The splitting problem for braid groups of the projective plane and a remarkable quotient of welded braid groups

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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHEMATIQUES

Préparée au sein de l'Université de Caen Normandie

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Thèse soutenue le 10/12/2021 devant le jury composé de		
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Acknowledgement

I would like to express my sincere gratitude and my warmest thanks to my advisor J. Guaschi, and my co-advisor P. Bellingeri. Their expertise, constant support and friendly guidance with perfect blend of insight and humor have been invaluable in every stage of this thesis and throughout my life as a Ph.D. student.

I would also wish to thank V. Bardakov, A. Cattabriga, S. Lambropoulou, S. Gervain and E. Wagner who generously took time out of their schedules for participating in my thesis defense committee. Special thanks are due to V. Bardakov and S. Lambropoulou for providing thorough reports of my thesis.

My gratitude extends to É. Ricard, as well as to every member of the Laboratoire de Mathématiques Nicolas Oresme for welcoming me in the department.

Very special thanks and appreciation to S. Z. Alterman (F) for holding helpful and fruitful mathematical discussions and for his multifaceted support.

Many thanks to my family for accompanying and supporting me these years in their own way.

My warmest thanks to the young members of LMNO, whom I got to know and with whom I spent beautiful moments during my Ph.D. thesis. That includes Arnaud, Étienne M., Julien, Angelot, Frank, Pablo, Guillaume, Coumba, Hung, Léonard, Vlerë, Thien, Nacer, Tin, Rubén, Tiphaine, Dorian, Njaka, Étienne E., Manu, Francesco, Giada, Édouard.

I could not omit thanking Anita and Marie for their constant assistance in administrative matters, always accompanied by positive vibes.

Contents

Introduction en français

Cette thèse est divisée en deux parties. Les trois premiers chapitres concernent des groupes de tresses des surfaces, tandis que les deux derniers traitent des groupes de tresses soudées et des groupes de tresses virtuelles sans restrictions.

Les groupes de tresses des surfaces sont à la fois une généralisation à toute surface connectée du groupe fondamental d'une surface et des groupes de tresses du plan, appelés groupes de tresses d'Artin, et qui ont été définis par Artin en 1925 dans [Art25]. Ils ont été initialement introduits par Zariski dans [Zar36], [Zar37] et plus tard, dans les années 1960, Fox en a donné une définition équivalente d'un point de vue topologique. Comme pour les groupes de tresses d'Artin, les groupes de tresses des surfaces peuvent être décrits de plusieurs points de vue, dont l'un est le suivant. Soit $n \in \mathbb{N}^*$; pour toute surface connexe Σ, on se donne un ensemble de n points distincts, $P = \{x_1, \dots, x_n\}$, à l'intérieur de Σ. Une n–tresse, $\beta = (\beta_1(t), \dots, \beta_n(t))_{t\in[0,1]}$, est une collection de n chemins distincts sur la surface Σ, $\beta_i(t) \neq \beta_j(t)$ pour tout $t \in [0,1]$ et tout $i \neq j$, où chaque chemin a pour point initial et final un point dans P, qui n'est pas forcément le même. Nous appelons ces n chemins les brins de la tresse. Maintenant, à homotopie près, on peut définir une structure de groupe sur cet ensemble de n–tresses. Dorénavant, on note le groupe de tresses de Σ à n brins par $B_n(\Sigma)$.

La théorie des groupes de tresses des surfaces a été largement étudiée, et il existe plusieurs résultats connus sur leur structure et leur présentations. Les premières présentations des groupes de tresses des surfaces compactes sans bord ont été données par Birman [Bir69] et Scott [Sco70]. Il existe d'autres présentations de groupes de tresses des surfaces dues à Bellingeri [Bel04], Gonçalves–Guaschi [GG04b],[GG10a], González-Meneses [GM01] et Lambropoulou [Lam00], entre autres. Parmi les surfaces, les groupes de tresses de la 2–sphère, $\mathbb{S}^2$, et du plan projectif, $\mathbb{R}P^2$, sont particulièrement intéressants, car elles sont les seules surfaces dont les groupes de tresses ont des éléments de torsion. Ce fait a été prouvé pour la 2–sphère par Fadell–Van Buskirk dans [FVB62] et pour le plan projectif par Van Buskirk dans [VB66].

Le groupe de tresses $B_n(\Sigma)$ est aussi lié au groupe symétrique S_n, puisqu'il existe un épimorphisme naturel $\sigma : B_n(\Sigma) \to S_n$, où à chaque n–tresse $\beta = (\beta_1(t), \dots, \beta_n(t))_{t\in[0,1]}$ on associe une permutation $\sigma(\beta) \in S_n$ définie par $\beta_i(1) = x_{\sigma(\beta)(i)} \in P$, pour tout $i \in \{1, \dots, n\}$. Le noyau du morphisme de groupes σ est ce qu'on appelle le groupe de tresses pures, et on le note $P_n(\Sigma)$.

Les groupes de tresses des surfaces sont aussi liés aux espaces de configuration des surfaces. On notera $F_n(\Sigma) = \{(p_1, \dots, p_n) \in \Sigma^n \mid p_i \neq p_j \text{ for all } i, j \in \{1, \dots, n\},\ i \neq j\}$ le n–ème espace de configurations ordonnées de la surface Σ. Compte tenu de l'action naturelle du groupe symétrique S_n sur $F_n(\Sigma)$, définie par permutation des coordonnées, on peut définir le n–ème espace de configurations non ordonnées, qui correspond à l'espace des orbites $F_n(\Sigma)/S_n$. De même, pour $n, m \in \mathbb{N}^*$, on peut considérer l'espace obtenu en prenant le quotient du $(n+m)$–ème espace de configurations de Σ par l'action du sous-groupe $S_n \times S_m$ de S_{n+m}, c'est-à-dire $F_{n+m}(\Sigma)/S_n \times S_m$. Fox–Neuwirth [FN62b] ont prouvé que $\pi_1(F_n(\Sigma))$ est isomorphe à $P_n(\Sigma)$, et $\pi_1(F_n(\Sigma)/S_n)$ est isomorphe à $B_n(\Sigma)$. De la même façon, les groupes $\pi_1(F_{n+m}(\Sigma)/(S_n \times S_m))$ sont isomorphes aux groupes notés $B_{n,m}(\Sigma)$, qui sont des sous-groupes des groupes $B_{n+m}(\Sigma)$ appelés groupes de tresses mixtes.

Les groupes de tresses mixtes sont utiles pour étudier la théorie des représentations des groupes de tresses des surfaces ainsi que pour la théorie des nœuds sur les 3-variétés. Ces groupes ont été étudiés par Sossinsky [Sos92], Manfredini [Man97], Paris–Rolfsen [PR99], Lambropoulou [Lam00], Gonçalves–Guaschi [GG04c], [GG05], [GG12] Bellingeri–Godelle–Guaschi [BGG] et Diamantis–Lambropoulou–Przytycki [DLP16], entre autres.

Un outil important dans l'étude des groupes de tresses est la suite exacte courte de Fadell–Neuwirth qui découle de la fibration de Fadell–Neuwirth, qui existe pour Σ sans bord. Toutefois, une fois passé à la suite en homotopie, pour $n, m \in \mathbb{N}^*$, et toute surface connexe Σ, nous avons aussi les suivantes suites exactes courtes de Fadell–Neuwirth:

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1, \text{ pour } 1 \leq m < n, \quad (1)$$

$$1 \longrightarrow B_m(\Sigma \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1, \quad (2)$$

qui sont connues sous le nom de suite exacte courte de Fadell–Neuwirth des groupes de tresses pures de surface et de suite exacte courte généralisée de Fadell–Neuwirth des groupes de tresses mixtes, respectivement. Les applications induites $\bar{p}_{n,m}$ et $\bar{q}_{n+m,n}$ peuvent être considérées géométriquement comme les épimorphismes qui oublient les $n-m$ et m derniers brins respectivement. On observe que lorsque Σ est la 2–sphère $\mathbb{S}^2$ ou le plan projectif $\mathbb{R}P^2$, il y a quelques restrictions sur les valeurs de n et m. On remarque que l'application $\bar{p}_{n,m}$ ne s'étend pas directement aux groupes de tresses $B_n(\Sigma) \to B_m(\Sigma)$, car elle n'est pas bien définie, mais $\bar{q}_{n+m,n}$ nous permet d'étendre $\bar{p}_{n,m}$ à certains sous-groupes de $B_{n+m}(\Sigma)$.

Le problème de scindement fait référence à la question de savoir si les suites exactes courtes (1) et (2) se scindent, ou de manière équivalente, si les applications $\bar{p}_{n,m}$ et $\bar{q}_{n+m,n}$ admettent une section, ce qui revient à demander s'il existe ou non des morphismes injectifs $\bar{s}_{n,m}$ et $\hat{s}_{n+m,n}$, respectivement, tel que $\bar{p}_{n,m} \circ \bar{s}_{n,m} = id_{P_m(\Sigma)}$ et $\bar{q}_{n,m} \circ \hat{s}_{n,m} = id_{B_n(\Sigma)}$. Un aspect intéressant de ce problème est qu'il peut être abordé algébriquement et géométriquement. L'équivalence entre le scindement des suites exactes courtes (1), (2) et l'existence d'une section géométrique pour les fibrations $p_{n,m} : F_{n+m}(\Sigma) \to F_n(\Sigma)$ et $q_{n+m,n} : F_{n+m}(\Sigma)/(S_n \times S_m) \to F_n(\Sigma)/S_n$, respectivement, a été donnée par Baues [Bau77] et Whitehead [Whi78] dans le cas que Σ est un espace asphérique, et par Gonçalves–Guaschi quand $\Sigma = \mathbb{S}^2$ ou $\Sigma = \mathbb{R}P^2$. Dans la Proposition 1.3.1 nous présentons cette équivalence de manière approfondie.

Le scindement de la suite exacte courte (1) dans le cas du plan, $\Sigma = \mathbb{R}^2$, est un résultat important dans la théorie classique des tresses. Comme conséquence, on obtient la forme normale d'Artin pour les groupes de tresses d'Artin purs P_n, c'est-à-dire la décomposition $P_n \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1$, où F_n est le groupe libre de rang n, qui est l'un des principaux outils dans l'étude de P_n. Par exemple, on utilise la forme normale d'Artin de P_n pour démontrer l'unicité des racines dans P_n, par Bardakov [Bar92], pour l'étude de la suite centrale descendante et de la nilpotence résiduelle de P_n, par Falk–Randell [FR85], ainsi que pour la preuve de la bi-ordonnabilité de P_n, par Kim–Rolfsen [KR03].

Il est donc important de savoir si le suites exactes courtes (1) et (2) se scindent ou non. Au cours des années 1960, qui a été la période de développement de la théorie des groupes de tresses des surfaces, de nombreux mathématiciens ont étudié ce problème pour des surfaces outres que le plan. Le problème de scindement pour (1) a été étudié notamment par Fadell [Fad62], Fadell–Neuwirth [FN62a], Fadell–Van Buskirk [FVB62], Van Buskirk [VB66] et Birman [Bir69], avec des approches géométriques et algébriques. Dans le cas des groupes de tresses pures des surfaces, le problème de scindement a été résolue par Gonçalves–Guaschi, dont la solution est donnée dans le Théorème 1.3.3. Pour donner une réponse positive au problème de scindement, il suffit de mettre en évidence une section explicite, soit géométrique, soit algébrique. Pour obtenir une réponse négative au problème de scindement, l'une des principales méthodes qui a été utilisée pour prouver le Théorème 1.3.3 est basée sur l'observation suivante : soit G un groupe, et soient K, H des sous-groupes distingués de G tels que H soit contenu dans K. Si la suite exacte courte $1 \to K \to G \to R \to 1$ est scindée alors la suite exacte courte $1 \to K/H \to G/H \to R \to 1$ l'est aussi. Le problème de scindement de l'extension de G/H est parfois plus facile à étudier que celui de G. Ainsi, en montrant que la deuxième extension ne se scinde pas, il suit que la première ne se scinde pas. Cependant, dans le cas des groupes de tresses mixtes, le problème de scindement n'a toujours pas de solution

complète. En particulier, à notre connaissance, la seule surface, à l'exception du plan $\mathbb{R}^2$, pour laquelle le scindement de la suite exacte courte (2) a été étudié, est la 2–sphère $\mathbb{S}^2$. Dans [GG05], Gonçalves–Guaschi, ont donné des résultats partiels pour le problème de scindement pour $\mathbb{S}^2$, voir le Théorème 1.3.4, et plus récemment Chen–Salter ont amélioré ces résultats, voir le Théorème 1.3.5.

Dans la première partie de cette thèse, nous étudions le problème de scindement des groupes de tresses mixtes du plan projectif $\mathbb{R}P^2$. Les groupes de tresses du plan projectif, $B_n(\mathbb{R}P^2)$, sont particulièrement intéressants car, avec le groupe de tresses de la 2–sphère, ce sont les seuls groupes de tresses qui contiennent des éléments de torsion. Une présentation de $B_n(\mathbb{R}P^2)$ a été donnée par Van Buskirk dans [VB66], voir le Théorème 2.1.1, dont l'ensemble des générateurs est constitué des $n-1$ générateurs d'Artin usuels, ainsi que des n éléments correspondant aux éléments du groupe fondamental du plan projectif basés respectivement dans les points de P. Les éléments de torsion de $B_n(\mathbb{R}P^2)$ ont d'abord été déterminés et caractérisés par Murasugi dans [Mur82]. Plus tard Gonçalves–Guaschi, en ont donné une caractérisation simplifiée dans [GG10b]. Les groupes $B_n(\mathbb{R}P^2)$ ont été largement étudiés par Gonçalves–Guaschi qui ont déterminé, à isomorphisme près, les sous-groupes finis maximaux de $P_n(\mathbb{R}P^2)$ ainsi que les sous-groupes finis de $B_n(\mathbb{R}P^2)$.

On précise maintenant la structure des trois premiers chapitres de la thèse.

Dans le Chapitre 1, nous rappelons les définitions et propriétés principales des groupes de tresses des surfaces. Nous donnons une définition géométrique des groupes de tresses des surfaces et nous présentons leur liens avec les espaces des configurations. De plus, dans la Section 1.2, nous donnons une description détaillée de la fibration de Fadell–Neuwith et des suites exactes courtes qui ont découlent

$$1 \longrightarrow P_{n-m}(\Sigma \smallsetminus \{x_1,\dots,x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1,$$

$$1 \longrightarrow B_m(\Sigma \smallsetminus \{x_1,\dots,x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1,$$

qui constituent le sujet d'étude principal de cette partie de la thèse. A la fin du Chapitre 1, dans la Section 1.3, on décrit en détail le problème de scindement et on en présentera les résultats connus là-dessus.

Au Chapitre 2, nous rappelons les groupes de tresses du plan projectif, $B_n(\mathbb{R}P^2)$, en donnant aussi des renseignements sur leur structure. Dans la Section 2.2, nous déterminerons des présentations de certains sous-groupes de $B_{n,m}(\mathbb{R}P^2)$ qui vont nous permettre d'obtenir une présentation des groupes de tresses mixtes $B_{n,m}(\mathbb{R}P^2)$. Nous donnons d'abord les présentations des groupes $P_n(\mathbb{R}P^2)$, $P_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})$ et $B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})$, en appliquant des techniques décrites dans [Joh97] (page 139) pour obtenir des présentations d'extensions de groupe. Ensuite, dans le Théorème 2.2.7, on obtient une présentation de $B_{n,m}(\mathbb{R}P^2)$ en appliquant les mêmes méthodes à la suivante suite exacte courte :

$$1 \to B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \to 1, \tag{3}$$

où l'application $\bar{q}_{n+m,n}$ peut être considérée géométriquement comme l'épimorphisme qui oublie les m derniers brins. À l'aide de ces présentations, nous donnons une présentation de l'abélianisé de chacun de ces groupes. Nous terminons le Chapitre 2 par la Section 2.3, où nous étudions les suites centrales descendantes et dérivées du groupe $B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})$, en prouvant en particulier, dans le Théorème 2.3.2, le résultat suivant:

Théorème 1. Soit $n \geq 1$. Alors:

- si $m \geq 3$, alors

$$\Gamma_2\big(B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})\big) = \Gamma_3\big(B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})\big).$$

- si $m \geq 5$, alors

$$\big(B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})\big)^{(1)} = \big(B_m(\mathbb{R}P^2 \smallsetminus \{x_1,\dots,x_n\})\big)^{(2)}.$$

En particulier, pour $m \geq 3$, $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$ n'est pas résiduellement nilpotent et pour $m \geq 5$, il n'est pas résiduellement résoluble.

En raison du fait suivant, le Théorème 1 est important dans l'étude du problème de scindement. Soit N un sous-groupe distingué de $B_{n,m}(\mathbb{R}P^2)$, qui est aussi contenu dans $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$. On obtient ainsi la suivante suite exacte courte quotientée :

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})/N \longrightarrow B_{n,m}(\mathbb{R}P^2)/N \xrightarrow{q} B_n(\mathbb{R}P^2) \longrightarrow 1,$$

où $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$ désigne l'homomorphisme induit par $\bar{q}_{n+m,n}$. On observe que si la suite exacte courte (3) se scinde alors cette suite exacte courte quotientée se scinde également. Plus précisément, considérons le diagramme commutatif suivant des suites exactes courtes :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) & \longrightarrow & B_{n,m}(\mathbb{R}P^2) & \xrightarrow{\bar{q}_{n+m,n}} & B_n(\mathbb{R}P^2) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle pr} & & \| & & \\
1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})/N & \longrightarrow & B_{n,m}(\mathbb{R}P^2)/N & \xrightarrow{q} & B_n(\mathbb{R}P^2) & \longrightarrow & 1.
\end{array}$$

Supposons que l'homomorphisme $\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ admette une section $\bar{s}_{n+m,n}$. Alors $s = pr \circ \bar{s}_{n+m,n}$ est une section pour $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$. Il suit que les suites centrales descendantes et dérivées de certains groupes fournissent un outil important dans l'étude du problème de scindement, puisqu'on peut étudier le quotient de $B_{n,m}(\mathbb{R}P^2)$ par N, pour n'importe quel élément N des suites centrales descendantes ou dérivées de $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$.

Nous terminons la première partie de la thèse au Chapitre 3, où nous étudions le scindement possible de (3). Dans la Section 3.1, on analyse l'existence d'une section pour l'homomorphisme

$$\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2),$$

pour $n = 1$ et $n = 2$, où $m \in \mathbb{N}^*$. En particulier, pour $m \in \mathbb{N}^*$, on montre dans la Proposition 3.1.1 que l'homomorphisme $\bar{q}_{1+m,n} : B_{1,m}(\mathbb{R}P^2) \to B_1(\mathbb{R}P^2)$ n'admet aucune section, tandis que pour $n = 2$, l'homomorphisme $\bar{q}_{2+m,n} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$ en admet une pour chaque $m \in \mathbb{N}^*$. Plus précisément, nous prouvons que la suite exacte courte

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow B_{2,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{2+m,2}} B_2(\mathbb{R}P^2) \longrightarrow 1$$

se scinde pour tout $m \in \mathbb{N}^*$. Dans la Proposition 3.1.2, nous fournissons une section géométrique explicite pour l'application $q_{2+m,2} : F_{2+m}(\Sigma)/(S_2 \times S_m) \to F_2(\Sigma)/S_2$, et dans la Proposition 3.1.3, nous mettons en évidence une section algébrique explicite pour l'homomorphisme $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$, où les homomorphismes $q_{2+m,2}, \bar{q}_{2+m,2}$ peuvent être considérés géométriquement comme les applications qui oublient respectivement les m derniers points et les m derniers brins. Pour le cas $n \geq 3$, dans la Section 3.2 nous donnons des conditions nécessaires sur m afin que la suite exacte courte

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1 \qquad (4)$$

se scinde. En particulier, dans le Théorème 3.2.4 nous montrons que $m = k(n-1)$, où $k \geq 1$, pour que (4) se scinde. De plus, en utilisant les éléments de torsion des groupes $B_{n,m}(\mathbb{R}P^2)$ et $B_n(\mathbb{R}P^2)$, dans la Proposition 3.2.7 nous obtenons des restrictions supplémentaires sur les valeurs de m, et en particulier, nous obtenons que $m \equiv 0 \bmod n$ ou $m \equiv 1 \bmod n$. Nous terminons le Chapitre 3 par la Section 3.3, où, pour certaines valeurs de m, nous donnons deux constructions différentes d'une section géométrique pour la fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$. Pour ces valeurs de m, l'existence de ces sections géométrique, donne une réponse positive au problème de scindement de la suite exacte courte (4). Réunissant le Théorème 3.2.4 et les Propositions 3.2.7, 3.3.1, 3.3.2, on obtient le résultat principal concernant le problème de scindement, qui est donné dans le Théorème 3.3.4 comme suit.

Théorème 2. Soit $m > 1$ et $n \geq 3$. Alors, la suite exacte courte

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

se scinde pour $m = 2n(n-1)$ et pour $m = kn(2n-1)(2n-2)$, où $k \geq 1$. De plus, si l'homomorphisme $\bar{q}_{n+m,n}$ admet une section, alors $m \equiv 0, (n+1)^2 \bmod n(n-1)$.

On remarque que les cas restants qui ne sont pas couverts par le Théorème 2 sont ouverts.

La deuxième partie de la thèse concerne les groupes de tresses soudées et les groupes de tresses virtuelles sans restrictions. Les groupes de tresses soudées, qu'on dénote par WB_n, pour $n \in \mathbb{N}^*$, sont un analogue tridimensionnel des groupes de tresses d'Artin B_n. Comme pour les groupes de tresses d'Artin, il en existe plusieurs interprétations, soit en termes du groupe de classe d'application, soit du groupe fondamental d'espaces de configuration spécifiques ou soit des automorphismes du groupe libre F_n. Le nom de groupes de tresses soudées a été introduit par Fenn–Rimányi–Rourke dans [FRR97]; de plus, ces groupes WB_n apparaissent dans la littérature sous de nombreux noms différents, par exemple comme automorphismes de conjugaison du groupe libre F_n grâce à Savushkina [Sav96], comme les groupes de tresses des cercles grâce à Baez–Crans–Wise [BCW07], et comme "untwisted ring groups" dû à Brendle–Hatcher [BH13]. Une présentation de WB_n a été donnée par Fenn–Rimányi–Rourke, dans [FRR97], qu'on présente dans le Théorème 4.1.3. Plus tard, Brendle–Hatcher, dans [BH13], ont donné une présentation du "untwisted ring group", UR_n, donné dans le Théorème 4.1.6, ce qui donne une interprétation géométrique du groupe de tresses soudées WB_n, puisque ces deux groupes sont isomorphes. Les générateurs σ_i, ρ_i, donnés dans les deux présentations ont été initialement considérés par Goldsmith dans [Gol81]. Le générateur σ_i agit comme une permutation en passant le cercle i–ème par le cercle $(i+1)$–ème et le générateur ρ_i les permute en passant le i–ème autour du $(i+1)$–ème.

Un sous-groupe important de WB_n est le groupe de tresses pures soudées WP_n, qui a également plusieurs interprétations. Dans cette thèse, nous nous intéressons principalement à la définition de WP_n en tant que sous-groupe du groupe des automorphismes du groupe libre F_n, constitué des automorphismes dits de conjugaison de bases, mais aussi en tant que groupe fondamental d'un espace de configurations spécifiques. Notez que tel que dans le cas des groupes de tresses des surfaces, WB_n est étroitement lié au groupe symétrique S_n, puisqu'on peut considérer l'application $\phi : WB_n \to S_n$, définie par $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, pour $i = 1, \ldots, n-1$. À partir de cela, il s'ensuit que le groupe de tresses pures soudées WP_n, est en fait le noyau de l'application ϕ. Dans [Sav96], Savushkina a prouvé que WB_n peut être vu comme le produit semi-direct de ses sous-groupes WP_n et S_n, et a déterminé l'action de S_n sur WP_n, comme décrit dans le Théorème 4.3.6. Pour être plus précis, S_n agit en permutant les indices des générateurs de WP_n.

Les groupes de tresses virtuelles sans restrictions, et que dans la suite on dénotera simplement par UVB_n, ont été introduites par Kauffman et Lambropoulou dans [KL04] et [KL06], où ils fournissent une nouvelle méthode pour convertir des nœuds virtuels et des entrelacs en tresses virtuelles, et ils prouvent un théorème de Markov pour les groupes de tresses virtuelles. De plus, le groupe UVB_n apparaît dans [KMRW17] comme quotient du groupe de tresses soudées WB_n et dans [BBD15], où Bardakov–Bellingeri–Damiani donnent une description de sa structure. Dans la Définition 5.1.1, nous voyons que UVB_n peut être défini en ajoutant une relation supplémentaire à la présentation du groupe WB_n, et donc, UVB_n a la même partie génératrice que WB_n. Encore une fois, en considérant l'application $\phi : UVB_n \to S_n$ définie par $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, pour $i = 1, \ldots, n-1$, le groupe de tresses pures virtuelles sans restrictions, que on dénote par UVP_n, correspond au noyau de l'application ϕ, étant un sous-groupe important de UVB_n.

Dans [BBD15], Bardakov–Bellingeri–Damiani ont étudié le groupe UVP_n, donnant une description de ses éléments et en donnant la présentation suivante :

$$UVP_n = \langle \lambda_{i,j},\ 1 \leq i \neq j \leq n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j},\ \text{pour } (k,l) \neq (j,i),\ 1 \leq i,j,k,l \leq n \rangle, \tag{5}$$

qu'on présente dans le Théorème 5.3.1. De cette présentation, il résulte que UVP_n est isomorphe au produit direct de $n(n-1)/2$ copies du groupe libre F_2. De la même manière que pour WB_n, Bardakov–Bellingeri–Damiani a montré dans [BBD15] que UVB_n peut être vu comme le produit semi-direct de ses sous-groupes UVP_n et S_n, où S_n agit en permutant le indices des générateurs de

UVP_n, comme décrit dans le Théorème 5.3.6. Notez que le groupe UVP_n est en fait un groupe d'Artin angle droit. On rappelle qu'un groupe d'Artin angle droit, aussi appelé groupe de graphes, est un groupe qui admet une présentation finie dans laquelle les seules relations sont des relations commutées entre les générateurs. Pour une étude générale sur les groupes d'Artin angle droit, nous renvoyons le lecteur à l'article [Cha07] de Charney.

Nous allons maintenant présenter la structure des quatrième et cinquième chapitres de cette thèse.

Dans le Chapitre 4, nous introduisons les groupes de tresses soudées, WB_n, en donnant plusieurs interprétations différentes ainsi qu'en donnant une présentation. Après avoir établi la théorie nécessaire pour WB_n, dans la Section 4.2, nous étudions leur suites centrales descedantes et nous montrons que, pour $n \geq 4$, WB_n n'est pas résiduellement nilpotent. En d'autres termes, dans la Proposition 4.2.1 nous montrons que, pour $n \geq 4$, $\Gamma_2(WB_n) \cong \Gamma_3(WB_n)$. De plus, pour $n \geq 5$, nous déterminons tous les homomorphismes possibles, à conjugaison près, de WB_n au groupe symétrique S_n, comme nous le décrivons dans le théorème suivant. Ici, on note l'automorphisme extérieur de S_6 par v_6, et par ϕ l'homomorphisme $WB_n \to S_n$ défini comme $\phi(\sigma_i) = \phi(\rho_i) = s_i$, où $s(i) = (i, i+1) \in S_n$ pour chaque $1 \leq i \leq n-1$.

Théorème 3. Soit $n \geq 5$ et soit $h : WB_n \to S_n$ un homomorphisme quelconque. Alors, à conjugaison près, l'une des conditions suivantes est vérifiée :

- L'homomorphisme h est l'homomorphisme ϕ.
- L'homomorphisme h est cyclique, dont l'image est d'ordre 2.
- L'homomorphisme h est abélien.
- Pour $n = 6$, l'homomorphisme h est $v_6 \circ \phi$.

On continue avec la Section 4.3, où l'on définit les groupes de tresses pures soudées, WP_n, et on prouve que c'est un sous-groupe caractéristique de WB_n et que le centralisateur de WP_n dans WB_n est trivial. Notez que pour $n = 2$ le groupe WP_2 n'est pas un sous-groupe caractéristique de WB_2 car nous fournissons un contre-exemple dans la Remarque 4.3.9. C'est Dyer–Grossman qui ont déterminé, dans [DG81], le groupe des automorphismes du groupe de tresses d'Artin et en particulier ils ont prouvé que, pour $n \geq 3$, $Out(B_n) \cong \mathbb{Z}_2$ généré par l'automorphisme $\epsilon_n : \sigma_i \mapsto \sigma_i^{-1}$, pour tout $1 \leq i \leq n-1$. Même si les groupes de tresses soudées WB_n sont des analogues tridimensionnels des groupes de tresses d'Artin B_n, leur groupe des automorphismes n'a pas encore été déterminé. Ainsi, nous concluons le quatrième chapitre avec la Section 4.4, où nous discutons sur le groupe des automorphismes de WB_n. En particulier, nous décrivons la difficulté que nous avons rencontrée en appliquant l'approche de Dyer–Grossman au cas de WB_n. Il s'agit d'un obstacle concernant l'action du groupe libre F_n dans WB_n. Nous aurions besoin de prouver que F_n est un sous-groupe caractéristique de WB_n; ce fait reste une conjecture et nous prouvons dans la Proposition 4.4.1 que, pour $n \geq 2$, F_n est un sous-groupe normal de WB_n. Enfin, pour $n \geq 3$, nous conjecturons que l'automorphisme

$$\alpha_n : \begin{cases} \sigma_i \mapsto \rho_i \sigma_i^{-1} \rho_i, \text{ pour } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ pour } 1 \leq i \leq n-1. \end{cases}$$

est le seul automorphisme extérieur de WB_n et, en particulier, que $Out(WB_n) \cong \mathbb{Z}_2$. Notez que l'automomorphisme α_n de WB_n peut être vu comme une composition des applications β_n, γ_n, définies par :

$$\beta_n : \begin{cases} \sigma_i \mapsto \sigma_i^{-1}, \text{ pour } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ pour } 1 \leq i \leq n-1, \end{cases} \tag{6}$$

et

$$\gamma_n : \begin{cases} \sigma_i \mapsto \rho_i \sigma_i \rho_i, \text{ pour } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ pour } 1 \leq i \leq n-1, \end{cases} \tag{7}$$

lesquelles sont des automorphismes de UVB_n. Néanmoins, β_n et γ_n ne sont pas des automorphismes de WB_n.

Cette thèse se termine par le Chapitre 5, où nous introduisons et étudions les groupes UVB_n. Ensuite, dans la Section 5.2 nous déterminons, pour $n \geq 5$, tous les homomorphismes possibles, à conjugaison près, de UVB_n au groupe symétrique S_n, ce qui est décrit dans le théorème suivant.

Théorème 4. Soit $n \geq 5$ et soit $h : UVB_n \to S_n$ un homomorphisme quelconque. Alors, à conjugaison près, l'une des conditions suivantes est vérifiée :

- L'homomorphisme h est l'homomorphisme ϕ.
- L'homomorphisme h est cyclique, dont l'image est d'ordre 2.
- L'homomorphisme h est abélien.
- Pour $n = 6$, l'homomorphisme h est $v_6 \circ \phi$.

Avec des arguments similaires à ceux pour WP_n, nous prouvons dans la Section 5.3 que, pour $n \geq 5$, UVP_n est un sous-groupe caractéristique de UVB_n et que son centralisateur dans UVB_n est trivial. Notez que pour $n = 2$, tel que dans le cas pour WP_n, le groupe UVP_2 n'est pas un sous-groupe caractéristique de UVB_2, puisqu'on donne un contre-exemple dans la remarque 5.3.9. Ensuite, dans la Section 5.4 nous nous concentrons sur les images finies de UVB_n et dans la Section 5.5 sur ses éléments de torsion. Pour être plus précis, avec la théorie des ensembles totalement symétriques, introduite par Kordek–Margalit dans [KM], pour $n \geq 3$ nous déterminons toutes les images possibles de UVB_n dans tout groupe fini G, sous n'importe quel homomorphisme.

Théorème 5. Soit $n \geq 3$ et $\phi : UVB_n \to G$ un homomorphisme de groupe à un groupe fini G. Alors, l'un des cas suivants doit se satisfaire :

- $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathbb{Z}_2$, pour quelque $m \in \mathbb{N}^*$. Dans ce cas, l'image de UVB_n est abélienne.
- $|\phi(UVB_n)| \geq 2^{\frac{n(n-1)}{2}-1}\left(\frac{n(n-1)}{2}\right)!$.
- $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathrm{Im}(\phi_{|S_n})$, pour quelque $m \in \mathbb{N}^*$. Dans ce cas, l'image de UVP_n est cyclique.

A propos des éléments de torsion de UVB_n, nous avons ce qui suit. Soit $\iota : S_n \to UVB_n$ l'application injective définie par $\iota\big((i, i+1)\big) = \rho_i \in UVB_n$, pour $1 \leq i \leq n-1$. En général, on sait que tout élément de torsion de UVB_n, ainsi que de WB_n, appartient à la fermeture de $\iota(S_n)$, mais pour le cas de UVB_n on a le résultat plus fort suivant.

Théorème 6. Soit $n \geq 2$. Pour tout élément de torsion, d'ordre r, w dans UVB_n il existe un élément $s \in S_n$, qui dépend de w, d'ordre r, et tel que w est conjugué à $\iota(s)$ par un élément de UVP_n.

On continue avec la Section 5.6, où l'on détermine l'image du groupe de tresses d'Artin B_n dans UVB_n, sous l'homomorphisme $\iota : B_n \to UVB_n$ défini par $\iota(\sigma_i) = \sigma_i$, pour $1 \leq i \leq n-1$. Nous savons que B_n a un plongement naturel dans WB_n, mais ce n'est pas ainsi dans le cas de UVB_n. Pour être plus précis, nous montrons dans la Proposition 5.6.3 que $\iota(B_n)$ est isomorphe à $B_n/[P_n, P_n]$, qui est un groupe cristallographique, comme le montrent Gonçalves–Guaschi–Ocampo dans [GGO17]. Avant de conclure ce dernier chapitre, nous étudions dans la Section 5.7 et la Section 5.8 le groupe des automorphismes de UVP_n et UVB_n, respectivement, et de plus, pour $n \geq 2$, nous prouvons dans le Corollaire 5.8.4 que les groupes UVP_n et UVB_n sont résiduellement finis et Hopfiens mais pas co-Hopfiens. Concernant le groupe des automorphismes de UVP_n, en utilisant la théorie des groupes d'Artin angle droit, aussi appelés groupes de graphes, nous en donnons une description complète. Sur la base de la présentation (5) de UVP_n, notons que le graphe qui correspond au groupe d'Artin angle droit UVP_n, est un graphe à $n(n-1)$ sommets, où l'ensemble de sommets est $V = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$ et il y a une arête reliant chaque paire de sommets à l'exception des paires $\{\lambda_{i,j}, \lambda_{j,i}\}$, car ce sont les seules paires de générateurs qui ne commutent pas. Par conséquent, en utilisant le résultat de Laurence [Lau95], qui a travaillé sur la théorie des automorphismes des groupes de graphes et étendu les travaux de Servatius [Ser89], nous obtenons un ensemble complet de générateurs pour le groupe des automorphismes de UVP_n, comme l'on décrit dans le théorème suivant.

Théorème 7. Soit $n \geq 2$ et $1 \leq i \neq j \leq n$. On a que

$$Aut(UVP_n) \cong \langle T_{\lambda_{j,i}},\ \mathbb{Z}_2^{n(n-1)},\ \mathbb{Z}_2^{n(n-1)/2} \rtimes S_{n(n-1)/2} \rangle,$$

où $T_{\lambda_{j,i}} : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i}$, tout en fixant les générateurs $\lambda_{k,l} \neq \lambda_{i,j}$, et $S_{n(n-1)/2}$ est le groupe symétrique de degré $n(n-1)/2$.

A partir de ce résultat et du fait que $UVP_n \cong \underbrace{F_2 \times \cdots \times F_2 \times \cdots \times F_2}_{n(n-1)/2\text{-fois}}$, nous pouvons en déduire que

$$Aut(UVP_n) \cong Aut(F_2)^{n(n-1)/2} \rtimes S_{n(n-1)/2},$$

où $S_{n(n-1)/2}$ agit sur $Aut(F_2)^{n(n-1)/2}$ en permutant les facteurs $n(n-1)/2$. Notez que ce résultat est en accord avec un cas particulier d'un résultat plus général prouvé par Zhang–Ventura–Wu dans [ZVW15], lesquels ont utilisé des techniques différentes des nôtres. À la fin, nous présentons des résultats partiels sur le groupe des automorphismes de UVB_n. Dans ce cas, on peut appliquer un résultat de Rose [Ros75], qui a donné une description du groupe des automorphismes des groupes qui possèdent un sous-groupe caractéristique propre et ont un centralisateur trivial. Ainsi, on obtient que, pour $n \geq 5$, $Aut(UVB_n) \cong N_{Aut(UVP_n)}(UVB_n)$, comme montré dans le Corollaire 5.8.2. De plus, dans la Proposition 5.8.3, pour β_n, γ_n défini dans (6), (7), nous prouvons que, pour $n \geq 3$, $\langle \beta_n, \gamma_n \rangle \subseteq Out(UVB_n)$, où $\langle \beta_n, \gamma_n \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Notons que dans [BP20], Bellingeri–Paris ont prouvé, pour les groupes de tresses virtuels VB_n, que $Out(VB_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Nous ne définirons pas ici les groupes de tresses virtuels, mais nous pourrions en dire qu'ils sont une extension des groupes de tresses classiques par le groupe symétrique. On a que WB_n est un quotient de VB_n et UVB_n un quotient de WB_n, et donc, nous espérons que le résultat de Bellingeri–Paris sur $Out(VB_n)$ avec nos résultats partiels sur $Out(UVB_n)$ pourraient être utiles pour les futurs travaux de détermination du groupe $Out(WB_n)$, ce qui est toujours un problème ouvert.

Introduction in english

This thesis is divided into two parts. The first three chapters concern surface braid groups, while the last two deal with welded and unrestricted virtual braid groups.

Surface braid groups are both a generalisation to any connected surface of the fundamental group of a surface and of the braid groups of the plane, which are known as Artin braid groups and were defined by Artin in 1925 in [Art25]. They were initially introduced by Zariski in [Zar36], [Zar37] and later during the 1960's, Fox gave an equivalent definition of surface braid groups from a topological point of view. As for Artin braid groups, surface braid groups can be described from several points of view, one of which is the following. Let $n \in \mathbb{N}$, for any connected surface Σ, we fix a set of n distinct points, $P = \{x_1, \dots, x_n\}$, in the interior of Σ. An n–braid, $\beta = (\beta_1(t), \dots, \beta_n(t))_{t\in[0,1]}$, is a collection of n distinct paths on the surface Σ, where each path has as initial and final point a point in P, which are possibly different. We call these n paths strands. Now, up to homotopy, one can define a group structure on this set of n–braids. From now on, we will denote a connected surface by Σ and the n-strand surface braid group of Σ by $B_n(\Sigma)$, and we call these groups full braid groups of Σ.

The theory of surface braid groups has been widely studied, and there are several known results about their structure and their presentations. The first presentations of braid groups of compact surfaces without boundary were found by Birman [Bir69] and Scott [Sco70]. Moreover, there exist further presentations of surface braid groups due to Bellingeri [Bel04], Gonçalves–Guaschi [GG04b],[GG10a], González-Meneses [GM01] and Lambropoulou [Lam00], among others. Among all the surfaces, the braid groups of the 2–sphere, $\mathbb{S}^2$ and the projective plane, $\mathbb{R}P^2$, are of particular interest, since they are the only surfaces whose braid groups contain torsion elements. This fact was proved for the 2–sphere by Fadell–Van Buskirk in [FVB62] and for the projective plane by Van Buskirk in [VB66].

The braid group $B_n(\Sigma)$ is closely related to the symmetric group, since there is a natural epimorphism $\sigma : B_n(\Sigma) \to S_n$, where to each n–braid $\beta = (\beta_1(t), \dots, \beta_n(t))_{t\in[0,1]}$ we associate a permutation $\sigma(\beta) \in S_n$ defined by $\beta_i(1) = x_{\sigma(\beta)(i)} \in P$, for all $i \in \{1, \dots, n\}$. The kernel of the map σ is the so-called pure braid group, which we denote by $P_n(\Sigma)$. In other words, $P_n(\Sigma)$ is the subgroup of $B_n(\Sigma)$, whose n–braids contains n strands each of which connects the initial point to the same final point.

Surface braid groups are closely related to configuration spaces of surfaces. We will denote by $F_n(\Sigma) = \{(p_1, \dots, p_n) \in \Sigma^n \mid p_i \neq p_j \text{ for all } i, j \in \{1, \dots, n\},\ i \neq j\}$ the n^{th} ordered configuration space of the surface Σ. Considering the natural action of the symmetric group S_n on $F_n(\Sigma)$, defined by permutation of coordinates, one can define the n^{th} unordered configuration space, which is the orbit space $F_n(\Sigma)/S_n$. Similarly, for $n, m \in \mathbb{N}$, we can consider the space obtained by quotienting the $(n+m)^{th}$ configuration space of Σ by the subgroup $S_n \times S_m$ of S_{n+m}, that is $F_{n+m}(\Sigma)/S_n \times S_m$. Fox–Neuwirth [FN62b] proved that $\pi_1(F_n(\Sigma))$ is isomorphic to $P_n(\Sigma)$, and $\pi_1(F_n(\Sigma)/S_n)$ is isomorphic to $B_n(\Sigma)$. Likewise, the groups $\pi_1(F_{n+m}(\Sigma)/(S_n \times S_m))$ are isomorphic to the groups that we denote by $B_{n,m}(\Sigma)$, which are subgroups of the full braid groups $B_{n+m}(\Sigma)$ known as mixed braid groups. Mixed braid groups are in general useful for studying representation theory of surface braid

groups and for knot theory in 3-manifolds. These groups have been studied, among others, by Sossinsky [Sos92], Manfredini [Man97], Paris–Rolfsen [PR99], Gonçalves–Guaschi [GG04c], [GG05], [GG12] Bellingeri–Godelle–Guaschi [BGG] and Lambropoulou [Lam00], where her approach differs from the previously mentioned, and the value n of the above mentioned mixed braid groups denotes that the first n strands are vertical, and it is proved that these braid groups are related to knots in handlebodies, in knot complements and in closed, connected, oriented 3-manifolds, see also references therein for prior works.

One important tool in the study of the braid groups is the Fadell–Neuwirth short exact sequence that arises from the Fadell–Neuwirth fibration, which exists for Σ with empty boundary. However, once passed to the long exact sequence in homotopy of the fibration, for $n, m \in \mathbb{N}$, and any connected surface Σ, we have the following Fadell–Neuwirth short exact sequences:

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1, \text{ for } 1 \leq m < n, \tag{8}$$

$$1 \longrightarrow B_m(\Sigma \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1, \tag{9}$$

known as the Fadell–Neuwirth short exact sequence of surface pure braid groups and the generalised Fadell–Neuwirth short exact sequence of mixed braid groups respectively. The induced maps $\bar{p}_{n,m}$ and $\bar{q}_{n+m,n}$ can be considered geometrically as the epimorphism that forgets the last $n-m$ and m strands respectively. Note that when Σ is the 2–sphere $\mathbb{S}^2$ or the projective plane $\mathbb{R}P^2$ there are some restrictions on the values of n and m. We remark that the map $\bar{p}_{n,m}$ does not extend directly to the full braid groups $B_n(\Sigma) \to B_m(\Sigma)$, since it is not well-defined, but $\bar{q}_{n+m,n}$ allows us to extend $\bar{p}_{n,m}$ to certain subgroups of $B_{n+m}(\Sigma)$.

The splitting problem refers to the question of whether or not the short exact sequences (8) and (9) split, or equivalently, whether or not the maps $\bar{p}_{n,m}$ and $\bar{q}_{n+m,n}$ admit a section, which is the same as asking whether or not there exist injective maps $\bar{s}_{n,m}$ and $\hat{s}_{n+m,n}$, respectively, such that $\bar{p}_{n,m} \circ \bar{s}_{n,m} = id_{P_m(\Sigma)}$ and $\bar{q}_{n,m} \circ \hat{s}_{n,m} = id_{B_n(\Sigma)}$. An interesting aspect of the splitting problem is that it can be approached algebraically and geometrically. The equivalence between the splitting of the short exact sequences (8), (9) and the existence of a geometric section for the fibrations $p_{n,m} : F_{n+m}(\Sigma) \to F_n(\Sigma)$ and $q_{n+m,n} : F_{n+m}(\Sigma)/(S_n \times S_m) \to F_n(\Sigma)/S_n$, respectively, is given by Baues [Bau77] and Whitehead [Whi78] if Σ is an aspherical space, and by Gonçalves–Guaschi when $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{R}P^2$. In Proposition 1.3.1 we present this equivalence in detail.

The splitting of the short exact sequence (8) in the case of the plane, $\Sigma = \mathbb{R}^2$, is an important result in classical braid theory. Based on this result, we have the Artin normal form for pure Artin groups P_n, that is $P_n \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1$, where F_n is the free group of rank n, and is one of the main tools in the study of P_n. The Artin normal form of P_n is used, for instance, for the uniqueness of roots in P_n, by Bardakov [Bar92], for the study of the lower central series and the residual nilpotence of P_n, by Falk–Randell [FR85], as well as for the proof of the bi-orderability of P_n, by Kim–Rolfsen [KR03].

It follows that it is important to know whether there is a splitting or not. During the 1960's, the period of the development of the theory of surface braid groups, many mathematicians studied this problem for surfaces besides the plane. The splitting problem for (8) was studied notably by Fadell [Fad62], Fadell–Neuwirth [FN62a], Fadell–Van Buskirk [FVB62], Van Buskirk [VB66] and Birman [Bir69], approaching it either geometrically or algebraically. In the case of the pure surface braid groups the splitting problem now has a complete solution, given in Theorem 1.3.3 by Gonçalves–Guaschi. To give a positive answer to the splitting problem it suffices to present an explicit section, either geometric or algebraic. To obtain a negative answer to the splitting problem, one of the main methods that was used to prove Theorem 1.3.3 is based on the following observation: let G be a group, and let K, H be normal subgroups of G such that H is contained in K. If the short exact sequence $1 \to K \to G \to R \to 1$ splits then so does the short exact sequence $1 \to K/H \to G/H \to R \to 1$. The study of the splitting problem of the second extension is sometimes easier from the one of the first extension, and thus, showing that this second extension does not split, implies that the first extension does not split either. However, in the case of the mixed braid groups the splitting problem does not yet have a complete solution. In particular, to the best of our knowledge, the only surface, besides the plane $\mathbb{R}^2$, for which the splitting of the short exact sequence (9) has been studied, is the 2–sphere $\mathbb{S}^2$. In [GG05], Gonçalves–Guaschi, gave

partial results for the splitting problem for $\mathbb{S}^2$, see Theorem 1.3.4, and more recently Chen–Salter strengthened these results, see Theorem 1.3.5.

In the first part of the thesis, we study the splitting problem of the mixed braid groups of the projective plane $\mathbb{R}P^2$. The braid groups of the projective plane, $B_n(\mathbb{R}P^2)$, are particularly interesting, since along with the braid group of the 2–sphere they are the only braid groups that contain torsion elements. A presentation of $B_n(\mathbb{R}P^2)$ was given by Van Buskirk in [VB66], see Theorem 2.1.1, where the set of generators consists of the $n-1$ standard Artin generators, together with the n elements that correspond to elements of the fundamental group of the projective plane based at the $1^{st}, \dots, n^{th}$ basepoint respectively. The torsion elements of $B_n(\mathbb{R}P^2)$ were first determined and characterised by Murasugi in [Mur82], and later Gonçalves–Guaschi gave a simplified characterisation of them in [GG10b]. The groups $B_n(\mathbb{R}P^2)$ were studied extensively by Gonçalves–Guaschi who determined, up to isomorphism, the maximal finite subgroups of $P_n(\mathbb{R}P^2)$ as well as the finite subgroups of $B_n(\mathbb{R}P^2)$.

We now present the structure of the first three chapters of the thesis.

In Chapter 1, we give an introduction to surface braid groups. We give a geometric definition of surface braid groups and we present their connection with the configuration spaces. Moreover in Section 1.2, we give a detailed description of the Fadell–Neuwith fibration and the resulting short exact sequences

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1,$$

$$1 \longrightarrow B_m(\Sigma \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1,$$

which constitute the main subject of study of the first part of the thesis. At the end of Chapter 1, in Section 1.3, we describe the splitting problem in detail and we present the existing results for this problem.

In Chapter 2, we introduce the braid groups of the projective plane, $B_n(\mathbb{R}P^2)$, giving also important details about their structure. In Section 2.2, we give presentations of certain subgroups of $B_{n,m}(\mathbb{R}P^2)$ that allow us to obtain a presentation of the mixed braid groups $B_{n,m}(\mathbb{R}P^2)$. In particular, we first exhibit presentations of the groups $P_n(\mathbb{R}P^2)$, $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ and $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, by applying standard techniques for obtaining presentations of group extensions as described in [Joh97] (page 139). Then, in Theorem 2.2.7, we obtain a presentation of $B_{n,m}(\mathbb{R}P^2)$ applying the same methods to the following short exact sequence:

$$1 \to B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \to 1, \tag{10}$$

where the map $\bar{q}_{n+m,n}$ can be considered geometrically as the epimorphism that forgets the last m strands. Moreover, using these presentations, we give a presentation of the Abelianisation of each of these groups. We conclude Chapter 2 with Section 2.3, where we study the lower central and derived series of the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, and in particular proving, in Theorem 2.3.2, that the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is neither residually nilpotent , for $m \geq 3$, nor residually solvable, for $m \geq 5$, as we see in the following theorem:

Theorem 1. *Let $n \geq 1$. Then:*

- *If $m \geq 3$, then*

$$\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big) = \Gamma_3\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big).$$

- *If $m \geq 5$, then*

$$\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)^{(1)} = \big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)^{(2)}.$$

In particular, for $m \geq 3$, $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is not residually nilpotent and for $m \geq 5$, it is not residually solvable.

Due to the following fact, Theorem 1 is important in the study of the splitting problem. Let N be a normal subgroup of $B_{n,m}(\mathbb{R}P^2)$, which is also contained in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$. We thus obtain the following quotiented short exact sequence:

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/N \longrightarrow B_{n,m}(\mathbb{R}P^2)/N \xrightarrow{q} B_n(\mathbb{R}P^2) \longrightarrow 1,$$

where $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$ denotes the homomorphism induced by $\bar{q}_{n+m,n}$. We observe that if the short exact sequence (10) splits then this quotiented short exact sequence also splits. More precisely, consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) & \longrightarrow & B_{n,m}(\mathbb{R}P^2) & \xrightarrow{\bar{q}_{n+m,n}} & B_n(\mathbb{R}P^2) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle pr} & & \| & & \\ 1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/N & \longrightarrow & B_{n,m}(\mathbb{R}P^2)/N & \xrightarrow{q} & B_n(\mathbb{R}P^2) & \longrightarrow & 1. \end{array}$$

Suppose that the homomorphism $\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ admits a section $\bar{s}_{n+m,n}$. Then $s = pr \circ \bar{s}_{n+m,n}$ is a section for $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$. It follows that knowing the lower central and derived series of certain groups is an important tool in the study of the splitting problem, since one could study the quotient of $B_{n,m}(\mathbb{R}P^2)$ by N, for N being an element of the lower central or the derived series of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$.

We complete the first part of the thesis in Chapter 3, w, where we study the possible splitting of (10). In Section 3.1, we analyse the existence of a section for the homomorsphism

$$\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2),$$

for $n = 1$ and $n = 2$, where $m \in \mathbb{N}$. In particular, for $m \in \mathbb{N}$ we prove in Proposition 3.1.1 that the homomorphism $\bar{q}_{1+m,n} : B_{1,m}(\mathbb{R}P^2) \to B_1(\mathbb{R}P^2)$ admits no section, while for $n = 2$, the homomorphism $\bar{q}_{2+m,n} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$ admits a section for every $m \in \mathbb{N}$. More precisely, we prove that the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow B_{2,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{2+m,2}} B_2(\mathbb{R}P^2) \longrightarrow 1$$

splits for all values of $m \in \mathbb{N}$. In Proposition 3.1.2 we provide an explicit geometric section for the map $q_{2+m,2} : F_{2+m}(\Sigma)/(S_2 \times S_m) \to F_2(\Sigma)/S_2$, and in Proposition 3.1.3, we exhibit an explicit algebraic section for the homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$, where the homomorphisms $q_{2+m,2}$ and $\bar{q}_{2+m,2}$ can be considered geometrically as the maps that forget the last m points and the last m strands respectively. For the cases $n \geq 3$, in Section 3.2 we provide necessary conditions for the values of m, in order to have that the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1 \qquad (11)$$

splits. In particular, in Theorem 3.2.4 we prove that it is necessary that $m = k(n-1)$, where $k \geq 1$, for (11) to split. Moreover, making use of the torsion elements of the groups $B_{n,m}(\mathbb{R}P^2)$ and $B_n(\mathbb{R}P^2)$, in Proposition 3.2.7 we obtain further restrictions on the values of m, and in particular, we obtain that either $m \equiv 0 \mod n$ or $m \equiv 1 \mod n$. We conclude Chapter 3 with Section 3.3, where, for certain values of m, we provide two different constructions of a cross-section for the fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$. The existence of these cross-sections, give a positive answer to the splitting problem of the short exact sequence (11), for these values of m. Combining these results, Theorem 3.2.4 and Propositions 3.2.7, 3.3.1, 3.3.2, we obtain the main result concerning the splitting problem, which is given in Theorem 3.3.4 and is the following.

Theorem 2. *Let $m > 1$ and $n \geq 3$. The short exact sequence*

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

splits for $m = 2n(n-1)$ and for $m = kn(2n-1)(2n-2)$, where $k \geq 1$. Moreover, if the homomorphism $\bar{q}_{n+m,n}$ admits a section, then $m \equiv 0, (n+1)^2 \mod n(n-1)$.

Note that the remaining cases that are not covered in Theorem 2 are open.

We continue with the second part of the thesis, which concerns the welded and unrestricted virtual braid groups. The welded braid groups, which we will denote by WB_n, for $n \in \mathbb{N}$, are a 3-dimensional analogue of the Artin braid groups B_n. Like for the Artin brain groups, there are several interpretations for the welded braid groups, such as in terms of mapping class groups, of fundamental group of specific configuration spaces and of automorphisms of the free group F_n. The name welded braid groups was introduced by Fenn–Rimányi–Rourke in [FRR97]. Moreover, the groups WB_n appear in the literature under many different names, for instance as conjugating automorphisms of the free group F_n due to Savushkina [Sav96], as loop braid groups due to Baez–Crans–Wise [BCW07] and as groups of untwisted rings due to Brendle–Hatcher [BH13]. A presentation of WB_n was given by Fenn–Rimányi–Rourke in [FRR97], given in Theorem 4.1.3. Later Brendle–Hatcher in [BH13] gave a presentation for the untwisted ring group, UR_n, given in Theorem 4.1.6, which gives a geometric interpretation of the welded braid group WB_n, since these two groups are isomorphic. The generators given in both presentations, σ_i, ρ_i, were initially considered by Goldsmith in [Gol81]. The generator σ_i permutes the i^{th} and the $(i+1)^{st}$ circles by passing the i^{th} circle through the $(i+1)^{st}$ and the generator ρ_i permutes them passing the i^{th} around the $(i+1)^{st}$.

An important subgroup of WB_n is the welded pure braid group WP_n, which has also several interpretations. In this thesis we are mainly interested in the definition of WP_n as the subgroup of the automorpism group of the free group F_n, that consists of the so-called basis-conjugating automorphisms, but also as the fundamental group of a space of specific configurations. Note that as in the case of the surface braid groups, WB_n is closely related to the symmetric group S_n, since one can consider the map $\phi : WB_n \to S_n$, defined by $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, for $i = 1, \dots, n-1$. From this, it follows that the welded pure braid group WP_n, is actually the kernel of this map ϕ. In [Sav96], Savushkina proved that WB_n can be seen as the semidirect product of its subgroups WP_n and S_n, and determined the action of S_n on WP_n, as described in Theorem 4.3.6. To be more precise, S_n acts by permuting the indices of the generators of WP_n.

The unrestricted virtual braids, UVB_n, were introduced by Kauffman and Lambropoulou in [KL04] and [KL06], where they provide a new method for converting virtual knots and links to virtual braids, and they prove a Markov Theorem for the virtual braid groups. Moreover, the group UVB_n appears in [KMRW17] as a quotient of the welded braid group WB_n and in [BBD15] where Bardakov–Bellingeri–Damiani give a description of its structure. In Defintion 5.1.1 we see that UVB_n, can be defined by adding one extra relation to the presentation of the group WB_n, and thus, UVB_n has the same generating set as WB_n. Once again, considering the map $\phi : UVB_n \to S_n$ defined by $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, for $i = 1, \dots, n-1$, the unrestricted virtual pure braid group, which we denote by UVP_n, is the kernel of the map ϕ, and is an important subgroup of UVB_n.

In [BBD15], Bardakov–Bellingeri–Damiani studied the group UVP_n, giving a description of its elements and moreover, giving the following presentation:

$$UVP_n = \langle \lambda_{i,j},\ 1 \le i \ne j \le n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j},\ \text{for } (k,l) \ne (j,i),\ 1 \le i,j,k,l \le n \rangle, \tag{12}$$

which we present in Theorem 5.3.1. From this presentation, it follows that UVP_n is isomorphic to the direct product of $n(n-1)/2$ copies of the free group F_2. Similarly to WB_n, Bardakov–Bellingeri–Damiani showed in [BBD15] that UVB_n can be seen as the semidirect product of its subgroups UVP_n and S_n, where S_n acts by permuting the indices of the generators of UVP_n, as described in Theorem 5.3.6. Note that the group UVP_n is actually a right-angled Artin group. We recall that a right-angled Artin group, also known as graph group, is a group which admits a finite presentation in which the only relations are commuting relations among the generators. For a general survey on the right-angled Artin groups we direct the reader to the article [Cha07] by Charney.

We will present now the structure of the fourth and fifth chapter of the thesis.

In Chapter 4 we introduce the welded braid groups, WB_n, giving a couple of different interpretations as well as giving a presentation of them. After having established the necessary theory of WB_n, in Section 4.2 we study their lower central series and we prove that, for $n \ge 4$, WB_n is not residually nilpotent. In other words, in Propostion 4.2.1 we show that for $n \ge 4$, $\Gamma_2(WB_n) \cong \Gamma_3(WB_n)$. Moreover, for $n \ge 5$, we determine all possible homomorphisms, up to conjugation, from WB_n to

the symmetric group S_n, as we describe in the following theorem. Note that we denote the outer automorphism of S_6 by v_6, and by ϕ the homomorphism $WB_n \to S_n$ defined by $\phi(\sigma_i) = \phi(\rho_i) = s_i$, where $s(i) = (i, i+1) \in S_n$, for every $1 \leq i \leq n-1$.

