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Study of gravitational waves emitted by a binary system of compact objects in the inspiral phase in the post-Newtonian approximation

Quentin Henry

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**THÈSE DE DOCTORAT
DE L'UNIVERSITÉ SORBONNE UNIVERSITÉ**

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École doctorale n°564: Physique en Île-de-France

réalisée

sous la direction de Luc BLANCHET en co-encadrement avec Guillaume FAYE

présentée par

Quentin HENRY

pour obtenir le grade de :

DOCTEUR DE L'UNIVERSITÉ SORBONNE UNIVERSITÉ

Sujet de la thèse :

Étude des ondes gravitationnelles émises par un système binaire d'objets compacts dans la phase spirale dans l'approximation post-newtonienne.

soutenue le 17 septembre 2021

devant le jury composé de :

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Introduction

At the beginning of the 20th century, mankind's conception of the Universe was very different from today. We did not know if matter was continuous or discrete, we thought about light only as waves, the only planets we knew were those of our Solar System and we did not know that there are other galaxies than the Milky Way. We thought the Universe was static and eternal. In 1905, *annus mirabilis* of physics, a young man of 26 years old, Albert Einstein, revolutionized our way of thinking the Universe thanks to four papers. The first of them put into light the photoelectric effect, *i.e.* the corpuscular aspect of light [1]. The second paper is about the Brownian motion [2] and allowed to give some convincing arguments to show that matter is made of atoms. The third paper is special relativity in which space and time are not dissociable [3]. The speed of light is the upper limit of propagation velocity in space. Finally, the fourth article contains, in an underlying way, the most famous equation of physics: $E = mc^2$ [4], giving a new meaning of the concept of mass.

After ten years of endeavour, Einstein published in 1915 his theory of general relativity [5], sometimes considered as the most important scientific creation made by one man. This relativistic theory of gravitation has passed each performed tests with flying colors, making it one of the most tested theories of physics. It allowed to explain the anomalous precession of the perihelion of Mercury. It also led to some theoretical curiosities such as black holes or gravitational waves, both predicted in 1916. They are solutions of the Einstein equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

The way to interpret them is the following: the left-hand side corresponds to the curvature of space-time while the right-hand side corresponds to the matter distribution in space-time. Varying one of these quantities will alter the value of the other one. Thus, space-time becomes a physical object itself.

As for each physical object, space-time has numerous properties. For instance, our Universe is expanding. The distance between two galaxies grows inexorably with time and the more distant the galaxies are, the faster they will move away from each other. It is the famous Hubble-Lemaître law. One other of its properties is that it is curved. Hence, if masses move in space-time, they can create waves. These waves, called gravitational waves (GW), change the value of the gravitational field as they propagate through space-time. This results in an oscillation of the relative separation of two free-falling points that we are today able to measure. Indeed, in 2015, the first ever direct detection of GW was made by the LIGO collaboration. It was created by the coalescence of two black holes. This new method of observing the sky marked a breakthrough in modern physics and allows to characterize objects that cannot be seen with telescopes. Today, 6 years after the first detection, more than 50 events of GWs have been confirmed to originate from binary systems of compact objects. These objects are black holes and neutron stars and we are now in position of measuring their masses, their spins and even have hints on the internal structure of neutron stars.

In order to detect them, we need to have accurate solutions of the Einstein equations. However, in the case of binary systems, we cannot solve them exactly. Thus, we need to use methods that provide approximate solutions. One of them is called the post-Newtonian approximation. This is the one that has been used throughout this PhD. This approximation is valid for small relative velocities

of the compact objects and for a weak gravitational field. For this reason, it is well suited to describe the inspiral phase of binary systems of compact objects. More specifically, it is an expansion using the small parameter v/c , v being the relative velocity and c the speed of light. It provides analytical results as a series of v/c . The n^{th} post-Newtonian (PN) order corresponds to a quantity of the order of $(v/c)^{2n}$.

My PhD has been composed of two main projects: the first one is computation of the impact of finite-size effects (via tidal interaction) in the phase of GW from 5PN order to 7.5PN order; the second one is the computation of the radiated energy flux of GW up to 4PN order. This manuscript is composed of three parts. Part **I** is a general introduction to the considered problem. Part **II** describes the tidal effect project and Part **III** the 4PN flux of GW.

More specifically, Chapter **1** sums up the basics of the GW theory. We briefly introduce general relativity and solve the linearized Einstein equations.

In Chapter **2**, we review the sources that can be detected by current detectors. We also briefly explain how the detectors work and we give a review of the detections made so far. Finally, we give a non-exhaustive list of approximation methods used to build templates.

In Chapter **3**, we explain the post-Newtonian formalism that has been used for the computations in Parts **II** and **III**. More specifically, it is a generalization of Chapter **1** in which we do not restrict ourselves to the linearized theory.

In Chapter **4**, we present the computation of the conservative sector of tidal effects. More precisely, we derive the conserved energy and the equations of motion induced by finite-size effects in binary systems.

Chapter **5** is dedicated to the radiative sector of the tidal effects project in which we derive the radiated energy flux for tidal effects. By combining this result with the conserved energy computed in the preceding chapter through a balance equation, we are able to provide the phase of GW for tidal effects up to 2.5PN relative order.

In Chapter **6**, we compute the mass quadrupole, necessary for the radiated energy flux, up to 4PN order. This chapter is filled with technical difficulties that we faced during this calculation.

Finally, in Chapter **7**, we present the current quadrupole moment, also required for the radiated energy flux, up to 3PN order. We also present the associated GW amplitude mode.

The research work done during this thesis led to the following publications.

On tidal effects:

- Quentin Henry, Guillaume Faye, Luc Blanchet; *Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order*, *Phys. Rev. D* **101**, 064047 (2020). arXiv: 1912.01920 [gr-qc]. [6]
- Quentin Henry, Guillaume Faye, Luc Blanchet; *Tidal effects in the gravitational-wave phase evolution of compact binary systems to next-to-next-to-leading post-Newtonian order*, *Phys. Rev. D* **102**, 044033 (2020). arXiv: 2005.13367 [gr-qc]. [7]
- Quentin Henry, Guillaume Faye, Luc Blanchet; *Hamiltonian for tidal interactions in compact binary systems to next-to-next-to-leading post-Newtonian order*, *Phys. Rev. D* **102**, 124074 (2020). arXiv: 2009.12332 [gr-qc]. [8]

On the 4PN phase:

- Tanguy Marchand, Quentin Henry, François Larrouturou, Sylvain Marsat, Guillaume Faye, Luc Blanchet; *The mass quadrupole moment of compact binary systems at the fourth post-Newtonian order*, *CQG* **37**, 215006 (2020). arXiv: 2003.13672 [gr-qc]. [9]
- Quentin Henry, Guillaume Faye, Luc Blanchet; *The current-type quadrupole moment and gravitational-wave mode $(\ell, m) = (2, 1)$ of compact binary systems at the third post-Newtonian order*, *CQG* **38**, 185004 (2021). arXiv: 2105.10876 [gr-qc]. [10]
- François Larrouturou, Quentin Henry, Guillaume Faye, Luc Blanchet; *The Quadrupole Moment of Compact Binaries to 4PN Order I. Non-Locality in Time and Infra-Red Divergencies*, In prep. [11]

Most of the computations presented in this dissertation have been implemented with the *Mathematica* software [12]. The heavy computations of Parts II and III have been analytically performed by using the *PNComBin* library of the *xtensor* package [13], developed by G. Faye, J. Laidet and S. Marsat, and its d -dimensional extension, *ExtendedPNComBin*, implemented by T. Marchand.

Part I

General introduction

Chapter 1

General relativity and linearized theory of gravitational waves

This chapter's purpose is to briefly remind the reader some of the aspects of general relativity as well as the fundamentals of gravitational waves which will be used in the rest of the manuscript. This introduction is greatly inspired on the following textbooks [14, 15, 16] in which much more details are provided.

1.1 General relativity in a nutshell

Consider a manifold \mathcal{M} of dimension 4 together with a metric $g_{\mu\nu}$, a symmetric non-degenerate rank-two tensor of signature +2. Throughout this thesis, Greek indices $\{\mu, \nu, \alpha, \dots\}$ will represent space-time indices (0, 1, 2, 3), while Latin indices $\{i, j, k, a, \dots\}$ will represent only spatial indices (1, 2, 3). We will use the Einstein summation convention.

Let us consider the tangent bundle $T\mathcal{M}$ to \mathcal{M} and two vector fields v^α and w^α living on $T\mathcal{M}$. Let us further define $\{\lambda_1, \lambda_2\} \in \mathbb{R}^2$ and ϕ a scalar function. We can define a covariant derivative as an operator ∇_μ verifying:

- Linearity: $\nabla_\mu(\lambda_1 v^\alpha + \lambda_2 w^\alpha) = \lambda_1 \nabla_\mu v^\alpha + \lambda_2 \nabla_\mu w^\alpha$
- Leibniz rule: $\nabla_\mu(v^\alpha w^\beta) = \nabla_\mu(v^\alpha)w^\beta + v^\alpha \nabla_\mu(w^\beta)$
- Commutativity with contractions: $\delta_\alpha^\nu \nabla_\mu v^\alpha = \nabla_\mu v^\nu$
- Reduction to partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$

The notation δ_α^ν stands for the Kronecker symbol and is 1 if $\alpha = \nu$ and 0 otherwise and $\partial_\mu \equiv \partial/\partial x^\mu$ is the usual partial derivative. For simplicity, we used these relations on vectors, but they hold for any tensor with arbitrary rank. One can link two covariant derivatives ∇_μ and $\tilde{\nabla}_\mu$ using a *connection* $C^\alpha_{\mu\nu}[\nabla, \tilde{\nabla}]$

$$\nabla_\mu v^\alpha = \tilde{\nabla}_\mu v^\alpha - C^\alpha_{\mu\nu}[\nabla, \tilde{\nabla}]v^\nu. \quad (1.1)$$

If we further impose that the manifold is torsion-free, meaning that $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$, and that ∇_μ is compatible with the metric $g_{\mu\nu}$, namely

$$\nabla_\sigma g_{\mu\nu} = 0, \quad (1.2)$$

then ∇_μ is unique. From now on, we will only consider this particular covariant derivative. It can be linked to the usual partial derivative, by using a coordinate system, through

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu} v^\nu, \quad (1.3)$$

where $\Gamma_{\mu\nu}^\alpha = C_{\mu\nu}^\alpha[\nabla, \partial]$ is called the *Christoffel* symbol. It can be seen as the connection between the covariant and the partial derivatives. Furthermore, it is uniquely defined in terms of the metric

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \quad (1.4)$$

The torsion-free condition imposes that the Christoffel symbols are symmetric on their contravariant indices $\{\mu, \nu\}$, *i.e.* $\Gamma_{\mu\nu}^\lambda = \Gamma_{(\mu\nu)}^\lambda$. Moreover, ∇_μ defines the Riemann tensor that we choose in several textbooks convention (which will be used throughout this thesis)

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\alpha = R^\alpha_{\lambda\mu\nu}v^\lambda. \quad (1.5)$$

We can also explicit the Riemann tensor coordinate components in terms of the Christoffel symbols and thus in terms of the metric

$$R^\lambda_{\rho\mu\nu} = \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\nu\rho}^\sigma \Gamma_{\sigma\mu}^\lambda - \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\lambda, \quad (1.6)$$

the Ricci tensor $R_{\mu\nu} \equiv g^{\lambda\rho}R_{\lambda\mu\rho\nu} = R^\lambda_{\lambda\mu\nu}$ and the Ricci scalar $R \equiv g^{\mu\nu}R_{\mu\nu} = R^\lambda_{\lambda}$.

The famous Einstein equations, without the cosmological constant, read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1.7)$$

and can be derived by varying the Einstein-Hilbert action with respect to the metric

$$S_{\text{EH}} = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-g}R + S_{\text{mat}}, \quad (1.8)$$

where G is the Newtonian gravitational constant, c the speed of light and S_{mat} is the action considered for the matter part. Note that the Einstein-Hilbert action also contains a surface term that is not written here. The stress-energy tensor is defined by varying the matter action with respect to the metric

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g_{\mu\nu}}. \quad (1.9)$$

The divergence of the stress-energy tensor vanishes, *i.e.* $\nabla_\nu T^{\mu\nu} = 0$, due to the Bianchi identities.

1.2 Linearized theory

In this section we perform the weak field approximation, meaning that the actual metric $g_{\mu\nu}$ is a small perturbation to the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. We can write

$$g_{\mu\nu} = \eta_{\mu\nu} + \ell_{\mu\nu}, \quad (1.10)$$

where symbolically $|\ell_{\mu\nu}| \ll 1$.¹ Indices will be raised and lowered using the Minkowski metric $\eta_{\mu\nu}$. In particular, $\square = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta$ is the flat d'Alembertian operator and $\Delta = \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ the flat Laplacian operator, where δ_{ij} is the Euclidean metric. In the following chapters, we will use a different definition for the perturbation metric. The goal is to linearize Eq. (1.7) in terms of ℓ . We expand the left-hand side in terms of ℓ and η then truncate it at linear order in $\ell = O(G)$. We find

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2} \left[\partial_{\mu\lambda}\ell^\lambda_{\nu} + \partial_{\nu\lambda}\ell^\lambda_{\mu} - \square\ell_{\mu\nu} - \partial_{\mu\nu}\ell - \eta_{\mu\nu}\partial_{\lambda\sigma}\ell^{\lambda\sigma} + \eta_{\mu\nu}\ell \right], \quad (1.11)$$

¹The reason we do not adopt the traditional $h_{\mu\nu}$ notation is because it will be used from Chapter 3 until the end of the manuscript. Its definition is slightly different than this one and is displayed in (3.1) but they coincide at lowest order.

where $\partial_{\mu\nu} \equiv \partial_\mu \partial_\nu$ and $\ell = \eta^{\alpha\beta} \ell_{\alpha\beta}$. We can simplify this expression by introducing the trace-reversed perturbation

$$\bar{\ell}_{\mu\nu} = \ell_{\mu\nu} - \frac{1}{2} \ell \eta_{\mu\nu}. \quad (1.12)$$

Note that $\bar{\ell} \equiv \eta_{\alpha\beta} \bar{\ell}^{\alpha\beta} = -\ell$, hence the name. Then Eq. (1.11) becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2} \left[\partial_{\mu\lambda} \bar{\ell}^\lambda{}_\nu + \partial_{\nu\lambda} \bar{\ell}^\lambda{}_\mu - \square \bar{\ell}_{\mu\nu} - \eta_{\mu\nu} \partial_{\lambda\sigma} \bar{\ell}^{\lambda\sigma} \right]. \quad (1.13)$$

At this stage, we would like to simplify the equation by imposing $\partial^\nu \bar{\ell}_{\mu\nu} = 0$. Let us show that this condition is always valid in some particular coordinate systems. We first perform a coordinate shift $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$. The new metric deviation in that coordinate system transforms as

$$\bar{\ell}_{\mu\nu}(x) \rightarrow \bar{\ell}'_{\mu\nu}(x') = \bar{\ell}_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\lambda \xi^\lambda), \quad (1.14)$$

at lowest order if $|\partial_\mu \xi_\nu|$ is of the order of $|\ell_{\mu\nu}| \ll 1$. This implies that the contracted derivative of $\bar{\ell}_{\mu\nu}$ transforms as

$$\partial^\nu \bar{\ell}_{\mu\nu}(x) \rightarrow \partial^\nu \bar{\ell}'_{\mu\nu}(x') = \partial^\nu \bar{\ell}_{\mu\nu}(x) - \square \xi_\mu. \quad (1.15)$$

By imposing the condition $\partial^\nu \bar{\ell}_{\mu\nu} = 0$, we have to solve $\square \xi_\mu = \partial^\nu \bar{\ell}_{\mu\nu}(x)$. We now use a Green function $G(x - x')$ of the d'Alembertian operator to find a particular solution for ξ_μ , namely

$$\xi_\mu(x)|_{\text{part}} = \square^{-1} \partial^\nu \bar{\ell}_{\mu\nu} = \int d^4 x' G(x - x') \partial^\nu \bar{\ell}_{\mu\nu}(x'). \quad (1.16)$$

In this coordinate system, we always have

$$\partial^\nu \bar{\ell}_{\mu\nu} = \mathcal{O}(\ell^2), \quad (1.17)$$

where ℓ is the order of magnitude of $\ell_{\mu\nu}$. This choice of gauge is called *harmonic gauge* (or *De Donder gauge* or *Lorenz gauge* or *Hilbert gauge*). This condition can be taken into account directly at the level of the action. One simply has to add a so-called gauge-fixing term in the action given explicitly in Eq. (4.10). Throughout this thesis, computations will be made in this particular gauge. Note that since we only exploited a particular solution, we still have a freedom to choose the homogeneous part, which we will do later on. Finally we get, the linearized Einstein equations

$$\square \bar{\ell}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} + \mathcal{O}(\ell^2). \quad (1.18)$$

We obviously recognize a wave equation that we will solve in the following sections. Furthermore, as a consequence of (1.17), the divergence with respect to the partial derivative of the stress-energy tensor vanishes at linear order in the perturbation metric in this specific coordinate system, namely

$$\partial_\nu T^{\mu\nu} = \mathcal{O}(\ell^2). \quad (1.19)$$

1.3 Gravitational waves propagation

For our study, the stress-energy tensor is compact supported, which means that it vanishes outside world-tubes enclosing the bodies. We can study the propagation of gravitational waves outside the bodies and thus consider

$$\square \bar{\ell}_{\mu\nu} = 0, \quad (1.20)$$

which is a planar wave equation. We now exploit the freedom on the homogeneous solution for $\square \xi_\mu = 0$ to further simplify the problem by choosing a convenient gauge. One can show that we can choose for ξ^0 an expression so that $\bar{\ell} = 0 = \ell$ (see *e.g.* [15] for proof) which means that $\bar{\ell}_{\mu\nu} = \ell_{\mu\nu}$.

Thus, until the end of this section we will drop the bar notation. We can also exploit some more degrees of freedom for ξ^i so that $\ell^{0i} = 0$. We find, using Eq. (1.17), the following relations

$$\ell^{0\mu} = 0, \quad \ell^i{}_i = 0, \quad \partial^j \ell_{ij} = 0. \quad (1.21)$$

This convenient coordinate system is called the transverse-traceless gauge, or TT gauge.

Note that to derive these conditions, we exploited the fact that $\square \ell_{\mu\nu} = 0$, which is valid only outside of the source of GW. This means that we cannot apply the TT-gauge everywhere in space. This will be crucial in the following sections. In this gauge, the solution for Eq. (1.20) for a monochromatic wave propagating along the z axis is

$$\ell_{\mu\nu}^{\text{TT}}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ell_+ & \ell_\times & 0 \\ 0 & \ell_\times & -\ell_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos(\omega(t - z/c) + \varphi) \quad (1.22)$$

The quantities $\ell_{+, \times}$ correspond to the amplitude of the two GW polarisations, ω and φ are constants corresponding to the pulsation and phase of the GW. Remark that the constant c is the speed of light. In other words, the theory predicts that the speed of GWs is the same as the speed of light.²

Now let us try to understand how a passing GW affects distances. Consider two events $P_1 = (t, x_1, 0, 0)$ and $P_2 = (t, x_2, 0, 0)$. In this coordinate system, the interval between these events is

$$\Delta s = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \simeq L \left(1 + \frac{1}{2} \ell_+ \cos(\omega(t - z/c) + \varphi) \right), \quad (1.23)$$

where $L = |x_2 - x_1|$. This means that two events separated by a spatial coordinate distance L will, to the eyes of an external observer, oscillate with the passing of a GW. The oscillation has the order of magnitude of $L \ell_{+, \times}$, a frequency ω and a phase φ . More generally, in Fig. 1.1 is displayed how the two polarisations of planar wave alter distances with time.

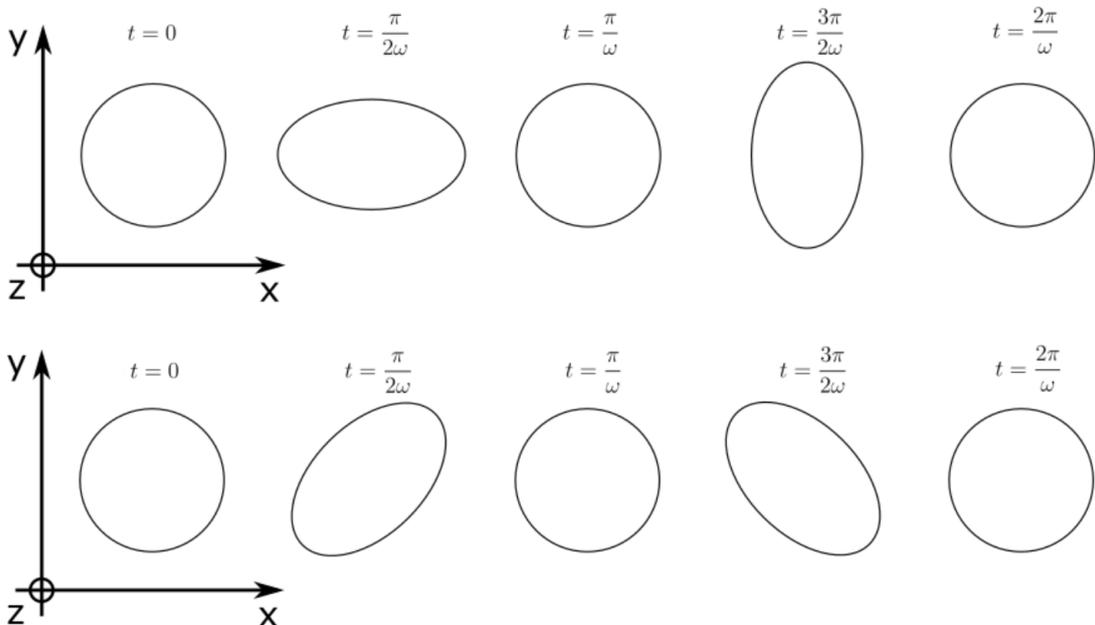


Figure 1.1: Effect of a plane GW of pulsation ω propagating along z direction on test particles. No deformation occur in the z direction. *Top*: ℓ_+ polarisation. *Bottom*: ℓ_\times polarisation. Courtesy to François Larrouturou.

²This has been confirmed in the detection of the binary neutron star event GW170817, in which an electromagnetic counterpart has been detected, constraining the speed of GWs as $|c_{\text{GW}} - c| \lesssim 10^{-15}c$ [17].

Note that the physical distance Δs is an observable quantity. What we want to do now is to determine the amplitude and frequency of the GW. Obviously, we have to solve Eq. (1.18) by making assumptions on the stress-energy tensor, which will determine the values of $\ell_{+,x}$, ω and φ .

1.4 Link with the source: the Einstein quadrupole formula

In this section we consider the full linearized wave equation Eq. (1.18). We assume that no incoming gravitational radiation enters the system. In other words, we apply the retarded Green function on the source

$$\bar{\ell}_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{c^4} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right). \quad (1.24)$$

At the lowest order in powers of $r = |\mathbf{x}|$, we have

$$\bar{\ell}_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{rc^4} \int d^3\mathbf{x}' T_{\mu\nu} \left(t - \frac{r}{c} + \frac{(\mathbf{x} \cdot \mathbf{x}')}{rc}, \mathbf{x}' \right) + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (1.25)$$

where $(\mathbf{x} \cdot \mathbf{x}')$ is the scalar product of \mathbf{x} and \mathbf{x}' . One can show using Eq. (1.19) and some integration by part that

$$\int d^3\mathbf{x}' T^{ij}(t, \mathbf{x}') = \frac{1}{2c^2} \frac{d^2}{dt^2} \int d^3\mathbf{x}' \left(x^i x'^j - \frac{1}{3} r'^2 \delta^{ij} \right) T^{00}(t, \mathbf{x}'), \quad (1.26)$$

where $r' = |\mathbf{x}'|$. At this stage we have to make assumptions on the form the stress-energy tensor. This is where the PN approximation enters. Remember that until now, we made two approximations: weak field because we restricted ourselves to the linear theory (first order in $\mathcal{O}(G)$) and far from the source (first order in $1/r$). Now we introduce the PN approximation, meaning that we take the first order in $1/c$ in the expression of $T^{\mu\nu}$. We suppose that the GW source is a perfect fluid in equilibrium. Such a matter system is described by the stress-energy tensor $T^{\mu\nu} = \rho(t, x) u^\mu u^\nu$ where $u^\mu(t, x) = (c, v^i(t, x))$ is the 4-velocity of the particles and ρ is the mass distribution of the system. In particular, $T^{00}(t, x) = \rho(t, x) c^2$.³ We can reexpress Eq. (1.25) in terms of a multipole, at leading order in powers of c , to obtain the solution for the perturbation metric

$$\bar{\ell}_{ij} = \frac{2G}{rc^4} \frac{d^2}{dt^2} I_{ij} \left(t - \frac{r}{c} \right), \quad (1.27)$$

where the *mass quadrupole* of the system is defined as

$$I_{ij}(t) \equiv \int d^3\mathbf{x} \hat{x}^{ij} \rho(t, x) + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (1.28)$$

The notation \hat{x}^{ij} represents the symmetric trace-free (STF) part of $x^i x^j$ namely $x^i x^j - r^2 \delta^{ij}/3$ in 3 dimensions. More generally, we note for any vector A^i , the STF part of a product $A^{i_1} \dots A^{i_\ell}$ $\hat{A}^{i_1 \dots i_\ell} = A^{(i_1 \dots i_\ell)}$. Eq. (1.28) is generalized in Eq. (3.17). Remark that sources which only admit a monopole and a dipole do not emit GWs. Furthermore, if the multipoles of a source are all stationary, no gravitational waves will be emitted either.

The GWs emitted by an isolated source carry energy and angular momentum. The fluxes of these quantities can be expressed in terms of the source quadrupole moment as follows

$$\langle \mathcal{F} \rangle = \frac{G}{5c^5} \left[I_{ab}^{(3)} I_{ab}^{(3)} + \mathcal{O}\left(\frac{1}{c^2}\right) \right], \quad (1.29a)$$

³Note that if we kept the LHS of Eq. (1.26), the solution would be incorrect for self gravitating systems. Indeed, Eq. (1.24) should be written in the full non-linear theory with $\tau_{\mu\nu}$, defined in Eq. (3.4), and not $T_{\mu\nu}$. At the lowest order in $1/c$, $\tau_{00} = T_{00}$, $\tau_{0i} = T_{0i}$ but $\tau_{ij} = T_{ij} + \partial_i U \partial_j U$, where U is the Newtonian potential.

$$\langle \mathcal{G}_i \rangle = \frac{2G}{5c^5} \epsilon_{iab} \left[\mathbb{I}_{ac}^{(2)} \mathbb{I}_{bc}^{(3)} + \mathcal{O}\left(\frac{1}{c^2}\right) \right], \quad (1.29b)$$

where ϵ_{iab} is the Levi-Civita tensor for which we take the convention $\epsilon_{123} = 1$ and $\mathbb{I}_{ab}^{(n)} = d^n \mathbb{I}_{ab} / dt^n$. The brackets correspond to the mean value over an orbital period. Both quantities $\langle \mathcal{F} \rangle$ and $\langle \mathcal{G}_i \rangle$ are functions of time t . The first equation (1.29a) is the famous Einstein quadrupole formula. The derivation of these formulae can be found in numerous textbooks, see *e.g.* [15] and of course, are only valid at leading order. The expressions (1.29) are generalised in Eqs. (3.18) and (3.19). As discussed in Sec. 3.3, the derivation of these quantities, especially the energy flux, is crucial to determine the GW phase.

Chapter 2

Detections

2.1 Possible sources

As shown in Eq. (1.27), systems that admit a time-dependent quadrupole generate GWs. Now, we turn to astrophysics to determine which kind of celestial objects can produce GWs. We can distinguish two types of astrophysical sources: isolated objects and self-gravitating systems containing at least two objects. Of course these two categories cover mostly all known astrophysical objects. However, very few of them can produce GWs that we are currently able to detect. Let us now list these sources.

2.1.1 Binary systems of compact objects

The amplitude of GW is linked to the compacity of the two compact objects. The notion of compacity will be widely discussed in Part II but we will briefly introduce it here. The compacity of a system of mass M and size R is defined as

$$\mathcal{C} = \frac{GM}{Rc^2}. \quad (2.1)$$

An object whose radius equals its Schwarzschild radius, namely a BH, has a compacity of $\mathcal{C} = 1/2$ which is the maximum one object can have. Objects that have a compacity of the order of $\gtrsim 0.1$ are also called compact. Among those, we find obviously BHs but also neutron stars (NS) that have a typical compacity of ~ 0.15 .

As we will see later, all the direct detections of GWs so far have been emitted from binary systems of compact objects, abbreviated as compact binaries. The simplest system that one can think of is an isolated binary black hole (BBH) because only gravitational interactions occur. Compact binaries lose energy through the emission of GWs. At the leading order, the energy flux of emitted GWs, derived using (1.29a) and (1.28), reads

$$\mathcal{F} = \frac{32}{5} \frac{G^4 m^5 \nu^2}{c^5 r_{12}^5}, \quad (2.2)$$

where r_{12} is the separation distance between the two compact objects, $m = m_1 + m_2$ the total mass and $\nu = \frac{m_1 m_2}{m^2}$ is the symmetric mass ratio of the system. On the other hand, this energy flux must correspond to the loss of some mechanical energy of the system E . Hence, while looking at the energy balance of the system we introduce the balance equation

$$\langle \mathcal{F} \rangle = -\frac{dE}{dt}. \quad (2.3)$$

This balance equation of the energy (as well as the balance equation for the angular momentum) of GWs plays a crucial role in the post-Newtonian approach as we will see in Chapter 3. Now at leading

order, meaning at Newtonian order, the well-known mechanical energy of the system reads

$$E = -\frac{Gm^2\nu}{2r_{12}}. \quad (2.4)$$

When we insert (2.2) and (2.4) into the balance equation, we find a differential equation for the separation distance

$$\dot{r}_{12} = -\frac{64 G^3 m^3 \nu}{5 r_{12}^3 c^5}. \quad (2.5)$$

The first implication of this differential equation is that the separation distance necessarily decreases over time due to the energy carried away by GWs. Solving this equation can also inform us on the lifetime of the system. Given an initial separation between the two bodies R_{12} , the system will roughly last a time

$$T = \frac{5}{256} \frac{R_{12}^4 c^5}{G^3 m^3 \nu}, \quad (2.6)$$

before the merger. Remark that this time depends on the symmetric mass ratio which $0 < \nu \leq 1/4$. In the case of two identical masses, this ratio has its highest possible value. So the bigger the asymmetry of the masses, the longer the system will live before its merger. Another implication is that the relative velocity of the two companions increases with time due to the third Kepler law.

Similar mass binary black hole coalescence

As we can see schematically in Fig. 2.1, we can distinguish three phases in the coalescence of a binary black hole (BBH) event. The first phase, called the inspiral phase, is the one in which the two objects orbit around each other at sufficiently far distance. Then, the phase in which the two bodies start touching is called the merger phase. Finally, the resulting black hole oscillates and relaxes to give a stationary rotating BH. This phase is called the ringdown. From (1.29), we see that a rotating BH in general relativity does not emit GWs which is why the signal goes to zero after the merger in Fig. 2.1.

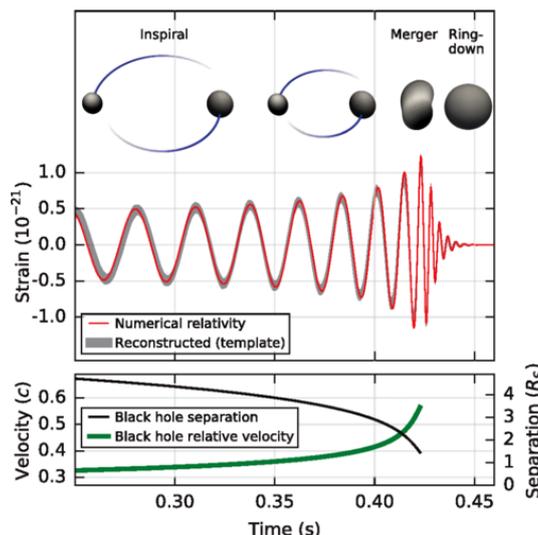


Figure 2.1: Figure extracted from [18]. Theoretical signal corresponding to the first merger ever detected of binary black holes event GW150914. *Top*: Illustration representing the three phases of a binary black hole coalescence. The red curve represents the signal constructed by numerical relativity. The grey curve corresponds to the signal given by phenomenological templates. *Bottom*: The black curve is the relative distance and the green curve is the relative velocity.

Note that we did not specify here which type of BHs we consider. Of course, the above description covers stellar BH coalescences, but it can also characterize binaries of supermassive BHs with similar masses. The difference between these two types of systems lies in the amplitude and mostly in the frequency of the GW.

Coalescences involving neutron stars

As previously said, NSs are also compact objects, thus, in a binary system, they are also susceptible to emit GWs that we can detect. In fact, coalescences of BH-NS or NS-NS have already been detected (see Sec. 2.3 for more details). However, the waveform can be very different than the one from a BBH merger. Indeed, NSs have an internal structure that BHs don't have. This additional complexity affects the scenarios in the binary system, see Fig. 2.2. In the inspiral phase, the internal structure of NSs barely affects their motion and the waveform. The discrepancy with BBHs is a small effect that is noticeable only in the late inspiral, right before the merger. It is due to the tidal interaction between the two companions. A part of my PhD was to compute the impact of the internal structure of compact objects in the emitted GW for the inspiral phase, see Part II.

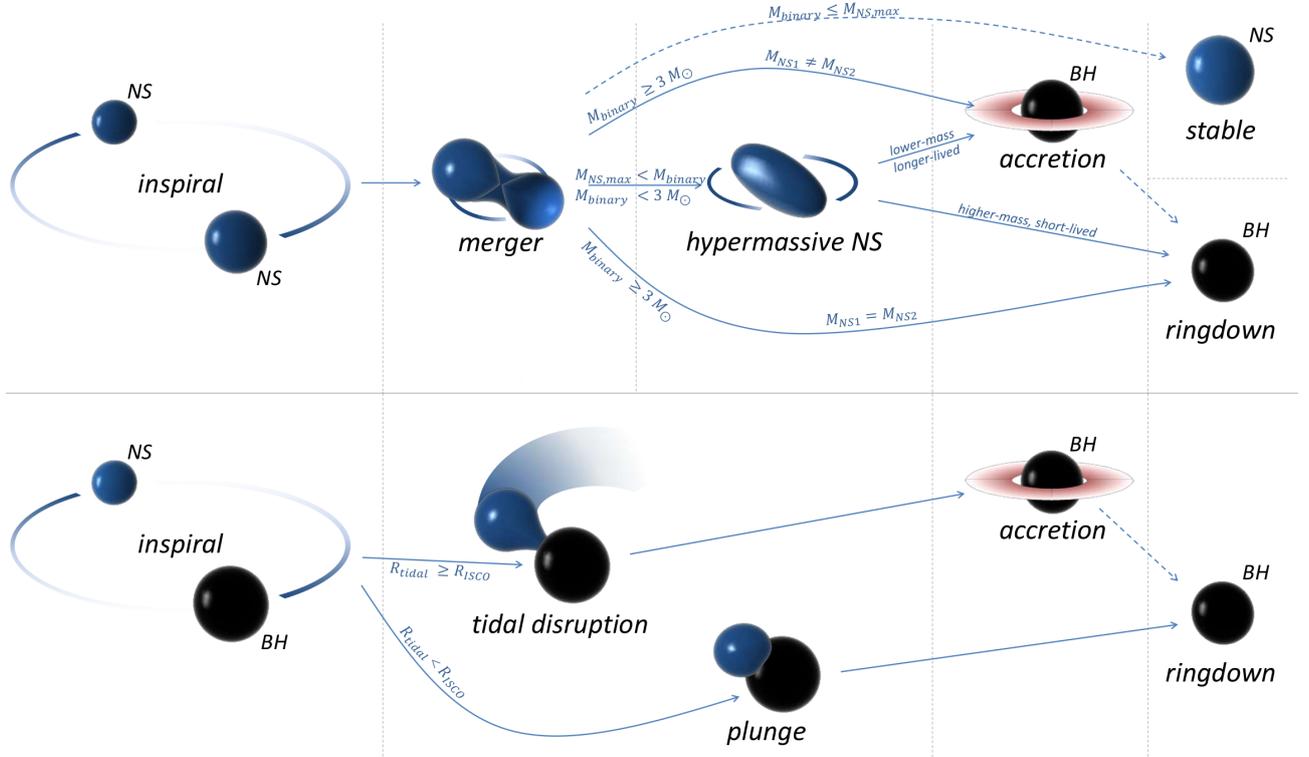


Figure 2.2: Figure extracted from [19]. Schematic diagram of the evolution of compact binary coalescences including at least one NS. *Top*: case of BNS systems. The scenarios after the merger depend on the masses of the two NSs. The notation $M_{NS,max}$ refers to the mass limit of non-rotating NSs. For $M_{binary} \gtrsim 3M_{\odot}$, the system rapidly collapses into a BH. For $M_{NS,max} < M_{binary} < 3M_{\odot}$, the system creates a hypermassive NS that collapses after a few milliseconds into a BH that may have an intermediate state with an accretion disk. For $M_{binary} \leq M_{NS,max}$, the system stabilises into a final NS. *Bottom*: case of a BH-NS system. R_{tidal} refers to the distance from the BH at which the NS gets tidally disrupted. R_{ISCO} refers to the innermost stable circular orbit. For $R_{tidal} \geq R_{ISCO}$, the NS gets tidally disrupted and the system becomes a BH with an accretion disk which will possibly fall into the BH later on. For $R_{tidal} < R_{ISCO}$, no tidal disruption occur and the system collapses into a final BH.

In Fig. 2.1, we see that once the two BHs have merged, they form a final BH that rapidly stabilizes

and stops emitting GWs after the merger. In the case of NSs, especially in the case of the creation of a hypermassive NS, the system keeps emitting GWs. In other words, the postmerger waveform of a binary NS system (BNS) can be qualitatively different from the one of a BBH. Furthermore, a final BH cannot emit GWs, but a stable final non-spherical NS can (see Sec. 2.1.2). In principle, this could allow to differentiate which scenario occurred. In Fig. 2.3 is shown a simulation of a merger of a BNS system.

