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Thomas Gerard

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Thomas GERARD

**Processus renforcés, représentation
comme mélange de processus de
Markov, et mouvement brownien
interagissant**

Devant le jury composé de :

Christophe Sabot
Bruno Schapira
Anne-Laure Basdevant
Silke Rolles

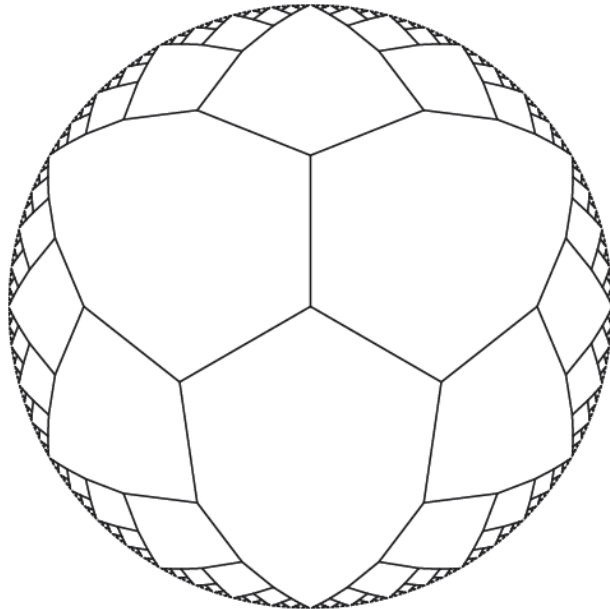
Guillaume Aubrun
Fabienne Castell
Titus Lupu

Université Lyon 1
Université Lyon 1
Université Paris Nanterre
Technische Universität
München
Université Lyon 1
Aix-Marseille Université
CNRS Paris

Directeur de thèse
Président du jury
Rapporteure
Rapporteure

Examineur
Examinatrice
Examineur

Processus renforcés, représentation comme mélange de
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Thomas GERARD

Thèse de doctorat

Université Claude Bernard Lyon 1
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Processus renforcés, représentation comme mélange de processus de Markov, et mouvement brownien interagissant

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soutenue publiquement le 10 juillet 2020 par
Thomas GERARD

devant le jury composé de :

Guillaume AUBRUN	(Université Lyon 1)	Examineur
Anne-Laure BASDEVANT	(Université Paris Nanterre)	Rapporteuse
Fabienne CASTELL	(Aix-Marseille Université)	Examinatrice
Titus LUPU	(CNRS Paris)	Examineur
Silke ROLLES	(Technische Universität München)	Rapporteuse
Christophe SABOT	(Université Lyon 1)	Directeur
Bruno SCHAPIRA	(Aix-Marseille Université)	Examineur

suite aux rapports de :

Anne-Laure BASDEVANT	(Université Paris Nanterre)
Silke ROLLES	(Technische Universität München)

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Introduction en français

Cette introduction présente de manière plus succincte le contenu du premier chapitre de ce manuscrit, écrit en anglais. Celui-ci porte sur des processus aléatoires renforcés, et leur représentation comme processus de Markov en environnement aléatoire. Plus particulièrement, on étudie la question de l'unicité d'une telle représentation pour le VRJP (processus de saut renforcé par sommets). De plus, on s'intéresse au lien entre l'environnement aléatoire correspondant au VRJP et un problème de mouvements Brownien en interaction.

On appelle « chaîne de Markov en environnement aléatoire », ou « mélange de chaînes de Markov », un processus aléatoire dont la loi est la moyenne de lois de chaînes de Markov, définies par une matrice de transition aléatoire. Un tel processus n'est pas nécessairement markovien, cependant il vérifie une propriété d'échangeabilité : la probabilité d'emprunter une certaine trajectoire ne dépend pas de l'ordre dans lequel cette trajectoire visite différents états. Une classe de théorèmes, dits de « de Finetti », permet d'obtenir des résultats réciproques : un processus vérifiant une certaine version de la condition d'échangeabilité peut s'écrire comme mélange de processus de Markov.

Certains processus non markoviens, présentant par exemple un comportement de renforcement, sont néanmoins échangeables et sont donc des processus de Markov en environnement aléatoire. C'est notamment le cas de l'urne de Pólya, un processus renforcé très simple, qui s'écrit comme un mélange de suites i.i.d. de variables de Bernoulli. Une contrepartie naturelle à l'urne de Pólya est la marche aléatoire renforcée par arêtes, ou ERRW. Ce processus, introduit par Diaconis, est défini comme une modification de la marche simple sur un graphe pondéré, à laquelle on ajoute le renforcement suivant : le poids d'une arête augmente chaque fois qu'elle est traversée par le processus. Grâce à un théorème de type de Finetti dû à Diaconis et Freedman, cette marche renforcée peut aussi être écrite comme un mélange de chaînes de Markov. La loi de l'environnement aléatoire associée a notamment été étudiée par Merkl et Rolles, qui en ont déduit des résultats de récurrence et transience pour l'ERRW.

Le sujet principal de ce manuscrit est le processus de saut renforcé par sommets, ou VRJP. Il s'agit d'un processus aléatoire en temps continu $(X_t)_{t \geq 0}$, défini sur un graphe pondéré, pour lequel le taux de saut d'un sommet i à un sommet j à un instant t est

$$W_{i,j} l_j(t), \text{ où } l_j(t) = 1 + \int_0^t \mathbf{1}_{\{X_s=j\}} ds,$$

et où $W_{i,j}$ est le poids de l'arête $\{i,j\}$. En d'autres termes, un sommet est d'autant plus attractif que le processus y a déjà passé du temps. Ce processus, introduit par Werner, a été étudié par Davis et Volkov, Collecchio, ainsi que Basdevant et Singh, sur des graphes tels que \mathbb{Z} , les arbres réguliers, et les arbres de Galton-Watson. Il est de plus très lié à l'ERRW, puisque la marche renforcée peut s'écrire comme un mélange de versions discrétisées du VRJP.

Le VRJP ne vérifie pas initialement de condition d'échangeabilité, puisque la hausse des taux de saut entraîne des sauts de plus en plus rapprochés. En revanche, un certain changement de variable temporelle rend le VRJP échangeable, et il est alors possible de l'écrire comme un mélange de processus de Markov. Pour un graphe fini, Sabot et Tarrès ont montré que l'environnement aléatoire associé peut être défini

grâce à des taux de saut de la forme

$$\frac{W_{i,j}}{2} e^{u_j - u_i},$$

où u est un champ aléatoire défini sur le graphe. La loi de ce champ est liée à un modèle de physique statistique étudié par Disertori, Spencer et Zirnbauer : le sigma-modèle hyperbolique supersymétrique. Des théorèmes de localisation et délocalisation pour cette loi ont permis de montrer des résultats de récurrence et de transience pour le VRJP ainsi que pour l'ERRW sur les réseaux \mathbb{Z}^d .

La loi de l'environnement aléatoire associé au VRJP sur un graphe fini, démarré en un point i_0 , peut être formulée autrement : grâce à un potentiel aléatoire noté β , et un opérateur de Schrödinger associé H_β . Ce potentiel peut aussi être défini pour un graphe infini, grâce à une bonne condition de bord. Cela a permis à Sabot et Zeng de construire un environnement aléatoire correspondant au VRJP sur un graphe infini. Les taux de saut aléatoires s'écrivent alors

$$\frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)},$$

où $G(x, y) = \hat{G}(x, y) + \frac{1}{2\gamma} \psi(x)\psi(y)$, \hat{G} étant la fonction de Green associée à H_β , ψ une fonction H_β -harmonique, et γ une variable aléatoire de loi Gamma(1/2, 1) indépendante du potentiel β . Cette nouvelle expression de l'environnement aléatoire, ainsi que des propriétés d'ergodicité du potentiel β , ont fourni d'autres résultats pour le VRJP et l'ERRW sur \mathbb{Z}^d .

C'est dans ce contexte que s'inscrit la première problématique de ce manuscrit : celle de l'unicité de la représentation du VRJP. En d'autres termes, existe-t-il plusieurs mesures de probabilité sur les environnements (appelées « mesures de mélange ») telles que les processus mélangés correspondants ont tous la loi du VRJP ? On dira alors qu'une telle mesure est une représentation du VRJP. La question se pose, car on sait déjà que l'unicité est vérifiée dès que le processus est récurrent, mais qu'il existe plusieurs représentations distinctes sur les arbres infinis, construites à partir de conditions au bord différentes. Pour y répondre, la première étape est de montrer que toute représentation du VRJP peut s'exprimer sous une forme similaire à celle de la représentation « standard » présentée plus haut. En effet, une représentation quelconque fait nécessairement intervenir le potentiel β ainsi qu'une fonction harmonique h pour l'opérateur H_β : les taux de saut s'écrivent alors

$$\frac{W_{i,j}}{2} \frac{G_h(i_0, j)}{G_h(i_0, i)},$$

où $G_h(i_0, x) = \hat{G}(i_0, x) + h(x)$.

La question de l'unicité pourrait donc être résolue en identifiant toutes les fonctions harmoniques pour l'opérateur aléatoire H_β , par exemple grâce à la notion de frontière de Martin. Cependant, la frontière de Martin correspondant à un environnement aléatoire n'est pas connue a priori. On étudie donc en particulier le cas de deux types de graphes infinis : le réseau \mathbb{Z}^d et les arbres infinis. Sur \mathbb{Z}^d , on utilise un théorème central limite en environnement aléatoire, et l'ergodicité du potentiel β , pour montrer que la frontière de Martin correspondante est triviale dans un certain régime, ce qui prouve l'unicité de la représentation dans ce régime. Sur les arbres infinis, on utilise des conditions de bord différentes pour construire une famille infinie de représentations, qui sont toutes distinctes. Elles fournissent une idée de ce qui pourrait constituer une classification de toutes les représentations sur l'arbre, en lien avec la frontière de celui-ci.

Cette thèse aborde aussi le lien entre le potentiel β introduit dans l'étude du VRJP, et une famille de mouvements Browniens en interaction, représentée comme la solution d'un système d'équations différentielles stochastiques (EDS). On fixe un graphe pondéré dont l'ensemble des sommets est V , et on lui associe une EDS. La solution de l'EDS est un processus à N dimensions, où N est le cardinal de V . Le lien évoqué plus haut, montré par Sabot et Zeng, s'exprime de la manière suivante : le vecteur aléatoire des

temps d'atteinte de 0 pour cette solution est $(1/2\beta_i)_{i \in V}$, où β est le potentiel associé au VRJP sur V . De plus, conditionnellement à ce vecteur, la famille de processus a la loi de ponts de Bessel tridimensionnels indépendants. Dans le cas $N = 1$, il s'agit d'un résultat classique concernant la temps d'atteinte de 0 pour un mouvement Brownien avec dérive, car $1/2\beta$ suit une loi inverse gaussienne. Un autre théorème, dû à Matsumoto et Yor et dit « des dérives opposées », est une conséquence de ce dernier résultat, et s'obtient à l'aide de la relation de Lamperti, qui fournit un changement de variable pertinent.

On montre ici une version N -dimensionnelle de la relation de Lamperti et du théorème de Matsumoto et Yor, en appliquant le changement de variable de Lamperti à la solution de l'EDS mentionnée plus haut. Une difficulté vient du fait que cette transformation n'est pas un changement de temps usuel : en effet, chaque coordonnée du processus doit être changée de temps de façon différente. Néanmoins, grâce à une propriété d'abélianité de l'EDS, les résultats de Lamperti et de Matsumoto-Yor admettent une généralisation naturelle.

Chapter 1

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Preamble

This manuscript presents my work concerning self-interacting random processes, in particular the Vertex-Reinforced Jump Process, and its representations as a random walk in random environment. The second and third chapter are modeled after the articles that constitute the main content of this PhD thesis. In this first chapter, I introduce the scientific context of the thesis, some prerequisite material, and the results that were obtained.

The first section tackles the general notion of exchangeability, and a related class of theorems named after de Finetti. A process is said to be exchangeable if its distribution is invariant under some class of permutations. According to de Finetti-type theorems, such a process can be written as the average of simpler processes (like i.i.d. sequences or Markov chains) with some random parameters. This kind of averaged process is called a mixture. We illustrate this principle through two examples of self-interacting processes: Pólya's urn, and the Edge-Reinforced Random Walk (ERRW). These present a form of reinforcing behavior, but still satisfy an exchangeability property, which means they can be represented as a mixture of Markov chains.

The second section presents another random process with reinforcement: the Vertex-Reinforced Jump Process (VRJP). It is defined as a continuous-time jump process on a graph, where the jump rate to a neighboring vertex is proportional to the time already spent by the process at this vertex. The VRJP is closely related to the ERRW. It is also exchangeable in some sense, so that it is a mixture of Markov processes, *i.e.* a Markov jump process with random jump rates. These can be expressed in a convenient way using a random potential β on the graph, and an associated Schrödinger operator H_β .

The third section presents the first part of my results, which concern the uniqueness of the representation of the VRJP as a mixture of Markov processes. In other words, is it possible to have several distinct distributions for random jump rates, for which the averaged process is the VRJP? This question is related to the set of H_β -harmonic functions on the graph, which we study using the notion of Martin boundary. This yields a uniqueness theorem on the lattice \mathbb{Z}^d . On infinite trees however, we are able to construct an infinite family of different representations. These results are stated and proved in the second chapter of the manuscript.

The fourth and final section of this introduction presents the link between the β field related to the VRJP, and hitting times of a family of Brownian motions with interacting drifts. This comes from the fact that the marginals of β are the inverse of Inverse Gaussian variables. Applying a time change to this family of Brownian motions, inspired by Lamperti's relation, we obtain a multi-dimensional version of Matsumoto and Yor's opposite drift theorem. The proof of this last result is detailed in the third chapter.

1.1 Reinforcement, exchangeability and random environment

Let us first present how processes defined by self-reinforcing behavior can be represented in a simpler way. For models with linear reinforcement, the process often satisfies some particular symmetry property called exchangeability. A class of theorems, named after Bruno de Finetti, can then be applied, yielding the following result: the distribution of the process is the average of simpler distributions (be it i.i.d. sequences or Markov processes), where the average is taken with respect to random parameters for these simpler processes (*e.g.* a random distribution for i.i.d. variables, or a random transition matrix for Markov chains). These random parameters will sometimes be referred to as a random environment.

We illustrate this principle through the examples of two self-reinforcing processes: Pólya's urn, and the Edge-Reinforced Random Walk.

1.1.1 Pólya's urn and de Finetti's theorem

Pólya's urn

Let us begin by presenting a classic model involving linear reinforcement: Pólya's urn. Consider an urn containing a black balls and b white balls. At every step of the process, draw a ball uniformly at random in the urn, then put it back along with an extra ball of the same color. This way, the probability of drawing a ball of this color at the next step increases, hence why we talk of reinforcement.

Remark 1.1.1. Depending on the initial quantities a and b , this effect can be more or less pronounced: if a and b are low, the first draws can quickly increase the proportion of balls of one color, and we talk about strong reinforcement; conversely, for high a and b we have weak reinforcement.

We will now describe this model in more detail. Let $(X_n)_{n \geq 1}$ be the canonical process in $\{0, 1\}^{\mathbb{N}^*}$, which represents successive draws in an urn: for $n \geq 1$, $X_n = 1$ means that the n th draw was a black ball. Let $\mathbb{P}_{a,b}$ be the probability distribution on $\{0, 1\}^{\mathbb{N}^*}$ under which X has the distribution of the draws in Pólya's urn. In other words, for $n \geq 0$ and $(x_1, \dots, x_n) \in \{0, 1\}^n$,

$$\mathbb{P}_{a,b}[X_{n+1} = 1 | X_1 = x_1, \dots, X_n = x_n] = \frac{a + \sum_{i=0}^n x_i}{a + b + n}$$

Indeed, before the $(n+1)$ th draw the urn contains $a+b+n$ balls, and if the first n draws were successively x_1, \dots, x_n , then the number of black balls that have been added to the urn is $\sum_{i=0}^n x_i$.

It is immediately apparent that under $\mathbb{P}_{a,b}$, the process X interacts heavily with its past, and therefore is not a sequence of independent variables. However, it satisfies another interesting property, which will relate it to families of i.i.d. variables: exchangeability.

Exchangeability and de Finetti's theorem

Let us define the exchangeability property for random sequences.

Definition 1.1.1. Let $(Y_n)_{n \in \mathbb{N}^*}$ be a sequence of random variables. Then Y is an exchangeable sequence if for all $n \geq 1$, for all $s \in \mathfrak{S}_n$, (Y_1, \dots, Y_n) and $(Y_{s(1)}, \dots, Y_{s(n)})$ have the same distribution.

An infinite sequence of i.i.d. random variables is obviously exchangeable, but the converse is not true: as stated previously, the sequence of draws in Pólya's urn is not independent, but it is exchangeable. Indeed, for $n \geq 1$ and $(x_1, \dots, x_n) \in \{0, 1\}^n$, we have

$$\mathbb{P}_{a,b}[X_1 = x_1, \dots, X_n = x_n] = \frac{\left(a(a+1)\dots(a-1 + \sum_{i=1}^n x_i)\right) \left(b(b+1)\dots(b-1 + n - \sum_{i=1}^n x_i)\right)}{(a+b)(a+b+1)\dots(a+b+n-1)},$$

which depends only on $\sum_{i=1}^n x_i$.

However, an exchangeable sequence of Bernoulli random variables can still be represented using i.i.d. variables, thanks to the following theorem (from [DF29]).

Theorem 1.1.1 (de Finetti's theorem). Let $(Y_k)_{k \in \mathbb{N}^*}$ be an exchangeable sequence of random variables taking their values in $\{0, 1\}$.

Then there is a probability distribution μ on $[0, 1]$ such that for all $n \geq 1$ and $y_1, \dots, y_n \in \{0, 1\}$,

$$\begin{aligned} \mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n] &= \int p^{\sum_{i=1}^n y_i} (1-p)^{n-\sum_{i=1}^n y_i} \mu(dp) \\ &= \int P_p^{\otimes n}[X_1 = y_1, \dots, X_n = y_n] \mu(dp), \end{aligned}$$

where P_p denotes the Bernoulli distribution with parameter p , and $(X_n)_{n \in \mathbb{N}}$ is the canonical process in $\{0, 1\}^{\mathbb{N}}$.

We can understand the theorem this way: if a sequence of Bernoulli variables is exchangeable, then its distribution is the same as if we had first taken a random parameter p according to some probability distribution μ , and then drawn a sequence of i.i.d. Bernoulli variables with parameter p . Such a sequence is called a mixture of i.i.d. Bernoulli variables. Note that conversely, a mixture of any i.i.d. variables is exchangeable.

Remark 1.1.2. The random parameter can also be recovered using the law of large numbers. Indeed thanks to Theorem 1.1.1, if $(Y_k)_{k \in \mathbb{N}^*}$ is an exchangeable sequence of $\{0, 1\}$ -valued random variables, the empirical mean

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$$

converges a.s. to a random variable \bar{Y}_∞ , with distribution μ . Moreover, conditionally on \bar{Y}_∞ , Y_1, Y_2, \dots are i.i.d. Bernoulli variables with parameter \bar{Y}_∞ .

Writing the distribution of a random sequence as a mixture of i.i.d. variables evokes the framework of Bayesian statistics. Let us give more details about the difference between frequentist and Bayesian statistics. When trying to estimate the parameters of some experiment, the frequentist approach consists in making a hypothesis on the value of the parameters, and then interpreting a sample (*i.e.* a sequence of results of i.i.d. experiments) in order to accept the hypothesis or reject it with some margin of error. The Bayesian approach relies on considering the parameters as random variables, following some *a priori* distribution, so that the sample is no longer the realization of an i.i.d. sequence, but of an exchangeable one. Conditioning on this sample gives another *a posteriori* distribution for the parameters. It is a refinement of the prior distribution, indicating what the parameters may be with higher probability. This type of reasoning will be applied later, in order to identify the mixing measure associated with an exchangeable process.

There exist generalizations of de Finetti's theorem: for instance, Hewitt and Savage ([HS55]) showed that the theorem still holds if the random variables in the exchangeable sequence take their values in a compact space. There are also versions of the theorem that rely on a weaker type of exchangeability, and characterize mixture of Markov processes, rather than mixture of i.i.d. sequences. This allows the representation of self-interacting processes as random walks in random environment, as we will see in the following section. Before this, let us go back to Pólya's urn.

Consequences for Pólya's urn

Recall that $(X_n)_{n \in \mathbb{N}^*}$ denotes the canonical process in $\{0, 1\}^{\mathbb{N}^*}$, and that $\mathbb{P}_{a,b}$ is the distribution of the sequence draws in Pólya's urn. We already know that under $\mathbb{P}_{a,b}$, the sequence $(X_n)_{n \in \mathbb{N}^*}$ is exchangeable. Therefore, according to de Finetti's theorem, there exists a random variable θ such that under $\mathbb{P}_{a,b}$, conditionally on θ , $(X_n)_{n \in \mathbb{N}^*}$ is a sequence of i.i.d. Bernoulli variables with parameter θ . Note that the proportion of black balls in the urn converges to θ .

The distribution of θ under $\mathbb{P}_{a,b}$, *i.e.* the mixing measure $\mu_{a,b}$, can be determined by computing its moments, since for $n \geq 1$,

$$\begin{aligned} \int_{[0,1]} p^n \mu_{a,b}(dp) &= \mathbb{P}_{a,b}[\forall i \in \llbracket 1, n \rrbracket, X_i = 1] = \prod_{i=1}^n \frac{a+i-1}{a+b+i-1} \\ &= \frac{\Gamma(a+b-1)\Gamma(a+n)}{\Gamma(a+b+n)\Gamma(a)}. \end{aligned}$$

This is the n th moment of the Beta distribution with parameters (a, b) , which is the distribution on \mathbb{R} with density

$$p \mapsto \frac{\mathbb{1}_{[0,1]}(p)}{B(a, b)} p^{a-1} (1-p)^{b-1} dp,$$

where

$$B(a, b) = \int_{[0,1]} p^{a-1} (1-p)^{b-1} dp = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Therefore, $\mu_{a,b}$ is the Beta(a, b) distribution.

Remark 1.1.3. Using the language of Bayesian statistics, we can consider $\mu_{a,b}$ as the prior distribution for the random parameter of the i.i.d. Bernoulli variables. For $n \geq 1$ and a sample $(x_1, \dots, x_n) \in \{0, 1\}^n$, we obtain the *a posteriori* distribution of the random parameter by conditioning on the event $\{X_1 = x_1, \dots, X_n = x_n\}$: let us define $s_n = \sum_{i=1}^n x_i$, then for all $A \in \sigma((X_k)_{k \geq n+1})$,

$$\begin{aligned} \mathbb{P}_{a,b}[A | X_1 = x_1, \dots, X_n = x_n] &= \frac{\int P_p^{\otimes N}[A] p^{s_n} (1-p)^{n-s_n} \mu_{a,b}(dp)}{\mathbb{P}_{a,b}[X_1 = x_1, \dots, X_n = x_n]} \\ &= \frac{\prod_{i=1}^n (a+b+i-1)}{\prod_{j=1}^{s_n} (a+j-1) \prod_{k=1}^{n-s_n} (b+k-1)} \int P_p^{\otimes N}[A] \frac{\mathbb{1}_{[0,1]}(p)}{B(a, b)} p^{a+s_n-1} (1-p)^{b+n-s_n-1} dp \\ &= \int P_p^{\otimes N}[A] \mu_{a+s_n, b+n-s_n}(dp), \end{aligned}$$

where the *a posteriori* distribution $\mu_{a+s_n, b+n-s_n}$ of the random parameter is the Beta distribution with parameters $(a+s_n, b+n-s_n)$. Note that this is the mixing measure of a Pólya urn starting with $a+s_n$ black balls and $b+n-s_n$ white balls, which is indeed the state of the current urn after these first n steps.

A more general model for Pólya's urn consists in having $m > 2$ colors for the balls. The exchangeability property is still verified, and the process of successive draws is a mixture of i.i.d. variables, following some random probability distribution on $[[1, m]]$. This random distribution can be considered as an element of the simplex $\mathcal{S}_m = \{(p_1, \dots, p_m) \in \mathbb{R}_+^m, \sum_{k=1}^m p_k = 1\}$, and follows a Dirichlet distribution with parameters (a_1, \dots, a_m) , where a_1, \dots, a_m are the initial amounts of balls of each color. The Dirichlet distribution on \mathcal{S}_m is a multivariate generalization of the Beta distribution, in fact Beta(a, b) is the first marginal of the Dirichlet distribution on \mathcal{S}_2 with parameters (a, b) .

Let us summarize this section: the process of draws in Pólya's urn is self-interacting, but since it is exchangeable, we can represent it as a mixture of i.i.d. sequences. The associated random parameter can be recovered (as an asymptotic object) from each realization of the experiment, thanks to Remark 1.1.2. We will encounter this type of representation for different self-interacting processes with reinforcement, which still verify some kind of exchangeability property. Thanks to de Finetti-type theorems, we are able to express them as a mixture of simpler processes, and are interested in the associated mixing measure.

1.1.2 De Finetti's theorem for Markov chains, and random walks in random environment

As seen in the previous section, de Finetti's theorem characterizes mixtures of i.i.d. sequences with the exchangeability property. What about mixtures of other types of processes? Another version of de Finetti's theorem, proved by Diaconis and Freedman ([DF80]), gives a characterization of mixtures of Markov chains in the case they are recurrent, involving a weaker exchangeability condition.

On a countable state space E , we say a process Y is partially exchangeable if the probability of Y taking a certain path depends only on its starting point, and on the number of transitions from i to j , for all $i, j \in E$. Then Diaconis and Freedman proved that if a process Y is recurrent and partially

exchangeable, then it is a mixture of recurrent Markov chains on E . In other words, there exists a distribution μ on the set of $|E| \times |E|$ transition matrices, such that the distribution of Y is the same as taking a random transition matrix according to μ , and then running a Markov chain on E using this (now fixed) transition matrix.

We will be interested in a slightly different version of this theorem, due to Rolles ([Rol03]), which characterizes mixtures of reversible Markov chains on finite state spaces. Let us present this theorem in the framework of random walks on graphs.

Random walks on weighted graphs

We begin with a few definitions and notations. Let $\mathcal{G} = (V, E)$ be a finite connected non-directed graph. For $i, j \in V$, we write $i \sim j$ if i and j are neighbors, *i.e.* if $\{i, j\} \in E$. We will only be interested in nearest-neighbor processes on \mathcal{G} , *i.e.* random sequences $(Y_n)_{n \in \mathbb{N}} \in V^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $\mathbb{P}[Y_n \sim Y_{n+1}] = 1$. We denote by $(X_n)_{n \geq 0}$ the canonical process in $V^{\mathbb{N}}$.

Let us endow the edges of \mathcal{G} with positive weights $(a_e)_{e \in E}$. For all $i, j \in V$, we denote $a_{i,j} = a_{\{i,j\}}$ if $\{i, j\} \in E$, and $a_{i,j} = 0$ otherwise. Let now $\mathbb{P}_{i_0}^{SRW(a)}$ be the distribution of the simple random walk on (\mathcal{G}, a) started at i_0 , defined as follows: $\mathbb{P}_{i_0}^{SRW(a)}[X_0 = i_0] = 1$, and for all $n \in \mathbb{N}$ and $i \in V$,

$$\mathbb{P}_{i_0}^{SRW(a)}[X_{n+1} = i | X_0, \dots, X_n] = \frac{a_{X_n, i}}{\sum_{j \sim X_n} a_{X_n, j}}.$$

In other words, the simple random walk on (\mathcal{G}, a) is a reversible Markov chain, in which the transition probability from any vertex to one of its neighbors is proportional to the weight of the corresponding edge.

The weights $(a_e)_{e \in E}$ are also referred to as "conductances". This comes from a formalism which consists in identifying weighted graphs with electrical networks, and formulating probabilistic problems regarding reversible random walks, using the language of currents, potentials, and energy. See [LP17] for more details on this formalism. Note that the transition matrix for the simple random walk on (\mathcal{G}, a) does not change if we multiply all a_e by a constant $\lambda > 0$. Therefore, the weights will usually be normalized by imposing $a_{e_0} = 1$ for some fixed edge e_0 .

Partial exchangeability and de Finetti's theorem for reversible Markov chains

Let us now present the de Finetti-type theorem which characterizes mixtures of recurrent reversible random walks, using a weaker version of exchangeability that is described below.

For $L \in \mathbb{N}$ and $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_L) \in V^{L+1}$, we will say that σ is a path in \mathcal{G} if for all $0 \leq k \leq L-1$, the successive vertices σ_k and σ_{k+1} are neighbors in \mathcal{G} . We denote by $|\sigma| = L$ the length of σ . Finally, for any path σ and for all $e \in E$, we define the number $N_e^c(\sigma)$ of non-directed crossings of e by σ :

$$N_e^c(\sigma) = \sum_{k=0}^{|\sigma|-1} \mathbb{1}_{\{\sigma_k, \sigma_{k+1}\} = e}.$$

Let us then introduce the following equivalence between paths: if σ and τ are two paths in \mathcal{G} , we have $\sigma \sim_{pe} \tau$ if and only if $\sigma_0 = \tau_0$ and for all $e \in E$, $N_e^c(\sigma) = N_e^c(\tau)$. Note that if $\sigma \sim_{pe} \tau$, we have necessarily $|\sigma| = |\tau| = L$, and $\sigma_L = \tau_L$. We can now define a variant of the notion of partial exchangeability.

Definition 1.1.2. *Let $(Y_n)_{n \in \mathbb{N}}$ be a nearest-neighbor random process on \mathcal{G} . Then Y is partially exchangeable in a reversible sense if, for all paths σ, τ in \mathcal{G} such that $\sigma \sim_{pe} \tau$, we have*

$$\mathbb{P}[Y_0 = \sigma_0, \dots, Y_{|\sigma|} = \sigma_{|\sigma|}] = \mathbb{P}[Y_0 = \tau_0, \dots, Y_{|\tau|} = \tau_{|\tau|}].$$

Once again, it is easy to see that mixtures of reversible random walks are themselves partially exchangeable in a reversible sense. The following theorem, proved by Rolles in [Rol03], states that the converse is true, providing that the process is recurrent. It is a variant of Diaconis and Freedman's theorem on mixture of Markov chains ([DF80]).

Let us first fix $e_0 \in E$ an arbitrary edge, and denote by $\mathcal{A}_E^{e_0}$ the set of edge weights on \mathcal{G} , normalized at e_0 : $\mathcal{A}_E^{e_0} = \{(a_e)_{e \in E} \in (\mathbb{R}_+^*)^E, a_{e_0} = 1\}$.

Theorem 1.1.2 (de Finetti's theorem for reversible random walks). *Let $(Y_n)_{n \in \mathbb{N}}$ be a nearest-neighbor process on a finite graph $\mathcal{G} = (V, E)$. We assume that $(Y_n)_{n \in \mathbb{N}}$ is partially exchangeable in a reversible sense, as well as recurrent, in the following sense: there are a.s. infinitely many $n \in \mathbb{N}$ such that $Y_n = Y_0$.*

Then there exists a probability distribution μ on $V \times \mathcal{A}_E^{e_0}$ such that for any path $\sigma = (\sigma_0, \dots, \sigma_L)$ in \mathcal{G} ,

$$\begin{aligned} \mathbb{P}[Y_0 = \sigma_0, \dots, Y_L = \sigma_L] &= \int \left(\mathbb{1}_{i=\sigma_0} \prod_{k=0}^{L-1} \frac{\alpha_{\sigma_k, \sigma_{k+1}}}{\sum_{j \sim \sigma_k} \alpha_{\sigma_k, j}} \right) \mu(di, d\alpha) \\ &= \int \mathbb{P}_i^{SRW(\alpha)}[X_0 = \sigma_0, \dots, X_L = \sigma_L] \mu(di, d\alpha). \end{aligned}$$

In other words, on a finite graph, a recurrent process which is partially exchangeable in a reversible sense is a mixture of reversible random walks.

Mixtures of random walks are also called random walks in random environment: their distribution corresponds to drawing a random transition matrix according to the mixing measure, then "freezing" this matrix, and running the corresponding random walk on the graph. The term "environment" refers to the transition matrix, which determines the law of the Markov chain. In the case of Theorem 1.1.2, we get a mixture of reversible random walks, therefore the environment can be expressed as weights on all non-directed edge, rather than with a transition matrix.

Remark 1.1.4. In the context of the theorem, the "annealed" process, *i.e.* the partially exchangeable process expressed as a mixture of random walks, is recurrent. Therefore, the "quenched" process, *i.e.* the random walk associated with a certain realization of the random environment, is μ -a.s. recurrent. As a result, the annealed process almost surely visits every vertex infinitely many times. Thanks to the law of large numbers, for all vertices $i \in V$, the proportion of times the process jumps from i to a neighbor j then converges to a probability transition $p_{i,j}$, which reveals the realization of the environment that this trajectory evolves in. Therefore, the mixing measure μ is uniquely determined by the distribution of the recurrent annealed process. We will later be interested in mixtures of transient Markov processes, and whether they can admit several distinct mixing measures.

1.1.3 The Edge-Reinforced Random Walk

Let us now introduce another model using reinforcement, the Edge-Reinforced Random Walk (ERRW). This self-interacting random walk was introduced by Diaconis in 1986, and is a natural counterpart to Pólya's urn in the context of random walks on graphs.

Presentation of the model

Let $\mathcal{G} = (V, E)$ be a connected non-directed graph. The vertex set V is not necessarily finite, but we assume that \mathcal{G} is locally finite, *i.e.* that each vertex has only a finite number of neighbors. If we let $(a_e)_{e \in E}$ be a family of weights on the edges of the graph, then we can still define the simple random walk on (\mathcal{G}, a) as in the previous section.

The ERRW on (\mathcal{G}, a) is constructed by adding reinforcing behavior to the simple random walk: we consider the family $(a_e)_{e \in E}$ as initial weights, and for every edge e , the weight of e is increased by 1 each time the random walk crosses e . More precisely, for $e \in E$ and $n \in \mathbb{N}$, we write

$$A_n(e) = a_e + \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k, X_{k+1}\}=e}.$$

Then for $i_0 \in V$, the ERRW on (\mathcal{G}, a) started at i_0 is the process with distribution $\mathbb{P}^{ERRW(a, i_0)}$, where: $\mathbb{P}^{ERRW(a, i_0)}[X_0 = i_0] = 1$ and for all $n \in \mathbb{N}$ and $i \in V$,

$$\mathbb{P}^{ERRW(a, i_0)}[X_{n+1} = i | X_0, \dots, X_n] = \frac{A_n(\{X_n, i\})}{\sum_{j \sim X_n} A_n(\{X_n, j\})}.$$

As described in Remark 1.1.1 for Pólya's urn, for low values of $(a_e)_{e \in E}$ we have "strong reinforcement", and for high values we have "weak reinforcement". Moreover, the ERRW also satisfies an exchangeability property: it is easy to see that it is partially exchangeable in the sense of Definition 1.1.2. On finite graphs, it can therefore be represented as a random walk in random environment.

The ERRW as a mixture of random walks on finite graphs

We now assume that \mathcal{G} is finite, and fix $(a_e)_{e \in E} \in (\mathbb{R}_+^*)^E$ and $i_0 \in V$. Using the Borel-Cantelli lemma, we can show that the ERRW on (\mathcal{G}, a) started at i_0 almost surely goes back an infinite amount of times to i_0 . Therefore from Theorem 1.1.2, it is a mixture of reversible random walks. The random environment is defined by a distribution \mathcal{M}^{a, i_0} on the set of conductances $\mathcal{A}_E^{e_0}$, where e_0 is a fixed edge. In other words, for any $(X_n)_{n \in \mathbb{N}}$ -measurable event A ,

$$\mathbb{P}^{ERRW(a, i_0)}[A] = \int \mathbb{P}_{i_0}^{SRW(\alpha)}[A] \mathcal{M}^{a, i_0}(d\alpha).$$

The expression of \mathcal{M}^{a, i_0} was computed by Diaconis and Coppersmith in [CD87], and is the object of the following theorem. For edge weights $(y_e)_{e \in E} \in (\mathbb{R}_+^*)^E$, we denote $y_i = \sum_{j \sim i} y_{i, j}$.

Theorem 1.1.3. *Let $\mathcal{G} = (V, E)$ be a finite connected graph, let $i_0 \in V$ and $e_0 \in E$ be fixed. Moreover, let $(a_e)_{e \in E} \in (\mathbb{R}_+^*)^E$ be initial weights on the graph. Then the mixing measure \mathcal{M}^{a, i_0} of the ERRW on (\mathcal{G}, a) started at i_0 has the following density:*

$$\mathcal{M}^{a, i_0}(d\alpha) = C(a, i_0) \frac{\sqrt{\alpha_{i_0}} \prod_{e \in E} \alpha_e^{a_e}}{\prod_{i \in V} \alpha_i^{\frac{1}{2}(a_i + 1)}} \sqrt{D(\alpha)} \prod_{e \neq e_0} \frac{d\alpha_e}{\alpha_e},$$

where

$$D(\alpha) = \sum_{T \in \mathcal{T}_G} \prod_{e \in T} \alpha_e,$$

with \mathcal{T}_G being the set of spanning trees of \mathcal{G} , and where

$$C(a, i_0) = \frac{2^{1-|V| + \sum_{e \in E} a_e}}{\sqrt{\pi}^{|V|-1}} \frac{\prod_{i \in V} \Gamma(\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0}))}{\prod_{e \in E} \Gamma(a_e)}.$$

At this stage, the fact that this is a probability density is a consequence of the proof that it is the mixing measure of the ERRW. We will later mention another way of showing that this indeed defines a probability distribution, in Remark 1.2.3.

The ERRW on infinite graphs

Let now \mathcal{G} be an infinite, locally finite graph. In this case, it is not as simple to show that the ERRW is a mixture of random walks. Even with Diaconis and Freedman's theorem for mixtures of Markov chains, which applies to processes on countable state spaces, the recurrence hypothesis is not necessarily verified by the ERRW. However, even without recurrence, we will see that the result is still valid.

A first special case is when \mathcal{G} is an infinite tree, *i.e.* a connected graph with no cycles. In [Pem88], Pemantle showed that thanks to the particular structure of trees, the reinforcement behaves in a very simple way. Indeed, for $i \in V$, let $n_1 \in \mathbb{N}$ be such that $X_{n_1} = i$, and let $n_2 = \inf\{n > n_1, X_n = i\}$ be the next instant that X comes back to i . Then the only edge weight adjacent to i that has changed between n_1 and n_2 is the weight of $\{i, X_{n_1+1}\}$, which has been increased by 2: once when leaving i , and once when coming back to it, since the process can not come back through any other edge. As a result, the edge weights incident to i evolve as Pólya's urn (with two balls added at each step instead of one), and are independent of what happens at other vertices. Therefore, the ERRW on the infinite tree is a mixture of random walks in random Dirichlet environments, meaning that the mixing measure is given by independent Dirichlet distributions at each vertex. One consequence is the following phase transition, shown by Pemantle:

Theorem 1.1.4. *Let \mathcal{G} be a d -ary tree with $d \geq 2$, endowed with constant initial weights, all equal to $a > 0$. Then there exists $a_0(d) > 0$ such that:*

- for $a < a_0(d)$, the ERRW on (\mathcal{G}, a) is a mixture of positive recurrent Markov chains.
- for $a > a_0(d)$, the ERRW on (\mathcal{G}, a) is a mixture of transient Markov chains.

In the more general case, Merkl and Rolles showed in [MR07] that the ERRW is in fact a mixture of reversible random walks on any infinite locally finite graph \mathcal{G} . They first proved some bounds for the mixing measure of ERRW on finite graphs. Introducing an exhausting sequence $(\mathcal{G}_n)_{n \in \mathbb{N}}$ of finite subgraphs of \mathcal{G} , they then applied the previous bounds to \mathcal{G}_n . Thanks to tightness and compactness arguments, taking the limit yields the existence of a mixing measure for the ERRW on the whole of \mathcal{G} . This also shows that the aforementioned bounds are still valid on \mathcal{G} .

In [MR⁺09], Merkl and Rolles also studied the ERRW on diluted versions of \mathbb{Z}^2 , meaning that each edge of the lattice \mathbb{Z}^2 is replaced by a path of r successive edges. For r large enough, and for strong enough reinforcement (*i.e.* for small initial weights), they showed a stronger bound on the random edge weights given by the mixing measure, which implies the recurrence of ERRW on the diluted version of \mathbb{Z}^2 .

Further results of recurrence and transience on the \mathbb{Z}^d lattice were later proved by Sabot and Tarrès in [ST15], as well as Angel, Crawford and Kozma in [ACK14], through links with some statistical physics models. In [ST15], this was realized using the relation between the ERRW and another self-reinforcing process on graphs: the Vertex-Reinforced Jump Process, which will be the focus of the remainder of this manuscript.

1.2 The Vertex-Reinforced Jump Process

The Vertex-Reinforced Jump Process (VRJP) was introduced by Werner. It is defined as a jump process on a graph, where the jumping rate to some vertex increases with the local time of the process at this vertex. The VRJP was first studied by Davis and Volkov on \mathbb{Z} in [DV02], who showed its recurrence. Later phase transitions between recurrence and transience were shown on regular trees by Davis and Volkov ([DV04]) and Collecchio ([Col09]), and on Galton-Watson trees by Basdevant and Singh ([BS10]). Similarly to the ERRW, the particular structure of trees makes the reinforcement behavior easier to study.

The VRJP satisfies a form of exchangeability condition, but only after applying a certain time change. In this exchangeable time scale, it can be written as a mixture of Markov jump processes. This section presents the way this representation was obtained on finite graphs by Sabot and Tarrès ([ST15]), and later extended to infinite graphs by the same authors and Zeng ([STZ17], [SZ19]). The VRJP is also closely related to the ERRW, as well as a statistical mechanics model, which has yielded many consequences for the study of both processes.

1.2.1 The VRJP in different time scales

With the same notations as before, let $\mathcal{G} = (V, E)$ be a locally finite graph with no loops, *i.e.* such that there is no edge from a vertex $i \in V$ to itself. The VRJP on \mathcal{G} will be defined as a continuous-time jump process taking values on V , such that the attractiveness of any vertex increases as the process spends time at this vertex. To define this more precisely, let us give some definitions and notations.

Jump processes on graphs

Let $(X_t)_{t \geq 0}$ be the canonical process on the space $\mathcal{C}_r(\mathbb{R}_+, V)$ of right-continuous functions $\mathbb{R}_+ \rightarrow V$. We denote by $(\mathcal{F}_t^X)_{t \geq 0}$ the filtration of the past of X , *i.e.* for all $t \geq 0$, $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$.

Let $r = (r_{i,j})_{i \sim j}$ be a family of positive weights on directed edges of \mathcal{G} , *i.e.* we allow $r_{i,j} \neq r_{j,i}$ for $\{i, j\} \in E$. For $i \not\sim j$, we also define $r_{i,j} = r_{j,i} = 0$. Let us fix a starting point $i_0 \in V$. Then the Markov jump process on \mathcal{G} with jump rates r , started at i_0 , is a right-continuous V -valued random process with distribution $\mathbb{P}_{i_0}^{MJP(r)}$, where: $\mathbb{P}_{i_0}^{MJP(r)}[X_0 = i_0] = 1$, and for $t \geq 0$ and $j \in V$,

$$\mathbb{P}_{i_0}^{MJP(r)}[X_{t+dt} = j | \mathcal{F}_t^X] = r_{X_t, j} dt + o(dt). \quad (1.1)$$

A more rigorous way to define this jump process is through its infinitesimal generator \mathcal{L}^r : if $g : V \times \mathbb{R}$ is bounded, for $i \in V$ we have

$$\mathcal{L}^r f(i) = \sum_{j \sim i} r_{i,j} (f(j) - f(i)).$$

Remark 1.2.1. Since the jump rates are constant, this distribution can also be described in the following way. While the process is at some vertex i , independent clocks are running for each neighbor of i . The clock associated with j rings at a random instant, which is an exponential variable with parameter $r_{i,j}$. As soon as the first of these clocks ring, the process jumps to the associated neighboring vertex. As a result, the "holding time" that the process spends at i is an exponential variable with parameter $r_i = \sum_{j \sim i} r_{i,j}$, and the probability that it jumps to j is $\frac{r_{i,j}}{r_i}$. The choice of which neighbor to jump to is independent from the waiting time at i , and all jumps occur independently from the past of the process, so that it is in fact a Markov process.

We will also be interested in self-interacting jump processes, defined by jump rates which evolve with time and depend on the trajectory of X . Such a process is no longer markovian, but we still define it through the expression of its jump rates, as in (1.1). Moreover, our focus will be on the VRJP and time-changed versions of the VRJP, in which the jump rates depend only on the local times $(l_i^X(t))_{i \in V, t \geq 0}$, *i.e.* the total time that X has spent at each vertex i up to the instant t :

$$l_i^X(t) = \int_0^t \mathbf{1}_{X_s = i} ds.$$

Let then $(r_{i,j})_{i \sim j}$ be a family of functions $r_{i,j} : (\mathbb{R}_+)^V \rightarrow \mathbb{R}_+^*$. The jump process which, at time t , jumps from i to j with rate $r_{i,j}(l^X(t))$, is defined in such a way that its distribution $\mathbb{P}^{JP(r)}$ satisfies

$$\mathbb{P}^{JP(r)}[X_{t+dt} = j | \mathcal{F}_t^X] = r_{X_t, j}(l^X(t)) dt + o(dt).$$

This expression of $\mathbb{P}^{JP(r)}$ is convenient for computations, in particular to identify the distribution of time-changed versions of X . However in this case, the jump rates are not constant, so the process can not be rigorously represented using exponential variables. To define it more precisely, we use the fact that under $\mathbb{P}^{JP(r)}$, $(X_t, (l_j^X(t))_{j \in V})_{t \geq 0}$ is a Markov process on $V \times \mathbb{R}_+^V$. Its infinitesimal generator \mathcal{L}^r is then given by

$$\mathcal{L}^r f(i, l) = \sum_{j \sim i} r_{i,j}(l) (f(j, l) - f(i, l)) + \partial_{l_i} f(i, l),$$

for $i \in V$ and $l = (l_j)_{j \in V} \in \mathbb{R}_+^V$, and for all bounded functions $f : V \times \mathbb{R}_+^V \rightarrow \mathbb{R}$ such that $f(i, \cdot)$ is \mathcal{C}^∞ for $i \in V$.

Finally, for such jump processes, it will also be convenient to introduce a discrete-time version, *i.e.* a V -valued random process $(\tilde{X}_n)_{n \in \mathbb{N}}$, registering only the successive vertices visited by X , and not the time spent at each vertex.

With these definitions out of the way, we can now introduce the model which is our main subject of interest.

The VRJP model and two relevant time changes

Let $(W_e)_{e \in E} \in (\mathbb{R}_+^*)^E$ be a family of positive weights on the edges of the graph \mathcal{G} , and let $i_0 \in V$ be a starting vertex. We define the distribution $\mathbb{P}^{VRJP(W, i_0)}$ of the VRJP on (\mathcal{G}, W) started at i_0 , such that: under $\mathbb{P}^{VRJP(W, i_0)}$, we have $X_0 = i_0$ almost surely, and for $i, j \in V$ and $t \geq 0$, X_t jumps from i to j at rate $W_{i,j}(1 + l_j^X(t))$.

Remark 1.2.2. As with Pólya's urn and the ERRW, depending on the starting parameters, the reinforcing behavior can be more or less pronounced (see Remark 1.1.1). For lower initial weights W , the first few jumps happen after a longer time, therefore the process stays longer at each vertex, resulting in stronger reinforcement. Conversely, for higher W , we have weaker reinforcement.

Our aim is once again to study the representations of this self-interacting process as a mixture of Markov processes. However, since the jump rates of the VRJP are always increasing, the process tends to jump faster and faster as time goes on. This makes it impossible for it to be a mixture of Markov jump processes, since the holding times would then be stationary. We will therefore need to apply a time change to the process, in order to slow it down. Another time change will also be useful to relate the VRJP to the ERRW.

Let us describe how these changes of variables are defined: for a continuous and increasing random function $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, adapted to the filtration $(\mathcal{F}_t^X)_{t \geq 0}$, we define the time-changed process \tilde{X}^A by $\tilde{X}_t^A = X_{A^{-1}(t)}$ for $t \geq 0$. We want for the local times of \tilde{X}^A to satisfy

$$1 + l_i^X(t) = f(l_i^{\tilde{X}^A}(A(t))) \tag{1.2}$$

for $i \in V$ and $t \geq 0$, where $f : [0, \infty) \rightarrow [1, \infty)$ is increasing and differentiable. This implies that at time $u \geq 0$, \tilde{X}^A jumps from i to j at rate

$$W_{i,j} f(l_j^{\tilde{X}^A}(u)) f'(l_i^{\tilde{X}^A}(u)). \tag{1.3}$$

Indeed, for $u \geq 0$ and $j \in V$

$$\begin{aligned} \mathbb{P}^{VRJP(W, i_0)} [\tilde{X}_{u+du}^A = j | \mathcal{F}_u^{\tilde{X}^A}] &= \mathbb{P}^{VRJP(W, i_0)} [X_{A^{-1}(u+du)} = j | \mathcal{F}_{A^{-1}(u)}^X] \\ &= W_{X_{A^{-1}(u)}, j} (1 + l_j^X(A^{-1}(u))) (A^{-1})'(u) du + o(du) \\ &= W_{\tilde{X}_u^A, j} f(l_j^{\tilde{X}^A}(u)) \frac{1}{A'(A^{-1}(u))} du + o(du), \end{aligned}$$

where we also get $\frac{1}{A'(A^{-1}(u))} = f'(l_{\tilde{X}_u^A}(u))$ by differentiating (1.2) and summing over all $i \in V$. This proves (1.3), moreover the time change is given by:

$$A(t) = \int_0^t (f^{-1})'(1 + l_{X_s^X}(s)) ds = \sum_{i \in V} f^{-1}(1 + l_i^X(t)).$$

We will be interested in two different choices of $f : [0, \infty) \rightarrow [1, \infty)$ in order to obtain satisfying jump rates in (1.3). These time changes were introduced by Sabot and Tarrès in [ST15]. The first one is $f_1 : x \mapsto \sqrt{1+x}$, which gives the time change

$$A_1(t) = \sum_{i \in V} (1 + l_i^X(t))^2 - 1,$$

and under $\mathbb{P}^{VRJP(W, i_0)}$, the time-changed process $Z^{(1)} = \tilde{X}^{A_1}$ jumps from i to j at time $t \geq 0$ with rate

$$\frac{W_{i,j} \sqrt{1 + l_j^{Z^{(1)}}(t)}}{2 \sqrt{1 + l_i^{Z^{(1)}}(t)}}.$$

In the next paragraph, we will see that this time scale is such that $Z^{(1)}$ satisfies an exchangeability condition.