Theorem 3. *Let $n \geq 5$ and let $h : WB_n \to S_n$ be any homomorphism. Then, up to conjugation, one of the following holds:*

- *The homomorphism h is the homomorphism ϕ.*
- *The homomorphism h is cyclic, whose image is of order 2.*
- *The homomorphism h is Abelian.*
- *For $n = 6$, the homomorphism h is $v_6 \circ \phi$.*

We continue with Section 4.3, where we define the welded pure braid groups, WP_n, and we prove that it is a characteristic subgroup of WB_n and that the centraliser of WP_n in WB_n is trivial. Note that for $n = 2$ the group WP_2 is not a characteristic subgroup of WB_2, as we provide a counter example in Remark 4.3.9. It was Dyer–Grossman who determined in [DG81] the automorphism group of the Artin braid group and in particular they proved that $Out(B_n) \cong \mathbb{Z}_2$, for $n \geq 3$, generated by the automorphism $\epsilon_n : \sigma_i \mapsto \sigma_i^{-1}$, for every $1 \leq i \leq n-1$. Even though the welded braid groups WB_n are a 3-dimensional analogue of the Artin braid groups B_n, the automorphism group of WB_n has not been yet determined. Thus, we conclude the fourth chapter with Section 4.4, where we discuss about the automorphism group of WB_n. In particular, we describe the difficulty that we encountered in applying the approach of Dyer–Grossman to the case of WB_n, which concerns the action of the free group F_n to WB_n. Even though we would like to prove that F_n is a characteristic subgroup of WB_n, something that we conjecture as well, we prove in Proposition 4.4.1 that F_n is a normal subgroup of WB_n, for $n \geq 2$. Lastly, for $n \geq 3$, we conjecture that the automorphism

$$\alpha_n : \begin{cases} \sigma_i \mapsto \rho_i \sigma_i^{-1} \rho_i, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1. \end{cases}$$

is the only outer automorphism of WB_n and, in particular, that $Out(WB_n) \cong \mathbb{Z}_2$. Note that the automomorphism α_n of WB_n can be seen as a composition of the maps β_n, γ_n, defined by:

$$\beta_n : \begin{cases} \sigma_i \mapsto \sigma_i^{-1}, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1, \end{cases} \tag{13}$$

and

$$\gamma_n : \begin{cases} \sigma_i \mapsto \rho_i \sigma_i \rho_i, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1, \end{cases} \tag{14}$$

which are actually automorphisms of UVB_n. Nevertheless, β_n and γ_n fail to be automorphisms of WB_n.

We complete this thesis with Chapter 5, where we introduce and study the unrestricted virtual braids UVB_n. Then, in Section 5.2 we determine all possible homomorphisms, up to conjugation, from UVB_n to the symmetric group S_n, for $n \geq 5$, as we describe in the following theorem.

Theorem 4. *Let $n \geq 5$ and let $h : UVB_n \to S_n$ be any homomorphism. Then, up to conjugation, one of the following holds:*

- *The homomorphism h is the homomorphism ϕ.*
- *The homomorphism h is cyclic, whose image is of order 2.*
- *The homomorphism h is Abelian.*
- *For $n = 6$, the homomorphism h is $v_6 \circ \phi$.*

With similar arguments to the case of WP_n, we prove in Section 5.3 that UVP_n is a characteristic subgroup of UVB_n, for $n \geq 5$, as well as that its centraliser in UVB_n is trivial. Note that for $n = 2$, as in the case of WP_n, the group UVP_2 is not a characteristic subgroup of UVB_2, as we provide a counter example in Remark 5.3.9. Then, in Section 5.4 we focus on the finite images of UVB_n and in Section 5.5 on its torsion elements. To be more precise, with the theory of totally symmetric sets, introduced by Kordek–Margalit in [KM], we determine all possible images of UVB_n in any finite group G, under any homomorphism, for $n \geq 3$.

Theorem 5. *Let $n \geq 3$ and $\phi : UVB_n \to G$ be a group homomorphism to a finite group G. Then, one of the following must hold:*

- $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathbb{Z}_2$, *for some $m \in \mathbb{N}$. In this case, the image of UVB_n is Abelian.*
- $|\phi(UVB_n)| \geq 2^{\frac{n(n-1)}{2}-1}\left(\frac{n(n-1)}{2}\right)!$.
- $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathrm{Im}(\phi_{|S_n})$, *for some $m \in \mathbb{N}$. In this case, the image of UVP_n is cyclic.*

About the torsion elements of UVB_n we have the following. Let $\iota : S_n \to UVB_n$ be the injective map defined by $\iota\big((i, i+1)\big) = \rho_i \in UVB_n$, for $1 \leq i \leq n-1$. In general, we know that any torsion element of UVB_n, as well as of WB_n, belongs to the closure of $\iota(S_n)$, but for the case of UVB_n we have the following stronger result.

Theorem 6. *Let $n \geq 2$. For any torsion element, of order r, w in UVB_n there exists an element $s \in S_n$, that depends on w, of order r and such that w is conjugate to $\iota(s)$ by an element of UVP_n.*

We continue with Section 5.6, where we determine the image of the Artin braid group B_n in UVB_n, under the homomorphism $\iota : B_n \to UVB_n$ defined by $\iota(\sigma_i) = \sigma_i$, for $1 \leq i \leq n-1$. We know that B_n has a natural embedding in WB_n, but this does not hold in the case of UVB_n. To be more precise, we show in Proposition 5.6.3 that $\iota(B_n)$ is isomorphic to $B_n/[P_n, P_n]$, which is a crystallographic group, as shown by Gonçalves–Guaschi–Ocampo in [GGO17]. Before concluding this last chapter, we study in Section 5.7 and Section 5.8 the automorphism group of UVP_n and UVB_n respectively and moreover we prove in Corollary 5.8.4 that the groups UVP_n and UVB_n are residually finite and Hopfian but not co-Hopfian, for $n \geq 2$. Concerning the automorphism group of UVP_n, using the theory of right-angled Artin groups, also known as graph groups, we give a complete description of it. Based on the presentation (12) of UVP_n, note that the graph that corresponds to the right-angled Artin group UVP_n, is a graph with $n(n-1)$ vertices, where the vertex set is $V = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$ and there is an edge connecting every pair of vertices except for the pairs $\{\lambda_{i,j}, \lambda_{j,i}\}$, since these are the only pairs of generators that do not commute. Therefore, using the result by Laurence [Lau95], who worked on the theory of the automorphisms of graph groups and extended the work of Servatius [Ser89], we obtained a complete set of generators for the automorphism group of UVP_n, as described in the following theorem.

Theorem 7. *Let $n \geq 2$ and $1 \leq i \neq j \leq n$. It holds that*

$$Aut(UVP_n) \cong \langle T_{\lambda_{j,i}},\ \mathbb{Z}_2^{n(n-1)},\ \mathbb{Z}_2^{n(n-1)/2} \rtimes S_{n(n-1)/2} \rangle,$$

where $T_{\lambda_{j,i}} : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i}$, while fixing the generators $\lambda_{k,l} \neq \lambda_{i,j}$, and $S_{n(n-1)/2}$ is the symmetric group of degree $n(n-1)/2$.

From this result and the fact that $UVP_n \cong \underbrace{F_2 \times \cdots \times F_2 \times \cdots \times F_2}_{n(n-1)/2\text{-times}}$, we deduce that

$$Aut(UVP_n) \cong Aut(F_2)^{n(n-1)/2} \rtimes S_{n(n-1)/2},$$

where $S_{n(n-1)/2}$ acts on $Aut(F_2)^{n(n-1)/2}$ by permuting the $n(n-1)/2$ factors. Note that this result agrees with a particular case of a more general result proved by Zhang–Ventura–Wu in [ZVW15], who used different techniques from ours. At the end we present partial results about the automorphism group of UVB_n. For the case of UVB_n, one can apply the result by Rose in [Ros75], who gave a description of the automorphism group of groups which possess a proper characteristic

subgroup and have trivial centraliser. Thus, we obtain that $Aut(UVB_n) \cong N_{Aut(UVP_n)}(UVB_n)$, for $n \geq 5$, as stated in Corollary 5.8.2. Moreover, in Proposition 5.8.3, for β_n, γ_n defined in (13), (14), we prove that $\langle \beta_n, \gamma_n \rangle \subseteq Out(UVB_n)$, for $n \geq 3$, where $\langle \beta_n, \gamma_n \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that in [BP20], Bellingeri–Paris proved, for the virtual braid groups VB_n, that $Out(VB_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We will not define here the virtual braid groups, but we could say that they are an extension of the classical braid groups by the symmetric group. It holds that WB_n is a quotient of VB_n and UVB_n a quotient of WB_n, and thus, we speculate that the result by Bellingeri–Paris about $Out(VB_n)$ together with our partial results about $Out(UVB_n)$ could be of help for future work in determining the group $Out(WB_n)$, which is still an open problem.

CHAPTER 1

Surface braid groups

In this chapter we outline the theory of surface braid groups, and we describe the main problem, that we will deal with in this part of the thesis, called the splitting problem. We start by defining the surface braid groups in two equivalent ways; from geometric and topological point of views, and we give some basic information about these groups. In Section 1.2, we introduce the Fadell–Neuwirth short exact sequence, which is a useful tool for the study of braid groups. We conclude the first chapter by describing the splitting problem, and by presenting the known solutions that have been given for certain surfaces.

1.1 Introduction to surface braid groups

Surface braid groups were first introduced by Zariski [Zar36], [Zar37], and generalise the braid groups B_n introduced by Artin in 1925 [Art25]. During the 1960's, Fox introduced an equivalent topological definition for surface braid groups in terms of the fundamental group of configuration spaces. An interesting aspect of surface braid groups is that they can be defined from several viewpoints, such as equivalence classes of geometric braids, as the fundamental group of configuration spaces and as trajectories of non-colliding particles. Moreover, they are closely related to mapping class groups. For a more detailed description of these different approaches, we refer the reader to the survey [GJP15] by Guaschi and Juan-Pineda.

In this thesis, we will interpret the surface braid groups as the fundamental group of configuration spaces but before discussing these spaces, we will provide a geometric definition of surface braids groups in order to get some intuition about this notion.

Definition 1.1.1. Let Σ denote a connected, compact or not, orientable or not, with or without boundary surface. Let $n \in \mathbb{N}$ and fix $P = \{p_1, ..., p_n\}$, a set of n distinct points in the interior of Σ. A geometric braid in Σ based at P is a collection of n arcs, $\beta = \{\beta_1, ..., \beta_n\}$, where the arcs $\beta_i : [0,1] \to \Sigma \times [0,1]$, $i = 1, \dots, n$, called strands or strings, satisfy the following conditions:

- For $i = 1, \dots, n$, $\beta_i(0) = (p_i, 0)$ and $\beta_i(1) \in P \times \{1\}$.
- For all $t \in [0,1]$ and for all $i \neq j \in \{1, \dots, n\}$, $\beta_i(t) \neq \beta_j(t)$.
- The strands $\beta_1, \dots, \beta_n$ are strictly monotone with respect to the coordinate $t \in [0,1]$. In other words, for all $t \in [0,1]$, each strand meets the subset $\Sigma \times \{t\}$ in exactly one point.

Two geometric braids are considered to be equivalent if they are isotopic, while keeping the endpoints of the strands fixed, and the equivalence classes are called n–braids. The concatenation of two n–braids $\beta, \bar{\beta}$, defined by glueing the endpoints of β to the respective initial points of $\bar{\beta}$, defines a group structure on the set of n–braids, which we call the n-strand braid group of Σ,

denoted by $B_n(\Sigma)$. The identity element of $B_n(\Sigma)$ is the braid all of whose n strands are vertical and the inverse of an n–braid $\beta = \{(\beta_1(t), ..., \beta_n(t))\}_{t\in[0,1]}$ is $\beta^{-1} = \{(\beta_1(1-t), ..., \beta_n(1-t))\}_{t\in[0,1]}$. Since Σ is a connected surface, note that up to isomophism, this group does not depend on the choice of the base point $P = \{p_1, ..., p_n\}$.

In the following figure we give an example of a 3–braid of the 2–torus.

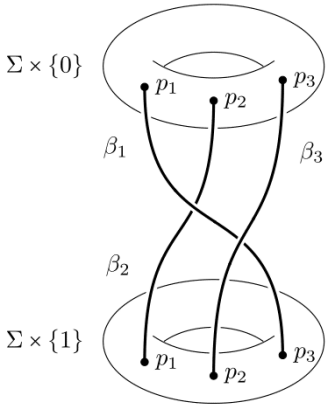


Figure 1.1 A geometric 3–braid when Σ is the 2–torus.

Remark 1.1.2. Projecting the stands lying in $\Sigma \times [0,1]$ onto the surface Σ we can consider the surface braids as collections of n paths. To be more precise, for a fixed set of n distinct points P in the interior of Σ, an n–braid can be considered as collections of n distinct paths on the surface Σ, where each path has as initial point a point in P and as final point again a point in P, but possibly different from its initial one.

The n–braids which have the property that $\beta_i(1) = p_i$, for all $i = 1, \ldots, n$, form a normal subgroup of $B_n(\Sigma)$, which we will denote by $P_n(\Sigma)$. The groups $B_n(\Sigma)$ will be called full braid groups and the groups $P_n(\Sigma)$ will be called pure braid groups.

Let S_n be the symmetric group over $P = \{p_1, ..., p_n\}$. There is a natural epimorphism $\sigma : B_n(\Sigma) \to S_n$, where to each n–braid β we associate a permutation $\sigma(\beta) \in S_n$ defined by $\beta_i(1) = (p_{\sigma(\beta)(i)}, 1)$, for all $i \in \{1, \ldots, n\}$. The kernel of the map σ is the pure braid group $P_n(\Sigma)$. Thus, we have the following short exact sequence:

$$1 \longrightarrow P_n(\Sigma) \longrightarrow B_n(\Sigma) \xrightarrow{\sigma} S_n \longrightarrow 1. \tag{1}$$

It follows that $P_n(\Sigma)$ is a normal subgroup of $B_n(\Sigma)$ of index $n!$. As we will see in detail in Chapter 2, this short exact sequence is an important tool for obtaining a presentation of subgroups of the full braid groups $B_n(\Sigma)$.

Remark 1.1.3. It is well known that if Σ is equal to $\mathbb{R}^2$ or to the 2-disc then the groups $B_n(\Sigma)$ and $P_n(\Sigma)$ are isomorphic to the usual Artin braid groups B_n and P_n respectively.

We continue by recalling the definition of the n^{th} configuration space of a surface Σ.

Definition 1.1.4. The n^{th} ordered configuration space of a surface Σ, which we denote by $F_n(\Sigma)$, is the set of n-tuples of pairwise distinct points in Σ:

$$F_n(\Sigma) = \{(p_1, \ldots, p_n) \in \Sigma^n \mid p_i \neq p_j \text{ for all } i, j \in \{1, \ldots, n\},\ i \neq j\}.$$

Considering the natural action of the symmetric group S_n on $F_n(\Sigma)$ defined by permutation of coordinates:

$$S_n \times F_n(\Sigma) \to F_n(\Sigma) \text{ defined by } \big(s, (p_1, \ldots, p_n)\big) \mapsto (p_{s(1)}, \ldots, p_{s(n)}),$$

we obtain the n^{th} unordered configuration space of Σ, which is the orbit space $F_n(\Sigma)/S_n$.

Remark 1.1.5. The notion of configuration space can be defined for any topological space X.

Remark 1.1.6. The space $F_n(\Sigma)$ may be equipped with the subspace topology from the inclusion of $F_n(\Sigma)$ in Σ^n. Moreover, $F_n(\Sigma)$ is a connected $2n$-dimensional open manifold.

Remark 1.1.7. The canonical projection $pr : F_n(\Sigma) \to F_n(\Sigma)/S_n$ is an $n!$-fold covering map.

We are ready now to present a fundamental theorem that describes the braid groups from a topological viewpoint.

Theorem 1.1.8 (Fox–Neuwirth, [FN62b]). *For $n \in \mathbb{N}$ we have the following isomorphisms:*

$$\pi_1(F_n(\Sigma)) \cong P_n(\Sigma) \text{ and } \pi_1(F_n(\Sigma)/S_n) \cong B_n(\Sigma).$$

From these isomorphisms, we obtain an important tool for the study of the surface braid groups: a short exact sequence involving surface braid groups, which we will describe in detail in Section 1.2.

Remark 1.1.9. Since $F_1(\Sigma) = F_1(\Sigma)/S_1 = \Sigma$, it holds that $\pi_1(F_1(\Sigma)) = \pi_1(F_1(\Sigma)/S_1) = \pi_1(\Sigma)$. In other words, we have that:

$$B_1(\Sigma) \cong P_1(\Sigma) \cong \pi_1(\Sigma).$$

At this point, we recall the classical presentation of the Artin braid groups, which coincides with the braid groups of the plane.

Theorem 1.1.10 (Artin, [Art25]). *For all $n \geq 1$, the braid group B_n admits the following presentation.*
Generators: $\sigma_1, \ldots, \sigma_{n-1}$.

Relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$, *for* $|i-j| > 1$, $1 \leq i, j \leq n-1$,

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, *for all* $1 \leq i \leq n-2$.

The generator σ_i can be seen geometrically as the braid with a single positive crossing of the i^{th} strand with the $(i+1)^{st}$ strand, while all other strands remain vertical. We illustrate the generator σ_i and its inverse σ_i^{-1} in the following figure.

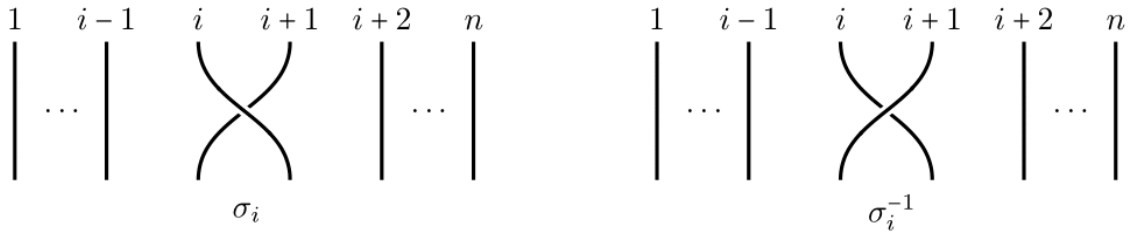


Figure 1.2 The generator σ_i and its inverse σ_i^{-1}.

Concerning the pure braid groups, for $1 \leq i < j \leq n$ the braid $A_{i,j}$ defined by:

$$A_{i,j} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1},$$

can be represented geometrically by a braid all of whose strings are vertical, with the exception of the jth strand, that wraps around the i^{th} strand. Elements of this type generate the pure braid group P_n, as we will see in the following proposition.

Proposition 1.1.11 (Hansen, [Han89]). *For all $n \geq 1$, the pure braid group P_n admits the following presentation.*
Generators: $\{A_{i,j} \mid 1 \leq i < j \leq n\}$.

Relations:

$$A_{r,s}^{-1}A_{i,j}A_{r,s} = \begin{cases} A_{i,j}, & \text{for } 1 < r < s < j \text{ or } r < s < i < j. \\ A_{r,j}A_{i,j}A_{r,j}^{-1}, & \text{for } r < i = s < j. \\ A_{r,j}A_{s,j}A_{i,j}A_{s,j}^{-1}A_{r,j}^{-1}, & \text{for } i = r < s < j. \\ A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1}A_{i,j}A_{s,j}A_{r,j}A_{s,j}^{-1}A_{r,j}^{-1}, & \text{for } r < i < s < j. \end{cases}$$

For surface braid groups there are several known presentations. The first presentations of braid groups of compact surfaces without boundary were found by Birman [Bir69] and Scott [Sco70]. Moreover, there exists further presentations of surface braid groups due to Bellingeri [Bel04], Gonçalves–Guaschi [GG04b],[GG10a], González-Meneses [GM01] and Lambropoulou [Lam00], among others.

Before concluding this section we discuss the centre and the torsion elements of the surface braid groups.

In the cases where the surface is the plane or the disc, the 2-sphere, the real projective plane, the annulus, the torus, the Möbius band and the Klein bottle their braid groups have non-trivial centre. In particular, these are the only surfaces whose braid groups have non-trivial centre. For more information regarding this subject we refer the reader to the survey [GJP15]. The braid

$$\Delta_n^2 = (\sigma_1 \dots \sigma_{n-1})^n \in P_n \subset B_n,$$

called the full twist of B_n, is a special braid, since it commutes with all of the generators $\sigma_1, \dots, \sigma_{n-1} \in B_n$. Due to the following result of Chow, the full twist generates the centre of B_n. More precisely, the full twist is the square of the well-known Garside element Δ_n of B_n, which is defined as follows:

$$\Delta_n = (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)\sigma_1.$$

Theorem 1.1.12 (Chow,[Cho48]). *It holds that* $Z(B_n) = Z(P_n) = \langle \Delta_n^2 \rangle$, *for* $n \geq 3$.

Remark 1.1.13. For $n = 1$ we have $B_1 = P_1 = 1$ and for $n = 2$ we have $Z(B_2) = \langle \sigma_1 \rangle \cong \mathbb{Z}$, while $Z(P_2) = \langle \Delta_2^2 \rangle$.

Together with the braid groups of the 2-sphere, $B_n(\mathbb{S}^2)$, the braid groups of the projective plane, $B_n(\mathbb{R}P^2)$, are of particular interest, since they contain torsion elements. In [FN62a], Fadell–Neuwirth state that neither the plane nor any compact surface, with the possible exceptions of the 2–sphere and the projective plane, has braid groups with finite order elements. It was proven in [FVB62] by Fadell–Van Buskirk that the braid groups of the sphere, $\mathbb{S}^2$, contain finite order elements. Moreover, Van Buskirk in [VB66] proved that the braid groups of the projective plane contain torsion elements. Thus, it follows that the only surface braid groups that contain torsion elements are $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$. For $B_n(\mathbb{R}P^2)$, we will describe the results in more detail in Chapter 2.

1.2 The Fadell–Neuwirth short exact sequence

One useful tool in the study of the braid groups is the Fadell–Neuwirth short exact sequence that arises from the Fadell–Neuwirth fibration, as we explain below. In [FN62a] by Fadell–Neuwirth, we obtain for $n, m \in \mathbb{N}$, where $1 \leq m < n$, and for any connected surface Σ with empty boundary, that the map

$$p_{n,m} : F_n(\Sigma) \to F_m(\Sigma), \text{ defined by } p_{n,m}(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = (x_1, \dots, x_m)$$

is a locally-trivial fibration. The fiber over a point $(x_1, \dots, x_m)$ of the base space $F_m(\Sigma)$ is the configuration space $F_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\})$, (the $(n-m)^{th}$ configuration space of Σ with m punctures), which we interpret as a subspace of the total space $F_n(\Sigma)$ via the injective map

$$i : F_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\}) \to F_n(\Sigma), \text{ defined by } i(y_1, \dots, y_{n-m}) = (x_1, \dots, x_m, y_1, \dots, y_{n-m}).$$

The map $p_{n,m}$ can be considered geometrically as forgetting the last $n-m$ points. For $1 \leq m < n$, the fibration $p_{n,m}$ gives rise to the following long exact sequence of the homotopy groups of these spaces:

$$\begin{aligned}\cdots \to &\pi_k(F_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\})) \to \pi_k(F_n(\Sigma)) \to \pi_k(F_m(\Sigma)) \to \\ &\pi_{k-1}(F_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\})) \to \pi_{k-1}(F_n(\Sigma)) \to \pi_{k-1}(F_m(\Sigma)) \to \\ &\cdots \to \pi_1(F_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\})) \to \pi_1(F_n(\Sigma)) \to \pi_1(F_m(\Sigma)) \to 1. \end{aligned} \tag{2}$$

For $m = n-1$, the long exact sequence (2) and the fact that $F_1(\Sigma \setminus \{x_1,\dots,x_{n-1}\}) = \Sigma \setminus \{x_1,\dots,x_{n-1}\}$ has the homotopy type of a one-dimensional complex, since it is homotopy equivalent to a bouquet of circles, give rise to the following isomorphisms:

$$\pi_k(F_n(\Sigma)) \cong \pi_k(F_{n-1}(\Sigma)) \cong \cdots \cong \pi_k(F_1(\Sigma)) = \pi_k(\Sigma), \text{ for } k \geq 3.$$

Moreover, it follows that $\pi_2(F_n(\Sigma)) \subset \pi_2(F_{n-1}(\Sigma)) \subset \cdots \subset \pi_2(F_1(\Sigma)) = \pi_2(\Sigma)$. With the exception of the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$, we know that for $k \geq 2$ any connected surface Σ has trivial higher homotopy groups $\pi_k(\Sigma) = 1$. Thus, from the above observations it follows that

$$\pi_k(F_n(\Sigma)) = 1, \text{ for } k \geq 2 \text{ and } \Sigma \neq \mathbb{S}^2, \mathbb{R}P^2.$$

From this result and from Theorem 1.1.8, it follows that, for Σ different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, the spaces $F_n(\Sigma)$ and $F_n(\Sigma)/S_n$ are Eilenberg–Mac Lane spaces of type $K(P_n(\Sigma),1)$ and $K(B_n(\Sigma),1)$ respectively.

Combining this result with the Fadell–Neuwirth fibration (2) we get the following result.

Proposition 1.2.1. *Let $n, m \in \mathbb{N}$, and let Σ be any connected surface different from the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$. For $1 \leq m < n$, there is a short exact sequence:*

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1.$$

Regarding the case of the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$, it was proved that $\pi_2(F_3(\mathbb{S}^2)) = \pi_2(F_2(\mathbb{R}P^2)) = 0$, in [FVB62] and [VB66], respectively. The following theorem sums up what we have seen so far regarding the Fadell–Neuwirth short exact sequence of surface braid groups.

Theorem 1.2.2. *Let $n, m \in \mathbb{N}$, where $1 \leq m < n$, and let Σ be any connected surface without boundary. When $\Sigma = \mathbb{S}^2$ we suppose that $m \geq 3$ and when $\Sigma = \mathbb{R}P^2$ we suppose that $m \geq 2$. The Fadell–Neuwirth fibration (2) induces the following short exact sequence:*

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1. \tag{3}$$

Remark 1.2.3. The short exact sequence in Theorem 1.2.2 is known as the Fadell–Neuwirth short exact sequence of surface pure braid groups.

Remark 1.2.4. The induced map $\bar{p}_{n,m}$ can be considered geometrically as the epimorphism that forgets the last $n-m$ strands.

Remark 1.2.5. An element of $P_{n-m}(\Sigma \setminus \{x_1,\dots,x_m\})$ may be interpreted as an n-strand braid whose first m strands are vertical.

Remark 1.2.6. To prove that $p_{n,m} : F_n(\Sigma) \to F_m(\Sigma)$ is a locally-trivial fibration, one needs to suppose that the connected surface Σ is without boundary. However, the long exact sequence (2) exists even if Σ has boundary. As a result, the Fadell–Neuwirth short exact sequence of surface pure braid groups exists for any connected surface. To justify this, let Σ be a surface with boundary, and let $\bar{\Sigma} = \Sigma \setminus \partial\Sigma$, where $\partial\Sigma$ stands for the boundary of Σ. The surface $\bar{\Sigma}$ is with empty boundary and thus for this surface the locally-trivial fibration $p_{n,m} : F_n(\bar{\Sigma}) \to F_m(\bar{\Sigma})$ exists. Therefore, for $\bar{\Sigma}$ the long exact sequence (2) and the Fadell–Neuwirth short exact sequence are valid. On the other hand, Σ and $\bar{\Sigma}$ are homotopy equivalent, since the inclusion of $\bar{\Sigma}$ in Σ is a homotopy equivalence between these two surfaces. Moreover, this inclusion induces a homotopy equivalence between their n^{th} configuration spaces. It follows that the long exact sequence (2) and the Fadell–Neuwirth short exact sequence in Theorem 1.2.2 are valid also for Σ, a connected surface with boundary, and that $P_n(\Sigma)$ are isomorphic to $P_n(\bar{\Sigma})$.

The Fadell–Neuwirth short exact sequence of surface pure braid groups does not extend directly to the full braid groups. It is clear that in the case of $B_n(\Sigma)$ the induced map $\bar{p}_{n,m}$, which geometrically can be considered as the epimorphism that forgets the last $n-m$ strands, is not well-defined. Nevertheless, the Fadell–Neuwirth short exact sequence of surface pure braid groups may be generalised to certain subgroups that lie between the pure braid groups $P_n(\Sigma)$ and the full braid groups $B_n(\Sigma)$, as we describe below.

Let $n, m \in \mathbb{N}$ and Σ be a connected surface. We consider the space obtained by quotienting the $(n+m)^{th}$ configuration space of Σ, $F_{n+m}(\Sigma)$, by the subgroup $S_n \times S_m$ of S_{n+m}. Once again if the surface Σ is without boundary we obtain a locally-trivial fibration $q_{n+m,n} : F_{n+m}(\Sigma)/(S_n \times S_m) \to F_n(\Sigma)/S_n$, defined by forgetting the last m coordinates, whose fiber can be identified with the unordered configuration space $F_m(\Sigma \setminus \{x_1, \dots, x_n\})/S_m$. We set

$$B_{n,m}(\Sigma) = \pi_1(F_{n+m}(\Sigma)/(S_n \times S_m)),$$

which are subgroups of the full braid groups $B_{n+m}(\Sigma)$ and we call them mixed braid groups. Notice that the mixed braid groups, $B_{n,m}(\Sigma)$, are defined whether or not Σ has boundary. As in the pure braid group case, we will see that we can obtain a generalised Fadell–Neuwirth short exact sequence, which arises from long exact sequence of the fibration $q_{n+m,n}$. Apart from allowing us to generalise the Fadell–Neuwirth short exact sequence, mixed braid groups are in general very useful and have been studied by Sossinsky [Sos92], Manfredini [Man97], Paris–Rolfsen [PR99], Lambropoulou [Lam00], Gonçalves–Guaschi [GG04c], [GG05], [GG12] and Bellingeri–Godelle–Guaschi [BGG], among others. In the following proposition we present a generalisation of Theorem 1.2.2.

Proposition 1.2.7. *Let $n, m \in \mathbb{N}$, and let Σ be any connected surface without boundary. When $\Sigma = \mathbb{S}^2$ we suppose that $n \geq 3$ and when $\Sigma = \mathbb{R}P^2$ we suppose that $n \geq 2$. The locally-trivial fibration $q_{n+m,n} : F_{n+m}(\Sigma)/(S_n \times S_m) \to F_n(\Sigma)/S_n$ induces the following short exact sequence:*

$$1 \longrightarrow B_m(\Sigma \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1. \tag{4}$$

Remark 1.2.8. The short exact sequence in Proposition 1.2.7 is known as the generalised Fadell–Neuwirth short exact sequence of mixed braid groups.

Remark 1.2.9. The induced map $\bar{q}_{n+m,n}$ can be considered geometrically as the epimorphism that forgets the last m strands.

Remark 1.2.10. Similarly to Remark 1.2.6, and using the same argument, the generalised Fadell–Neuwirth short exact sequence of mixed braid groups in Proposition 1.2.7 exists also for connected surfaces with boundary.

A natural question that arises from these two short exact sequences, 3 and 4, is whether or not they split, or in other words whether or not there is a section for the maps $\bar{p}_{n,m}$ and $\bar{q}_{n+m,m}$. We describe this problem in more detail in the following section.

1.3 The splitting problem

Let $n, m \in \mathbb{N}$, $1 \leq m < n$, and let Σ be any connected surface. When $\Sigma = \mathbb{S}^2$ we suppose that $m \geq 3$ and when $\Sigma = \mathbb{R}P^2$ we suppose that $m \geq 2$. As we explained in the previous section, we have the following Fadell–Neuwirth short exact sequence:

$$1 \longrightarrow P_{n-m}(\Sigma \setminus \{x_1, \dots, x_m\}) \longrightarrow P_n(\Sigma) \xrightarrow{\bar{p}_{n,m}} P_m(\Sigma) \longrightarrow 1. \tag{5}$$

Similarly, let $n, m \in \mathbb{N}$, and let Σ any connected surface. When $\Sigma = \mathbb{S}^2$ we suppose that $n \geq 3$ and when $\Sigma = \mathbb{R}P^2$ we suppose that $n \geq 2$. We have the followig generalised Fadell–Neuwirth short exact sequence:

$$1 \longrightarrow B_m(\Sigma \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\Sigma) \xrightarrow{\bar{q}_{n+m,n}} B_n(\Sigma) \longrightarrow 1. \tag{6}$$

Note that, even in the cases where $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{R}P^2$, the homomorphisms $\bar{p}_{n,m}$ and $\bar{q}_{n,m}$, given geometrically by forgetting the last $(n-m)$ and the last m strings respectively, are defined for all $n \in \mathbb{N}$.

One can naturally ask whether or not the maps $\bar{p}_{n,m}$ and $\bar{q}_{n+m,n}$ admit a section, which is the same as asking, whether or not there exists an injective map $\bar{s}_{n,m}$ and $\hat{s}_{n+m,n}$, respectively, such that $\bar{p}_{n,m} \circ \bar{s}_{n,m} = id_{P_m(\Sigma)}$ and $\bar{q}_{n,m} \circ \hat{s}_{n,m} = id_{B_n(\Sigma)}$. If a short exact sequence admits such a section we say that the short exact sequence splits. The question, to which we shall refer as the splitting problem, of whether the short exact sequences (5) and (6) split, can be approached algebraically and geometrically. For instance, to obtain a positive answer to the splitting problem, it suffices to provide an explicit section, either on the level of configuration spaces or on the algebraic level. A geometrical approach would be to show the existence of a geometric cross-section on the level of configuration spaces, which in turn provides a section on the algebraic level. To be more precise, in the following proposition we present the equivalence between the algebraic section and the geometric section (cross-section). For Σ an aspherical space the following result is due to Baues [Bau77] and Whitehead [Whi78], while for $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{R}P^2$ we refer the reader to [GG05] and [GG04a] respectively.

Proposition 1.3.1. *Let $n, m \in \mathbb{N}$, and let Σ be a compact, connected surface. When $\Sigma = \mathbb{S}^2$ we suppose that $m \geq 3$ and when $\Sigma = \mathbb{R}P^2$ we suppose that $m \geq 2$. The Fadell–Neuwirth fibration $p_{n,m} : F_n(\Sigma) \to F_m(\Sigma)$, for $1 \leq m < n$, admits a cross-section if and only if the short exact sequence (5) splits. Similarly, the Fadell–Neuwirth fibration $q_{n+m,n} : F_{n+m}(\Sigma)/(S_n \times S_m) \to F_n(\Sigma)/S_n$, where for $\Sigma = \mathbb{S}^2$ we suppose that $n \geq 3$ and for $\Sigma = \mathbb{R}P^2$ we suppose that $n \geq 2$, admits a cross-section if and only if the short exact sequence (6) splits.*

Let F_n denote the free group of rank n. In the case where $\Sigma = \mathbb{R}^2$, $n \geq 2$ and $m = n-1$, the short exact sequence (5) becomes

$$1 \longrightarrow F_{n-1} \longrightarrow P_n \xrightarrow{\bar{p}_{n,n-1}} P_{n-1} \longrightarrow 1, \tag{7}$$

since $P_1(\mathbb{R}^2 \setminus \{x_1, \ldots, x_{n-1}\}) = \pi_1(F_1(\mathbb{R}^2 \setminus \{x_1, \ldots, x_{n-1}\})) = \pi_1(\mathbb{R}^2 \setminus \{x_1, \ldots, x_{n-1}\}) \cong F_{n-1}$. Considering the natural inclusion of P_{n-1} into P_n, where any generator $A_{i,j}$, $1 \leq i < j \leq n-1$, as element in P_{n-1}, is mapped to $A_{i,j}$, $1 \leq i < j \leq n-1$, as element in P_n, we see that the map $\bar{p}_{n,n-1}$ admits a section, and thus the short exact sequence splits. Geometrically this section can be thought as adding a vertical strand completely unlinked from the rest strands. Therefore, P_n is isomorphic to the semi-direct product of F_{n-1} and P_{n-1}. By induction on n, P_n may be written as an iterated semi-direct product of free groups, known as the Artin normal form:

$$P_n \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1.$$

The procedure for obtaining the Artin normal form of a pure braid $\alpha \in P_n$ is known as Artin combing, and involves writing α in the form $\alpha = \alpha_{n-1} \cdots \alpha_1$, where $\alpha_i \in F_i$, $1 \leq i \leq n-1$. By the uniqueness of normal forms with respect to semi-direct products, the combed normal form of a braid is unique. Braid combing is a procedure, defined by Artin [Art47b], to solve the word problem in braid groups for the first time. Further details about this decomposition, based on the short exact sequence, can be found, for instance, in [KT08] by Kassel–Turaev.

This decomposition is an important result in classical braid theory. The word problem in a free group of finite rank is well-known and solvable, and thus the decomposition (7) provides a finite algorithm to solve the word problem in P_n. Moreover, since P_n is of finite index in B_n, it follows that it makes it easy to solve the word problem in B_n as well. The splitting of the short exact sequence (5) and the resulting decomposition (7) have been frequently used to prove several claims about P_n, using induction. For instance the result about P_n, that taking roots is unique, by Bardakov [Bar92], the study of the lower central series and the residual nilpotence of P_n, by Farrell–Rushon [FR00], as well as the proof of the bi-orderability of P_n, by Kim–Rolfsen [KR03], make use of decomposition (7). Thus, the problem of deciding whether such a decomposition (7) exists for surface braid groups, which is connected to the splitting problem, is fundamental. Indeed, this was a central question, during the foundation and the development of the theory of braid groups during the 1960's, studied by Fadell [Fad62], Fadell–Neuwirth [FN62a], Fadell–Van Buskirk [FVB62], Van Buskirk [VB66] and Birman [Bir69], among others.

In the case of the pure braid groups, the splitting problem for the short exact sequence (5) has been studied for other surfaces besides the plane. To begin with, Fadell–Neuwirth [FN62a] gave several sufficient conditions for the fibration $p_{n,m} : F_n(\Sigma) \to F_m(\Sigma)$ to admit a cross-section. For surfaces with boundary we see, in [GG04b] and [GG10a] by Gonçalves–Guaschi, that the fibration $p_{n,m} : F_n(\Sigma) \to F_m(\Sigma)$ always admits a cross-section, and thus the homomorphism $\bar{p}_{n,m} : P_n(\Sigma) \to P_m(\Sigma)$ does too. For surfaces without boundary, we have the following results. In [FVB62], Fadell and Van Buskirk proved that for $\Sigma = \mathbb{S}^2$ and $m \geq 3$ the fibration $p_{n,m}$ admits a cross-section, and thus the short exact sequence (5), in this case, splits. Also, Van Buskirk [VB66] worked on the case where $\Sigma = \mathbb{R}P^2$ and proved that the fibration $p_{3,2} : F_3(\mathbb{R}P^2) \to F_2(\mathbb{R}P^2)$ admits a cross-section, and thus the short exact sequence (5) for $n = 3$ and $m = 2$ splits, while for $n \geq 2$ neither the fibration $p_{n,1} : F_n(\mathbb{R}P^2) \to F_1(\mathbb{R}P^2)$ nor the homomorphism $\bar{p}_{n,1} : P_n(\mathbb{R}P^2) \to P_1(\mathbb{R}P^2)$ admits a section. Moreover, for Σ being the 2-torus and $n \geq 2$, Birman in [Bir69] presented an explicit cross-section $p_{n,n-1} : F_n(\mathbb{T}^2) \to F_{n-1}(\mathbb{T}^2)$, and thus it follows that the homomorphism $\bar{p}_{n,1} : P_n(\mathbb{T}^2) \to P_{n-1}(\mathbb{T}^2)$ admits a section too, for all $n \geq 2$. As a result, the short exact sequence (5) for $\Sigma = \mathbb{T}^2$ splits for all $1 \leq m < n$. Finally, a more general result, as stated below, covering the case where Σ is an orientable surface without boundary of genus at least two, was given by Gonçalves–Guaschi in [GG04b].

Theorem 1.3.2 (Gonçalves–Guaschi, [GG04b]). *Let Σ be a compact, connected, orientable surface without boundary of genus $g \geq 2$. Then, the short exact sequence (5) splits if and only if $m = 1$.*

In addition, Gonçalves and Guaschi studied in [GG04a], [GG05] and [GG07] the remaining cases and at the end they gave a complete solution to the splitting problem of the short exact sequence (5) in [GG10a].

Theorem 1.3.3 (Gonçalves–Guaschi, [GG10a]). *Let $n, m \in \mathbb{N}$ where $1 \leq m < n$, and let Σ be a connected surface.*

i. If Σ has non-empty boundary then the short exact sequence (5) admits a section for all n, m.

ii. If Σ is without boundary then the short exact sequence (5) admits a section if and only if one of the following conditions holds:

   a. Σ is the 2-sphere $\mathbb{S}^2$, the 2-torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$.

   b. Σ is the projective plane $\mathbb{R}P^2$, where $n = 3$ and $m = 2$.

   c. Σ is different from the projective plane $\mathbb{R}P^2$, the 2-sphere $\mathbb{S}^2$, the 2-torus $\mathbb{T}^2$, the Klein bottle $\mathbb{K}^2$ and $m = 1$.

To give a positive answer to the splitting problem it suffices to present an explicit section, either geometrically or algebraically. To obtain a negative answer to the splitting problem, one of the main methods, that was used in proving Theorem 1.3.3, is to study the splitting problem of certain quotients of the braid groups and then extend the result to the initial short exact sequence. We will describe this technique in detail in Section 3.2.

The splitting problem for the generalised Fadell–Neuwirth short exact sequence (6) does not have a complete solution yet. More precisely, to the best of our knowledge, the surface for which we have an answer for the splitting problem of the generalised Fadell–Neuwirth short exact sequence (6), besides the plane, $\mathbb{R}^2$, is the 2–sphere, $\Sigma = \mathbb{S}^2$. The splitting problem for $\Sigma = \mathbb{S}^2$ was studied by Gonçalves–Guaschi in [GG05] and later by Chen–Salter in [CS20]. To be more precise we have the following results.

Theorem 1.3.4 (Gonçalves–Guaschi, [GG05]). *Let $n, m \in \mathbb{N}$. The short exact sequence*

$$1 \longrightarrow B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \longrightarrow B_{3,m}(\mathbb{S}^2) \xrightarrow{\bar{q}_{3+m,3}} B_3(\mathbb{S}^2) \longrightarrow 1,$$

splits if and only if $m \equiv 0, 2 \bmod 3$. Moreover, for $n \geq 4$, if the homomorphism $\bar{q}_{n+m,n} : B_{n,m}(\mathbb{S}^2) \to B_n(\mathbb{S}^2)$ admits a section then m is congruent modulo $n(n-1)(n-2)$ to one of the four values $0, -n(n-2), (n-1)(n-2), (n-1)(n-2) - n(n-2)$.

For the cases where $n = 1$ and $n = 2$ Gonçalves and Guaschi, [GG05], obtained the following results. The homomorphism $\bar{q}_{1+m,1} : B_{1,m}(\mathbb{S}^2) \to B_1(\mathbb{S}^2)$ admits a section for every m, but the homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{S}^2) \to B_2(\mathbb{S}^2)$ does not admit any section. Finally, we can also see that the fibration $q_{2,1} : F_{1,1}(\mathbb{S}^2) \to F_1(\mathbb{S}^2)$ admits a cross-section, since one can just consider the antipodal map.

In [CS20], Chen–Salter give a substantial strengthening of Theorem 1.3.4, by exhibiting explicit cross-sections, among other things.

Theorem 1.3.5 (Chen–Salter, [CS20])**.** *Let $n, m \in \mathbb{N}$. The following hold:*

i. For any $n \geq 3$ and any $m \equiv 0$ mod $n(n-1)(n-2)$, the fibration $q_{n+m,n} : F_{n,m}(\mathbb{S}^2) \to F_n(\mathbb{S}^2)$ admits a cross-section.

ii. For $n = 3$ and $m \geq 0$ such that $m \equiv 0, 2$ mod 3 the fibration $q_{3+m,3} : F_{3,m}(\mathbb{S}^2) \to F_3(\mathbb{S}^2)$ admits a cross-section.

iii. For $n = 4$ and $m \geq 0$ such that $m \equiv 0, 6, 16, 22$ mod 24 the fibration $q_{4+m,4} : F_{4,m}(\mathbb{S}^2) \to F_4(\mathbb{S}^2)$ admits a cross-section.

iv. For $n \geq 6$, the fibration $q_{n+m,n} : F_{n,m}(\mathbb{S}^2) \to F_n(\mathbb{S}^2)$ admits a cross-section if and only if $m \equiv 0$ mod $n(n-1)(n-2)$.

CHAPTER 2

Braid groups of the projective plane, $B_n(\mathbb{R}P^2)$

This chapter is devoted to the study of the braid groups of the projective plane, $B_n(\mathbb{R}P^2)$. We start by introducing these groups and stating important known results about their presentations, their torsion elements and their subgroups. In Section 2.2, we give a presentation of certain subgroups of $B_n(\mathbb{R}P^2)$, the goal being to obtain a presentation of the group $B_{n,m}(\mathbb{R}P^2) \subset B_{n+m}(\mathbb{R}P^2)$. In Section 2.3, based on these presentations, we study the lower central and derived series of the subgroup $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ of $B_{n,m}(\mathbb{R}P^2)$.

2.1 The group $B_n(\mathbb{R}P^2)$

Along with the braid groups of the 2–sphere, $B_n(\mathbb{S}^2)$, the braid groups of the projective plane, $B_n(\mathbb{R}P^2)$, possess a special place among the braid groups of surfaces due to the fact that they have a non-trivial centre and contain torsion elements. Note that throughout this thesis we consider the projective plane as the quotient space of the closed 2–disc, $\mathbb{D}^2$, obtained by identifying the antipodal points on the boundary circle of $\mathbb{D}^2$, as in Figure 2.1. To begin, we give a presentation of these groups due to Van Buskirk.

Theorem 2.1.1 (Van Buskirk, [VB66])**.** *Let $n \in \mathbb{N}$. The following constitutes a presentation of $B_n(\mathbb{R}P^2)$.*

Generators: $\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_n$.

Relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$, *for* $|i-j| > 1$,

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, *for* $1 \le i \le n-2$,

(iii) $\sigma_i\rho_j = \rho_j\sigma_i$, *for* $j \ne i, i+1$,

(iv) $\rho_i = \sigma_i\rho_{i+1}\sigma_i$, *for* $1 \le i \le n-1$,

(v) $\sigma_i^2 = \rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i$, *for* $1 \le i \le n-1$,

(vi) $\rho_1^2 = \sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1$.

The generators $\sigma_1, \dots, \sigma_{n-1}$ are the standard Artin generators, and each of the generators $\rho_1, \dots, \rho_n$ corresponds geometrically to an element of the fundamental group of the projective plane based at the $1^{st}, 2^{nd}, \dots, n^{th}$ point, respectively, as we see in Figure 2.1. From this presentation, we observe that $B_1(\mathbb{R}P^2) = \langle \rho_1 \mid \rho_1^2 = 1 \rangle \cong \mathbb{Z}_2$, and from [VB66] (page 87), we know that $B_2(\mathbb{R}P^2)$ is

isomorphic to the dicyclic group of order 16 and in particular $B_2(\mathbb{R}P^2) = \langle \sigma_1, \rho_1 \mid (\sigma_1\rho_1^{-1})^4 = \rho_1^2 = \sigma_1^2 \rangle$. If $n \geq 3$, the group $B_n(\mathbb{R}P^2)$ is infinite.

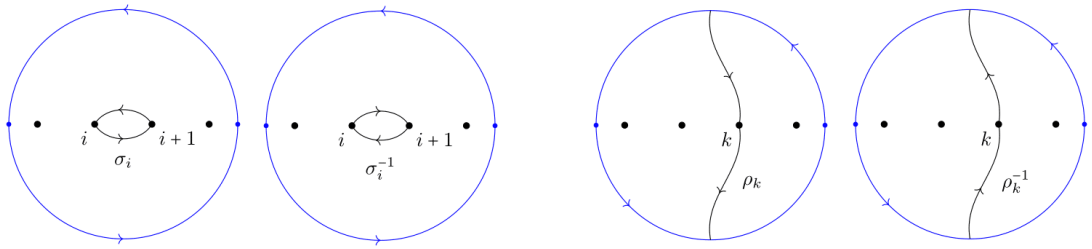


Figure 2.1 The generators σ_i and ρ_k, and their inverses, of $B_n(\mathbb{R}P^2)$.

Regarding the pure braid group of the projective plane, $P_n(\mathbb{R}P^2)$, a presentation was given in [GG05], by Gonçalves–Guaschi, but we will give a different one in Section 2.2.

As we already mentioned, the centre of the braid groups of $\mathbb{R}P^2$ is non-trivial. More precisely, we have the following theorem by Murasugi.

Proposition 2.1.2 (Murasugi, [Mur82]). *For $n \geq 2$, the full twist*

$$\Delta_n^2 = (\sigma_1 \dots \sigma_{n-1})^n \in P_n(\mathbb{R}P^2) \subset B_n(\mathbb{R}P^2),$$

generates the centre of $B_n(\mathbb{R}P^2)$.

The full twist, as well as generating the centre of $B_n(\mathbb{R}P^2)$, is also the unique torsion element of order 2 in $B_n(\mathbb{R}P^2)$, as we will see in the following proposition. The torsion elements of $B_n(\mathbb{R}P^2)$ were first determined by Murasugi in [Mur82]. Later Gonçalves–Guaschi, with methods different to those of Murasugi, obtained the following result.

Proposition 2.1.3. *(Gonçalves–Guaschi, [GG04a]) For $n \geq 2$ the following hold:*

- *The group $B_n(\mathbb{R}P^2)$ contains a torsion element of order k if and only if k divides either $4n$ or $4(n-1)$.*
- *The (non-trivial) torsion elements of $P_n(\mathbb{R}P^2)$ are of order precisely 2 and 4.*
- *The full twist Δ_n^2 is the unique element of $B_n(\mathbb{R}P^2)$ of order 2.*

Moreover, a concrete example of torsion elements in $B_n(\mathbb{R}P^2)$ of order $4n$ and $4(n-1)$ was given in [GG04a]. To be more precise, for $n \geq 2$, the elements

$$a_n = \sigma_{n-1}^{-1} \dots \sigma_1^{-1} \cdot \rho_1 \text{ and } b_n = \sigma_{n-2}^{-1} \dots \sigma_1^{-1} \cdot \rho_1 \tag{1}$$

are of order $4n$ and $4(n-1)$, respectively.

The characterisation of the finite-order elements of $B_n(\mathbb{R}P^2)$ was given by Murasugi in [Mur82]. Later, in [GG10b] a simplified characterisation was given.

Proposition 2.1.4 (Gonçalves–Guaschi, [GG10b]). *Let $n \geq 2$ and let x be an element in $B_n(\mathbb{R}P^2)$. The element x is of finite order if and only if there exists $i \in \{1, 2\}$ and $0 \leq r \leq n+1-i$ such that x is a power of a conjugate of the following element:*

$$A_i(n, r, 2r/l, p/l) = (\rho_r \sigma_{r-1} \dots \sigma_1)^{2r/l} (\sigma_{r+1} \dots \sigma_{n-1} \sigma_{r+1}^{i-1})^{p/l},$$

where $p = (n+1-i) - r$ and $l = gcd(p, 2r)$. Moreover, the order of the element $A_i(n, r, 2r/l, p/l)$ is $2l$.

From this explicit description of the torsion elements in $B_n(\mathbb{R}P^2)$ we can obtain the elements $a_n = \sigma_{n-1}^{-1} \dots \sigma_1^{-1} \cdot \rho_1$ and $b_n = \sigma_{n-2}^{-1} \dots \sigma_1^{-1} \cdot \rho_1$ from $A_1(n, n, 1, 0)$ and $A_2(n, n-1, 1, 0)$, respectively. To be more precise, using relation $\rho_i = \sigma_i \rho_{i+1} \sigma_i$, that holds in $B_n(\mathbb{R}P^2)$, we have that $A_1(n, n, 1, 0) = (\rho_n \sigma_{n-1} \dots \sigma_1) = \sigma_{n-1}^{-1} \dots \sigma_1^{-1} \cdot \rho_1$ and $A_2(n, n-1, 1, 0) = (\rho_{n-1} \sigma_{n-2} \dots \sigma_1) = \sigma_{n-2}^{-1} \dots \sigma_1^{-1} \cdot \rho_1$.

Before concluding this section, we will present the classification of the finite subgroups of $B_n(\mathbb{R}P^2)$ and of $P_n(\mathbb{R}P^2)$.

Proposition 2.1.5 (Gonçalves–Guaschi, [GG10b]). *The maximal finite subgroups, up to isomorphism, of $P_n(\mathbb{R}P^2)$ are the following:*

- *The group $\mathbb{Z}_2$ if $n = 1$;*
- *The quaternion group of order 8, $\mathcal{Q}_8$ if $n = 2, 3$;*
- *The group $\mathbb{Z}_4$ if $n \geq 4$.*

Theorem 2.1.6 (Gonçalves–Guaschi, [GG12]). *Let $n \geq 2$. The finite subgroups of the braid groups $B_n(\mathbb{R}P^2)$ are isomorphic to the subgroups of the following groups:*

- *The dicyclic group of order $8n$, $Dic_{8n} = \langle x, y \mid x^{2n} = y^2,\ yxy^{-1} = x^{-1} \rangle$;*
- *The dicyclic group of order $8(n-1)$, $Dic_{8(n-1)} = \langle x, y \mid x^{2(n-1)} = y^2,\ yxy^{-1} = x^{-1} \rangle$ if $n \geq 3$;*
- *The binary octahedral group of order 48, O^* if $n \equiv 0, 1 \mod 3$;*
- *The binary icosahedral group of order 120, I^* if $n \equiv 0, 1, 6, 10 \mod 15$.*

Further information about the binary polyhedral groups O^* and I^* could be found in [AM04]. Note that the centre of Dic_{8n}, $Dic_8(n-1)$, O^* and I^* is isomorphic to the cyclic group of order 2, $\mathbb{Z}_2$. Moreover, their quotient by their centre is either the dihedral group, the symmetric group of degree 4, S_4 or the alternating group of degree 5, A_5. To be more precise, it holds that:

$$Dic_{8n}/\mathbb{Z}^2 \cong Dih_{4n},\ Dic_{8(n-1)}/\mathbb{Z}^2 \cong Dih_{4(n-1)},\ O^*/\mathbb{Z}^2 \cong S_4, \text{ and } I^*/\mathbb{Z}^2 \cong A_5.$$

In [GG12] (Proposition 15), Gonçalves–Guaschi gave explicit algebraic realisations of the groups Dic_{8n} and $Dic_{8(n-1)}$ as subgroups of $B_n(\mathbb{R}P^2)$, for all $n \geq 2$. To be more precise, we have the following proposition.

Proposition 2.1.7 ([GG12], Gonçalves–Guaschi). *Let $n \geq 2$. Then the following statements hold:*

1. *$\langle a_n, \Delta_n \rangle \cong Dic_{8n}$, where a_n is defined in (1).*
2. *$\langle b_n, \Delta_n a_n^{-1} \rangle \cong Dic_{8(n-1)}$, where b_n is defined in (1).*

2.2 Presentation of $B_{n,m}(\mathbb{R}P^2)$ and of certain subgroups

Let $m \in \mathbb{N}$ and $n \geq 2$. In order to study the splitting problem of the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1, \tag{2}$$

we require a presentation of the groups $B_{n,m}(\mathbb{R}P^2)$. We recall that the map $\bar{q}_{n+m,n}$ can be considered geometrically as the map that forgets the last m strands and that the groups $B_{n,m}(\mathbb{R}P^2)$, defined by $B_{n,m}(\mathbb{R}P^2) = \pi_1(F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m))$, are called mixed braid groups, which are subgroups of the full braid groups $B_{n+m}(\mathbb{R}P^2)$.

In order to obtain a presentation of $B_{n,m}(\mathbb{R}P^2)$, we will use the short exact sequence (2) and standard results about presentations of group extensions. Thus, we first need a presentation of the groups $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$, which we can obtain making use of the short exact sequence

$$1 \to P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \to B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \to S_m \to 1,$$

and for making use of this short exact sequence, we first have to determine the groups $P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$. Note that there is already one presentation of the group $P_n(\mathbb{R}P^2)$ in [GG07], but we give a different one here, where our generators ρ_i differ from those in [GG07].

We start with the following proposition that provides a presentation of the pure braid groups $P_n(\mathbb{R}P^2)$. Note that, for $\mathbb{D}^2$ a topological disc, the inclusion of $\mathbb{D}^2$ into the projective plane, $\mathbb{R}P^2$, induces a non-injective homomorphism $\iota : B_n \to B_n(\mathbb{R}P^2)$.

Proposition 2.2.1. *The following constitutes a presentation of $P_n(\mathbb{R}P^2)$, for $n \in \mathbb{N}$.*

Generators: *$B_{i,j}$, for $1 \le i < j \le n$ and ρ_k, for $1 \le k \le n$.*

Relations:

(i) The Artin relations between the generators $B_{i,j}$ coming from those of P_n. For $1 \le i < j \le n$ and $1 \le r < s \le n$,

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j}, & \text{for } i < r < s < j \text{ or } r < s < i < j. \\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j}, & \text{for } r < i = s < j. \\ B_{s,j}^{-1}B_{i,j}B_{s,j}, & \text{for } i = r < s < j. \\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j}, & \text{for } r < i < s < j. \end{cases}$$

(ii) $\rho_i(B_{1,i}\dots B_{i-1,i}) = (B_{i,i+1}\dots B_{i,n})\rho_i^{-1}$, *for $1 \le i \le n$, called surface relations.*

(iii) $\rho_i\rho_j\rho_i^{-1} = \Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)^{-1} B_{i,j}\Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)\rho_j$, *for $1 \le i < j \le n$.*

(iv) For $1 \le i < j \le n$, $1 \le k \le n$ and $k \ne j$,

$$\rho_k B_{i,j}\rho_k^{-1} = \begin{cases} B_{i,j}, & \text{for } j < k \text{ or } k < i. \\ \rho_j^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)^{-1} B_{i,j}^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)\rho_j & \text{for } k = i. \\ A_{j,k}B_{i,j}A_{j,k}^{-1} & \text{for } i < k < j, \end{cases}$$

where $A_{j,k} := \rho_j^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)^{-1} B_{k,j}^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)\rho_j$.

Remark 2.2.2. The generator $B_{i,j}$ is given by $B_{i,j} = \sigma_i^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_i$ and it can be seen geometrically as a braid all of whose strands are vertical, with the exception of the i^{th} strand, that wraps around the j^{th} strand. Equivalently, $B_{i,j}$ can be given by $B_{i,j} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$ and it can be seen geometrically as a braid all of whose strands are vertical, with the exception of the j^{th} strand, that wraps around the i^{th} strand. Note that in the following figure, as well as in all those that follow, we will illustrate the generator $B_{i,j}$ and its inverse using the geometric illustration of the generator σ_i given in Figure 2.1.

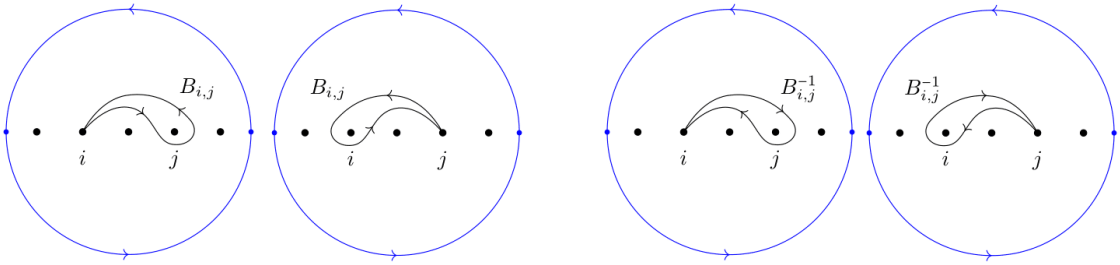


Figure 2.2 Equivalent illustrations of the generators $B_{i,j}$ and $B_{i,j}^{-1}$ of $P_n(\mathbb{R}P^2)$.