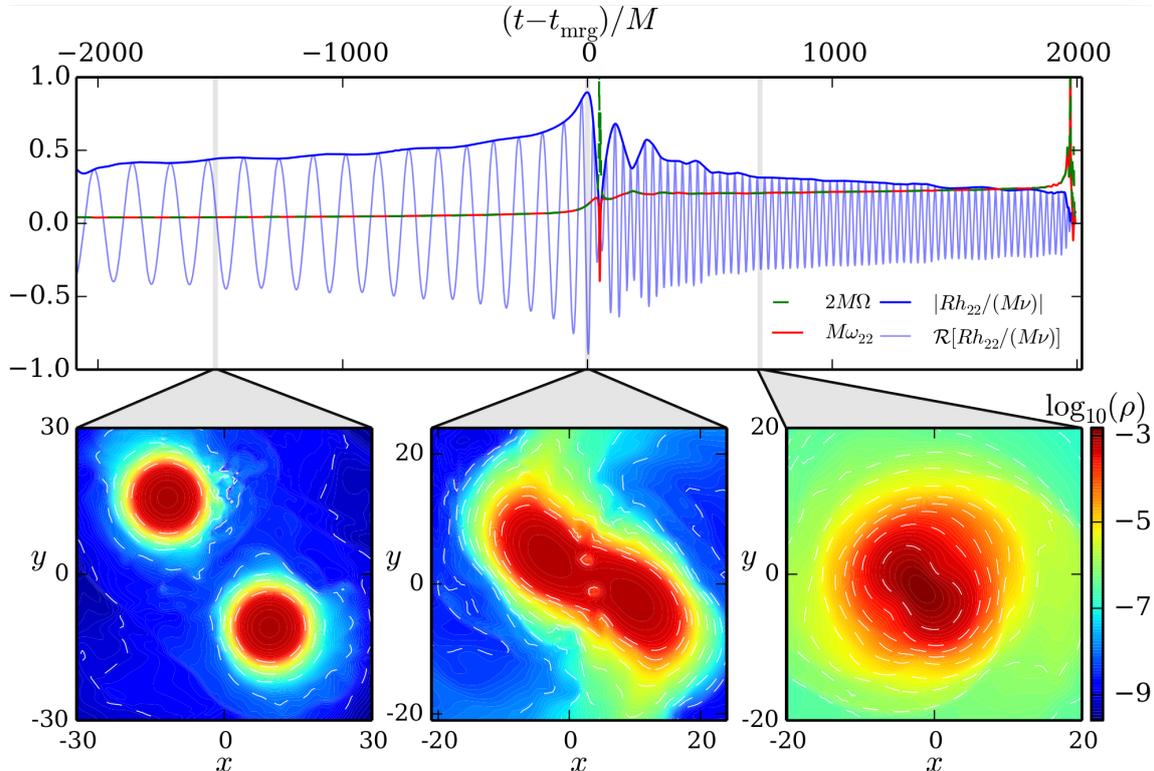


Figure 2.3: Figure extracted from [20]. Simulation of GWs emitted by a BNS merger. *Top*: real part and amplitude of the GW mode $Rh_{22}/(\nu M)$ and the associated dimensionless frequency $M\omega_{22}$ versus the mass-normalized retarded time t/M for a fiducial configuration. The signal is shifted to the moment of merger, t_{mrg} , defined by the amplitude’s peak (end of chirping). Also shown is (twice) the dynamical frequency $M\Omega \sim M\omega_{22}/2$. *Bottom*: Snapshots of $\log_{10}\rho$ on the orbital plane, during the late inspiral (left panel), at simulation time corresponding to t_{mrg} (middle panel), and during the postmerger (right panel). We refer to [20] for more details.

We see that the post-merger amplitude of the GW is qualitatively different from the BBH one. However, as we can expect, its shape highly depends on other parameters such as the spins and the internal structure of the NSs. As shown in [21], the spin highly affects the post-merger signal. To date, no post-merger remnant has been detected but such an observation would indicate that at least one of the two companions is a NS.

Extreme mass ratio inspiral

In the two previous points, we have described the waveform and different scenarios for compact binaries in which the two companions have similar masses. However, in compact binaries, we can also encounter a pair of two bodies which have a significant mass difference. Such systems are called extreme mass ratio inspiral (EMRI). They are composed of a central supermassive BH paired with a small mass BH, a NS or a white dwarf. Most galaxies contain in their center a supermassive BH and those who have active nuclei are likely to host EMRIs. Although such systems are not in the

sensitivity band of the current detectors, we will be able to detect them with the next generation detector LISA.

2.1.2 Other types of sources

Supernovae

At the end of their life, stars can collapse as supernovae. In general, these phenomena are not isotropic and thus, they can generate GWs. These astrophysical processes are complex and many effects have to be taken into account such as electromagnetic, relativistic hydrodynamic, neutrino and general relativity physics. However, studies on the GW signature of core-collapse supernovae have been performed and show that they would be detectable by the current detectors [22]. Furthermore, such events can create compact stars, such as fast rotating NSs, can also leave a remnant of GW emission.

Rotating anisotropic objects

As said above, when we look at (1.27), we see that rotating non-axisymmetric objects can also emit GWs. The astrophysical objects that could be detectable by our current detectors are nearby, spinning and slightly non-axisymmetric isolated NSs in our galaxy. However, no signal of such events has been detected so far [23].

Stochastic background

A stochastic background of gravitational waves can be created by the superposition of a large number of independent sources [24]. Of course, in those sources, we find the ones listed above together with pulsar timing arrays (PTAs) [25]. In addition, the relic gravitational waves from the early evolution of the universe plays a role to the background. We can also find other types of sources created by new physics such as dark matter or cosmic strings [26].

2.2 The detectors

2.2.1 Second generation detectors

Ground based GW detectors running observations are of second generation. There are 4 currently in operation: the two Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors, the Virgo interferometer and Kamioka Gravitational Wave Detector (KAGRA). The LIGO detectors are based the USA, Virgo in Italy and KAGRA in Japan. The same type of detector is being built in India too. All second generation detectors work in the same way and are designed to detect the merger phase of binary systems of compact objects in the stellar mass range.

In Fig. 2.4 is displayed a schematic view of their structure. The principle is the following: a source emits a laser beam that reaches a beam splitter such that half of the beam goes in the x direction and the other half in the y direction. Each beam enters an arm at the end of which stands a free falling test mass which is a mirror suspended and isolated. Then, the two beams recombine at the level of the beam splitter and their interference is measured by a photodetector. This interferometer is not exactly a Michelson interferometer due to the presence of additional mirrors at the beginning of the two arms which form Fabry-Perot cavities.

Now, what is measured is the difference of length between both arms continuously. We saw in Sec. 1.3 how a planar wave affects the distances in a plane perpendicular to its direction of propagation. We also saw that it does not affect distances along its direction of propagation. So if its direction of propagation is along *e.g.* the y axis of the detector, only the length of the arm on the

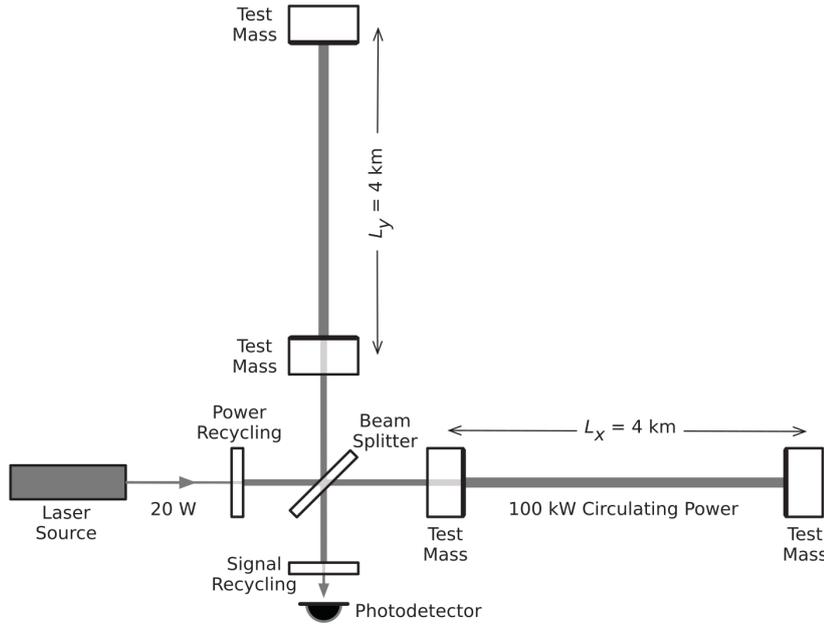


Figure 2.4: Figure extracted from [18]. Schematic representation of the LIGO detectors. Virgo and KAGRA are of the same type.

x axis will oscillate and a signal will be detected. However, if it is *e.g.* along the bisection of the x and y axes of the detector, both arms will be altered exactly in the same way and no signal will be detected. In other words, such detectors have blind spots. Thus, the more unaligned detectors we have on Earth, the less we take the risk of missing events that come from their blind spots.

2.2.2 Third generation detectors

Next generation detectors are currently being designed. The two main projects are LISA and Einstein Telescope. However, they work slightly differently than the second generation detectors.

LISA (Laser Interferometer Space Antenna) is a project led by ESA together with NASA. It is a space-based detector which is made of three spacecrafts forming an equilateral triangle with sides of 2.5 millions kilometers. The barycentre of this triangle will orbit around the Sun on the Earth orbit. Each of the spacecraft will emit two laser beams in the direction of the others. LISA aims mainly at detecting supermassive black hole mergers, EMRIs and the inspiral phase of stellar mass compact binaries. Of course this list is non exhaustive and many other effects are expected to be detected.

Einstein telescope, on the other hand, is a European project of a ground-based detector. It is also shaped as a triangle with arms of 10 kilometers. Its frequency band is roughly the same as the second generation but its sensitivity is drastically increased, see *e.g.* [27]. This will notably allow us to better constrain the finite-size effects in BNS systems.

2.3 Direct detections of GWs

The network of GW detectors started with the two LIGO detectors (Hanford and Livingston) in its first run called O1. This run lasted from 2015/09/12 to 2016/01/19. After this run, the detectors were upgraded in order to launch the second observation run O2 that started from 2016/10/30 to 2017/08/25. In the mean time, Virgo joined the collaboration on 2017/08/01. The third run was split in two runs called O3a and O3b. O3a started on 2019/04/01 to 2019/10/01 while O3b from

2019/11/01 to 2020/03/27. KAGRA also joined the GW network on 2020/02/25. Unfortunately, the O3b had to be stopped due to the COVID-19 pandemic, only a month after KAGRA started.

During these runs, more than 50 events have been detected. As mentioned above, the detectors were designed to detect binary systems of compact objects that have stellar masses, so all the detections are in this mass range. Two catalogs are available [28, 29] listing the confirmed events by the collaborations for the O1, O2 and O3a runs. Although some events in the O3b run have been confirmed, the O3b catalog is not yet published.

2.3.1 The O1 run

The catalog [28] regroups the O1 and O2 events confirmed by the collaboration. The first event ever detected [18], GW150914, on 2015/09/14, marked a breakthrough in modern physics. Not only did it show that we were indeed able to detect GWs, but also the high value of the estimated masses of the two companions being of $m_1 = 36^{+5}_{-4}M_\odot$ and $m_2 = 29^{+4}_{-4}M_\odot$ allowed the signal to be extremely clear as we can see in Fig. 2.5. The whole O1 run accounts 3 BBH events of similar mass.

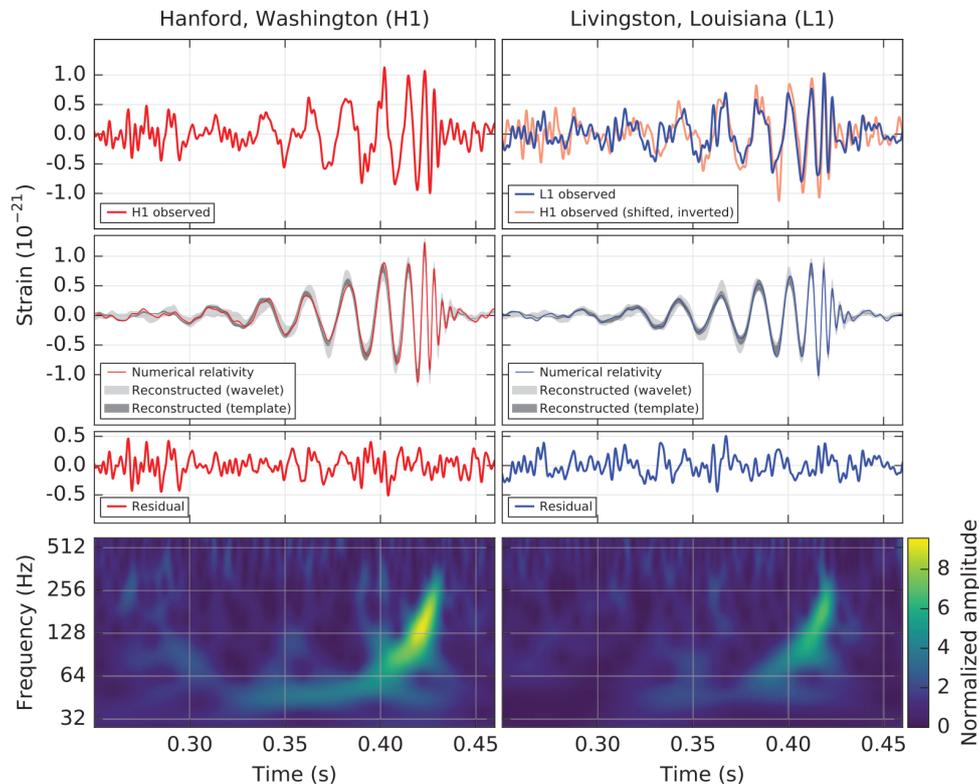


Figure 2.5: Figure extracted from [18]. H1 stands for the Hanford detector, L1 for Livingstone. *Top row, left* : H1 strain. *Top row, right* : L1 strain. GW150914 arrived first at L1 and $6.9^{+0.5}_{-0.4}$ ms later at H1; for a visual comparison, the H1 data are also shown, shifted in time by this amount and inverted (to account for the detectors’ relative orientations). *Second row* : Gravitational-wave strain projected onto each detector in the 35–350 Hz band. Solid lines show a numerical relativity waveform for a system with parameters consistent with those recovered from GW150914. Shaded areas show 90% credible regions for two independent waveform reconstructions. One (dark gray) models the signal using binary black hole template waveforms. The other (light gray) does not use an astrophysical model, but instead calculates the strain signal as a linear combination of sine-Gaussian wavelets. *Third row* : Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. *Bottom row* : A time-frequency representation of the strain data, showing the signal frequency increasing over time.

2.3.2 The O2 run

During the second run, 8 events have been confirmed. Among those, 7 BBH mergers in which the two BHs had similar masses. They were all stellar BHs and the chirp masses of the events ranged from $8M_{\odot}$ to $36M_{\odot}$.

The most interesting event of this run is undoubtedly the remaining event, GW170817, which was coming most likely from a BNS system. Indeed, it was the first ever detection of a BNS merger. In addition, an electromagnetic counterpart has been detected with a gamma ray burst 1.7s after the GW signal. Furthermore, the BNS was in a blind spot of Virgo which started collecting data only two weeks before the event. This allowed to restrain the sky area in which the event took place and the first telescope to locate its position, 10.9h after the event, was the Swope Supernova Survey [30]. This detection has many physical implications notably due to the electromagnetic counterpart. The first of them is the very precise constraint on the speed of GWs which is $|c_{\text{GW}} - c| \lesssim 10^{-15}c$. This single constraint allowed to discard many alternative theories of gravitation and further confirms general relativity. Moreover, the detection allowed to constrain the possible models on the internal structure of NSs through the equation of state [31]. This topic is widely discussed in Part II. An additional feature of this detection is that the observation of the electromagnetic counterpart allowed us to measure an independant value of the Hubble-Lemaître constant H_0 [17]. The resulting object has a mass of $2.74_{-0.01}^{+0.04}M_{\odot}$ for a low-spin prior. We do not know if it is a BH or a NS, but it is either the lowest mass BH or the highest mass NS known.

2.3.3 The O3 run

As mentionned above, the O3 run has been split in two distinct runs: O3a and O3b. In O3a, the confirmed events are listed in [29]. In this catalog, 39 events are displayed and the total duration of the run was ~ 26 weeks, which means that ~ 1.5 event per week has been detected. Out of the 39 events, possibly 3 come from systems containing at least one NS. However, there was no constraint on the tidal parameters and no electromagnetic counterpart has been detected, so we cannot rule out the BBH hypothesis. As an example, the event GW190425 is likely BNS due to the small estimated masses of the companions $m_1 = 2.0_{-0.3}^{+0.6}M_{\odot}$ and $m_2 = 1.4_{-0.3}^{+0.3}M_{\odot}$.

Interestingly, GW190814 has the highest recorded mass ratio: $m_1 = 23.2_{-1.0}^{+1.1}M_{\odot}$ and $m_2 = 2.59_{-0.09}^{+0.08}M_{\odot}$ [32]. It is the first significant mass ratio that has been recorded.

The O3b catalog is not yet available, but two BH-NS events, GW200105 and GW200115, have been confirmed [33]. Once again, no electromagnetic counterpart has been detected.

Although numerous events have been detected from compact binaries, the LIGO, Virgo and KAGRA collaborations try to extract possible signals coming from other types of sources such as short GW bursts [34] or continuous waves [23], cosmic strings [35] or dark matter [36]. No such signal has been found so far. Furthermore, tests on general relativity can be performed [37], *e.g.* by seeking additional GW modes that are not predicted by general relativity.

2.4 Approximation methods for compact binary systems

Let us now turn to the theory and review the different ways to solve the Einstein equations for compact binaries. As we will see in Secs. 3.2.1 and 4.1, we can separate the different physical effects (such as point-particle, spins, finite-size effects...) that enter the waveforms. For example, spins of compact objects affect the waveform but the effect is small when compared with the waveform generated by static BHs in a binary system. As we will see in the following chapters, the computations I performed during my PhD were on two topics: The extension of the precision on the point-particle and on the finite-size effects in the inspiral of compact binaries. Below are listed different methods

to solve, either analytically either numerically, the Einstein equations and we give the state of the art in these methods. However, we do not present the results regarding spin effects.

2.4.1 Post-Newtonian

The post-Newtonian (PN) formalism is an approximation method that provides analytical solutions for the description of the inspiral of compact binaries. This formalism is the one that I used throughout my PhD and is detailed in Chapter 3. However, let us briefly mention here in what it consists. The PN approximation assumes a slow motion of the bodies. It is a Taylor expansion of powers of v/c , v being the relative velocity of the companions and c the speed of light. One PN order corresponds to a quantity of the order $\mathcal{O}(1/c^2)$.

For the point-particle part of the waveform, meaning that the two BHs have no spin and no internal structure, the conservative dynamics is known to 4PN order [38, 39, 40]. Recent works also derived some of the coefficients up to 6PN [41, 42]. The radiative sector on the other hand is known up to 3.5PN [43] and 4PN is currently being investigated (see Part III for more details). Moreover, the 4.5PN term in the GW flux is already known and has been published in [44].

Finite-size effects arise in the PN expansion at 5PN order. Even though it is a high order, the numerical value of the leading order is high enough to be measurable by current ground-based detectors. Computations on finite-size effects have been among the projects studied during my PhD (see Part II). More specifically, I derived the tidal effects in the GW phase (in the absence of spins) from 5PN to 7.5PN, so 2.5PN relative order. This problem has been tackled by multiple groups. For references, see Table 4.1 for the conservative sector and Table 5.1 for the radiative sector. The result for the GW phase including tidal effects is provided in (5.41)-(5.42). The coupling to spins has been derived up to 6.5PN [45]. For more details on tidal effects in the PN expansion, see *e.g.* the recent review [46].

2.4.2 Effective field theory

An effective field theory (EFT) is a theory that does not attempt to be valid at all scales but only in the regime that we are interested in. It exploits the existence of a separation of scales and has been mainly developed in the context of particle physics. The EFT approach to the binary inspiral problem was originally proposed by Goldberger and Rothstein in the context of gravitationally bound non-rotating extended objects [47]. However these methods differ from the original particle physics point of view due to the fact that general relativity is a classical theory. They use a (Feynman) diagrammatic approach to tackle the problem. For detailed reviews on the topic, see *e.g.* [48, 49]. Computations in this approach are often done in the post-Minkowskian (PM) approximation, in which we perform a Taylor expansion in powers of G . One PM order corresponds to a quantity of the order $\mathcal{O}(G)$.

Regarding the conservative dynamics of non-spinning point particles, they derived the 4PN equations of motion using a Lagrangian approach [50, 40] as well as a Hamiltonian approach [51] at the same order. Furthermore, the 3PM dynamics has also been derived in [52]. In the radiative sector, the 2PN quantities are available [53] and some more work on the formalism has to be done to perform the computations at higher orders. Some work on finite-size effects has also been performed and the 2PM Hamiltonian for tidal effects is available [54]. Furthermore, new synergies between the traditional PN approach and EFT led to the determination of some new coefficients in the conserved energy at high PN orders [55].

2.4.3 Scattering amplitudes

The scattering amplitudes method is a computational framework that aims at computing the dynamics of gravitationally bound compact objects. The idea behind this method is to relate the

scattering amplitudes of two interacting BHs to the orbital dynamics of the system. Indeed, both processes are governed by the same underlying theory [56]. Furthermore, this approach allows to use the well-understood methods used in particle physics. The scattering of two BHs is, computationally speaking, similar to the scattering of two quarks because their interactions via gluons or gravitons are analogous. The link between these two processes is called the double copy [57]. For a detailed overview of this method see *e.g.* [56].

The conservative dynamics of non-spinning BBHs has been derived with this method up to 3PM order [58] and partially up to 4PM [59]. Some work has also been done on finite-size effects. They derived conservative dynamics including tidal effects up to 2PM order beyond leading order [60]. We will see in Sec. 4.5 that the overlap with the results in the PN approach is in full agreement with these results.

2.4.4 Gravitational self force

Gravitational self force (GSF) is an approximation method that aims at describing EMRIs. The small parameter in this approach is the mass ratio of the two companions $m/M \ll 1$, see *e.g.* [61] for a review. At the lowest order in the expansion, the small object follows geodesics around the massive one as does a test particle. However, at the following orders, this is no longer the case and the test particle is deviated from the initial geodesic as if it felt a force which is then called self force. The results provided by this method will be very useful for LISA which will be notably sensitive to EMRIs.

The first order of the development in m/M is known and provides results for physical observables at extremely high PN orders: 21.5PN for analytical results [62] and 22PN with numerical integration [63]. These results are extremely useful for the computations in the PN approach. Indeed, they allow us to check the terms entering our expressions in the limit $\nu \rightarrow 0$. The second order is now being tackled but technical difficulties arise due to some strong but unphysical divergencies on the worldline of the small object [64].

2.4.5 Numerical relativity

The methods listed above are approximations which are only valid in particular domains. They provide analytical solutions that are crucial in the modelling of the waveforms. However, as the two compact objects get closer, the Einstein equations start to be highly non linear. No analytical methods are available to describe the dynamics of the system or the GWs emitted during the last orbits. Today, the only method available to solve the Einstein equations during the merger phase is to solve them numerically. For a review on this method, see *e.g.* [65]. A breakthrough in this domain has been achieved in 2005 [66] where Pretorius managed to simulate the last few orbits during the coalescence of two BHs. Today numerous catalogs on the merger of BBHs are available, see *e.g.* [67, 68].

Although numerical relativity is valid at each step of the coalescence, and especially the merger, it requires a lot of computational power to provide waveforms. Some signals, like GW170817, lasted for roughly a minute and recorded ~ 3000 cycles in the detector band. Numerical relativity is inadequate for such a high number of cycles. Therefore, templates need to be build by combining analytical solutions with numerical relativity for the merger part.

2.4.6 Effective one body

Effective one body (EOB) is a resummation method developed in 1999 by Alexandra Buonanno and Thibault Damour [69] that allows to build templates. The basic idea is to map the two-body problem onto an effective one-body problem, i.e. the motion of a test particle in some effective

external metric by means of an effective Hamiltonian. It allows to solve exactly the effective problem of the test particle in this deformed Schwarzschild metric by introducing a particular non-perturbative method for re-summing the PN expansion of the equations of motion. Some coefficients in the expansion corresponding to high PN order remain unknown. They are fitted by comparing with numerical relativity. This clever method allows to build templates used for data analysis of the LIGO/Virgo/KAGRA collaborations.

One of many results obtained in the EOB approach is the consideration of dynamical tides. Indeed, the results discussed above on finite-size effects in binary systems assume that the two bodies interact through static (or adiabatic) tidal effects. However, it has been shown in [70], using an EOB Hamiltonian, that matter resonances inside the NS can arise and that these effects cannot always be negligible compared to adiabatic tides.

Chapter 3

The post-Newtonian formalism

The traditional post-Newtonian (PN) formalism that has been used throughout my PhD is in fact a diminutive of the post-Newtonian multipolar-post-Minkowskian formalism (PN-MPM). This formalism is constituted of two main approximations: slow motion and weak field which correspond to, respectively, the PN and post-Minkowskian (PM) approximations. They are valid in two overlapping regions of space-time and they allow to perform Taylor expansions in their respective validity regions using different small parameters.

The PN approximation is valid in the so-called *near zone* defined by a distance to the source much smaller than the typical wavelength of the GW ($r \ll \lambda$). In this region, we can perform Taylor expansions using the small parameter v/c where v is the relative velocity of the bodies in the considered system. A n PN order corresponds to a quantity of the order $\mathcal{O}(c^{-2n})$.

The PM approximation is valid in the *exterior zone* defined by a distance to the source bigger than the typical size of the system a ($r > a$). In this region, we can perform a Taylor expansion using the small parameter the gravitational constant G . A n PM order corresponds to a quantity of the order $\mathcal{O}(G^n)$.

The two regions overlap in the *buffer zone* ($a < r \ll \lambda$). In this zone, both PN and PM expansions are valid. By means of multipole decompositions and matching, we are able to link the two approximations to get the information of the system up to spatial infinity. This constitutes the PN-MPM formalism [71, 72, 73, 74, 75, 76, 77, 78], in which we apply specifically the harmonic-coordinates condition. This is a completion of the linearized theory presented briefly in Chapter 1. Until the rest of this manuscript, we consider a space-time of $d+1$ dimensions. In the present chapter, which aims at giving an overview of the PN-MPM formalism, the results are presented for $d = 3$. However, as we will see in Sec. 3.3, the computations have also been done for arbitrary complex d .

3.1 The PN-MPM formalism

The definition of the gravitational-field amplitude is slightly different from the one in Chapter 1, namely

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \quad (3.1)$$

which is the perturbation of the so-called gothic metric $\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and where g is the determinant of the metric $g_{\mu\nu}$. Note that this definition is equivalent to $\bar{\ell}_{\mu\nu}$ used in Eq. (1.10) at the lowest order up to a minus sign. We recall that the harmonic gauge condition

$$\partial_\nu h^{\mu\nu} = 0, \quad (3.2)$$

fixes a class of coordinates called harmonic coordinates. Without doing any approximation, one can rewrite the Einstein equations in this coordinate system as

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (3.3)$$

where $\tau^{\mu\nu}$ is called the pseudo stress-energy tensor and is constituted of the stress-energy tensor and the non linearities in the gravitational field $h^{\mu\nu}$

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}. \quad (3.4)$$

The second term in the definition of the pseudo stress-energy tensor is exactly given by

$$\Lambda^{\mu\nu} = -h^{\alpha\beta}\partial_{\alpha\beta}h^{\mu\nu} + \partial_{\alpha}h^{\mu\beta}\partial_{\beta}h^{\nu\alpha} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_{\lambda}h^{\alpha\sigma}\partial_{\sigma}h^{\beta\lambda} \quad (3.5)$$

$$- g^{\mu\alpha}g_{\beta\sigma}\partial_{\lambda}h^{\nu\sigma}\partial_{\alpha}h^{\beta\lambda} - g^{\nu\alpha}g_{\beta\sigma}\partial_{\lambda}h^{\mu\sigma}\partial_{\alpha}h^{\beta\lambda} + g_{\alpha\beta}g^{\lambda\sigma}\partial_{\lambda}h^{\mu\alpha}\partial_{\sigma}h^{\nu\beta} \quad (3.6)$$

$$+ \frac{1}{8}(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})(2g_{\lambda\tau}g_{\rho\sigma} - g_{\tau\rho}g_{\lambda\sigma})\partial_{\alpha}h^{\lambda\sigma}\partial_{\beta}h^{\tau\rho}. \quad (3.7)$$

In the linearized theory, $\Lambda^{\mu\nu} = \mathcal{O}(h^2) = \mathcal{O}(G^2)$ and is thus neglected. The PN-MPM formalism allows going beyond this approximation and its "philosophy" is the following: we first solve the linearized theory that gives us a solution for $h_{(1)}^{\mu\nu}$ at the first order. Then, we insert this solution in $\Lambda^{\mu\nu}$ and truncate at the second order in G . We recover an equation of the same form as Eq. (3.3) in which the RHS does not depend on $h_{(2)}^{\mu\nu}$. Next, we reiterate this procedure order by order. Let us first detail properly the linearized theory using a multipole approach.

3.1.1 Linearized theory

Here we consider the Einstein equations in the exterior zone so that, at linear order in G , together with the harmonic gauge condition, they read

$$\square h_{(1)}^{\mu\nu} = 0, \quad (3.8a)$$

$$\partial_{\nu}h_{(1)}^{\mu\nu} = 0. \quad (3.8b)$$

It has been shown [79, 71] that the most general solution of these linearized field equations outside some time-like world tube enclosing the source and stationary in the past, reads

$$h_{(1)}^{\mu\nu} = k_{(1)}^{\mu\nu} + \partial^{\mu}\varphi_{(1)}^{\nu} + \partial^{\nu}\varphi_{(1)}^{\mu} - \eta^{\mu\nu}\partial_{\lambda}\varphi_{(1)}^{\lambda} \quad (3.9)$$

where the first term depends on two STF-tensorial multipole moments, I_L and J_L , called *source multipole moments* which are arbitrary functions of the time. Furthermore, with the no-incoming wave condition, they depend only on the retarded time in harmonic coordinates $u = t - r/c$ in the expression of $k_{(1)}^{\mu\nu}$. Its general expression reads

$$k_{(1)}^{00} = -\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^{\ell}}{\ell!}\partial_L\left(\frac{I_L(u)}{r}\right), \quad (3.10a)$$

$$k_{(1)}^{0i} = \frac{4}{c^3}\sum_{\ell\geq 1}\left[\partial_{L-1}\left(\frac{I_{iL-1}^{(1)}(u)}{r}\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{J_{bL-1}(u)}{r}\right)\right], \quad (3.10b)$$

$$k_{(1)}^{ij} = -\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^{\ell}}{\ell!}\left[\partial_{L-2}\left(\frac{I_{ijL-2}^{(2)}(u)}{r}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\epsilon_{ab(i}\frac{J_{j)bL-2}^{(1)}(u)}{r}\right)\right]. \quad (3.10c)$$

The gauge vector depends also on four sets of STF-tensorial multipole moments called *gauge moments*

$$\varphi_{(1)}^0 = \frac{4}{c^3}\sum_{\ell\geq 0}\frac{(-)^{\ell}}{\ell!}\partial_L\left(\frac{1}{r}W_L(u)\right), \quad (3.11a)$$

$$\begin{aligned} \varphi_{(1)}^i &= -\frac{4}{c^4} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_{iL} \left(\frac{1}{r} X_L(u) \right) \\ &\quad - \frac{4}{c^4} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left(\frac{1}{r} Y_{iL-1}(u) \right) + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} Z_{bL-1}(u) \right) \right\}. \end{aligned} \quad (3.11b)$$

where $L = i_1 \cdots i_\ell$ is a multi-index composed of ℓ multipolar indices i_1, \dots, i_ℓ ; $iL = ii_1 \cdots i_\ell$ is a multi-index with $\ell + 1$ indices. Similarly, $\partial_L = \partial_{i_1} \cdots \partial_{i_\ell}$ is the product of ℓ partial derivatives $\partial_i = \partial/\partial x^i$, and $\hat{\partial}_L = \text{STF}(\partial_{i_1} \cdots \partial_{i_\ell})$. The superscript (n) denotes n time derivatives, and an overbar indicates a PN-expanded quantity.

The fact that the multipoles are STF is crucial because it ensures that the decomposition of the metric in terms of these functions is unique. We will present how they are linked to the mass distribution of the system in Sec. 3.1.4. Now the next step is to solve (3.3) beyond the linear approximation.

3.1.2 The MPM algorithm

In Sec. 3.1.1, we give a solution of the linearized theory in terms of generic source multipole moments. The MPM algorithm allows finding the solution of Eq. (3.3) at any order in G knowing the explicit solution of the preceding orders. This procedure is iterative, so the equation that we want to solve is

$$\square h_{(n)}^{\mu\nu} = \Lambda_{(n)}^{\mu\nu}[h_{(1)}^{\mu\nu}, h_{(2)}^{\mu\nu}, \dots, h_{(n-1)}^{\mu\nu}], \quad (3.12a)$$

$$\partial_\nu h_{(n)}^{\mu\nu} = 0, \quad (3.12b)$$

assuming that $h_{(1)}^{\mu\nu}, \dots, h_{(n-1)}^{\mu\nu}$ are already known (in terms of the source multipole moments). Since the source term in Eq. (3.12a) is fully known, the solution should be $\square_{\text{ret}}^{-1} \Lambda_{(n)}^{\mu\nu}$ with the no incoming-waves condition. However, the retarded propagator cannot be taken as such because $\Lambda_{(n)}^{\mu\nu}$ diverges when $r \rightarrow 0$. Thus, one has to introduce a regularisation together with a cut-off r_0 , in the way that we define

$$\tilde{r} \equiv \frac{r}{r_0}. \quad (3.13)$$

We can now deduce an intermediate solution $u_{(n)}^{\mu\nu}$ of (3.12a), which is the regularised retarded propagator acting on the source

$$u_{(n)}^{\mu\nu} = \mathcal{FP}_{B=0} \square_{\text{ret}}^{-1} [\tilde{r}^B \Lambda_{(n)}^{\mu\nu}], \quad (3.14)$$

where B is an arbitrary complex number. The adopted regularisation is the Hadamard partie finie regularisation, detailed in Sec. 3.2.2 and Sec. 6.3.2. As we said, the introduction of this regulator comes from the fact that $\Lambda_{(n)}^{\mu\nu}$ diverges in $r \rightarrow 0$ because it is expressed in terms of the metric (3.9). Thus, one has to choose B sufficiently high to compensate the divergencies in $r \rightarrow 0$ in the explicit expressions of $k_{(1)}^{\mu\nu}$ and $\varphi_{(1)}^\mu$ in (3.10) and (3.11). Once the integration is performed, we take the limit $B \rightarrow 0$.

Now, the source term in (3.12a) is completely known in terms of the source multipole moments and is well defined. However, $u_{(n)}^{\mu\nu}$ is solution of (3.12a) but not of (3.12b) because it is generally not divergenceless due to the introduction of \tilde{r}^B . In order to build the full solution, we need to find another quantity $v_{(n)}^{\mu\nu}$ that needs to be added to $u_{(n)}^{\mu\nu}$. This quantity has to be a homogeneous solution of (3.12a) and has to compensate the divergence of $u_{(n)}^{\mu\nu}$ so that the full solution verifies the harmonic condition (3.12b). The homogeneous solution of (3.12a), namely the solution of $\square h_{(n)}^{\mu\nu} = 0$, has already been solved and the general solution is provided in (3.9). More specifically, we know the general form of $v_{(n)}^{\mu\nu}$ expressed in terms of a generic set of multipole moments, *a priori* different from $\{I_L, J_L, \dots\}$. Imposing that the divergence of $v_{(n)}^{\mu\nu}$ compensates the divergence of $u_{(n)}^{\mu\nu}$ allows to

express $v_{(n)}^{\mu\nu}$ in terms of the source multipole moments by identification. In the end the solution of Eq. (3.12) is

$$h_{(n)}^{\mu\nu} = u_{(n)}^{\mu\nu} + v_{(n)}^{\mu\nu}, \quad (3.15)$$

and once an order is known, we can reiterate the procedure to the next order. The detailed algorithm to derive $v_{(n)}^{\mu\nu}$ can be found in [80]. It is also proved in this review that at each order, $u_{(n)}^{\mu\nu}$ and $v_{(n)}^{\mu\nu}$ are analytical functions of B . Hence, we can use analytic continuation at each order so that we are allowed to repeat the above algorithm.

Finally, it is also proved in [80] that the most general solution of the harmonic coordinates Einstein field equations in the exterior zone is given by

$$h_{\text{ext}}^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu} [I_L, J_L, \dots, Z_L]. \quad (3.16)$$

In the following sections we want to link the physical source to the observables, especially the TT perturbed metric at spatial infinity. Hence, on the one hand we have to link the source multipole moments to the pseudo stress-energy tensor, which is done in Secs. 3.1.4 and 3.1.5. On the other hand, we need to link the source multipoles to the TT metric and also the energy and angular momentum fluxes in the so-called radiative zone, which is done in Sec. 3.1.3.