The other relevant time change corresponds to $f_2 : x \mapsto e^x$, *i.e.*

$$A_2(t) = \sum_{i \in V} \log(1 + l_i^X(t)).$$

The associated time-changed process $Z^{(2)} = \tilde{X}^{A_2}$ has the following jump rates under $\mathbb{P}^{VRJP(W, i_0)}$:

$$W_{i,j} e^{l_i^{Z^{(2)}}(t) + l_j^{Z^{(2)}}(t)}.$$

This process will help relate the VRJP to the ERRW, and also appears in the proof of the representation of $Z^{(1)}$ as a mixture of Markov processes (Theorem 1.2.2).

Partial exchangeability in continuous time

In [Fre96], Freedman showed a continuous-time version of de Finetti's theorem for Markov processes. Similarly to Diaconis and Freedman's theorem for discrete-time Markov chains, it relies on a partial exchangeability condition, defined as follows. If $(Y_t)_{t \geq 0}$ is a random process on a countable state space E , it is partially exchangeable if for all $h > 0$, the discrete-time process $(Y_{nh})_{n \in \mathbb{N}}$ is partially exchangeable in the sense of Diaconis and Freedman (rather than the sense of Definition 1.1.2). In other words, the probability that $(Y_{nh})_{n \in \mathbb{N}}$ follows some given path depends only on the number of times this path goes from i to j , for all $i, j \in E$. Then Freedman's theorem is the following.

Theorem 1.2.1. *Let $(Y_t)_{t \geq 0}$ be a random process on a countable state space E , started at some point $i_0 \in E$. We assume that Y is partially exchangeable, as well as recurrent, *i.e.* there are almost surely infinitely many integers k such that $Y_k = i_0$. Moreover, we assume that Y has no point of discontinuity, *i.e.* for all $t \geq 0$, if $t_n \rightarrow t$ then $Y_{t_n} \xrightarrow{a.s.} Y_t$.*

Then X is a mixture of Markov jump processes.

Another formulation of partial exchangeability was introduced by Zeng in [Zen16], for the type of processes that we are interested in, *i.e.* jump processes on graphs with jump rates depending only on local times. Such a process admits a density function, defined on the set of trajectories. Then the process is partially exchangeable if and only if the density is the same for trajectories with equal transition counts and local times. Let us describe this more precisely.

We first introduce a notation for trajectories on a graph $\mathcal{G} = (V, E)$: for $t \geq 0$, we define a trajectory up to time t by its number of jumps, jump times and visited vertices. Therefore, for $n \in \mathbb{N}$, for $0 < s_1 < \dots < s_n < t$ and for $i_0, \dots, i_n \in V$, we denote by $\sigma_{t,n}^{(i,s)}$ the function $[0, t] \rightarrow V$ such that for all $k \in \llbracket 1, n \rrbracket$, for $u \in [s_k, s_{k+1})$, $\sigma_{t,n}^{(i,s)}(u) = i_k$ (with $s_0 = 0$ and $s_{n+1} = t$), and $\sigma_{t,n}^{(i,s)}(t) = i_n$. Moreover, for $\sigma = \sigma_{t,n}^{(i,s)}$, we define:

$$\begin{cases} \forall i \in V, l_i^\sigma(t) = \int_0^t \mathbb{1}_{\sigma=u} du, \\ \forall i, j \in V, N_{i,j}(\sigma) = \sum_{k=0}^{n-1} \mathbb{1}_{(Y_{s_k}, Y_{s_{k+1}}) = (i,j)}. \end{cases}$$

We also denote by \mathcal{P}_t the set of trajectories up to time t :

$$\mathcal{P}_t = \left\{ \sigma_{t,n}^{(i,s)}, n \in \mathbb{N}, 0 < s_1 < \dots < s_n < t \text{ and } i_1, \dots, i_n \in V \right\},$$

as well as $\mathcal{P} = \bigcup_{t \geq 0} \mathcal{P}_t$ the set of all trajectories.

The following propositions, showed in [Zen16], give the existence of a density of trajectories for some jump processes on graphs, and a characterization of partial exchangeability through this density.

Proposition 1.2.1. *Let Y be a jump process on $\mathcal{G} = (V, E)$ with jump rates depending only on local times, *i.e.* for $i \sim j$ there are functions $r_{i,j} : \mathbb{R}_+^V \rightarrow \mathbb{R}_+^*$ such that Y jumps at time t from i to j with rate $r_{i,j}(l^Y(t))$.*

Then the distribution of Y admits a density function F defined on trajectories \mathcal{P} , such that for all $t \geq 0$ and for all bounded measurable functions $\Phi : \mathcal{P}_t \rightarrow \mathbb{R}$,

$$\mathbb{E}[\Phi(X_s, 0 \leq s \leq t)] = \sum_{n \geq 0} \sum_{i_0, \dots, i_n} \int_{0 < s_1 < s_2 < \dots < s_n < t} \Phi \left(\sigma_{t,n}^{(i,s)} \right) F \left(\sigma_{t,n}^{(i,s)} \right) \prod_{k=1}^n ds_k.$$

Moreover, for $\sigma = \sigma_{t,n}^{(i,s)}$ where $n \geq 0$, $i_0, \dots, i_n \in V$ and $0 < s_1 < \dots < s_n < t$, we have

$$F(\sigma) = \exp \left(- \int_0^t \sum_{j \sim \sigma(u)} r_{\sigma(u),j}(l^\sigma(u)) du \right) \prod_{k=1}^n r_{i_{k-1}, i_k}(l^\sigma(s_k)).$$

This is easily proved for jump processes with constant jump rates, or in the case where the jump rate $r_{i,j}(l^Y(t))$ does not depend on $l_i^Y(t)$. Indeed, in this case the holding time at a vertex i between two jumps is an exponential variable with parameter $\sum_{j \sim i} r_{i,j}(l^Y(t))$, which gives the exponential term in $F(\sigma)$. The more general case can be obtained through a time change.

We can now give an equivalent formulation of partial exchangeability using the density of the process. Let us first define an equivalence relation on \mathcal{P} . For $\sigma, \tau \in \mathcal{P}$, we write $\sigma \sim_{pe} \tau$ if and only if: there exists $t \geq 0$ such that $\sigma, \tau \in \mathcal{P}_t$, and for all $i, j \in V$, $l_i^\sigma(t) = l_i^\tau(t)$ and $N_{i,j}(\sigma) = N_{i,j}(\tau)$.

Proposition 1.2.2. *Let Y be as in Proposition 1.2.1 and let F be its density.*

Then Y is partially exchangeable if and only if: for all $\sigma, \tau \in \mathcal{P}$, if $\sigma \sim_{pe} \tau$, then $F(\sigma) = F(\tau)$.

Let us return to the time-changed VRJP on a graph $\mathcal{G} = (V, E)$. Fix edge weights $(W_e)_{e \in E}$ as well as a starting point $i_0 \in V$. Under $\mathbb{P}^{VRJP(W, i_0)}$, the process $Z^{(1)}$ has the following density F (see for instance [ST16]): for $n \in \mathbb{N}$, $0 < s_1 < \dots < s_n < t$ and $i_1, \dots, i_n \in V$, for $\sigma = \sigma_{t, n}^{(i, s)}$,

$$F(\sigma) = \exp\left(-\sum_{i \sim j} W_{i,j} \left(\sqrt{(1 + l_i^\sigma(t))(1 + l_j^\sigma(t))} - 1\right)\right) \frac{2^{-n} \prod_{k=0}^{n-1} W_{i_k, i_{k+1}}}{\prod_{i \in V \setminus \{i_n\}} \sqrt{1 + l_i^\sigma(t)}}.$$

Since this density depends only on $l_i^\sigma(t)$ and $N_{i,j}(\sigma)$ for $i, j \in V$, Proposition 1.2.2 yields the fact that under $\mathbb{P}^{VRJP(W, i_0)}$, $Z^{(1)}$ is partially exchangeable. On finite graphs, it is possible to show that the VRJP is recurrent, and then apply Theorem 1.2.1 to show that this time-changed version of the VRJP is a mixture of Markov processes.

However, the first proof of this representation in [ST15] did not use Theorem 1.2.1, but a more explicit approach. We will describe this method in the next section, but let us first present a link between the ERRW and the VRJP, through the time-changed process $Z^{(2)}$.

How the ERRW relates to the VRJP

In [ST15], Sabot and Tarrès showed that in some sense, the ERRW on a weighted graph is a mixture of VRJP with random initial weights. Let us describe how.

Fix a locally finite graph (\mathcal{G}, E) , endowed with initial edge weights $(a_e)_{e \in E}$. The first step is to introduce a continuous-time version Y of the ERRW on (\mathcal{G}, a) , with a construction due to Davis and Sellke. To do this, simply define a jump process with jump rate $a_{\{i,j\}} + N_{\{i,j\}}^c(t)$ from i to j at time t , where $N_{\{i,j\}}^c(t)$ is the number of times that Y has jumped from i to j or from j to i up to time t . The associated discrete-time process is then effectively the ERRW on (\mathcal{G}, a) .

Another way of defining this continuous-time ERRW is the following: to every edge $e \in E$, assign independent clocks that run only when the process is at one of the ends of the edge e . Each clock rings after an exponential time, with parameter $a_e + N_e^c(t)$, at which point Y jumps through e , and the clock restarts, with an updated parameter. Thanks to a result by Kendall ([Ken66]), each clock can be represented as a mixture of time-changed Poisson point processes with parameter 1, where the time change is characterized by a random parameter W_e distributed according to a Gamma($a_e, 1$) distribution. As a consequence, there exist independent Gamma variables $(W_e)_{e \in E}$, such that conditionally on $(W_e)_{e \in E}$, Y is a jump process with jumping rates $W_{i,j} e^{l_i^Y + l_j^Y}$.

In other words, Y has the same distribution as a mixture of time-changed process $Z^{(2)}$ under $\mathbb{P}^{VRJP(W, i_0)}$, where $(W_e)_{e \in E}$ are independent and W_e is a Gamma($a_e, 1$) variable for $e \in E$. By observing only the discrete-time process \hat{Y} , we get the ERRW, which is therefore itself a mixture of discrete-time versions of the VRJP with random initial weights $(W_e)_{e \in E}$, that are independent and Gamma-distributed.

1.2.2 The VRJP as a mixture of Markov jump processes on finite graphs

The representation of the VRJP (in the exchangeable time scale) as a mixture of Markov jump processes was shown by Sabot and Tarrès in [ST15]. Their proof does not use de Finetti's theorem for Markov jump processes, but a more explicit method: identifying a converging functional of the process, and showing that conditionally on its limit, $Z^{(1)}$ is indeed a Markov jump process. This functional, depending only on the local times of the process, has a distribution that appears in the study of the supersymmetric hyperbolic sigma model. This relation provided results for the recurrence and transience of the VRJP and the ERRW, thanks to localization and delocalization results due to Disertori, Spencer and Zirnbauer ([DSZ10],[DS10]).

The mixing measure of the VRJP on finite graphs

Let us first describe the aforementioned distribution, which will provide a mixing measure for the VRJP in Theorem 1.2.2(iii). Let (\mathcal{G}, W) be a finite weighted graph, and $i_0 \in V$ a fixed vertex. Define $\mathcal{H}_{i_0} = \{u \in \mathbb{R}^V, u_{i_0} = 0\}$. Then the following measure is a probability distribution on \mathcal{H}_{i_0} :

$$\mathcal{Q}_V^{W, i_0}(du) = \frac{e^{-\sum_{i \in V} u_i}}{\sqrt{2\pi}^{|V|-1}} \exp\left(-\sum_{\{i, j\} \in E} W_{i, j} (\cosh(u_i - u_j) - 1)\right) \sqrt{D(W, u)} \prod_{i \in V \setminus \{i_0\}} du_i$$

where $D(W, u) = \sum_{T \in \mathcal{T}_{\mathcal{G}}} \prod_{\{i, j\} \in T} W_{i, j} e^{u_i + u_j}$ (using the same notations as in Theorem 1.1.3). Thanks to the matrix-tree theorem, $D(W, u)$ is also equal to any diagonal minor of the $|V| \times |V|$ matrix $M(W, u)$, where

$$M(W, u)_{i, j} = \begin{cases} W_{i, j} e^{u_i + u_j} & \text{if } i \neq j \\ -\sum_{k \in V} W_{i, k} e^{u_i + u_k} & \text{if } i = j \end{cases}$$

The fact that \mathcal{Q}_V^{W, i_0} is a probability distribution is a significant result. It was obtained as a consequence of supersymmetry in [DSZ10]. It also results from the fact that \mathcal{Q}_V^{W, i_0} is the distribution of an asymptotic functional of the VRJP, as stated in the following theorem from [ST15].

Theorem 1.2.2. *Let $\mathcal{G} = (V, E)$ be a finite graph endowed with initial weights $(W_e)_{e \in E}$, and $i_0 \in V$ a starting point.*

(i) *For all $i \in V$, the following limit exists $\mathbb{P}^{VRJP(W, i_0)}$ -a.s.:*

$$U_i = \lim_{t \rightarrow \infty} \frac{1}{2} \log \left(\frac{l_i^{Z^{(1)}}(t)}{l_{i_0}^{Z^{(1)}}(t)} \right) = \lim_{t \rightarrow \infty} \left(l_i^{Z^{(2)}}(t) - l_{i_0}^{Z^{(2)}}(t) \right).$$

(ii) *Under $\mathbb{P}^{VRJP(W, i_0)}$, $(U_i)_{i \in V}$ is distributed according to \mathcal{Q}_V^{W, i_0} .*

(iii) *Under $\mathbb{P}^{VRJP(W, i_0)}$, conditionally on $(U_i)_{i \in V}$, the time-changed process $Z^{(1)}$ is a Markov jump process with jump rate $\frac{W_{i, j}}{2} e^{U_j - U_i}$ from i to j .*

The proof of (i) relies on martingale arguments, using the expression of the infinitesimal generator of the process $Z^{(2)}$. For (ii), the distribution of U is identified by showing that the Laplace transform of \mathcal{Q}_V^{W, i_0} is solution of a Feynman-Kac equation, involving the same generator.

Finally, (iii) is proved thanks to arguments inspired by Bayesian statistics. Let \tilde{Z} be a mixture of Markov jump processes with jump rates $\frac{W_{i, j}}{2} e^{U_j - U_i}$, where U is distributed according to $\mathcal{Q}_V^{W, i_0}(du)$. It is then possible to compute the distribution of U conditionally on the trajectory of \tilde{Z} up to a time $t \geq 0$. Indeed, this conditioning makes some extra terms appear, that can be interpreted as a modification of the parameters of \mathcal{Q}_V^{W, i_0} . Using the fact that it is indeed a probability distribution, adjusting the normalizing terms gives an explicit expression of the conditional distribution of U , and therefore the distribution of the future of \tilde{Z} conditionally on its past. It turns out that \tilde{Z} has the same jump rates as the time-changed process $Z^{(1)}$ under $\mathbb{P}^{VRJP(W, i_0)}$.

Remark 1.2.3. The representation of the VRJP with the mixing measure \mathcal{Q}_V^{W, i_0} and the link between ERRW and VRJP also provide another way of computing the mixing measure of the ERRW described in Theorem 1.1.3, and showing that it is in fact a probability measure.

The supersymmetric hyperbolic sigma model, and the VRJP and ERRW on \mathbb{Z}^d

The mixing measure \mathcal{Q}_V^{W,i_0} appears in the work of Disertori, Spencer and Zirnbauer ([DSZ10], [DS10]) concerning the supersymmetric hyperbolic sigma model. This is a spin model on a weighted graph (\mathcal{G}, W) where the spins take their values in $\mathbb{H}^{2|2}$, a supermanifold extension of the hyperbolic plane, with two added anticommuting variables. However, it can be reduced to a single random field on the graph, which has the distribution \mathcal{Q}_V^{W,i_0} . As a result, some estimates proved for the supersymmetric hyperbolic sigma model yield recurrence and transience results for the VRJP and the ERRW.

Consider the d -dimensional lattice \mathbb{Z}^d , and define $\Lambda_n^{(d)} = \mathbb{Z}^d \cap [-n, n]^d$ for all $n \geq 0$, as well as E_n the set of edges contained in $\Lambda_n^{(d)}$. We endow the edges of \mathbb{Z}^d , and therefore those of all $\Lambda_n^{(d)}$, with constant weights W .

In [DS10], Disertori and Spencer showed a localization result for the supersymmetric hyperbolic sigma model for low weights W , which assures the exponential decay of $e^{u_i/2}$ under $\mathcal{Q}_{\Lambda_n^{(d)}}^{W,0}(du)$, uniformly in n : there exist constants $\underline{W}(d) > 0$ and $C(d), k(W, d) > 0$, which are explicitly known, such that for all $0 < W < \underline{W}(d)$, $n \in \mathbb{N}$ and $i \in \Lambda_n^{(d)}$,

$$\int e^{u_i/2} \mathcal{Q}_{\Lambda_n^{(d)}}^{W,0}(du) \leq C(d) e^{-k(W,d)|i|},$$

where $|\cdot|$ is the Euclidean norm on \mathbb{R}^d . This exponential decay allowed Sabot and Tarrès to show in [ST15] that for all $d \geq 2$, the VRJP on (\mathbb{Z}^d, W) is recurrent for strong reinforcement, *i.e.* constant W low enough, in the following sense: in this context, the discrete-time version of the VRJP is a mixture of positive recurrent reversible random walks. They also generalized Disertori and Spencer's result to random i.i.d. weights $(W_e)_{e \in E}$ with distribution Gamma($a, 1$), which yields in the same way that the ERRW on \mathbb{Z}^d , with constant initial weights a small enough, is positive recurrent. This proof is still valid for any graph with bounded degree. This was also independently proved by Angel, Crawford and Kozma in [ACK14], who gave a direct proof of the exponential localization of the ERRW at strong reinforcement.

In the case where $d \geq 3$, Disertori, Spencer and Zirnbauer showed in [DSZ10] that the fluctuations of the potential with distribution $\mathcal{Q}_{\Lambda_n^{(d)}}^{W,0}$ are bounded uniformly in $n \in \mathbb{N}$, for W large enough. In other words: for any $m > 0$, there exists $\overline{W}(d, m) > 0$ such that for all $W > \overline{W}(d, m)$, for any $n \in \mathbb{N}$, $i, j \in \Lambda_n^{(d)}$,

$$\int \cosh^m(u_x - u_y) \mathcal{Q}_{\Lambda_n^{(d)}}^{W,0}(du) \leq 2.$$

Sabot and Tarrès used these estimates in [ST15] to show that for $d \geq 3$, the VRJP is transient on \mathbb{Z}^d for weak enough reinforcement. With Disertori, they also showed in [DST15] a version of this delocalization result for random i.i.d. weights with Gamma distribution, implying the transience of the ERRW on \mathbb{Z}^d at weak reinforcement.

More results for the VRJP and ERRW on \mathbb{Z}^d will be described later, and were obtained thanks to a representation of the VRJP on infinite graphs.

The random β potential

Theorem 1.2.2(iii) provides a representation of the VRJP as a mixture of Markov jump processes on finite graphs. In order to extend this result to infinite graphs, a natural approach is to proceed as with the ERRW: take an exhausting sequence of finite subgraphs (V_n, E_n) , and try to show that the environment provided by the mixing measure $\mathcal{Q}_{V_n}^{W,i_0}$ converges when $n \rightarrow \infty$. It is possible to obtain a weak convergence of the random field u , but the limit loses properties (like translation invariance or ergodicity) that are needed to obtain more results for the VRJP on infinite graphs. There exists however a random potential

on V , which can be used to define the right random environment, and has a more convenient distribution. Let us describe this potential.

Let $\mathcal{G} = (V, E)$ be a finite connected graph endowed with edge weights $(W_e)_{e \in E}$. As before, for $i, j \in V$, we use the notation $W_{i,j} = W_{\{i,j\}}$ if $\{i, j\} \in E$, and $W_{i,j} = 0$ otherwise. Note that we do not exclude the case where $W_{i,i} \neq 0$ for some vertex $i \in V$.

For $\beta = (\beta_i)_{i \in V} \in \mathbb{R}^V$, we denote by $H_\beta = 2\beta - W$ the $|V| \times |V|$ matrix such that for $i, j \in V$,

$$(H_\beta)_{i,j} = \begin{cases} -W_{i,j} & \text{if } i \neq j \\ 2\beta_i - W_{i,i} & \text{if } i = j. \end{cases}$$

We also consider $\langle \cdot, \cdot \rangle$ to be the canonical scalar product on \mathbb{R}^V , and $\mathbf{1} \in \mathbb{R}^V$ to be the vector such that $\mathbf{1}_i = 1$ for all $i \in V$. Moreover, we denote by $\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, H_\beta > 0\}$ the event on which H_β is positive definite. With these notations, we can introduce the random potential β which will provide another expression of the mixing measure of the VRJP.

Proposition 1.2.3. *Let ν_V^W be the measure on \mathcal{D}_V^W defined by:*

$$\nu_V^W(d\beta) = \left(\frac{2}{\pi}\right)^{\frac{|V|}{2}} \exp\left(-\frac{1}{2}\langle \mathbf{1}, H_\beta \mathbf{1} \rangle\right) \frac{\prod_{i \in V} d\beta_i}{\sqrt{\det(H_\beta)}}$$

Then ν_V^W is a probability distribution. Its Laplace transform is

$$\int e^{-\langle \lambda, \beta \rangle} \nu_V^W(d\beta) = \exp\left(-\sum_{\{i,j\} \in E} W_{i,j} \left(\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1\right)\right) \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}}$$

for $\lambda \in \mathbb{R}_+^V$.

Moreover, under $\nu_V^W(d\beta)$:

- for $i \in V$, $\frac{1}{2\beta_i}$ is an Inverse Gaussian variable, with parameters $((\sum_{j \sim i} W_{i,j})^{-1}, 1)$.
- β is 1-dependent, i.e. if $U, U' \subset V$ are such that $d_{\mathcal{G}}(U, U') \geq 2$ (where $d_{\mathcal{G}}$ is the graph distance on \mathcal{G}), then $(\beta_i)_{i \in U}$ and $(\beta_j)_{j \in U'}$ are independent.

This distribution was introduced by Sabot, Tarrès and Zeng in [STZ17]. They showed the relation between ν_V^W and \mathcal{Q}_V^{W, i_0} that we present in Proposition 1.2.4, which provides an expression of the mixing measure of the VRJP using the β potential.

Remark 1.2.4. The matrix $H_\beta = 2\beta - W$ can be interpreted as a random Schrödinger operator. Indeed, if we denote by Δ the discrete Laplacian on the weighted graph (\mathcal{G}, W) , then $H_\beta = \mathcal{V} - \Delta$, where $(\mathcal{V}f)(i) = (2\beta_i - \sum_{j \sim i} W_{i,j})f(i)$. The operator \mathcal{V} is the multiplication by a random potential on \mathcal{G} , which is not i.i.d. as in the Anderson model, but at least 1-dependent.

Let us describe in more detail the link between this random potential and the mixing measure of the VRJP. Let $i_0 \in V$ be a fixed starting point, and let F_{i_0} be the density (in the sense of Proposition 1.2.1) of the time-changed process $Z^{(1)}$ under $\mathbb{P}^{VRJP(W, i_0)}$: for $n \in \mathbb{N}$, $0 < s_1 < \dots < s_n < t$ and $i_1, \dots, i_n \in V$, for $\sigma = \sigma_{t,n}^{(i,s)}$,

$$F_{i_0}(\sigma) = \exp\left(-\sum_{i \sim j} W_{i,j} \left(\sqrt{(1+l_i^\sigma(t))(1+l_j^\sigma(t))} - 1\right)\right) \frac{2^{-n} \prod_{k=0}^{n-1} W_{i_k, i_{k+1}}}{\prod_{i \in V \setminus \{i_n\}} \sqrt{1+l_i^\sigma(t)}}.$$

However, we have another expression for this density, since the time-changed VRJP is a mixture of Markov jump processes with jump rates $\frac{W_{i,j}}{2}e^{u_j-u_i}$, where the mixing measure is $\mathcal{Q}_V^{W,i_0}(du)$. Therefore, we have

$$F_{i_0}(\sigma) = \int_{\mathcal{H}_{i_0}} \exp\left(-\sum_{i \in V} \left(\sum_{j \sim i} \frac{W_{i,j}}{2} e^{u_j - u_i}\right) l_i^\sigma(t)\right) e^{u_{i_n}} \prod_{k=0}^{n-1} \frac{W_{i_k, i_{k+1}}}{2} \mathcal{Q}_V^{W,i_0}(du).$$

Let us set $\tilde{\beta}_i = \sum_{j \sim i} \frac{W_{i,j}}{2} e^{u_j - u_i}$ for $i \in V$. This is the parameter of the exponential time that the Markov jump process with rates $\frac{W_{i,j}}{2} e^{u_j - u_i}$ spends at i before jumping away. Thanks to the two expressions of F_{i_0} , we can identify the distribution of $(\tilde{\beta}_i)_{i \in V}$ under \mathcal{Q}_V^{W,i_0} . Indeed, let σ be a cyclical distribution, *i.e.* such that $i_n = i_0$. Then, we get

$$\int_{\mathcal{H}_{i_0}} e^{-\sum_{i \in V} \tilde{\beta}_i l_i^\sigma(t)} \mathcal{Q}_V^{W,i_0}(du) = \exp\left(-\sum_{i \sim j} W_{i,j} \left(\sqrt{(1+l_i^\sigma(t))(1+l_j^\sigma(t))} - 1\right)\right) \prod_{i \in V \setminus \{i_0\}} \frac{1}{\sqrt{1+l_i^\sigma(t)}},$$

which is the Laplace transform of the distribution of $\tilde{\beta}$. It is almost the same as for ν_V^W , but the term $i = i_0$ is missing in the last product. To obtain a more symmetrical expression, which is in fact the Laplace transform of ν_V^W , we add an independent Gamma(1/2, 1) variable to $\tilde{\beta}_{i_0}$.

Conversely, for all $i_0 \in V$ the mixing measure \mathcal{Q}_V^{W,i_0} can be recovered from a random field with distribution ν_V^W .

Proposition 1.2.4. *For all $\beta \in \mathcal{D}_V^W$, let us denote by $G = (H_\beta)^{-1}$ the Green function associated with the random Schrödinger operator H_β . Fix $i_0 \in V$, and define $u(i_0, \cdot) \in \mathcal{H}_{i_0}$ by*

$$u(i_0, i) = \log\left(\frac{G(i_0, i)}{G(i_0, i_0)}\right)$$

for $i \in V$. Then under $\nu_V^W(d\beta)$, we have:

- (i) $(u(i_0, i))_{i \in V}$ is $(\beta_i)_{i \in V \setminus \{i_0\}}$ -measurable, and distributed according to \mathcal{Q}_V^{W,i_0} .
- (ii) $G(i_0, i_0)$ is independent from $(\beta_i)_{i \neq i_0}$, and therefore from $u(i_0, \cdot)$. Moreover, $\frac{1}{2G(i_0, i_0)}$ is a Gamma variable with parameter (1/2, 1).
- (iii) For $i \in V$,

$$\beta_i = \sum_{j \sim i} \frac{W_{i,j}}{2} e^{u(i_0, j) - u(i_0, i)} + \mathbf{1}_{i=i_0} \frac{1}{2G(i_0, i_0)}.$$

As a consequence, the random potential with distribution ν_V^W can be used to construct a random environment for the time-changed VRJP started at any point. Indeed, for $\beta \in \mathcal{D}_V^W$ we denote by \mathbb{P}^{W,β,i_0} the distribution of the Markov jump process with jump rates

$$\frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}.$$

Then according to Theorem 1.2.2 and Proposition 1.2.4, under $\mathbb{P}^{VRJP(W,i_0)}$, the time-changed process $\tilde{Z}^{(1)}$ is a mixture of the Markov processes with distribution $\mathbb{P}_{i_0}^{W,\beta,i_0}$, and the mixing measure is $\nu_V^W(d\beta)$.

Remark 1.2.5. Note that having $W_{i,i} \neq 0$ for some $i \in V$ does not change the distribution of H_β , and therefore the distribution of the random jump rates remains unchanged.

This representation using the β potential will be convenient when studying the VRJP on infinite graphs. Indeed, the potential is 1-dependent, and the distribution ν_V^W behaves well with respect to restriction, as detailed below.

For $U, U' \subset V$, and for any vector $x \in \mathbb{R}^V$ and matrix $M \in \mathbb{R}^{V \times V}$, let us denote by $x_U = (x_i)_{i \in U}$ the restriction of x to U , and in the same way, $M_{U, U'} = (M_{i, j})_{i \in U, j \in U'}$. Then for finite $U \subset V$, under $\nu_V^W(d\beta)$, the distribution of β_U is explicitly known: it is a perturbation of $\nu_U^{W_{U, U}}$, introduced by Letac in [LW19], and involves a boundary term η .

Proposition 1.2.5. *Let $\mathcal{G} = (V, E)$ be a finite connected graph.*

(i) *For $\eta \in (\mathbb{R}_+)^V$, let us define the measure $\nu_V^{W, \eta}$ on \mathcal{D}_V^W by*

$$\nu_V^{W, \eta}(d\beta) = \left(\frac{2}{\pi}\right)^{\frac{|V|}{2}} \exp\left(-\frac{1}{2}\langle \mathbf{1}, H_\beta \mathbf{1} \rangle - \frac{1}{2}\langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \mathbf{1} \rangle\right) \frac{\prod_{i \in V} d\beta_i}{\sqrt{\det(H_\beta)}}.$$

Then $\nu_V^{W, \eta}$ is a probability distribution. Note that $\nu_V^W = \nu_V^{W, 0}$.

(ii) *Let $U \subset V$. Then under $\nu_V^{W, \eta}(d\beta)$, β_U is distributed according to $\nu_U^{W_{U, U}, \hat{\eta}}$, where*

$$\hat{\eta} = \eta_U + W_{U, U^c} \mathbf{1}_{U^c}.$$

(iii) *Moreover under $\nu_V^{W, \eta}(d\beta)$, conditionally on β_U , the distribution of β_{U^c} is $\nu_{U^c}^{\tilde{W}, \tilde{\eta}}$, where*

$$\tilde{W} = W_{U^c, U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} W_{U, U^c} \text{ and } \tilde{\eta} = \eta_{U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} \eta_U.$$

In particular, this result implies that under $\nu_V^W(d\beta)$, the distribution of a restriction β_U depends only on the weights $W_{U, U}$ inside of U , and the edges W_{U, U^c} coming out of U . This will be especially useful to extend the representation of the VRJP as a mixture of Markov processes to infinite graphs.

1.2.3 Representation on infinite graphs

The results in the upcoming section come from [SZ19], where Sabot and Zeng showed that the VRJP is a mixture of Markov processes on infinite graphs. As with the result of Merkl and Rolles for the ERRW ([MR07]), the proof relies on the existence of a mixing measure on finite restrictions of the graph, and the convergence of this mixing measure on the infinite graph. The key ingredient is the choice of a particular boundary condition on finite restrictions, which will help both in the construction of a β potential on the whole graph, and in the proof that the associated random environment converges.

Wired boundary condition

Let $\mathcal{G} = (V, E)$ be an infinite graph, that we assume to be locally finite and connected, as well as endowed with edge weights W . As usual, we have $W_{i, j} = W_{j, i} > 0$ for $\{i, j\} \in E$, and $W_{i, j} = 0$ otherwise. Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of finite connected subsets of V , such that $\bigcup_{n \in \mathbb{N}} V_n = V$.

For $n \in \mathbb{N}$, we define the restriction of (\mathcal{G}, W) to V_n with a *wired* boundary condition, as follows. Let us introduce a new vertex δ_n , that will be considered as a boundary point. Define then the graph $\mathcal{G}^{(n)} = (\tilde{V}^{(n)}, \tilde{E}^{(n)})$ with

$$\begin{aligned} \tilde{V}^{(n)} &= V_n \cup \{\delta_n\} \\ \text{and } \tilde{E}^{(n)} &= \{\{i, j\} \in E, i, j \in V_n\} \cup \{\{i, \delta_n\}, i \in \partial V_n\}, \end{aligned}$$

where for $U \subset V$, $\partial U = \{i \in U, \exists j \in V \setminus U, \{i, j\} \in E\}$. This corresponds to identifying all vertices outside of V_n to a single vertex δ_n . The edges between to vertices of V_n are conserved, and for $i \in V_n$, the edges between i and vertices outside V_n are fused as a single edge $\{i, \delta_n\}$. Moreover, we endow $\mathcal{G}^{(n)}$ with appropriate weights $\tilde{W}^{(n)}$, so that the weights inside of V_n are the same, and the weights coming out of V_n are conserved:

$$\begin{aligned} \tilde{W}_{i,j}^{(n)} &= W_{i,j} \text{ for } i, j \in V_n, \\ \text{and } \tilde{W}_{i,\delta_n}^{(n)} &= \sum_{j \sim i, j \notin V_n} W_{i,j} = W_{\{i\}, V_n^c} \mathbb{1}_{V_n^c}. \end{aligned}$$

Then, $(\mathcal{G}^{(n)}, \tilde{W}^{(n)})$ is called the restriction of (\mathcal{G}, W) to V_n with a *wired* boundary condition.

The β potential on infinite graphs

Let us first see how this boundary condition allows for the definition of a distribution ν_V^W on the whole infinite graph.

Fix $n \in \mathbb{N}$, and let $\beta^{(n)} = (\beta_i^{(n)})_{i \in \tilde{V}^{(n)}}$ be a random potential on $\mathcal{G}^{(n)}$, with distribution $\nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$. Then according to Proposition 1.2.5(ii), the restriction $\beta_{V_n}^{(n)}$ has distribution $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$, where:

$$W^{(n)} = W_{V_n, V_n} \text{ and } \eta^{(n)} = W_{V_n, V_n^c} \mathbb{1}_{V_n^c} = \left(\tilde{W}_{i,\delta_n}^{(n)} \right)_{i \in V_n}.$$

However, for $m > n$, if $\beta^{(m)}$ is a random potential on $\mathcal{G}^{(m)}$ with distribution $\nu_{\tilde{V}^{(m)}}^{\tilde{W}^{(m)}}$, then the restriction $\beta_{V_n}^{(m)}$ has the same distribution as $\beta_{V_n}^{(n)}$, since the edge weights coming out of V_n are still given by $\eta^{(n)}$. Thanks to Kolmogorov's extension theorem, this compatibility implies the existence of a β potential on the whole infinite graph \mathcal{G} .

Proposition 1.2.6. *For $\beta \in \mathbb{R}^V$, let us still denote by $H_\beta = 2\beta - W$ the Schrödinger operator associated with β , i.e. such that $(H_\beta f)_i = 2\beta_i f_i - \sum_{j \sim i} W_{i,j} f_j$, for $f \in \mathbb{R}^V$. We also define*

$$\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, (H_\beta)_{U,U} > 0 \text{ for all finite } U \subset V\}.$$

Then there exists a unique distribution ν_V^W on \mathcal{D}_V^W such that: for all finite subset U of V , under $\nu_V^W(d\beta)$, the restriction β_U is distributed according to $\nu_{U, U^c}^{W_{U, U^c}, \eta}$, where $\eta = W_{U, U^c} \mathbb{1}_{U^c}$.

This distribution can now be used to construct a coupling of $\beta^{(n)}$ fields on each restriction $\mathcal{G}^{(n)}$. Indeed under $\nu_V^W(d\beta)$, for $n \in \mathbb{N}$ the restriction β_{V_n} has the distribution $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$. Therefore, it can be completed into a random potential $\beta^{(n)}$ on $\mathcal{G}^{(n)}$, such that $\beta_{V_n}^{(n)} = \beta_{V_n}$, and $\beta^{(n)}$ is distributed according to $\nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$. The distribution of $\beta_{\delta_n}^{(n)}$ can be obtained thanks to Proposition 1.2.5(iii), but we will express it in another way.

Let $H_\beta^{(n)} = 2\beta^{(n)} - \tilde{W}^{(n)}$ be the Schrödinger operator associated with $\beta^{(n)}$, and $G^{(n)} = (H_\beta^{(n)})^{-1}$ the corresponding Green function. According to Proposition 1.2.4 with $i_0 = \delta_n$, we then have

$$\beta_{\delta_n}^{(n)} = \sum_{j \in V_n} \frac{\tilde{W}_{\delta_n, j}^{(n)}}{2} \frac{G^{(n)}(\delta_n, j)}{G^{(n)}(\delta_n, \delta_n)} + \frac{1}{2G^{(n)}(\delta_n, \delta_n)},$$

where $\left(\frac{G^{(n)}(\delta_n, j)}{G^{(n)}(\delta_n, \delta_n)} \right)$ is $\beta_{V_n}^{(n)}$ -measurable, and $\frac{1}{2G^{(n)}(\delta_n, \delta_n)}$ is a Gamma(1/2, 1) variable, independent from $\beta_{V_n}^{(n)}$. We can therefore couple all potentials $\beta^{(n)}$ by taking $\frac{1}{2G^{(n)}(\delta_n, \delta_n)} = \gamma$, where γ is a random

Gamma(1/2, 1) variable, independent from β . From now on, we will denote by $\nu_V^W(d\beta, d\gamma)$ the distribution on $\mathcal{D}_V^W \times \mathbb{R}$ defined as:

$$\nu_V^W(d\beta, d\gamma) = \nu_V^W(d\beta) \otimes \frac{\mathbb{1}_{\gamma > 0}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma.$$

This coupling of $\beta^{(n)}$ potentials provides a coupling of the representations of the VRJP on the finite restrictions $\mathcal{G}^{(n)}$, which is then used to prove that these representations converge when $n \rightarrow \infty$.

Convergence of the representation

Fix $i_0 \in V$ a starting point, that we assume to be in $\bigcap_{n \in \mathbb{N}} V_n$. For all $n \in \mathbb{N}$, the VRJP on $(\mathcal{G}^{(n)}, \tilde{W}^{(n)})$ started at i_0 is a mixture of the Markov processes with distribution $\mathbb{P}_{i_0}^{\tilde{W}^{(n)}, \beta^{(n)}, i_0}$, where $\beta^{(n)}$ is defined from β and γ as in the previous paragraph, and the mixing measure is $\nu_V^W(d\beta, d\gamma)$. Recall that the jump rates defining $\mathbb{P}_{i_0}^{\tilde{W}^{(n)}, \beta^{(n)}, i_0}$ are, for $i \sim j$,

$$\frac{\tilde{W}_{i,j}^{(n)} G^{(n)}(i_0, j)}{2 G^{(n)}(i_0, i)}.$$

To show that the representation on $\mathcal{G}^{(n)}$ converges to a representation on \mathcal{G} , we first need to prove that the Green function $G^{(n)}$ converges. In order to do this, it is convenient to write $G^{(n)}$ as a sum over all paths in $\mathcal{G}^{(n)}$.

We denote by $\mathcal{P}_{i,j}^{\tilde{V}^{(n)}}$ the set of paths σ from i to j in $\mathcal{G}^{(n)}$, i.e. $\sigma = (\sigma_0, \dots, \sigma_l) \in (\tilde{V}^{(n)})^{l+1}$ where $l = |\sigma|$ is the length of σ , $\sigma_0 = i$ and $\sigma_l = j$, and for $k \in \llbracket 0, l-1 \rrbracket$, $\sigma_k \sim \sigma_{k+1}$. Then $G^{(n)}$ can be written in the following way: for $i, j \in \tilde{V}^{(n)}$,

$$G^{(n)}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{\tilde{V}^{(n)}}} \frac{\tilde{W}_\sigma^{(n)}}{(2\beta^{(n)})_\sigma},$$

where $\tilde{W}_\sigma^{(n)} = \prod_{k=0}^{|\sigma|-1} \tilde{W}_{\sigma_k, \sigma_{k+1}}^{(n)}$ and $(2\beta^{(n)})_\sigma = \prod_{k=0}^{|\sigma|} 2\beta_{\sigma_k}^{(n)}$. This sum can be separated in two, by distinguishing paths that stay inside V_n , and paths that visit $\delta^{(n)}$ at some point. These last paths can be decomposed into three parts: a first path from i to a neighbor z_1 of δ_n , then a path from δ_n to δ_n , and finally a path from another neighbor z_2 of δ_n to j . For $i, j \in V_n$, this yields

$$\begin{aligned} G^{(n)}(i, j) &= \sum_{\sigma \in \mathcal{P}_{i,j}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma} + \sum_{z_1, z_2 \sim \delta_n} \left(\sum_{\sigma \in \mathcal{P}_{i,z_1}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \tilde{W}_{z_1, \delta_n}^{(n)} G^{(n)}(\delta_n, \delta_n) \tilde{W}_{\delta_n, z_2}^{(n)} \left(\sum_{\sigma \in \mathcal{P}_{z_2, j}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \\ &= \hat{G}^{(n)}(i, j) + \psi^{(n)}(i) \frac{1}{2\gamma} \psi^{(n)}(j), \end{aligned}$$

where $\hat{G}^{(n)} : V \times V \rightarrow \mathbb{R}_+$ and $\psi^{(n)} : V \rightarrow \mathbb{R}_+$ are β -measurable functions defined as follows:

- $(\hat{G}^{(n)})_{V_n, V_n} = ((H_\beta)_{V_n, V_n})^{-1}$ and $\hat{G}^{(n)}(i, j) = 0$ if $i \notin V_n$ or $j \notin V_n$.
- For $i \in V_n$, $\psi^{(n)}(i) = \frac{G^{(n)}(\delta_n, i)}{G^{(n)}(\delta_n, \delta_n)} = e^{u^{(n)}(\delta_n, i)}$ with the notations of Proposition 1.2.4 (where $u^{(n)}(\delta_n, \cdot)$ has distribution $\mathcal{Q}_{\tilde{V}^{(n)}, \delta_n}^{\tilde{W}^{(n)}}$), and $\psi^{(n)}(i) = 1$ for $i \notin V_n$. Note that $(H_\beta^{(n)} \psi^{(n)})_{V_n} = 0$.

Under $\nu_V^W(d\beta, d\gamma)$, Sabot and Zeng showed that $\hat{G}^{(n)}$ and $\psi^{(n)}$ converge almost surely. In particular, if $\mathcal{F}_{V_n}^\beta = \sigma((\beta_i)_{i \in V_n})$, then for all $i \in V$, $(\psi^{(n)}(i))_{n \in \mathbb{N}}$ is a $(\mathcal{F}_{V_n}^\beta)$ -martingale. We denote by \hat{G} and ψ the respective limits of these objects, which are β -measurable. For $\beta \in \mathcal{D}_V^W$, ψ is H_β -harmonic, *i.e.* $(H_\beta\psi)(i) = 0$ for all $i \in V$. Finally, define $G : V \times V \rightarrow \mathbb{R}_+$ such that for $i, j \in V$,

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma} \psi(i)\psi(j).$$

This leads to the following theorem, which gives a representation of the time-changed VRJP as a mixture of Markov processes on infinite graphs.

Theorem 1.2.3. *For $\beta \in \mathcal{D}_V^W$, $\gamma > 0$ and a starting point $i_0 \in V$, let $\mathbb{P}^{W, \beta, \gamma, i_0}$ be the distribution of the Markov jump process with jump rates*

$$\frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}$$

from i to j .

Then under $\mathbb{P}^{VRJP(W, i_0)}$, the time-changed process $Z^{(1)}$ is a mixture of Markov jump processes with distribution $\mathbb{P}_{i_0}^{W, \beta, \gamma, i_0}$, and the mixing measure is $\nu_V^W(d\beta, d\gamma)$.

Moreover, for all $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, the Markov jump process $\mathbb{P}_{i_0}^{W, \beta, \gamma, i_0}$ is recurrent if and only if $\psi(i) = 0$ for all $i \in V$, and transient if and only if $\psi(i) > 0$ for all $i \in V$.

The proof relies on using the representations on restrictions $\mathcal{G}^{(n)}$. Indeed, for all finite subset $U \subset V$, let n_0 be such that $U \subset V_{n_0}$. Then for $n \geq n_0$ the VRJP, up to the first time it leaves U , is a mixture of Markov processes with jump rates $\frac{W_{i,j}}{2} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}$. The distribution of the stopped process can therefore be written as an integral involving the density of trajectories for these Markov processes. Taking $n \rightarrow \infty$, and using a uniform integrability argument on the family $\left(\frac{G^{(n)}(i_0, i)}{G^{(n)}(i_0, i_0)} \right)_{n \in \mathbb{N}}$ for all $i \in V$, this integral converges to the desired result.

Theorem 1.2.3 implies that the discrete-time version of the VRJP, which is not modified by the time changes applied to X , is a mixture of reversible random walks on \mathcal{G} , with the random conductance between i and j equal to $W_{i,j}G(i_0, i)G(i_0, j)$, under the mixing measure $\nu_V^W(d\beta, d\gamma)$. Recall also that the ERRW on (\mathcal{G}, a) , with initial weights $(a_e)_{e \in E}$, is a mixture of discrete-time VRJP with independent starting weights $(W_e)_{e \in E}$ following Gamma($a_e, 1$) distributions. This provides the following expression of a mixing measure for the ERRW: it is also a mixture of reversible random walks, with the same conductances $W_{i,j}G(i_0, i)G(i_0, j)$, under the distribution $\tilde{\nu}_V^a(dW, d\beta, d\gamma)$, where

$$\tilde{\nu}_V^a(dW, d\beta, d\gamma) = \int \nu_V^W(d\beta, d\gamma) \bigotimes_{e \in E} \frac{W_e^{a_e-1}}{\Gamma(a_e)} \exp(-W_e) dW_e.$$

Remark 1.2.6. The martingale property of $\psi^{(n)}$ plays a central role in the proof of the convergence of $G^{(n)}$. Moreover, the quadratic variation of the vectorial martingale $(\psi^{(n)}(i))_{i \in V}$ is given by $\hat{G}^{(n)}$, *i.e.* $\langle \psi(i), \psi(j) \rangle_n = \hat{G}^{(n)}(i, j)$. This was generalized by Disertori, Merkl and Rolles in [DMR17], where they introduced an infinite hierarchy of martingales, thanks to symmetry arguments derived from the $\mathbb{H}^{2|2}$ model. They expanded this family further in [DMR19], using the antisymmetric variables of the $\mathbb{H}^{2|2}$ model.

Consequences for the VRJP and ERRW on \mathbb{Z}^d

The fact that the β -potential is 1-dependent also yields ergodicity properties for the representation described above. More precisely, let (\mathcal{G}, W) be vertex transitive, *i.e.* the group \mathcal{A} of automorphisms of \mathcal{G}

that leave W invariant is vertex transitive: for $i, j \in V$, there exists $f \in \mathcal{A}$ such that $f(i) = f(j)$. Then under $\nu_V^W(d\beta)$, the potential $(\beta_i)_{i \in V}$ is stationary and ergodic with respect to the group \mathcal{A} . This is also true of \hat{G} and ψ , since they are β -measurable. This has a number of consequences for the VRJP on \mathbb{Z}^d with constant initial weights W , which is vertex transitive.

First, on vertex transitive weighted graphs, the event $\{\forall i \in V, \psi(i) = 0\}$ is invariant by the group \mathcal{A} , and therefore has probability 0 or 1. According to Theorem 1.2.3, this implies that the Markov jump process $\mathbb{P}^{W, \beta, i_0}$ is either ν_V^W -a.s. recurrent, or ν_V^W -a.s. transient.

Moreover, the VRJP satisfies a functional central limit theorem at weak enough reinforcement on \mathbb{Z}^d , $d \geq 3$. In this regime, the transience of the VRJP was already known, but this is a stronger result. Let $(\hat{Z}_n)_{n \in \mathbb{N}}$ denote the discrete-time version of the VRJP, and for $t \geq 0$, define $B_t^{(m)} = \frac{1}{\sqrt{m}} \hat{Z}_{[mt]}$. Then there exists $\bar{W} > 0$ such that for $W > \bar{W}$, under $\mathbb{P}^{VRJP(W, 0)}$, $B^{(m)}$ converges in distribution to a d -dimensional Brownian motion with covariance matrix $\sigma^2 I_d$, where $0 < \sigma^2 < \infty$.