Remark 2.2.3. The products $\Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)^{-1} B_{i,j}\Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)$, $\Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)^{-1} B_{i,j}^{-1}\Big(\prod_{l=i+1}^{j-1} B_{l,j}\Big)$ and $\Big(\prod_{l=i+1}^{j} B_{i,l}\Big)$, $\Big(\prod_{l=i+1}^{j} B_{i,l}\Big)^{-1}$, for $i \le j$, that appear frequently in the presentation of $P_n(\mathbb{R}P^2)$, can be seen geometrically as in the following figures.

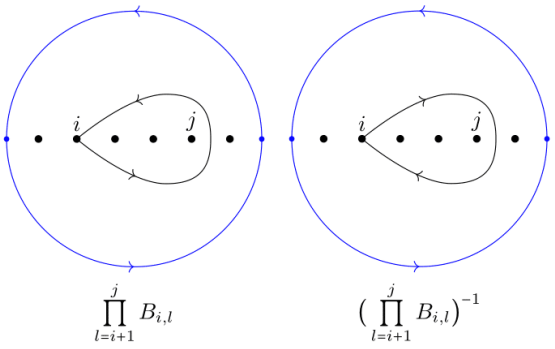


Figure 2.3 The products $\big(\prod_{l=i+1}^{j} B_{i,l}\big)$, $\big(\prod_{l=i+1}^{j} B_{i,l}\big)^{-1}$, for $i \leq j$.

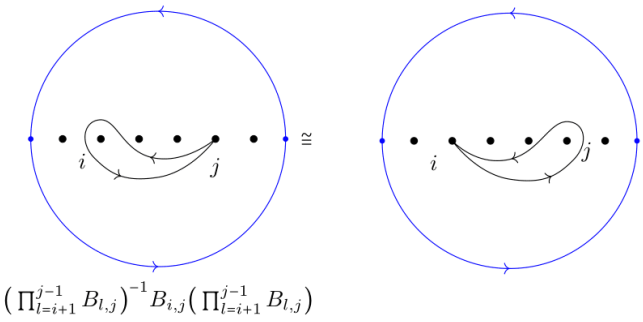


Figure 2.4 The product $\big(\prod_{l=i+1}^{j-1} B_{l,j}\big)^{-1} B_{i,j} \big(\prod_{l=i+1}^{j-1} B_{l,j}\big)$, for $i \leq j$.

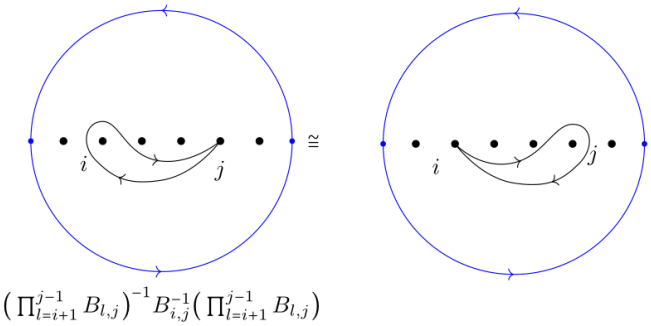


Figure 2.5 The product $\big(\prod_{l=i+1}^{j-1} B_{l,j}\big)^{-1} B_{i,j}^{-1} \big(\prod_{l=i+1}^{j-1} B_{l,j}\big)$, for $i \leq j$.

Proof. We will prove the claim by applying induction on $n \in \mathbb{N}$ and using standard results concerning the presentation of an extension.

For $n = 1$ the given presentation yields $P_1(\mathbb{R}P^2) = \langle \rho_1 \mid \rho_1^2 = 1 \rangle \cong \mathbb{Z}_2$, which holds, since $P_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. Moreover, for $n = 2$ we obtain that $P_2(\mathbb{R}P^2) = \langle \rho_1, \rho_2 \mid \rho_1^2 = \rho_2^2 = (\rho_1\rho_2)^2 \rangle \cong Q_8$, the quaternion group of order 8, which is indeed the case since Van Buskirk, [VB66], proved that $P_2(\mathbb{R}P^2)$ is isomorphic Q_8.

Suppose that, for some $n \geq 2$, the group $P_n(\mathbb{R}P^2)$ has the given presentation.

We consider the corresponding Fadell–Neuwirth short exact sequence, where the map $\bar{p}_{n+1,n}$ can be considered geometrically as the epimorphism that forgets the last strand:

$$1 \longrightarrow \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow P_{n+1}(\mathbb{R}P^2) \stackrel{\bar{p}_{n+1,n}}{\longrightarrow} P_n(\mathbb{R}P^2) \longrightarrow 1.$$

For reasons of symmetry we take the free group $\ker(\bar{p}_{n+1,n}) \cong \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ to have the following presentation with $n+1$ generators and a single relation:

$$\langle \rho_{n+1}, B_{1,n+1}, B_{2,n+1}, \dots, B_{n,n+1} \mid \rho_{n+1}(B_{1,n+1}B_{2,n+1}\cdots B_{n,n+1})\rho_{n+1} = 1 \rangle.$$

In order to obtain a presentation of $P_{n+1}(\mathbb{R}P^2)$, we will apply standard techniques for obtaining presentations of group extensions as described in [Joh97] (page 139).

Based on these techniques, the required generating set of $P_{n+1}(\mathbb{R}P^2)$ is obtained by taking the union of the generators of $\pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ together with coset representatives of the generators of $P_n(\mathbb{R}P^2)$, namely the generators ρ_k, for $1 \le k \le n$ and $B_{i,j}$, for $1 \le i < j \le n$. Thus the resulting generating set of $P_{n+1}(\mathbb{R}P^2)$ is

$$\langle B_{i,j}, \rho_k \mid \text{for } 1 \le i < j \le n+1,\ 1 \le k \le n+1 \rangle.$$

The relations that we obtain for the presentation of $P_{n+1}(\mathbb{R}P^2)$ arise from the following three classes of relations.

The first class consists of the relations of $\ker(\bar{p}_{n+1,n})$, which in this case, it is the following single relation: $\rho_{n+1}(B_{1,n+1}B_{2,n+1}\cdots B_{n,n+1})\rho_{n+1} = 1$.

The second class of relations is obtained by rewriting the relations of $P_n(\mathbb{R}P^2)$ in terms of the corresponding coset representatives in $P_{n+1}(\mathbb{R}P^2)$, and then expressing the resulting elements as a word in the generators of $\ker(\bar{p}_{n+1,n})$. In this way, one may see that all of the relations of $P_n(\mathbb{R}P^2)$ are lifted directly to relations of $P_{n+1}(\mathbb{R}P^2)$, except for the relation

$$\rho_i(B_{1,i}\cdots B_{i-1,i}) = (B_{i,i+1}\cdots B_{i,n})\rho_i^{-1}, \text{ for } 1 \le i \le n,$$

which may be rewritten as $\rho_i(B_{1,i}\cdots B_{i-1,i})\rho_i(B_{i,i+1}\cdots B_{i,n})^{-1} = 1$, for $1 \le i \le n$. For the same values of i we have:

$$\rho_i(B_{1,i}\cdots B_{i-1,i})\rho_i(B_{i,i+1}\cdots B_{i,n})^{-1} = \Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big)^{-1} B_{i,n+1} \Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big),$$

see Figure 2.6. It follows that $\rho_i(B_{1,i}\cdots B_{i-1,i})\rho_i = \Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big)^{-1} B_{i,n+1} \Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big)(B_{i,i+1}\cdots B_{i,n})$, for $1 \le i \le n$. Moreover, $\Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big)^{-1} B_{i,n+1} \Big(\prod_{l=i+1}^{n} B_{l,n+1}\Big)(B_{i,i+1}\cdots B_{i,n}) = (B_{i,i+1}\cdots B_{i,n})B_{i,n+1}$, see Figure 2.7. Thus, $\rho_i(B_{1,i}\cdots B_{i-1,i})\rho_i = (B_{i,i+1}\cdots B_{i,n})B_{i,n+1}$ and finally we get that $\rho_i(B_{1,i}\cdots B_{i-1,i}) = (B_{i,i+1}\cdots B_{i,n})B_{i,n+1}\rho_i^{-1}$, for $1 \le i \le n$. Combining this relation with the single relation of the presentation of $\ker(\bar{p}_{n+1,n})$, we obtain the following relation:

$$\rho_i(B_{1,i}\cdots B_{i-1,i}) = (B_{i,i+1}\cdots B_{i,n}B_{i,n+1})\rho_i^{-1}, \text{ for } 1 \le i \le n+1.$$

This is relation (ii) of the given presentation of $P_{n+1}(\mathbb{R}P^2)$.

The third class of relations is obtained by rewriting the conjugates of the generators of $\ker(\bar{p}_{n+1,n})$ by the coset representatives of the generators of $P_n(\mathbb{R}P^2)$ in $P_{n+1}(\mathbb{R}P^2)$ in terms of the generators of $\ker(\bar{p}_{n+1,n})$. Thus, we obtain:

(a) The Artin relations for $B_{i,j}$. For all $1 \le i < j \le n$ and $1 \le l \le n$:

$$B_{i,j}B_{l,n+1}B_{i,j}^{-1} = \begin{cases} B_{l,n+1}, & \text{for } l < i \text{ or } j < l. \\ B_{l,n+1}^{-1}B_{i,n+1}^{-1}B_{l,n+1}B_{i,n+1}B_{l,n+1}, & \text{for } l = j. \\ B_{j,n+1}^{-1}B_{l,n+1}B_{j,n+1}, & \text{for } l = i. \\ B_{j,n+1}^{-1}B_{i,n+1}^{-1}B_{j,n+1}B_{i,n+1}B_{l,n+1}B_{i,n+1}^{-1}B_{j,n+1}^{-1}B_{i,n+1}B_{j,n+1}, & \text{for } i < l < j. \end{cases}$$

We can obtain these relations geometrically or using the presentation of P_n given in Proposition 1.1.11, but with conjugation on the right. One can consider the generators $B_{i,j}$ to be words in P_n and using the homomorphism $P_n \to P_n(\mathbb{R}P^2)$ induced by an inclusion $\mathbb{D}^2 \subseteq \mathbb{R}P^2$, these same relations also hold in $P_n(\mathbb{R}P^2)$.

(b) For all $1 \le i < j \le n$:

$$B_{i,j}\rho_{n+1}B_{i,j}^{-1} = \rho_{n+1}.$$

(c) For all $1 \le k \le n$:

$$\rho_k\rho_{n+1}\rho_k^{-1} = \Big(\prod_{l=k+1}^{n} B_{l,n+1}\Big)^{-1} B_{k,n+1}\Big(\prod_{l=k+1}^{n} B_{l,n+1}\Big)\rho_{n+1}.$$

(d) For all $1 \le k \le n,\ 1 \le l \le n$:

$$\rho_k B_{l,n+1}\rho_k^{-1} = \begin{cases} B_{l,n+1}, & \text{for } k < l. \\ \rho_{n+1}^{-1}\Big(\prod_{j=l+1}^{n} B_{j,n+1}\Big)^{-1} B_{l,n+1}^{-1}\Big(\prod_{j=l+1}^{n} B_{j,n+1}\Big)\rho_{n+1} & \text{for } k = l. \\ A_{n+1,k}B_{l,n+1}A_{n+1,k}^{-1} & \text{for } l < k, \end{cases}$$

where $A_{n+1,k} \coloneqq \rho_{n+1}^{-1}\Big(\prod_{j=k+1}^{n} B_{j,n+1}\Big)^{-1} B_{k,n+1}^{-1}\Big(\prod_{j=k+1}^{n} B_{j,n+1}\Big)\rho_{n+1}$.

Relations (b), (c), (d) are obtained geometrically, as one can see, for the general case, Figures 2.8, 2.9, 2.10, respectively.

We can now clearly see that relation (i) of the presentation of $P_{n+1}(\mathbb{R}P^2)$ is obtained from the lifted relation (i) of $P_n(\mathbb{R}P^2)$, together with relation (a) from the third class of relations. Relation (ii), namely the surface relations, of the presentation of $P_{n+1}(\mathbb{R}P^2)$ has already been obtained by combining a relation from the second class with the single relation from the first class of relations. Relation (iii) is obtained from the lifted relation (iii) of $P_n(\mathbb{R}P^2)$, together with relation (c) from the third class of relations. Finally, relation (iv) of the presentation of $P_{n+1}(\mathbb{R}P^2)$ can be obtained from the lifted relation (iv) of $P_n(\mathbb{R}P^2)$, together with the relations (b) and (d) of the third class.

To sum up, the set of generators and the set of relations, that we have obtained, coincide with the given presentation for $P_{n+1}(\mathbb{R}P^2)$ and it follows by induction that the given presentation is valid for $P_n(\mathbb{R}P^2)$, for every $n \in \mathbb{N}$. □

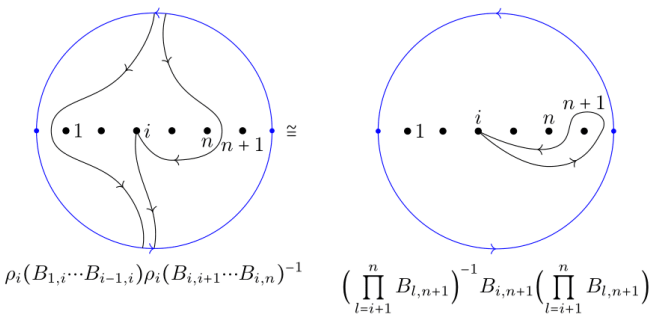


Figure 2.6 Relation (iia).

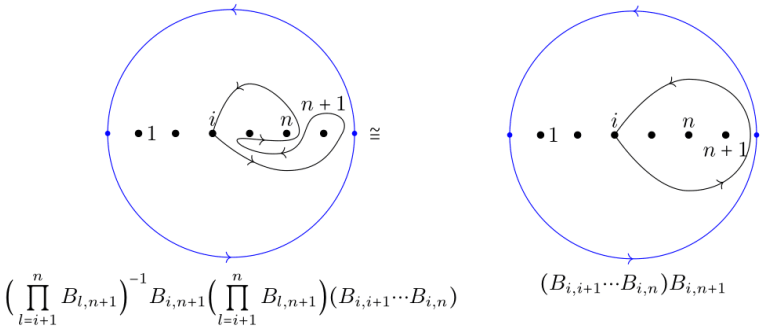


Figure 2.7 Relation (iib).

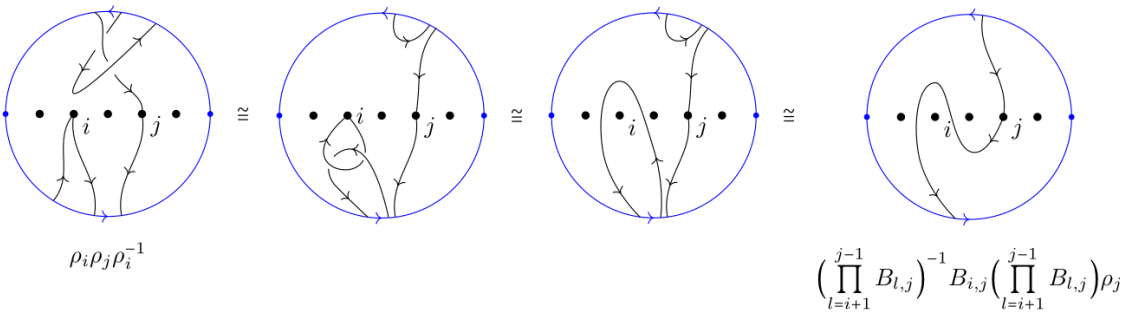


Figure 2.8 Relation (iii).

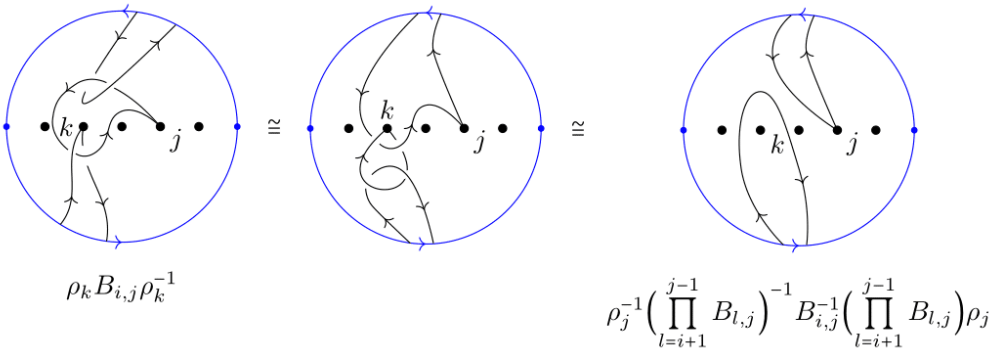


Figure 2.9 Relation (iv) for $k = i$.

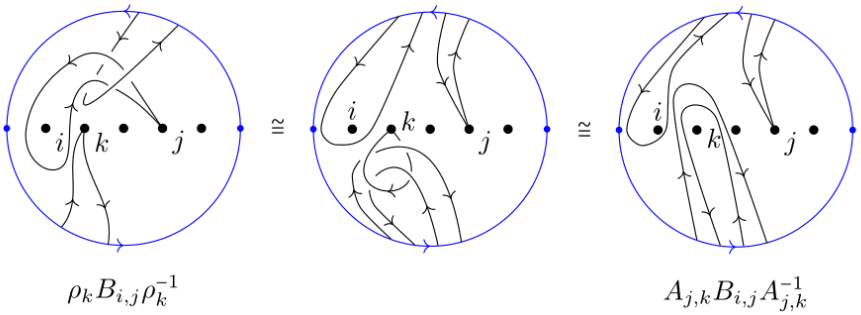


Figure 2.10 Relation (iv) for $i < k < j$.

Next, we give a presentation of the group $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$.

Proposition 2.2.4. *For $n, m \geq 1$, the following constitutes a presentation of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$.*

Generators: *$B_{i,j}$, for $1 \leq i < j$, $n+1 \leq j \leq n+m$ and ρ_k, for $n+1 \leq k \leq n+m$.*

Relations:

(i) *For $1 \leq i < j$ and $1 \leq k < l$, where $n+1 \leq j < l \leq n+m$,*

$$B_{i,j}B_{k,l}B_{i,j}^{-1} = \begin{cases} B_{k,l}, & \text{for } k < i \text{ or } j < k. \\ B_{k,l}^{-1}B_{i,l}^{-1}B_{k,l}B_{i,l}B_{k,l}, & \text{for } i < k = j < l. \\ B_{j,l}^{-1}B_{k,l}B_{j,l}, & \text{for } k = i < j < l. \\ B_{j,l}^{-1}B_{i,l}^{-1}B_{j,l}B_{i,l}B_{k,l}B_{i,l}^{-1}B_{j,l}^{-1}B_{i,l}B_{j,l}, & \text{for } i < k < j < l. \end{cases}$$

(ii) *For $n+1 \leq k \leq n+m$,*
$\rho_k(B_{1,k} \cdots B_{k-1,k}) = (B_{k,k+1} \cdots B_{k,n+m})\rho_k^{-1}$, *called surface relations.*

(iii) *For $n+1 \leq k < l \leq n+m$,*
$$\rho_k\rho_l\rho_k^{-1} = \Big(\prod_{i=k+1}^{l-1} B_{i,l}\Big)^{-1} B_{k,l}\Big(\prod_{i=k+1}^{l-1} B_{i,l}\Big)\rho_l.$$

(iv) *For $1 \leq i < j$, $n+1 < j \leq n+m$, $n+1 \leq k \leq n+m$ and $k \neq j$,*

$$\rho_k B_{i,j}\rho_k^{-1} = \begin{cases} B_{i,j}, & \text{for } j < k \text{ or } k < i. \\ \rho_j^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)^{-1} B_{i,j}^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)\rho_j & \text{for } k = i. \\ A_{j,k}B_{i,j}A_{j,k}^{-1} & \text{for } i < k < j, \end{cases}$$

where $A_{j,k} := \rho_j^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)^{-1} B_{k,j}^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)\rho_j$.

Proof. We proceed using the same techniques as in the proof of 2.2.1; applying induction on m, where $m \in \mathbb{N}$ and using standard results concerning the presentation of group extensions [Joh97] (page 139).

Let $n \geq 1$. For $m = 1$ we have that

$$\begin{aligned} P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) &= \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \\ &= \langle \rho_{n+1}, B_{1,n+1}, B_{2,n+1}, \dots, B_{n,n+1} \mid \rho_{n+1}(B_{1,n+1}B_{2,n+1} \cdots B_{n,n+1})\rho_{n+1} = 1 \rangle. \end{aligned}$$

This presentation of $P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ coincides with that given in the statement and therefore the presentation is valid for $m = 1$.

Suppose that $m \geq 2$, and that for some m the group $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ has the given presentation. We will obtain a presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, based on the following short exact sequence:

$$1 \to P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{n+m}\}) \longrightarrow P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \xrightarrow{\bar{p}_{m+1,m}} P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to 1,$$

where $\bar{p}_{m+1,m}$ can be considered geometrically as the epimorphism that forgets the last strand.

Based on standard techniques for obtaining presentations of group extensions, the required generating set of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is obtained by taking the union of the generators of $\ker(\bar{p}_{m+1,m})$, namely of the group $P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{n+m}\}) =$
$\langle \rho_{n+m+1}, B_{1,n+m+1}, \dots, B_{n+m,n+m+1} \mid \rho_{n+m+1}(B_{1,n+m+1} \cdots B_{n+m,n+m+1})\rho_{n+m+1} = 1 \rangle$, together with coset representatives of the generators of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, namely the generators ρ_k for $n+1 \leq k \leq n+m$ and $B_{i,j}$, for $1 \leq i < j$, $n+1 \leq j \leq n+m$. Thus the resulting generating set of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is

$$\langle B_{i,j},\ \rho_k \mid \text{for } 1 \leq i < j \leq n+m+1,\ 1 \leq k \leq n+m+1 \rangle.$$

The relations that we attain for the presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ arise from the following three classes of relations.

The first class consists of the relations of $\ker(\bar{p}_{m+1,m})$, which in this case, it is the following single relation: $\rho_{n+m+1}(B_{1,n+m+1}\cdots B_{n+m,n+m+1})\rho_{n+m+1} = 1$.

The second class of relations is obtained by rewriting the relations of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ in terms of the corresponding coset representatives in $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, and then expressing the resulting elements as a word in the generators of $\ker(\bar{p}_{m+1,m})$. In this way, one may see that all of the relations of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ are lifted directly to relations of $P_{n+1}(\mathbb{R}P^2)$, except for the surface relations, $\rho_k(B_{1,k}\cdots B_{k-1,k}) = (B_{k,k+1}\cdots B_{k,n+m})\rho_k^{-1}$, for $n+1 \le k \le n+m$. Expressing the relation $\rho_k(B_{1,k}\cdots B_{k-1,k})\rho_k(B_{k,k+1}\cdots B_{k,n+m})^{-1} = 1$, for $n+1 \le k \le n+m$ as a word in the generators of $\ker(\bar{p}_{m+1,m})$, we obtain that $\rho_k(B_{1,k}\cdots B_{k-1,k})\rho_k(B_{k,k+1}\cdots B_{k,n+m})^{-1} = \Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)^{-1} B_{k,n+m+1}\Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)$, for $n+1 \le k \le n+m$. It follows $\rho_k(B_{1,k}\cdots B_{k-1,k})\rho_k = \Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)^{-1} B_{k,n+m+1}\Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)(B_{k,k+1}\cdots B_{k,n+m})$, for $n+1 \le i \le n+m$. It holds that $\Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)^{-1} B_{k,n+m+1}\Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)(B_{k,k+1}\cdots B_{k,n+m}) = (B_{k,k+1}\cdots B_{k,n+m})B_{k,n+m+1}$, a relation which is obtained geometrically. Thus, $\rho_k(B_{1,k}\cdots B_{k-1,k})\rho_k = (B_{k,k+1}\cdots B_{k,n+m})B_{k,n+m+1}$ and finally we get that $\rho_k(B_{1,k}\cdots B_{k-1,k}) = (B_{k,k+1}\cdots B_{k,n+m})B_{k,n+m+1}\rho_k^{-1}$, for $n+1 \le k \le n+m$. Combining this relation with the single relation of the presentation of $\ker(\bar{p}_{m+1,m})$, we obtain the relation:

$$\rho_k(B_{1,k}\cdots B_{k-1,k}) = (B_{k,k+1}\cdots B_{k,n+m+1})\rho_k^{-1}, \text{ for } 1 \le k \le n+m+1.$$

This is relation (ii) of the given presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$.

The third class of relations is obtained by rewriting the conjugates of the generators of $\ker(\bar{p}_{m+1,m})$ by the coset representatives of the generators of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ in $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, in terms of the generators of $\ker(\bar{p}_{m+1,m})$. Thus, we obtain:

(a) The Artin relations for $B_{i,j}$. For all $1 \le i \le j$, $n+1 \le j \le n+m$ and $1 \le l \le n+m$:

$$B_{i,j}B_{l,n+m+1}B_{i,j}^{-1} = \begin{cases} B_{l,n+m+1}, \text{ for } l < i \text{ or } j < l. \\ B_{l,n+m+1}^{-1}B_{i,n+m+1}^{-1}B_{l,n+m+1}B_{i,n+m+1}B_{l,n+m+1}, \text{ for } l = j. \\ B_{j,n+m+1}^{-1}B_{l,n+m+1}B_{j,n+m+1}, \text{ for } l = i. \\ B_{j,n+m+1}^{-1}B_{i,n+m+1}^{-1}B_{j,n+m+1}B_{i,n+m+1}B_{l,n+m+1}\cdot \\ \qquad \cdot B_{i,n+m+1}^{-1}B_{j,n+m+1}^{-1}B_{i,n+m+1}B_{j,n+m+1}, \text{ for } i < l < j. \end{cases}$$

(b) For all $1 \le i < j$, $n+1 \le j \le n+m$:
$B_{i,j}\rho_{n+m+1}B_{i,j}^{-1} = \rho_{n+m+1}$.

(c) For all $n+1 \le k \le n+m$:
$\rho_k\rho_{n+m+1}\rho_k^{-1} = \Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)^{-1} B_{k,n+m+1}\Big(\prod_{l=k+1}^{n+m} B_{l,n+m+1}\Big)\rho_{n+m+1}$.

(d) For all $n+1 \le k \le n+m$, $1 \le l \le n+m$:

$$\rho_k B_{l,n+m+1}\rho_k^{-1} = \begin{cases} B_{l,n+m+1}, & \text{for } k < l. \\ \rho_{n+m+1}^{-1}\Big(\prod_{s=k+1}^{n+m} B_{s,n+m+1}\Big)^{-1} B_{l,n+m+1}^{-1}\Big(\prod_{s=k+1}^{n+m} B_{s,n+m+1}\Big)\rho_{n+m+1} & \text{for } k = l. \\ A_{n+m+1,k}B_{l,n+m+1}A_{n+m+1,k}^{-1} & \text{for } l < k, \end{cases}$$

where $A_{n+m+1,k} := \rho_{n+m+1}^{-1}\Big(\prod_{s=k+1}^{n+m} B_{s,n+m+1}\Big)^{-1} B_{k,n+m+1}^{-1}\Big(\prod_{s=k+1}^{n+m} B_{s,n+m+1}\Big)\rho_{n+m+1}$.

Once again, relations (a), (b), (c), (d) are obtained geometrically.

We can now clearly see that relation (i) of the presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$ is obtained from the lifted relation (i) of $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, together with relation (a) from the third class of relations. Relation (ii), namely the surface relations, of the presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$ has already been obtained as we already saw, by combining a relation from the second class with the single relation from the first class of relations. Relation (iii) is obtained from the lifted relation (iii) of $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, together with relation (c) from the third class of relations. Finally, relation (iv) of the presentation of $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$ can be obtained from the lifted relation (iv) of $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, together with the relations (b) and (d) of the third class.

To sum up, the set of generators and the set of relations, that we have obtained, coincide with the given presentation applied to $P_{m+1}(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$ and it follows by induction that the given presentation is valid for $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, for every $m \in \mathbb{N}$. □

In order to give a presentation for $B_{n,m}(\mathbb{R}P^2)$, we first determine a presentation of the group $B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$.

Proposition 2.2.5. *For $n, m \geq 1$, the following constitutes a presentation of $B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$.*

Generators: *$B_{i,j}$, for $1 \leq i \leq n$, $n+1 \leq j \leq n+m$, ρ_k, for $n+1 \leq k \leq n+m$, and σ_l, for $1 \leq l \leq m-1$.*

Relations:

(i) For $1 \leq i,k \leq n$ and $n+1 \leq j < l \leq n+m$,

$$B_{i,j}B_{k,l}B_{i,j}^{-1} = \begin{cases} B_{k,l}, & \text{for } k < i. \\ B_{j,l}^{-1}B_{k,l}B_{j,l}, & \text{for } k = i. \\ B_{j,l}^{-1}B_{i,l}^{-1}B_{j,l}B_{i,l}B_{k,l}B_{i,l}^{-1}B_{j,l}^{-1}B_{i,l}B_{j,l}, & \text{for } i < k. \end{cases}$$

Moreover, for $k = j$,

$$\begin{aligned} B_{i,j}B_{k,l}B_{i,j}^{-1} =& (\sigma_{l-1-n}\cdots\sigma_{j+1-n}\sigma_{j-n}^{-2}\sigma_{j+1-n}^{-1}\cdots\sigma_{l-1-n}^{-1})B_{i,l}^{-1}(\sigma_{l-1-n}\cdots\sigma_{j+1-n}\sigma_{j-n}^{2}\sigma_{j+1-n}^{-1}\cdots\sigma_{l-1-n}^{-1}) \\ & B_{i,l}(\sigma_{l-1-n}\cdots\sigma_{j+1-n}\sigma_{j-n}^{2}\sigma_{j+1-n}^{-1}\cdots\sigma_{l-1-n}^{-1}). \end{aligned}$$

(ii) For $n+1 \leq k \leq n+m$,
$\rho_k\Big(\prod_{i=1}^{n} B_{i,k}\Big)(\sigma_{k-1-n}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{k-1-n}) = (\sigma_{k-n}\cdots\sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2}\cdots\sigma_{k-n})\rho_k^{-1}$.

(iii) For $n+1 \leq k < j \leq n+m$,
$\rho_k\rho_j\rho_k^{-1} = (\sigma_{j-1-n}^{-1}\cdots\sigma_{k+1-n}^{-1}\sigma_{k-n}^2\sigma_{k+1-n}\cdots\sigma_{j-1-n})\rho_j$,
$\rho_k^{-1}\rho_j\rho_k = \rho_j(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^2\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1})$.

(iv) For $1 \leq i \leq n$, $n+1 \leq j \leq n+m$, $n+1 \leq k \leq n+m$ and $k \neq j$

$$\rho_k B_{i,j}\rho_k^{-1} = \begin{cases} B_{i,j}, & \text{for } j < k. \\ T_{k,j}^{-1}B_{i,j}T_{k,j}, & \text{for } n+1 \leq k < j, \end{cases}$$

where $T_{k,j} := \rho_j^{-1}(\sigma_{j-1-n}^{-1}\cdots\sigma_{k+1-n}^{-1}\sigma_{k-n}^2\sigma_{k+1-n}\cdots\sigma_{j-1-n})\rho_j$.

(v) For $1 \leq r,s \leq m-1$,
$\sigma_r\sigma_s = \sigma_s\sigma_r$, *for* $|r-s| > 1$,
$\sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}$.

(vi) *For* $1 \le i \le n,\ n+1 \le j \le n+m,\ 1 \le r \le m-1$,

$$\sigma_r B_{i,j} \sigma_r^{-1} = \begin{cases} B_{i,j}, & \text{for } r \neq j-n-1, j-n. \\ \sigma_{j-n-1}^2 B_{i,j-1} \sigma_{j-n-1}^{-2}, & \text{for } r = j-n-1,\ n+2 \le j \le n+m. \\ B_{i,j+1}, & \text{for } r = j-n,\ n+1 \le j \le n+m-1. \end{cases}$$

(vii) *For* $1 \le r \le m-1,\ n+1 \le k \le n+m$,

$$\sigma_r \rho_k \sigma_r^{-1} = \begin{cases} \rho_k, & \text{for } r \neq k-n, k-1-n. \\ \sigma_{k-n}^2 \rho_{k+1}, & \text{for } r = k-n. \\ \rho_{k-1} \sigma_{k-1-n}^{-2}, & \text{for } r = k-1-n. \end{cases}$$

Remark 2.2.6. In the above relations, for $n+1 \le i < j \le n+m$ the element $B_{i,j}$, which does not appear in the list of generators, should be rewritten as:

$$B_{i,j} = \sigma_{j-1-n} \cdots \sigma_{i+1-n} \sigma_{i-n}^2 \sigma_{i+1-n}^{-1} \cdots \sigma_{j-1-n}^{-1},$$

and in particular $B_{n+r,n+r+1} = \sigma_r^2$, for $1 \le r \le m-1$. Moreover, for $n+1 \le k < j \le n+m$ it holds that $\prod_{i=k}^{j-1} B_{i,j} = \sigma_{j-1-n} \cdots \sigma_{k+1-n} \sigma_{k-n}^2 \sigma_{k+1-n} \cdots \sigma_{j-1-n}$.

Proof. Once again, we will proceed by applying standard results concerning the presentation of an extension, [Joh97] (page 139), based on the following short exact sequence:

$$1 \to P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to S_m \to 1.$$

We have already a presentation of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, from Proposition 2.2.4, and of the symmetric group given as follows:

$$S_m = \langle \sigma, \dots, \sigma_{m-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \text{for } 1 \le i \le n-2,\ \sigma_i^2 = 1,\ \sigma_i \sigma_j = \sigma_j \sigma_i,\ \text{for } 1 \le i \neq j \le n-1 \rangle.$$

So, the generators of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ together with coset representatives of the generators of S_m, form a generating set of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$. It follows that this generating set is that given in the statement, with σ_l, for $1 \le l \le m-1$, being the coset representatives of the generators of the symmetric group. Observe that, from Remark 2.2.2, $B_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$, and so, for $n+1 \le i < j \le n+m$ we can rewrite the generator $B_{i,j}$ as $B_{i,j} = \sigma_{j-1-n} \cdots \sigma_{i+1-n} \sigma_{i-n}^2 \sigma_{i+1-n}^{-1} \cdots \sigma_{j-1-n}^{-1}$, as we see in Remark 2.2.6. Thus, one may delete these elements from the set and one obtains the required set of generators.

We now determine the three classes of relations. The first class is obtained by taking the relations of $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ given in Proposition 2.2.4:

- For $1 \le i, k \le n$ and $n+1 \le j < l \le n+m$,

$$B_{i,j} B_{k,l} B_{i,j}^{-1} = \begin{cases} B_{k,l}, & \text{for } k < i. \\ B_{j,l}^{-1} B_{k,l} B_{j,l}, & \text{for } k = i. \\ B_{j,l}^{-1} B_{i,l}^{-1} B_{j,l} B_{i,l} B_{k,l} B_{i,l}^{-1} B_{j,l}^{-1} B_{i,l} B_{j,l}, & \text{for } i < k. \end{cases}$$

  For $k = j$, which means that $n+1 \le k$, then:

$$\begin{aligned} B_{i,j} B_{k,l} B_{i,j}^{-1} =& (\sigma_{l-1-n} \cdots \sigma_{j+1-n} \sigma_{j-n}^{-2} \sigma_{j+1-n}^{-1} \cdots \sigma_{l-1-n}^{-1}) B_{i,l}^{-1} (\sigma_{l-1-n} \cdots \sigma_{j+1-n} \sigma_{j-n}^2 \sigma_{j+1-n}^{-1} \cdots \sigma_{l-1-n}^{-1}) B_{i,l} \\ & (\sigma_{l-1-n} \cdots \sigma_{j+1-n} \sigma_{j-n}^2 \sigma_{j+1-n}^{-1} \cdots \sigma_{l-1-n}^{-1}). \end{aligned}$$

- For $n+1 \le k < j \le n+m$,

$$\begin{aligned} \rho_k \rho_j \rho_k^{-1} &= \Big(\prod_{i=k+1}^{j-1} B_{i,j} \Big)^{-1} B_{k,j} \Big(\prod_{i=k+1}^{j-1} B_{i,j} \Big) \rho_j = \Sigma_{k,j}^{-1} (\sigma_{j-1-n} \cdots \sigma_{k+1-n} \sigma_{k-n}^2 \sigma_{k+1-n}^{-1} \cdots \sigma_{j-1-n}^{-1}) \Sigma_{k,j} \rho_j \\ &= (\sigma_{j-1-n}^{-1} \cdots \sigma_{k+1-n}^{-1} \sigma_{k-n}^2 \sigma_{k+1-n} \cdots \sigma_{j-1-n}) \rho_j, \end{aligned} \tag{3}$$

  where $\Sigma_{k,j} := \sigma_{j-1-n} \cdots \sigma_{k+2-n} \sigma_{k+1-n}^2 \sigma_{k+2-n} \cdots \sigma_{j-1-n}$.

- For $1 \le i \le n,\ n+1 \le j,k \le n+m,\ k \ne j$,

$$\rho_k B_{i,j} \rho_k^{-1} = \begin{cases} B_{i,j}, & \text{for } j < k. \\ T_{k,j}^{-1} B_{i,j} T_{k,j}, & \text{for } n+1 \le k < j, \end{cases}$$

where $T_{k,j} := \rho_j^{-1}(\sigma_{j-1-n}^{-1} \cdots \sigma_{k+1-n}^{-1} \sigma_{k-n}^{2} \sigma_{k+1-n} \cdots \sigma_{j-1-n})\rho_j$.
If $i = k$, which means that $n+1 \le i$, then the relation

$$\rho_k B_{k,j} \rho_k^{-1} = \rho_j^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)^{-1} B_{k,j}^{-1}\Big(\prod_{l=k+1}^{j-1} B_{l,j}\Big)\rho_j$$

becomes as follows:

$$\begin{aligned} \rho_k(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^2\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1})\rho_k^{-1} = &\rho_j^{-1}(\sigma_{j-1-n}^{-1}\cdots\sigma_{k+2-n}^{-1}\sigma_{k+1-n}^{-2}\sigma_{k+2-n}^{-1}\cdots\sigma_{j-1-n}^{-1}) \\ &(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^{-2}\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1}) \\ &(\sigma_{j-1-n}\cdots\sigma_{k+2-n}\sigma_{k+1-n}^{2}\sigma_{k+2-n}\cdots\sigma_{j-1-n})\rho_j. \end{aligned}$$

Thus,

$$\rho_k(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^2\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1})\rho_k^{-1} = \rho_j^{-1}(\sigma_{j-1-n}^{-1}\cdots\sigma_{k+1-n}^{-1}\sigma_{k-n}^{-2}\sigma_{k+1-n}\cdots\sigma_{j-1-n})\rho_j.$$

Using relation (3) it follows that

$$\rho_k(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^2\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1})\rho_k^{-1} = \rho_j^{-1}(\rho_j\rho_k\rho_j^{-1}\rho_k^{-1})\rho_j,$$

and we conclude the following:

$$\rho_k^{-1}\rho_j\rho_k = \rho_j(\sigma_{j-1-n}\cdots\sigma_{k+1-n}\sigma_{k-n}^2\sigma_{k+1-n}^{-1}\cdots\sigma_{j-1-n}^{-1}).$$

This relation is a right conjugation of ρ_j by ρ_k, while relation (3) is a left conjugation.

- For $n+1 \le k \le n+m$, the surface relations $\rho_k(B_{1,k}\cdots B_{k-1,k}) = (B_{k,k+1}\cdots B_{k,n+m})\rho_k^{-1}$ become $\rho_k\Big(\prod_{i=1}^{n} B_{i,k}\Big)\Big(\prod_{j=n+1}^{k-1} B_{j,k}\Big) = \Big(\prod_{l=k+1}^{n+m} B_{k,l}\Big)\rho_k^{-1}$, and finally we obtain the following:

$$\rho_k\Big(\prod_{i=1}^{n} B_{i,k}\Big)(\sigma_{k-1-n}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{k-1-n}) = (\sigma_{k-n}\cdots\sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2}\cdots\sigma_{k-n})\rho_k^{-1}.$$

The second class of relations is obtained by rewriting the relations of S_m in terms of the coset representatives and expressing the corresponding elements as a word in $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$. Doing so, we have the following relations:

(a) $\sigma_r\sigma_s = \sigma_s\sigma_r$, for $|r-s| > 1$,

(b) $\sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}$,

(c) $\sigma_r^2 = B_{n+r,n+r+1}$, for $1 \le r \le m-1$, which appears in Remark 2.2.6.

The third class of relations is obtained by rewriting the conjugates of the generators of $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, by the coset representatives, in terms of the generators of $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$. We obtain the following relations:

- For $1 \le r \le m-1$ and $1 \le i \le n,\ n+1 \le j \le n+m$,

$$\sigma_r B_{i,j} \sigma_r^{-1} = \begin{cases} B_{i,j}, & \text{for } r \ne j-n-1, j-n. \\ B_{i,j+1}, & \text{for } r = j-n \text{ and } n+1 \le j \le n+m-1. \\ \sigma_{j-n-1}^2 B_{i,j-1} \sigma_{j-n-1}^{-2}, & \text{for } r = j-n-1 \text{ and } n+2 \le j \le n+m. \end{cases} \tag{4}$$

- For $1 \le r \le m-1$, $n+1 \le k \le n+m$,

$$\sigma_r \rho_k \sigma_r^{-1} = \begin{cases} \rho_k, & \text{for } r \neq k-n, k-1-n. \\ \sigma_{k-n}^2 \rho_{k+1}, & \text{for } r = k-n. \\ \rho_{k-1}\sigma_{k-1-n}^{-2}, & \text{for } r = k-1-n. \end{cases}$$

The relations of the third class are obtained geometrically

The resulting relations are those given in the statement and this completes the proof. □

Finally, we are ready to give a presentation of the group $B_{n,m}(\mathbb{R}P^2)$.

Theorem 2.2.7. *For $n \ge 2$ and $m \ge 1$, the following constitutes a presentation of $B_{n,m}(\mathbb{R}P^2)$.*

Generators: $B_{i,j}$, *for* $1 \le i \le n$ *and* $n+1 \le j \le n+m$,
ρ_k, *for* $n+1 \le k \le n+m$,
σ_l, *for* $1 \le l \le m-1$,
τ_s, *for* $1 \le s \le n-1$ *and*
q_t, *for* $1 \le t \le n$.

Relations:

(I) *The relations (i)-(vii) of Proposition 2.2.5.*

(II) *For* $1 \le i \le n-1$, $1 \le j \le n$,
$\tau_i\tau_j = \tau_i\tau_j$, *for* $|i-j| > 1$,
$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$,
$\tau_i q_j = q_j\tau_i$, *for* $j \neq i, i+1$,
$q_i = \tau_i q_{i+1}\tau_i$,
$\tau_i^2 = q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i$,
$q_1^2 = (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})(B_{n,n+1}B_{n,n+2}\cdots B_{n,n+m-1}B_{n,n+m})(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1)$.

(III) *(a)* *For* $1 \le l \le m-1$, $1 \le s \le n-1$ *and* $1 \le t \le n$,
$\sigma_l\tau_s = \tau_s\sigma_l$,
$\sigma_l q_t = q_t\sigma_l$.

(b) *For* $n+1 \le k \le n+m$, $1 \le s \le n-1$ *and* $1 \le t \le n$,
$\rho_k\tau_s = \tau_s\rho_k$,
$q_t\rho_k q_t^{-1} = E_{t,k}\rho_k$,
where $E_{t,k} := N_{t,k}^{-1}B_{t,k}N_{t,k}$ *and* $N_{t,k} := \Big(\prod_{l=t+1}^{n} B_{l,k}\Big)(\sigma_{k-1-n}\cdots\sigma_1^2\cdots\sigma_{k-1-n})$.

(c) *For* $1 \le s \le n-1$ *and* $1 \le i \le n < j \le n+m$,

$$\tau_s B_{i,j}\tau_s^{-1} = \begin{cases} B_{i,j}, & \text{for } s \neq i-1, i. \\ B_{i,j}^{-1}B_{i-1,j}B_{i,j}, & \text{for } s = i-1. \\ B_{i+1,j}, & \text{for } s = i. \end{cases}$$

(d) *For* $1 \le t \le n$ *and* $1 \le i \le n < k \le n+m$,

$$q_t B_{i,k} q_t^{-1} = \begin{cases} B_{i,k}, & \text{for } t < i. \\ (N_{t,k}\rho_k)^{-1}B_{i,k}^{-1}(N_{t,k}\rho_k), & \text{for } t = i. \\ (\rho_k^{-1}E_{t,k}\rho_k)^{-1}B_{i,k}(\rho_k^{-1}E_{t,k}\rho_k), & \text{for } i < t. \end{cases}$$

Proof. Once more we will apply the same methods concerning the presentation of an extension, [Joh97] (page 139) based on the following short exact sequence:

$$1 \to B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \to 1,$$

where the map $\bar{q}_{n+m,n}$ can be considered geometrically as the epimorphism that forgets the last m strands. Let $q_1, \ldots, q_n$ and $\tau_1, \ldots, \tau_{n-1}$ be the standard generators of $B_n(\mathbb{R}P^2)$ as described in

Theorem 2.1.1. We keep the same notation and for simplicity we denote the corresponding coset representatives in $B_{n,m}(\mathbb{R}P^2)$ by $q_1,\dots,q_n$ and $\tau_1,\dots,\tau_{n-1}$. The union of these elements together with the generators of $B_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_n\})$ of Proposition 2.2.5 gives us the set of generators of $B_{n,m}(\mathbb{R}P^2)$.

As before we will obtain three classes of relations in $B_{n,m}(\mathbb{R}P^2)$. The first class of relations is the set of relations of $\ker(\bar{q}_{n+m,n})$, meaning of $B_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_n\})$, which are relations (I) of the statement.

The second class of relations is obtained by rewriting the relations of $B_n(\mathbb{R}P^2)$ in terms of the chosen coset representatives in $B_{n,m}(\mathbb{R}P^2)$ and expressing the corresponding elements as a word in $B_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_n\})$. Therefore we get the following:
for $1\leq i\leq n-1$, $1\leq j\leq n$ and $|i-j|>1$ we have $\tau_i\tau_j=\tau_i\tau_j$, $\tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1}$, and for $j\neq i,i+1$ we have $\tau_i q_j=q_j\tau_i$. Moreover, we obtain $q_i=\tau_i q_{i+1}\tau_i$ and $\tau_i^2=q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i$. The relation $q_1^2(\tau_1\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_1)^{-1}=1$ of $B_n(\mathbb{R}P^2)$ can be expressed as a word in $B_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_n\})$ as follows:

$$q_1^2(\tau_1\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_1)^{-1}=\Big(\prod_{j=n+1}^{n+m}(B_{n,j}^{-1}\cdots B_{2,j}^{-1})B_{1,j}(B_{2,j}\cdots B_{n,j})\Big). \tag{5}$$

We will come back to this relation later, as we will use some relations that we will see in the third class of relations, in order to simplify it.

The third class of relations is derived from conjugating the generators of $B_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_n\})$ by the coset representatives $q_1,\dots,q_n$ and $\tau_1,\dots,\tau_{n-1}$.

- By conjugating the generators σ_l, for $1\leq l\leq m-1$ and by using the standard Artin relations in $B_{n,m}(\mathbb{R}P^2)$, we obtain the following relations:
  for all $1\leq l\leq m-1$ and $1\leq s\leq n-1$, we have $\sigma_l\tau_s=\tau_s\sigma_l$. Moreover, we have $\sigma_l q_t=q_t\sigma_l$, for all $1\leq l\leq m-1$ and $1\leq t\leq n$. These relations correspond to relations $(III)(a)$ of the statement.

- By conjugating the generators ρ_k, for $n+1\leq k\leq n+m$, we have the following:
  for $n+1\leq k\leq n+m$, $1\leq s\leq n-1$ and $1\leq t\leq n$,
  $\rho_k\tau_s=\tau_s\rho_k$ and
  $q_t\rho_k q_t^{-1}=\Big(\big(\prod_{l=t+1}^{n}B_{l,k}\big)(\sigma_{k-1}\cdots\sigma_1^2\cdots\sigma_{k-1})\Big)^{-1}B_{t,k}\Big(\big(\prod_{l=t+1}^{n}B_{l,k}\big)(\sigma_{k-1}\cdots\sigma_1^2\cdots\sigma_{k-1})\Big)\rho_k$,
  where the term $\prod_{l=t+1}^{n}B_{l,k}$ corresponds to the first n points and the term $(\sigma_{k-1-n}\cdots\sigma_1^2\cdots\sigma_{k-1-n})$ corresponds to the following $k-n$ points. These relations correspond to relations $(III)(b)$ of the statement.

- By conjugating the generators $B_{i,j}$, for $1\leq i\leq n$ and $n+1\leq j\leq n+m$ the following relations arise:
  for $1\leq s\leq n-1$ and $1\leq i\leq n<j\leq n+m$,

$$\tau_s B_{i,j}\tau_s^{-1}=\begin{cases}B_{i,j}, & \text{for } s\neq i-1,i.\\ B_{i,j}^{-1}B_{i-1,j}B_{i,j}, & \text{for } s=i-1.\\ B_{i+1,j}, & \text{for } s=i.\end{cases} \tag{6}$$

  For $1\leq t\leq n$ and $1\leq i\leq n<k\leq n+m$,

$$q_t B_{i,k}q_t^{-1}=\begin{cases}B_{i,k}, & \text{for } t<i.\\ \rho_k^{-1}N_{t,k}^{-1}B_{i,k}^{-1}N_{t,k}\rho_k, & \text{for } t=i.\\ (\rho_k^{-1}N_{t,k}^{-1}B_{t,k}N_{t,k}\rho_k)^{-1}B_{i,k}(\rho_k^{-1}N_{t,k}^{-1}B_{t,k}N_{t,k}\rho_k). & \text{for } i<t.\end{cases}$$

  These relations correspond to relations $(III)(c)$ and $(III)(d)$ of the statement.

The relations of the third class are obtained geometrically.

We come back to relation (5). From relations (6), for any $n+1\leq j\leq n+m$ we obtain

$$(B_{n,j}^{-1}\cdots B_{2,j}^{-1})B_{1,j}(B_{2,j}\cdots B_{n,j})(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})=(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})B_{n,j},$$

and thus we get the following: $\Big(\prod_{j=n+1}^{n+m}(B_{n,j}^{-1}\cdots B_{2,j}^{-1})B_{1,j}(B_{2,j}\cdots B_{n,j})\Big)(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}) = (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})(B_{n,n+1}B_{n,n+2}\cdots B_{n,n+m-1}B_{n,n+m})$. Thus, relation (5) becomes

$$\begin{aligned} q_1^2 &= \Big(\prod_{j=n+1}^{n+m}(B_{n,j}^{-1}\cdots B_{2,j}^{-1})B_{1,j}(B_{2,j}\cdots B_{n,j})\Big)(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1) \\ &= (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})(B_{n,n+1}B_{n,n+2}\cdots B_{n,n+m-1}B_{n,n+m})(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1), \end{aligned}$$

which is the last relation of relations (II) of the statement.

As a result, we have obtained a generating set and a complete set of relations, which coincide with those given in the statement, and this completes the proof. □

Remark 2.2.8. Theorem 2.2.7, which gives a presentation of $B_{n,m}(\mathbb{R}P^2)$, holds for $n \geq 2$ and $m \geq 1$, since the short exat sequence

$$1 \to B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \to 1,$$

that we use in the proof, holds for $n \geq 2$. For $n = m = 1$, we have $B_{1,1}(\mathbb{R}P^2) = P_2(\mathbb{R}P^2)$, which is isomorphic to the quaternion group of order 8, see [VB66]. To obtain a presentation of $B_{n,m}(\mathbb{R}P^2)$ for $n = 1$ and $m \geq 2$, one can use the short exact sequence,

$$1 \to B_1(\mathbb{R}P^2 \setminus \{x_1,\dots,x_m\}) \to B_{1,m}(\mathbb{R}P^2) \xrightarrow{q_m} B_m(\mathbb{R}P^2) \to 1,$$

where q_m can be considered geometrically as forgetting the first strand. And once more one can apply standard techniques for obtaining presentations of group extensions as described in [Joh97] (page 139), which we have explicitely described in the proofs of Propositions 2.2.1, 2.2.4, 2.2.5 and Theorem 2.2.7.

From Propositions 2.2.1, 2.2.4, 2.2.5 and Theorem 2.2.7 we are able to obtain a presentation of the Abelianisation of $P_n(\mathbb{R}P^2)$, $P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$, $B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})$ and $B_{n,m}(\mathbb{R}P^2)$, respectively. In the following Corollaries, we denote by $\Gamma_2(G)$, for G a group, the commutator subgroup of G.

Corollary 2.2.9. *For $n \geq 1$, the following constitutes a presentation of $P_n(\mathbb{R}P^2)/\Gamma_2\big(P_n(\mathbb{R}P^2)\big)$.*

Generators: *ρ_k, for $1 \leq k \leq n$.*
Relations:

(i) $\rho_k^2 = 1$, *for* $1 \leq k \leq n$.

Hence, $P_n(\mathbb{R}P^2)/\Gamma_2(P_n(\mathbb{R}P^2)) \cong \mathbb{Z}_2^n$, which is generated by $\rho_1,\dots,\rho_n$.

Corollary 2.2.10. *For $n, m \geq 1$, the following constitutes a presentation of*

$$P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})/\Gamma_2\big(P_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})\big).$$

Generators: *ρ_k, for $1 \leq k \leq m$.*
Relations:

(i) $\rho_k^2 = 1$, *for* $1 \leq k \leq m$.

Hence $P_n(\mathbb{R}P^2)/\Gamma_2(P_n(\mathbb{R}P^2)) \cong \mathbb{Z}_2^m$, which is generated by $\rho_1,\dots,\rho_m$.

Corollary 2.2.11. *For $n, m \geq 1$, the following constitutes a presentation of*

$$B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})/\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1,\dots,x_n\})\big).$$

Generators: *ρ, σ, $\beta_1,\dots,\beta_n$.*

Relations:

(i) $\sigma^2 = 1$,

(ii) $\rho^2 \prod_{i=1}^{n} \beta_i = 1$,

(iii) $\beta_i \beta_j = \beta_j \beta_i$, *for* $1 \leq i, j \leq n$,

(iv) $\sigma \beta_i = \beta_i \sigma$, *and* $\rho \beta_i = \beta_i \rho$, *for* $1 \leq i \leq n$,

(v) $\sigma \rho = \rho \sigma$.

In particular is holds that

$$B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})) \cong \mathbb{Z}^n \times \mathbb{Z}_2,$$

where ρ, $\beta_1, \dots, \beta_{n-1}$ *generate the* $\mathbb{Z}^n$*-component and* σ *the* $\mathbb{Z}_2$*-component.*

Note that the Artin generators, $\sigma_1, \dots, \sigma_{m-1}$, are pairwise conjugate and thus are sent to the same element in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)$. We observe that the elements ρ, σ, $\beta_1, \dots, \beta_{n-1}$ form a generating set for $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$, since we can obtain the element β_n from relation (ii).

Corollary 2.2.12. *For* $n \geq 2$ *and* $m \geq 1$*, the following constitutes a presentation of* $B_{n,m}(\mathbb{R}P^2)/\Gamma_2\big(B_{n,m}(\mathbb{R}P^2)\big)$.

Generators*:* ρ, σ, τ, q.

Relations*:*

(i) $\rho^2 = 1$, $\sigma^2 = 1$, $\tau^2 = 1$, $q^2 = 1$,

(ii) $\rho\sigma = \sigma\rho$, $\rho\tau = \tau\rho$, $\rho q = q\rho$,

(iii) $\sigma\tau = \tau\sigma$, $\sigma q = q\sigma$,

(iv) $\tau q = q\tau$.

Hence, $B_{n,m}(\mathbb{R}P^2)/\Gamma_2\big(B_{n,m}(\mathbb{R}P^2)\big) \cong \mathbb{Z}_2^4$*, which is generated by* ρ, σ, τ *and* q.

2.3 Lower central and derived series of the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$

As we will see in Chapter 3, to study the splitting problem, it is helpful to know the lower central and derived series of certain subgroups of the surface braid groups. Before stating the result about the lower central and derived series of the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, we recall the definition of these notions.

Definition 2.3.1. Let $i \in \mathbb{N}$.

The lower central series of a group G is defined as the descending normal series

$$\Gamma_1(G) = G \supseteq \Gamma_2(G) \supseteq \cdots \supseteq \Gamma_n(G) \supseteq \dots,$$

where $\Gamma_i(G) = [\Gamma_{i-1}(G), G]$ is the subgroup of G generated by the commutators $[x, y] = xyx^{-1}y^{-1}$, for $x \in \Gamma_{i-1}(G)$ and $y \in G$.

The derived series of a group G is defined as the descending normal series

$$G^{(0)} = G \supseteq G^{(1)} \supseteq \cdots \supseteq G^{(n)} \supseteq \dots,$$

where $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ is the subgroup of G generated by the commutators $[x, y]$, for $x, y \in G^{(i-1)}$. Note that $\Gamma_2(G) = G^{(1)}$.

Let X be some property of groups. We recall that a group G is residually X if for every non-trivial element $g \in G$ there is a homomorphism h from G to a group with property X such that $h(g) \neq 1$.

Let $m = 1$ and $n \geq 1$. It holds that $B_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) = F_n$, where F_n denotes the free group on n generators. So the lower central and derived series of $B_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ are those of free groups of finite rank.

The following theorem presents the lower central and derived series of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ for $n \geq 2$ and for most values of m.

Theorem 2.3.2. *Let $n \geq 1$. Then:*

- *If $m \geq 3$, then*

$$\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big) = \Gamma_3\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big).$$

- *If $m \geq 5$, then*

$$\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)^{(1)} = \big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)^{(2)}.$$

In particular, for $m \geq 3$, $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is not residually nilpotent and for $m \geq 5$, it is not residually solvable.

Proof. Let $n \geq 1$. We will prove that $\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big) = \Gamma_3\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})\big)$, for $m \geq 3$. Let $m \geq 3$ and let $\Gamma_2 := \Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$ and $\Gamma_3 := \Gamma_3(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$. We have the following short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_2 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) & \xrightarrow{ab} & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle pr|_{\Gamma_2}} & & \downarrow{\scriptstyle pr} & & \| & & \\
1 & \longrightarrow & \Gamma_2/\Gamma_3 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3 & \xrightarrow{p} & B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2 & \longrightarrow & 1.
\end{array}$$

The map ab is the quotient map that sends $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ to its Abelianisation $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$. The map pr is the canonical projection from $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ to $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$ and the map $pr|_{\Gamma_2}$ is the restriction of pr to Γ_2. We have that the map ab factors throught $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$ and thus, the map p is such that $ab = p \circ pr$. We will obtain relations in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$ by projecting the relations of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ given in Proposition 2.2.5. Moreover, from Corollary 2.2.11, we know that $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$ is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_2$. In particular, under the map ab, the generators $\sigma_1, \dots, \sigma_{m-1}$ are sent to $\bar{\sigma}_1 \in B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$, where $\bar{\sigma}_1 = \sigma_1\Gamma_2$. It follows that

$$pr(\sigma_i) = t_i \cdot \sigma_1\Gamma_3 \in B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3,$$

for $1 \leq i \leq m-1$, where $t_i \in \ker(p) = \Gamma_2/\Gamma_3$ and t_1 may be taken to be 1.