3.1.3 The radiative zone

At this stage, we managed to express the perturbed metric $h_{\text{ext}}^{\mu\nu}$ in terms of the source multipole moments in a general way. Now we are interested in the TT version of the perturbed metric at spatial infinity. More precisely, in the so-called radiative coordinates (T, \mathbf{X}) in which the retarded time $U \equiv T - R/c$ becomes asymptotically a null coordinate at future null infinity (where $R = |\mathbf{X}|$ is the distance to the source and T the time). It has been shown that the TT perturbed metric in the radiative coordinates can be generally written at the leading order in $1/R$ in terms of general STF multipole moments U_L and V_L as [79, 80]

$$H_{ij}^{\text{TT}}(U, \mathbf{X}) = \frac{4G}{Rc^2} \mathcal{P}_{ijab}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left[N_{L-2} U_{abL-2}(U) - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} V_{b)dL-2}(U) \right] + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (3.17)$$

where $\mathcal{P}_{ijab}(\mathbf{N})$ is the TT projector and $\mathbf{N} = \mathbf{X}/R$. Note that (3.17) defines U_L and V_L which are called *radiative multipole moments*. In the same coordinate system, we can write the energy flux of GWs $\mathcal{F} = (dE/dU)^{\text{GW}}$ in terms of those multipoles as [79]

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} U_L^{(1)} U_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)(2\ell+1)!!} V_L^{(1)} V_L^{(1)} \right]. \quad (3.18)$$

Furthermore, we can also express the angular momentum flux $\mathcal{G}_i = (dJ_i/dU)^{\text{GW}}$ as

$$\mathcal{G}_i = \epsilon_{iab} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} U_{aL-1} U_{bL-1}^{(1)} + \frac{4\ell^2(\ell+2)}{c^2(\ell-1)(\ell+1)(2\ell+1)!!} V_{aL-1} V_{bL-1}^{(1)} \right]. \quad (3.19)$$

By expressing the radiative multipole moments in terms of the source ones, we will be able to know the TT perturbed metric and the fluxes in terms of the source multipoles and thus to the physical PN source. Remark that by comparing the quadrupole formula (1.29a) to (3.18), we see that at the leading order, the radiative quadrupole can be written in terms of the source quadrupole as

$$U_{ij}(U) = I_{ij}^{(2)}(U) + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (3.20)$$

It is shown in [80] that the general structure of the perturbed metric $h_{\mu\nu}$ in the far zone can be written at the leading order in $1/r$ as

$$h_{(n)} = \sum_{\ell,p} \hat{n}^L \frac{(\ln r)^p}{r} F_{L,p}(u) + o\left(\frac{1}{r}\right), \quad (3.21)$$

where $F_{L,p}$ are functions of the retarded time in harmonic coordinates. The idea is to identify the radiative multipoles to the source ones in the TT metric. However, the presence of logarithms in h_{ij}^{TT} prevents such identification. We first have to perform a coordinate shift that breaks the harmonic condition. More precisely, we express the retarded time in radiative coordinates U in terms of the retarded time in harmonic coordinates u as

$$U = u - \frac{2GM}{c^3} \ln\left(\frac{r}{r_0}\right) + \mathcal{O}(G^2), \quad (3.22)$$

where M here denotes the Arnowitt-Deser-Misner (ADM) mass of the system.¹ It is shown in [80] that this coordinate shift absorbs all the $\ln(R)$ at each PM order in H_{ij}^{TT} . We are now able to identify the two sets of multipoles and, *e.g.* at 2PN order we have

$$U_L(U) = I_L^{(\ell)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau I_L^{(\ell+2)}(U - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \kappa_\ell \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (3.23a)$$

$$V_L(U) = J_L^{(\ell)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau J_L^{(\ell+2)}(U - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \pi_\ell \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (3.23b)$$

where κ_ℓ and π_ℓ are coefficients given by

$$\kappa_\ell = \frac{2\ell^2 + 5\ell + 4}{\ell(\ell+1)(\ell+2)} + \sum_{k=1}^{\ell-2} \frac{1}{k}, \quad (3.24a)$$

$$\pi_\ell = \frac{\ell-1}{\ell(\ell+1)} + \sum_{k=1}^{\ell-1} \frac{1}{k}. \quad (3.24b)$$

The source multipoles in (3.23) constitute the instantaneous part of the radiative moments while the other terms are the non-linear effects which depend on the past of the source. In particular, at 1.5PN the so-called tail terms appear. Very briefly, the tail terms correspond to the backscatter of the GW on the space-time background which creates a tail in the GW signal after the passage of the instantaneous part of the wave. At higher orders, other effects arise, such as tails-of-tails (when the GW scatters twice), memory effects or even tails-of-memory. Notably, the tails-of-tails appear at 3PN. In Eq. (7.20), we give a generalisation of (3.23b) up to $\mathcal{O}(1/c^7)$ (3PN) for $\ell = 2$.

3.1.4 Matching the source moments to a Post-Newtonian source

At this stage, we have an algorithm that allows to express the solution at an order n in terms of the source multipole moments defined in the linear case. This section aims at linking the source multipole moments to the source of the system itself, *i.e.* to the stress-energy tensor. We have to start with the so-called matching equation. As mentioned before, the MPM expansion is valid in the exterior zone while the PN expansion is valid in the near zone. We denote \mathcal{M} the multipolar expansion in the near zone and an overbar represents the PN expansion of a given quantity. In the buffer zone, both PN and PM expansions are valid and so

$$\mathcal{M}(h) = \bar{h}. \quad (3.25)$$

¹Normally, the scale constant entering the logarithm should be independent from r_0 but it is taken as such for simplicity. For more details, see Sec. 7.3.

In other words, in the buffer zone the MPM expansion of the gravitational field equals its PN expansion. Now, we want to give a formal link between the source moments and the pseudo-stress energy tensor. Thus, we impose that the MPM expansion of the PN expanded field equals the PN expansion of the MPM expanded field. This is translated mathematically in the matching equation

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h}). \quad (3.26)$$

This equation has to be understood in terms of formal functional identities. One can show that under this condition, the multipole expansion of the field equation outside a PN source is

$$\mathcal{M}(h^{\mu\nu}) = \mathcal{F}\mathcal{P}_{B=0} \square_{\text{ret}}^{-1} \left[\tilde{r}^B \mathcal{M}(\Lambda^{\mu\nu}) \right] - \frac{4G}{c^4} \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \mathcal{F}_L^{\mu\nu}(t - r/c) \right], \quad (3.27)$$

where $\mathcal{F}_L^{\mu\nu}$ is STF on its L indices and reads

$$\mathcal{F}_L^{\mu\nu}(u) = \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_L \int_{-1}^1 dz \delta_\ell(z) \bar{\tau}^{\mu\nu}(\mathbf{x}, u + zr/c), \quad (3.28)$$

and

$$\delta_\ell(z) = \frac{(2\ell + 1)!!}{2^{\ell+1}\ell!} (1 - z^2)^\ell, \quad \int_{-1}^1 dz \delta_\ell(z) = 1. \quad (3.29)$$

Note the presence of the regulator $\mathcal{F}\mathcal{P}$ called Hadamard Partie finie regularisation which was introduced in Sec. 3.1.2 and more details are given in Sec. 3.2.2. Furthermore, the first term in (3.27) is exactly $u^{\mu\nu}$ already defined in Eq. (3.14). At this point, we have to impose the harmonic-gauge condition on $\mathcal{M}(h^{\mu\nu})$ in order to decompose $\mathcal{F}_L^{\mu\nu}$ into a set irreducible multipole moments which are given here

$$\mathcal{F}_L^{00} = R_L, \quad (3.30a)$$

$$\mathcal{F}_L^{0i} = {}^{(+)}T_{iL} + \epsilon_{ai<i_\ell} {}^{(0)}T_{L-1>a} + \delta_{i<i_\ell} {}^{(-)}T_{L-1>,} \quad (3.30b)$$

$$\mathcal{F}_L^{ij} = {}^{(+2)}U_{ijL} + \text{STF}_L \text{STF}_{ij} [\epsilon_{aii_\ell} {}^{(+1)}U_{ajL-1} + \delta_{ii_\ell} {}^{(0)}U_{jL-1} \quad (3.30c)$$

$$+ \delta_{ii_\ell} \epsilon_{ajj_{\ell-1}} {}^{(-1)}U_{aL-2} + \delta_{ii_\ell} \delta_{jj_{\ell-1}} {}^{(-2)}U_{L-2}] + \delta_{ij} V_L. \quad (3.30d)$$

The different tensors R_L, \dots, V_L are STF and uniquely defined in terms of $\mathcal{F}_L^{\mu\nu}$. Note that this derivation has been made in 3 dimensions but had to be generalized in d dimensions (see Sec. I and II of [10]) in order to apply dimensional regularisation, discussed in Sec. 3.2.2.

3.1.5 Explicit expression of the source multipole moments for a PN source

Let us consider Eq. (3.27), on the one hand, we know from (3.16) the expression of the multipolar expansion of the perturbed metric in terms of the source multipole moments. On the other hand, from (3.28) and the decomposition (3.30a), we know the multipole moments in terms of the PN source. An identification of the sets of irreducible moments leads to the link between the source moments and the PN expansion of the pseudo stress-energy tensor $\tau^{\mu\nu}$ in the following way [76, 77]

$$\begin{aligned} I_L(t) = \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \left[\delta_\ell \hat{x}_L \Sigma - \frac{4(2\ell + 1)\delta_{\ell+1}}{c^2(\ell + 1)(2\ell + 3)} \hat{x}_{iL} \Sigma_i^{(1)} \right. \\ \left. + \frac{2(2\ell + 1)\delta_{\ell+2}}{c^4(\ell + 1)(\ell + 2)(2\ell + 5)} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right] (\mathbf{x}, t + zr/c), \quad (3.31a) \end{aligned}$$

$$J_L(t) = \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \varepsilon_{ab<i_\ell} \left[\delta_\ell \hat{x}_{L-1>a} \Sigma_b - \frac{2\ell + 1}{c^2(\ell + 2)(2\ell + 3)} \delta_{\ell+1} \hat{x}_{L-1>ac} \Sigma_{bc}^{(1)} \right] (\mathbf{x}, t + zr/c), \quad (3.31b)$$

together with the 4 gauge moments

$$W_L(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \left\{ \frac{2\ell+1}{(\ell+1)(2\ell+3)} \delta_{\ell+1} \hat{x}_{iL} \Sigma_i - \frac{2\ell+1}{2c^2(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL} \Sigma_{ij}^{(1)} \right\} (\mathbf{x}, t + zr/c), \quad (3.32a)$$

$$X_L(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \left\{ \frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL} \Sigma_{ij} \right\} (\mathbf{x}, t + zr/c), \quad (3.32b)$$

$$Y_L(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \left\{ -\delta_\ell \hat{x}_L \Sigma_{ii} + \frac{3(2\ell+1)}{(\ell+1)(2\ell+3)} \delta_{\ell+1} \hat{x}_{iL} \Sigma_i^{(1)} - \frac{2(2\ell+1)}{c^2(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} (\mathbf{x}, t + zr/c), \quad (3.32c)$$

$$Z_L(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \int_{-1}^1 dz \epsilon_{ab\langle i\ell} \left\{ -\frac{2\ell+1}{(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1)bc} \Sigma_{ac} \right\} (\mathbf{x}, t + zr/c). \quad (3.32d)$$

Here, $\hat{x}_L \equiv \text{STF}(x_{i_1} x_{i_2} \cdots x_{i_\ell})$ is the multipolar factor, the brackets surrounding indices refer to the STF projection, and the Σ 's (or their partial time-derivatives $\Sigma^{(n)}$'s), that must be evaluated at position \mathbf{x} and at time $t+z|\mathbf{x}|/c$, are defined in terms of the PN expansion of the pseudo stress-energy tensor $\bar{\tau}^{\mu\nu}$ in harmonic coordinates by

$$\Sigma \equiv \frac{\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2}, \quad \Sigma_i \equiv \frac{\bar{\tau}^{0i}}{c}, \quad \Sigma_{ij} \equiv \bar{\tau}^{ij}. \quad (3.33)$$

Recall that the overbar refers to the PN expansion. The expressions of the source moments (3.31) are formally valid up to any PN order. In practice, their PN-expanded expressions are to be computed by means of the infinite PN series²

$$\int_{-1}^1 dz \delta_\ell(z) \Sigma(\mathbf{x}, t + zr/c) = \sum_{k=0}^{+\infty} \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!} \left(\frac{r}{c}\right)^{2k} \Sigma^{(2k)}(\mathbf{x}, t). \quad (3.34)$$

These expressions are valid for any type of PN-expanded source. During the majority of my PhD, I computed the source multipole moments for different effects. Note that these expressions are given in 3 dimensions. However, as we will see in Part III, these formulae had to be derived in arbitrary d dimensions and the derivations are presented in [10]. For more details on the derivation of Eq. (3.31) and Eq. (3.32) in 3 dimensions, see [80].

3.1.6 The PN metric

The metric, in the PN approximation, is fully parametrized by a set of so-called potentials. As an example, we present here the 2PN metric expressed with respect to the 5 potentials $\{V, V_i, \hat{W}_{ij}, \hat{X}, \hat{R}_i\}$ entering at this order

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \frac{8}{c^6} \left(\hat{X} + V_i V_i + \frac{V^3}{6} \right) + \mathcal{O}\left(\frac{1}{c^8}\right), \quad (3.35a)$$

$$g_{0i} = -\frac{4V_i}{c^3} - \frac{8\hat{R}_i}{c^5} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (3.35b)$$

²The function $\delta_\ell(z)$ is defined (with $\delta(z)$ denoting the one-dimensional Dirac distribution) by

$$\delta_\ell(z) \equiv \frac{(2\ell+1)!!}{2^{\ell+1}\ell!} (1-z^2)^\ell, \quad \text{so that} \quad \int_{-1}^1 dz \delta_\ell(z) = 1 \quad \text{and} \quad \lim_{\ell \rightarrow +\infty} \delta_\ell(z) = \delta(z).$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2V}{c^2} + \frac{2V^2}{c^4} \right) + \frac{4\hat{W}_{ij}}{c^4} + \mathcal{O}\left(\frac{1}{c^6}\right). \quad (3.35c)$$

These potentials solve the flat-space wave equations (with $\square = \eta^{\mu\nu} \partial_{\mu\nu}$)

$$\square V = -4\pi G \sigma, \quad (3.36a)$$

$$\square V_i = -4\pi G \sigma_i, \quad (3.36b)$$

$$\square \hat{W}_{ij} = -4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V, \quad (3.36c)$$

$$\square \hat{R}_i = -4\pi G (V \sigma_i - V_i \sigma) - 2\partial_k V \partial_i V_k - \frac{3}{2} \partial_t V \partial_i V, \quad (3.36d)$$

$$\square \hat{X} = -4\pi G V \sigma_{kk} + 2V_k \partial_t \partial_k V + V \partial_t^2 V + \frac{3}{2} (\partial_t V)^2 - 2\partial_i V_j \partial_j V_i + \hat{W}_{ij} \partial_{ij} V, \quad (3.36e)$$

where the matter source densities are defined in terms of the components of the matter stress-energy tensor as

$$\sigma = \frac{T^{00} + T^{ii}}{c^2}, \quad \sigma_i = \frac{T^{0i}}{c}, \quad \sigma_{ij} = T^{ij}, \quad (3.37)$$

with $T^{ii} = \delta_{ij} T^{ij}$. As we can see, there is a hierarchy in the potentials. Indeed, in the case of a source made of compact objects, the simplest ones $\{V, V_i\}$ have compact support sources that are easy to integrate. For example, at Newtonian order in the metric, only the potential V is required and corresponds to the Newtonian gravitational potential (since $\square = \Delta - \partial_t^2/c^2 = \Delta + \mathcal{O}(c^{-2})$). The other potentials depend on the simpler ones, and the higher order we go, the more complex the sources of the potentials become. As we will discuss in Sec. 3.2.2, starting at 3PN we need to consider dimensional regularisation. Thus, in App. B are displayed the full d -dimensional 4PN metric as well as the required potentials.

3.2 Binary systems of compact objects

In this section, we specify the matter distribution for binary systems of compact objects. In particular, the conventions we adopt are displayed in Fig. A.1. We model our system of compact objects by point-particles given in terms Dirac distributions $\delta_A \equiv \delta^{(3)}(\mathbf{x} - \mathbf{y}_A)$. To get physical results with this model, we have to introduce also an ultraviolet regularisation. Accordingly, we have to make computations in terms of distributions and not functions.

3.2.1 Choice of stress-energy tensor

Up to now, we have shown how to link the observables, *e.g.* the GW flux, to the source multipole moments. These multipoles are expressed generally in terms of the pseudo stress-energy tensor and are valid for any type of PN source. We need to specify our model and apply it to binary systems of compact objects. This means that we have to express the form of the stress energy tensor, or, equivalently, of the matter action.

At the lowest order, we can consider that the compact objects are "points" so that their matter action would be the point-particle action

$$S_{\text{pp}} = - \sum_{A=1,2} m_A c^2 \int d\tau_A, \quad (3.38)$$

where m_A is the mass of body A and τ_A its proper time. This effective action is defined only on the worldlines of the bodies and describes the minimal coupling of matter to gravity. This action, after varying it with respect to the metric and using Eq. (1.9), leads to the following stress-energy tensor

$$T^{\mu\nu} = \sum_{A=1,2} \mu_A(t) v_A^\mu v_A^\nu \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (3.39)$$

where the three-dimensional Dirac distribution is confined to the worldline $\mathbf{y}_A(t)$ and we pose for the effective time-varying masses (with m_A the constant PN mass)

$$\mu_A(t) = \frac{m_A c u_A^0}{\sqrt{-[g]_A}}, \quad \tilde{\mu}_A(t) = \left(1 + \frac{v_A^2}{c^2}\right) \mu_A(t), \quad (3.40)$$

the worldlines are parametrized by the coordinate time $t = x^0/c$ of the harmonic coordinates; the coordinate velocities are $v_A^\mu = (c, v_A^i)$, with $v_A^i = cu_A^i/u_A^0 = dy_A^i/dt$, the relativistic Lorentz factor reads $u_A^0 = (-[g_{\mu\nu}]_A v_A^\mu v_A^\nu / c^2)^{-1/2}$ and g is the determinant of the metric. The labels A and the angular brackets $[\dots]_A$ denote that the quantities are regularized on the location of the bodies. In previous sections, we introduced an infra-red (IR) regularisation, here we introduce the ultra-violet (UV) because of the point-particle model. More details are given in Sec. 3.2.2. Considering the point particle action, the employed PN sources read

$$\sigma(\mathbf{x}, t) = \sum_A \tilde{\mu}_A(t) \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (3.41a)$$

$$\sigma_i(\mathbf{x}, t) = \sum_A \mu_A(t) v_A^i \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)), \quad (3.41b)$$

$$\sigma_{ij}(\mathbf{x}, t) = \sum_A \mu_A(t) v_A^i v_A^j \delta^{(3)}(\mathbf{x} - \mathbf{y}_A(t)). \quad (3.41c)$$

In the effective approach we can go beyond the minimal coupling to gravity approximation and consider other effects, such as spins or finite-size effects. Note that even if we consider other effects, the stress-energy tensor would still be proportional to (derivatives of) Dirac distributions. In Sec. 4.1, we discuss the effective action for finite-size effects in systems of compact objects, which describes the tidal interactions between the bodies (in the adiabatic limit).

3.2.2 The regularisation schemes

In the PN-MPM formalism, we have to deal with two types of divergencies: the UV and the IR. However, they are required for fundamentally different reasons. On the one hand, the compact bodies are considered as points so the UV divergencies come from the model itself because of the infinite value of self-field of point particles. If we were to employ a model of a perfect fluid for example, no divergencies would appear and thus no UV regularisation would be needed. On the other hand, the IR divergencies come from the formalism itself because we define quantities (such as the source multipole moments) as integrals over all space of integrands only valid in the near zone (*e.g.* the PN expansion of the pseudo stress-energy tensor). The UV and IR divergencies, being of very different nature, can in principle be treated with two different regularisations. Originally, the PN-MPM formalism has been developed using the Hadamard partie finie regularisation [71, 81] for both IR and UV divergencies. However at 3PN, the EoM in this approach contained an ambiguity [82] and it turned out that the problem came from the Hadamard regularisation. In particular, this regularisation breaks the diffeomorphism invariance. The choice was made to use dimensional regularisation (DR) instead of the Hadamard regularisation in the UV. This choice comes from the fact that DR has been well known essentially because of its use in QFT. This choice removed the ambiguity in the EoM at 3PN [83]. Nonetheless, Hadamard was still employed for the IR. At 4PN in the conservative sector, the computation with Hadamard regularisation in the IR also failed so DR has been used to derive the EoM. By consistency, we have to use this regularisation in the radiative sector. Thus, we now use full DR at high order. Moreover, we have recently shown that employing dimensional or Hadamard regularisations for the treatment of IR divergencies at 3PN were strictly equivalent, see Secs. 6.4.6, 6.4.7 and 7.2.4.

The computations using pure DR are more complicated than the ones using Hadamard and some of them are not doable in pure DR. For example, in d dimensions, we cannot derive the analytic

expressions of some potentials. Thus, in practice we first perform the computations in $3d$ and then compute the difference between the two regularisations. Saying that the computations are performed in $3d$ means that we employ Hadamard partie finie. Furthermore, the DR scheme is also called, notably in Chapter 6, the $B\varepsilon$ procedure (see Sec. 6.4.3 for more details on this procedure). The details of these regularisations are presented in Sec. 6.3 for Hadamard and dimensional regularisation in the UV and in Sec. 6.4 in for dimensional regularisation in the IR.

3.3 Practical computations

As mentioned before, we want to compute the phase and amplitude of GWs. It is sufficient to know the radiative multipoles in order to derive the amplitude $h_{+, \times}$. However, to compute the phase in the quasi-circular orbits approximation, we need to take into account the balance equation for the energy

$$\langle \mathcal{F} \rangle = -\frac{dE}{dt}, \quad (3.42)$$

where the brackets $\langle \dots \rangle$ represent the mean value over an orbital period. We can decompose the problem in a conservative and dissipative part. Therefore in practice, we always perform two derivations: first in the conservative sector, where we derive the EoM and conserved quantities, and notably the energy; second the flux through the computation of the radiative multipole moments. However, the emission of GW alters the conservative EoM. This effect arises at 2.5PN and is inserted in the EoM *a posteriori* by a radiation-reaction force. The treatment of the conservative sector, notably using an effective Lagrangian approach, is discussed in 4.2.

If we were to relax the quasi-circular orbits approximation, and thus consider eccentricities, we would also have to consider the balance equation for the angular momentum

$$\langle \mathcal{G}_i \rangle = -\frac{dJ_i}{dt}. \quad (3.43)$$

As for the balance equation for the energy, the LHS is computed in the radiative sector by means of multipole moments and the RHS is computed in the conservative sector.

In practice, the results of the flux and energy are in the end written in terms of the PN parameter x defined as

$$x = \left(\frac{Gm\omega}{c^3} \right)^{2/3}, \quad (3.44)$$

where $m = m_1 + m_2$ is the total mass of the system and ω is the orbital frequency. Note that it is very convenient to write the final observables in terms of x because it is a 1PN quantity.

To compute the flux, we need the radiative multipole moments (see Eq. (3.18)). They can be computed by knowing the source multipole moments and the non-linear effects (themselves derived using the source multipoles). These source moments are integrals of the potentials and the PN expanded stress-energy tensor. Since the potentials are sourced by the stress-energy tensor, there is a straightforward way to do the computations. As it was mentioned in Sec. 3.2.2, the regularisation that we use starts to be incorrect at 3PN and we have to use DR. Since DR is more complicated than the Hadamard Partie finie, in practice we perform the computation in $3d$ and then compute the difference between dimensional and Hadamard regularisations. The procedure to compute the GW flux is as follows, we compute the:

1. Stress-energy tensor decomposed in 3+1: $\{T^{00}, T^{0i}, T^{ij}\}$
2. Potentials $\{V, V_i, \hat{W}_{ij}, \dots\}$
 - \hookrightarrow 3d computation
 - \hookrightarrow Difference with dimensional regularisation

3. Source multipole moments $\{I_L, J_L, \dots\}$
 - \hookrightarrow 3d computation
 - \hookrightarrow Difference with dimensional regularisation
4. Non-linear effects (tails, memory...)
 - \hookrightarrow 3d computation
 - \hookrightarrow Difference with dimensional regularisation
5. Radiative multipole moments $\{U_L, V_L\}$
6. Flux

In some intermediate computations, we obtain some poles in $\frac{1}{d-3}$ but since U_L and V_L are observables, the poles have to vanish because the physical result should not depend on the regularisation method. Step 5, in which we add the source multipole moments to the non-linear effects, establishes a very good test on the explicit results. Indeed, the fact that the different cut-offs are compensated in the radiative moments is absolutely not trivial since we introduced them on the one hand in the conservative sector, through the EoM (needed to derive the source multipoles); and on the other hand in the radiative sector. In the end, when we have the flux and the energy, we can insert their values in Eq. (3.42). In the case where we perform the quasi-circular orbit approximations, we are able to derive the GW phase

$$\varphi = \int dt \omega = \int \omega(x) \frac{dE/dx}{dE/dt} dx = -\frac{c^3}{Gm} \int x^{3/2} \frac{dE/dx}{\langle \mathcal{F}(x) \rangle} dx, \quad (3.45)$$

where x is the PN parameter defined in (3.44).

This formalism is now going to be applied to the problem of tidal effects, presented in Part II and to the computation of the 4PN flux for point particles in Part III.

Part II

Tidal effects

Introduction

The discovery of GWs generated by the inspiral and merger of two NSs [17, 84] marked a breakthrough in fundamental physics, notably by allowing for the first time a direct constraint on the equation of state (EoS) of cold matter at supranuclear densities deep inside NSs. This important test excluded some of the stiffest EoS, for which the pressure increases a lot for a given increase in density, and which therefore offer more resistance to the gravitational collapse, resulting in a NS that is less compact. This finding is consistent with known constraints on the radius of NSs from electromagnetic-based observations [85]. However, the majority of soft EoS, which are more easy to compress and thus predict a more compact NS, is still viable [86] (see also [87, 46] for reviews). Studying tidal effects also permits to distinguish between BHs, NSs or, possibly, more exotic entities like boson stars [88, 87].

The tidal interaction affects both the conservative equations of motion (EoM) and the GW emission of the compact binary system. This results in a modification of the time evolution of the binary's orbital frequency and phase which is directly observable (see *e.g.* [89, 90, 91, 92]). The tidal distortion depends on the Love numbers [93], characterizing the rigidity and the deformability of the body, *i.e.* its capacity to change shape under the influence of an external tidal field. Those Love numbers depend in turn on the internal EoS of the body, which is uncertain at high densities [94, 95]. They decrease as the compactness of the body increases, reaching zero in the limit of a non-spinning maximally compact object, *i.e.*, for a Schwarzschild BH [96, 97, 98].

The leading tidal contributions to the orbital dynamics are due to quadrupolar deformations and, for compact binaries, manifest themselves as very small formal corrections in the accelerations, of the order of 5PN or $\sim (v/c)^{10}$, where v denotes the relative orbital velocity. However, the coefficient appearing in front of the small 5PN factor $(v/c)^{10}$ can be quite large and the effect is measurable.³ It scales like the dimensionless parameter

$$\Lambda^{(2)} = \frac{2}{3} k^{(2)} \left(\frac{Rc^2}{Gm} \right)^5, \quad (3.46)$$

where $k^{(2)}$ denotes the mass-type quadrupolar second Love number of the body, while m and R represent its mass and radius. Typically, the compactness parameter $\mathcal{C} = Gm/(Rc^2)$ is of order 0.15 for neutron stars while the Love number is $k^{(2)} \sim 0.1$ (depending on the EoS) [97, 98], hence we expect $\Lambda^{(2)} \sim 1000$. With the binary neutron star event GW170817 [17], the detectors LIGO and Virgo have already been able to put an observational constraint on a particular combination of $\Lambda_1^{(2)}$, $\Lambda_2^{(2)}$ and the masses that enter the orbital phase evolution of the two neutron stars [90, 92]. As already said, this constraint permitted excluding some of the stiffest EoS, for which the neutron stars are less compact [99, 28].

During the inspiral phase of coalescing NS binaries the orbital dynamics is dominated by point-mass contributions and the waveform is essentially identical to that of black holes; but closer to the merger small corrections arise due to the finite-size effects of NS. These can be described by resorting to a tidal expansion in the small parameter $\sim R_A/r_{AB}$, where R_A is the size of one of

³One can speculate that the tidal 5PN coefficient is larger than the purely orbital 5PN contribution to the orbital phase for point particles, which is currently unknown.

the NS and r_{AB} is the typical orbital separation. The tides arise from the response of the NS to the gradient of the companion's gravitational field across the matter distribution. The tidal expansion is a multipole expansion with the tidal mass quadrupole moment of the object dominant, and higher mass- or current-type moments sub-dominant. The deformation and finite size effects are parametrized by a series of coefficients associated with each multipole moments and referred to as the tidal deformabilities (or polarizabilities) of the NS.

For GW detectors, the main observable is the so-called binary's chirp, *i.e.*, the time evolution of the compact binary's orbital frequency $\omega(t)$ and phase $\varphi(t) = \int dt \omega(t)$ through GW radiation reaction during the inspiral. The detectors are sensitive to some particular combination of the two deformabilities of the NSs and the two masses that enters the binary's chirp. To the lowest tidal mass quadrupole order, the chirp is given by the combination of the two relations,

$$x = \frac{1}{4} \theta^{-1/4} \left[1 + \frac{39}{8192} \tilde{\Lambda}^{(2)} \theta^{-5/4} \right], \quad (3.47a)$$

$$\varphi = \varphi_0 - \frac{x^{-5/2}}{32\nu} \left[1 + \frac{39}{8} \tilde{\Lambda}^{(2)} x^5 \right], \quad (3.47b)$$

where $\nu = m_1 m_2 / m^2$ is the symmetric mass ratio, $m = m_1 + m_2$ is the total mass, and where we use the dimensionless frequency $x \equiv (\frac{Gm\omega}{c^3})^{2/3}$ and time $\theta \equiv \frac{\nu c^3}{5Gm} (t_c - t)$ variables, with t_c denoting the instant of coalescence — at which the distance between the particles formally vanishes while the frequency diverges —, φ_0 being an reference constant phase. The most commonly used approximant for GW data analysis in the Fourier domain is defined by using the stationary phase approximation (SPA), for which the phase of the dominant mode at twice the orbital frequency reads

$$\psi = 2\pi f t_c - 2\varphi_0 - \frac{\pi}{4} + \frac{3x^{-5/2}}{128\nu} \left[1 - \frac{39}{2} \tilde{\Lambda}^{(2)} x^5 \right]. \quad (3.48)$$

Since x is a small PN parameter of the order of $\mathcal{O}(c^{-2})$, we see that the effect of the internal structure of NS (in the non-spinning case) is comparable to relativistic orbital effects occurring at the 5PN order beyond the point-particle contribution computed with the usual Einstein quadrupole formula. Of course, the latter estimate is just formal, since we have to take into account, besides the small factor x^5 , the numerical value of the 5PN coefficient parametrizing the finite-size effect in (3.47)–(3.48).

The tidal polarizabilities (3.49) are physical parameters to the extent that they directly parametrize the effective matter action (5.1)–(5.2) we adopt in this work, following Refs. [100, 6], as an efficient and elegant tool to describe tidal effects in the case of compact bodies. In Chapters 4 and 5, we analyze the tidal response of NS binaries and the modification of the GW phase to higher order, corresponding to mass quadrupole, current quadrupole and mass octupole tidal interactions. Accordingly, we introduce three tidal polarizability coefficients, conveniently denoted, using standard normalization [94, 95], as

$$G\mu_A^{(2)} \equiv \left(\frac{Gm_A}{c^2} \right)^5 \Lambda_A^{(2)} = \frac{2}{3} k_A^{(2)} R_A^5, \quad (3.49a)$$

$$G\sigma_A^{(2)} = \frac{1}{48} j_A^{(2)} R_A^5, \quad (3.49b)$$

$$G\mu_A^{(3)} = \frac{2}{15} k_A^{(3)} R_A^7, \quad (3.49c)$$

and related to corresponding relativistic generalizations $k_A^{(2)}$, $j_A^{(2)}$ and $k_A^{(3)}$ of Love numbers for the mass quadrupole, current quadrupole and mass octupole moments of the body, with R_A denoting its radius in a coordinate system such that the area of the sphere of radius R_A is $4\pi R_A^2$.

Note that we consider *static* tides, however, recent work [101] have shown that dynamical tides coupled to spins can induce resonances in binary NS systems that are not negligible compared to the leading tidal deformation $\Lambda^{(2)}$.

Chapter 4

Conservative sector

This chapter aims at computing the EoM and conserved quantities up to next-to-next-to-leading order or NNL/2PN order (meaning 2PN beyond leading order) for tidal effects. It is organized as follows. In Sec. 4.1, we define the general effective action with appropriate non-minimal matter couplings describing finite-size effects. In Sec. 4.2, we give some general methods that were used at the level of the action and Lagrangian to derive the EoM and conserved quantities. The quantities entering this action are determined by the 2PN metric, presented in Sec. 4.3.1 and computed off-shell, *i.e.*, without replacement of accelerations by the EoM. The final Lagrangian, accurate to NNL/2PN order, is displayed in Sec. 4.3, together with the associated NL/1PN center-of-mass (CoM) position. We then derive, in Sec. 4.4, the tidal dynamics in the CoM frame for general orbits. From the Lagrangian derived in the CoM frame we deduce the associated Hamiltonian in isotropic coordinates in Sec. 4.5. We then provide the EoM as well as the conserved quantities for quasi-circular orbits in Sec. 4.6. In App. C we show that the tidal multipole moments up to the NNL/2PN order can be defined equivalently by means of either the Riemann tensor or the Weyl tensor. Finally, we give in App. D.1 the complete tidal acceleration in a general frame for arbitrary orbits to NNL/2PN order. This chapter is based on the two papers [6, 8].

4.1 The effective matter action for static tides

4.1.1 Generalities

As mentioned in Sec. 3.2.1, we model the compact objects as point-like sources using the point-particle action. This action is an effective action describing the minimal coupling of matter to gravity. The first step is to go beyond the minimal coupling in order to define finite-size effects in the action, called non-minimal action S_{nm} . This is done using effective field theory. Note that in Part II, we consider *static tides*, meaning that we assume that each body always stays in static equilibrium, and we neglect their coupling to spins. In order to define such an action, we need to introduce a local inertial coordinate frame along each body worldline, together with the associated local tetrad $e_{\hat{\alpha}}^{\mu}$ where the hat indices denote the tetradic components. More details on the choice of tetrad are given in Sec. 5.1.3. In its own local frame, the body feels the tidal multipole moments generated by the other bodies at its very location, namely the ℓ -th order mass-type moments $G_{\hat{L}}$ and the current-type ones $H_{\hat{L}}$, where those quantities refer to the spatial tetradic components of the moments, *i.e.* projected along the local tetrad, with $\hat{L} = \hat{i}_1 \cdots \hat{i}_{\ell}$ denoting a multi-spatial index composed of ℓ spatial tetradic indices.

The internal structure is then entirely determined by the mass and the EoS. Thus, the elementary bricks that are allowed to construct S_{nm} are tensors defined from the metric only and evaluated at the given particle position, with all indices contracted so as to preserve the invariance under rotation and parity in the corresponding constant-time hypersurface of the local rest frame. In our case, it is

sufficient to consider the same non-minimal terms as in Ref. [102], built from quadratic couplings in the tidal moments \hat{G}_L and \hat{H}_L . Hence the form of the matter action (adding also the particle's label $A \in \{1, 2\}$)¹

$$S_{\text{nm}} = \sum_A \sum_{\ell \geq 2} \frac{1}{2\ell!} \left[\mu_A^{(\ell)} \int d\tau_A \left(G_L^A(\tau_A) \right)^2 + \frac{\ell}{(\ell+1)c^2} \sigma_A^{(\ell)} \int d\tau_A \left(H_L^A(\tau_A) \right)^2 + \frac{\mu_A'^{(\ell)}}{c^2} \int d\tau_A \left(\dot{G}_L^A(\tau_A) \right)^2 + \frac{\ell}{(\ell+1)c^4} \sigma_A'^{(\ell)} \int d\tau_A \left(\dot{H}_L^A(\tau_A) \right)^2 + \dots \right]. \quad (4.1)$$

The ellipsis indicates many higher-order non-linear combinations of the tidal moments and their covariant (proper-time) derivatives, which we do not need to include here (see *e.g.* Eq. (2.3) of [100]). For more insight and motivation about the non-minimal action, see Refs. [103, 104, 105, 100] and the treatment of tidal effects in the Newtonian model as recalled in App. A of [6].

The above tidal moments are given by appropriate covariant derivatives of the Weyl tensor. We define first the spatial tetradic components of the moments appearing in Eq. (4.11) below (for $\ell \geq 2$) as

$$G_{\hat{L}}^A = -c^2 \left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C_{\hat{i}_{\ell-1} \hat{0} \hat{i}_\ell} \rangle_A \right], \quad (4.2a)$$

$$G_{\hat{L}}^A = 2c^3 \left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C_{\hat{i}_{\ell-1} \hat{0} \hat{i}_\ell}^* \rangle_A \right]. \quad (4.2b)$$

The angle brackets over the ℓ free spatial indices $\hat{L} = \hat{i}_1 \cdots \hat{i}_\ell$ of the above tensor expressions means that they must be replaced by their symmetric and trace-free (STF) parts over those indices, the underlined indices being excluded from the STF projection. We denote by $\nabla_{\hat{\alpha}}$ the usual covariant tetradic derivative (we pose $\hat{\alpha} = (\hat{0}, \hat{i})$), whereas $C_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ and $C_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}^*$ represent the tetradic components of the Weyl tensor (whose definition is recalled in Eq. (C.2)) and its dual respectively.² By construction, the tidal moments (4.2) are symmetric over their spatial indices \hat{L} and all their traces are zero, *i.e.*, $\delta_{\hat{i}_1 \hat{i}_2} G_{\hat{i}_1 \hat{i}_2 \cdots \hat{i}_\ell} = 0$.

Next, we introduce the covariant versions of the previous tidal tensors. Since $u^\mu = e_0^\mu$, this is achieved by imposing that they live in the particle's local spatial hypersurface, which is orthogonal to the four velocity. Thus, we complete the definition of the tidal moments (4.2) by requiring them to obey

$$G_{\hat{0}\hat{\alpha}_2 \cdots \hat{\alpha}_\ell}^A = H_{\hat{0}\hat{\alpha}_2 \cdots \hat{\alpha}_\ell}^A = 0. \quad (4.3)$$

In this way, $G_{\hat{\alpha}_1 \cdots \hat{\alpha}_\ell}$ and $H_{\hat{\alpha}_1 \cdots \hat{\alpha}_\ell}$ are both Lorentz tensors and covariant scalars, while their covariant versions in an arbitrary coordinate system $\{x^\mu\}$ read

$$G_{\mu_1 \cdots \mu_\ell}^A = -c^2 \left[\nabla_{\langle \mu_1}^\perp \cdots \nabla_{\mu_{\ell-2}}^\perp C_{\mu_{\ell-1} \rho \mu_\ell} \rangle_\sigma \right]_A u_A^\rho u_A^\sigma, \quad (4.4a)$$

$$H_{\mu_1 \cdots \mu_\ell}^A = 2c^3 \left[\nabla_{\langle \mu_1}^\perp \cdots \nabla_{\mu_{\ell-2}}^\perp C_{\mu_{\ell-1} \rho \mu_\ell}^* \rangle_\sigma \right]_A u_A^\rho u_A^\sigma. \quad (4.4b)$$

¹The constant mass of body A is denoted m_A and its proper time $d\tau_A = (-[g_{\mu\nu}]_A dy_A^\mu dy_A^\nu / c^2)^{1/2}$, where $y_A^\mu(\tau_A)$ is the particle's worldline. The four velocity $u_A^\mu = dy_A^\mu / (c d\tau_A)$ is such that $[g_{\mu\nu}]_A u_A^\mu u_A^\nu = -1$, with $[g_{\mu\nu}]_A$ denoting the metric regularized at the location of body A ; this is of course nothing but the time-time component of the orthonormalizing condition of the tetrad, $\eta_{\hat{\alpha}\hat{\beta}} = [g_{\mu\nu}]_A e_{\hat{\alpha}}^{A\mu} e_{\hat{\beta}}^{A\nu}$.