This is a consequence of a similar theorem from [DMFGW89] on random walks in random conductances, which is valid when the conductances are ergodic and satisfy an integrability condition. For \hat{Z} , the conductances are $W_{i,j}G(0, i)G(0, j)$, and are therefore not ergodic. However, once conditioned to never coming back to 0, \hat{Z} has the same distribution as a random walk with conductances $W_{i,j}\psi(i)\psi(j)$, similarly conditioned, to which the theorem applies. The integrability hypothesis is verified at weak reinforcement, thanks to estimates by Disertori, Spencer and Zirnbauer ([DSZ10]). The random walk in random conductances $W_{i,j}\psi(i)\psi(j)$ will reappear later in this manuscript, to study H_β -harmonic functions.

This proof of a functional central limit theorem is also valid for the ERRW at weak reinforcement on \mathbb{Z}^d , $d \geq 3$, with constant initial weights $(a_e)_{e \in E}$. Indeed, the ERRW is a mixture of (discrete-time) VRJP with i.i.d. starting weights $(W_e)_{e \in E}$, which conserves the ergodicity of the random conductances $W_{i,j}\psi(i)\psi(j)$. The estimates implying the integrability condition were shown in [DST15] to prove the transience of the ERRW at this regime.

Another consequence of the ergodicity properties of β is the following fact: on \mathbb{Z}^2 endowed with constant initial weights $(a_e)_{e \in E}$, the ERRW is recurrent for any $a > 0$. Indeed, it is a mixture of reversible random walks, which are transient if and only if the associated ψ is positive everywhere on \mathbb{Z}^d . Since ψ is also stationary and ergodic, it cannot satisfy the polynomial decay shown by Merkl and Rolles in [MR06] for the random environment of the ERRW on \mathbb{Z}^2 . Therefore the process is a.s. recurrent.

A corresponding result for the VRJP was also shown by Sabot ([Sab19]) as well as Kozma and Peled ([KP19]), both using Mermin-Wagner-type arguments to show bounds on e^{u_i} under the mixing measure $\mathcal{Q}_V^{W, i_0}(du)$. With the same argument concerning the ergodicity of ψ , this implies the a.s. recurrence of the VRJP on \mathbb{Z}^2 with constant initial weights.

This was also proved in another way by Poudevigne in [Pou19]. His method involves a coupling of β potentials for different values of the initial weights W , and a martingale with respect to the varying W . This yields a monotonicity result relative to the initial weights, which has several consequences. Firstly, if a weighted graph (\mathcal{G}, W) is such that the associated simple random walk is recurrent, then the VRJP on (\mathcal{G}, W) is recurrent as well. In particular, this is true for the VRJP on \mathbb{Z}^2 endowed with constant initial weights. Secondly, there is a unique phase transition between transience and recurrence of the VRJP on \mathbb{Z}^d , for all $d \geq 3$.

Another representation on infinite trees

As with random walks in random environment, if a mixture of Markov jump processes is recurrent, the associated jump rates can be recovered as asymptotic functionals of the trajectory. This implies the uniqueness of the representation, *i.e.* of the distribution of the random jump rates. However, if the random environment has a nonzero probability of defining a transient Markov process, there is no reason to have uniqueness of the representation. A first counterexample appears in the case of infinite trees.

Indeed, it is possible to construct a different representation, using another boundary condition on finite restrictions: the *free* boundary condition, *i.e.* removing any edge with at least one end out of the subgraph. This restriction is then a finite tree, on which the random jump rates given by Theorem 1.2.2 are independent for each edge. Therefore, the distributions of the environment for any finite subtree are compatible. This yields a representation on the whole infinite tree, with independent random jump rates, which was described by Chen and Zeng in [CZ16]. This result is consistent with the observations of Basdevant and Singh in [BS10], who proved that the phase transition between recurrence and transience of the VRJP on a regular tree was the same as for a random walk in random i.i.d. environment.

On a d -regular tree for $d \geq 3$, and with constant initial weights large enough so that the VRJP is transient (thanks to results from [DV02], [BS10]), this representation is different from the one defined in Theorem 1.2.3. Indeed, for any infinite self-avoiding path ω starting at the root of the tree, the sequence $(\psi(\omega_n))_{n \in \mathbb{N}}$ almost surely does not converge to 0, since ψ is stationary and ergodic. However the analogous quantity converges to 0 in the representation with independent jump rates.

The existence of several distinct distributions for a random environment representing the VRJP motivated the first part of my work, presented in the next section.

1.3 The question of uniqueness of the representation

As mentioned above, on some infinite weighted graphs (specifically infinite trees) the VRJP admits several distinct representations as a mixture of Markov processes. This begs the following question: in what context is there uniqueness of the representation? The following section presents results from [Ger19], in which I tackled this issue.

A first step was to show that all random environments representing the VRJP are related to the β potential introduced in the previous section. This shifts the problem to studying H_β -harmonic functions on the graph, using the notion of Martin boundary. On the lattice \mathbb{Z}^d , this approach leads to a uniqueness result. On infinite trees however, we construct an infinite family of distinct representations.

1.3.1 Common form for all representations

We still consider an infinite graph $\mathcal{G} = (V, E)$, connected and locally finite, endowed with positive weights $(W_e)_{e \in E}$. From now on, we will denote by $\tilde{\mathbb{P}}^{VRJP(W, i_0)}$ the distribution of the time-changed process $Z^{(1)}$ under $\mathbb{P}^{VRJP(W, i_0)}$. According to Theorem 1.2.3, it is a mixture of Markov jump processes in the following sense:

$$\tilde{\mathbb{P}}^{VRJP(W, i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{W, \beta, \gamma, i_0}[\cdot] \nu_V^W(d\beta, d\gamma),$$

where $\mathbb{P}^{W, \beta, \gamma, i_0}$ is the distribution of the Markov process on \mathcal{G} with jump rate $\frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}$ from i to j . We are interested in whether this is the only way to write $\tilde{\mathbb{P}}^{VRJP(W, i_0)}$ as a mixture of Markov processes.

Let us introduce more formally the notion of representations of the VRJP. We denote by \mathcal{J}_V^E the set of jump rates on \mathcal{G} , *i.e.* the set of $(r_{i,j})_{i,j \in V} \in \mathbb{R}_+^{V \times V}$ such that $r_{i,j} = 0$ if $\{i, j\} \notin E$. Let now \mathcal{R} be a probability distribution on \mathcal{J}_V^E , and $i_0 \in V$. We will say that \mathcal{R} is a representation of the time-changed VRJP starting at i_0 (as a mixture of Markov jump processes) if

$$\tilde{\mathbb{P}}^{VRJP(W, i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{MJJP(r)} \mathcal{R}(dr),$$

where we still denote by $\mathbb{P}^{MJJP(r)}$ the distribution of the Markov jump process on \mathcal{G} with jump rates $(r_{i,j})_{i \sim j}$. The distribution of jump rates appearing in Theorem 1.2.3 will be called "standard representation".

General representations and the β potential

The first question concerning arbitrary representations of the VRJP is the following: can they all be expressed in the same way as the standard representation, using the β potential? The answer is positive, as described below.

For $r \in \mathcal{J}_V^E$ and $i \in V$, we will denote by $r_i = \sum_{i \sim j} r_{i,j}$. The following theorem assures that for any representation \mathcal{R} of the VRJP, under $\mathcal{R}(dr)$ the holding time rates $(r_i)_{i \in V}$ have the same distribution as the analogous quantities $(\beta_i)_{i \in V}$ in the standard representation. By adding an independent Gamma(1/2, 1) variable at the starting point, we therefore get a potential with distribution ν_V^W .

Theorem 1.3.1. *Let \mathcal{R} be a representation of the VRJP starting at i_0 . We define the following distribution on $\mathcal{J}_V^E \times \mathbb{R}$: $\mathcal{R}(dr, d\gamma) = \mathcal{R}(dr) \otimes \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$.*

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we define a potential $\beta \in \mathbb{R}_+^V$ on \mathcal{G} by:

$$\beta_i = r_i + \mathbb{1}_{\{i=i_0\}}\gamma.$$

Then under $\mathcal{R}(dr, d\gamma)$, β is distributed according to ν_V^W .

Moreover there exists a random H_β -harmonic function $h : V \rightarrow \mathbb{R}_+$, such that for all $i \sim j$,

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G_h(i_0, j)}{G_h(i_0, i)},$$

where for $i \in V$, $G_h(i_0, i) = \hat{G}(i_0, i) + h(i)$, and \hat{G} still denotes the β -measurable Green function defined on $V \times V$.

The expression of the jump rates using G_h results easily from the fact that $(\beta_i)_{i \in V}$ has distribution ν_V^W . To prove this, we proceed in the same manner as with the standard representation: by expressing the density of cyclical trajectories of the time-changed VRJP as the Laplace transform of the distribution \mathcal{R} .

Indeed, take $n \in \mathbb{N}$, $0 < s_1 < \dots < s_n < t$ and $i_1, \dots, i_n \in V$, and let $\sigma = \sigma_{i,n}^{(i,s)}$ the trajectory defined as in Proposition 1.2.1. Let also F_{i_0} be the density of trajectories for the VRJP started at i_0 . Then we have two ways of writing $F_{i_0}(\sigma)$: the explicit expression

$$F_{i_0}(\sigma) = \exp \left(- \sum_{i \sim j} W_{i,j} \left(\sqrt{(1 + l_i^\sigma(t))(1 + l_j^\sigma(t))} - 1 \right) \right) \frac{2^{-n} \prod_{k=0}^{n-1} W_{i_k, i_{k+1}}}{\prod_{i \in V \setminus \{i_n\}} \sqrt{1 + l_i^\sigma(t)}},$$

and the averaged expression using the representation \mathcal{R} ,

$$F_{i_0}(\sigma) = \int \exp \left(- \sum_{i \in V} r_i l_i^\sigma(t) \right) \prod_{k=0}^{n-1} r_{i_k, i_{k+1}} \mathcal{R}(dr).$$

In order to identify the distribution of $(r_i)_{i \in V}$, we take $i_0 = i_n$ so that σ is a cyclic trajectory. We then only need to show that

$$\prod_{k=0}^{n-1} r_{i_k, i_{k+1}} = \prod_{k=0}^{n-1} \frac{W_{i_k, i_{k+1}}}{2},$$

which we obtain by observing the density of the n -fold concatenated trajectory when $n \rightarrow \infty$ and $t \rightarrow 0$.

This common form for all representations of the VRJP is a first step in answering the question of uniqueness of the representation. To go further, we have to know more about H_β -harmonic functions. We will be able to characterize them using the notion of Martin boundary.

Harmonic functions and Martin boundary

We will consider two cases for the VRJP on (\mathcal{G}, W) : when it is a mixture of a.s. recurrent Markov processes, or a.s. transient ones. In the first case, as with random walks in random environment, every vertex is visited infinitely many times and the law of large numbers yields the uniqueness of the representation. Therefore, the standard representation is the only one, and the H_β -harmonic function h in Theorem 1.3.1 is zero everywhere.

From now on, we assume that the VRJP on (\mathcal{G}, W) is a.s. transient. According to Theorem 1.2.3, we then have $\nu_V^W(d\beta)$ -a.s. $\psi(i) > 0$ for all $i \in V$. This will allow us to define a Markov operator Δ^ψ , such that H_β -harmonic functions and Δ^ψ -harmonic functions are related.

For $i, j \in V$ and $\beta \in \mathcal{D}_V^W$, we define the following random conductances: $c_{i,j}^\psi = W_{i,j}\psi(i)\psi(j)$. Note that the associated random walk is transient. These conductances first appeared in the proof of the central limit theorem for the VRJP (from [SZ19]). Moreover, let Δ^ψ be the discrete Laplacian associated with the conductances c^ψ , i.e. for $f \in \mathbb{R}^V$ and $i \in V$,

$$(\Delta^\psi f)(i) = \sum_{j \sim i} c_{i,j}^\psi (f(j) - f(i)).$$

Then for any function $\varphi : V \rightarrow \mathbb{R}$, φ is Δ^ψ -harmonic (i.e. $\Delta^\psi \varphi \equiv 0$) if and only if $\psi\varphi$ is H_β -harmonic. In order to identify H_β -harmonic functions, we then simply have to study harmonic functions for a transient random walk, which is the purpose of the notion of Martin boundary.

Let us give more details about this particular type of boundary, that can be found for instance in [Woe00]: let $(Y_n)_{n \in \mathbb{N}}$ be a transient random walk on a locally finite graph $\mathcal{G} = (V, E)$. We denote by $g : V^2 \rightarrow \mathbb{R}_+$ the Green function associated with the walk, i.e. $g(i, j) = \sum_{n \in \mathbb{N}} \mathbb{P}_i[Y_n = j]$. Let us fix an arbitrary point ϕ , and introduce the kernel $K : V^2 \rightarrow \mathbb{R}_+$ such that $K(i, j) = \frac{g(i, j)}{g(\phi, j)}$. Then the Martin boundary \mathcal{M} is defined so that $\hat{V} = V \cup \mathcal{M}$ is the smallest compactification of V such that K extends continuously to $V \times \hat{V}$. As a consequence, for $\alpha \in \mathcal{M}$ and $(x_k)_{k \in \mathbb{N}} \in V^{\mathbb{N}}$, we have $x_k \rightarrow \alpha$ if and only if $x_k \rightarrow \infty$ and $K(i, x_k) \rightarrow K(i, \alpha)$ for all $i \in V$. $K : V \times \mathcal{M} \rightarrow \mathbb{R}_+$ is then called the Martin kernel.

This notion of boundary is closely related to positive harmonic functions for the walk Y . Indeed, for all $\alpha \in \mathcal{M}$, $K(\cdot, \alpha)$ is harmonic. Conversely, the following representation theorem assures that any positive harmonic function can be expressed in terms of the Martin boundary.

Theorem 1.3.2. *Let Y be a transient random walk on $\mathcal{G} = (V, E)$, and let \mathcal{M} and K be the associated Martin boundary and Martin kernel. Let $h : V \rightarrow \mathbb{R}_+$ be a positive harmonic function for the random walk Y . Then there exists a measure μ^h on \mathcal{M} such that for $i \in V$,*

$$h(i) = \int K(i, \alpha) \mu^h(d\alpha).$$

A particular case is the one where the Martin boundary is trivial, i.e. reduced to a single point. In this case, the only harmonic functions are constant. If we manage to show that the Martin boundary associated with Δ^ψ is trivial, the H_β -harmonic function h in Theorem 1.3.1 has to be proportional to ψ , which implies the uniqueness of the representation.

However, generally speaking we have no way of identifying the Martin boundary for a random walk in random environment. We still obtain results in two particular cases: on the lattice \mathbb{Z}^d , where we have a local central limit theorem, which provides precise estimates of the associated Green function; and on the infinite tree, where the Martin boundary is deterministic, and equal to the geometric boundary of the tree (the set of infinite rays).

1.3.2 Uniqueness of the representation on \mathbb{Z}^d

Let us present here the result we obtained for the VRJP on the lattice \mathbb{Z}^d with constant initial weights W . We take $\mathcal{G} = (\mathbb{Z}^d, E_d)$, where $E_d = \{\{i, j\} \subset \mathbb{Z}^d, |i - j| = 1\}$, and $|\cdot|$ is the Euclidean norm on \mathbb{Z}^d .

Recall the following results on transience and recurrence of the VRJP on \mathbb{Z}^d : for $d = 2$, the VRJP on (\mathcal{G}, W) is a mixture of positive recurrent Markov processes for any constant initial weights W ; for $d \geq 3$, there exists $\underline{W} > 0$ such that for $W < \underline{W}$, the VRJP is a mixture of positive recurrent Markov processes, and for $W > \underline{W}$, a mixture of transient Markov processes.

In the recurrent case, the uniqueness of the representation is already known. We show that for $d \geq 3$, there exists $\overline{W} \geq \underline{W}$ such that if $W > \overline{W}$, the VRJP on (\mathbb{Z}^d, W) admits a unique representation as a mixture of Markov chains. As announced in the previous section, we prove this by showing that the Martin boundary associated with Δ^ψ is $\nu_{\mathbb{Z}^d}^W(d\beta)$ -almost surely trivial in this regime.

Martin boundary in random conductances

In order to identify the Martin boundary associated with some random environment, we need to control the long-range behavior of the Green function corresponding to the quenched process. This is provided by a local central limit theorem, which was proved by Andres, Deuschel and Slowik in [ADS16a]. This result states that if the random conductances satisfy some ergodicity and integrability conditions, the associated Green function has almost surely the same long-range behavior as for a standard d -dimensional Brownian motion. Since we know that the Martin boundary for such a Brownian motion is trivial, this yields the following result.

Proposition 1.3.1. *Let $d \geq 3$ and $(\omega_e)_{e \in E_d} \in (\mathbb{R}_+^*)^{E_d}$ be random conductances on (\mathbb{Z}^d, E_d) . We assume that the distribution of ω is stationary and ergodic with respect to translations of \mathbb{Z}^d , and that there exist $1 < p, q \leq \infty$ such that $\mathbb{E}[\omega_e^p] < \infty$ and $\mathbb{E}[\omega_e^{-q}] < \infty$ for all $e \in E_d$.*

Then the walk associated with ω is a.s. transient, and the corresponding Martin boundary is a.s. trivial.

Consequences for the representations of the VRJP on \mathbb{Z}^d

Take constant weights W large enough so that the VRJP on (\mathbb{Z}^d, W) is a.s. transient. Therefore we also have $\nu_{\mathbb{Z}^d}^W$ -a.s. $\psi(i) > 0$ for all $i \in \mathbb{Z}^d$, and $c_{i,j}^\psi$ is well-defined for all $i, j \in \mathbb{Z}^d$. The previous result can then be applied to the random conductances c^ψ under $\nu_{\mathbb{Z}^d}^W(d\beta)$. In this context, the potential β is stationary and ergodic for the translations of \mathbb{Z}^d , so this is also true for ψ , as well as the conductances c^ψ . The integrability condition of Proposition 1.3.1 is satisfied for W large enough, thanks to the delocalization result from Disertori, Spencer and Zirnbauer ([DSZ10]). In this regime, the Martin boundary associated with c^ψ is then a.s. trivial, and therefore the H_β -harmonic functions are a.s. proportional to ψ . Using Theorem 1.3.1, this yields the uniqueness of the representation of the VRJP.

The following theorem summarizes the uniqueness results that we have on \mathbb{Z}^d .

Theorem 1.3.3. *Let $\mathcal{G} = (\mathbb{Z}^d, E_d)$ be endowed with constant edge weights $W > 0$. We consider representations of the time-changed VRJP, with distribution $\tilde{\mathbb{P}}^{VRJP(W,0)}$, as a mixture of Markov processes.*

Then:

- *If $d \in \{1, 2\}$, there is a unique representation.*
- *If $d \geq 3$, there are constants \underline{W} and \overline{W} such that for $0 < W < \underline{W}$ or for $W > \overline{W}$, there is a unique representation.*

We do not yet know if the representation is still unique when $\underline{W} \leq W \leq \overline{W}$.

1.3.3 A family of representations on infinite trees

For a transient random walk on an infinite rooted tree, the Martin boundary is always the set of ends of the tree, *i.e.* the set of infinite self-avoiding paths (or rays) started at the root. Therefore, the boundary is far richer than in the case of \mathbb{Z}^d , and there should be a lot more possibilities for harmonic functions. This is a first hint at the fact that we will find many distinct representations on infinite trees (with some regularity assumptions).

Consider then the VRJP on an infinite tree. We already know that the process admits two distinct representations, built using different boundary conditions: *wired* and *free*. To construct a family of different representations, we introduce a new type of boundary condition, which is a variant of the *wired* case, taking advantage of the tree structure.

A variant of the *wired* boundary condition

Let us first define some notations on trees. Let $\mathcal{T} = (T, E)$ be an infinite tree, and let $\phi \in T$ be a fixed arbitrary root. For all $i \in T$, we denote by $|i| = d(\phi, i)$ the graph distance between ϕ and i , also called depth of i . Then for $n \in \mathbb{N}$, $D^{(n)} = \{i \in T, |i| = n\}$ will be called the n th generation of \mathcal{T} . Finally, for $i \in T$, we denote by $S(i)$ the set of children of i , and by T_i the subtree of descendants of i .

We endow \mathcal{T} with weights $(W_e)_{e \in E}$, with the same notations as usual for $(W_{i,j})_{i,j \in T}$. We will construct new representations of the VRJP on (\mathcal{T}, W) by using new boundary conditions on finite restrictions of \mathcal{T} . As in the case of the standard representation, the distribution of the jump rates representing the VRJP on these finite graphs will converge to a representation on the infinite tree.

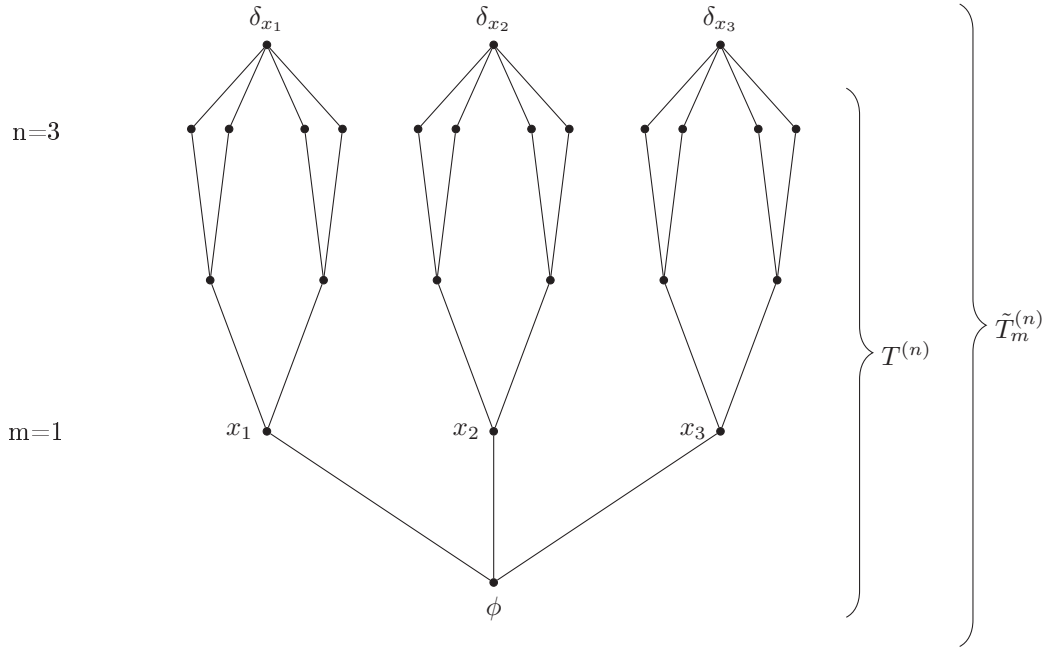
Let $0 \leq m \leq n$ be fixed. The restriction of \mathcal{T} to the first n generations, with *free* boundary conditions, is $(T^{(n)}, E^{(n)})$ where

$$T^{(n)} = \bigcup_{0 \leq k \leq n} D^{(k)} \text{ and } E^{(n)} = \{\{i, j\} \in E, i, j \in T^{(n)}\}.$$

We define a new boundary condition on this restriction by adding some boundary points, one for each branch of the tree rooted at the m th generation: for all $x \in D^{(m)}$, we introduce a boundary point δ_x , and we define the boundary set $B_m = \{\delta_x, x \in D^{(m)}\}$. We can then define the graph $\mathcal{G}_m^{(n)} = (\tilde{T}_m^{(n)}, \tilde{E}_m^{(n)})$, where

$$\tilde{T}_m^{(n)} = T^{(n)} \cup B_m \text{ and } \tilde{E}_m^{(n)} = E^{(n)} \cup \bigcup_{x \in D^{(m)}} \{\{i, \delta_x\}, i \in T_x \cap D^{(n)}\}.$$

In other words, for $x \in D^{(m)}$, all points in $T_x \setminus T^{(n)}$ are fused together into a single point δ_x , which represents a boundary point for the branch T_x .



As with the *wired* boundary condition, in order to preserve the compatibility of distributions for the β potential, we endow these restrictions with weights $\tilde{W}_m^{(n)}$, where

$$(\tilde{W}_m^{(n)})_e = \begin{cases} W_e & \text{if } e \in E^{(n)} \\ \sum_{j \in S(i)} W_{i,j} & \text{if } e = \{i, \delta_x\} \text{ where } i \in D^{(n)} \cap T_x. \end{cases}$$

This way, the weights coming out of $T^{(n)}$ are still $\eta^{(n)} = W_{T^{(n)}, (T^{(n)})^c} \mathbf{1}_{(T^{(n)})^c}$.

Remark 1.3.1. Note that for $m = 0$, this construction yields the *wired* boundary condition.

On $(\mathcal{G}_m^{(n)}, \tilde{W}_m^{(n)})$, the representation of the VRJP can be expressed using a potential $\beta_m^{(n)}$ with distribution $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$. As with the standard representation, and thanks to the choice of weights $\tilde{W}_m^{(n)}$, we can obtain such a potential thanks to the distribution ν_T^W . Indeed, under $\nu_T^W(d\beta)$, the restriction $\beta_{T^{(n)}}$ has distribution $\nu_{T^{(n)}}^{W^{(n)}, \eta^{(n)}}$, where $W^{(n)} = W_{T^{(n)}, T^{(n)}}$. According to Proposition 1.2.5, this can be extended into a potential $\beta_m^{(n)}$ with distribution $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$. This is done by taking $(\beta_m^{(n)})_{B_m}$ distributed according to $\nu_{B_m}^{\tilde{W}_m^{(n)}}$ (conditionally on $\beta_{T^{(n)}}$), where $\tilde{W}_m^{(n)}$ is a $|B_m| \times |B_m|$ weights matrix defined by

$$\tilde{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{B_m, T^{(n)}} (2\beta_{T^{(n)}} - W^{(n)})^{-1} (\tilde{W}_m^{(n)})_{T^{(n)}, B_m}.$$

Contrary to the standard case, we can not couple the potentials $\beta_m^{(n)}$ for all $n \geq m$ using an independent Gamma random variable. However, we will see that $(2\beta_m^{(n)})_{B_m} - \tilde{W}_m^{(n)}$ converges in distribution, which will be enough.

From the potential $\beta_m^{(n)}$ on $\mathcal{G}_m^{(n)}$, the random jump rates defining the representation are expressed through the Green function $G_m^{(n)} = (2\beta_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$. For $i, j \in T^{(n)}$, $G_m^{(n)}(i, j)$ can once again be

written as a sum over paths between i and j in $\mathcal{G}_m^{(n)}$. As with the standard representation, we separate this sum between paths that stay inside of $T^{(n)}$ (which sum up to $\hat{G}^{(n)}(i, j) = (2\beta_{T^{(n)}} - W^{(n)})^{-1}$), and paths that visit at least one boundary point. Since the boundary B_m can now consist of several points, we have to specify the first and last ones that are visited by the path. This gives the following decomposition:

$$G_m^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \sum_{b, b' \in B_m} \chi_m^{(n)}(i, b) G_m^{(n)}(b, b') \chi_m^{(n)}(j, b'),$$

where for $i \in T^{(n)}$, $b \in B_m$, $\chi_m^{(n)}(i, b)$ is the sum over paths that start from i , and stop when they hit the boundary B_m for the first time, at point b . More precisely, $\chi_m^{(n)} : T^{(n)} \times B_m \rightarrow \mathbb{R}_+$ is defined such that, for all $x \in D^{(m)}$,

$$\begin{cases} (H_\beta \chi_m^{(n)}(\cdot, \delta_x))_{T^{(n)}} = 0, \\ \chi_m^{(n)}(i, \delta_x) = 1 & \text{if } i \in T_x \setminus T_x^{(n)}, \\ \chi_m^{(n)}(i, \delta_x) = 0 & \text{if } i \in T \setminus (T_x \cup T^{(n)}). \end{cases}$$

Note that

$$\sum_{x \in D^{(m)}} \chi_m^{(n)}(i, \delta_x) = \psi^{(n)}(i) \quad (1.4)$$

for all $i \in T$, where $\psi^{(n)}$ is the martingale introduced in the case of the *wired* condition. The function $\chi_m^{(n)}$ does not satisfy a martingale property, however we are still able to show that it converges, which will provide the convergence of the representation associated with $G_m^{(n)}$.

Convergence of the representations

We want to show that $G_m^{(n)}$ converges when $n \rightarrow \infty$. From the construction of the standard representation, we already know that $\hat{G}^{(n)}$ and $\psi^{(n)}$ converge $\nu_T^W(d\beta)$ -almost surely.

If $\beta \in \mathcal{D}_T^W$ is such that $\psi \equiv 0$, then from (1.4), $\chi_m^{(n)}$ converges a.s. to 0. Otherwise, using the expression of $\chi_m^{(n)}(i, \delta_x)$ as a sum over paths, and the tree structure, we can show that it converges as well, to a function $\chi_m : T \times B_m \rightarrow \mathbb{R}_+$. Moreover χ_m is such that for all $b \in B_m$, $\chi_m(\cdot, b)$ is H_β -harmonic.

Let us come back to the Martin boundary associated with the random Markov operator Δ^ψ . As mentioned earlier, this Martin boundary is deterministic, and is the set Ω of ends of the tree, *i.e.* the set of infinite self avoiding paths (also called rays) starting at the root ϕ . For $i \in T$, we denote by Ω_i the set of rays that pass through i , so that they go to infinity in T_i . We also introduce the family of harmonic measures $(\mu_i^\psi)_{i \in T}$ on Ω , defined as follows. Since the random walk associated with Δ^ψ is transient, we can define its limit as a Ω -valued random variable, and μ_i^ψ is the distribution of this limit for the walk started at i . Moreover, for $A \in \mathcal{B}(\Omega)$, the function $i \mapsto \mu_i^\psi(A)$ is Δ^ψ -harmonic.

For a tree, the expression of the harmonic measure is explicit, and in the present case we have: for $x \in D^{(m)}$ and $i \in T$, $\chi_m(i, \delta_x) = \psi(i) \mu_i^\psi(\Omega_x)$. From now on, for all $i \in T$ we will denote by $\chi(i, \cdot) = \psi(i) \mu_i^\psi$, which defines a family of measures on Ω .

Finally, we have to show that $(G_m^{(n)})_{B_m, B_m}$ converges in distribution. Writing it as a Schur complement, we have

$$(G_m^{(n)})_{B_m, B_m} = ((2\beta_m^{(n)})_{B_m} - \check{W}_m^{(n)})^{-1},$$

where conditionally on $\beta_{T^{(n)}}$, $(\beta_m^{(n)})_{B_m}$ is distributed according to $\nu_{B_m}^{\check{W}_m^{(n)}}$. Moreover, $\check{W}_m^{(n)}$ has an expression as a sum over paths, so that for $x, y \in D^{(m)}$ with $x \neq y$, $(\check{W}_m^{(n)})_{\delta_x, \delta_y}$ can be written using $\hat{G}^{(n)}$ and $\chi_m^{(n)}$:

$$(\check{W}_m^{(n)})_{\delta_x, \delta_y} = \frac{\chi_m^{(n)}(x \wedge y, \delta_x) \chi_m^{(n)}(x \wedge y, \delta_y)}{\hat{G}^{(n)}(x \wedge y, x \wedge y)}.$$

Here $x \wedge y$ denotes the closest common ancestor of x and y , *i.e.* the vertex $i \in T$ with the largest depth such that $x, y \in T_i$. This expression shows that the off-diagonal terms of $\check{W}_m^{(n)}$ converge a.s., which is enough to show the convergence in distribution of $(G_m^{(n)})_{B_m, B_m}$. Indeed, let $\check{C}_m^{(n)}$ be the $|B_m| \times |B_m|$ matrix with zero diagonal coefficients, and equal to $\check{W}_m^{(n)}$ outside of the diagonal. Then

$$(G_m^{(n)})_{B_m, B_m} = ((2\rho_m^{(n)}) - \check{C}_m^{(n)})^{-1}$$

where $(\rho_m^{(n)})_b = (\beta_m^{(n)})_b - (\check{W}_m^{(n)})_{b,b}$ for $b \in B_m$. Conditionally on $\beta_{T^{(n)}}$, $\rho_m^{(n)}$ is distributed according to $\nu_{B_m}^{\check{C}_m^{(n)}}$, and therefore converges in distribution.

These results lead to the following theorem, which states that this construction yields a representation of the VRJP for all $m \geq 0$. The proof is the same as for Theorem 1.2.3, using the convergences presented above, and a uniform integrability argument.

Theorem 1.3.4. *Let $m \geq 0$ be fixed. For ν_T^W -almost all $\beta \in \mathcal{D}_T^W$, we define the $|B_m| \times |B_m|$ matrix \check{C}_m by*

$$(\check{C}_m)_{\delta_x, \delta_{x'}} = \begin{cases} 0 & \text{if } x = x', \\ \frac{\chi(x \wedge x', \Omega_x) \chi(x \wedge x', \Omega_{x'})}{\hat{G}(x \wedge x', x \wedge x')} & \text{otherwise.} \end{cases}$$

From now on, let us write: $\nu_{T, B_m}^W(d\beta, d\rho_m) = \nu_T^W(d\beta) \nu_{B_m}^{\check{C}_m}(d\rho_m)$.

Moreover for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, we define $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, and for $i, j \in T$,

$$G_m(i, j) = \hat{G}(i, j) + \sum_{b, b' \in B_m} \chi_m(i, b) \chi_m(j, b') \check{G}_m(b, b'),$$

Finally, denote by $\mathbb{P}^{W, \beta, \rho_m, i_0}$ the distribution of the Markov jump process where the jump rate from i to j is $\frac{W_{i,j}}{2} \frac{G_m(i_0, j)}{G_m(i_0, i)}$.

Then the time-changed VRJP with distribution $\tilde{\mathbb{P}}^{VRJP(W, i_0)}$ is a mixture of these Markov jump processes under $\nu_{T, B_m}^W(d\beta, d\rho_m)$, *i.e.*

$$\tilde{\mathbb{P}}^{VRJP(W, i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{W, \beta, \rho_m, i_0}[\cdot] \nu_{T, B_m}^W(d\beta, d\rho_m).$$

Increasing values of m correspond to richer and richer boundary conditions in the construction of these representations. In fact, when $m \rightarrow \infty$, we can show that the distribution of the random jump rates converge weakly, using a tightness argument. The limit is in fact the representation built using *free* boundary conditions, with independent jump rates for every edge.

Moreover, we prove that these representations can all be different. On a d -regular tree with $d \geq 3$, endowed with constant weights W large enough so that the VRJP is a.s. transient, the distributions of jump rates associated with G_m are all distinct for different values of m . The proof relies on observing the behavior of the quenched process at infinity, more precisely by comparing the associated harmonic measures.

Remark 1.3.2. We can rewrite the expression of G_m to involve an integral on the Martin boundary Ω : for $i, j \in T$,

$$G_m(i, j) = \hat{G}(i, j) + \int_{\Omega^2} \chi(i, d\omega) \chi(j, d\tau) \check{g}_m(\omega, \tau),$$

where $\check{g}_m(\omega, \tau) = \check{G}(\delta_x, \delta_y)$ if $\omega \in \Omega_x$ and $\tau \in \Omega_y$, for $x, y \in D^{(m)}$. Moreover, we can construct more distinct representations by choosing other boundary conditions based on simple partitions of the Martin

boundary Ω . This begs the following question: can all representations of the VRJP on an infinite tree be obtained through the choice of such a boundary condition (or as a limit of such representations)? Some monotonicity arguments, proved by Poudevigne in [Pou19], could help in addressing this issue. We plan to investigate this in future work.

1.4 The β potential and interacting Brownian motions

The distribution ν_V^W that appears in the study of the VRJP also relates to the Brownian motion. Indeed, the β field can be viewed as a multi-variate version of the Inverse Gaussian distribution, which was introduced as the distribution of the first time a drifted Brownian motion hits a certain level. In [SZ17], Sabot and Zeng showed that β is in fact the vector of hitting times for a family of interacting Brownian motions.

Another result related to the hitting time of the Brownian motion is Matsumoto and Yor's opposite drift theorem. They showed that a Brownian motion with negative drift can be represented as a mixture of Brownian motions with the opposite drift and an added corrective term. Using Sabot and Zeng's result for the β field, we can obtain a multi-dimensional version of this theorem as well.

1.4.1 Hitting times of interacting Brownian motions

Let us first present the following classical result concerning the hitting time of a drifted Brownian motion. Let $\theta > 0$ and $\eta \geq 0$ be fixed, and let B be a standard 1-dimensional Brownian motion. We define the drifted Brownian motion X by

$$X(t) = \theta + B(t) - \eta t$$

for $t \geq 0$, and denote by T^0 the first hitting time of 0 by $X(t)$. Then the distribution of T^0 has the following density for $t \in \mathbb{R}$:

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbb{1}_{\{t \geq 0\}}. \quad (1.5)$$

Moreover, the distribution of $(X(t))_{0 \leq t \leq T^0}$ conditionally on T^0 is that of a 3-dimensional Bessel bridge from θ to 0 on $[0, T^0]$.

For $\eta = 0$, *i.e.* if X is a Brownian motion with no drift, then (1.5) gives the distribution of $\frac{1}{2\gamma}$, where γ is a Gamma random variable with parameter $(\frac{1}{2}, \theta^2)$. If $\eta > 0$, (1.5) is the density of the Inverse Gaussian distribution with parameter $(\frac{\theta}{\eta}, \theta^2)$. This motivated the proof of a multi-dimensional version of this result in [SZ17]. For a finite graph $\mathcal{G} = (V, E)$ endowed with weights $(W_e)_{e \in E}$, a family $(X_i)_{i \in V}$ of Brownian motions with interacting drifts have the following hitting times of 0: $T_i^0 = \frac{1}{2\beta_i}$, where $(\beta_i)_{i \in V}$ is the potential introduced in the study of the VRJP.

Let us present the aforementioned theorem. Let $\mathcal{G} = (V, E)$ be a finite connected graph, endowed with conductances $(W_e)_{e \in E} \in (\mathbb{R}_+^*)^E$. For $i, j \in V$, we employ the usual notation: $W_{i,j} = W_{\{i,j\}}$ if $\{i, j\} \in E$, and $W_{i,j} = 0$ otherwise. Note that it is possible to have $W_{i,i} > 0$. For $\beta \in \mathbb{R}^V$, we still define the $|V| \times |V|$ matrix $H_\beta = 2\beta - W$, where β denotes the diagonal matrix with coefficients $(\beta_i)_{i \in V}$, as well as the set $\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, H_\beta > 0\}$. The following proposition introduces a generalization of the measure $\nu_V^{W,\eta}$ introduced in 1.2.5.

Proposition 1.4.1. *For all $\theta \in (\mathbb{R}_+^*)^V$ and $\eta \in (\mathbb{R}_+)^V$, the measure $\nu_V^{W,\theta,\eta}$ defined by*

$$\nu_V^{W,\theta,\eta}(d\beta) = \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle \theta, H_\beta \theta \rangle - \frac{1}{2}\langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \theta \rangle\right) \frac{\prod_{i \in V} \theta_i}{\sqrt{|H_\beta|}} d\beta$$

is a probability distribution on \mathcal{D}_V^W . Moreover, for all $i \in V$, the random variable $\frac{1}{2\beta_i - W_{i,i}}$ has Inverse Gaussian distribution with parameter $(\frac{\theta_i}{\eta_i + \sum_{j \neq i} W_{i,j} \theta_j}, \theta_i^2)$.

For $t \in (\mathbb{R}_+)^V$, we also define the matrix $K_t = \text{Id} - tW$, where t denotes the diagonal matrix with coefficients $(t_i)_{i \in V}$. Note that if $t \in (\mathbb{R}_+^*)^V$, we have $K_t = tH_{\frac{1}{2t}}$, where $\frac{1}{2t} = (\frac{1}{2t_i})_{i \in V}$. Finally, for $t \in (\mathbb{R}_+)^V$ and $T \in (\overline{\mathbb{R}_+})^V$, we define the vector $t \wedge T = (t_i \wedge T_i)_{i \in V}$.

The main result from [SZ17] is presented below. It states that a family of interacting Brownian motions, introduced as the solution of a system of stochastic differential equations, can be represented as a mixture of independent Bessel bridges. Their hitting times of 0 are determined by the β potential.

Theorem 1.4.1. *Let $\theta \in (\mathbb{R}_+^*)^V$ and $\eta \in (\mathbb{R}_+)^V$ be fixed, and let $(B_i(t))_{i \in V, t \geq 0}$ be a standard $|V|$ -dimensional Brownian motion.*

(i) *The following stochastic differential equation (SDE) has a unique pathwise solution:*

$$X_i(t) = \theta_i + \int_0^t \mathbf{1}_{s < T_i^0} dB_i(s) - \int_0^t \mathbf{1}_{s < T_i^0} ((W\psi)(s) + \eta)_i ds \quad (E_V^{W, \theta, \eta}(X))$$

for $i \in V$ and $t \geq 0$, where for $i \in V$, T_i^0 is the first hitting time of 0 by X_i , and for $t \geq 0$,

$$\psi(t) = K_{t \wedge T^0}^{-1}(X(t) + (t \wedge T^0)\eta).$$

(ii) *If $(X_i)_{i \in V}$ is solution of $(E_V^{W, \theta, \eta}(X))$, the vector $(\frac{1}{2T_i^0})_{i \in V}$ is distributed according to $\nu_V^{W, \theta, \eta}$. Moreover, conditionally on $(T_i^0)_{i \in V}$, the interrupted trajectories $(X_i(t))_{0 \leq t \leq T_i^0}$ are distributed as independent 3-dimensional Bessel bridges.*

The idea of the proof is to show that a mixture of Bessel bridges with such hitting times does in fact satisfy the SDE $(E_V^{W, \theta, \eta}(X))$. This is proved by identifying the Radon-Nikodym derivative (relative to a standard Brownian motion) of the mixture of Bessel bridges. It is an integral with respect to $\nu_V^{W, \theta, \eta}$, and the computation yields an exponential martingale. Applying Girsanov's theorem gives the desired outcome.

Remark 1.4.1. The function $\psi(t)$ appearing in $(E_V^{W, \theta, \eta}(X))$ is related to the martingale $\psi^{(n)}$ involved in the standard representation. Indeed, taking $t \rightarrow \infty$, $\psi(t)$ becomes $(H_\beta)^{-1}\eta$, which corresponds to $\psi^{(n)}$ in the case where $V = V_n$ and $\eta = \eta^{(n)}$, using the notations from the construction of the standard representation.

1.4.2 Opposite drift theorem and Lamperti's relation

The classical result on the hitting time of a Brownian motion is a particular case of a more general result concerning Bessel processes. For $\mu > 0$, if X is a Bessel process with index $-\mu$ started at 1, the first hitting time of 0 by X is $T_\mu^0 = \frac{1}{2\gamma_\mu}$, where γ_μ is a Gamma($\mu, 1$) variable. Moreover, conditionally on T_μ^0 , $(X_t)_{0 \leq t \leq T_\mu^0}$ is a Bessel bridge with index μ . The case $\mu = \frac{1}{2}$ corresponds to the initial result concerning the Brownian motion (with no drift), since the modulus of a Brownian motion is a Bessel process with index $-\frac{1}{2}$, and since a 3-dimensional Bessel process has index $\frac{1}{2}$.

In [YM01], Matsumoto and Yor showed a theorem relating Brownian motion with opposite drifts μ and $-\mu$, which is a consequence of the previous result on Bessel processes. Indeed, there exists a correspondence between a Bessel process with index μ and a Brownian motion with drift μ , thanks to a certain time change. This is called Lamperti's relation, and is stated below.

Proposition 1.4.2. *Let $(\rho(u))_{u \geq 0}$ be a drifted Brownian motion with drift $\mu \in \mathbb{R}$. For $u \geq 0$, define*

$$T(u) = \int_0^u \exp(2\rho(v)) dv.$$

Then there exists a Bessel process $(X(t))_{t \geq 0}$ with index μ , starting from 1, such that for $u \geq 0$,

$$e^{\rho(u)} = X(T(u)).$$

Matsumoto and Yor's theorem results from applying the time change from Lamperti's relation to a Bessel process with index $-\mu$ on one hand, and a Bessel bridge with index μ on the other hand.

Let us present a particular version of the opposite drift theorem, in the case where $\mu = \frac{1}{2}$, and the Bessel process with index $-\frac{1}{2}$ is replaced by a Brownian motion with a negative drift $-\eta$. In other words, it is obtained by applying Lamperti's time change to the initial result concerning drifted Brownian motion and 3-dimensional Bessel bridges.

Theorem 1.4.2. *Let $\theta > 0$ and $\eta \geq 0$ be fixed, and let B be a standard 1-dimensional Brownian motion. We define the process ρ as the solution of the following SDE:*

$$\rho(u) = \log(\theta) + B(u) - \frac{1}{2}u - \int_0^u \eta e^{\rho(v)} dv$$

for $u \geq 0$.

Let us define $T(u) = \int_0^u e^{2\rho(v)} dv$. Then:

(i) We have

$$T(u) \xrightarrow[u \rightarrow \infty]{a.s.} T^0,$$

where T^0 is distributed according to

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbf{1}_{\{t \geq 0\}} dt.$$

(ii) Conditionally on T^0 , there exists a standard 1-dimensional Brownian motion \hat{B} such that for $u \geq 0$,

$$\rho(u) = \log(\theta) + \hat{B}(u) + \frac{1}{2}u + \log\left(\frac{T^0 - T(u)}{T^0}\right).$$

Our aim is to prove a multi-dimensional version of this result thanks to Theorem 1.4.1, by using the time change from Lamperti's relation on solutions of $(E_V^{W, \theta, \eta}(X))$.

1.4.3 A multi-dimensional version of the opposite drift theorem

The result proved in [GSZ20] is the following.

Theorem 1.4.3. *Let $\mathcal{G} = (V, E)$ be a finite connected graph endowed with weights $(W_e)_{e \in E}$. We fix $\eta \in (\mathbb{R}_+)^V$ and $\theta \in (\mathbb{R}_+^*)^V$.*

Let \tilde{B} be a $|V|$ -dimensional standard Brownian motion, then the following system of SDEs has a unique pathwise solution, defined on \mathbb{R}_+ :

$$\begin{cases} \rho_i(u) = \log(\theta_i) + \tilde{B}_i(u) + \int_0^u \left(-\frac{1}{2} - e^{\rho_i(v)} \left(\tilde{W}^{(v)}(e^{\rho(v)} + T(v)\eta) + \eta\right)_i\right) dv, \\ T_i(u) = \int_0^u e^{2\rho_i(v)} dv, \end{cases} \quad (E_V^{W, \theta, \eta}(\rho))$$

for $i \in V$ and $u \geq 0$, where $e^{\rho(u)}$ denotes the vector $(e^{\rho_i(u)})_{i \in V}$, and

$$\tilde{W}^{(u)} = WK_{T(u)}^{-1} = W(\text{Id} - T(u)W)^{-1}.$$

Let (ρ, T) be the solution of $(E_V^{W, \theta, \eta}(\rho))$. Then

(i) For all $i \in V$, we have

$$T_i(u) = \int_0^u e^{2\rho_i(v)} dv \xrightarrow[u \rightarrow \infty]{\text{a.s.}} T_i^0,$$

where $\left(\frac{1}{2T_i^0}\right)_{i \in V}$ is distributed according to $\nu_V^{W, \theta, \eta}$.

(ii) Moreover, conditionally on T^0 , there exists a standard $|V|$ -dimensional Brownian motion \hat{B} such that for $i \in V$ and $u \geq 0$,

$$\rho_i(u) = \log(\theta_i) + \hat{B}_i(u) + \frac{1}{2}u + \log\left(\frac{T_i^0 - T_i(u)}{T_i^0}\right).$$

In particular, the processes $(\rho_i, T_i)_{i \in V}$ are independent conditionally on T^0 .

As mentioned before, we obtain this result by applying Lamperti's time-change to solutions of $(E_V^{W, \theta, \eta}(X))$, denoted by $(X_i(s))_{i \in V, s \geq 0}$, so that $\rho_i(v) = \log(X_i(T_i(v)))$ for $i \in V$ and $v \geq 0$. The main difficulty of the proof is to show that these $(\rho_i)_{i \in V}$ are indeed solutions of $(E_V^{W, \theta, \eta}(\rho))$. Since the time change increases at a different speed on each coordinate $i \in V$, the interaction term $(W\psi)(s)$ in $(E_V^{W, \theta, \eta}(X))$ does not immediately correspond to $\tilde{W}^{(u)}(e^{\rho(u)} + T(u)\eta)$ in $(E_V^{W, \theta, \eta}(\rho))$. We have two ways of overcoming this issue.

The first proof mirrors that of Theorem 1.4.1. Consider a mixture of independent Bessel bridges between 0 and T_i^0 , where $T_i^0 = \frac{1}{2\beta_i}$ under $\nu_V^{W, \theta, \eta}(d\beta)$. Apply to them Lamperti's time change and use Girsanov's theorem to compute their Radon-Nikodym derivative relative to the Brownian motion. Then integrate this with respect to $\nu_V^{W, \theta, \eta}$ to obtain the Radon-Nikodym derivative of the annealed process, and use Girsanov's theorem again to identify the corresponding SDE.