For simplicity, we will denote $\sigma_1\Gamma_3$ by $\tilde{\sigma}_1$. Projecting now relation (v) of Proposition 2.2.5, $\sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}$, into $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$ we obtain $t_r\tilde{\sigma}_1 t_{r+1}\tilde{\sigma}_1 t_r\tilde{\sigma}_1 = t_{r+1}\tilde{\sigma}_1 t_r\tilde{\sigma}_1 t_{r+1}\tilde{\sigma}_1$, and since $t_i \in \Gamma_2/\Gamma_3$, where Γ_2/Γ_3 is a central subgroup of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$, it follows that

$$t_r = t_{r+1},$$

for $1 \leq r \leq m-1$, meaning that $1 = t_1 = \cdots = t_{m-1}$. We conclude that

$$pr(\sigma_1) = \cdots = pr(\sigma_{m-1}) = \tilde{\sigma}_1.$$

Similarly, since every generator $\rho_{n+1}, \dots, \rho_{n+m}$ is projected to the same element $\bar{\rho}_{n+1}$ in the Abelianisation $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$, where $\bar{\rho}_{n+1} = \rho_{n+1}\Gamma_2$, it follows that

$$pr(\rho_i) = s_i \cdot \rho_{n+1}\Gamma_3 \in B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3,$$

for $n+1 \leq i \leq n+m$, where $s_i \in \ker(p) = \Gamma_2/\Gamma_3$ and s_{n+1} may be taken to be 1.

We will denote $\rho_{n+1}\Gamma_3$ by $\tilde{\rho}_{n+1}$. Projecting relation (iii) of Proposition 2.2.5,

$$\rho_k \rho_j \rho_k^{-1} = (\sigma_{j-1-n}^{-1} \dots \sigma_{k+1-n}^{-1} \sigma_{k-n}^2 \sigma_{k+1-n} \dots \sigma_{j-1-n}) \rho_j,$$

for $n+1 \leq k < j \leq n+m$, under pr, gives $s_k \tilde{\rho}_{n+1} s_j \tilde{\rho}_{n+1} \tilde{\rho}_{n+1}^{-1} s_k^{-1} = \tilde{\sigma}_1^2 s_j \tilde{\rho}_{n+1}$, and since $s_i \in \Gamma_2/\Gamma_3$, for $n+1 \leq i \leq n+m$, where Γ_2/Γ_3 is central in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$, it follows that $\tilde{\sigma}_1^2 = 1$.

Since $m \geq 3$, the relation $\sigma_1 \rho_{n+3} \sigma_1^{-1} = \rho_{n+3}$ $((vii)$ of Proposition 2.2.5) exists in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, and projecting it into $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$, it follows that $\tilde{\sigma}_1 s_{n+3} \tilde{\rho}_{n+1} \tilde{\sigma}_1^{-1} = s_{n+3} \tilde{\rho}_{n+1}$. Since $s_{n+3} \in \Gamma_2/\Gamma_3$ we obtain

$$\tilde{\sigma}_1 \tilde{\rho}_{n+1} = \tilde{\rho}_{n+1} \tilde{\sigma}_1.$$

Furthermore, the projection of the relation $\sigma_{k-n} \rho_k \sigma_{k-n}^{-1} = B_{k,k+1} \rho_{k+1}$ $((vii)$ of Proposition 2.2.5), for $n+1 \leq k \leq n+m-1$, yields $\tilde{\sigma}_1 s_k \tilde{\rho}_{n+1} \tilde{\sigma}_1^{-1} = 1 \cdot s_{k+1} \tilde{\rho}_{n+1}$. We know that $\tilde{\sigma}_1 \tilde{\rho}_{n+1} = \tilde{\rho}_{n+1} \tilde{\sigma}_1$, $s_{n+3} \in \Gamma_2/\Gamma_3$ and that $s_k \tilde{\rho}_{n+1} = s_{k+1} \tilde{\rho}_{n+1}$. Thus, $1 = s_{n+1} = \dots = s_{n+m-1} = s_{n+m}$, and it follows that

$$pr(\rho_{n+1}) = \dots = pr(\rho_{n+m}) = \tilde{\rho}_{n+1}.$$

From Corollary 2.2.11, we have that $ab(B_{i,j}) = \bar{B}_{i,n+1} \in B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$, $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$, where $\bar{B}_{i,n+1} = B_{i,n+1}\Gamma_2$. Note that in Corollary 2.2.11 the element $ab(B_{i,j}) = \bar{B}_{i,n+1}$ is denoted by β_i. Thus, $pr(B_{i,j}) = a_{i,j} \cdot B_{i,n+1}\Gamma_3 \in B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, for $1 \leq i \leq n$, $n+1 \leq j \leq n+m$, where $a_{i,j} \in \ker(p) = \Gamma_2/\Gamma_3$ and $a_{i,n+1}$ may be taken to be 1.

We will denote $B_{i,n+1}\Gamma_3$ by $\tilde{B}_{i,n+1}$. Since $m \geq 3$, the relation $\sigma_1 B_{i,n+3} \sigma_1^{-1} = B_{i,n+3}$ $((vi)$ of Proposition 2.2.5), for $1 \leq i \leq n$, exists in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, and projecting it into $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$, it follows that $\tilde{\sigma}_1 a_{i,n+3} \tilde{B}_{i,n+1} \tilde{\sigma}_1^{-1} = a_{i,n+3} \tilde{B}_{i,n+1}$. Since $a_{i,n+3} \in \Gamma_2/\Gamma_3$ we obtain

$$\tilde{\sigma}_1 \tilde{B}_{i,n+1} = \tilde{B}_{i,n+1} \tilde{\sigma}_1,$$

for $1 \leq i \leq n$. Moreover, $pr(\sigma_{j-n} B_{i,j} \sigma_{j-n}^{-1}) = pr(B_{i,j+1})$, for $1 \leq i \leq n$, $n+1 \leq j \leq n+m-1$, yields $\tilde{\sigma}_1 a_{i,j} \tilde{B}_{i,n+1} \tilde{\sigma}_1^{-1} = a_{i,j+1} \tilde{B}_{i,n+1}$. We know that $\tilde{\sigma}_1 \tilde{B}_{i,n+1} = \tilde{B}_{i,n+1} \tilde{\sigma}_1$ and that $a_{i,j} \in \Gamma_2/\Gamma_3$, and thus $1 = a_{i,n+1} = \dots = a_{i,n+m-1} = a_{i,n+m}$, and

$$pr(B_{i,j}) = \tilde{B}_{i,n+1},$$

for $1 \leq i \leq n$, $n+1 \leq j \leq n+m$.

We conclude that the projection of the generating set of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ gives rise to the following generating set for $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$:

$$\{\tilde{\sigma}_1, \tilde{\rho}_{n+1}, \tilde{B}_{1,n+1}, \dots, \tilde{B}_{n,n+1}\},$$

subjected to the relations $\tilde{\sigma}_1^2 = 1$, $\tilde{\sigma}_1 \tilde{\rho}_{n+1} = \tilde{\rho}_{n+1} \tilde{\sigma}_1$ and $\tilde{\sigma}_1 \tilde{B}_{i,n+1} = \tilde{B}_{1,n+1} \tilde{\sigma}_1$, for $1 \leq i \leq n$.

In order to determine further relations among these generators we will project now some other relations of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ into $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$. Projecting relation (iv) of Proposition 2.2.5 we obtain $\tilde{\rho}_{n+1} \tilde{B}_{i,n+1} \tilde{\rho}_{n+1}^{-1} = \tilde{B}_{i,n+1}$, for $1 \leq i \leq n$, which implies that, $\tilde{\rho}_{n+1} \tilde{B}_{i,n+1} = \tilde{B}_{i,n+1} \tilde{\rho}_{n+1}$, for $1 \leq i \leq n$. Similarly, projecting relation (i) of Proposition 2.2.5 we obtain $\tilde{B}_{i,n+1} \tilde{B}_{k,n+1} = \tilde{B}_{k,n+1} \tilde{B}_{i,n+1}$, for $1 \leq i, k \leq n$. Finally, the projection of relation $\rho_k \prod_{i=1}^{n} B_{i,k} (\sigma_{k-1-n} \dots \sigma_1^2 \dots \sigma_{k-1-n}) = (\sigma_{k-n} \dots \sigma_{m-1}^2 \dots \sigma_{k-n}) \rho_k^{-1}$ $((ii)$ of Proposition 2.2.5), for $n+1 \leq k \leq n+m$, yields

$$\tilde{\rho}_{n+1}^2 \prod_{i=1}^{n} \tilde{B}_{i,n+1} = 1.$$

Summing up, we have shown that the generators

$$\{\tilde{\sigma}_1, \tilde{\rho}_{n+1}, \tilde{B}_{1,n+1}, \dots, \tilde{B}_{n,n+1}\},$$

of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$, commute pairwise. That means that the quotient group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_3$ is an Abelian group. By the universal property of the Abelianisation, it follows that Γ_2 is a subgroup of Γ_3, since any homomorphism of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ onto an Abelian

group factors through $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma_2$, which gives $\Gamma_2 \subseteq \Gamma_3$. But by the definition of the lower central series, we know that $\Gamma_3 \subseteq \Gamma_2$, and therefore we conclude that $\Gamma_3 = \Gamma_2$. As a result, $\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})) = \Gamma_3(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$, for $m \geq 3$.

We now prove the second statement using similar arguments. Let $m \geq 5$. For simplicity, we will denote the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ by G. Recall that for a group G, $\Gamma_2(G) = G^{(1)}$. We consider the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^{(1)} & \longrightarrow & G & \xrightarrow{ab} & G/G^{(1)} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle pr|_{G^{(1)}}} & & \downarrow{\scriptstyle pr} & & \| & & \\ 1 & \longrightarrow & G^{(1)}/G^{(2)} & \longrightarrow & G/G^{(2)} & \xrightarrow{p} & G/G^{(1)} & \longrightarrow & 1. \end{array}$$

The map ab is the quotient map that sends G to its Abelianisation $G/G^{(1)}$. The map pr is the canonical projection from G to $G/G^{(2)}$ and the map $pr|_{G^{(1)}}$ is the restriction of pr to $G^{(1)}$. We have that the map ab factors throught $G/G^{(2)}$ and thus, the map p is such that $ab = p \circ pr$. Using the same argument as before, we have:

$$pr(\sigma_i) = t_i \cdot \sigma_1 G^{(2)} \in G/G^{(2)},$$

for $1 \leq i \leq m-1$, where $t_i \in \ker(p) = G^{(1)}/G^{(2)}$ and t_1 may be taken to be 1.

Let $\sigma_1 G^{(2)}$ be denoted by $\bar{\sigma}_1$. For $m \geq 5$, the projection, by pr, of the relation $\sigma_r\sigma_s = \sigma_s\sigma_r$ ((v) of Proposition 2.2.5), for $|r-s| > 1$ implies that $\bar{\sigma}_1 t_k \bar{\sigma}_1 = t_k \bar{\sigma}_1 \bar{\sigma}_1$, for $3 \leq k \leq m-1$, and so $\bar{\sigma}_1$ commutes with t_k for $3 \leq k \leq m-1$. Moreover, for $4 \leq l \leq m-1$, the projection of the same relation yields $t_l \bar{\sigma}_1 t_2 \bar{\sigma}_1 = t_2 \bar{\sigma}_1 t_l \bar{\sigma}_1$. Since $\bar{\sigma}_1$ commutes with t_l and $G^{(1)}/G^{(2)}$ is an Abelian group, it follows that $\bar{\sigma}_1 t_2 = t_2 \bar{\sigma}_1$. Hence t_k commutes with $\bar{\sigma}_1$, for $2 \leq k \leq m-1$ and $m \geq 5$. Now, projecting the relation $\sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}$ ((v) of Proposition 2.2.5) into $G/G^{(2)}$ it follows that $t_r\bar{\sigma}_1 t_{r+1}\bar{\sigma}_1 t_r \bar{\sigma}_1 = t_{r+1}\bar{\sigma}_1 t_r \bar{\sigma}_1 t_{r+1}\bar{\sigma}_1$, and so for $2 \leq i \leq m-1$ we have $t_2 = \cdots = t_{m-1}$, since t_k commutes with $\bar{\sigma}_1$, for $2 \leq k \leq m-1$. If $r = 1$, the projection of the same relation yields $\bar{\sigma}_1 t_2 \bar{\sigma}_1 \bar{\sigma}_1 = t_2 \bar{\sigma}_1 \bar{\sigma}_1 t_2 \bar{\sigma}_1$, thus $t_1 = t_2 = 1$ and $t_1 = \cdots = t_{m-1} = 1$. We conclude that

$$pr(\sigma_1) = \cdots = pr(\sigma_{m-1}) = \bar{\sigma}_1.$$

Once again, as in the previous case we have

$$pr(\rho_i) = s_i \cdot \rho_{n+1} G^{(2)} \in G/G^{(2)},$$

for $n+1 \leq i \leq n+m$, where $s_i \in \ker(p) = G^{(1)}/G^{(2)}$ and s_{n+1} may be taken to be 1.

We will denote $\rho_{n+1}G^{(2)}$ by $\bar{\rho}_{n+1}$. The proof that $\bar{\sigma}_1^2 = 1$ differs here, since we can no longer use the argument of central elements. Projecting relation (iii) of Proposition 2.2.5, $\rho_k\rho_j\rho_k^{-1} = (\sigma_{j-1-n}^{-1} \dots \sigma_{k+1-n}^{-1}\sigma_{k-n}^2\sigma_{k+1-n} \dots \sigma_{j-1-n})\rho_j$, into $G/G^{(2)}$, for $k = n+1$ and $j = n+2$, we obtain $\bar{\rho}_{n+1}s_{n+2}\bar{\rho}_{n+1}\bar{\rho}_{n+1}^{-1}\bar{\rho}_{n+1}^{-1}s_{n+2}^{-1} = \bar{\sigma}_1^2$. Thus, $\bar{\rho}_{n+1}s_{n+2}\bar{\rho}_{n+1}^{-1}s_{n+2}^{-1} = \bar{\sigma}_1^2$. We continue with the projection into $G/G^{(2)}$ of the relation $\sigma_2\rho_{n+1}\sigma_2^{-1} = \rho_{n+1}$ ((vii) of Proposition 2.2.5), which implies that $\bar{\sigma}_1\bar{\rho}_{n+1}\bar{\sigma}_1^{-1} = \bar{\rho}_{n+1}$, and thus $\bar{\rho}_{n+1}$ commutes with $\bar{\sigma}_1$. Furthermore, the projection of the relation $\sigma_1\rho_{n+1}\sigma_1^{-1} = \sigma_1^2\rho_{n+2}$ ((vii) of Proposition 2.2.5) into $G/G^{(2)}$, gives rise to $\bar{\sigma}_1\bar{\rho}_{n+1}\bar{\sigma}_1^{-1} = \bar{\sigma}_1^2 s_{n+2}\bar{\rho}_{n+1}$. Since $\bar{\rho}_{n+1}$ commutes with $\bar{\sigma}_1$, $\bar{\sigma}_1^2 s_{n+2} = 1$, which yields $s_{n+2} = \bar{\sigma}_1^{-2}$. Combining this result with the relation $\bar{\rho}_{n+1}s_{n+2}\bar{\rho}_{n+1}^{-1}s_{n+2}^{-1} = \bar{\sigma}_1^2$, we conclude that $\bar{\sigma}_1^2 = 1$, but also that $\sigma_{n+2} = 1$.

We now prove that the generators $\rho_{n+1}, \dots, \rho_{n+m}$ all have the same image in $G/G^{(2)}$ under pr. Since $m \geq 5$, the relation $\sigma_l\rho_k\sigma_l^{-1} = \rho_k$ ((vii) of Proposition 2.2.5), which holds for $l \neq k-n, k-n-1$ and $n+1 \leq k \leq n+m$ exists in G. We fix $n+1 \leq k \leq n+m-1$. Thus, there exists some $1 \leq l \leq m-1$, such that $\sigma_l\rho_k\sigma_l^{-1} = \rho_k$. So, projecting this relation into $G/G^{(2)}$ it follows that $\bar{\sigma}_1 s_k \bar{\rho}_{n+1}\bar{\sigma}_1^{-1} = s_k\bar{\rho}_{n+1}$. Thus, $\bar{\sigma}_1$ commutes with $s_k\bar{\rho}_{n+1}$, for $n+1 \leq k \leq n+m-1$. Furthermore, the projection of the relation $\sigma_{k-n}\rho_k\sigma_{k-n}^{-1} = \sigma_{k-n}^2\rho_{k+1}$ ((vii) of Proposition 2.2.5) in $G/G^{(2)}$, for $n+1 \leq k \leq n+m-1$, implies that $\bar{\sigma}_1 s_k \bar{\rho}_{n+1}\bar{\sigma}_1^{-1} = 1 \cdot s_{k+1}\bar{\rho}_{n+1}$. Since $\bar{\sigma}_1$ commutes with $s_k\bar{\rho}_{n+1}$, we obtain $s_k\bar{\rho}_{n+1} = s_{k+1}\bar{\rho}_{n+1}$ for $n+1 \leq k \leq n+m-1$ and therefore $s_k = s_{k+1}$ for $n+1 \leq k \leq n+m-1$. Hence, $pr(\rho_{n+1}) = \cdots = pr(\rho_{n+m}) = \bar{\rho}_{n+1}$, and $\bar{\sigma}_1\bar{\rho}_{n+1} = \bar{\rho}_{n+1}\bar{\sigma}_1$.

As before, we have $pr(B_{i,j}) = a_{i,j} \cdot B_{i,n+1}G^{(2)} \in G/G^{(2)}$, for $1 \leq i \leq n,\ n+1 \leq j \leq n+m$, where $a_{i,j} \in \ker(p) = G^{(1)}/G^{(2)}$ and $a_{i,n+1}$ may be taken to be 1. We will denote $B_{i,n+1}G^{(2)}$ by $\bar{B}_{i,n+1}$.

Since $m \geq 5$, the relation $\sigma_r B_{i,j} \sigma_r^{-1} = B_{i,j}$ $((vi)$ of Proposition 2.2.5), for $r \neq j-n-1, j-n$, where $n+1 \leq j \leq n+m$, $1 \leq r \leq m-1$ and $1 \leq i \leq n$, exists in G. We fix $n+1 \leq j \leq n+m-1$. Thus, there exists some $1 \leq l \leq m-1$, such that $\sigma_l B_{i,j} \sigma_l^{-1} = B_{i,j}$. So, projecting this relation into $G/G^{(2)}$ it follows that $\bar{\sigma}_1 a_{i,j} \bar{B}_{i,n+1} \bar{\sigma}_1^{-1} = a_{i,j} \bar{B}_{i,n+1}$. Hence, $\bar{\sigma}_1$ commutes with $a_{i,j} \bar{B}_{i,n+1}$ for $1 \leq i \leq n$, $n+1 \leq j \leq n+m-1$. Furthermore, the projection of the relation $\sigma_{j-n} B_{i,j} \sigma_{j-n}^{-1} = B_{i,j+1}$ $((vi)$ of Proposition 2.2.5) in $G/G^{(2)}$, for $1 \leq i \leq n$ and $n+1 \leq j \leq n+m-1$, yields $\bar{\sigma}_1 a_{i,j} \bar{B}_{i,n+1} \bar{\sigma}_1^{-1} = a_{i,j+1} \bar{B}_{i,n+1}$. Hence, $a_{i,j} = a_{i,j+1}$, for $1 \leq i \leq n$ and $n+1 \leq j \leq n+m-1$. As a result, $pr(B_{i,j}) = \bar{B}_{i,n+1}$, for or $1 \leq i \leq n$, $n+1 \leq j \leq n+m$, and moreover $\bar{\sigma}_1 \bar{B}_{i,n+1} = \bar{B}_{i,n+1} \bar{\sigma}_1$, for $1 \leq i \leq n$.

We conclude that the projection of the generating set of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ given in Proposition 2.2.5 gives rise to the following generating set for $G/G^{(2)}$:

$$\{\bar{\sigma}_1, \bar{\rho}_{n+1}, \bar{B}_{1,n+1}, \dots, \bar{B}_{n,n+1}\}.$$

Once again, projecting relations (i) and (iv) of Proposition 2.2.5, that hold in G into $G/G^{(2)}$, we obtain $\bar{B}_{i,n+1} \bar{B}_{k,n+1} = \bar{B}_{k,n+1} \bar{B}_{i,n+1}$, for $1 \leq i,k \leq n$ and $\bar{\rho}_{n+1} \bar{B}_{i,n+1} = \bar{B}_{i,n+1} \bar{\rho}_{n+1}$, for $1 \leq i \leq n$, respectively.

Finally, the projection of relation (ii) of Proposition 2.2.5,

$$\rho_k \prod_{i=1}^{n} B_{i,k} (\sigma_{k-1-n} \dots \sigma_1^2 \dots \sigma_{k-1-n}) = (\sigma_{k-n} \dots \sigma_{m-1}^2 \dots \sigma_{k-n}) \rho_k^{-1},$$

in $G/G^{(2)}$, for $n+1 \leq k \leq n+m$, implies that $\bar{\rho}_{n+1}^2 \prod_{i=1}^{n} \bar{B}_{i,n+1} = 1$.

We observe that the generators $\{\bar{\sigma}_1, \bar{\rho}_{n+1}, \bar{B}_{1,n+1}, \dots, \bar{B}_{n,n+1}\}$ of $G/G^{(2)}$ commute pairwise. That means that this quotient group is an Abelian group. Applying the same argument as before, we conclude that $G^{(1)} \subseteq G^{(2)}$. By the definition of the derived series, we know that $G^{(2)} \subseteq G^{(1)}$, and therefore we conclude that $G^{(2)} = G^{(1)}$, which completes the proof. □

Remark 2.3.3. Let $n \geq 1$. We saw that Theorem 2.3.2 holds for $m \geq 3$, as for the lower central series, and for $m \geq 5$, as for the derived series. To determine the lower central and derived series of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}$ for the values $m = 2$ and $m = 2, 3, 4$, respectively, one cannot use the same arguments as we did for proving Theorem 2.3.2, since couple of relations of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}$ that we used during the proof do not hold for such small values of m, and so these cases become harder to treat.

CHAPTER 3

The splitting of the Fadell–Neuwirth short exact sequence for $B_n(\mathbb{R}P^2)$

In this Chapter we study the splitting problem of the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1,$$

for $n \geq 2$ and $m \in \mathbb{N}$. Note that even though this short exact sequence exists for $n \geq 2$, the homomorphism $\bar{q}_{n+m,n}$ exists for all values of $n, m \in \mathbb{N}$. In Section 3.1 we prove that if $n = 1$ the homomorphism $\bar{q}_{n+m,n}$ does not admit a section for any value of m, but if $n = 2$ it admits a section for every value of m. In Section 3.2, we present necessary conditions for this short exact sequence to split, by shifting the problem to a quotiented short exact sequence. We conclude the third chapter by constructing, for certain values of m, a geometric section for the Fadell–Neuwirth fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$, which implies that, for these values of m, the homomorphism $\bar{q}_{n+m,n}$ admits a section.

3.1 The problem of the existence of a section for the cases $n = 1, 2$

Let $m \in \mathbb{N}$. In this section we will prove the non-existence of an algebraic section for the homomorphism $B_{1,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{1+m,1}} B_1(\mathbb{R}P^2)$, where the map $\bar{q}_{1+m,1}$ can be considered geometrically as the maps that forget the last m strands. Moreover, we will prove that the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow B_{2,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{2+m,2}} B_2(\mathbb{R}P^2) \longrightarrow 1$$

splits for all values of $m \in \mathbb{N}$. We will provide an explicit geometric section for the map $q_{2+m,2} : F_{2+m}(\Sigma)/(S_2 \times S_m) \to F_2(\Sigma)/S_2$ as well as an explicit algebraic section for the homomorphism $B_{2,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{2+m,2}} B_2(\mathbb{R}P^2)$, where the maps $q_{2+m,2}$ and $\bar{q}_{2+m,2}$ can be considered geometrically as the maps that forget the last m points and the last m strands respectively.

Proposition 3.1.1. *Let $m \in \mathbb{N}$. The homomorphism $\bar{q}_{1+m,1} : B_{1,m}(\mathbb{R}P^2) \to B_1(\mathbb{R}P^2)$ admits no section.*

Proof. Let $m \in \mathbb{N}$. We have $B_1(\mathbb{R}P^2) = \langle \rho_1 \mid \rho_1^2 = 1 \rangle \cong \mathbb{Z}_2$. Under a possible section $s : B_1(\mathbb{R}P^2) \to B_{1,m}(\mathbb{R}P^2)$, the only non-trivial element of $B_1(\mathbb{R}P^2)$, ρ_1, is of order two and thus it has to be mapped to the full twist Δ^2_{1+m} in $B_{1,m}(\mathbb{R}P^2)$, since Δ^2_{1+m} is the unique element of order two in $B_{1,m}(\mathbb{R}P^2) \subset B_{1+m}(\mathbb{R}P^2)$, see Proposition 2.1.3. The full twist $\Delta^2_{1+m} = (\sigma_1 \cdots \sigma_m)^{1+m}$ is mapped by $\bar{q}_{1+m,1}$ to the trivial element of $B_1(\mathbb{R}P^2)$. Thus, we obtain $(\bar{q}_{1+m,1} \circ s)(\rho_1) = \bar{q}_{1+m,1}(\Delta^2_{1+m}) = 1 \neq \rho_1$ and as a result, such a section $s : B_1(\mathbb{R}P^2) \to B_{1,m}(\mathbb{R}P^2)$ cannot exist. □

Proposition 3.1.2. *Let $m \in \mathbb{N}$. The map $q_{2+m,2} : F_{2+m}(\Sigma)/(S_2 \times S_m) \to F_2(\Sigma)/S_2$ admits a geometric section, for every $m \in \mathbb{N}$.*

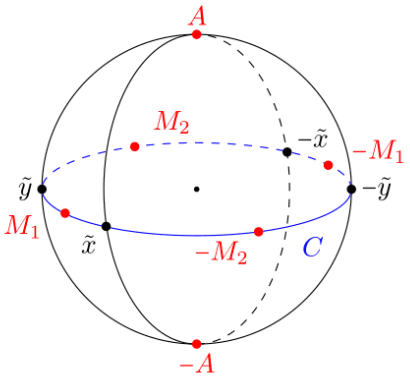


Figure 3.1

Proof. Let x, y be two distinct unordered points in $\mathbb{R}P^2$. We consider the quotient map from the sphere onto the projective plane, which is a two-sheeted covering map. These two points, x, y, are lifted to two pairs of distinct antipodal points on the sphere $\mathbb{S}^2$. Consider the elements $\tilde{x}, -\tilde{x} \in \mathbb{S}^2$ and $\tilde{y}, -\tilde{y} \in \mathbb{S}^2$ whose projection under the quotient map into $\mathbb{R}P^2$ is x and y respectively. For any two distinct non-antipodal points on $\mathbb{S}^2$, there is a unique plane which passes through these two points and the centre of the sphere. This plane intersects the sphere in a great circle. Whichever choice of two non-antipodal points we make among the points $\tilde{x}, -\tilde{x}, \tilde{y}, -\tilde{y} \in \mathbb{S}^2$, the plane Π that passes through these two points and the centre of the sphere intersects the sphere in a great circle C, which clearly contains the points $\tilde{x}, -\tilde{x}, \tilde{y}, -\tilde{y}$. Consider now the unique unit vectors $\vec{A}, -\vec{A}$, that are orthogonal to the plane Π. These two vectors intersect the sphere in two antipodal points $A, -A \in \mathbb{S}^2$. Consider the geodesics $A\tilde{x}, A(-\tilde{x}), A\tilde{y}, A(-\tilde{y})$, and $-A\tilde{x}, -A(-\tilde{x}), -A\tilde{y}, -A(-\tilde{y})$. On each of these geodesics, take n equally-spaced distinct new points on $\mathbb{S}^2$. We observe that these $8n$ new distinct points on $\mathbb{S}^2$ are projected to $4n$ new distinct points on $\mathbb{R}P^2$, since the $4n$ equally spaced distinct new points on $A\tilde{x}, A(-\tilde{x}), A\tilde{y}, A(-\tilde{y})$ are antipodal to the $4n$ equally spaced distinct new points on $-A\tilde{x}, -A(-\tilde{x}), -A\tilde{y}, -A(-\tilde{y})$. Thus based on this construction, from two distinct unordered points x, y in $\mathbb{R}P^2$ we obtain $4n$ distinct unordered points in $\mathbb{R}P^2$, different from x and y. We will denote these $4n$ distinct unordered new points in $\mathbb{R}P^2$ by N. This is an explicit cross-section $s_{4n} : F_2(\mathbb{R}P^2)/S_2 \to F_{2+4n}(\mathbb{R}P^2)/(S_2 \times S_{4n})$, for $n \in \mathbb{N}$.

Moreover, if we project the points $A, -A$ from $\mathbb{S}^2$ into $\mathbb{R}P^2$ we obtain one new point on $\mathbb{R}P^2$, which we denote by A for simplicity, different from x, y and from the other points of N on $\mathbb{R}P^2$. This yields an explicit cross-section $s_{4n+1} : F_2(\mathbb{R}P^2)/S_2 \to F_{2+(4n+1)}(\mathbb{R}P^2)/(S_2 \times S_{4n+1})$, for $n \in \mathbb{N}$.

We recall that the points $\tilde{x}, -\tilde{x}, \tilde{y}, -\tilde{y}$ lie on the great circle C on $\mathbb{S}^2$. These four points give rise to four geodesics; $\tilde{x}\tilde{y}$, $\tilde{y}(-\tilde{x}), -\tilde{x}(-\tilde{y}), -\tilde{y}\tilde{x}$ on C. Taking the midpoints of each of these four geodesics, we observe that the midpoint of the geodesic $\tilde{x}\tilde{y}$ is the antipodal point of the midpoint of the geodesic $-\tilde{x}(-\tilde{y})$ and the midpoint of the geodesic $\tilde{y}(-\tilde{x})$ is the antipodal point of the midpoint of the geodesic $-\tilde{y}\tilde{x}$. We denote these four midpoints by $M_1, -M_1, M_2, -M_2$, as we see in Figure 3.1. As a result, projecting $M_1, -M_1, M_2, -M_2$ from $\mathbb{S}^2$ into $\mathbb{R}P^2$ we obtain two new distinct unordered points, which we denote by M_1 and M_2 for simplicity. Note that M_1 and M_2 are different from x, y and from any other new point that we have already constructed. Thus the points of N that we have already explicitly constructed together with these two new distinct unordered points M_1 and M_2, provide an explicit cross-section $s_{4n+2} : F_2(\mathbb{R}P^2)/S_2 \to F_{2+(4n+2)}(\mathbb{R}P^2)/(S_2 \times S_{4n+2})$, for $n \in \mathbb{N}$.

Finally, if we consider the N points together with the two unordered distinct points, M_1, M_2 and the point A on $\mathbb{R}P^2$, we obtain $4n+3$ new unordered distinct points different from x, y on the projective plane. This leads to an explicit cross-section $s_{4n+3} : F_2(\mathbb{R}P^2)/S_2 \to F_{2+(4n+3)}(\mathbb{R}P^2)/(S_2 \times S_{4n+3})$, for $n \in \mathbb{N}$.

Summing up, there exists a cross-section, $s_m : F_2(\mathbb{R}P^2)/S_2 \to F_{2+m}(\mathbb{R}P^2)/(S_2 \times S_m)$, for the fibration $q_{2+m,2} : F_{2+m}(\mathbb{R}P^2)/(S_2 \times S_m) \to F_2(\mathbb{R}P^2)/S_2$, for any $m \equiv 0, 1, 2, 3 \bmod 4$ and thus for

any $m \in \mathbb{N}$. We underline that the new points that we have obtained are independent from any choice of the representatives on $\mathbb{S}^2$ of x, y and that this construction depends continuously on x and y. □

As well as the existence of a cross-section $s_m : F_2(\mathbb{R}P^2)/S_2 \to F_{2+m}(\mathbb{R}P^2)/(S_2 \times S_m)$, for any $m \in \mathbb{N}$ we can also provide an explicit algebraic section $\bar{s}_m : B_2(\mathbb{R}P^2) \to B_{2,m}(\mathbb{R}P^2)$ for the homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$, as we describe in the following proposition.

Proposition 3.1.3. *Let $m \in \mathbb{N}$. The homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$ admits a section, for every $m \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$. We will define explicitly an algebraic section $\bar{s}_m : B_2(\mathbb{R}P^2) \to B_{2,m}(\mathbb{R}P^2)$ for the homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$. We will do so by considering two different cases, when $m = 2k$ and when $m = 2k + 1$, where $k \in \mathbb{N}$. In Section 2.1 we mentioned that Van Buskirk, ([VB66] page 87), proved that $B_2(\mathbb{R}P^2)$ is isomorphic to the dicyclic group of order 16 and in particular that $B_2(\mathbb{R}P^2) = \langle \sigma_1, \rho_1 \mid (\sigma_1\rho_1^{-1})^4 = \rho_1^2 = \sigma_1^2 \rangle$. To be compatible with our upcoming presentation, we give the following presentation of the group $B_2(\mathbb{R}P^2)$:

$$B_2(\mathbb{R}P^2) = \langle a_2,\ \Delta_2 \mid a_2^8 = 1,\ \Delta_2^2 = a_2^4,\ \Delta_2 a_2 \Delta_2^{-1} = a_2^{-1} \rangle \cong Dic_{16},$$

where $a_2 = \sigma_1^{-1}\rho_1$ and $\Delta_2 = \sigma_1$. Note that the isomorphism follows from Proposition 2.1.7. Before examining the two different cases, we will prove a general statement which we will use later. Let

$$Dic_{8(2+2k)} = \langle x, y \mid x^{4(2k+2)} = 1,\ y^2 = x^{2(2k+2)},\ yxy^{-1} = x^{-1} \rangle,$$

then

$$Dic_{16} \cong \langle x^{k+1}, x^k y \mid x^{8(k+1)} = 1,\ (x^k y)^2 = x^{4(k+1)},\ (x^k y)x^{k+1}(x^k y)^{-1} = x^{-(k+1)} \rangle. \tag{1}$$

To begin with, note that from the given presentation of $Dic_{8(2+2k)}$ it follows that

$$x^{k+1} \text{ is of order 8.} \tag{2}$$

Moreover, using the relation $yxy^{-1} = x^{-1}$ we have

$$(x^k y)^2 = x^k y x^k y^{-1} y^2 = x^k x^{-k} y^2 = y^2.$$

From the given presentation of $Dic_{8(2+2k)}$ we also have

$$x^{4(k+1)} = y^2.$$

Thus,

$$(x^k y)^2 = x^{4(k+1)}. \tag{3}$$

Lastly, from the relation $yxy^{-1} = x^{-1}$, we deduce that

$$(x^k y)x^{k+1}(x^k y)^{-1} = x^{-(k+1)}. \tag{4}$$

From relations (2), (3) and (4) we conclude that $\langle x^{k+1}, x^k y \rangle$ is isomorphic to a quotient of the dicyclic group of order 16. But

$$Dic_{8(2+2k)} = \langle x \rangle \sqcup \langle x \rangle y,$$

where $x^{k+1} \in \langle x \rangle$ while $x^k y \in \langle x \rangle y$. Thus, $\langle x^{k+1} \rangle \cap \{x^k y\} = \varnothing$, which implies that $\langle x^{k+1}, x^k y \rangle$ contains at least 16 distinct elements, and so

$$\langle x^{k+1}, x^k y \rangle \cong Dic_{16}.$$

(a) First, let $m = 2k$, for $k \in \mathbb{N}$. From Proposition 2.1.6, for $n \geq 2$, the group $B_n(\mathbb{R}P^2)$, contains the dicyclic group of order $8n$ as a subgroup,

$$Dic_{8n} = \langle x, y \mid x^{2n} = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Thus, $Dic_{8(2+2k)}$ is a subgroup of $B_{2+2k}(\mathbb{R}P^2)$. From Proposition 2.1.7 we have

$$Dic_{8(2+2k)} = \langle a_{2+2k},\ \Delta_{2+2k} \mid a_{2+2k}^{4(2k+2)} = 1,\ \Delta_{2+2k}^2 = a_{2+2k}^{2(2k+2)},\ \Delta_{2+2k} a_{2+2k} \Delta_{2+2k}^{-1} = a_{2+2k}^{-1} \rangle,$$

where $a_{2+2k} = \sigma_{1+2k}^{-1} \cdots \sigma_1^{-1} \rho_1$ and $\Delta_{2+2k} = (\sigma_1\sigma_2\cdots\sigma_{1+2k})(\sigma_1\sigma_2\cdots\sigma_{2k})\cdots(\sigma_1\sigma_2)\sigma_1$. We consider the subgroup of $Dic_{8(2+2k)}$ generated by the elements a_{2+2k}^{k+1} and $a_{2+2k}^{k}\Delta_{2+2k}$ and it follows from (1) that $\langle a_{2+2k}^{k+1}, a_{2+2k}^{k}\Delta_{2+2k} \rangle \cong Dic_{16}$. Therefore, the homomorphism

$$i_1 : B_2(\mathbb{R}P^2) \to B_{2+2k}(\mathbb{R}P^2)$$

defined by

$$i_1(a_2) = a_{2+2k}^{k+1} \text{ and } i_1(\Delta_2) = a_{2+2k}^{k}\Delta_{2+2k}$$

is an injective homomorphism of $B_2(\mathbb{R}P^2)$ in $B_{2+2k}(\mathbb{R}P^2)$. Note that the elements a_{2+2k}^{k+1} and $a_{2+2k}^{k}\Delta_{2+2k}$ have the correct permutation type, but no transposition corresponds to the first and second strand. In order to obtain an embedding of $B_2(\mathbb{R}P^2)$ in $B_{2,2k}(\mathbb{R}P^2)$, we need to conjugate a_{2+2k}^{k+1} and $a_{2+2k}^{k}\Delta_{2+2k}$ by a suitable element. Thus, we conjugate a_{2+2k}^{k+1} and $a_{2+2k}^{k}\Delta_{2+2k}$ by $(\sigma_2\sigma_3\cdots\sigma_{k+1})$. We set

$$c := (\sigma_2\sigma_3\cdots\sigma_{k+1}).$$

Note that the projection of the elements c and c^{-1} in the symmetric group S_{2+2k} is the permutation given by $(k+1)$-cycles

$$\big(2\ (k+2)\ (k+1)\ldots 5\ 4\ 3\big) \text{ and } \big((k+2)\ 2\ 3\ldots(k-1)\ k\ (k+1)\big) \text{ respectively.}$$

Moreover, the projection of element a_{2+2k} in S_{2+2k} is the permutation given by $(2k+2)$-cycle $\big(1\ 2\ldots\ (2k+1)\ (2k+2)\big)$. Thus, the projection of element a_{2+2k}^{k+1} in S_{2+2k} is the permutation that corresponds to the following product of $(k+1)$ transpositions:

$$\big(1\ (k+2)\big)\big(2\ (k+3)\big)\cdots\big(k\ (2k+1)\big)\big((k+1)\ (2k+2)\big).$$

It follows that $c \cdot a_{2+2k}^{k+1} \cdot c^{-1} \in B_{2,2k}(\mathbb{R}P^2)$, since its projection in S_{2+2k} is the permutation given by the following product of $(k+1)$ transpositions:

$$\big(1\ 2\big)\big(3\ (k+3)\big)\cdots\big(p\ (k+p)\big)\cdots\big((k+1)\ (2k+1)\big)\big((k+2)\ (2k+2)\big),\ \text{for } 3 \le p \le k+2.$$

We have a similar result for the element $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$. For simplicity, we will focus only on the 1^{st} and 2^{nd} points, regarding the permutation that corresponds to the projection of the element $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$ in S_{2+2k}. We have the following:

$$1 \xrightarrow{c} 1 \xrightarrow{a_{2+2k}^{k}} k+1 \xrightarrow{\Delta_{2+2k}} k+2 \xrightarrow{c^{-1}} 2 \text{ and } 2 \xrightarrow{c} k+2 \xrightarrow{a_{2+2k}^{k}} 2k+2 \xrightarrow{\Delta_{2+2k}} 1 \xrightarrow{c^{-1}} 1.$$

Thus, the projection of the element $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$ in S_{2+2k} is a permutation that contains the transposition $(1\ 2)$. Therefore, $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1} \in B_{2,2k}(\mathbb{R}P^2)$. As a result, we conclude that the homomorphism

$$\bar{i}_1 : B_2(\mathbb{R}P^2) \to B_{2,2k}(\mathbb{R}P^2)$$

defined by

$$\bar{i}_1(a_2) = c \cdot a_{2+2k}^{k+1} \cdot c^{-1} \text{ and } \bar{i}_1(\Delta_2) = c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$$

is an embedding of $B_2(\mathbb{R}P^2)$ in $B_{2,2k}(\mathbb{R}P^2)$, since $B_2(\mathbb{R}P^2)$ is generated by a_2 and Δ_2. To prove that $\bar{q}_{2+2k,2} \circ \bar{i}_1 = id_{B_2(\mathbb{R}P^2)}$, which means that the homomorphism $\bar{i}_1$ is indeed a section for $\bar{q}_{2+2k,2} : B_{2,2k}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$, it suffices to make the following observation, which is also illustrated in Figures 3.2 and 3.3, in the case $k = 3$.

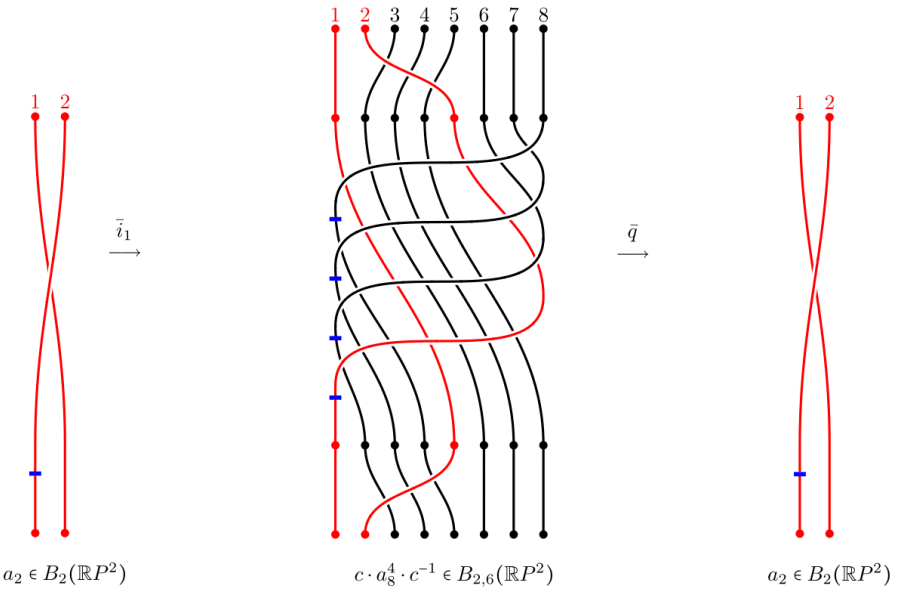


Figure 3.2 The image of $a_2 = \sigma_1^{-1}\rho_1$ under the map $\bar{i}_1 : B_2(\mathbb{R}P^2) \to B_{2,2k}(\mathbb{R}P^2)$, for $k = 3$. Note that with the blue dash we illustrate the element ρ_1.

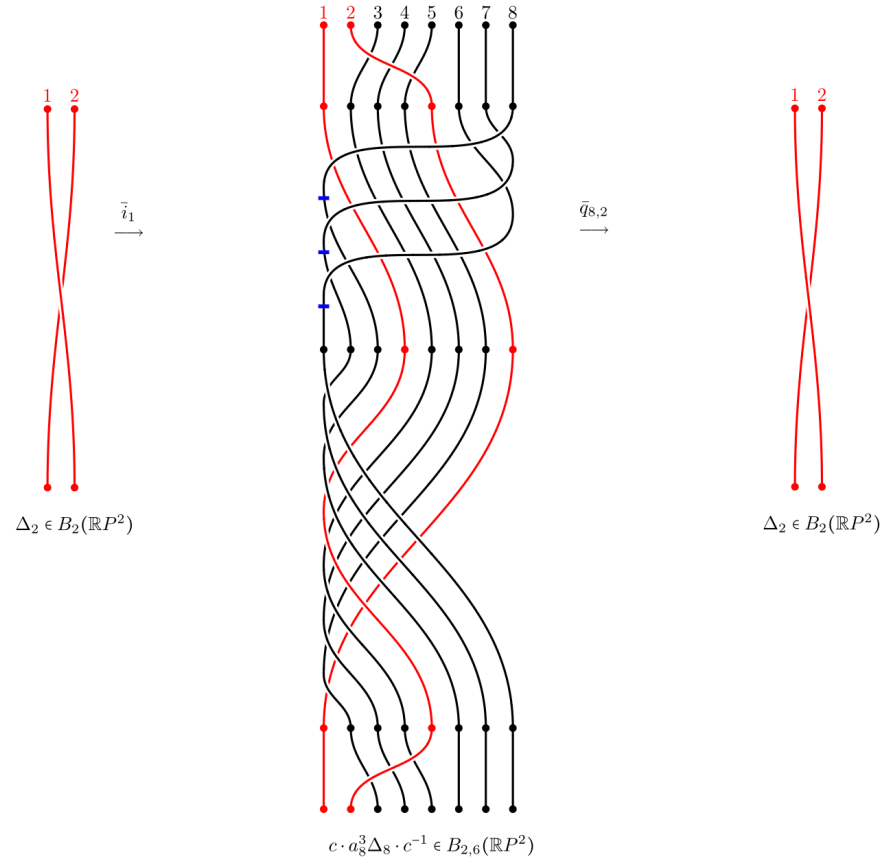


Figure 3.3 The image of $\Delta_2 = \sigma_1$ under the map $\bar{i}_1 : B_2(\mathbb{R}P^2) \to B_{2,2k}(\mathbb{R}P^2)$, for $k = 3$. Note that with the blue dash we illustrate the element ρ_1.

Note that in the braid c, the 1^{st} and the 2^{nd} strands have no crossing between them, and that

the 2^{nd} strand ends at the $(k+2)^{th}$ point, while the 1^{st} strand ends at the 1^{st} point. Then, in the braid a_{2+2k}^{k+1} there is a negative crossing, σ_{k+1}^{-1}, between the 1^{st} strand and the $(k+2)^{th}$, as the 1^{st} strand ends at the $(k+2)^{th}$ point and the $(k+2)^{th}$ strand ends at the 1^{st} point under ρ_1. Finally, the braid c^{-1} has no crossing between the 1^{st} and the $(k+2)^{th}$ strands and the $(k+2)^{th}$ strand ends at the 2^{nd} point. Thus, the braid, $c \cdot a_{2+2k}^{k+1} \cdot c^{-1}$, concerning the first and the second strands, consists precisely of a crossing σ_1^{-1} and then ρ_1. This implies that the homomorphism $\bar{q}_{2+2k,2}$, that can be considered geometrically as forgetting the last $2k$ strands, maps the element $c \cdot a_{2+2k}^{k+1} \cdot c^{-1}$ to $\sigma_1^{-1}\rho_1$. Therefore, $(\bar{q}_{2+2k,2} \circ \bar{i}_1)(a_2) = a_2 \in B_2(\mathbb{R}P^2)$. Similarly, in the braid c, the 1^{st} and the 2^{nd} strands have no crossing between them, and the 2^{nd} strand ends at the $(k+2)^{th}$ point, while the first stand ends at the first point. Then, in the braid a_{2+2k}^{k} there is no crossing between the 1^{st} strand and the $(k+2)^{th}$, where the 1^{st} strand ends at the $(k+1)^{th}$ point and the $(k+2)^{th}$ strand ends at the $(2k+2)^{th}$ point. The braid Δ_{2+2k} contains only one positive crossing σ_{k+1} between the $(k+1)^{th}$ strand and the $(2k+2)^{th}$, as the $(k+1)^{th}$ strand ends at the $(k+2)^{th}$ point and the $(2k+2)^{th}$ ends at the 1^{st} point. Finally, the braid c^{-1} has no crossing between the 1^{st} and the $(k+2)^{th}$ strands and the $(k+2)^{th}$ strand ends at the 2^{nd} point. Thus, the braid, $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$, concerning the first and the second strands, consists precisely of a crossing σ_1. This implies that the homomorphism $\bar{q}_{2+2k,2}$, that can be considered geometrically as forgetting the last $2k$ strands, maps the element $c \cdot a_{2+2k}^{k}\Delta_{2+2k} \cdot c^{-1}$ to σ_1. Therefore, $(\bar{q}_{2+2k,2} \circ \bar{i}_1)(\Delta_2) = \Delta_2 \in B_2(\mathbb{R}P^2)$. Summing up, the homomorphism $\bar{i}_1$ is a section for the homomorphism $\bar{q}_{2+2k,2}$.

(b) Now let $m = 2k+1$, for $k \in \mathbb{N}$. From Proposition 2.1.6, for $n \geq 3$, the group $B_n(\mathbb{R}P^2)$, contains the dicyclic group of order $8(n-1)$ as a subgroup,

$$Dic_{8(n-1)} = \langle x, y \mid x^{2(n-1)} = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Thus, $Dic_{8(2+2k)}$ is a subgroup of $B_{2+(2k+1)}(\mathbb{R}P^2)$, where from Proposition 2.1.7,

$$Dic_{8(2+2k)} = \langle b, \Delta a^{-1} \mid b^{4(2k+2)} = 1,\ (\Delta a^{-1})^2 = b^{2(2k+2)}, (\Delta a^{-1})b(\Delta a^{-1})^{-1} = b^{-1} \rangle,$$

and where $b := b_{2+(2k+1)} = \sigma_{2k+1}^{-1}\cdots\sigma_1^{-1}\rho_1$, $a := a_{2+(2k+1)} = \sigma_{2k+2}^{-1}\cdots\sigma_1^{-1}\rho_1$ and $\Delta := \Delta_{2+(2k+1)} = (\sigma_1\sigma_2\cdots\sigma_{2k+2})(\sigma_1\sigma_2\cdots\sigma_{2k+1})\cdots(\sigma_1\sigma_2)\sigma_1$. We consider the subgroup of $Dic_{8(2+2k)}$ generated by the elements b^{k+1} and $b^k\Delta a^{-1}$ and it follows from (1) that $\langle b^{k+1}, b^k\Delta a^{-1} \rangle \cong Dic_{16}$. Therefore, the homomorphism

$$i_2 : B_2(\mathbb{R}P^2) \to B_{2+(2k+1)}(\mathbb{R}P^2)$$

defined by

$$i_2(a_2) = b^{k+1} \text{ and } i_2(\Delta_2) = b^k\Delta a^{-1}$$

is an injective homomorphism of $B_2(\mathbb{R}P^2)$ in $B_{2+(2k+1)}(\mathbb{R}P^2)$. In order to obtain an embedding of $B_2(\mathbb{R}P^2)$ in $B_{2,(2k+1)}(\mathbb{R}P^2)$, we need to conjugate b^{k+1} and $b^k\Delta a^{-1}$ by a suitable element. Thus, as in the case (a), we conjugate b^{k+1} and $b^k\Delta a^{-1}$ by $c := (\sigma_2\sigma_3\cdots\sigma_{k+1})$. Moreover, note that the braid $b = \sigma_{2k+1}^{-1}\cdots\sigma_1^{-1}\rho_1 \in B_{2+(2k+1)}(\mathbb{R}P^2)$ differs from the braid $a_{2+2k} = \sigma_{2k+1}^{-1}\cdots\sigma_1^{-1}\rho_1 \in B_{2+2k}(\mathbb{R}P^2)$ by just one extra vertical strand that corresponds to the $(2+(2k+1))^{th}$ point. That means that the element b^{k+1} and the element a_{2+2k}^{k+1} have the same braiding concerning the first $(2+2k)$ points. Therefore, as we saw in case (a), the projection of element a_{2+2k} in S_{2+2k} is the permutation given by the $(2k+2)$-cycle $\big(1\ 2 \ldots\ (2k+1)\ (2k+2)\big)$, and the same holds for the projection of the element b in $S_{2+(2k+1)}$. Moreover, as we saw in case (a), the projection of the element a_{2+2k}^{k+1} in S_{2+2k} is the permutation given by the following product of $(k+1)$-transpositions:

$$\big(1\ (k+2)\big)\big(2\ (k+3)\big)\cdots\big(k\ (2k+1)\big)\big((k+1)\ (2k+2)\big),$$

and therefore, the element b^{k+1} corresponds to the same permutation under the projection in $S_{2+(2k+1)}$.

It follows that $c \cdot b^{k+1} \cdot c^{-1} \in B_{2,(2k+1)}(\mathbb{R}P^2)$, since its projection in $S_{2+(2k+1)}$ is the permutation given by the following product of $(k+1)$-transpositions:

$$\big(1\ 2\big)\big(3\ (k+3)\big)\cdots\big(p\ (k+p)\big)\cdots\big((k+1)\ (2k+1)\big)\big((k+2)\ (2k+2)\big),\ \text{for } 3 \leq p \leq k+2.$$

We have a similar result for the element $c \cdot b^k \Delta a^{-1} \cdot c^{-1}$. For simplicity, we will focus only on the 1^{st} and 2^{nd} points, regarding the permutation that corresponds to the projection of the element $c \cdot b^k \Delta a^{-1} \cdot c^{-1}$ in $S_{2+(2k+1)}$. We have the following:

$$1 \xrightarrow{c} 1 \xrightarrow{b^k} k+1 \xrightarrow{\Delta} k+3 \xrightarrow{a^{-1}} k+2 \xrightarrow{c^{-1}} 2$$

and

$$2 \xrightarrow{c} k+2 \xrightarrow{b^k} 2k+2 \xrightarrow{\Delta} 2 \xrightarrow{a^{-1}} 1 \xrightarrow{c^{-1}} 1.$$

Thus, the projection of the element $c \cdot b^k \Delta a^{-1} \cdot c^{-1}$ in $S_{2+(2k+1)}$ is a permutation that contains the transposition $(1\ 2)$. Therefore, $c \cdot b^k \Delta a^{-1} \cdot c^{-1} \in B_{2,2k+1}(\mathbb{R}P^2)$. As a result, we conclude that the homomorphism

$$\bar{i}_2 : B_2(\mathbb{R}P^2) \to B_{2,2k+1}(\mathbb{R}P^2),$$

defined by $\bar{i}_2(a_2) = c \cdot b^{k+1} \cdot c^{-1}$ and $\bar{i}_2(\Delta_2) = c \cdot b^k \Delta a^{-1} \cdot c^{-1}$ is an embedding of $B_2(\mathbb{R}P^2)$ in $B_{2,2k+1}(\mathbb{R}P^2)$, since $B_2(\mathbb{R}P^2)$ is generated by a_2 and Δ_2. To prove that

$$\bar{q}_{2+(2k+1),2} \circ \bar{i}_2 = id_{B_2(\mathbb{R}P^2)},$$

which means that the homomorphism $\bar{i}_2$ is indeed a section for

$$\bar{q}_{2+(2k+1),2} : B_{2,2k+1}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2),$$

it suffices to make the following observation, which is also illustrated in Figures 3.4 and 3.5, in the case $k = 3$.

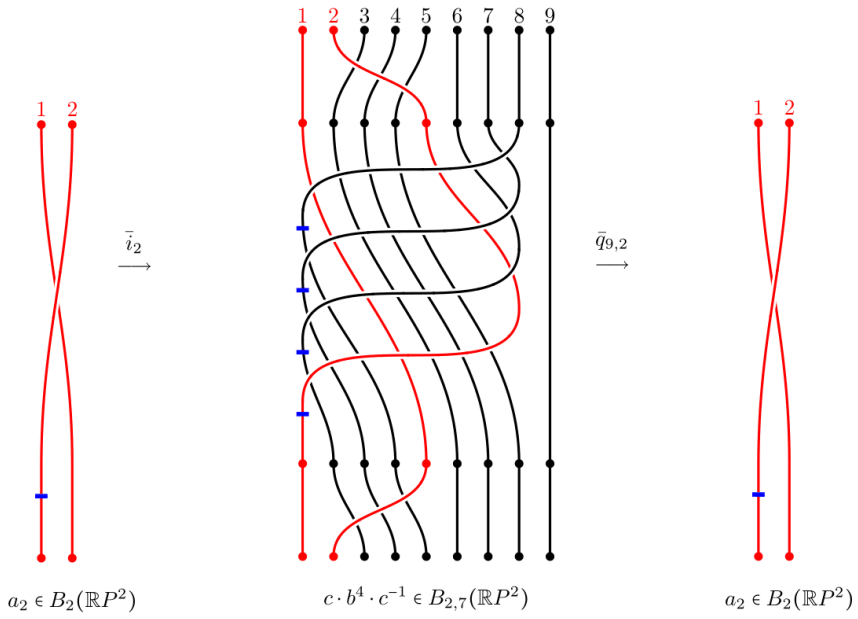


Figure 3.4 The image of $a_2 = \sigma^{-1}\rho_1$ under the map $\bar{i}_2 : B_2(\mathbb{R}P^2) \to B_{2,2k+1}(\mathbb{R}P^2)$, for $k = 3$. Note that with the blue dash we illustrate the element ρ_1.

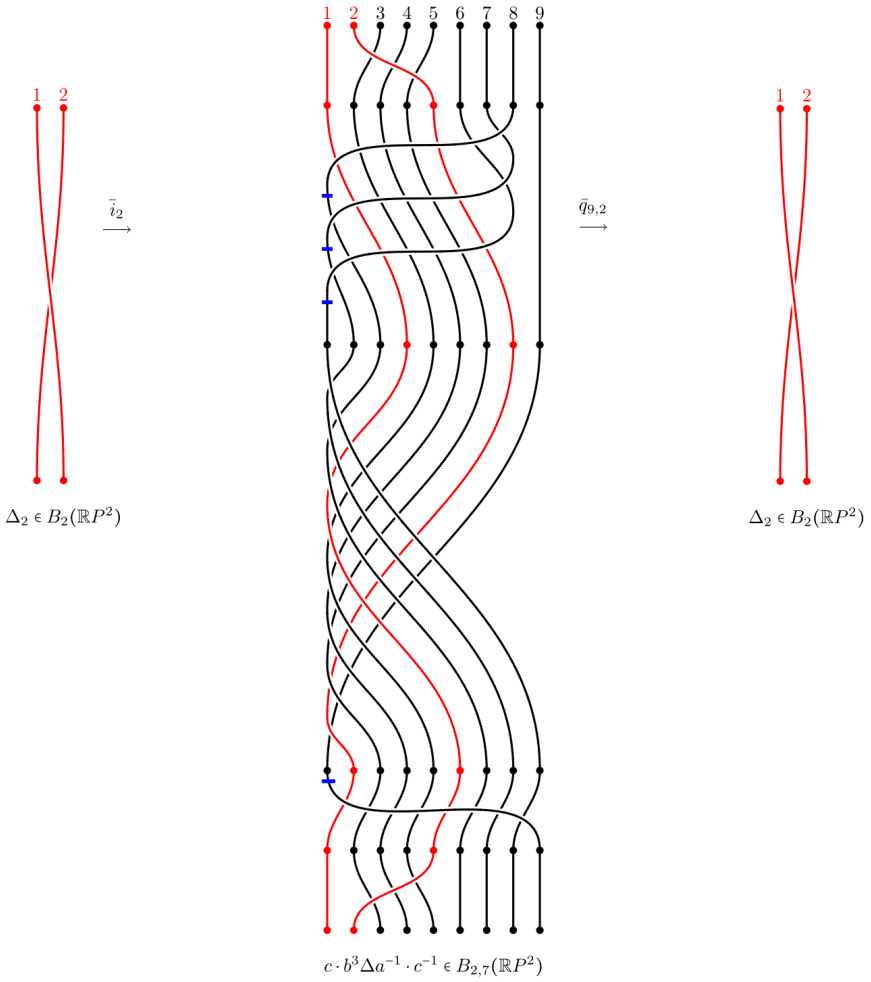


Figure 3.5 The image of $\Delta_2 = \sigma_1$ under the map $\bar{i}_2 : B_2(\mathbb{R}P^2) \to B_{2,2k+1}(\mathbb{R}P^2)$, for $k = 3$. Note that the first three blue dashes illustrate the element ρ_1, while the last one illustrates the element ρ_1^{-1}.