²In our convention, $C_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}^* \equiv \frac{1}{2} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} C_{\hat{\gamma}\hat{\delta}}^{\hat{\eta}\hat{\zeta}}$ or, in covariant form, $C_{\mu\nu\rho\sigma}^* \equiv \frac{1}{2} \varepsilon_{\mu\nu\lambda\kappa} C^{\lambda\kappa}_{\rho\sigma}$, where $\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ denotes the tetradic components of the completely anti-symmetric Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$, defined by $\varepsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = 1$ and $\varepsilon_{0123} = \sqrt{-g}$. The tetradic covariant derivative obeys, *e.g.*, $\nabla_{\hat{\alpha}} V^{\hat{\beta}} = e_{\hat{\alpha}}^\mu e^{\hat{\beta}}_\nu \nabla_\mu V^\nu$.

Here, we denote $\nabla_\mu^\perp = \perp_\mu^\nu \nabla_\nu$, with $\perp_\mu^\nu = \delta_\mu^\nu + u_\mu u^\nu$ being the projector onto the hypersurface orthogonal to the four velocity (notice that $\perp_{\hat{\alpha}}^\mu = (0, e_i^\mu)$). Also, the operators ∇^\perp commute, and the operator $\nabla_{\mu_1}^\perp \dots \nabla_{\mu_\ell}^\perp$ has to be understood as $\perp_{\mu_1}^{\nu_1} \dots \perp_{\mu_\ell}^{\nu_\ell} \nabla_{\nu_1} \dots \nabla_{\nu_\ell}$. The tidal moments are both STF over all their space-time indices and transverse to the four velocity, namely $u^\mu G_{\mu\mu_2\dots\mu_\ell} = u^\mu H_{\mu\mu_2\dots\mu_\ell} = 0$, which is equivalent to (4.3).

Very important to the formalism is the fact that the Weyl tensor and its covariant derivatives in (4.4) are to be evaluated at the location of the particle A following the regularisation, as indicated by the square brackets $[\dots]_A$. Physically, the regularisation is crucial because it removes the self field of the particle A , and therefore permits automatically selecting the external (tidal) field due to the other particles $B \neq A$. We know one regularisation able to give a complete, consistent and physical answer in high PN approximations, namely DR (see *e.g.* [106, 83]). In our practical calculations at the relatively low NNL/2PN order, it is simpler to use the Hadamard partie finie regularisation, since it has been shown [100] to be equivalent to DR (see also discussions in Ref. [83]).

As the tidal moments are transverse to the velocity, the action (4.1) can be rewritten in covariant form as

$$S_{\text{nm}} = \sum_A \sum_{\ell \geq 2} \frac{1}{2\ell!} \left[\mu_A^{(\ell)} \int d\tau_A G_{\alpha_1 \dots \alpha_\ell}^A G_A^{\alpha_1 \dots \alpha_\ell} + \frac{\ell}{(\ell+1)c^2} \sigma_A^{(\ell)} \int d\tau_A H_{\alpha_1 \dots \alpha_\ell}^A H_A^{\alpha_1 \dots \alpha_\ell} + \dots \right]. \quad (4.5)$$

We observe that the reference to the local tetrad has completely disappeared from the action. For convenience, we will work only with the global (tensorial) components $G_{\mu_1 \dots \mu_\ell}$ and $H_{\mu_1 \dots \mu_\ell}$ of the moments henceforth.

The coefficients $\mu^{(\ell)}$ and $\sigma^{(\ell)}$ entering the non-minimal action characterize the deformability and polarizability of the body under the influence of the external tidal field. They are linked to the dimensionless mass-type $k^{(\ell)}$ and current-type $j^{(\ell)}$ second Love numbers as [100]

$$G\mu_A^{(\ell)} = \frac{2}{(2\ell-1)!!} k_A^{(\ell)} R_A^{2\ell+1}, \quad G\sigma_A^{(\ell)} = \frac{\ell-1}{4(\ell+2)(2\ell-1)!!} j_A^{(\ell)} R_A^{2\ell+1}, \quad (4.6)$$

where R_A is the radius of body A. In the effective description Eq. (4.5) of compact objects, only the coefficients $\mu_A^{(\ell)}$ and $\sigma_A^{(\ell)}$ are measurable. The normalization constants in $\mu_A^{(\ell)}$ are chosen to match usual Newtonian definitions.

The polarizability coefficients (4.6) actually determine the formal PN order at which the tidal effects appear. For compact objects the compactness parameter defined as the ratio $\mathcal{C} = Gm/(Rc^2)$ is of the order of one. Inserting $\mathcal{C} \sim 1$ in (4.6), we recover the fact that the dominant tidal effect is due to the mass quadrupole and is formally of the order of

$$\epsilon_{\text{tidal}} = \frac{1}{c^{10}}, \quad (4.7)$$

i.e., is comparable to a 5PN orbital effect. With the notation (4.7) for the dominant effect, we see that the deformability coefficients in the action scale like

$$\{\mu_A^{(\ell)}, \sigma_A^{(\ell)}\} = \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^{4\ell-8}}\right). \quad (4.8)$$

We are now able to truncate the action (4.5) in order to consider only the quantities contributing up to NNL/2PN.

4.1.2 The total 2PN action for static tides

To sum up, the total action that we consider is then

$$S = S_g + S_m, \quad (4.9)$$

where the gravitational part S_g is the standard Einstein-Hilbert action, to which we add the appropriate harmonic-gauge fixing term

$$S_g = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu \right], \quad (4.10)$$

with $\Gamma^\mu = g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu$. Then the matter part up to NNL/2PN is

$$S_m = \sum_A \int d\tau_A \left[-m_A c^2 + \frac{\mu_A^{(2)}}{4} G_{\mu\nu}^A G^{\mu\nu}_A + \frac{\sigma_A^{(2)}}{6c^2} H_{\mu\nu}^A H^{\mu\nu}_A + \frac{\mu_A^{(3)}}{12} G_{\lambda\mu\nu}^A G^{\lambda\mu\nu}_A + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right) \right]. \quad (4.11)$$

As discussed above, the tidal tensors are normalized in such a way that their leading order in the PN expansion is 0. So the order of the different contributions are determined by the tidal polarizabilities whose orders are given in Eq. (4.8). In particular, the only tidal polarizabilities that have an order lower than $1/c^{14}$ are

$$\mu_A^{(2)} \sim \sigma_A^{(2)} \sim \frac{1}{c^{10}} = \epsilon_{\text{tidal}}, \quad \mu_A^{(3)} \sim \frac{1}{c^{14}} = \frac{\epsilon_{\text{tidal}}}{c^4}. \quad (4.12)$$

Then the relevant tidal tensors, generally defined in Eq. (4.4), are given by

$$G_{\mu\nu}^A = -c^2 R_{\mu\rho\nu\sigma}^A u_A^\rho u_A^\sigma, \quad (4.13a)$$

$$H_{\mu\nu}^A = 2c^3 R_{(\mu\rho\nu)\sigma}^{*A} u_A^\rho u_A^\sigma, \quad (4.13b)$$

$$G_{\lambda\mu\nu}^A = -c^2 \nabla_{(\lambda}^\perp R_{\mu\rho\nu)\sigma}^A u_A^\rho u_A^\sigma. \quad (4.13c)$$

The Riemann tensor and its dual are evaluated at point A following the Hadamard regularisation, and we denote $\nabla_\lambda^\perp R_{\mu\nu\rho\sigma}^A \equiv (\nabla_\lambda^\perp R_{\mu\nu\rho\sigma})_A$ its projected covariant derivative evaluated at point A , with $(\nabla_\lambda^\perp)_A = (\perp_\lambda^\kappa \nabla_\kappa)_A$ with $(\perp_\lambda^\nu)_A = (\delta_\lambda^\nu + u_\lambda u^\nu)_A$. Note that they are defined with respect to the Riemann tensor and not the Weyl tensor. In App. C, I show the equivalence between these two definitions. Furthermore, replacing the STF operator by the symmetrization operator in the definitions (4.4) for the mass quadrupole, current quadrupole and mass octupole moments does not affect the values of those tensors. The resulting expressions are simpler than the original formulae. However, the off-shell mass-type tidal moments defined in this manner are no longer trace-free, contrary to their Weyl counterparts.

4.2 General methods

Now that the matter action is defined, we wish to compute its associated Lagrangian in order to derive the EoM and conserved quantities. In this section, we show some methods to simplify the problem.

4.2.1 The Fokker method

Let us consider an action S that is a functional of the metric $g_{\mu\nu}$ and the dynamical variables of the system $q = \{\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)}\}$. The Fokker method is a way of defining another equivalent action in a perturbative problem. It consists in the following steps:

1. we start from the total action $S = S_g + S_m = S[q, g_{\mu\nu}]$
2. we solve the Einstein field equations resulting from the metric variation. This yields a solution $g_{\mu\nu}^{(\text{sol})}[q]$
3. we insert $g_{\mu\nu}^{(\text{sol})}$ back into S , which defines the Fokker action $S_F[q] \equiv S[q, g_{\mu\nu}^{(\text{sol})}[q]]$

Moreover, in our problem the so-called solution metric $g_{\mu\nu}^{(\text{sol})}$ is the PN metric which is already known in terms of potentials up to 4PN. This method, quite simple in practice, drastically simplifies the problem as shown right below.

4.2.2 Properties of the Fokker action

The Fokker action has two powerful properties:

1. the dynamics for the matter is identical to that of the original action $\delta S = 0 \Leftrightarrow \delta S_F = 0$
2. in a perturbative scheme, the Fokker action in which we insert the first order metric solution is equivalent to the Fokker action in which we insert zero-th order metric solution up to first order

Proof of 1.

Recall that $g_{\mu\nu}^{(\text{sol})}$ satisfies $\frac{\delta S}{\delta g_{\mu\nu}}[q, g_{\mu\nu}^{(\text{sol})}] = 0$. By varying the Fokker action we get

$$\begin{aligned} \delta S_F &= \int d^4x \left[\left(\frac{\delta S}{\delta q} \right) [q, g_{\mu\nu}^{(\text{sol})}] \delta q + \underbrace{\left(\frac{\delta S}{\delta g_{\mu\nu}} \right) [q, g_{\mu\nu}^{(\text{sol})}]}_{0 \text{ by definition}} \frac{\delta g_{\mu\nu}}{\delta q} \delta q \right] \\ &= \int d^4x \left(\frac{\delta S}{\delta q} \right) [q, g_{\mu\nu}^{(\text{sol})}] \delta q. \end{aligned} \quad (4.14)$$

If $\delta S = 0$, then $\frac{\delta S}{\delta q}[q, g_{\mu\nu}^{(\text{sol})}] = 0$ and so $\delta S_F = 0$ and reciprocally. In the end $\delta S = 0 \Leftrightarrow \delta S_F = 0$.

Proof of 2.

Let us consider the action of the form $S[q, g_{\mu\nu}] = S^{(0)}[q, g_{\mu\nu}] + \varepsilon S^{(1)}[q, g_{\mu\nu}]$, where $\varepsilon \ll 1$. In our case, $\varepsilon = \varepsilon_{\text{tidal}}$ because tidal effects are at least a 5PN contribution. Then the metric can also be split in two parts $g_{\mu\nu}^{(\text{sol})} = g_{\mu\nu}^{(\text{sol})0} + \varepsilon g_{\mu\nu}^{(\text{sol})1}$. Then we can compute the difference between the Fokker action defined right above with the Fokker action in which we replaced the metric by the 0th order one

$$\begin{aligned} S_F[q] - S[q, g_{\mu\nu}^{(\text{sol})0}] &= S[q, g_{\mu\nu}^{(\text{sol})}] - S[q, g_{\mu\nu}^{(\text{sol})0}] \\ &= \int d^4x \underbrace{\left(\frac{\delta S}{\delta g_{\mu\nu}} \right) [q, g_{\mu\nu}^{(\text{sol})0}]}_{\mathcal{O}(\varepsilon) \text{ by definition}} \varepsilon g_{\mu\nu}^{(\text{sol})1} + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{O}(\varepsilon^2), \end{aligned} \quad (4.15)$$

which shows that S_F and $S[q, g_{\mu\nu}^{(\text{sol})0}]$ are equivalent up to $\mathcal{O}(\varepsilon)$.

In the case of tidal effects, the 0th order metric is the PN point-particle part and the perturbation is the tidal one. We want to compute the EoM and conserved quantities including tidal effects up to NNL/2PN, which corresponds to a 7PN effect ($1/c^{14}$). Since $\mathcal{O}(\varepsilon_{\text{tidal}}^2) = 1/c^{20}$, it is amply sufficient to inject only the point-particle part of the metric in our action. The same argument has also been shown and used in Sec. II.E of Ref. [100].

To sum up, our starting action to derive the Lagrangian is $S_g + S_m$ (given in Eq. (4.10) and (4.11)) in which we replace the metric by the already known point-particle PN metric. Even though we do not control entirely the PN metric at the requested order (7PN), we have shown that the matter dynamics using this action is equivalent to the original one. Once we have this particular Fokker action $S'_F[q] = S[q, g_{\mu\nu}^{\text{pp}}[q]]$, it is straightforward to compute the associated Fokker Lagrangian $L_F = L_F(\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)})$.

4.2.3 The reduction method

At this stage, we know the Fokker Lagrangian $L_F(\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)})$, which is a generalized Lagrangian. We want to reduce its expression by finding another Lagrangian that gives equivalent

matter dynamics and limiting as much as possible the number of derivatives of the accelerations while remaining in the harmonic coordinates. The reason why we want to reduce the Lagrangian is that it becomes simpler to derive the conserved quantities (see Sec. 4.3.3). It is possible [107] to manage that the new Lagrangian then depends on the positions, velocities, accelerations and merely is linear in the accelerations. We generally cannot reduce it further without losing the harmonic gauge condition. The method is the following

1. We replace each derivative of the acceleration appearing in the Lagrangian by the sum of its on shell value $\vec{A}_A^{(k)} = \vec{A}_A^{(k)}(\vec{y}_A, \vec{v}_A)$ and a parameter $\delta\vec{a}_A^{(k)}$, namely $\vec{a}_A^{(k)} = \vec{A}_A^{(k)} + \delta\vec{a}_A^{(k)}$. Note that by definition, $\delta\vec{a}_A^{(k)}|_{\text{on shell}} = 0$.
2. After the first step we end up with three types of terms:
 - The ones that do not contain any $\delta\vec{a}_A^{(k)}$, these ones are to be left unaltered.
 - The ones that are linear in $\delta\vec{a}_A^{(k)}$ which we need to keep.
 - The ones at least quadratic in $\delta\vec{a}_A^{(k)}$, called double-zero terms. We can drop them because they do not affect the EoM. Indeed, the variation of a product of $\delta\vec{a}_A^{(k)}$ is of the form $\delta(\delta\vec{a}_A^{(k_1)})\delta\vec{a}_A^{(k_2)}|_{\text{on shell}} + \delta(\delta\vec{a}_A^{(k_2)})\delta\vec{a}_A^{(k_1)}|_{\text{on shell}} = 0$, by definition of $\delta\vec{a}_A^{(k)}$. Thus, they always vanish.
3. We consider the terms linear in $\delta\vec{a}_A^{(k)}$. In those terms, we replace back $\delta\vec{a}_A^{(k)} = \vec{a}_A^{(k)} - \vec{A}_A^{(k)}$. By doing so, the part proportional to the acceleration, which is necessarily a scalar product $\vec{C}(\vec{y}_A, \vec{v}_A) \cdot \vec{a}_A^{(n)}$, can be replaced, using an integration by part and dropping total time derivatives, by

$$\vec{C}(\vec{y}_A, \vec{v}_A) \cdot \vec{a}_A^{(n)} \rightarrow -\dot{\vec{C}}(\vec{y}_A, \vec{v}_A, \vec{a}_A) \cdot \vec{a}_A^{(n-1)}. \quad (4.16)$$

At this stage we eliminated all the products of derivatives of the accelerations and lowered the number of time derivatives by one.

4. We reiterate the three first points until the obtention of a Lagrangian $L(\vec{y}_A, \vec{v}_A, \vec{a}_A)$ depending only on the positions, velocities, accelerations and that is linear in the accelerations. This last point comes from the fact that we can drop double zero terms.

4.2.4 Derivation of EoM and conserved quantities from a generalized Lagrangian

Let us consider a generalized Lagrangian $L(\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)})$. We can derive [82] the EoM of such a Lagrangian using the generalized Euler-Lagrangian formula

$$\frac{\delta L}{\delta y_A^i} \equiv \sum_{\ell=0}^{n+2} (-)^{\ell} \frac{d^{\ell}}{dt^{\ell}} \left(\frac{\partial L}{\partial (y_A^i)^{(\ell)}} \right) = 0, \quad (4.17)$$

where $(y_A^i)^{(\ell)}$ is the ℓ^{th} time derivative of the positions. Eq. (4.17) gives an expression of the accelerations with respect to the other dynamical variables, namely $a_A^i = a_A^i(\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)})$. However, the acceleration entering in the right-hand side are of lower order (because they are multiplied by inverse powers of c). This implies that we can replace iteratively its value in order to find $a_A^i(\vec{y}_A, \vec{v}_A)$.

The computation of the conserved quantities requires some more work [82]. As already said, reducing the Lagrangian simplifies the following expressions. We define the conjugate momenta p_A^i and q_A^i of the positions and velocities of particles A

$$p_A^i \equiv \frac{\delta L}{\delta v_A^i} = \frac{\partial L}{\partial v_A^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial a_A^i} \right), \quad (4.18a)$$

$$q_A^i \equiv \frac{\delta L}{\delta a_A^i} = \frac{\partial L}{\partial a_A^i}. \quad (4.18b)$$

These quantities allow deriving the 10 conserved quantities associated to the reduced Fokker Lagrangian L_F in the following way

$$E = \sum_A \left(v_A^i p_A^i - a_A^i q_A^i \right) - L, \quad (4.19a)$$

$$P^i = \sum_A p_A^i, \quad (4.19b)$$

$$J^i = \epsilon_{ijk} \sum \left(y_A^j p_A^k + v_A^j q_A^k \right), \quad (4.19c)$$

$$G^i = -Z^i + \sum_A \left[-q_A^i + \frac{1}{c^2} \left(y_A^i p_A^k v_A^k + y_A^i q_A^k a_A^k + v_A^i q_A^k v_A^k \right) \right], \quad (4.19d)$$

where E is the energy, P^i the linear momentum, J^i the angular momentum and G^i the CoM position. The CoM position is tricky to compute because of the definition of Z^i . Indeed, under a Lorentz boost W^i , the particle trajectory transforms as $y_A^i \rightarrow y_A^i + \delta y_A^i$ and the Lagrangian as $L \rightarrow L + \delta L$. Explicitly, they read $\delta y_A^i = -W^i t + \frac{1}{c^2} W^j y_A^j v_A^i + O(W^2)$ and $\delta L = W^i dZ^i/dt + O(W^2)$ which defines Z^i . After applying the boost on the Lagrangian, one needs to find the expression of Z^i in order to compute the CoM and so has to "integrate" the linear in W^i part of δL .

4.3 The NNL reduced Fokker Lagrangian in a general frame

With the above methods in hand, we are now able to apply them to our problem. We first need to compute the tidal tensors using the unreduced values of the point-particle potentials. Then we can insert them in the definition of the Lagrangian in order to reduce it. Finally, we will be able to derive the EoM and conserved quantities.

4.3.1 Computation of the regularized tidal tensors

At the 2PN order, the metric of a general matter system in harmonic coordinates, given in Eq. (3.35), can be parametrized by the set of potentials $\{V, V_i, \hat{W}_{ij}, \hat{R}_i, \hat{X}\}$ whose definitions are given in Eq. (3.36). To perform a consistent Fokker reduction of the original action, the solutions of Eqs. (3.36) must be in principle constructed with the symmetric Green function, which kills all contributions of odd powers of $1/c$ at the current approximation level. As discussed above, thanks to the properties of the Fokker action, we only need the metric produced by point particles and can neglect tidal effects when inserting the metric (3.35) into the Fokker action. Therefore, we just require the potentials for point particles without including any internal structure effect. The relevant potentials have already been published in, *e.g.*, [108], except that we compute here their off-shell values, without replacement of accelerations by means of the EoM (we then call them the "unreduced" potentials). However, it is known that the replacement of accelerations in the action is equivalent to performing an unphysical shift of the particles' worldlines [109]. I have checked that, indeed, by inserting the reduced (on shell) versions of the potentials into the action, the final gauge invariant result for the conserved energy reduced to circular orbits, comes out the same.

As already said, in practical calculations, I used the Hadamard regularisation, which is equivalent to DR up to the relatively low NNL/2PN order [100, 83]. The method to regularise quantities at point A using Hadamard partie finie is explained in Sec. 6.3.2. After performing a $(3 + 1)$ splitting of the tidal tensors and injecting the 2PN point-particle metric, I found³

³The notation $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$ represents the Euclidean distance between the two bodies (at constant time $y_1^0 = y_2^0 = ct$); the unit direction from body 2 to body 1 is then $n_{12}^i = (y_1^i - y_2^i)/r_{12}$; $v_{12}^i = v_1^i - v_2^i$ stands for the relative velocity; the usual Euclidean scalar product of vectors is denoted with parentheses, *e.g.* $(n_{12} v_1) = \mathbf{n}_{12} \cdot \mathbf{v}_1$; the cross product is denoted, *e.g.* $(n_{12} \times v_{12})_i$, and the mixed product, *e.g.* $(n_{12}, v_1, v_2) = (n_{12} v_1 \times v_2)$.

$$\begin{aligned}
[G_{ij}]_1 = & \frac{Gm_2}{r_{12}^3} \left[3n_{12\langle i}n_{12j\rangle} + \frac{1}{c^2} \left[n_{12\langle i}n_{12j\rangle} \left(-\frac{15}{2}(n_{12}v_2)^2 + 6v_{12}^2 - \frac{3}{2}r_{12}(n_{12}a_2) - \frac{3Gm_1}{r_{12}} - \frac{3Gm_2}{r_{12}} \right) \right. \right. \\
& - 6n_{12\langle i}v_{1j\rangle}(n_{12}v_{12}) + 2v_{1\langle i}v_{1j\rangle} + n_{12\langle i}v_{2j\rangle} \left(12(n_{12}v_1) - 6(n_{12}v_2) \right) - 6v_{1\langle i}v_{2j\rangle} + 3v_{2\langle i}v_{2j\rangle} - 3a_{2\langle i}n_{12j\rangle}r_{12} \\
& + \delta_{ij} \left((n_{12}v_1)^2 - \frac{1}{3}v_1^2 \right) \left. \right] + \frac{1}{c^4} \left\{ n_{12\langle i}n_{12j\rangle} \left[\frac{105}{8}(n_{12}v_2)^4 + 30(n_{12}v_2)^2(v_1v_2) + 6(v_1v_2)^2 - 15(n_{12}v_2)^2v_1^2 \right. \right. \\
& - 12(v_1v_2)v_1^2 + 6v_1^4 - \frac{45}{2}(n_{12}v_2)^2v_2^2 - 12(v_1v_2)v_2^2 + 6v_1^2v_2^2 + 6v_2^4 + Gm_2(n_{12}a_2) \\
& + \frac{Gm_1}{r_{12}} \left(-\frac{291}{2}(n_{12}v_1)^2 + 291(n_{12}v_1)(n_{12}v_2) - \frac{273}{2}(n_{12}v_2)^2 + 35v_{12}^2 \right) + Gm_1 \left(14(n_{12}a_1) - 10(n_{12}a_2) \right) \\
& + \frac{Gm_2}{r_{12}} \left(9(n_{12}v_2)^2 + 18v_{12}^2 \right) + \frac{1}{8}r_{12}^3(\ddot{a}_2n_{12}) - \frac{15G^2m_1^2}{14r_{12}^2} + \frac{35G^2m_1m_2}{r_{12}^2} + \frac{5G^2m_2^2}{r_{12}^2} + r_{12} \left(12(v_1a_2)(n_{12}v_2) \right. \\
& - \frac{27}{2}(v_2a_2)(n_{12}v_2) + \frac{45}{4}(n_{12}a_2)(n_{12}v_2)^2 + 6(n_{12}a_2)(v_1v_2) - 3(n_{12}a_2)v_1^2 - \frac{9}{2}(n_{12}a_2)v_2^2) \\
& + r_{12}^2 \left(\frac{9}{8}(n_{12}a_2)^2 - \frac{15}{8}a_2^2 + \frac{3}{2}(n_{12}v_2)(n_{12}\dot{a}_2) + 2(v_1\dot{a}_2) - 2(v_2\dot{a}_2) \right) \left. \right] + n_{12\langle i}v_{1j\rangle} \left[\frac{62Gm_1}{r_{12}}(n_{12}v_{12}) \right. \\
& - \frac{18Gm_2}{r_{12}}(n_{12}v_{12}) + 15(n_{12}v_1)(n_{12}v_2)^2 - 15(n_{12}v_2)^3 + 6(n_{12}v_2)(v_1v_2) + 6(n_{12}v_2)v_{12}^2 - 6(n_{12}v_1)v_1^2 \\
& + r_{12} \left(-(v_{12}a_2) + 3(n_{12}a_2)(n_{12}v_1) - 9(n_{12}a_2)(n_{12}v_2) \right) - r_{12}^2(n_{12}\dot{a}_2) \left. \right] + v_{1\langle i}v_{1j\rangle} \left(-3(n_{12}v_2)^2 + 2v_1^2 \right. \\
& - r_{12}(n_{12}a_2) - \frac{3Gm_1}{r_{12}} + \frac{6Gm_2}{r_{12}} \left. \right) + n_{12\langle i}v_{2j\rangle} \left[-30(n_{12}v_1)(n_{12}v_2)^2 + 15(n_{12}v_2)^3 - 12(n_{12}v_1)(v_1v_2) \right. \\
& + 12(n_{12}v_1)v_1^2 + 12(n_{12}v_1)v_2^2 - 6(n_{12}v_2)v_2^2 + \frac{Gm_1}{r_{12}} \left(-68(n_{12}v_1) + 62(n_{12}v_2) \right) + \frac{Gm_2}{r_{12}} \left(12(n_{12}v_1) \right. \\
& - 18(n_{12}v_2) \left. \right) + r_{12}^2(n_{12}\dot{a}_2) + r_{12} \left(-2(v_1a_2) - 6(n_{12}a_2)(n_{12}v_1) - (v_2a_2) + 9(n_{12}a_2)(n_{12}v_2) \right) \left. \right] \\
& + v_{1\langle i}v_{2j\rangle} \left(-6(n_{12}v_1)(n_{12}v_2) + 15(n_{12}v_2)^2 - 6(v_1v_2) - 6v_{12}^2 + 5r_{12}(n_{12}a_2) + \frac{8Gm_1}{r_{12}} - \frac{10Gm_2}{r_{12}} \right) \\
& + v_{2\langle i}v_{2j\rangle} \left(6(n_{12}v_1)^2 - \frac{15}{2}(n_{12}v_2)^2 + 3v_2^2 - \frac{5}{2}r_{12}(n_{12}a_2) - \frac{4Gm_1}{r_{12}} + \frac{5Gm_2}{r_{12}} \right) + 4Gm_1a_{1\langle i}n_{12j\rangle} \\
& + a_{2\langle i}n_{12j\rangle} \left[r_{12} \left(-12(n_{12}v_1)(n_{12}v_2) + \frac{27}{2}(n_{12}v_2)^2 + 4(v_1v_2) - 2v_1^2 - 5v_2^2 \right) + \frac{9}{2}r_{12}^2(n_{12}a_2) - 3Gm_1 \right. \\
& - Gm_2 \left. \right] + a_{2\langle i}v_{1j\rangle}r_{12} \left(-(n_{12}v_1) + 7(n_{12}v_2) \right) + a_{2\langle i}v_{2j\rangle}r_{12} \left(-2(n_{12}v_1) - 7(n_{12}v_2) \right) - \frac{5}{4}a_{2\langle i}a_{2j\rangle}r_{12}^2 \\
& + n_{12\langle i}\dot{a}_{2j\rangle}r_{12}^2 \left(-2(n_{12}v_1) + 5(n_{12}v_2) \right) + 3v_{1\langle i}\dot{a}_{2j\rangle}r_{12}^2 - 3v_{2\langle i}\dot{a}_{2j\rangle}r_{12}^2 + \frac{7}{4}n_{12\langle i}\ddot{a}_{2j\rangle}r_{12}^3 + \delta_{ij} \left[-\frac{5}{2}(n_{12}v_1)^2(n_{12}v_2)^2 \right. \\
& - 2(n_{12}v_1)(n_{12}v_2)(v_1v_2) + (v_1v_2)^2 + (n_{12}v_1)^2v_1^2 + \frac{3}{2}(n_{12}v_2)^2v_1^2 - \frac{1}{3}v_1^4 + 2(n_{12}v_1)^2v_2^2 - v_1^2v_2^2 \\
& - \frac{4}{3}Gm_1(n_{12}a_2) + \frac{Gm_1}{r_{12}} \left(-\frac{16}{3}(n_{12}v_{12})^2 - (n_{12}v_1)^2 + \frac{4}{3}v_{12}^2 + \frac{1}{3}v_1^2 \right) + \frac{Gm_2}{r_{12}} \left(4(n_{12}v_{12})^2 - (n_{12}v_1)^2 \right. \\
& - \frac{4}{3}v_{12}^2 + \frac{1}{3}v_1^2 \left. \right) + r_{12} \left(\frac{4}{3}(v_{12}a_2)(n_{12}v_{12}) - (v_1a_2)(n_{12}v_1) - \frac{1}{2}(n_{12}a_2)(n_{12}v_1)^2 + \frac{1}{2}(n_{12}a_2)v_1^2 \right) \\
& \left. \left. - \frac{16G^2m_1m_2}{3r_{12}^2} + \frac{2G^2m_2^2}{3r_{12}^2} + r_{12}^2 \left(\frac{4}{3}a_2^2 - \frac{4}{3}(v_1\dot{a}_2) + \frac{4}{3}(v_2\dot{a}_2) \right) \right] \right\} + \mathcal{O} \left(\frac{1}{c^6} \right), \tag{4.20a}
\end{aligned}$$

$$\begin{aligned}
[H_{ij}]_1 = & \frac{Gm_2}{r_{12}^3} \left\{ 12(n_{12} \times v_{12})_{\langle i}n_{12j\rangle} + \frac{1}{c^2} \left[(n_{12} \times v_{12})_{\langle i}n_{12j\rangle} \left(-30(n_{12}v_2)^2 + 12(v_1v_2) + 12v_{12}^2 - 6r_{12}(n_{12}a_2) \right) \right. \right. \\
& + \frac{4Gm_1}{r_{12}} + \frac{12Gm_2}{r_{12}} \left. \right) - 12(a_2 \times n_{12})_{\langle i}n_{12j\rangle}r_{12}(n_{12}v_2) + 12(n_{12} \times v_{12})_{\langle i}v_{2j\rangle}(n_{12}v_1) - 2(a_2 \times v_{12})_{\langle i}n_{12j\rangle}r_{12} \\
& - 2a_{2\langle i}(n_{12} \times v_{12})_{j\rangle}r_{12} + 2(n_{12} \times \dot{a}_2)_{\langle i}n_{12j\rangle}r_{12}^2 + 4\delta_{ij}(n_{12}, v_1, v_2)(n_{12}v_1) \left. \right\} + \mathcal{O} \left(\frac{1}{c^4} \right), \tag{4.20b}
\end{aligned}$$

$$[G_{ijk}]_1 = -\frac{15Gm_2n_{12\langle i}n_{12j}n_{12k\rangle}}{r_{12}^4} + \mathcal{O} \left(\frac{1}{c^2} \right). \tag{4.20c}$$

The other components of the tidal moments are readily obtained from, *e.g.*, the relations $[G_{0i}]_1 = -v_1^j [G_{ij}]_1/c$ and $[G_{00}]_1 = v_1^i v_1^j [G_{ij}]_1/c^2$, which are equivalent to $[G_{\hat{0}\hat{0}}]_1 = [G_{\hat{0}\hat{i}}]_1 = 0$ in tetradic notation. In Eqs. (4.20), most of the terms are STF, which we denote by angular brackets surrounding the indices. Note however the appearance of pure trace contributions, due to the fact that we have not resorted here to tetradic projections and have used the Riemann tensor instead of the Weyl tensor (see the discussion in App. C).

4.3.2 The Lagrangian

After computing the values of the tidal tensors, I injected them in the total Lagrangian, which led to a Lagrangian $L' = L'(\vec{y}_A, \vec{v}_A, \vec{a}_A, \vec{a}_A^{(1)}, \vec{a}_A^{(2)})$. I applied the reduction method as explained in Sec. 4.2.3 and obtained the reduced NNL/2PN Lagrangian in a general frame

$$L = L_{\text{pp}} + L_{\text{tidal}}, \quad (4.21)$$

where I recovered the well-known point-particle Lagrangian

$$\begin{aligned} L_{\text{pp}} = & \frac{m_1 v_1^2}{2} + \frac{G m_1 m_2}{2 r_{12}} \\ & + \frac{1}{c^2} \left\{ -\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}} \left(-\frac{1}{4} (n_{12} v_1) (n_{12} v_2) + \frac{3}{2} v_1^2 - \frac{7}{4} (v_1 v_2) \right) \right\} \\ & + \frac{1}{c^4} \left\{ \frac{G^3 m_1^3 m_2}{2 r_{12}^3} + \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} \right. \\ & \quad + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left(\frac{7}{2} (n_{12} v_1)^2 - \frac{7}{2} (n_{12} v_1) (n_{12} v_2) + \frac{1}{2} (n_{12} v_2)^2 + \frac{1}{4} v_1^2 - \frac{7}{4} (v_1 v_2) + \frac{7}{4} v_2^2 \right) \\ & \quad + \frac{G m_1 m_2}{r_{12}} \left(\frac{3}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 - \frac{7}{8} (n_{12} v_2)^2 v_1^2 + \frac{7}{8} v_1^4 + \frac{3}{4} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) \right. \\ & \quad \left. \left. - 2 v_1^2 (v_1 v_2) + \frac{1}{8} (v_1 v_2)^2 + \frac{15}{16} v_1^2 v_2^2 \right) + \frac{m_1 v_1^6}{16} \right. \\ & \left. + G m_1 m_2 \left(-\frac{7}{4} (a_1 v_2) (n_{12} v_2) - \frac{1}{8} (n_{12} a_1) (n_{12} v_2)^2 + \frac{7}{8} (n_{12} a_1) v_2^2 \right) \right\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right), \end{aligned} \quad (4.22)$$

and the expression of the tidal part of the Lagrangian up to the NNL/2PN order in harmonic coordinates

$$\begin{aligned} L_{\text{tidal}} = & \frac{G^2 m_2^2}{r_{12}^6} \left\{ \frac{3}{2} \mu_1^{(2)} + \frac{1}{c^2} \left[\mu_1^{(2)} \left(-\frac{9}{2} (n_{12} v_1)^2 - 18 (n_{12} v_1) (n_{12} v_2) + 18 (n_{12} v_2)^2 - \frac{9}{2} (v_1 v_2) + \frac{15}{4} v_1^2 \right) \right. \right. \\ & \left. \left. + \sigma_1^{(2)} \left(-12 (n_{12} v_{12})^2 + 12 v_{12}^2 \right) - \frac{3 G m_1 \mu_1^{(2)}}{r_{12}} - \frac{21 G m_2 \mu_1^{(2)}}{2 r_{12}} \right] + \frac{1}{c^4} \left[\mu_1^{(2)} \left(\frac{9}{2} (n_{12} v_1)^4 \right. \right. \right. \\ & \left. \left. - 18 (n_{12} v_1)^3 (n_{12} v_2) + 45 (n_{12} v_1)^2 (n_{12} v_2)^2 - 54 (n_{12} v_1) (n_{12} v_2)^3 + \frac{63}{2} (n_{12} v_2)^4 + 9 (n_{12} v_1) (n_{12} v_2) (v_1 v_2) \right. \right. \\ & \left. \left. - 18 (n_{12} v_2)^2 (v_1 v_2) + \frac{9}{2} (v_1 v_2)^2 - 9 (n_{12} v_1)^2 v_{12}^2 + 27 (n_{12} v_1) (n_{12} v_2) v_{12}^2 - 36 (n_{12} v_2)^2 v_{12}^2 \right. \right. \\ & \left. \left. + 9 (v_1 v_2) v_{12}^2 + 9 v_{12}^4 - \frac{9}{4} (n_{12} v_1)^2 v_1^2 - \frac{9}{2} (n_{12} v_1) (n_{12} v_2) v_1^2 + \frac{27}{2} (n_{12} v_2)^2 v_1^2 - 9 (v_1 v_2) v_1^2 \right. \right. \\ & \left. \left. - \frac{27}{4} v_{12}^2 v_1^2 + \frac{69}{16} v_1^4 \right) + \mu_1^{(2)} r_{12} \left(-12 (v_{12} a_2) (n_{12} v_1) + 60 (n_{12} a_2) (n_{12} v_1)^2 + 21 (v_{12} a_2) (n_{12} v_2) \right. \right. \\ & \left. \left. - \frac{9}{2} (v_1 a_2) (n_{12} v_2) - 102 (n_{12} a_2) (n_{12} v_1) (n_{12} v_2) + 60 (n_{12} a_2) (n_{12} v_2)^2 + \frac{69}{2} (n_{12} a_2) (v_1 v_2) - \frac{69}{4} (n_{12} a_2) v_1^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{39}{2}(n_{12}a_2)v_2^2) + \sigma_1^{(2)} \left(60(n_{12}v_{12})^4 - 96(n_{12}v_{12})^3(n_{12}v_1) + 48(n_{12}v_{12})^2(n_{12}v_1)^2 - 24(n_{12}v_{12})^2(v_1v_2) \right. \\
& + 24(n_{12}v_{12})(n_{12}v_1)(v_1v_2) + 12(v_1v_2)^2 - 84(n_{12}v_{12})^2v_{12}^2 + 96(n_{12}v_{12})(n_{12}v_1)v_{12}^2 - 36(n_{12}v_1)^2v_{12}^2 \\
& + 24(v_1v_2)v_{12}^2 + 24v_{12}^4 + 18(n_{12}v_{12})^2v_1^2 - 24(n_{12}v_{12})(n_{12}v_1)v_1^2 - 24(v_1v_2)v_1^2 - 18v_{12}^2v_1^2 + 12v_1^4 \left. \right) \\
& + \sigma_1^{(2)} r_{12} \left(16(n_{12}a_2)(n_{12}v_{12})^2 + 24(v_{12}a_2)(n_{12}v_1) - 24(n_{12}a_2)(n_{12}v_{12})(n_{12}v_1) - 16(n_{12}a_2)v_{12}^2 \right) \\
& + \frac{Gm_1\mu_1^{(2)}}{r_{12}} \left(\frac{807}{8}(n_{12}v_1)^2 + \frac{381}{8}(n_{12}v_1)(n_{12}v_2) - 138(n_{12}v_2)^2 - \frac{387}{8}(v_1v_2) + \frac{63}{8}v_1^2 + 42v_2^2 \right) \\
& + \frac{Gm_2\mu_1^{(2)}}{r_{12}} \left(\frac{27}{2}(n_{12}v_1)^2 + \frac{1051}{8}(n_{12}v_1)(n_{12}v_2) - \frac{865}{8}(n_{12}v_2)^2 + \frac{83}{8}(v_1v_2) - \frac{45}{4}v_1^2 + \frac{49}{8}v_2^2 \right) \\
& + \frac{Gm_1\sigma_1^{(2)}}{r_{12}} \left(-8(n_{12}v_{12})^2 + 8v_{12}^2 \right) + \frac{Gm_2\sigma_1^{(2)}}{r_{12}} \left(36(n_{12}v_{12})^2 - 36v_{12}^2 \right) - \frac{60G^2m_1^2\mu_1^{(2)}}{7r_{12}^2} \\
& + \left. \frac{707G^2m_1m_2\mu_1^{(2)}}{8r_{12}^2} + \frac{165G^2m_2^2\mu_1^{(2)}}{4r_{12}^2} \right] + \frac{15\mu_1^{(3)}}{2r_{12}^2} \left. \right\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right). \tag{4.23}
\end{aligned}$$

Note that the last term, although it does not contain any explicit $1/c$ -factor, is actually a NNL term (see Eq. (4.8)). The notation $1 \leftrightarrow 2$ means that we add the symmetric counterpart in the exchange of the two particles of the expression.

