon)^{\frac{5}{6}}\right)^{\frac{6}{5}}\right)(1+|x|)^{-1}\right) \\
&\quad + o\left(\frac{\varepsilon^3 (r_1^\varepsilon)^4}{\mu_1^\varepsilon} \left(1 + \frac{|x - a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\right) \\
&= o\left(\left(\varepsilon^3 + \rho_\varphi^\varepsilon + \rho_\nu^\varepsilon\right)(1+|x|)^{-1}\right) \\
&\quad + o\left(\frac{\varepsilon^3 (r_1^\varepsilon)^4}{\mu_1^\varepsilon} \left(1 + \frac{|x - a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\right)
\end{aligned} \tag{5.7.19}$$

□

Let us write now that

$$(\Phi^\varepsilon - r_i\mathcal{B}_i^\varepsilon)(x) = (\Phi^\varepsilon - r_i\mathcal{B}_i^\varepsilon)(x) - (\Phi^\varepsilon - r_i\mathcal{B}_i^\varepsilon)(a_i^\varepsilon) + O(\varepsilon^2(r_i^\varepsilon)^3),$$

thanks to Proposition 5.6.2. Thus we have that

$$|(\Phi^\varepsilon - r_i^\varepsilon\mathcal{B}_i^\varepsilon)(x)| \leq C\left(\varepsilon^2(r_i^\varepsilon)^3 + |\varphi^\varepsilon(x) - \varphi^\varepsilon(a_i^\varepsilon)| + r_{1-i}^\varepsilon|\mathcal{B}_{1-i}^\varepsilon(x) - \mathcal{B}_{1-i}^\varepsilon(a_i^\varepsilon)|\right). \tag{5.7.20}$$

We clearly have, thanks to the definition (5.7.1) of $\rho_\varphi^\varepsilon$ that

$$|\varphi^\varepsilon(x) - \varphi^\varepsilon(a_i^\varepsilon)| \lesssim \rho_\varphi^\varepsilon|x - a_i^\varepsilon| \lesssim \rho_\varphi^\varepsilon\left(|x - a_i^\varepsilon| + \mu_i^\varepsilon\right).$$

Thus, by straightforward computations, we can already write that

$$\begin{aligned}
&\varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_i^\varepsilon(y)|^2|(\Phi^\varepsilon - r_i^\varepsilon\mathcal{B}_i^\varepsilon)|(y)\,dy\\
&\lesssim \varepsilon^4(r_i^\varepsilon)^3\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(\frac{\mu_i^\varepsilon}{(\mu_i^\varepsilon)^2 + |y-a_i^\varepsilon|^2}\right)^2 dy\\
&\quad + \varepsilon^2\rho_\varphi^\varepsilon\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon)^3 + |y-a_i^\varepsilon|^3}\,dy\\
&\quad + \varepsilon^2 r_{1-i}^\varepsilon\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^4}|\mathcal{B}_{1-i}^\varepsilon(y) - \mathcal{B}_{1-i}^\varepsilon(a_i^\varepsilon)|dy\\
&\lesssim \frac{\varepsilon^4(r_i^\varepsilon)^3}{\mu_i^\varepsilon + |x-a_i^\varepsilon|} + \varepsilon^2\rho_\varphi^\varepsilon\frac{\mu_i^\varepsilon}{\mu_i^\varepsilon + |x-a_i^\varepsilon|}\\
&\quad + \varepsilon^2 r_{1-i}^\varepsilon\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^4}|\mathcal{B}_{1-i}^\varepsilon(y) - \mathcal{B}_{1-i}^\varepsilon(a_i^\varepsilon)|dy\\
&\lesssim \left(\frac{\varepsilon^4(r_i^\varepsilon)^3}{\mu_i^\varepsilon} + \varepsilon^2\rho_\varphi^\varepsilon\right)\left(1 + \frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1}\\
&\quad + \varepsilon^2 r_{1-i}^\varepsilon\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^4}|\mathcal{B}_{1-i}^\varepsilon(y) - \mathcal{B}_{1-i}^\varepsilon(a_i^\varepsilon)|\,dy.
\end{aligned} \tag{5.7.21}$$

Estimating the last integral is more tricky. By distinguishing appropriate cases, we claim the following:

**Claim 5.7.5.** *For $i = 0, 1$, we have that, for all $x \in \mathbb{R}^2$,*

$$\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^4}|\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)|dy \lesssim \left(\mu_j^\varepsilon\left(1 + \frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1} + \mu_i^\varepsilon\left(1 + \frac{|x-a_j^\varepsilon|}{\mu_j^\varepsilon}\right)^{-1}\right),$$

*where $j = 1 - i$ and $a_i^\varepsilon$ is chosen as in Proposition 5.6.2.*

*Proof of Claim 5.7.5:* We first write that

$$\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^4}|\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)|dy = A_1^\varepsilon(x) + A_2^\varepsilon(x) + A_3^\varepsilon(x),$$

where

$$A_1^\varepsilon(x) = \int_{B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_i^\varepsilon)} \frac{1}{|x-y|} \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y - a_i^\varepsilon|)^4} |\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| dy,$$

$$A_2^\varepsilon(x) = \int_{B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon)} \frac{1}{|x-y|} \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y - a_i^\varepsilon|)^4} |\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| dy \text{ and}$$

$$A_3^\varepsilon(x) = \int_{B^c_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(b_j^\varepsilon) \bigcap B^c_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon)} \frac{1}{|x-y|} \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y - a_i^\varepsilon|)^4} |\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| dy.$$

We now estimate $A_1^\varepsilon(x)$, $A_2^\varepsilon(x)$ and $A_3^\varepsilon(x)$.

- Estimating $A_1^\varepsilon(x)$: One can verify that

$$|\nabla \mathcal{B}_j^\varepsilon(y)| \leq C \frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |a_i^\varepsilon - a_j^\varepsilon|^2} \leq C\mu_j^\varepsilon \text{ for all } y \in B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_i^\varepsilon),$$

since $\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon| = O\left(|a_i^\varepsilon - a_j^\varepsilon|\right)$, thanks to Proposition 5.6.2. Thus we have that

$$|\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| \leq C\mu_j^\varepsilon \left(|y - a_i^\varepsilon| + \mu_i^\varepsilon\right),$$

from which we deduce that

$$A_1^\varepsilon(y) \leq C\mu_j^\varepsilon \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |y - a_i^\varepsilon|)^3} dy \leq C\mu_j^\varepsilon \left(1 + \frac{|x - a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1}. \tag{5.7.22}$$

- Estimating $A_2^\varepsilon(x)$: Either $|a_i^\varepsilon - a_j^\varepsilon| = O(\mu_i^\varepsilon)$ or not, thanks to the definition of $a_i^\varepsilon$, we can verify that

$$\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon| \leq C(\mu_i^\varepsilon + |y - a_i^\varepsilon|) \text{ for all } y \in B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon),$$

which leads to

$$A_2^\varepsilon(x) \leq C \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|)^4} \int_{B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon)} \frac{1}{|x-y|} |\mathcal{B}_j^\varepsilon(y) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| dy. \tag{5.7.23}$$

For all $y \in B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon)$, let $y_\top \in [a_j^\varepsilon, a_i^\varepsilon]$ be such that $|y - a_j^\varepsilon| = |y_\top - a_j^\varepsilon|$. We can write that

$$\begin{aligned} |\mathcal{B}_i^\varepsilon(y) - \mathcal{B}_i^\varepsilon(a_i^\varepsilon)| &\leq |\mathcal{B}_i^\varepsilon(y) - \mathcal{B}_i^\varepsilon(y_\top)| + |\mathcal{B}_i^\varepsilon(y_\top) - \mathcal{B}_i^\varepsilon(a_i^\varepsilon)| \\ &\leq C \left( \frac{\mu_j^\varepsilon |y - a_j^\varepsilon|}{\left(\mu_j^\varepsilon\right)^2 + |y - a_j^\varepsilon|^2} + \frac{\mu_j^\varepsilon}{\mu_j^\varepsilon + |y - a_j^\varepsilon|} \frac{|a_j^\varepsilon - a_i^\varepsilon|}{\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|} \right) \\ &\leq C \frac{\mu_j^\varepsilon}{\mu_j^\varepsilon + |y - a_j^\varepsilon|} \frac{|a_j^\varepsilon - a_i^\varepsilon|}{\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|}. \end{aligned} \tag{5.7.24}$$

Indeed, we have that

$$|\mathcal{B}_i^\varepsilon(y) - \mathcal{B}_i^\varepsilon(y_\top)| \leq C \max_{\widehat{yy_\top}} |\nabla \mathcal{B}_i^\varepsilon| \times |y - y_\top| \leq C \frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |y - a_j^\varepsilon|^2} |y - a_j^\varepsilon|,$$

where $\widehat{yy_\top}$ is the circular arc centered on $a_j^\varepsilon$ joining $y$ and $y_\top$. By distinguishing appropriate cases, we also get that

$$|\mathcal{B}_j^\varepsilon(y_\top) - \mathcal{B}_j^\varepsilon(a_i^\varepsilon)| \leq C \int_{|a_j^\varepsilon - y|}^{|a_j^\varepsilon - a_i^\varepsilon|} \frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + t^2} dt \leq \frac{\mu_j^\varepsilon}{\mu_j^\varepsilon + |y - a_j^\varepsilon|} \frac{|a_j^\varepsilon - a_i^\varepsilon|}{\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|}$$

Coming back to (5.7.23) together with (5.7.24), direct computations show that

$$A_2^\varepsilon(x) \leq \frac{(\mu_i^\varepsilon)^2}{(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|)^4} \frac{|a_j^\varepsilon - a_i^\varepsilon|\mu_j^\varepsilon}{\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|} \int_{B_{\frac{|a_j^\varepsilon - a_i^\varepsilon|}{2}}(a_j^\varepsilon)} \frac{1}{|x-y|} \frac{1}{\mu_j^\varepsilon + |y - a_j^\varepsilon|} dy$$

$$\leq C \frac{(\mu_i^\varepsilon)^2 \mu_j^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4 \left(\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|\right)^2} \frac{1}{\left(|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|\right)} \ln\left(2 + \frac{|a_j^\varepsilon - a_i^\varepsilon|}{\mu_j^\varepsilon + |x - a_j^\varepsilon|}\right). \tag{5.7.25}$$

(i) If we have

$$\frac{|x - a_j^\varepsilon| + \mu_j^\varepsilon}{|a_j^\varepsilon - a_i^\varepsilon|} \to 0 \text{ as } \varepsilon \to 0,$$

we can write that

$$A_2^\varepsilon(x) \leq C \frac{(\mu_i^\varepsilon)^2 \mu_j^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4 \left(\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|\right)^2} \frac{|a_j^\varepsilon - a_i^\varepsilon|}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|} \frac{1}{\mu_j^\varepsilon + |x - a_j^\varepsilon|}$$

$$\leq C \frac{\mu_i^\varepsilon}{\left(\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|\right)^2} \frac{\mu_j^\varepsilon}{\mu_j^\varepsilon + |x - a_j^\varepsilon|} \frac{\mu_i^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4} \frac{|a_j^\varepsilon - a_i^\varepsilon|}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|}$$

$$\leq C\mu_i^\varepsilon \left(1 + \frac{|x - a_j^\varepsilon|}{\mu_j^\varepsilon}\right)^{-1},$$

since we have $\frac{\mu_i^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4} \leq 1$ and $\frac{|a_j^\varepsilon - a_i^\varepsilon|}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|} \leq 1$.

(ii) Else, if we have

$$\frac{|x - a_j^\varepsilon| + \mu_j^\varepsilon}{|a_j^\varepsilon - a_i^\varepsilon|} \nrightarrow 0 \text{ as } \varepsilon \to 0,$$

we can write that

$$A_2^\varepsilon(x) \leq C \frac{(\mu_i^\varepsilon)^2 \mu_j^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon |a_i^\varepsilon - a_j^\varepsilon|\right)^4 \left(\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|\right)^2} \frac{1}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|}$$

$$\leq C \frac{\mu_j^\varepsilon}{\left(\mu_j^\varepsilon + |a_j^\varepsilon - a_i^\varepsilon|\right)^2} \frac{\mu_i^\varepsilon}{\mu_i^\varepsilon + |x - a_i^\varepsilon|} \times \frac{\mu_i^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4} \frac{\mu_i^\varepsilon + |x - a_i^\varepsilon|}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|}.$$

Using the fact that $|x - a_i^\varepsilon| \leq |x - a_j^\varepsilon| + |a_j^\varepsilon - a_i^\varepsilon| + \mu_i^\varepsilon$, we get that

$$A_2^\varepsilon(x) \leq C\mu_j^\varepsilon \left(1 + \frac{|x - a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1} \times \frac{\mu_i^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4} \left(1 + \frac{\mu_i^\varepsilon}{|a_j^\varepsilon - a_i^\varepsilon| + |x - a_j^\varepsilon|}\right)$$

$$\leq C\mu_j^\varepsilon \left(1 + \frac{|x - a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1} \left(\frac{\mu_i^\varepsilon |a_j^\varepsilon - a_i^\varepsilon|^3}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4} + \frac{(\mu_i^\varepsilon)^2 |a_j^\varepsilon - a_i^\varepsilon|^2}{\left(\mu_i^\varepsilon + |a_i^\varepsilon - a_j^\varepsilon|\right)^4}\right)$$

$$\leq C\mu_j^\varepsilon \left(1 + \frac{|x - a_i^\varepsilon|}{\mu_i^\varepsilon}\right).$$

Hence, we show that in all cases,

$$A_2^\varepsilon(x) \leq C\left(\mu_j^\varepsilon \left(1 + \frac{|x - a_i^\varepsilon|}{\mu_i^\varepsilon}\right) + \mu_i^\varepsilon \left(1 + \frac{|x - a_j^\varepsilon|}{\mu_j^\varepsilon}\right)\right). \tag{5.7.26}$$

- Estimating $A_3^\varepsilon(x)$: Proceeding as we did in (5.7.24), we have that

$$|\mathcal{B}_j^\varepsilon(y)-\mathcal{B}_j^\varepsilon(a_i^\varepsilon)| \le C\frac{\mu_j^\varepsilon}{\mu_j^\varepsilon+|a_j^\varepsilon-a_i^\varepsilon|}\frac{|y-a_j^\varepsilon|}{\mu_j^\varepsilon+|y-a_j^\varepsilon|} \text{ for all } y\in B^c_{\frac{|a_j^\varepsilon-a_i^\varepsilon|}{2}}(a_i^\varepsilon)\cap B^c_{\frac{|a_j^\varepsilon-a_i^\varepsilon|}{2}}(a_j^\varepsilon), \quad (5.7.27)$$

which leads to

$$A_3^\varepsilon(x) \le C\frac{\mu_j^\varepsilon}{\mu_j^\varepsilon+|a_j^\varepsilon-a_i^\varepsilon|}\int_{\mathbb{R}^2\setminus B_{\frac{|a_j^\varepsilon-a_i^\varepsilon|}{2}}(a_j^\varepsilon)}\frac{1}{|x-y|}\frac{(\mu_i^\varepsilon)^2|y-a_j^\varepsilon|}{\left(\mu_i^\varepsilon+|y-a_j^\varepsilon|\right)^5}dy.$$

By straightforward computations together with the fact that $\mu_i^\varepsilon = O\left(|a_j^\varepsilon-a_i^\varepsilon|\right)$, we get that

$$\begin{aligned} A_3^\varepsilon(x) &\le C\frac{(\mu_i^\varepsilon)^2\mu_j^\varepsilon}{\mu_j^\varepsilon+|a_j^\varepsilon-a_i^\varepsilon|}\frac{1}{(\mu_j^\varepsilon)^2+|a_j^\varepsilon-a_i^\varepsilon|^2}\frac{1}{\mu_j^\varepsilon+|x-a_i^\varepsilon|+|a_j^\varepsilon-a_i^\varepsilon|} \\ &\le C\frac{\mu_i^\varepsilon}{\mu_j^\varepsilon+|a_j^\varepsilon-a_i^\varepsilon|}\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2+|a_j^\varepsilon-a_i^\varepsilon|^2}\frac{\mu_i^\varepsilon}{\mu_i^\varepsilon+|x-a_i^\varepsilon|} \\ &\le C\mu_j^\varepsilon\left(1+\frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1}. \end{aligned} \quad (5.7.28)$$

Hence, the claim is proved, thanks to (5.7.22), (5.7.26) and (5.7.28). □

Thanks to Claim 5.7.5 together with (5.7.21), we thus get that

$$\begin{aligned} &\varepsilon^2 r_i^\varepsilon\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_i^\varepsilon(y)|^2|(\Phi^\varepsilon-r_i\mathcal{B}_i^\varepsilon)|(y)\,dy \\ &\lesssim\left(\frac{\varepsilon^4(r_i^\varepsilon)^4}{\mu_i^\varepsilon}+{+}\varepsilon^2 r_i^\varepsilon\rho_\varphi^\varepsilon+\varepsilon^2 r_i^\varepsilon r_{1-i}^\varepsilon\mu_{1-i}^\varepsilon\right)\left(1+\frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1}. \end{aligned} \quad (5.7.29)$$

At last, it remains to write that

$$\varepsilon^3\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\mathcal{B}_i^\varepsilon(y)|^2\,dy \lesssim \frac{\varepsilon^3}{\mu_i^\varepsilon}\left(1+\frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1} \quad (5.7.30)$$

by direct computations.

Coming back to (5.7.4) with (5.7.5), (5.7.9), (5.7.10), (5.7.12), (5.7.17), (5.7.29), and (5.7.30), we obtain Claim 5.7.1. □

**Claim 5.7.6.** *For every $R>0$ and $x\in\mathbb{D}_0(R)$, we have that*

$$\begin{aligned} (1+|x|)|\nabla\nu^\varepsilon(x)| \lesssim{}& \left(\rho_{0,R}^\varepsilon+\varepsilon^3+R^{-1}\rho^\varepsilon\right)(1+|x|)^{-1} \\ &+\left(\left(\frac{1}{r_1^\varepsilon}\rho_{\varphi,1,R}^\varepsilon+\rho_{\nu,1,R}^\varepsilon+r_1^\varepsilon\rho_{\eta,1,R}^\varepsilon\right)\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right) \\ &+\left(\frac{1}{r_1^\varepsilon}\rho_\varphi^\varepsilon+\rho_\nu^\varepsilon+r_1^\varepsilon\rho_\eta^\varepsilon\right)\left(R+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right) \\ &+\left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}+\frac{1}{r_1^\varepsilon}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right) \\ &+o\left(\left(\rho^\varepsilon+\frac{1}{r_1^\varepsilon}\right)(1+|x|)^{-1}\right). \end{aligned} \quad (5.7.31)$$

*Proof of Claim 5.7.6:* Applying the Green's representation formula in Lemma 6.2.2 to the second equation of Proposition 5.7.10, and estimating various terms as in Claim 5.7.1, see in particular (5.7.9), (5.7.10), (5.7.12), (5.7.17), (5.7.29) and (5.7.30), we can write that

$$\begin{aligned}(1+|x|)\,|\nabla\nu^\varepsilon(x)| \lesssim\ & (1+|x|)\,|\nabla\psi^\varepsilon_{\nu,\nu}(x)| + (1+|x|)\,|\nabla\psi^\varepsilon_{\eta,\varphi}(x)| \\ & + \varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\nu(y)|^2\,dy \\ & + \varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(r_0^\varepsilon\,|\vec n^\varepsilon-\mathcal{N}_0^\varepsilon| + (r_0^\varepsilon)^2\left|\vec H^\varepsilon-\frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon\right|\right)|\nabla\mathcal{B}_0^\varepsilon|^2(y)\,dy \\ & + \varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(r_1^\varepsilon\,|\vec n^\varepsilon-\mathcal{N}_1^\varepsilon| + (r_1^\varepsilon)^2\left|\vec H^\varepsilon-\frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon\right|\right)|\nabla\mathcal{B}_1^\varepsilon|^2(y)\,dy \\ & + \left(\frac{1}{r_0^\varepsilon}\rho^\varepsilon_{\varphi,0,R}+\rho^\varepsilon_{\nu,0,R}+\rho^\varepsilon_{\eta,0,R}+\varepsilon^3+R^{-1}(\rho^\varepsilon_\varphi+\rho^\varepsilon_\nu+\rho^\varepsilon_\eta)\right)(1+|x|)^{-1} \\ & + \left(\left(\frac{1}{r_1^\varepsilon}\rho^\varepsilon_{\varphi,1,R}+\rho^\varepsilon_{\nu,1,R}+r_1^\varepsilon\rho^\varepsilon_{\eta,1,R}\right)\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right) \\ & + \left(\frac{1}{r_1^\varepsilon}\rho^\varepsilon_\varphi+\rho^\varepsilon_\nu+r_1^\varepsilon\rho^\varepsilon_\eta\right)\left(R+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon\,(1+|x|)}\right) \\ & + \left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}+\frac{1}{r_1^\varepsilon}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon\,(1+|x|)}\right)\end{aligned} \tag{5.7.32}$$

Proceeding as in Claim 5.7.2, since $\nabla\nu^\varepsilon\to 0$ in $L^2(\mathbb{R}^2)$ as $\varepsilon\to 0$, we have that, for all $x\in\mathbb{R}^2$

$$(1+|x|)|\nabla\psi^\varepsilon_{\nu,\nu}(x)| \lesssim \|\nabla\nu^\varepsilon\|_2\rho_\nu^\varepsilon\,(1+|x|)^{-1} = o(\rho_\nu^\varepsilon\,(1+|x|)^{-1}). \tag{5.7.33}$$

Proceeding as in Claim 5.7.3, we have

$$\varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\nu^\varepsilon(y)|^2\,dy \lesssim \left((\varepsilon^3+\rho_\nu^\varepsilon)(1+|x|)^{-1}\right)+o\left(\frac{\varepsilon^3}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right). \tag{5.7.34}$$

Proceeding as in (5.7.20), thanks to Proposition 5.6.2, we get that, for $i=0,1$,

$$\begin{aligned}\varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(r_i^\varepsilon\,|\vec n^\varepsilon-\mathcal{N}_i^\varepsilon| + (r_i^\varepsilon)^2\left|\vec H^\varepsilon-\frac{1}{r_i^\varepsilon}\mathcal{H}_i^\varepsilon\right|\right)|\nabla\mathcal{B}_i^\varepsilon|^2(y)\,dy & \\ \lesssim o\left(\left(\frac{\varepsilon^3(r_i^\varepsilon)^3}{\mu_i^\varepsilon}+\rho_\nu^\varepsilon+1\right)\left(1+\frac{|x-a_i|^\varepsilon}{\mu_i^\varepsilon}\right)^{-1}\right) & \\ + o\left(\left(\frac{\varepsilon^3(r_i^\varepsilon)^3}{\mu_i^\varepsilon}+\left(\rho_\eta^\varepsilon+\frac{(r_i^\varepsilon)^2}{r_{1-i}^\varepsilon}\right)\right)\right)\left(1+\frac{|x-a_i^\varepsilon|}{\mu_i^\varepsilon}\right)^{-1}. &\end{aligned} \tag{5.7.35}$$

In order to control $|\nabla\psi^\varepsilon_{\eta,\varphi}(x)|$, since we do not necessarily have $\nabla\eta^\varepsilon\to 0$ in $L^2(\mathbb{R}^2)$ as $\varepsilon\to 0$, we must exploit estimates similar to those obtained in Claim 5.7.1.