The second proof uses a form of strong Markov property satisfied by solutions $(X_i)_{i \in V}$ of $(E_V^{W, \theta, \eta}(X))$, showed in [SZ17]. It states that for any multi-stopping time $(T_i)_{i \in V}$ with respect to $(X_i)_{i \in V}$, the shifted process $(X_i(T_i + t))_{i \in V, t \geq 0}$ is solution of a similar SDE. This new equation has the same form as $(E_V^{W, \theta, \eta}(X))$, with modified parameters which depend on the trajectory of X_i up to time T_i , for all $i \in V$. We apply this Markov property to the time change $(T_i(u))_{i \in V}$ for all $u \geq 0$. This allows us to identify the infinitesimal generator of (ρ, T) at time u , which is in fact the generator associated with $(E_V^{W, \theta, \eta}(\rho))$.

Remark 1.4.2. In the case where V is a single point and E is empty, Theorem 1.4.3 gives the result of Theorem 1.4.2. For $\eta = 0$, this becomes the original Matsumoto-Yor opposite-drift theorem: a Brownian motion with drift $-\frac{1}{2}$ is written as a mixture of Brownian motions with the opposite drift $\frac{1}{2}$, and a corrective term involving an exponential functional of the trajectory.

The multi-dimensional opposite-drift theorem that we proved is limited to the case of drifts $-\frac{1}{2}$ and $\frac{1}{2}$, since it results from Theorem 1.4.1, which concerns Bessel processes with index $-\frac{1}{2}$ and $\frac{1}{2}$ (*i.e.* Brownian motion and 3-dimensional Bessel bridges). We could try and obtain a similar result for other values of the drift. This necessitates the use of a random potential analogous to β , whose marginals would relate to the hitting times of Bessel processes with other indices, *i.e.* generalized Inverse Gaussian variables. The case of index $\frac{3}{2}$ might be solvable, thanks to recent developments by Bauerschmidt, Crawford, Helmuth and Swan in [BCHS19], and by Crawford in [Cra19]. These articles concern other sigma models, in particular $\mathbb{H}^{2|4}$, which is related to random spanning forests, and could provide a generalization of the β potential corresponding to index $\frac{3}{2}$.

Chapter 2

Representations of the VRJP and Martin boundary

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Abstract

This chapter is modeled after [Ger19], and written to be as self-contained as possible. In it, we show that all possible representations of the VRJP as a mixture of Markov processes can be expressed in a similar form as in the standard representation, using the random field β and harmonic functions for an associated operator H_β . This allows for the proof that the VRJP on \mathbb{Z}^d (with certain initial conditions) has a unique representation, by showing that an associated Martin boundary is trivial. Moreover, on infinite trees, we construct a family of representations, that are all different when the VRJP is transient and the tree is d -regular (with $d \geq 3$).

In Section 2.1, we present past results on the VRJP and the β potential, as well as the main results of this chapter. Section 2.2 exposes some useful technical results concerning the β field, as well as basic definitions and properties of the Martin boundary. In Section 2.3, we prove how all representations of the VRJP have a common form, *i.e.* Theorem 2.1.3. We use these results in Section 2.4 to study the case of the graph \mathbb{Z}^d , and show Theorem 2.1.4 using a local limit theorem in random environment. In Sections 2.5 and 2.6, we study the β field on trees with our new boundary condition, and show the convergence of the associated Green function. We use this in Section 2.7, to show that this construction provides representations of the VRJP (Theorem 2.1.6), and that they are different in the case of a regular tree (Proposition 2.1.8).

2.1 Statement of the results

2.1.1 Previous results

Let $\mathcal{G} = (V, E)$ be a finite connected non-directed graph, endowed with conductances $(W_e)_{e \in E}$. We describe $(W_e)_{e \in E}$ with a matrix $(W_{i,j})_{i,j \in V}$, where

$$W_{i,j} = \begin{cases} W_{\{i,j\}} & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In [ST15], Sabot and Tarrès proved that the time-changed VRJP on \mathcal{G} with respect to W could be represented as a mixture of Markov processes, *i.e.* as a random walk in random environment. With Zeng, they showed in [STZ17] that this environment could be related to a random Schrödinger operator H_β , constructed from a random potential $\beta = (\beta_i)_{i \in V}$, in the following way.

For $\beta \in \mathbb{R}^V$, we will denote by $H_\beta = 2\beta - W$ the $|V| \times |V|$ symmetrical matrix such that $(H_\beta)_{i,j} = 2\beta_i \mathbf{1}_{i=j} - W_{i,j}$ for $i, j \in V$. Let us define the set $\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, H_\beta > 0\}$, where $H_\beta > 0$ means that the matrix H_β is positive definite. Note that if $\beta \in \mathcal{D}_V^W$, then $\beta_i > 0$ for all $i \in V$. The following proposition describes the probability distribution of the random potential that will be used to represent the VRJP.

Proposition 2.1.1. [Theorem 1 in [STZ17], Lemma 4 in [SZ19]]

(i) Let $\mathcal{G} = (V, E)$ be a finite connected graph, endowed with conductances W , and let $\eta \in \mathbb{R}_+^V$. We define by $\nu_V^{W,\eta}$ the measure on $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W))$ such that

$$\nu_V^{W,\eta}(d\beta) = \left(\frac{2}{\pi}\right)^{\frac{|V|}{2}} e^{-\frac{1}{2}(\langle \mathbf{1}, H_\beta \mathbf{1} \rangle + \langle \eta, (H_\beta)^{-1} \eta \rangle)} e^{\langle \eta, \mathbf{1} \rangle} \frac{\prod_{i \in V} d\beta_i}{\sqrt{\det(H_\beta)}}.$$

Then $\nu_V^{W,\eta}$ is a probability distribution. Its Laplace transform is

$$\int e^{-\langle \lambda, \beta \rangle} \nu_V^{W,\eta}(d\beta) = e^{-\sum_{i \in V} \eta_i (\sqrt{1+\lambda_i} - 1) - \sum_{i \sim j} W_{i,j} (\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1)} \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}},$$

for $\lambda \in \mathbb{R}_+^V$. When $\eta = 0$, we will write $\nu_V^W = \nu_V^{W,0}$.

(ii) Let us denote by d_G the graph distance in \mathcal{G} . Under $\nu_V^{W,\eta}(d\beta)$, if $V_1, V_2 \subset V$ are such that $d_G(V_1, V_2) \geq 2$, then $(\beta_i)_{i \in V_1}$ and $(\beta_j)_{j \in V_2}$ are independent. We will say that the potential with distribution ν_V^W is 1-dependent.

Let $\mathcal{C}_r(\mathbb{R}_+, V)$ be the space of right-continuous functions from \mathbb{R}_+ to V . This will be the space of trajectories of the random processes we will study in this paper. These processes will be described by probability distributions on $\mathcal{C}_r(\mathbb{R}_+, V)$.

We will use slightly different notations from the introductory chapter : let us denote by (Z_t) the canonical process in $\mathcal{C}_r(\mathbb{R}_+, V)$, where $Z_t(\omega) = \omega(t)$ for $\omega \in \mathcal{C}_r(\mathbb{R}_+, V)$. Moreover for $i_0 \in V$, let $\mathbb{P}^{VRJP(i_0)}$ denote the distribution of the time-changed VRJP on (\mathcal{G}, W) , in the exchangeable time scale described in the introduction. The following theorem describes how to represent this process as a mixture of Markov processes, using a random environment that can be constructed from the β field under $\nu_V^W(d\beta)$.

Theorem 2.1.1. [Theorem 2 in [ST15], Theorem 3 in [STZ17]] Let $\mathcal{G} = (V, E)$ be a finite graph, endowed with conductances W . We fix a vertex $i_0 \in V$. For $\beta \in \mathcal{D}_V^W$, we denote by $G = (H_\beta)^{-1}$ the Green function associated with β , and by $\mathbb{P}_x^{\beta, i_0}$ the distribution of the Markov jump process started at $x \in V$, with jump rate from i to j given by $\frac{1}{2}W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$.

Then for all $i_0 \in V$, the time-changed VRJP on (\mathcal{G}, W) , started at i_0 , is a mixture of these Markov jump processes under the distribution $\nu_V^W(d\beta)$. In other words,

$$\mathbb{P}^{VRJP(W, i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, i_0}[\cdot] \nu_V^W(d\beta).$$

An interesting property of the distribution ν_V^W is its behavior with respect to restriction. For $\beta \in \mathbb{R}^V$ and $V_1, V_2 \subset V$, let us denote $\beta_{V_1} = (\beta_i)_{i \in V_1}$, and $W_{V_1, V_2} = (W_{i,j})_{i \in V_1, j \in V_2}$.

Proposition 2.1.2. [Lemma 4 in [SZ19]] Let us fix $U \subset V$, and set $\hat{\eta}_i = \sum_{j \in U^c} W_{i,j}$ for $i \in U$, i.e. $\hat{\eta} = W_{U, U^c} \mathbb{1}_{U^c}$. Then under $\nu_V^W(d\beta)$, β_U is distributed according to $\nu_U^{W_{U, U^c}, \hat{\eta}}$.

Hence under $\nu_V^W(d\beta)$, the distribution of β_U depends only on the weights of edges inside U , and coming out of U . This is useful to define the β field on infinite graphs.

Let now $\mathcal{G} = (V, E)$ be an infinite connected non-directed graph, that is locally finite, i.e. each vertex $v \in V$ has finite degree. We endow \mathcal{G} with conductances W . To study the associated VRJP, we want to define an analogue of the β field on \mathcal{G} . In [SZ19], Sabot and Zeng did this by using a *wired* boundary condition, defined as follows.

Let $(V_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite connected subsets of V , such that

$$\bigcup_{n \in \mathbb{N}} V_n = V.$$

For $n \in \mathbb{N}$, we introduce a new vertex δ_n , and define a new graph $\mathcal{G}^{(n)} = (\tilde{V}^{(n)}, \tilde{E}^{(n)})$, where

$$\begin{aligned} \tilde{V}^{(n)} &= V_n \cup \{\delta_n\} \\ \text{and } \tilde{E}^{(n)} &= \{\{i, j\} \in E, i, j \in V_n\} \cup \{\{i, \delta_n\}, i \in V_n \text{ and } \exists j \notin V_n, i \sim j\}. \end{aligned}$$

The graph $\mathcal{G}^{(n)}$ is called the restriction of \mathcal{G} to V_n with *wired* boundary condition. We endow this graph with the conductances $\tilde{W}^{(n)}$ defined by $\tilde{W}_{i,j}^{(n)} = W_{i,j}$ if $i, j \in V_n$, and $\tilde{W}_{i, \delta_n}^{(n)} = \sum_{j \sim i, j \notin V_n} W_{i,j}$.

For all $n \in \mathbb{N}$, let $(\beta_i^{(n)})_{i \in \tilde{V}^{(n)}}$ be a random potential on the graph $\mathcal{G}^{(n)}$ distributed according to $\nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$. Then from Proposition 2.1.2, we know that the restriction $\beta_{V_n}^{(n)}$ is distributed according to $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$,

where $W^{(n)} = W_{V_n, V_n}$ and $\eta^{(n)} := \tilde{W}_{V_n, \{\delta_n\}}^{(n)} = W_{V_n, V_n^c} \mathbf{1}_{V_n^c}$. In fact, for a fixed $n \in \mathbb{N}$ and any $n' \geq n$, the restrictions $\beta_{V_n}^{(n')}$ have the same distribution $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$. By Kolmogorov extension theorem, this allows the construction of a distribution ν_V^W for infinite V .

For $\beta \in \mathbb{R}^V$, let us still denote by $H_\beta = 2\beta - W$ the Schrödinger operator associated with $(\beta_i)_{i \in V}$, i.e. for all $f \in \mathbb{R}^V$ and $i \in V$, $(H_\beta f)_i = 2\beta_i f_i - \sum_{j \sim i} W_{i,j} f_j$. We also define

$$\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, (H_\beta)_{U,U} > 0 \text{ for all finite subset } U \text{ of } V\}.$$

Proposition 2.1.3. [Proposition 1 in [SZ19]] *Let $\mathcal{G} = (V, E)$ be an infinite locally finite graph, endowed with conductances W . There exists a unique probability distribution ν_V^W on \mathcal{D}_V^W such that under $\nu_V^W(d\beta)$, for all finite subset $U \subset V$, $\beta_U \sim \nu_U^{W_{U,U}, \eta}$ where $\eta = W_{U, U^c} \mathbf{1}_{U^c}$. Its Laplace transform is*

$$\int e^{-\langle \lambda, \beta \rangle} \nu_V^W(d\beta) = e^{-\sum_{i \sim j} W_{i,j} (\sqrt{1+\lambda_i} \sqrt{1+\lambda_j} - 1)} \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}}$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

The wired boundary condition is not only useful to define ν_V^W on infinite graphs, but also to link this distribution to representations of the VRJP, by applying Theorem 2.1.1 to the graph $\mathcal{G}^{(n)}$. Indeed from Proposition 2.1.3, for any $n \in \mathbb{N}$, under $\nu_V^W(d\beta)$ we have $\beta_{V_n} \sim \nu_{V_n}^{W^{(n)}, \eta^{(n)}}$. Hence, from Proposition 2.1.2, we can extend β_{V_n} into a potential $\beta^{(n)} \sim \nu_{V^{(n)}}^{\tilde{W}^{(n)}}$ such that $\beta_{V_n}^{(n)} = \beta_{V_n}$. We denote $H_\beta^{(n)} = 2\beta^{(n)} - \tilde{W}^{(n)}$ and $G^{(n)} = (H_\beta^{(n)})^{-1}$. From Theorem 2.1.1, we know that $G^{(n)}$ gives a representation of the VRJP on $\mathcal{G}^{(n)}$.

Definition 2.1.1. (i) For $\beta \in \mathcal{D}_V^W$, let us define $\hat{G}^{(n)} : V \times V \rightarrow \mathbb{R}_+$ by $(\hat{G}^{(n)})_{V_n, V_n} = ((H_\beta)_{V_n, V_n})^{-1}$, and $\hat{G}^{(n)}(i, j) = 0$ if $i \notin V_n$ or $j \notin V_n$.

(ii) For $\beta \in \mathcal{D}_V^W$, let $\psi^{(n)} \in \mathbb{R}_+^{V_n}$ be defined by

$$\begin{cases} (H_\beta \psi^{(n)})_{V_n} = 0 \\ \psi_{V_n^c}^{(n)} = 1. \end{cases}$$

Note that $\psi_{V_n}^{(n)} = (\hat{G}_{V_n, V_n}^{(n)}) \eta^{(n)}$.

It is possible, using a decomposition of the Green function as a sum over paths (see [SZ19], or Proposition 2.2.3), to write

$$G^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \psi^{(n)}(i) G^{(n)}(\delta_n, \delta_n) \psi^{(n)}(j)$$

for $i, j \in V_n$. Under $\nu_V^W(d\beta)$, $G^{(n)}(\delta_n, \delta_n)$ is independent of β_{V_n} , and $1/2G^{(n)}(\delta_n, \delta_n)$ is always distributed according to a Gamma(1/2, 1) distribution (see Proposition 2.2.1 (ii)). The following theorem describes how taking $n \rightarrow \infty$ in this previous expression gives a representation of the VRJP on infinite graphs.

Theorem 2.1.2. [Theorem 1 in [SZ19]]

(i) Under $\nu_V^W(d\beta)$, for $i, j \in V$, the increasing sequence $\hat{G}^{(n)}(i, j)$ converges almost surely to a finite random variable $\hat{G}(i, j)$.

(ii) Let \mathcal{F}_n be the σ -field generated by β_{V_n} . Then under $\nu_V^W(d\beta)$, for all $i \in V$, $\psi^{(n)}(i)$ is a nonnegative (\mathcal{F}_n) -martingale which converges almost surely to an integrable random variable $\psi(i)$. Moreover, ψ is H_β -harmonic on V , i.e. $H_\beta \psi(i) = 2\beta_i \psi(i) - \sum_{j \sim i} W_{i,j} \psi(j) = 0$ for $i \in V$.

(iii) From now on, we will denote $\nu_V^W(d\beta, d\gamma) = \nu_V^W(d\beta) \otimes \frac{\mathbf{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$, where $\frac{\mathbf{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$ is a Gamma(1/2, 1) distribution.

Let now $i_0 \in V$ be fixed. For $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, we define

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma} \psi(i)\psi(j),$$

and denote by $\mathbb{P}_x^{\beta, \gamma, i_0}$ the distribution of the Markov jump process started at $x \in V$, where the jump rate from i to j is $\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$.

Then the time-changed VRJP on (\mathcal{G}, W) , started at i_0 , is a mixture of these Markov jump processes under $\nu_V^W(d\beta, d\gamma)$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, \gamma, i_0}[\cdot] \nu_V^W(d\beta, d\gamma).$$

(iv) For ν_V^W -almost all β and all $i_0 \in V$, we have:

- The Markov process with law $\mathbb{P}_x^{\beta, \gamma, i_0}$ is recurrent if and only if $\psi(i) = 0$ for all $i \in V$.
- The Markov process with law $\mathbb{P}_x^{\beta, \gamma, i_0}$ is transient if and only if $\psi(i) > 0$ for all $i \in V$.

Note that for $i_0 \in V$ fixed, in this representation of the VRJP started at i_0 , the β field cannot be expressed as a function of the random jump rates $\frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}$ that define the environment. However, we can define the $\tilde{\beta}$ field rooted at i_0 , where $\tilde{\beta}_i$ is the rate of the exponential holding time at i for the associated Markov process.

Proposition 2.1.4. For all $i \in V$, $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, we define

$$\tilde{\beta}_i = \sum_{j \sim i} \frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)} = \beta_i - \mathbf{1}_{\{i=i_0\}} \frac{1}{2G(i_0, i_0)}.$$

Then under $\nu_V^W(d\beta, d\gamma)$, $1/2G(i_0, i_0)$ has distribution Gamma(1/2, 1) and is independent from $\tilde{\beta}$. Moreover the Laplace transform of $\tilde{\beta}$ is

$$\int e^{-\langle \lambda, \tilde{\beta} \rangle} \nu_V^W(d\beta, d\gamma) = e^{-\sum_{i \sim j} W_{i,j} (\sqrt{1+\lambda_i} \sqrt{1+\lambda_j} - 1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1+\lambda_i}}$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

2.1.2 A common form for all representations

We still consider $\mathcal{G} = (V, E)$ to be an infinite connected graph, locally finite and endowed with conductances $(W_{i,j})_{i,j \in V}$. Thanks to Theorem 2.1.2, we already know that the time-changed VRJP with distribution $\mathbb{P}^{VRJP(i_0)}$ can be written as a mixture of Markov jump processes, using the distribution ν_V^W . We will refer to this as the standard representation. We are now interested in other possible random environments, that would represent the VRJP in the same sense, and whether they can be expressed in a form similar to the standard representation.

We will denote by \mathcal{J}_V^E the set of jump rates on \mathcal{G} , i.e. the set of $(r_{i,j})_{i,j \in V} \in \mathbb{R}_+^{V \times V}$ such that $r_{i,j} = 0$ if $\{i, j\} \notin E$.

Definition 2.1.2. Let $\mathcal{R}(dr)$ be a probability distribution on \mathcal{J}_V^E . For $i_0 \in V$ fixed, we will say that $\mathcal{R}(dr)$ is the distribution of a random environment representing the time-changed VRJP started at i_0 if

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^r[\cdot] \mathcal{R}(dr),$$

where for $r \in \mathcal{J}_V^E$, \mathbb{P}^r is the distribution of the Markov jump process with jump rate from i to j given by $r_{i,j}$. We will also say that $\mathcal{R}(dr)$ defines a representation of $\mathbb{P}^{VRJP(i_0)}$.

The following result tells us that in fact, any representation of the VRJP can be expressed in a similar form as the standard representation, using a β field as well as H_β -harmonic functions.

For $i \in V$ and $r \in \mathcal{J}_V^E$, we define $r_i = \sum_{j \sim i} r_{i,j}$.

Theorem 2.1.3. Let $i_0 \in V$ be fixed, and let $\mathcal{R}(dr)$ be the distribution of a random environment representing the time-changed VRJP with law $\mathbb{P}^{VRJP(i_0)}$. We write $\mathcal{R}(dr, d\gamma) = \mathcal{R}(dr) \otimes \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we define $\beta \in (\mathbb{R}_+)^V$ by $\beta_i = r_i + \mathbb{1}_{\{i=i_0\}}\gamma$ for $i \in V$. Then under $\mathcal{R}(dr, d\gamma)$, $\beta \sim \nu_V^W$, and there exists a random H_β -harmonic function $h : V \rightarrow \mathbb{R}_+$, such that for all $i \sim j$,

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)},$$

where $G(i_0, i) = \hat{G}(i_0, i) + h(i)$ for $i \in V$, and \hat{G} is the function of β defined in Theorem 2.1.2.

In order to try and classify all representations of the VRJP, we now need to identify H_β -harmonic functions, and to determine which ones can appear in the expression of a representation, as in Theorem 2.1.3. Two interesting cases arise, depending on (\mathcal{G}, W) : when the VRJP is almost surely recurrent, or almost surely transient.

In the first case, we can use the law of large numbers to show that the representation of the VRJP as a mixture of Markov processes is unique.

Proposition 2.1.5. If (\mathcal{G}, W) is such that the VRJP is almost surely recurrent, then the representation of the time-changed VRJP started at i_0 as a mixture of Markov processes is unique, i.e. if $\mathcal{R}(dr)$ and $\mathcal{R}'(dr)$ define representations of $\mathbb{P}^{VRJP(i_0)}$, then $\mathcal{R}(dr) = \mathcal{R}'(dr)$.

Note that in this case, according to Theorem 2.1.2 (iv), under $\nu_V^W(d\beta)$, we have a.s. $\psi(i) = 0$ for all $i \in V$, and the jump rates in the standard representation are given by $\frac{W_{i,j}}{2} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i)}$. Therefore, the H_β -harmonic function associated with the representation (by Theorem 2.1.3) is $h \equiv 0$.

In the second case, i.e. when the VRJP is almost surely transient, we can introduce a random conductance model, associated with ψ .

Proposition 2.1.6. If (\mathcal{G}, W) is such that the VRJP is almost surely transient, then under $\nu_V^W(d\beta)$:

- (i) We have a.s. $\psi(i) > 0$ for all $i \in V$, where ψ is defined in Theorem 2.1.2.
- (ii) We define the random conductances $c_{i,j}^\psi = W_{i,j}\psi(i)\psi(j)$ for all $i, j \in V$. Then the associated reversible random walk is a.s. transient.
- (iii) Let Δ^ψ be the discrete Laplacian associated with the random conductances $c_{i,j}^\psi$. Then a function $\varphi : V \rightarrow \mathbb{R}$ is Δ^ψ -harmonic if and only if $i \mapsto \psi(i)\varphi(i)$ is H_β -harmonic.

Remark 2.1.1. This allows a more convenient expression of representation in the transient case. Indeed, if $\mathcal{R}(dr)$ defines a representation of $\mathbb{P}^{VRJP(i_0)}$, Theorem 2.1.3 allows us to construct a β field distributed according to ν_V^W , and to express the jump rates $r_{i,j}$ using β and a H_β -harmonic function h . According to Proposition 2.1.6 (iii), we have $h = \psi\varphi$, where φ is a Δ^ψ -harmonic function, *i.e.* harmonic for a transient random walk.

The notion of Martin boundary is a useful tool to represent harmonic functions with respect to a transient random walk on a graph $\mathcal{G} = (V, E)$. Indeed, V admits a boundary \mathcal{M} so that $V \cup \mathcal{M}$ is compact for a certain topology, and there is a kernel $K : V \times \mathcal{M}$ so that any positive harmonic function h can be written as

$$h(x) = \int_{\mathcal{M}} K(x, \alpha) \mu^h(d\alpha)$$

for $x \in V$, where μ^h is a positive measure on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. \mathcal{M} is called the Martin boundary of V with respect to the random walk, and K is the Martin kernel, which is defined using the Green kernel associated with the random walk. For more details on Martin boundaries, see Section 2.2.3.

In order to study representations of the VRJP in the transient case, we want to describe Δ^ψ -harmonic functions, according to Remark 2.1.1. We will therefore need to identify the Martin boundary \mathcal{M}^ψ associated with Δ^ψ . This will be possible when \mathcal{G} is \mathbb{Z}^d , or an infinite tree.

2.1.3 Main results

2.1.3.1 Representations of the VRJP on \mathbb{Z}^d

Let us consider the case where \mathcal{G} is the lattice \mathbb{Z}^d , *i.e.* $\mathcal{G} = (V, E)$ with

$$V = \mathbb{Z}^d \text{ and } E = E_d := \{\{x, y\}, |x - y| = 1\}$$

where $|x|$ is the Euclidean norm of x . Let us endow \mathcal{G} with constant initial conductances W . Since this model is invariant by isometries of \mathbb{Z}^d , we will only consider the VRJP started at 0.

We can identify several situations in which the representation is unique. For $d = 2$, or if W is small enough, the VRJP is almost surely recurrent (see [Sab19], and Corollary 1 in [ST15]), so that the representation of $\mathbb{P}^{VRJP(0)}$ is unique according to Proposition 2.1.5. For $d \geq 3$ and W large enough, the VRJP is almost surely transient (see Corollary 3 in [ST15]), hence we can introduce Δ^ψ defined in Proposition 2.1.6. Since (\mathcal{G}, W) is vertex transitive, from Proposition 3 of [SZ19], under $\nu_V^W(d\beta)$, ψ is stationary and ergodic. This allows us to apply a local limit theorem for random walks in random conductances (from [ADS16a]), and show that the Martin boundary \mathcal{M}^ψ associated with Δ^ψ is almost surely trivial for W large enough. These cases are regrouped in the following result.

Theorem 2.1.4. *Let \mathcal{G} be the \mathbb{Z}^d lattice, endowed with constant edge weights, *i.e.* $W_{i,j} = W > 0$ for all $i \sim j$. We consider representations of $\mathbb{P}^{VRJP(0)}$ as a mixture of Markov processes.*

Then:

- *If $d \in \{1, 2\}$, there is a unique representation of $\mathbb{P}^{VRJP(0)}$.*
- *If $d \geq 3$, there are constants \underline{W} and \overline{W} such that for $0 < W < \underline{W}$ or for $W > \overline{W}$, there is a unique representation of $\mathbb{P}^{VRJP(0)}$.*

2.1.3.2 A family of representations on infinite trees

Let us now consider the case where the graph is an infinite tree $\mathcal{T} = (T, E)$, that we assume to be locally finite, and endow with conductances W . In [CZ16], Chen and Zeng described a representation of the

time-changed VRJP with a different expression than the standard representation. Indeed, if $(T_n)_{n \in \mathbb{N}}$ is an increasing and exhausting sequence of finite connected subsets of T , the subgraphs $\mathcal{T}^{(n)} = (V_n, E_n)$ of \mathcal{G} are finite trees (where $E_n = \{\{i, j\} \in E, i, j \in V_n\}$). These are called restrictions of \mathcal{G} with *free* boundary conditions.

Moreover, on finite trees, Theorem 2.1.1 gives a representation of the VRJP where jump rates are independent. Therefore, a representation of the VRJP on \mathcal{T} can be obtained from representations on $\mathcal{T}^{(n)}$, using independent jump rates.

Theorem 2.1.5. [Theorem 3 in [CZ16]] *Let ϕ be an arbitrary root for \mathcal{T} . For all $i \in T \setminus \{\phi\}$, we denote by \tilde{i} the parent of i . Let also $(A_i)_{i \in T \setminus \{\phi\}}$ be independent random variables where A_i is an inverse Gaussian random variable with parameter $(W_{\tilde{i}, i}^-, 1)$, i.e.*

$$\mathbb{P}[A_i \in ds] = \mathbf{1}_{s \geq 0} \sqrt{\frac{W_{\tilde{i}, i}^-}{2\pi s^3}} e^{-W_{\tilde{i}, i}^- \frac{(s-1)^2}{2s}} ds.$$

Then the process with law $\mathbb{P}^{VRJP(\phi)}$ on \mathcal{T} is a mixture of Markov jump processes, in which the jump rate from \tilde{i} to i is $\frac{1}{2} W_{\tilde{i}, i}^- A_i$, and the jump rate from i to \tilde{i} is $\frac{1}{2} \frac{W_{\tilde{i}, i}^-}{A_i}$, for all $i \in T \setminus \{\phi\}$.

In some cases, this representation is different from the standard representation.

Proposition 2.1.7. *Let $\mathcal{T} = (T, E)$ be an infinite d -regular tree with $d \geq 3$, i.e. such that each vertex in T has exactly d neighbors. We endow \mathcal{T} with constant conductances W . Then for W large enough, the distribution of the random environment described in Theorem 2.1.5 is different from the distribution of the standard representation.*

We now know two ways of constructing a representation of the VRJP on \mathcal{T} , that are associated with different boundary conditions on restrictions to finite graphs, and can have distinct distributions. This leads us to introduce new boundary conditions in order to construct a family of different representations of the VRJP, following the same method as for the standard representation.

Let us start by giving a few notations on trees. For all $x, y \in T$, we denote by $d(x, y)$ the graph distance between x and y , and by $[x, y]$ the unique shortest path between x and y :

$$[x, y] = (x = [x, y]_0, [x, y]_1, \dots, [x, y]_{d(x, y)-1}, y = [x, y]_{d(x, y)}).$$

Note that any path σ from x to y necessarily crosses all vertices $[x, y]_k$ for $0 \leq k \leq d(x, y)$.

Let us fix an arbitrary root ϕ in \mathcal{T} . Then, for all $x \in T$, we denote by $|x| = d(\phi, x)$ the depth of the vertex $x \in T$. If $x \neq \phi$, we also denote by $\tilde{x} = [\phi, x]_{|x|-1}$ the parent of x . Finally, for any $x \in T$, we define the set $S(x) = \{y \in T, x = \tilde{y}\}$ of x 's children, and the set $T_x = \{y \in T, \exists k \geq 0, [\phi, y]_k = x\}$ of its descendants.

For $x, y \in T$, we denote by $x \wedge y$ the "closest common ancestor" of x and y , i.e. $x \wedge y = [\phi, x]_{N_{x, y}}$ where $N_{x, y} = \max\{k \geq 0, [\phi, x]_k = [\phi, y]_k\}$. Note that we also have $x \wedge y = [x, y]_{k_0}$, where k_0 is such that $|[x, y]_{k_0}| = \min\{|[x, y]_k|, 0 \leq k \leq d(x, y)\}$.

For $n \in \mathbb{N}$, we denote by $D^{(n)} = \{x \in T, |x| = n\}$ the tree's n th generation. Let us then define $T^{(n)} = \bigcup_{0 \leq k \leq n} D^{(k)}$, as well as $E^{(n)} = \{\{i, j\} \in E, i, j \in T^{(n)}\}$. We also denote $T_x^{(n)} = T_x \cap T^{(n)}$ for $x \in T$ and $\bar{n} \geq |x|$. The restriction of the tree to the first n generations, with *free* boundary conditions, is the graph $(T^{(n)}, E^{(n)})$, that we endow with the induced conductances $W^{(n)} = W_{T^{(n)}, T^{(n)}}$.

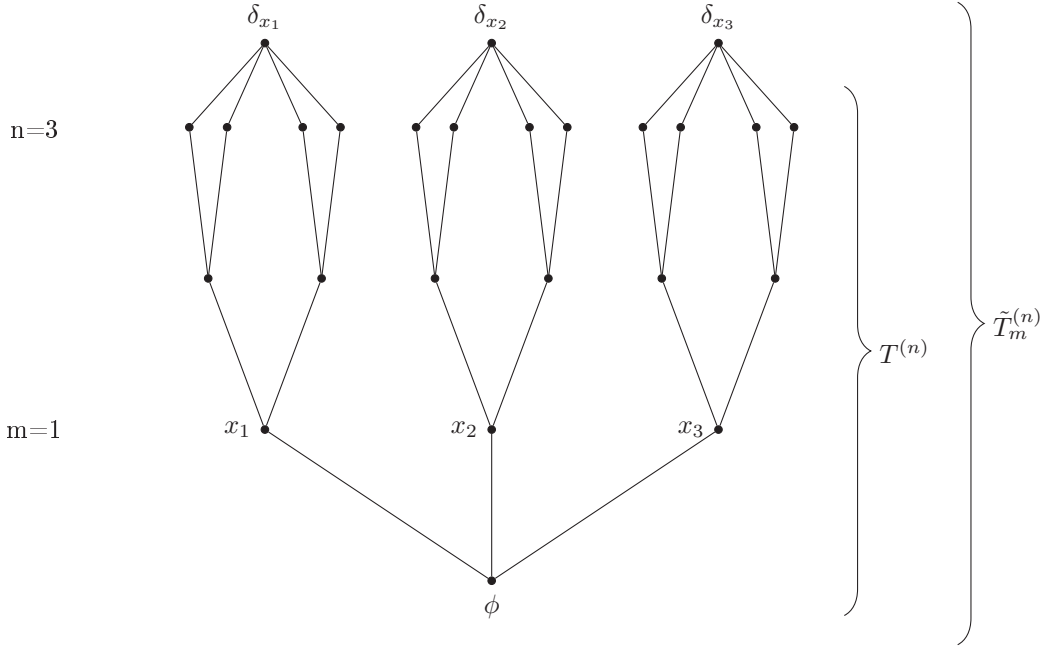
Finally, we define the set Ω of ends of \mathcal{T} , i.e. the set of infinite self-avoiding paths (or rays) in \mathcal{T} starting at ϕ . For $x \in T$, we denote by Ω_x the subset of Ω corresponding to the branch T_x , i.e. the set of rays in \mathcal{T} that cross x . Note that the Martin boundary associated with a transient walk on a tree is always Ω , which depends only on the geometry of the tree. This will be convenient to express Δ^ψ -harmonic functions, where Δ^ψ is the random Laplace operator introduced in Proposition 2.1.6.

In the construction of the standard representation, the *wired* boundary condition was defined by adding a single boundary point δ to a finite graph, where δ could be interpreted as a point at infinity for the graph. We will now introduce a variant of this boundary condition, by adding multiple boundary points, each being a point at infinity for a different branch of the tree.

Let us first fix a generation $m \geq 0$, and to each vertex $x \in D^{(m)}$, we associate a boundary point δ_x , that will be the point at infinity for T_x . We denote by $B_m = \{\delta_x, x \in D^{(m)}\}$ the boundary set associated to this generation. For all $n \geq m$, let us then define the graph

$$\mathcal{G}_m^{(n)} = \left(\tilde{T}_m^{(n)}, \tilde{E}_m^{(n)} \right), \text{ where } \tilde{T}_m^{(n)} = T^{(n)} \cup B_m$$

$$\text{and } \tilde{E}_m^{(n)} = E^{(n)} \cup \bigcup_{x \in D^{(m)}} \{ \{y, \delta_x\}, y \in T_x \cap D^{(n)} \}.$$



This graph is the restriction of T to $T^{(n)}$ with a variant of the *wired* boundary condition. Note that we get the standard *wired* boundary condition by taking $m = 0$. We endow $\mathcal{G}_m^{(n)}$ with the conductances $\tilde{W}_m^{(n)}$, defined for $e \in \tilde{E}^{(n)}$ by

$$\left(\tilde{W}_m^{(n)} \right)_e = \begin{cases} W_e^{(n)} = W_e & \text{if } e \in E^{(n)} \\ \sum_{j \in S(i)} W_{i,j} & \text{if } e = \{i, \delta_x\}, \text{ where } i \in T_x \cap D^{(n)}. \end{cases}$$

As with the *wired* boundary condition, these weights are defined so that for $n \geq m$, the weights coming out of $T^{(n)}$ are given by $W_{T^{(n)}, (T^{(n)})^c} \mathbf{1}_{(T^{(n)})^c} = \eta^{(n)}$. This will allow for the compatibility of $\beta_m^{(n)}$ fields defined on $\mathcal{G}_m^{(n)}$ for $n \geq m$. Note that these weights do not depend on m , *i.e.* do not depend on the choice of the boundary condition.

From Proposition 2.1.3, under $\nu_T^W(d\beta)$, for all $n \geq m$ we have $\beta_{T^{(n)}} \sim \nu_{T^{(n)}}^{W^{(n)}, \eta^{(n)}}$. Hence, from Proposition 2.1.2, we can extend $\beta_{T^{(n)}}$ into a potential $\beta_m^{(n)} \sim \nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$ on $\mathcal{G}_m^{(n)}$ such that $(\beta_m^{(n)})_{T^{(n)}} = \beta_{T^{(n)}}$.

Let us then denote by $G_m^{(n)} = (2\beta_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$ the Green function associated with $\beta_m^{(n)}$. From Theorem 2.1.1, we know how to represent the time-changed VRJP on $\mathcal{G}_m^{(n)}$ using $G_m^{(n)}$. In order to obtain a result on the infinite tree \mathcal{T} , we will see that $G_m^{(n)}$ converges when $n \rightarrow \infty$.

For $\beta \in \mathcal{D}_T^W$, we still define $H_\beta = 2\beta - W$ and take $V_n = T^{(n)}$ for all $n \in \mathbb{N}$ in Definition 2.1.1, we get $\hat{G}^{(n)} = ((H_\beta)_{T^{(n)}, T^{(n)}})^{-1}$ and $\psi^{(n)} = \hat{G}^{(n)}\eta^{(n)}$, which converge ν_V^W -a.s. to \hat{G} and ψ respectively, according to Theorem 2.1.2.

Definition 2.1.3. For $n \geq m \geq 0$, let $\chi_m^{(n)} \in \mathbb{R}_+^{T^{(n)} \times B_m}$ be defined by

$$\begin{cases} (H_\beta \chi_m^{(n)}(\cdot, \delta_x))_{T^{(n)}} = 0, \\ \chi_m^{(n)}(i, \delta_x) = 1 & \text{if } i \in T_x \setminus T_x^{(n)}, \\ \chi_m^{(n)}(i, \delta_x) = 0 & \text{if } i \in T \setminus (T_x \cup T^{(n)}). \end{cases}$$

for $x \in D^{(m)}$. Note that $(\chi_m^{(n)}(\cdot, \delta_x))_{T^{(n)}} = (\hat{G}^{(n)})_{T^{(n)}, T^{(n)}} W_{T^{(n)}, (T^{(n)})^c} \mathbf{1}_{T_x \setminus T_x^{(n)}}$.

Remark 2.1.2. For $n \geq m$, $\chi_m^{(n)}$ is $\beta_{T^{(n)}}$ -measurable, and for $x \in D^{(m)}$ and $y \in T^{(n)}$,

$$\begin{aligned} \sum_{b \in B_m} \chi_m^{(n)}(y, b) &= \sum_{x \in D^{(m)}} \chi_m^{(n)}(y, \delta_x) = \hat{G}^{(n)}(y, \cdot) W_{T^{(n)}, (T^{(n)})^c} \mathbf{1}_{(T^{(n)})^c} \\ &= \hat{G}^{(n)}(y, \cdot) \eta^{(n)} = \psi^{(n)}(y). \end{aligned}$$

It is possible to decompose $G_m^{(n)}$ as a sum over paths in $\mathcal{G}_m^{(n)}$, which gives, for $i, j \in T^{(n)}$,

$$G_m^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \sum_{x, x' \in D^{(m)}} \chi_m^{(n)}(i, \delta_x) G_m^{(n)}(\delta_x, \delta_{x'}) \chi_m^{(n)}(j, \delta_{x'}).$$

Once again, we will study the limit of this expression when $n \rightarrow \infty$, to obtain a representation of the VRJP on (\mathcal{T}, W) . However under $\nu_T^W(d\beta)$, contrary to $\psi^{(n)}$, $\chi_m^{(n)}(\cdot, \delta_x)$ is not a martingale when $m \neq 0$. Moreover, the term $(G_m^{(n)})_{B_m, B_m}$ is not independent of $\beta_{T^{(n)}}$ for $m \neq 0$. Therefore, we cannot use the same argument as in the proof of Theorem 2.1.2.

As with ψ , we expect $\chi_m^{(n)}(\cdot, \delta_x)$ to converge to a H_β -harmonic function on T , for all $x \in D^{(m)}$ and ν_V^W -almost all β . When $\psi > 0$, we can once again introduce the operator Δ^ψ in order to study H_β -harmonic functions (see Proposition 2.1.6). We can characterise Δ^ψ -harmonic functions with the corresponding Martin boundary \mathcal{M}^ψ and the Martin kernel K^ψ . Since the graph is a tree, the Martin boundary is equal to the set Ω of ends of \mathcal{T} , which is deterministic. Note that the boundary condition used to define $\mathcal{G}_m^{(n)}$ corresponds to the identification of Ω_x to a single point δ_x , for all $x \in D^{(m)}$. We will see that the limit of $\chi_m^{(n)}(\cdot, \delta_x)$ can be expressed with the family of harmonic measures associated with Δ^ψ , defined as follows.

For a transient random walk $(X_k)_{k \in \mathbb{N}}$ on a graph $\mathcal{G} = (V, E)$, it is possible to define the limit X_∞ of the trajectory as a \mathcal{M} -valued random variable, where \mathcal{M} is the Martin boundary associated with the random walk. Then the family of harmonic measures is defined as $(\mu_x)_{x \in V}$, where μ_x is the distribution of X_∞ when the walk starts at x . For all $x \in V$, μ_x is a probability measure on \mathcal{M} , and for $A \in \mathcal{B}(\mathcal{M})$, $x \mapsto \mu_x(A)$ is harmonic for the random walk.

In our case, we denote by $(\mu_x^\psi)_{x \in T}$ the harmonic measures associated with Δ^ψ . For $\beta \in \mathcal{D}_T^W$ such that $\psi \equiv 0$, we adopt the convention that μ_x^ψ is the null measure on Ω for all $x \in T$. We will now see how under $\nu_T^W(d\beta)$, for all $x \in D^{(m)}$, $\chi_m^{(n)}(\cdot, \delta_x)$ converges to a H_β -harmonic function related to the harmonic measures $(\mu_y^\psi)_{y \in T}$, and how this gives us a representation of the VRJP on (\mathcal{T}, W) for each $m \geq 0$.

Theorem 2.1.6. (i) For all $m \geq 0$, for $y \in T$ and $x \in D^{(m)}$, we have ν_T^W -almost surely $\chi_m^{(n)}(y, \delta_x) \rightarrow \psi(y) \mu_y^\psi(\Omega_x)$. For all $y \in T$, we define the measure $\chi(y, \cdot) = \psi(y) \mu_y^\psi(\cdot)$ on Ω .

(ii) Let $m \geq 0$ be fixed. For ν_T^W -almost all β , we define the $|B_m| \times |B_m|$ matrix \check{C}_m by

$$(\check{C}_m)_{\delta_x, \delta_{x'}} = \begin{cases} 0 & \text{if } x = x', \\ \frac{\chi(x \wedge x', \Omega_x) \chi(x \wedge x', \Omega_{x'})}{\hat{G}(x \wedge x', x \wedge x')} & \text{otherwise.} \end{cases}$$

From now on, let us write: $\nu_{T, B_m}^W(d\beta, d\rho_m) = \nu_T^W(d\beta) \nu_{B_m}^{\check{C}_m}(d\rho_m)$.

For ν_T^W -almost all β and for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, we define $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, as well as $\check{g}_m : \Omega^2 \rightarrow \mathbb{R}_+$ a locally constant function, such that for $x, x' \in D^{(m)}$ and $\omega \in \Omega_x, \tau \in \Omega_{x'}$, we have $\check{g}_m(\omega, \tau) = \check{G}_m(\delta_x, \delta_{x'})$. Finally, for ν_T^W -almost all β and for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, for $i, j \in T$, we define

$$G_m(i, j) = \hat{G}(i, j) + \int_{\Omega^2} \chi(i, d\omega) \chi(j, d\tau) \check{g}_m(\omega, \tau),$$

and denote by $\mathbb{P}_x^{\beta, \rho_m, i_0}$ the distribution of the Markov jump process started at $x \in V$, where the jump rate from i to j is $\frac{1}{2} W_{i,j} \frac{G_m(i_0, j)}{G_m(i_0, i)}$.

Then the process with law $\mathbb{P}^{VRJP(i_0)}$ is a mixture of these Markov jump processes, under the mixing measure $\nu_{T, B_m}^W(d\beta, d\rho_m)$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, \rho_m, i_0}[\cdot] \nu_{T, B_m}^W(d\beta, d\rho_m).$$

(iii) The distribution under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ of the jump rates $(\frac{1}{2} W_{i,j} \frac{G_m(i_0, j)}{G_m(i_0, i)})_{i \sim j}$ converges weakly to the distribution of jump rates in the representation described in Theorem 2.1.5.

Let us now consider the case where \mathcal{T} is a d -regular tree, with $d \geq 3$, endowed with constant conductances, i.e. $W_e = W > 0$ for all $e \in E$. Then (\mathcal{T}, W) is vertex transitive, and from Proposition 3 of [SZ19], we know that under $\nu_T^W(d\beta)$, ψ is stationary and ergodic. Therefore, depending on d and W , we either have $\mathbb{P}[\forall i \in T, \psi(i) = 0] = 1$, or $\mathbb{P}[\forall i \in T, \psi(i) > 0] = 1$.

In the first case, from Theorem 2.1.2 (iv), this means that the VRJP is a.s. recurrent, and therefore admits a unique representation (see Proposition 2.1.5). Note that in Theorem 2.1.6, we have a.s. $G_m = \hat{G}$ for all $m \in \mathbb{N}$, so that all the corresponding representations are indeed equal. The following proposition describes the second case, i.e. when the VRJP is a.s. transient. According to a result from [DV04], this is true for large enough initial weights W .

Proposition 2.1.8. *Let \mathcal{T} be a d -regular tree, with $d \geq 3$, endowed with constant conductances W large enough so that the VRJP on (\mathcal{T}, W) is a.s. transient. Then the representations of the VRJP given in Theorem 2.1.6 are different for distinct values of m , i.e. if $m \neq m'$, the distributions of jump rates $(\frac{1}{2} W_{i,j} \frac{G_m(i_0, j)}{G_m(i_0, i)})_{i \sim j}$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ and $(\frac{1}{2} W_{i,j} \frac{G_{m'}(i_0, j)}{G_{m'}(i_0, i)})_{i \sim j}$ under $\nu_{T, B_{m'}}^W(d\beta, d\rho_{m'})$ are different for all $i_0 \in T$.*

2.1.4 Open questions

A first question concerns the case of \mathbb{Z}^d with constant conductances W : is it possible to show that the Martin boundary associated with Δ^ψ is a.s. trivial for any W such that the VRJP is transient? In this case, it would prove the uniqueness of the representation of the VRJP on \mathbb{Z}^d for any constant initial conductances W .

Another question concerns a possible classification of all representations on trees using partitions of the Martin boundary. We have constructed a family of representations from different boundary conditions on

the tree, corresponding to some finite partitions of the Martin boundary Ω , more precisely the partition $\Omega = \bigcup_{x \in D^{(m)}} \Omega_x$ for $m \in \mathbb{N}$. It should be possible to define more representations using the same method, with boundary conditions associated with other finite partitions of Ω , where each set in the partition can be written as a finite union of sets Ω_x . To generalise this, we can ask if it is possible to determine which partitions give us a valid representation, and whether all representations can be written in this form, or as a limit of such representations, as in Theorem 2.1.6 (iii).

2.2 Technical prerequisite

2.2.1 The random potential β on finite graphs

Let $\mathcal{G} = (V, E)$ be a finite connected non-directed graph, endowed with conductances $(W_e)_{e \in E}$. Let us give some useful properties on the distribution ν_V^W .

Proposition 2.2.1. [Proposition 2, Theorem 3 in [STZ17]] For $\beta \in \mathcal{D}_V^W$, let $G = (H_\beta)^{-1}$ be the Green function associated with β . We define $F : V \times V \rightarrow \mathbb{R}$ by

$$F(i, j) = \frac{G(i, j)}{G(j, j)}.$$

Then under $\nu_V^W(d\beta)$, for all $i_0 \in V$, we have the following properties:

- (i) $(F(i, i_0))_{i \in V}$ is $(\beta_i)_{i \in V \setminus \{i_0\}}$ -measurable.
- (ii) If we denote $\gamma = \frac{1}{2G(i_0, i_0)}$, then γ is a Gamma random variable with parameter $(1/2, 1)$. Moreover, γ is independent of $(\beta_i)_{i \neq i_0}$, and therefore independent of $(F(i, i_0))_{i \in V}$.

This proposition explains the presence of γ in the expression on G in Theorem 2.1.2. Moreover, it allows us to prove Proposition 2.1.4, describing the distribution of the $\tilde{\beta}$ field.

Proof of Proposition 2.1.4. Let $\mathcal{G} = (V, E)$ be an infinite connected non-directed graph, and $(V_n)_{n \in \mathbb{N}}$ an increasing exhausting sequence of finite connected subsets of V . For $n \in \mathbb{N}$, let $\mathcal{G}^{(n)} = (\tilde{V}^{(n)}, \tilde{E}^{(n)})$ be the restriction of \mathcal{G} to V_n with wired boundary condition, endowed with conductances $\tilde{W}^{(n)}$, defined as in section 2.1.1. Moreover, for $n \in \mathbb{N}$, we still define $\hat{G}^{(n)}$ and $\psi^{(n)}$ as in Definition 2.1.1.