4.3.3 The EoM and conserved quantities

The method for computing the EoM and conserved quantities are detailed in Sec. 4.2.4. The generalized Euler-Lagrange formula has been used to derive the NNL/2PN accelerations. Since the problem is perturbative, I replaced iteratively the on shell accelerations with their value in terms y_A^i and v_A^i only. The result is displayed in Eq. (D.1). I have checked that their value is in agreement with the literature (up to NL/1PN, since the 2PN was not derived yet) and that they are invariant under a Lorentz boost as they should be.

The structure of the leading order EoM and energy allows to compute the corresponding CoM quantities at NNL/2PN order without requesting G^i itself at that order. So it is sufficient to know the CoM at the NL/1PN for this calculation given by $G^i = G_{\text{pp}}^i + G_{\text{tidal}}^i$, where

$$G_{\text{pp}}^i = m_1 y_1^i + \frac{m_1}{2c^2} \left(v_1^2 - \frac{Gm_2}{r_{12}} \right) y_1^i + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.24}$$

and where the dominant tidal piece appears only at the NL/1PN order and reads

$$G_{\text{tidal}}^i = \frac{3G^2m_2^2}{2r_{12}^5c^2} \mu_1^{(2)} \left(3n_{12}^i - \frac{y_1^i}{r_{12}} \right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^4}\right). \tag{4.25}$$

The other conserved quantities in a general frame are quite lengthy, they are given in the CoM in Eqs. (4.42) and (4.43).

4.4 Lagrangian and conserved quantities in the CoM frame

The CoM frame is defined as the frame in which the equation $G^i = 0$ holds. Solving for $G^i = 0$ yields the CoM position of the particle 1 as a function of the relative separation and velocity. I found

$y_1^i = (y_1^i)_{\text{pp}} + (y_1^i)_{\text{tidal}}$, where⁴

$$(y_1^i)_{\text{pp}} = \left[X_2 + \frac{\nu \Delta}{2c^2} \left(v^2 - \frac{Gm}{r} \right) \right] x^i + \mathcal{O} \left(\frac{1}{c^4} \right), \quad (4.26a)$$

$$(y_1^i)_{\text{tidal}} = -\frac{3G^2 m \nu}{2r^6 c^2} \left[\Delta \mu_+^{(2)} + 5\mu_-^{(2)} \right] x^i + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^4} \right), \quad (4.26b)$$

with the position of the particle 2 obtained by the exchange $1 \leftrightarrow 2$. The velocities $v_1^i = (v_1^i)_{\text{pp}} + (v_1^i)_{\text{tidal}}$ are found by differentiating Eqs. (4.26a)–(4.26b), using in that process the full EoM, which include the tidal effect. Here and below, we define the following convenient combinations of the tidal polarizabilities:

$$\mu_{\pm}^{(\ell)} = \frac{1}{2} \left(\frac{m_2}{m_1} \mu_1^{(\ell)} \pm \frac{m_1}{m_2} \mu_2^{(\ell)} \right), \quad \sigma_{\pm}^{(\ell)} = \frac{1}{2} \left(\frac{m_2}{m_1} \sigma_1^{(\ell)} \pm \frac{m_1}{m_2} \sigma_2^{(\ell)} \right), \quad (4.27)$$

where the chosen normalisation is such that $\mu_+^{(\ell)} = \mu_1^{(\ell)} = \mu_2^{(\ell)}$ and $\mu_-^{(\ell)} = 0$ when the two bodies are identical, with the same mass and internal structure. Likewise for $\sigma_{\pm}^{(\ell)}$. It is also convenient to define their adimensioned value that will be used later on

$$\tilde{\mu}_{\pm}^{(\ell)} = \left(\frac{c^2}{Gm} \right)^{2\ell+1} G \mu_{\pm}^{(\ell)}, \quad \tilde{\sigma}_{\pm}^{(\ell)} = \left(\frac{c^2}{Gm} \right)^{2\ell+1} G \sigma_{\pm}^{(\ell)}. \quad (4.28)$$

At this stage, the EoM in the CoM frame can be derived in two possible ways: either by computing the CoM acceleration $a^i = a_1^i - a_2^i$ directly, based on the replacement rules (4.26a)–(4.26b); or by deriving first the expression of the Lagrangian in the CoM frame, varying it and then recover the EoM. I resorted to the two methods and the results are in full agreement (see also [38] for further details on the second method). The CoM Lagrangian may be decomposed as $L = L_{\text{pp}} + L_{\text{tidal}}$, where L_{pp} is *e.g.* given by Eq. (4.2) in [110] while the tidal part is, up to NNL/2PN order

$$\begin{aligned} \frac{L_{\text{tidal}}}{\mu} = & \frac{G^2 m}{r^6} \left\{ 3\mu_+^{(2)} + \frac{1}{c^2} \left\{ \left[\mu_+^{(2)} \left(\frac{27}{2} + 9\nu \right) + \frac{45}{2} \Delta \mu_-^{(2)} - 24\sigma_+^{(2)} \right] \dot{r}^2 + \left[\mu_+^{(2)} \left(\frac{15}{4} + \frac{3}{2}\nu \right) - \frac{15}{4} \Delta \mu_-^{(2)} \right. \right. \right. \\ & + 24\sigma_+^{(2)} \left. \right] v^2 + \frac{Gm}{r} \left(-\frac{27}{2} \mu_+^{(2)} + \frac{15}{2} \Delta \mu_-^{(2)} \right) \left. \right\} + \frac{1}{c^4} \left[r \left\{ \left[\mu_+^{(2)} \left(21 - \frac{45}{2}\nu \right) + \Delta \mu_-^{(2)} \left(21 - \frac{9}{2}\nu \right) \right. \right. \right. \\ & - 48\nu \sigma_+^{(2)} \left. \right] a_v \dot{r} + \left[\mu_+^{(2)} \left(-60 + 18\nu \right) + \Delta \mu_-^{(2)} \left(-60 + 18\nu \right) + \sigma_+^{(2)} \left(-16 + 48\nu \right) - 16\Delta \sigma_-^{(2)} \right] a_n \dot{r}^2 \\ & + \left[\mu_+^{(2)} \left(\frac{39}{2} - \frac{27}{4}\nu \right) + \Delta \mu_-^{(2)} \left(\frac{39}{2} - \frac{9}{4}\nu \right) + 16\sigma_+^{(2)} + 16\Delta \sigma_-^{(2)} \right] a_n v^2 \left. \right\} + \left[\mu_+^{(2)} \left(36 - 72\nu + 18\nu^2 \right) \right. \\ & + \Delta \mu_-^{(2)} \left(27 - 18\nu \right) + \sigma_+^{(2)} \left(72 - 96\nu \right) + 48\Delta \sigma_-^{(2)} \left. \right] \dot{r}^4 + \left[\mu_+^{(2)} \left(-\frac{189}{4} + 72\nu - \frac{45}{2}\nu^2 \right) + \Delta \mu_-^{(2)} \left(-\frac{99}{4} \right. \right. \\ & - \frac{27}{2}\nu \left. \right) + \sigma_+^{(2)} \left(-114 + 132\nu \right) - 54\Delta \sigma_-^{(2)} \left. \right] \dot{r}^2 v^2 + \left[\mu_+^{(2)} \left(\frac{249}{16} - 12\nu - \frac{27}{8}\nu^2 \right) + \Delta \mu_-^{(2)} \left(\frac{39}{16} + \frac{27}{8}\nu \right) \right. \\ & + \sigma_+^{(2)} \left(42 - 36\nu \right) + 6\Delta \sigma_-^{(2)} \left. \right] v^4 + \frac{Gm}{r} \left\{ \left[\mu_+^{(2)} \left(-\frac{249}{2} + \frac{355}{2}\nu + 39\nu^2 \right) + \Delta \mu_-^{(2)} \left(-\frac{303}{2} + \frac{135}{2}\nu \right) \right. \right. \\ & + 28\sigma_+^{(2)} - 44\Delta \sigma_-^{(2)} \left. \right] \dot{r}^2 + \left[\mu_+^{(2)} \left(\frac{123}{4} - 41\nu + 3\nu^2 \right) + \frac{213}{4} \Delta \mu_-^{(2)} - 28\sigma_+^{(2)} + 44\Delta \sigma_-^{(2)} \right] v^2 \left. \right\} \end{aligned}$$

⁴We pose $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt$; $r = |\mathbf{x}| = r_{12}$ denotes the separation, $n^i = x^i/r$ the unit direction, and we have $\dot{r} = (nv) = \mathbf{n} \cdot \mathbf{v}$; mass parameters are: the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = m_1 m_2 / m^2 = X_1 X_2$ and the mass difference $\Delta = X_1 - X_2$, with $X_A = m_A / m$.

$$+ \frac{G^2 m^2}{r^2} \left[\mu_+^{(2)} \left(\frac{915}{28} + \frac{3119}{28} \nu \right) - \frac{1395}{28} \Delta \mu_-^{(2)} \right] + \mu_+^{(3)} \frac{15}{r^2} \left. \right\} + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.29)$$

Note again that the last term is actually a NNL/2PN contribution. The corresponding relative CoM acceleration is displayed in App. D.1. Similarly, the tidal part of the conserved energy $E = E_{\text{pp}} + E_{\text{tidal}}$ is

$$\begin{aligned} \frac{E_{\text{tidal}}}{m\nu} = & -3 \frac{G^2 m}{r^6} \mu_+^{(2)} + \frac{1}{c^2} \left\{ \frac{G^2 m}{r^6} \left[\left(\frac{27}{2} + 9\nu \right) \mu_+^{(2)} + \frac{45}{2} \Delta \mu_-^{(2)} - 24\sigma_+^{(2)} \right] \dot{r}^2 \right. \\ & + \left. \left(\left(\frac{15}{4} + \frac{3}{2}\nu \right) \mu_+^{(2)} - \frac{15}{4} \Delta \mu_-^{(2)} + 24\sigma_+^{(2)} \right) v^2 \right] + \frac{G^3 m^2}{r^7} \left[\frac{27}{2} \mu_+^{(2)} - \frac{15}{2} \Delta \mu_-^{(2)} \right] \left. \right\} \\ & + \frac{1}{c^4} \left\{ \frac{G^2 m}{r^6} \left[\left((-372 - 72\nu + 54\nu^2) \mu_+^{(2)} + (-399 + 90\nu) \Delta \mu_-^{(2)} + (88 + 96\nu) \sigma_+^{(2)} + 16\Delta \sigma_-^{(2)} \right) \dot{r}^4 \right. \right. \\ & + \left. \left(\left(\frac{1125}{4} - \frac{27}{2}\nu - \frac{135}{2}\nu^2 \right) \mu_+^{(2)} + \left(\frac{1395}{4} - 135\nu \right) \Delta \mu_-^{(2)} + (-198 - 36\nu) \sigma_+^{(2)} - 18\Delta \sigma_-^{(2)} \right) \dot{r}^2 v^2 \right. \\ & + \left. \left(\left(\frac{99}{16} - \frac{27}{4}\nu - \frac{81}{8}\nu^2 \right) \mu_+^{(2)} + \left(-\frac{531}{16} + \frac{135}{8}\nu \right) \Delta \mu_-^{(2)} + (110 - 60\nu) \sigma_+^{(2)} + 2\Delta \sigma_-^{(2)} \right) v^4 \right] \\ & + \frac{G^3 m^2}{r^7} \left[\left(\left(-\frac{213}{2} + \frac{499}{2}\nu + 39\nu^2 \right) \mu_+^{(2)} + \left(-\frac{267}{2} + \frac{135}{2}\nu \right) \Delta \mu_-^{(2)} + (60 + 48\nu) \sigma_+^{(2)} - 12\Delta \sigma_-^{(2)} \right) \dot{r}^2 \right. \\ & + \left. \left(\left(\frac{51}{4} - 113\nu + 3\nu^2 \right) \mu_+^{(2)} + \frac{141}{4} \Delta \mu_-^{(2)} + (-60 - 48\nu) \sigma_+^{(2)} + 12\Delta \sigma_-^{(2)} \right) v^2 \right] \\ & + \left. \frac{G^4 m^3}{r^8} \left[\left(-\frac{915}{28} - \frac{3119}{28}\nu \right) \mu_+^{(2)} + \frac{1395}{28} \Delta \mu_-^{(2)} \right] \right\} - 15 \frac{G^2 m}{r^8} \mu_+^{(3)} + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.30) \end{aligned}$$

Finally, for the CoM angular momentum $J^i = J_{\text{pp}}^i + J_{\text{tidal}}^i$, the tidal part reads (denoting $L^i = \epsilon_{ijk} x^j v^k$)

$$\begin{aligned} \frac{J_{\text{tidal}}^i}{m\nu} = & \frac{G^2 m}{c^2 r^6} L^i \left[\mu_+^{(2)} \left(\frac{15}{2} + 3\nu \right) - \frac{15}{2} \Delta \mu_-^{(2)} + 48\sigma_+^{(2)} + \frac{1}{c^2} \left\{ \left[\mu_+^{(2)} \left(\frac{303}{2} - 27\nu - 45\nu^2 \right) + \Delta \mu_-^{(2)} \left(\frac{393}{2} - 90\nu \right) \right. \right. \right. \\ & + \left. \left. \sigma_+^{(2)} \left(-196 - 120\nu \right) - 76\Delta \sigma_-^{(2)} \right] \dot{r}^2 + \left[\mu_+^{(2)} \left(\frac{9}{4} - 12\nu - \frac{27}{2}\nu^2 \right) + \Delta \mu_-^{(2)} \left(-\frac{201}{4} + \frac{45}{2}\nu \right) + \sigma_+^{(2)} \left(136 \right. \right. \right. \\ & - \left. \left. 96\nu \right) - 8\Delta \sigma_-^{(2)} \right] v^2 + \frac{Gm}{r} \left[\mu_+^{(2)} \left(\frac{87}{2} - 154\nu + 6\nu^2 \right) + \frac{177}{2} \Delta \mu_-^{(2)} + \sigma_+^{(2)} \left(-88 - 48\nu \right) + 56\Delta \sigma_-^{(2)} \right] \left. \right\} \left. \right] \\ & + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.31) \end{aligned}$$

The point-particle pieces E_{pp} and J_{pp}^i are depicted in Eqs. (4.8) and (4.9) of Ref. [110].

4.5 The tidal Hamiltonian in isotropic coordinates

After the publication of our first paper on the conservative part of tidal effects [6], some other groups, using different techniques, such as scattering amplitude [60] and EFT [54] had computed the Hamiltonian including tidal effects up to 2PM (second post-Minkowskian order beyond the leading order, *i.e.* from order G^6 to G^8). However, the comparisons of our results were not trivial since they used a Hamiltonian approach while we used a Lagrangian approach. As their Hamiltonians were given in isotropic coordinates, in order to check the consistency of our three results, we derived the Hamiltonian in isotropic starting from our Lagrangian (4.21) and then compared our result with the PN expansion of their Hamiltonian. The overlap of our results turned out to be in complete agreement.

To construct a Hamiltonian from the Lagrangian (4.22)-(4.23), we need to remove the accelerations at NNL/2PN orders by means of shifts of the particles' trajectories also known as "contact" transformations. For a generalized Lagrangian $L(y_A^i, v_A^i, a_A^i)$ that is linear in accelerations and such that the accelerations appear at the highest considered PN order (in our case NNL/2PN order), the contact transformation can be taken to be $y_A^i \rightarrow Y_A^i = y_A^i + \delta y_A^i$ with [82]

$$\delta y_A^i = \frac{1}{m_A} \left(q_A^i + \frac{\partial F}{\partial v_A^i} \right) + \mathcal{O} \left(\frac{1}{c^6}, \frac{\epsilon_{\text{tidal}}}{c^6} \right), \quad (4.32)$$

where, $q_A^i = q_{A\text{pp}}^i + q_{A\text{tidal}}^i$ is defined in Eq. (4.18b) and $F = F_{\text{pp}} + F_{\text{tidal}}$ is an arbitrary function of the positions and velocities present at the highest 2PN and NNL/2PN levels. The effect of this contact transformation, combined with the addition of a total time derivative, yields the physically equivalent Lagrangian $L \rightarrow L + \delta L$ with the extra contribution

$$\delta L = \sum_{A=1,2} \frac{\delta L}{\delta y_A^i} \delta y_A^i + \frac{dF}{dt} + \mathcal{O} \left(\frac{1}{c^6}, \frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.33)$$

Here, $\delta L/\delta y_A^i$ is defined in Eq. (4.17) evaluated off shell, without replacement of the accelerations. With the choice (4.32), the new Lagrangian is now ordinary, *i.e.*, depends only on positions and velocities but is no more written in the harmonic coordinates.

As already said, the function F can be adjusted at will and, for the point-particle case, it can be chosen in such a way that, starting from the harmonic-coordinates Lagrangian, the target Lagrangian uses position variables corresponding to ADM coordinates. For convenience, I adopted for F the point-particle part leading to ADM coordinates, without tidal terms and up to NNL/2PN order. Thus, according to Eq. (4.15) in [82], I took

$$F = \frac{Gm_1 m_2}{c^4} \left\{ \frac{Gm_1}{r_{12}} \left(\frac{7}{4}(n_{12}v_1) - \frac{1}{4}(n_{12}v_2) \right) + \frac{1}{4}(n_{12}v_2)v_1^2 \right\} + 1 \leftrightarrow 2 + \mathcal{O} \left(\frac{1}{c^6} \right). \quad (4.34)$$

After the specific contact transformation (4.32)-(4.34), the Lagrangian $L' = L + \delta L$ is a functional of trajectories $Y_A^i = y_A^i + \delta y_A^i$ and velocities $V_A^i = dY_A^i/dt$. A Hamiltonian H' follows from the usual Legendre transformation. We denote the conjugate momenta $P_A^i = \partial L'/\partial V_A^i$, and also pose $R_{12} = |\mathbf{Y}_1 - \mathbf{Y}_2|$, $N_{12}^i = (Y_1^i - Y_2^i)/R_{12}$, $(N_{12}P_1) = \mathbf{N}_{12} \cdot \mathbf{P}_1$ *etc.* I derived $H' = H'_{\text{pp}} + H'_{\text{tidal}}$, given in App. D.1, where the point-particle part reproduces the Hamiltonian in ADM coordinates to the considered order [111].

From now on, we restrict attention to the frame of the CoM, for which the relative canonical momentum is simply $\mathbf{P} \equiv \mathbf{P}_1 = -\mathbf{P}_2$. Setting $R = R_{12}$ and $\mathbf{N} = N_{12}^i$, we further change notation to introduce appropriate reduced variables

$$\hat{H}' = \frac{H'}{m\nu}, \quad \hat{\mathbf{P}} = \frac{\mathbf{P}}{m\nu}, \quad \hat{R} = \frac{R}{Gm}. \quad (4.35)$$

With the Hamiltonian (D.3)-(D.4) in hand, one has a large freedom of variables provided by arbitrary canonical transformations. On the other hand, the community of scattering amplitudes and the EFT are deriving Hamiltonians in the PM approximation using isotropic coordinates, say $(\boldsymbol{\rho}, \mathbf{p})$ [58]. Isotropic coordinates drastically simplify the expression of the Hamiltonian, which then depends on the momentum \mathbf{p} only through its norm $p^2 \equiv \mathbf{p}^2$ and not on the radial component $(\mathbf{p} \cdot \mathbf{n})$ separately (with $\mathbf{n} = \boldsymbol{\rho}/\rho$). Thus, I performed a canonical transformation from the reduced variables $(\hat{\mathbf{X}}, \hat{\mathbf{P}})$ to (reduced versions of) new isotropic variables $(\hat{\boldsymbol{\rho}}, \hat{\mathbf{p}})$ where the generating function $\hat{G}(\hat{\mathbf{X}}, \hat{\mathbf{p}})$ has been conveniently chosen to get the Hamiltonian in isotropic coordinates

$$\hat{G}_{\text{pp}} = (\hat{N}\hat{p}) \left\{ \hat{R} + \frac{\nu}{2c^2} + \frac{1}{c^4} \left[\frac{1}{\hat{R}} \left(-\frac{1}{4}\nu + \frac{1}{4}\nu^2 \right) + \frac{1}{8}\nu^2 (\hat{N}\hat{p})^2 + \left(\frac{1}{4}\nu - \frac{1}{8}\nu^2 \right) \hat{p}^2 \right] \right\} + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (4.36a)$$

$$\begin{aligned}
\hat{G}_{\text{tidal}} = (\hat{N}\hat{p}) & \left\{ \frac{1}{c^{12}\hat{R}^5} \left[\tilde{\mu}_+^{(2)} \left(\frac{9}{4} + \frac{3}{2}\nu \right) + \frac{15}{4} \tilde{\mu}_-^{(2)} \Delta - 4\tilde{\sigma}_+^{(2)} \right] + \frac{1}{c^{14}} \left[\frac{1}{\hat{R}^6} \left[\tilde{\mu}_+^{(2)} \left(-18 + \frac{71}{4}\nu - \frac{9}{4}\nu^2 \right) \right. \right. \right. \\
& + \tilde{\mu}_-^{(2)} \Delta \left(-\frac{363}{14} - \frac{15}{2}\nu \right) + \tilde{\sigma}_+^{(2)} \left(\frac{114}{7} + 16\nu \right) - \frac{34}{7} \tilde{\sigma}_-^{(2)} \Delta \left. \right] + \frac{1}{\hat{R}^5} \left(\left[\tilde{\mu}_+^{(2)} \left(\frac{9}{2} - 9\nu + \frac{9}{4}\nu^2 \right) \right. \right. \\
& + \tilde{\mu}_-^{(2)} \Delta \left(\frac{27}{8} - \frac{9}{4}\nu \right) + \tilde{\sigma}_+^{(2)} \left(9 - 12\nu \right) + 6\tilde{\sigma}_-^{(2)} \Delta \left. \right] (\hat{N}\hat{p})^2 + \left[\tilde{\mu}_+^{(2)} \left(-\frac{27}{4} + \frac{81}{8}\nu - \frac{3}{8}\nu^2 \right) \right. \\
& \left. \left. + \tilde{\mu}_-^{(2)} \Delta \left(-\frac{69}{16} + \frac{9}{4}\nu \right) + \tilde{\sigma}_+^{(2)} \left(-\frac{25}{2} + 10\nu \right) - 6\tilde{\sigma}_-^{(2)} \Delta \right] \hat{p}^2 \right\} + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right), \quad (4.36b)
\end{aligned}$$

with the canonical transformation being specified by $\hat{P}^i = \partial\hat{G}/\partial\hat{X}^i$ and $\hat{p}^i = \partial\hat{G}/\partial\hat{p}^i$. The Hamiltonian in isotropic coordinates is obtained as $\hat{H}^{\text{iso}}(\hat{\rho}, \hat{\mathbf{p}}) = \hat{H}'(\hat{\mathbf{X}}, \hat{\mathbf{P}})$. To NNL/2PN order, I found

$$\begin{aligned}
\hat{H}_{\text{pp}}^{\text{iso}} = & -\frac{1}{\hat{\rho}} + \frac{1}{2}\hat{p}^2 + \frac{1}{c^2} \left[\frac{1}{\hat{\rho}^2} \left(\frac{1}{2} + \frac{1}{2}\nu \right) + \frac{\hat{p}^2}{\hat{\rho}} \left(-\frac{3}{2} - \nu \right) + \left(-\frac{1}{8} + \frac{3}{8}\nu \right) \hat{p}^4 \right] \\
& + \frac{1}{c^4} \left[\frac{1}{\hat{\rho}^3} \left(-\frac{1}{4} - \frac{3}{2}\nu \right) + \frac{\hat{p}^2}{\hat{\rho}^2} \left(\frac{5}{2} + \frac{27}{4}\nu + \frac{3}{4}\nu^2 \right) + \frac{\hat{p}^4}{\hat{\rho}} \left(\frac{5}{8} - \frac{5}{2}\nu - \nu^2 \right) \right. \\
& \left. + \left(\frac{1}{16} - \frac{5}{16}\nu + \frac{5}{16}\nu^2 \right) \hat{p}^6 \right] + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (4.37a)
\end{aligned}$$

$$\begin{aligned}
\hat{H}_{\text{tidal}}^{\text{iso}} = & -\frac{3\tilde{\mu}_+^{(2)}}{c^{10}\hat{\rho}^6} + \frac{1}{c^{12}} \left(\frac{\hat{p}^2}{\hat{\rho}^6} \left[\tilde{\mu}_+^{(2)} \left(-6 - 3\nu \right) - 20\tilde{\sigma}_+^{(2)} \right] + \frac{1}{\hat{\rho}^7} \left[\tilde{\mu}_+^{(2)} \left(\frac{63}{4} + \frac{21}{2}\nu \right) - \frac{15}{4} \tilde{\mu}_-^{(2)} \Delta - 4\tilde{\sigma}_+^{(2)} \right] \right) \\
& + \frac{1}{c^{14}} \left(\frac{\hat{p}^4}{\hat{\rho}^6} \left[\tilde{\mu}_+^{(2)} \left(-\frac{63}{16} - \frac{21}{2}\nu - 3\nu^2 \right) + \tilde{\sigma}_+^{(2)} \left(-\frac{15}{2} - 40\nu \right) \right] + \frac{\hat{p}^2}{\hat{\rho}^7} \left[\tilde{\mu}_+^{(2)} \left(\frac{261}{8} + \frac{699}{8}\nu + \frac{81}{8}\nu^2 \right) \right. \right. \\
& + \tilde{\mu}_-^{(2)} \Delta \left(-\frac{1989}{112} - \frac{15}{4}\nu \right) + \tilde{\sigma}_+^{(2)} \left(\frac{1977}{14} + 126\nu \right) - \frac{204}{7} \tilde{\sigma}_-^{(2)} \Delta \left. \right] + \frac{1}{\hat{\rho}^8} \left[\tilde{\mu}_+^{(2)} \left(-\frac{339}{7} - \frac{666}{7}\nu \right. \right. \\
& \left. \left. - 15\nu^2 \right) + \tilde{\mu}_-^{(2)} \Delta \left(\frac{141}{7} + 15\nu \right) + \tilde{\sigma}_+^{(2)} \left(\frac{142}{7} + 20\nu \right) - \frac{34}{7} \tilde{\sigma}_-^{(2)} \Delta - 15\tilde{\mu}_+^{(3)} \right] \right) + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.37b)
\end{aligned}$$

This Hamiltonian can be compared to the PN expansion of the Hamiltonian derived by the other groups. Of course, we recover for $\hat{H}_{\text{pp}}^{\text{iso}}$ the 2PN expansion of the 3PM Hamiltonian given in Eq. (10.8) of [56].⁵ Gladly, we also find a complete agreement for $\hat{H}_{\text{tidal}}^{\text{iso}}$ with the PN expansion of the NL PM tidal Hamiltonians in Eq. (7) of [60] and Eqs. (19-20) of [54]. Namely, the results agree with the overlapping terms of the mass and current quadrupoles up to order G^3 and up to the NL/6PN order $\mathcal{O}(\epsilon_{\text{tidal}}/c^2)$. They also agree with the leading G^2 order of the mass octupole in [54] up to NL/1PN order. Note that the NL/1PM approximation computed in [60, 54] gives all the PN tidal terms at orders G^2 and G^3 up to NL/1PN order but overlooks those in G^4 arising at NNL/2PN (see the last line of Eq. (D.4)).

4.6 Tidal effects for quasi-circular orbits

Going back to the conserved quantities, we continue the computations in the case of quasi-circular orbits, *i.e.* orbits that are quasi-circular in our harmonic coordinate system. The relative acceleration

⁵For point-particles without internal structure, the n PM Hamiltonian permits controlling the $(n-1)$ PN approximation. Thus, the 3PM conservative Hamiltonian is sufficient to completely control the 2PN conservative dynamics.

of the binary system is of the form

$$a^i = -\omega^2 x^i + \frac{1}{c^5} F_{\text{react}}^i, \quad (4.38)$$

where ω is the orbital angular frequency and F_{react}^i is a radiation reaction force which appears at 2.5PN. This means that $\dot{r} = \mathcal{O}(c^{-5})$. However, our Lagrangian approach is purely conservative so the radiation reaction in (4.38) has to be added *ad hoc* using another method. Furthermore, one can explicitly see in Eq. (D.2) that the CoM acceleration is purely radial, namely $a^i = -\omega^2 x^i$, thus we can read off ω from this expression. Relevant quantities will then depend only on the bodies' separation r or equivalently, on the orbital frequency. In the case of circular orbits, it is convenient to introduce the dimensionless PN parameters associated with the separation and orbital frequency as

$$\gamma = \frac{Gm}{rc^2}, \quad x = \left(\frac{Gm\omega}{c^3} \right)^{2/3}. \quad (4.39)$$

By identifying the expression of ω^2 from the circular-orbit EoM as explained above and expressing in terms of γ iteratively, I recovered the well-known formula for point masses at the 2PN order, with a non-trivial NNL/2PN relative tidal contribution

$$(\omega^2)_{\text{pp}} = \frac{Gm}{r^3} \left[1 + (-3 + \nu)\gamma + \left(6 + \frac{61}{4}\nu + \nu^2 \right) \gamma^2 \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (4.40a)$$

$$\begin{aligned} (\omega^2)_{\text{tidal}} = \frac{Gm}{r^3} & \left\{ 18\tilde{\mu}_+^{(2)}\gamma^5 + \left[\left(-\frac{249}{2} + 51\nu \right) \tilde{\mu}_+^{(2)} + \frac{75}{2}\Delta\tilde{\mu}_-^{(2)} + 96\tilde{\sigma}_+^{(2)} \right] \gamma^6 \right. \\ & + \left[\left(\frac{34317}{56} + \frac{2976}{7}\nu + 54\nu^2 \right) \tilde{\mu}_+^{(2)} + \left(-\frac{12051}{56} + 90\nu \right) \Delta\tilde{\mu}_-^{(2)} \right. \\ & \left. \left. + \left(-616 + 264\nu \right) \tilde{\sigma}_+^{(2)} + 200\Delta\tilde{\sigma}_-^{(2)} + 120\tilde{\mu}_+^{(3)} \right] \gamma^7 \right\} + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right). \end{aligned} \quad (4.40b)$$

Note that, as mentioned before, the remainders are of the order $\mathcal{O}(1/c^6)$ and not $\mathcal{O}(1/c^5)$. The next step consists in determining the relation between γ and x , by inverting Eqs. (4.40), which gives

$$\gamma_{\text{pp}} = x \left[1 + \left(1 - \frac{\nu}{3} \right) x + \left(1 - \frac{65}{12}\nu \right) x^2 \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (4.41a)$$

$$\begin{aligned} \gamma_{\text{tidal}} = x & \left\{ -6\tilde{\mu}_+^{(2)}x^5 + \left[\left(-\frac{37}{2} + 3\nu \right) \tilde{\mu}_+^{(2)} - \frac{25}{2}\Delta\tilde{\mu}_-^{(2)} - 32\tilde{\sigma}_+^{(2)} \right] x^6 \right. \\ & + \left[\left(-\frac{4355}{56} + \frac{1105}{21}\nu + 15\nu^2 \right) \tilde{\mu}_+^{(2)} + \left(-\frac{3683}{56} + \frac{95}{6}\nu \right) \Delta\tilde{\mu}_-^{(2)} \right. \\ & \left. \left. + \left(-\frac{440}{3} + \frac{88}{3}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{200}{3}\Delta\tilde{\sigma}_-^{(2)} - 40\tilde{\mu}_+^{(3)} \right] x^7 \right\} + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right). \end{aligned} \quad (4.41b)$$

The conserved energy for circular orbits can now be computed. To do so, I took Eq. (4.30) to which I added the point-particle part, set $\dot{r} = 0$ and replaced $v^2 = r^2\omega^2$ by its expression in terms of the parameter γ using Eqs. (4.40). This yields E first as a function of γ . I finally inserted there the previous relation (4.41) to get an important result, namely the expression of the circular energy as a function of the frequency-dependent parameter x

$$E_{\text{pp}} = -\frac{1}{2}m\nu x c^2 \left[1 + \left(-\frac{3}{4} - \frac{\nu}{12} \right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24} \right) x^2 \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (4.42a)$$

$$\begin{aligned} E_{\text{tidal}} = -\frac{1}{2}m\nu x c^2 & \left\{ -18\tilde{\mu}_+^{(2)}x^5 + \left[\left(-\frac{121}{2} + 33\nu \right) \tilde{\mu}_+^{(2)} - \frac{55}{2}\Delta\tilde{\mu}_-^{(2)} - 176\tilde{\sigma}_+^{(2)} \right] x^6 \right. \\ & + \left[\left(-\frac{20865}{56} + \frac{5434}{21}\nu - \frac{91}{4}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(-\frac{11583}{56} + \frac{715}{12}\nu \right) \tilde{\mu}_-^{(2)} \right. \end{aligned}$$

$$+ \left(-\frac{2444}{3} + \frac{1768}{3}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{884}{3} \Delta \tilde{\sigma}_-^{(2)} - 130 \tilde{\mu}_+^{(3)} \Big] x^7 \Big\} + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.42b)$$

This is the quantity we will insert in the energy balance equation in order to derive the GW phase. I also computed by the same method the constant angular momentum for circular orbits, which reads

$$J_{\text{pp}} = \frac{Gm^2\nu}{cx^{1/2}} \left[1 + \left(\frac{3}{2} + \frac{\nu}{6} \right) x + \left(\frac{27}{8} - \frac{19}{8}\nu + \frac{\nu^2}{24} \right) x^2 \right] + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (4.43a)$$

$$J_{\text{tidal}} = \frac{Gm^2\nu}{cx^{1/2}} \left\{ 12\tilde{\mu}_+^{(2)} x^5 + \left[\left(\frac{77}{2} - 21\nu \right) \tilde{\mu}_+^{(2)} + \frac{35}{2} \Delta \tilde{\mu}_-^{(2)} + 112\tilde{\sigma}_+^{(2)} \right] x^6 + \left[\left(\frac{1605}{7} - \frac{3344}{21}\nu + 14\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{891}{7} - \frac{110}{3}\nu \right) \tilde{\mu}_-^{(2)} + \left(\frac{1504}{3} - \frac{1088}{3}\nu \right) \tilde{\sigma}_+^{(2)} + \frac{544}{3} \Delta \tilde{\sigma}_-^{(2)} + 80\tilde{\mu}_+^{(3)} \right] x^7 \right\} + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \quad (4.43b)$$

I have verified that the energy E and angular momentum J for circular orbits are linked by the relation

$$\frac{\partial E}{\partial \omega} = \omega \frac{\partial J}{\partial \omega} + \mathcal{O} \left(\frac{1}{c^6}, \frac{\epsilon_{\text{tidal}}}{c^6} \right), \quad (4.44)$$

which is just one aspect of the ‘‘first law of binary point-particle mechanics’’ [112].

To conclude, I compared the invariant energy as given in (4.42) with existing results in the literature. In Table 4.1 is provided for each order and for each tidal multipole contributing to the conserved energy $E_{\text{tidal}}(x)$ the references which we agree with.

$E_{\text{tidal}}(x)$	Mass quadrupole	Current quadrupole	Mass octupole
5PN (L)	[90, 113, 100, 114, 92]		
6PN (NL)	[100, 114, 45]	[100, 45, 115]	
7PN (NNL)	[100]	[100]	[100, 116]

Table 4.1: Comparison with the existing literature. We indicate for each order and each tidal multipolar piece contributing to the conserved energy the references to which we agree with.

Note in particular that there is full agreement with all results of Ref. [100]. I have checked, notably, that by re-expanding the tidal effects entering the EOB Hamiltonian [100] in the form of a PN Taylor series, I recover exactly Eq. (4.42). However, I do not recover the 6PN coefficient for the current quadrupole piece in the preprint [116], where the discrepancy is by a factor 2. This must be due to a mistake in this paper.