**Claim 5.7.7.** *We have*

$$\begin{aligned}(1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi}(x)| = & \ o\left(\left(\varepsilon^3+\rho^\varepsilon_\varphi+\rho^\varepsilon_\nu+\rho^\varepsilon_\eta\right)(1+|x|)^{-1}\right) \\ & + o\left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right).\end{aligned} \tag{5.7.36}$$

*Proof of Claim 5.7.7.* Thanks to Proposition 5.6.3 and the definition of $\psi^\varepsilon_{\eta,\varphi}$, using Green's representation formula (6.2.4), we can write that

$$\begin{aligned}
(1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi}(x)| &\lesssim \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)||\nabla\varphi^\varepsilon(y)|\,dy\\
&\lesssim \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)||\nabla\psi^\varepsilon_{\varphi,\nu}(y)|\,dy\\
&\quad+\sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\nabla\psi^\varepsilon_{i,\varphi}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,\nu}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,1-i}(y)|\right)\,dy\\
&\quad+\varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi^\varepsilon_{\varphi,\varphi}(y)|+|\chi^\varepsilon_{\varphi,\nu}(y)|\right)\,dy\\
&\quad+\varepsilon^2\sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(r_i^\varepsilon|\chi^\varepsilon_i(y)|+|\chi^\varepsilon_{i,\varphi}(y)|+r_i^\varepsilon|\chi^\varepsilon_{i,\nu}(y)|+r_i^\varepsilon|\chi^\varepsilon_{i,1-i}(y)|\right)dy\\
&\quad+\varepsilon^3\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi^\varepsilon_0(y)|+(r_1^\varepsilon)^4|\chi^\varepsilon_1(y)|\right)dy,
\end{aligned}\tag{5.7.37}$$

where

$$\begin{aligned}
\chi^\varepsilon_{\varphi,\varphi}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\varphi^\varepsilon(y)|^2dy,\\
\chi^\varepsilon_{\varphi,\nu}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\varphi^\varepsilon(y)||\nabla\nu^\varepsilon(y)|\,dy,\\
\chi^\varepsilon_{i}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\mathcal{B}^\varepsilon_i(y)|^2|\Phi^\varepsilon-r_i\mathcal{B}^\varepsilon_i(y)|dy,\\
\chi^\varepsilon_{i,\varphi}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\mathcal{B}^\varepsilon_i(y)||\nabla\varphi^\varepsilon(y)|dy,\\
\chi^\varepsilon_{i,\nu}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\mathcal{B}^\varepsilon_i(y)||\nabla\nu^\varepsilon(y)|dy \text{ and}\\
\chi^\varepsilon_{i,1-i}(x) &:= \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}|\nabla\mathcal{B}^\varepsilon_i(y)||\nabla\mathcal{B}^\varepsilon_{1-i}(y)|dy.
\end{aligned}$$

Thanks to (5.5.53) and the generalized Wente inequality in Lorentz spaces, see Lemma 6.3.6, we have

$$\begin{aligned}
&\|\nabla\psi^\varepsilon_{\varphi,\nu}\|_{2,1}+\|\nabla\psi^\varepsilon_{i,\varphi}\|_{2,1}+\|\nabla\psi^\varepsilon_{i,\nu}\|_{2,1}\\
&\quad\le C\left(\|\nabla\varphi^\varepsilon\|_2+\|\nabla\nu^\varepsilon\|_2+|\nabla\mathcal{B}^\varepsilon_i\|_2\right)\left(\|\nabla\varphi^\varepsilon\|_2+\|\nabla\nu^\varepsilon\|_2\right)=o(1).
\end{aligned}$$

Therefore, using Hölder inequality in Lorentz spaces, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\nabla\psi^\varepsilon_{\varphi,\nu}(y)|+\sum_{i=0,1}|\nabla\psi^\varepsilon_{i,\varphi}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,\nu}(y)|\right)dy\\
&\lesssim\rho^\varepsilon_\eta\int_{\mathbb{R}^2}\frac{1}{|x-y|\,(1+|y|)}\left(|\nabla\psi^\varepsilon_{\varphi,\nu}(y)|+\sum_{i=0,1}|\nabla\psi^\varepsilon_{i,\varphi}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,\nu}(y)|\right)dy\\
&\lesssim\rho^\varepsilon_\eta\sup_{x\in\mathbb{R}^2}\left\|\frac{1}{|x-\cdot|}\right\|_{2,+\infty}\left(\|\nabla\psi^\varepsilon_{\varphi,\nu}\|_{2,1}+\sum_{i=0,1}\|\nabla\psi^\varepsilon_{i,\varphi}\|_{2,1}+r_i^\varepsilon\|\nabla\psi^\varepsilon_{i,\nu}\|_{2,1}\right)\\
&=o(\rho^\varepsilon_\eta).
\end{aligned}\tag{5.7.38}$$

Let us now use (5.7.10) to write that

$$\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\nabla \psi^\varepsilon_{0,1}(y)|\, dy \lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|} |\nabla \eta^\varepsilon(y)| \frac{\ln\left(2+\frac{|y-a^\varepsilon_1|}{\mu^\varepsilon_1(1+|y|)}\right)}{1+\frac{|y-a^\varepsilon_1|}{\mu^\varepsilon_1}}\, dy.$$

Let us write now thanks to definition (5.7.1) that,

$$|\nabla \eta^\varepsilon(y)| \lesssim \rho^\varepsilon_\eta \frac{1}{1+|y|^2},$$

so that,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\nabla \psi^\varepsilon_{0,1}(y)| dy \\
&\lesssim \rho^\varepsilon_\eta \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(1+|y|^2)} \frac{\ln\left(2+\frac{|y-a^\varepsilon_1|}{\mu^\varepsilon_1(1+|y|)}\right)}{1+\frac{|y-a^\varepsilon_1|}{\mu^\varepsilon_1}}\, dy \\
&= o\left(\rho^\varepsilon_\eta (1+|x|)^{-1}\right).
\end{aligned} \tag{5.7.39}$$

Thanks to (5.7.11) in Claim 5.7.3 and the definition (5.7.1) of $\rho^\varepsilon_\eta$, we can write

$$|\nabla \eta^\varepsilon(y)|^{1-\gamma} \leq (\rho^\varepsilon_\eta)^{1-\gamma} (1+|y|)^{-2(1-\gamma)} \tag{5.7.40}$$

and we obtain that, thanks to the weak estimates (5.5.41) on $|\nabla \eta^\varepsilon|$, for every $1<\beta<2$ and $0<\alpha,\gamma<1$,

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\chi^\varepsilon_{\varphi,\varphi}(y)| dy \\
&= o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} |\nabla \eta^\varepsilon(y)|^\gamma (1+|y|)^{2\gamma-3}\, dy\right) \\
&+ o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} (r^\varepsilon_1+\mu^\varepsilon_1)^\beta \int_{\mathbb{R}^2} \frac{1}{|x-y|} |\nabla \eta^\varepsilon(y)|^\gamma (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{1-\beta} (1+|y|)^{2\gamma+\beta-4}\, dy\right) \\
&= o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (r^\varepsilon_1)^{-\gamma} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{-\gamma} (1+|y|)^{\gamma-3}\, dy\right) \\
&+ o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{-\alpha\gamma} (1+|y|)^{\alpha\gamma-3}\, dy\right) \\
&+ o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} (r^\varepsilon_1+\mu^\varepsilon_1)^\beta \int_{\mathbb{R}^2} \frac{1}{|x-y|} (r^\varepsilon_1)^{-\gamma} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{1-(\beta+\gamma)} (1+|y|)^{\gamma+\beta-4}\, dy\right) \\
&+ o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} (r^\varepsilon_1+\mu^\varepsilon_1)^\beta \int_{\mathbb{R}^2} \frac{1}{|x-y|} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{1-(\beta+\alpha\gamma)} (1+|y|)^{\alpha\gamma+\beta-4}\, dy\right).
\end{aligned} \tag{5.7.41}$$

Since $0<\gamma<1$ and $3-\gamma>2$, we get

$$\begin{aligned}
&o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (r^\varepsilon_1)^{-\gamma} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{-\gamma} (1+|y|)^{\gamma-3}\, dy\right) \\
&\quad + o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (\mu^\varepsilon_1+|y-a^\varepsilon_1|)^{-\alpha\gamma} (1+|y|)^{\alpha\gamma-3}\, dy\right) \\
&= o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{1-\gamma} (\rho^\varepsilon_\varphi)^{2-\beta} (1+|x|)^{-1}\right).
\end{aligned} \tag{5.7.42}$$

Take $\beta = \frac{3}{2}$, $\gamma = \frac{2}{3}$ and $0 < \alpha < 1$ such that $\alpha\gamma + \beta < 1$, then we have

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\chi^\varepsilon_{\varphi,\varphi}(y)|dy &= o\left(\varepsilon^2 \left(\rho^\varepsilon_\eta\right)^{1/3} \left(\rho^\varepsilon_\varphi\right)^{1/2} (1+|x|)^{-1}\right) \\ &+ o\left(\varepsilon^2 \left(\rho^\varepsilon_\eta\right)^{1/3} \left(\rho^\varepsilon_\varphi\right)^{1/2} (r^\varepsilon_1 + \mu^\varepsilon_1)^{3/2} (r^\varepsilon_1)^{-2/3} (\mu^\varepsilon_1 + |x - a^\varepsilon_1|)^{-1/6} (1+|x|)^{-5/6}\right).\end{aligned} \tag{5.7.43}$$

Proceeding as in (5.7.19), thanks to Young's inequality, we obtain that

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\chi^\varepsilon_{\varphi,\varphi}(y)|dy &= o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{\frac{1}{3}} (\rho^\varepsilon_\varphi)^{\frac{1}{2}} (1+|x|)^{-1}\right) \\ &+ o\left(\varepsilon^2 (\rho^\varepsilon_\eta)^{\frac{1}{3}} (\rho^\varepsilon_\varphi)^{\frac{1}{2}} (r^\varepsilon_1 + \mu^\varepsilon_1)^{\frac{3}{2}} (\mu^\varepsilon_1 + |x - a^\varepsilon_1|)^{-\frac{1}{6}} (1+|x|)^{-\frac{5}{6}}\right) \\ &= o\left(\left(\varepsilon^{12} + \left((\rho^\varepsilon_\eta)^{\frac{1}{3}} (\rho^\varepsilon_\varphi)^{\frac{1}{2}}\right)^{\frac{6}{5}}\right) (1+|x|)^{-1}\right) \\ &+ o\left(\left(\varepsilon^{\frac{3}{2}} (\rho^\varepsilon_\eta)^{\frac{1}{3}} (\rho^\varepsilon_\varphi)^{\frac{1}{2}}\right)^{\frac{6}{5}} (1+|x|)^{-1}\right) \\ &+ o\left(\varepsilon^3 (r^\varepsilon_1 + \mu^\varepsilon_1)^9 (\mu^\varepsilon_1 + |x - a^\varepsilon_1|)^{-1}\right) \\ &= o\left(\left(\varepsilon^3 + \left((\rho^\varepsilon_\eta)^{\frac{5}{6}} + (\rho^\varepsilon_\varphi)^{\frac{5}{6}}\right)^{\frac{6}{5}}\right) (1+|x|)^{-1}\right) \\ &+ o\left(\frac{\varepsilon^3 (r^\varepsilon_1)^9}{\mu^\varepsilon_1} \left(1 + \frac{|x - a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1}\right) \\ &= o\left(\left(\varepsilon^3 + \rho^\varepsilon_\eta + \rho^\varepsilon_\varphi\right) (1+|x|)^{-1}\right) \\ &+ o\left(\frac{\varepsilon^3 (r^\varepsilon_1)^9}{\mu^\varepsilon_1} \left(1 + \frac{|x - a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1}\right)\end{aligned} \tag{5.7.44}$$

Similarly to (5.7.44), we can also deduce from Claim 5.7.4 that

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla \eta^\varepsilon(y)||\chi^\varepsilon_{\varphi,\nu}(y)|dy &= o\left(\left(\varepsilon^3 + \rho^\varepsilon_\eta + \rho^\varepsilon_\varphi + \rho^\varepsilon_\nu\right) (1+|x|)^{-1}\right) \\ &+ o\left(\frac{\varepsilon^3 (r^\varepsilon_1)^4}{\mu^\varepsilon_1} \left(1 + \frac{|x - a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1}\right).\end{aligned} \tag{5.7.45}$$

Now, we establish estimates on $\chi^\varepsilon_i$, $\chi^\varepsilon_{i,j}$, $\chi^\varepsilon_{i,\varphi}$ and $\chi^\varepsilon_{i,\nu}$. Since $\sup_{\varepsilon>0} \|\Phi^\varepsilon_i - r_i \mathcal{B}^\varepsilon_i\|_\infty < +\infty$, we can write that

$$\begin{aligned}|\chi^\varepsilon_i(x)| &\leq \int_{\mathbb{R}^2} \frac{1}{|x-y|} \left(\frac{\mu^\varepsilon_i}{(\mu^\varepsilon_i + |y - a^\varepsilon_i|)^2}\right)^2 dy \\ &\lesssim \frac{1}{(\mu^\varepsilon_i + |x - a^\varepsilon_i|)^\gamma} \quad \text{for } 0 \leq \gamma < 3,\end{aligned} \tag{5.7.46}$$

while

$$\begin{aligned}|\chi^\varepsilon_{i,1-i}(x)| &\lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{\mu^\varepsilon_i}{(\mu^\varepsilon_i + |y - a^\varepsilon_i|)^2} \frac{\mu^\varepsilon_{1-i}}{(\mu^\varepsilon_{1-i} + |y - a^\varepsilon_{1-i}|)^2} dy \\ &\lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|} \left(\frac{\mu^\varepsilon_i}{(\mu^\varepsilon_i + |y - a^\varepsilon_i|)^2}\right)^2 dy + \int_{\mathbb{R}^2} \frac{1}{|x-y|} \left(\frac{\mu^\varepsilon_{1-i}}{(\mu^\varepsilon_{1-i} + |y - a^\varepsilon_{1-i}|)^2}\right)^2 dy \\ &\lesssim \left(\frac{1}{(\mu^\varepsilon_i + |x - a^\varepsilon_i|)^\gamma} + \frac{1}{(\mu^\varepsilon_{1-i} + |x - a^\varepsilon_{1-i}|)^\gamma}\right) \quad \text{for } 0 \leq \gamma < 3, .\end{aligned} \tag{5.7.47}$$

Thanks to (5.7.46), (5.7.47) and the weak estimates (5.5.41) on $|\nabla\eta^\varepsilon|$, we can write that, for $0 \leq \gamma_1, \gamma_2 < 3$ and for $0 < \alpha < 1$,

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\eta^\varepsilon(y)| \left( r_i^\varepsilon |\chi_i^\varepsilon(y)| + r_i^\varepsilon |\chi_{i,1-i}^\varepsilon(y)| \right) dy \\
&= o\left( \varepsilon^2 (\rho_\eta^\varepsilon)^{1-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(1+|y|)^{1-\alpha\beta}} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{\alpha\beta+\gamma_1}} dy \right) \\
&\quad + o\left( \varepsilon^2 (\rho_\eta^\varepsilon)^{1-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(1+|y|)^{1-\beta}} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{\beta+\gamma_2}} dy \right)
\end{aligned} \tag{5.7.48}$$

By taking $\alpha = \frac{1}{2}, \beta = \frac{2}{3}, \gamma_1 = \frac{4}{3}$ and $\gamma_2 = 1$ in (5.7.48), thanks to Young's inequality, we get that

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\eta^\varepsilon(y)| \left( r_i^\varepsilon |\chi_i^\varepsilon(y)| + r_i^\varepsilon |\chi_{i,1-i}^\varepsilon(y)| \right) dy \\
&= o\left( \varepsilon^2 (\rho_\eta^\varepsilon)^{1/3} \left( 1 + (r_i^\varepsilon)^{-2/3} \right) (\mu_1^\varepsilon + |x-a_1^\varepsilon|)^{-2/3} (1+|x|)^{-1/3} \right) \\
&= o\left( \frac{\varepsilon^3}{r_i^\varepsilon \mu_1^\varepsilon} \left( 1 + \frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon} \right)^{-1} \right) + o\left( \rho_\eta^\varepsilon (1+|x|)^{-1} \right).
\end{aligned} \tag{5.7.49}$$

Next, thanks to (5.5.7) and (5.7.1) we can write that, for every $0 < \gamma < 1$,

$$\begin{aligned}
|\chi_{i,\varphi}^\varepsilon(x)| + r_i^\varepsilon |\chi_{i,\nu}^\varepsilon(x)| &\lesssim \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} \frac{\mu_i^\varepsilon}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^2} \left( |\nabla\varphi^\varepsilon(y)| + r_i^\varepsilon |\nabla\nu^\varepsilon(y)| \right) dy \\
&\lesssim \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} \frac{\mu_i^\varepsilon}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^2} \left( (\rho_\varphi^\varepsilon)^{1-\gamma} |\nabla\varphi^\varepsilon(y)|^\gamma \right) \frac{1}{(1+|y|)^{1-2\gamma}} dy \\
&\quad + \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} \frac{\mu_i^\varepsilon}{(\mu_i^\varepsilon + |y-a_i^\varepsilon|)^2} \left( r_i^\varepsilon (\rho_\nu^\varepsilon)^{1-\gamma} |\nabla\nu^\varepsilon(y)|^\gamma \right) \frac{1}{(1+|y|)^{1-2\gamma}} dy \\
&= o\left( \left( (\rho_\varphi^\varepsilon)^{1-\gamma} + r_i^\varepsilon (\rho_\nu^\varepsilon)^{1-\gamma} \right) \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{\mu_1^\varepsilon}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{2+\gamma}} \frac{1}{(1+|y|)^{1-\gamma}} dy \right) \\
&= o\left( \left( (\rho_\varphi^\varepsilon)^{1-\gamma} + r_i^\varepsilon (\rho_\nu^\varepsilon)^{1-\gamma} \right) (\mu_1^\varepsilon)^{1-\gamma} (\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{-1} (1+|y|)^{\gamma-1} \right).
\end{aligned} \tag{5.7.50}$$

Thanks to (5.7.50) and the weak estimates (5.5.41) on $|\nabla\eta^\varepsilon|$, we can write that, for every $0 < \beta < 1$, $0 < \alpha < 1$, $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$,

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\eta^\varepsilon(y)| \left( |\chi_{i,\varphi}^\varepsilon(y)| + r_i^\varepsilon |\chi_{i,\nu}^\varepsilon(y)| \right) dy \\
&= o\left( \varepsilon^2 (\rho_\eta^\varepsilon)^{1-\beta} \int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|} |\nabla\eta^\varepsilon(y)|^\beta \left( |\chi_{i,\varphi}^\varepsilon(y)| + r_i^\varepsilon |\chi_{i,\nu}^\varepsilon(y)| \right) \frac{1}{(1+|y|)^{2(1-\beta)}} dy \right) \\
&= o\left( \varepsilon^2 \left( (\rho_\varphi^\varepsilon)^{1-\gamma_1} + r_i^\varepsilon (\rho_\nu^\varepsilon)^{1-\gamma_1} \right) (\rho_\eta^\varepsilon)^{1-\beta} (\mu_1^\varepsilon)^{1-\gamma_1} (r_1^\varepsilon)^{-\beta} \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{1+\beta}} \frac{1}{(1+|y|)^{3-(\beta+\gamma_1)}} dy \right) \\
&\quad + o\left( \varepsilon^2 \left( (\rho_\varphi^\varepsilon)^{1-\gamma_2} + r_i^\varepsilon (\rho_\nu^\varepsilon)^{1-\gamma_2} \right) (\rho_\eta^\varepsilon)^{1-\beta} \mu_1^\varepsilon \int_{\mathbb{R}^2} \frac{1}{|x-y|} \frac{1}{(\mu_1^\varepsilon + |y-a_1^\varepsilon|)^{1+(\alpha\beta+\gamma_2)}} \frac{1}{(1+|y|)^{3-(\alpha\beta+\gamma_2)}} dy \right)
\end{aligned} \tag{5.7.51}$$

Taking $\beta = \frac{1}{2}$ and $\gamma_1 = \frac{1}{2}$ into (5.7.51), we obtain that

$$\begin{aligned}
&o\left(\varepsilon^2\left((\rho_\varphi^\varepsilon)^{1-\gamma_1} + r_i^\varepsilon(\rho_\nu^\varepsilon)^{1-\gamma_1}\right)(\rho_\eta^\varepsilon)^{1-\beta}(\mu_1^\varepsilon)^{1-\gamma_1}(r_1^\varepsilon)^{-\beta}\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{1}{(\mu_1^\varepsilon+|y-a_1^\varepsilon|)^{1+\beta}}\frac{1}{(1+|y|)^{1/2}}\,dy\right)\\
&= o\left(\varepsilon^2\left((\rho_\varphi^\varepsilon)^{\frac{1}{2}}(\rho_\eta^\varepsilon)^{\frac{1}{2}} + r_i^\varepsilon(\rho_\nu^\varepsilon)^{\frac{1}{2}}(\rho_\eta^\varepsilon)^{\frac{1}{2}}\right)(r_1^\varepsilon)^{-\frac{1}{2}}\left(\mu_1^\varepsilon+|x-a_1^\varepsilon|\right)^{-\frac{1}{2}}(1+|x|)^{-\frac{1}{2}}\right)\\
&= o\left(\frac{\varepsilon^3}{\mu_1^\varepsilon(r_1^\varepsilon)}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right) + o\left(\rho^\varepsilon(1+|x|)^{-1}\right)
\end{aligned} \tag{5.7.52}$$