The proof of Theorem 2.1.2 (iii) uses the fact that under $\nu_V^W(d\beta, d\gamma)$, there exists a coupling of random fields $(\beta^{(n)})_{n \in \mathbb{N}}$, such that for all $n \in \mathbb{N}$: $\beta^{(n)} \sim \nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$; $\beta_{V_n}^{(n)} = \beta_{V_n}$; and $G^{(n)} = (2\beta^{(n)} - \tilde{W}^{(n)})^{-1}$, where for $i, j \in \tilde{V}^{(n)}$,

$$G^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \frac{1}{2\gamma} \psi^{(n)}(i) \psi^{(n)}(j).$$

We can then apply Proposition 2.2.1 to $\beta^{(n)}$ at i_0 : since $G^{(n)} = (2\beta^{(n)} - \tilde{W}^{(n)})^{-1}$, we have

$$\beta_i^{(n)} = \tilde{\beta}_i^{(n)} + \mathbf{1}_{\{i=i_0\}} \frac{1}{2G^{(n)}(i_0, i_0)}$$

for all $i \in \tilde{V}^{(n)}$, where

$$\tilde{\beta}_i^{(n)} = \sum_{j \sim i} \frac{\tilde{W}_{i,j}^{(n)}}{2} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i)}.$$

According to Proposition 2.2.1, $1/2G^{(n)}(i_0, i_0)$ is a random Gamma(1/2, 1) variable, which is independent of $(\beta_i^{(n)})_{i \in \tilde{V}^{(n)} \setminus \{i_0\}}$. Moreover, for $i \neq i_0$, $\tilde{\beta}_i^{(n)} = \beta_i^{(n)}$, and

$$\tilde{\beta}_{i_0}^{(n)} = \sum_{j \sim i_0} \frac{\tilde{W}_{i_0, j}^{(n)}}{2} \frac{G^{(n)}(i_0, j)}{G^{(n)}(i_0, i_0)} = \sum_{j \sim i_0} \frac{\tilde{W}_{i_0, j}^{(n)}}{2} F^{(n)}(j, i_0),$$

so that $\tilde{\beta}^{(n)}$ is $(\beta_i^{(n)})_{i \in \tilde{V}^{(n)} \setminus \{i_0\}}$ -measurable, and therefore independent of $G^{(n)}(i_0, i_0)$.

Taking the limit when $n \rightarrow \infty$, we deduce that $1/2G(i_0, i_0)$ is a random Gamma(1/2, 1) variable, independent from the $\tilde{\beta}$ field, where for $i \in V$,

$$\tilde{\beta}_i = \sum_{j \sim i} \frac{W_{i, j}}{2} \frac{G(i, j)}{G(i, i)} = \beta_i - \mathbf{1}_{\{i=i_0\}} \frac{1}{2G(i_0, i_0)}.$$

Since the Laplace transform of a Gamma(1/2, 1) variable is, for $t \geq 0$,

$$\int e^{-t\gamma} \frac{\mathbf{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma = \frac{1}{\sqrt{1+t}},$$

and given the Laplace transform of ν_V^W in Proposition 2.1.3, we now know that the Laplace transform of $\tilde{\beta}$ is, for $\lambda \in \mathbb{R}_+^V$ with finite support,

$$\mathbb{E}[e^{-\langle \lambda, \tilde{\beta} \rangle}] = \frac{1}{\sqrt{1 + \lambda_{i_0}}} e^{-\sum_{i \sim j} W_{i, j} (\sqrt{1 + \lambda_i} \sqrt{1 + \lambda_j} - 1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1 + \lambda_i}}.$$

□

On finite graphs, the distribution ν_V^W , and more generally $\nu_V^{W, \eta}$ for $\eta \in \mathbb{R}_+^V$, behaves well with respect to restriction, as shown in the next proposition, which is a generalization of Proposition 2.1.2.

Proposition 2.2.2. [Lemma 4 in [SZ19]] *Let us fix $U \subset V$ and $\eta \in (\mathbb{R}_+)^V$. Then, under $\nu_V^{W, \eta}(d\beta)$, we have:*

(i) β_U is distributed according to $\nu_U^{W_{U, U}, \hat{\eta}}$, where

$$\hat{\eta} = \eta_U + W_{U, U^c} \mathbf{1}_{U^c}.$$

(ii) Conditionally on β_U , β_{U^c} is distributed according to $\nu_{U^c}^{\tilde{W}, \tilde{\eta}}$, where

$$\tilde{W} = W_{U^c, U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} W_{U, U^c} \text{ and } \tilde{\eta} = \eta_{U^c} + W_{U^c, U} ((H_\beta)_{U, U})^{-1} \eta_U.$$

Proposition 2.1.2 is a direct consequence of (i), in the case where $\eta = 0$. Moreover, (ii) is useful to extend a potential $\beta_U \sim \nu_U^{W_{U, U}, \hat{\eta}}$ where $\hat{\eta} = W_{U, U^c} \mathbf{1}_{U^c}$ into a potential $\beta_V \sim \nu_V^W$, using the distribution of β_{U^c} conditionally on β_U .

2.2.2 Green function and sums over paths

Let us still consider a finite connected non-directed graph $\mathcal{G} = (V, E)$ endowed with conductances W . For $\beta \in \mathcal{D}_V^W$, it will be useful to express the Green function $G = H_\beta^{-1}$ as a sum over paths in \mathcal{G} . We first introduce some notations for sets of paths.

Definition 2.2.1. (i) For $i, j \in V$, we denote by $\mathcal{P}_{i,j}^V$ the set of paths σ from i to j in V , i.e. the set of finite sequences $\sigma = (\sigma_0, \dots, \sigma_l)$ in V , where $\sigma_0 = i$, $\sigma_l = j$ and $\sigma_k \sim \sigma_{k+1}$ for $0 \leq k \leq l-1$. We denote by $|\sigma| = l$ the length of the path σ .

(ii) For $U \subset V$, $i \in U$ and $j \notin U$, we denote by $\overline{\mathcal{P}}_{i,j}^U$ the set of paths $\sigma \in \mathcal{P}_{i,j}^V$ such that $\sigma_k \in U$ for $0 \leq k \leq |\sigma| - 1$.

(iii) For $U \subset V$ and $i, j \in V$, we denote by $\mathcal{P}_{i,U,j}^V$ the set of paths $\sigma \in \mathcal{P}_{i,j}^V$ such that $\sigma_k \in U$ for some $k \in \llbracket 0, |\sigma| \rrbracket$.

(iv) For $i, j \in V$ and $\sigma \in \mathcal{P}_{i,j}^V$, we define the following notations:

$$W_\sigma = \prod_{k=0}^{|\sigma|-1} W_{\sigma_k, \sigma_{k+1}}, (2\beta)_\sigma = \prod_{k=0}^{|\sigma|} 2\beta_{\sigma_k} \text{ and } (2\beta)_\sigma^- = \prod_{k=0}^{|\sigma|-1} 2\beta_{\sigma_k}.$$

We get the following expressions, in terms of sums over paths, for G and related quantities.

Proposition 2.2.3. [Proposition 6 in [STZ17]] Let $\beta \in \mathcal{D}_V^W$. Then:

(i) For $i, j \in V$,

$$G(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}.$$

In particular, for $U \subset V$ we denote $\hat{G}^U = ((H_\beta)_{U,U})^{-1}$, then for $i, j \in U$, we obtain

$$\hat{G}^U(i, j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^U} \frac{W_\sigma}{(2\beta)_\sigma}.$$

(ii) For $i, j \in V$,

$$F(i, j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma^-} = \sum_{z \sim j} \hat{G}^{V \setminus \{j\}}(i, z) W_{z,j}.$$

(iii) For $U \subset V$ and $i, j \in U^c$,

$$\sum_{\sigma \in \mathcal{P}_{i,U,j}^V} \frac{W_\sigma}{(2\beta)_\sigma^-} = \sum_{z_1, z_2 \in U} \left(\sum_{\sigma \in \overline{\mathcal{P}}_{i,z_1}^{U^c}} \frac{W_\sigma}{(2\beta)_\sigma^-} \right) G(z_1, z_2) \left(\sum_{\sigma \in \overline{\mathcal{P}}_{z_2,j}^{U^c}} \frac{W_\sigma}{(2\beta)_\sigma^-} \right).$$

In particular, if $U = \{z\}$, this becomes

$$\sum_{\sigma \in \mathcal{P}_{i,\{z\},j}^V} \frac{W_\sigma}{(2\beta)_\sigma^-} = F(i, z) G(z, z) F(j, z) = F(i, z) G(z, j).$$

Remark 2.2.1. If $\mathcal{G} = (V, E)$ is now an infinite graph, let $(V_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite connected subsets of V such that $V = \bigcup_{n \in \mathbb{N}} V_n$. For $\beta \in \mathcal{D}_V^W$ and $n \geq 0$, we define $\hat{G}^{(n)} = \hat{G}^{V_n} = ((H_\beta)_{V_n, V_n})^{-1}$ as in Definition 2.1.1. Then from Proposition 2.2.3 (i), we get

$$\hat{G}^{(n)}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma}$$

for $n \geq 0$ and $i, j \in V_n$. From Theorem 2.1.2 (i), under $\nu_V^W(d\beta)$ the increasing sequence $(\hat{G}^{(n)}(i, j))_{n \in \mathbb{N}}$ converges almost surely to $\hat{G}(i, j)$. Hence, we get

$$\hat{G}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}$$

for $i, j \in V$.

Let us also define $\hat{F}^{(n)}(i, j) = \frac{\hat{G}^{(n)}(i, j)}{\hat{G}^{(n)}(j, j)}$ and $\hat{F}(i, j) = \frac{\hat{G}(i, j)}{\hat{G}(j, j)}$, for all $i, j \in V$ and $n \geq \max(|i|, |j|)$. Then, from Proposition 2.2.3 (ii) we have

$$\hat{F}^{(n)}(i, j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V_n \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma} \xrightarrow{n \rightarrow \infty} \hat{F}(i, j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma},$$

where the convergence is true ν_V^W -almost surely.

2.2.3 Martin boundary and harmonic functions

Let us give more details about the theory of Martin boundaries. The following results can be found in [Woe00].

Let $\mathcal{G} = (V, E)$ be an infinite graph, we consider an irreducible random walk $(X_n)_{n \in \mathbb{N}}$ on \mathcal{G} , whose transition matrix is P , where $P_{i,j} = 0$ if $\{i, j\} \notin E$ (i.e. we assume that (X_n) is a nearest-neighbor random walk). Moreover, we assume that (X_n) is transient.

Let us denote by \mathbb{P}_x the distribution of the random walk started at $x \in V$, and by g the associated Green function, i.e.

$$g(x, y) = \sum_{n \in \mathbb{N}} \mathbb{P}_x[X_n = y].$$

We also denote

$$f(x, y) = \mathbb{P}_x[\exists n \in \mathbb{N}, X_n = y] = \frac{g(x, y)}{g(y, y)}.$$

For all $y \in V$, $g(\cdot, y)$ is harmonic at any $x \in V \setminus \{y\}$, i.e. for $x \neq y$, $g(x, y) = \sum_{z \sim x} P_{x,z} g(z, y)$. This is still true for $f(\cdot, y)$. The Martin Kernel, defined below using f , as well as the Martin boundary, will allow us to represent all positive harmonic functions for the random walk.

Definition 2.2.2. *Let us fix a reference point $\phi \in V$.*

(i) *The Martin kernel is the function $K : V^2 \rightarrow \mathbb{R}_+$ defined by*

$$K(x, y) = \frac{f(x, y)}{f(\phi, y)} = \frac{g(x, y)}{g(\phi, y)}.$$

(ii) *The Martin compactification is the smallest compactification \hat{V} of V , which is unique up to a homeomorphism, so that $K(\cdot, \cdot)$ extends continuously to $V \times \hat{V}$. The Martin boundary is defined as $\mathcal{M} = \hat{V} \setminus V$.*

Theorem 2.2.1. *Let us denote by \mathcal{H}^+ the set of positive harmonic functions on V . Then for all $h \in \mathcal{H}^+$, there is a Borel measure μ^h on \mathcal{M} such that for all $x \in V$,*

$$h(x) = \int_{\mathcal{M}} K(x, \alpha) \mu^h(d\alpha).$$

Remark 2.2.2. If, for all $x \in V$ and for all sequences $(y_n)_{n \geq 1}$ going to infinity, we have $K(x, y_n) \rightarrow 1$, then the Martin boundary is trivial, *i.e.* reduced to a single point. According to Theorem 2.2.1, in this case, all positive harmonic functions are constant.

Since (X_n) is transient, we almost surely have $X_n \rightarrow \infty$, in the sense that for all finite subset $U \subset V$, $\{n \in \mathbb{N}, X_n \in U\}$ is almost surely finite. Thanks to the Martin boundary, we can now describe this convergence more precisely.

Theorem 2.2.2. For all $x \in V$, (X_n) converges \mathbb{P}_x -a.s. to a \mathcal{M} -valued random variable X_∞ . The distribution of X_∞ under \mathbb{P}_x , denoted by μ_x , verifies

$$\mu_x(B) = \int_B K(x, \alpha) \mu_\phi(d\alpha)$$

for all $B \subset \mathcal{M}$ and $x \in V$.

The space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), (\mu_x)_{x \in V})$ is called Poisson boundary. Moreover, we call harmonic measures, or exiting measures, the family $(\mu_x)_{x \in V}$.

In the case where $\mathcal{T} = (T, E)$ is an infinite tree, the Martin compactification will coincide with another, which does not depend on the random walk defined by P , but simply on the geometry of the tree \mathcal{T} .

Definition 2.2.3. Let us fix an arbitrary root ϕ for \mathcal{T} .

- (i) We call infinite ray in T an infinite self-avoiding path starting at ϕ , *i.e.* a sequence $\omega = (\omega_k)_{k \in \mathbb{N}}$ of distinct vertices in T , such that $\omega_k \sim \omega_{k+1}$ for $k \in \mathbb{N}$ and $\omega_0 = \phi$. The set of infinite rays, also called the set of ends of T , is denoted by Ω .
- (ii) If $\omega, \xi \in \Omega$, we denote $N_{\omega, \xi} = \max\{k \in \mathbb{N}, \omega_k = \xi_k\}$. We can also define, if $x \in T$, $N_{\omega, x} = \max\{k \leq |x|, \omega_k = [\phi, x]_k\}$. We then set $O_\omega^k = \{\xi \in \Omega, N_{\omega, \xi} \geq k\} \cup \{x \in T, N_{\omega, x} \geq k\}$.
- (iii) We define the end topology on $T \cup \Omega$, which is discrete on T , and such that $(O_\omega^k)_{k \in \mathbb{N}}$ is a basis of neighborhoods at $\omega \in \Omega$.

Proposition 2.2.4. The end topology on $T \cup \Omega$ does not depend on the choice of ϕ , and is induced by the following metric:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ e^{-N_{x, y}} & \text{otherwise,} \end{cases}$$

for $x, y \in T \cup \Omega$. Moreover $T \cup \Omega$ is compact, and called the end compactification.

Theorem 2.2.3. (i) Let (X_n) be a nearest-neighbor random walk on \mathcal{T} , that we assume to be transient. Then the Martin compactification coincides with the end compactification, and we can identify \mathcal{M} to Ω , and set $\hat{T} = T \cup \Omega$.

- (ii) The Martin kernel on $T \times \hat{T}$ is locally constant, with

$$K(x, \omega) = K(x, x \wedge \omega), \text{ where } x \wedge \omega = \omega_{N_{\omega, x}}$$

for $x \in T, \omega \in \Omega$.

We also have an expression of harmonic measures μ_x on the tree. For $x \in T$, we denote by Ω_x the set of ends for the subtree T_x , *i.e.* $\Omega_x = \{\omega \in \Omega, \exists k \in \mathbb{N}, \omega_k = x\}$. Moreover, we denote by $U_x = T_x \setminus \{x\}$. Then:

Proposition 2.2.5. For $x \neq \phi$ and $i \in T$,

$$\mu_i(\Omega_x) = \mathbf{1}_{\{i \in U_x\}}(1 - f(i, x)) + f(i, x) \frac{1 - f(x, \tilde{x})}{1 - f(\tilde{x}, x)f(x, \tilde{x})}.$$

Remark 2.2.3. From Carathéodory's extension theorem, this entirely describes the expression of μ_ϕ . From Theorem 2.2.2, we can then describe all harmonic measures using f .

2.3 Distributions of arbitrary representations

2.3.1 A common expression for jump rates: Proof of Theorem 2.1.3

Let $\mathcal{G} = (V, E)$ be a locally finite connected graph, endowed with conductances $(W_{i,j})_{i,j \in V}$ such that $W_{i,j} = W_{j,i} > 0$ if $\{i, j\} \in E$, and $W_{i,j} = 0$ otherwise. We still denote by $\mathbb{P}^{VRJP(i_0)}$ the law of the VRJP on (\mathcal{G}, W) , in the exchangeable time scale, started at $i_0 \in V$. Let us first show that the distribution of the $\tilde{\beta}$ field (see Proposition 2.1.4) appears in all representations of the VRJP.

Recall that for all $r \in \mathcal{J}_V^E$ and $i \in V$, we define $r_i = \sum_{j \sim i} r_{i,j}$.

Proposition 2.3.1. Let $i_0 \in V$ be fixed, and let $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$, in the sense of Definition 2.1.2.

Then under $\mathcal{R}(dr)$, $(r_i)_{i \in V}$ has the same distribution as the field $\tilde{\beta}$ rooted at i_0 , i.e. its Laplace transform is

$$\int e^{-\langle \lambda, r \rangle} \mathcal{R}(dr) = e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+\lambda_i}\sqrt{1+\lambda_j}-1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1+\lambda_i}},$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

Proof. Let $i_0 \in V$ be fixed, let $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^r[\cdot] \mathcal{R}(dr),$$

where \mathbb{P}^r is the distribution of the Markov jump process with jump rate from i to j given by $r_{i,j}$.

Let us prove that under $\mathcal{R}(dr)$, $(r_i)_{i \in V}$ has the same distribution as the $\tilde{\beta}$ field from the standard representation.

Lemma 2.3.1. There exists a random field $(u_i)_{i \in V} \in \mathbb{R}^V$ such that \mathcal{R} -almost surely, $r_{i,j} = \frac{W_{i,j}}{2} e^{u_j - u_i}$ for $i \sim j$.

Remark 2.3.1. Since the random field $(u_i)_{i \in V}$ is defined up to an additive constant, we can set $u_{i_0} = 0$ a.s. without loss of generality.

Proof of Lemma 2.3.1. For $r \in \mathcal{J}_V^E$, let us define $t_{i,j} = \frac{2}{W_{i,j}} r_{i,j}$ for all $i \sim j$. Then to prove this lemma, it is enough to show that for any cycle $\sigma = (\sigma_0, \dots, \sigma_n)$, we have \mathcal{R} -a.s. $t_\sigma := \prod_{k=0}^{n-1} t_{\sigma_k, \sigma_{k+1}} = 1$. Since \mathcal{G} is connected, we only need to prove this for cycles σ such that $\sigma_0 = i_0$.

Recall that we denote by $(Z_t)_{t \geq 0}$ the canonical process on $\mathcal{C}_r(\mathbb{R}_+, V)$. Let \mathbb{P}^{MJJP} be the distribution of the Markov jump process with jump rates $\frac{1}{2}W_{i,j}$. Then, according to Theorem 3 from [ST16], for all $T \geq 0$ the law of $(Z_t)_{t \leq T}$ under $\mathbb{P}^{VRJP(i_0)}$ is absolutely continuous with respect to its law under $\mathbb{P}_{i_0}^{MJJP}$, and its Radon-Nikodym derivative is

$$\frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2}W_{i,i}l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}},$$

where $W_i = \sum_{j \sim i} W_{i,j}$, and $l_i = \int_0^T \mathbb{1}_{\{Z_t=i\}} dt$ is the local time at i .

Let σ be a cycle such that $\sigma_0 = \sigma_{|\sigma|} = i_0$. We denote by σ^n the n th concatenation of σ , and for $T \geq 0$ and τ a path in \mathcal{G} , by $\{(Z_t)_{t \leq T} \sim \tau\}$ the event where the discrete trajectory of $(Z_t)_{t \geq T}$ follows the path τ . Then we have, for $n \geq 1$ and $T \geq 0$,

$$\mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] = \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} \mathbb{P}_{i_0}^{MJJP}(dz).$$

However, since the random environment $(r_{i,j})_{i \sim j}$ gives a representation of the VRJP as a mixture of Markov processes, we also have

$$\begin{aligned} \mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] &= \int \mathbb{P}_{i_0}^r[(Z_t)_{t \leq T} \sim \sigma^n] \mathcal{R}(dr) \\ &= \int \left(\int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \frac{e^{-\sum_{i \in V} r_i l_i} \left(\prod_{k=0}^{|\sigma|-1} r_{\sigma_k, \sigma_{k+1}} \right)^n}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i} \left(\prod_{k=0}^{|\sigma|-1} \frac{1}{2} W_{\sigma_k, \sigma_{k+1}} \right)^n} \mathbb{P}_{i_0}^{MJJP}(dz) \right) \mathcal{R}(dr) \\ &= \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \int \frac{e^{-\sum_{i \in V} r_i l_i}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} (t_\sigma)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJJP}(dz) \end{aligned}$$

Let us fix $\varepsilon > 0$, and define the event $A_{\sigma, \varepsilon} = \{t_\sigma \geq 1 + \varepsilon\}$. Then we get

$$\mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] \geq \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \int \mathbb{1}_{A_{\sigma, \varepsilon}} e^{-\sum_{i \in V} r_i l_i} (1 + \varepsilon)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJJP}(dz).$$

Let $M > 0$ be such that under $\mathcal{R}(dr)$, $\mathbb{P}[A_{\sigma, \varepsilon} \cap B_M] \geq \mathbb{P}[A_{\sigma, \varepsilon}]/2$, where $B_M = \{\forall i \in V, r_i \leq M\}$. Note that $T = \sum_{i \in V} l_i$, so that

$$\begin{aligned} \mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] &\geq \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \int \mathbb{1}_{A_{\sigma, \varepsilon} \cap B_M} e^{-MT} (1 + \varepsilon)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJJP}(dz) \\ &\geq \frac{e^{-MT}}{2} (1 + \varepsilon)^n \mathbb{P}[A_{\sigma, \varepsilon}] \mathbb{P}_{i_0}^{MJJP}[(Z_t)_{t \leq T} \sim \sigma^n]. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} \mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] &= \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} \mathbb{P}_{i_0}^{MJJP}(dz) \\ &\leq e^{M'T} \mathbb{P}_{i_0}^{MJJP}[(Z_t)_{t \leq T} \sim \sigma^n], \end{aligned}$$

where $M' = \max\{\frac{1}{2} W_{\sigma_k}, 0 \leq k < |\sigma|\}$. Since $\mathbb{P}_{i_0}^{MJJP}[(Z_t)_{t \leq T} \sim \sigma^n] > 0$ for all $T > 0$ and $n \in \mathbb{N}$, we get

$$\frac{e^{-MT}}{2} (1 + \varepsilon)^n \mathbb{P}[A_{\sigma, \varepsilon}] \leq e^{M'T}.$$

Taking $n \rightarrow \infty$ for fixed $T > 0$ shows that $\mathbb{P}[A_{\sigma, \varepsilon}] = 0$. As a result, we have almost surely $t_\sigma \leq 1$.

For $\varepsilon > 0$, we now set $A'_{\sigma, \varepsilon} = \{t_\sigma \leq 1 - \varepsilon\}$. Using the same notations as before, and the fact that a.s. $t_\sigma \leq 1$, we get

$$\begin{aligned} \mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] &\leq \int \mathbb{1}_{\{(z_t)_{t \leq T} \sim \sigma^n\}} \int e^{M'T} \left(\mathbb{1}_{A'_{\sigma, \varepsilon}^c} + \mathbb{1}_{A'_{\sigma, \varepsilon}} (1 - \varepsilon)^n \right) \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJJP}(dz) \\ &\leq e^{M'T} \left(\mathbb{P}[A'_{\sigma, \varepsilon}^c] + \mathbb{P}[A'_{\sigma, \varepsilon}] (1 - \varepsilon)^n \right) \mathbb{P}_{i_0}^{MJJP}[(Z_t)_{t \leq T} \sim \sigma^n]. \end{aligned}$$

On the other hand, on the event $\{(Z_t)_{t \leq T} \sim \sigma^n\}$, we have $l_i \leq T$ for all $i \in \{\sigma_k, 0 \leq k < |\sigma|\}$ and $l_i = 0$ for all other $i \in V$. As a result, for such trajectories,

$$\frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2}W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} \geq \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}},$$

where $M'' = \sum_{i,j \in \{\sigma_k\}} W_{i,j}$, so that,

$$\mathbb{P}^{VRJP(i_0)}[(Z_t)_{t \leq T} \sim \sigma^n] \geq \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}} \mathbb{P}_{i_0}^{MJP}[(Z_t)_{t \leq T} \sim \sigma^n].$$

As before, this yields

$$e^{M'T} (\mathbb{P}[A'_{\sigma,\varepsilon}{}^c] + \mathbb{P}[A'_{\sigma,\varepsilon}](1-\varepsilon)^n) \geq \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}}$$

for all $T > 0$ and $n \in \mathbb{N}$. Taking first $n \rightarrow \infty$, then $T \rightarrow 0$, we get that under $\mathcal{R}(dr)$, $\mathbb{P}[A'_{\sigma,\varepsilon}{}^c] = 1$. Therefore, we can conclude that \mathcal{R} -almost surely, $t_\sigma = 1$. \square

In order to identify the distribution of $(r_i)_{i \in V}$ under $\mathcal{R}(dr)$, we obtain their Laplace transform as the density of cyclic trajectories of $(Z_t)_{t \geq 0}$ under $\mathbb{P}^{VRJP(i_0)}$ with respect to $\mathbb{P}_{i_0}^{MJP}$. Indeed, given a cyclic trajectory $(z_t)_{t \geq 0}$ in \mathcal{G} , started at i_0 , we denote by σ the associated cyclic path in \mathcal{G} , and $(l_i)_{i \in V}$ the local times, so that $T = \sum_{i \in V} l_i$, and $l_i > 0$ if and only if $i \in \{\sigma_k, 0 \leq k < |\sigma|\}$. Then the Radon-Nikodym derivative at $(z_t)_{t \geq 0}$ of $\mathbb{P}^{VRJP(i_0)}$ with respect to $\mathbb{P}_{i_0}^{MJP}$ is almost surely

$$\frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2}W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}},$$

but also

$$\int \frac{e^{-\sum_{i \in V} r_i l_i}}{e^{-\sum_{i \in V} \frac{1}{2}W_i l_i}} \mathcal{R}(dr)$$

since $t_\sigma = 1$ \mathcal{R} -almost surely. Therefore, for all finite connected subset U of V , and almost all $(l_i)_{i \in V} \in (\mathbb{R}_+^*)^U \times \{0\}^{V \setminus U}$, we have

$$e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} = \int e^{-\sum_{i \in V} r_i l_i} \mathcal{R}(dr) = \mathbb{E}[e^{-\sum_{i \in V} r_i l_i}].$$

Since these are continuous functions of $(l_i)_{i \in V}$, this equality is true for all $(l_i)_{i \in V} \in \mathbb{R}_+^V$ with finite support. As a result, under $\mathcal{R}(dr)$, $(r_i)_{i \in V}$ has the same Laplace transform as the field $\tilde{\beta}$, associated with the standard representation of the VRJP started at i_0 (see Proposition 2.1.4), and therefore the same distribution. \square

Proof of Theorem 2.1.3. Let $i_0 \in V$ be fixed, and $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$. Thanks to Proposition 2.3.1, we know the distribution of $(r_i)_{i \in V}$ under $\mathcal{R}(dr)$. Note that the distribution of a Gamma(1/2, 1) variable is $\frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$, and that its Laplace transform is given by

$$\int e^{-t\gamma} \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma = \frac{1}{\sqrt{1+t}}$$

for $t \geq 0$. From now on, we denote $\mathcal{R}(dr, d\gamma) = \mathcal{R}(dr) \otimes \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$, which is a distribution on $\mathcal{J}_V^E \times \mathbb{R}$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we now define $(\beta_i)_{i \in V}$ by $\beta_i = r_i + \mathbf{1}_{\{i=i_0\}}\gamma$ for $i \in V$. Then under $\mathcal{R}(dr, d\gamma)$, the Laplace transform of β is, for $\lambda \in \mathbb{R}_+^V$ with finite support,

$$\begin{aligned} \int e^{-\langle \lambda, \beta \rangle} \mathcal{R}(dr) \frac{\mathbf{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma &= \int e^{-\sum_{i \in V} \lambda_i r_i} \mathcal{R}(dr) \int e^{-\lambda_{i_0} \gamma} \frac{\mathbf{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma \\ &= \frac{1}{\sqrt{1 + \lambda_{i_0}}} e^{-\sum_{i \sim j} W_{i,j} (\sqrt{1+l_i} \sqrt{1+l_j} - 1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}}, \end{aligned}$$

i.e. β is distributed according to ν_V^W (see Proposition 2.1.3). We can then define $\mathcal{R}(dr, d\gamma)$ -a.s. $\hat{G} : V \times V \rightarrow \mathbb{R}_+$ and $\psi : V \rightarrow \mathbb{R}_+$ thanks to Theorem 2.1.2. Moreover, by analogy with the standard representation, let $G(i_0, \cdot) : V \rightarrow \mathbb{R}_+$ be defined by:

$$G(i_0, i) = \frac{1}{2\gamma} e^{u_i},$$

where $(u_i)_{i \in V}$ was introduced in Lemma 2.3.1. This way, under $\mathcal{R}(dr, d\gamma)$, for all $i \neq j \in V$ the jump rate $r_{i,j}$ can be written as

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}.$$

Let us set $h(i) = G(i_0, i) - \hat{G}(i_0, i)$ for all $i \in V$. Then $H_\beta h = 0$. Indeed, for $i \neq i_0$, we have

$$2\beta_i G(i_0, i) - \sum_{j \sim i} W_{i,j} G(i_0, j) = 2r_i G(i_0, i) - \sum_{j \sim i} 2r_{i,j} G(i_0, i) = 0,$$

and for $i = i_0$,

$$2\beta_{i_0} G(i_0, i_0) - \sum_{j \sim i_0} W_{i_0,j} G(i_0, j) = \frac{r_{i_0} + \gamma}{\gamma} - \sum_{j \sim i_0} \frac{r_{i_0,j}}{\gamma} = 1,$$

therefore for $i \in V$, $H_\beta G(i_0, i) = \mathbf{1}_{\{i=i_0\}} = H_\beta \hat{G}(i_0, i)$.

Let us finally show that $h(i) \geq 0$ for all $i \in V$. We define $\tau_{i_0}^+ = \inf\{k \geq 1, \tilde{Z}_k = i_0\}$, where $(\tilde{Z}_k)_{k \in \mathbb{N}}$ is the discrete version of $(Z_t)_{t \geq 0}$. Then for all $i \neq i_0$,

$$\begin{aligned} \mathbb{P}_i^r[\tau_{i_0}^+ < \infty] &= \sum_{\sigma \in \overline{\mathcal{P}}_{i, i_0}^{V \setminus \{i_0\}}} \mathbb{P}_i^r[(\tilde{Z}_0, \dots, \tilde{Z}_{|\sigma|}) = (\sigma_0, \dots, \sigma_{|\sigma|})] \\ &= \sum_{\sigma \in \overline{\mathcal{P}}_{i, i_0}^{V \setminus \{i_0\}}} \prod_{k=0}^{|\sigma|-1} \frac{r_{\sigma_k, \sigma_{k+1}}}{r_{\sigma_k}} = \sum_{\sigma \in \overline{\mathcal{P}}_{i, i_0}^{V \setminus \{i_0\}}} \prod_{k=0}^{|\sigma|-1} \frac{W_{\sigma_k, \sigma_{k+1}}}{2\beta_{\sigma_k}} \frac{G(i_0, \sigma_{k+1})}{G(i_0, \sigma_k)} \\ &= \frac{G(i_0, i_0)}{G(i_0, i)} \sum_{\sigma \in \overline{\mathcal{P}}_{i, i_0}^{V \setminus \{i_0\}}} \frac{W_\sigma}{(2\beta)_\sigma} = \frac{G(i_0, i_0)}{G(i_0, i)} \frac{\hat{G}(i_0, i)}{\hat{G}(i_0, i_0)}, \end{aligned}$$

from Proposition 2.2.3. Therefore, we have $G(i_0, i) \leq \frac{G(i_0, i_0)}{\hat{G}(i_0, i_0)} \hat{G}(i_0, i)$. Moreover, thanks to the Markov

property we have

$$\begin{aligned} \mathbb{P}_{i_0}^r[\tau_{i_0}^+ < \infty] &= \sum_{j \sim i_0} \mathbb{P}_{i_0}^r[\tilde{Z}_1 = j] \mathbb{P}_j^r[\tau_{i_0}^+ < \infty] = \sum_{j \sim i_0} \frac{r_{i_0,j}}{r_{i_0}} \frac{G(i_0, i_0)}{G(i_0, j)} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i_0)} \\ &= \sum_{j \sim i_0} \frac{W_{i_0,j}}{2(\beta_{i_0} - \gamma)} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i_0)} = \frac{1}{2\beta_{i_0} - \frac{1}{G(i_0, i_0)}} \sum_{j \sim i_0} W_{i_0,j} \frac{\hat{G}(i_0, j)}{\hat{G}(i_0, i_0)} \\ &= \frac{2\beta_{i_0} - \frac{1}{G(i_0, i_0)}}{2\beta_{i_0} - \frac{1}{G(i_0, i_0)}}, \end{aligned}$$

therefore we have $G(i_0, i_0) \geq \hat{G}(i_0, i_0)$, as well as $G(i_0, i) \geq \hat{G}(i_0, i)$ for all $i \in V$.

As a result, $G(i_0, \cdot)$ can be written, for all $i \in V$, as

$$G(i_0, i) = \hat{G}(i_0, i) + h(i),$$

where $h : V \rightarrow \mathbb{R}_+$ is a non-negative H_β -harmonic function. \square

2.3.2 The recurrent and transient cases: Proofs of Propositions 2.1.5 and 2.1.6

Proof of Proposition 2.1.5. We assume that (\mathcal{G}, W) is such that the VRJP is almost surely recurrent.

Let $(r_{i,j})_{i \sim j}$ be fixed jump rates on V , such that the associated Markov chain is recurrent. We denote by $\mathbb{P}_{i_0}^r$ its distribution when started at i_0 . Note that under $\mathbb{P}_{i_0}^r$, the time spent at a vertex i before jumping is an exponential variable with parameter r_i , and the probability to then jump to a specific neighbor j is $\frac{r_{i,j}}{r_i}$.

Let us then define the following functions of the trajectory (Z_t) : for $i \in V$ and $n \geq 1$, we define $\delta t_i^{(n)}$ as the time spent by (Z_t) at the vertex i during its n th visit to i , and $v_i^{(n)}$ the neighbor of i towards which the process jumps after its n th visit to i . Under $\mathbb{P}_{i_0}^r$, since the process is recurrent, these random variables are well-defined for all $i \in V$ and $n \geq 1$. Moreover, the sequences $(\delta t_i^{(n)})_{n \geq 1}$ and $(v_i^{(n)})_{n \geq 1}$ are independent, so thanks to the law of large numbers, we have almost surely

$$\bar{\delta t}_i := \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \delta t_i^{(k)}}{n} = \frac{1}{r_i} \text{ and } \bar{p}_{i,j} := \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathbb{1}_{\{v_i^{(k)}=j\}}}{n} = \frac{r_{i,j}}{r_i},$$

for all $i, j \in V$.

Let now $\mathcal{R}(dr)$ be the distribution of a random environment representing the VRJP on (\mathcal{G}, W) , *i.e.* $\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^r[\cdot] \mathcal{R}(dr)$. Since the VRJP is a.s. recurrent, then under $\mathcal{R}(dr)$, $\mathbb{P}_{i_0}^r$ is a.s. the distribution of a recurrent Markov chain. Moreover, under $\mathbb{P}^{VRJP(i_0)}$, $\bar{\delta t}_i$ and $\bar{p}_{i,j}$ are a.s. well-defined for all $i, j \in V$, and $\left(\frac{\bar{p}_{i,j}}{\bar{\delta t}_i}\right)_{i \sim j}$ is distributed according to \mathcal{R} . Since these functions of the trajectory do not depend on the chosen representation, the distribution \mathcal{R} is uniquely determined. \square

Proof of Proposition 2.1.6. Let (\mathcal{G}, W) be such that the VRJP is almost surely transient. Since, according to Theorem 2.1.2 (iii), we have $\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, \gamma, i_0}[\cdot] \nu_V^W(d\beta, d\gamma)$, then under $\nu_V^W(d\beta, d\gamma)$, the Markov process with distribution $\mathbb{P}_{i_0}^{\beta, \gamma, i_0}$ is a.s. transient. From Theorem 2.1.2 (iv), this means that under $\nu_V^W(d\beta)$, we have a.s. $\psi(i) > 0$ for all $i \in V$, which proves (i).

Let us now consider the random conductance model with conductances $c_{i,j}^\psi = W_{i,j} \psi(i) \psi(j)$. We denote by $\pi_i^\psi = \sum_{j \sim i} c_{i,j}^\psi$ the corresponding invariant measure, where $\pi_i^\psi = \psi(i) \sum_{j \sim i} W_{i,j} \psi(j) = 2\beta_i \psi(i)^2$ since

ψ is H_β -harmonic. Let \mathbb{P}^ψ be the distribution of the associated random walk, whose transition probability from i to j is

$$p_{i,j}^\psi = \frac{c_{i,j}^\psi}{\pi_i^\psi} = \frac{W_{i,j}\psi(j)}{2\beta_i\psi(i)}$$

for $i, j \in V$. Moreover, let us denote by g^ψ the Green kernel associated with \mathbb{P}^ψ , defined for $i, j \in V$ as $g^\psi(i, j) = \sum_{k \in \mathbb{N}} \mathbb{P}_i^\psi[X_k = j]$, where $(X_k)_{k \in \mathbb{N}}$ denotes the canonical process on $V^{\mathbb{N}}$. Then we have

$$\begin{aligned} g^\psi(i, j) &= \sum_{k \in \mathbb{N}} \sum_{\sigma \in \mathcal{P}_{i,j}^V, |\sigma|=k} \mathbb{P}_i^\psi[(X_0, \dots, X_k) = \sigma] = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \prod_{k=0}^{|\sigma|-1} \frac{W_{\sigma_k, \sigma_{k+1}}\psi(\sigma_{k+1})}{2\beta_{\sigma_k}\psi(\sigma_k)} \\ &= \frac{\psi(j)}{\psi(i)} \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma} = \frac{\psi(j)}{\psi(i)} 2\beta_j \hat{G}(i, j), \end{aligned}$$

where under $\nu_V^W(d\beta)$, $\hat{G}(i, j)$ is a.s. finite for all $i, j \in V$, from Theorem 2.1.2 (i). As a result, we have almost surely $g^\psi(i, j) < \infty$, therefore the random walk \mathbb{P}^ψ is transient almost surely, proving (ii).

Let $\Delta^\psi = (p_{i,j}^\psi - \mathbb{1}_{\{i=j\}})_{i,j \in V}$ be the discrete Laplacian associated with \mathbb{P}^ψ . We will say that a function $\varphi : V \rightarrow \mathbb{R}$ is Δ^ψ -harmonic if $(\Delta^\psi \varphi)(i) = \left(\sum_{j \sim i} p_{i,j}^\psi \varphi(j) \right) - \varphi(i) = 0$ for all $i \in V$. Therefore, a function φ is Δ^ψ -harmonic if and only if for any $i \in V$,

$$2\beta_i \psi(i) \varphi(i) - \sum_{j \sim i} W_{i,j} \psi(j) \varphi(j) = 0,$$

i.e. if and only if $\psi\varphi$ is H_β -harmonic, which concludes the proof of (iii). \square

2.4 Representations of the VRJP on \mathbb{Z}^d : Proof of Theorem 2.1.4

Let us now consider the case where $\mathcal{G} = (V, E)$ is the \mathbb{Z}^d lattice, endowed with constant edge weights, *i.e.* $W_{i,j} = W > 0$ for all $i \sim j$. For $x \in \mathbb{R}^d$, we will denote by $|x|$ its Euclidean norm. We fix $i_0 = 0$.

2.4.1 Recurrence and transience of the VRJP on \mathbb{Z}^d

For $d = 2$, the VRJP on (\mathcal{G}, W) is a.s. recurrent for all $W > 0$, according to Theorem 2 in [Sab19]. Therefore, the representation of $\mathbb{P}^{VRJP(0)}$ as a mixture of Markov jump processes is unique (see Proposition 2.1.5). If $d \geq 3$, Corollary 1 in [ST15] tells us that for small enough W , the VRJP is a.s. recurrent, in which case the representation of $\mathbb{P}^{VRJP(0)}$ is once again unique. Let us now show that for large enough W , even though the VRJP is almost surely transient, the representation is still unique.

From Corollary 3 in [ST15], we know that for W large enough, the VRJP is a.s. transient. From now on, we consider such W . Then thanks to Proposition 2.1.6, under $\nu_V^W(d\beta)$ we have a.s. $\psi(i) > 0$ for all $i \in V$. Moreover, we can define the Markov operator Δ^ψ and, for $h : V \rightarrow \mathbb{R}_+$, h is H_β -harmonic if and only if h/ψ is Δ^ψ -harmonic. In light of Remark 2.1.1, in order to show that the representation of the VRJP is unique, we need to show that the only positive Δ^ψ -harmonic functions are constants, *i.e.* that the Martin boundary \mathcal{M}^ψ associated with Δ^ψ is almost surely trivial. To do this, we will need a local limit theorem in random environment, found in [ADS16a].

2.4.2 Local limit theorem for random walk in random conductances

Let us consider the random conductances model on $\mathcal{G} = (\mathbb{Z}^d, E_d)$, with $d \geq 2$. Let \mathbb{P} be a distribution on the set of conductances $(\mathbb{R}_+^*)^{E_d}$, such that under $\mathbb{P}(d\omega)$, we have a.s. $0 < \omega_{i,j} < \infty$ for all $i \sim j$. For

$\omega \in (\mathbb{R}_+^*)^{E_d}$, let P^ω be the distribution of the continuous-time constant speed random walk associated with ω . This is the Markov jump process with jump rate from i to j given by $\frac{\omega_{i,j}}{\pi_i^\omega}$, where $\pi_i^\omega = \sum_{j \sim i} \omega_{i,j}$. This way under P^ω , the holding time of $(Z_t)_{t \geq 0}$ at each point is an exponential variable of parameter 1, which justifies the term "constant speed". Finally, we denote by q^ω the heat kernel, *i.e.* the transition density of the walk with respect to π^ω : for $x, y \in \mathbb{Z}^d$ and $t \geq 0$,

$$q^\omega(t, x, y) = \frac{P_x^\omega[Z_t = y]}{\pi_y^\omega}.$$

The following theorem from [ADS16a] is a local limit theorem for q^ω , under ergodicity and integrability assumptions.

Theorem 2.4.1. [Theorem 1.11 in [ADS16a]] *Let us assume that $\mathbb{P}(d\omega)$ is stationary and ergodic with respect to translations of \mathbb{Z}^d , and that there exist $p, q \in (1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that $\mathbb{E}[\omega_{i,j}^p] < \infty$ and $\mathbb{E}[\omega_{i,j}^{-q}] < \infty$ for all $i \sim j$.*

Then for $0 < T_1 < T_2$ and $K > 0$, we have \mathbb{P} -a.s.

$$\lim_{n \rightarrow \infty} \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)| = 0,$$

where $\lfloor nx \rfloor = (\lfloor nx_1 \rfloor, \dots, \lfloor nx_d \rfloor)$, $a = 1/\mathbb{E}[\pi_0^\omega]$ and k_t is the Gaussian heat kernel with some deterministic covariance matrix Σ^2 , *i.e.*

$$k_t(x) = \frac{1}{\sqrt{(2\pi t)^d \det(\Sigma^2)}} e^{-\frac{x^t (\Sigma^2)^{-1} x}{2t}}.$$

Remark 2.4.1. If $\mathbb{P}(d\omega)$ is also stationary with respect to all isometries of \mathbb{Z}^d , then the limiting Brownian motion must be as well, therefore its deterministic covariance matrix has the form $\Sigma^2 = \sigma^2 I_d$, where $\sigma^2 > 0$.

This also provides a local limit theorem for the Green kernel g^ω , defined for $\omega \in (\mathbb{R}_+^*)^{E_d}$ and $x, y \in \mathbb{Z}^d$ by

$$g^\omega(x, y) = \int_0^\infty q^\omega(t, x, y) dt.$$

This result was also mentioned in [ADS16a], we give here the details for the proof of a slightly stronger result¹, that insures the uniform convergence for x in an annulus.

Theorem 2.4.2 (Variant of Theorem 1.14 in [ADS16a]). *For $d \geq 3$, under the assumptions of Theorem 2.4.1, we have \mathbb{P} -a.s.*

$$\lim_{n \rightarrow \infty} \sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| = 0,$$

where g_{BM} is the Green kernel associated with the Brownian motion with covariance matrix Σ^2 , *i.e.*

$$g_{BM}(0, x) = \int_0^\infty k_t(x) dt = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2} \det(\Sigma^2)^{1/2}} (x^t (\Sigma^2)^{-1} x)^{1-d/2}.$$

Proof. This result is obtained by integrating in Theorem 2.4.1. Moreover, we will need the following bounds on q^ω , which are true almost surely.

Firstly, Theorem 1.6 in [ADS16b] gives a short-range bound, which also applies to k_t : \mathbb{P} -a.s. there are constants $C, c_1, c_2 > 0$ such that for $t \geq Cn|x|$,

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq c_1 t^{-d/2} e^{-\frac{c_2 (n|x|)^2}{t}},$$

¹I would like to thank Sebastian Andres for his help regarding the details of this proof.

and for all $t \geq 0$,

$$k_t(x) \leq c_1 t^{-d/2} e^{-\frac{c_2|x|^2}{t}}.$$

Now, for a long-range bound: using Corollaries 11 and 12 from [Dav93], there exists \mathbb{P} -a.s. a constant $c_3 > 0$ such that for all $t \leq 2Cn$ and $|x| \geq 1$, we have

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq \frac{1}{\sqrt{\pi_0^\omega \pi_{\lfloor nx \rfloor}^\omega}} e^{-c_3 n |x|}.$$

Note that the integrability assumption implies that $\mathbb{E}[\rho_0^\omega] < \infty$, where $\rho_i^\omega = \sum_{l \sim i} \frac{1}{\omega_{i,l}}$. Therefore, for $|x| \leq 2$,

$$\frac{1}{\pi_{\lfloor nx \rfloor}^\omega} \leq \rho_{\lfloor nx \rfloor}^\omega \leq \sum_{|y| \leq 2n} \rho_y^\omega, \text{ as well as } \frac{1}{\pi_0^\omega} \leq \sum_{|y| \leq 2n} \rho_y^\omega,$$

and thanks to the ergodic theorem, \mathbb{P} -a.s. there exist $c_4 > 0$ and $N_0 \leq 1$ such that for $n \geq N_0$, $\sum_{|y| \leq 2n} \rho_y^\omega \leq c_4 (2n)^d \mathbb{E}[\rho_0^\omega]$. For such n and for $1 \leq |x| \leq 2$ and $t \leq 2Cn$, we get

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq c_5 n^d e^{-c_3 n |x|}.$$

Using these bounds, we now know that for $n \geq N_0$ and $1 \leq |x| \leq 2$, we have \mathbb{P} -a.s.

$$\begin{aligned} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| &= \left| n^d \int_0^\infty q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt - a \int_0^\infty k_t(x) dt \right| \\ &\leq n^d \int_0^{2C/n} q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + n^d \int_{2C/n}^{T_1} q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + a \int_0^{T_1} k_t(x) dt \\ &\quad + \int_{T_1}^{T_2} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)| dt + n^d \int_{T_2}^\infty q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + a \int_{T_2}^\infty k_t(x) dt \\ &\leq C' n^{2d-1} e^{-c_3 n} + (1+a) \int_0^{T_1} c_1 t^{-d/2} e^{-c_2/t} dt + (1+a) \int_{T_2}^\infty c_1 t^{-d/2} e^{-c_2/t} dt \\ &\quad + (T_2 - T_1) \sup_{|x| \leq 2} \sup_{t \in [T_1, T_2]} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)|. \end{aligned}$$

Let $\varepsilon > 0$. Since $t \mapsto c_1 t^{-d/2} e^{-c_2/t}$ is integrable on $(0, \infty)$, we can fix $T_1, T_2 > 0$ independently of x such that

$$\int_0^{T_1} c_1 t^{-d/2} e^{-c_2/t} dt + \int_{T_2}^\infty c_1 t^{-d/2} e^{-c_2/t} dt < \frac{\varepsilon}{2(1+a)}.$$

Then

$$\begin{aligned} \sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| &\leq (T_2 - T_1) \sup_{|x| \leq 2} \sup_{t \in [T_1, T_2]} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)| \\ &\quad + C' n^{2d-1} e^{-c_3 n} + \frac{\varepsilon}{2}, \end{aligned}$$

so that from Theorem 2.4.1, there exists $N \geq N_0$ independent of x such that for $n \geq N$,

$$\sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| \leq \varepsilon,$$

which is true \mathbb{P} -almost surely. \square

Remark 2.4.2. Let us fix conductances $\omega \in (\mathbb{R}_+^*)^{E_d}$. We denote by $(\tilde{Z}_n)_{n \in \mathbb{N}}$ the discrete version of $(Z_t)_{t \geq 0}$. Then, for $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} g^\omega(x, y) &= \int_0^\infty \frac{P_x^\omega[Z_t = y]}{\pi_y^\omega} dt = \frac{1}{\pi_y^\omega} E_x^\omega \left[\int_0^\infty \mathbf{1}_{\{Z_t = y\}} dt \right] \\ &= \frac{1}{\pi_y^\omega} E_x^\omega \left[\sum_{n=0}^\infty \mathbf{1}_{\{\tilde{Z}_n = y\}} \right] = \frac{1}{\pi_y^\omega} \sum_{n=0}^\infty P_x^\omega[\tilde{Z}_n = y], \end{aligned}$$

where $\sum_{n=0}^\infty P_x^\omega[\tilde{Z}_n = \cdot]$ is the Green kernel associated with $(\tilde{Z}_n)_{n \in \mathbb{N}}$ under P_x^ω . Indeed, since under P_x^ω the holding time of Z at each point is an exponential variable of parameter 1, the expected time spent by $(Z_t)_{t \geq 0}$ at y is exactly the expected number of visits of y by $(\tilde{Z}_n)_{n \in \mathbb{N}}$.