Chapter 5

Radiative sector

In this chapter, we compute the tidal effects in the GW energy flux up to NNL/2PN order within the PN-MPM formalism discussed in Chapter 3. The PN-MPM approach describes the waveform by means of mass and current radiative multipole moments defined in the asymptotic region, which are themselves related to some appropriate source-type multipole moments defined in the near zone for the whole matter system. Beware that we work with two different kinds of multipole moments: the tidal moments, describing the individual deformation of the bodies, and the source multipole moments describing the matter distribution of the overall system. The summary of the steps of the computations in the radiative sector is given in Sec. 3.3. The chapter is organized accordingly as follows. In Sec. 5.1, we recall the matter action we start with (details of its construction are given in *e.g.* [100, 6]) and compute the stress-energy tensor of the system as well as its 3+1 decomposition rewritten in a convenient form. Next, we calculate, in Sec. 5.2, the potentials sourced by the previous stress-energy tensor (some long formulas are relegated to App. D.2). In Sec. 5.3 we apply the GW generation formalism, which yields the expressions of the source multipole moments of the binary system in a general frame. Those are then specialized to the CoM frame and, in a last stage, for circular orbits (while the moments written in a general frame are too long, their CoM expression are displayed in App. D.2). As for the instantaneous GW flux, it is computed in Sec. 5.4 in a modal form based on the mode decomposition of the source multipole moments. The missing tail part is obtained from those flux modes. In Sec. 5.5 we present the result for the phase evolution, both in the standard Taylor form and in the Fourier domain, using the stationary phase approximation. Finally, in Sec. 5.6, we conclude and make comparisons with the existing literature. This chapter is based on the paper [7].

5.1 The stress-energy tensor from the tidal effective action

In the preceding chapter, we analyzed the motion of a compact binary system including tidal interactions. To do so, we considered the gravitational Einstein-Hilbert action endowed with the standard harmonic gauge fixing term 4.10, to which we added the effective matter action for a system of N massive gravitationally interacting compact bodies with internal structure in the adiabatic approximation (4.11). The motion was obtained by varying the associated Fokker action. The next important step in this approach, pursued in this section, consists in the computation of the matter stress-energy tensor, whose vocation is to be inserted into the adopted GW generation formalism. For that purpose, we need only the matter part of the action which admits the general form

$$S_m = \sum_{A=1}^N \int d\tau_A L_A, \quad (5.1)$$

where the term associated with particle A integrates over its proper time variation $d\tau_A$ which is such that the four velocity $cu_A^\mu = dy_A^\mu/d\tau_A$ is normalized to $g_{\mu\nu}^A u_A^\mu u_A^\nu = -1$. Here, $g_{\mu\nu}^A$ means that the

metric is evaluated at the location of the particle A , with the self-field contribution from A removed with the help of an appropriate self-field regularisation, namely DR.¹

In the approximation of point particles (pp) deprived of internal structure and unresponsive to tidal fields, the action is given by the standard mass term. To describe the response of the internal structure of the compact objects to tidal interactions, we add to the point-particle action the already truncated specific non-minimally coupled piece:²

$$L_A = -m_A c^2 + \frac{\mu_A^{(2)}}{4} G_{\mu\nu}^A G_A^{\mu\nu} + \frac{\sigma_A^{(2)}}{6c^2} H_{\mu\nu}^A H_A^{\mu\nu} + \frac{\mu_A^{(3)}}{12} G_{\lambda\mu\nu}^A G_A^{\lambda\mu\nu}. \quad (5.2)$$

To the NNL/2PN order investigated in this chapter, it is sufficient to consider the above three terms, made of quadratic products of mass and current tidal multipole moments, namely the mass quadrupole tidal moment $G_{\mu\nu}^A$, the current quadrupole $H_{\mu\nu}^A$ and the mass octupole $G_{\lambda\mu\nu}^A$.

The motivation for writing the Lagrangian (5.2) stems from the fact that the matter action for a given body, in the limit of small radius relevant for compact objects, in the post-Newtonian regime, can be expanded near the worldline of a representative point on which the resulting action is then localized, and that, in the absence of spins, it can be only built from the metric and its derivatives in a way that preserves parity and general covariance. We already emphasized the crucial role played by the self-field regularisation, which must properly be the DR in this framework.

5.1.1 Derivation of the covariant stress-energy tensor

In order to compute the stress-energy tensor, we first shift from the action (5.1) parametrized by the particle's proper time τ_A to an action defined in terms of an arbitrary parametrization $\bar{\tau}$. For instance, this parametrization can be the same for all particles. Once this is done, the ensuing expression for the action is manifestly invariant by reparametrization. We pose (with $\bar{u}_A^2 \equiv g_{\mu\nu}^A \bar{u}_A^\mu \bar{u}_A^\nu$)

$$d\tau_A = d\bar{\tau} \sqrt{-\bar{u}_A^2}, \quad u_A^\mu = \frac{\bar{u}_A^\mu}{\sqrt{-\bar{u}_A^2}}, \quad L_A = \frac{\bar{L}_A}{\sqrt{-\bar{u}_A^2}}. \quad (5.3)$$

The original action (5.1) now becomes (for simplicity's sake, we drop the particle's label until the end of Sec. 5.1.1)

$$S_m = \sum \int d\bar{\tau} \bar{L}(\bar{u}^\mu, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}). \quad (5.4)$$

As it is written, the Lagrangian \bar{L} is an (ordinary) function of the following variables, regarded as independent: the arbitrary parametrized four-velocity \bar{u}^μ , the covariant metric, the Riemann tensor and the covariant derivative of the Riemann tensor. The configuration variables are just the particle's positions $y^\mu(\bar{\tau})$ and their derivatives $\bar{u}^\mu(\bar{\tau}) = dy^\mu/d\bar{\tau}$. We thus define the linear momentum p_μ as the conjugate momentum of the position, *i.e.*,

$$p_\mu = \frac{\partial \bar{L}}{\partial \bar{u}^\mu}. \quad (5.5)$$

¹As discussed in Sec. 5.2 and proved in App. A of [7], at this order Hadamard and DR are equivalent. Thus I used Hadamard regularisation for the computations.

²We use the same conventions and notation as in Ref. [6]. See [100, 6] for more details, as well as [103, 104, 105, 96, 90, 97, 98, 102] for preceding fundamental works and alternative discussions. See also [117, 118] for general definitions of the Dixon moments, including spins, or [119, 120, 121] for a more practical approach at the level of the action.

Following Refs. [117, 118, 119, 121], we further introduce the quadrupole current $J^{\mu\nu\rho\sigma}$ and octupole current $J^{\lambda\mu\nu\rho\sigma}$ as³

$$J^{\mu\nu\rho\sigma} = -6 \frac{\partial \bar{L}}{\partial R_{\mu\nu\rho\sigma}}, \quad J^{\lambda\mu\nu\rho\sigma} = -12 \frac{\partial \bar{L}}{\partial \nabla_\lambda R_{\mu\nu\rho\sigma}}. \quad (5.6)$$

The current $J^{\mu\nu\rho\sigma}$, and the current $J^{\lambda\mu\nu\rho\sigma}$ on its four last indices, have the same symmetries as the Riemann tensor. In addition, $J^{\lambda\mu\nu\rho\sigma}$ satisfies the cyclic symmetry $J^{[\lambda\mu\nu]\rho\sigma} = 0$ as a consequence of the Bianchi identity.

By varying the action with respect to the worldline of the particle we obtain the EoM [118]

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{6} J^{\nu\rho\sigma\kappa} \nabla_\mu R_{\nu\rho\sigma\kappa} - \frac{1}{12} J^{\lambda\nu\rho\sigma\kappa} \nabla_\mu \nabla_\lambda R_{\nu\rho\sigma\kappa}. \quad (5.7)$$

Next, the stress-energy tensor is obtained by variation with respect to the metric. With the action depending on the Riemann tensor and its first covariant derivative, we get it as the sum of monopole, quadrupole and octupole pieces [121],

$$T^{\mu\nu} = T_{\text{mono}}^{\mu\nu} + T_{\text{quad}}^{\mu\nu} + T_{\text{oct}}^{\mu\nu}. \quad (5.8)$$

Note that there is no dipole contribution. The reason is because the dipolar part of the stress-energy tensor comes effectively only from spins, which are not considered in this computation. The monopole part takes the usual form of the stress-energy tensor of a particle with worldline y^μ , four-linear momentum p_μ and four-velocity u^μ (parametrized by τ), namely

$$T_{\text{mono}}^{\mu\nu} = \int d\tau p^{(\mu} u^{\nu)} \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g}}, \quad (5.9)$$

(with $\delta^{(4)}(x - y(\tau))$ the four-dimensional Dirac distribution), while the quadrupolar and octupolar pieces are given by

$$T_{\text{quad}}^{\mu\nu} = \int d\tau \left[\frac{1}{3} R^{(\mu}{}_{\lambda\rho\sigma} J^{\nu)\lambda\rho\sigma} \right] \frac{\delta^{(4)}(x - y)}{\sqrt{-g}} + \nabla_\rho \nabla_\sigma \int d\tau \left[-\frac{2}{3} J^{\rho(\mu\nu)\sigma} \right] \frac{\delta^{(4)}(x - y)}{\sqrt{-g}}, \quad (5.10a)$$

$$\begin{aligned} T_{\text{oct}}^{\mu\nu} &= \int d\tau \left[\frac{1}{6} \nabla^\lambda R^{(\mu}{}_{\xi\rho\sigma} J^{\nu)\xi\rho\sigma} + \frac{1}{12} \nabla^{(\mu} R_{\xi\tau\rho\sigma} J^{\nu)\xi\tau\rho\sigma} \right] \frac{\delta^{(4)}(x - y)}{\sqrt{-g}} \\ &+ \nabla_\rho \int d\tau \left[-\frac{1}{6} R^{(\mu}{}_{\xi\lambda\sigma} J^{\rho\nu)\xi\lambda\sigma} - \frac{1}{3} R^{(\mu}{}_{\xi\lambda\sigma} J^{\nu)\rho\xi\lambda\sigma} + \frac{1}{3} R^{\rho}{}_{\xi\lambda\sigma} J^{(\mu\nu)\xi\lambda\sigma} \right] \frac{\delta^{(4)}(x - y)}{\sqrt{-g}} \\ &+ \nabla_\lambda \nabla_\rho \nabla_\sigma \int d\tau \left[\frac{1}{3} J^{\sigma\rho(\mu\nu)\lambda} \right] \frac{\delta^{(4)}(x - y)}{\sqrt{-g}}. \end{aligned} \quad (5.10b)$$

As these formulas are general [117, 118, 119, 121], they can be applied to the specific case of the Lagrangian (5.2). With this in mind, I derived the stress-energy tensor using two different methods: first inserting (5.5) and (5.6) in (5.9) and (5.10); second by directly varying the original matter action using (1.9) and then indentifying the linear momentum as well as the two currents $J^{\mu\nu\rho\sigma}$ and $J^{\lambda\mu\nu\rho\sigma}$. The two methods gave the same results. For simplicity, the results are presented setting $c = 1$. I found for the linear momentum

$$p_\mu = m u_\mu + \mu^{(2)} \left[-R_{\mu\alpha\gamma\beta} u^\gamma G^{\alpha\beta} + \frac{3}{4} u_\mu G^{\alpha\beta} G_{\alpha\beta} \right] + \sigma^{(2)} \left[\frac{1}{3} R_{(\mu\alpha\gamma)\beta}^* u^\gamma H^{\alpha\beta} + \frac{1}{2} H^{\alpha\beta} H_{\alpha\beta} u_\mu \right] \quad (5.11)$$

³The chosen prefactors match previous definitions in the literature [121]. As shown in the Appendix A of [121], they are such that $J^{\mu\nu\rho\sigma}$ and $J^{\lambda\mu\nu\rho\sigma}$ coincide with the Dixon quadrupole and octupole moments [117, 118], respectively, at the considered approximation level. We refer to (5.6) as multipole ‘‘currents’’ in order to reduce the possible confusion with the tidal moments G_L , H_L as well as with the source multipole moments I_L , J_L considered in Sec. 5.3.

$$+ \mu^{(3)} \left[\frac{1}{4} G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} u_\mu - \frac{1}{3} G^{\alpha\beta\gamma} \nabla_\alpha^\perp R_{\beta\mu\gamma\rho} u^\rho - \frac{1}{6} G^{\alpha\beta\gamma} u_\alpha \nabla_\mu^\perp R_{\beta\rho\gamma\sigma} u^\rho u^\sigma - \frac{1}{6} \perp_{\mu\alpha} G^{\alpha\beta\gamma} u^\kappa \nabla_\kappa R_{\beta\rho\gamma\sigma} u^\rho u^\sigma \right],$$

and for the quadrupole and octupole currents

$$J^{\mu\nu\rho\sigma} = \mu^{(2)} \left(-3u^{[\mu} G^{\nu]\rho} u^{\sigma]} \right) + \sigma^{(2)} \left(\varepsilon^{\mu\nu}{}_{\alpha\beta} u^\alpha H^{\beta[\rho} u^{\sigma]} + \varepsilon^{\rho\sigma}{}_{\alpha\beta} u^\alpha H^{\beta[\mu} u^{\nu]} \right), \quad (5.12a)$$

$$J^{\lambda\mu\nu\rho\sigma} = \mu^{(3)} \left(-2 \perp_\kappa^\lambda u^{[\mu} G^{\nu]\kappa} u^{\rho} u^{\sigma]} \right). \quad (5.12b)$$

As a verification, I checked explicitly a relation that must hold as a consequence of the invariance of the action by worldline reparametrization, namely

$$p_\mu u^\mu = -m + \frac{\mu^{(2)}}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\sigma^{(2)}}{6} H_{\mu\nu} H^{\mu\nu} + \frac{\mu^{(3)}}{12} G_{\lambda\mu\nu} G^{\lambda\mu\nu} = L, \quad (5.13)$$

see Eq. (2.9) of [121] for more details on this relation.

5.1.2 Ready-to-use expressions

The stress-energy tensor is written in Eqs. (5.9) and (5.10) together with the explicit expressions (5.11) of the linear momentum and (5.12) of the currents, in terms of the tidal multipole moments. In turn, the tidal moments are given in terms of the metric, curvature and matter variables by Eqs. (4.13). In this section, we need to rephrase the previous results in a more suitable way in order to insert the stress-energy tensor in the PN-MPM formalism. We first need to split the indices into spatial and temporal ones, called (3+1) decomposition, so that the factors of c become explicit. Furthermore, we can get rid of the integral over τ in (5.8) by splitting the 4-dimensional Dirac to a 3-dimensional spatial Dirac and a 1-dimensional temporal one. By doing so, we can write the stress-energy tensor in the form

$$T^{\mu\nu} = \sum_A \left[U_A^{\mu\nu} \delta_A + \nabla_\alpha (U_A^{\mu\nu\alpha} \delta_A) + \nabla_\alpha \nabla_\beta (U_A^{\mu\nu\alpha\beta} \delta_A) + \nabla_\alpha \nabla_\beta \nabla_\gamma (U_A^{\mu\nu\alpha\beta\gamma} \delta_A) \right], \quad (5.14)$$

where we use the coordinate-time t parametrization and denote $\delta_A \equiv \delta^{(3)}(x^i - y_A^i(t))$ the usual three-dimensional Dirac distribution, and where (for $u_A^0 = dt/d\tau_A$)

$$U_A^{\mu\nu} = \frac{1}{u_A^0 \sqrt{-g}} \left(p_A^{(\mu} u_A^{\nu)} + \frac{1}{3} R_A^{(\mu}{}_{\lambda\rho\sigma} J_A^{\nu)\lambda\rho\sigma} + \frac{1}{6} \nabla_\lambda R_A^{(\mu}{}_{\xi\rho\sigma} J_A^{\lambda\nu)\xi\rho\sigma} + \frac{1}{12} \nabla^{(\mu} R_{A\xi\tau\rho\sigma} J_A^{\nu)\xi\tau\rho\sigma} \right), \quad (5.15a)$$

$$U_A^{\mu\nu\alpha} = \frac{1}{3u_A^0 \sqrt{-g}} \left(-\frac{1}{2} R_A^{(\mu}{}_{\xi\lambda\sigma} J_A^{\alpha\nu)\xi\lambda\sigma} - R_A^{(\mu}{}_{\xi\lambda\sigma} J_A^{\nu)\alpha\xi\lambda\sigma} + R_A^\alpha{}_{\xi\lambda\sigma} J_A^{(\mu\nu)\xi\lambda\sigma} \right), \quad (5.15b)$$

$$U_A^{\mu\nu\alpha\beta} = -\frac{2}{3u_A^0 \sqrt{-g}} J_A^{\alpha(\mu\nu)\beta}, \quad (5.15c)$$

$$U_A^{\mu\nu\alpha\beta\gamma} = -\frac{1}{3u_A^0 \sqrt{-g}} J_A^{\gamma\beta(\mu\nu)\alpha}. \quad (5.15d)$$

Note that all the U 's are symmetric over μ and ν ; in addition, $U_A^{\mu\nu\alpha\beta}$ and $U_A^{\mu\nu\alpha\beta\gamma}$, over their 4 first indices, have the same symmetries as the Jacobi tensor $R^{\alpha(\mu\nu)\beta}$ (due to their definitions (5.6)).

Next, by expanding the covariant derivatives in (5.14) as the sum of partial derivatives and Christoffel symbols, we get some ready-to-use formulas that are then directly entered into computational codes, namely

$$T^{\mu\nu} = \sum_A \left[\mathcal{T}_M^{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_\alpha (\mathcal{T}_D^{\mu\nu\alpha} \delta_A) + \frac{1}{\sqrt{-g}} \partial_{\alpha\beta} (\mathcal{T}_Q^{\mu\nu\alpha\beta} \delta_A) + \frac{1}{\sqrt{-g}} \partial_{\alpha\beta\gamma} (\mathcal{T}_O^{\mu\nu\alpha\beta\gamma} \delta_A) \right], \quad (5.16)$$

where

$$\mathcal{T}_M^{\mu\nu} = U^{\mu\nu} + 2\Gamma_{\lambda\rho}^{(\mu} U^{\nu)\lambda\rho} + \left[\partial_\lambda \Gamma_{\rho\sigma}^{(\mu} + \Gamma_{\rho\sigma}^\kappa \Gamma_{\lambda\kappa}^{(\mu} \right] U^{\nu)\lambda\rho\sigma} - \Gamma_{\lambda\kappa}^{(\mu} \Gamma_{\rho\sigma}^{\nu)} U^{\kappa\lambda\rho\sigma} \quad (5.17a)$$

$$\begin{aligned}
& + \left[\partial_\lambda \Gamma_{\gamma\delta}^{(\mu} \Gamma_{\rho\sigma}^{\nu)} + \partial_\sigma \left(\Gamma_{\rho\lambda}^{(\mu} \Gamma_{\gamma\delta}^{\nu)} \right) - 2\Gamma_{\lambda\sigma}^\kappa \Gamma_{\kappa\rho}^{(\mu} \Gamma_{\gamma\delta}^{\nu)} + \Gamma_{\gamma\delta}^\kappa \Gamma_{\kappa\rho}^{(\mu} \Gamma_{\lambda\sigma}^{\nu)} - 2\Gamma_{\sigma\gamma}^\kappa \Gamma_{\rho\lambda}^{(\mu} \Gamma_{\kappa\delta}^{\nu)} \right] U^{\rho\lambda\gamma\delta\sigma} \\
& + \left[2\Gamma_{\sigma\gamma}^\kappa \partial_\lambda \Gamma_{\kappa\delta}^{(\mu} + \Gamma_{\sigma\lambda}^\kappa \partial_\kappa \Gamma_{\gamma\delta}^{(\mu} - \partial_\sigma \left(\Gamma_{\gamma\delta}^\kappa \Gamma_{\kappa\lambda}^{(\mu} \right) + \Gamma_{\gamma\delta}^\rho \Gamma_{\sigma\lambda}^\kappa \Gamma_{\rho\kappa}^{(\mu} + 2\Gamma_{\kappa\delta}^\rho \Gamma_{\sigma\gamma}^\kappa \Gamma_{\rho\lambda}^{(\mu} - \partial_{\lambda\sigma} \Gamma_{\gamma\delta}^{(\mu} \right] U^{\nu)\lambda\gamma\delta\sigma}, \\
\mathcal{T}_D^{\mu\nu\alpha} & = \sqrt{-g} \left[U^{\mu\nu\alpha} + \Gamma_{\sigma\kappa}^\alpha U^{\mu\nu\sigma\kappa} - 2\Gamma_{\sigma\kappa}^{(\mu} U^{\nu)\alpha\sigma\kappa} - \partial_\sigma \Gamma_{\rho\lambda}^\alpha U^{\mu\nu\rho\lambda\sigma} + \partial_\sigma \Gamma_{\rho\lambda}^{(\mu} \left(2U^{\nu)\alpha\rho\lambda\sigma} + U^{\nu)\sigma\rho\lambda\alpha} \right) \right. \\
& \quad + \Gamma_{\lambda\gamma}^{(\mu} \left(U^{\nu)\lambda\sigma\rho\alpha} \Gamma_{\rho\sigma}^\gamma - \Gamma_{\rho\sigma}^{\nu)} U^{\gamma\lambda\sigma\rho\alpha} \right) + 2\Gamma_{\sigma\kappa}^\rho \left(\Gamma_{\rho\lambda}^\alpha U^{\mu\nu\kappa\lambda\sigma} - \Gamma_{\lambda\rho}^{(\mu} U^{\nu)\alpha\kappa\lambda\sigma} \right) \\
& \quad \left. - 2\Gamma_{\rho\lambda}^{(\mu} \left(U^{\nu)\alpha\kappa\lambda\sigma} \Gamma_{\sigma\kappa}^\rho - \Gamma_{\sigma\kappa}^{\nu)} U^{\alpha\kappa\rho\lambda\sigma} \right) + 2 \left(\Gamma_{\rho\lambda}^\alpha \Gamma_{\sigma\kappa}^{(\mu} - \Gamma_{\sigma\kappa}^\alpha \Gamma_{\rho\lambda}^{(\mu} \right) U^{\nu)\kappa\rho\lambda\sigma} \right], \quad (5.17b)
\end{aligned}$$

$$\mathcal{T}_Q^{\mu\nu\alpha\beta} = \sqrt{-g} \left[U^{\mu\nu\alpha\beta} + 2\Gamma_{\sigma\kappa}^{(\mu} \left(U^{\nu)\sigma\alpha\beta\kappa} - U^{\nu)\alpha\sigma\kappa\beta} \right) + \Gamma_{\sigma\kappa}^\alpha \left(2U^{\mu\nu\sigma\beta\kappa} + U^{\mu\nu\sigma\kappa\beta} \right) \right], \quad (5.17c)$$

$$\mathcal{T}_O^{\mu\nu\alpha\beta\gamma} = \sqrt{-g} U^{\mu\nu\alpha\beta\gamma}. \quad (5.17d)$$

I then inserted the above lengthy expression in Mathematica notebooks in order to find the explicit expression of $T^{\mu\nu}$ and especially its (3+1) decomposition.

In order to compute the multipole moments of the system I_L, J_L defined in (3.31), we require σ to be known at NNL/2PN order, σ_i at NL/1PN and σ_{ij} at leading order (see (3.37) for their definitions). For their derivation, I used the (3+1) decomposition of (5.16). This led to the complete, ready-to-use expressions for the sources σ 's that are split into a point-particle part

$$\sigma_{\text{pp}} = \frac{m_1 u_1^0}{\sqrt{-g_1}} \left(1 + \frac{v_1^2}{c^2} \right) \delta_1 + 1 \leftrightarrow 2, \quad (5.18a)$$

$$(\sigma_i)_{\text{pp}} = \frac{m_1 u_1^0}{\sqrt{-g_1}} v_1^i \delta_1 + 1 \leftrightarrow 2, \quad (5.18b)$$

$$(\sigma_{ij})_{\text{pp}} = \frac{m_1 u_1^0}{\sqrt{-g_1}} v_1^i v_1^j \delta_1 + 1 \leftrightarrow 2, \quad (5.18c)$$

and a ‘‘direct’’ tidal part $\sigma_{\text{tidal}}, (\sigma_i)_{\text{tidal}}$ and $(\sigma_{ij})_{\text{tidal}}$ expressed in terms of the tidal multipole moments; these are reported in Eqs. (D.5) due to their lengthy expressions. Beware that the point-particle part (5.18) will actually involve ‘‘indirectly’’ tidal effects contained into the potentials parametrizing the metric as computed in Sec. 5.2.

5.1.3 The tetrad choice

The tidal moments G_{ij}, H_{ij} and G_{ijk} (when evaluated at point 1) have been computed in (4.20). These expressions are not projected onto a tetrad. However, in order to present the expressions of $\sigma_{\text{tidal}}, (\sigma_i)_{\text{tidal}}$ and $(\sigma_{ij})_{\text{tidal}}$ as shown in (D.5) and everywhere henceforth, like for instance in (5.23), it is more convenient to use the tetradic components of these moments.

We pose $e_{\hat{\alpha}}{}^\mu = \partial x^\mu / \partial X^{\hat{\alpha}}$, where $\{x^\mu\}$ is a global coordinate system and $\{X^{\hat{\alpha}}\}$ is a local inertial frame in the vicinity of a given body. We choose $\{X^{\hat{\alpha}}\}$ to be a Fermi local normal coordinate system [122, 123], so that the tetrad is orthonormal on the worldline, the time coordinate of the Fermi coordinates coincides with the proper time along the worldline, and the zero-th time-like tetrad vector is the four velocity of the particle. More specifically, $e_{\hat{\alpha}}{}^\mu = (e_{\hat{0}}{}^\mu, e_{\hat{a}}{}^\mu)$ constructed as follows:

$$e_{\hat{0}}{}^\mu \equiv u^\mu, \quad (5.19a)$$

$$e_{\hat{a}}{}^\mu = \left(\gamma^{\mu i} - \gamma^{\mu 0} \frac{v^i}{c} \right) e_{\hat{a}i} \quad \text{with} \quad e_{\hat{a}i} \equiv (\sqrt{\gamma})_{\hat{a}i}. \quad (5.19b)$$

Here $\gamma^{\mu\nu} = g^{\mu\lambda} \perp_\lambda^\nu$ is the inverse of the positive-definite metric $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ induced by $g_{\mu\nu}$ on the hypersurface orthogonal to u^μ at the intersection point with the worldline, and the spatial tetrad vectors are defined from the square root $(\sqrt{\gamma})_{\hat{a}i}$ of the positive definite symmetric matrix γ_{ij} . One can show that this basis is complete and orthonormal (for more details, see [124]).

For simplicity, in the rest of this chapter, we will switch notation for the tidal tensors projected on the tetrad so that $G_{\hat{a}\hat{b}} \equiv \hat{G}_{ab}$, $H_{\hat{a}\hat{b}} \equiv \hat{H}_{ab}$ and $G_{\hat{a}\hat{b}\hat{c}} \equiv \hat{G}_{abc}$, dropping the hats on the tetradic indices. Remembering that the tidal moments are defined in the particle's local frame orthogonal to the four velocity, *i.e.* $\hat{G}_{0\alpha} = \hat{H}_{0\alpha} = \hat{G}_{0\alpha\beta} = 0$, we have for instance (similarly for H_{ij} and G_{ijk})

$$G_{ij} = e^a{}_i e^b{}_j \hat{G}_{ab}, \quad (5.20a)$$

$$\hat{G}_{ab} = \left[e_a{}^i e_b{}^j - 2 \frac{v_1^i}{c} e_{(a}{}^0 e_b)^j + \frac{v_1^i v_1^j}{c^2} e_a{}^0 e_b{}^0 \right] G_{ij}, \quad (5.20b)$$

where $e^\beta{}_\nu$ denotes the (transposed) inverse of $e_\alpha{}^\mu$. The projection of the tidal tensors onto this tetrad simplifies significantly the computations, mostly because the projected three-dimensional tidal tensors become traceless. However, the final results are independent on a particular choice of tetrad ($e_0{}^\mu, e_a{}^\mu$) used in intermediate calculations. Other groups [114] may have used different conventions for the tetrad with equivalent final results (see Table 5.1).

5.2 The potentials

For the calculation of the source multipole moments of the system (5.27), the metric, including tidal contributions, is required up to NNL/2PN order. This is in contrast with Chapter 4 on the NNL/2PN dynamics and EoM, where it was sufficient to insert the 2PN metric just for point particles, discarding internal structure effects. More specifically, the only quantity required to be known up to NNL/2PN in the source multipole moments is $\propto \sigma_{2\text{PN}}$, in which the metric only appears through $\sqrt{-g}$ and the Lorentz factor u^0 . This implies that, even though the 2PN metric (3.35) is required, we do not need to know all the potentials that parametrize it. In particular, in this calculation, \hat{X} and \hat{R}_i do not appear, meaning that only V at NL/1PN order as well as V_i and \hat{W}_{ij} at leading order will be necessary. We recall here their definitions

$$\square V = -4\pi G \sigma, \quad (5.21a)$$

$$\square V_i = -4\pi G \sigma_i, \quad (5.21b)$$

$$\square \hat{W}_{ij} = -4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V. \quad (5.21c)$$

Since the source terms involve both point particle and tidal contributions, the potentials themselves will involve a tidal part and the well-known pp part. The techniques I used to compute the potentials are well documented in *e.g.* [108, 9] and described partly in Sec. 6.3.2. In this work, dissipative radiation reaction effects can be ignored since they do not contribute to the flux until the 2.5PN order, so that the Green function will be taken to be the symmetric one. As explained in Sec. 3.2.2, it is essential to use a proper UV-type regularisation, namely DR. In fact, we showed in App. A of [7] that Hadamard and DR are equivalent also at NNL/2PN for tidal effects due to the structure of the terms entering the potentials and the multipoles. In practice I used the Hadamard regularisation for this computation.

The tidal contributions to the metric obtained in present formalism show an interesting feature: the tidal part of the potential V contains distributional terms,⁴ which arise because of the distributional multi-derivatives in the expressions of the matter sources (D.5), see Sec. 6.3.1 for more details on the treatment of distributional terms. To the lowest order, σ_{tidal} is proportional to $\hat{G}_{1ab} \partial_{ab}(1/r_1)$, and, since $\partial_{ab}(1/r_1) = 3\hat{n}_1^{ab} r_1^{-3} - \frac{4\pi}{3} \delta_{ab} \delta_1$, this leads to a distributional term for V proportional to the trace $\delta^{ab} \hat{G}_{1ab}$, but this quantity vanishes because the tidal tensors are projected onto the tetrad

⁴This does not happen for the point-particle part of the potentials even up to 4PN. This is because the point-particle sources do not contain derivatives of Dirac distributions.

and are traceless. At NL/1PN order, though, the distributional piece is non-zero. It is given by the Gel'fand-Shilov formula (see Eq. (6.27)) as

$$V^{\text{distr}} = \frac{2\pi}{5} \frac{G\mu_1^{(2)}}{c^2} \hat{G}_{1ab} v_1^a v_1^b \delta_1 + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^4}\right). \quad (5.22)$$

In the end, this term does not contribute to the calculation because, in the adopted approach, the NL potential V is only needed in a surface term at infinity where the UV regularisation is irrelevant. However, it would be important to take into account if we were to evaluate the equivalent volume integral. For the ordinary part of the complete potential V at the NL/1PN order, computed with Hadamard's regularisation, I found

$$\begin{aligned} V = & \frac{Gm_1}{r_1} + \frac{3G\mu_1^{(2)} \hat{G}_{1ab} n_1^a n_1^b}{2r_1^3} \\ & + \frac{1}{c^2} \left\{ Gm_1 \left[-\frac{(n_1 v_1)^2}{2r_1} + \frac{2v_1^2}{r_1} + Gm_2 \left(-\frac{r_1}{4r_{12}^3} - \frac{5}{4r_1 r_{12}} + \frac{r_2^2}{4r_1 r_{12}^3} \right) \right] \right. \\ & + \mu_1^{(2)} \left[G \left(\frac{3}{4r_1} \hat{G}_{1ab} \hat{G}_{1ab} + \left(3v_1^2 - \frac{15}{4}(n_1 v_1)^2 \right) \frac{\hat{G}_{1ab} n_1^a n_1^b}{r_1^3} + \frac{3}{2}(n_1 v_1) \left(\frac{\hat{G}_{1ab} n_1^a v_1^b}{r_1^3} - \frac{n_1^a n_1^b \partial_t \hat{G}_{1ab}}{r_1^2} \right) \right. \right. \\ & \quad \left. \left. + \frac{2n_1^a v_1^b \partial_t \hat{G}_{1ab}}{r_1^2} - \frac{n_1^a n_1^b \partial_t^2 \hat{G}_{1ab}}{4r_1} \right) \right. \\ & + \frac{G^2 m_2}{r_{12}^3} \left(\frac{3r_{12}}{2r_1^2} \hat{G}_{1ab} n_{12}^a n_1^b + \left(-\frac{6}{r_1} - \frac{3}{2r_2} \right) \hat{G}_{1ab} n_{12}^a n_{12}^b + \left(-\frac{3}{8r_1} - \frac{39r_{12}^2}{8r_1^3} + \frac{3r_2^2}{8r_1^3} \right) \hat{G}_{1ab} n_1^a n_1^b \right) \\ & \left. \left. + \frac{9}{2} \frac{G^3 m_2^2}{r_{12}^7} \left((n_{12} n_1) \left(1 - \frac{r_1}{r_2} \right) - \frac{r_{12}}{r_2} \right) \right] + \frac{4G\sigma_1^{(2)} \varepsilon_{bij} \hat{H}_{1aj} n_1^a n_1^b v_1^i}{r_1^3} \right\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (5.23) \end{aligned}$$

I also computed V_i at leading order, which reads

$$V_i = \frac{Gm_1}{r_1} v_1^i + \frac{3G\mu_1^{(2)} \hat{G}_{1ab} n_1^a n_1^b}{2r_1^3} v_1^i + \frac{G\mu_1^{(2)} n_1^a \partial_t \hat{G}_{1ai}}{2r_1^2} + \frac{G\sigma_1^{(2)} \varepsilon_{iab} \hat{H}_{1ak} n_1^b n_1^k}{r_1^3} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c}\right). \quad (5.24)$$

Note that, due to the way the leading term of $(\sigma_i)_{\text{tidal}}$ is written in Eq. (D.5b), some non-zero distributional terms are generated by multi-derivatives, but they cancel out in the end, so the potential V_i does not contain any. For the potential \hat{W}_{ij} at leading order, I found

$$\begin{aligned} \hat{W}_{ij} = & \frac{Gm_1}{r_1} (v_1^i v_1^j - \delta^{ij} v_1^2) + \frac{G^2 m_1^2}{4r_1^2} (n_1^i n_1^j - \delta_{ij}) - G^2 m_1 m_2 \partial_{1(i} \partial_{2j)} \ln S \\ & + \mu_1^{(2)} \left[\frac{G^2 m_2}{r_1 r_{12}^3} \left(\hat{G}_{1ij} - 3n_{12}^{(i} \hat{G}_{1j)a} n_{12}^a + 3\delta^{ij} \hat{G}_{1ab} n_{12}^a n_{12}^b \right) - G^2 m_2 \hat{G}_{1ab} \partial_{2(i} \partial_{1j)ab} \ln S \right. \\ & \quad \left. + \frac{G}{r_1^3} \left(\frac{r_1^2}{2} \partial_t^2 \hat{G}_{1ij} + r_1 n_1^a v_1^i \partial_t \hat{G}_{1ja} + \frac{3}{2} \hat{G}_{1ab} n_1^a n_1^b (v_1^i v_1^j - \delta^{ij} v_1^2) - \delta^{ij} r_1 n_1^a v_1^b \partial_t \hat{G}_{1ab} \right) \right] \\ & + \frac{2G\sigma_1^{(2)}}{r_1^3} \left[\varepsilon_{ab}^{(i} (v_1^j) \hat{H}_{1ak} n_1^b n_1^k + \frac{r_1}{3} \partial_t \hat{H}_{1ja} n_1^b) - \delta^{ij} \varepsilon_{abk} \hat{H}_{1kl} n_1^a n_1^l v_1^b \right] \\ & - G^2 \mu_1^{(2)} m_1 \hat{G}_{1ab} \left[\frac{5}{128} \partial_{ijab} \left[\ln \left(\frac{r_1}{r_0} \right) \right] + \frac{5}{16} \frac{n_1^a n_1^b}{r_1^4} \delta_{ij} + \frac{1}{4r_1^4} \delta^{a(i} \left(n_1^j) n_1^b - \frac{3}{8} \delta^{j)b} \right) \right] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c}\right). \quad (5.25) \end{aligned}$$

The pp part is depicted in the first line, where we denote $\partial_{A_i} \equiv \partial/\partial y_A^i$ and $S \equiv (r_1 + r_2 + r_{12})/r_0$.

These potentials satisfy $\square V = \mathcal{O}(1/c^4)$, $\Delta V_i = \mathcal{O}(1/c^2)$ and $\Delta \hat{W}_{ij} = -\partial_i V \partial_j V + \mathcal{O}(1/c^2)$ outside the particles. I also checked that they obey the harmonic gauge constraints formulated in terms of the potentials

$$\partial_t \left\{ V + \frac{1}{c^2} \left[\frac{1}{2} \hat{W} + 2V^2 \right] \right\} + \partial_i \left\{ V_i + \frac{2}{c^2} \left[\hat{R}_i + V V_i \right] \right\} = \mathcal{O} \left(\frac{1}{c^3} \right), \quad (5.26a)$$

$$\partial_t V_i + \partial_j \left\{ \hat{W}_{ij} - \frac{1}{2} \delta_{ij} \hat{W} \right\} = \mathcal{O} \left(\frac{1}{c} \right), \quad (5.26b)$$

which yield at the NL/1PN order the same EoM as obtained in the conservative sector, *i.e.* in (D.1). This test confirms the values of all potentials that are required for the integration of the source multipole moments in Sec. 5.3. Note that, for this verification, I had to compute V_i at NL/1PN order as well as \hat{R}_i at lowest order, where \hat{R}_i is defined in Eq. (3.36). I do not give their values here since they do not enter later calculations.

In order to derive the multipole moments, we have to also know the value of the potentials at the location of the two bodies. If one would take the limit $x^i \rightarrow y_1^i$ in the potentials in the whole space, $1/r_1$ would diverge and thus would find infinity. Of course, we have to use a UV regularisation. I computed the values of the potentials at point A using Hadamard. The procedure to get the value of an expression regularised at point A is explained in Sec. 6.3.2.