Taking $\gamma_2 = \frac{2}{3}$ and $\alpha = \frac{2}{3}$ into (5.7.51), we have $\alpha\beta + \gamma_2 = 1$ and thanks to Young's inequality, we get that

$$\begin{aligned}
&o\left(\varepsilon^2\left((\rho_\varphi^\varepsilon)^{1-\gamma_2} + r_i^\varepsilon(\rho_\nu^\varepsilon)^{1-\gamma_2}\right)(\rho_\eta^\varepsilon)^{1-\beta}\mu_1^\varepsilon\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{1}{(\mu_1^\varepsilon+|y-a_1^\varepsilon|)^{1+(\alpha\beta+\gamma_2)}}\frac{1}{(1+|y|)^{3-(\alpha\beta+\gamma_2)}}\,dy\right)\\
&= o\left(\varepsilon^2\left((\rho_\varphi^\varepsilon)^{\frac{1}{3}}(\rho_\eta^\varepsilon)^{\frac{1}{2}} + r_i^\varepsilon(\rho_\nu^\varepsilon)^{\frac{1}{3}}(\rho_\eta^\varepsilon)^{\frac{1}{2}}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\right)\\
&= o\left(\left(\varepsilon^{12} + \left[(\rho_\varphi^\varepsilon)^{\frac{1}{3}}(\rho_\eta^\varepsilon)^{\frac{1}{2}}\right]^{\frac{6}{5}} + r_i^\varepsilon\left[(\rho_\nu^\varepsilon)^{\frac{1}{3}}(\rho_\eta^\varepsilon)^{\frac{1}{2}}\right]^{\frac{6}{5}}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\right)\\
&= o\left(\left(\varepsilon^{12} + \left[(\rho_\varphi^\varepsilon)^{\frac{5}{6}} + (\rho_\eta^\varepsilon)^{\frac{5}{6}}\right]^{\frac{6}{5}} + r_i^\varepsilon\left[(\rho_\nu^\varepsilon)^{\frac{5}{6}} + (\rho_\eta^\varepsilon)^{\frac{5}{6}}\right]^{\frac{6}{5}}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\right)\\
&= o\left(\left(\varepsilon^3+\rho^\varepsilon\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\right)
\end{aligned} \tag{5.7.53}$$

Coming back to (5.7.51) with (5.7.52) and (5.7.53), we get that

$$\begin{aligned}
&\varepsilon^2\sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi_{i,\varphi}^\varepsilon(y)| + r_i^\varepsilon|\chi_{i,\nu}^\varepsilon(y)|\right)\,dy\\
&= o\left(\rho^\varepsilon(1+|x|)^{-1}\right) + o\left(\left(\frac{\varepsilon^\varepsilon}{\mu_1^\varepsilon}+\rho^\varepsilon\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\right).
\end{aligned} \tag{5.7.54}$$

At last, proceeding as in (5.7.49), we can use (5.7.1), (5.7.34) and (5.7.30) to write that, for $0 < \gamma < 3$,

$$\begin{aligned}
&\varepsilon^3\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi_{0,0}^\varepsilon(y)| + (r_i^\varepsilon)^4|\chi_{1,1}^\varepsilon(y)|\right)dy\\
&= O\left(\varepsilon^3\left(\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\nabla\eta^\varepsilon(y)|\frac{1}{1+|y|}\,dy + (r_i^\varepsilon)^4\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\nabla\eta^\varepsilon(y)|\frac{1}{(\mu_1^\varepsilon+|y-a_1^\varepsilon|)^\gamma}\,dy\right)\right)\\
&= O\left(\varepsilon^3\rho_\eta^\varepsilon\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{1}{1+|y|^3}\,dy\right) + O\left(\varepsilon^3(r_1^\varepsilon)^4(\rho_\eta^\varepsilon)^{1/3}(r_1^\varepsilon)^{-2/3}\left(\mu_1^\varepsilon+|x-a_1^\varepsilon|\right)^{-2/3}(1+|x|)^{-1/3}\right)\\
&= O\left(\varepsilon^3\rho_\eta^\varepsilon(1+|x|)^{-1}\right) + o\left(\rho_\eta^\varepsilon(1+|x|)^{-1}\right) + O\left(\frac{\varepsilon^4(r_1^\varepsilon)^3}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right).
\end{aligned} \tag{5.7.55}$$

Coming back to (5.7.37) with (5.7.37), (5.7.38), (5.7.39), (5.7.44), (5.7.45), (5.7.49), (5.7.54) and (5.7.55), we get Claim 5.7.7. □

Hence, combining (5.7.32) with (5.7.33) and Claim 5.7.7, we get Claim 5.7.6. □

**Claim 5.7.8.** *For every $R>0$ and $x\in\mathbb{D}_0(R)$, we have that*

$$\begin{aligned}(1+|x|)|\nabla\eta^\varepsilon(x)| \lesssim\ & \left(\rho^\varepsilon_{\nu,0,R}+\rho^\varepsilon_{\eta,0,R}+\varepsilon^3+R^{-1}(\rho^\varepsilon_\varphi+\rho^\varepsilon_\nu+\rho^\varepsilon_\eta)\right)(1+|x|)^{-1}\\ &+\left(\left(\frac{1}{r_1^\varepsilon}\rho^\varepsilon_{\nu,1,R}+\rho^\varepsilon_{\eta,1,R}\right)\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\\ &+\left(\frac{1}{r_1^\varepsilon}\rho^\varepsilon_\nu+\rho^\varepsilon_\eta\right)\left(R+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\\ &+\left(\frac{\varepsilon^3(r_1^\varepsilon)^2}{\mu_1^\varepsilon}+\frac{1}{r_1^\varepsilon}\right)\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right)\\ &+o\left(\left(\rho^\varepsilon+\frac{1}{r_1^\varepsilon}\right)(1+|x|)^{-1}\right).\end{aligned}\tag{5.7.56}$$

*Proof of Claim 5.7.8:* First, thanks to Hodge decomposition theorem, we can write that

$$\nabla\eta^\varepsilon-3\left(\nabla\eta^\varepsilon\cdot_{h^\varepsilon}\vec{n}^\varepsilon\right)\vec{n}^\varepsilon=\nabla C^\varepsilon+\nabla^\perp D^\varepsilon,\tag{5.7.57}$$

where $C^\varepsilon, D^\varepsilon\in C^\infty(\mathbb{R}^2,\mathbb{R}^3)$ satisfy

$$\left(|\nabla C^\varepsilon|+|\nabla D^\varepsilon|\right)(x)=O\left(\frac{1}{|x|^2}\right)\ \text{as}\ |x|\to+\infty.$$

Thanks to the third equation of the system (5.6.31) in Proposition 5.6.3, we have that

$$\begin{cases}\Delta C^\varepsilon=-\nabla^\perp\nu^\varepsilon\wedge\nabla\eta^\varepsilon-\nabla^\perp\mathcal{N}_0^\varepsilon\wedge\nabla\eta^\varepsilon-\nabla^\perp\mathcal{N}_1^\varepsilon\wedge\nabla\eta^\varepsilon\\ \qquad-\frac{1}{r_0^\varepsilon}\nabla^\perp(\nu^\varepsilon+\mathcal{N}_1^\varepsilon)\wedge\nabla\mathcal{H}_0^\varepsilon-\frac{1}{r_1^\varepsilon}\nabla^\perp(\nu^\varepsilon+\mathcal{N}_0^\varepsilon)\wedge\nabla\mathcal{H}_1^\varepsilon\\ \qquad+3\left(div\left(\left(\nabla\frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon\cdot_{h_0^\varepsilon}(\vec{n}-\mathcal{N}_0^\varepsilon)\right)\vec{n}\right)+div\left(\left(\nabla\frac{1}{r_0^\varepsilon}\mathcal{H}_0^\varepsilon\cdot_{h_0^\varepsilon}\mathcal{N}_0^\varepsilon\right)(\vec{n}-\mathcal{N}_0^\varepsilon)\right)\right)\\ \qquad+3\left(div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon\cdot_{h_1^\varepsilon}(\vec{n}-\mathcal{N}_1^\varepsilon)\right)\vec{n}\right)+div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\mathcal{H}_1^\varepsilon\cdot_{h_1^\varepsilon}\mathcal{N}_1^\varepsilon\right)(\vec{n}-\mathcal{N}_1^\varepsilon)\right)\right)\\ \qquad+R^\varepsilon(\eta)\\ \Delta D^\varepsilon=3div\left(\left(\nabla^\perp\eta^\varepsilon\cdot_{h^\varepsilon}\vec{n}^\varepsilon\right)\vec{n}^\varepsilon\right)\\ \qquad=h^\varepsilon(\Phi^\varepsilon)_{ij}div\left((\nabla^\perp\eta^\varepsilon)^i(\vec{n}^\varepsilon)^j\vec{n}^\varepsilon\right)+\partial_l h^\varepsilon_{ij}(\Phi^\varepsilon)\left(\nabla\Phi^\varepsilon\right)^l(\nabla^\perp\eta^\varepsilon)^i(\vec{n}^\varepsilon)^j)\vec{n}^\varepsilon.\end{cases}\tag{5.7.58}$$

We apply the Green's representation formula (6.2.4) to (5.7.58). Most of the terms have already been estimated in Claims 5.7.1 and 5.7.6, see in particular (5.7.9), (5.7.10), (5.7.12), (5.7.29), (5.7.34) and

(5.7.30). We thus get that

$$
\begin{aligned}
(1+|x|)\,|\nabla C^{\varepsilon}(x)| \lesssim\ & (1+|x|)\,|\nabla \psi^{\varepsilon}_{\eta,\nu}(x)| \\
& + \left(\rho^{\varepsilon}_{0,R} + \varepsilon^3 + R^{-1}(\rho^{\varepsilon})\right)(1+|x|)^{-1} \\
& + \left(\left(\frac{1}{r^{\varepsilon}_1}\rho^{\varepsilon}_{\nu,1,R} + \rho^{\varepsilon}_{\eta,1,R}\right)\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right) \\
& + \left(\frac{1}{r^{\varepsilon}_1}\rho^{\varepsilon}_{\nu} + \rho^{\varepsilon}_{\eta}\right)\left(R+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right) \\
& + \left(\frac{\varepsilon^3 (r^{\varepsilon}_1)^2}{\mu^{\varepsilon}_1} + \frac{1}{r^{\varepsilon}_1}\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right) \\
& + \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(\left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_0}\mathcal{H}^{\varepsilon}_0\cdot_{h^{\varepsilon}_0}(\vec{n}-\mathcal{N}^{\varepsilon}_0)\right)\vec{n}\right)\right| + \left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_0}\mathcal{H}^{\varepsilon}_0\cdot_{h^{\varepsilon}_0}\mathcal{N}^{\varepsilon}_0\right)(\vec{n}-\mathcal{N}^{\varepsilon}_0)\right)\right|\right)(y)\,dy \\
& + \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(\left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_1}\mathcal{H}^{\varepsilon}_1\cdot_{h^{\varepsilon}_1}(\vec{n}-\mathcal{N}^{\varepsilon}_1)\right)\vec{n}\right)\right| + \left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_1}\mathcal{H}^{\varepsilon}_1\cdot_{h^{\varepsilon}_1}\mathcal{N}^{\varepsilon}_1\right)(\vec{n}-\mathcal{N}^{\varepsilon}_1)\right)\right|\right)(y)\,dy \\
& + o\left(\left(\rho^{\varepsilon}+\frac{1}{r^{\varepsilon}_1}\right)(1+|x|)^{-1}\right) \text{ and} \\
(1+|x|)\,|\nabla D^{\varepsilon}(x)| \lesssim\ & (1+|x|)\,|\nabla \psi^{\varepsilon}_{\eta,\nu}(x)| \\
& + \left(\rho^{\varepsilon}_{0,R} + \varepsilon^3 + R^{-1}(\rho^{\varepsilon}) + o\left(\frac{1}{r^{\varepsilon}_1}\right)\right)(1+|x|)^{-1} \\
& + \left(\rho^{\varepsilon}_{\eta,1,R}\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right) \\
& + \left(\rho^{\varepsilon}_{\eta}\right)\left(R+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right) \\
& + \left(\frac{\varepsilon^3 (r^{\varepsilon}_1)^2}{\mu^{\varepsilon}_1} + \frac{1}{r^{\varepsilon}_1}\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right) \\
& + \varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\varphi^{\varepsilon}(y)||\nabla\eta^{\varepsilon}(y)|\,dy. \\
& + o\left(\left(\rho^{\varepsilon}+\frac{1}{r^{\varepsilon}_1}\right)(1+|x|)^{-1}\right).
\end{aligned}
\tag{5.7.59}
$$

In order to get estimates on $|\nabla\psi^{\varepsilon}_{\eta,\nu}(x)|$, we show that we have the following claim:

**Claim 5.7.9.**

$$
(1+|x|)\,|\nabla\psi^{\varepsilon}_{\eta,\nu}(x)| = o\left(\left(\varepsilon^3+\rho^{\varepsilon}+\frac{1}{(r^{\varepsilon}_1)}+\frac{\varepsilon^3(r^{\varepsilon}_1)}{\mu^{\varepsilon}_1}\right)(1+|x|)^{-1}\right) + o\left(\frac{\varepsilon^3(r^{\varepsilon}_1)^2}{\mu^{\varepsilon}_1}\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\right). \tag{5.7.60}
$$

*Proof of Claim 5.7.9:* Thanks to Proposition 5.7.10 and the definition of $\psi^{\varepsilon}_{\eta,\nu}$, together with (5.7.44),

(5.7.54) and (5.7.55), we can write with the Green's representation formula that

$$\begin{aligned}
(1+|x|)|\nabla\psi^\varepsilon_{\eta,\nu}(x)| \lesssim\ & (1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi,\eta}(x)| \\
&+ \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)||\nabla\psi^\varepsilon_{\nu,\nu}(y)|dy \\
&+ \sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(\frac{1}{r_i^\varepsilon}|\nabla\psi^\varepsilon_{i,\varphi}(y)|+|\nabla\psi^\varepsilon_{i,\nu}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,\eta}(y)|+\frac{1}{r_i^\varepsilon}|\nabla\psi^\varepsilon_{i,1-i}(y)|\right)dy \\
&+ \varepsilon^2\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi^\varepsilon_{\varphi,\varphi}(y)|+|\chi^\varepsilon_{\varphi,\nu}(y)|+|\chi^\varepsilon_{\varphi,\eta}(y)|++|\chi^\varepsilon_{\nu,\eta}(y)|\right)dy \\
&+ \varepsilon^2\sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi^\varepsilon_{i}(y)|+\frac{1}{r_i^\varepsilon}|\chi^\varepsilon_{i,\varphi}(y)|+|\chi^\varepsilon_{i,\nu}(y)|+r_i^\varepsilon|\chi^\varepsilon_{i,\eta}(y)|+\frac{1}{r_i^\varepsilon}|\chi^\varepsilon_{i,1-i}(y)|\right)dy \\
&+ \varepsilon^3\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(|\chi^\varepsilon_{0,0}(y)|+(r_1^\varepsilon)^2|\chi^\varepsilon_{1,1}(y)|\right)dy \\
\leq\ & (1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi,\eta}(x)| \\
&+ \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)||\nabla\psi^\varepsilon_{\nu,\nu}(y)|dy \\
&+ \sum_{i=0,1}\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^\varepsilon(y)|\left(\frac{1}{r_i^\varepsilon}|\nabla\psi^\varepsilon_{i,\varphi}(y)|+|\nabla\psi^\varepsilon_{i,\nu}(y)|+r_i^\varepsilon|\nabla\psi^\varepsilon_{i,\eta}(y)|+\frac{1}{r_i^\varepsilon}|\nabla\psi^\varepsilon_{i,1-i}(y)|\right)dy \\
&+ o\left(\left(\varepsilon^3+\rho^\varepsilon\right)(1+|x|)^{-1}\right)+o\left(\frac{\varepsilon^3(r_1^\varepsilon)^2}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right),
\end{aligned} \tag{5.7.61}$$

where $\psi^\varepsilon_{\eta,\varphi,\eta}$ is the bounded solution in $C^2(\mathbb{R}^2,\mathbb{R}^3)$ of

$$\begin{cases}\Delta\psi^\varepsilon_{\eta,\varphi,\eta}=\nabla\eta^\varepsilon\wedge\nabla^\perp\psi^\varepsilon_{\varphi,\eta}\\ \lim\limits_{|x|\to+\infty}\psi^\varepsilon_{\eta,\varphi,\eta}(x)=0\end{cases}. \tag{5.7.62}$$

Note here that we estimated the terms in $|\chi^\varepsilon_{\varphi,\eta}|$ and in $|\chi^\varepsilon_{\nu,\eta}|$ similarily as we did for the terms in $|\chi^\varepsilon_{\varphi,\varphi}|$ in (5.7.44).

Thanks to the weak estimates (5.5.41) on $|\nabla\eta^\varepsilon|$ together with (5.5.53), we have that

$$r_1^\varepsilon\|\nabla\eta^\varepsilon\|_{2,+\infty}+\|\nabla\varphi^\varepsilon\|_2=o(1), \tag{5.7.63}$$

which, with the help of generalized Wente inequality in Lemma 6.3.6, implies that $r_1^\varepsilon\|\nabla\psi^\varepsilon_{\varphi,\eta}\|_2=o\left(1\right)$ and then that $(r_1^\varepsilon)^2\|\nabla\psi^\varepsilon_{\eta,\varphi,\eta}\|_2=o(1)$ by the definition (5.7.62) of $\psi^\varepsilon_{\eta,\varphi,\eta}$. Therefore, thanks to Lemma 6.3.8, we have

$$\begin{aligned}
(1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi,\eta}(x)| &\lesssim \|\nabla\psi^\varepsilon_{\eta,\varphi,\eta}\|_2^{1/2}(\rho_\eta^\varepsilon)^{1/2}\|(1+|x|^2)\nabla\psi^\varepsilon_{\varphi,\eta}\|_\infty^{1/2}(1+|x|)^{-1} \\
&= o\left(\frac{1}{r_1^\varepsilon}(\rho_\eta^\varepsilon)^{1/2}\|(1+|x|^2)\nabla\psi^\varepsilon_{\varphi,\eta}\|_\infty^{1/2}\left(1+|x|\right)^{-1}\right). \\
&= o\left(\rho_\eta^\varepsilon(1+|x|)^{-1}\right)+o\left(\frac{1}{(r_1^\varepsilon)^2}\|(1+|x|^2)\nabla\psi^\varepsilon_{\varphi,\eta}\|_\infty(1+|x|)^{-1}\right)
\end{aligned} \tag{5.7.64}$$

Moreover, thanks to Claim 5.7.7, we have

$$\begin{aligned}
(1+|x|)|\nabla\psi^\varepsilon_{\eta,\varphi}(x)| \lesssim\ & o\left((\varepsilon^3+\rho^\varepsilon)(1+|x|)^{-1}\right) \\
&+ o\left(\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\right)
\end{aligned} \tag{5.7.65}$$

which, together with (5.7.64), we get that

$$(1+|x|)|\nabla\psi^{\varepsilon}_{\eta,\varphi,\eta}(x)| = o\left(\rho^{\varepsilon}_{\eta}(1+|x|)^{-1}\right) + o\left(\left(\frac{\varepsilon^3}{(r^{\varepsilon}_1)^2}+\frac{\rho^{\varepsilon}}{(r^{\varepsilon}_1)^2}+\frac{\varepsilon^3(r^{\varepsilon}_1)}{\mu^{\varepsilon}_1}\right)(1+|x|)^{-1}\right) \tag{5.7.66}$$

Again, thanks to generalized Wente inequality, we state that

$$\|\nabla\psi^{\varepsilon}_{\nu,\nu}\|_{2,1} + \|\nabla\psi^{\varepsilon}_{i,\varphi}\|_{2,1} + \|\nabla\psi^{\varepsilon}_{i,\nu}\|_{2,1} + r^{\varepsilon}_i\|\nabla\psi^{\varepsilon}_{i,\eta}\|_{2,1} = o(1) \tag{5.7.67}$$

as $\varepsilon\to 0$, since we have that $\|\nabla\varphi^{\varepsilon}\|_2 + \|\nabla\nu^{\varepsilon}\|_2 + \|\nabla\eta^{\varepsilon}\|_{2,+\infty} = o(1)$ and $\|\nabla\mathcal{B}^{\varepsilon}_i\|_{2,1} = O(1)$. Therefore, using Hölder's inequality in Lorentz spaces, we can write that

$$\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}|\nabla\eta^{\varepsilon}(y)|\left(|\nabla\psi^{\varepsilon}_{\nu,\nu}(y)| + |\nabla\psi^{\varepsilon}_{i,\varphi}(y)| + |\nabla\psi^{\varepsilon}_{i,\nu}(y)| + r^{\varepsilon}_i|\nabla\psi^{\varepsilon}_{i,\eta}(y)|\right)dy = o\left(\rho^{\varepsilon}_{\eta}\right). \tag{5.7.68}$$

Coming back to (5.7.61) with (5.7.68) and (5.7.66), we obtain Claim 5.7.9. □

Let us now write thanks to the definition of $\nu^{\varepsilon}$ that

$$\begin{aligned}
&\left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_i}\mathcal{H}^{\varepsilon}_i\cdot_{h^{\varepsilon}_i}(\vec{n}-\mathcal{N}^{\varepsilon}_i)\right)\vec{n}\right)\right| + \left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_i}\mathcal{H}^{\varepsilon}_i\cdot_{h^{\varepsilon}_i}\mathcal{N}^{\varepsilon}_i\right)(\vec{n}-\mathcal{N}^{\varepsilon}_i)\right)\right| \\
&\qquad\lesssim \frac{1}{r^{\varepsilon}_i}\left(|\Delta\mathcal{H}^{\varepsilon}_i| + |\nabla\mathcal{H}^{\varepsilon}_i||\nabla\mathcal{N}^{\varepsilon}_i| + \varepsilon^2|\nabla\mathcal{H}^{\varepsilon}_i||\nabla\Phi^{\varepsilon}|\right)|\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i| \\
&\qquad\quad + \frac{1}{r^{\varepsilon}_i}|\nabla\mathcal{H}^{\varepsilon}_i||\nabla\nu^{\varepsilon}| + \frac{1}{r^{\varepsilon}_i}|\nabla\mathcal{H}^{\varepsilon}_i||\nabla\mathcal{N}^{\varepsilon}_{1-i}| \\
&\qquad\lesssim \frac{1}{r^{\varepsilon}_i}(|\nabla\mathcal{B}^{\varepsilon}_i|^2 + \varepsilon^2|\nabla\mathcal{B}^{\varepsilon}_i||\nabla\Phi^{\varepsilon}|)|\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i| + \frac{1}{r^{\varepsilon}_i}|\nabla\mathcal{B}^{\varepsilon}_i||\nabla\nu^{\varepsilon}| \\
&\qquad\quad + \frac{1}{r^{\varepsilon}_i}|\nabla\mathcal{B}^{\varepsilon}_i||\nabla\mathcal{B}^{\varepsilon}_{1-i}|.
\end{aligned} \tag{5.7.69}$$

Here we used the fact that $|\vec{n}^{\varepsilon}| + |\mathcal{N}^{\varepsilon}_i| + |\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i| = O(1)$ and that

$$|\Delta\mathcal{H}^{\varepsilon}_i| + |\nabla\mathcal{N}^{\varepsilon}_i|^2 + |\nabla\mathcal{H}^{\varepsilon}_i|^2 = O\left(|\nabla\mathcal{B}^{\varepsilon}_i|^2\right)$$

thanks to Proposition 5.6.1.