2.4.3 Martin boundary associated with Δ^ψ

We return to the VRJP on \mathbb{Z}^d , $d \geq 3$, with constant initial conductances W large enough so that the VRJP is almost surely transient. From Proposition 2.1.6, under $\nu_V^W(d\beta)$, we then have a.s. $\psi(i) > 0$ for all $i \in V$. Moreover, the random conductance model associated with conductances $c_{i,j}^\psi = W_{i,j}\psi(i)\psi(j)$ defines almost surely a transient random walk. We still denote by Δ^ψ the discrete Laplacian, and define $\pi_i^\psi = \sum_{j \sim i} c_{i,j}^\psi = 2\beta_i\psi(i)^2$, as well as g^ψ the corresponding Green kernel:

$$g^\psi(x, y) = \sum_{k=0}^\infty \mathbb{P}_x^\psi[X_k = y] = \frac{\psi(y)}{\psi(x)} 2\beta_j \hat{G}(i, j).$$

We want to identify the Martin boundary \mathcal{M}^ψ associated with Δ^ψ , by studying the behavior at infinity of the Martin kernel K^ψ , defined by

$$K^\psi(x, y) = \frac{g^\psi(x, y)}{g^\psi(0, y)}$$

for all $x, y \in \mathbb{Z}^d$. In order to do this, we will use Theorem 2.4.2.

Proposition 2.4.1. *There exists $\bar{W} > 0$ such that for $W > \bar{W}$, under $\nu_V^W(d\beta)$, the Martin boundary \mathcal{M}^ψ is almost surely trivial.*

Proof. Note that under $\nu_V^W(d\beta)$, the distribution of the random conductances $c_{i,j}^\psi$ is stationary and ergodic with respect to all isometries of \mathbb{Z}^d , thanks to Proposition 3 of [SZ19]. Moreover, for W large enough, the integrability assumption of Theorem 2.4.2 will be verified.

Lemma 2.4.1. *Consider the graph $\mathcal{G} = (V = \mathbb{Z}^d, E = E_d)$, with $d \geq 3$, with constant initial conductances W . Then for all $p \geq 1$, there exists $W_p > 0$ such that for $W > W_p$, for all $i \sim j$, under $\nu_V^W(d\beta)$ we have*

$$\mathbb{E}[(\psi(i)\psi(j))^p] < \infty \text{ and } \mathbb{E}[(\psi(i)\psi(j))^{-p}] < \infty.$$

Proof. The proof is the same as for Lemma 9 (i) in [SZ19], and uses Theorem 1 of [DSZ10], which gives a control on moments of arbitrarily large order. \square

Let $W^t > 0$ be such that for $W > W^t$, the VRJP on (\mathbb{Z}^d, W) is almost surely transient. Moreover, we define $\bar{W} = \max(W^t, W_{d+1})$. From now on, we assume that $W > \bar{W}$, so that thanks to Lemma 2.4.1, under $\nu_V^W(d\beta)$, for all $i \sim j$ we have

$$\mathbb{E}[(c_{i,j}^\psi)^{d+1}] < \infty \text{ and } \mathbb{E}[(c_{i,j}^\psi)^{-(d+1)}] < \infty,$$

Therefore, according to Theorem 2.4.2 and Remarks 2.4.1 and 2.4.2, there exists $\sigma^2 > 0$ such that ν_V^W -almost surely,

$$\sup_{1 \leq |z| \leq 2} |n^{d-2}(\pi_{[nz]}^\psi)^{-1} g^\psi(0, [nz]) - a g_{BM}(0, z)| \xrightarrow[n \rightarrow \infty]{} 0,$$

where $a = 1/\mathbb{E}[\pi_0^\psi]$ and g_{BM} is the Green kernel associated with a Brownian motion with covariance matrix $\sigma^2 I_d$, i.e.

$$g_{BM}(0, z) = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}\sigma^2} |z|^{2-d}.$$

Using this result, we have ν_V^W -almost surely: $g^\psi(0, y_n) \sim_{n \rightarrow \infty} a\pi_{y_n}^\psi g_{BM}(0, y_n)$, for any sequence $(y_n)_{n \geq 1}$ such that $|y_n| \rightarrow \infty$. Indeed, for such a sequence (y_n) , let us define $m_n = \lfloor |y_n| \rfloor$ and $z_n = y_n/m_n$. Then, since $1 \leq |z_n| \leq 2$ for all $n \geq 1$, we have

$$\begin{aligned} \left| \frac{g^\psi(0, y_n)}{a\pi_{y_n}^\psi g_{BM}(0, y_n)} - 1 \right| &= \left| \frac{(\pi_{[m_n z_n]}^\psi)^{-1} g^\psi(0, m_n z_n)}{m_n^{2-d} a g_{BM}(0, z_n)} - 1 \right| \\ &= \left| \frac{m_n^{d-2} (\pi_{[m_n z_n]}^\psi)^{-1} g^\psi(0, [m_n z_n]) - a g_{BM}(0, z_n)}{a g_{BM}(0, z_n)} \right| \\ &\leq \frac{\sup_{1 \leq |z| \leq 2} |m_n^{d-2} (\pi_{[m_n z]}^\psi)^{-1} g^\psi(0, [m_n z]) - a g_{BM}(0, z)|}{a \inf_{1 \leq |z| \leq 2} g_{BM}(0, z)} \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

ν_V^W -almost surely.

Moreover, for $x \in \mathbb{Z}^d$ fixed, let ψ^x be the translated function defined by $\psi^x(y) = \psi(y - x)$. Then ψ^x and ψ have the same distribution under $\nu_V^W(d\beta)$, therefore we have ν_V^W -a.s., for all $(y_n)_{n \geq 1}$ such that $|y_n| \rightarrow \infty$,

$$g^\psi(x, y_n) = g^{\psi^x}(0, y_n - x) \sim_{n \rightarrow \infty} a\pi_{y_n}^\psi g_{BM}(0, y_n - x),$$

since $|y_n - x| \rightarrow \infty$ and $\pi_{y_n - x}^{\psi^x} = \pi_{y_n}^\psi$. Let us denote by A_x the ν_V^W -almost sure event where this is true. Since \mathbb{Z}^d is denumerable, $\bigcap_{x \in \mathbb{Z}^d} A_x$ is still ν_V^W -almost sure. Therefore, we have ν_V^W -a.s. that for all $x \in \mathbb{Z}^d$, for all $(y_n)_{n \geq 1}$ such that $|y_n| \rightarrow \infty$,

$$K^\psi(x, y_n) \sim_{n \rightarrow \infty} \frac{a\pi_{y_n}^\psi g_{BM}(0, y_n - x)}{a\pi_{y_n}^\psi g_{BM}(0, y_n)} = \frac{|y_n - x|^{2-d}}{|y_n|^{2-d}} \xrightarrow[n \rightarrow \infty]{} 1.$$

As a result, from Remark 2.2.2, the Martin boundary associated with Δ^ψ is ν_V^W -a.s. trivial. \square

Let $\mathcal{R}(dr)$ be the distribution of an environment representing $\mathbb{P}^{VRJP(0)}$ on \mathbb{Z}^d endowed with constant initial conductances $W > \bar{W}$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we define β by $\beta_i = r_i + \mathbb{1}_{\{i=0\}}\gamma$. According to Theorem 2.1.3, under $\mathcal{R}(dr, d\gamma)$ we then have $\beta \sim \nu_V^W$. We define \hat{G} and ψ as functions of β , as in Theorem 2.1.2, and we can write

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(0, j)}{G(0, i)},$$

where $G(0, i) = \hat{G}(0, i) + h(i)$ for all $i \in \mathbb{Z}^d$, with h a H_β -harmonic function. Since W is large enough so that under $\nu_V^W(d\beta)$, $\psi(i) > 0$ for all $i \in \mathbb{Z}^d$ almost surely, the operator Δ^ψ is well-defined, and h/ψ is Δ^ψ -harmonic. However, according to Proposition 2.4.1 the Martin boundary associated with Δ^ψ is ν_V^W -a.s. trivial, therefore positive Δ^ψ -harmonic functions are almost surely constant. As a result, there is a nonnegative random variable g such that for all $i \in \mathbb{Z}^d$, we have \mathcal{R} -a.s.

$$G(0, i) = \hat{G}(0, i) + g\psi(i).$$

In particular, $g = (G(0, 0) - \hat{G}(0, 0))/\psi(0)$, so g can be written as a function of $(\beta, \frac{1}{2G(0,0)})$, and therefore has a function of $((r_i)_{i \in \mathbb{Z}^d}, \gamma)$. Since according to Proposition 2.3.1, under $\mathcal{R}(dr, d\gamma)$ the distribution of $((r_i)_{i \in \mathbb{Z}^d}, \gamma)$ does not depend on the chosen representation \mathcal{R} , this shows that the distribution of the jump rates $r_{i,j} = \frac{W_{i,j}}{2} \frac{G(0,j)}{G(0,i)}$ is uniquely determined, *i.e.* that the representation is unique.

Remark 2.4.3. Note that we can identify the distribution of g using the standard representation. This shows that under $\mathcal{R}(dr), d\gamma$, we have $g = \psi(0)/2\gamma'$, where γ' is a Gamma(1/2, 1) random variable independent from $(\beta_i)_{i \in \mathbb{Z}^d}$.

2.5 The potential $\beta_m^{(n)}$ on trees

Let T be an infinite tree, that is locally finite. We still fix an arbitrary root ϕ . In this section, we define a β field on the restrictions of T with the boundary conditions introduced in Section 2.1.3.2. This way, we can apply Theorem 2.1.1 to these finite restrictions.

2.5.1 Construction of $\beta_m^{(n)}$ on $\mathcal{G}_m^{(n)}$

From Proposition 2.1.3, under $\nu_T^W(d\beta)$, for all $n \in \mathbb{N}$ we have $\beta_{T^{(n)}} \sim \nu_{T^{(n)}}^{W^{(n)}, \eta^{(n)}}$, where

$$W^{(n)} = W_{T^{(n)}, T^{(n)}} \text{ and } \eta^{(n)} = W_{T^{(n)}, (T^{(n)})^c} \mathbf{1}_{(T^{(n)})^c}.$$

As usual, for $\beta \in \mathcal{D}_T^W$, we define $H_\beta = 2\beta - W$. For $n \in \mathbb{N}$, let us take $V_n = T^{(n)}$ in Definition 2.1.1, so that we get $\hat{G}^{(n)} = \hat{G}^{T^{(n)}} = ((H_\beta)_{T^{(n)}, T^{(n)}})^{-1}$ and $\psi^{(n)} = \hat{G}^{(n)} \eta^{(n)}$.

Let us fix $n \geq m \geq 0$. To represent the VRJP on $(\mathcal{G}_m^{(n)}, \tilde{W}_m^{(n)})$ using Theorem 2.1.1, we need to introduce a potential $\beta_m^{(n)}$ on $\tilde{T}_m^{(n)} = T^{(n)} \cup B_m$, distributed according to $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$. According to Proposition 2.2.2, this is true if and only if: the restriction $(\beta_m^{(n)})_{T^{(n)}}$ is distributed according to $\nu_{T^{(n)}}^{\hat{W}_m^{(n)}, \hat{\eta}_m^{(n)}}$, and conditionally on $(\beta_m^{(n)})_{T^{(n)}}$, the restriction $(\beta_m^{(n)})_{B_m}$ is distributed according to $\nu_{B_m}^{\tilde{W}_m^{(n)}}$, with the following notations :

$$\begin{aligned} \hat{W}_m^{(n)} &= (\tilde{W}_m^{(n)})_{T^{(n)}, T^{(n)}}, \hat{\eta}_m^{(n)} = (\tilde{W}_m^{(n)})_{T^{(n)}, B_m} \mathbf{1}_{B_m} \\ \text{and } \check{W}_m^{(n)} &= (\tilde{W}_m^{(n)})_{B_m, B_m} + (\tilde{W}_m^{(n)})_{B_m, T^{(n)}} ((H_\beta^{(n)})_{T^{(n)}, T^{(n)}})^{-1} (\tilde{W}_m^{(n)})_{T^{(n)}, B_m}, \end{aligned}$$

where we define by $H_\beta^{(n)} = 2\beta_m^{(n)} - \tilde{W}_m^{(n)}$ the Schrödinger operator associated with $\beta_m^{(n)}$. Given the definition of $\tilde{W}_m^{(n)}$, note that we have $\hat{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{T^{(n)}, T^{(n)}} = W^{(n)}$. Moreover, for $i \in T^{(n)}$,

$$(\hat{\eta}_m^{(n)})_i = \sum_{b \in B} (\tilde{W}_m^{(n)})_{i,b} = \mathbf{1}_{i \in D^{(n)}} \sum_{j \in S(i)} W_{i,j} = W_{\{i\}, D^{(n+1)}} \mathbf{1}_{D^{(n+1)}} = \eta_i^{(n)}.$$

Therefore, let us define $(\beta_m^{(n)})_{T^{(n)}} = \beta_{T^{(n)}}$. Hence, under $\nu_T^W(d\beta)$ we have $(\beta_m^{(n)})_{T^{(n)}} \sim \nu_{T^{(n)}}^{W^{(n)}, \eta^{(n)}}$, where $\nu_{T^{(n)}}^{W^{(n)}, \eta^{(n)}} = \nu_{T^{(n)}}^{\hat{W}_m^{(n)}, \hat{\eta}_m^{(n)}}$. It remains to extend $(\beta_m^{(n)})_{T^{(n)}}$ into a field $\beta_m^{(n)}$ over all of $\tilde{T}_m^{(n)}$. To do this, for $\beta' \in (\mathbb{R}_+^*)^{B_m}$, we define $(\beta_m^{(n)})_{B_m} = \beta'$. Then under $\nu_T^W(d\beta) \nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$, the field $\beta_m^{(n)}$ constructed this way on $\tilde{T}_m^{(n)}$ is distributed according to $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$, from Proposition 2.2.2. Note that $(\tilde{W}_m^{(n)})_{B_m, B_m} = 0$ and

$$(H_\beta^{(n)})_{T^{(n)}, T^{(n)}} = 2(\beta_m^{(n)})_{T^{(n)}} - (\tilde{W}_m^{(n)})_{T^{(n)}, T^{(n)}} = 2\beta_{T^{(n)}} - W^{(n)} = (H_\beta)_{T^{(n)}, T^{(n)}},$$

therefore we have

$$\begin{aligned}\check{W}_m^{(n)} &= (\check{W}_m^{(n)})_{B_m, B_m} + (\check{W}_m^{(n)})_{B_m, T^{(n)}} ((H_\beta^{(n)})_{T^{(n)}, T^{(n)}})^{-1} (\check{W}_m^{(n)})_{T^{(n)}, B_m} \\ &= (\check{W}_m^{(n)})_{B_m, T^{(n)}} \hat{G}^{(n)} (\check{W}_m^{(n)})_{T^{(n)}, B_m}.\end{aligned}$$

For $n \geq m$, let us denote by $G_m^{(n)} = (H_\beta^{(n)})^{-1} = (2\beta_m^{(n)} - \check{W}_m^{(n)})^{-1}$ the Green function associated with $\beta_m^{(n)}$ on $\mathcal{G}_m^{(n)}$. Then, from Theorem 2.1.1, we know that the law of the time-changed VRJP on $(\mathcal{G}_m^{(n)}, \check{W}_m^{(n)})$, started at $i_0 \in T^{(n)}$, is a mixture of Markov jump processes under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, where the jump rate from i to j is $\frac{1}{2}(\check{W}_m^{(n)})_{i,j} \frac{G_m^{(n)}(i_0, j)}{G_m^{(n)}(i_0, i)}$. In order to obtain a result on the infinite tree \mathcal{T} , we will show that $(G_m^{(n)}(i, j))_{i, j \in T}$ converges in distribution when $n \rightarrow \infty$.

2.5.2 Expression of $G_m^{(n)}(i, j)$ as a sum over paths

To show the convergence of $G_m^{(n)}$, it will be useful to express it as a sum over paths in \mathcal{T} . We will use notations and results presented in Section 2.2.2.

Let $\beta \in \mathcal{D}_T^W$ and $\beta' \in \mathcal{D}_{B_m}^{\check{W}_m^{(n)}}$, and $\beta_m^{(n)}$ be defined as previously. For $i, j \in T$, let us fix some $n_0 > m$ such that $i, j \in T^{(n_0)}$. For $n \geq n_0$, by applying Proposition 2.2.3 (i) to $\beta_m^{(n)}$ in the graph $\mathcal{G}_m^{(n)}$, we get

$$G_m^{(n)}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{\check{T}_m^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma}.$$

This sum over paths can be decomposed as follows: a path $\sigma \in \mathcal{P}_{i,j}^{\check{T}_m^{(n)}}$ can either cross some vertex in B_m , in which case $\sigma \in \mathcal{P}_{i, B_m, j}^{\check{T}_m^{(n)}}$, or never cross any vertex in B_m , in which case $\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}$. As a result, we have $\mathcal{P}_{i,j}^{\check{T}_m^{(n)}} = \mathcal{P}_{i,j}^{T^{(n)}} \cup \mathcal{P}_{i, B_m, j}^{\check{T}_m^{(n)}}$, where $\mathcal{P}_{i,j}^{T^{(n)}} \cap \mathcal{P}_{i, B_m, j}^{\check{T}_m^{(n)}} = \emptyset$, so

$$\begin{aligned}G_m^{(n)}(i, j) &= \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} + \sum_{\sigma \in \mathcal{P}_{i, B_m, j}^{\check{T}_m^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} \\ &= \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} + \sum_{b, b' \in B_m} \left(\sum_{\sigma \in \mathcal{P}_{i,b}^{T^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} \right) G_m^{(n)}(b, b') \left(\sum_{\sigma \in \mathcal{P}_{j,b'}^{T^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} \right),\end{aligned}$$

from Proposition 2.2.3. Note that for any $\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}$, $\frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} = \frac{W_\sigma}{(2\beta)_\sigma}$, since $(\check{W}_m^{(n)})_{T^{(n)}, T^{(n)}} = W_{T^{(n)}, T^{(n)}}$ and $(\beta_m^{(n)})_{T^{(n)}} = \beta_{T^{(n)}}$. As a result,

$$\sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\check{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} = \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} = \hat{G}^{(n)}(i, j).$$

Moreover, note that for $x \in D^{(m)}$ and $y \in T^{(n)}$,

$$\begin{aligned} \sum_{\sigma \in \overline{\mathcal{P}}_{y, \delta_x}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} &= \sum_{z \sim \delta_x} \left(\sum_{\sigma_1 \in \mathcal{P}_{y, z}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_{\sigma_1}}{(2\beta_m^{(n)})_{\sigma_1}} \right) (\tilde{W}_m^{(n)})_{z, \delta_x} = \sum_{z \in T_x \cap D^{(n)}} \hat{G}^{(n)}(y, z) \sum_{z' \in S(z)} W_{z, z'} \\ &= \hat{G}^{(n)}(y, \cdot) W_{T^{(n)}, (T^{(n)})^c} \mathbb{1}_{T_x \setminus T_x^{(n)}} = \chi_m^{(n)}(y, \delta_x), \end{aligned}$$

from Definition 2.1.3. As a result, we have

$$G_m^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \sum_{b, b' \in B_m} \chi_m^{(n)}(i, b) G_m^{(n)}(b, b') \chi_m^{(n)}(j, b').$$

We will show that under $\nu_T^W(d\beta)$, this expression converges almost surely when $n \rightarrow \infty$. From Theorem 2.1.2 (i), we already know that $\hat{G}^{(n)}(i, j)$ converges to $\hat{G}(i, j)$, let us now study the respective limits of $\chi_m^{(n)}$ and $(G_m^{(n)})_{B_m, B_m}$.

2.6 Convergence of $G_m^{(n)}$

The goal of this section is to prove the convergence of $\chi_m^{(n)}$ and $(G_m^{(n)})_{B_m, B_m}$. We will first describe the Martin boundary associated with Δ^ψ , and the harmonic measures $(\mu_i^\psi)_{i \in T}$.

2.6.1 Process associated with Δ^ψ and Martin boundary

First, note that for $\beta \in \mathcal{D}_T^W$, we have either $\psi(i) > 0$ for all $i \in T$, or $\psi \equiv 0$. In the first case, we can do as in Proposition 2.1.6: the random walk associated with the conductances $(c_{i,j}^\psi)_{i \sim j}$ is transient for ν_T^W -almost all β , since the associated Green function $g = g^\psi$ is given by

$$g^\psi(i, j) = \frac{\psi(j)}{\psi(i)} 2\beta_j \hat{G}(i, j),$$

for $i, j \in T$. Moreover, we define the Markov operator Δ^ψ by

$$(\Delta^\psi h)(i) = \left(\sum_{j \sim i} \frac{W_{i,j} \psi(j)}{2\beta_i \psi(i)} h(j) \right) - h(i) = 0$$

for $h : T \rightarrow \mathbb{R}$ and $i \in T$, and a function h is Δ^ψ -harmonic if and only if ψh is H_β -harmonic.

Let us fix $\beta \in \mathcal{D}_T^W$ such that $\psi(i) > 0$ for all $i \in T$, and such that the random walk associated with $(c_{i,j}^\psi)_{i \sim j}$ is transient. This allows us to apply results regarding the Martin boundary of a tree. From Theorem 2.2.3, the Martin boundary \mathcal{M}^ψ associated with Δ^ψ is the set Ω of ends of T . Note that it does not depend on β . We also get the Martin kernel $K = K^\psi$: for $x \in T$ and $\omega \in \Omega$,

$$K(x, \omega) = K(x, x \wedge \omega) = \frac{f(x, x \wedge \omega)}{f(\phi, x \wedge \omega)} = \frac{\psi(\phi) \hat{F}(x, x \wedge \omega)}{\psi(x) \hat{F}(\phi, x \wedge \omega)},$$

where $f(i, j) = \frac{g(i, j)}{g(j, j)} = \frac{\psi(j)}{\psi(i)} \hat{F}(i, j)$ for $i, j \in T$. Moreover, we denote by $(\mu_i^\psi)_{i \in T}$ the associated family of harmonic measures on Ω . From Proposition 2.2.5, we have, for $i, x \in T$,

$$\mu_i^\psi(\Omega_x) = \mathbb{1}_{\{i \in U_x\}} (1 - f(i, x)) + f(i, x) \frac{1 - f(x, \bar{x})}{1 - f(x, \bar{x}) f(\bar{x}, x)}.$$

Note that we have only defined $(\mu_y^\psi)_{y \in T}$ for $\beta \in \mathcal{D}_T^W$ such that $\psi > 0$ and the walk in conductances c^ψ is transient. In remaining cases, we adopt the convention that μ_y^ψ is the null measure on Ω for all $y \in T$.

2.6.2 Convergence of $\chi_m^{(n)}$: Proof of Theorem 2.1.6 (i)

From Theorem 2.1.2, we know that ν_T^W -almost surely, for all $i, j \in T$, $\hat{G}^{(n)}(i, j)$ converges to $\hat{G}(i, j)$ and $\psi^{(n)}(i)$ converges to $\psi(i)$. Let $\beta \in \mathcal{D}_T^W$ be such that these convergences hold. Let us show that for such β , for all $m \in \mathbb{N}$, $x \in D^{(m)}$ and for all $i \in T$, $\chi_m^{(n)}(i, \delta_x)$ converges to $\psi(i)\mu_i^\psi(\Omega_x)$, and we will have shown that this convergence holds ν_T^W -almost surely.

If β is such that $\psi \equiv 0$, we know that for all $i \in T$, $x \in D^{(m)}$, $0 \leq \chi_m^{(n)}(i, \delta_x) \leq \psi^{(n)}(i)$ from Remark 2.1.2, so $\chi_m^{(n)}(i, \delta_x) \xrightarrow{n \rightarrow \infty} 0 = \psi(i)\mu_i^\psi(\Omega_x)$. We now suppose that β is such that $\psi(i) > 0$ for all $i \in T$.

Let us fix $i \in T$ and $x \in D^{(m)}$. Recall that for $n \geq \max(|i|, m)$,

$$\chi_m^{(n)}(i, \delta_x) = \sum_{y \in T_x \cap D^{(n)}} \hat{G}^{(n)}(i, y) \eta_y^{(n)} = \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{i,y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}.$$

Let us decompose the paths $\sigma \in \mathcal{P}_{i,y}^{T^{(n)}}$, in order to write $\chi_m^{(n)}(i, \delta_x)$ as a function of $\hat{F}^{(n)}$ and $\psi^{(n)}$. We will distinguish two cases.

First, if $i \notin U_x = T_x \setminus \{x\}$, then for all $y \in T_x \cap D^{(n)}$, any path from i to y in $T^{(n)}$ necessarily visits x , i.e. $\mathcal{P}_{i,y}^{T^{(n)}} = \mathcal{P}_{i,\{x\},y}^{T^{(n)}}$. Therefore, from Proposition 2.2.3 (iii), $\hat{G}^{(n)}(i, y) = \hat{F}^{(n)}(i, x) \hat{G}^{(n)}(x, y)$. In order to decompose $\hat{G}^{(n)}(x, y)$, let us introduce the quantity $c_x(\sigma)$, defined as the number of times the path σ crosses the directed edge (x, \bar{x}) , i.e.

$$c_x(\sigma) = \#\{k \in \llbracket 0, |\sigma| - 1 \rrbracket, (\sigma_k, \sigma_{k+1}) = (x, \bar{x})\}.$$

Then we have

$$\hat{G}^{(n)}(x, y) = \sum_{C \in \mathbb{N}} \sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

If $\sigma \in \mathcal{P}_{x,y}^{T^{(n)}}$ is such that $c_x(\sigma) = C \geq 1$, then it has to visit \bar{x} at least once. As a result, σ can be written as the concatenation of a path $\sigma_1 \in \overline{\mathcal{P}}_{x,\bar{x}}^{T^{(n)} \setminus \{\bar{x}\}}$ with a path $\sigma'_1 \in \mathcal{P}_{\bar{x},y}^{T^{(n)}}$ such that $c_x(\sigma'_1) = C - 1$. Since $\bar{x} \notin U_x$, the path σ'_1 has to visit x , so it can be written as the concatenation of a path $\sigma_2 \in \overline{\mathcal{P}}_{\bar{x},x}^{T^{(n)} \setminus \{x\}}$ with a path $\sigma_3 \in \mathcal{P}_{x,y}^{T^{(n)}}$ such that $c_x(\sigma_3) = C - 1$. Therefore, for all $C \geq 1$,

$$\begin{aligned} \sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C}} \frac{W_\sigma}{(2\beta)_\sigma} &= \left(\sum_{\sigma_1 \in \overline{\mathcal{P}}_{x,\bar{x}}^{T^{(n)} \setminus \{\bar{x}\}}} \frac{W_{\sigma_1}}{(2\beta)_{\sigma_1}} \right) \left(\sum_{\sigma_2 \in \overline{\mathcal{P}}_{\bar{x},x}^{T^{(n)} \setminus \{x\}}} \frac{W_{\sigma_2}}{(2\beta)_{\sigma_2}} \right) \left(\sum_{\substack{\sigma_3 \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma_3) = C-1}} \frac{W_{\sigma_3}}{(2\beta)_{\sigma_3}} \right) \\ &= \hat{F}^{(n)}(x, \bar{x}) \hat{F}^{(n)}(\bar{x}, x) \sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C-1}} \frac{W_\sigma}{(2\beta)_\sigma}. \end{aligned}$$

Moreover, note that the paths $\sigma \in \mathcal{P}_{x,y}^{T^{(n)}}$ such that $c_x(\sigma) = 0$ are those that stay in the subtree $T_x^{(n)}$, *i.e.* the set $\mathcal{P}_{x,y}^{T_x^{(n)}}$. By induction, we get:

$$\hat{G}^{(n)}(x, y) = \sum_{C \in \mathbb{N}} \left(\hat{F}^{(n)}(x, \tilde{x}) \hat{F}^{(n)}(\tilde{x}, x) \right)^C \sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

Since $\hat{G}^{(n)}(x, y) < \infty$, we have $\hat{F}^{(n)}(x, \tilde{x}) \hat{F}^{(n)}(\tilde{x}, x) < 1$, which gives

$$\begin{aligned} \chi_m^{(n)}(i, \delta_x) &= \sum_{y \in T_x \cap D^{(n)}} \hat{F}^{(n)}(i, x) \hat{G}^{(n)}(x, y) \eta_y^{(n)} \\ &= \hat{F}^{(n)}(i, x) \frac{1}{1 - \hat{F}^{(n)}(x, \tilde{x}) \hat{F}^{(n)}(\tilde{x}, x)} \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}. \end{aligned}$$

In order to express this last sum, recall that

$$\begin{aligned} \psi^{(n)}(x) &= \sum_{y \in D^{(n)}} \hat{G}^{(n)}(x, y) \eta_y^{(n)} \\ &= \sum_{y \in D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x, \{\tilde{x}\}, y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} + \sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}, \end{aligned}$$

where we have separated the paths that go from x to y by visiting \tilde{x} , and those that stay in $T_x^{(n)}$, since $\mathcal{P}_{x,y}^{T^{(n)}} = \mathcal{P}_{x, \{\tilde{x}\}, y}^{T^{(n)}} \cup \mathcal{P}_{x,y}^{T_x^{(n)}}$. From Proposition 2.2.3 (iii), we have

$$\sum_{\sigma \in \mathcal{P}_{x, \{\tilde{x}\}, y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} = \hat{F}^{(n)}(x, \tilde{x}) \hat{G}^{(n)}(\tilde{x}, y).$$

Moreover, if $y \in D^{(n)} \setminus T_x$, then $\mathcal{P}_{x,y}^{T_x^{(n)}}$ is empty. As a result, we get

$$\sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)} = \psi^{(n)}(x) - \hat{F}^{(n)}(x, \tilde{x}) \psi^{(n)}(\tilde{x}),$$

which finally gives

$$\chi_m^{(n)}(i, \delta_x) = \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \tilde{x}) \psi^{(n)}(\tilde{x})}{1 - \hat{F}^{(n)}(x, \tilde{x}) \hat{F}^{(n)}(\tilde{x}, x)}.$$

In the second case, *i.e.* if $i \in U_x$, then for $y \in T_x \cap D^{(n)}$, there are paths from i to y in $T^{(n)}$ that do not visit x . More precisely, we have the following partition: $\mathcal{P}_{i,y}^{T^{(n)}} = \mathcal{P}_{i, \{x\}, y}^{T^{(n)}} \cup \mathcal{P}_{i,y}^{U_x^{(n)}}$, where $U_x^{(n)} = U_x \cap T^{(n)}$. As a result, we have

$$\chi_m^{(n)}(i, \delta_x) = \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \tilde{x}) \psi^{(n)}(\tilde{x})}{1 - \hat{F}^{(n)}(x, \tilde{x}) \hat{F}^{(n)}(\tilde{x}, x)} + \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{i,y}^{U_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}.$$

In the same way we did above, we can show that

$$\sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{i,y}^{U_x^{(n)}}} \frac{W_\sigma}{(2\beta)^\sigma} \right) \eta_y^{(n)} = \psi^{(n)}(i) - \hat{F}^{(n)}(i, x) \psi^{(n)}(x).$$

In conclusion, we have established the following:

$$\begin{aligned} \chi_m^{(n)}(i, \delta_x) &= \mathbf{1}_{\{i \in U_x\}} (\psi^{(n)}(i) - \hat{F}^{(n)}(i, x) \psi^{(n)}(x)) + \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \bar{x}) \psi^{(n)}(\bar{x})}{1 - \hat{F}^{(n)}(x, \bar{x}) \hat{F}^{(n)}(\bar{x}, x)} \\ &= \psi^{(n)}(i) \left(\mathbf{1}_{\{i \in U_x\}} (1 - f^{(n)}(i, x)) + f^{(n)}(i, x) \frac{1 - f^{(n)}(x, \bar{x})}{1 - f^{(n)}(x, \bar{x}) f^{(n)}(\bar{x}, x)} \right), \end{aligned}$$

where

$$f^{(n)}(i, j) = \frac{\psi^{(n)}(j)}{\psi^{(n)}(i)} \hat{F}^{(n)}(i, j) \xrightarrow{n \rightarrow \infty} \frac{\psi(j)}{\psi(i)} \hat{F}(i, j) = f(i, j)$$

for all $i, j \in T$. As a result, we finally have

$$\chi_m^{(n)}(i, \delta_x) \xrightarrow{n \rightarrow \infty} \psi(i) \left(\mathbf{1}_{\{i \in U_x\}} (1 - f(i, x)) + f(i, x) \frac{1 - f(x, \bar{x})}{1 - f(x, \bar{x}) f(\bar{x}, x)} \right) = \psi(i) \mu_i^\psi(\Omega_x).$$

We can now define, for all $i \in T$, the measure $\chi(i, \cdot) = \psi(i) \mu_i^\psi$. Note that $\chi(i, \cdot)$ is absolutely continuous with respect to $\chi(\phi, \cdot)$, and its Radon-Nikodym derivative is $\omega \mapsto \frac{\hat{F}(i, i \wedge \omega)}{\hat{F}(\phi, i \wedge \omega)}$. Moreover, for all $A \in \mathcal{B}(\Omega)$, $i \mapsto \mu_i^\psi(A)$ is Δ^ψ -harmonic, so $\chi(\cdot, A) : i \mapsto \psi(i) \mu_i^\psi(A)$ is H_β -harmonic.

2.6.3 Convergence of $(G_m^{(n)})_{B_m, B_m}$

Let us show that under $\nu_T^W(d\beta) \nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, the distribution of $(G_m^{(n)})_{B_m, B_m}$ converges weakly when $n \rightarrow \infty$.

We can write $(G_m^{(n)})_{B_m, B_m}$ as the inverse of a Schur complement. Indeed,

$$\begin{aligned} (G_m^{(n)})_{B_m, B_m} &= (H_\beta^{(n)})_{B_m, B_m}^{-1} = \left((H_\beta^{(n)})_{B_m, B_m} - (\check{W}_m^{(n)})_{B_m, T^{(n)}} ((H_\beta^{(n)})_{T^{(n)}, T^{(n)}})^{-1} (\check{W}_m^{(n)})_{T^{(n)}, B_m} \right)^{-1} \\ &= \left((2\beta_m^{(n)})_{B_m} - \check{W}_m^{(n)} \right)^{-1} = (\check{H}_\beta^{(n)})^{-1}, \end{aligned}$$

where $\check{H}_\beta^{(n)} = 2(\beta_m^{(n)})_{B_m} - \check{W}_m^{(n)}$. We apply the following change of variables: for $\beta \in \mathcal{D}_T^W$, $\beta' \in \mathcal{D}_{B_m}^{\check{W}_m^{(n)}}$ and $b \in B_m$, let us define $(\rho_m^{(n)})_b = (\beta_m^{(n)})_b - \frac{1}{2}(\check{W}_m^{(n)})_{b,b}$. Then $\check{H}_\beta^{(n)} = 2\rho_m^{(n)} - \check{C}_m^{(n)}$, where if $b, b' \in B_m$,

$$(\check{C}_m^{(n)})_{b,b'} = \begin{cases} (\check{W}_m^{(n)})_{b,b'} & \text{if } b \neq b' \\ 0 & \text{if } b = b'. \end{cases}$$

Under $\nu_T^W(d\beta) \nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, the vector $\rho_m^{(n)}$ is then distributed according to $\nu_{B_m}^{\check{C}_m^{(n)}}$ conditionally on β_T . Let us show that the matrix $\check{C}_m^{(n)}$ converges ν_T^W -almost surely, to prove that $\rho_m^{(n)}$ converges in distribution.

Let us fix $\beta \in \mathcal{D}_T^W$, as well as $x \neq y \in D^{(m)}$, and $i \sim \delta_x, j \sim \delta_y$. A path from i to j in $T^{(n)}$ necessarily crosses $x \wedge y$, since $i \in T_x$ and $j \in T_y$. Therefore, $\mathcal{P}_{i,j}^{T^{(n)}} = \mathcal{P}_{i,\{x \wedge y\},j}^{T^{(n)}}$, so

$$\begin{aligned} (\check{C}_m^{(n)})_{\delta_x, \delta_y} &= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i} \hat{G}^{(n)}(i, j) (\tilde{W}_m^{(n)})_{j, \delta_y} \\ &= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i} \left(\sum_{\sigma \in \mathcal{P}_{i,\{x \wedge y\},j}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) (\tilde{W}_m^{(n)})_{j, \delta_y} \\ &= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i} \hat{F}^{(n)}(i, x \wedge y) (\tilde{W}_m^{(n)})_{\delta_y, j} \hat{F}^{(n)}(j, x \wedge y) \hat{G}^{(n)}(x \wedge y, x \wedge y) \\ &= \frac{1}{\hat{G}^{(n)}(x \wedge y, x \wedge y)} \left(\sum_{i \sim \delta_x} (\tilde{W}_m^{(n)})_{\delta_x, i} \hat{G}^{(n)}(i, x \wedge y) \right) \left(\sum_{j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_y, j} \hat{G}^{(n)}(j, x \wedge y) \right) \\ &= \frac{\chi^{(n)}(x \wedge y, \delta_x) \chi^{(n)}(x \wedge y, \delta_y)}{\hat{G}^{(n)}(x \wedge y, x \wedge y)}, \end{aligned}$$

and $(\check{C}_m^{(n)})_{\delta_x, \delta_x} = 0$. Since $\chi^{(n)}$ converges ν_T^W -almost surely, the matrix $\check{C}_m^{(n)}$ also converges to a matrix \check{C}_m , where for $x, x' \in D^{(m)}$,

$$(\check{C}_m)_{\delta_x, \delta_{x'}} = \begin{cases} \frac{\chi(x \wedge y, \Omega_x) \chi(x \wedge y, \Omega_{x'})}{\hat{G}(x \wedge y, x \wedge y)} & \text{if } x \neq x' \\ 0 & \text{if } x = x'. \end{cases}$$

Under $\nu_T^W(d\beta) \nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$, and conditionally on β_T , $\rho_m^{(n)}$ is distributed according to $\nu_{B_m}^{\check{C}_m^{(n)}}$, which ν_T^W -almost surely converges weakly to $\nu_{B_m}^{\check{C}_m}$ by Lévy's theorem.

Therefore, the distribution of $(G_m^{(n)})_{B_m, B_m} = (2\rho_m^{(n)} - \check{C}_m^{(n)})^{-1}$ under $\nu_T^W(d\beta) \nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$ converges weakly to the distribution of $\check{G}_m := (2\rho_m - \check{C}_m)^{-1}$ under $\nu_T^W(d\beta) \nu_{B_m}^{\check{C}_m}(d\rho_m) =: \nu_{T, B_m}^W(d\beta, d\rho_m)$.

2.7 Representations of the VRJP on infinite trees

2.7.1 Representation of the VRJP using G_m : Proof of Theorem 2.1.6 (ii)

Recall that for $0 \leq m \leq n$ and $i, j \in T^{(n)}$,

$$G_m^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \sum_{b, b' \in B_m} \chi_m^{(n)}(i, b) G_m^{(n)}(b, b') \chi_m^{(n)}(j, b'),$$

where under $\nu_T^W(d\beta) \nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$, $G_m^{(n)} = (2\hat{\beta}_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$ provides a representation on the VRJP on $\mathcal{G}_m^{(n)}$, from Theorem 2.1.1. We have shown that when $n \rightarrow \infty$, $\chi_m^{(n)}$ converges almost surely, and $(\check{C}_m^{(n)})_{B_m, B_m}$ converges in distribution. As a result for all $i, j \in T$, the distribution of $G_m^{(n)}(i, j)$ converges weakly to the distribution of $G_m(i, j)$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$, where

$$G_m(i, j) = \hat{G}(i, j) + \sum_{x, x' \in D^{(m)}} \chi(i, \Omega_x) \chi(j, \Omega_y) \check{G}_m(\delta_x, \delta_y).$$

The second term in $G_m(i, j)$ can be rewritten as an integral on Ω^2 . Indeed, let us define $\check{g}_m : \Omega^2 \rightarrow \mathbb{R}$ in the following way: for $\omega, \tau \in \Omega$, if $x, y \in D^{(m)}$ are such that $\omega \in \Omega_x$ and $\tau \in \Omega_y$, we set $\check{g}_m(\omega, \tau) = \check{G}_m(\delta_x, \delta_y)$. With these new notations, we can now write, for $i, j \in T$,

$$\begin{aligned} G_m(i, j) &= \hat{G}(i, j) + \chi_m(i, \cdot) \check{G}_m^t(\chi_m(j, \cdot)) \\ &= \hat{G}(i, j) + \int_{\Omega^2} \chi(i, d\omega) \chi(j, d\tau) \check{g}_m(\omega, \tau). \end{aligned}$$

For $\beta \in \mathcal{D}_T^W$, $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, and all $i_0, i, j \in T$, we denote $r_{i,j}^{\beta, \rho_m, i_0} = \frac{W_{i,j} G_m(i_0, j)}{2 G_m(i_0, i)}$. To prove Theorem 2.1.6 (ii), we have to see that the process with distribution $\mathbb{P}^{VRJP(i_0)}$ is a mixture of Markov processes $\mathbb{P}_{i_0}^{\beta, \rho_m, i_0}$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$. The proof is the same than that of Theorem 2.1.2 (iii) (see [SZ19]). It consists in studying trajectories of the time-changed VRJP, stopped when they leave a finite subgraph included in $T^{(n)}$. They can be considered as trajectories of the time-changed VRJP on $\mathcal{G}_m^{(n)}$, and represented using $G_m^{(n)}$ thanks to Theorem 2.1.1. Taking the limit in distribution when $n \rightarrow \infty$ then gives the result. Note that the proof needs an argument of uniform integrability on the family $\left(\frac{G_m^{(n)}(i_0, j)}{G_m^{(n)}(i_0, i)} \right)_{n \geq m}$ for all $i, j \in T$, which is given by Proposition 7 and Corollary 2 from [SZ19].

2.7.2 Convergence to another representation: Proof of Theorem 2.1.6 (iii)

Let us show that the representations of the VRJP built with G_m converge in distribution when $m \rightarrow \infty$ to the representation described in Theorem 2.1.5. To show this, we use a tightness argument, based on the following lemma regarding the distribution ν_V^W .

Lemma 2.7.1. *Let $\mathcal{G} = (V, E)$ be a finite graph, endowed with conductances W . We denote $G = (H_\beta)^{-1}$ for $\beta \in \mathcal{D}_V^W$. Then for all $\eta \in \mathbb{R}_+^V$, under $\nu_V^W(d\beta)$, $\langle \eta, G\eta \rangle$ has the same distribution as $\frac{\langle \eta, \mathbf{1} \rangle^2}{2\gamma}$, where γ is a Gamma random variable with parameter $(1/2, 1)$.*

Proof. Let $\eta \in \mathbb{R}_+^V$ be fixed. We will compute the Laplace transform of $\langle \eta, G\eta \rangle$: for $\lambda \in \mathbb{R}_+$,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \langle \eta, G\eta \rangle} \right] &= \int e^{-\lambda \langle \eta, (H_\beta)^{-1} \eta \rangle} \nu_V^W(d\beta) \\ &= e^{-\sqrt{2\lambda} \langle \eta, \mathbf{1} \rangle} \int e^{-\frac{1}{2} \langle \sqrt{2\lambda} \eta, (H_\beta)^{-1} \sqrt{2\lambda} \eta \rangle} e^{\langle \sqrt{2\lambda} \eta, \mathbf{1} \rangle} \nu_V^W(d\beta) \\ &= e^{-\sqrt{2\lambda} \langle \eta, \mathbf{1} \rangle} \int \nu_V^{W, \sqrt{2\lambda} \eta}(d\beta) = e^{-\sqrt{2\lambda} \langle \eta, \mathbf{1} \rangle}, \end{aligned}$$

since $\nu_V^{W, \sqrt{2\lambda} \eta}$ is a probability measure. Let us now compute, for $\gamma \sim \text{Gamma}(1/2, 1)$, the Laplace transform of $\frac{1}{2\gamma}$: for $\lambda \in \mathbb{R}_+$,

$$\mathbb{E} \left[e^{-\frac{\lambda}{2\gamma}} \right] = \int \mathbf{1}_{u>0} \frac{e^{-u}}{\sqrt{\pi u}} e^{-\frac{\lambda}{2u}} du = \int_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{1}{2}(\frac{1}{v} + 2\lambda v)} dv = e^{-\sqrt{2\lambda}} \int_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{2\lambda}{v} (v - \frac{1}{\sqrt{2\lambda}})^2} dv,$$

by taking $v = 1/2u$. Since $\mathbf{1}_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{2\lambda}{v} (v - \frac{1}{\sqrt{2\lambda}})^2} dv$ is the density of an Inverse Gaussian distribution with parameter $(1, 1/\sqrt{2\lambda})$, we finally get $\mathbb{E}[e^{-\frac{\lambda}{2\gamma}}] = e^{-\sqrt{2\lambda}}$. Therefore, for all $\lambda \geq 0$,

$$\mathbb{E} \left[e^{-\lambda \langle \eta, G\eta \rangle} \right] = e^{-\sqrt{2\lambda} \langle \eta, \mathbf{1} \rangle} = \mathbb{E} \left[e^{-\frac{\lambda \langle \eta, \mathbf{1} \rangle^2}{2\gamma}} \right],$$

which proves the result. \square

For $m \geq 0$ and $\beta \in \mathcal{D}_T^W$, let us define, for $i \in T$ and $m \geq 0$, the vector $\bar{\mu}_i^{(m)} \in \mathbb{R}^{B_m}$ by $(\bar{\mu}_i^{(m)})(\delta_x) = \mu_i^\psi(\Omega_x)$ for all $x \in D^{(m)}$. Then, for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$ and $i, j \in T$,

$$\begin{aligned} \int_{\Omega^2} \chi(i, d\omega) \chi(j, d\tau) \check{g}_m(\omega, \tau) &= \psi(i) \psi(j) \sum_{x, x' \in D^{(m)}} \int_{\Omega_x \times \Omega_{x'}} \mu_i(d\omega) \mu_j(d\tau) \check{G}_m(\delta_x, \delta_{x'}) \\ &= \psi(i) \psi(j) \langle \bar{\mu}_i^{(m)}, \check{G}_m \bar{\mu}_j^{(m)} \rangle. \end{aligned}$$

We denote, for $m \geq 0$ and $i, j \in T$,

$$a_{i,j}^{(m)} = \frac{1}{4} \langle \bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}, \check{G}_m(\bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}) \rangle.$$

so that we have

$$\begin{aligned} G_m(i, j) &= \hat{G}(i, j) + \psi(i) \psi(j) \langle \bar{\mu}_i^{(m)}, \check{G}_m \bar{\mu}_j^{(m)} \rangle \\ &= \hat{G}(i, j) + \frac{\psi(i) \psi(j)}{2} \left(4a_{i,j}^{(m)} - a_i^{(m)} - a_j^{(m)} \right). \end{aligned}$$

Therefore, we can write $(G_m(i, j))_{i,j \in T} = \Phi \left((\hat{G}(i, j))_{i,j \in T}, (\psi(i))_{i \in T}, (a_{i,j}^{(m)})_{i,j \in T} \right)$, where Φ is a continuous function.

We will denote by $\bar{\nu}_T^W(d\beta, d\rho)$ the distribution of a coupling of distributions $\nu_{T, B_m}^W(d\beta, d\rho_m)$ for all $m \geq 0$.