Note that in the braid c, the 1^{st} and the 2^{nd} strands have no crossing between them, and that the 2^{nd} strand ends at the $(k+2)^{th}$ point, while the 1^{st} strand ends at the 1^{st} point. Then, in the braid b^{k+1} there is a negative crossing, σ_{k+1}^{-1}, between the 1^{st} strand and the $(k+2)^{th}$, as the 1^{st} strand ends at the $(k+2)^{th}$ point and the $(k+2)^{th}$ strand ends at the 1^{st} point under ρ_1. Finally, the braid c^{-1} has no crossing between the 1^{st} and the $(k+2)^{th}$ strands and the $(k+2)^{th}$ strand ends at the 2^{nd} point. Thus, the braid, $c \cdot b^{k+1} \cdot c^{-1}$, concerning the first and the second strands, consists precisely of a crossing σ_1^{-1} and then ρ_1. This implies that the homomorphism $\bar{q}_{2+(2k+1),2}$, that can be considered geometrically as forgetting the last $(2k+1)$ strands, maps the element $c \cdot b^{k+1} \cdot c^{-1}$ to $\sigma_1^{-1}\rho_1$. Therefore, $(\bar{q}_{2+(2k+1),2} \circ \bar{i}_2)(a_2) = a_2 \in B_2(\mathbb{R}P^2)$. Similarly, in the braid c, the 1^{st} and the 2^{nd} strands have no crossing between them, and the 2^{nd} strand ends at the $(k+2)^{th}$ point, while the 1^{st} stand ends at the 1^{st} point. Then, in the braid b^k there is no crossing between the 1^{st} strand and the $(k+2)^{th}$, where the 1^{st} strand ends at the $(k+1)^{th}$ point and the $(k+2)^{th}$ strand ends at the $(2k+2)^{th}$ point. The braid Δ contains only one positive crossing σ_{k+1} between the $(k+1)^{th}$ strand and the $(2k+2)^{th}$, as the $(k+1)^{th}$ strand ends at the $(k+3)^{th}$ point and the $(2k+2)^{th}$ ends at the 2^{nd} point. Then, the braid a^{-1} has no crossing between the 2^{nd} and the $(k+3)^{th}$ strands, where the 2^{nd} strand ends at the 1^{st} point and the $(k+3)^{th}$ and the $(k+2)^{th}$ point. Finally, the braid c^{-1} has no crossing between the 1^{st} and the $(k+2)^{th}$ strands and the $(k+2)^{th}$ strand ends at the 2^{nd} point. Thus, the braid, $c \cdot b^k \Delta a^{-1} \cdot c^{-1}$, concerning the first and the second strands, consists precisely of a crossing σ_1. This implies that the homomorphism $\bar{q}_{2+(2k+1),2}$, that can be considered

geometrically as forgetting the last $(2k+1)$ strands, maps the element $c \cdot b^k \Delta a^{-1} \cdot c^{-1}$ to σ_1. Therefore, $(\bar{q}_{2+(2k+1),2} \circ \bar{i}_2)(\Delta_2) = \Delta_2 \in B_2(\mathbb{R}P^2)$. Summing up, the homomorphism $\bar{i}_2$ is a section for the homomorphism $\bar{q}_{2+(2k+1),2}$.

We conclude that, for every $m \in \mathbb{N}$, we have an explicit algebraic section

$$\bar{s} : B_2(\mathbb{R}P^2) \to B_{2,m}(\mathbb{R}P^2)$$

for the homomorphism $\bar{q}_{2+m,2} : B_{2,m}(\mathbb{R}P^2) \to B_2(\mathbb{R}P^2)$ defined as follows:

$$\bar{s} = \begin{cases} \bar{i}_1, & \text{for } m = 2k, \\ \bar{i}_2, & \text{for } m = 2k+1. \end{cases}$$

□

3.2 The splitting problem for the cases $n \geq 3$

Let $m \in \mathbb{N}$ and $n \geq 3$. We consider the following short exact sequence:

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1. \tag{5}$$

In this Section we study the splitting problem of the short exact sequence (5), for $n \geq 3$ and $m \in \mathbb{N}$. In particular, we first consider the case $m = 1$, where we prove that this short exact sequence does not split for any value of $n \geq 3$, and later the consider the case $m > 1$.

Proposition 3.2.1. *For $n \geq 3$, the short exact sequence*

$$1 \longrightarrow B_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,1}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+1,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

does not split.

Proof. Let $n \geq 3$. We have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{n+1}(\mathbb{R}P^2) & \longrightarrow & B_{n,1}(\mathbb{R}P^2) & \longrightarrow & S_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \bar{q}_{n+1,n}|_{P_{n+1}(\mathbb{R}P^2)}} & & \downarrow{\scriptstyle \bar{q}_{n+1,n}} & & \| & & \\ 1 & \longrightarrow & P_n(\mathbb{R}P^2) & \longrightarrow & B_n(\mathbb{R}P^2) & \longrightarrow & S_n & \longrightarrow & 1. \end{array}$$

The horizontal short exact sequences are those described in (1). Suppose on the contrary that for $n \geq 3$, the homomorphism $\bar{q}_{n+1,n} : B_{n,1}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ admits an algebraic section $\bar{s}_{n+1,n} : B_n(\mathbb{R}P^2) \to B_{n,1}(\mathbb{R}P^2)$. Recall that the homomorphism $\bar{q}_{n+1,n}$ can be considered geometrically as forgetting the last strand. Therefore, from the commutative diagram it follows that there exists an algebraic section $\bar{s}_{n+1,n}|_{P_n(\mathbb{R}P^2)}$ for the homomorphism $\bar{q}_{n+1,n}|_{P_{n+1}(\mathbb{R}P^2)}$, as well. But from Theorem 1.3.3, we know that for $n \geq 3$ and $m = 1$ the homomorphism $P_{n+1}(\mathbb{R}P^2) \to P_n(\mathbb{R}P^2)$ does not admit an algebraic section. We thus reach a contradiction, and this concludes the proof. □

We consider now the case $m > 1$. For $n \geq 3$ and $m > 1$ we will provide a necessary condition, in order to have that the short exact sequence (5) splits. Let N be a normal subgroup of $B_{n,m}(\mathbb{R}P^2)$, which is contained also in $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$. The obtained quotiented short exact sequence is the following:

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})/N \longrightarrow B_{n,m}(\mathbb{R}P^2)/N \xrightarrow{q} B_n(\mathbb{R}P^2) \longrightarrow 1, \tag{6}$$

where $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$ denotes the induced homomorphism. We observe that if the short exact sequence (5) splits then this quotiented short exact sequence has to split as well. To be more precise, consider the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) & \longrightarrow & B_{n,m}(\mathbb{R}P^2) & \xrightarrow{\bar{q}_{n+m,n}} & B_n(\mathbb{R}P^2) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle pr} & & \| & & \\ 1 & \longrightarrow & B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})/N & \longrightarrow & B_{n,m}(\mathbb{R}P^2)/N & \xrightarrow{q} & B_n(\mathbb{R}P^2) & \longrightarrow & 1. \end{array} \tag{7}$$

Suppose that the homomorphism $\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ admits a section $\bar{s}_{n+m,n} : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)$. Then $q : B_{n,m}(\mathbb{R}P^2)/N \to B_n(\mathbb{R}P^2)$ admits a section $s = pr \circ \bar{s}_{n+m,n} : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)/N$. Based on this observation, we see that it is helpful to know the lower central and derived series of the braid groups appearing in these group extensions, since possible choices of the group N could be elements of either the lower central series of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ or of the derived series of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$. The choice that we make for N, in order to study the splitting of the quotiented short exact sequence (6) is $\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$. We know that the subgroups of the lower central series of a group are characteristic subgroups. Thus, $\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$ is a characteristic subgroup of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, and since $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is a normal subgroup of $B_{n,m}(\mathbb{R}P^2)$, it follows that $\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$ is a normal subgroup of $B_{n,m}(\mathbb{R}P^2)$. For $\Gamma := \Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$, which is a normal subgroup of $B_{n,m}(\mathbb{R}P^2)$ contained in $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$, we get the following short exact sequence:

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma \longrightarrow B_{n,m}(\mathbb{R}P^2)/\Gamma \xrightarrow{q} B_n(\mathbb{R}P^2) \longrightarrow 1. \tag{8}$$

In order to study the short exact sequence (8) we need to know a presentation of the quotients $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma$ and $B_{n,m}(\mathbb{R}P^2)/\Gamma$. We already have a presentation of the Abelian group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma$, given by Corollary 2.2.11. We recall that $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma \cong \mathbb{Z}^n \times \mathbb{Z}_2$, where $\rho, \beta_1, \dots, \beta_{n-1}$ generate the $\mathbb{Z}^n$-component and σ the $\mathbb{Z}_2$-component. Regarding a possible presentation of $B_{n,m}(\mathbb{R}P^2)/\Gamma$, one can obtain one using the presentation of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma$, given by Corollary 2.2.11, the given presentation of $B_n(\mathbb{R}P^2)$ in Theorem 2.1.1, and applying standard techniques for obtaining a presentation of group extensions as described in [Joh97] (page 139) and presented in detail in Propositions 2.2.1, 2.2.4, 2.2.5 and Theorem 2.2.7.

However, for $N = \Gamma$ the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma$ is the Abelianisation of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$. Thus, from commutative diagram of short exact sequences (7), a presentation of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ may also be obtained straightforwardly by considering as set of generators the union of the generators of $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})/\Gamma$ and the coset representatives of the generators of $B_n(\mathbb{R}P^2)$, and as set of relations the relations of $B_{n,m}(\mathbb{R}P^2)$, given in Theorem 2.2.7, projected into $B_{n,m}(\mathbb{R}P^2)/\Gamma$. Thus, we obtain the following proposition.

Proposition 3.2.2. *For $n \geq 2$, $m \geq 1$, the following constitutes a presentation of $B_{n,m}(\mathbb{R}P^2)/\Gamma$, where $\Gamma = \Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}))$.*

Generators: ρ, σ, $\beta_1, \dots, \beta_n$, $\tau_1, \dots, \tau_{n-1}$, $q_1, \dots, q_n$.

Relations:

(I) *The relations (i)-(vi) from Corollary 2.2.11,*

(II) *For $1 \leq i, k \leq n-1$, $1 \leq j \leq n$,*
$\tau_i\tau_k = \tau_k\tau_i$, *for* $|i-k| > 1$,
$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$,
$\tau_i q_j = q_j\tau_i$, *for* $j \neq i, i+1$,
$q_i = \tau_i q_{i+1}\tau_i$,
$\tau_i^2 = q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i$,
$q_1^2 = (\tau_1\tau_2 \dots \tau_{n-2}\tau_{n-1})\beta_n^m(\tau_{n-1}\tau_{n-2} \dots \tau_2\tau_1)$.

(III) *(a)* *For $1 \leq i \leq n-1$, $1 \leq j \leq n$,*
$\sigma\tau_i = \tau_i\sigma$,
$\sigma q_j = q_j\sigma$.

(b) *For $1 \leq i \leq n-1$, $1 \leq j \leq n$,*
$\rho\tau_i = \tau_i\rho$,
$q_j\rho = \beta_j\rho q_j$.

(c) *For $1 \leq i \leq n-1$, $1 \leq k \leq n$,*

$$\tau_i \beta_k \tau_i^{-1} = \begin{cases} \beta_k, & \text{for } i \neq k-1, k. \\ \beta_{k-1}, & \text{for } i = k-1. \\ \beta_{k+1}, & \text{for } i = k. \end{cases}$$

(d) For $1 \leq j, k \leq n$,

$$q_j \beta_k q_j^{-1} = \begin{cases} \beta_k, & \text{for } j \neq k. \\ \beta_k^{-1}, & \text{for } j = k. \end{cases}$$

Remark 3.2.3. In Proposition 3.2.2 we denote by τ_i and q_j the coset representatives of the generators σ_i and ρ_j of $B_n(\mathbb{R}P^2)$, given by Theorem 2.1.1, for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Now, suppose that there exists a section $\bar{s}_{n+m,m} : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)$ for the map $\bar{q}_{n+m,m} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$. As we already saw, it follows that there exists a section $s : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)/\Gamma$ for $q : B_{n,m}(\mathbb{R}P^2)/\Gamma \to B_n(\mathbb{R}P^2)$. From Corollary 2.2.11, the set $\{\beta_1, \ldots, \beta_{n-1}, \rho, \sigma\}$ forms a generating set of $\ker(q)$, which is the group $B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})/\Gamma \cong \mathbb{Z}^n \times \mathbb{Z}_2$. This allows us to consider the image of the generating set of $B_n(\mathbb{R}P^2)$, under the section s, as follows:

$$s(\tau_i) = \tau_i \cdot \beta_1^{k_{i,1}} \beta_2^{k_{i,2}} \cdots \beta_{n-1}^{k_{i,n-1}} \rho^{l_i} \sigma^{m_i}, \text{ for } 1 \leq i \leq n-1, \tag{9}$$

$$s(q_j) = q_j \cdot \beta_1^{\bar{k}_{j,1}} \beta_2^{\bar{k}_{j,2}} \cdots \beta_{n-1}^{\bar{k}_{j,n-1}} \rho^{\bar{l}_j} \sigma^{\bar{m}_j}, \text{ for } 1 \leq j \leq n, \tag{10}$$

where $k_{i,1}, k_{i,2}, \ldots, k_{i,n-1}, l_i, \bar{k}_{j,1}, \bar{k}_{j,2}, \bar{k}_{j,n-1}, \bar{l}_j \in \mathbb{Z}$ and $m_i, \bar{m}_j \in \{0, 1\}$. Note that these integers are unique for every $s(\tau_i)$ and $s(q_j)$, where $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Under the assumption that there exists a section $s : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)/\Gamma$, the images under s of the relations in $B_n(\mathbb{R}P^2)$ are also relations in $B_{n,m}(\mathbb{R}P^2)/\Gamma$. In this way, we will obtain further information regarding the exponents in the formulas (9), (10) and possible restrictions for the value of m, under the assumption that the short exact sequence (5) splits.

Based on the presentation of $B_n(\mathbb{R}P^2)$ for $n \geq 3$ given by Theorem 2.1.1, we have the following six relations, which hold in $B_{n,m}(\mathbb{R}P^2)/\Gamma$:

- **R1.** $s(\tau_i q_j) = s(q_j \tau_i)$, for $j \neq i, i+1$, $1 \leq i \leq n-1$, $1 \leq j \leq n$.
- **R2.** $s(q_i) = s(\tau_i q_{i+1} \tau_i)$, for $1 \leq i \leq n-1$.
- **R3.** $s(\tau_i \tau_{i+1} \tau_i) = s(\tau_{i+1} \tau_i \tau_{i+1})$, for $1 \leq i \leq n-1$.
- **R4.** $s(\tau_i^2) = s(q_{i+1}^{-1} q_i^{-1} q_{i+1} q_i)$, for $1 \leq i \leq n-1$.
- **R5.** $s(\tau_i \tau_j) = s(\tau_j \tau_i)$, for $|i-j| > 1$ and $1 \leq i, j \leq n-1$.
- **R6.** $s(q_1^2) = s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}^2 \tau_{n-2} \cdots \tau_2 \tau_1)$.

In order to prove the theorem that follows, we will make use of the following relations in $B_{n,m}(\mathbb{R}P^2)/\Gamma$ that appear in Proposition 3.2.2:

1. $\sigma^2 = 1$, relation (i) of Corollary 2.2.11,
2. $\beta_n = \beta_1^{-1} \cdots \beta_i^{-1} \cdots \beta_{n-1}^{-1} \rho^{-2}$, relation (ii) of Corollary 2.2.11,
3. β_j, σ, ρ commute pairwise, relations (iii), (iv), (v) of Corollary 2.2.11,
4. τ_i commutes with σ, ρ, relations $(III)(a)$, $(III)(b)$ of Proposition 3.2.2,
5. $\beta_i \tau_i = \tau_i \beta_{i+1}$ and $\beta_{i+1} \tau_i = \tau_i \beta_i$, relation $(III)(c)$ of Proposition 3.2.2,
6. $q_i = \tau_i q_{i+1} \tau_i$ and $\tau_i q_j = q_j \tau_i$, for $j \neq i, i+1$, relation (II) of Proposition 3.2.2,

7. q_j commutes with σ, relation $(III)(a)$ of Proposition 3.2.2,

8. $\beta_j q_j = q_j \beta_j^{-1}$ and $\beta_i q_j = q_j \beta_i$, for $j \neq i$, relation $(III)(d)$ of Proposition 3.2.2,

9. $\rho q_j = q_j \beta_j \rho$, which is deduced from $q_j \rho = \beta_j \rho q_j$ and $\beta_j q_j = q_j \beta_j^{-1}$ given by $(III)(d)$ and $(III)(b)$ of Proposition 3.2.2 respectively. Thus, using also the fact that ρ commutes with β_i, it holds that $\rho^k q_j = \rho^{k-1}(\rho q_j) = \rho^{k-1}(q_j \beta_j \rho) = \rho^{k-2}(\rho q_j)(\beta_j \rho) = \rho^{k-2}(q_j \beta_j \rho)(\beta_j \rho) = \rho^{k-2}(q_j \beta_j^2 \rho^2) = \cdots = \rho(q_j \beta_j^{k-1} \rho^{k-1}) = q_j \beta_j^k \rho^k$, which means that

$$\rho^k q_j = q_j \beta_j^k \rho^k, \text{ for } k \in \mathbb{Z},$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Theorem 3.2.4. *Let $m \geq 1$ and $n \geq 3$. If the following short exact sequence*

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1 \tag{11}$$

splits, then $m = k(n-1)$, where $k \geq 1$.

Proof. Based on the above discussion and on the assumption that the short exact sequence (11) splits, we will examine the relations **R1.-R6.**, which hold in $B_{n,m}(\mathbb{R}P^2)/\Gamma$, from which we will deduce that m is a multiple of $(n-1)$.

We start with the relation **R1.** where $s(\tau_i q_j) = s(q_j \tau_i)$, for $j \neq i, i+1$, $1 \leq i \leq n-1$ and $1 \leq j \leq n$. We focus on the case where $i = n-1$ and $1 \leq j \leq n-2$. Using the formulas (9), (10) and the above relations, 1.-9., we have the following:

$$\begin{aligned}
s(\tau_{n-1} q_j) &= \tau_{n-1} \beta_1^{k_{n-1,1}} \cdots \beta_j^{k_{n-1,j}} \cdots \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{m_{n-1}} q_j \beta_1^{\bar{k}_{j,1}} \cdots \beta_j^{\bar{k}_{j,j}} \cdots \beta_{n-1}^{\bar{k}_{j,n-1}} \rho^{\bar{l}_j} \sigma^{\bar{m}_j} \\
&= \tau_{n-1} q_j \beta_1^{k_{n-1,1}} \cdots \beta_j^{-k_{n-1,j}} \cdots \beta_{n-1}^{k_{n-1,n-1}} \beta_j^{l_{n-1}} \rho^{l_{n-1}} \sigma^{m_{n-1}} \beta_1^{\bar{k}_{j,1}} \cdots \beta_j^{\bar{k}_{j,j}} \cdots \beta_{n-1}^{\bar{k}_{j,n-1}} \rho^{\bar{l}_j} \sigma^{\bar{m}_j} \\
&= \tau_{n-1} q_j \beta_1^{k_{n-1,1} + \bar{k}_{j,1}} \cdots \beta_j^{-k_{n-1,j} + l_{n-1} + \bar{k}_{j,j}} \cdots \beta_{n-1}^{k_{n-1,n-1} + \bar{k}_{j,n-1}} \rho^{l_{n-1} + \bar{l}_j} \sigma^{m_{n-1} + \bar{m}_j}
\end{aligned}$$

and

$$\begin{aligned}
s(q_j \tau_{n-1}) &= q_j \beta_1^{\bar{k}_{j,1}} \cdots \beta_j^{\bar{k}_{j,j}} \cdots \beta_{n-1}^{\bar{k}_{j,n-1}} \rho^{\bar{l}_j} \sigma^{\bar{m}_j} \tau_{n-1} \beta_1^{k_{n-1,1}} \cdots \beta_j^{k_{n-1,j}} \cdots \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{m_{n-1}} \\
&= q_j \tau_{n-1} \beta_1^{\bar{k}_{j,1}} \cdots \beta_j^{\bar{k}_{j,j}} \cdots \beta_{n-2}^{\bar{k}_{j,n-2}} \beta_n^{\bar{k}_{j,n-1}} \rho^{\bar{l}_j} \sigma^{\bar{m}_j} \beta_1^{k_{n-1,1}} \cdots \beta_j^{k_{n-1,j}} \cdots \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{m_{n-1}} \\
&= q_j \tau_{n-1} \beta_1^{\bar{k}_{j,1}} \cdots \beta_j^{\bar{k}_{j,j}} \cdots \beta_{n-2}^{\bar{k}_{j,n-2}} (\beta_1^{-\bar{k}_{j,n-1}} \cdots \beta_j^{-\bar{k}_{j,n-1}} \cdots \beta_{n-1}^{-\bar{k}_{j,n-1}} \rho^{-2\bar{k}_{j,n-1}}) \rho^{\bar{l}_j} \sigma^{\bar{m}_j} \\
&\quad \beta_1^{k_{n-1,1}} \cdots \beta_j^{k_{n-1,j}} \cdots \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{m_{n-1}} \\
&= q_j \tau_{n-1} \beta_1^{\bar{k}_{j,1} - \bar{k}_{j,n-1} + k_{n-1,1}} \cdots \beta_j^{\bar{k}_{j,j} - \bar{k}_{j,n-1} + k_{n-1,j}} \cdots \beta_{n-2}^{\bar{k}_{j,n-2} - \bar{k}_{j,n-1} + k_{n-1,n-2}} \beta_{n-1}^{-\bar{k}_{j,n-1} + k_{n-1,n-1}} \\
&\quad \rho^{\bar{l}_j - 2\bar{k}_{j,n-1} + l_{n-1}} \sigma^{\bar{m}_j + m_{n-1}}.
\end{aligned}$$

Comparing the coefficients of the elements ρ and β_j of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$\bar{k}_{j,n-1} = 0, \text{ for } 1 \leq j \leq n-2. \tag{12}$$

$$l_{n-1} = 2k_{n-1,j}, \text{ for } 1 \leq j \leq n-2. \tag{13}$$

In the case where $j = n$ and $1 \leq i < n-1$ we have the following:

$$\begin{aligned}
s(\tau_i q_n) &= \tau_i \beta_1^{k_{i,1}} \cdots \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \cdots \beta_{n-1}^{k_{i,n-1}} \rho^{l_i} \sigma^{m_i} q_n \beta_1^{\bar{k}_{n,1}} \cdots \beta_i^{\bar{k}_{n,i}} \beta_{i+1}^{\bar{k}_{n,i+1}} \cdots \beta_{n-1}^{\bar{k}_{n,n-1}} \rho^{\bar{l}_n} \sigma^{\bar{m}_n} \\
&= \tau_i q_n \beta_1^{k_{i,1}} \cdots \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \cdots \beta_{n-1}^{k_{i,n-1}} \beta_n^{l_i} \rho^{l_i} \sigma^{m_i} \beta_1^{\bar{k}_{n,1}} \cdots \beta_i^{\bar{k}_{n,i}} \beta_{i+1}^{\bar{k}_{n,i+1}} \cdots \beta_{n-1}^{\bar{k}_{n,n-1}} \rho^{\bar{l}_n} \sigma^{\bar{m}_n} \\
&= \tau_i q_n \beta_1^{k_{i,1} - l_i + \bar{k}_{n,1}} \cdots \beta_i^{k_{i,i} - l_i + \bar{k}_{n,i}} \beta_{i+1}^{k_{i,i+1} - l_i + \bar{k}_{n,i+1}} \cdots \beta_{n-1}^{k_{i,n-1} - l_i + \bar{k}_{n,n-1}} \rho^{l_i - 2l_i + \bar{l}_n} \sigma^{m_i + \bar{m}_n}
\end{aligned}$$

and

$$\begin{aligned} s(q_n\tau_i) &= q_n\beta_1^{\bar{k}_{n,1}}\cdots\beta_i^{\bar{k}_{n,i}}\beta_{i+1}^{\bar{k}_{n,i+1}}\cdots\beta_{n-1}^{\bar{k}_{n,n-1}}\rho^{\bar{l}_n}\sigma^{\bar{m}_n}\tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\rho^{l_i}\sigma^{m_i} \\ &= q_n\tau_i\beta_1^{\bar{k}_{n,1}}\cdots\beta_{i+1}^{\bar{k}_{n,i}}\beta_i^{\bar{k}_{n,i+1}}\cdots\beta_{n-1}^{\bar{k}_{n,n-1}}\rho^{\bar{l}_n}\sigma^{\bar{m}_n}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\rho^{l_i}\sigma^{m_i} \\ &= q_n\tau_i\beta_1^{\bar{k}_{n,1}+k_{i,1}}\cdots\beta_i^{\bar{k}_{n,i+1}+k_{i,i}}\beta_{i+1}^{\bar{k}_{n,i}+k_{i,i+1}}\cdots\beta_{n-1}^{\bar{k}_{n,n-1}+k_{i,n-1}}\rho^{\bar{l}_n+l_i}\sigma^{\bar{m}_n+m_i}. \end{aligned}$$

Comparing the coefficients of the element β_1 of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$l_i = 0, \text{ for } 1 \leq i \leq n-2. \tag{14}$$

We continue with **R2.** for which $s(q_i) = s(\tau_i q_{i+1}\tau_i)$, for $1 \leq i \leq n-1$. We will discuss the case $i = n-1$ separately. So, for $1 \leq i \leq n-2$ we have from (14) that $l_i = 0$, and once again, based on the formulas (9), (10) and the previously mentioned relations, 1.-9., we have the following:
$s(q_i) = q_i\beta_1^{\bar{k}_{i,1}}\cdots\beta_i^{\bar{k}_{i,i}}\beta_{i+1}^{\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i,n-1}}\rho^{\bar{l}_i}\sigma^{\bar{m}_i}$

and

$$\begin{aligned} s(\tau_i q_{i+1}\tau_i) &= \tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}q_{i+1}\beta_1^{\bar{k}_{i+1,1}}\cdots\beta_i^{\bar{k}_{i+1,i}}\beta_{i+1}^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\rho^{\bar{l}_{i+1}}\sigma^{\bar{m}_{i+1}} \\ &\quad \tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i} \\ &= \tau_i q_{i+1}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{-k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\tau_i\beta_1^{\bar{k}_{i+1,1}}\cdots\beta_{i+1}^{\bar{k}_{i+1,i}}\beta_i^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\rho^{\bar{l}_{i+1}}\sigma^{\bar{m}_{i+1}} \\ &\quad \beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i} \\ &= \tau_i q_{i+1}\tau_i\beta_1^{k_{i,1}}\cdots\beta_{i+1}^{k_{i,i}}\beta_i^{-k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\beta_1^{\bar{k}_{i+1,1}}\cdots\beta_{i+1}^{\bar{k}_{i+1,i}}\beta_i^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\rho^{\bar{l}_{i+1}}\sigma^{\bar{m}_{i+1}} \\ &\quad \beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i} \\ &= \tau_i q_{i+1}\tau_i\beta_1^{2k_{i,1}+\bar{k}_{i+1,1}}\cdots\beta_i^{-k_{i,i+1}+\bar{k}_{i+1,i+1}+k_{i,i}}\beta_{i+1}^{k_{i,i}+\bar{k}_{i+1,i}+k_{i,i+1}}\cdots\beta_{n-1}^{2k_{i,n-1}+\bar{k}_{i+1,n-1}}\rho^{\bar{l}_{i+1}}\sigma^{2m_i+\bar{m}_{i+1}} \\ &= \tau_i q_{i+1}\tau_i\beta_1^{2k_{i,1}+\bar{k}_{i+1,1}}\cdots\beta_i^{-k_{i,i+1}+\bar{k}_{i+1,i+1}+k_{i,i}}\beta_{i+1}^{k_{i,i}+\bar{k}_{i+1,i}+k_{i,i+1}}\cdots\beta_{n-1}^{2k_{i,n-1}+\bar{k}_{i+1,n-1}}\rho^{\bar{l}_{i+1}}\sigma^{\bar{m}_{i+1}}. \end{aligned}$$

Comparing the coefficients of the elements σ and ρ of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$\bar{m}_i = \bar{m}_{i+1}, \text{ for } 1 \leq i \leq n-2. \tag{15}$$

$$\bar{l}_i = \bar{l}_{i+1}, \text{ for } 1 \leq i \leq n-2. \tag{16}$$

In the case where $i = n-1$ we have the following:
$s(q_{n-1}) = q_{n-1}\beta_1^{\bar{k}_{n-1,1}}\cdots\beta_i^{\bar{k}_{n-1,i}}\cdots\beta_{n-1}^{\bar{k}_{n-1,n-1}}\rho^{\bar{l}_{n-1}}\sigma^{\bar{m}_{n-1}}$

and

$$\begin{aligned} s(\tau_{n-1}q_n\tau_{n-1}) &= \tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{m_{n-1}}q_n\beta_1^{\bar{k}_{n,1}}\cdots\beta_{n-1}^{\bar{k}_{n,n-1}}\rho^{\bar{l}_n}\sigma^{\bar{m}_n} \\ &\quad \tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{m_{n-1}}. \end{aligned}$$

We know that the element σ commutes with the other elements that appear in the second equation. This implies that the coefficient of σ in the second equation is $\bar{m}_n$, since $\sigma^2 = 1$. Thus, comparing the coefficients of the element σ of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$\bar{m}_n = \bar{m}_{n-1}. \tag{17}$$

We continue with relation **R3.** where $s(\tau_i\tau_{i+1}\tau_i) = s(\tau_{i+1}\tau_i\tau_{i+1})$, for $1 \leq i \leq n-1$. We will discuss the case $i = n-2$ separately. So, for $1 \leq i < n-2$ we have from (14) that $l_i = 0$, and once again,

based on the formulas (9), (10) and the previously mentioned relations, 1.-9., we have the following:

$$\begin{aligned}
s(\tau_i\tau_{i+1}\tau_i) &= \tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\beta_{i+2}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\tau_{i+1}\beta_1^{k_{i+1,1}}\cdots\beta_i^{k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}}\\
&\quad \tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\beta_{i+2}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\\
&= \tau_i\tau_{i+1}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+2}^{k_{i,i+1}}\beta_{i+1}^{k_{i,i+2}}\beta_i^{k_{i,i}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\tau_i\beta_1^{k_{i+1,1}}\cdots\beta_{i+1}^{k_{i+1,i}}\\
&\quad \beta_i^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\beta_{i+2}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i+m_{i+1}}\\
&= \tau_i\tau_{i+1}\tau_i\beta_1^{k_{i,1}}\cdots\beta_{i+1}^{k_{i,i}}\beta_{i+2}^{k_{i,i+1}}\beta_i^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\beta_1^{k_{i+1,1}}\cdots\beta_{i+1}^{k_{i+1,i}}\beta_i^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\\
&\quad \beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\beta_{i+2}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i+m_{i+1}}\\
&= \tau_i\tau_{i+1}\tau_i\beta_1^{2k_{i,1}+k_{i+1,1}}\cdots\beta_{i-1}^{2k_{i,i-1}+k_{i+1,i-1}}\beta_i^{k_{i,i+2}+k_{i+1,i+1}+k_{i,i}}\beta_{i+1}^{k_{i,i}+k_{i+1,i}+k_{i,i+1}}\\
&\quad \beta_{i+2}^{k_{i,i+1}+k_{i+1,i+2}+k_{i,i+2}}\beta_{i+3}^{2k_{i,i+3}+k_{i+1,i+3}}\cdots\beta_{n-1}^{2k_{i,n-1}+k_{i+1,n-1}}\sigma^{m_{i+1}}
\end{aligned}$$

and

$$\begin{aligned}
s(\tau_{i+1}\tau_i\tau_{i+1}) &= \tau_{i+1}\beta_1^{k_{i+1,1}}\cdots\beta_i^{k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}}\tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\\
&\quad \beta_{i+2}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^{m_i}\tau_{i+1}\beta_1^{k_{i+1,1}}\cdots\beta_i^{k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}}\\
&= \tau_{i+1}\tau_i\beta_1^{k_{i+1,1}}\cdots\beta_{i+1}^{k_{i+1,i}}\beta_i^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}}\tau_{i+1}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\\
&\quad \beta_{i+2}^{k_{i,i+1}}\beta_{i+1}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\beta_1^{k_{i+1,1}}\cdots\beta_i^{k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}+m_i}\\
&= \tau_{i+1}\tau_i\tau_{i+1}\beta_1^{k_{i+1,1}}\cdots\beta_{i+2}^{k_{i+1,i}}\beta_i^{k_{i+1,i+1}}\beta_{i+1}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}}\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\\
&\quad \beta_{i+2}^{k_{i,i+1}}\beta_{i+1}^{k_{i,i+2}}\cdots\beta_{n-1}^{k_{i,n-1}}\beta_1^{k_{i+1,1}}\cdots\beta_i^{k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+1}}\beta_{i+2}^{k_{i+1,i+2}}\cdots\beta_{n-1}^{k_{i+1,n-1}}\sigma^{m_{i+1}+m_i}\\
&= \tau_{i+1}\tau_i\tau_{i+1}\beta_1^{2k_{i+1,1}+k_{i,1}}\cdots\beta_{i-1}^{2k_{i+1,i-1}+k_{i,i-1}}\beta_i^{k_{i+1,i+1}+k_{i,i}+k_{i+1,i}}\beta_{i+1}^{k_{i+1,i+2}+k_{i,i+2}+k_{i+1,i+1}}\\
&\quad \beta_{i+2}^{k_{i+1,i}+k_{i,i+1}+k_{i+1,i+2}}\beta_{i+3}^{2k_{i+1,i+3}+k_{i,i+3}}\cdots\beta_{n-1}^{2k_{i+1,n-1}+k_{i,n-1}}\sigma^{m_i}.
\end{aligned}$$

Comparing the coefficients of the element β_i of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, it follows that $k_{i+1,i} = k_{i,i+2}$, for $1 \le i < n-2$. Using this relation and comparing the coefficients of the element β_{i+1} of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, it follows that $k_{i,i} + k_{i,i+1} = k_{i+1,i+2} + k_{i+1,i+1}$, for $1 \le i < n-2$. Moreover, comparing the coefficients of σ we obtain $m_i = m_{i+1}$, for $1 \le i < n-2$. Thus, we have the following:

$$m_i = m_{i+1}, \text{ for } 1 \le i < n-2. \tag{18}$$

$$k_{i,i} + k_{i,i+1} = k_{i+1,i+2} + k_{i+1,i+1}, \text{ for } 1 \le i < n-2. \tag{19}$$

In the case where $i = n-2$ we have the following:

$$\begin{aligned}
s(\tau_{n-2}\tau_{n-1}\tau_{n-2}) =& \tau_{n-2}\beta_1^{k_{n-2,1}}\cdots\beta_{n-2}^{k_{n-2,n-2}}\beta_{n-1}^{k_{n-2,n-1}}\sigma^{m_{n-2}}\tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_{n-2}^{k_{n-1,n-2}}\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{m_{n-1}}\\
&\tau_{n-2}\beta_1^{k_{n-2,1}}\cdots\beta_{n-2}^{k_{n-2,n-2}}\beta_{n-1}^{k_{n-2,n-1}}\sigma^{m_{n-2}}
\end{aligned}$$

and

$$\begin{aligned}
s(\tau_{n-1}\tau_{n-2}\tau_{n-1}) =& \tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_{n-2}^{k_{n-1,n-2}}\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{m_{n-1}}\tau_{n-2}\beta_1^{k_{n-2,1}}\cdots\beta_{n-2}^{k_{n-2,n-2}}\beta_{n-1}^{k_{n-2,n-1}}\sigma^{m_{n-2}}\\
&\tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_{n-2}^{k_{n-1,n-2}}\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{m_{n-1}}.
\end{aligned}$$

As σ commutes with the other elements that appear in these two equations, it follows that the coefficient of σ in the first equation is m_{n-1} and in second equation is m_{n-2}, since $\sigma^2 = 1$. Thus, we obtain the following:

$$m_{n-1} = m_{n-2}. \tag{20}$$

We continue with the relation **R4.** where $s(\tau_i^2) = s(q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i)$, for $1 \le i \le n-1$. From (15) and (17), we have that $\bar{m}_1 = \bar{m}_2 = \cdots = \bar{m}_{n-1} = \bar{m}_n =: \bar{M}$ and from (18) and (20) that $m_1 = m_2 = \cdots = m_{n-2} = m_{n-1} := M$. From the mentioned relations, 1.-9., it holds that $\rho^k q_j = q_j\beta_j^k\rho^k$,

and thus we obtain $\rho^k q_j^{-1} = q^{-j}\beta_j^k\rho^k$ and $\rho^{-k}q_j^{-1} = q^{-j}\beta_j^{-k}\rho^{-k}$, for $k \in \mathbb{Z}$, and $1 \leq j \leq n$. Moreover, using (14) and (16), we obtain the following, for $1 \leq i < n-1$:

$$\begin{aligned}
s(\tau_i^2) &= \tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^M\tau_i\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^M\\
&= \tau_i^2\beta_1^{k_{i,1}}\cdots\beta_{i+1}^{k_{i,i}}\beta_i^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^M\beta_1^{k_{i,1}}\cdots\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\beta_{n-1}^{k_{i,n-1}}\sigma^M\\
&= \tau_i^2\beta_1^{2k_{i,1}}\cdots\beta_{i-1}^{2k_{i,i-1}}\beta_i^{k_{i,i+1}+k_{i,i}}\beta_{i+1}^{k_{i,i}+k_{i,i+1}}\beta_{i+2}^{2k_{i,i+2}}\cdots\beta_{n-1}^{2k_{i,n-1}}
\end{aligned}$$

and

$$\begin{aligned}
s(q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i) &= \beta_1^{-\bar{k}_{i+1,1}}\cdots\beta_i^{-\bar{k}_{i+1,i}}\beta_{i+1}^{-\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{-\bar{k}_{i+1,n-1}}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}q_{i+1}^{-1}\\
&\quad \beta_1^{-\bar{k}_{i,1}}\cdots\beta_i^{-\bar{k}_{i,i}}\beta_{i+1}^{-\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{-\bar{k}_{i,n-1}}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}q_i^{-1}\\
&\quad q_{i+1}\beta_1^{\bar{k}_{i+1,1}}\cdots\beta_i^{\bar{k}_{i+1,i}}\beta_{i+1}^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&\quad q_i\beta_1^{\bar{k}_{i,1}}\cdots\beta_i^{\bar{k}_{i,i}}\beta_{i+1}^{\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i,n-1}}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&= q_{i+1}^{-1}\beta_1^{-\bar{k}_{i+1,1}}\cdots\beta_i^{-\bar{k}_{i+1,i}}\beta_{i+1}^{\bar{k}_{i+1,i+1}}\beta_{i+2}^{-\bar{k}_{i+1,i+2}}\cdots\beta_{n-1}^{-\bar{k}_{i+1,n-1}}\beta_{i+1}^{-\bar{l}_i}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}\\
&\quad q_i^{-1}\beta_1^{-\bar{k}_{i,1}}\cdots\beta_{i-1}^{-\bar{k}_{i,i-1}}\beta_i^{\bar{k}_{i,i}}\beta_{i+1}^{-\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{-\bar{k}_{i,n-1}}\beta_i^{-\bar{l}_i}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}\\
&\quad q_{i+1}q_i\beta_1^{\bar{k}_{i+1,1}}\cdots\beta_{i-1}^{\bar{k}_{i+1,i-1}}\beta_i^{-\bar{k}_{i+1,i}}\beta_{i+1}^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\beta_i^{\bar{l}_i}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&\quad \beta_1^{\bar{k}_{i,1}}\cdots\beta_i^{\bar{k}_{i,i}}\beta_{i+1}^{\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i,n-1}}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&= q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i\beta_1^{-\bar{k}_{i+1,1}}\cdots\beta_i^{-\bar{k}_{i+1,i}}\beta_{i+1}^{-\bar{k}_{i+1,i+1}}\beta_{i+2}^{-\bar{k}_{i+1,i+2}}\cdots\beta_{n-1}^{-\bar{k}_{i+1,n-1}}\beta_{i+1}^{\bar{l}_i}\beta_i^{\bar{l}_i}\beta_{i+1}^{-\bar{l}_i}\beta_i^{-\bar{l}_i}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}\\
&\quad \beta_1^{-\bar{k}_{i,1}}\cdots\beta_{i-1}^{-\bar{k}_{i,i-1}}\beta_i^{-\bar{k}_{i,i}}\beta_{i+1}^{\bar{k}_{i,i+1}}\beta_{i+2}^{-\bar{k}_{i,i+2}}\cdots\beta_{n-1}^{-\bar{k}_{i,n-1}}\beta_i^{\bar{l}_i}\beta_{i+1}^{-\bar{l}_i}\beta_i^{-\bar{l}_i}\rho^{-\bar{l}_i}\sigma^{-\bar{M}}\\
&\quad \beta_1^{\bar{k}_{i+1,1}}\cdots\beta_{i-1}^{\bar{k}_{i+1,i-1}}\beta_i^{-\bar{k}_{i+1,i}}\beta_{i+1}^{\bar{k}_{i+1,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i+1,n-1}}\beta_i^{\bar{l}_i}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&\quad \beta_1^{\bar{k}_{i,1}}\cdots\beta_i^{\bar{k}_{i,i}}\beta_{i+1}^{\bar{k}_{i,i+1}}\cdots\beta_{n-1}^{\bar{k}_{i,n-1}}\rho^{\bar{l}_i}\sigma^{\bar{M}}\\
&= q_{i+1}^{-1}q_i^{-1}q_{i+1}q_i\beta_1^0\cdots\beta_{i-1}^0\beta_i^{-2\bar{k}_{i+1,i}+\bar{l}_i}\beta_{i+1}^{2\bar{k}_{i,i+1}-\bar{l}_i}\beta_{i+2}^0\cdots\beta_{n-1}^0\rho^0.
\end{aligned}$$

Comparing the coefficients of $\beta_1,\ldots,\beta_{i-1},\beta_{i+2},\ldots\beta_{n-1}$ and of β_{i+1} of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$k_{i,s} = 0, \text{ for } s = 1,2,\ldots,n-1,\ s \neq i, i+1,\ 1 \leq i \leq n-2. \tag{21}$$

$$k_{i,i} + k_{i,i+1} = -\bar{l}_i + 2\bar{k}_{i,i+1},\ 1 \leq i \leq n-2. \tag{22}$$

From now onwards we suppose that $n \geq 4$, so that the relation that we will use exists. In the last but one relation **R5.**, where $s(\tau_i\tau_j) = s(\tau_j\tau_i)$, for $|i-j| > 1$ and $1 \leq i,j \leq n-1$, we will focus on the case where $i = n-1$ and $1 \leq j \leq n-3$. Thus, for these values, using (14) and (18), we have the following:

$$\begin{aligned}
s(\tau_{n-1}\tau_j) &= \tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_j^{k_{n-1,j}}\beta_{j+1}^{k_{n-1,j+1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^M\tau_j\beta_1^{k_{j,1}}\cdots\beta_j^{k_{j,j}}\beta_{j+1}^{k_{j,j+1}}\cdots\beta_{n-1}^{k_{j,n-1}}\sigma^M\\
&= \tau_{n-1}\tau_j\beta_1^{k_{n-1,1}}\cdots\beta_{j+1}^{k_{n-1,j}}\beta_j^{k_{n-1,j+1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^M\beta_1^{k_{j,1}}\cdots\beta_j^{k_{j,j}}\beta_{j+1}^{k_{j,j+1}}\cdots\beta_{n-1}^{k_{j,n-1}}\sigma^M\\
&= \tau_{n-1}\tau_j\beta_1^{k_{n-1,1}+k_{j,1}}\cdots\beta_j^{k_{n-1,j+1}+k_{j,j}}\beta_{j+1}^{k_{n-1,j}+k_{j,j+1}}\cdots\beta_{n-1}^{k_{n-1,n-1}+k_{j,n-1}}\rho^{l_{n-1}}
\end{aligned}$$

and

$$\begin{aligned}
s(\tau_j\tau_{n-1}) &= \tau_j\beta_1^{k_{j,1}}\cdots\beta_j^{k_{j,j}}\beta_{j+1}^{k_{j,j+1}}\cdots\beta_{n-1}^{k_{j,n-1}}\sigma^M\tau_{n-1}\beta_1^{k_{n-1,1}}\cdots\beta_j^{k_{n-1,j}}\beta_{j+1}^{k_{n-1,j+1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^M\\
&= \tau_j\tau_{n-1}\beta_1^{k_{j,1}}\cdots\beta_j^{k_{j,j}}\beta_{j+1}^{k_{j,j+1}}\cdots\beta_{n-2}^{k_{j,n-2}}\beta_n^{k_{j,n-1}}\sigma^M\beta_1^{k_{n-1,1}}\cdots\beta_j^{k_{n-1,j}}\beta_{j+1}^{k_{n-1,j+1}}\cdots\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^M\\
&= \tau_j\tau_{n-1}\beta_1^{k_{j,1}+k_{n-1,1}-k_{j,n-1}}\cdots\beta_j^{k_{j,j}+k_{n-1,j}-k_{j,n-1}}\beta_{j+1}^{k_{j,j+1}+k_{n-1,j+1}-k_{j,n-1}}\cdots\beta_{n-2}^{k_{j,n-2}+k_{n-1,n-2}-k_{j,n-1}}\\
&\quad \beta_{n-1}^{k_{n-1,n-1}-k_{j,n-1}}\rho^{l_{n-1}-2k_{j,n-1}}.
\end{aligned}$$

Comparing the coefficients of the element ρ of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, it follows that $k_{j,n-1} = 0$, for $1 \leq j \leq n-3$. Using this result and comparing the coefficients of β_i in these two equations, we obtain the following:

$$k_{n-1,j} = k_{n-1,j+1}, \text{ for } 1 \leq j \leq n-3. \tag{23}$$

We now analyse the results that we obtained so far. From (18), (20) we obtain $m_1 = m_2 = \cdots = m_{n-2} = m_{n-1} =: M$. From (14) we get $l_1 = l_2 = \cdots = l_{n-2} = 0$. From (13) we get $l_{n-1} = 2k_{n-1,j}$, for $1 \leq j \leq n-2$. From (16) it follows that $\bar{l}_1 = \bar{l}_2 = \cdots = \bar{l}_{n-2} = \bar{l}_{n-1} =: \bar{l}$. From (21) we get $k_{i,s} = 0$, for $1 \leq i \leq n-2$ and $s = 1, 2, \ldots, n-1, s \neq i, i+1$. That means that for $1 \leq i \leq n-2$ and $s = 1, 2, \ldots, n-1$, the coefficients $k_{i,s}$ are zero except for $k_{i,i}$ and $k_{i,i+1}$. Finally, from (23) we have $k_{n-1,1} = k_{n-1,2} = \cdots = k_{n-1,n-3} = k_{n-1,n-2} := \pi$. Gathering together all these results, the image of the elements τ_i, for $1 \leq i \leq n-1$, under the section s is follows:

$$s(\tau_i) = \tau_i \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \sigma^M, \text{ for } 1 \leq i \leq n-2$$

and

$$s(\tau_{n-1}) = \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^M.$$

Finally, we examine the relation **R6.**, where $s(q_1^2) = s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}^2 \tau_{n-2} \cdots \tau_2 \tau_1)$.

$$s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}^2 \tau_{n-2} \cdots \tau_2 \tau_1) = s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}) s(\tau_{n-1} \tau_{n-2} \cdots \tau_2 \tau_1)$$

We will compute $s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1})$ and $s(\tau_{n-1} \tau_{n-2} \cdots \tau_2 \tau_1)$ separately.

$$\begin{aligned}
s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}) &= \tau_1 \beta_1^{k_{1,1}} \beta_2^{k_{1,2}} \sigma^M \tau_2 \beta_2^{k_{2,2}} \beta_3^{k_{2,3}} \sigma^M \cdots \tau_i \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \sigma^M \cdots \tau_{n-2} \beta_{n-2}^{k_{n-2,n-2}} \beta_{n-1}^{k_{n-2,n-1}} \sigma^M \\
&\quad \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^M \\
&= \tau_1 \beta_1^{k_{1,1}} \beta_2^{k_{1,2}} \tau_2 \beta_2^{k_{2,2}} \beta_3^{k_{2,3}} \cdots \tau_i \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \cdots \tau_{n-2} \beta_{n-2}^{k_{n-2,n-2}} \beta_{n-1}^{k_{n-2,n-1}} \\
&\quad \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{(n-1)M} \\
&= \tau_1 \tau_2 \beta_1^{k_{1,1}} \beta_3^{k_{1,2}+k_{2,3}} \beta_2^{k_{2,2}} \cdots \tau_i \beta_i^{k_{i,i}} \beta_{i+1}^{k_{i,i+1}} \cdots \tau_{n-2} \beta_{n-2}^{k_{n-2,n-2}} \beta_{n-1}^{k_{n-2,n-1}} \\
&\quad \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{(n-1)M} \\
&= \tau_1 \tau_2 \cdots \tau_i \beta_1^{k_{1,1}} \beta_{i+1}^{k_{1,2}+k_{2,3}+\cdots+k_{i,i+1}} (\beta_2^{k_{2,2}} \cdots \beta_i^{k_{i,i}}) \\
&\quad \tau_{i+1} \beta_{i+1}^{k_{i+1,i+1}} \beta_{i+2}^{k_{i+1,i+2}} \cdots \tau_{n-2} \beta_{n-2}^{k_{n-2,n-2}} \beta_{n-1}^{k_{n-2,n-1}} \\
&\quad \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{(n-1)M} \\
&= \tau_1 \tau_2 \cdots \tau_i \cdots \tau_{n-2} \beta_1^{k_{1,1}} \beta_{n-1}^{k_{1,2}+k_{2,3}+\cdots+k_{i,i+1}+\cdots+k_{n-2,n-1}} (\beta_2^{k_{2,2}} \cdots \beta_i^{k_{i,i}} \cdots \beta_{n-2}^{k_{n-2,n-2}}) \\
&\quad \tau_{n-1} \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{(n-1)M} \\
&= \tau_1 \tau_2 \cdots \tau_i \cdots \tau_{n-2} \tau_{n-1} \beta_1^{k_{1,1}} \beta_n^{k_{1,2}+k_{2,3}+\cdots+k_{i,i+1}+\cdots+k_{n-2,n-1}} \\
&\quad (\beta_2^{k_{2,2}} \cdots \beta_i^{k_{i,i}} \cdots \beta_{n-2}^{k_{n-2,n-2}} \beta_{n-1}^{k_{n-1,n-1}}) \beta_1^{\pi} \beta_2^{\pi} \cdots \beta_{n-2}^{\pi} \rho^{l_{n-1}} \sigma^{(n-1)M}.
\end{aligned}$$

Setting $\lambda := k_{1,2} + k_{2,3} + \cdots + k_{i,i+1} + \cdots + k_{n-2,n-1}$, we have:

$$\begin{aligned}
s(\tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1}) &= \tau_1 \tau_2 \cdots \tau_i \cdots \tau_{n-2} \tau_{n-1} \\
&\quad \beta_1^{k_{1,1}+\pi} \beta_n^{\lambda} \beta_2^{k_{2,2}+\pi} \cdots \beta_i^{k_{i,i}+\pi} \cdots \beta_{n-2}^{k_{n-2,n-2}+\pi} \beta_{n-1}^{k_{n-1,n-1}} \rho^{l_{n-1}} \sigma^{(n-1)M} \\
&= \tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1} \\
&\quad \beta_1^{k_{1,1}+\pi-\lambda} \beta_2^{k_{2,2}+\pi-\lambda} \cdots \beta_i^{k_{i,i}+\pi-\lambda} \cdots \beta_{n-2}^{k_{n-2,n-2}+\pi-\lambda} \beta_{n-1}^{k_{n-1,n-1}-\lambda} \rho^{l_{n-1}-2\lambda} \sigma^{(n-1)M}.
\end{aligned}$$

We continue by computing $s(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1)$.

$$
\begin{aligned}
s(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1) &= \tau_{n-1}\beta_1^{\pi}\beta_2^{\pi}\cdots\beta_{n-2}^{\pi}\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^{M}\tau_{n-2}\beta_{n-2}^{k_{n-2,n-2}}\beta_{n-1}^{k_{n-2,n-1}}\sigma^{M}\cdots\tau_i\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\sigma^{M}\cdots\\
&\quad \tau_2\beta_2^{k_{2,2}}\beta_3^{k_{2,3}}\sigma^{M}\tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\sigma^{M}\\
&= \tau_{n-1}\beta_1^{\pi}\beta_2^{\pi}\cdots\beta_{n-2}^{\pi}\beta_{n-1}^{k_{n-1,n-1}}\tau_{n-2}\beta_{n-2}^{k_{n-2,n-2}}\beta_{n-1}^{k_{n-2,n-1}}\cdots\tau_i\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\\
&\quad \tau_2\beta_2^{k_{2,2}}\beta_3^{k_{2,3}}\tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\rho^{l_{n-1}}\sigma^{(n-1)M}\\
&= \tau_{n-1}\tau_{n-2}\beta_1^{\pi}\beta_2^{\pi}\cdots\beta_{n-1}^{\pi+k_{n-2,n-1}}\beta_{n-2}^{k_{n-1,n-1}+k_{n-2,n-2}}\cdots\tau_i\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\cdots\\
&\quad \tau_2\beta_2^{k_{2,2}}\beta_3^{k_{2,3}}\tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\rho^{l_{n-1}}\sigma^{(n-1)M}\\
&= \tau_{n-1}\tau_{n-2}\cdots\tau_i\beta_1^{\pi}\beta_2^{\pi}\cdots\beta_{i+1}^{k_{i,i+1}+\pi}\cdots\beta_{n-1}^{\pi+k_{n-2,n-1}}\beta_i^{k_{n-1,n-1}+k_{n-2,n-2}+\cdots+k_{i,i}}\cdots\\
&\quad \tau_2\beta_2^{k_{2,2}}\beta_3^{k_{2,3}}\tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\rho^{l_{n-1}}\sigma^{(n-1)M}\\
&= \tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\beta_1^{\pi}\beta_3^{\pi+k_{2,3}}\cdots\beta_{i+1}^{\pi+k_{i,i+1}}\cdots\beta_{n-1}^{\pi+k_{n-2,n-1}}\beta_2^{k_{n-1,n-1}+k_{n-2,n-2}+\cdots+k_{i,i}\cdots+k_{2,2}}\\
&\quad \tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\rho^{l_{n-1}}\sigma^{(n-1)M}\\
&= \tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1\\
&\quad \beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_{i+1}^{\pi+k_{i,i+1}}\cdots\beta_{n-1}^{\pi+k_{n-2,n-1}}\beta_1^{k_{n-1,n-1}+k_{n-2,n-2}+\cdots+k_{i,i}\cdots+k_{2,2}+k_{1,1}}\\
&\quad \rho^{l_{n-1}}\sigma^{(n-1)M}.
\end{aligned}
$$

Setting $\omega := k_{1,1} + k_{2,2} + \cdots + k_{i,i} + \cdots + k_{n-2,n-2} + k_{n-1,n-1}$, we have:

$$
\begin{aligned}
s(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1) &= \tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega}\beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_i^{\pi+k_{i-1,i}}\cdots\beta_{n-2}^{\pi+k_{n-3,n-2}}\beta_{n-1}^{\pi+k_{n-2,n-1}}\rho^{l_{n-1}}\sigma^{(n-1)M}.
\end{aligned}
$$

And now we are able to compute $s(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1)$.

$$
\begin{aligned}
s(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1) &= s(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})s(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1)\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_2^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i,i}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-2,n-2}+\pi-\lambda}\beta_{n-1}^{k_{n-1,n-1}-\lambda}\rho^{l_{n-1}-2\lambda}\\
&\quad \sigma^{(n-1)M}\tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega}\beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_i^{\pi+k_{i-1,i}}\cdots\beta_{n-2}^{\pi+k_{n-3,n-2}}\beta_{n-1}^{\pi+k_{n-2,n-1}}\rho^{l_{n-1}}\sigma^{(n-1)M}\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_2^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i,i}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-2,n-2}+\pi-\lambda}\beta_{n-1}^{k_{n-1,n-1}-\lambda}\\
&\quad \tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega}\beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_i^{\pi+k_{i-1,i}}\cdots\beta_{n-2}^{\pi+k_{n-3,n-2}}\beta_{n-1}^{\pi+k_{n-2,n-1}}\rho^{2l_{n-1}-2\lambda}\sigma^{2(n-1)M}.
\end{aligned}
$$

Setting

$$
\begin{aligned}
A &:= \beta_1^{\omega}\beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_i^{\pi+k_{i-1,i}}\cdots\beta_{n-2}^{\pi+k_{n-3,n-2}}\beta_{n-1}^{\pi+k_{n-2,n-1}}\rho^{2l_{n-1}-2\lambda}\sigma^{2(n-1)M}\\
&= \beta_1^{\omega}\beta_2^{\pi+k_{1,2}}\beta_3^{\pi+k_{2,3}}\cdots\beta_i^{\pi+k_{i-1,i}}\cdots\beta_{n-2}^{\pi+k_{n-3,n-2}}\beta_{n-1}^{\pi+k_{n-2,n-1}}\rho^{2l_{n-1}-2\lambda},
\end{aligned}
$$

we have:

$$\begin{aligned}
s(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1) &= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_2^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i,i}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-2,n-2}+\pi-\lambda}\beta_{n-1}^{k_{n-1,n-1}-\lambda}\\
&\quad \tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1 A\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\tau_{n-1}\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_2^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i,i}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-2,n-2}+\pi-\lambda}\beta_n^{k_{n-1,n-1}-\lambda}\\
&\quad \tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1 A\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\tau_{n-1}\tau_{n-2}\cdots\tau_i\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_2^{k_{2,2}+\pi-\lambda}\cdots\beta_{i+1}^{k_{i,i}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-3,n-3}+\pi-\lambda}\beta_{n-1}^{k_{n-2,n-2}+\pi-\lambda}\beta_n^{k_{n-1,n-1}-\lambda}\\
&\quad \tau_{i-1}\cdots\tau_2\tau_1 A\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\\
&\quad \beta_1^{k_{1,1}+\pi-\lambda}\beta_3^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i-1,i-1}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-3,n-3}+\pi-\lambda}\beta_{n-1}^{k_{n-2,n-2}+\pi-\lambda}\\
&\quad \beta_n^{k_{n-1,n-1}-\lambda}\tau_1 A\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}\tau_{n-1}\tau_{n-2}\cdots\tau_i\cdots\tau_2\tau_1\\
&\quad \beta_2^{k_{1,1}+\pi-\lambda}\beta_3^{k_{2,2}+\pi-\lambda}\cdots\beta_i^{k_{i-1,i-1}+\pi-\lambda}\cdots\beta_{n-2}^{k_{n-3,n-3}+\pi-\lambda}\beta_{n-1}^{k_{n-2,n-2}+\pi-\lambda}\\
&\quad \beta_n^{k_{n-1,n-1}-\lambda}A\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega}\beta_2^{k_{1,1}+2\pi-\lambda+k_{1,2}}\beta_3^{k_{2,2}+2\pi-\lambda+k_{2,3}}\cdots\beta_i^{k_{i-1,i-1}+2\pi-\lambda+k_{i-1,i}}\cdots\\
&\quad \beta_{n-2}^{k_{n-3,n-3}+2\pi-\lambda+k_{n-3,n-2}}\beta_{n-1}^{k_{n-2,n-2}+2\pi-\lambda+k_{n-2,n-1}}\beta_n^{k_{n-1,n-1}-\lambda}\rho^{2l_{n-1}-2\lambda}\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega-k_{n-1,n-1}+\lambda}\beta_2^{k_{1,1}+2\pi-\lambda+k_{1,2}-k_{n-1,n-1}+\lambda}\\
&\quad \cdots\beta_i^{k_{i-1,i-1}+2\pi-\lambda+k_{i-1,i}-k_{n-1,n-1}+\lambda}\cdots\\
&\quad \beta_{n-2}^{k_{n-3,n-3}+2\pi-\lambda+k_{n-3,n-2}-k_{n-1,n-1}+\lambda}\beta_{n-1}^{k_{n-2,n-2}+2\pi-\lambda+k_{n-2,n-1}-k_{n-1,n-1}+\lambda}\\
&\quad \rho^{2l_{n-1}-2\lambda-2k_{n-1,n-1}+2\lambda}\\
&= \tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1\\
&\quad \beta_1^{\omega-k_{n-1,n-1}+\lambda}\beta_2^{k_{1,1}+2\pi+k_{1,2}-k_{n-1,n-1}}\cdots\beta_i^{k_{i-1,i-1}+2\pi+k_{i-1,i}-k_{n-1,n-1}}\cdots\\
&\quad \beta_{n-2}^{k_{n-3,n-3}+2\pi+k_{n-3,n-2}-k_{n-1,n-1}}\beta_{n-1}^{k_{n-2,n-2}+2\pi+k_{n-2,n-1}-k_{n-1,n-1}}\\
&\quad \rho^{2l_{n-1}-2k_{n-1,n-1}}.
\end{aligned}$$

Now:

$$\begin{aligned}
s(q_1^2) &= q_1\beta_1^{\bar{k}_{1,1}}\cdots\beta_{n-1}^{\bar{k}_{1,n-1}}\rho^{\bar{l}}\sigma^{\bar{m}_1}q_1\beta_1^{\bar{k}_{1,1}}\cdots\beta_{n-1}^{\bar{k}_{1,n-1}}\rho^{\bar{l}}\sigma^{\bar{m}_1}\\
&= q_1q_1\beta_1^{-\bar{k}_{1,1}}\beta_2^{\bar{k}_{1,2}}\cdots\beta_{n-1}^{\bar{k}_{1,n-1}}\beta_1^{\bar{l}}\rho^{\bar{l}}\sigma^{\bar{m}_1}\beta_1^{\bar{k}_{1,1}}\cdots\beta_{n-1}^{\bar{k}_{1,n-1}}\rho^{\bar{l}}\sigma^{\bar{m}_1}\\
&= q_1^2\beta_1^{\bar{l}}\beta_2^{2\bar{k}_{1,2}}\cdots\beta_i^{2\bar{k}_{1,i}}\cdots\beta_{n-1}^{2\bar{k}_{1,n-1}}\rho^{2\bar{l}}\\
&= (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1)\beta_1^m\beta_1^{\bar{l}}\beta_2^{2\bar{k}_{1,2}}\cdots\beta_i^{2\bar{k}_{1,i}}\cdots\beta_{n-1}^{2\bar{k}_{1,n-1}}\rho^{2\bar{l}}\\
&= (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1)\beta_1^{m+\bar{l}}\beta_2^{2\bar{k}_{1,2}}\cdots\beta_i^{2\bar{k}_{1,i}}\cdots\beta_{n-1}^{2\bar{k}_{1,n-1}}\rho^{2\bar{l}}.
\end{aligned}$$

We recall that $q_1^2 = (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})\beta_n^m(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1)$, given in Proposition 3.2.2, (II). The last but one equality comes from relation

$$q_1^2 = (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})\beta_n^m(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1) = (\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1})(\tau_{n-1}\tau_{n-2}\cdots\tau_2\tau_1)\beta_1^m$$

that holds in $B_{n,m}(\mathbb{R}P^2)/\Gamma$. Comparing the coefficients of the element β_1 of $B_{n,m}(\mathbb{R}P^2)/\Gamma$ in these two equations, we obtain the following:

$$m + \bar{l} = \omega + \lambda - k_{n-1,n-1}. \tag{24}$$

From (24) we have that $m + \bar{l} = \omega + \lambda - k_{n-1,n-1}$. From the definitions of ω and λ we obtain:

$$\begin{aligned} m + \bar{l} &= (k_{1,1} + k_{2,2} + \cdots + k_{n-2,n-2} + k_{n-1,n-1}) + (k_{1,2} + k_{2,3} + \cdots + k_{i,i+1} + \cdots + k_{n-2,n-1}) - k_{n-1,n-1} \\ &= (k_{1,1} + k_{2,2} + \cdots + k_{i,i} + \cdots + k_{n-2,n-2}) + (k_{1,2} + k_{2,3} + \cdots + k_{i,i+1} + \cdots + k_{n-2,n-1}) \\ &= (k_{1,1} + k_{1,2}) + (k_{2,2} + k_{2,3}) + \cdots + (k_{i,i} + k_{i,i+1}) + \cdots + (k_{n-2,n-2} + k_{n-2,n-1}). \end{aligned}$$

In addition, from (19) we have $k_{i,i} + k_{i,i+1} = k_{i+1,i+1} + k_{i+1,i+2}$, for $1 \leq i < n-2$. Therefore we obtain the following:

$$\begin{cases} k_{1,1} + k_{1,2} = k_{2,2} + k_{2,3} \\ k_{2,2} + k_{2,3} = k_{3,3} + k_{3,4} \\ \vdots \\ k_{n-4,n-4} + k_{n-4,n-3} = k_{n-3,n-3} + k_{n-3,n-2} \\ k_{n-3,n-3} + k_{n-3,n-2} = k_{n-2,n-2} + k_{n-2,n-1}, \end{cases}$$

which implies that:

$$k_{1,1} + k_{1,2} = k_{2,2} + k_{2,3} = \cdots = k_{n-3,n-3} + k_{n-3,n-2} = k_{n-2,n-2} + k_{n-2,n-1}. \tag{25}$$

We conclude that

$$m + \bar{l} = (n-2)(k_{1,1} + k_{1,2}). \tag{26}$$

Moreover, from (22) we have $k_{i,i} + k_{i,i+1} = -\bar{l}_i + 2\bar{k}_{i,i+1}$, for $1 \leq i \leq n-2$. Letting $i = n-2$, and using (16) we get:

$$k_{n-2,n-2} + k_{n-2,n-1} = -\bar{l} + 2\bar{k}_{n-2,n-1}.$$

However, from (12) we know that $\bar{k}_{n-2,n-1} = 0$, which in turn gives that

$$k_{n-2,n-2} + k_{n-2,n-1} = -\bar{l}. \tag{27}$$

In conclusion, from (25), (26) and (27) it follows that $m = (n-1)(k_{1,1} + k_{1,2})$ for $n \geq 4$, and therefore that $m = k(n-1)$ for $k \geq 1$, since m is a natural number. The conclusion is valid in the case when $n \geq 4$ since we have used the relation **R5.**, $s(\tau_i\tau_j) = s(\tau_j\tau_i)$ for $|i-j| > 1$ and $1 \leq i, j \leq n-1$, which only holds when $n \geq 4$.