Proceeding as in (5.7.20), thanks to Proposition 5.6.2, we can write that

$$\begin{aligned}
|(\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i)(x)| &\le |\left(\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i\right)(x) - \left(\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i\right)(a^{\varepsilon}_i)| + |\vec{n}^{\varepsilon}-\mathcal{N}^{\varepsilon}_i|(a^{\varepsilon}_i) \\
&\lesssim \left(\varepsilon^3 + |\nu^{\varepsilon}(x)-\nu^{\varepsilon}(a^{\varepsilon}_i)| + |\mathcal{N}^{\varepsilon}_{1-i}(x)-\mathcal{N}^{\varepsilon}_{1-i}(a^{\varepsilon}_i)|\right)
\end{aligned}$$

which leads together with (5.7.69) to

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(\left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_0}\mathcal{H}^{\varepsilon}_0\cdot_{h^{\varepsilon}_0}(\vec{n}-\mathcal{N}^{\varepsilon}_0)\right)\vec{n}\right)\right| + \left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_0}\mathcal{H}^{\varepsilon}_0\cdot_{h^{\varepsilon}_0}\mathcal{N}^{\varepsilon}_0\right)(\vec{n}-\mathcal{N}^{\varepsilon}_0)\right)\right|\right)(y)\,dy \\
&\quad + \int_{\mathbb{R}^2}\frac{1+|y|}{|x-y|}\left(\left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_1}\mathcal{H}^{\varepsilon}_1\cdot_{h^{\varepsilon}_1}(\vec{n}-\mathcal{N}^{\varepsilon}_1)\right)\vec{n}\right)\right| + \left|div\left(\left(\nabla\frac{1}{r^{\varepsilon}_1}\mathcal{H}^{\varepsilon}_1\cdot_{h^{\varepsilon}_1}\mathcal{N}^{\varepsilon}_1\right)(\vec{n}-\mathcal{N}^{\varepsilon}_1)\right)\right|\right)(y)\,dy \\
&\lesssim \left(\rho^{\varepsilon}_{0,R}+\varepsilon^3+R^{-1}(\rho^{\varepsilon})\right)(1+|x|)^{-1} \\
&\quad + \left(\left(\frac{1}{r^{\varepsilon}_1}\rho^{\varepsilon}_{\nu,1,R}+\rho^{\varepsilon}_{\eta,1,R}\right)\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right) \\
&\quad + \left(\frac{1}{r^{\varepsilon}_1}\rho^{\varepsilon}_{\nu}+\rho^{\varepsilon}_{\eta}\right)\left(R+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right) \\
&\quad + \left(\frac{\varepsilon^3(r^{\varepsilon}_1)^2}{\mu^{\varepsilon}_1}+\frac{1}{r^{\varepsilon}_1}\right)\left(1+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1}\right)^{-1}\ln\left(2+\frac{|x-a^{\varepsilon}_1|}{\mu^{\varepsilon}_1(1+|x|)}\right).
\end{aligned} \tag{5.7.70}$$

Note here that we estimated the terms in $|\mathcal{N}^\varepsilon_{1-i}(x) - \mathcal{N}^\varepsilon_{1-i}(a^\varepsilon_i)|$ as we did for the terms in $|\mathcal{B}^\varepsilon_{1-i}(x) - \mathcal{B}^\varepsilon_{1-i}(a^\varepsilon_i)|$, see Claim 5.7.5.

Coming back to (5.7.59) together with Claim 5.7.9 and (5.7.70), we get that

$$\begin{aligned}|\nabla C^\varepsilon(x)| + |\nabla D^\varepsilon(x)| \leq C \sum_{i,j\neq i} \left( \left( \rho^\varepsilon_{i,R} + \frac{\varepsilon^3}{\mu^\varepsilon_i} + \tau^\varepsilon_{ij} \right) \left( 1 + \frac{|x - a^\varepsilon_i|}{\mu^\varepsilon_i} \right)^{-2/3} \right) + \delta^\varepsilon_R \rho^\varepsilon \\ + o\left( \sum_{i,j\neq i} \left( \frac{\varepsilon^3}{\mu^\varepsilon_i} + \tau^\varepsilon_{ij} \right) \right).\end{aligned} \tag{5.7.71}$$

Finally, using the fact that $|\nabla \eta^\varepsilon| \leq |\nabla \eta^\varepsilon - 3(\nabla \eta^\varepsilon \cdot_{h^\varepsilon} \vec{n}^\varepsilon)\vec{n}^\varepsilon| \lesssim |\nabla C^\varepsilon| + |\nabla D^\varepsilon|$ together with (5.7.71), we get Claim 5.7.8. □

Thanks to Claims 5.7.1, 5.7.6 and 5.7.8, we get the following strong pointwise estimates on $|\nabla \varphi^\varepsilon|$, $|\nabla \nu^\varepsilon|$ and $|\nabla \eta^\varepsilon|$:

**Proposition 5.7.10.** *We have that, for every $R > 0$,*

$$\begin{aligned}(1+|x|)|\nabla \varphi^\varepsilon|(x) \lesssim \left(\rho^\varepsilon_{0,R} + \varepsilon^3 + R^{-1}\rho^\varepsilon\right)(1+|x|)^{-1} \\ + \left(\left(\rho^\varepsilon_{\varphi,1,R} + r^\varepsilon_1 \rho^\varepsilon_{\nu,1,R}\right)\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right) \\ + \left(\left(\rho^\varepsilon_\varphi + r^\varepsilon_1 \rho^\varepsilon_\nu\right)\right) \left(R + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right) \\ + \left(\frac{\varepsilon^3 (r^\varepsilon_1)^4}{\mu^\varepsilon_1} + 1\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right) \\ + o\left(\rho^\varepsilon (1+|x|)^{-1}\right)\end{aligned} \tag{5.7.72}$$

$$\begin{aligned}(1+|x|)|\nabla \nu^\varepsilon(x)| \lesssim \left(\rho^\varepsilon_{0,R} + \varepsilon^3 + R^{-1}\rho^\varepsilon\right)(1+|x|)^{-1} \\ + \left(\left(\frac{1}{r^\varepsilon_1}\rho^\varepsilon_{\varphi,1,R} + \rho^\varepsilon_{\nu,1,R} + r^\varepsilon_1 \rho^\varepsilon_{\eta,1,R}\right)\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right) \\ + \left(\frac{1}{r^\varepsilon_1}\rho^\varepsilon_\varphi + \rho^\varepsilon_\nu + r^\varepsilon_1 \rho^\varepsilon_\eta\right) \left(R + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right) \\ + \left(\frac{\varepsilon^3 (r^\varepsilon_1)^3}{\mu^\varepsilon_1} + \frac{1}{r^\varepsilon_1}\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right) \\ + o\left(\left(\rho^\varepsilon + \frac{1}{r^\varepsilon_1}\right)(1+|x|)^{-1}\right)\end{aligned} \tag{5.7.73}$$

$$\begin{aligned}(1+|x|)|\nabla \eta^\varepsilon(x)| \lesssim \left(\rho^\varepsilon_{0,R} + \varepsilon^3 + R^{-1}(\rho^\varepsilon_\varphi + \rho^\varepsilon_\nu + \rho^\varepsilon_\eta)\right)(1+|x|)^{-1} \\ + \left(\left(\frac{1}{r^\varepsilon_1}\rho^\varepsilon_{\nu,1,R} + \rho^\varepsilon_{\eta,1,R}\right)\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right) \\ + \left(\frac{1}{r^\varepsilon_1}\rho^\varepsilon_\nu + \rho^\varepsilon_\eta\right) \left(R + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right) \\ + \left(\frac{\varepsilon^3 (r^\varepsilon_1)^2}{\mu^\varepsilon_1} + \frac{1}{r^\varepsilon_1}\right) \left(1 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1}\right)^{-1} \ln\left(2 + \frac{|x-a^\varepsilon_1|}{\mu^\varepsilon_1 (1+|x|)}\right) \\ + o\left(\left(\rho^\varepsilon + \frac{1}{r^\varepsilon_1}\right)(1+|x|)^{-1}\right).\end{aligned} \tag{5.7.74}$$

### 5.7.2 Interactions between bubble terms and Riemannian terms

Thanks to the pointwise estimates (5.5.7), (5.5.41), we get that

**Lemma 5.7.11.** *For every $R > 0$, we have that*

$$\frac{1}{r_1^\varepsilon}(\rho^\varepsilon_{\varphi,0,R} + \rho^\varepsilon_{\nu,0,R}) + \rho^\varepsilon_{\eta,0,R} = o\left(\frac{1}{r_1^\varepsilon}\right). \tag{5.7.75}$$

Before proving our main theorem 5.2.3, we state that the dominating term in our strong pointwise estimates (5.7.72) is $\frac{\varepsilon^3}{\mu_1^\varepsilon}$ due to the derivatives of the curvature of the space $(\mathbb{R}^3, h^\varepsilon)$. The main goal of this section is thus to prove the following Lemma:

**Lemma 5.7.12.** *We have that*

$$\left(\rho^\varepsilon + \frac{1}{r_1^\varepsilon}\right) = O\left(\frac{\varepsilon^3 (r_1^\varepsilon)^3}{\mu_1^\varepsilon}\right). \tag{5.7.76}$$

By contradiction, we assume that

$$\frac{\varepsilon^3 (r_1^\varepsilon)^3}{\mu_1^\varepsilon} = o\left(\frac{1}{r_1^\varepsilon} + \rho^\varepsilon\right).$$

**Claim 5.7.13.** *We have*

$$\rho^\varepsilon = O\left(\frac{1}{r_1^\varepsilon}\right) \tag{5.7.77}$$

*Proof.* By contradiction, we assume that

$$\frac{1}{r_1^\varepsilon} = o(\rho^\varepsilon). \tag{5.7.78}$$

Combining (5.7.78) with (5.7.72), (5.7.73) and (5.7.74), together with Lemma 5.7.11, we get that, for every $R > 0$,

$$\begin{aligned}(1+|x|)\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi| + |\nabla\nu^\varepsilon| + r_1^\varepsilon|\nabla\eta^\varepsilon|\right)(x) \lesssim{}& \left(R^{-1}\rho^\varepsilon\right)(1+|x|)^{-1} \\ &+ \rho^\varepsilon\left(1 + \frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1} \ln\left(2 + \frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right) \\ &+ o\left(\rho^\varepsilon(1+|x|)^{-1}\right)\end{aligned} \tag{5.7.79}$$

in $\mathbb{D}_0(R)$. Let $x^\varepsilon \in \mathbb{R}^2$ be such that

$$\left(1+|x^\varepsilon|^2\right)\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi^\varepsilon| + |\nabla\nu^\varepsilon| + r_1^\varepsilon|\nabla\eta^\varepsilon|\right)(x^\varepsilon) = \frac{\rho^\varepsilon}{2}.$$

**Claim 5.7.14.** *Up to a subsequence, there exists $R > 0$ such that $x^\varepsilon \in \Omega_1^\varepsilon(R)$. In particular, we have $\frac{\rho^\varepsilon}{2} \leq \rho^\varepsilon_{1,R}$*

*Proof.* First, we cannot have $|x^\varepsilon| \to +\infty$ as $\varepsilon \to 0$, since, thanks to Theorem 5.5.1, we have that

$$\sup_{|x|\geq R} (1+|x|^2)\left(|\nabla\varphi^\varepsilon|(x)+|\nabla\nu^\varepsilon(x)+|\nabla\eta^\varepsilon(x)|\right) = o(1) \text{ for every } R>0. \tag{5.7.80}$$

Therefore, we have $\sup_{\varepsilon>0}|x^\varepsilon|<+\infty$.

Next, if we have $\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\to+\infty$ as $\varepsilon\to 0$, thanks to (5.7.79), we get that, for $R>0$ big enough,

$$\begin{aligned}\frac{\rho^\varepsilon}{2} &\lesssim (1+|x^\varepsilon|)\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi^\varepsilon|+|\nabla\nu^\varepsilon|+r_1^\varepsilon|\nabla\eta^\varepsilon|\right)(x^\varepsilon)\\ &\lesssim \frac{\rho^\varepsilon}{R}+o(\rho^\varepsilon)+o(\rho^\varepsilon)\\ &= \frac{\rho^\varepsilon}{R}+o(\rho^\varepsilon),\end{aligned} \tag{5.7.81}$$

which gives contradition by taking $R>0$ large enough and thus proves Claim 5.7.14. □

On every compact set of $\mathbb{R}^2$, we have that

$$\begin{aligned}&\frac{1}{r_1^\varepsilon}|\nabla\tilde\varphi^\varepsilon|+|\nabla\tilde\nu^\varepsilon|+r_1^\varepsilon|\nabla\tilde\eta^\varepsilon| = O(\mu_1^\varepsilon\rho^\varepsilon),\\ &|\nabla\tilde{\mathcal{B}}_1^\varepsilon|+|\nabla\tilde{\mathcal{N}}_1^\varepsilon|+|\nabla\tilde{\mathcal{H}}_1^\varepsilon| = O(1),\\ &\frac{1}{r_1^\varepsilon}|\nabla r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon|+|\nabla\tilde{\mathcal{N}}_0^\varepsilon|+\frac{r_1^\varepsilon}{r_0^\varepsilon}|\nabla\tilde{\mathcal{H}}_0^\varepsilon| = O\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right)=o(\mu_1^\varepsilon\rho^\varepsilon),\end{aligned} \tag{5.7.82}$$

where $\tilde f := f(\mu_1^\varepsilon\cdot+a_1^\varepsilon)$.

Thanks to (5.7.82) and Proposition 5.6.3, by straightforward computations, we get that

$$\begin{cases}\Delta\varphi^\varepsilon = \nabla\mathcal{N}_1^\varepsilon\wedge\nabla^\perp\varphi^\varepsilon + r_1^\varepsilon\left(\nabla\nu^\varepsilon\wedge\nabla^\perp\mathcal{B}_1^\varepsilon\right)+o\left(\mu_1^\varepsilon r_1^\varepsilon\rho^\varepsilon\right)\\ \Delta\nu^\varepsilon = -2\nabla^\perp\mathcal{N}_1^\varepsilon\wedge\nabla\nu^\varepsilon - 2\nabla\eta^\varepsilon\wedge\left(r_1^\varepsilon\nabla^\perp\mathcal{B}_1^\varepsilon\right)\\ \qquad\qquad -2\left(\frac{1}{r_1^\varepsilon}\nabla\mathcal{H}_1^\varepsilon\right)\wedge\nabla^\perp\varphi^\varepsilon+o\left(\mu_1^\varepsilon\rho^\varepsilon\right),\\ div\left(\nabla\eta^\varepsilon-3\left(\nabla\eta^\varepsilon\cdot\mathcal{N}_1^\varepsilon\right)\mathcal{N}_1^\varepsilon\right) = -\nabla^\perp\mathcal{N}_1^\varepsilon\wedge\nabla\eta^\varepsilon\\ \qquad\qquad -\frac{1}{r_1^\varepsilon}\nabla^\perp\nu^\varepsilon\wedge\nabla\mathcal{H}_1^\varepsilon+o\left(\frac{\mu_1^\varepsilon\rho^\varepsilon}{r_1^\varepsilon}\right).\end{cases} \tag{5.7.83}$$

For convenience reasons, we also write

$$\mathcal{L}_{\tilde{\mathcal{B}}_1^\varepsilon}\left(\frac{1}{r_1^\varepsilon}\tilde\varphi^\varepsilon,\tilde\nu^\varepsilon,r_1^\varepsilon\tilde\eta^\varepsilon\right)=o(\mu_1^\varepsilon\rho^\varepsilon), \tag{5.7.84}$$

where the meaning of $\mathcal{L}_{\tilde{\mathcal{B}}_1^\varepsilon}$ is given before the beginning of Subsection 6.5 in Appendix 6. Thanks to Proposition 5.6.4 and Propsition 5.6.2, we also get the relation of quasi-conformality:

$$\begin{aligned}&\left\langle(\tilde{\mathcal{B}}_1^\varepsilon)_x,\frac{1}{r_1^\varepsilon}\tilde\varphi_y^\varepsilon\right\rangle+\left\langle\frac{1}{r_1^\varepsilon}\tilde\varphi_x^\varepsilon,(\tilde{\mathcal{B}}_1^\varepsilon)_y\right\rangle=o(\mu_1^\varepsilon\rho^\varepsilon),\\ &\left\langle(\tilde{\mathcal{B}}_1^\varepsilon)_x,\frac{1}{r_1^\varepsilon}\tilde\varphi_x^\varepsilon\right\rangle-\left\langle\frac{1}{r_1^\varepsilon}\tilde\varphi_y^\varepsilon,(\tilde{\mathcal{B}}_1^\varepsilon)_y\right\rangle=o(\mu_1^\varepsilon\rho^\varepsilon),\end{aligned} \tag{5.7.85}$$

supplemented with the conditions

$$\begin{aligned}
&\left|\nabla\left(\frac{r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon}{r_1^\varepsilon}\right)\right|(0)=o(\mu_1^\varepsilon\rho^\varepsilon) \text{ and}\\
&\left|\nabla^2\left(\frac{r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon}{r_1^\varepsilon}\right)(\nabla\tilde{\mathcal{B}}_1^\varepsilon)\right|(0)=o(\mu_1^\varepsilon\rho^\varepsilon)\\
&\left|\nabla^2\left(\frac{r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon}{r_1^\varepsilon}\right)(\nabla\tilde{\mathcal{B}}_1^\varepsilon)\right|(0)=o(\mu_1^\varepsilon\rho^\varepsilon).
\end{aligned}\tag{5.7.86}$$

Then, by standard elliptic theory, $\frac{\tilde{\varphi}^\varepsilon-\tilde{\varphi}^\varepsilon(0)}{\mu_1^\varepsilon\rho^\varepsilon r_1^\varepsilon}$, $\frac{\tilde{\nu}^\varepsilon}{\mu_1^\varepsilon\rho^\varepsilon}$ and $\frac{r_1^\varepsilon\tilde{\eta}^\varepsilon}{\mu_1^\varepsilon\rho^\varepsilon}$ respectively converge to some $\tilde{\varphi}$, $\tilde{\nu}$ and $\tilde{\eta}$ in $C^2_{loc}(\mathbb{R}^2)$ which satisfy

$$\mathcal{L}_{\omega_1}(\tilde{\varphi},\tilde{\nu},\tilde{\eta})=0 \text{ in } \mathbb{R}^2,\tag{5.7.87}$$

and the relation of conformality

$$\begin{aligned}
&\langle(\omega_1)_x,\tilde{\varphi}_y\rangle+\langle\tilde{\varphi}_x,(\omega_1)_y\rangle=0,\\
&\langle(\omega_1)_x,\tilde{\varphi}_x\rangle-\langle\tilde{\varphi}_y,(\omega_1)_y\rangle=0,
\end{aligned}\tag{5.7.88}$$

supplemented with the conditions

$$|\nabla\tilde{\varphi}|(0)=0 \text{ and } |\nabla^2\tilde{\varphi}(\nabla\omega_1)|(0)=0.\tag{5.7.89}$$

Therefore, thanks to the classification of linearized solution in Proposition 6.4.3, see Appendix 6, we get that

$$\nabla\tilde{\varphi}\equiv\nabla\tilde{\nu}\equiv\nabla\tilde{\eta}\equiv 0,$$

which indicates $\rho^\varepsilon_{1,R}=o(\rho^\varepsilon)$ and thus contradicts the fact that $\frac{\rho^\varepsilon}{2}\leq\rho^\varepsilon_{1,R}$. Hence, we prove (5.7.77). □

Thanks to Lemma 5.7.11, Proposition 5.7.10, and Claim 5.7.13, we now get that

$$\begin{aligned}
(1+|x|)\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi|+|\nabla\nu^\varepsilon|+r_1^\varepsilon|\nabla\eta^\varepsilon|\right)(x)\lesssim\ & o\left(\frac{1}{r_1^\varepsilon}(1+|x|)^{-1}\right)\\
&+\frac{1}{r_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)^{-1}\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right) \text{ on } \mathbb{D}_0(R)\\
&+o\left(\frac{1}{r_1^\varepsilon}(1+|x|)^{-1}\right)
\end{aligned}\tag{5.7.90}$$

Again we rescale around $a_1^\varepsilon$ by setting $\tilde{f}:=f(\mu_1^\varepsilon\cdot+a_1^\varepsilon)$. Thanks to (5.7.90), we get that

$$\begin{aligned}
&\frac{1}{r_1^\varepsilon}|\nabla\tilde{\varphi}^\varepsilon|+|\nabla\tilde{\nu}^\varepsilon|+r_1^\varepsilon|\nabla\tilde{\eta}^\varepsilon|=O\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right),\\
&|\nabla\tilde{\mathcal{B}}_1^\varepsilon|+|\nabla\tilde{\mathcal{N}}_1^\varepsilon|+|\nabla\mathcal{H}_1^\varepsilon|=O(1) \text{ and}\\
&\frac{1}{r_1^\varepsilon}|\nabla(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon)|+|\nabla\tilde{\mathcal{N}}_0^\varepsilon|+r_1^\varepsilon\left|\nabla\left(\frac{1}{r_0^\varepsilon}\tilde{\mathcal{H}}_0^\varepsilon\right)\right|=O\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right)
\end{aligned}\tag{5.7.91}$$

Thanks to (5.7.90) and Claim (5.7.13), together with (5.6.11) and Proposition 5.6.3, on every compact subset of $\mathbb{R}^2$, by straightforward computations and weak estimates (5.5.7), we get that

$$\mathcal{L}_{\tilde{\mathcal{B}}_1^\varepsilon}\left(\frac{1}{r_1^\varepsilon}(\tilde{\varphi}^\varepsilon+r_0\tilde{\mathcal{B}}_0^\varepsilon),\tilde{\nu}^\varepsilon+\tilde{\mathcal{N}}_0^\varepsilon,r_1^\varepsilon\left(\tilde{\eta}^\varepsilon+\frac{1}{r_0^\varepsilon}\tilde{\mathcal{H}}_0^\varepsilon\right)\right)=o\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right),\tag{5.7.92}$$

and the relation of quasi-conformality

$$\left\langle \left(\tilde{\mathcal{B}}_1^\varepsilon\right)_x, \frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)_y\right\rangle + \left\langle \frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)_x, \left(\tilde{\mathcal{B}}_1^\varepsilon\right)_y\right\rangle = o\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right),$$
$$\left\langle \left(\tilde{\mathcal{B}}_1^\varepsilon\right)_x, \frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)_x\right\rangle - \left\langle \frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)_y, \left(\tilde{\mathcal{B}}_1^\varepsilon\right)_y\right\rangle = o\left(\frac{\mu_1^\varepsilon}{r_1^\varepsilon}\right),$$

supplemented with the conditions

$$\left|\nabla\frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)\right|(0) = o(\mu_1^\varepsilon\tau^\varepsilon),$$
$$\left|\nabla^2\frac{1}{r_1^\varepsilon}\left(r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon+\tilde{\varphi}^\varepsilon\right)(\nabla\tilde{\mathcal{B}}_1^\varepsilon)\right|(0) = o(\mu_1^\varepsilon\tau^\varepsilon).$$

Thanks to standard elliptic theory, $\frac{\tilde{\varphi}^\varepsilon+r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon-(\tilde{\varphi}^\varepsilon+r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon)(0)}{\mu_1^\varepsilon}$, $\frac{r_1^\varepsilon(\tilde{\nu}^\varepsilon+\tilde{\mathcal{N}}_0^\varepsilon)}{\mu_1^\varepsilon}$ and $\frac{(r_1^\varepsilon)^2\left(\tilde{\eta}^\varepsilon+\frac{1}{r_0^\varepsilon}\tilde{\mathcal{H}}_0^\varepsilon\right)}{\mu_1^\varepsilon}$ respectively converge in $C^2_{loc}(\mathbb{R}^2)$ to some $\tilde{\sigma}_\varphi$, $\tilde{\sigma}_\nu$ and $\tilde{\sigma}_\eta$, which satisfy

$$\mathcal{L}_{\omega_1}(\tilde{\sigma}_\varphi,\tilde{\sigma}_\nu,\tilde{\sigma}_\eta) = 0 \text{ on } \mathbb{R}^2, \tag{5.7.93}$$

and the relation of conformality

$$\langle(\omega_1)_x,(\tilde{\sigma}_\varphi)_y\rangle + \langle(\tilde{\sigma}_\varphi)_x,(\omega_1)_y\rangle = 0,$$
$$\langle(\omega_1)_x,(\tilde{\sigma}_\varphi)_x\rangle - \langle(\tilde{\sigma}_\varphi)_y,(\omega_1)_y\rangle = 0,$$

supplemented with the initial conditions

$$|\nabla(\tilde{\sigma}_\varphi)|(0) = 0,$$
$$\left|\nabla^2(\tilde{\sigma}_\varphi)(\nabla\omega_1)\right|(0) = 0.$$

Thanks to Proposition 6.4.3, we get that

$$\nabla\tilde{\sigma}_\varphi \equiv \nabla\tilde{\sigma}_\nu \equiv \nabla\tilde{\sigma}_\eta \equiv 0. \tag{5.7.94}$$

On the other hand, thanks to the definition of $r_0^\varepsilon\mathcal{B}_0^\varepsilon$, we have that $\nabla\left(\frac{r_0^\varepsilon\tilde{\mathcal{B}}_0^\varepsilon}{\mu_1^\varepsilon}\right)$ converges in $C^2_{loc}(\mathbb{R}^2)$ to some non-zero constant 2-vector $\nabla\tilde{\mathcal{B}}_0$, and thus

$$|\nabla\tilde{\mathcal{B}}_0| > 0. \tag{5.7.95}$$

Coming back to (5.7.94) together with (5.7.95), we obtain that, for all $x^\varepsilon \in B(a_1^\varepsilon, R\mu_1^\varepsilon)$ with $R$ large enough, we have

$$\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi^\varepsilon| + |\nabla\nu^\varepsilon| + r_1^\varepsilon|\nabla\eta^\varepsilon|\right)(x^\varepsilon) \geq C\frac{1}{r_1^\varepsilon}, \tag{5.7.96}$$

for some $C > 0$.