Lemma 2.7.2. *Let us set $Z_m = \left((\hat{G}(i, j))_{i,j \in T}, (\psi(i))_{i \in T}, (a_{i,j}^{(m)})_{i,j \in T} \right)$ for $m \geq 0$, which takes its values in $\mathbb{R}^{T^2} \times \mathbb{R}^T \times \mathbb{R}^{T^2}$. Then under $\bar{\nu}_T^W(d\beta, d\rho)$, $(Z_m)_{m \geq 0}$ is tight.*

Proof. For $\varepsilon > 0$, let K_ε be a compact subset of \mathbb{R} such that $0 \in K_\varepsilon$ and

$$\mathbb{P} \left[\frac{1}{2\gamma} \in K_\varepsilon \right] > 1 - \varepsilon$$

when $\gamma \sim \text{Gamma}(1/2, 1)$. Let us now fix $m \geq 0$. Recall that $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, where \check{C}_m is a β -measurable matrix of conductances on B_m , and conditionally on β , ρ_m is distributed according to $\nu_{B_m}^{\check{C}_m}$. Therefore for $i, j \in T$, from Lemma 2.7.1, $\langle \bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}, \check{G}_m(\bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}) \rangle$ has conditionally on β the same distribution as $\frac{(\mu_i^\psi(\Omega) + \mu_j^\psi(\Omega))^2}{2\gamma}$, where $\gamma \sim \text{Gamma}(1, 1/2)$. This implies that conditionally on $\{\psi \neq 0\}$, $a_{i,j}^{(m)}$ has the same distribution as $\frac{1}{2\gamma}$, and conditionally on $\{\psi \equiv 0\}$, $a_{i,j}^{(m)} = 0$. As a result, for all $\varepsilon > 0$,

$$\mathbb{P} \left[a_{x,x'}^{(m)} \in K_\varepsilon \right] = \mathbb{P}[\psi \equiv 0] + \mathbb{P}[\psi \neq 0] \mathbb{P} \left[\frac{1}{2\gamma} \in K_\varepsilon \right] > 1 - \varepsilon.$$

Let now $(\tilde{a}_k^{(m)})_{k \in \mathbb{N}}$ be an enumeration of $(a_{i,j}^{(m)})_{i,j \in T}$. Then for $\varepsilon > 0$,

$$\mathbb{P} \left[\forall k \in \mathbb{N}, \tilde{a}_k^{(m)} \in K_{2^{-n-1}\varepsilon} \right] \geq 1 - \sum_{k \in \mathbb{N}} 2^{-n-1}\varepsilon = 1 - \varepsilon,$$

where $\tilde{K}_\varepsilon = \prod_{k \in \mathbb{N}} K_{2^{-n-1}\varepsilon}$ is a compact subset of $\mathbb{R}^{\mathbb{N}}$. Moreover, the β -measurable random variable $\left((\hat{G}(i, j))_{i,j \in T}, (\psi(i))_{i \in T} \right)$ takes its values in $\mathbb{R}^{T^2} \times \mathbb{R}^T$, where T is countable. As a result, for all $\varepsilon > 0$, there is a compact subset $K'_\varepsilon \subset \mathbb{R}^{T^2} \times \mathbb{R}^T$ such that

$$\mathbb{P} \left[\left((\hat{G}(i, j))_{i,j \in T}, (\psi(i))_{i \in T} \right) \in K'_\varepsilon \right] > 1 - \varepsilon.$$

We can now conclude that for all $\varepsilon > 0$,

$$\mathbb{P} \left[\left((\hat{G}(i, j))_{i, j \in T}, (\psi(i))_{i \in T}, (a_{i, j}^{(m)})_{i, j \in T} \right) \in \tilde{K}_{\varepsilon/2} \times K'_{\varepsilon/2} \right] > 1 - \varepsilon,$$

where $\tilde{K}_{\varepsilon/2} \times K'_{\varepsilon/2}$ is compact, and does not depend on m . \square

As a result, there is an extraction $(m_k)_{k \in \mathbb{N}}$ such that $(Z_{m_k})_{k \in \mathbb{N}}$ converges in distribution under $\bar{\nu}_T^W(d\beta, d\rho)$. Since $G_{m_k} = \Phi(Z_{m_k})$ where Φ is continuous, $(G_{m_k}(i, j))_{i, j \in T}$ also converges in distribution under $\bar{\nu}_T^W(d\beta, d\rho)$, as do the random jump rates $(r_{i, j}^{(m_k), \phi})_{i, j \in T}$. Let us show that the limit distribution of the environment does not depend on the extraction, which will mean that $((r_{i, j}^{(m), \phi})_{i, j \in T})_{m \in \mathbb{N}}$ converges in distribution, since it is tight.

Lemma 2.7.3. *For $m \geq 1$ and for all $n \geq m$, under the distribution $\nu_T^W(d\beta)\nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$, the random variables $\left(\frac{G_m^{(n)}(\phi, i)}{G_m^{(n)}(\phi, \tilde{i})} \right)_{i \in T^{(m)} \setminus \{\phi\}}$ are independent inverse Gaussian variables, where $\frac{G_m^{(n)}(\phi, \tilde{i})}{G_m^{(n)}(\phi, \tilde{i})}$ has parameter $(W_{\tilde{i}, \tilde{i}}^{\tilde{i}}, 1)$ for $i \in T^{(m)} \setminus \{\phi\}$.*

Proof. Let us fix $1 \leq m \leq n$. For $i \in T^{(m)} \setminus \{\phi\}$, we denote $g_i = \frac{G_m^{(n)}(\phi, i)}{G_m^{(n)}(\phi, \tilde{i})}$. Since $|i| \leq m$, any path in $\mathcal{G}_m^{(n)}$ from ϕ to i crosses \tilde{i} , so from Proposition 2.2.3 (ii) and (iii),

$$g_i = \frac{G_m^{(n)}(\tilde{i}, i)}{G_m^{(n)}(\tilde{i}, \tilde{i})} = \sum_{j \sim \tilde{i}} W_{\tilde{i}, j} \sum_{\sigma \in \mathcal{P}_{\tilde{i}, i}^{\tilde{T}_m^{(n)} \setminus \{\tilde{i}\}}} \frac{(\tilde{W}_m^{(n)})_{\sigma}}{(2\beta_m^{(n)})_{\sigma}}.$$

For $i \in T^{(m)} \setminus \{\phi\}$, let us denote by \tilde{T}_i the connected component of i in $\tilde{T}_m^{(n)} \setminus \{\tilde{i}\}$, i.e. $\tilde{T}_i = T_m^{(n)} \cup \{\delta_x, x \in D^{(m)} \cap T_i\}$, endowed with the restriction of $\tilde{W}_m^{(n)}$. This way, we get

$$g_i = W_{\tilde{i}, i} \sum_{\sigma \in \mathcal{P}_{\tilde{i}, i}^{\tilde{T}_i}} \frac{(\tilde{W}_m^{(n)})_{\sigma}}{(2\beta_m^{(n)})_{\sigma}} = W_{\tilde{i}, i} \left((H_{\beta}^{(n)})_{\tilde{T}_i, \tilde{T}_i} \right)^{-1}(i, i),$$

so g_i is $(\beta_m^{(n)})_{\tilde{T}_i}$ -measurable.

To prove that $(g_i)_{i \in T^{(m)} \setminus \{\phi\}}$ are independent, it will be enough to see that for $i \in T^{(m)} \setminus \{\phi\}$, g_i is independent of $g_{U_i^{(m)}}$, and that for $x \in T^{(m-1)}$, the restrictions $(g_{\tilde{T}_i})_{i \in S(x)}$ are independent.

Writing $\left((H_{\beta}^{(n)})_{\tilde{T}_i, \tilde{T}_i} \right)^{-1}(i, i)$ as a Schur complement, we see that, if we set $\tilde{U}_i = \tilde{T}_i \setminus \{i\}$,

$$g_i = \frac{W_{\tilde{i}, i}}{2(\beta_m^{(n)})_i - \sum_{j, j' \in S(i)} W_{i, j} W_{i, j'} \left((H_{\beta}^{(n)})_{\tilde{U}_i, \tilde{U}_i} \right)^{-1}(j, j')} = \frac{W_{\tilde{i}, i}}{\tilde{H}_{\beta}^{\{i\}}}.$$

From Proposition 2.2.2, conditionally on $(\beta_m^{(n)})_{\tilde{U}_i}$, the distribution of $(\beta_m^{(n)})_i$ is given by

$$\sqrt{\frac{2}{\pi}} \frac{\mathbb{1}_{\tilde{H}_{\beta}^{\{i\}} > 0}}{\sqrt{\det(\tilde{H}_{\beta}^{\{i\}})}} e^{-\frac{1}{2}(\tilde{H}_{\beta}^{\{i\}} + W_{\tilde{i}, i}^2 (\tilde{H}_{\beta}^{\{i\}})^{-1})} e^{W_{\tilde{i}, i} d(\beta_m^{(n)})_i},$$

so by a change of variables, the distribution of g_i conditionally on $(\beta_m^{(n)})_{\tilde{U}_i}$ is

$$\mathbb{1}_{g_i > 0} \sqrt{\frac{W_{\tilde{i},i}^-}{2\pi g_i^3}} e^{-\frac{W_{\tilde{i},i}^-}{2g_i} (g_i - 1)^2} dg_i,$$

i.e. $g_i \sim IG(W_{\tilde{i},i}^-, 1)$. Since this distribution does not depend on $(\beta_m^{(n)})_{\tilde{U}_i}$, g_i is independent of $(\beta_m^{(n)})_{\tilde{U}_i}$. For all $j \in U_i^{(m)}$, $\tilde{T}_j \subset \tilde{U}_i$, so g_j is $(\beta_m^{(n)})_{\tilde{U}_i}$ -measurable. Therefore, g_i is independent of $g_{U_i^{(m)}}$.

Moreover, for $x \in T^{(m-1)}$, the sets $(\tilde{T}_i)_{i \in S(x)}$ are all at distance 2 from one another in $\mathcal{G}_m^{(n)}$. Since $\beta^{(n)}$ is 1-dependent, the restrictions $(\beta_{\tilde{T}_i}^{(n)})_{i \in S(x)}$ are independent. For $j \in \tilde{T}_i$, we have $\tilde{T}_j \subset \tilde{T}_i$, so g_j is $\beta_{\tilde{T}_i}^{(n)}$ -measurable. Therefore the restrictions $(g_{\tilde{T}_i})_{i \in S(x)}$ are independent, which concludes the proof. \square

For $m \geq 1$, the distribution of $G_m^{(n)}$ under $\nu_T^W(d\beta)\nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$ converges weakly to the distribution of G_m under $\nu_{T, B_m}^W(d\beta, d\rho_m)$. If we denote $g_i^{(m)} = \frac{G_m(\phi, i)}{G_m(\phi, \tilde{i})}$ for $i \in T^{(m)} \setminus \{\phi\}$, and take the limit in Lemma 2.7.3, we get that under $\nu_{T, B_m}^W(d\beta, d\rho_m)$, $(g_i^{(m)})_{i \in T^{(m)} \setminus \{\phi\}}$ are independent, and $g_i^{(m)} \sim IG(W_{\tilde{i},i}^-, 1)$ for $i \in T^{(m)} \setminus \{\phi\}$. Recall that the random environment associated with G_m is given by the following jump rates:

$$r_{\tilde{i},i}^{\beta, \rho_m, \phi} = \frac{W_{\tilde{i},i}^- G_m(\phi, i)}{2 G_m(\phi, \tilde{i})} \text{ and } r_{i, \tilde{i}}^{\beta, \rho_m, \phi} = \frac{W_{\tilde{i},i}^- G_m(\phi, \tilde{i})}{2 G_m(\phi, i)},$$

for all $i \in T \setminus \{\phi\}$, and $r_{i,j}^{\beta, \rho_m, \phi} = 0$ if $i \not\sim j$.

Let now $(m_k)_{k \in \mathbb{N}}$ be an extraction such that under $\bar{\nu}_T^W(d\beta, d\rho)$, $(r_{i,j}^{\beta, \rho_{m_k}, \phi})_{i,j \in T}$ converges in distribution to a limit environment $(r_{i,j}^{(\infty), \phi})_{i,j \in T}$. Then, we have $r_{i,j}^{(\infty), \phi} = 0$ for $i \not\sim j$. Moreover, let us set $g_i^{(\infty)} = \frac{2}{W_{\tilde{i},i}^-} r_{\tilde{i},i}^{(\infty), \phi}$ for $i \in T \setminus \{\phi\}$. Note that for all $m \in \mathbb{N}$, if k is such that $m_k \geq m$, we have $T^{(m)} \subset T^{(m_k)}$ so for all $i \in T^{(m)} \setminus \{\phi\}$,

$$r_{\tilde{i},i}^{\beta, \rho_{m_k}, \phi} = \frac{W_{\tilde{i},i}^- g_i^{(m_k)}}{2} \text{ and } r_{i, \tilde{i}}^{\beta, \rho_{m_k}, \phi} = \frac{W_{\tilde{i},i}^-}{2g_i^{(m_k)}}.$$

Taking $k \rightarrow \infty$, we get that $(g_i^{(m_k)})_{i \in T^{(m)} \setminus \{\phi\}}$ converges in distribution to $(g_i^{(\infty)})_{i \in T^{(m)} \setminus \{\phi\}}$, which implies that $(g_i^{(\infty)})_{i \in T^{(m)} \setminus \{\phi\}}$ are independent. Since this is true for all $m \geq 0$, $(g_i^{(\infty)})_{i \in T \setminus \{\phi\}}$ are independent. Moreover, for all $i \in T \setminus \{\phi\}$, $g_i^{(\infty)} \sim IG(W_{\tilde{i},i}^-, 1)$ and

$$r_{\tilde{i},i}^{(\infty), \phi} = \frac{W_{\tilde{i},i}^-}{2} g_i^{(\infty)} \text{ and } r_{i, \tilde{i}}^{(\infty), \phi} = \frac{W_{\tilde{i},i}^-}{2g_i^{(\infty)}}.$$

The random environment given by these jump rates is in fact the one described in Theorem 2.1.5, hence its distribution does not depend on the extraction $(m_k)_{k \in \mathbb{N}}$. Since the sequence of jump rates $((r_{i,j}^{\beta, \rho_{m_k}, \phi})_{i,j \in T})_{m \geq 1}$ is tight, this implies that under $\bar{\nu}_T^W(d\beta, d\rho)$, it converges in distribution to the random environment given in Theorem 2.1.5.

2.7.3 Distinct representations on a regular tree: Proofs of Propositions 2.1.7 and 2.1.8

Let us start by proving that on regular trees where the VRJP is transient, the standard representation and the one given in Theorem 2.1.5 are different.

Proof of Proposition 2.1.7. Let $\mathcal{T} = (T, E)$ be a d -regular tree, where $d \geq 3$. It was shown in [DV04] that there exists a $\bar{W} > 0$ such that for $W > \bar{W}$, the VRJP on \mathcal{T} endowed with constant conductances W is almost surely transient. Note that the VRJP is defined in a slightly different manner in [DV04], but it can be related to the definition used here, thanks to a time rescaling described in Appendix B of [STZ17]. From now on, we take $W > \bar{W}$.

We consider jump rates $(r_{i,j})_{i \sim j}$ on the tree \mathcal{T} . Let ϕ be an arbitrary root for \mathcal{T} , and let $(i_k)_{k \geq 0}$ be an infinite self-avoiding path (or ray) in \mathcal{T} , such that for $k \geq 0$, $|i_k| = k$. Let us define $S_n = \prod_{k=1}^n \frac{2}{W} r_{i_{k-1}, i_k}$. We will compare the distribution of S_n under two distribution of jump rates.

Let $\mathcal{R}_{ind}(dr)$ be the distribution of jump rates in the representation described in Theorem 2.1.5. Under $\mathcal{R}_{ind}(dr)$, we know that S_n has the distribution of $\prod_{i=1}^n A_{i_k}$, where A_{i_k} are independent inverse Gaussian variables with parameter $(W, 1)$. Note that $\mathbb{E}[A_1] = 1$, so by Jensen's inequality, $\mathbb{E}[\log(A_1)] < 0$. By the law of large numbers, we then have a.s. that $\sum_{k=1}^n \log(A_i) \xrightarrow[n \rightarrow \infty]{} -\infty$, so that $S_n \xrightarrow[n \rightarrow \infty]{a.s.} 0$.

Let now $\mathcal{R}_{st}(dr)$ be the distribution of jump rates in the standard representation of the VRJP started at $\phi = i_0$. Under $\mathcal{R}_{st}(dr)$, Theorem 2.1.2 tells us that S_n has the same distribution as

$$\prod_{k=1}^n \frac{G(i_0, i_k)}{G(i_0, i_{k-1})} = \frac{G(i_0, i_n)}{G(i_0, i_0)} = \frac{\hat{G}(i_0, i_n) + \frac{1}{2\gamma} \psi(i_0) \psi(i_n)}{G(i_0, i_0)}$$

under $\nu_V^W(d\beta, d\gamma)$, where according to Proposition 2.1.6, $\psi(i) > 0$ a.s. for all $i \in T$. Moreover, since the distribution of ψ under $\nu_V^W(d\beta)$ is stationary for the group of transformations of \mathcal{T} (see Proposition 3 in [SZ19]), $\psi(i_n)$ has the same distribution as $\psi(i_0)$ for all $n \in \mathbb{N}$, and cannot tend to 0 a.s. when $n \rightarrow \infty$. Therefore, neither can S_n under $\mathcal{R}_{st}(dr)$, which proves that \mathcal{R}_{st} and \mathcal{R}_{ind} are different. \square

We will now prove Proposition 2.1.8, *i.e.* that on a d -regular tree with $d \geq 3$, for constant W large enough so that the VRJP is transient, the representations in the family given by Theorem 2.1.6 are all different. In order to do this, we will compare the distribution of the random harmonic measures for each representation.

The following Proposition gives an expression for the measure of sets Ω_x , for $x \in T$. We will see how this expression behaves differently whether or not $|x| > m$.

Proposition 2.7.1. *Let $m \in \mathbb{N}$ be fixed. We also fix $\beta \in \mathcal{D}_T^W$ and $\rho_m \in \mathcal{D}_{B_m}^{\tilde{C}_m}$. We denote by $\mu_\phi^{\beta, \rho_m, \phi}$ the exiting measure of the transient Markov process $\mathbb{P}_\phi^{\beta, \rho_m, \phi}$ defined in Theorem 2.1.6. Then for $x \in T$,*

$$\mu_\phi^{\beta, \rho_m, \phi}(\Omega_x) = \frac{\int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}{\int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}.$$

Proof. We denote $\mu = \mu_\phi^{\beta, \rho_m, \phi}$. Let g be the Green function associated with the discrete Markov chain associated with $\mathbb{P}_i^{\beta, \rho_m, \phi}$, *i.e.* with jump rates $r_{i,j}^{\beta, \rho_m, \phi} = \frac{W_{i,j}}{2} \frac{G_m(\phi, j)}{G_m(\phi, i)}$. Let us denote, for $i, j \in T$, $f(i, j) = \frac{g(i, j)}{g(j, j)}$. Then from Proposition 2.2.5, we get the following expression for $x \neq \phi$:

$$\mu(\Omega_x) = f(\phi, x) \frac{1 - f(x, \bar{x})}{1 - f(x, \bar{x})f(\bar{x}, x)}.$$

For $i, j \in T$, we have

$$g(i, j) = \sum_{k \in \mathbb{N}} \mathbb{P}_i^{\beta, \rho_m, \phi}[X_k = j] = \sum_{\sigma \in \mathcal{P}_{i,j}^T} \prod_{k=0}^{|\sigma|-1} \frac{r_{\sigma_k, \sigma_{k+1}}^{\beta, \rho_m, \phi}}{\tilde{\beta}_{\sigma_k}} = \frac{G_m(\phi, j)}{G_m(\phi, i)} \sum_{\sigma \in \mathcal{P}_{i,j}^T} \frac{W_\sigma}{(2\tilde{\beta})^\sigma},$$

where $\tilde{\beta}_i = \sum_{j \sim i} r_{i,j}^{\beta, \rho_m, \phi}$ are the rates of the corresponding holding times. Note that $\tilde{\beta}_i = \beta_i - \mathbb{1}_{\{i=\phi\}} \frac{1}{2G_m(\phi, \phi)}$ for $i \in T$. In particular, if a path σ never crosses ϕ , then $(2\tilde{\beta})_\sigma = (2\beta)_\sigma$.

Let us denote $\tilde{G}(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma}$ and $\tilde{F}(i, j) = \frac{\tilde{G}(i, j)}{\tilde{G}(j, j)}$, then

$$g(i, j) = \frac{G_m(\phi, j)}{G_m(\phi, i)} \tilde{G}(i, j) 2\tilde{\beta}_j, \text{ and } f(i, j) = \frac{G_m(\phi, j)}{G_m(\phi, i)} \frac{\tilde{G}(i, j)}{\tilde{G}(j, j)} = \frac{G_m(\phi, j)}{G_m(\phi, i)} \tilde{F}(i, j).$$

The expression for the measure becomes

$$\begin{aligned} \mu(\Omega_x) &= \frac{G_m(\phi, x)}{G_m(\phi, \phi)} \frac{\tilde{G}(\phi, x)}{\tilde{G}(x, x)} \left(\frac{1 - \frac{G_m(\phi, \bar{x})}{G_m(\phi, x)} \tilde{F}(x, \bar{x})}{1 - \tilde{F}(x, \bar{x}) \tilde{F}(\bar{x}, x)} \right) \\ &= \frac{\tilde{G}(\phi, x)}{G_m(\phi, \phi)} \left(\frac{G_m(\phi, x) - G_m(\phi, \bar{x}) \tilde{F}(x, \bar{x})}{\tilde{G}(x, x) - \tilde{F}(x, \bar{x}) \tilde{G}(\bar{x}, x)} \right). \end{aligned}$$

Let us compute the following terms: firstly,

$$\begin{aligned} \tilde{G}(x, x) - \tilde{F}(x, \bar{x}) \tilde{G}(\bar{x}, x) &= \tilde{G}(x, x) - \tilde{F}(x, \bar{x}) \tilde{G}(\bar{x}, \bar{x}) \tilde{F}(\bar{x}, x) \\ &= \sum_{\sigma \in \mathcal{P}_{x,x}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma} - \sum_{\sigma \in \mathcal{P}_{x, \{\bar{x}\}, x}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma} = \sum_{\sigma \in \mathcal{P}_{x, \bar{x}}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma}. \end{aligned}$$

Indeed, paths from x to x that do not cross \bar{x} have to stay in the connected component of x in $T \setminus \{\bar{x}\}$, which is T_x , *i.e.* $\mathcal{P}_{x,x}^T \setminus \mathcal{P}_{x, \{\bar{x}\}, x}^T = \mathcal{P}_{x,x}^{T_x}$. Moreover, $\phi \notin T_x$, so for $\sigma \in \mathcal{P}_{x,x}^{T_x}$, $(2\tilde{\beta})_\sigma = (2\beta)_\sigma$. As a result,

$$\tilde{G}(x, x) - \tilde{F}(x, \bar{x}) \tilde{G}(\bar{x}, x) = \sum_{\sigma \in \mathcal{P}_{x, \bar{x}}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma} = \hat{G}^{T_x}(x, x).$$

Secondly,

$$\begin{aligned} G_m(\phi, x) - G_m(\phi, \bar{x}) \tilde{F}(x, \bar{x}) &= \hat{G}(\phi, x) - \hat{G}(\phi, \bar{x}) \tilde{F}(x, \bar{x}) \\ &\quad + \int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \left(\chi(x, d\tau) - \tilde{F}(x, \bar{x}) \chi(\bar{x}, d\tau) \right). \end{aligned}$$

Note that if $\sigma \in \overline{\mathcal{P}}_{x, \bar{x}}^T$, then $\sigma_k \in T_x$ for $k \leq |\sigma| - 1$, so $(2\tilde{\beta})_\sigma^- = (2\beta)_\sigma^-$. Therefore, $\tilde{F}(x, \bar{x}) = \hat{F}(x, \bar{x})$. Moreover, since $\mathcal{P}_{\phi, x}^T = \mathcal{P}_{\phi, \{\bar{x}\}, x}^T$, we have

$$\hat{G}(\phi, x) - \hat{G}(\phi, \bar{x}) \hat{F}(x, \bar{x}) = 0.$$

Recall also that the density of $\chi(x, \cdot)$ with respect to $\chi(\phi, \cdot)$ is $\tau \mapsto \frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)}$. As a result,

$$\begin{aligned} G_m(\phi, x) - G_m(\phi, \bar{x}) \tilde{F}(x, \bar{x}) &= \int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \left(\chi(x, d\tau) - \hat{F}(x, \bar{x}) \chi(\bar{x}, d\tau) \right) \\ &= \int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau) \left(\frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \bar{x}) \frac{\hat{F}(\bar{x}, \bar{x} \wedge \tau)}{\hat{F}(\phi, \bar{x} \wedge \tau)} \right). \end{aligned}$$

For $\tau \notin \Omega_x$, $x \wedge \tau = \bar{x} \wedge \tau$ and paths from x to $\bar{x} \wedge \tau$ cross \bar{x} . Therefore,

$$\frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \bar{x}) \frac{\hat{F}(\bar{x}, \bar{x} \wedge \tau)}{\hat{F}(\phi, \bar{x} \wedge \tau)} = \frac{\hat{F}(x, \bar{x} \wedge \tau) - \hat{F}(x, \bar{x}) \hat{F}(\bar{x}, \bar{x} \wedge \tau)}{\hat{F}(\phi, \bar{x} \wedge \tau)} = 0.$$

For $\tau \in \Omega_x$, $x \wedge \tau = x$ and $\bar{x} \wedge \tau = \bar{x}$, so

$$\begin{aligned} \frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \bar{x}) \frac{\hat{F}(\bar{x}, \bar{x} \wedge \tau)}{\hat{F}(\phi, \bar{x} \wedge \tau)} &= \frac{\hat{G}(x, x)}{\hat{G}(\phi, x)} - \hat{F}(x, \bar{x}) \frac{\hat{G}(\bar{x}, \bar{x})}{\hat{G}(\phi, \bar{x})} \\ &= \frac{\hat{G}(x, x) - \hat{F}(x, \bar{x}) \hat{G}(\bar{x}, \bar{x}) \hat{F}(\bar{x}, x)}{\hat{G}(\phi, x)} = \frac{\hat{G}^{T_x}(x, x)}{\hat{G}(\phi, x)}. \end{aligned}$$

As a result, we have

$$G_m(\phi, x) - G_m(\phi, \bar{x}) \tilde{F}(x, \bar{x}) = \frac{\hat{G}^{T_x}(x, x)}{\hat{G}(\phi, x)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau).$$

For $x \neq \phi$, we finally get

$$\begin{aligned} \mu(\Omega_x) &= \frac{\tilde{G}(\phi, x)}{G_m(\phi, \phi) \hat{G}(\phi, x)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau) \\ &= \frac{\tilde{G}(\phi, \phi)}{\hat{G}(\phi, \phi)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau) \end{aligned}$$

since $\tilde{F}(x, \phi) = \hat{F}(x, \phi)$. Moreover, by summing over $x \in S(\phi)$, we have the same expression for $\mu(\Omega_\phi) = \mu(\Omega)$:

$$1 = \mu(\Omega) = \frac{\tilde{G}(\phi, \phi)}{G_m(\phi, \phi) \hat{G}(\phi, \phi)} \int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau).$$

As a result, for all $x \in T$,

$$\mu(\Omega_x) = \frac{\int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}{\int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}.$$

□

Proof of Proposition 2.1.8. Let \mathcal{T} be a d -regular tree, with $d \geq 3$, endowed with constant conductances W such that $\mathbb{P}[\forall i \in T, \psi(i) > 0] = 1$. Note that (\mathcal{T}, W) is vertex transitive, so it is enough to show the proposition for $i_0 = \phi$. The following lemma is a consequence of the symmetries of (\mathcal{T}, W) , and guarantees that almost surely, the exiting measure gives weight to the whole boundary Ω .

Lemma 2.7.4. *Almost surely under $\nu_T^W(d\beta)$, for all $x \neq \phi$, $\chi(\phi, \Omega_x) > 0$.*

Proof. For all $x \neq \phi$, we define $\hat{\chi}_x = \psi(x) - \hat{F}(x, \bar{x})\psi(\bar{x})$. Then $\chi(\phi, \Omega_x) = \hat{F}(\phi, x) \frac{\hat{\chi}_x}{1 - \hat{F}(x, \bar{x})\hat{F}(\bar{x}, x)}$, and $\mathbb{P}[\chi(\phi, \Omega_x) > 0] = \mathbb{P}[\hat{\chi}_x > 0]$.

Note that

$$\psi^{(n)}(x) - \hat{F}^{(n)}(x, \bar{x})\psi^{(n)}(\bar{x}) = \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x, y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}$$

is β_{T_x} -measurable. Therefore, taking the limit when $n \rightarrow \infty$ shows that $\hat{\chi}_x$ is also β_{T_x} -measurable. As a result, given a fixed $m \leq 1$, the random variables $(\hat{\chi}_x)_{x \in D^{(m)}}$ are independent, since ν_V^W is 1-dependent, and have the same distribution, since ν_V^W is invariant under the group of automorphisms of \mathcal{T} .

Moreover, we have $\psi(\phi) = \sum_{y \in D^{(m)}} \hat{F}(\phi, y) \frac{\hat{\chi}_y}{1 - \hat{F}(y, \bar{y})\hat{F}(\bar{y}, y)}$, so

$$\mathbb{P}[\psi(\phi) = 0] = \mathbb{P}[\forall y \in D^{(m)}, \hat{\chi}_y = 0] = \mathbb{P}[\hat{\chi}_x = 0]^{|D^{(m)}|}$$

for any $x \in D^{(m)}$. Since $\mathbb{P}[\psi(\phi) = 0] = 0$, we get $\mathbb{P}[\hat{\chi}_x = 0] = 0$ for all $x \in D^{(m)}$ and all $m \geq 1$, which implies that almost surely, for all $x \neq \phi$, $\chi(\phi, \Omega_x) > 0$. \square

Let us fix $m > m'$, and denote $\mu^{(m)} = \mu_\phi^{\beta, \rho_m, \phi}$ and $\mu^{(m')} = \mu_\phi^{\beta, \rho_{m'}, \phi}$. Let $x \in D^{(m)}$ be fixed, note that $x \neq \phi$. We define the following events:

$$A_x^{(m)} = \left\{ \frac{\mu^{(m)}(\Omega_x)}{\mu^{(m)}(\Omega_{\bar{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})} \right\} \text{ and } A_x^{(m')} = \left\{ \frac{\mu^{(m')}(\Omega_x)}{\mu^{(m')}(\Omega_{\bar{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})} \right\}.$$

Let us first show that the event $A_x^{(m)}$ is $r^{\beta, \rho_m, \phi}$ -measurable. Note that the exiting measure $\mu^{(m)}$ is measurable with respect to the corresponding environment $r^{\beta, \rho_m, \phi}$. Moreover, for $i \neq \phi$, $\beta_i = \tilde{\beta}_i = \sum_{j \sim i} r_{i,j}^{\beta, \rho_m, \phi}$ is $r^{\beta, \rho_m, \phi}$ -measurable. Therefore, we just have to show that $\frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})}$ is $\beta_{T \setminus \{\phi\}}$ -measurable. Since $\chi(\phi, \Omega_x) = \hat{G}(\phi, x) \sum_{y \in S(x)} W_{x,y} \hat{\chi}_y$, we have

$$\frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})} = \frac{\hat{G}(\phi, x) \sum_{y \in S(x)} W_{x,y} \hat{\chi}_y}{\hat{G}(\phi, \bar{x}) \sum_{z \in S(\bar{x})} W_{\bar{x},z} \hat{\chi}_z} = \hat{F}(x, \bar{x}) \frac{\sum_{y \in S(x)} W_{x,y} \hat{\chi}_y}{\sum_{z \in S(\bar{x})} W_{\bar{x},z} \hat{\chi}_z},$$

which is $\beta_{U_{\bar{x}}}$ -measurable and therefore $\beta_{T \setminus \{\phi\}}$ -measurable. We can conclude that $A_x^{(m)}$ is $r^{\beta, \rho_m, \phi}$ -measurable, and in the same way, $A_x^{(m')}$ is $r^{\beta, \rho_{m'}, \phi}$ -measurable. We are now going to show that under $\nu_{T, B_m}^W(d\beta, d\rho_{m'})$ we have $\mathbb{P}[A_x^{(m')}] = 1$, while under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ we have $\mathbb{P}[A_x^{(m)}] = 0$. This will prove that the distributions of $r^{\beta, \rho_m, \phi}$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ and $r^{\beta, \rho_{m'}, \phi}$ under $\nu_{T, B_m}^W(d\beta, d\rho_{m'})$ are different.

Since $|x| = m > m'$, we have $|\bar{x}| \geq m'$, so there exists $z \in D^{(m')}$ such that $\bar{x} \in T_z$, i.e. $\Omega_{\bar{x}} \subset \Omega_z$. Then for all $\tau \in \Omega_{\bar{x}}$, $\int_\Omega \chi(\phi, d\omega) \check{g}_{m'}(\omega, \tau) = \sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z)$. As a result,

$$\frac{\mu^{(m')}(\Omega_x)}{\mu^{(m')}(\Omega_{\bar{x}})} = \frac{\int_{\Omega_x} \left(\sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z) \right) \chi(\phi, d\tau)}{\int_{\Omega_{\bar{x}}} \left(\sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z) \right) \chi(\phi, d\tau)} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})},$$

so $\mathbb{P}[A_x^{(m')}] = 1$ under $\nu_{T, B_{m'}}^W(d\beta, d\rho_{m'})$.

We will finally show that ν_{T, B_m}^W -almost surely, $\mathbb{P}[A_x^{(m)} | \beta] = 0$. Since $|x| = m$, we have

$$\begin{aligned} \mu^{(m)}(\Omega_x) &= \chi_m(\phi, \delta_x) \sum_{b \in B_m} \check{G}_m(\delta_x, b) \chi_m(\phi, b), \\ \text{and } \mu^{(m)}(\Omega_{\bar{x}}) &= \sum_{y \in S(\bar{x})} \chi_m(\phi, \delta_y) \sum_{b \in B_m} \check{G}_m(\delta_y, b) \chi_m(\phi, b). \end{aligned}$$

Let us denote, for $y \in D^{(m)}$, $u_y = \sum_{b \in B_m} \check{G}_m(\delta_y, b) \chi_m(\phi, b)$. Then

$$A_x^{(m)} = \left\{ \frac{\mu^{(m)}(\Omega_x)}{\mu^{(m)}(\Omega_{\bar{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})} \right\} = \left\{ \sum_{y \in S(\bar{x})} \frac{\chi(\phi, \Omega_y)}{\chi(\phi, \Omega_{\bar{x}})} u_y = u_x \right\} = \{u \in \ker(f_\beta)\},$$

where $f_\beta : (v_y)_{y \in D^{(m)}} \mapsto \sum_{y \in S(\bar{x})} \frac{\chi(\phi, \Omega_y)}{\chi(\phi, \Omega_{\bar{x}})} v_y - v_x$ is a linear form conditionally on β , which has almost surely rank 1 according to Lemma 2.7.4, so that $\ker(f_\beta)$ is a hyperplane of $\mathbb{R}^{|D^{(m)}|}$. Let us show that

conditionally on β , the distribution of $(u_y)_{y \in D^{(m)}}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{|D^{(m)}|}$, and therefore $\mathbb{P}[A_x^{(m)}|\beta] = \mathbb{P}[u \in \ker(f_\beta)|\beta] = 0$.

Recall that $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, where conditionally on β , ρ_m is distributed according to $\nu_{B_m}^{\check{C}_m}$, which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{|B_m|} = \mathbb{R}^{|D^{(m)}|}$. Let us define

$$\Phi : \begin{array}{ccc} \mathbb{R}^{|D^{(m)}|} & \longrightarrow & \mathbb{R}^{|D^{(m)}|} \\ \rho_m & \longmapsto & (u_y)_{y \in D^{(m)}} = \check{G}_m \chi_m(\phi, \cdot) \end{array} .$$

For all ρ_m such that $2\rho_m - \check{C}_m > 0$, Φ is differentiable, and its differential is

$$d_{\rho_m} \Phi(v) = -2\check{G}_m \text{diag}(v) \check{G}_m \chi_m(\phi, \cdot) = -2\check{G}_m \text{diag}(v)u,$$

which is invertible, with $(d_{\rho_m} \Phi)^{-1}(w) = \left(-\frac{(\check{G}_m^{-1}w)_y}{2u_y} \right)_{y \in D^{(m)}}$. Note that this is well-defined since $u_y > 0$ for all $y \in D^{(m)}$, thanks to Lemma 2.7.4. As a result, Φ is a local diffeomorphism. Therefore, the distribution of $u = \Phi(\rho_m)$, conditionally on β , admits a density with respect to the Lebesgue measure on $\mathbb{R}^{|D^{(m)}|}$. We deduce that almost surely, $\mathbb{P}[A_x^{(m)}|\beta] = \mathbb{P}[u \in \ker(f_\beta)|\beta] = 0$, and therefore $\mathbb{P}[A_x^{(m)}] = 0$, which concludes the proof. \square

Chapter 3

The β potential and a multi-dimensional opposite-drift theorem

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Abstract

This chapter is once again self-contained, and presents results from [GSZ20]. They concern a multi-dimensional version of the Matsumoto-Yor opposite-drift theorem, obtained in the case where the drift is $\frac{1}{2}$. The proof relies on applying the time change from Lamperti's relation, to a result from [SZ17], which relates the β potential to a family of Brownian motions with interacting drifts.

Section 3.1 presents prerequisite material as well as the main result of this chapter. In Section 3.2, we study the Lamperti time change on Bessel bridges. Finally, in Section 3.3, we present two ways of applying a multi-dimensional version of this time change to the interacting Brownian motions of Theorem 3.1.2, to obtain Theorem 3.1.4.

3.1 Statement of the results

3.1.1 A first classical result

Let us first recall a well-known result regarding hitting times of the drifted Brownian motion (see [Wil74] and [Val91]).

Proposition 3.1.1. *Let $\theta > 0$ and $\eta \geq 0$ be fixed, and let B be a standard 1-dimensional Brownian motion. We define the drifted Brownian motion X by*

$$X(t) = \theta + B(t) - \eta t$$

for $t \geq 0$, and denote by T^0 the first hitting time of 0 by $X(t)$. Then the distribution of T^0 is given by

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbf{1}_{\{t \geq 0\}} dt. \quad (3.1)$$

Moreover, conditionally on T^0 , $(X(t))_{0 \leq t \leq T^0}$ has the same distribution as a 3-dimensional Bessel bridge from θ to 0 on $[0, T^0]$.

Note that for $\eta = 0$, i.e. if X is a Brownian motion with no drift, then (3.1) is the distribution of $\frac{1}{2\gamma}$, where γ is a Gamma random variable with parameter $(\frac{1}{2}, \theta^2)$. If $\eta > 0$, (3.1) is the Inverse Gaussian distribution with parameter $(\frac{\theta}{\eta}, \theta^2)$.

We will now present two related results. One is the Matsumoto-Yor opposite-drift theorem from [YM01], obtained from Proposition 3.1.1 thanks to a time change appearing in Lamperti's relation. The second one is a multivariate version of Proposition 3.1.1 from [SZ17], where the hitting times of a family of interacting Brownian motions are given by the β potential introduced in [STZ17] to study the VRJP.

3.1.2 Opposite drift theorem and Lamperti's relation

Let us present a version of the Matsumoto-Yor opposite drift theorem from [YM01], in the specific case where the drift μ is $\frac{1}{2}$, and with an added term depending on η .

Theorem 3.1.1. *[Theorem 2.2 and Proposition 3.1 in [YM01]] Let $\theta > 0$ and $\eta \geq 0$ be fixed, and let B be a standard 1-dimensional Brownian motion. We define the process ρ as the solution of the following SDE :*

$$\rho(u) = \log(\theta) + B(u) - \frac{1}{2}u - \int_0^u \eta e^{\rho(v)} dv$$

for $u \geq 0$.

Let us define $T(u) = \int_0^u e^{2\rho(v)} dv$. Then :

(i) We have

$$T(u) \xrightarrow[u \rightarrow \infty]{a.s.} T^0,$$

where T^0 is distributed according to

$$\frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2} \frac{(\theta - \eta t)^2}{t}\right) \mathbf{1}_{\{t \geq 0\}} dt.$$

(ii) Conditionally on T^0 , there exists a standard 1-dimensional Brownian motion \hat{B} such that for $u \geq 0$,

$$\rho(u) = \log(\theta) + \hat{B}(u) + \frac{1}{2}u + \log\left(\frac{T^0 - T(u)}{T^0}\right).$$

One proof of Theorem 3.1.1 relies on applying a time change to Proposition 3.1.1. The relevant time change is the one that appears in Lamperti's relation, presented below (see [RY13] p.452) :

Proposition 3.1.2. [Lamperti's relation] Let $(\rho(u))_{u \geq 0}$ be a drifted Brownian motion with drift $\mu \in \mathbb{R}$. For $u \geq 0$, define

$$T(u) = \int_0^u \exp(2\rho(v)) dv.$$

Then there exists a Bessel process $(X(t))_{t \geq 0}$ with index μ , starting from 1, such that for $u \geq 0$,

$$e^{\rho(u)}(u) = X(T(u)).$$

Let us sketch the proof of Theorem 3.1.1 using this time change. We use the same notations as in Proposition 3.1.1: let $\theta > 0$, $\eta > 0$, let B be a Brownian motion and define $X(t) = \theta + B(t) + \eta t$. We define $U(t) = \int_0^t \frac{1}{X(s)^2} ds$, and $T = U^{-1}$. Then T is the analogue of the time change featured in Lamperti's relation, where X plays the role of a Bessel process with index $-\frac{1}{2}$ (with an added drift η). We define $\rho(u) = \log(X(T(u)))$, so that $e^{\rho(u)} = X(T(u))$, and ρ has the distribution described in Theorem 3.1.1, i.e. a Brownian motion with drift $-\frac{1}{2}$, and an extra term when $\eta > 0$. Moreover, when $u \rightarrow \infty$, we have $T(u) \rightarrow T^0$, where T^0 is the first hitting time of 0 by X . Conditionally on T^0 , X has the distribution of a 3-dimensional Bessel bridge, i.e. a Bessel bridge with index $\frac{1}{2}$, and ρ has the distribution described in Theorem 3.1.1 (ii).

Remark 3.1.1. In [YM01], the opposite drift theorem is stated in a different form, where $\eta = 0$ and the drift μ can be different from $\frac{1}{2}$. Its proof still relies on applying Lamperti's relation, but this time to a result concerning hitting times of Bessel processes with any index $-\mu$ (see [Law18] and [PY81]).

The aim of this article is now to obtain a multi-dimensional version of Theorem 3.1.1, by applying the time change from Lamperti's relation to Theorem 3.1.2, which gives a generalization of Proposition 3.1.1 to a multi-dimensional Brownian motion with interacting drifts.

3.1.3 Brownian motions with interacting drifts and the random β potential

Let $\mathcal{G} = (V, E)$ be a finite connected graph, endowed with conductances $(W_e)_{e \in E} \in (\mathbb{R}_+^*)^E$. For $i, j \in V$, we denote by $W_{i,j} = W_{\{i,j\}}$ if $\{i,j\} \in E$, and $W_{i,j} = 0$ otherwise. Note that it is possible to have $W_{i,i} > 0$. For $\beta \in \mathbb{R}^V$, we define the matrix $H_\beta = 2\beta - W$, where β denotes the diagonal matrix with coefficients $(\beta_i)_{i \in V}$.

Proposition 3.1.3 (Theorem 4 in [STZ17], Theorem 2.2 in [LW19]). *For all $\theta \in (\mathbb{R}_+^*)^V$ and $\eta \in (\mathbb{R}_+)^V$, the measure $\nu_V^{W,\theta,\eta}$ defined by*

$$\nu_V^{W,\theta,\eta}(d\beta) = \mathbf{1}_{H_\beta > 0} \left(\frac{2}{\pi} \right)^{|V|/2} \exp \left(-\frac{1}{2} \langle \theta, H_\beta \theta \rangle - \frac{1}{2} \langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \theta \rangle \right) \prod_{i \in V} \frac{\theta_i}{\sqrt{|H_\beta|}} d\beta$$

is a probability distribution. Moreover, for all $i \in V$, the random variable $\frac{1}{2\beta_i - W_{i,i}}$ has Inverse Gaussian distribution with parameter $(\frac{\theta_i}{\eta_i + \sum_{j \neq i} W_{i,j} \theta_j}, \theta_i^2)$.

For $t \in (\mathbb{R}_+)^V$, we also denote by K_t the matrix $K_t = \text{Id} - tW$, where t still denotes the diagonal matrix with coefficients $(t_i)_{i \in V}$. Note that if $t \in (\mathbb{R}_+^*)^V$, we have $K_t = tH_{\frac{1}{2t}}$, where $\frac{1}{2t} = (\frac{1}{2t_i})_{i \in V}$. Finally, for $t \in (\mathbb{R}_+)^V$ and $T \in (\overline{\mathbb{R}_+})^V$, we define the vector $t \wedge T = (t_i \wedge T_i)_{i \in V}$.

Theorem 3.1.2. [Lemma 1 and Theorem 1 in [SZ17]] *Let $\theta \in (\mathbb{R}_+^*)^V$ and $\eta \in (\mathbb{R}_+)^V$ be fixed, and let $(B_i(t))_{i \in V, t \geq 0}$ be a standard $|V|$ -dimensional Brownian motion.*

(i) *The following stochastic differential equation (SDE) has a unique pathwise solution :*

$$X_i(t) = \theta_i + \int_0^t \mathbf{1}_{s < T_i^0} dB_i(s) - \int_0^t \mathbf{1}_{s < T_i^0} ((W\psi)(s) + \eta)_i ds \quad (E_V^{W,\theta,\eta}(X))$$

for $i \in V$ and $t \geq 0$, where for $i \in V$, T_i^0 is the first hitting time of 0 by X_i , and for $t \geq 0$,

$$\psi(t) = K_{t \wedge T^0}^{-1} (X(t) + (t \wedge T^0) \eta).$$

(ii) *If $(X_i)_{i \in V}$ is solution of $(E_V^{W,\theta,\eta}(X))$, the vector $(\frac{1}{2T_i^0})_{i \in V}$ has distribution $\nu_V^{W,\theta,\eta}$, and conditionally on $(T_i^0)_{i \in V}$, the paths $(X_i(t))_{0 \leq t \leq T_i^0}$ are independent 3-dimensional Bessel bridges.*

To obtain an analogue of the opposite drift theorem as in Section 3.1.2, we want to apply the time change from Lamperti's relation to solutions $(X_i)_{i \in V}$ of $(E_V^{W,\theta,\eta}(X))$. A problem will arise in the interaction term, since the time change will be different on every coordinate of X . To solve this, we will use a form of strong Markov property verified by solutions of $(E_V^{W,\theta,\eta}(X))$, which is a consequence of Theorem 3.1.2(ii). This Markov property will be true with respect to multi-stopping times, defined as follows.

Definition 3.1.1. *Let X be a solution of $(E_V^{W,\theta,\eta}(X))$. A random vector $T = (T_i)_{i \in V} \in \overline{\mathbb{R}_+}^V$ will be called a multi-stopping time with respect to X , if : for all $t \in \mathbb{R}_+^V$, the event $\cap_{i \in V} \{T_i \leq t_i\}$ is \mathcal{F}_t^X -measurable, where*

$$\mathcal{F}_t^X = \sigma((X_i(s))_{0 \leq s \leq t_i, i \in V}).$$

In this case, we denote by \mathcal{F}_T^X the σ -algebra of events anterior to T , i.e.

$$\mathcal{F}_T^X = \{A \in \mathcal{F}_\infty^X, \forall t \in \mathbb{R}_+^V, A \cap \{T_i \leq t_i\} \in \mathcal{F}_t^X\}$$

Let us now formulate the strong Markov property for solutions of $(E_V^{W,\theta,\eta}(X))$.

Theorem 3.1.3. [Theorem 2 (iv) in [SZ17]] *Let X be a solution of $(E_V^{W,\theta,\eta}(X))$, and $T = (T_i)_{i \in V}$ be a multi-stopping time with respect to X .*

Define the shifted process Y by

$$Y_i(t) = X_i(T_i + t)$$

for $i \in V$ and $t \geq 0$. Moreover, we denote

$$\tilde{W}^{(T)} = W (K_{T \wedge T^0})^{-1}, \tilde{\eta}^{(T)} = \eta + \tilde{W}^{(T)}((T \wedge T^0) \eta), \text{ and } X(T) = (X_i(T_i))_{i \in V}.$$

Then on the event $\cap_{i \in V} \{T_i < \infty\}$, conditionally on T and \mathcal{F}_T^X , the process Y has the same distribution as the solution of $(E_V^{\tilde{W}^{(T)}, X(T), \tilde{\eta}^{(T)}}(X))$.

3.1.4 Main results : A multi-dimensional version of the opposite drift theorem

Let $(X_i)_{i \in V}$ be a solution of $(E_V^{W, \theta, \eta}(X))$. As in the usual case of Lamperti's relation, let us introduce the functional that will define the time change. For $i \in V$ and $t \geq 0$, we set

$$U_i(t) = \int_0^t \frac{\mathbb{1}_{s < T_i^0}}{X_i(s)^2} ds.$$

We first have to show that this time change goes to infinity.

Lemma 3.1.1. *Let $i \in V$ be fixed. Then*

$$\lim_{t \rightarrow T_i^0} U_i(t) = +\infty,$$

so that $U_i : [0, T_i^0) \rightarrow [0, +\infty)$ is a bijection.

For all $i \in V$, we then define $T_i = (U_i|_{[0, T_i^0]})^{-1}$. Therefore, for all $u \geq 0$, $T_i(u) < T_i^0$ and $\lim_{u \rightarrow \infty} T_i(u) = T_i^0$.