Nevertheless, we can obtain the same result for $n = 3$. Indeed, using the results of relations (13), (14), (20), which also hold in the case $n = 3$, we obtain $l_2 = 2k_{2,1}$, $l_1 = 0$ and that $m_1 = m_2 := M$, respectively. Thus, $s(\tau_1) = \tau_1\beta_1^{k_{1,1}}\beta_2^{k_{1,2}}\sigma^M$ and $s(\tau_2) = \tau_2\beta_1^{k_{2,1}}\beta_2^{k_{2,2}}\rho^{2k_{2,1}}\sigma^M$. It holds that $s(\tau_1)$ and $s(\tau_2)$ have the same structure as the equations $s(\tau_i) = \tau_i\beta_i^{k_{i,i}}\beta_{i+1}^{k_{i,i+1}}\sigma^M$ and $s(\tau_{n-1}) = \tau_{n-1}\beta_1^{\pi}\beta_2^{\pi}\cdots\beta_{n-2}^{\pi}\beta_{n-1}^{k_{n-1,n-1}}\rho^{l_{n-1}}\sigma^M$, since $\pi = k_{2,1}$ for $n = 3$. Therefore, we can indeed apply (24), which follows from relation **R6**, $s(q_1^2) = s(\tau_1\tau_2\cdots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\cdots\tau_2\tau_1)$, that also holds for $n = 3$. Thus, we obtain from (24) that $m + \bar{l} = k_{1,1} + k_{1,2}$, which is (26), and from (27), it follows that $m = 2(k_{1,1} + k_{1,2})$, which is the result in the case $n = 3$. As a result we obtain that if the short exact sequence (5) splits then $m = k(n-1)$, for $n \geq 3$ and $k \geq 1$. □

Remark 3.2.5. Note that Theorem 3.2.4 also holds for $m = 1$. However, in Proposition 3.2.1 we treat the case $m = 1$ seperately, proving that for $m = 1$ and $n \geq 3$ the homomorphism $\bar{q} : B_{n,1}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ does not admit a section. Even thought we could ignore Proposition 3.2.1, since we could obtain the same result from Theorem 3.2.4 for $m = 1$, we present it, as in that proof we illustrate the relation of the splitting of the mixed braids with the splitting of the pure braids.

Remark 3.2.6. Concerning the quotiented short exact sequence that we used to prove Theorem 3.2.4, except for quotienting by $\Gamma_2(B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}))$, we have tried other quotiens as well, for example quotienting by $\Gamma_2(P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}))$, but they did not impove the result of Theorem 3.2.4. Moreover, note that from Theorem 2.3.2, we have $\Gamma_2\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})\big) = \Gamma_3\big(B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})\big)$, which shows that it would be of no use to try any quotient by $\Gamma_k(B_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}))$, for $k \geq 3$.

We can obtain further restrictions for the value of m, examining the torsion elements of the group $B_n(\mathbb{R}P^2)$ and $B_{n,m}(\mathbb{R}P^2)$.

Proposition 3.2.7. *Let $m > 1$ and $n \geq 3$. If the following short exact sequence*

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

splits, then either $m \equiv 0$ mod n or $m \equiv 1$ mod n.

Proof. From Proposition 2.1.3, we know that the group $B_n(\mathbb{R}P^2)$ contains a torsion element of order k if and only if k divides $4n$ or $4(n-1)$. Similarly, the group $B_{n+m}(\mathbb{R}P^2)$ contains a torsion element of order l if and only if l divides $4(n+m)$ or $4(n-1+m)$.

Suppose that the homomorphism $\bar{q}_{n+m,n}$ admits a section $\bar{s}_{n,n+m} : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)$, where $B_{n,m}(\mathbb{R}P^2) \subset B_{n+m}(\mathbb{R}P^2)$. Since, $\bar{s}_{n,n+m}$ is a section, it follows that any torsion element from $B_n(\mathbb{R}P^2)$ is mapped under $\bar{s}_{n,n+m}$ to a torsion element in $B_{n,m}(\mathbb{R}P^2)$ of the same order. Since $B_n(\mathbb{R}P^2)$ contains a torsion element of order $4n$, it follows that its image under $\bar{s}_{n,n+m}$ is a torsion element of order $4n$ in $B_{n,m}(\mathbb{R}P^2)$. Therefore, $4n$ divides $4(n+m)$ or $4(n-1+m)$ by Proposition 2.1.3. Thus,

$$4n \mid (4n+4m) \text{ or } 4n \mid [4(n-1)+4m].$$

As a result $4n + 4m = p \cdot 4n$ or $4(n-1) + 4m = \bar{p} \cdot 4n$, for $p, \bar{p} \in \mathbb{Z}$. It follows that $m = (p-1) \cdot n$ or $m = (\bar{p}-1) \cdot n + 1$, or equivalently that either $m \equiv 0 \bmod n$ or $m \equiv 1 \bmod n$. □

Remark 3.2.8. Arguing with the same manner for the torsion element of order $4(n-1)$ in $B_n(\mathbb{R}P^2)$, it follows that $(n-1)$ divides $(n+m)$ or $[(n-1)+m]$. In other words, it follows that $m = l(n-1)+n$ or $m = \bar{l}(n-1)$, for some $l, \bar{l} \in \mathbb{Z}$, but in fact from Theorem 3.2.4, we know that $m = k(n-1)$, which is a stronger result.

Remark 3.2.9. From Theorem 3.2.4 we have $m \equiv 0 \bmod (n-1)$ and from Proposition 3.2.7 we have $m \equiv 0, 1 \bmod n$. Combining these two results, if the short exact sequence

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

splits, then it follows that

$$m \equiv 0, (n-1)^2 \bmod n(n-1).$$

3.3 Construction of a geometric section for the case $n \geq 3$

In the previous section, for $n \geq 3$ and $m > 1$, we saw that if the homomorphism $\bar{q}_{n+m,n} : B_{n,m}(\mathbb{R}P^2) \to B_n(\mathbb{R}P^2)$ admits a section $\bar{s}_{n+m,n} : B_n(\mathbb{R}P^2) \to B_{n,m}(\mathbb{R}P^2)$, then $m = k(n-1)$, for $k \in \mathbb{N}$. Moreover, from Proposition 1.3.1, the existence of such a section $\bar{s}_{n+m,n}$ is equivalent to the existence of a cross-section for the Fadell–Neuwirth fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$. In this section, we will present two different constructions of a geometric cross-section $s_m : F_n(\mathbb{R}P^2)/S_n \to F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m)$ for this fibration and for certain values of m.

In [CS20], Chen–Salter constructed geometric cross-sections using Möbius transformations in the case of the 2–sphere, $\mathbb{S}^2$, for some of the cases given by Gonçalves–Guaschi in Theorem 1.3.4. We will carry out a similar construction for the case of the projective plane, $\mathbb{R}P^2$, as we describe in the proof of the following proposition.

Proposition 3.3.1. *Let $n \geq 3$ and $m > 1$. The fibration*

$$q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$$

admits a cross-section for $m = kn(2n-1)(2n-2)$, where $k \geq 1$.

Proof. Let $n \geq 3$. We consider the two-sheeted covering map $p : \mathbb{S}^2 \to \mathbb{R}P^2$. Let $\{y_1, \dots, y_n\}$ be an element of $F_n(\mathbb{R}P^2)/S_n$. Using p, $\bigcup_{i=1}^{n} p^{-1}(\{y_i\}) = \{x_1, x_2, \dots, x_{2n}\}$, which is an element in $A_n(\mathbb{S}^2)$, where by $A_n(\mathbb{S}^2)$ we denote the set of n pairs of antipodal points, which is a subset of $F_{2n}(\mathbb{S}^2)/S_{2n}$. So, $\{x_1, x_2, \dots, x_{2n}\}$ consists of n pairs of antipodal points of S^2. We can transform the problem of the construction of a cross-section on the configuration space of the projective plane to the construction of a cross-section on the configuration space of the 2–sphere, as we will

see in what follows. We will generalise a particular construction, using Möbius transformations, of a cross-section on the configuration space of the 2–sphere, given by Chen–Salter in [CS20] (Remark 3.6), which induces the cabling map $B_n(\mathbb{S}^2) \to B_{n,m}(\mathbb{S}^2)$, given in Theorem B of [CS20]. Considering the element $\{x_1, x_2, \dots, x_{2n}\} \in A_n(\mathbb{S}^2) \subset F_{2n}(\mathbb{S}^2)/S_{2n}$, we will construct new distinct points on $\mathbb{S}^2$ around each point x_i, for $i = 1, \dots, 2n$.

Let (x_p, x_q, x_r) be an ordered triple of distinct points of the set of points $\{x_1, x_2, \dots, x_{2n}\} \in A_n(\mathbb{S}^2)$. We consider the unique Möbius transformation of the Riemann sphere, defined by $M_{x_p,x_q,x_r}(x) = \frac{(x-x_p)(x_q-x_r)}{(x-x_r)(x_q-x_p)}$, for $x \in \mathbb{S}^2$, which sends (x_p, x_q, x_r) to $(0, 1, \infty)$. We recall that the Möbius transformations are automorphisms of the Riemann sphere. For every $x_i \in \{x_1, x_2, \dots, x_{2n}\}$ we define the following map:

$$R_{x_i}(x) := \prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i\}} M_{x_i,x_j,x_k}(x) = \prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i\}} \frac{(x - x_i)(x_j - x_k)}{(x - x_k)(x_j - x_i)}.$$

For every $x_i \in \{x_1, x_2, \dots, x_{2n}\}$, the map $R_{x_i} : \mathbb{S}^2 \to \mathbb{S}^2$ is a rational map and in particular it is a product of $(2n-1)(2n-2)$ rational fractions, and thus by fundamental Theorem of Algebra, the map R_{x_i} is of degree $(2n-1)(2n-2)$. It follows that the preimage of any regular value of R_{x_i} is a set of $(2n-1)(2n-2)$ distinct points. Moreover, by the definition of R_{x_i}, it follows that x_i is the only zero of R_{x_i}.

Note that by the definition of $M_{x_p,x_q,x_r}(x)$ we have $M_{x_p,x_q,x_r}(x) = M_{-x_p,-x_q,-x_r}(-x)$, for every $x \in \mathbb{S}^2$. For simplicity, in what follows we will denote by i', the index k for which $x_k = -x_i$. Moreover, note that for any pair of j, k, where $j, k \neq i'$, and for any factor $M_{x_i,x_j,x_k}(x)$ included in the product $\prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i\}} M_{x_i,x_j,x_k}(x)$ the factor $M_{x_i,-x_j,-x_k}(x)$ is included as well, since the source of x_j, x_k is a set of exactly $n-1$ pairs of antipodal points. Furthermore, for $j = i'$ (respectively for $k = i'$), for any factor $M_{x_i,-x_i,x_k}(x)$, where $k \in \{1, \dots, 2n\} \setminus \{i, i'\}$ (respectively $M_{x_i,x_j,-x_i}(x)$, where $j \in \{1, \dots, 2n\} \setminus \{i, i'\}$), included in the product $\prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i\}} M_{x_i,x_j,x_k}(x)$, the factor $M_{x_i,-x_i,-x_k}(x)$ (respectively $M_{x_i,-x_j,-x_i}(x)$), is included as well. Similarly, note that for any pair of j, k, where $j, k \neq i$, and for any factor $M_{-x_i,x_j,x_k}(-x)$ included in $\prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i'\}} M_{-x_i,x_j,x_k}(-x)$, the factor $M_{-x_i,-x_j,-x_k}(-x)$ is included as well, since the source of x_j, x_k is a set of exactly $n-1$ pairs of antipodal points. Furthermore, for $j = i$ (respectively for $k = i$), for any factor $M_{-x_i,x_i,x_k}(-x)$, where $k \in \{1, \dots, 2n\} \setminus \{i, i'\}$ (respectively $M_{-x_i,x_j,x_i}(-x)$, where $j \in \{1, \dots, 2n\} \setminus \{i, i'\}$), included in the product $\prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i\}} M_{-x_i,x_j,x_k}(-x)$, the factor $M_{-x_i,x_i,-x_k}(-x)$ (respectively $M_{-x_i,-x_j,x_i}(-x)$), is included as well. Therefore,

$$R_{x_i}(x) = \prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{x_i,x_j,x_k}(x) \prod_{k \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{x_i,-x_i,x_k}(x) \prod_{j \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{x_i,x_j,-x_i}(x)$$

and

$$R_{-x_i}(-x) = \\ = \prod_{j \neq k \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{-x_i,x_j,x_k}(-x) \prod_{k \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{-x_i,x_i,x_k}(-x) \prod_{j \in \{1,\dots,2n\} \setminus \{i,i'\}} M_{-x_i,x_j,x_i}(-x).$$

Thus, from $M_{x_p,x_q,x_r}(x) = M_{-x_p,-x_q,-x_r}(-x)$, we obtain:

$$R_{x_i}(x) = R_{-x_i}(-x), \text{ for every } x \in \mathbb{S}^2. \tag{28}$$

Near each point $x_i \in \{x_1, x_2, \dots, x_{2n}\}$, we will now construct $(2n-1)(2n-2)$ new distinct unordered points. To do so, we consider a small regular value ϵ_i of R_{x_i} close to zero and we consider its preimage. The preimage, $R_{x_i}^{-1}(\epsilon_i)$, will be a set of $(2n-1)(2n-2)$ distinct points, and by continuity close to x_i and different from it. One needs to verify that the $(2n-1)(2n-2)$ new distinct points close to the point x_i are also distinct from the $(2n-1)(2n-2)$ new distinct points close to any other point x_j, for $j \neq i$. In other words, we have to chose each regular value in such a way that $R_{x_i}^{-1}(\epsilon_i) \cap R_{x_j}^{-1}(\epsilon_j) = \varnothing$, for every $i \neq j \in \{1, \dots, 2n\}$. In order to achieve this, we have to refine the choice of the regular value of every R_{x_i}, as we describe in what follows.

To begin with, we denote the spherical metric by $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^+$, which is the distance function on the sphere based on the central angle between two points, and we set

$$m := \frac{1}{3} \min_{i \neq j \in \{1,\dots,2n\}} d(x_i, x_j).$$

Moreover, we can define a map $f : \{x_1, x_2, \dots, x_{2n}\} \to \mathbb{R}^+$, that assigns a positive small number r_i to every element $x_i \in \{x_1, x_2, \dots, x_{2n}\}$, which is a regular value of the map R_{x_i}, such that any x with $0 < |x| \leq r_i$ is a regular value of R_{x_i} and moreover

$$R_{x_i}^{-1}(\bar{D}(0, r_i)) := \{x \in \mathbb{S}^2 \mid R_{x_i}(x) \in \bar{D}(0, r_i)\} \subset \bar{D}(x_i, m),$$

where $\bar{D}(z, a) = \{y \in \mathbb{S}^2 \mid d(z, y) \leq a\}$. The inclusion $R_{x_i}^{-1}(\bar{D}(0, r_i)) \subset \bar{D}(x_i, m)$ follows by continuity of R_{x_i} and by the fact that x_i is the only zero of R_{x_i}. Note that we can always find such a regular value r_i of R_{x_i}, since the critical values of R_{x_i} are finite. To be more precise, one could consider the distance, p_i, from zero to the critical value c_i of R_{x_i}, for every $1 \leq i \leq l$, where $l \in \mathbb{N}$ is the number of critical values of R_{x_i}. Then, by considering the minimum of these distances, $p := \min(p_1, \dots, p_l) \in \mathbb{R}^+$, all points, except for zero, with distance from zero less that p are regular values of R_{x_i}. Finally, we set r to be the smallest value among r_i, where $1 \leq i \leq 2n$, that is

$$r := \min(r_1, \dots, r_{2n}) \in \mathbb{R}^+. \tag{29}$$

Therefore, for every point $x_i \in \{x_1, x_2, \dots, x_{2n}\}$ we have defined a rational map R_{x_i} and a small positive number r, which is a regular value of R_{x_i} and moreover

$$R_{x_i}^{-1}(r) = \{\zeta_1^i, \dots, \zeta_{(2n-1)(2n-2)}^i\} \subset \bar{D}(x_i, m), \text{ for every } i \in \{1, \dots, 2n\},$$

where $\zeta_1^i, \dots, \zeta_{(2n-1)(2n-2)}^i$ are pairwise distinct. This implies that

$$R_{x_i}^{-1}(r) \cap R_{x_j}^{-1}(r) = \varnothing, \text{ for every } i \neq j \in \{1, \dots, 2n\}.$$

In addition, from (28), we have that

$$R_{-x_i}^{-1}(r) = \{-\zeta_1^i, \dots, -\zeta_{(2n-1)(2n-2)}^i\} = -R_{x_i}^{-1}(r) = \{\zeta_1^i, \dots, \zeta_{(2n-1)(2n-2)}^i\}.$$

In other words, the $(2n-1)(2n-2)$ new distinct points around x_i are antipodal to the $(2n-1)(2n-2)$ new distinct points around $-x_i$. Now, set

$$S_r := \bigcup_{1 \leq i \leq 2n} R_{x_i}^{-1}(r),$$

which is the set of the $2n(2n-1)(2n-2)$ new distinct points of $\mathbb{S}^2$ that we constructed, based on the points of the set $\{x_1, x_2, \dots, x_{2n}\} \in A_n(\mathbb{S}^2)$. Note that $S_r \in A_{n(2n-1)(2n-2)}(\mathbb{S}^2)$.

Note that if $r', r'' \in \mathbb{R}^+$ satisfy $0 < r' < r'' < r$, then by construction, $S_{r'} \cap S_{r''} = \varnothing$. So if $k \in \mathbb{N}$, in the same way, we may obtain $k\big((2n(2n-1)(2n-2))\big)$ new points on $\mathbb{S}^2$, taking for example $S_r \cup S_{\frac{1}{2}r} \cup \dots \cup S_{\frac{1}{k}r}$, where once again the $k(2n-1)(2n-2)$ new points close to x_i will be antipodal to the $k(2n-1)(2n-2)$ new points close to $-x_i$.

Projecting now the set of points $S_r \cup S_{\frac{1}{2}r} \cup \dots \cup S_{\frac{1}{k}r}$ into $\mathbb{R}P^2$, by the covering map p, around each point $y_1, \dots, y_n$ of $\mathbb{R}P^2$ we obtain $k((2n-1)(2n-2))$ new distinct unordered points. In total, we have constructed $kn((2n-1)(2n-2))$ new distinct unordered points, that depend continuously on the element $\{y_1, \dots, y_n\} \in F_n(\mathbb{R}P^2)/S_n$. This yields a cross-section $s_m : F_n(\mathbb{R}P^2)/S_n \to F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m)$ of the fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$, for $m = kn(2n-1)(2n-2)$, where $k \geq 1$. □

We continue by presenting another construction of a cross-section, which covers smaller values of m.

Proposition 3.3.2. *Let $n \geq 3$ and $m > 1$. For $m = 2n(n-1)$ the fibration*

$$q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$$

admits a cross-section.

Proof. Let $n \geq 3$. As in the proof of Proposition 3.3.1, we consider the two-sheeted covering map $p : \mathbb{S}^2 \to \mathbb{R}P^2$. Let $\{y_1, \ldots, y_n\}$ be an element of $F_n(\mathbb{R}P^2)/S_n$. Using p, $\bigcup_{i=1}^{n} p^{-1}(\{y_i\}) = \{x_1, x_2, \ldots, x_{2n}\}$, which is an element in $A_n(\mathbb{S}^2)$, where by $A_n(\mathbb{S}^2)$ we denote the set of n pairs of antipodal points, which is a subset of $F_{2n}(\mathbb{S}^2)/S_{2n}$. So, $\{x_1, x_2, \ldots, x_{2n}\}$ consists of n pairs of antipodal points of S^2. We denote the distance function on the sphere based on the central angle between two points by $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^2$. Clearly, we have $0 \leq d(x_i, x_j) \leq \pi$, with $d(x_i, x_i) = 0$ and $d(x_i, -x_i) = \pi$, for any $x_i, x_j \in \mathbb{S}^2$. We define the following positive numbers m and M_{x_i}:

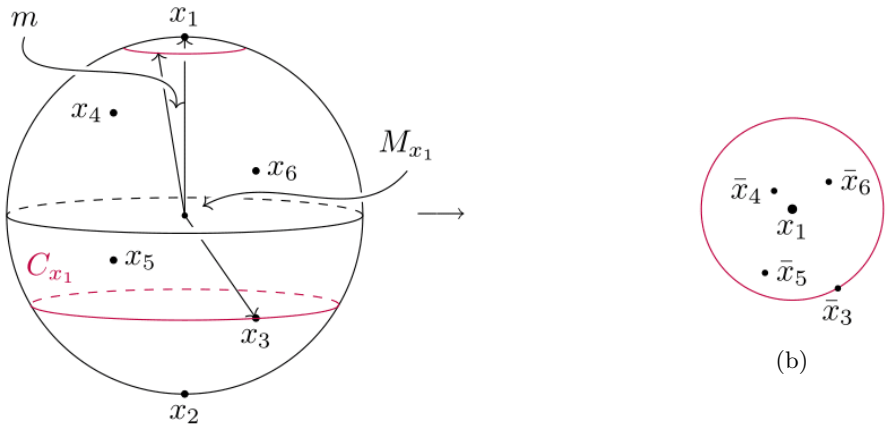
$$m := \frac{1}{3} \min_{i \neq j} d(x_i, x_j), \text{ for every } 1 \leq i \neq j \leq 2n.$$

and

$$M_{x_i} := \max_{x_i \neq -x_j} d(x_i, x_j) < \pi, \text{ for } 1 \leq i \leq 2n,$$

which is the largest distance between the point x_i and the points in the set $\{x_1, \ldots, x_{2n}\} \setminus \{-x_i\}$.

We consider the closed discs $\bar{D}(x_i, m) = \{y \in \mathbb{S}^2 \mid d(x_i, y) \leq m\}$, for all $1 \leq i \leq 2n$. Clearly, by definition of m these discs are pairwise disjoint. Our aim is to construct $2n-2$ new distinct unordered points inside each closed disc $\bar{D}(x_i, m)$. Without loss of generality, we will describe this construction for the point x_1, the construction will be the same for the other points. We want to move all of the points of the set $\{x_1, \ldots x_{2n}\} \setminus \{-x_1\}$ to pairwise distinct points inside the closed disc $\bar{D}(x_1, m)$. The distance from x_1 to any point of the set $\{x_1, \ldots x_{2n}\} \setminus \{-x_1\}$ is at most M_{x_1}, and so $\{x_1, \ldots x_{2n}\} \setminus \{-x_1\} \subset P_1$, where $P_1 = \{z \in \mathbb{S}^2 \mid d(x_1, z) \leq M_{x_1}\}$. Let $C_{x_1} = \{z \in \mathbb{S}^2 \mid d(x_1, z) = M_{x_1}\}$. Moving now this circle C_{x_1} and shrinking the space P_1 toward the point x_1, along the longitudes with respect to the axis that passes through the centre of the sphere and x_1, we move the points of the set $\{x_1, \ldots, x_{2n}\} \setminus \{-x_1\}$ to pairwise distinct points inside the closed disc $\bar{D}(x_1, m)$. To do so, we shrink the circle C_{x_1} until it coincides with the boundary of the closed disc $\bar{D}(x_1, m)$. In other words, at the end of the process, the angle M_{x_1} becomes m. In this way, we have constructed $2n-2$ new distinct unordered points close to the point x_1. For $n = 3$ we illustrate an example of how we obtain 4 new distinct unordered points close to x_1, and in particular inside $\bar{D}(x_1, m)$. In Figure 3.6a we see that P_1 is the subset of $\mathbb{S}^2$ that lies above the circle C_{x_1}, including C_{x_1}. Shrinking the space P_1 towards the point x_1, along the vertical axis that passes through the centre of the sphere and x_1, until C_{x_1} to coincide with the boundary of the closed disc $\bar{D}(x_1, m)$, we obtain Figure 3.6b that represents $\bar{D}(x_1, m)$, where we have constructed 4 new distinct unordered points. We denote the points at which the points x_3, x_4, x_5, x_6 will end up after the shrinking by $\bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6$ respectively.



(a) (b)

Figure 3.6

We apply the same method for every point of $\{x_2, \ldots, x_{2n}\}$ and thus we obtain $2n(2n-2)$ new distinct unordered points on the sphere. Note that these $2n(2n-2)$ new points are distinct,

since we constructed $(2n-2)$ distinct points inside $2n$ pairwise disjoint closed discs. Moreover, choosing the same shrinking process for x_i and $-x_i$, the $(2n-2)$ new points that we obtain inside the closed disc $\bar{D}(x_i, m)$ are the antipodal points of the $(2n-2)$ new points that we obtain inside the closed disc $\bar{D}(-x_i, m)$. Thus, by projecting these $2n(2n-2)$ new distinct points into $\mathbb{R}P^2$, via the covering map p, we obtain $2n(n-1)$ new distinct unordered points, that depend continuously on the element $\{y_1, \dots, y_n\} \in F_n(\mathbb{R}P^2)/S_n$. We conclude that there exists a cross-section $s_m : F_n(\mathbb{R}P^2)/S_n \to F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m)$ for the fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$, for $m = 2n(n-1)$. □

Remark 3.3.3. From Proposition 1.3.1 and Proposition 3.2.4 we know that if the fibration $q_{n+m,n} : F_{n+m}(\mathbb{R}P^2)/(S_n \times S_m) \to F_n(\mathbb{R}P^2)/S_n$ admits a cross-section, then $m = k(n-1)$, for $k \geq 1$. In Propositions 3.3.1 and 3.3.2, we have shown that we have a cross-section s_m for $q_{n+m,n}$, for $m = 2n(n-1)$ and $m = kn(2n-1)(2n-2)$, where $k \geq 1$. These results are compatible since in both cases the value of m is a multiple of $(n-1)$.

We conclude this section by presenting the main result of this part of the thesis.

Theorem 3.3.4. *Let $m > 1$ and $n \geq 3$. The short exact sequence*

$$1 \longrightarrow B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}_{n+m,n}} B_n(\mathbb{R}P^2) \longrightarrow 1$$

splits for $m = 2n(n-1)$ and for $m = kn(2n-1)(2n-2)$, where $k \geq 1$. Moreover, if the homomorphism $\bar{q}_{n+m,n}$ admits a section, then $m \equiv 0, (n-1)^2 \mod n(n-1)$.

Proof. This result follows from Propositions 1.3.1, 3.3.1, 3.3.2 and Remark 3.2.9. □

Remark 3.3.5. For $n = 3$, based on Theorem 3.3.4, the smallest value of m, for which the homomorphism $\bar{q}_{3+m,3} : B_{3,m}(\mathbb{R}P^2) \to B_3(\mathbb{R}P^2)$ could possibly admit a section is $m = 6$, and moreover, $m = 12$ is the smallest known value for which the homomorphism $\bar{q}_{3+m,3} : B_{3,m}(\mathbb{R}P^2) \to B_3(\mathbb{R}P^2)$ admits a section.

Remark 3.3.6. To the best of our knowledge the remaining cases that are not covered by Theorem 3.3.4 are open.

CHAPTER 4

The welded braid groups WB_n

The chapter on welded braid groups is structured as follows. In Section 4.1 we give a couple of different interpretations of the welded braid groups WB_n as well as a presentation of them. We continue in Section 4.2, where we prove that the subgroups of their lower central series coincide, for $n \geq 4$. Moreover, we determine all possible homomorphisms from WB_n to the symmetric group S_n. A discussion about an important subgroup of WB_n, the welded pure braid group WP_n, follows in Section 4.3, where we also provide a proof that WP_n is a characteristic subgroup of WB_n with a trivial centralizer. Furthermore, we show that, the free group F_n, seen as a subgroup of WB_n, is a normal one. Lastly, in Section 4.4 we discuss about the automorphism group of WB_n.

4.1 Introduction

Let $n \in \mathbb{N}$. The welded braid groups WB_n appear in the literature under many different names; conjugating automorphisms of the free group F_n due to Savushkina, [Sav96], loop braid groups due to Baez–Crans–Wise, [BCW07] and groups of untwisted rings due to Brendle–Hatcher, [BH13]. The name of welded braid groups, which we will be using and denoting by WB_n, was introduced by Fenn–Rimányi–Rourke, [FRR97].

The groups WB_n are actually a 3–dimensional analogue of the Artin braid groups B_n and there are several interpretations of it; in terms of mapping class groups, fundamental group of specific configuration spaces and automorphisms of the free group F_n. We refer the reader to [Dam17] by Damiani for a complete presentation of the equivalent definitions of the welded braid groups.

In this thesis we will work on the welded braid groups as automorphisms of the free group F_n, but also as the fundamental group of the spaces of specific configurations, as we describe in the following definitions.

Definition 4.1.1. Let $n \in \mathbb{N}$ and let F_n denote the free group of rank n with generators $\{x_1, \dots, x_n\}$ and let $Aut(F_n)$ denote its automorphism group. The group WB_n is the subgroup of $Aut(F_n)$ that consists of the so-called conjugating automorphisms, $a : x_i \mapsto W_i^{-1} x_{\pi(i)} W_i$, where π is a permutation and W_i is a word in F_n. To be more precise, let $\sigma_i \in Aut(F_n),\ i = 1, \dots, n-1$ be given by

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j,\ j \neq i, i+1 \end{cases} \tag{1}$$

and let $\rho_i \in Aut(F_n),\ i = 1, \dots, n-1$ be given by

$$\rho_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j, \ j \neq i, i+1 \end{cases} \tag{2}$$

The welded braid group WB_n is the one generated by both sets of elements σ_i and ρ_i, $i = 1, \dots, n$.

Remark 4.1.2. The elements σ_i, $i = 1, \dots, n-1$ generate the braid subgroup of $Aut(F_n)$, which is isomorphic to the Artin braid group B_n. Moreover, the elements ρ_i, $i = 1, \dots, n-1$ generate the permutation subgroup of $Aut(F_n)$ which is a copy of the symmetric group S_n.

The following theorem gives a presentation of WB_n, due to Fenn–Rimányi–Rourke, [FRR97].

Theorem 4.1.3 (Fenn–Rimányi–Rourke, [FRR97])**.** *Let $n \in \mathbb{N}$. The welded braid group WB_n admits the following presentation:*

$$\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \mid R \rangle,$$

where R is the set of the following relations:

i. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, *for* $i = 1, \dots, n-2$,

ii. $\sigma_i \sigma_j = \sigma_j \sigma_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

iii. $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$, *for* $i = 1, \dots, n-2$,

iv. $\rho_i \rho_j = \rho_j \rho_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

v. $\rho_i^2 = 1$, *for* $i = 1, \dots, n-1$,

vi. $\sigma_i \rho_j = \rho_j \sigma_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

vii. $\sigma_{i+1} \rho_i \rho_{i+1} = \rho_i \rho_{i+1} \sigma_i$, *for* $i = 1, \dots, n-2$,

viii. $\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i$, *for* $i = 1, \dots, n-2$.

Remark 4.1.4. In Theorem 4.1.3 we see that relations (i)-(ii) are braid group relations, relations (iii)-(v) are permutation group relations and relations $(vi) - (viii)$ are mixed relations.

We define now the untwisted ring group, UR_n, which gives a geometric interpretation of the welded braid group WB_n, as we will see in the following proposition that these two groups are isomorphic. The following definition appears in [Gol81], by Goldsmith, under the name of motion groups.

Definition 4.1.5. Let $n \in \mathbb{N}$ and let $\mathcal{UR}_n$ be the space of all configurations of n disjoint pairwise unlinked unordered Euclidean circles in $\mathbb{R}^3$ lying in planes parallel to a fixed one. The untwisted ring group UR_n is its fundamental group.

Due to Brendle–Hatcher, [BH13], we have the following presentation of UR_n.

Theorem 4.1.6 (Brendle–Hatcher, [BH13])**.** *Let $n \in \mathbb{N}$. The untwisted ring group UR_n admits the following presentation:*

$$\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \mid R \rangle,$$

where R is the set of the following relations:

i. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, *for* $i = 1, \dots, n-2$,

ii. $\sigma_i \sigma_j = \sigma_j \sigma_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

iii. $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$, *for* $i = 1, \dots, n-2$,

iv. $\rho_i \rho_j = \rho_j \rho_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

v. $\rho_i^2 = 1$, *for* $i = 1, \dots, n-1$,

vi. $\sigma_i \rho_j = \rho_j \sigma_i$, *for* $|i-j| > 1$, *where* $1 \leq i, j \leq n-1$,

vii. $\sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1}$, *for* $i = 1, \dots, n-2$,

viii. $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$, *for* $i = 1, \dots, n-2$.

Note that the generators σ_i, ρ_i, were initially considered by *Goldsmith* in [Gol81]. The generator σ_i permutes the i^{th} and the $(i+1)^{st}$ circles by passing the i^{th} circle through the $(i+1)^{st}$ and the generator ρ_i permutes them passing the i^{th} around the $(i+1)^{st}$, as depicted in the following figure.

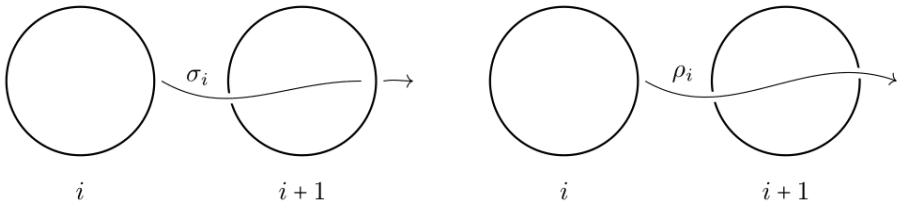


Figure 4.1 The generators σ_i and ρ_i.

Due to Damiani, [Dam17], we have the following proposition, where we see that the groups WB_n and UR_n are actually equivalent formulations of the same group. This result follows as a consequence of a result that was established for more general groups in [Gol81] by Goldsmith.

Proposition 4.1.7 (Damiani, [Dam17])**.** *For* $n \in \mathbb{N}$, *there is a natural isomorphism between the untwisted ring group* UR_n *and the welded braid group* WB_n.

In this chapter from now onwards, we will use as presentation of WB_n the one given in Theorem 4.1.6.

4.2 Homomorphisms from WB_n to the symmetric group S_n

The lower central series of a group G is defined as the descending series

$$\Gamma_1(G) = G \supseteq \Gamma_2(G) \supseteq \cdots \supseteq \Gamma_n \supseteq \dots,$$

where $\Gamma_i(G) = [\Gamma_{i-1}(G), G]$; the subgroup of G generated by all commutators $[x, y] = xyx^{-1}y^{-1}$, for $x \in \Gamma_{i-1}(G)$ and $y \in G$.

Proposition 4.2.1. *Let* $n \in \mathbb{N}$. *For the welded braid group* WB_n *we have the following results:*

- *For* $n \geq 2$ *the Abelianisation of* WB_n *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}_2$.
- *For* $n = 3$, $\Gamma_2(WB_3)/\Gamma_3(WB_3) \cong \mathbb{Z}_2$ *and for* $n \geq 4$, $\Gamma_2(WB_n) \cong \Gamma_3(WB_n)$.

Proof. Based on the presentation that we gave in Theorem 4.1.6 we have that the Abelianisation of WB_n is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$; the copy of the group $\mathbb{Z}$ is generated by the elements σ_i, the generators of the Artin braid group B_n and the copy of the group $\mathbb{Z}_2$ is generated by the elements ρ_i, the generators of symmetric group S_n.

Let $n \geq 3$. We have the following short exact sequence:

$$1 \to \Gamma_2(WB_n)/\Gamma_3(WB_n) \to \Gamma_1(WB_n)/\Gamma_3(WB_n) \xrightarrow{p} \Gamma_1(WB_n)/\Gamma_2(WB_n) \to 1.$$

Let $\Gamma_1/\Gamma_2 := \Gamma_1(WB_3)/\Gamma_2(WB_3)$ and $\Gamma_1/\Gamma_3 := \Gamma_1(WB_3)/\Gamma_3(WB_3)$. For $n = 3$, the group $\Gamma_1/\Gamma_3 := \Gamma_1(WB_3)/\Gamma_3(WB_3)$ is generated by the set

$$\{\bar{\sigma}_1 = \sigma_1\Gamma_3,\ \bar{\sigma}_2 = \sigma_2\Gamma_3,\ \bar{\rho}_1 = \rho_1\Gamma_3,\ \bar{\rho}_2 = \rho_2\Gamma_3\}.$$

Since $\sigma_2 = \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1} = \sigma_1[\sigma_2, \sigma_1]$ and $\rho_2 = \rho_1\rho_2\rho_1\rho_2^{-1}\rho_1^{-1} = \rho_1[\rho_2, \rho_1]$, it follows that σ_1, σ_2 project into the same element inside Γ_1/Γ_2; $\sigma_1\Gamma_2 = \sigma_1\Gamma_2$ and similarly, ρ_1, ρ_2 project into the same

element inside Γ_1/Γ_2; $\rho_1\Gamma_2 = \rho_2\Gamma_2$. Thus, $\bar{\sigma}_1 = \sigma_1\Gamma_3$, $\bar{\sigma}_2 = t\cdot\sigma_1\Gamma_3$ for $t \in \ker(p) = \Gamma_2/\Gamma_3$ and similarly $\bar{\rho}_1 = \rho_1\Gamma_3$, $\bar{\rho}_2 = s\cdot\rho_1\Gamma_3$ for $s \in \ker(p) = \Gamma_2/\Gamma_3$. Projecting now relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ into Γ_1/Γ_3 it follows that $\bar{\sigma}_1 t\bar{\sigma}_1\bar{\sigma}_1 = t\bar{\sigma}_1\bar{\sigma}_1 t\bar{\sigma}_1$, and since $t \in \Gamma_2/\Gamma_3$, where Γ_2/Γ_3 is a central subgroup of Γ_1/Γ_3, it follows that $t = 1$. As a result, $\bar{\sigma}_1 = \bar{\sigma}_2$. Also, projecting relation $\rho_1\rho_2\rho_1 = \rho_2\rho_1\rho_2$ into Γ_2/Γ_3 and using the same arguments it follows that $\bar{\rho}_1 = \bar{\rho}_2$. We conclude that Γ_1/Γ_3 is generated by these two elements $\{\bar{\sigma}_1, \bar{\rho}_1\}$. Now, the element $[\bar{\sigma}_1, \bar{\rho}_1]$ generates Γ_2/Γ_3. We observe that $[\bar{\sigma}_1, \bar{\rho}_1]^2 = [\bar{\sigma}_1, \bar{\rho}_1^2] = 1$ inside Γ_2/Γ_3. The first equality follows from the fact that $[\bar{\sigma}_1, \bar{\rho}_1]^2 = [\bar{\sigma}_1, \bar{\rho}_1]\bar{\rho}_1[\bar{\sigma}_1, \bar{\rho}_1]\bar{\rho}_1^{-1} = [\bar{\sigma}_1, \bar{\rho}_1^2]$ and that $[\bar{\sigma}_1, \bar{\rho}_1] \in \Gamma_2/\Gamma_3$ is central in Γ_1/Γ_3. The last equality holds because $\rho_1^2 = 1$ and therefore $(\bar{\rho}_1)^2 = 1$. Thus, the group Γ_2/Γ_3 is cyclic of order 2, which means $\Gamma_2(WB_3)/\Gamma_3(WB_3) \cong \mathbb{Z}_2$.

For $n \geq 4$, let $\Gamma_1/\Gamma_2 := \Gamma_1(WB_n)/\Gamma_2(WB_n)$ and $\Gamma_1/\Gamma_3 := \Gamma_1(WB_n)/\Gamma_3(WB_n)$. We have the following situation: all generators $\bar{\sigma}_i = \sigma_i\Gamma_3$, for $i = 1, \ldots, n-1$, in Γ_1/Γ_3 are sent to the same element in Γ_1/Γ_2; the Abelianisation of WB_n, since the elements $\sigma_i \in WB_n$, for $1 \leq i \leq n-1$, are pairwise conjugate and similarly for the generators $\bar{\rho}_j = \rho_j\Gamma_3$, for $1 \leq j \leq n-1$. Therefore, without loss of generality, we can assume that for any $\bar{\sigma}_i \in \Gamma_1/\Gamma_3$, $1 \leq i \leq n-1$, there exists $t_i \in \Gamma_2/\Gamma_3$ such that $\bar{\sigma}_i = t_i\bar{\sigma}_1$; where $t_1 = 1$, and similarly for $1 \leq j \leq n$ there exists $s_j \in \Gamma_2/\Gamma_3$ such that $\bar{\rho}_j = s_j\bar{\rho}_1$; where $s_1 = 1$. Projecting relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $1 \leq i \leq n-2$, into Γ_1/Γ_3 we get $t_i\bar{\sigma}_1 t_{i+1}\bar{\sigma}_1 t_i\bar{\sigma}_1 = t_{i+1}\bar{\sigma}_1 t_i\bar{\sigma}_1 t_{i+1}\bar{\sigma}_1$. We know that the elements $t_i, t_{i+1} \in \Gamma_2/\Gamma_3$ are central in Γ_1/Γ_3, and therefore the last relation gives that $t_i = t_{i+1} = 1$, since $t_1 = 1$. As a result we have that $\bar{\sigma}_1 = \cdots = \bar{\sigma}_{n-1} =: \bar{\sigma} \in \Gamma_1/\Gamma_3$. With a similar argument we obtain $\bar{\rho}_1 = \cdots = \bar{\rho}_{n-1} =: \bar{\rho} \in \Gamma_1/\Gamma_3$. We deduce that the group Γ_1/Γ_3 has two generators, $\bar{\sigma}, \bar{\rho}$, where $(\bar{\rho})^2 = 1$, since $\rho_i^2 = 1 \in WB_n$. Moreover, projecting relation $\sigma_i\rho_j = \rho_j\sigma_i$, $|i-j| > 1$ into Γ_1/Γ_3, we obtain that $\bar{\sigma}\bar{\rho} = \bar{\rho}\bar{\sigma}$, which means that the two generators of Γ_1/Γ_3 commute. Therefore, the group Γ_1/Γ_3 is an Abelian group with two generator, $\bar{\sigma}, \bar{\rho}$, satisfying $(\bar{\rho})^2 = 1$. By the universal property of the Abelianisation it follows that Γ_2 has to be a subgroup of Γ_3, since any projection of WB_n into an Abelian group has to factor through Γ_1/Γ_2, and therefore $\Gamma_2 \subseteq \Gamma_3$. But, by the definition of the lower central series we know that $\Gamma_3 \subseteq \Gamma_2$ and therefore we conclude that $\Gamma_2 = \Gamma_3$. □

For further results about the commutator subgroups of the welded braid groups WB_n, we refer the reader to [BGN19], by Bardakov–Gongopadhyay–Neshchadim, where among others, they proved that, for $m \geq 5$, the commutator subgroups, $[WB_n, WB_n]$, are perfect, which means that $[WB_n, WB_n] = \big[[WB_n, WB_n], [WB_n, WB_n]\big]$.

We recall the following definitions, which we will use in the following theorem.

- Let G, H be two groups. For every $x \in H$ we have the group homomorphism $h_x : H \to H$, defined by $h_x(y) = xyx^{-1}$. Two group homomorphisms $h_1, h_2 : G \to H$ are said to be conjugate if there exists an element $x \in H$ such that $h_2 = h_x \circ h_1$, which means that $h_2(g) = xh_1(g)x^{-1}$, for every $g \in G$.
- A group homomorphism $h : G \to H$ is said to be Abelian if its image $h(G)$ is an Abelian subgroup of H.
- A group homomorphism $h : G \to H$ is said to be cyclic if its image $h(G)$ is a cyclic subgroup of H.

Before stating the main theorem of this section we recall a known fact about the homomorphisms of the symmetric group.

It is well known that the symmetric group S_6 has an outer automorphism, which we will denote by v_6, unlike all other symmetric groups. Due to Artin, [Art47a] and Lin, [Lin], the following known result can be deduced.

Proposition 4.2.2. *Let $n, m \in \mathbb{Z}$ with $n \geq m$, such that $n \geq 5$, $m \geq 2$. For any homomorphism $h : S_n \to S_m$ one of the following holds.*

1. *The homomorphism h is Abelian and therefore cyclic.*
2. *For $n = m$ the homomorphism h is, up to conjugation, the identity.*
3. *For $n = m = 6$ the homomorphism h is, up to conjugation, v_6.*

We define now some maps that we will use in the proof of the following theorem.

1. Let α be the homomorphism $S_n \to WB_n$ defined by $\alpha(s_i) = \rho_i$, for $1 \leq i \leq n-1$.

2. Let ϕ be the homomorphism $WB_n \to S_n$ defined by $\phi(\sigma_i) = \phi(\rho_i) = s_i$, where $s(i) = (i, i+1) \in S_n$, for every $1 \leq i \leq n-1$.

Theorem 4.2.3. *Let $n \geq 5$ and let $h : WB_n \to S_n$ be any homomorphism. Then, up to conjugation, one of the following holds:*

- *The homomorphism h is the homomorphism ϕ.*
- *The homomorphism h is cyclic, whose image is of order 2.*
- *The homomorphism h is Abelian.*
- *For $n = 6$, the homomorphism h is $v_6 \circ \phi$.*

Proof. For $n \geq 5$, let $h : WB_n \to S_n$ be any homomorphism. From Proposition 4.2.2, the composition map $h \circ \alpha : S_n \to S_n$ is, up to conjugation, either Abelian or the identity homomorphism, and only in the case $n = 6$ we could also have that $h \circ \alpha$ is the homomorphism v_6. We will examine this case separately.

Suppose that $h \circ \alpha$ is the identity homomorphism. We have that $(h \circ \alpha)(s_i) = h(\alpha(s_i)) = h(\rho_i) = s_i$, for $1 \leq i \leq n-1$. It follows that $h(\rho_i) = s_i$, for $1 \leq i \leq n-1$. Moreover, from relation $\rho_i\sigma_j = \sigma_j\rho_i$, $|i-j| > 1$, we get that $h(\sigma_1) = h(\rho_i^{-1}\sigma_1\rho_i) = s_i^{-1}h(\sigma_1)s_i$ for $3 \leq i \leq n-1$, which means that $s_ih(\sigma_1) = h(\sigma_1)s_i$, for $3 \leq i \leq n-1$. For the same values of i, since $h(\sigma_1)$ commutes with s_i, we get that $h(\sigma_1)$ belongs to the centraliser of $\langle s_3, \ldots, s_{n-1}\rangle$ inside S_n; but this centraliser is $\{1, s_1\}$. As a result we have either that $h(\sigma_1) = 1$ or that $h(\sigma_1) = s_1$. We shall check each case separately.

Suppose that $h(\sigma_1) = 1$. In this case, because of relation $\sigma_i = (\sigma_1 \ldots \sigma_{n-1})^{i-1}\sigma_1(\sigma_1 \ldots \sigma_{n-1})^{1-i}$, we have that $h(\sigma_i) = 1$, $1 \leq i \leq n-1$. Therefore, we get $h(\rho_i) = s_i$ and $h(\sigma_i) = 1$, for $1 \leq i \leq n-1$. Now, from relation $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$, for $1 \leq i \leq n-2$, it follows that $h(\rho_i)h(\sigma_{i+1})h(\sigma_i) = h(\sigma_{i+1})h(\sigma_i)h(\rho_{i+1})$, which gives that $h(\rho_i) = h(\rho_{i+1}) = s_i = s_{i+1}$, for $1 \leq i \leq n-2$. Thus, the image of the homomorphism h in S_n is cyclic of order 2. Thus, $h(WB_n) \cong \mathbb{Z}_2$.

Suppose that $h(\sigma_1) = s_1$. By induction we can show that $h(\sigma_i) = s_i$, for $1 \leq i \leq n-1$. For $i = 1$ the hypothesis holds. Suppose that for some $i \geq 2$ we have $h(\sigma_i) = s_i$. From relation $\rho_{i+1}\rho_i\sigma_{i+1} = \sigma_i\rho_{i+1}\rho_i$ it follows that $h(\sigma_{i+1}) = h(\rho_i)^{-1}h(\rho_{i+1})^{-1}h(\sigma_i)h(\rho_{i+1})h(\rho_i)$. Thus, $h(\sigma_{i+1}) = s_i^{-1}s_{i+1}^{-1}s_is_{i+1}s_i = s_{i+1}$, which completes the induction. As a result we have that $h(\rho_i) = s_i$ and $h(\sigma_i) = s_i$, for $1 \leq i \leq n-1$. This implies that h is, up to conjugation, the homomorphism ϕ.

We will consider now the case where the homomorphism $h \circ \alpha : S_n \to S_n$ is Abelian and therefore cyclic, since the generators of S_n satisfy relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $1 \leq i \leq n-1$. We have that for all $1 \leq i \leq n-1$, $(h \circ \alpha)(s_i) = w$, for an element $w \in S_n$, with $w^2 = 1$. Therefore, $(h \circ \alpha)(s_i) = h(\alpha(s_i)) = h(\rho_i) = w$, which gives that $h(\rho_i) = w$, where $w^2 = 1$, $1 \leq i \leq n-1$. Relation $\sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1}$, under h, becomes $h(\sigma_i)w^2 = w^2h(\sigma_{i+1})$, and therefore, $h(\sigma_i) = h(\sigma_{i+1})$, for $1 \leq i \leq n-1$. Thus, we have that $h(\sigma_1) = \cdots = h(\sigma_{n-1}) =: \tau \in S_n$. Moreover, from relation $\rho_i\sigma_j = \sigma_j\rho_i$, for $|i-j| > 1$ and $1 \leq i, j \leq n-1$, we get $h(\rho_i)h(\sigma_j) = h(\sigma_j)h(\rho_i)$, which implies that $w\tau = \tau w$. As a result we have that the image of the homomorphism h is the Abelian group generated by the elements w and τ, with $w^2 = 1$. We conclude that in this case the homomorphism h is Abelian.

Lastly, suppose that $n = 6$ and that the homomorphism $h \circ \alpha : S_6 \to S_6$ is, up to conjugation, the homomorphism v_6; $h \circ \alpha = v_6$. The map $v_6^{-1} \circ h \circ \alpha$ becomes the identity homomorphism $S_6 \to S_6$ and it follows that $(v_6^{-1} \circ h \circ \alpha)(s_i) = (v_6^{-1} \circ h)(\rho_i) = s_i$, for $1 \leq i \leq 5$. Using relation $\rho_i\sigma_j = \sigma_j\rho_i$, $|i-j| > 1$ we get that, for $i \in \{3, 4, 5\}$, $(v_6^{-1} \circ h)(\sigma_1) = (v_6^{-1} \circ h)(\rho_i^{-1}\sigma_1\rho_i) = s_i^{-1}(v_6^{-1} \circ h)(\sigma_1)s_i$, and we obtain $s_i(v_6^{-1} \circ h)(\sigma_1) = (v_6^{-1} \circ h)(\sigma_1)s_i$. In other words, we have that $(v_6^{-1} \circ h)(\sigma_1)$ belongs to the centraliser of $\langle s_3, s_4, s_5\rangle$ in S_6, but the centraliser of $\langle s_3, s_4, s_5\rangle$ in S_6 is $\{1, s_1\}$. Therefore, either $(v_6^{-1} \circ h)(\sigma_1) = 1$ or $(v_6^{-1} \circ h)(\sigma_1) = s_1$. Following the same arguments as before, we conclude that in the case where $(v_6^{-1} \circ h)(\sigma_1) = 1$, it follows that $(v_6^{-1} \circ h)(\sigma_i) = 1$ and that $(v_6^{-1} \circ h)(\rho_i) = (v_6^{-1} \circ h)(\rho_{i+1}) = s_i = s_{i+1}$, for $1 \leq i \leq n-2$. Thus, $(v_6^{-1} \circ h)$ is a cyclic homomorphism, and as a result h is again a cyclic homomorphism, whose image is of order 2. In the case where $(v_6^{-1} \circ h)(\sigma_1) = s_1$, it follows that $(v_6^{-1} \circ h)(\sigma_i) = s_i$, and as a result $(v_6^{-1} \circ h) = \phi$, since we already have that $(v_6^{-1} \circ h)(\rho_i) = s_i$, for $1 \leq i \leq 5$. All together we get that the homomorphism h can be $v_6 \circ \phi$. □

Remark 4.2.4. For $n = 3, 4$, to determine all possible homomorphisms from WB_n to S_n seems to be trickier. Note that for $n = 2$ it holds that $WB_2 = \langle \sigma_1, \rho_1 \mid \rho_1^2 = 1 \rangle$, which is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}_2$, and thus the image of any homomorphism $h : WB_2 \to S_2$ is either the trivial group or S_2.