Combining (5.7.96) with the estimates (5.7.90), for every $R > 0$ large enough, by taking any $|x^\varepsilon - a_1^\varepsilon| = R\mu_1^\varepsilon$, we get that

$$C\frac{1}{r_1^\varepsilon} \lesssim \frac{1}{r_1^\varepsilon}(1+R)^{-1}\ln(2+R) + o\left(\frac{1}{r_1^\varepsilon}\right), \tag{5.7.97}$$

which gives contradiction as $R \to +\infty$. Therefore, we prove Lemma 5.7.12.

Thanks to Lemma 5.7.12 and Proposition 5.7.10, we have the following proposition:

**Proposition 5.7.15.** *For $R > 0$ and $x \in \mathbb{D}_0(R)$, we have that*

$$(1+|x|)\left(\frac{1}{r_1^\varepsilon}|\nabla\varphi^\varepsilon(x)| + |\nabla\nu^\varepsilon(x)| + r_1^\varepsilon|\nabla\eta^\varepsilon(x)|\right) \lesssim \frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}\left(1+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon}\right)\ln\left(2+\frac{|x-a_1^\varepsilon|}{\mu_1^\varepsilon(1+|x|)}\right) + \delta_R^\varepsilon\frac{\varepsilon^3(r_1^\varepsilon)^3}{\mu_1^\varepsilon}, \tag{5.7.98}$$

*where* $\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\delta_R^\varepsilon = 0$.

### 5.7.3 How to conclude with these strong estimates ?

In order to prove our main theorem 5.2.3, we first place ourselves close to the bubble $r_1^\varepsilon\mathcal{B}_1^\varepsilon$. Set $\tilde{f} = f(\mu_1^\varepsilon\cdot + a_1^\varepsilon)$. Thanks to Proposition 5.7.15, we have that

$$\begin{aligned} |\nabla(\tilde{\mathcal{B}}_0^\varepsilon)| + |\nabla\tilde{\mathcal{N}}_0^\varepsilon| + |\nabla\tilde{\mathcal{H}}_0^\varepsilon| &= O(\mu_1^\varepsilon) \\ &= O\left(\varepsilon^3(r_1^\varepsilon)^4\right) \\ |\nabla(\tilde{\mathcal{B}}_1^\varepsilon)| + |\nabla\tilde{\mathcal{N}}_1^\varepsilon| + |\nabla\tilde{\mathcal{H}}_1^\varepsilon| &= |\nabla\tilde{\omega}_1^\varepsilon|(1+o(1)) \end{aligned} \tag{5.7.99}$$

Set

$$\begin{aligned} \tilde{\varphi}^\varepsilon &= \tilde{\Phi}^\varepsilon - r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon \\ \tilde{\nu}^\varepsilon &= \tilde{n}^\varepsilon - \tilde{\mathcal{N}}_1^\varepsilon \\ \tilde{\eta}^\varepsilon &= \tilde{\eta}^\varepsilon - \frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon. \end{aligned} \tag{5.7.100}$$

Thanks to (5.7.99) and (5.6.31), we get that, on every compact set of $\mathbb{R}^2$,

$$\begin{cases} \Delta\tilde{\varphi}^\varepsilon = \nabla\tilde{\mathcal{N}}_1^\varepsilon\wedge\nabla^\perp\tilde{\varphi}^\varepsilon + \nabla\tilde{\nu}^\varepsilon\wedge\nabla^\perp r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon \\ \qquad + O\left(|\nabla\tilde{\nu}^\varepsilon||\nabla\tilde{\varphi}^\varepsilon|\right) \\ \qquad + R^\varepsilon(\tilde{\varphi}) \\ \Delta\tilde{\nu}^\varepsilon = -2\nabla^\perp\tilde{\mathcal{N}}_1^\varepsilon\wedge\nabla\tilde{\nu}^\varepsilon - 2\nabla\tilde{\eta}^\varepsilon\wedge\nabla^\perp r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon - 2\frac{1}{r_1^\varepsilon}\nabla\tilde{\mathcal{H}}_1^\varepsilon\wedge\nabla^\perp\tilde{\varphi}^\varepsilon \\ \qquad + O\left(|\nabla\tilde{\nu}^\varepsilon|^2 + |\nabla\tilde{\eta}^\varepsilon||\nabla\tilde{\varphi}^\varepsilon|\right) \\ \qquad + R^\varepsilon(\tilde{\nu}), \\ div\left(\nabla\tilde{\eta}^\varepsilon - 3\left(\nabla\tilde{\eta}^\varepsilon\cdot\tilde{n}^\varepsilon\right)\tilde{n}^\varepsilon\right) = -\nabla^\perp\tilde{\mathcal{N}}_1^\varepsilon\wedge\nabla\tilde{\eta}^\varepsilon - \nabla^\perp\tilde{\nu}^\varepsilon\wedge\nabla\frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon \\ \qquad + 3\left(div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon\cdot(\tilde{\nu}^\varepsilon)\right)\tilde{n}^\varepsilon\right) + div\left(\left(\nabla\frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon\cdot\tilde{\mathcal{N}}_1^\varepsilon\right)(\tilde{n}^\varepsilon - \tilde{\mathcal{N}}_1^\varepsilon)\right)\right) \\ \qquad + O\left(|\nabla\tilde{\nu}^\varepsilon||\nabla\tilde{\eta}^\varepsilon|\right) \\ \qquad + R^\varepsilon(\tilde{\eta}), \end{cases} \tag{5.7.101}$$

where

$$
\begin{aligned}
R^{\varepsilon}(\tilde{\varphi})_k := & \, o\left(|\nabla\tilde{\mathcal{N}}_1^{\varepsilon}||\nabla\tilde{\varphi}^{\varepsilon}| + |\nabla\tilde{\nu}^{\varepsilon}||\nabla r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}|\right) \\
& + O\left(\varepsilon^2 r_1^{\varepsilon}|\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}||\nabla\tilde{\omega}_1^{\varepsilon}|^2\right) \\
& - \varepsilon^3 (r_1^{\varepsilon})^4 B_{ikjmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n \langle \nabla(\tilde{\omega}^{\varepsilon})^i, \nabla(\tilde{\omega}^{\varepsilon})^j \rangle \\
& + \varepsilon^3 \frac{(r_1^{\varepsilon})^4 |\nabla\tilde{\omega}^{\varepsilon}|^2}{12} Ric_{mn,\,j}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^j \tilde{\omega}^{\varepsilon} \\
& + \varepsilon^3 \frac{(r_1^{\varepsilon})^4 |\nabla\tilde{\omega}^{\varepsilon}|^2}{6} R_{ijmk,\,n}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^j (\tilde{\omega}^{\varepsilon})^i \\
& + \varepsilon^3 (r_1^{\varepsilon})^4 \left(B_{iljmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\omega^{\varepsilon})^i (\omega^{\varepsilon})^l (\omega_x^{\varepsilon})^j\right) \tilde{\omega}_x^{\varepsilon} \\
& + \varepsilon^3 (r_1^{\varepsilon})^4 \left(B_{iljmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\omega^{\varepsilon})^i (\omega^{\varepsilon})^l (\omega_y^{\varepsilon})^j\right) \tilde{\omega}_y^{\varepsilon} \\
& - \varepsilon^3 (r_1^{\varepsilon})^4 \left(B_{iljmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\omega^{\varepsilon})^i \langle \nabla(\tilde{\omega}^{\varepsilon})^j, \nabla(\tilde{\omega}^{\varepsilon})^l \rangle\right) \tilde{\omega}^{\varepsilon} \\
& + O\left(\varepsilon^3 (|\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}|^2 |\nabla\tilde{\Phi}^{\varepsilon}|^2 + |\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}|^3 |\nabla\tilde{n}^{\varepsilon}||\nabla\tilde{\Phi}^{\varepsilon}|)\right) \\
& + O\left(\varepsilon^4 (r_1^{\varepsilon})^4 |\nabla\tilde{\omega}_1^{\varepsilon}|^2\right)
\end{aligned}
\tag{5.7.102}
$$

$$
\begin{aligned}
R^{\varepsilon}(\tilde{\nu})_k := & \, o\left(|\nabla\tilde{\mathcal{N}}_1^{\varepsilon}||\nabla\tilde{\nu}^{\varepsilon}| + |\nabla\tilde{\eta}^{\varepsilon}||\nabla r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}| + |\frac{1}{r_1^{\varepsilon}}\nabla\tilde{\mathcal{H}}_1^{\varepsilon}||\nabla\tilde{\varphi}^{\varepsilon}|\right) \\
& + O\left(\varepsilon^2 |\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}||\nabla\tilde{\omega}_1^{\varepsilon}|^2\right) \\
& + O\left(\varepsilon^2 \left|\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}\right| |\tilde{H}^{\varepsilon}||\nabla(r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon})|^2\right) \\
& + O\left(\varepsilon^2 \left|\tilde{H}^{\varepsilon} - \frac{1}{r_1^{\varepsilon}}\tilde{\mathcal{H}}_1^{\varepsilon}\right| |\nabla(r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon})|^2\right) \\
& + 2\varepsilon^3 (r_1^{\varepsilon})^3 B_{ikjmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n \langle \nabla(\tilde{\omega}^{\varepsilon})^i, \nabla(\tilde{\omega}^{\varepsilon})^j \rangle \\
& - \varepsilon^3 \frac{(r_1^{\varepsilon})^3 |\nabla\tilde{\omega}^{\varepsilon}|^2}{12} Ric_{mn,\,j}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^j \tilde{\omega}^{\varepsilon} \\
& - \varepsilon^3 \frac{(r_1^{\varepsilon})^3 |\nabla\tilde{\omega}^{\varepsilon}|^2}{6} R_{ijmk,\,n}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^j (\tilde{\omega}^{\varepsilon})^i \\
& - \varepsilon^3 (r_1^{\varepsilon})^3 |\nabla\tilde{\omega}^{\varepsilon}|^2 B_{ikjmn}(p^{\varepsilon})(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m (\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n (\tilde{\omega}^{\varepsilon})^i (\tilde{\omega}^{\varepsilon})^j \\
& + \varepsilon^3 (r_1^{\varepsilon})^3 B_{ikjmn}(p^{\varepsilon})(\delta^{ml}(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^n + \delta^{nl}(\tilde{\omega}_1^{\varepsilon} + c_1^{\varepsilon})^m)(\tilde{\omega}^{\varepsilon})^i \langle \nabla(\tilde{\omega}^{\varepsilon})^j, \nabla(\tilde{\omega}^{\varepsilon})^l \rangle \\
& + O\left(\varepsilon^3 |\tilde{\Phi}^{\varepsilon} - r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon}|^3 (|\nabla\tilde{n}^{\varepsilon}|^2 + |\nabla\tilde{H}^{\varepsilon}||\nabla\tilde{\Phi}^{\varepsilon}|)\right) + O\left(\varepsilon^4 (r_1^{\varepsilon})^3 |\nabla\tilde{\omega}_1^{\varepsilon}|^2\right) \text{ and}
\end{aligned}
\tag{5.7.103}
$$

$$
\begin{aligned}
R^\varepsilon(\tilde\eta)_k := {} & o\left(|\nabla\tilde{\mathcal{N}}_1^\varepsilon||\nabla\tilde\eta^\varepsilon| + |\nabla\tilde\nu^\varepsilon||\nabla\frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon|\right)\\
& + 2\varepsilon^3(r_1^\varepsilon)^2 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n\langle\nabla(\tilde\omega^\varepsilon)^i, \nabla(\tilde\omega^\varepsilon)^j\rangle\\
& - \varepsilon^3\frac{(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2}{12}Ric_{mn,\,j}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j\tilde\omega^\varepsilon\\
& - \varepsilon^3\frac{(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2}{6}R_{ijmk,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^i\\
& - \varepsilon^3(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega^\varepsilon)^i(\tilde\omega^\varepsilon)^j\\
& + \varepsilon^3(r_1^\varepsilon)^2 B_{ikjmn}(p^\varepsilon)(\delta^{ml}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n + \delta^{nl}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m)(\tilde\omega^\varepsilon)^i\langle\nabla(\tilde\omega^\varepsilon)^j, \nabla(\tilde\omega^\varepsilon)^l\rangle\\
& - \varepsilon^3\frac{(r_1^\varepsilon)^2}{2}R_{ijml,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^l(\tilde\omega_x^\varepsilon)^i\tilde\omega_x^\varepsilon\\
& - \varepsilon^3\frac{(r_1^\varepsilon)^2}{2}R_{ijml,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^l(\tilde\omega_y^\varepsilon)^i\tilde\omega_y^\varepsilon\\
& - 5\varepsilon^3(r_1^\varepsilon)^2\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\omega^\varepsilon)^i(\omega^\varepsilon)^l(\tilde\omega_x^\varepsilon)^j\right)\tilde\omega_x^\varepsilon\\
& - 5\varepsilon^3(r_1^\varepsilon)^2\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega^\varepsilon)^i(\tilde\omega^\varepsilon)^l(\tilde\omega_y^\varepsilon)^j\right)\tilde\omega_y^\varepsilon\\
& + \varepsilon^3 A^\varepsilon\tilde\omega_1^\varepsilon\\
& + O\left(\varepsilon^2\frac{1}{r_\theta^\varepsilon}|\tilde\Phi^\varepsilon - r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon||\nabla\tilde\omega_\theta^\varepsilon|^2\right)\\
& + O\left(\varepsilon^2|\tilde\Phi^\varepsilon - r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon|\left|div\left(\left(\left(\frac{1}{r_\theta^\varepsilon}\nabla\tilde{\mathcal{H}}_\theta^\varepsilon\right)\cdot\tilde{\mathcal{N}}_\theta^\varepsilon\right)\tilde{\mathcal{N}}_\theta^\varepsilon\right)\right|\right)\\
& + O\left(\varepsilon^2|\tilde\Phi^\varepsilon - r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon|\left|\nabla(r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon)\right|\left|\frac{1}{r_\theta^\varepsilon}\nabla\tilde{\mathcal{H}}_\theta^\varepsilon\right|\right)\\
& + O\left(\varepsilon^2|\tilde\Phi^\varepsilon - r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon|\left|div\left(\left(\left(\frac{1}{r_\theta^\varepsilon}\tilde{\mathcal{H}}_\theta^\varepsilon\nabla(r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon)\right)\cdot\tilde{\mathcal{N}}_\theta^\varepsilon\right)\tilde{\mathcal{N}}_\theta^\varepsilon\right)\right|\right)\\
& + O\left(\varepsilon^2|\tilde\Phi^\varepsilon - r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon|\left|\nabla(r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon)\right|\left|\frac{1}{r_\theta^\varepsilon}\tilde{\mathcal{H}}_\theta^\varepsilon\nabla(r_\theta^\varepsilon\tilde{\mathcal{B}}_\theta^\varepsilon)\right|\right)\\
& + O\left(\varepsilon^2\frac{1}{r_\theta^\varepsilon}|\tilde n^\varepsilon - \tilde{\mathcal{N}}_\theta^\varepsilon||\nabla\tilde\omega_\theta^\varepsilon|^2\right)\\
& + O\left(\varepsilon^3|\tilde\Phi^\varepsilon - r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon|^3\left(|\nabla\tilde H^\varepsilon||\nabla\tilde n^\varepsilon|\right) + \varepsilon^3|\tilde\Phi^\varepsilon - r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon|^2|\nabla\tilde H^\varepsilon||\nabla\tilde\Phi^\varepsilon|\right) + O\left(\varepsilon^4(r_1^\varepsilon)^2|\nabla\tilde\omega_1^\varepsilon|^2\right),
\end{aligned}
\tag{5.7.104}
$$

for some scalar function $A^\varepsilon$.

Thanks to (5.7.99), (5.7.98) and Proposition 5.6.2, we have that

$$
\begin{aligned}
\frac{1}{r_1^\varepsilon}|\nabla\tilde\varphi^\varepsilon| + |\nabla\tilde\nu^\varepsilon| + r_1^\varepsilon|\nabla\tilde\eta^\varepsilon| &= O(\varepsilon^3(r_1^\varepsilon)^3)\\
O\left(|\nabla\tilde\nu^\varepsilon||\nabla\tilde\varphi^\varepsilon|\right) &= o(\varepsilon^3(r_1^\varepsilon)^4)\\
O\left(|\nabla\tilde\nu^\varepsilon|^2 + |\nabla\tilde\eta^\varepsilon||\nabla\tilde\varphi^\varepsilon|\right) &= o\left(\varepsilon^3(r_1^\varepsilon)^3\right)\\
O\left(|\nabla\tilde\nu^\varepsilon||\nabla\tilde\eta^\varepsilon\right) &= o\left(\varepsilon^3(r_1^\varepsilon)^2\right)\\
\varepsilon^2|\tilde\Phi^\varepsilon - r_1^\varepsilon\tilde{\mathcal{B}}_1^\varepsilon| &= o\left(\varepsilon^3(r_1^\varepsilon)^3\right)\\
\varepsilon^2|\tilde n^\varepsilon - \tilde{\mathcal{N}}_1^\varepsilon| &= o\left(\varepsilon^3(r_1^\varepsilon)^3\right)\\
\varepsilon^2|\tilde H^\varepsilon - \frac{1}{r_1^\varepsilon}\tilde{\mathcal{H}}_1^\varepsilon| &= o\left(\varepsilon^3(r_1^\varepsilon)^2\right).
\end{aligned}
\tag{5.7.105}
$$