We can now show that the time-changed solution $(X_i \circ T_i)_{i \in V}$ can be written as

$$X_i(T_i(u)) = e^{\rho_i(u)}$$

for $u \geq 0$, where $(\rho_i)_{i \in V}$ is solution of a new system of stochastic differential equations :

Theorem 3.1.4. (i) *For $i \in V$ and $u \geq 0$, let us define $\rho_i(u) = \log(X_i(T_i(u)))$. Then (ρ, T) is solution of the following system of SDEs :*

$$\begin{cases} \rho_i(v) = \log(\theta_i) + \tilde{B}_i(v) + \int_0^v \left(-\frac{1}{2} - e^{\rho_i(u)} \left(\tilde{W}^{(u)}(e^{\rho(u)} + T(u)\eta) + \eta \right)_i \right) du, \\ T_i(v) = \int_0^v e^{2\rho_i(u)} du, \end{cases} \quad (E_V^{W, \theta, \eta}(\rho))$$

for $i \in V$ and $v \geq 0$, where $(\tilde{B}_i)_{i \in V}$ is a $|V|$ -dimensional standard Brownian motion, $e^{\rho(u)}$ denotes the vector $(e^{\rho_i(u)})_{i \in V}$, and

$$\tilde{W}^{(u)} = WK_{T(u)}^{-1} = W(\text{Id} - T(u)W)^{-1}.$$

(ii) *The equation $(E_V^{W, \theta, \eta}(\rho))$ admits a unique pathwise solution $u \mapsto (\rho(u), T(u))$, which is a.s. well defined on all of \mathbb{R}_+ .*

As a consequence of Theorems 3.1.2(ii) and 3.1.4, we can relate the solutions of $(E_V^{W, \theta, \eta}(\rho))$ to time-changed Bessel bridges and the distribution $\nu_V^{W, \theta, \eta}$. This is stated in Theorem 3.1.5 below, which is the multi-dimensional version of Theorem 3.1.1.

Theorem 3.1.5. *Let \tilde{B} be a $|V|$ -dimensional standard Brownian motion, and let (ρ, T) be solution of $(E_V^{W, \theta, \eta}(\rho))$.*

(i) *For all $i \in V$, we have*

$$T_i(u) = \int_0^u e^{2\rho_i(v)} dv \xrightarrow[u \rightarrow \infty]{a.s.} T_i^0,$$

where $\left(\frac{1}{2T_i^0}\right)_{i \in V}$ is distributed according to $\nu_V^{W, \theta, \eta}$.

(ii) Conditionally on T^0 , there exists a standard $|V|$ -dimensional Brownian motion \hat{B} such that for $i \in V$ and $u \geq 0$,

$$\rho_i(u) = \log(\theta_i) + \hat{B}_i(u) + \frac{1}{2}u + \log\left(\frac{T_i^0 - T_i(u)}{T_i^0}\right).$$

In particular, the processes $(\rho_i, T_i)_{i \in V}$ are independent conditionally on T^0 .

3.1.5 An open question

The Gamma and Inverse Gaussian distributions, as well as the distribution of the inverse of a Gamma or Inverse Gaussian variable, all fall into the family of the so-called generalized Inverse Gaussian distributions.

A random variable is said to have generalized Inverse Gaussian distribution with parameter (q, a, b) where $q \in \mathbb{R}$ and $a, b > 0$, and denoted $\text{GIG}(q, a, b)$ if it has the following density:

$$\left(\frac{a}{b}\right)^{q/2} \frac{1}{2K_q(\sqrt{ab})} t^{q-1} e^{-\frac{1}{2}(at+b/t)} \mathbf{1}_{t>0}. \quad (3.2)$$

In particular, we have the following special cases (where zero parameter is understood as in [Nor94]):

$$\text{IG}\left(\frac{\theta}{\eta}, \theta^2\right) = \text{GIG}\left(-\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2}\right), \quad \text{Gamma}\left(\frac{1}{2}, \theta^2\right) = \text{GIG}\left(-\frac{1}{2}, 0, \frac{\theta^2}{2}\right)$$

and

$$X \sim \text{GIG}\left(-\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2}\right) \Leftrightarrow 1/X \sim \text{GIG}\left(\frac{1}{2}, \frac{\theta^2}{2}, \frac{\eta^2}{2}\right).$$

Define the last visit of 0 of our drifted Brownian motion to be $T^1 = \sup\{t \geq 0 : B_t + \theta - \eta t = 0\}$. By a time inversion argument, i.e. setting

$$\tilde{B}_t = \begin{cases} -tB_{1/t} & t > 0 \\ 0 & t = 0 \end{cases},$$

the Gaussian process \tilde{B} is also a Brownian motion and we deduce that $\frac{1}{T^1}$ is the first visit time to 0 of $\tilde{B}_t + \eta - \theta t$, hence T^1 is $\text{GIG}(\frac{1}{2}, \frac{\theta^2}{2}, \frac{\eta^2}{2})$ distributed, moreover, by Strong Markov property of Brownian motion, we deduce that $T^1 - T^0$ is $\text{Gamma}(\frac{1}{2}, \theta^2)$ distributed, and it is independent of T^0 .

More generally, we have the following identity in distribution, which is known as the Matsumoto-Yor property [WW07, MY03]:

Proposition 3.1.4. *Let $T = (T^0, T^1)$ be a random vector, then there is equivalence between the following statements:*

$$(i) \left(\frac{1}{T^0}, T^1 - T^0\right) \sim \text{GIG}\left(\frac{1}{2}, \frac{\theta^2}{2}, \frac{\eta^2}{2}\right) \otimes \text{Gamma}\left(\frac{1}{2}, \theta^2\right)$$

$$(ii) \left(\frac{1}{T^0} - \frac{1}{T^1}, T^1\right) \sim \text{Gamma}\left(\frac{1}{2}, \eta^2\right) \otimes \text{GIG}\left(\frac{1}{2}, \frac{\eta^2}{2}, \frac{\theta^2}{2}\right).$$

It is natural to ask whether such an identity holds true in the case of the distribution in Proposition 3.1.3, and whether one is able to define the process in the time inversion.

3.2 Time change on Bessel bridges

3.2.1 Limit of the time change : Proof of Lemma 3.1.1

Let $X = (X_i)_{i \in V}$ be a solution of $(E_V^{W, \theta, \eta}(X))$. According to Theorem 3.1.2, conditionally on $(T_i^0)_{i \in V}$, the trajectories $(X_i(t))_{0 \leq t \leq T_i^0}$ are independent 3-dimensional Bessel bridges. As a result, in order to prove Lemma 3.1.1, it is enough to show the same result for a 3-dimensional Bessel bridge.

Let us then fix $\theta > 0$ and $T^0 > 0$, and let X be a 3-dimensional Bessel bridge from θ to 0 between 0 and T^0 . We want to show that

$$U(t) = \int_0^t \frac{ds}{X(s)^2} \xrightarrow[t \rightarrow T^0]{a.s.} +\infty.$$

We will do so by applying the time change from Lamperti's relation.

Since X is a 3-dimensional Bessel bridge, there exists a standard Brownian motion B such that

$$dX(t) = dB(t) + \frac{1}{X(t)}dt - \frac{X(t)}{T^0 - t}dt,$$

therefore by Ito's lemma, for $t < T^0$,

$$\begin{aligned} d \log(X(t)) &= \frac{dB(t)}{X(t)} + \frac{dt}{X(t)^2} - \frac{dt}{T^0 - t} - \frac{1}{2} \frac{dt}{X(t)^2} \\ &= dM(t) + \frac{1}{2}dU(t) + d \log(T^0 - t), \end{aligned}$$

where $M(t) = \int_0^t \frac{dB(s)}{X(s)}$ is a martingale, and $\langle M \rangle_t = U(t)$. Therefore, there exists a standard Brownian motion \hat{B} such that $M(t) = \hat{B}(U(t))$. Finally, for $t \geq 0$, we have

$$\log(X(t)) = \log(\theta) + \hat{B}(U(t)) + \frac{1}{2}U(t) + \log\left(\frac{T^0 - t}{T^0}\right), \quad (3.3)$$

i.e.

$$\frac{X(t)}{T^0 - t} = \frac{\theta}{T^0} e^{\hat{B}(U(t)) + \frac{1}{2}U(t)}.$$

However, since X is a 3-dimensional Bessel bridge, there also exists a 3-dimensional Bessel process Y such that for $t \geq 0$,

$$X(t) = (T^0 - t)Y\left(\frac{t}{T^0(T^0 - t)}\right)$$

(see [RY13] p.467). Therefore, when $t \rightarrow T^0$, we have a.s. $\frac{X(t)}{T^0 - t} \rightarrow +\infty$. Since $u \mapsto \hat{B}(u) + \frac{1}{2}u$ cannot explode in finite time, we have necessarily

$$U(t) \xrightarrow[t \rightarrow T^0]{a.s.} +\infty.$$

3.2.2 Time change on the conditional process : Proof of Theorem 3.1.5

Let $(\tilde{B}_i)_{i \in V}$ be a $|V|$ -dimensional standard Brownian motion. According to Theorem 3.1.4, there exists a $|V|$ -dimensional standard Brownian motion $(B_i)_{i \in V}$ such that, if $(X_i)_{i \in V}$ is the solution of $(E_V^{W, \theta, \eta}(X))$ with the Brownian motion B , and if T_i is the inverse function of $U_i : t \mapsto \int_0^t \frac{ds}{X_i(s)^2}$ for all $i \in V$, then (ρ, T) is the solution of $(E_V^{W, \theta, \eta}(\rho))$ with the Brownian motion \tilde{B} , where $\rho_i(u) = \log(X_i(T_i(u)))$ for $u \geq 0$.

Therefore, according to Lemma 3.1.1, we have a.s. for all $i \in V$:

$$\lim_{u \rightarrow +\infty} T_i(u) = T_i^0,$$

where T_i^0 is the hitting time of 0 by X_i . Moreover, we can apply Theorem 3.1.2 (ii) to X : the vector $\left(\frac{1}{2T_i^0}\right)_{i \in V}$ is distributed according to $\nu_V^{W, \theta, \eta}$, and conditionally on $(T_i^0)_{i \in V}$, the trajectories $(X_i(t))_{0 \leq t \leq T_i^0}$ are independent 3-dimensional Bessel bridges from θ_i to 0 respectively. Since $\rho_i(u) = \log(X_i(T_i(u)))$ for $u \geq 0$, conditionally on $(T_i^0)_{i \in V}$, the processes $(\rho_i, T_i)_{i \in V}$ are independent, and their distribution is given by applying the time change from Lamperti's relation to a 3-dimensional Bessel bridge. This time-change was already realized in the proof of Lemma 3.1.1 (see (3.3)), and the result is as follows : conditionally on $(T_i^0)_{i \in V}$, for all $i \in V$, there exists a standard Brownian motion \hat{B}_i such that for $u \geq 0$.

$$\begin{cases} \rho_i(u) = \log(\theta_i) + \hat{B}_i(u) + \frac{1}{2}u + \log\left(\frac{T_i^0 - T_i(u)}{T_i^0}\right) \\ T_i(u) = \int_0^u e^{2\rho_i(w)} dw. \end{cases}$$

3.3 Multi-dimensional time change : Proof of Theorem 3.1.4

Let us first assume that Theorem 3.1.4 (i) is proven, and show (ii), *i.e.* that $(E_V^{W, \theta, \eta}(\rho))$ has a.s. a unique pathwise solution defined on all of \mathbb{R}_+ . Let \tilde{B} be a $|V|$ -dimensional Brownian motion. Thanks to Theorem 3.1.4 (i), we know that $(E_V^{W, \theta, \eta}(\rho))$ admits a solution that is well defined on \mathbb{R}_+ . Let us now show that this solution is necessarily unique.

Let (ρ^*, T^*) be another solution of $(E_V^{W, \theta, \eta}(\rho))$ with the Brownian motion \tilde{B} . Let also \mathcal{K} be a compact subset of $\mathbb{R}^V \times \{t \in \mathbb{R}_+, K_t > 0\}$ containing $(\log(\theta_i), 0)_{i \in V}$. Then the function

$$\begin{cases} \mathcal{K} \rightarrow \mathbb{R}^V \times \mathbb{R}^V \\ (\rho, t) \mapsto \left(-\frac{1}{2} - e^{\rho_i} \left(W K_t^{-1}(e^\rho + t\eta) + \eta \right)_i, e^{2\rho_i} \right)_{i \in V} \end{cases}$$

is bounded and Lipschitz. Therefore, up to the stopping time $U_{\mathcal{K}} = \inf\{u \geq 0, (\rho(u), T(u)) \notin \mathcal{K}\}$, we have $(\rho(u), T(u)) = (\rho^*(u), T^*(u))$ from Theorem 2.1, p.375 of [RY13]. Since this is true for all compact subset \mathcal{K} of $\mathbb{R}^V \times \{t \in \mathbb{R}_+, K_t > 0\}$, we have a.s. $(\rho, T) = (\rho^*, T^*)$. This concludes the proof of Theorem 3.1.4 (ii).

Theorem 3.1.4 (i) remains to be proven. Let B be a standard $|V|$ -dimensional Brownian motion, and let $(X_i)_{i \in V}$ be a solution of $(E_V^{W, \theta, \eta}(X))$. For $i \in V$, recall that T_i is the inverse function of

$$U_i : \begin{cases} [0, T_i^0[\rightarrow [0, +\infty[\\ t \mapsto \int_0^t \frac{ds}{X_i(s)^2} \end{cases}$$

and $\rho_i(u) = \log(X_i(T_i(u)))$ for $u \geq 0$.

In order to show that (ρ, T) is solution of $(E_V^{W, \theta, \eta}(\rho))$, we want to apply the same time change as in Lamperti's relation. However, in the equation $(E_V^{W, \theta, \eta}(X))$, the term $\psi(t)$ represents an interaction between the coordinates $X_i(t)$ at the same time $t \geq 0$, which correspond to different times $U_i(t)$ when writing $X_i(t) = e^{\rho_i(U_i(t))}$.

We present here two different ways of overcoming this problem. The first proof relies on identifying the infinitesimal generator of the process $(\rho_i, T_i)_{i \in V}$, using the strong Markov property presented in Theorem

3.1.3. The second one uses Theorem 3.1.2 (ii), so that we can write X as a mixture of independent Bessel bridges, to which we can apply the time change separately, and then identify the law of the annealed process using Girsanov's theorem.

3.3.1 First proof of (i) : using the strong Markov property of Theorem 3.1.3

Firstly, let $u \geq 0$ be fixed, and $f : \mathbb{R}^V \times \mathbb{R}^V \rightarrow \mathbb{R}$ be a compactly-supported \mathcal{C}^2 function. To identify the infinitesimal generator of (ρ, T) , let us compute

$$\lim_{v \rightarrow u^+} \frac{\mathbb{E}[f(\rho(v), T(v)) | \mathcal{F}_u^{(\rho, T)}] - f(\rho(u), T(u))}{v - u}.$$

Note that $(T_i(u))_{i \in V}$ is a multi-stopping time in the sense of Definition 3.1.1 and that $\mathcal{F}_u^{(\rho, T)} = \mathcal{F}_{T(u)}^X$. Let us then define

$$\tilde{W}^{(u)} = W(K_{T(u)})^{-1}, \quad \tilde{K}_t^{(u)} = \text{Id} - t\tilde{W}^{(u)}, \quad \text{and} \quad \tilde{\eta}^{(u)} = \eta + \tilde{W}^{(u)}(T(u)\eta).$$

Thanks to Theorem 3.1.3, conditionally on $\mathcal{F}_{T(u)}^X$, the shifted process

$$Y = Y^{(u)} : t \mapsto (X_i(T_i(u) + t))_{i \in V}$$

is solution of the following equation :

$$\begin{cases} dY_i(t) = \mathbf{1}_{t \leq \hat{T}_i^0} d\hat{B}_i(t) - \mathbf{1}_{t \leq \hat{T}_i^0} \left(\tilde{W}^{(u)}(\tilde{K}_t^{(u)})^{-1} \left(Y(t) + (t \wedge \hat{T}_i^0) \tilde{\eta}^{(u)} \right) + \tilde{\eta}^{(u)} \right)_i dt, \\ Y_i(0) = X_i(T_i(u)), \end{cases}$$

for $i \in V$, $t \geq 0$, where \hat{B} is a $|V|$ -dimensional standard Brownian motion independent from $\mathcal{F}_{T(u)}^X$, and \hat{T}_i^0 is the first hitting time of 0 by Y_i .

Let us now fix $v > u$, and define the interrupted process

$$Z = Z^{(u, v)} : t \mapsto \left(X_i((T_i(u) + t) \wedge T_i(v)) \right)_{i \in V}.$$

For all $i \in V$ and $t \geq 0$, we then have $Z_i(t) = Y_i(t \wedge \hat{T}_i(v))$, where $\hat{T}_i(v) = T_i(v) - T_i(u)$. Therefore, Z is solution of

$$\begin{cases} dZ_i(t) = \mathbf{1}_{t \leq \hat{T}_i(v)} d\hat{B}_i(t) - \mathbf{1}_{t \leq \hat{T}_i(v)} \left(\tilde{W}^{(u)}(\tilde{K}_t^{(u)})^{-1} \left(Y(t) + (t \wedge \hat{T}_i^0) \tilde{\eta}^{(u)} \right) + \tilde{\eta}^{(u)} \right)_i dt, \\ Z_i(0) = X_i(T_i(u)), \end{cases}$$

for $i \in V$ and $t \geq 0$. Moreover, since $\hat{T}_i(v) < \hat{T}_i^0 < \infty$ a.s. for all $i \in V$, there exists a.s. \hat{T}^∞ large enough so that $Z_i(t) = Y_i(\hat{T}_i(v)) = X_i(T_i(v))$ for all $i \in V$ and $t \geq \hat{T}^\infty$.

According to Ito's lemma, for all $t \geq 0$ we have

$$\begin{aligned} d \log(Z_i(t)) &= \mathbf{1}_{t \leq \hat{T}_i(v)} \frac{d\hat{B}_i(t)}{Z_i(t)} - \mathbf{1}_{t \leq \hat{T}_i(v)} \frac{dt}{2Z_i(t)^2} \\ &\quad - \mathbf{1}_{t \leq \hat{T}_i(v)} \left(\tilde{W}^{(u)}(\tilde{K}_t^{(u)})^{-1} \left(Y(t) + (t \wedge \hat{T}_i^0) \tilde{\eta}^{(u)} \right) + \tilde{\eta}^{(u)} \right)_i \frac{dt}{Z_i(t)}, \end{aligned}$$

where we can also replace $t \wedge \hat{T}^0$ with t , since $\hat{T}_i(v) < \hat{T}^0$. Moreover, we denote by \hat{M}_i the following martingale for $i \in V$: $\hat{M}_i(t) = \int_0^t \frac{d\hat{B}_i(s)}{Z_i(s)}$ for $t \geq 0$.

Let us now denote

$$\Phi(t) = \left(\log(Z_i(t)), (T_i(u) + t) \wedge T_i(v) \right)_{i \in V} \in \mathbb{R}^V \times \mathbb{R}^V$$

for $t \geq 0$. Then, applying Ito's lemma to $t \mapsto f(\Phi(t))$, we get

$$\begin{aligned} f(\Phi(t)) - f(\Phi(0)) &= \sum_{i \in V} \int_0^{t \wedge \hat{T}_i(v)} \frac{\partial f}{\partial \rho_i}(\Phi(s)) d\hat{M}_i(s) + \sum_{i \in V} \int_0^{t \wedge \hat{T}_i(v)} \frac{1}{2} \frac{\partial^2 f}{\partial \rho_i^2}(\Phi(s)) \frac{ds}{Z_i(s)^2} \\ &+ \sum_{i \in V} \int_0^{t \wedge \hat{T}_i(v)} \frac{\partial f}{\partial \rho_i}(\Phi(s)) \left(-\frac{1}{2} - Z_i(s) \left(\tilde{W}^{(u)}(\tilde{K}_s^{(u)})^{-1} (Y(s) + s\tilde{\eta}^{(u)}) + \tilde{\eta}^{(u)} \right)_i \right) \frac{ds}{Z_i(s)^2} \\ &+ \sum_{i \in V} \int_0^{t \wedge \hat{T}_i(v)} \frac{\partial f}{\partial t_i}(\Phi(s)) ds. \end{aligned}$$

Taking $t \geq \hat{T}^\infty$, we get $t \wedge \hat{T}_i(v) = \hat{T}_i(v)$ for all $i \in V$, and

$$f(\Phi(t)) - f(\Phi(0)) = f(\rho(v), T(v)) - f(\rho(u), T(u)),$$

since $\rho_i(w) = \log(X_i(T_i(w)))$ for $w \in \mathbb{R}_+$ and $i \in V$. For all $i \in V$, we can now use the following time change in the corresponding integrals above: $s = T_i(w) - T_i(u) = \hat{T}_i(w)$, *i.e.* $w = U_i(T_i(u) + s)$. Note that

$$\begin{aligned} \text{for } 0 \leq s \leq \hat{T}_i(v), \quad \frac{d}{ds} U_i(T_i(u) + s) &= \frac{1}{X_i(T_i(u) + s)^2} = \frac{1}{Z_i(s)^2}, \\ \text{and for } u \leq w \leq v, \quad \frac{d}{dw} \hat{T}_i(w) &= X_i(T_i(w))^2 = e^{2\rho_i(w)}. \end{aligned}$$

As a result, we obtain :

$$\begin{aligned} &f(\rho(v), T(v)) - f(\rho(u), T(u)) \\ &= \sum_{i \in V} \left(\int_u^v \frac{\partial f}{\partial \rho_i}(\rho(w), T(w)) d\hat{M}_i(\hat{T}_i(w)) + \int_u^v \frac{1}{2} \frac{\partial^2 f}{\partial \rho_i^2}(\rho(w), T(w)) dw \right. \\ &+ \int_u^v \frac{\partial f}{\partial \rho_i}(\rho(w), T(w)) \left(-\frac{1}{2} - e^{\rho_i(w)} \left(\tilde{W}^{(u)}(\tilde{K}_{\hat{T}_i(w)}^{(u)})^{-1} (X(T_i(w)) + \hat{T}_i(w)\tilde{\eta}^{(u)}) + \tilde{\eta}^{(u)} \right)_i \right) dw \\ &\left. + \int_u^v \frac{\partial f}{\partial t_i}(\rho(w), T(w)) e^{2\rho_i(w)} dw \right). \end{aligned}$$

Here, the vector $X(T_i(w)) = (X_j(T_i(w)))_{j \in V}$ is different from $e^{\rho(w)} = (X_j(T_j(w)))_{j \in V}$. This is why we need to take $v \rightarrow u$ and identify the generator.

Note that since \hat{B} is independent from $\mathcal{F}_{T(u)}^X$, we have

$$\mathbb{E} \left[\int_u^v \frac{\partial f}{\partial \rho_i}(\rho(w), T(w)) d\hat{M}_i(\hat{T}_i(w)) \Big| \mathcal{F}_u^{(\rho, T)} \right] = \mathbb{E} \left[\int_0^{\hat{T}_i(v)} \frac{\partial f}{\partial \rho_i}(\Phi(t)) \frac{d\hat{B}_i(s)}{Z_i(s)} \Big| \mathcal{F}_{T(u)}^X \right] = 0$$

for all $i \in V$, and therefore

$$\begin{aligned} \mathbb{E}\left[f(\rho(v), T(v)) - f(\rho(u), T(u)) \middle| \mathcal{F}_u^{(\rho, T)}\right] &= \mathbb{E}\left[\sum_{i \in V} \left(\int_u^v \frac{1}{2} \frac{\partial^2 f}{\partial \rho_i^2}(\rho(w), T(w)) dw \right. \right. \\ &\quad \left. \left. + \int_u^v \frac{\partial f}{\partial \rho_i}(\rho(w), T(w)) \left(-\frac{1}{2} - e^{\rho_i(w)} \left(\tilde{W}^{(u)}(\tilde{K}_{\hat{T}_i(w)}^{(u)})^{-1}(X(T_i(w)) + \hat{T}_i(w)\tilde{\eta}^{(u)}) + \tilde{\eta}^{(u)}\right)_i\right) dw \right. \right. \\ &\quad \left. \left. + \int_u^v \frac{\partial f}{\partial t_i}(\rho(w), T(w)) e^{2\rho_i(w)} dw\right) \middle| \rho(u), T(u)\right]. \end{aligned}$$

By continuity and dominated convergence, we can then conclude that

$$\begin{aligned} \lim_{v \rightarrow u^+} \frac{\mathbb{E}\left[f(\rho(v), T(v)) \middle| \mathcal{F}_u^{(\rho, T)}\right] - f(\rho(u), T(u))}{v - u} &= \sum_{i \in V} \left(\frac{1}{2} \frac{\partial^2 f}{\partial \rho_i^2}(\rho(u), T(u)) \right. \\ &\quad \left. + \frac{\partial f}{\partial \rho_i}(\rho(u), T(u)) \left(-\frac{1}{2} - e^{\rho_i(u)} (\tilde{W}^{(u)} e^{\rho(u)} + \tilde{\eta}^{(u)})_i\right) + \frac{\partial f}{\partial t_i}(\rho(u), T(u)) e^{2\rho_i(u)}\right), \end{aligned}$$

which is in fact $\mathcal{L}f(u)$, where \mathcal{L} is the infinitesimal generator associated with the system of SDEs $(E_V^{W, \theta, \eta}(\rho))$.

3.3.2 Second proof of (i) : using the mixing measure and Girsanov's theorem

This proof follows the same structure as that of Theorem 3.1.2 : starting from the distribution of the process as a mixture of simpler quenched processes, and computing the integral in order to identify the annealed distribution, using Girsanov's theorem.

Let us denote by $X = (X_i(t))_{i \in V, t \geq 0}$ the canonical process in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^V)$, and by \mathbb{P} the distribution on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^V)$ under which X is solution of $(E_V^{W, \theta, \eta}(X))$. According to Theorem 3.1.2 (ii), the vector $(\beta_i)_{i \in V} = \left(\frac{1}{2T_i^0}\right)$ has distribution $\nu_V^{W, \theta, \eta}$. Moreover, conditionally on $(T_i^0)_{i \in V}$, the marginal processes X_i for $i \in V$ are independent 3-dimensional Bessel bridges from θ_i to 0 on $[0, T_i^0]$. In other words, we can write

$$\mathbb{P}[\cdot] = \int \left(\bigotimes_{i \in V} \mathbb{P}_i^{\beta_i}[\cdot]\right) \nu_V^{W, \theta, \eta}(d\beta),$$

where for $i \in V$, $\mathbb{P}_i^{\beta_i}$ is the distribution on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ under which the canonical process X_i is a 3-dimensional Bessel bridge from θ_i to 0 on $[0, T_i^0]$.

We can now apply the time change independently on each marginal X_i for all $i \in V$. According to the computations done in the proof of Lemma 3.1.1 (see (3.3)), we know that under $\mathbb{P}_i^{\beta_i}$, there exists a Brownian motion \hat{B}_i such that

$$\rho_i(u) = \log(\theta_i) + \hat{B}_i(u) + \frac{1}{2}u + \log\left(\frac{T_i^0 - T_i(u)}{T_i^0}\right)$$

for $u \geq 0$, where $T_i^0 = \frac{1}{2\beta_i}$ and $T_i(u) = \int_0^u e^{2\rho_i(v)} dv$, *i.e.*

$$\rho_i(u) = \log(\theta_i) + \hat{B}_i(u) + \frac{1}{2}u + \log\left(1 - 2\beta_i \int_0^u e^{2\rho_i(v)} dv\right).$$

For each $i \in V$, let us now define a martingale L_i by

$$L_i(u) = \int_0^u \left(-\frac{1}{2} + \frac{2\beta_i e^{2\rho_i(v)}}{1 - 2\beta_i \int_0^v e^{2\rho_i(s)} ds} \right) d\hat{B}_i(v)$$

for $u \geq 0$, so that $\rho_i(u) = \hat{B}_i(u) - \langle \hat{B}_i, L_i \rangle_u$. We can then introduce a probability distribution $\hat{\mathbb{P}}_i$ such that for all $u \geq 0$,

$$\mathbb{E} \left[\frac{d\hat{\mathbb{P}}_i}{d\mathbb{P}_i^{\beta_i}} \middle| \mathcal{F}_u^i \right] = \mathcal{E}(L_i)(u),$$

where $\mathcal{F}_u^i = \sigma(\hat{B}_i(v), 0 \leq v \leq u)$, and $\mathcal{E}(L_i)(u) = e^{L_i(u) - \frac{1}{2}\langle L_i, L_i \rangle_u}$ is the exponential martingale associated with L_i . Then by Girsanov's theorem, ρ_i is a standard Brownian motion under $\hat{\mathbb{P}}_i$. Note that $\hat{\mathbb{P}}_i$ does not depend on β_i .

From now on, let us write $\phi_i(u) = 1 - 2\beta_i \int_0^u e^{2\rho_i(v)} dv = \frac{T_i^0 - T_i(u)}{T_i^0}$ for $u \geq 0$ and $i \in V$. The following lemma gives an expression of $\mathcal{E}(L_i)$.

Lemma 3.3.1. *For $i \in V$ and $u \geq 0$, define*

$$E_i(u) = \exp \left(-\theta_i^2 \beta_i + \frac{\beta_i e^{2\rho_i(u)}}{\phi_i(u)} - \frac{1}{2} \rho_i(u) + \frac{1}{8} u \right) \phi_i(u)^{3/2} \sqrt{\theta_i}.$$

Then $\mathcal{E}(L_i) = E_i$.

Proof. It suffices to show that $\frac{dE_i(u)}{E_i(u)} = dL_i(u)$ for all $u \geq 0$, which will imply that $E_i = \mathcal{E}(L_i)$, since $E_i(0) = 1$ almost surely. Note that $\rho_i(u) = \hat{B}_i(u) + \frac{1}{2}u + \log(\phi_i(u))$, so that

$$E_i(u) = \exp \left(-\theta_i^2 \beta_i + \beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} - \frac{1}{2} \hat{B}_i(u) - \frac{1}{8} u \right) \phi_i(u) \sqrt{\theta_i}.$$

By Ito's lemma, for $u \geq 0$ we have

$$\begin{aligned} dE_i(u) &= \left(2\beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} - \frac{1}{2} \right) E_i(u) d\hat{B}_i(u) \\ &\quad + \frac{1}{2} \left(\left(2\beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} - \frac{1}{2} \right)^2 + 4\beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} \right) E_i(u) du \\ &\quad + \left(\beta_i (\phi_i(u) + \phi_i'(u)) e^{2\hat{B}_i(u)+u} - \frac{1}{8} + \frac{\phi_i'(u)}{\phi_i(u)} \right) E_i(u) du. \end{aligned}$$

Since $\phi_i'(u) = -2\beta_i e^{2\rho_i(u)} = -2\beta_i \phi_i(u)^2 e^{2\hat{B}_i(u)+u}$, we get

$$\begin{aligned} \frac{dE_i(u)}{E_i(u)} &= \left(-\frac{1}{2} + \frac{2\beta_i e^{2\rho_i(u)}}{\phi_i(u)} \right) d\hat{B}_i(u) + \left(2\beta_i^2 \phi_i(u)^2 e^{4\hat{B}_i(u)+2u} + \frac{1}{8} + \beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} \right. \\ &\quad \left. + \beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} - 2\beta_i^2 \phi_i(u)^2 e^{4\hat{B}_i(u)+2u} - \frac{1}{8} - 2\beta_i \phi_i(u) e^{2\hat{B}_i(u)+u} \right) du \\ &= dL_i(u). \end{aligned}$$

□

Fix $u \geq 0$, then for any event $A_u \in \mathcal{F}_u^\rho = \sigma(\rho(v), 0 \leq v \leq u)$, we have

$$\begin{aligned} \mathbb{P}[A_u] &= \int \left(\bigotimes_{i \in V} \mathbb{P}_i^{\beta_i} [A_u] \right) \nu_V^{W, \theta, \eta} = \int \int_{A_u} \prod_{i \in V} \left(E_i(u)^{-1} d\hat{\mathbb{P}}_i \right) \nu_V^{W, \theta, \eta}(d\beta) \\ &= \int_{A_u} D(u) d\hat{\mathbb{P}}, \end{aligned}$$

where $\hat{\mathbb{P}} = \bigotimes_{i \in V} \hat{\mathbb{P}}_i$ and for $u \geq 0$,

$$D(u) = \int \left(\prod_{i \in V} E_i(u)^{-1} \right) \nu_V^{W, \theta, \eta}(d\beta).$$

We now have to compute $D(u)$, and express it as an exponential martingale, in order to apply Girsanov's theorem once again, and identify the distribution of ρ under \mathbb{P} .

For $u \geq 0$, we have

$$\begin{aligned} D(u) &= \int \exp \left(\sum_{i \in V} \left(\theta_i^2 \beta_i - \frac{\beta_i e^{2\rho_i(u)}}{\phi_i(u)} + \frac{1}{2} \rho_i(u) - \frac{1}{8} u \right) \right) \frac{1}{\prod_{i \in V} \phi_i(u)^{3/2} \sqrt{\theta_i}} \\ &\quad \mathbb{1}_{H_\beta > 0} \left(\frac{2}{\pi} \right)^{|V|/2} \exp \left(-\frac{1}{2} \langle \theta, H_\beta \theta \rangle - \frac{1}{2} \langle \eta, (H_\beta)^{-1} \eta \rangle + \langle \eta, \theta \rangle \right) \frac{\prod_{i \in V} \theta_i d\beta_i}{\sqrt{|H_\beta|}}. \end{aligned}$$

In order to compute this integral, we will introduce a change of variables, and obtain an integral against the distribution $\nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}}$, where $\tilde{W}^{(u)}$, $\tilde{\theta}^{(u)}$ and $\tilde{\eta}^{(u)}$ are new parameters depending on the trajectory of ρ up to time u .

Let us introduce the following notations : for $u \geq 0$,

$$\begin{cases} \beta_i^{(u)} = \frac{1}{2T_i(u)} \text{ for } i \in V \\ H^{(u)} = 2\beta^{(u)} - W \\ K^{(u)} = T(u)H^{(u)} = \text{Id} - T(u)W. \end{cases}$$

From there, we define the new following parameters :

$$\begin{cases} \tilde{W}^{(u)} = W(K^{(u)})^{-1} = W + W(H^{(u)})^{-1}W \\ \tilde{\eta}^{(u)} = \tilde{W}^{(u)}T(u)\eta + \eta \\ \tilde{\theta}_i^{(u)} = e^{\rho_i(u)} \text{ for } i \in V \end{cases}$$

as well as these associated quantities :

$$\begin{cases} \tilde{T}_i(u) = \frac{1}{2\beta_i} - T_i(u) = \frac{\phi_i(u)}{2\beta_i} \text{ for } i \in V \\ \tilde{\beta}_i^{(u)} = \frac{1}{2\tilde{T}_i(u)} = \frac{\beta_i}{\phi_i(u)} \text{ for } i \in V \\ \tilde{H}^{(u)} = 2\tilde{\beta}^{(u)} - \tilde{W}^{(u)} \\ \tilde{K}^{(u)} = \tilde{T}(u)\tilde{H}^{(u)} = \text{Id} - \tilde{T}(u)\tilde{W}^{(u)}. \end{cases}$$

Using these new notations, we can already write

$$\sum_{i \in V} \frac{\beta_i e^{2\rho_i(u)}}{\phi_i(u)} = \sum_{i \in V} (\tilde{\theta}_i^{(u)})^2 \tilde{\beta}_i^{(u)} = \frac{1}{2} \left\langle \tilde{\theta}^{(u)}, \left(\tilde{H}^{(u)} + \tilde{W}^{(u)} \right) \tilde{\theta}^{(u)} \right\rangle \quad (3.4)$$

for $u \geq 0$. Moreover, we will need the following technical lemma in order to express $D(u)$ as an integral against $\nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}}$.

Lemma 3.3.2 (Lemma 2 in [SZ17]). *For $u \geq 0$, we have :*

- (i) $K_{1/2\beta} = \tilde{K}^{(u)} K^{(u)}$
- (ii) $\tilde{\eta}^{(u)} = T(u)^{-1} (H^{(u)})^{-1} \eta$
- (iii) $\langle \tilde{\eta}^{(u)}, (\tilde{H}^{(u)})^{-1} \tilde{\eta}^{(u)} \rangle = \langle \eta, H_\beta^{-1} \eta \rangle - \langle \eta, (H^{(u)})^{-1} \eta \rangle$.

Using Lemma 3.3.2 (i), we get that for $u \geq 0$,

$$H_\beta = 2\beta K_{1/2\beta} = 2\beta \tilde{K}^{(u)} K^{(u)} = 2\beta \tilde{T}^{(u)} \tilde{H}^{(u)} K^{(u)},$$

where $2\beta_i \tilde{T}_i^{(u)} = 1 - \frac{\beta_i}{\beta_i^{(u)}} = \phi_i(u)$ for $i \in V$. Therefore, we have

$$\prod_{i \in V} \phi_i(u)^{3/2} \sqrt{|H_\beta|} = \prod_{i \in V} \phi_i(u)^2 \sqrt{|\tilde{H}^{(u)}|} \sqrt{|K^{(u)}|}, \quad (3.5)$$

where

$$\frac{d\tilde{\beta}_i^{(u)}}{d\beta_i} = \frac{1}{\left(1 - \frac{\beta_i}{\beta_i^{(u)}}\right)^2} = \frac{1}{\phi_i(u)^2}.$$

Moreover, for all $u \geq 0$ we have :

$$\mathbf{1}_{H_\beta > 0} = \mathbf{1}_{H^{(u)} > 0} \mathbf{1}_{\tilde{H}^{(u)} > 0}. \quad (3.6)$$

Combining equations (3.4), (3.5) and (3.6), as well as Lemma 3.3.2 (iii), we finally obtain :

$$\begin{aligned} D(u) &= \left(\int \mathbf{1}_{\tilde{H}^{(u)} > 0} \left(\frac{2}{\pi} \right)^{|V|/2} \exp \left(-\frac{1}{2} \langle \tilde{\theta}^{(u)}, \tilde{H}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2} \langle \tilde{\eta}^{(u)}, (\tilde{H}^{(u)})^{-1} \tilde{\eta}^{(u)} \rangle + \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle \right) \right. \\ &\quad \left. \frac{\prod_{i \in V} \tilde{\theta}_i^{(u)}}{\sqrt{|\tilde{H}^{(u)}|}} \prod_{i \in V} \frac{d\tilde{\beta}_i^{(u)}}{d\beta_i} d\beta_i \right) \mathbf{1}_{H^{(u)} > 0} \exp \left(\frac{1}{2} \langle \theta, W\theta \rangle + \langle \eta, \theta \rangle \right) \prod_{i \in V} \sqrt{\theta_i} \\ &\quad \exp \left(-\frac{1}{2} \langle \tilde{\theta}^{(u)}, \tilde{W}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2} \langle \eta, (H^{(u)})^{-1} \eta \rangle - \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle \right) \frac{\prod_{i \in V} \exp \left(\frac{1}{2} \rho_i(u) - \frac{1}{8} u \right)}{\sqrt{|K^{(u)}|} \prod_{i \in V} \tilde{\theta}_i^{(u)}} \\ &= \mathbf{1}_{H^{(u)} > 0} \exp \left(-\frac{1}{2} \langle \tilde{\theta}^{(u)}, \tilde{W}^{(u)} \tilde{\theta}^{(u)} \rangle - \frac{1}{2} \langle \eta, (H^{(u)})^{-1} \eta \rangle - \langle \tilde{\eta}^{(u)}, \tilde{\theta}^{(u)} \rangle \right) \\ &\quad \frac{\prod_{i \in V} \exp \left(-\frac{1}{2} \rho_i(u) - \frac{1}{8} u \right)}{\sqrt{|K^{(u)}|}} \exp \left(\frac{1}{2} \langle \theta, W\theta \rangle + \langle \eta, \theta \rangle \right) \prod_{i \in V} \sqrt{\theta_i}, \end{aligned}$$

since the integral between brackets becomes

$$\int \nu_V^{\tilde{W}^{(u)}, \tilde{\theta}^{(u)}, \tilde{\eta}^{(u)}} (d\tilde{\beta}^{(u)}) = 1.$$

Let us now show that D is the exponential martingale associated with a certain \mathcal{F}_u^ρ -martingale. By Ito's lemma, for $u \geq 0$ we have

$$\begin{aligned} dD(u) &= \sum_{i \in V} \left(-(\tilde{W}^{(u)} e^{\rho(u)})_i e^{\rho_i(u)} - \tilde{\eta}_i^{(u)} e^{\rho_i(u)} - \frac{1}{2} \right) D(u) d\rho_i(u) \\ &\quad + \frac{1}{2} \sum_{i \in V} \left(\left(-(\tilde{W}^{(u)} e^{\rho(u)})_i e^{\rho_i(u)} - \tilde{\eta}_i^{(u)} e^{\rho_i(u)} - \frac{1}{2} \right)^2 \right. \\ &\quad \left. + \left(-(\tilde{W}^{(u)} e^{\rho(u)})_i e^{\rho_i(u)} - \tilde{W}_{i,i}^{(u)} e^{2\rho_i(u)} - \tilde{\eta}_i^{(u)} e^{\rho_i(u)} \right) \right) D(u) du \\ &\quad + \left(-\frac{1}{2} \langle e^{\rho(u)}, \partial_u(\tilde{W}^{(u)} e^{\rho(u)}) \rangle - \frac{1}{2} \langle \eta, \partial_u(H^{(u)})^{-1} \eta \rangle \right. \\ &\quad \left. - \langle \partial_u \tilde{\eta}^{(u)}, e^{\rho(u)} \rangle - \frac{|V|}{8} - \frac{1}{2} \frac{\partial_u |K^{(u)}|}{|K^{(u)}|} \right) D(u) du. \end{aligned}$$

Since $H^{(u)} = 2\beta^{(u)} - W = 1/T(u) - W$, we have

$$\partial_u(H^{(u)})^{-1} = (H^{(u)})^{-1} T(u)^{-1} \partial_u(T(u)) T(u)^{-1} (H^{(u)})^{-1},$$

so that, using Lemma 3.3.2 (ii), we get

$$\begin{aligned} \langle \eta, \partial_u(H^{(u)})^{-1} \eta \rangle &= \langle T(u)^{-1} (H^{(u)})^{-1} \eta, e^{2\rho(u)} T(u)^{-1} (H^{(u)})^{-1} \eta \rangle \\ &= \langle \tilde{\eta}^{(u)}, e^{2\rho(u)} \tilde{\eta}^{(u)} \rangle = \sum_{i \in V} (\tilde{\eta}_i^{(u)})^2 e^{2\rho_i(u)}. \end{aligned}$$

Moreover, $\tilde{W}^{(u)} = W + W(H^{(u)})^{-1}W$, therefore

$$\begin{aligned} \langle e^{\rho(u)}, \partial_u(\tilde{W}^{(u)} e^{\rho(u)}) \rangle &= \langle e^{\rho(u)}, W(H^{(u)})^{-1} T(u)^{-1} e^{2\rho(u)} T(u)^{-1} (H^{(u)})^{-1} W e^{\rho(u)} \rangle \\ &= \langle e^{\rho(u)}, \tilde{W}^{(u)} e^{2\rho(u)} \tilde{W}^{(u)} e^{\rho(u)} \rangle = \sum_{i \in V} (\tilde{W}^{(u)} e^{\rho(u)})_i^2 e^{2\rho_i(u)}, \end{aligned}$$

and $\tilde{\eta}^{(u)} = \tilde{W}^{(u)} T(u) \eta + \eta$, so

$$\begin{aligned} \langle \partial_u \tilde{\eta}^{(u)}, e^{\rho(u)} \rangle &= \langle \partial_u(\tilde{W}^{(u)}) T(u) \eta + \tilde{W}^{(u)} \partial_u(T(u)) \eta, e^{\rho(u)} \rangle \\ &= \langle \tilde{W}^{(u)} e^{2\rho(u)} \tilde{W}^{(u)} T(u) \eta + \tilde{W}^{(u)} e^{2\rho(u)} \eta, e^{\rho(u)} \rangle \\ &= \langle \tilde{W}^{(u)} e^{2\rho(u)} \tilde{\eta}^{(u)}, e^{\rho(u)} \rangle = \sum_{i \in V} \left(\tilde{W}^{(u)} e^{\rho(u)} \right)_i \tilde{\eta}_i^{(u)} e^{2\rho_i(u)} \end{aligned}$$

Finally, we have

$$\begin{aligned} \partial_u |K^{(u)}| &= \text{Tr}(|K^{(u)}| (K^{(u)})^{-1} \partial_u K^{(u)}) = -|K^{(u)}| \text{Tr}(W (K^{(u)})^{-1} e^{2\rho(u)}) \\ &= -|K^{(u)}| \sum_{i \in V} \tilde{W}_{i,i}^{(u)} e^{2\rho_i(u)}. \end{aligned}$$

Therefore, we get :

$$\begin{aligned}
\frac{dD(u)}{D(u)} &= \sum_{i \in V} \left(- \left(\tilde{W}^{(u)} \left(e^{\rho(u)} + T(u)\eta \right) + \eta \right)_i e^{\rho_i(u)} - \frac{1}{2} \right) d\rho_i(u) \\
&\quad + \frac{1}{2} \sum_{i \in V} \left((\tilde{W}^{(u)} e^{\rho(u)})_i^2 e^{2\rho_i(u)} + (\tilde{\eta}_i^{(u)})^2 e^{2\rho_i(u)} + \frac{1}{4} + 2 \left(\tilde{W}^{(u)} e^{\rho(u)} \right)_i \tilde{\eta}_i^{(u)} e^{2\rho_i(u)} \right. \\
&\quad \left. + (\tilde{W}^{(u)} e^{\rho(u)})_i e^{\rho_i(u)} + \tilde{\eta}_i^{(u)} e^{\rho_i(u)} - (\tilde{W}^{(u)} e^{\rho(u)})_i e^{\rho_i(u)} - \tilde{W}_{i,i}^{(u)} e^{2\rho_i(u)} - \tilde{\eta}_i^{(u)} e^{\rho_i(u)} \right. \\
&\quad \left. - (\tilde{W}^{(u)} e^{\rho(u)})_i^2 e^{2\rho_i(u)} - (\tilde{\eta}_i^{(u)})^2 e^{2\rho_i(u)} - 2 \left(\tilde{W}^{(u)} e^{\rho(u)} \right)_i \tilde{\eta}_i^{(u)} e^{2\rho_i(u)} - \frac{1}{4} + \tilde{W}_{i,i}^{(u)} e^{2\rho_i(u)} \right) du \\
&= \sum_{i \in V} \left(- \left(\tilde{W}^{(u)} \left(e^{\rho(u)} + T(u)\eta \right) + \eta \right)_i e^{\rho_i(u)} - \frac{1}{2} \right) d\rho_i(u) = d\tilde{L}(u),
\end{aligned}$$

where for $i \in V$ and $u \geq 0$,

$$\tilde{L}_i(u) = \int_0^u \left(-\frac{1}{2} - \left(\tilde{W}^{(u)} \left(e^{\rho(u)} + T(u)\eta \right) + \eta \right)_i e^{\rho_i(u)} \right) d\rho_i(u).$$

Therefore, D is the exponential martingale associated with \tilde{L} .

Recall that for $u \geq 0$ and any event $A_u \in \mathcal{F}_u^\rho = \sigma(\rho(v), 0 \leq v \leq u)$, we have

$$\mathbb{P}[A_u] = \int_{A_u} D(u) d\hat{\mathbb{P}},$$

i.e. \mathbb{P} is such that

$$\mathbb{E} \left[\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \middle| \mathcal{F}_u^\rho \right] = \mathcal{E}(\tilde{L})(u)$$

for all $u \geq 0$. Moreover, $\hat{\mathbb{P}} = \otimes_{i \in V} \hat{\mathbb{P}}_i$, therefore ρ is a $|V|$ -dimensional standard Brownian motion under $\hat{\mathbb{P}}$. According to Girsanov's theorem, the process $\tilde{B}(u) = \rho(u) - \langle \rho, \tilde{L} \rangle_u$ is a standard Brownian motion under \mathbb{P} . In other words, under \mathbb{P} , the process ρ verifies the following SDE : for all $i \in V$ and $u \geq 0$,

$$d\rho_i(u) = d\tilde{B}_i(u) - \frac{1}{2} du - \left(\tilde{W}^{(u)}(e^{\rho(u)} + T(u)\eta) + \eta \right)_i e^{\rho_i(u)} du.$$

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Processus renforcés, représentation comme mélange de processus de Markov, et mouvement brownien interagissant

Résumé. Cette thèse porte sur les processus aléatoires renforcés, en particulier le VRJP (processus de saut renforcé par sommet) et ses représentations comme un processus de Markov en environnement aléatoire.

On s'intéresse d'abord à la question de l'unicité de cette représentation. Pour cela, on montre que toute représentation s'exprime à l'aide d'un même champ aléatoire β . On se ramène alors à étudier les fonctions harmoniques pour un opérateur de Schrödinger aléatoire associé, grâce à la notion de frontière de Martin. On obtient des résultats dans le cas du réseau à coordonnées entières, et celui des arbres infinis.

Dans un second temps, on considère le lien entre le champ β et les temps d'atteinte d'une famille de mouvements browniens interagissant. On utilise ce lien, ainsi qu'un changement de variable temporelle, pour montrer une version multi-dimensionnelle du théorème dit « des dérives opposées ».

Mots-clés : processus renforcés, processus de Markov en environnement aléatoire, frontière de Martin, équations différentielles stochastiques, relation de Lamperti.

Reinforced processes, representation as a mixture of Markov processes, and interacting Brownian motions

Abstract. This thesis concerns reinforced random processes, in particular the VRJP (vertex-reinforced jump process) and its representations as a Markov process in random environment.

We first consider the issue of uniqueness of this representation. To this aim, we show that any representation can be expressed using the same random field β . We are then reduced to studying harmonic functions for an associated random Schrödinger operator, thanks to the notion of Martin boundary. We get results for the integer lattice as well as infinite trees.

We are then interested in the link between this β field and hitting times for a family of interacting Brownian motions. We use this link, as well as a change of time variable, to show a multi-dimensional version of the so-called "opposite drift" theorem.

Keywords: reinforced processes, Markov processes in random environment, Martin boundary, stochastic differential equations, Lamperti's relation.



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