Remark 4.2.5. The only possible surjective homomorphisms from WB_n to S_n , for $n \geq 5$, are the homomorphism ϕ and, in the case of $n = 6$, also the homomorphism $v_6 \circ \phi$.

4.3 The welded pure braid groups WP_n

Let $n \in \mathbb{N}$. The welded pure braid group WP_n, as a subgroup of the welded braid group WB_n has also several interpretations. We now present two definitions of WP_n. One that involves automorphisms of the free group F_n and another in terms of the fundamental group of the space of specific configurations.

Definition 4.3.1. Let $n \in \mathbb{N}$ and let F_n denote the free group of rank n with generators $\{x_1, \ldots, x_n\}$ and let $Aut(F_n)$ denote its automorphism group. The welded pure braid group WP_n is the subgroup of $WB_n \subset Aut(F_n)$, which is generated by the automorphisms of the form

$$\epsilon_{i,j} : \begin{cases} x_i \mapsto x_j^{-1} x_i x_j, \\ x_k \mapsto x_k, \text{ for } k \neq i, j. \end{cases} \tag{3}$$

Thus, WP_n is the subgroup of $Aut(F_n)$, which consists of the so-called basis-conjugating automorphisms, $a : x_i \mapsto W_i^{-1} x_i W_i$, where W is a word in F_n.

Definition 4.3.2. Let $n \in \mathbb{N}$ and let $\mathcal{PUR}_n$ be the space of all configurations of n disjoint pairwise unlinked ordered Euclidean circles in $\mathbb{R}^3$ lying in planes parallel to a fixed one. The pure untwisted ring group PUR_n is its fundamental group.

In the previous section, we saw that $WB_n \cong UR_n$. Similarly we have $WP_n \cong PUR_n$, which means that WP_n and PUR_n are actually equivalent formulations of the same group. For a complete proof of the equivalence of these formulations of WP_n, as automorphisms of the free group and as the fundamental group of the space of certain configurations, we refer the reader to [Dam17].

A presentation of WP_n is known due to Humphries, [Hum85] and McCool, [McC86], who determined the generating set and the defining relations respectively, but also due to Brendle–Hatcher, [BH13], who used a different approach. To be more precise, Humphries and McCool considered WP_n as in Definition 4.3.1, while Brendle–Hatcher as in Definition 4.3.2.

Theorem 4.3.3. *Let $n \in \mathbb{N}$. The welded pure braid group WP_n admits the following presentation:*

$$\langle \epsilon_{i,j}, \text{ for, } 1 \leq i \neq j \leq n \mid R \rangle,$$

where R is the set of the following relations:

1. $\epsilon_{i,j}\epsilon_{k,l} = \epsilon_{k,l}\epsilon_{i,j}$,
2. $\epsilon_{i,k}\epsilon_{j,k} = \epsilon_{j,k}\epsilon_{i,k}$,
3. $\epsilon_{i,j}(\epsilon_{i,k}\epsilon_{j,k}) = (\epsilon_{i,k}\epsilon_{j,k})\epsilon_{i,j}$.

Based on the presentation of WP_n, given in Theorem 4.3.3, where the relations are commutation relations we obtain the following result about the Abelianisation of WP_n.

Corollary 4.3.4. *For $n \geq 2$ the Abelianisation of WP_n is isomorphic to $\mathbb{Z}^{n(n-1)}$.*

In terms of elements of the fundamental group of space of configurations, the element $\epsilon_{i,j}$ represents the movement of the i^{th} circle passing through the j^{th} circle and going back to its position, as shown in the following figure:

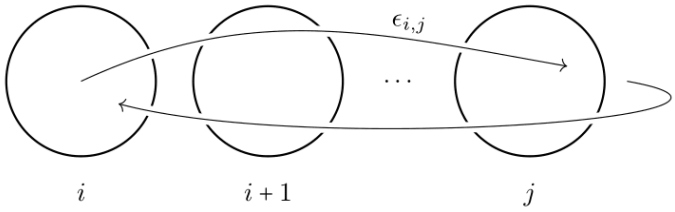


Figure 4.2 The element $\epsilon_{i,j}$.

Remark 4.3.5. With a geometric approach, based on Figure 4.1 and Figure 4.2, one can see that the generator $\epsilon_{i,i+1}$ is actually equal to $\rho_i\sigma_i^{-1}$ and the generator $\epsilon_{i+1,i}$ equal to $\sigma_i^{-1}\rho_i$. Thus we have

$$\epsilon_{i,i+1} = \rho_i\sigma_i^{-1} \text{ and } \epsilon_{i+1,i} = \sigma_i^{-1}\rho_i, \text{ for } 1 \leq i \leq n-1.$$

Considering the map $\phi : WB_n \to S_n$, defined by $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, for $i = 1, \ldots, n-1$, it follows that the welded pure braid group WP_n, is actually the kernel of this map ϕ. In [Sav96], Savushkina proved that WB_n can be seen as the semidirect product of its subgroups WP_n and S_n, and determined the action of S_n on WP_n, as described in the following theorem.

Theorem 4.3.6 (Savushkina, [Sav96])**.** *Let $n \in \mathbb{N}$. The group WB_n is isomorphic to the semidirect product $WP_n \rtimes S_n$, where the action of the symmetric group on WP_n is defined as follows:*

- $\epsilon_{i,j}s_k = s_k\epsilon_{i,j}$,
- $\epsilon_{i,j}s_i = s_i\epsilon_{i+1,j}$,
- $\epsilon_{i,j}s_j = s_j\epsilon_{i,j+1}$,
- $\epsilon_{i,i+1}s_i = s_i\epsilon_{i+1,i}$.

Remark 4.3.7. Note that the action of the symmetric group S_n on WP_n, in Theorem 4.3.6, can be written more compactly as

$$s\epsilon_{i,j}s^{-1} = \epsilon_{s(i),s(j)},$$

for any elemement $s \in S_n$ and $\epsilon_{i,j} \in WP_n$.

We are ready now to show that the group WP_n is actually a characteristic subgroup of WB_n, for $n \geq 5$, and that its centraliser in WB_n is trivial.

Proposition 4.3.8. *For $n \geq 5$ the group WP_n is a characteristic subgroup of WB_n.*

Proof. In order to prove that the group WP_n is a characteristic subgroup of WB_n, we have to show that for any automorphism of WB_n the subgroup WP_n stays invariant, that is for any $f \in Aut(WB_n)$ then $f(WP_n) = WP_n$.

Let f be any automorphism of the group WB_n and h be any surjective homomorphisms from WB_n to the symmetric group S_n. We consider the following composition map:

$$h \circ f : WB_n \xrightarrow{f} WB_n \xrightarrow{h} \!\!\!\!\rightarrow S_n.$$

From Remark 4.2.5 we know that the only possible surjective homomorphisms from WB_n to S_n are the homomorphism ϕ and, in the case of $n = 6$, also the homomorphism $v_6 \circ \phi$. These homomorphisms have as kernel the group WP_n. Therefore, it follows that $\ker(h) = WP_n$ and $\ker(h \circ f) = WP_n$, as both of them are epimorphisms from WB_n to S_n.

We have $\ker(h \circ f) = f^{-1}(WP_n)$, since f is an automorphism and $\ker(h) = WP_n$. Moreover, we have $\ker(h \circ f) = WP_n$. It follows that $f^{-1}(WP_n) = WP_n$, for any $f \in Aut(WB_n)$, which completes the proof. □

Remark 4.3.9. For $n = 2$ the group WP_2 is not a characteristic subgroup of WB_2. This is the case because the automorphism $\alpha : WB_2 \to WB_2$, defined by

$$\alpha : \begin{cases} \sigma_1 \mapsto \sigma_1^{-1}\rho_1, \\ \rho_1 \mapsto \rho_1, \end{cases} \tag{4}$$

does not send the element $\epsilon_{1,2} \in WP_2$ to an element in WP_2. In particular, $\epsilon_{1,2} = \rho_1\sigma_1^{-1}$, from Remark 4.3.5, and thus $\alpha(\epsilon_{1,2}) = \sigma_1 \notin WP_2$.

In [Sav96], Savushkina proved that WP_n has trivial centre, and moreover we show that the centraliser of WP_n in WB_n is also trivial.

Theorem 4.3.10 (Savushkina, [Sav96]). *Let $n \in \mathbb{N}$. The group WP_n has trivial centre.*

Proposition 4.3.11. *Let $n \in \mathbb{N}$. The centraliser of WP_n in WB_n is trivial.*

Proof. Let $n \in \mathbb{N}$. We want to calculate the centraliser

$$C_{WB_n}(WP_n) = \{g \in WB_n \mid gp = pg, \text{ for every } p \in WP_n\}.$$

We know that $WB_n = WP_n \rtimes S_n$, which means that any $g \in WB_n$ can be expressed as $g = w(\epsilon_{i,j})w(s_k)$, for $w(\epsilon_{i,j})$ and $w(s_k)$ a word in WP_n and S_n respectively.

Let $g \in C_{WB_n}(WP_n)$ be a non-trivial element, where $g = ES$, for E, S fixed words in WP_n and S_n respectively. It holds that $gp = pg$ for every element $p \in WP_n$. Therefore,

$$\epsilon_{k,l} \cdot ES = ES \cdot \epsilon_{k,l}, \text{ for every } k \neq l \in \{1, \dots, n\}.$$

From Remark 4.3.7, and in particular based on the action of the symmetric group on the generator $\epsilon_{k,l} \in WP_n$, we obtain

$$\epsilon_{k,l} \cdot ES = E \cdot \epsilon_{S(k),S(l)}S, \text{ for every } k \neq l \in \{1, \dots, n\}.$$

Thus,

$$\epsilon_{k,l} = E \cdot \epsilon_{S(k),S(l)} \cdot E^{-1} \in WP_n, \text{ for every } k \neq l \in \{1, \dots, n\}. \tag{5}$$

From relation (5), if $\epsilon_{k,l} = \epsilon_{S(k),S(l)}$, for every $k \neq l \in \{1, \dots, n\}$, it follows that the element E commutes with every generator $\epsilon_{k,l} \in WP_n$, and then we obtain a contradiction, since WP_n has a trivial centre, Theorem 4.3.10. Therefore, it holds that there exists $k, l \in \{1, \dots, n\}$ such that $\epsilon_{k,l} \neq \epsilon_{S(k),S(l)}$ and $\epsilon_{k,l} = E \cdot \epsilon_{S(k),S(l)} \cdot E^{-1}$. It follows that for such k, l there exists a pair $(r, t) \in \{1, \dots, n\}$, such that $\big(S(k), S(l)\big) = (r, t)$ and $\epsilon_{k,l} = E \cdot \epsilon_{r,t} \cdot E^{-1}$. This means that under the Abelianisation map the two distinct generators of the group WP_n, $\epsilon_{k,l}$ and $\epsilon_{r,t}$, would coincide. But, as stated in Corollary 4.3.4, the Abelianisation of WP_n is isomorphic to the free Abelian group of rank $n(n-1)$ generated by the elements $\epsilon_{i,j}$, for $1 \leq i, j \leq n$. Therefore, relation (5) can not hold.

Now, suppose that $g = S$; meaning that E is a trivial word in WP_n. It has to hold that

$$\epsilon_{k,l} \cdot S = S \cdot \epsilon_{k,l}, \text{ for every } k, l \in \{1, \dots, n\}. \tag{6}$$

But relation (6) implies that the word S in the symmetric group fixes all the elements of the set $\{1, \dots, n\}$. This is possible only when S is the trivial element, which leads once more to a contradiction, since g is a non-trivial element in $C_{WB_n}(WP_n)$. Since the centre of WP_n is trivial we do not need to check the case where $g = E$.

We conclude that a non-trivial element g in $C_{WB_n}(WP_n)$ does not exist, and therefore the centraliser $C_{WB_n}(WP_n)$ is trivial. □

4.4 About the automorphism group of WB_n

In [DG81], Dyer–Grossman proved that the group of the outer automorphisms of the Artin braid group, B_n, is isomorphic to $\mathbb{Z}_2$. We recall that for a group G, the outer automorphisms group of G is defined by the following quotient: $Out(G) = Aut(G)/Inn(G)$, where $Inn(G)$ is the group of the inner automorphisms of G. More precisely, for the Artin braid group we have $Out(B_n) \cong \mathbb{Z}_2$, for $n \geq 3$, generated by the automorphism ϵ_n, defined as follows:

$$\epsilon_n : \sigma_i \mapsto \sigma_i^{-1}, \text{ for every } 1 \leq i \leq n-1.$$

The automorphism group of the welded braid group, $Aut(WB_n)$, has not been determined yet. Even though the welded braid group, WB_n, is precisely a generalisation of the Artin braid group

B_n in the three dimensional space, the method which Dyer–Grossman used in their proof can not be applied directly for the group WB_n. The main problem that we encountered concerns the free group, F_n. To be more precise, it is needed, at some part of their proof, to show that the free group, F_{n-1}, is a characteristic subgroup of a quotient of B_n. Thus, we wanted to prove that F_n is a characteristic subgroup of WB_n, but the techniques that they use are not applicable in the case of the welded braid group. This is the case because the action of F_n, as a subgroup of WB_n, to WB_n has more freedom in comparison to the case of the Artin braid group, B_n. As we will see in the following proposition, we are able to show that F_n is actually a normal subgroup of WB_n. Nevertheless, we speculate, a stronger result, that the free group F_n is a characteristic subgroup of WB_n, which seems harder to obtain.

Proposition 4.4.1. *Let $n \geq 2$. The free group of rank n, F_n, is a normal subgroup of WB_n.*

Proof. Identifying the free group F_n as the group of inner automorphisms inside $Aut(F_n)$ we have $F_n \subseteq WP_n \subset WB_n \subset Aut(F_n)$. More precisely, we have that for any $\beta \in F_n$ it holds that $\beta(x_i) = wx_iw^{-1}$, for every $x_i \in F_n$ and for $w \in F_n$ a fixed element in F_n.

Let $\gamma \in WB_n \subset Aut(F_n)$. We consider the composition $\gamma\beta\gamma^{-1} : F_n \to F_n$. For any $x_i \in F_n$ we have that $\gamma\beta\gamma^{-1}(x_i) = \gamma\beta(\gamma^{-1}(x_i)) = \gamma(w\gamma^{-1}(x_i)w^{-1}) = \gamma(w)\gamma\gamma^{-1}(x_i)\gamma(w^{-1}) = \gamma(w)x_i(\gamma(w))^{-1} \in F_n \subset Aut(F_n)$, for $\gamma(w)$ a fixed element in F_n. As a result, $\gamma\beta\gamma^{-1} \in F_n \subset Aut(F_n)$, which implies that $\beta \in F_n$ is invariant under conjugation by any element of WB_n. We conclude that the group F_n is a normal subgroup of WB_n. □

The conjecture that we have regarding the outer automorphism group of WB_n is that it is not trivial, and more precisely, we conjecture that, for $n \geq 3$, the group $Out(WB_n)$ is generated by the automorphism α_n, which is of order two defined as follows:

$$\alpha_n : \begin{cases} \sigma_i \mapsto \rho_i\sigma_i^{-1}\rho_i, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1. \end{cases} \tag{7}$$

Therefore, we conjecture that $Out(WB_n) \cong \mathbb{Z}_2$, for $n \geq 3$.

The automorpism α_n, defined in (7), can be seen as the composition of two other maps; the map β_n and γ_n, defined as follows:

$$\beta_n : \begin{cases} \sigma_i \mapsto \sigma_i^{-1}, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1, \end{cases} \tag{8}$$

and

$$\gamma_n : \begin{cases} \sigma_i \mapsto \rho_i\sigma_i\rho_i, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1. \end{cases} \tag{9}$$

These two maps, although their composition is an automorphism of the welded braid group, they are not automorphisms of the group WB_n. Actually, they are not even homomorphisms, since relation $R8$, $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$, for $i = 1, \ldots, n-2$, in WB_n, is not preserved neither under the map β_n nor under the map γ_n. These two maps send the group WB_n to a different group, which we will denote by $\widetilde{WB}_n$. The group $\widetilde{WB}_n$ differs from the group WB_n only in the last relation $R8$, $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$, for $i = 1, \ldots, n-2$, which becomes $\sigma_i\sigma_{i+1}\rho_i = \rho_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, \ldots, n-2$. Based on this observation, we provide a motivation to explore the group which contains both relations $R8$ and $\widetilde{R8}$. This group appears to be an already known group, the unrestricted virtual braid group, which we explore in the following chapter.

CHAPTER 5

The unrestricted virtual braid groups UVB_n

In this chapter we explore the unrestricted virtual braid groups UVB_n. We begin by defining the group UVB_n and providing several references about the unrestricted virtual braid group. In Section 5.2 we determine all possible homomorphisms, up to conjugation, from UVB_n to the symmetric group S_n, for $n \geq 5$. We continue with Section 5.3 where we introduce an important subgroup of UVB_n, the group UVP_n, and we show that it is a characteristic subgroup as well as that it has a trivial centraliser in UVB_n. The Section 5.4 is devoted to the study of all possible images of UVB_n in any finite group G, under any group homomorphism. In this study, the theory of totally symmetric sets plays a main role and so we introduce these sets and we present several results which involve them. In Section 5.5, we give a characterisation of the torsion element in UVB_n, and in particular we show that any torsion element in UVB_n is a conjugate of an element of the symmetric group S_n. In Section 5.6 we prove that the Artin braid group B_n embeds in UVB_n as the crystallographic group $B_n/[P_n, P_n]$. At the last but one Section 5.7, we provide the necessary theory about the right-angled Artin groups, which we use for giving a complete description of the automorphism group of the group UVP_n. We complete this chapter with Section 5.8, where we give partial results about the automorphism group of UVB_n and prove that the groups UVB_n and UVP_n are Hopfian but not co-Hopfian.

5.1 Introduction

The group of unrestricted virtual braids, which we will denote throughout this text by UVB_n, was introduced by Kauffman and Lambropoulou in [KL04] and [KL06], where, respectively, they provide a new method for converting virtual knots and links to virtual braids and they prove a Markov Theorem for the virtual braid groups. The group UVB_n also appears in [KMRW17] as a quotient of the welded braid group WB_n and in [BBD15] where Bardakov–Bellingeri–Damiani give a description of the structure of this group. Therefore, it is worth studying the group of unrestricted virtual braids, UVB_n, for being a quotient of the welded braid group WB_n as well as for its own sake.

The unrestricted virtual braid group is defined as follows.

Definition 5.1.1. Let $n \in \mathbb{N}$. The group of unrestricted virtual braids UVB_n is defined by the group presentation

$$\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \mid R \rangle,$$

where R is the set of the following relations:

1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, \dots, n-2$,
2. $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| > 1$, where $1 \leq i, j \leq n-1$,

3. $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$, for $i = 1, \dots, n-2$,

4. $\rho_i\rho_j = \rho_j\rho_i$, for $|i-j| > 1$, where $1 \le i, j \le n-1$,

5. $\rho_i^2 = 1$, for $i = 1, \dots, n-1$,

6. $\sigma_i\rho_j = \rho_j\sigma_i$, for $|i-j| > 1$, where $1 \le i, j \le n-1$,

7. $\sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1}$, for $i = 1, \dots, n-2$,

8. $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$, for $i = 1, \dots, n-2$,

9. $\rho_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\rho_i$, for $i = 1, \dots, n-2$.

Remark 5.1.2. We see that, in order to define the group UVB_n, the only extra relation that we added to the group WB_n is relation 9.

Remark 5.1.3. Let $n \ge 2$. Based on the presentation of UVB_n, Definition 5.1.1, it follows that the Abelianisation of UVB_n is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, where $\mathbb{Z}$ is generated by $[\sigma_1]$ and $\mathbb{Z}_2$ is generated by $[\rho_1]$.

5.2 Homomorphisms from UVB_n to the symmetric group S_n

Before presenting all possible homomorphisms from UVB_n to the symmetric group S_n, we recall the map ϕ.

- Let ϕ be the homomorphism $UVB_n \to S_n$ defined by $\phi(\sigma_i) = \phi(\rho_i) = s_i$, where $s_i = (i, i+1) \in S_n$, for every $1 \le i \le n-1$.

We recall that with v_6 we denote the outer automorphism of the symmetric group S_6.

Theorem 5.2.1. *Let $n \ge 5$ and let $h : UVB_n \to S_n$ be any homomorphism. Then, up to conjugation, one of the following holds:*

- *The homomorphism h is the homomorphism ϕ.*
- *The homomorphism h is cyclic, whose image is of order 2.*
- *The homomorphism h is Abelian.*
- *For $n = 6$, the homomorphism h is $v_6 \circ \phi$.*

Proof. To prove this theorem we follow the same arguments as in Theorem 4.2.3; the equivalent theorem for the group WB_n. The same arguments hold here as well, since the relations that are used are valid also in this group, and the new relation, $\rho_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\rho_i$, that holds in UVB_n, does not provide further results. □

Remark 5.2.2. For $n = 3, 4$, to determine all possible homomorphisms from UVB_n to S_n seems to be trickier. Note that for $n = 2$ it holds that $UVB_2 = \langle \sigma_1, \rho_1 \mid \rho_1^2 = 1 \rangle = WB_2$, which is isomorphic to $\mathbb{Z} * \mathbb{Z}_2$, and thus the image of any homomorphism $h : UVB_2 \to S_2$ is either the trivial group or S_2.

Remark 5.2.3. The only possible surjective homomorphisms from UVB_n to S_n , for $n \ge 5$, are the homomorphism ϕ and, in the case of $n = 6$, also the homomorphism $v_6 \circ \phi$.

5.3 The unrestricted virtual pure braid groups UVP_n

Let $\phi : UVB_n \to S_n$ be the map defined by $\phi(\sigma_i) = \phi(\rho_i) = (i, i+1) \in S_n$, for $i = 1, \dots, n-1$. The unrestricted virtual pure braid group, which we denote by UVP_n, is the kernel of the map ϕ.

In [BBD15], Bardakov–Bellingeri–Damiani defined the following elements of UVP_n:

$$\begin{aligned}
&\lambda_{i,i+1} = \rho_i \sigma_i^{-1}, \text{ for } i = 1, \dots, n-1,\\
&\lambda_{i+1,i} = \sigma_i^{-1} \rho_i, \text{ for } i = 1, \dots, n-1,\\
&\lambda_{i,j} = \rho_{j-1}\rho_{j-2} \dots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \dots \rho_{j-2}\rho_{j-1}, \text{ for } 1 \le i < j-1 \le n-1,\\
&\lambda_{j,i} = \rho_{j-1}\rho_{j-2} \dots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \dots \rho_{j-2}\rho_{j-1}, \text{ for } 1 \le i < j-1 \le n-1.
\end{aligned} \tag{1}$$

Moreover, they gave a presentation of UVP_n, which is presented in the following theorem.

Theorem 5.3.1 (Bardakov–Bellingeri–Damiani, [BBD15]). *Let $n \in \mathbb{N}$. The group UVP_n admits the following presentation:*
Generators: $\lambda_{i,j}$, for $1 \le i \ne j \le n$.
Relations: The generators commute pairwise except for the couples $\lambda_{i,j}, \lambda_{j,i}$.

To give a better insight into UVP_n we present the following remarks.

Remark 5.3.2. The generators $\lambda_{i,j}$, for $1 \le i \ne j \le n$ of UVB_n are actually the same elements as the generators $\epsilon_{i,j}$, for, $1 \le i \ne j \le n$ in Theorem 4.3.3. In other words, one can define a map that sends each $\lambda_{i,j}$ to $\epsilon_{i,j}$.

Remark 5.3.3. The group

$$UVP_n = \langle \lambda_{i,j},\ 1 \le i \ne j \le n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j},\ \text{for } (k,l) \ne (j,i),\ 1 \le i,j,k,l \le n \rangle,$$

can be equivalently seen as the direct product of the following $n(n-1)/2$ factors:

$$UVP_n = \langle \lambda_{1,2}, \lambda_{2,1} \rangle \times \dots \times \langle \lambda_{i,j}\lambda_{j,i} \rangle \times \dots \times \langle \lambda_{n-1,n}, \lambda_{n,n-1} \rangle, \text{ for } 1 \le i \ne j \le n.$$

Thus, UVP_n is isomorphic to the direct product of $n(n-1)/2$ copies of the free group of rank 2:

$$UVP_n \cong \underbrace{F_2 \times \dots \times F_2 \times \dots \times F_2}_{n(n-1)/2\text{-times}}, \text{ for } n \ge 2.$$

Remark 5.3.4. The group UVP_n has trivial centre, $Z(UVP_n) = e$, since it is isomorphic to the direct product of free groups. For the same reason it follows that UVP_n is torsion free as well.

Based on the presentation of UVP_n, given in Theorem 5.3.1, where the relations are commutation relations we obtain the following result about the Abelianisation of UVP_n.

Corollary 5.3.5. *For $n \ge 2$, the Abelianisation of UVP_n is isomorphic to $\mathbb{Z}^{n(n-1)}$.*

The question that was posed in [KL06] about the non-trivial structure of UVB_n was answered by Bardakov–Bellingeri–Damiani in [BBD15], where they gave a decomposition of UVB_n into its subgroup UVP_n and the symmetric group S_n, as presented in the following theorem.

Theorem 5.3.6 (Bardakov–Bellingeri–Damiani, [BBD15]). *The group UVB_n is isomorphic to the semi-direct product $UVP_n \rtimes S_n$, where S_n acts by permuting the indices of the generators of UVP_n.*

More precisely, for all $\lambda_{i,j} \in UVP_n$, where $1 \le i \ne j \le n$, and for any $s \in S_n$ we have the following conjugating rule:

$$\iota(s)\lambda_{i,j}\iota(s)^{-1} = \lambda_{s(i),s(j)},$$

where ι is the injective map $\iota : S_n \to UVB_n$ defined by $\iota\big((i,i+1)\big) = \rho_i$; it is the natural section for the map ϕ that we defined in the beginning of this section. Moreover, the action of the symmetric group S_n on the generating set of UVP_n is transitive, see [[BBD15], Corollary 2.6].

Note that Theorem 5.3.6 is an equivalent result to Theorem 4.3.6, where we have a decomposition of WB_n into $WP_n \rtimes S_n$.

Remark 5.3.7. Having that $UVB_n \cong UVP_n \rtimes S_n$ it follows that the centre of UVB_n is trivial, $Z(UVB_n) = e$. This is indeed the case because the centre of the symmetric group and the centre of UVP_n are trivial, see Remark 5.3.4.

We now continue with proving that the group UVP_n is actually a characteristic subgroup of UVB_n, for $n \geq 5$, and that its centraliser in UVB_n is trivial.

Proposition 5.3.8. *For $n \geq 5$, the group UVP_n is a characteristic subgroup of UVB_n.*

Proof. In order to prove that the group UVP_n is a characteristic subgroup of UVB_n, we have to show that for any automorphism of UVB_n the subgroup UVP_n stays invariant. That is for any $f \in Aut(UVB_n)$ then $f(UVP_n) = UVP_n$.

Let f be any automorphism of the group UVB_n and h be any surjective homomorphisms from UVB_n to the symmetric group S_n. We consider the following composition map:

$$h \circ f : UVB_n \xrightarrow{f} UVB_n \xrightarrow{h} S_n.$$

From Remark 5.2.3 we know that the only possible surjective homomorphisms from UVB_n to S_n are the homomorphism ϕ and, in the case of $n = 6$, also the homomorphism $v_6 \circ \phi$; which homomorphisms have clearly as kernel the group UVP_n. Therefore, it follows that $\ker(h) = UVP_n$ and $\ker(h \circ f) = UVP_n$, as both of them are epimorphisms from UVB_n to S_n.

We have that $\ker(h \circ f) = f^{-1}(UVP_n)$, since f is an automorphism and $\ker(h) = UVP_n$. Moreover, we have that $\ker(h \circ f) = UVP_n$. It follows that $f^{-1}(UVP_n) = UVP_n$, for any $f \in Aut(UVB_n)$, which completes the proof. □

Remark 5.3.9. For $n = 2$ the group UVP_2 is not a characteristic subgroup of UVB_2. This is the case because the automorphism $\alpha : UVB_2 \to UVB_2$, defined by

$$\alpha : \begin{cases} \sigma_1 \mapsto \sigma_1^{-1}\rho_1, \\ \rho_1 \mapsto \rho_1, \end{cases}$$

does not send the element $\lambda_{1,2} \in UVP_2$ to an element in UVP_2. In particular, $\lambda_{1,2} = \rho_1\sigma_1^{-1}$, from Remark 1, and thus $\alpha(\lambda_{1,2}) = \sigma_1 \notin WP_2$.

Proposition 5.3.10. *Let $n \in \mathbb{N}$. The centraliser of UVP_n in UVB_n is trivial.*

Proof. The proof is the same as in the proof of Proposition 4.3.11, where we prove that the centraliser of WP_n in WB_n is trivial. □

5.4 Finite image of UVB_n

The main tool that we will use in order to determine all possible images of UVB_n, under a group homomorphism, in any finite group G is the theory of totally symmetric sets, which was introduced by Kordek and Margalit in [KM].

We start by introducing the notion of totally symmetric sets and presenting several results about them.

Definition 5.4.1 (Kordek–Margalit, [KM])**.** A subset X of a group G is called a totally symmetric set of G if it satisfies the following two conditions:

- The elements of the set $X = \{x_1, \ldots, x_n\}$ commute pairwise.
- Each permutation of X can be achieved via conjugation by an element of G.

We will present now a couple of facts about the totally symmetric sets, presented in [KM] and [CKLP20].

Lemma 5.4.2 (Kordek–Margalit, [KM])**.** *Let X be a totally symmetric set, of a group G, which has size k. For any homomorphism $h : G \to H$ it holds that $h(X)$ is either a singleton or a totally symmetric set of size k. In other words, $|f(X)|$ is either 1 or $|X|$.*

Proposition 5.4.3 (Chudnovsky, Kordek, Li, Partin, [CKLP20])**.** *Suppose that X is a totally symmetric set of a group G, with size $|X| = k$, whose elements have finite order. Then it follows that $|G| \geq 2^{k-1}k!$.*

Remark 5.4.4. Note that all elements of a totally symmetric set, X, are conjugate to each other, and thus every element of X has the same order. In particular, if one element of X has finite order p, then every other element of X has also order p.

Remark 5.4.5. Proposition 5.4.3 can be restated in an equivalent way and that is the following: Let $h : G \to H$ be a group homomorphsim from G to a finite group H. Suppose that X is a totally symmetric set of G with size $|X|$, then $|h(G)| \geq 2^{|X|-1}|X|!$.

From Lemma 5.4.2 we know that either $|h(X)| = |X|$ or $|h(X)| = 1$. Thus, the equivalence between these two statements comes from the fact that, for any finitely generated Abelian group S, then S is a torsion group if and only if the group S is a finite group.

We are ready now to define some totally symmetric sets of UVB_n, for $n \geq 3$. Based on the presentation of UVP_n given in Theorem 5.3.1, we define the following totally symmetric sets.

Lemma 5.4.6. *Let $n \geq 3$. The following n sets, whose cardinality is $n(n-1)/2$ are totally symmetric sets of UVB_n.*

$$A_i := \left\{\lambda_{i,1}, \ldots, \lambda_{i,n}, B_i, C_i\right\}, \text{ for } 1 \leq i \leq n,$$

where B_i, C_i are the following families of generators:
$B_i = \bigcup_{j=n}^{i+1} \{\lambda_{j,k}\}_{1 \leq k \neq i \leq j-1}$ *and* $C_i = \bigcup_{s=i-1}^{2} \{\lambda_{s,t}\}_{1 \leq t \leq i-2}$.

Proof. Let $n \geq 3$. The sets A_i do not contain any pair of elements of the form $\{\lambda_{k,l}, \lambda_{l,k}\}$, and therefore the elements inside each A_i commute pairwise, by Theorem 5.3.1. Moreover, by Theorem 5.3.6, UVB_n can be seen as a semi-direct product, $UVP_n \rtimes S_n$, where the symmetric group S_n acts by conjugation on the elements of UVP_n permuting the set of the elements $\lambda_{i,j}$. Thus, it follows that indeed the elements $\lambda_{i,j}$ are pairwise conjugate in UVB_n. As a result we get that indeed the sets A_i are totally symmetric sets. Moreover, the size of every totally symmetric set A_i is:

$$|A_i| = (n-1) + (n-2) + \cdots + \big(n - (n-2)\big) + \big(n - (n-1)\big) = n(n-1)/2.$$

□

At this point we will provide an example in order to make clearer the construction of these totally symmetric sets in UVB_n. For $n = 5$ we have the following five totally symmetric sets in UVB_5:

$$A_1 = \{\lambda_{1,2}, \lambda_{1,3}, \lambda_{1,4}, \lambda_{1,5}, \lambda_{5,2}, \lambda_{5,3}, \lambda_{5,4}, \lambda_{4,2}, \lambda_{4,3}, \lambda_{3,2}\}, \ |A_1| = 10.$$

$$A_2 = \{\lambda_{2,1}, \lambda_{2,3}, \lambda_{2,4}, \lambda_{2,5}, \lambda_{5,1}, \lambda_{5,3}, \lambda_{5,4}, \lambda_{4,1}, \lambda_{4,3}, \lambda_{3,1}\}, \ |A_2| = 10.$$

$$A_3 = \{\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,4}, \lambda_{3,5}, \lambda_{5,1}, \lambda_{5,2}, \lambda_{5,4}, \lambda_{4,1}, \lambda_{4,2}, \lambda_{2,1}\}, \ |A_3| = 10.$$

$$A_4 = \{\lambda_{4,1}, \lambda_{4,2}, \lambda_{4,3}, \lambda_{4,5}, \lambda_{5,1}, \lambda_{5,2}, \lambda_{5,3}, \lambda_{3,1}, \lambda_{3,2}, \lambda_{2,1}\}, \ |A_4| = 10.$$

$$A_5 = \{\lambda_{5,1}, \lambda_{5,2}, \lambda_{5,3}, \lambda_{5,4}, \lambda_{4,1}, \lambda_{4,2}, \lambda_{4,3}, \lambda_{3,1}, \lambda_{3,2}, \lambda_{2,1}\}, \ |A_5| = 10.$$

Remark 5.4.7. We observe that for any $1 \leq i, j \leq n$ where $i \neq j$ we have that $A_i \neq A_j$, since $\lambda_{i,j} \in A_i$ but $\lambda_{i,j} \notin A_j$, and also that $A_i \cap A_j \neq \varnothing$. Moreover, the set $\bigcup_{i=1}^{n} A_i$ is equal to the generating set of UVP_n. That is $\bigcup_{i=1}^{n} A_i = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$.

We shall now state and prove the main theorem of this section.

Theorem 5.4.8. *Let $n \geq 3$ and $\phi : UVB_n \to G$ be a group homomorphism to a finite group G. Then, one of the following must hold:*

- *$\phi(UVB_n) \cong \mathbb{Z}_m \times \mathbb{Z}_2$, for some $m \in \mathbb{N}$. In this case, the image of UVB_n is Abelian.*

- $|\phi(UVB_n)| \geq 2^{\frac{n(n-1)}{2}-1}\big(\frac{n(n-1)}{2}\big)!$.
- $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathrm{Im}(\phi_{|S_n})$, *for some* $m \in \mathbb{N}$. *In this case, the image of* UVP_n *is cyclic.*

Proof. Suppose that ϕ is Abelian. That means that the image of UVB_n is an Abelian subgroup of G. In this case we have the following situation: relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $1 \leq i \leq n-2$, that holds in UVB_n implies $\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i) = \phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1})$. Since $\phi(UVB_n)$ is an Abelian subgroup of G it follows that $\phi(\sigma_i)^2\phi(\sigma_{i+1}) = \phi(\sigma_i)\phi(\sigma_{i+1})^2$, and therefore that $\phi(\sigma_i) = \phi(\sigma_{i+1})$, for any $1 \leq i \leq n-2$. Therefore, there exists an element g in G such that $\phi(\sigma_i) = g \in UVB_n$, for all values $1 \leq i \leq n-1$. Similarly, we have relation $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ in UVB_n, for $1 \leq i \leq n-2$. Applying the same argument it follows that $\phi(\rho_i) = \phi(\rho_{i+1})$. Moreover, relation $\rho_i^2 = 1 \in UVB_n$ gives that $\phi(\rho_i)^2 = 1$, for any $1 \leq i \leq n-1$. As a result we conclude that $\phi(\sigma_i)$ generates a finite subgroup, $\mathbb{Z}_m$, inside G, and $\phi(\rho_i)$ generates the subgroup $\mathbb{Z}_2$ inside G. All together, we get that

$$\phi(UVB_n) \cong \mathbb{Z}_m \times \mathbb{Z}_2.$$

Suppose that ϕ is not Abelian. Moreover, suppose that at least one of the images of the totally symmetric sets A_i, defined in Lemma 5.4.6, under the homomorphism ϕ, is not a singleton. From Lemma 5.4.2 we have that either $|\phi(A_i)| = n(n-1)/2$ or $|\phi(A_i)| = 1$, and therefore, for some $k \in \{1, \ldots, n\}$, we have a totally symmetric set A_k for which $|\phi(A_k)| = n(n-1)/2$. From Proposition 5.4.3 we obtain that

$$|\phi(UVB_n)| \geq 2^{\frac{n(n-1)}{2}-1}\Big(\frac{n(n-1)}{2}\Big)!.$$

For the last possible case we consider that ϕ is not Abelian and also that $|\phi(A_i)| = 1$, for all $1 \leq i \leq n$. The fact that $|\phi(A_i)| = 1$ implies that $\phi(A_i) = g_i \in G$, for every $1 \leq i \leq n$. From Remark 5.4.7 we have $A_i \cap A_j \neq \varnothing$ for every pair i, j. Now, without loss of generality, we set $i = 1$ and therefore, from the fact that $A_1 \cap A_j \neq \varnothing$, for all $2 \leq j \leq n$ and that $\phi(A_1) = g_1 \in G$, we conclude that $\phi(A_j) = g_1 \in G$, for all $2 \leq j \leq n$. This means that every generator of UVP_n is mapped to the same element $g_1 \in G$, since $\bigcup_{i=1}^{n} A_i = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$, from Remark 5.4.7. Note that g could be possibly be the trivial element. From Theorem 5.3.6 we have that UVB_n is isomorphic to the semi-direct product $UVP_n \rtimes S_n$ and that for any generator $s \in S_n$ it holds that $s\lambda_{i,j}s^{-1} = \lambda_{s(i),s(j)}$. Under the homomorphism ϕ we obtain $\phi(s)\phi(\lambda_{i,j})(\phi(s))^{-1} = \phi(\lambda_{s(i),s(j)})$ and therefore $\phi(s)\phi(\lambda_{i,j}) = \phi(\lambda_{s(i),s(j)})\phi(s)$. Finally we get $\phi(s)g_1 = g_1\phi(s)$. We conclude that the image of any generator s of the symmetric group, $\phi(s)$, commutes with g_1. In other words, the image of UVP_n is cyclic, where every generator of UVP_n is mapped to the same element which commutes with the image of the symmetric group. Thus, $\phi(UVB_n) \cong \mathbb{Z}_m \times \mathrm{Im}(\phi_{|S_n})$, for some $m \in \mathbb{N}$, and in particular, $\phi(UVB_n) \cong \mathrm{Im}(\phi_{|S_n})$ in the case when $g = 1$. □

5.5 Torsion elements of UVB_n

In this section we characterise the torsion elements of UVB_n. Let ι be the injective map $\iota : S_n \to UVB_n$ defined by $\iota\big((i, i+1)\big) = \rho_i \in UVB_n$.

Proposition 5.5.1. *Let* $n \geq 2$. *Any torsion element* w *in* UVB_n *belongs to the normal closure of* $\iota(S_n)$.

Proof. Let g be a torsion element in UVB_n. The image of g in the quotient of UVB_n by the normal closure of $\iota(S_n)$ is the identity since this quotient is isomorphic to UVP_n which is torsionless. Therefore g belongs to the normal closure of $\iota(S_n)$. □

Note that this proposition is true also for WB_n since WP_n is torsionless. In the case of UVB_n we have however a stronger result, as described in the following theorem.

Theorem 5.5.2. *Let* $n \geq 2$. *For any torsion element* w, *of order* r, *in* UVB_n *there exists an element* $s \in S_n$, *that depends on* w, *of order* r *and such that* w *is conjugate to* $\iota(s)$ *by an element of* UVP_n.

Before moving on to the proof of the Theorem 5.5.2, we fix some notation and we make some observations about the action of the symmetric group S_n on the group UVP_n, which we will use in the proof. Let $s \in S_n$ and let G_s denote the cyclic group generated by the element s. From Theorem 5.3.6 we have that an element $s \in S_n$ acts on $UVP_n = \langle \lambda_{i,j} \mid 1 \le i \ne j \le n \rangle$ by permuting the indices of the elements $\lambda_{i,j}$. For simplicity, we write the element $s\lambda_{i,j}s^{-1}$ as $s(\lambda_{i,j})$ (and more generally sgs^{-1} as $s(g)$ for $g \in UVP_n$) and we set $\lambda_{s(i,j)}$ to be the element $\lambda_{s(i),s(j)}$. We will also identify a permutation s with the element $\iota(s)$ in UVB_n.

Let G_s act on the set $\{\lambda_{i,j} \mid 1 \le i \ne j \le n\}$. We can express the orbit of the element $\lambda_{i,j}$ as follows:

$$G_s \cdot \lambda_{i,j} := \{\lambda_{s^\iota(i,j)},\ 0 \le \iota \le k_{i,j} - 1,\ k_{i,j} \in \mathbb{N}_{>0}\},$$

where $k_{i,j} \ge 1$ is the cardinality of the orbit of the element $\lambda_{i,j}$. We observe that the cardinality of the orbit $G_s \cdot \lambda_{i,j}$ coincides with the cardinality of the orbit $G_s \cdot \lambda_{j,i}$. We define $O_{i,j}$ to be the union of the orbits $G_s \cdot \lambda_{i,j}$ and $G_s \cdot \lambda_{i,j}$. By abuse of notation, we call the sets $O_{i,j}$ orbits. Notice that orbits $G_s \cdot \lambda_{i,j}$ and $G_s \cdot \lambda_{i,j}$ possibly coincide: in this case $G_s \cdot \lambda_{i,j} = \{\lambda_{i,j}, \lambda_{j,i}\}$, $G_s \cdot \lambda_{j,i} = \{\lambda_{j,i}, \lambda_{i,j}\}$ and $O_{i,j} = \{\lambda_{i,j}, \lambda_{j,i}\}$ (and therefore $k_{i,j} = k_{j,i} = 2$). Notice that if $G_s \cdot \lambda_{i,j} = \{\lambda_{i,j}\}$ and $G_s \cdot \lambda_{j,i} = \{\lambda_{j,i}\}$ we have still that $O_{i,j} = \{\lambda_{i,j}, \lambda_{j,i}\}$ but in this case $k_{i,j} = k_{j,i} = 1$. Therefore the cardinality of $O_{i,j}$ is $2k_{i,j}$ if $G_s \cdot \lambda_{i,j}$ and $G_s \cdot \lambda_{i,j}$ are distinct and 2 otherwise. In the following we set $\langle O_{i,j} \rangle$ to be the group generated by the set $O_{i,j}$.

We present the following two lemmas, which we will use in the proof of Theorem 5.5.2.

Lemma 5.5.3. *Let $w \in UVB_2$. If $w^2 = 1$, then w is a conjugate to s_1 by an element of UVP_2.*

Proof. Recall that $UVB_2 = UVP_2 \rtimes S_2$, where UVP_2 is the group freely generated by $\{\lambda_{1,2}, \lambda_{2,1}\}$ and where $s_1 \in S_2$ permutes $\lambda_{1,2}$ and $\lambda_{2,1}$. Therefore, if $w \in UVB_2$, w can be written as $w = g_{1,2}s_1$, where $g_{1,2} \in F_{1,2} = \langle \lambda_{1,2}, \lambda_{2,1} \rangle$.

Suppose that $w^2 = 1$. If $g_{1,2} = 1$ the statement is trivially verified. Otherwise, since $w^2 = 1$ implies that $g_{1,2}s(g_{1,2}) = 1$, the word $g_{1,2}$ cannot be written in a reduced form as a word starting and ending with a non trivial power of the same generator: more precisely we can suppose w.l.o.g. that $g_{1,2}$ can be written as $g_{1,2} = \lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{1,2}^{\epsilon_{t-1}}\lambda_{2,1}^{\epsilon_t} \ne 1$, for $\epsilon_1, \epsilon_2, \dots, \epsilon_t$ non zero integers, and we can rewrite $g_{1,2}s(g_{1,2}) = 1$ as follows:

$$g_{1,2}s(g_{1,2}) = \underbrace{\left(\lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{1,2}^{\epsilon_{t-1}}\lambda_{2,1}^{\epsilon_t}\right)}_{\ne 1} \cdot \underbrace{\left(\lambda_{2,1}^{\epsilon_1}\lambda_{1,2}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t-1}}\lambda_{1,2}^{\epsilon_t}\right)}_{\ne 1} = 1 \in F_{1,2}.$$

Since $g_{1,2}s(g_{1,2}) = 1$ in the free group $F_{1,2}$ we have that $\epsilon_{t-j} = -\epsilon_j$ for $j = 0, \dots, t-1$. Therefore,

$$g_{1,2} = \lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t/2}}\lambda_{1,2}^{-\epsilon_{t/2}}\cdots\lambda_{1,2}^{-\epsilon_2}\lambda_{2,1}^{-\epsilon_1}.$$

Now, we can indeed express $w = g_{1,2}s$ as a conjugation of s by an element in UVP_2.

$$\begin{aligned} w = g_{1,2}s &= \lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t/2}}\lambda_{1,2}^{-\epsilon_{t/2}}\cdots\lambda_{1,2}^{-\epsilon_2}\lambda_{2,1}^{-\epsilon_1} \cdot s \\ &= \lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t/2}} \cdot s \cdot \lambda_{2,1}^{-\epsilon_{t/2}}\cdots\lambda_{2,1}^{-\epsilon_2}\lambda_{1,2}^{-\epsilon_1} \\ &= \left(\lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t/2}}\right) \cdot s \cdot \left(\lambda_{1,2}^{\epsilon_1}\lambda_{2,1}^{\epsilon_2}\cdots\lambda_{2,1}^{\epsilon_{t/2}}\right)^{-1} \\ &= \Lambda_1 \cdot s \cdot \Lambda_1^{-1}, \end{aligned}$$

where $\Lambda_1 \in F_{1,2}$.

We conclude that:

$$w = \Lambda_1 \cdot s \cdot \Lambda_1^{-1}, \text{ where } \Lambda_1 \in UVP_2.$$

□

Lemma 5.5.4. *Let $n > 2$ and $s \in S_n$. Let $O_1, \dots, O_i, \dots, O_{l_s}$, the disjoint orbits of the action of G_s on the pairs $\{\lambda_{i,j}, \lambda_{j,i}\}$. Let $1 \le i \le l_s$ and $\gamma_j \in \langle O_j \rangle$ such that $\gamma_j s$ has finite order in UVB_n. Then $\gamma_j s$ is conjugate to s by an element of UVP_n.*

Proof. Let $s \in S_n$. It is enough to prove the statement for the orbit of the pair $\{\lambda_{1,2}, \lambda_{2,1}\}$. We set $O_1 = \{\lambda_{s^\iota(1,2)}, \lambda_{s^\iota(2,1)}\}$. Let γ_1 be a non trivial element of $\langle O_1 \rangle$ such that $\gamma_1 s$ is of finite order in UVB_n.

Let first suppose that $G_s \cdot \lambda_{1,2} = G_s \cdot \lambda_{2,1}$; therefore $\langle O_1 \rangle = UVP_2$, the free group generated by $\{\lambda_{1,2}, \lambda_{2,1}\}$ and $s = s_1 \cdot s'$ where s' has disjoint support from s_1. Note that $(\gamma_1 s)^m = \underbrace{\gamma_1 s \cdot \gamma_1 s \cdots \gamma_1 s}_{m-\text{times}} = \gamma_1 s \gamma_1 s^{-1} s^2 \gamma_1 s^{-2} \cdots s^{m-1} \gamma_1 s^{-(m-1)} s^m = \gamma_1 s(\gamma_1) \cdots s^{m-1}(\gamma_1) s^m$, and thus $(\gamma_1 s)^m = \gamma_1 s(\gamma_1) \cdots s^{m-1}(\gamma_1) s^m$ for any integer m; if $\gamma_1 \in O_1$ then $s^2(\gamma_1) = \gamma_1$ and if $\gamma_1 s(\gamma_1)$ is of finite order then $\gamma_1 s(\gamma_1) = 1$ since $\langle O_1 \rangle$ is the free group generated by $\{\lambda_{1,2}, \lambda_{2,1}\}$. Therefore if $\gamma_1 s$ has finite order in UVB_n, $\gamma_1 s_1$ is of order 2. We can therefore apply previous lemma to $\gamma_1 s_1$ and $\gamma_1 s_1 = \gamma' s_1 \gamma'^{-1}$ for some $\gamma' \in \langle O_1 \rangle$. Then we have that $\gamma_1 s = \gamma_1 s_1 s' = \gamma' s_1 \gamma'^{-1} s' = \gamma' s_1 s' \gamma'^{-1} = \gamma' s \gamma'^{-1}$ since s' acts trivially on $\{\lambda_{1,2}, \lambda_{2,1}\}$.

So let suppose from now on that $G_s \cdot \lambda_{1,2} \neq G_s \cdot \lambda_{2,1}$; We set k_1 the cardinality of $G_s \cdot \lambda_{1,2}$; therefore O_1 has cardinality $2k_1$. Then $\gamma_1 = g_{1,2} g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)}$, where $g_{1,2} \in F_{1,2} = \langle \lambda_{1,2}, \lambda_{2,1} \rangle$ and $g_{s^p(1,2)} \in F_{s^p(1,2)} = \langle \lambda_{s^p(1,2)}, \lambda_{s^p(2,1)} \rangle$, for $1 \leq p \leq k_1 - 1$. Clearly, it can happen that for some $1 \leq p \leq k_1 - 1$ the factor $g_{s^p(1,2)}$ does not exist in γ_1, which means that γ_1 does not contain a subword in $F_{s^p(1,2)} = \langle \lambda_{s^p(1,2)}, \lambda_{s^p(2,1)} \rangle$, or that after possible commutations this factor becomes trivial. The elements $g_{1,2}, g_{s(1,2)}, \ldots, g_{s^{k_1-1}(1,2)}$ commute pairwise, since they belong to different factors of UVP_n. Notice that if $(\gamma_1 s)$ is of finite order, since UVP_n is torsionless it follows that $\gamma_1 s(\gamma_1) \cdots s^{k_1-1}(\gamma_1) = 1$ and thus that

$$g_{1,2} g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)} s\big(g_{1,2} g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)}\big) \cdots s^{k_1-1}\big(g_{1,2} g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)}\big) = 1.$$

We deduce that the product of the elements that belong to $F_{1,2}$ and to $F_{s^p(1,2)}$, $1 \leq p \leq k_1 - 1$ are trivial. More precisely, the elements that belong to $F_{1,2}$ are $g_{1,2}, s\big(g_{s^{k_1-1}(1,2)}\big), \ldots, s^{k_1-1}\big(g_{s(1,2)}\big)$ and we have that $g_{1,2} s\big(g_{s^{k_1-1}(1,2)}\big) \cdots s^{k_1-1}\big(g_{s(1,2)}\big) = 1$. As a result, we get that

$$g_{1,2} = s^{k_1-1}\big(g^{-1}_{s(1,2)}\big) \cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big).$$

The element $\gamma_1 = g_{1,2} g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)}$ gets the following form:

$$\gamma_1 = s^{k_1-1}\big(g^{-1}_{s(1,2)}\big) \cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big) \cdot g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)}.$$

We are ready to prove that $\gamma_1 s$ is a conjugation of s by an element of UVP_n. We have that

$$\gamma_1 \cdot s = s^{k_1-1}\big(g^{-1}_{s(1,2)}\big) \cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big) \cdot g_{s(1,2)} \cdots g_{s^{k_1-1}(1,2)} \cdot s.$$

Based on this expression of $\gamma_1 \cdot s$ we consider the pairs $s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)$, $g_{s^t(1,2)}$, for $1 \leq t \leq k_1 - 1$. We will prove that the element $s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big) g_{s^t(1,2)} \cdot s$ can be written as a conjugation of s by an element of UVP_n, for any $1 \leq t \leq k_1 - 1$.

We set $E_t = \prod_{c=k_1-(t-1)}^{k_1-1} s^c\big(g^{-1}_{s^t(1,2)}\big)$ for $1 \leq t \leq k_1 - 1$. Notice that $E_1 = 1$ and that $E_2 = s^{k_1-1}\big(g^{-1}_{s^2(1,2)}\big)$.