Therefore, we obtain that

$$
\begin{cases}
\Delta\tilde{\varphi}^{\varepsilon} = \nabla\tilde{\mathcal{N}}_1^{\varepsilon} \wedge \nabla^{\perp}\tilde{\varphi}^{\varepsilon} + \nabla\tilde{\nu}^{\varepsilon} \wedge \nabla^{\perp} r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon} \\
\qquad\quad + R^{\varepsilon}(\tilde{\varphi}) + o(\varepsilon^3 (r_1^{\varepsilon})^4) \\
\Delta\tilde{\nu}^{\varepsilon} = -2\nabla^{\perp}\tilde{\mathcal{N}}_1^{\varepsilon} \wedge \nabla\tilde{\nu}^{\varepsilon} - 2\nabla\tilde{\eta}^{\varepsilon} \wedge \nabla^{\perp} r_1^{\varepsilon}\tilde{\mathcal{B}}_1^{\varepsilon} - 2\dfrac{1}{r_1^{\varepsilon}}\nabla\tilde{\mathcal{H}}_1^{\varepsilon} \wedge \nabla^{\perp}\tilde{\varphi}^{\varepsilon} \\
\qquad\quad + R^{\varepsilon}(\tilde{\nu}) + o\left(\varepsilon^3 (r_1^{\varepsilon})^3\right), \\
div\left(\nabla\tilde{\eta}^{\varepsilon} - 3\left(\nabla\tilde{\eta}^{\varepsilon} \cdot \tilde{n}^{\varepsilon}\right)\tilde{n}^{\varepsilon}\right) = -\nabla^{\perp}\tilde{\mathcal{N}}_1^{\varepsilon} \wedge \nabla\tilde{\eta}^{\varepsilon} - \nabla^{\perp}\tilde{\nu}^{\varepsilon} \wedge \nabla\dfrac{1}{r_1^{\varepsilon}}\tilde{\mathcal{H}}_1^{\varepsilon} \\
\qquad\quad + 3\left(div\left(\left(\nabla\dfrac{1}{r_1^{\varepsilon}}\tilde{\mathcal{H}}_1^{\varepsilon} \cdot (\tilde{\nu}^{\varepsilon})\right)\tilde{n}^{\varepsilon}\right) + div\left(\left(\nabla\dfrac{1}{r_1^{\varepsilon}}\tilde{\mathcal{H}}_1^{\varepsilon} \cdot \tilde{\mathcal{N}}_1^{\varepsilon}\right)(\tilde{n}^{\varepsilon} - \tilde{\mathcal{N}}_1^{\varepsilon})\right)\right) \\
\qquad\quad + R^{\varepsilon}(\tilde{\eta}) + o(\varepsilon^2 (r_1^{\varepsilon})^2),
\end{cases}
\tag{5.7.106}
$$

where

$$
\begin{aligned}
R^\varepsilon(\tilde\varphi)_k &= -\varepsilon^3(r_1^\varepsilon)^4 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n\langle\nabla(\tilde\omega^\varepsilon)^i, \nabla(\tilde\omega^\varepsilon)^j\rangle \\
&\quad + \varepsilon^3\frac{(r_1^\varepsilon)^4|\nabla\tilde\omega^\varepsilon|^2}{12} Ric_{mn,\,j}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j\tilde\omega^\varepsilon \\
&\quad + \varepsilon^3\frac{(r_1^\varepsilon)^4|\nabla\tilde\omega^\varepsilon|^2}{6} R_{ijmk,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^i \\
&\quad + \varepsilon^3(r_1^\varepsilon)^4\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\omega^\varepsilon)^i(\omega^\varepsilon)^l(\omega_x^\varepsilon)^j\right)\tilde\omega_x^\varepsilon \\
&\quad + \varepsilon^3(r_1^\varepsilon)^4\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\omega^\varepsilon)^i(\omega^\varepsilon)^l(\omega_y^\varepsilon)^j\right)\tilde\omega_y^\varepsilon \\
&\quad - \varepsilon^3(r_1^\varepsilon)^4\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\omega^\varepsilon)^i\langle\nabla(\tilde\omega^\varepsilon)^j, \nabla(\tilde\omega^\varepsilon)^l\rangle\right)\tilde\omega^\varepsilon \\
&\quad + o(\varepsilon^3(r_1^\varepsilon)^4) \\
R^\varepsilon(\tilde\nu)_k &= 2\varepsilon^3(r_1^\varepsilon)^3 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n\langle\nabla(\tilde\omega^\varepsilon)^i, \nabla(\tilde\omega^\varepsilon)^j\rangle \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^3|\nabla\tilde\omega^\varepsilon|^2}{12} Ric_{mn,\,j}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j\tilde\omega^\varepsilon \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^3|\nabla\tilde\omega^\varepsilon|^2}{6} R_{ijmk,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^i \\
&\quad - \varepsilon^3(r_1^\varepsilon)^3|\nabla\tilde\omega^\varepsilon|^2 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega^\varepsilon)^i(\tilde\omega^\varepsilon)^j \\
&\quad + \varepsilon^3(r_1^\varepsilon)^3 B_{ikjmn}(p^\varepsilon)(\delta^{ml}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n + \delta^{nl}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m)(\tilde\omega^\varepsilon)^i\langle\nabla(\tilde\omega^\varepsilon)^j, \nabla(\tilde\omega^\varepsilon)^l\rangle \\
&\quad + o(\varepsilon^3(r_1^\varepsilon)^3) \text{ and} \\
R^\varepsilon(\tilde\eta)_k &= 2\varepsilon^3(r_1^\varepsilon)^2 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n\langle\nabla(\tilde\omega^\varepsilon)^i, \nabla(\tilde\omega^\varepsilon)^j\rangle \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2}{12} Ric_{mn,\,j}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j\tilde\omega^\varepsilon \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2}{6} R_{ijmk,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^i \\
&\quad - \varepsilon^3(r_1^\varepsilon)^2|\nabla\tilde\omega^\varepsilon|^2 B_{ikjmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega^\varepsilon)^i(\tilde\omega^\varepsilon)^j \\
&\quad + \varepsilon^3(r_1^\varepsilon)^2 B_{ikjmn}(p^\varepsilon)(\delta^{ml}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n + \delta^{nl}(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m)(\tilde\omega^\varepsilon)^i\langle\nabla(\tilde\omega^\varepsilon)^j, \nabla(\tilde\omega^\varepsilon)^l\rangle \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^2}{2} R_{ijml,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^l(\tilde\omega_x^\varepsilon)^i\tilde\omega_x^\varepsilon \\
&\quad - \varepsilon^3\frac{(r_1^\varepsilon)^2}{2} R_{ijml,\,n}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^j(\tilde\omega^\varepsilon)^l(\tilde\omega_y^\varepsilon)^i\tilde\omega_y^\varepsilon \\
&\quad - 5\varepsilon^3(r_1^\varepsilon)^2\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\omega^\varepsilon)^i(\omega^\varepsilon)^l(\tilde\omega_x^\varepsilon)^j\right)\tilde\omega_x^\varepsilon \\
&\quad - 5\varepsilon^3(r_1^\varepsilon)^2\left(B_{iljmn}(p^\varepsilon)(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^m(\tilde\omega_1^\varepsilon + c_1^\varepsilon)^n(\tilde\omega^\varepsilon)^i(\tilde\omega^\varepsilon)^l(\tilde\omega_y^\varepsilon)^j\right)\tilde\omega_y^\varepsilon \\
&\quad + \varepsilon^3 A^\varepsilon\tilde\omega_1^\varepsilon \\
&\quad + o(\varepsilon^3(r_1^\varepsilon)^2).
\end{aligned}
\tag{5.7.107}
$$

Dividing respectively the equations in (5.7.106) by $\varepsilon^3(r_1^\varepsilon)^4$, $\varepsilon^3(r_1^\varepsilon)^3$ and $\varepsilon^3(r_1^\varepsilon)^2$ , thanks to (5.7.107) and standard elliptic theory, we get that, up to a subsequence, $\nabla\left(\frac{\tilde\varphi^\varepsilon}{\varepsilon^3(r_1^\varepsilon)^4}\right)$, $\nabla\left(\frac{\tilde\nu^\varepsilon}{\varepsilon^3(r_1^\varepsilon)^3}\right)$ and $\nabla\left(\frac{\tilde\eta^\varepsilon}{\varepsilon^3(r_1^\varepsilon)^2}\right)$

respectively converge to $\nabla\varphi$, $\nabla\nu$ and $\nabla\eta$ in $C^0_{loc}(\mathbb{R}^2)$ for some $\varphi,\nu,\eta\in C^2(\mathbb{R}^2)$ satisfying

$$\begin{aligned}\Delta\varphi+\nabla\omega\wedge\nabla^\perp\varphi-\nabla\omega\wedge\nabla^\perp\nu=&-B_{ikjmn}(p)(\omega+c)^m(\omega+c)^n\langle\nabla\omega^i,\nabla\omega^j\rangle\\&+\frac{|\nabla\omega|^2}{12}Ric_{mn,\,j}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega\\&+\frac{|\nabla\omega|^2}{6}R_{ijmk,\,n}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega^i\\&+\left(B_{iljmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^l(\omega_x)^j\right)\omega_x\\&+\left(B_{iljmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^l(\omega_y)^j\right)\omega_y\\&-\left(B_{iljmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i\langle\nabla\omega^j,\nabla\omega^l\rangle\right)\omega,\end{aligned}\tag{5.7.108}$$

$$\begin{aligned}\Delta\nu-2\nabla\omega\wedge\nabla^\perp\nu-2\nabla\omega\wedge\nabla^\perp\varphi+2\nabla\omega\wedge\nabla^\perp\eta&\\=2B_{ikjmn}(p)(\omega+c)^m(\omega+c)^n\langle\nabla\omega^i,\nabla\omega^j\rangle&\\-\frac{|\nabla\omega|^2}{12}Ric_{mn,\,j}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega&\\-\frac{|\nabla\omega|^2}{6}R_{ijmk,\,n}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega^i&\\-|\nabla\omega|^2B_{ikjmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^j&\\+B_{ikjmn}(p)(\delta^{ml}(\omega+c)^n+\delta^{nl}(\omega+c)^m)(\omega)^i\langle\nabla\omega^j,\nabla\omega^l\rangle\text{ and}&\end{aligned}\tag{5.7.109}$$

$$\begin{aligned}&div\left(\nabla\eta-3\left(\nabla\omega\cdot\nu+\nabla\eta\cdot\omega\right)\omega\right)\\&+\nabla\omega\wedge\nabla^\perp\eta+\nabla\omega\wedge\nabla^\perp\nu=2B_{ikjmn}(p)(\omega+c)^m(\omega+c)^n\langle\nabla\omega^i,\nabla\omega^j\rangle\\&-\frac{|\nabla\omega|^2}{12}Ric_{mn,\,j}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega\\&-\frac{|\nabla\omega|^2}{6}R_{ijmk,\,n}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j\omega^i\\&-|\nabla\omega|^2B_{ikjmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^j\\&+B_{ikjmn}(p)(\delta^{ml}(\omega+c)^n+\delta^{nl}(\omega+c)^m)(\omega)^i\langle\nabla\omega^j,\nabla\omega^l\rangle\\&-\frac{}{2}R_{ijml,\,n}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j(\omega)^l(\omega_x)^i\omega_x\\&-\frac{}{2}R_{ijml,\,n}(p)(\omega+c)^m(\omega+c)^n(\omega+c)^j(\omega)^l(\omega_y)^i\omega_y\\&-5\left(B_{iljmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^l(\omega_x)^j\right)\omega_x\\&-5\left(B_{iljmn}(p)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^l(\omega_y)^j\right)\omega_y\\&+A\omega,\end{aligned}\tag{5.7.110}$$

where $\omega=\lim_{\varepsilon\to0}\tilde{\omega}_1^\varepsilon$, $c=\lim_{\varepsilon\to0}c_1^\varepsilon$ and $A^\varepsilon\to A$ in $C^0_{loc}(\mathbb{R}^2)$ for some scalar function $A$. In addition, since we have $h^\varepsilon(\tilde{\Phi}^\varepsilon)(\tilde{n}^\varepsilon,\tilde{n}^\varepsilon)=1$, thanks to Lemma 6.5.1, we also get that

$$\begin{aligned}\nabla\omega\cdot\nu+\nabla\nu\cdot\omega=&\frac{1}{12}\left(R_{ijmk,\,n}(p)+R_{kjmi,\,n}(p)\right)(\omega+c)^m(\omega+c)^n(\omega+c)^j(\omega)^i(\nabla\omega)^k\\&+\frac{1}{12}\left(R_{ikmj,\,n}(p)+R_{imkj,\,n}(p)+R_{inmj,\,k}(p)\right)(\omega+c)^m(\omega+c)^n(\omega)^i(\omega)^j(\nabla\omega)^k.\end{aligned}\tag{5.7.111}$$

Multiplying (5.7.108) by $-2$, (5.7.109) by $1$ and (5.7.110) by $-1$ and summing them up, together

with (5.7.111), we get that

$$
\begin{aligned}
&-2\left(\Delta\varphi+2\nabla\omega\wedge\nabla^{\perp}\varphi\right)\\
&\quad+\left(\Delta\nu-\nabla\omega\wedge\nabla^{\perp}\nu-3div(\pi_{\omega}(\nabla\nu))\right)\\
&\quad-\left(\Delta\eta-\nabla\omega\wedge\nabla^{\perp}\eta-3div(\pi_{\omega}(\nabla\eta))\right)\\
&=2B_{ikjmn}(p)(\omega+c)^{m}(\omega+c)^{n}\langle\nabla\omega^{i},\nabla\omega^{j}\rangle\\
&\quad-\frac{|\nabla\omega|^{2}}{3}R_{ijmk,\,n}(p)(\omega+c)^{m}(\omega+c)^{n}(\omega+c)^{j}(\omega)^{i}\\
&\quad+\frac{1}{2}R_{ijml,\,n}(p)(\omega+c)^{m}(\omega+c)^{n}(\omega+c)^{j}(\omega)^{l}(\omega_{x})^{i}\omega_{x}\\
&\quad+\frac{1}{2}R_{ijml,\,n}(p)(\omega+c)^{m}(\omega+c)^{n}(\omega+c)^{j}(\omega)^{l}(\omega_{y})^{i}\omega_{y}\\
&\quad-\frac{1}{4}\left(R_{ijml,\,n}(p)+R_{kjmi,\,n}(p)\right)(\omega+c)^{m}(\omega+c)^{n}(\omega+c)^{j}(\omega)^{i}(\omega_{x})^{l}\omega_{x}\\
&\quad-\frac{1}{4}\left(R_{ijml,\,n}(p)+R_{kjmi,\,n}(p)\right)(\omega+c)^{m}(\omega+c)^{n}(\omega+c)^{j}(\omega)^{i}(\omega_{y})^{l}\omega_{y}\\
&\quad-\frac{1}{4}\left(R_{ilmj,\,n}(p)+R_{imlj,\,n}(p)+R_{inmj,\,l}(p)\right)(\omega+c)^{m}(\omega+c)^{n}(\omega)^{i}(\omega)^{j}(\omega_{x})^{l}\omega_{x}\\
&\quad-\frac{1}{4}\left(R_{ilmj,\,n}(p)+R_{imlj,\,n}(p)+R_{inmj,\,l}(p)\right)(\omega+c)^{m}(\omega+c)^{n}(\omega)^{i}(\omega)^{j}(\omega_{y})^{l}\omega_{y}\\
&\quad+3\left(B_{iljmn}(p)(\omega+c)^{m}(\omega+c)^{n}(\omega)^{i}(\omega)^{l}(\omega_{x})^{j}\right)\omega_{x}\\
&\quad+3\left(B_{iljmn}(p)(\omega+c)^{m}(\omega+c)^{n}(\omega)^{i}(\omega)^{l}(\omega_{y})^{j}\right)\omega_{y}\\
&\quad+A'\omega,
\end{aligned}
\tag{5.7.112}
$$

for some scalar function $A'$, where $\pi_{\omega}(\nabla\nu):=(\nabla\nu\cdot\omega)\omega$ and $\pi_{\omega}(\nabla\eta):=(\nabla\eta\cdot\omega)\omega$. Note that we also know, thanks to Proposition 5.7.10 and Lemma 5.7.12 that,

$$
\begin{aligned}
\left(|\nabla\varphi|+|\nabla\nu|+|\nabla\eta|\right)(x)&=O\left(|x|^{-2/3}\right)\ \text{and}\\
\left(|\varphi|+|\nu|+|\eta|\right)(x)&=o(|x|)\ \text{as}\ |x|\to+\infty.
\end{aligned}
\tag{5.7.113}
$$

For all $q=1,2,3$, we denote $Z^{q}$ by $Z^{1}=\omega_{x}$, $Z^{2}=\omega_{y}$ and $Z^{3}=x\omega_{x}+y\omega_{y}$. Notice that we have

$$
\begin{aligned}
&\Delta Z^{q}+2\nabla\omega\wedge\nabla^{\perp}Z^{q}=0,\\
&\nabla\omega\wedge\nabla^{\perp}Z^{q}+div(\pi_{\omega}(\nabla Z^{q}))=0.
\end{aligned}
$$

Multiplying (5.7.112) by $Z^{q}$, and integrating by parts on $\mathbb{R}^{2}$, thanks to the fact that integrals containing an odd number of coordinates of $\omega$ vanish and the fact that

$$
\langle\nabla\omega^{i},\nabla\omega^{j}\rangle=\frac{|\nabla\omega|^{2}}{2}\left(\delta^{ij}-\omega^{i}\omega^{j}\right),
$$

By direct computations, we obtain that, for $q = 1, 2, 3$,

$$\begin{aligned}
0 = &\int_{\mathbb{R}^2} -2\varphi \cdot \left(\Delta Z^q + 2\nabla\omega \wedge \nabla^\perp Z^q\right) \\
&+ \nu \cdot \left(\Delta Z^q - \nabla\omega \wedge \nabla^\perp Z^q - 3div(\pi_\omega(\nabla Z^q))\right) \\
&- \eta \cdot \left(\Delta Z^q - \nabla\omega \wedge \nabla^\perp Z^q - 3div(\pi_\omega(\nabla Z^q))\right) dx \\
= &\int_{\mathbb{R}^2} (2B_{ikjmn}(p)\omega^m\omega^n \left(\delta^{ij} - \omega^i\omega^j\right) (Z^q)^k) \frac{|\nabla\omega|^2}{2} dx \\
&+ \sum_{m,n} (c)^m (c)^n \int_{\mathbb{R}^2} (2B_{ikjmn}(p) \left(\delta^{ij} - \omega^i\omega^j\right) (Z^q)^k \\
&- \frac{2}{3}(R_{ijmk,\,n}(p) + R_{imjk,\,n}(p) + R_{inmk,\,j}(p))\omega^i\omega^j (Z^q)^k \\
&- \frac{1}{4}\left(R_{ikmj,\,n}(p) + R_{imkj,\,n}(p) + R_{inmj,\,k}(p)\right)\omega^i\omega^j (Z^q)^k \\
&+ 3B_{ijkmn}(p)\omega^i\omega^j (Z^q)^k) \frac{|\nabla\omega|^2}{2} dx \\
= &\int_{\mathbb{R}^2} (2B_{ikjmn}(p)\omega^m\omega^n \left(\delta^{ij} - \omega^i\omega^j\right) (Z^q)^k) \frac{|\nabla\omega|^2}{2} dx,
\end{aligned} \tag{5.7.114}$$

where the integrability on $\mathbb{R}^2$ and validity of integration by parts are insured by (5.7.113) and by the fact that

$$\left(|Z^q| + |\nabla Z^q|\right)(z) = O\left(\frac{1}{1+|z|^2}\right) \text{ for all } q = 1, 2, 3.$$

With help of stereographic projection and the following identities on the sphere:

$$\int_{\mathbb{S}^2} y^m y^n d\sigma_{g_0} = \frac{4\pi}{3}\delta^{mn} \text{ and } \int_{\mathbb{S}^2} y^m y^n y^q y^k d\sigma_{g_0} = \frac{4\pi}{15}(\delta^{mn}\delta^{qk} + \delta^{mq}\delta^{nk} + \delta^{mk}\delta^{nq}),$$

together with symmetries of Riemannian curvature tensor, from (5.7.114) we get, for all $q = 1, 2, 3$,

$$\int_{\mathbb{S}^2} B_{ikjmn}(p)\omega^m\omega^n(\delta^{ij} - \omega^i\omega^j)(\delta^{qk} - \omega^q\omega^k) d\sigma_{g_0} = 0,$$

which gives

$$Ric_{mq}{}^{,m}(p) = 0. \tag{5.7.115}$$

It is natural that (5.7.115) is independent of $c$, since our choice on system of coordinates is arbitrary. Finally, thanks to Bianchi's second identity, (5.7.115) implies

$$\nabla Scal(p) = 0.$$

Hence, we complete the proof of the main Theorem 5.2.3.

# Chapter 6

# Appendix

## 6.1 Reformulation of Willmore equation

In this section, we briefly explain how we obtain the reformulation of the third equation of the Willmore system (5.3.5) from (5.3.4):

The left hand side of the third equation in (5.3.4) can be rewritten as

$$\begin{aligned} D^*\left[D\vec{H} - 3\pi_{\vec{n}}(D\vec{H}) - \star_h\left(\vec{H}\wedge D^\perp\vec{n}\right)\right] &= D^*D\vec{H} - 3D^*\left(h(D\vec{H},\vec{n})\vec{n}\right) - \star_h\left(D\vec{H}\wedge D^\perp\vec{n}\right) \\ &= D^*D\vec{H} - 3D^*\left(\nabla H\vec{n}\right) - \star_h\left(D\vec{H}\wedge D^\perp\vec{n}\right). \end{aligned} \tag{6.1.1}$$

Here we use the fact that

$$\begin{aligned} D\vec{H} &= \nabla H\vec{n} + HD\vec{n}, \\ h(D\vec{H},\vec{n}) &= \nabla H\vec{n} \end{aligned} \tag{6.1.2}$$

since $h(\vec{n},\vec{n}) = 1$, $h(D\vec{n},\vec{n}) = 0$, and that $D$ commute with the Hodge operator $\star_h$ and $DD^\perp\vec{n} = 0$.

In normal coordinates, we can write that

$$\begin{aligned} D\vec{n} &= \nabla\vec{n} + \vec{\Gamma}_{ij}(\Phi)(\vec{n})^j(\nabla\Phi)^j, \\ D\vec{H} &= \nabla\vec{H} + \vec{\Gamma}_{ij}(\Phi)(\vec{H})^j(\nabla\Phi)^j, \\ D^*D\vec{H} &= \Delta\vec{H} + 2\vec{\Gamma}_{ij}(\Phi)(\nabla\vec{H})^i(\nabla\Phi)^j + \vec{\Gamma}_{ij}(\Phi)(\vec{H})^i(\Delta\Phi)^j \\ &\quad + \partial_l\vec{\Gamma}_{ij}(\Phi)(\vec{H})^i(\nabla\Phi)^j(\nabla\Phi)^l \text{ and} \\ \star_h\left(D\vec{H}\wedge D^\perp\vec{n}\right)^k &= \sqrt{|h|}(h^{-1})^{ik}(\Phi)\left(D\vec{H}\wedge D^\perp\vec{n}\right)_i \end{aligned} \tag{6.1.3}$$

Moreover, thanks to (6.1.2) and (6.1.3), we have that

$$\begin{aligned} D^*\left(\nabla H\vec{n}\right) &= \Delta H\vec{n} + \nabla H\bullet D\vec{n} \\ &= \Delta H\vec{n} + \nabla H\bullet\nabla\vec{n} + \nabla H\bullet\left(\vec{\Gamma}_{ij}(\Phi)(\vec{n})^i(\nabla\Phi)^j\right) \\ &= div\left(\nabla H\vec{n}\right) + \nabla H\bullet\left(\vec{\Gamma}_{ij}(\Phi)(\vec{n})^i(\nabla\Phi)^j\right) \\ &= div\left(h\left(D\vec{H},\vec{n}\right)\vec{n}\right) + \nabla H\bullet\left(\vec{\Gamma}_{ij}(\Phi)(\vec{n})^i(\nabla\Phi)^j\right). \end{aligned} \tag{6.1.4}$$

Again thanks to (6.1.3), we also get that

$$\begin{aligned} div\left(h\left(D\vec{H},\vec{n}\right)\vec{n}\right) &= div\left(\left(\left(\nabla\vec{H}\right)\cdot\vec{n}\right)\vec{n}\right) + div\left((h-\delta)_{ij}(\Phi)(\nabla\vec{H})^i(\vec{n})^j\vec{n}\right) \\ &\quad + div\left(\left[h_{\alpha\beta}(\Phi)\left(\Gamma^\alpha_{ij}(\Phi)(\vec{H})^i(\nabla\Phi)^j\right)(\vec{n})^\beta\right]\vec{n}\right) \end{aligned} \tag{6.1.5}$$

Combining (6.1.1), (6.1.3), (6.1.4) and (6.1.5), we obtain the third equation of the system (5.3.5) in normal coordinates.