5.3 The source multipole moments

As discussed in Sec. 3.1.5, the STF multipole moments of an isolated PN radiative source are obtained by performing a matching between the inner PN expansion in the system near zone and the outer MPM expansion in the far zone. Their expressions are given in (3.31). For convenience, we can decompose I_L into three pieces corresponding to the three terms entering (3.31a), referred to as scalar (S), vector (V) and tensor (T) terms. Applying the formula (3.34), we further split each of these pieces into various parts labelled I, II, III, ... according to their PN order. This leads to the decomposition of the ℓ -th order mass-type moment to NNL/2PN order as

$$I_L = \text{SI}_L + \text{SII}_L + \text{SIII}_L + \text{VI}_L + \text{VII}_L + \text{TI}_L, \quad (5.27a)$$

$$\begin{aligned} \text{SI}_L = \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_L \left\{ \sigma + \frac{4V}{c^4} \sigma_{ii} - \frac{2}{\pi G c^4} V_i \partial_t \partial_i V - \frac{1}{\pi G c^4} \hat{W}_{ij} \partial_{ij}^2 V - \frac{1}{2\pi G c^4} (\partial_t V)^2 \right. \\ \left. + \frac{2}{\pi G c^4} \partial_i V_j \partial_j V_i - \frac{1}{2\pi G c^2} \Delta(V^2) - \frac{2}{3\pi G c^4} \Delta(V^3) - \frac{1}{2\pi G c^4} \Delta(V \hat{W}) \right\}, \end{aligned} \quad (5.27b)$$

$$\text{SII}_L = \frac{1}{2c^2(2\ell+3)} \frac{d^2}{dt^2} \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_L r^2 \left\{ \sigma + \frac{4}{c^2} \sigma V - \frac{1}{\pi G c^2} \partial_i V \partial_i V \right\}, \quad (5.27c)$$

$$\text{SIII}_L = \frac{1}{8c^4(2\ell+3)(2\ell+5)} \frac{d^4}{dt^4} \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_L r^4 \sigma, \quad (5.27d)$$

$$\begin{aligned} \text{VI}_L = -\frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \frac{d}{dt} \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_{iL} \left\{ \sigma_i + \frac{2}{c^2} \sigma_i V - \frac{2}{c^2} \sigma V_i + \frac{1}{\pi G c^2} \partial_j V \partial_i V_j \right. \\ \left. + \frac{3}{4\pi G c^2} \partial_t V \partial_i V - \frac{1}{2\pi G c^2} \Delta(V V_i) \right\}, \end{aligned} \quad (5.27e)$$

$$\text{VII}_L = -\frac{2(2\ell+1)}{c^4(\ell+1)(2\ell+3)(2\ell+5)} \frac{d^3}{dt^3} \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_{iL} r^2 \sigma_i, \quad (5.27f)$$

$$\text{TI}_L = \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \frac{d^2}{dt^2} \mathcal{F}\mathcal{P} \int_{B=0} d^3 \mathbf{x} \tilde{r}^B \hat{x}_{ijL} \left\{ \sigma_{ij} + \frac{1}{4\pi G} \partial_i V \partial_j V \right\}. \quad (5.27g)$$

Similarly, for the ℓ -th order current moments to NNL/2PN order,

$$J_L = \text{VI}_L + \text{VII}_L + \text{TI}_L, \quad (5.28a)$$

$$\text{VI}_L = \varepsilon_{ab\langle i\ell} \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_{L-1\rangle a} \left\{ \sigma_b + \frac{1}{c^2} \left[2(\sigma_b V - \sigma V_b) + \frac{1}{\pi G} \left(\partial_i V \partial_b V_i + \frac{3}{4} \partial_t V \partial_b V - \frac{1}{2} \Delta(V V_b) \right) \right] \right\} \quad (5.28b)$$

$$\text{VII}_L = \frac{1}{2c^2(2\ell+3)} \varepsilon_{ab\langle i\ell} \frac{d^2}{dt^2} \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B r^2 \hat{x}_{L-1\rangle a} \sigma_b, \quad (5.28c)$$

$$\text{TI}_L = -\frac{(2\ell+1)}{c^2(\ell+2)(2\ell+3)} \varepsilon_{ab\langle i\ell} \frac{d}{dt} \mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_{L-1\rangle ac} \left\{ \sigma_{bc} + \frac{1}{4\pi G} \partial_b V \partial_c V \right\}. \quad (5.28d)$$

In the above expressions, three types of terms can be distinguished:

- compact support (C) terms, whose integrands are proportional to the matter currents σ 's
- non-compact (NC) support terms, whose volume integrals extend up to infinity

These two types of terms can be integrated exactly. As mentioned above, at this order, Hadamard regularisation is equivalent to DR and can thus be employed systematically. The method to compute the C and NC terms is widely explained in Secs. 6.3.1 and 6.3.2.

- “surface” terms, also NC, but whose integrands are products of \hat{x}_L and pure Laplacians

To integrate these terms, assuming that the expansion of F when $r \rightarrow \infty$ is power-like (without logarithms), it can be proved that

$$\mathcal{F}\mathcal{P}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_L \Delta F = -4\pi(2\ell+1) (F r^{\ell+1} \hat{n}_L)_\infty, \quad (5.29)$$

where the notation $(\dots)_\infty$ means the Hadamard partie finie regularisation at infinity, see the general formula (6.48). With this formula, I have shown that, at NNL/2PN order, all the terms of this type for $F = \{V^2, V^3, V\hat{W}, VV_i\}$ vanish.

The explicit expressions of the multipole moments of the system to NNL/2PN are too long to be listed here. However, they are substantially shortened in the CoM frame and are displayed in App. D.2. I used the same method as in Sec. 4.4 to perform the frame transformation. The total source moments for circular orbits including both point-particle and tidal parts read

$$\begin{aligned} \text{I}_{ij} = & m r^2 \left[n^{(i} n^{j)} \left\{ \nu \left[1 + \left(-\frac{1}{42} - \frac{13}{14} \nu \right) \gamma + \left(-\frac{461}{1512} - \frac{23435}{1512} \nu - \frac{241}{1512} \nu^2 \right) \gamma^2 \right] \right. \right. \\ & + \left(3\tilde{\mu}_+^{(2)} + 3\Delta\tilde{\mu}_-^{(2)} \right) \gamma^5 + \left[\tilde{\mu}_+^{(2)} \left(-\frac{3}{2} + \frac{1}{7} \nu - \frac{222}{7} \nu^2 \right) + \Delta\tilde{\mu}_-^{(2)} \left(-\frac{3}{2} - \frac{67}{7} \nu \right) + \frac{160}{3} \nu \tilde{\sigma}_+^{(2)} \right] \gamma^6 \\ & + \left[\tilde{\mu}_+^{(2)} \left(\frac{871}{56} + \frac{907}{168} \nu + \frac{9643}{168} \nu^2 + \frac{929}{42} \nu^3 \right) + \Delta\tilde{\mu}_-^{(2)} \left(\frac{871}{56} + \frac{1853}{24} \nu - \frac{7201}{168} \nu^2 \right) \right. \\ & + \left. \tilde{\sigma}_+^{(2)} \left(\frac{388}{9} \nu - \frac{2504}{7} \nu^2 \right) + \frac{1732}{63} \Delta\nu \tilde{\sigma}_-^{(2)} \right] \gamma^7 \left. \right\} + \lambda^{(i} \lambda^{j)} \left\{ \nu \left[\left(\frac{11}{21} - \frac{11}{7} \nu \right) \gamma \right. \right. \\ & + \left(\frac{1013}{378} + \frac{299}{378} \nu - \frac{365}{378} \nu^2 \right) \gamma^2 \left. \right] + \left[\tilde{\mu}_+^{(2)} \left(3 + \frac{104}{7} \nu - \frac{198}{7} \nu^2 \right) \right. \\ & + \left. \Delta\tilde{\mu}_-^{(2)} \left(3 - \frac{38}{7} \nu \right) + \frac{128}{3} \nu \tilde{\sigma}_+^{(2)} \right] \gamma^6 + \left[\tilde{\mu}_+^{(2)} \left(-\frac{19}{2} + \frac{617}{42} \nu + \frac{5039}{42} \nu^2 + \frac{260}{21} \nu^3 \right) \right. \\ & + \left. \Delta\tilde{\mu}_-^{(2)} \left(-\frac{19}{2} + \frac{1291}{42} \nu - \frac{1649}{42} \nu^2 \right) + \tilde{\sigma}_+^{(2)} \left(-\frac{64}{9} \nu - \frac{1696}{7} \nu^2 \right) + \frac{2048}{63} \Delta\nu \tilde{\sigma}_-^{(2)} \right] \gamma^7 \left. \right\}, \quad (5.30a) \end{aligned}$$

$$\begin{aligned}
I_{ijk} = m\nu r^3 & \left[n^{(i} n^j n^{k)} \left\{ -\Delta (1 - \nu\gamma) + 18\tilde{\mu}_-^{(2)} \gamma^5 + \left[\Delta\tilde{\mu}_+^{(2)} \left(-\frac{3}{2} + 48\nu \right) + \tilde{\mu}_-^{(2)} \left(-\frac{39}{2} - 60\nu \right) - 84\Delta\tilde{\sigma}_+^{(2)} \right. \right. \right. \\
& \left. \left. + 84\tilde{\sigma}_-^{(2)} \right] \gamma^6 \right\} + n^{(i} \lambda^j \lambda^{k)} \left\{ -\Delta (1 - 2\nu)\gamma + \left[\Delta\tilde{\mu}_+^{(2)} (-39 + 36\nu) + \tilde{\mu}_-^{(2)} (39 - 42\nu) \right. \right. \\
& \left. \left. - 72\Delta\tilde{\sigma}_+^{(2)} + 72\tilde{\sigma}_-^{(2)} \right] \gamma^6 \right\} \right], \tag{5.30b}
\end{aligned}$$

$$I_{ijkl} = m\nu r^4 n^{(i} n^j n^k n^{l)} \left[1 - 3\nu + \left(18\tilde{\mu}_+^{(2)} - 18\Delta\tilde{\mu}_-^{(2)} \right) \gamma^5 \right], \tag{5.30c}$$

$$\begin{aligned}
J_{ij} = \sqrt{G} (mr)^{3/2} \ell^{(i} n^{j)} & \left\{ -\Delta\nu \left[1 + \left(\frac{25}{28} + \frac{3}{14}\nu \right) \gamma \right] + \left(-9\Delta\nu\tilde{\mu}_+^{(2)} + 9\nu\tilde{\mu}_-^{(2)} + 12\Delta\tilde{\sigma}_+^{(2)} + 12\tilde{\sigma}_-^{(2)} \right) \gamma^5 \right. \\
& + \left[\Delta\tilde{\mu}_+^{(2)} \left(\frac{663}{28}\nu + \frac{117}{7}\nu^2 \right) + \tilde{\mu}_-^{(2)} \left(-\frac{177}{7}\nu + \frac{477}{14}\nu^2 \right) + \Delta\tilde{\sigma}_+^{(2)} \left(-10 - \frac{690}{7}\nu \right) \right. \\
& \left. \left. + \tilde{\sigma}_-^{(2)} \left(-10 - \frac{346}{7}\nu \right) \right] \gamma^6 \right\}, \tag{5.30d}
\end{aligned}$$

$$J_{ijk} = \sqrt{G} (mr)^{5/2} \frac{\nu}{m} \ell^{(i} n^j n^{k)} \left\{ 1 - 3\nu + \left[\tilde{\mu}_+^{(2)} (21 - 27\nu) - 12\Delta\tilde{\mu}_-^{(2)} + 64\tilde{\sigma}_+^{(2)} \right] \gamma^5 \right\}. \tag{5.30e}$$

We remind here some previously introduced notations: the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = \frac{m_1 m_2}{m^2}$, the normalized mass difference $\Delta \equiv \frac{m_1 - m_2}{m}$, the PN parameter $\gamma = \frac{Gm}{rc^2}$, the relative velocity $v^i = v_1^i - v_2^i$, the adimensionalized tidal polarizabilities of Eq. (4.28). We also denote by \mathbf{n} the unit direction pointing from body 2 to 1, $\boldsymbol{\lambda}$ the unit vector perpendicular to \mathbf{n} in the orbital plane pointing to the same direction than the velocity, and $\boldsymbol{\ell}$ the unit vector perpendicular to the orbital plane, such that $(\mathbf{n}, \boldsymbol{\lambda}, \boldsymbol{\ell})$ forms an oriented orthonormal triad. This means notably that $\lambda^i = \frac{v^i}{r\omega}$ for exactly circular orbits.

5.4 The GW flux

The general expression of the GW flux in terms of radiative multipole moments is given in Eq. (3.18). Its truncated expression at 2PN reads

$$\mathcal{F} = \frac{G}{c^5} \left\{ \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} + \frac{1}{c^2} \left[\frac{1}{189} U_{ijk}^{(1)} U_{ijk}^{(1)} + \frac{16}{45} V_{ij}^{(1)} V_{ij}^{(1)} \right] + \frac{1}{c^4} \left[\frac{1}{9072} U_{ijkm}^{(1)} U_{ijkm}^{(1)} + \frac{1}{84} V_{ijk}^{(1)} V_{ijk}^{(1)} \right] + \mathcal{O} \left(\frac{1}{c^6} \right) \right\}. \tag{5.31}$$

In order to compute it, we need to compute the difference between the radiative and the source multipole moments, which is, at this order, only composed of simple tail terms (see (5.32)).

5.4.1 Computation of the tail terms

The general expression of the radiative multipoles with respect to the source ones up to 2PN are (see Sec. 3.1.3)

$$U_L(t) = I_L^{(\ell)}(t) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau I_L^{(\ell+2)}(t - \tau) \ln \left(\frac{\tau}{\tau_\ell} \right) + \mathcal{O} \left(\frac{1}{c^5} \right), \tag{5.32a}$$

$$V_L(t) = J_L^{(\ell)}(t) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau J_L^{(\ell+2)}(t - \tau) \ln \left(\frac{\tau}{\lambda_\ell} \right) + \mathcal{O} \left(\frac{1}{c^5} \right), \tag{5.32b}$$

where τ_ℓ and λ_ℓ are some constant coefficients⁵ and M is the ADM mass which in the case of circular orbits reads

$$M = m + \frac{E}{c^2} = m \left[1 - \frac{\nu x}{2} \left(1 - 18 \tilde{\mu}_+^{(2)} x^5 \right) \right] + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^4} \right), \quad (5.33)$$

where x is defined in (3.44). To compute the flux (5.31) for circular orbits, it is simpler to compute the tail part directly in the flux because a lot of terms coming from the tail contribution cancel out in the end. As an example, let us compute the tail part of $U_{ij}^{(1)} U_{ij}^{(1)}$ which is $\propto I_{ij}^{(3)} \int d\tau I_{ij}^{(5)} \ln(\tau/\tau_\ell)$. For circular orbits, $I_{ij} = A n^{(i} n^{j)} + B \lambda^{(i} \lambda^{j)} + O(1/c^4)$ where A and B are time-independent expressions. These two unit vectors lying in the orbital plane read $n^i = (\cos(\omega\tau), \sin(\omega\tau), 0)$ and $\lambda^i = (-\sin(\omega\tau), \cos(\omega\tau), 0)$. Thus, the time derivatives act only on the vectors n^i and λ^i . They do it in such a way that $dn^i/dt = \omega \lambda^i$ and $d\lambda^i/dt = -\omega n^i$. In the end, the computation of the tail terms in the flux only require to know the following integral (given in 4.331 page 571 of [125])

$$\int_0^{+\infty} d\tau e^{i\omega\tau} \ln \left(\frac{\tau}{2p} \right) = \frac{1}{\omega} \left[-\frac{\pi}{2} \text{sgn}(\omega) - i \left(\ln(2p|\omega|) + \gamma_E \right) \right]. \quad (5.34)$$

Finally, I applied the method to compute the tail contributions of the NL/1PN source mass quadrupole, as well as the leading orders of the source current quadrupole and source mass octupole. I recovered the well-known point-particle part of the tails in the flux and found for the tidal part

$$\begin{aligned} \mathcal{F}_{\text{tidal tail}} = \frac{768\pi c^5}{5G} \nu x^{23/2} & \left\{ (1 + 4\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} + \left[\left(-\frac{22}{21} - \frac{5053}{1344} \nu - \frac{2029}{48} \nu^2 \right) \tilde{\mu}_+^{(2)} \right. \right. \\ & \left. \left. + \Delta \left(-\frac{22}{21} - \frac{351}{64} \nu \right) \tilde{\mu}_-^{(2)} + \left(-\frac{1}{18} + \frac{226}{9} \nu \right) \tilde{\sigma}_+^{(2)} - \frac{\Delta}{18} \tilde{\sigma}_-^{(2)} \right] x \right\}. \quad (5.35) \end{aligned}$$

Note that in Eq. (5.32), the tail term contains a prefactor $1/c^3$. Since the source multipole moments contain only even powers of c (up to 2PN), the odd powers of c correspond to dissipative effects. Consequently, the tail contributions do not mix with the instantaneous ones in the flux and phase. This is why they are half-integer powers of x in (5.35).

5.4.2 Computation of the flux

For the instantaneous part, we need to differentiate the source multipoles with respect to time using the EoM (D.1). Then, we combine the instantaneous and tail parts to get the total flux $\mathcal{F} = \mathcal{F}_{\text{pp}} + \mathcal{F}_{\text{tidal}}$. The part generated by point particles without internal structure, \mathcal{F}_{pp} , already known [126, 127, 128, 129], is given to consistent order by

$$\begin{aligned} \mathcal{F}_{\text{pp}} = \frac{32c^5 \nu^2 x^5}{5G} & \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12} \nu \right) x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 \right. \\ & \left. + \left(-\frac{8191}{672} - \frac{583}{24} \nu \right) \pi x^{5/2} \right\}. \quad (5.36) \end{aligned}$$

On the other hand, the tidal contribution to the flux complete to NNL/2PN order reads

$$\begin{aligned} \mathcal{F}_{\text{tidal}} = \frac{192c^5 \nu x^{10}}{5G} & \left\{ (1 + 4\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} \right. \\ & \left. + \left[\left(-\frac{22}{21} - \frac{1217}{168} \nu - \frac{155}{6} \nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(-\frac{22}{21} - \frac{23}{24} \nu \right) \tilde{\mu}_-^{(2)} + \left(-\frac{1}{9} + \frac{76}{3} \nu \right) \tilde{\sigma}_+^{(2)} - \frac{1}{9} \Delta \tilde{\sigma}_-^{(2)} \right] x \right\} \quad (5.37) \end{aligned}$$

⁵Explicitly $\tau_\ell = 2r_0 e^{-\kappa_\ell}/c$ and $\lambda_\ell = 2r_0 e^{-\pi_\ell}/c$ with κ_ℓ and π_ℓ are defined in (3.24)

$$\begin{aligned}
& + 4\pi \left[(1 + 4\nu)\tilde{\mu}_+^{(2)} + \Delta\tilde{\mu}_-^{(2)} \right] x^{3/2} \\
& + \left[\left(\frac{167}{54} - \frac{722429}{18144}\nu + \frac{15923}{336}\nu^2 + \frac{965}{12}\nu^3 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{167}{54} + \frac{66719}{2016}\nu - \frac{2779}{144}\nu^2 \right) \tilde{\mu}_-^{(2)} \right. \\
& \quad \left. + \left(-\frac{173}{756} + \frac{145}{3}\nu - 208\nu^2 \right) \tilde{\sigma}_+^{(2)} + \Delta \left(-\frac{173}{756} + \frac{1022}{27}\nu \right) \tilde{\sigma}_-^{(2)} + \frac{80}{3}\nu\tilde{\mu}_+^{(3)} \right] x^2 \\
& + 4\pi \left[\left(-\frac{22}{21} - \frac{5053}{1344}\nu - \frac{2029}{48}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(-\frac{22}{21} - \frac{351}{64}\nu \right) \tilde{\mu}_-^{(2)} + \left(-\frac{1}{18} + \frac{226}{9}\nu \right) \tilde{\sigma}_+^{(2)} \right. \\
& \quad \left. - \frac{\Delta}{18}\tilde{\sigma}_-^{(2)} \right] x^{5/2} \Bigg\}.
\end{aligned}$$

At this stage, we have in hand the total GW energy flux given by (5.36)-(5.37) and also the conserved energy of the system (4.42). We are now able to insert both in the balance equation (3.42) in order to derive the GW phase.

5.5 The GW phase

Together with the conservative energy of the system (4.42), the above energy flux permits determining the frequency and phase evolution for circular orbits through the two ordinary differential equations

$$\frac{d\omega}{dt} = -\frac{\mathcal{F}(\omega)}{dE/d\omega}, \quad \frac{d\varphi}{dt} = \omega. \quad (5.38)$$

There are various ways to solve those equations approximately, called PN approximants, yielding significant deviations from numerical relativity at small separations, *i.e.*, outside the domain of validity of the PN expansion [87]. Following the simplest adiabatic Taylor PN approximant, I obtained the phase in the time domain as $\varphi = \varphi_{\text{pp}} + \varphi_{\text{tidal}}$, where I recovered the point-particle part at 2.5PN,

$$\begin{aligned}
\varphi_{\text{pp}} = & -\frac{1}{32\nu x^{5/2}} \left\{ 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu \right) x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2 \right) x^2 \right. \\
& \left. + \left(\frac{38645}{1344} - \frac{65}{16}\nu \right) \pi x^{5/2} \ln\left(\frac{x}{x_0}\right) \right\}, \quad (5.39)
\end{aligned}$$

and found the tidal contribution to be

$$\begin{aligned}
\varphi_{\text{tidal}} = & -\frac{3x^{5/2}}{16\nu^2} \left\{ (1 + 22\nu)\tilde{\mu}_+^{(2)} + \Delta\tilde{\mu}_-^{(2)} \right. \\
& + \left[\left(\frac{195}{56} + \frac{1595}{14}\nu + \frac{325}{42}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{195}{56} + \frac{4415}{168}\nu \right) \tilde{\mu}_-^{(2)} + \left(-\frac{5}{63} + \frac{3460}{21}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{5}{63}\Delta\tilde{\sigma}_-^{(2)} \right] x \\
& - \frac{5\pi}{2} \left[(1 + 22\nu)\tilde{\mu}_+^{(2)} + \Delta\tilde{\mu}_-^{(2)} \right] x^{3/2} \\
& + \left[\left(\frac{136190135}{9144576} + \frac{975167945}{1524096}\nu - \frac{281935}{2016}\nu^2 + 5\nu^3 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{136190135}{9144576} + \frac{211985}{864}\nu + \frac{1585}{432}\nu^2 \right) \tilde{\mu}_-^{(2)} \right. \\
& \quad \left. + \left(-\frac{745}{1512} + \frac{1933490}{1701}\nu - \frac{3770}{27}\nu^2 \right) \tilde{\sigma}_+^{(2)} + \Delta \left(-\frac{745}{1512} + \frac{19355}{81}\nu \right) \tilde{\sigma}_-^{(2)} + \frac{1000}{9}\nu\tilde{\mu}_+^{(3)} \right] x^2 \\
& + \pi \left[\left(-\frac{397}{32} - \frac{5343}{16}\nu + \frac{1315}{12}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(-\frac{397}{32} - \frac{6721}{96}\nu \right) \tilde{\mu}_-^{(2)} + \left(\frac{1}{3} - \frac{4156}{9}\nu \right) \tilde{\sigma}_+^{(2)} + \frac{\Delta}{3}\tilde{\sigma}_-^{(2)} \right] x^{5/2} \Bigg\}. \quad (5.40)
\end{aligned}$$

Next, motivated by data analysis applications, I derived the phase in the Fourier domain within the stationary-phase approximation in the form $\psi^{\text{SPA}} = 2\pi f t_c + \psi_{\text{pp}} + \psi_{\text{tidal}}$ where

$$\psi_{\text{pp}} = \frac{3}{128\nu x^{5/2}} \left\{ 1 + \left(\frac{3715}{756} + \frac{55}{9}\nu \right) x - 16\pi x^{3/2} + \left(\frac{15293365}{508032} + \frac{27145}{504}\nu + \frac{3085}{72}\nu^2 \right) x^2 \right. \quad (5.41)$$

$$+ \left(\frac{38645}{252} - \frac{65}{3} \nu \right) \pi x^{5/2} \ln \left(\frac{x}{x_0} \right) \Big\},$$

and

$$\begin{aligned} \psi_{\text{tidal}} = & - \frac{9x^{5/2}}{16\nu^2} \left\{ (1 + 22\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} \right. & (5.42) \\ & + \left[\left(\frac{195}{112} + \frac{1595}{28} \nu + \frac{325}{84} \nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{195}{112} + \frac{4415}{336} \nu \right) \tilde{\mu}_-^{(2)} + \left(-\frac{5}{126} + \frac{1730}{21} \nu \right) \tilde{\sigma}_+^{(2)} - \frac{5}{126} \Delta \tilde{\sigma}_-^{(2)} \right] x \\ & - \pi \left[(1 + 22\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} \right] x^{3/2} \\ & + \left[\left(\frac{136190135}{27433728} + \frac{975167945}{4572288} \nu - \frac{281935}{6048} \nu^2 + \frac{5}{3} \nu^3 \right) \tilde{\mu}_+^{(2)} + \Delta \left(\frac{136190135}{27433728} + \frac{211985}{2592} \nu + \frac{1585}{1296} \nu^2 \right) \tilde{\mu}_-^{(2)} \right. \\ & \quad \left. + \left(-\frac{745}{4536} + \frac{1933490}{5103} \nu - \frac{3770}{81} \nu^2 \right) \tilde{\sigma}_+^{(2)} + \Delta \left(-\frac{745}{4536} + \frac{19355}{243} \nu \right) \tilde{\sigma}_-^{(2)} + \frac{1000}{27} \nu \tilde{\mu}_+^{(3)} \right] x^2 \\ & + \pi \left[\left(-\frac{397}{112} - \frac{5343}{56} \nu + \frac{1315}{42} \nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left(-\frac{397}{112} - \frac{6721}{336} \nu \right) \tilde{\mu}_-^{(2)} + \left(\frac{2}{21} - \frac{8312}{63} \nu \right) \tilde{\sigma}_+^{(2)} \right. \\ & \quad \left. + \frac{2}{21} \Delta \tilde{\sigma}_-^{(2)} \right] x^{5/2} \Big\}. \end{aligned}$$

The tidal phase is now known completely for non-spinning compact objects in inspiralling binary systems up to 2.5PN beyond the leading order.

5.6 Conclusion of Part II

In this chapter and the preceding one, I have solved the problem of the dynamics and GW emission of compact binary systems without spins for (adiabatic) tidal, internal structure-dependent effects at the NNL/2PN order, meaning formally the order 7.5PN (taking into account tails) in the GW phase evolution. I used the formalism of the effective matter action of Ref. [100], which describes massive point-like particles with internal structure by introducing specific non-minimal couplings to the space-time curvature that model the finite size effects of the compact bodies due to the tidal interactions. Since the matter action is localized on the worldline of the particles, it is sometimes referred to as a “skeletonized” action. Up to NNL/2PN order there appear three polarizability coefficients corresponding to mass quadrupole, current quadrupole and mass octupole tidal interactions. In Chapter 4, summing up [6] and [8], I derived the associated effective Fokker action to obtain the conservative dynamics, *i.e.*, the EoM and conserved integrals of the motion. I also derived the associated Hamiltonian in isotropic coordinates in order to compare it with literature and I found total agreement on the overlap between their PM result and the present PN result. In Chapter 5, summing up [7], I computed the matter stress-energy tensor of the compact binary from the same effective action, and inserted it into a GW generation formalism based on MPM approximations for the external field [71], which are matched to the PN expansion of the inner field [76, 77]. The PN-MPM approach constitutes a very general way for computing the GW emission (and radiation reaction onto the source) once one is given the matter stress-energy-tensor. In particular, I resorted to general ready-to-use expressions for the source multipole moments and nonlinear interactions between those moments (tails, *etc.*) leading to the observable waveform at infinity and, thus, the energy flux. At last, once the flux up to NNL/2PN for tidal effects had been obtained and reduced for circular orbits, I combined it with the result for the conservative energy found in [6]. Namely, I employed the standard flux-balance argument to determine the binary’s chirp, *i.e.*, the orbital phase and frequency evolution through GW emission for compact binaries on quasi-circular orbits.

These results extend and complete several previous results in the literature. In Table 5.1 are summarized the previous achievements in the field for each PN order and tidal multipole component.

The result agree with all the previous results quoted in Table 5.1. Finally the tidal phase of non-spinning NS binaries is complete up to the NNL order including NL tails, which means formally up to the high 7.5PN level. Remark that three new coefficients in the GW phase have been computed in this

φ_{tidal}	Mass quadrupole	Current quadrupole	Mass octupole
5PN (L)	[90, 113, 91, 114, 92] ✓		
6PN (NL)	[91, 114, 45] ✓	[45, 115] ✓	
6.5PN (tail)	[91, 45] ✓		
7PN (NNL)	✓	✓	[45, 116] ✓
7.5PN (tail)	✓	✓	✓

Table 5.1: Comparison with the existing literature. The checkmarks ✓ indicate the contributions I computed. The citations correspond to the papers our result agree with for the tidal phase.

work. However, I disagree with some coefficients in the literature. First, with the 6PN coefficient due to the current quadrupole moment computed in the preprint [116]. Indeed, as mentioned in Sec. 4.6, the mistake in the computation of the conserved energy of this paper changes also the value of the phase. Second, I also disagree with the mass quadrupole contribution to the tail term at the 7.5PN order as reported in Ref. [91]. The latter reference obtains for the mass quadrupole contributions to the SPA phase of two identical NS (see Eq. (31) in [91]):

$$\psi_{\text{tidal}}^{\text{DNV}} = -\kappa_2^T \frac{39}{4} v^5 \left[1 + \frac{3115}{1248} v^2 - \pi v^3 + \left(\frac{23073805}{3302208} + \frac{20}{81} \bar{\alpha}_2^{(2)} + \frac{20}{351} \beta_2^{22} \right) v^4 - \frac{4283}{1092} \pi v^5 \right],$$

with $\kappa_2^T = 6\tilde{\mu}_+^{(2)}$ in their notation (recall that we have $\tilde{\mu}_-^{(2)} = 0$ for identical NS). Further work [100] fixed $\bar{\alpha}_2^{(2)} = 85/14$ to be the contribution of the NNL equations of motion to the phasing. Now the present work permitted to fix $\beta_2^{22} = 642083/1016064$. However, we found disagreement with the 2.5PN (tail) term in the SPA phase in Ref. [91]. This is under investigation following some interesting discussions with Alessandro Nagar.

The result (5.41)-(5.42) completes the contribution of static tidal effects to the GW phase emitted by a binary system of non-spinning compact objects up to 2.5PN beyond the leading order in the quasi-circular orbits approximation. Although such precision is too high for LIGO/Virgo, it will be useful for Einstein Telescope and also reducing biases in the EOB models.

Perspectives

Some of the assumptions performed in this computation can be relaxed in order to give more realistic models. First of all, the effective action (4.5) has been built using EFT methods under the assumption of static (or adiabatic) tides. This means we assume that the two compact objects are always at thermodynamical equilibrium. This assumption can be relaxed and have been studied in [70, 130]. They showed, using an EOB approach, that dynamical tides can induce matter resonances. For some equations of state, they are not negligible compared to adiabatic tides and thus, dynamical tides should be included in the templates.

Another assumption was performed at the level of the phase where we imposed the quasi-circular orbit condition. The energy and flux are both known for generic orbits. However, in order to derive the phase, the problem starts to be difficult if one includes eccentricities. This approximation is quite realistic due to the circularisation of the orbits in binary systems, however in principle it shouldn't be the case for all BNS systems.

This computation has been done for spinless compact objects. As mentioned above, some work has tackled the coupling of tidal effects with (aligned) spins up to 6.5PN order in the quasi-circular orbits approximation [45]. This coupling occurs between the leading order in spin term with the

tidal mass quadrupole and hits at 6.5PN. However, to be consistent at 7.5PN order, one has also to include the coupling the leading order in spin with the tidal current quadrupole that hits at 7.5PN.

Part III

Towards the 4PN phase

Introduction

The second project of my PhD aimed at computing the 4PN phase of the GW signal emitted by binary systems of spinless compact objects without internal structure and is currently known up to 3.5PN [43]. As said in Sec. 2.4, this result has been long awaited by the community to build more precise templates especially for the building of EOB models. It will be important notably for data analysis in LISA or 3rd generation ground-based detectors such as Einstein Telescope. As summarized in Sec. 3.3, in order to derive the 4PN phase, we have to know at 4PN both the conserved energy and the radiated flux of GWs. The conservative sector has already been tackled by several groups using different methods, which agree on the EoM and the conserved quantities, and notably the energy. More specifically, Blanchet et al. derived them (before the start of my PhD) from the Fokker Lagrangian in harmonic coordinates [131, 132, 133, 38, 134]. This result has also been obtained beforehand by means of the canonical Hamiltonian formalism of general relativity [135, 111, 136, 39, 137], and by later the effective field theory (EFT) [47, 138, 139, 140, 49, 50, 40, 51]. However, the computation of the 4PN flux has never been achieved. In the PN-MPM formalism, the flux is expressed in terms of the radiative multipole moments of the system as given in Eq. (3.18). Moreover, this formula tells us that we need to know the radiative mass quadrupole up to 4PN, the current quadrupole and mass octupole up to 3PN and some other lower order multipoles. The hardest part is, by far, the computation of the radiative mass quadrupole because of numerous subtleties, notably due to the regularisation scheme, which are detailed in Chapter 6. Furthermore, and as summed up in Sec. 3.3, in order to compute the radiative multipoles, one needs to know the source multipoles and also the non-linear pieces, such as tails, tails-of-tails, memory or tails-of-memory effects.

The state of the art for the computation of the multipole moments is the following. Previous computations of multipole moments for compact binaries (without spins) included the mass-type quadrupole moment at 1PN order [141], 2PN order [127, 128] and 3PN order [142, 143, 144]. The current quadrupole moment has been determined at 1PN order [145, 127, 146] and 2.5PN order [147]. The mass octupole moment was computed up to 3PN order [148] and the current octupole moment to 2PN order [149, 147]. Other moments are known, such as the mass hexadecapole (2^4) one at 2PN order and the 2^5 mass and 2^4 current ones at 1PN order [147]. So the missing source multipole moments were, at the beginning of my PhD in 2018, the two most complicated: the 4PN mass quadrupole and the 3PN current quadrupole. As said in Sec. 3.2.2, dimensional regularisation (DR) must be used starting from 3PN. Thus, the expression of the multipoles, derived in Sec. 3.1.5 had also to be obtained in d dimensions. The d -dimensional expression of the mass quadrupole, provided in (6.1), was found in [144]. However, we had to derive the d -dimensional generalisation of the source current quadrupole, which was harder because in arbitrary dimension, one cannot use the Levi-Civita tensor which plays a key role in 3 dimensions.

In Chapter 6, I present the computation of the 4PN source mass quadrupole. This chapter is divided in two parts: the first one corresponds to its computation with DR in the UV, which is based in our already published paper [9], while the second part corresponds to our yet unpublished work on DR for the IR. All the technical aspects of the computations are described there. Although the source quadrupole is now fully known, the radiative multipole is not and this part is missing in order to derive the 4PN flux. In Chapter 7, I present the full computation of the current quadrupole fully regularised with DR. I also present the associated GW amplitude mode.

Chapter 6

The 4PN source mass quadrupole

The aim of this chapter is to describe the work performed on the computation of the 4PN source mass quadrupole for compact binaries. This multipole has to be regularised with DR in the UV and in the IR. As mentioned below, the general d -dimensional expression of the quadrupole in terms of the pseudo stress-energy tensor has been derived in [144]. The plan of this chapter goes as follows: in Secs. 6.1 and 6.2 we derive the expression of the mass quadrupole in terms of the elementary potentials that parametrize the PN metric. In Sec. 6.3, we derive the 4PN mass quadrupole using Hadamard regularisation to which we add the UV correction coming from DR. This section is based on our paper [9]. In Sec. 6.4, we compute the IR correction from DR in order to get the final fully dimensional-regularised 4PN source mass quadrupole. The results in the last section of this chapter is yet unpublished [11].

6.1 The general expression of the source mass quadrupole

In Secs. 3.1.4 and 3.1.5, we showed how to derive the expression of the source multipole moments in terms of the pseudo stress-energy tensor in 3 dimensions. The formula for the general ℓ -th mass-type multipole moment in $d \equiv 3 + \varepsilon$ dimensions reads [144]

$$\begin{aligned}
 I_L(t) = & \frac{d-1}{2(d-2)} \mathcal{FP}_{B=0} \int \frac{d^d \mathbf{x}}{\ell_0^\varepsilon} \tilde{r}^B \left\{ \hat{x}_L \bar{\Sigma}_{[\ell]} - \frac{4(d+2\ell-2)}{c^2(d+\ell-2)(d+2\ell)} \hat{x}_{iL} \bar{\Sigma}_{i[\ell+1]}^{(1)} \right. \\
 & + \frac{2(d+2\ell-2)}{c^4(d+\ell-1)(d+\ell-2)(d+2\ell+2)} \hat{x}_{ijL} \bar{\Sigma}_{ij[\ell+2]}^{(2)} \\
 & \left. - \frac{4(d-3)(d+2\ell-2)}{c^2(d-1)(d+\ell-2)(d+2\ell)} B \hat{x}_{iL} \frac{x_j}{r^2} \bar{\Sigma}_{ij[\ell+1]} \right\}(\mathbf{x}, t). \quad (6.1)
 \end{aligned}$$

The overall d -dependent factor in front is such that (6.1) reduces to the usual Newtonian-looking expression of the multipole moments in the Newtonian approximation, given by $I_L = m_1 \hat{y}_1^L + 1 \leftrightarrow 2 + \mathcal{O}(c^{-2})$. The constant ℓ_0 has to be introduced to preserve homogeneity and is the characteristic length scale associated with DR.

The last term of (6.1) will not contribute because of the B and the $d-3 = \varepsilon$ factors appearing simultaneously. To see this one splits the integral into a near-zone contribution $r < \mathcal{R}$ and a far-zone one $r > \mathcal{R}$. In the UV part of the integral, one has to apply the limit $B \rightarrow 0$ since there are no IR divergences (hence no poles $\propto 1/B$), thus the $r < \mathcal{R}$ part vanishes. The argument that this term does not contribute in the IR is detailed in Sec. 6.4.4.