For $t = 1$ we have that

$$s^{k_1-1}\big(g^{-1}_{s(1,2)}\big) g_{s(1,2)} \cdot s = s^{k_1-1}\big(g^{-1}_{s(1,2)}\big) \cdot s \cdot s^{k_1-1}\big(g_{s(1,2)}\big), \text{ since } s \cdot s^{k_1-1}\big(g_{s(1,2)}\big) \cdot s^{-1} = g_{s(1,2)}.$$

For $2 \leq t \leq k_1 - 1$, $E_t = s^{k_1-(t-1)}\big(g^{-1}_{s^t(1,2)}\big) \cdots s^{k_1-2}\big(g^{-1}_{s^t(1,2)}\big) s^{k_1-1}\big(g^{-1}_{s^t(1,2)}\big) = \prod_{c=k_1-(t-1)}^{k_1-1} s^c\big(g^{-1}_{s^t(1,2)}\big)$. The elements $s^{k_1-(t-1)}\big(g^{-1}_{s^t(1,2)}\big), \ldots, s^{k_1-2}\big(g^{-1}_{s^t(1,2)}\big), s^{k_1-1}\big(g^{-1}_{s^t(1,2)}\big)$ commute pairwise since they belong to $F_{s(1,2)}, \ldots, F_{s^{t-2}(1,2)}, F_{s^{t-1}(1,2)}$, respectively. We have that

$$\begin{aligned} s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big) g_{s^t(i,j)} \cdot s &= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big) E_t E_t^{-1} g_{s^t(i,j)} \cdot s \\ &= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big) E_t \cdot s \cdot s^{k_1-1}(E_t^{-1}) s^{k_1-1}\big(g_{s^t(i,j)}\big) \end{aligned}$$

The element $s^{k_1-1}\big(E_t^{-1}\big)$ is $\big(s^{k_1-t}\big(g_{s^t(1,2)}\big) \cdots s^{k_1-3}\big(g_{s^t(1,2)}\big) s^{k_1-2}\big(g_{s^t(1,2)}\big)\big) = \prod_{c=k_1-t}^{k_1-2} s^c\big(g_{s^t(1,2)}\big)$. In $s^{k_1-1}\big(E_t^{-1}\big)$ the elements $s^{k_1-t}\big(g_{s^t(1,2)}\big), \ldots, s^{k_1-3}\big(g_{s^t(1,2)}\big), s^{k_1-2}\big(g_{s^t(1,2)}\big)$ commute pairwise, since

they belong to $F_{1,2},\dots,F_{s^{t-3}(1,2)},F_{s^{t-2}(1,2)}$, respectively. Moreover, the element $s^{k_1-1}\big(g_{s^t(i,j)}\big)$ belongs in $F_{s^{t-1}(1,2)}$ and thus it commutes with $s^{k_1-1}\big(E_t\big)$. As a result, continuing the above equalities,we obtain that

$$\begin{aligned}
s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)g_{s^t(1,2)}\cdot s &= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_tE_t^{-1}g_{s^t(1,2)}\cdot s\\
&= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_t\cdot s\cdot s^{k_1-1}(E_t)s^{k_1-1}\big(g_{s^t(1,2)}\big)\\
&= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-(t-1)}\big(g^{-1}_{s^t(1,2)}\big)\cdots s^{k_1-2}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-1}\big(g^{-1}_{s^t(1,2)}\big)\cdot s\cdot\\
&\quad s^{k_1-t}\big(g_{s^t(1,2)}\big)\cdots s^{k_1-3}\big(g_{s^t(1,2)}\big)s^{k_1-2}\big(g_{s^t(1,2)}\big)s^{k_1-1}\big(g_{s^t(1,2)}\big)\\
&= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-(t-1)}\big(g^{-1}_{s^t(1,2)}\big)\cdots s^{k_1-2}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-1}\big(g^{-1}_{s^t(1,2)}\big)\cdot s\cdot\\
&\quad \big(s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-(t-1)}\big(g^{-1}_{s^t(1,2)}\big)\cdots s^{k_1-2}\big(g^{-1}_{s^t(1,2)}\big)s^{k_1-1}\big(g^{-1}_{s^t(1,2)}\big)\big)^{-1}\\
&= s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_t\cdot s\cdot E_t^{-1}s^{k_1-t}\big(g_{s^t(1,2)}\big).
\end{aligned}$$

We conclude that indeed $s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)g_{s^t(1,2)}\cdot s$ can be written as a conjugation of s by an element of UVP_n, for any $1\le t\le k_1-1$. More precisely, we have that

$$s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)g_{s^t(1,2)}\cdot s = s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_t\cdot s\cdot E_t^{-1}s^{k_1-t}\big(g_{s^t(1,2)}\big),\ \text{for any } 1\le t\le k_1-1.$$

Based on this result we will show now that $\gamma_1 s=\Lambda_1\cdot s\cdot\Lambda_1^{-1}$, where $\Lambda_1\in UVP_n$. We recall that

$$\gamma_1\cdot s = s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)\cdot g_{s(1,2)}\cdots g_{s^{k_1-1}(1,2)}\cdot s.$$

We will proceed as follows: at every step we will be considering the pair $s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big),g_{s^t(1,2)}$, for $1\le t\le k_1-1$. Starting with the pair $s^{k_1-1}\big(g^{-1}_{s(1,2)}\big),g_{s(1,2)}$ we simply move at the right hand side of s the element $g_{s(1,2)}$, which becomes $s^{k_1-1}\big(g_{s(1,2)}\big)$. We can indeed move the element $g_{s(1,2)}$, since it commutes with all the elements on its right, as none of them belong to $F_{s(1,2)}$. We continue with the pair $s^{k_1-2}\big(g^{-1}_{s^2(1,2)}\big),g_{s^2(1,2)}$, where we add and substract the element E_2 besides $s^{k_1-2}\big(g^{-1}_{s^2(1,2)}\big)$. Then, we move on the right hand side of s first the element $g_{s^2(1,2)}$, which becomes $s^{k_1-1}\big(g_{s^2(1,2)}\big)$ and then the element E_2^{-1}, which becomes $s^{k_1-1}(E_2^{-1})$. We can indeed make these moves, since at every moment that we move an element on the right, till we arrive on the right hand side of s, the element that we move does not meet any other element that belongs in the same factor of UVP_n, and therefore they indeed commute. We continue doing the same at every step with all the pairs till we do the same with the last pair $s\big(g^{-1}_{s^{k_1-1}(1,2)}\big),g_{s^{k_1-1}(1,2)}$. Finally, we observe that on the left hand side and the right hand side of s, we have the same elements with opposite signs, and with eligible permutations we achive also the right ordering of elements and we indeed get that $\gamma_1 s=\Lambda_1\cdot s\cdot\Lambda_1^{-1}$, where $\Lambda_1\in UVP_n$. We present now the formulation of this method.

$$\begin{aligned}
\gamma_1\cdot s &= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)\cdot g_{s(1,2)}\cdots g_{s^t(i,j)}\cdots g_{s^{k_1-1}(1,2)}\cdot s\\
&= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)\cdot g_{s^2(i,j)}\cdots g_{s^t(i,j)}\cdots g_{s^{k_1-1}(1,2)}\cdot s\cdot s^{k_1-1}\big(g_{s(1,2)}\big)\\
&\ \ \vdots\\
&= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_tE_t^{-1}s^{k_1-t}\big(g^{-1}_{s^{t+1}(1,2)}\big)\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)\cdot g_{s^t(i,j)}\cdots g_{s^{k_1-1}(1,2)}\cdot s\cdot\\
&\quad s^{k_1-1}\big(E_{t-1}^{-1}\big)s^{k_1-1}\big(g_{s^{t-1}(1,2)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big)\\
&= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_ts^{k_1-t}\big(g^{-1}_{s^{t+1}(1,2)}\big)\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)\cdot g_{s^{t+1}(1,2)}\cdots g_{s^{k_1-1}(1,2)}\cdot s\cdot\\
&\quad s^{k_1-1}\big(E_t^{-1}\big)s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big)\\
&\ \ \vdots\\
&= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_t\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)E_{k_1-1}E_{k_1-1}^{-1}\cdot g_{s^{k_1-1}(1,2)}\cdot s\cdot\\
&\quad s^{k_1-1}\big(E_{k-2}^{-1}\big)s^{k_1-1}\big(g_{s^{k-2}(i,j)}\big)\cdots s^{k_1-1}\big(E_t^{-1}\big)s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big)\\
&= s^{k_1-1}\big(g^{-1}_{s(1,2)}\big)\cdots s^{k_1-t}\big(g^{-1}_{s^t(1,2)}\big)E_t\cdots s\big(g^{-1}_{s^{k_1-1}(1,2)}\big)E_{k_1-1}\cdot s\cdot\\
&\quad s^{k_1-1}\big(E_{k_1-1}^{-1}\big)s^{k_1-1}\big(g_{s^{k_1-1}(1,2)}\big)\cdots s^{k_1-1}\big(E_t^{-1}\big)s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big).
\end{aligned}$$

At each step we can move the product E_t^{-1} on the right hand side of s, since $E_t^{-1} \in F_{s(1,2)} \times \cdots \times F_{s^{t-2}(1,2)} \times F_{s^{t-1}(1,2)}$ and all the elements that it meets belong in the factors $F_{1,2}, F_{s^t(1,2)}, \ldots, F_{s^{k_1-1}(1,2)}$, and therefore they indeed commute. Now, replacing the values of E_{k_1-1} and E_t we obtain the following:

$$
\begin{aligned}
\gamma_1 \cdot s = {} & s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots s^{k_1-t}\big(g_{s^t(1,2)}^{-1}\big)E_t\cdots s\big(g_{s^{k_1-1}(1,2)}^{-1}\big)E_{k_1-1}\cdot s\cdot \\
& s^{k_1-1}\big(E_{k_1-1}^{-1}\big)s^{k_1-1}\big(g_{s^{k_1-1}(1,2)}\big)\cdots s^{k_1-1}\big(E_t^{-1}\big)s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big) \\
= {} & s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots s^{k_1-t}\big(g_{s^t(1,2)}^{-1}\big)\Big(\prod_{c=k_1-(t-1)}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdots s\big(g_{s^{k_1-1}(1,2)}^{-1}\big)\Big(\prod_{c=2}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdot s\cdot \\
& s^{k_1-1}\Big(\prod_{c=2}^{k_1-1} s^c\big(g_{s^t(1,2)}\big)\Big)s^{k_1-1}\big(g_{s^{k_1-1}(1,2)}\big)\cdots s^{k_1-1}\Big(\prod_{c=k_1-(t-1)}^{k_1-1} s^c\big(g_{s^t(1,2)}\big)\Big) \\
& s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big) \\
= {} & s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots s^{k_1-t}\big(g_{s^t(1,2)}^{-1}\big)\Big(\prod_{c=k_1-(t-1)}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdots s\big(g_{s^{k_1-1}(1,2)}^{-1}\big)\Big(\prod_{c=2}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdot s\cdot \\
& \Big(\prod_{c=1}^{k_1-2} s^c\big(g_{s^t(1,2)}\big)\Big)s^{k_1-1}\big(g_{s^{k_1-1}(1,2)}\big)\cdots\Big(\prod_{c=k_1-t}^{k_1-2} s^c\big(g_{s^t(1,2)}\big)\Big)s^{k_1-1}\big(g_{s^t(i,j)}\big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big) \\
= {} & s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots\Big(\prod_{c=k_1-t}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdots\Big(\prod_{c=1}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdot s\cdot \\
& \Big(\prod_{c=1}^{k_1-1} s^c\big(g_{s^t(1,2)}\big)\Big)\cdots\Big(\prod_{c=k_1-t}^{k_1-1} s^c\big(g_{s^t(1,2)}\big)\Big)\cdots s^{k_1-1}\big(g_{s(1,2)}\big) \\
= {} & s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots\Big(\prod_{c=k_1-t}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdots\Big(\prod_{c=1}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdot s\cdot \\
& \left(s^{k_1-1}\big(g_{s(1,2)}^{-1}\big)\cdots\Big(\prod_{c=k_1-t}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\cdots\Big(\prod_{c=1}^{k_1-1} s^c\big(g_{s^t(1,2)}^{-1}\big)\Big)\right)^{-1} = \Lambda_1\cdot s\cdot\Lambda_1^{-1}.
\end{aligned}
$$

Therefore, $\gamma_1 \cdot s = \Lambda_1 \cdot s \cdot \Lambda_1^{-1}$, where $\Lambda_1 \in \langle O_1\rangle = \langle \lambda_{s^\iota(1,2)}, \lambda_{s^\iota(2,1)}\rangle_{0\le\iota\le k_1-1}$, an element in UVP_n. With the same techniques we obtain that $\gamma_i \cdot s = \Lambda_i \cdot s \cdot \Lambda_i^{-1}$, where Λ_i belongs in $\langle O_i\rangle$, for every $1 \le i \le l_s$. □

Based on this discussion, we proceed to the proof of Theorem 5.5.2.

Proof of Theorem 5.5.2. Let $w \in UVB_n$ be a torsion element of order r. We know that UVB_n is isomorphic to $UVP_n \rtimes S_n$ and that $UVP_n \cong F_{1,2} \times \cdots \times F_{i,j} \times \cdots \times F_{n-1,n}$, where $F_{i,j} = \langle \lambda_{i,j}, \lambda_{j,i}\rangle$. We set $w = us$, where $u \in UVP_n$ and $s \in S_n$. Based on the above discussion, the pairs of generators $\{\lambda_{i,j}, \lambda_{j,i}\}$ are partitioned into the pairwise disjoint orbits $O_1, \ldots, O_i, \ldots, O_{l_s}$, for $1 \le i \le l_s$, where the number of orbits depends on s.

We recall that the elements $\lambda_{i,j}$, for $1 \le i \ne j \le n$, except for the couples $\lambda_{i,j}, \lambda_{j,i}$, commute pairwise. Therefore, we can see that, by the definition of O_i, for $1 \le i \le l_s$, the elements inside an orbit O_i may not commute, but elements from different orbits commute pairwise, as elements of UVP_n, which means that the orbits commute pairwise, as subsets of UVP_n. Based on this partition, we can rewrite the element $w = us$ in an equivalent way, which is simply a reordering of the elements $\lambda_{i,j}$ in u. We have that u is an element in UVP_n. We want to reorder the elements in u based on the orbit in which they belong. That is

$$u = \gamma_1 \cdots \gamma_i \cdots \gamma_{l_s},$$

where γ_i is a word generated by the elements of O_i and their inverses. It is possible that for some indices $\gamma_i = 1$, which means that u does not contain any element from O_i, after possible cancellations.

By hypothesis $w^r = (us)^r = 1$, and then

$$\underbrace{us \cdot us \cdots us}_{r-\text{times}} = usus^{-1}s^2us^{-2}\cdots s^{r-1}us^{-(r-1)} \cdot s^r = 1.$$

Therefore $usus^{-1}s^2us^{-2}\cdots s^{r-1}us^{-(r-1)} = u \cdot s(u)\cdots s^{r-1}(u) = 1$ in UVP_n and $s^r = 1$. Since $u = \gamma_1\cdots\gamma_i\cdots\gamma_{l_s}$, where $\gamma_i \in \langle O_i \rangle$, for $1 \leq i \leq l_s$, it has to hold that

$$\gamma_i s(\gamma_i)\cdots s^{r-1}(\gamma_i) = 1,$$

for any $1 \leq i \leq l_s$. Previous lemmas allows to conclude that the element $\gamma_i s$ is a conjugation of $s \in S_n$ by an element of UVP_n, for any $1 \leq i \leq l_s$.

We will prove that this result suffices to complete the proof. Since $w = us$, with $u = \gamma_1\cdots\gamma_i\cdots\gamma_{l_s}$, where the elements γ_i commute pairwise, it follows that

$$\begin{aligned} w = us &= \gamma_1\cdots\gamma_i\cdots\gamma_{l_s} \cdot s \\ &= \gamma_1\cdots\gamma_i\cdots\gamma_{l_s-1}\Lambda_{l_s} \cdot s \cdot \Lambda_{l_s}^{-1} \\ &\vdots \\ &= \gamma_1\cdots\gamma_{i-1}\Lambda_{l_s}\cdots\Lambda_i \cdot s \cdot \Lambda_i^{-1}\cdots\Lambda_{l_s}^{-1} \\ &\vdots \\ &= \Lambda_{l_s}\cdots\Lambda_i\cdots\Lambda_1 \cdot s \cdot \Lambda_1^{-1}\cdots\Lambda_i^{-1}\cdots\Lambda_{l_s}^{-1} \\ &= \Lambda \cdot s \cdot \Lambda^{-1}. \end{aligned}$$

To obtain the above equalities, we just apply the equalities $\gamma_i s = \Lambda_i \cdot s \cdot \Lambda_i^{-1}$, starting with γ_{l_s} and finishing with γ_1, which hold for every $1 \leq i \leq l_s$, and using the fact that Λ_i belongs in $\langle O_i \rangle$, and thus γ_i commutes with every Λ_j, for $i \neq j$, $1 \leq i, j \leq l_s$. Therefore, we have that $w = \Lambda s \Lambda^{-1}$, where $\Lambda \in UVP_n$ and this completes the proof. □

5.6 Crystallographic Subgroup of UVB_n

The Artin braid group B_n has a natural embedding in the welded braid group WB_n in contrast to UVB_n. We will give a proof that the Artin braid group on n strands, B_n, under the group homomorphism ι that sends the generator σ_i of B_n to the generator σ_i of UVB_n, does not have a natural embedding in UVB_n. More precisely, we show that the image of B_n, under the homomorphism ι, into UVB_n, is isomorphic to the crystallographic group $B_n/[P_n, P_n]$. The fact that the crystallographic group is a subgroup of UVB_n may give some information about the distribution of the torsion elements in UVB_n and about its subgroups.

We denote by ι the canonical map from B_n to UVB_n:

$$\iota : B_n \to UVB_n \text{ defined by } \iota(\sigma_i) = \sigma_i \in UVB_n, \text{ for } 1 \leq i \leq n-1.$$

We obtain the following short exact sequences:

$$1 \longrightarrow \ker(\iota\big|_{P_n}) \longrightarrow P_n \xrightarrow{\iota\big|_{P_n}} \mathrm{Im}(\iota\big|_{P_n}) \longrightarrow 1,$$

$$1 \longrightarrow \ker(\iota) \longrightarrow B_n \xrightarrow{\iota} \mathrm{Im}(\iota) \longrightarrow 1.$$

Moreover, we have that $P_n = \ker(\pi)$, where $\pi : B_n \to S_n$, $\pi(\sigma_i) = s_i = (i, i+1) \in S_n$, for $1 \leq i \leq n-1$. Similarly, we know that $UVP_n = \ker(\bar{\pi})$, where $\bar{\pi} : UVB_n \to S_n$, $\bar{\pi}(\sigma_i) = \bar{\pi}(\rho_i) = s_i$, for the same values of i. The same follows for the groups $\mathrm{Im}(\iota\big|_{P_n}), \mathrm{Im}(\iota) \subset UVB_n$. That is $\mathrm{Im}(\iota\big|_{P_n}) = \ker(\bar{\pi}\big|_{\mathrm{Im}(\iota)})$. As a result we obtain the following commutative diagram, where each column and each row is a short exact sequence:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(\iota\big|_{P_n}) & \longrightarrow & P_n & \xrightarrow{\iota\big|_{P_n}} & \mathrm{Im}(\iota\big|_{P_n}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(\iota) & \longrightarrow & B_n & \xrightarrow{\iota} & \mathrm{Im}(\iota) & \longrightarrow & 1. \\
 & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}\big|_{\mathrm{Im}(\iota)}} & & \\
 & & & & S_n & = & S_n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

By diagram chasing we obtain $\ker(\iota) \subseteq \ker(\iota\big|_{P_n})$. But it also holds that $\ker(\iota\big|_{P_n}) \subseteq \ker(\iota)$, and therefore we conclude that

$$\ker(\iota) = \ker(\iota\big|_{P_n}). \tag{2}$$

We now state a result by Bardakov–Bellingeri–Damiani, who proved that $\mathrm{Im}(\iota\big|_{P_n}) \cong P_n/[P_n, P_n]$, which we will use to prove our claim. We recall that the pure braid group P_n is the kernel of the homomorphism from B_n to the symmetric group S_n sending every generator σ_i to the permutation $(i\ i+1) \in S_n$. As we described in Proposition 1.1.11, P_n is generated by the set $\{A_{i,j} \mid 1 \leq i < j \leq n\}$, where $A_{i,j} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$, for $1 \leq i < j \leq n$.

Proposition 5.6.1 (Bardakov–Bellingeri–Damiani, [BBD15])**.** *Let $n \in \mathbb{N}$ and let $\iota : P_n \to UVP_n$ be the canonical map of the pure braid group P_n in UVP_n. Then $\iota(P_n)$ is isomorphic to the Abelianisation of P_n .*

Remark 5.6.2. Note that this result is based on the fact that the generators $A_{i,j} \in P_n$ can be rewritten in UVP_n as follows, ([BBD15], p. 7):

$$\iota(A_{i,i+1}) = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \text{ for } i = 1, \ldots n-1$$

and

$$\iota(A_{i,j}) = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1}\cdots\lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j}\cdots\lambda_{j-2,j}\lambda_{j-1,j}, \text{ for } 2 \leq i+1 < j \leq n.$$

So, since $\mathrm{Im}(\iota\big|_{P_n}) \cong P_n/[P_n, P_n]$, the short exact sequence of the upper row, from our commutative diagram, implies that $\ker(\iota\big|_{P_n}) = [P_n, P_n]$. Now, combining this fact with relation (2) we conclude the following result.

Proposition 5.6.3. *Let $n \geq 2$ and let $\iota : B_n \to UVB_n$ be the canonical map, $\iota(\sigma_k) = \sigma_k$, for $1 \leq k \leq n-1$. Then, the image of the Artin braid group inside UVB_n is isomorphic to the group $B_n/[P_n, P_n]$. That is $\iota(B_n) \cong B_n/[P_n, P_n]$.*

In [GGO17], Gonçalves–Guaschi–Ocampo have shown that the group $B_n/[P_n, P_n]$ is actually a crystallographic group. Thus, we deduce the following corollary.

Corollary 5.6.4. *Let $n \geq 2$. The unrestricted virtual braid group, UVB_n, contains a crystallographic group as a subgroup, and that is the group $B_n/[P_n, P_n]$.*

We now state some results concerning the crystallographic group $B_n/[P_n, P_n]$. For further results about the group $B_n/[P_n, P_n]$, we direct the reader to [GGO17].

Proposition 5.6.5 (Gonçalves–Guaschi–Ocampo, [GGO17])**.** *For $n \geq 2$ we have the following short exact sequence:*

$$1 \longrightarrow \mathbb{Z}^{n(n-1)/2} \longrightarrow B_n/[P_n, P_n] \xrightarrow{\hat{\pi}} S_n \longrightarrow 1,$$

where $\hat{\pi}$ is the homomorphism induced by $\pi : B_n \to S_n$.

Theorem 5.6.6 (Gonçalves–Guaschi–Ocampo, [GGO17]). *For $n \geq 3$ the crystallographic group $B_n/[P_n, P_n]$ has only odd order torsion elements.*

Corollary 5.6.7 (Gonçalves–Guaschi–Ocampo, [GGO17]). *Let $n \geq 3$, the torsion elements of $B_n/[P_n, P_n]$ are equal to the torsion elements of S_n. Moreover, for any element s in S_n, of odd order r, there exists an element $b \in B_n/[P_n, P_n]$, of order r such that $\hat{\pi}(b) = s$.*

Thus, combining Corollary 5.6.7 with Theorem 5.5.2 we obtain the following result.

Corollary 5.6.8. *Let $n \geq 3$. For any element $s \in S_n$, of odd order r, there exists an element $b \in B_n/[P_n, P_n]$, of order r such that b is conjugated to $\iota(s)$ by an element of UVP_n, where $B_n/[P_n, P_n]$ is considered as subgroup of UVB_n,*

5.7 The automorphism group of UVP_n

In this section we determine the automorphism group of UVP_n. First, we introduce the notion of right-angled Artin groups, which will be our main tool for this section. A right-angled Artin group, also known as graph group, is a group which admits a finite presentation in which the only relations are commuting relations among the generators.

Every right-angled Artin group defines a graph whose vertices are the generators of the group and for every two generators that commute there is an edge connecting these two vertices. The converse also holds. For every graph Γ, with V its vertex set, there is a right-angled Artin group, graph group, associated to Γ, R_Γ, defined as follows:

$$R_\Gamma = \langle v_1, \ldots, v_n \in V \mid v_i v_j = v_j v_i, \text{ if } v_i, v_j \text{ are joined by an edge in } \Gamma \rangle.$$

From this association we can see that the right-angled Artin group that corresponds to the complete graph on n vertices is the free Abelian group $\mathbb{Z}$ and that the graph on n vertices with no edges corresponds to the free group F_n of rank n. For a general survey on the right-angled Artin groups we direct the reader to the article [Cha07] by Charney.

We can see that the group UVP_n is a right-angled Artin group, since it admits the following presentation, as stated in Theorem 5.3.1:

$$UVP_n = \langle \lambda_{i,j},\ 1 \leq i \neq j \leq n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j},\ \text{for } (k,l) \neq (j,i),\ 1 \leq i,j,k,l \leq n \rangle.$$

From this presentation, the graph, Γ, which corresponds to the right-angled Artin group UVP_n, is a graph with $n(n-1)$ vertices, where the vertex set is $V = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$ and there is an edge connecting every pair of vertices except for the pairs $\{\lambda_{i,j}, \lambda_{j,i}\}$, since these are only pairs of generators that do not commute.

We continue now with providing the theory around the automorphisms of graph groups. Extending the work of Servatius [Ser89], a complete set of generators for the automorphism group of a graph group was found by Laurence [Lau95]. Before giving the main result we present some notions that will be needed.

Let Γ be a graph with V being its vertex set.

- The link of a vertex $v \in V$, $lk(v)$, is the set of all vertices that are connected to v with an edge.
- The star of a vertex $v \in V$, $st(v)$, is the union $lk(v) \cup \{v\}$.
- For any $w \neq v$, $w, v \in V$, we say that v dominates w, $w \leq v$, if $lk(w) \subseteq st(v)$.

The theorem that follows is due to Laurence [Lau95], who proved the conjecture that had been stated, and in certain special cases proved, by Servatius [Ser89].

Theorem 5.7.1 (Laurence, [Lau95]). *Let Γ be a finite graph defining a graph group R_Γ. Then the following automorphisms generate the automorphism group of R_Γ, $Aut(R_\Gamma)$:*

- *Inversions, $I_v : v \to v^{-1}$, which inverts a generator $v \in V$ and fix the rest.*

- *Dominated Transvections, $T_v : w \to wv$, for $w, v \in V$ such that v dominates w, $w \leq v$, and fix the rest.*
- *Graph Automorphisms, G: Any bijection of the graph to itself that preserves the relation vertices-edges.*
- *Locally Inner Automorphisms: $L_{v,Y} : y \to vyv^{-1}$, for all $y \in Y$, where Y is a connected component of $\Gamma - st(v)$ and $|Y| > 1$.*

Remark 5.7.2. The condition $|Y| > 1$, in the Locally Inner Automorphisms, is placed in order to eliminate redundancy. This is so because in the case where $|Y| = 1$, the automorphism $L_{v,Y}$ can be obtained by composition of dominated transvections (since the single vertex $y \in Y$ is dominated by v) and inversions.

Based on the graph that corresponds to the group UVP_n, we shall make the following remarks.

Remark 5.7.3. The only domination relations that occur in the graph that corresponds to UVP_n are

$$\lambda_{i,j} \leq \lambda_{j,i} \text{ and } \lambda_{j,i} \leq \lambda_{i,j}.$$

This is the case because $lk(\lambda_{i,j}) = V \setminus \{\lambda_{i,j}, \lambda_{j,i}\} \subseteq st(\lambda_{j,i}) = V \setminus \{\lambda_{i,j}\}$, and similarly $lk(\lambda_{j,i}) = V \setminus \{\lambda_{j,i}, \lambda_{i,j}\} \subseteq st(\lambda_{i,j}) = V \setminus \{\lambda_{j,i}\}$. As a result, $\lambda_{i,j}$ and $\lambda_{j,i}$ dominate each other. It would not have been possible that $\lambda_{i,j}$ is dominated by another generator $\lambda_{k,l} \neq \lambda_{j,i}$, since $lk(\lambda_{j,i}) = V \setminus \{\lambda_{j,i}, \lambda_{i,j}\} \nsubseteq st(\lambda_{k,l}) = V \setminus \{\lambda_{l,k}\}$.

Remark 5.7.4. We observe that for every vertex $\lambda_{i,j} \in V$ the subgraph $\Gamma \setminus st(\lambda_{i,j})$ is just the vertex $\lambda_{j,i}$, since $st(\lambda_{i,j}) = V \setminus \{\lambda_{j,i}\}$.

We are now ready to determine the group $Aut(UVP_n)$, for $n \geq 2$.

Theorem 5.7.5. *Let $n \geq 2$ and $1 \leq i \neq j \leq n$. It holds that*

$$Aut(UVP_n) \cong \langle T_{\lambda_{j,i}},\ \mathbb{Z}_2^{n(n-1)},\ \mathbb{Z}_2^{n(n-1)/2} \rtimes S_{n(n-1)/2} \rangle,$$

where $T_{\lambda_{j,i}}$ stands for the Dominated Transvection, and $S_{n(n-1)/2}$ is the symmetric group of degree $n(n-1)/2$.

Proof. Let $n \geq 2$, $1 \leq i \neq j \leq n$ and Γ be the graph associated to the right-angled Artin group UVP_n. We recall that the graph Γ, which corresponds to UVP_n, is a graph with $n(n-1)$ vertices, where the vertex set is $V = \{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$ and there is an edge connecting every pair of vertices except for the pairs $\{\lambda_{i,j}, \lambda_{j,i}\}$. From Theorem 5.7.1 we see that the automorphism group of UVP_n is generated by the following four families of automorphisms; the Inversions, the Dominated Transvections, the Graph Automorphisms and the Locally Inner Automorphisms.

From Remark 5.7.2 and Remark 5.7.4 we conclude that in the case of UVP_n we do not have any Locally Inner Automorphism.

From Remark 5.7.3 it follows that any Dominated Transvection in UVP_n is generated by $T_{\lambda_{i,j}}$ and $T_{\lambda_{j,i}}$, and they are defined as follows:

$$T_{\lambda_{i,j}} : \lambda_{j,i} \mapsto \lambda_{j,i}\lambda_{i,j}, \text{ while fixing the rest generators}$$

and

$$T_{\lambda_{j,i}} : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i}, \text{ while fixing the rest generators.}$$

It remains to determine the Inversions and the Graph Automorphisms of UVP_n. Clearly, the Inversions in UVP_n, are $I_{\lambda_{i,j}}$, where

$$I_{\lambda_{i,j}} : \lambda_{i,j} \mapsto \lambda_{i,j}^{-1}, \text{ while fixing the rest generators.}$$

We know that the size of the set the of generators of UVP_n, $\{\lambda_{i,j}\}_{1 \leq i \neq j \leq n}$, is $n(n-1)$. Moreover the Inversion Automorphisms have order two. Therefore, for $I := \{I_{\lambda_{i,j}}\}_{1 \leq i \neq j \leq n}$, it follows that

$$\langle I \rangle \cong \mathbb{Z}_2^{n(n-1)}.$$

Finally, we will describe all possible Graph Automorphisms of the graph Γ. We recall that a Graph Automorphism is a symmetry of the graph, that is a bijection to itself while preserving the edge-vertex connectivity. In our case, we see that there are only two possible symmetries of our graph. One that exchanges the generators $\lambda_{i,j}$ and $\lambda_{j,i}$, while fixing the rest generators and another that exchanges the pair of generators $\{\lambda_{i,j}, \lambda_{j,i}\}$ with another pair of generators $\{\lambda_{k,l}, \lambda_{l,k}\}$, while fixing the rest generators. We denote these two Graph Automorphisms as follows:

$$E_{i,j} : \lambda_{i,j} \leftrightarrow \lambda_{j,i}, \text{ while fixing the rest generators}$$

and

$$P_{ij,kl} : \lambda_{i,j} \leftrightarrow \lambda_{k,l}, \ P_{ij,kl} : \lambda_{j,i} \leftrightarrow \lambda_{l,k}, \text{ while fixing the rest generators,}$$

where $\lambda_{i,j} \leftrightarrow \lambda_{j,i}$ means that this map exchanges the generators $\lambda_{i,j}$ and $\lambda_{j,i}$. These two Graph Automorphisms are defined in such a way that it follows that $E_{i,j} = E_{j,i}$ and $P_{ij,kl} = P_{kl,ij}$. In Γ all vertices are pairwise connected with an edge except for the pairs $\{\lambda_{i,j}, \lambda_{j,i}\}$, and the vertices $\lambda_{i,j}$ and $\lambda_{j,i}$ are connected with the exact same vertices. Therefore, if we want to preserve the edge-vertex connectivity we have to permute the vertex set in such a way that each vertex is connected with the same vertices before and after the permutation. Therefore, it is clear that only these two type of maps, $E_{i,j}$ and $P_{ij,kl}$, preserve the edge-vertex connectivity, and thus they are indeed the only symmetries of the graph.

For $E := \{E_{i,j}\}$ and $P := \{P_{ij,kl}\}$ we have that $|E| = n(n-1)/2$, since $E_{i,j} = E_{j,i}$ and that $|P| = n(n-1)/2$, since $P_{ij,kl} = P_{kl,ij}$. The elements $E_{i,j} \in E$ and $P_{ij,kl} \in P$ do not commute, since $(E_{i,j} \circ P_{ij,kl})(\lambda_{i,j}) = \lambda_{k,l}$ and $(P_{ij,kl} \circ E_{i,j})(\lambda_{i,j}) = \lambda_{l,k}$, where $\lambda_{k,l} \neq \lambda_{l,k}$. More precisely, we have that P acts on E by permuting the elements $E_{i,j}$ inside E. To be more precise, $P_{ij,kl} \circ E_{i,j} \circ P_{ij,kl} = E_{k,l}$, since $P_{ij,kl} \circ E_{i,j} \circ P_{ij,kl}(\lambda_{i,j}) = \lambda_{i,j}$ and $P_{ij,kl} \circ E_{i,j} \circ P_{ij,kl}(\lambda_{k,l}) = \lambda_{l,k}$. We conclude that $\langle E \rangle \cong \mathbb{Z}_2^{n(n-1)/2}$ and that $\langle P \rangle \cong S_{n(n-1)/2}$. All together we obtain

$$\langle E, P \rangle \cong \mathbb{Z}_2^{n(n-1)/2} \rtimes S_{n(n-1)/2},$$

where the symmetric group $S_{n(n-1)/2}$ acts on $\mathbb{Z}_2^{n(n-1)/2}$ by permuting the components of the product.

Moreover, it holds that $E_{i,j} T_{\lambda_{j,i}} E_{i,j} = T_{\lambda_{i,j}}$, and thus we keep only the Dominated Transvection $T_{\lambda_{j,i}}$ in the generating set of $Aut(UVP_n)$ and this completes the proof. □

Having a concrete set of generators of the automorphism group of UVP_n we can analyse this result a bit further. We recall that, from Remark 5.3.3, the group UVP_n is isomorphic to the direct product of $n(n-1)/2$ copies of the free group of rank 2:

$$UVP_n = \langle \lambda_{1,2}, \lambda_{2,1} \rangle \times \cdots \times \langle \lambda_{i,j}\lambda_{j,i} \rangle \times \cdots \times \langle \lambda_{n-1,n}, \lambda_{n,n-1} \rangle, \text{ for } 1 \leq i \neq j \leq n,$$

$$UVP_n \cong \underbrace{F_2 \times \cdots \times F_2 \times \cdots \times F_2}_{n(n-1)/2\text{-times}}, \text{ for } n \geq 2.$$

Based on this isomorhism, we can make some observations about the generators of the automorphism group of UVP_n, given in Theorem 5.7.5. For $1 \leq i \neq j \leq n$ the group $Aut(UVP_n)$ is generated by the following four automorphisms:

$T_{\lambda_{j,i}} : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i}$, while fixing the rest generators,
$I_{\lambda_{i,j}} : \lambda_{i,j} \mapsto \lambda_{i,j}^{-1}$, while fixing the rest generators,
$E_{i,j} : \lambda_{i,j} \leftrightarrow \lambda_{j,i}$, while fixing the rest generators,
$P_{ij,kl} : \lambda_{i,j} \leftrightarrow \lambda_{k,l}, \ P_{ij,kl} : \lambda_{j,i} \leftrightarrow \lambda_{l,k}$, while fixing the rest generators.

With the exception of the automorphisms $P_{ij,kl}$, it follows that all these automorphisms do not permute the F_2-factors of UVP_n, but they rather take an element from a factor and send it to the same factor. In other words, each F_2-factors of UVP_n stay invariant under the automorphisms $T_{\lambda_{j,i}}$, $I_{\lambda_{i,j}}$ and $E_{i,j}$. Moreover, the image of a generator $\lambda_{i,j}$, under these automorphisms, belongs to the set generated by the elements $\{\lambda_{i,j}\lambda_{j,i}, \lambda_{i,j}^{-1}, \lambda_{j,i}\}$. These are the three automorphisms that generate the group $Aut(F_2)$, as we will see shortly. On the contrary, the automorphism $P_{ij,kl}$ permutes the $n(n-1)/2$ F_2-factors of UVP_n.

It is well known, and proved by Nielsen in [Nie24], that $Aut(F_2)$, for $F_2 = \langle x_1, x_2 \rangle$, is generated by the following three automorphisms:
$\alpha_1 : x_1 \leftrightarrow x_2$,
$\alpha_2 : x_1 \mapsto x_1^{-1},\ x_2 \mapsto x_2$,
$\alpha_3 : x_1 \mapsto x_1 x_2,\ x_2 \mapsto x_2$.

It turns out that $Aut(UVP_n)$ is the semi-direct product of $Aut(F_2)^{n(n-1)/2}$ and the symmetric group $S_{n(n-1)/2}$, which acts on $Aut(F_2)^{n(n-1)/2}$ by permuting the $n(n-1)/2$ factors of $Aut(F_2)$. And so we obtain the following corollary.

Corollary 5.7.6. *For $n \geq 2$ it holds that*

$$Aut(UVP_n) \cong Aut(F_2)^{n(n-1)/2} \rtimes S_{n(n-1)/2},$$

where $S_{n(n-1)/2}$ acts on $Aut(F_2)^{n(n-1)/2}$ by permuting the $n(n-1)/2$ factors.

This result agrees with a particular case of a more general result proved by Zhang–Ventura–Wu in [ZVW15], where they obtained the same result using different techniques. We state their result below and we see that for $m = 1$ and $n_1 = n(n-1)/2$ their result coincides with Corollary 5.7.6.

Proposition 5.7.7 (Zhang–Ventura–Wu, [ZVW15])**.** *Let $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$ be a product group, where $m \geq 1$, $n_i \geq 1$, $G_i \ncong G_j$, for $i \neq j$, and each G_i is a free group or a surface group. If G is a hyperbolic type then, for every $\phi \in Aut(G)$, there exists automorphisms $\phi_{i,j} \in Aut(G_i)$ and permutations $\sigma_i \in S_{n_i}$, such that*

$$\phi = \sigma_1 \circ \cdots \circ \sigma_m \circ \Big(\prod_{i=1}^{m} \prod_{j=1}^{n_i} \phi_{i,j} \Big) = \prod_{i=1}^{m} \Big(\sigma_i \circ \prod_{j=1}^{n_i} \phi_{i,j} \Big).$$

5.8 About the automorphism group of UVB_n

After having obtained several results about the group UVP_n and after having determined the group $Aut(UVP_n)$, we shall now present partial results about the automorphism group of UVB_n. In [Ros75], Rose gave a description of the automorphism group of groups which possess a proper characteristic subgroup that have trivial centraliser.

Proposition 5.8.1 (Rose, [Ros75])**.** *Let G be a group with a characteristic subgroup H such that $C_G(H) = e$. Then G is naturally embedded in $Aut(H)$ by means of conjugation of H by the elements of G. Moreover, there is a natural isomorphism between $Aut(G)$ and the normaliser of G in $Aut(H)$. That is $Aut(G) \cong N_{Aut(H)}(G)$.*

We proved that UVP_n is a characteristic subgroup of UVB_n, Proposition 5.3.8, and that $C_{UVB_n}(UVP_n) = e$, Proposition 5.3.10. Therefore, applying Proposition 5.8.1 for $G = UVB_n$ and $H = UVP_n$, we obtain the following corollary.

Corollary 5.8.2. *For $n \geq 5$ it holds that*

$$Aut(UVB_n) \cong N_{Aut(UVP_n)}(UVB_n).$$

In Section 4.4, we defined two maps, β_n and γ_n, which fail to be automorphisms of WB_n, but they are actually automorphisms of the group UVB_n. More precisely, the maps β_n and γ_n are defined as follows:

$$\beta_n : \begin{cases} \sigma_i \mapsto \sigma_i^{-1}, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1, \end{cases}$$

and

$$\gamma_n : \begin{cases} \sigma_i \mapsto \rho_i \sigma_i \rho_i, \text{ for } 1 \leq i \leq n-1, \\ \rho_i \mapsto \rho_i, \text{ for } 1 \leq i \leq n-1, \end{cases}$$

where $\beta_n, \gamma_n \in Aut(UVB_n)$.

In particular, note that β_n and γ_n are of order two and moreover $\beta_n \circ \gamma_n = \gamma_n \circ \beta_n$. Thus, $\langle \beta_n, \gamma_n \rangle$ generates a subgroup of $Aut(UVB_n)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, as we will see in the following proposition, the automorpsims β_n, γ_n and $\beta_n \circ \gamma_n$ are actually elements of the outer automorphisms group of UVB_n, $Out(UVB_n)$, where $Out(UVB_n) = Aut(UVB_n)/Inn(UVB_n)$.

Proposition 5.8.3. *Let $n \geq 3$. It holds that*

$$\langle \beta_n, \gamma_n \rangle \subseteq Out(UVB_n),$$

where $\langle \beta_n, \gamma_n \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Let $n \geq 3$. We will show that the automorphisms β_n, γ_n and $\beta_n \circ \gamma_n$ are not inner automorphisms. Note that any inner automorphism of a group G acts trivially on the Abelianisation of G. That is if $h : G \to G$ is an inner automorphism then the induced automorphism $\bar{h} : G^{ab} \to G^{ab}$ is the trivial one. From Remark 5.1.3, we have that the Abelianisation of UVB_n is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, where $\mathbb{Z}$ is generated by the image of σ_1, $[\sigma_1]$ and $\mathbb{Z}_2$ is generated by the image of ρ_1, $[\rho_1]$. The automorphisms β_n and $\beta_n \circ \gamma_n$ act non-trivially on the Abelianisation of UVB_n, since $\beta_n(\sigma_1) = \sigma_1^{-1}$ and $(\beta_n \circ \gamma_n)(\sigma_1) = \rho_1 \sigma_1^{-1} \rho_1$, for $1 \leq i \leq n-1$.

By contradiction we will prove that γ_n is not an inner automorphism. First, notice that from the set of relations (1) we have that $\gamma_n(\lambda_{k,l}) = \lambda_{l,k}$, for every $\lambda_{k,l} \in UVP_n$, where $k \neq l$ and $k, l = 1, \ldots, n$. Suppose that γ_n is an inner automorphism. It follows that there exists some non-trivial element $g \in UVB_n$ such that $\gamma_n(u) = i_g(u) = gug^{-1}$, for every $u \in UVB_n$. In particular, it has to hold that $\gamma_n(\lambda_{k,l}) = g\lambda_{k,l}g^{-1} = \lambda_{l,k}$, for every $k, l = 1, \ldots, n$, where $k \neq l$. Using the same arguments as we did in the proof of Proposition 5.3.10, we will show that the relation $\lambda_{l,k} = g\lambda_{k,l}g^{-1}$, for every $k, l = 1, \ldots, n$, $k \neq l$ and for a fixed non-trivial element $g \in UVB_n$, can not hold.

We know that $UVB_n = UVP_n \rtimes S_n$ and thus, $g = \Lambda S$, for Λ, S fixed words in UVP_n and S_n respectively. Therefore, it has to hold that

$$\lambda_{l,k} \cdot \Lambda S = \Lambda S \cdot \lambda_{k,l}, \text{ for every } k, l \in \{1, \ldots, n\}, k \neq l.$$

From Theorem 5.3.6, and in particular based on the action of the symmetric group on every generator $\lambda_{k,l} \in UVP_n$, we obtain

$$\lambda_{l,k} \cdot \Lambda S = \Lambda \cdot \lambda_{S(k),S(l)} S, \text{ for every } k, l \in \{1, \ldots, n\}, k \neq l.$$

Thus,

$$\lambda_{l,k} = \Lambda \cdot \lambda_{S(k),S(l)} \cdot \Lambda^{-1} \in UVP_n, \text{ for every } k, l \in \{1, \ldots, n\}, k \neq l. \tag{3}$$

Suppose that either $S(k) \neq l$ or $S(l) \neq k$. Then, from relation (3), we see that under the Abelianisation map these distinct generators, $\lambda_{l,k}$ and $\lambda_{S(k),S(l)}$, of UVP_n, would coincide. But this leads to a contradiction since we know from Corollary 5.3.5 that the Abelianisation of UVP_n is isomorphic to the free Abelian group of rank $n(n-1)$ generated by the images of the elements $\lambda_{i,j}$, for $1 \leq i \neq j \leq n$. Therefore, it has to hold that $S(k) = l$ and $S(l) = k$ for every $k, l \in \{1, \ldots, n\}$, $k \neq l$, where S is a fixed element in S_n. But once again, this leads to a contradiction because a permutation on n elements cannot permute all possible couples (l, k), for $1 \leq l \neq k \leq n$.

Now, suppose that $g = S$, meaning that Λ is a trivial word in UVP_n. Then, it has to hold that

$$\lambda_{l,k} \cdot S = S \cdot \lambda_{k,l}, \text{ for every } k, l \in \{1, \ldots, n\}, k \neq l. \tag{4}$$

But relation (4) implies that the word S in the symmetric group fixes all the elements of the set $\{1, \ldots, n\}$. This is possible only when S is the trivial element, which leads once more to a contradiction, since g is a non-trivial element. Since the centre of UVP_n is trivial, Remark 5.3.4, we do not need to check the case where $g = \Lambda$.

Thus, we conclude that γ_n is not an inner automorphism and this completes the proof. □

We speculate that this result about the outer automorhism group of UVB_n, $Out(UVB_n)$, could be of help in determining the group $Out(WB_n)$, since UVB_n is a quotient of WB_n by an extra relation.

Finally, we complete this section by proving that the groups UVB_n and UVP_n are residually finite and Hopfian, but not co-Hopfian.

We recall that a group G is called Hopfian if every surjective homomorphism $G \to G$ is also an injective homomorphism and it is called co-Hopfian if every injective homomorphism $G \to G$ is also a surjective homomorphism.

Corollary 5.8.4. *Let $n \geq 2$. The groups UVB_n and UVP_n are residually finite and Hopfian, but not co-Hopfian.*

Proof. By Theorem 5.3.6, we have that $UVB_n \cong UVP_n \rtimes S_n$. We already know that the group UVP_n is a right-angled Artin group and it is known that every right-angled Artin group is linear; see [HW99]. Moreover, it is also known that a finitely generated linear group is residually finite. It clearly follows now that UVP_n is a residually finite group. Moreover, UVB_n is a residually finite group, since it is an extension of UVP_n by S_n, which is a finite group. Finally, the groups UVB_n and UVP_n are Hopfian, since they are finitely generated, residually finite goups.

In order to show that the group UVP_n is not co-Hopfian, we just provide the following homomorphism:

$$h : UVP_n \to UVP_n \text{ defined by } h : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i}, \text{ for every } 1 \leq i \neq j \leq n,$$

which is injective but not surjective, since the elements $\lambda_{i,j}$, for $1 \leq i \neq j \leq n$ do not have a preimage under h. Now, we extend the map h to a homomorphism of $UVB_n \cong UVP_n \rtimes S_n$ as follows:

$$\bar{h} : UVB_n \to UVB_n \text{ defined by } \bar{h} : \lambda_{i,j} \mapsto \lambda_{i,j}\lambda_{j,i} \text{ and } \bar{h} : s_k \mapsto s_k,$$

for every $1 \leq i \neq j \leq n$ and $1 \leq k \leq n-1$, where $s_k \in S_n$. Similarly, this homomorphism is injective but not surjective, since the elements $\lambda_{i,j}$, for $1 \leq i \neq j \leq n$ do not have a preimage under $\bar{h}$, and therefore the group UVB_n is not co-Hopfian. □

Bibliography

[AM04] A. Adem and R. J. Milgram. *Cohomology of finite groups*. Grundlehren der Mathematischen Wissenschaften, Vol. 309. Springer-Verlag, Berlin, (2004).

[Art25] E. Artin. Theorie der Zöpfe. *Abh. Math. Sem. Univ. Hamburg*, 4:47–72, (1925).

[Art47a] E. Artin. Braids and permutations. *Annals of Mathematics*, 48:643–649, (1947).

[Art47b] E. Artin. Theory of braids. *Annals of Mathematics*, 48:101–126, (1947).

[Bar92] V. G. Bardakov. On the theory of braid groups. *Mat. Sb.*, 183:3–42, (1992).

[Bau77] H. J. Baues. *Obstruction theory on homotopy classification of maps*. Lecture Notes in Mathematics, Vol. 628. Springer-Verlag, Berlin-New York, (1977).

[BBD15] V. G. Bardakov, P. Bellingeri, and C. Damiani. Unrestricted virtual braids, fused links and other quotients of virtual braid groups. *J. Knot Theory Ramifications*, 24:23, (2015).

[BCW07] J. C. Baez, A. S. Crans, and D. K. Wise. Exotic statistics for strings in 4D BF theory. *Advances in Theoretical and Mathematical Physics*, 11:707–749, (2007).

[Bel04] P. Bellingeri. On presentations of surface braid groups. *J. Algebra*, 274:543–563, (2004).

[BGG] P. Bellingeri, E. Godelle, and J. Guaschi. Exact sequences, lower central series and representations of surface braid groups. *arXiv:1106.4982*.

[BGN19] V. G. Bardakov, K. Gongopadhyay, and M.V. Neshchadim. Commutator subgroups of virtual and welded braid groups. *Internat. J. Algebra Comput.*, 29:507–533, (2019).

[BH13] T. E. Brendle and A. Hatcher. Configuration spaces of rings and wickets. *Comment. Math. Helv.*, 88:131–162, (2013).

[Bir69] J. S. Birman. On braid groups. *Comm. Pure Appl. Math.*, 22:41–72, (1969).

[BP20] P. Bellingeri and L. Paris. Virtual braids and permutations. *Ann. Inst. Fourier*, 70:1341–1362, (2020).

[Cha07] R. Charney. An introduction to right-angled artin groups. *Geometriae Dedicata*, 125:141–158, (2007).

[Cho48] W.-L. Chow. On the algebraical braid group. *Ann. of Math.*, 49:654–658, (1948).

[CKLP20] A. Chudnovsky, K. Kordek, Q. Li, and C. Partin. Finite quotients of braid groups. *Geometriae Dedicata*, 207:409–416, (2020).

[CS20] L. Chen and N. Salter. Section problems for configurations of points on the Riemann sphere. *Algebraic & Geometric Topology*, 20:3047–3082, (2020).

[Dam17] C. Damiani. A journey through loop braid groups. *Expo. Math.*, 35:252–285, (2017).

[DG81] J. L. Dyer and E. K. Grossman. The automorphism groups of the braid groups. *American Journal of Mathematics*, 103:1151–1169, (1981).

[DLP16] I. Diamantis, S. Lambropoulou, and J. Przytycki. Topological steps toward the Homflypt skein module of the lens spaces $L(p, 1)$ via braids. *J. Knot Theory Ramifications*, 25:26, (2016).

[Fad62] E. Fadell. Homotopy groups of configuration spaces and the string problem of Dirac. *Duke Mathematical Journal*, 29:231–242, (1962).

[FN62a] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, (1962).

[FN62b] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, (1962).

[FR85] M. Falk and R. Randell. The lower central series of a fiber-type arrangement. *Invent. Math.*, 82:77–88, (1985).

[FR00] F. T. Farrell and S. K. Roushon. The Whitehead groups of braid groups vanish. *Internat. Math. Res. Notices*, 10:515–526, (2000).

[FRR97] R. Fenn, R. Rimányi, and C. Rourke. The braid-permutation group. *Topology*, 36:123–135, (1997).

[FVB62] E. Fadell and J. Van Buskirk. The braid groups of E^2 and S^2. *Duke Math. J.*, 29:243–257, (1962).

[GG04a] D. L. Gonçalves and J. Guaschi. The braid groups of the projective plane. *Algebraic & Geometric Topology*, 4:757–780, (2004).

[GG04b] D. L. Gonçalves and J. Guaschi. On the structure of surface pure braid groups. *Journal of pure and applied algebra*, 186:187–218, (2004).

[GG04c] D. L. Gonçalves and J. Guaschi. The roots of the full twist for surface braid groups. *Math. Proc. Camb. Phil. Soc.*, 137:307–320, (2004).

[GG05] D. L. Gonçalves and J. Guaschi. The braid group $B_{n,m}(\mathbb{S}^2)$ and a generalisation of the Fadell-Neuwirth short exact sequence. *Journal of Knot Theory and its Ramifications*, 14:375–403, (2005).

[GG07] D. L. Gonçalves and J. Guaschi. The braid groups of the projective plane and the Fadell–Neuwirth short exact sequence. *Geometriae Dedicata*, 130:93–107, (2007).

[GG10a] D. L. Gonçalves and J. Guaschi. Braid groups of non-orientable surfaces and the Fadell–Neuwirth short exact sequence. *Journal of Pure and Applied Algebra*, 214:667–677, (2010).

[GG10b] D. L. Gonçalves and J. Guaschi. Classification of the virtually cyclic subgroups of the pure braid groups of the projective plane. *Journal of Group Theory*, 13:277–294, (2010).

[GG12] D. L. Gonçalves and J. Guaschi. Surface braid groups and coverings. *Journal of the London Mathematical Society*, 85:855–868, (2012).

[GGO17] D. L. Gonçalves, J. Guaschi, and O. Ocampo. A quotient of the Artin braid groups related to crystallographic groups. *Journal of Algebra*, 474:393–423, (2017).

[GJP15] J. Guaschi and D. Juan-Pineda. A survey of surface braid groups and the lower algebraic K-theory of their group rings. In *Handbook of group actions. Vol. II*, Adv. Lect. Math., Vol. 32, pages 23–75. Int. Press, Somerville, MA, (2015).

[GM01] J. González-Meneses. New presentations of surface braid groups. *J. Knot Theory Ramif.*, 10:431–451, (2001).

[Gol81] D. L. Goldsmith. The theory of motion groups. *Michigan Mathematical Journal*, 28:3–17, (1981).

[Han89] V. L. Hansen. *Braids and coverings: selected topics.* London Mathematical Society Student Texts, Vol. 18. Cambridge University Press, (1989).

[Hum85] S. P. Humphries. On weakly distinguished bases and free generating sets of free groups. *Quart. J. Math. Oxford Ser. (2)*, 36:215–219, (1985).

[HW99] T. Hsu and D. T. Wise. On linear and residual properties of graph products. *Michigan Math. J.*, 46:251–259, (1999).

[Joh97] D. L. Johnson. *Presentations of groups.* London Mathematical Society Student Texts, Vol. 15. Cambridge University Press, Cambridge, (1997).

[KL04] L. H. Kauffman and S. Lambropoulou. Virtual braids. *Fund. Math.*, 184:159–186, (2004).

[KL06] L. H. Kauffman and S. Lambropoulou. Virtual braids and the *L*-move. *J. Knot Theory Ramifications*, 15:773–811, (2006).

[KM] K. Kordek and D. Margalit. Homomorphisms of commutator subgroups of braid groups. *arXiv:1910.06941.*

[KMRW17] Z. Kádár, P. Martin, E. Rowell, and Z. Wang. Local representations of the loop braid group. *Glasgow Mathematical Journal*, 59:359–378, (2017).

[KR03] D. M. Kim and D. Rolfsen. An ordering for groups of pure braids and fibre-type hyperplane arrangements. *Canadian Journal of Mathematics*, 55:822–838, (2003).

[KT08] C. Kassel and V. Turaev. *Braid groups.* Graduate Texts in Mathematics, Vol. 247. Springer, New York, (2008).

[Lam00] S. Lambropoulou. Braid structures in knot complements, handlebodies and 3-manifolds. In *Knots in Hellas '98 (Delphi)*, Ser. Knots Everything, Vol. 24, pages 274–289. World Sci. Publ., River Edge, NJ, (2000).

[Lau95] M. R. Laurence. A generating set for the automorphism group of a graph group. *Journal of the London Mathematical Society*, 52:318–334, (1995).

[Lin] V. Lin. Braids and permutations. *arXiv math/0404528.*

[Man97] S. Manfredini. Some subgroups of Artin's braid group. *Topology and its Applications*, 78:123–142, (1997).

[McC86] J. McCool. On basis-conjugating automorphisms of free groups. *Canadian Journal of Mathematics*, 38:1525–1529, (1986).

[Mur82] K. Murasugi. Seifert fibre spaces and braid groups. *Proc. London Math. Soc.*, 44:71–84, (1982).

[Nie24] J. Nielsen. Die isomorphismengruppe der freien gruppen. *Mathematische Annalen*, 91:169–209, (1924).

[PR99] L. Paris and D. Rolfsen. Geometric subgroups of surface braid groups. *Ann. Inst. Fourier*, 49:417–472, (1999).

[Ros75] J. S. Rose. Automorphism groups of groups with trivial centre. *Proceedings of the London Mathematical Society*, 3:167–193, (1975).

[Sav96] A. G. Savushkina. On the group of conjugating automorphisms of a free group. *Mathematical Notes*, 60:68–80, (1996).

[Sco70] G. P. Scott. Braid groups and the group of homeomorphisms of a surface. *Proc. Cambridge Philos. Soc.*, 68:605–617, (1970).

[Ser89] H. Servatius. Automorphisms of graph groups. *Journal of Algebra*, 126:34–60, (1989).

[Sos92] A. B. Sossinsky. Preparation theorems for isotopy invariants of links in 3-manifolds. In *Quantum groups (Leningrad, 1990)*, Lecture Notes in Math., Vol. 1510, pages 354–362. Springer, Berlin, (1992).

[VB66] J. Van Buskirk. Braid groups of compact 2-manifolds with elements of finite order. *Trans. Amer. Math. Soc.*, 122:81–97, (1966).

[Whi78] G. W. Whitehead. *Elements of homotopy theory.* Graduate Texts in Mathematics, Vol. 61. Springer-Verlag, New York-Berlin, (1978).

[Zar36] O. Zariski. On the Poincaré group of rational plane curves. *American Journal of Mathematics*, 58:607–619, (1936).

[Zar37] O. Zariski. The topological discriminant group of a Riemann surface of genus p. *American Journal of Mathematics*, 59:335–358, (1937).

[ZVW15] Q. Zhang, E. Ventura, and J. Wu. Fixed subgroups are compressed in surface groups. *International Journal of Algebra and Computation*, 25:865–887, (2015).

Résumé

Le problème de scindement pour les groupes de tresses du plan projectif et un quotient remarquable des groupes de tresses soudées

Cette thèse est divisée en deux parties. La première partie concerne des groupes de tresses des surfaces, tandis que la deuxième traite des groupes de tresses soudées et des groupes de tresses virtuelles sans restrictions.

Soient $n, m \in \mathbb{N}^*$ et $B_{n,m}(\mathbb{R}P^2)$ l'ensemble des $(n+m)$-tresses du plan projectif dont la permutation associée appartient au sous-groupe $S_n \times S_m$ du groupe symétrique S_{n+m}. Dans la première partie de cette thèse, nous étudions le problème de scindement de la suivante suite exacte courte généralisée de Fadell–Neuwirth:

$$1 \to B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}} B_n(\mathbb{R}P^2) \to 1,$$

où l'application $\bar{q}$ peut être considérée géométriquement comme l'épimorphisme qui oublie les m derniers brins, ainsi que l'existence d'une section de la fibration correspondante $q : F_{n+m}(\mathbb{R}P^2)/S_n \times S_m \to F_n(\mathbb{R}P^2)/S_n$, où on note par $F_n(\mathbb{R}P^2)$ le n–ème espace de configurations ordonnées du plan projectif $\mathbb{R}P^2$.

Nos principaux résultats sont les suivants : si $n = 1$ l'homomorphisme $\bar{q}$ et la fibration correspondante q n'admettent aucune section, tandis que si $n = 2$, alors $\bar{q}$ et q admettent une section. Pour $n \geq 3$, on montre que si $\bar{q}$ et q admettent une section alors $m \equiv 0, (n-1)^2 \bmod n(n-1)$. De plus, l'homomorphisme $\bar{q}$ et la fibration q admettent une section pour $m = kn(2n-1)(2n-2)$, où $k \geq 1$, et pour $m = 2n(n-1)$. En outre, nous prouvons que pour $m \geq 3$, $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ n'est pas résiduellement nilpotent et pour $m \geq 5$, il n'est pas résiduellement résoluble.

Soit $n \in \mathbb{N}$. Dans la deuxième partie de la thèse, nous étudions les groupes de tresses soudées WB_n et les groupes de tresses virtuelles sans restrictions UVB_n, ainsi que leurs sous-groupes purs, c'est-à-dire les groupes de tresses pures soudées WP_n et les groupes de tresses pures virtuelles sans restrictions UVP_n.

Nos principaux résultats sont les suivants : pour $n \geq 5$, nous donnons une description complète, à conjugaison près, des homomorphismes possibles de WB_n et UVB_n dans le groupe symétrique S_n. Pour $n \geq 3$, on donne une caractérisation complète des homomorphismes de UVB_n dans tout groupe fini G. Pour $n \geq 5$, nous montrons que WP_n et UVP_n sont des sous-groupes caractéristiques de WB_n et UVB_n respectivement. De plus, nous déterminons le groupe des automorphismes de UVP_n, et nous prouvons que $\mathbb{Z}_2 \times \mathbb{Z}_2$ est un sous-groupe du groupe des automorphismes extérieurs de UVB_n. Enfin, nous montrons que UVB_n et UVP_n sont résiduellement finis et Hopfiens mais pas co-Hopfiens.

Mots-clés: Groupes de tresses des surfaces ; Présentation de groupe ; Suite exacte courte de Fadell–Neuwirth ; Problème de scindement ; Fibration ; Résiduellement nilpotent ; Résiduellement résoluble ; Groupes de tresses soudées ; Groupes de tresses virtuelles sans restrictions ; Groupes d'Artin angle droit ; Groupe des automorphismes ; Résiduellement fini ; Hopfien ; Co-Hopfien.

Abstract

The splitting problem for braid groups of the projective plane and a remarkable quotient of welded braid groups

This thesis is divided into two parts. The first part concerns surface braid groups, while the second deals with welded and unrestricted virtual braid groups.

Let $n, m \in \mathbb{N}$, and let $B_{n,m}(\mathbb{R}P^2)$ be the set of $(n+m)$-braids of the projective plane whose associated permutation lies in the subgroup $S_n \times S_m$ of the symmetric group S_{n+m}. In the first part of this work, we study the splitting problem of the following generalised Fadell–Neuwirth short exact sequence:

$$1 \to B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \to B_{n,m}(\mathbb{R}P^2) \xrightarrow{\bar{q}} B_n(\mathbb{R}P^2) \to 1,$$

where the map $\bar{q}$ can be considered geometrically as the epimorphism that forgets the last m strands, as well as the existence of a section of the corresponding fibration $q : F_{n+m}(\mathbb{R}P^2)/S_n \times S_m \to F_n(\mathbb{R}P^2)/S_n$, where we denote by $F_n(\mathbb{R}P^2)$ the n^{th} ordered configuration space of the projective plane $\mathbb{R}P^2$.

Our main results are the following: if $n = 1$ the homomorphism $\bar{q}$ and the corresponding fibration q admits no section, while if $n = 2$, then $\bar{q}$ and q admit a section. For $n \geq 3$, we show that if $\bar{q}$ and q admit a section then $m \equiv 0, (n-1)^2 \bmod n(n-1)$. Moreover, using geometric constructions, we show that the homomorphism $\bar{q}$ and the fibration q admit a section for $m = kn(2n-1)(2n-2)$, where $k \geq 1$, and for $m = 2n(n-1)$. In addition, we show that for $m \geq 3$, $B_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is not residually nilpotent and for $m \geq 5$, it is not residually solvable.

Let $n \in \mathbb{N}$. In the second part of this work, we study the welded braid groups WB_n, the unrestricted virtual braid groups UVB_n, as well as their pure subgroups, namely the welded pure braid groups WP_n and unrestricted virtual pure braid groups UVP_n.

Our main results are as follows: for $n \geq 5$, we give a complete description, up to conjugation, of all possible homomorphisms from WB_n and UVB_n to the symmetric group S_n. For $n \geq 3$, we give a complete characterisation of any group homomorphism from UVB_n to any finite group G. For $n \geq 5$, we prove that WP_n and UVP_n are characteristic subgroups of WB_n and UVB_n respectively. In addition, we determine the automorphism group of UVP_n, and we prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of the outer automorphism group of UVB_n. Lastly, we show that UVB_n and UVP_n are residually finite and Hopfian but not co-Hopfian.

Keywords: Surface braid groups; Group presentation; Fadell–Neuwirth short exact sequence; Section problem; Fibration; Residually nilpotent; Residually solvable; Welded braid groups; Unrestricted virtual braid groups; Right-angled Artin groups; Automorphism group; Residually finite; Hopfian; Co-Hopfian.