## 6.2 Green's representation formula

Let $G$ be a two-variable function on $\mathbb{R}^2 \times \mathbb{R}^2$ defined as

$$G(x,y) = G(|x-y|) = \frac{1}{2\pi} \ln |x-y| \text{ for all } x \neq y.$$

It is well-known that $G$ is the fundamental solution of the Laplacian on the plane, namely that, for every fixed $x$,

$$\Delta_y G(x,y) = \delta_x.$$

**Lemma 6.2.1.** *Let $\psi, f \in C^2(\mathbb{R}^2)$ be two maps such that $\nabla\psi \in L^\infty(\mathbb{R}^2)$, $f = O\left(\frac{1}{|x|^2}\right)$ and*

$$\Delta\psi = f.$$

*Then, we have*

$$\nabla\psi(x) = \int_{\mathbb{R}^2} \nabla_y G(x,y) f(y) dy \text{ for all } x \in \mathbb{R}^2. \tag{6.2.1}$$

*In particular, if $f(x) = \frac{1}{1+|x|^2}$, then, by straightforward computations, we have that*

$$|\nabla\psi(x)| \leq C \frac{\ln(2+|x|)}{1+|x|}, \text{ for some } C > 0. \tag{6.2.2}$$

*Proof of Lemma 6.2.1:* Let $x \in \mathbb{R}^2$ and $R > 0$ be such that $x \in B_R(0)$. Applying the Green's representation formula to the equation and integrating by parts, we obtain that

$$\begin{aligned} \nabla\psi(x) &= \int_{B_R(0)} G_R(x,y) \nabla f(y) dy + \int_{\partial B_R(0)} \partial_\nu G_R(x,y) \nabla\psi(z) dl \\ &= -\int_{B_R(0)} \nabla_y G_R(x,y) f(y) dy + \int_{\partial B_R(0)} G_R(x,y) f(y) dl + \int_{\partial B_R(0)} \partial_\nu G_R(x,y) \nabla\psi(y) dl, \end{aligned} \tag{6.2.3}$$

where $\nu$ is the outter normal vector of $\partial B_R(0)$ and $G_R(x,y)$ is defined by

$$G_R(x,y) := \begin{cases} G(|x-y|) - G\left(\frac{|x|}{R}\left|y - \frac{R^2 x}{|x|^2}\right|\right) & \text{if } x \neq 0 \\ G(|y|) - G(R) & \text{if } x = 0 \end{cases}.$$

Using the fact that, for $x$ fixed, we have that

$$\nabla G_R(x,y) = O\left(\frac{1}{|y|}\right), \; G_R(x,y) = O(\ln|y|) \text{ as } |y| \to +\infty$$

and that

$$\nabla(\partial_\nu G_R(x,y)) = O\left(\frac{1}{R^2}\right) \text{ for } y \in \partial B_R(0) \text{ and as } R \to +\infty,$$

we obtain (6.2.1) by taking the limit of (6.2.3) as $R \to +\infty$.

□

**Lemma 6.2.2.** *Let $\varphi \in C^\infty(\mathbb{R}^2)$ with $|\nabla\varphi(y)| \lesssim (1+|y|)^{-2}$ and $|\nabla^2\varphi(y)| \lesssim (1+|y|)^{-3}$. Then we have that*

$$(1+|x|)\,|\nabla\varphi(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|}|\Delta\varphi(y)|\,dy. \tag{6.2.4}$$

*Proof.* Let $x \in \mathbb{R}^2$. For every $R > |x|$, we have

$$\begin{aligned}\varphi(x) = &\int_{\mathbb{D}_0(R)} G(x,y)\Delta\varphi(y)\,dy \\ &+ \int_{\partial\mathbb{D}_0(R)} G(x,y)\partial_\nu\varphi(y)\,dy \\ &+ \int_{\partial\mathbb{D}_0(R)} \frac{\langle y, y-x\rangle}{|x-y|^2|y|}\varphi(y)\,dy.\end{aligned} \tag{6.2.5}$$

Therefore, we have

$$\begin{aligned}\nabla\varphi(x) = &\frac{1}{2\pi}\int_{\mathbb{D}_0(R)} \frac{y-x}{|y-x|}\Delta\varphi(y)\,dy \\ &+ \frac{1}{2\pi}\int_{\partial\mathbb{D}_0(R)} \frac{y-x}{|y-x|}\partial_\nu\varphi(y)\,dy \\ &+ \frac{1}{2\pi}\int_{\partial\mathbb{D}_0(R)} \frac{\left(R^2-|x|^2\right)y + 2\left(\langle x,y\rangle - R^2\right)x}{R|x-y|^4}\varphi(y)\,dy.\end{aligned} \tag{6.2.6}$$

Thanks to the decreasing condition on $\varphi$ and (6.2.6), as $R \to +\infty$, we get

$$\nabla\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{y-x}{|y-x|}\Delta\varphi(y)\,dy. \tag{6.2.7}$$

Notice that again thanks to the decreasing condition, one can write

$$\int_{\mathbb{R}^2} \Delta\varphi(y)\,dy = 0. \tag{6.2.8}$$

Therefore, for every $x \neq 0$, we also have

$$\nabla\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\frac{y-x}{|y-x|^2} + \frac{x}{|x|^2}\right)\Delta\varphi(y)\,dy. \tag{6.2.9}$$

Thanks to (6.2.7) and (6.2.9), we get

$$|\nabla\varphi(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|\Delta\varphi(y)|\,dy \tag{6.2.10}$$

and

$$|\nabla\varphi(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|y|}{|x-y||x|}|\Delta\varphi(y)|\,dy,\ \forall x \neq 0. \tag{6.2.11}$$

Multiply (6.2.11) by $|x|$ and sum up with (6.2.10), we obtain that

$$(1+|x|)|\nabla\varphi(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1+|y|}{|x-y|}|\Delta\varphi(y)|\,dy. \tag{6.2.12}$$

□

## 6.3 Lorentz-Wente type inequality

Here we recall the notion of Lorentz spaces, which can be seen as interpolation spaces between $L^p$ spaces, and some general estimates developed in Tartar [131].

**Definition 6.3.1.** *Let $\Omega$ be an open subset in a measured space $(X, \mu)$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. Then the* Lorentz space $L^{p,q}(\Omega)$ *is defined as the space of measurable fonctions $f$ on $\Omega$ such that, when $q < +\infty$,*

$$\|f\|_{p,q} := p^{\frac{1}{q}} \left( \int_0^{+\infty} t^{q-1} \left(\mu(\{|f| \geq t\})\right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} < +\infty$$

*and when $q = +\infty$,*

$$\|f\|_{p,+\infty} := \sup_{t>0} t \left(\mu(\{|f| \geq t\})\right)^{\frac{1}{p}} < +\infty.$$

*Remark* 6.3.2. Notice that for $p \geq 1$, we have that

$$L^p = L^{p,p}$$

and that

$$L^{p,q} \subset L^{p,r} \text{ whenever } 1 \leq q \leq r \leq +\infty.$$

**Lemma 6.3.3** (Theorem 3.3.6., Hélein [69])**.** *Let $\phi$ be solution of*

$$\Delta_g \phi = f \in L^1(\Sigma).$$

*Then, $\nabla_g \phi \in L^{2,+\infty}(\Sigma)$ and*

$$\|\nabla_g \phi\|_{L^{2,+\infty}(\Sigma)} \leq C \|f\|_{L^1(\Sigma)}. \tag{6.3.1}$$

*where $C > 0$ only depends on $\Sigma$.*

**Lemma 6.3.4** (see Wente [135])**.** *Let $a, b \in W^{1,2}(\mathbb{D})$ and $\phi$ be solution of*

$$\begin{cases} \Delta \phi = \nabla^\perp a \cdot \nabla b \text{ on } \mathbb{D} \\ \phi = 0 \text{ on } \partial \mathbb{D}. \end{cases} \tag{6.3.2}$$

*Then $\phi \in C^0(\mathbb{D}) \cap W^{1,2}(\mathbb{D})$ and there exists $C > 0$ such that*

$$\|\phi\|_{L^\infty(\mathbb{D})} + \|\nabla \phi\|_{L^2(\mathbb{D})} \leq C \|\nabla a\|_{L^2(\mathbb{D})} \|\nabla b\|_{L^2(\mathbb{D})}. \tag{6.3.3}$$

**Lemma 6.3.5** (Neumann boundary condition version of Wente's lemma, see Lemma 3.1.2 in Hélein [69] and Lemma A.6 in Rivière [121])**.** *Let $a, b \in W^{1,1}(\mathbb{D})$ such that $\nabla a \in L^{2,+\infty}(\mathbb{D})$ and $\nabla b \in L^{2,1}(\mathbb{D})$. Let $\phi$ be solution of*

$$\begin{cases} \Delta \phi = \nabla a \cdot \nabla^\perp b \text{ on } \mathbb{D} \\ \dfrac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \mathbb{D}. \end{cases} \tag{6.3.4}$$

*Then we have*

$$\|\nabla \phi\|_{L^{2,+\infty}(\mathbb{D})} \leq C \|\nabla a\|_{L^{2,+\infty}(\mathbb{D})} \|\nabla b\|_{L^{2,1}(\mathbb{D})} \tag{6.3.5}$$

*for some $C > 0$.*

**Lemma 6.3.6** (Theorem 3., Ge [64]). *Let $(\Sigma, g)$ be a smooth compact surface. Assume that $a, b \in W^{1,1}(\Sigma)$ such that $\nabla a \in L^{2,p}(\Sigma)$ and $\nabla b \in L^{2,q}(\Sigma)$ for $p \geq 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. Let $\phi$ be solution of*

$$\Delta_g \phi = \nabla_g^{\perp} a \cdot_g \nabla_g b \text{ on } \Sigma.$$

*Then we have $\nabla \phi \in L^{2,r}(\Sigma)$ and*

$$\|\nabla \phi\|_{L^{2,r}(\Sigma)} \leq C(\Sigma, p, q) \|\nabla a\|_{L^{2,p}(\Sigma)} \|\nabla b\|_{L^{2,q}(\Sigma)} \tag{6.3.6}$$

*Remark* 6.3.7. The inequality (6.3.6) still remains true for $\Sigma = \Omega$ a bounded domain of $\mathbb{R}^2$, provided that

$$\phi = 0 \text{ on } \partial\Omega.$$

**Lemma 6.3.8** (Lemma A.2., Bethuel-Brezis-Hélein [22]). *Let $\Omega = \mathbb{R}^2$ or be a bounded domain of $\mathbb{R}^2$. Assume $\psi$ satisfies*

$$\begin{cases} \Delta \psi = f \text{ on } \Omega \\ \psi = 0 \text{ on } \partial\Omega \end{cases}. \tag{6.3.7}$$

*Then,*

$$\|\nabla \psi\|_{\infty}^2 \leq C \|f\|_{\infty} \|\psi\|_{\infty}, \tag{6.3.8}$$

*where $C$ is a constant depending only on $\Omega$.*

Also, notice that the $L^{2,+\infty}$-norm of the derivative of the Green function (seen as map on $y \in \mathcal{R}^2$) is uniformly bounded, namely, we have that, for every $x \in \mathbb{R}^2$,

$$\|\nabla_y G(x, y)\|_{L^{2,+\infty}(\mathbb{R}^2)} \leq C,$$

where $C > 0$ is a constant independent of $x$.

## 6.4 Global solutions for the linearized Willmore system

In this section, we establish and solve the linearized Willmore system around a sphere: $\Phi = r\omega + \varphi$, $\vec{n} = -\hat{\omega} + \nu$ and $\vec{H} = -\frac{1}{r}\hat{\omega} + \eta$, where $\omega$ is defined in (5.4.1), Remark 5.4.2.

First, we can derive the relations between the components of derivatives of $\varphi$ under the orthogonal basis $(\hat{\omega}_x, \hat{\omega}_y, \hat{\omega})$.

**Lemma 6.4.1.** *Let $\varphi \in H^2(\mathbb{R}^2, \mathbb{R}^3)$ satisfy the conformal condition with respect to the sphere $r\omega$:*

$$\begin{cases} \varphi_x \cdot \hat{\omega}_y + \hat{\omega}_x \cdot \varphi_y = 0 \\ \varphi_x \cdot \hat{\omega}_x - \varphi_y \cdot \hat{\omega}_y = 0 \end{cases}. \tag{6.4.1}$$

*Then, there exist $a, b, c, d \in H^2(\mathbb{R}^2, \mathbb{R})$ such that*

$$\begin{aligned} \varphi_x &= a\hat{\omega}_x + b\hat{\omega}_y + c\hat{\omega} \\ \varphi_y &= -b\hat{\omega}_x + a\hat{\omega}_y + d\hat{\omega}. \end{aligned}$$

*Moreover, we have*

$$\begin{aligned} a_y &= -b_x + d \\ a_x &= b_y + c \\ c_y &= |\nabla \hat{\omega}|^2 b + d_x \end{aligned} \tag{6.4.2}$$

*and in particular,*

$$\begin{aligned} \Delta b &= -|\nabla \omega|^2 b \\ \Delta a &= c_x + d_y. \end{aligned} \tag{6.4.3}$$

*Proof.* The existence of $(a, b, c, d)$ directly stems from (6.4.1). By using the fact that $\phi_{xy} = \phi_{yx}$, we obtain (6.4.2). □

With help of conformal parametrization, we can now classify the solutions of the linearized Willmore system.

**Proposition 6.4.2.** *Let $\varphi, \nu, \eta \in H^2(\mathbb{R}^2, \mathbb{R}^3)$ satisfy the linearized Willmore system:*

$$\begin{cases} \Delta\varphi = -\nabla\omega \wedge \nabla^\perp\varphi + r\nabla\nu \wedge \nabla^\perp\omega \\ \Delta\nu = -2\nabla^\perp\omega \wedge \nabla\nu + 2\frac{1}{r}\nabla\omega \wedge \nabla^\perp\varphi - 2r\nabla\eta \wedge \nabla^\perp\omega \\ \Delta\eta = 3div(\left(-\frac{1}{r}\nabla\omega \cdot \nu - \nabla\eta \cdot \omega\right)(-\omega)) + \nabla^\perp\omega \wedge \nabla\eta + \frac{1}{r}\nabla^\perp\nu \wedge \nabla\omega \end{cases} \tag{6.4.4}$$

*where $r > 0$ and*

$$\begin{aligned} \nu &= \frac{2}{r|\nabla\hat{\omega}|^2}\left(\hat{\omega}_x \wedge \varphi_y + \varphi_x \wedge \hat{\omega}_y + (\hat{\omega}_x \cdot \varphi_x + \hat{\omega}_y \cdot \varphi_y)\hat{\omega}\right) \\ \eta &= \frac{\Delta\varphi}{r^2|\nabla\hat{\omega}|^2} + \frac{2(\varphi_x \cdot \hat{\omega}_x + \varphi_y \cdot \hat{\omega}_y)}{r^2|\nabla\hat{\omega}|^2}\hat{\omega}. \end{aligned} \tag{6.4.5}$$

*Assume $\varphi$ verifies (6.3.7). Then using the same notation as in Lemma 6.4.1, we have*

$$\Delta a = -|\nabla\omega|^2 a.$$

*Proof.* Set $Y = \frac{1}{r|\nabla\hat{\omega}|^2}\left(\Delta a + |\nabla\hat{\omega}|^2 a\right)$. The second identity in (6.4.5) gives that

$$\eta = \frac{1}{r}(\nu + Y\hat{\omega}). \tag{6.4.6}$$

Next we multiply the third equation in (6.4.4) by $r$, using (6.4.6) and the fact that $\nabla\hat{\omega} \cdot \nu + \nabla\nu \cdot \omega = 0$, we get that

$$\Delta\nu + \Delta(Y\hat{\omega}) = 3div((\nabla Y)\hat{\omega}) + 2\nabla^\perp\hat{\omega} \wedge \nabla\nu + \nabla^\perp\hat{\omega} \wedge \nabla(Y\omega). \tag{6.4.7}$$

Combined with the second equation in (6.4.4), we obtain that

$$\Delta(Y\hat{\omega}) = 3div((\nabla Y)\hat{\omega}) - 2\nabla\nu \wedge \nabla^\perp\hat{\omega} - \frac{2}{r}\nabla\hat{\omega} \wedge \nabla^\perp\hat{\omega} + \nabla(Y\hat{\omega}) \wedge \nabla^\perp\hat{\omega}. \tag{6.4.8}$$

A straightforward computation shows that

$$Y\Delta\hat{\omega} = (\Delta Y)\hat{\omega} - |\nabla\hat{\omega}|^2 Y\hat{\omega}, \tag{6.4.9}$$

which implies

$$\Delta Y = 0 \text{ on } \mathbb{R}^2,$$

and hence, $Y \equiv 0$ since $\varphi \in H^2(\mathbb{R}^2, \mathbb{R}^3)$. □

The following proposition shows that under proper initial conditions on $\varphi$, there exist only trivial solutions for the linearized Willmore system.

**Proposition 6.4.3.** *Let $\varphi, \nu, \eta \in C^\infty(\mathbb{R}^2, \mathbb{R}^3) \cap H^2(\mathbb{R}^2, \mathbb{R}^3)$ be as in Proposition 6.4.2 complemented with initial conditions $\nabla\varphi(0) = 0$ and $\langle\nabla^2\varphi, \nabla\omega\rangle(0) = 0$. Then we get*

$$\nabla\varphi \equiv 0 \text{ and } \nu \equiv \eta \equiv 0.$$

*Proof.* Thanks to Proposition 6.4.2, using the same notation as in Lemma 6.4.1, we have that

$$\Delta a = -|\nabla\omega|^2 a \text{ and } \Delta b = -|\nabla\omega|^2 b.$$

Moreover, thanks to Lemma 6.2.1, since $|\nabla\omega(x)| \leq C\frac{1}{1+|x|^2}$, we have that

$$|\nabla\varphi(x)| \leq C\frac{\ln(2+|x|)}{1+|x|},$$

which, together with the initial conditions on $\varphi$, leads to

$$\begin{cases} a(0) = b(0) = 0, \\ \nabla a(0) = \nabla b(0) = 0 \text{ and} \\ |a| + |b| = O\left(1 + |x|^{3/2}\right) \end{cases}. \tag{6.4.10}$$

Since $\varphi \in H^2(\mathbb{R}^2, \mathbb{R}^3)$, $a$ and $b$ can be extended as maps in $H^1(\mathbb{S}^2)$ satisfying

$$\Delta_{\mathbb{S}^2} a = -2a, \ \Delta_{\mathbb{S}^2} b = -2b.$$

Hence, $a$ and $b$ can be written as linear combinations of the first eigenvectors of $\Delta_{\mathbb{S}^2}$. Then, together with (6.4.10), (see Lemma C.1 of [83] for more details), we get that

$$a \equiv b \equiv 0,$$

which proves Proposition 6.4.3. □

For convenience reasons, we usually replace $\nu$ by $r\nu$ and $\eta$ by $r^2\eta$, so we can omit the radius $r$. In order to simplify the notations, for every given triplet $(\varphi, \nu, \eta)$ satisfying (6.4.5), we denote by $\mathcal{L}_\omega(\varphi, \nu, \eta)$ the linearized Willmore operator (6.4.4) with respect to $\omega$.

$$\boxed{\mathcal{L}_\omega(\varphi,\nu,\eta) := \begin{cases} \Delta\varphi + \nabla\hat{\omega} \wedge \nabla^\perp\varphi - \nabla\nu \wedge \nabla^\perp\hat{\omega} \\ \Delta\nu + 2\nabla^\perp\hat{\omega} \wedge \nabla\nu - 2\nabla\hat{\omega} \wedge \nabla^\perp\varphi + 2\nabla\eta \wedge \nabla^\perp\hat{\omega} \\ \Delta\eta + 3div((\nabla\hat{\omega}\cdot\nu + \nabla\eta\cdot\hat{\omega})(-\hat{\omega})) - \nabla^\perp\hat{\omega} \wedge \nabla\eta - \nabla^\perp\nu \wedge \nabla\hat{\omega} \end{cases}}.$$

## 6.5 Taylor expansion of the metric

The following lemma is a simple consequence of the expansion of the metric under a normal coordinates centered at $p \in M$, see [126].

**Lemma 6.5.1.**

$$\begin{aligned} (h_\varepsilon)_{ik} &= \delta_{ik} + \frac{\varepsilon^2}{3}R_{ijmk}(p)y^j y^m + \frac{\varepsilon^3}{6}R_{ijmk,n}(p)y^j y^m y^n + o(\varepsilon^3) \\ \sqrt{|h_\varepsilon|} &= 1 - \frac{\varepsilon^2}{6}Ric_{mn}(p)y^m y^n - \frac{\varepsilon^3}{12}Ric_{mn,j}y^m y^n y^j + o(\varepsilon^3) \\ (h^\varepsilon)^{ik}(y) &= \delta_{ik} - \frac{\varepsilon^2}{3}R_{ijmk}(p)y^j y^m - \frac{\varepsilon^3}{6}R_{ijmk,n}(p)y^j y^m y^n + o(\varepsilon^3) \\ (\Gamma_\varepsilon)^k_{ij}(y) &= \varepsilon^2 A_{ikjm}(p)y^m + \varepsilon^3 B_{ikjmn}(p)y^m y^n + o(\varepsilon^3) \\ (\Gamma_\varepsilon)^k_{ij,l}(y) &= \varepsilon^2 A_{ikjl}(p) + \varepsilon^3 B_{ikjmn}(p)(\delta^{ml}y^n + \delta^{nl}y^m) + o(\varepsilon^3), \end{aligned} \tag{6.5.1}$$

*where* $(h^\varepsilon)^{ik}$ *is the inverse matrix of* $(h_\varepsilon)_{ik}$ *and*

$$\begin{aligned} A_{ikjm}(p) &= \frac{1}{3}\left(R_{jmik}(p) + R_{imjk}(p)\right) \\ B_{ikjmn}(p) &= \frac{1}{12}\left(2R_{jmik,n}(p) + 2R_{imjk,n}(p) + R_{jmnk,i}(p) + R_{imnk,j}(p) - R_{imnj,k}(p)\right). \end{aligned}$$

# Bibliography


[1] G. Alberti, S. Baldo, and G. Orlandi. Functions with prescribed singularities. *Journal of the European Mathematical Society*, 5(3):275–311, Sept. 2003.

[2] R. Alessandroni and E. Kuwert. Local solutions to a free boundary problem for the Willmore functional. *Calc. Var. Partial Differential Equations*, 55(2):55:24, 2016.

[3] N. D. Alikakos, P. W. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Archive for Rational Mechanics and Analysis*, 128(2):165–205, June 1994.

[4] L. Ambrosio. Geometric evolution problems, distance function and viscosity solutions. In *Calculus of variations and partial differential equations (Pisa, 1996)*, pages 5–93. Springer, Berlin, 2000.

[5] L. Ambrosio and N. Dancer. *Calculus of variations and partial differential equations*. Springer-Verlag, Berlin, 2000. Topics on geometrical evolution problems and degree theory, Papers from the Summer School held in Pisa, September 1996, Edited by G. Buttazzo, A. Marino and M. K. V. Murthy.

[6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs, 2000.

[7] L. Ambrosio, N. Gigli, and G. Savare. *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2 edition, 2008.

[8] L. Ambrosio and C. Mantegazza. Curvature and distance function from a manifold. *J. Geom. Anal.*, 8(5):723–748, 1998. Dedicated to the memory of Fred Almgren.

[9] L. Ambrosio and S. Masnou. A direct variational approach to a problem arising in image reconstruction. *Interfaces and Free Boundaries*, 5:63–81, 2003.

[10] P. Aviles and Y. Giga. A mathematical problem related to the physical theory of liquid crystal configurations. In *Miniconference on Geometry/Partial Differential Equations, 2*, pages 1–16. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, 1987.

[11] P. Aviles and Y. Giga. The distance function and defect energy. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 126(5):923–938, 1996.

[12] P. Aviles and Y. Giga. On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg–Landau type energy for gradient fields. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 129(1):1–17, 1999/ed.

[13] C. Ballester, M. Bertalmío, V. Caselles, G. Sapiro, and J. Verdera. Filling-in by joint interpolation of vector fields and gray levels. *IEEE Trans. Image Process.*, 10(8):1200–1211, 2001.

[14] M. Bauer and E. Kuwert. Existence of minimizing Willmore surfaces of prescribed genus. *Int. Math. Res. Not.*, 2003(10):553–576, 2003.

[15] G. Bellettini. *Lecture notes on mean curvature flow, barriers and singular perturbations*, volume 12 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2013.

[16] G. Bellettini, G. Dal Maso, and M. Paolini. Semicontinuity and relaxation properties of a curvature depending functional in 2d. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 20(2):247–297, 1993.

[17] G. Bellettini and L. Mugnai. Approximation of Helfrich's Functional via Diffuse Interfaces. *SIAM Journal on Mathematical Analysis*, 42(6):2402–2433, 2010-01-01.

[18] G. Bellettini and M. Paolini. Approssimazione variazionale di funzionali con curvatura. In *Seminario Analisi Matematica, Univ. Bologna*, pages 87–97, 1993.

[19] Y. Bernard and T. Rivière. Local Palais-Smale sequences for the Willmore functional. *Comm. Anal. Geom.*, 19(3):563–599, 2011.

[20] Y. Bernard and T. Rivière. Singularity removability at branch points for Willmore surfaces. *Pacific J. Math.*, 265(2):257–311, 2013.

[21] Y. Bernard and T. Rivière. Energy quantization for Willmore surfaces and applications. *Ann. of Math. (2)*, 180(1):87–136, 2014.

[22] F. Bethuel, H. Brezis, and F. Hélein. Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. Partial Differential Equations*, 1(2):123–148, 1993.

[23] W. Blaschke. *Vorlesungen über Differentialgeometrie III*. Springer, Berlin, 1929.

[24] M. Bonafini, G. Orlandi, and E. Oudet. Variational approximation of functionals defined on 1-dimensional connected sets: the planar case. *SIAM J. Math. Anal.*, 50(6):6307–6332, 2018.

[25] M. Bonafini and E. Oudet. A convex approach to the Gilbert-Steiner problem. *Interfaces Free Bound.*, 22(2):131–155, 2020.

[26] M. Bonnivard, E. Bretin, and A. Lemenant. Numerical approximation of the Steiner problem in dimension 2 and 3. *Math. Comp.*, 89(321):1–43, 2020.

[27] M. Bonnivard, A. Lemenant, and F. Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015.

[28] A. Braides. *Gamma-Convergence for Beginners*. Oxford University Press, July 2002.

[29] E. Bretin, F. Dayrens, and S. Masnou. Volume reconstruction from slices. *SIAM J. Imaging Sci.*, 10(4):2326–2358, 2017.

[30] E. Bretin, S. Masnou, and É. Oudet. Phase-field approximations of the Willmore functional and flow. *Numerische Mathematik*, 131(1):115–171, 2015-09-01.

[31] H. Brezis, J.-M. Coron, and E. H. Lieb. Harmonic maps with defects. *Communications in Mathematical Physics*, 107(4):649–705, 1986.

[32] L. Bronsard and R. V. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *Journal of Differential Equations*, 90(2):211–237, 1991-04-01.

[33] R. L. Bryant. A duality theorem for Willmore surfaces. *J. Differential Geom.*, 20(1):23–53, 1984.

[34] J. W. Cahn, C. M. Elliott, and A. Novick-Cohen. The Cahn–Hilliard equation with a concentration dependent mobility: Motion by minus the Laplacian of the mean curvature. *European Journal of Applied Mathematics*, 7(3):287–301, 1996-06.

[35] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

[36] F. Cao, Y. Gousseau, S. Masnou, and P. Pérez. Geometrically guided exemplar-based inpainting. *SIAM J. Imaging Sci.*, 4(4):1143–1179, 2011.

[37] A. Chambolle, L. A. D. Ferrari, and B. Merlet. A phase-field approximation of the Steiner problem in dimension two. *Adv. Calc. Var.*, 12(2):157–179, 2019.

[38] A. Chambolle, L. A. D. Ferrari, and B. Merlet. Variational approximation of size-mass energies for $k$-dimensional currents. *ESAIM Control Optim. Calc. Var.*, 25:Paper No. 43, 39, 2019.

[39] A. Chambolle and T. Pock. Total roto-translational variation. *Numer. Math.*, 142(3):611–666, 2019.

[40] J. Chen and Y. Li. Bubble tree of branched conformal immersions and applications to the Willmore functional. *Amer. J. Math.*, 136(4):1107–1154, 2014.

[41] X. Chen. Generation and Propagation of Interfaces in Reaction-Diffusion Systems. *Transactions of the American Mathematical Society*, 334(2):877–913, 1992.

[42] X. Chen. Global asymptotic limit of solutions of the Cahn-Hilliard equation. *Journal of Differential Geometry*, 44(2):262–311, 1996.

[43] P. Colli and P. Laurençot. A phase-field approximation of the Willmore flow with volume constraint. *Interfaces and Free Boundaries*, 13(3):341–351, Oct. 2011.

[44] P. Colli and P. Laurençot. A Phase-Field Approximation of the Willmore Flow with Volume and Area Constraints. *SIAM Journal on Mathematical Analysis*, 44(6):3734–3754, 2012-01-01.

[45] K. Crane, U. Pinkall, and P. Schröder. Robust fairing via conformal curvature flow. *ACM Trans. Graph.*, 32(4), July 2013.

[46] A. Dahan-Dalmédico. Mécanique et théorie des surfaces: les travaux de Sophie Germain. *Historia Math.*, 14(4):347–365, 1987.

[47] G. Dal Maso. *An Introduction to* Γ*-Convergence*. Birkhäuser Boston, 1993.

[48] A. Dall'Acqua. Uniqueness for the homogeneous Dirichlet Willmore boundary value problem. *Ann. Global Anal. Geom.*, 42(3):411–420, 2012.

[49] C. DAmbrosio, M. Fampa, J. Lee, and S. Vigerske. On a nonconvex MINLP formulation of the Euclidean Steiner tree problem in n-space. In *International Symposium on Experimental Algorithms*, page 122–133. Springer, 2015.

[50] H. Dang, P. C. Fife, and L. A. Peletier. Saddle solutions of the bistable diffusion equation. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 43(6):984–998, 1992-11-01.

[51] E. De Giorgi. Some remarks on Γ-convergence and least square methods. In G. D. Maso and G. Dell'Antonio, editors, *Composite Media and Homogenization Theory*, pages 135–142. Birkhaüser, Boston, 1991.

[52] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 58(6):842–850, 1975.

[53] P. De Mottoni and M. Schatzman. Geometrical Evolution of Developed Interfaces. *Transactions of the American Mathematical Society*, 347(5):1533–1589, 1995.

[54] M. C. Delfour and J.-P. Zolésio. Shape analysis via oriented distance functions. *J. Funct. Anal.*, 123(1):129–201, 1994.

[55] M. Droske and M. Rumpf. A level set formulation for Willmore flow. *Interfaces Free Bound.*, 6(3):361–378, 2004.

[56] O. Druet. Sharp local isoperimetric inequalities involving the scalar curvature. *Proc. Amer. Math. Soc.*, 130(8):2351–2361 (electronic), 2002.

[57] O. Druet, E. Hebey, and F. Robert. A $C^0$-theory for the blow-up of second order elliptic equations of critical Sobolev growth. *Electron. Res. Announc. Amer. Math. Soc.*, 9:19–25, 2003.

[58] Q. Du and X. Feng. Chapter 5 - The phase field method for geometric moving interfaces and their numerical approximations. In A. Bonito and R. H. Nochetto, editors, *Handbook of Numerical Analysis*, volume 21 of *Geometric Partial Differential Equations - Part I*, pages 425–508. Elsevier, 2020-01-01.

[59] Q. Du, C. Liu, R. Ryham, and X. Wang. A phase field formulation of the Willmore problem. *Nonlinearity*, 18(3):1249–1267, 2005-02.

[60] Q. Du, C. Liu, and X. Wang. A phase field approach in the numerical study of the elastic bending energy for vesicle membranes. *Journal of Computational Physics*, 198(2):450–468, 2004-08-10.

[61] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992.

[62] D. J. Eyre. Unconditionally Gradient Stable Time Marching the Cahn-Hilliard Equation. *MRS Online Proceedings Library Archive*, 529, 1998/ed.

[63] M. Fei and Y. Liu. Sharp interface limit of a phase field model for elastic bending energy. *arXiv:1904.11139 [math]*, Apr. 2019.

[64] Y. Ge. A remark on generalized harmonic maps into spheres. *Nonlinear Anal.*, 36(4, Ser. A: Theory Methods):495–506, 1999.

[65] S. Germain. *Recherches sur la théorie des surfaces élastiques*. Courcier, Paris, 1821.

[66] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[67] H.-C. Grunau. The asymptotic shape of a boundary layer of symmetric Willmore surfaces of revolution. In *Inequalities and applications 2010*, volume 161 of *Internat. Ser. Numer. Math.*, pages 19–29. Birkhäuser/Springer, Basel, 2012.

[68] S. W. Hawking. Gravitational radiation in an expanding universe. *J. Math. Phys.*, 9:598–604, Apr. 1968.

[69] F. Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 2002. Translated from the 1996 French original, With a foreword by James Eells.

[70] R. Ignat. Two-dimensional unit-length vector fields of vanishing divergence. *Journal of Functional Analysis*, 262(8):3465–3494, 2012-04-15.

[71] R. Ignat. Singularities of divergence-free vector fields with values into s1 or s2: Applications to micromagnetics. *Confluentes Mathematici*, 04(03):1230001, 2012-09-01.

[72] R. Ignat and B. Merlet. Lower Bound for the Energy of Bloch Walls in Micromagnetics. *Archive for Rational Mechanics and Analysis*, 199(2):369–406, 2011-02-01.

[73] R. Ignat and B. Merlet. Entropy method for line-energies. *Calculus of Variations and Partial Differential Equations*, 44(3):375–418, July 2012.

[74] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *Journal of Differential Geometry*, 38(2):417–461, 1993.

[75] R. Jerrard and H. Soner. Functions of Bounded Higher Variation. *Indiana University Mathematics Journal*, 51(3):645–677, 2002.

[76] D. Juhl, D. Warme, P. Winter, and M. Zachariasen. The geosteiner software package for computing steiner trees in the plane: an updated computational study, 2014. 11th DIMACS Implementation Challenge ; Conference date: 04-12-2014 Through 05-12-2014.

[77] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972.

[78] R. V. Kohn and P. Sternberg. Local minimisers and singular perturbations*. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 111(1-2):69–84, 1989/ed.

[79] R. Kusner. Comparison surfaces for the Willmore problem. *Pac. J. Math.*, 138(2):317–345, 1989.

[80] T. Lamm and J. Metzger. Small surfaces of Willmore type in Riemannian manifolds. *Int. Math. Res. Not. IMRN*, 2010(19):3786–3813, 2010.

[81] T. Lamm and J. Metzger. Minimizers of the Willmore functional with a small area constraint. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(3):497–518, 2013.

[82] T. Lamm, J. Metzger, and F. Schulze. Foliations of asymptotically flat manifolds by surfaces of Willmore type. *Math. Ann.*, 350(1):1–78, 2011.

[83] P. Laurain. Concentration of $CMC$ surfaces in a 3-manifold. *Int. Math. Res. Not. IMRN*, 2012(24):5585–5649, 2012.

[84] P. Laurain and A. Mondino. Concentration of small Willmore spheres in Riemannian 3-manifolds. *Anal. PDE*, 7(8):1901–1921, 2014.

[85] P. Laurain and T. Rivière. Energy quantization of willmore surfaces at the boundary of the moduli space, 2016.

[86] H. B. Lawson, Jr. Complete minimal surfaces in $S^3$. *Ann. of Math.*, 92:335–374, 1970.

[87] H. G. Lee and J.-Y. Lee. A semi-analytical Fourier spectral method for the Allen–Cahn equation. *Computers & Mathematics with Applications*, 68(3):174–184, Aug. 2014.

[88] G. Leonardi and S. Masnou. Locality of the mean curvature of rectifiable varifolds. *Adv. Calc. of Var.*, 2(1):17–42, 2009.

[89] P. Li and S. T. Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.*, 69(2):269–291, 1982.

[90] Y. Li. Some remarks on willmore surfaces embedded in $\mathbb{R}^3$. *J. Geom. Anal.*, 25:1–14, 2015.

[91] Y. Li, H. G. Lee, D. Jeong, and J. Kim. An unconditionally stable hybrid numerical method for solving the Allen–Cahn equation. *Computers & Mathematics with Applications*, 60(6):1591–1606, Sept. 2010.

[92] P. Loreti and R. March. Propagation of fronts in a nonlinear fourth order equation. *European Journal of Applied Mathematics*, 11(2):203–213, 2000-04.

[93] C. Mantegazza and A. C. Mennucci. Hamilton-Jacobi equations and distance functions on Riemannian manifolds. *Appl. Math. Optim.*, 47(1):1–25, 2003.

[94] F. C. Marques and A. Neves. Min-Max theory and the Willmore conjecture. *Annals of Mathematics*, 179(2):683–782, 2014.

[95] S. Masnou and J. Morel. Level lines based disocclusion. In *Proceedings of the 1998 IEEE International Conference on Image Processing, ICIP-98, Chicago, Illinois, USA, October 4-7, 1998*, pages 259–263. IEEE Computer Society, 1998.

[96] S. Masnou and J. M. Morel. On a variational theory of image amodal completion. *Rend. Semin. Mat. Univ. Padova*, 116:211–252, 2006.

[97] U. Menne. Second order rectifiability of integral varifolds of locally bounded first variation. *J. Geom. Anal.*, 23(2):709–763, 2013.

[98] B. Merlet. A highly anisotropic nonlinear elasticity model for vesicles. I: Eulerian formulation, rigidity estimates and vanishing energy limit. *Arch. Ration. Mech. Anal.*, 217(2):651–680, 2015.

[99] B. Merlet. A highly anisotropic nonlinear elasticity model for vesicles. II: Derivation of the thin bilayer bending theory. *Arch. Ration. Mech. Anal.*, 217(2):681–740, 2015.

[100] A. Miranville. *The Cahn–Hilliard Equation: Recent Advances and Applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2019.

[101] M. Mizuno and Y. Tonegawa. Convergence of the Allen–Cahn Equation with Neumann Boundary Conditions. *SIAM Journal on Mathematical Analysis*, 47(3):1906–1932, 2015-01-01.

[102] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[103] L. Modica and S. Mortola. Un esempio di $\Gamma-$-convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[104] A. Mondino. Some results about the existence of critical points for the Willmore functional. *Math. Z.*, 266(3):583–622, 2010.

[105] A. Mondino. The conformal Willmore functional: a perturbative approach. *J. Geom. Anal.*, 23(2):764–811, 2013.

[106] A. Mondino and T. Rivière. Willmore spheres in compact Riemannian manifolds. *Adv. Math.*, 232:608–676, 2013.

[107] A. Mondino and T. Rivière. Immersed spheres of finite total curvature into manifolds. *Adv. Calc. Var.*, 7(4):493–538, 2014.

[108] L. Mugnai. Gamma-convergence results for phase-field approximations of the 2D-Euler elastica functional. *ESAIM Control Optim. Calc. Var.*, 19(3):740–753, 2013.

[109] L. Mugnai. Gamma-convergence results for phase-field approximations of the 2d-euler elastica functional. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(3):740–753, 2013.

[110] L. Mugnai and M. Röger. Convergence of Perturbed Allen-Cahn Equations to Forced Mean Curvature Flow. *Indiana University Mathematics Journal*, 60, Mar. 2009.

[111] Y. Nagase and Y. Tonegawa. A singular perturbation problem with integral curvature bound. *Hiroshima Mathematical Journal*, 37(3):455–489, 2007-11.

[112] J. C. C. Nitsche. Boundary value problems for variational integrals involving surface curvatures. *Quart. Appl. Math.*, 51(2):363–387, 1993.

[113] A. Novick-Cohen. The Cahn-Hilliard equation. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 201–228. Elsevier/North-Holland, Amsterdam, 2008.

[114] M. Ortiz and G. Gioia. The morphology and folding patterns of buckling-driven thin-film blisters. *Journal of the Mechanics and Physics of Solids*, 42(3):531–559, Mar. 1994.

[115] B. Palmer. Uniqueness theorems for Willmore surfaces with fixed and free boundaries. *Indiana Univ. Math. J.*, 49(4):1581–1601, 2000.

[116] R. L. Pego and O. Penrose. Front migration in the nonlinear Cahn-Hilliard equation. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 422(1863):261–278, 1989-04-08.

[117] S. D. Poisson. *Mémoire sur les Surfaces Élastiques*. Mem. Cl. Sci. Math. Phys. Inst. de France, 1812.

[118] Y. Qi and G.-F. Zheng. Convergence of solutions of the weighted Allen–Cahn equations to Brakke type flow. *Calculus of Variations and Partial Differential Equations*, 57(5):133, Aug. 2018.

[119] G. E. Raynor. Isolated singular points of harmonic functions. *Bull. Amer. Math. Soc.*, 32(5):537–544, 1926.

[120] T. Rivière. Conservation laws for conformally invariant variational problems. *Invent. Math.*, 168(1):1–22, 2007.

[121] T. Rivière. Analysis aspects of Willmore surfaces. *Invent. Math.*, 174(1):1–45, 2008.

[122] T. Rivière. Variational principles for immersed surfaces with $L^2$-bounded second fundamental form. *J. Reine Angew. Math.*, 695:41–98, 2014.

[123] J. S. Rowlinson. Translation of J. D. van der Waals' "The thermodynamik theory of capillarity under the hypothesis of a continuous variation of density". *Journal of Statistical Physics*, 20(2):197–200, Feb. 1979.

[124] J. Rubinstein, P. Sternberg, and J. B. Keller. Fast Reaction, Slow Diffusion, and Curve Shortening. *SIAM Journal on Applied Mathematics*, 49(1):116–133, 1989-02-01.

[125] M. Röger and R. Schätzle. On a Modified Conjecture of De Giorgi. *Mathematische Zeitschrift*, 254(4):675–714, 2006-12-01.

[126] T. Sakai. *Riemannian geometry*, volume 149 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.

[127] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[128] L. Simon. Existence of surfaces minimizing the Willmore functional. *Comm. Anal. Geom.*, 1(2):281–326, 1993.

[129] L. Simon. Existence of surfaces minimizing the Willmore functional. *Comm. Anal. Geom.*, 1(2):281–326, 1993.

[130] L. Tartar. The compensated compactness method applied to systems of conservation laws. In *Systems of nonlinear partial differential equations (Oxford, 1982)*, volume 111 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 263–285. Reidel, Dordrecht, 1983.

[131] L. Tartar. Imbedding theorems of Sobolev spaces into Lorentz spaces. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(3):479–500, 1998.

[132] G. Thomsen. Über die Auswertung der Quantenintegrale für den unsymmetrischen Kreisel. *Math. Ann.*, 94(1):146–162, 1925.

[133] X. Wang. Asymptotic Analysis of Phase Field Formulations of Bending Elasticity Models. *SIAM Journal on Mathematical Analysis*, 39(5):1367–1401, 2008-01-01.

[134] J. L. Weiner. On a problem of Chen, Willmore, et al. *Indiana Univ. Math. J.*, 27(1):19–35, 1978.

[135] H. C. Wente. An existence theorem for surfaces of constant mean curvature. *J. Math. Anal. Appl.*, 26:318–344, 1969.

[136] T. J. Willmore. Note on embedded surfaces. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)*, 11B:493–496, 1965.

[137] T. J. Willmore. *Riemannian geometry*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[138] X. Yang. Error analysis of stabilized semi-implicit method of Allen-Cahn equation. *Discrete & Continuous Dynamical Systems - B*, 11(4):1057, Apr. 2009.

[139] C. Zwilling. *The Diffuse Interface Approximation of the Willmore Functional in Configurations with Interacting Phase Boundaries*. Technische Universität Dortmund, 2018.

# Questions d'approximation et de compacité pour des problèmes variationnels géométriques

**Résumé :** La première partie de cette thèse est consacrée à l'étude théorique et numérique de l'approximation par méthode de champ de phase de deux flots géométriques, le flot de courbure moyenne et le flot de Willmore. L'analyse d'un modèle particulier d'approximation du flot de Willmore nous amène à proposer un nouveau terme de réaction qui charge les singularités du champ normal associé à une forme en évolution. On en déduit un nouveau modèle d'approximation du flot de courbure moyenne qui empêche les changements de topologie. Ce modèle est en particulier bien adapté à l'approximation numérique de solutions en 3D du problème de Steiner et du problème de Plateau.
Dans la deuxième partie de la thèse, on étudie le comportement asymptotique de petites sphères de Willmore plongées dans une variété riemannienne de dimension 3. En utilisant la formulation de l'équation de Willmore donnée par Rivière en un système triple d'EDPs elliptiques, on montre que, dans le cas où seules deux sphères apparaissent dans la décomposition asymptotique, les petites sphères de Willmore se concentrent nécessairement en un point critique de la courbure scalaire de la variété ambiante.

**Mots clés :** Approximation par méthode de champ de phase, Flots géométriques, Contraintes topologiques, Energie de Willmore, Phénomènes de concentration, Analyse non linéaire.

## *Questions of approximation and compactness for some geometric variational problems*

**Abstract :** The first part of this thesis is devoted to the theoretical and numerical study of the phase field approximation of two geometric flows, the mean curvature flow and the Willmore flow. The analysis of a particular model of approximation of the Willmore flow leads us to propose a new reaction term which charges the singularities of the normal field associated to an evolving shape. We derive a new model of approximation of the mean curvature flow which prevents topology changes. This model is in particular well adapted to the numerical approximation of 3D solutions of the Steiner problem and the Plateau problem.
In the second part of the thesis, we study the asymptotic behavior of small embedded Willmore spheres in a Riemannian manifold of dimension 3. Using the formulation of Willmore equation derived by Rivière in terms of a triple system of elliptic PDEs, we show that, in the case where only two spheres appear in the asymptotic decomposition, small embedded Willmore spheres necessarily concentrate at a critical point of the scalar curvature of the ambient manifold.

**Keywords :** Phase-field approximation, Geometric flows, Topological constraints, Willmore energy, Concentration phenomena, Nonlinear Analysis.

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