As in $3d$, the source terms are defined from the PN expansion of the pseudo stress-energy tensor $\bar{\tau}^{\mu\nu}$. It enters the right-hand side of the Einstein field equations in harmonic coordinates. It takes the same form as in Eqs. (3.3)–(3.4), except that the Newton constant there reads $G = \ell_0^\varepsilon G_N$, where

G_N is the Newton constant in 3 dimensions. We have

$$\bar{\Sigma} = \frac{2}{d-1} \frac{(d-2)\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2}, \quad \bar{\Sigma}_i = \frac{\bar{\tau}^{0i}}{c}, \quad \bar{\Sigma}_{ij} = \bar{\tau}^{ij}. \quad (6.2)$$

The matter terms are still given by the point-mass expressions (3.39)–(3.40) but with now $\delta = \delta^{(d)}$, the Dirac function in d dimensions. The generalization of Eqs. (3.29)–(3.34) to d dimensions reads as

$$\bar{\Sigma}_\ell(\mathbf{x}, t) = \int_{-1}^1 dz \delta_\ell^{(\varepsilon)}(z) \bar{\Sigma}(\mathbf{x}, t + zr/c), \quad (6.3)$$

where

$$\delta_\ell^{(\varepsilon)}(z) \equiv \frac{\Gamma\left(\ell + \frac{3}{2} + \frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\ell + 1 + \frac{\varepsilon}{2}\right)} (1 - z^2)^{\ell + \frac{\varepsilon}{2}}, \quad \int_{-1}^1 dz \delta_\ell^{(\varepsilon)}(z) = 1. \quad (6.4)$$

In practice we only need the formal PN expansion

$$\bar{\Sigma}_{[\ell]}(\mathbf{x}, t) = \sum_{k=0}^{+\infty} \alpha_\ell^k \left(\frac{r}{c} \frac{\partial}{\partial t}\right)^{2k} \bar{\Sigma}(\mathbf{x}, t), \quad (6.5)$$

with the numerical coefficients now being given by

$$\alpha_\ell^k = \frac{1}{2^{2k} k!} \frac{\Gamma(\ell + \frac{d}{2})}{\Gamma(\ell + \frac{d}{2} + k)}, \quad (6.6)$$

where Γ is the Euler gamma function. It is very useful to define for the matter stress-energy tensor the following matter currents

$$\sigma = \frac{2}{d-1} \frac{(d-2)T^{00} + T^{ii}}{c^2}, \quad \sigma_i = \frac{T^{0i}}{c}, \quad \sigma_{ij} = T^{ij}, \quad (6.7)$$

which are given in the case of compact binary systems by

$$\sigma = \tilde{\mu}_1 \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2, \quad (6.8a)$$

$$\sigma_i = \mu_1 v_1^i \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2, \quad (6.8b)$$

$$\sigma_{ij} = \mu_1 v_1^i v_1^j \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2, \quad (6.8c)$$

where $\mathbf{y}_1 = (y_1^i)$ is the particle's position, $\mathbf{v}_1 = d\mathbf{y}_1/dt = (v_1^i)$ the coordinate velocity, and we have introduced, besides μ_1 which keeps the same expression as in 3 dimensions, see Eq. (3.40), the useful tilded version

$$\tilde{\mu}_1 = \frac{2}{d-1} \left(d - 2 + \frac{v_1^2}{c^2}\right) \mu_1. \quad (6.9)$$

The matter current densities (6.8) generate all the compact-support terms in the expression of the quadrupole moment.

6.2 The quadrupole moment as a function of potentials

6.2.1 The elementary potentials

To start the derivation of the mass quadrupole moment we need to inject into (6.1) the PN metric $\bar{h}^{\mu\nu}$ which is an explicit solution of the Einstein field equations (3.3) valid in the near zone (recall that the overbar means PN expansion). The metric components \bar{h}^{00} , \bar{h}^{0i} and \bar{h}^{ij} are respectively to be expanded up to orders c^{-8} , c^{-7} and c^{-8} included to reach the required accuracy. Thus, we need

\bar{h}^{00} and \bar{h}^{0i} at 3PN order (remind that c^{-8} in \bar{h}^{00} actually corresponds to 3PN), and \bar{h}^{ij} at the 4PN order.

Building up on [142, 143, 144, 148], the metric has been parametrized with appropriate PN elementary retarded potentials, namely scalar potentials V , K , \hat{X} and \hat{T} , vector potentials V_i , \hat{R}_i and \hat{Y}_i , and tensor ones \hat{W}_{ij} , \hat{Z}_{ij} and \hat{M}_{ij} . The structure of the parametrization in 3 dimensions is

$$\bar{h}^{00} = -\frac{4V}{c^2} - \frac{2}{c^4} (\hat{W} + 4V^2) - \frac{8}{c^6} (\hat{X} + \dots) - \frac{64}{c^8} (\hat{T} + \dots) + \mathcal{O}(c^{-10}), \quad (6.10a)$$

$$\bar{h}^{0i} = \frac{4V_i}{c^3} + \frac{8}{c^5} (\hat{R}_i + V_i V) + \frac{16}{c^7} (\hat{Y}_i + \dots) + \mathcal{O}(c^{-9}), \quad (6.10b)$$

$$\bar{h}^{ij} = -\frac{4}{c^4} \left(\hat{W}_{ij} - \frac{1}{2} \delta_{ij} \hat{W} \right) - \frac{16}{c^4} \left(\hat{Z}_{ij} - \frac{1}{2} \delta_{ij} \hat{Z} \right) - \frac{32}{c^8} (\hat{M}_{ij} + \dots) + \mathcal{O}(c^{-10}). \quad (6.10c)$$

The ellipsis symbolizes non-linear products of the elementary potentials introduced at lower orders. The complete expression of the metric at 4PN order in d dimensions is given in Eqs. (B.2). As mentioned in Sec. 3.1.6, the potentials obey some flat space-time wave equations. The expressions given in the rest of this section are in $3d$, for complete d dimensional definitions see App. B. Some have a compact support, like

$$\square V = -4\pi G \sigma, \quad \square V_i = -4\pi G \sigma_i, \quad (6.11)$$

while there are many quadratic non-linear terms (sometimes called “ $\partial V \partial V$ ”) such as in

$$\square \hat{W}_{ij} = -4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V, \quad (6.12)$$

and higher order terms (called “non-compact”) such as the cubic term $\hat{W}_{ij} \partial_{ij} V$ in

$$\square \hat{X} = -4\pi G V \sigma_{ii} + \hat{W}_{ij} \partial_{ij} V + 2V_i \partial_t \partial_i V + V \partial_t^2 V + \frac{3}{2} (\partial_t V)^2 - 2\partial_i V_j \partial_j V_i. \quad (6.13)$$

See Eqs. (B.4) for thorough definitions of all these potentials in d dimensions. Among these, the only purely 4PN potential which is needed for the 4PN quadrupole moment is \hat{M}_{ij} , which obeys in 3 dimensions:

$$\begin{aligned} \square \hat{M}_{ij} = G\pi \left[\left(-4V_i V_j + \delta^{ij} (2V_a V_a + \hat{X}) \right) \sigma + 4(\hat{R}_{(i} + V V_{(i}) \sigma_{j)}) - 2\hat{W}_{a(i} \sigma_{j)a} \right. \\ \left. - 4V^2 \sigma_{ij} + \delta^{ij} \left(-2\hat{R}_b \sigma_b - 2V V_m \sigma_m + \frac{1}{2} \hat{W}_{kl} \sigma_{kl} + V^2 \sigma_{pp} - \frac{1}{2} \hat{W} \sigma_{qq} \right) \right] \\ - \partial_t V_i \partial_t V_j + V_a \partial_t \partial_a \hat{W}_{ij} + 2V_{(i} \partial_a V_{j)} \partial_a V - \partial_t \hat{W}_{(ia} \partial_a V_{j)} \\ + \frac{1}{2} \hat{W}_{ab} \partial_{ab} \hat{W}_{ij} - \frac{1}{2} \partial_a \hat{W}_{ib} \partial_b \hat{W}_{ja} + \frac{1}{2} \hat{W}_{(ia} \partial_a V \partial_j) V - \frac{1}{4} \partial_i \hat{W}_{ab} \partial_j \hat{W}_{ab} \\ - 2\partial_a V_{(i} (\partial_j) \hat{R}_a + V_a \partial_j) V) - 2\partial_a \hat{R}_{(i} \partial_j) V_a + \partial_t \hat{W}_{(ia} \partial_j) V_a + \partial_b \hat{W}_{a(i} \partial_j) \hat{W}_{ab} \\ + 2\partial_{(i} V_a \partial_j) \hat{R}_a + \left(-2\partial_t \hat{R}_{(i} + \frac{1}{2} V_{(i} \partial_t V - \partial_{(i} \hat{X}) \right) \partial_j) V + V \left(\frac{1}{2} \partial_t^2 \hat{W}_{ij} - 2\partial_t V_{(i} \partial_j) V \right) \\ + \delta_{ij} \left[-\frac{1}{2} V_a \partial_t \partial_a \hat{W} - \frac{1}{4} V^2 \partial_t^2 V + \partial_t \hat{R}_a \partial_a V - \frac{1}{4} \hat{W}_{ab} \partial_{ab} \hat{W} + \partial_a V_b \partial_b \hat{R}_a \right. \\ \left. - \frac{1}{8} \hat{W}_{ab} \partial_a V \partial_b V - \frac{1}{2} \partial_t \hat{W}_{ab} \partial_a V_b - \frac{1}{4} \partial_a \hat{W}_{bc} \partial_c \hat{W}_{ab} - \frac{1}{4} V_a \partial_t V \partial_a V \right. \\ \left. + V \left(\frac{3}{8} (\partial_t V)^2 - \frac{1}{2} V_a \partial_t \partial_a V - \frac{1}{4} \partial_t^2 \hat{W} + \partial_t V_a \partial_a V - \frac{1}{4} \hat{W}_{ab} \partial_{ab} V + \frac{1}{2} \partial_a V_b \partial_b V_a \right) \right]. \end{aligned} \quad (6.14)$$

By inserting the PN metric (B.2) into the mass quadrupole moment, one obtains the full expression in terms of the latter PN potentials. However, we do not know explicitly all the required potentials (either in 3 or d dimensions), since they are solutions of complicated wave equations such as (6.14). Thus, crucial simplifications of the result have to be performed first, in order to put the expression into computable form; see the complete result for all the terms in App. E.2.

6.2.2 The method of super-potentials

The first technique we used in order to be able to compute all the terms composing the source mass quadrupole at 4PN is the method of “super-potentials”. Many of the most difficult of those terms are of the form ϕP where ϕ is a simple potential or derivative of a simple potential, and P is a complicated potential whose expression in the whole space is not known. For instance P could be the 4PN potential \hat{M}_{ij} entering the spatial components of the metric and obeying the equation (6.14). On the other hand, in our case, ϕ will be one of the following potentials: $\partial_{ab}V$, $\partial_t\partial_aV$, ∂_aV_b or ∂_aV .

To compute the integral $\int d^3\mathbf{x} r^B \hat{x}_L \phi P$ (in 3 or d dimensions) we notice that, at leading order, $\hat{x}_L \phi$ may be recast in the form of a Laplace operator acting on some solution Ψ_L^ϕ :

$$\Delta\Psi_L^\phi = \hat{x}_L\phi. \quad (6.15)$$

Assuming that Ψ_L^ϕ can be constructed analytically, a mere integration by part yields a volume integral whose source is known explicitly, namely $-\int d^3\mathbf{x} r^B \Psi_L^\phi \Delta P$, plus terms that are essentially surface integrals at infinity when the Hadamard partie finie is applied.

Now, as it turns out, it is possible to construct the solution Ψ_L^ϕ by means of the super-potentials of ϕ , defined as the hierarchy of solutions ϕ_{2k} of the sequence of Poisson equations

$$\Delta\phi_{2k+2} = \phi_{2k}, \quad (6.16)$$

together with $\phi_0 = \phi$ so that we have $\Delta^k\phi_{2k} = \phi$. The solution of Eq. (6.15) is then given in analytic closed form as [150]

$$\Psi_L^\phi = \Delta^{-1}(\hat{x}_L\phi) = \sum_{k=0}^{\ell} \frac{(-2)^k \ell!}{(\ell-k)!} x_{\langle L-K} \partial_{K\rangle} \phi_{2k+2}. \quad (6.17)$$

This formula has been derived by induction in 3 dimensions in [150] but the proof works as well in d dimensions and no extra factor needs to be inserted. The precise choice of the Poisson solutions involved in the above algorithms is irrelevant for this particular problem, hence the operator Δ^{-1} has not been precisely defined in Eq. (6.17). However, it is convenient in practice to take $\Delta^{-1} = \widetilde{\Delta}^{-1}$, where $\widetilde{\Delta}^{-1} = \mathcal{FP}_{B=0}\Delta^{-1}\tilde{r}^B$ represents the Poisson integral regularised at infinity by means of the Hadamard finite part prescription.

With this tool in hands, we can thus transform the integral we were looking for into the much more tractable form

$$\int d^3\mathbf{x} r^B \hat{x}_L \phi P = \int d^3\mathbf{x} r^B \left(\Psi_L^\phi \Delta P + \partial_i \left[\partial_i \Psi_L^\phi P - \Psi_L^\phi \partial_i P \right] \right). \quad (6.18)$$

The first term involves the source of the potential P , because $\square P = \Delta P + \mathcal{O}(1/c^2)$ and is therefore computable. The second one is a surface term, which is also computable (see Sec. 6.3.4 for the $3d$ computation and Sec. 6.4.4 for its dimensional treatment). For instance, in the case where $P = \hat{M}_{ij}$, with \hat{M}_{ij} being the 4PN tensor potential, we will replace $\Delta\hat{M}_{ij}$ by the source given explicitly by Eq. (6.14), which is correct since we are already at the maximal 4PN order and thus \hat{M}_{ij} is merely Newtonian. For more details on the computation of the super-potentials, see Sec. III. B. of [9].

6.2.3 Integrations by part and surface terms

The second technique to simplify the expression of the quadrupole moment consists in integrating some terms by part in order to transform volume integrals into simpler volume integrals together with surface and compact-supported integrals. For instance, we systematically rewrote integrals involving the double gradient of a simple compact-support potential like V , defined in (6.11), and a difficult one P (with non-compact support) as

$$\partial_i V \partial_i P = \frac{1}{2} \left[\Delta(VP) - V\Delta P - P\Delta V \right]. \quad (6.19)$$

The second term in (6.19) is much simpler because it contains (modulo higher PN corrections) the source of the potentials P , *i.e.* $\Delta P = S + \mathcal{O}(c^{-2})$. The third term is also easy to evaluate because it depends only on the value of the potential P at the location of the particles, since V has a compact support: $\Delta V = -4\pi G\sigma + \mathcal{O}(c^{-2})$.

As for the first term in (6.19), it yields an example of a so-called ‘‘Laplacian term’’, coming after integration by parts from the derivation of the regularisation factor $(r/r_0)^B$. The surface integrals of the Laplacian terms, as well as the analogous so-called ‘‘divergence terms’’, are very easy to integrate within the Hadamard finite part prescription. Therefore, we kept as much as possible the terms into Laplacian or divergence form.

Once the latter two techniques — super-potentials and surface integrals — have been applied, one obtains an extremely long expression for the quadrupole moment as a function of potentials and super-potentials, where all terms can be explicitly integrated.

6.2.4 Expression in terms of potentials

In d dimensions, the mass-type quadrupole moment is given by Eq. (6.1) for $\ell = 2$. The first three terms of (6.1) will be denoted respectively by S_{ij} , V_{ij} and T_{ij} with S , V and T standing for scalar, vector and tensor. We recall that the fourth term in (6.1) does not contribute in the considered prescription: DR for the UV and Hadamard for the IR¹. Thus we write

$$I_{ij} = S_{ij} + V_{ij} + T_{ij}. \quad (6.20)$$

Furthermore, we know from the series expansion (6.5) that each term S , V and T corresponds to a finite sum at 4PN order indexed by the integer k in (6.5). At the 4PN order this sum has 5 terms ($k = 0, 1, 2, 3, 4$) for S , 4 for V and 3 for T . We denote these terms using capital roman numerals hence

$$S = SI + SII + SIII + SIV + SV, \quad (6.21a)$$

$$V = VI + VII + VIII + CIV, \quad (6.21b)$$

$$T = TI + TII + TIII. \quad (6.21c)$$

In these notations, we ignore the always understood quadrupole indices ij . In practice, I performed the computations on the non-STF mass quadrupole and then took the STF projection of the result. Since there are poles $\propto 1/\varepsilon$ in the expressions to be projected, it is crucial to apply the STF projection in d dimensions, *i.e.* $\hat{T}^{ij} = T^{ij} - \frac{1}{d}\delta^{ij}T^{kk}$. However, alternatively, one can wait until the end of the calculation when the poles have been removed by the appropriate shifts (see Appendix E.1) to apply finally the usual STF projection in 3 dimensions. We have checked explicitly that the STF operation commutes with the integration and notably with the different regularisation procedures.

In addition, each of the 654 terms contains non-compact support terms which we denote with the suffix NC (such as SINC), compact support terms (suffix C) and surface terms of two sorts, either surface Laplacian terms (SL) or surface divergence terms (SD). Hence we write for the term SI:

$$SI = SIC + SINC + SISL + SISD, \quad (6.22)$$

and similarly for the other terms. The full expressions in terms of potentials of SIC, SINC, SISL and SISD are given in App. E.2.

6.3 The properly UV regularised quadrupole

As mentioned above, the first steps for the computation of the mass quadrupole consist in deriving the potentials in $3d$ (with Hadamard partie finie prescription) and, then, in d dimensions. Some

¹And also DR for the IR, as we will see in Sec. 6.4.4.

potentials are too complicated to compute in the whole space so we used the previously described techniques to avoid this problem by transforming the NC terms into C + Surface terms. This allows computing the same integrals but it requires to know only the potentials regularised on the location of the bodies or at spatial infinity. However, I did not compute the potentials myself, since it had already been done, notably by Tanguy Marchand, Luc Blanchet, François Larrouturou and Guillaume Faye. Thus, my contribution to the computation of the d -dimensional UV regularised mass quadrupole started from the computation of the integrals in terms of the potentials, already knowing their expressions. For more details on the derivation of the potentials required for the computation of the mass quadrupole, see Sec. IV B and Sec. V of [9]. Furthermore, I focused on the C and NC terms. Because of the length and complexity of the computations, the *Mathematica* codes I ran had to be parallelized on the cluster of IAP. I used the *xAct* library [13] for tensor manipulations.

6.3.1 Computation of C and distributional terms

Compact terms

The C terms are the simplest ones to integrate if we assume that we already know the potentials. Indeed, their most general form is

$$\lambda \int d^d \mathbf{x} \hat{x}^L \partial_{L_1} P_1 \dots \partial_{L_n} P_n \delta_A, \quad (6.23)$$

where P_k are potentials, $\delta_A = \delta^{(d)}(\mathbf{x} - \mathbf{y}_A)$ and λ is an arbitrary constant coefficient. The result for such an integral is then

$$\lambda \hat{y}_A^L (\partial_{L_1} P_1)_A \dots (\partial_{L_n} P_n)_A, \quad (6.24)$$

where $(\partial_{L_k} P_k)_A$ is the dimensional-regularised value of $\partial_{L_k} P_k$ at the location of body A . Thus, C terms can be computed directly in d -dimensions. Since I already knew the values of the regularised potentials and their derivatives, these terms were trivial to compute. Note that some potentials contain a pole $1/\varepsilon$ when computed in d -dimensions. This means that C terms generate poles.

Distributional terms

Furthermore, terms of the form (6.23) can also come from NC terms. Indeed, the computations are made in terms of distributions and not functions due to the fact that we use Dirac delta distributions to model the physical system. Such a consideration complicates the computations because if one applies a sufficient amount of derivatives on an expression, distributions will appear. A simple example in $3d$ is the following:

$$\partial_{ab} \left(\frac{1}{r_1} \right) = 3 \frac{n_1^{\langle a} n_1^{b \rangle}}{r_1^3} - \frac{4\pi}{3} \delta^{ab} \delta_1, \quad (6.25)$$

where the first term is the ordinary part of the derivative and the second is the additional part due to distributions. In general, we can decompose a derivative in an ordinary and a purely distributional part

$$\partial_i P = (\partial_i P)_{\text{ord}} + D_i [P]. \quad (6.26)$$

The value of the distributional part D_i is given by the Gel'fand Shilov formula [9, 151, 152]. This formula, adapted to our notations, reads

$$D_i \left[\frac{n^L}{r^{m+\varepsilon}} \right] = 4\pi^{1+\frac{\varepsilon}{2}} (-)^m 2^{m+\varepsilon} (\ell+1)! \frac{\Gamma\left(\frac{\ell+m+1+\varepsilon}{2}\right)}{\Gamma(\ell+m+1+\varepsilon)} \sum_{p=p_0}^{\lfloor m/2 \rfloor} \frac{\Delta^{p-1} \partial_{(M-2P} \delta_{iL+2P-M)} \delta^{(3+\varepsilon)}}{2^{2p} (p-1)! (m-2p)! \left(\frac{\ell+1-m}{2} - p\right)!} \quad (6.27)$$

where $p_0 = \text{Max}\left(\frac{m-\ell-1}{2}, 1\right)$ and $\delta_L = \delta_{i_1 i_2} \dots \delta_{i_{\ell-1} i_\ell}$.

Non-compact terms that contain a sufficiently high number of derivatives can generate distributional contributions. As an example, using (6.25), the term $\int d^d x \hat{x}^{ij} \hat{W}_{ab} \partial_{ab} V$ will create a distributional term of the form

$$\propto \int d^d \mathbf{x} \hat{x}^{ij} \hat{W}_{ab} \delta_{ab} \delta_A, \quad (6.28)$$

which is exactly of the same form as (6.23) and therefore can be integrated in the same way. Note that derivatives of Dirac distributions can also arise from (6.27). So one has first to perform IBPs in order to rewrite the integral in the form of (6.23). For the mass quadrupole I computed all the contributions of the C and distributional parts using these methods.

6.3.2 Hadamard regularisation: computation of NC terms in 3d

In this section, we present the Hadamard partie finie regularisation. For a detailed review, see [153]. We first consider a function $F(\mathbf{x})$ which is assumed to be smooth and well-defined everywhere except at the positions of the two bodies. We are interested in the computation of integrals of the form

$$I = \mathcal{FP} \int d^3 \mathbf{x} F(\mathbf{x}). \quad (6.29)$$

This integral can then diverge at $\mathbf{x} = \mathbf{y}_1$, $\mathbf{x} = \mathbf{y}_2$ and $|\mathbf{x}| \rightarrow +\infty$. The divergencies at the locations of the bodies are treated differently from the one at spatial infinity. We can split the integral in two parts: one for $r < \mathcal{R}$ and one for $r > \mathcal{R}$, where \mathcal{R} is a fixed constant sufficiently large, so that

$$I = I_{\mathcal{R}}^- + I_{\mathcal{R}}^+, \quad (6.30)$$

where $I_{\mathcal{R}}^-$ is defined in (6.33) and $I_{\mathcal{R}}^+$ in (6.37). The first step of the Hadamard partie finie procedure is to perform a Laurent expansion of F around $\mathbf{x} = \mathbf{y}_1$,

$$F(\mathbf{x}) = \sum_{p_0 \leq p \leq N} r_1^p \varphi_p(\mathbf{n}_1) + o(r_1^N), \quad (6.31)$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and the coefficients φ_{1p} depend on the unit direction $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ of approach to the singularity. Note that the Laurent-expansion of F can contain logarithms and its full form is given in (6.82). However, considering logarithms here is not crucial for the understanding of the procedure. Then, we define the regularised value of the function F at point 1 to be the average with respect to the direction \mathbf{n}_1 of the term with zeroth power in r_1 , namely

$$(F)_1 \equiv \int \frac{d\Omega}{4\pi} \varphi_p(\mathbf{n}_1), \quad (6.32)$$

where $d\Omega$ is the infinitesimal solid angle on the 3-dimensional sphere. With this in hand, we can compute the Hadamard partie finie value (in the rest of this thesis we also call it 3d value) of the integral (6.29).

Hadamard partie finie for UV divergencies

In Hadamard regularisation, the 3-dimensional spatial integral is defined by the following \mathcal{FP} prescription, depending on two constants s_1 and s_2 associated with logarithmic divergences at the two singular points, say

$$I_{\mathcal{R}}^- = \mathcal{FP}_{s_1, s_2} \int_{r < \mathcal{R}} d^3 \mathbf{x} F(\mathbf{x}). \quad (6.33)$$

We define the region of space \mathcal{V} which is inside the radius \mathcal{R} and excludes two balls $B_1(s)$ and $B_2(s)$ centered on \mathbf{y}_1 and \mathbf{y}_2 with a radius s , so that $\mathcal{V} = \mathcal{V}(\mathcal{R}, s)$. The integral over \mathcal{V} is convergent, by virtue of the properties of F while the integrals inside the balls can be divergent. Let us now

define $I_{\mathcal{R},s}^-$ as the integral over \mathcal{V} . If the integrals over the balls were convergent, then we would have directly $I_{\mathcal{R}}^- = \lim_{s \rightarrow 0} I_{\mathcal{R},s}^-$. In the general case, we have to study the potential divergencies of $I_{\mathcal{R},s}^-$ in order to subtract them before taking the limit $s \rightarrow 0$. To do so, we insert the Laurent expansion of F (6.31) in $I_{\mathcal{R},s}^-$. Then by splitting this integral into a radial and an angular one, we find that its singular part can be written as

$$-4\pi \left(\sum_{p+3 < 0} \frac{s^{p+3}}{p+3} \left(\frac{F}{r_1^p} \right)_1 + \ln \left(\frac{s}{s_1} \right) (r_1^3 F)_1 \right) + 1 \leftrightarrow 2. \quad (6.34)$$

In the end, we subtract this quantity to $I_{\mathcal{R}}^-$ and then take the limit $s \rightarrow 0$. The final result for the treatment of the UV divergencies in the Hadamard prescription for such integrals is

$$I_{\mathcal{R}}^- = \lim_{s \rightarrow 0} \left[\int_{\mathcal{V}(\mathcal{R},s)} d^3\mathbf{x} F(\mathbf{x}) + 4\pi \left(\sum_{p+3 < 0} \frac{s^{p+3}}{p+3} \left(\frac{F}{r_1^p} \right)_1 + \ln \left(\frac{s}{s_1} \right) (r_1^3 F)_1 \right) + 1 \leftrightarrow 2 \right]. \quad (6.35)$$

Remark that, as \mathcal{V} depends on \mathcal{R} , the first term in (6.35) involves some powers and logarithms of \mathcal{R} . The contributions in \mathcal{R} will be canceled by the ones in $I_{\mathcal{R}}^+$. The previous method is also equivalent to introducing the two cut-offs s_1 and s_2 in the following way

$$I_{\mathcal{R}}^- = \mathcal{FP} \int_{r < \mathcal{R}} d^3\mathbf{x} \left(\frac{r_1}{s_1} \right)^\alpha \left(\frac{r_2}{s_2} \right)^\beta F(\mathbf{x}), \quad (6.36)$$

where we compute the integral in the whole space by conveniently choosing the values of α and β so that the integral converges and then taking their limit towards 0 by analytic continuation.

Hadamard partie finie for IR divergencies

To treat the IR divergencies of I , complementary to the previous section, we can limit the integration domain to be $r = |\mathbf{x}| > \mathcal{R}$

$$I_{\mathcal{R}}^+ = \mathcal{FP} \int_{r > \mathcal{R}} d^3\mathbf{x} \left(\frac{r}{r_0} \right)^B F(\mathbf{x}). \quad (6.37)$$

For simplicities sake, we assume that the multipolar expansion of F at spatial infinity does not contain logarithms *i.e.* $F(\mathbf{x}) = \sum_p r^p \varphi_p(\mathbf{n})$. After following the same kind of reasoning as for the UV, we find that the Hadamard regularised value of the outer part of (6.29) reads

$$I_{\mathcal{R}}^+ = 4\pi \left[\sum_{p < -3} \frac{\mathcal{R}^{p+3}}{p+3} \langle \varphi_p(\mathbf{n}) \rangle + \ln \left(\frac{\mathcal{R}}{r_0} \right) \langle \varphi_{-3}(\mathbf{n}) \rangle \right]. \quad (6.38)$$

As mentioned above, the \mathcal{R} are compensated by the ones in the first term of (6.35).

6.3.3 Computation of the UV DDR for NC terms

In order to obtain the dimensional regularised (for the UV) mass quadrupole, it is now sufficient to compute the so-called difference of Hadamard and DR abbreviated as DDR. In DR, the integral is regularized by means of analytic continuation in $d = 3 + \varepsilon$, so that

$$I_{\mathcal{R}}^{(d)-} = \int_{r < \mathcal{R}} \frac{d^d\mathbf{x}}{\ell_0^\varepsilon} F^{(d)}(\mathbf{x}). \quad (6.39)$$

We assume here that we can compute the expansion of the source $F^{(d)}$ in d dimensions in the vicinity of the particles, say \mathbf{y}_1 . While in 3 dimensions, we have the expansion around \mathbf{y}_1 as in (6.31), in d dimensions, we have a similar albeit more complicated expansion

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0 \leq p \leq N \\ q_0 \leq q \leq q_1}} \frac{r_1^{p+q\varepsilon}}{\ell_0^{q\varepsilon}} \varphi_{1,p,q}^{(\varepsilon)}(\mathbf{n}_1) + o(r_1^N), \quad (6.40)$$

where the coefficients now depend on an extra integer q reflecting the more complicated structure of the expansion involving powers $p + q\varepsilon$ (with both $p, q \in \mathbb{Z}$). Since the two expansions (6.31) and (6.40) must agree in the limit $\varepsilon \rightarrow 0$, the relation

$$\sum_{q_0 \leq q \leq q_1} \varphi_{1,p,q}^{(0)} = \varphi_1^p, \quad (6.41)$$

must hold for any p .

Now, we are interested in the *difference* between DR and Hadamard partie finie, because this is precisely what we have to add to the Hadamard result (6.35) in order to get the correct d -dimensional result (6.39). This difference is

$$\mathcal{D}I^{\text{UV}} = I_{\mathcal{R}}^{(d)-} - I_{\mathcal{R}}^-, \quad (6.42)$$

of which we merely compute the pole part $\propto 1/\varepsilon$ and the finite part $\propto \varepsilon^0$ in the Laurent expansion when $\varepsilon \rightarrow 0$, the other terms vanishing in that limit. The key point is that the difference (6.42) does not depend on \mathcal{R} but only on the coefficients of the expansion around the two singularities as defined by (6.40), modulo the neglected $\mathcal{O}(\varepsilon)$ terms. We have [144]

$$\mathcal{D}I^{\text{UV}} = \frac{\Omega_{d-1}}{\varepsilon} \sum_{q_0 \leq q \leq q_1} \left[\frac{1}{q+1} + \varepsilon \ln \left(\frac{s_1}{\ell_0} \right) \right] \langle \varphi_{1,-3,q}^{(\varepsilon)} \rangle + 1 \leftrightarrow 2, \quad (6.43)$$

where it is crucial that the angular average is performed in d dimensions, *i.e.*,

$$\langle \varphi_{1,p,q}^{(\varepsilon)} \rangle = \int \frac{d\Omega_{d-1}}{\Omega_{d-1}} \varphi_{1,p,q}^{(\varepsilon)}(\mathbf{n}_1), \quad \Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \quad (6.44)$$

Here, $d\Omega_{d-1}$ is the infinitesimal solid angle on the $(d-1)$ -dimensional sphere and $\Omega_{d-1} = \int d\Omega_{d-1}$. In actual calculations, we have verified that there is no problem with the potentially dangerous value $q = -1$ in (6.43) since we always have $\langle \varphi_{1,-3,-1}^{(\varepsilon)} \rangle = 0$.

As we see from (6.43), the calculation generates many UV-type poles $1/\varepsilon$, but we proved that all those poles are canceled by the specific shift determined from the 4PN equations of motion in Refs. [133, 38]. However, at 4PN order, all the poles do not come from (6.43), because, as mentioned in Sec. 6.3.1, they are also can also come from C terms.

6.3.4 Computation of surface terms in 3d

Surface terms can be split in two categories : Laplacian terms, having the form of (6.45) and divergence terms, having the form of (6.49). In this section we show how they are treated in 3d.

Treatment of Laplacian terms

Let us consider a product of derivatives of potentials G . Following [143], a generic Laplacian term reads

$$T_L = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \hat{x}_L \Delta G. \quad (6.45)$$

We recall that $\tilde{r} \equiv r/r_0$ where r_0 is the IR cut-off for Hadamard regularisation. Integrating the Laplacian by parts, we find

$$T_L = \mathcal{FP}_{B=0} B(B+2\ell+1) r_0^{-B} \int_{r>\mathcal{R}} d^3\mathbf{x} r^{B-2} \hat{x}_L G. \quad (6.46)$$

due to the prefactor B , the above integral can be restricted to the outer zone: $r > \mathcal{R}$, where \mathcal{R} is an arbitrary length, the integral for $r < \mathcal{R}$ is finite. Indeed, the matching equation which leads to the expressions of the multipole moments is originally applied for smooth matter distributions, so that the metric is smooth everywhere in the near zone. As such, G is also smooth there and, due to the factor B , the near zone contribution is zero after the \mathcal{FP} procedure. In practice, the point-particle approximation leads to UV divergences, but these are separately treated by DR, in which the \mathcal{FP} plays no role.

Because of the B prefactor in (6.46), we need to look at the $1/B$ pole in the integral. This pole can only come from a radial integral of the form $\int_{\mathcal{R}}^{+\infty} dr r^{B-1} = -\mathcal{R}^B/B$. Thus, considering the asymptotic expansion of G when $r \rightarrow \infty$, which we denote by $\mathcal{M}(G)$ as it is identical to the multipole expansion, we find that the pole comes only from the term of order $r^{-\ell-1}$ in that expansion. At 4PN order, we also obtain a logarithmic dependence for some of the potentials. Hence, if we define $X_p(\mathbf{n})$ and $X_p^{\ln}(\mathbf{n})$ to be the coefficients of r^{-p-1} and $r^{-p-1} \ln r$ in the multipole expansion, we have

$$\mathcal{M}(G) = \dots + \frac{1}{r^{\ell+1}} \left[X_\ell(\mathbf{n}) + X_\ell^{\ln}(\mathbf{n}) \ln\left(\frac{r}{r_0}\right) \right] + o\left(r^{-\ell-1}\right), \quad (6.47)$$

so that we finally obtain (applying the definition of the \mathcal{FP})

$$T_L = \int d\Omega \hat{n}_L \left[-(2\ell+1) X_\ell(n^i) + X_\ell^{\ln}(n^i) \right]. \quad (6.48)$$

Treatment of divergence terms

The other surface integrals occurring are the “divergence terms”, for instance the second term in the right-hand side of (6.18). Most of these terms come from the method of super-potentials. They are of the form

$$K = \mathcal{FP}_{B=0} \int d^3\mathbf{x} \tilde{r}^B \partial_i H_i. \quad (6.49)$$

A similar reasoning to the one before shows that they depend only on the $1/r^2$ coefficient, say $Y_i(\mathbf{n})$, in the asymptotic or multipole expansion $\mathcal{M}(H_i)$ when $r \rightarrow \infty$:

$$\mathcal{M}(H_i) = \dots + \frac{1}{r^2} \left[Y_i(\mathbf{n}) + Y_i^{\ln}(\mathbf{n}) \ln\left(\frac{r}{r_0}\right) \right] + o\left(r^{-2}\right), \quad (6.50)$$

and the \mathcal{FP} procedure yields simply

$$K = \int d\Omega n_i Y_i(\mathbf{n}). \quad (6.51)$$

There is no contribution from the logarithm in (6.50).

In order to apply the previous method, we have to obtain the asymptotic expansion when $r \rightarrow +\infty$ of G or H_i [Eqs. (6.47) or (6.50)], where G and H_i are made of products of derivatives of potentials, involving in general one potential which is not known in the whole space, for instance the 4PN potential $P = \hat{M}_{ij}$ or its trace. To find the expansion when $r \rightarrow +\infty$ of such potential, we relied on the method explained in Sec. V D of [9].

6.3.5 Final result for the UV regularised mass quadrupole

We applied the methods described above to compute the source mass quadrupole moment at the 4PN order in the case of circular orbits. As for the Fokker Lagrangian computation of the equations of motion [131], we first used Hadamard’s *partie finie* to cure the UV divergences and obtained an initial result depending on $\ln s_1$ and $\ln s_2$. Then, we computed the difference between the DR and the Hadamard *partie finie* regularisation for the UV divergences, which yielded a new result free of $\ln s_1$ and $\ln s_2$ but containing poles in $1/\varepsilon$, as well as the DR scale ℓ_0 .

Let us recall that these poles should cancel out when expressing physical observables such as the energy flux or the orbital phase of the system, but can still be present in intermediate non-gauge invariant results such as the equations of motion or the expression of the source multipole moments. However, in that case, it is extremely useful to remove the UV poles by applying a shift of the particle’s trajectories. This provides an important test of the result and also a substantial simplification. At 3PN, it was already shown that applying the same shift as used for the 3PN equations of motion to the 3PN source mass quadrupole moment, consistently removes all the UV poles [83].

At 4PN, the situation is a bit more complicated. The shift that was applied to the Fokker Lagrangian in order to get the final result for the 4PN equations of motion as obtained in [133, 38], and from which all the conserved quantities were derived in [134], is composed of three terms:

1. The shift $\xi_{1,2}$ given in the Appendix C of [131]² and which removes all the UV-type $1/\varepsilon$ poles in the Fokker Lagrangian
2. The shift $\chi_{1,2}$ that was applied in [133] and removes all the IR-type $1/\varepsilon$ poles of the Fokker Lagrangian
3. Finally, the shift $\eta_{1,2}$ given in the Appendix A of [134] that does not contain any pole and was merely used for convenience

For completeness, we provide in App. E.1 the full expressions of the shifts $\xi_{1,2}$ and $\eta_{1,2}$. Note that the IR shift $\chi_{1,2}$ will be used in the next section.

We have applied the sum of the shifts $\xi_{1,2}$ and $\eta_{1,2}$ to the 4PN quadrupole moment and checked that all the UV-type poles $1/\varepsilon$ (as well as the usual concomitant constants such as Euler’s constant γ_E) cancel out as they should. Those shifts have been determined from the separate calculation of the Fokker Lagrangian and equations of motion. Furthermore, as we already mentioned, at 4PN, some of the potentials entering the compact support terms do contain poles. In the end, these poles combine with those coming from the DR of the volume integrals of non-compact support terms. The proper cancellation of all the poles constitutes a robust check of the UV DR computations and a major confirmation that we understand the connection between the conservative equations of motion and the multipole moments within the framework of the MPM-PN approach.

The next steps are to reduce this result to the CoM frame and then to the case of quasi-circular orbits. We only need the 3PN expressions of the CoM coordinates, and the 3PN equations of motion, in order to express the mass quadrupole moment at the 4PN order in the CoM frame for circular orbits. Therefore, even if the result does not yet use DR for the IR divergences, we can still consistently express it in the CoM frame for circular orbits. The result, then much more compact, is given as follows

$$I_{ij} = \mu \left(A x_{\langle i} x_{j \rangle} + B \frac{r^2}{c^2} v_{\langle i} v_{j \rangle} + \frac{G^2 m^2 \nu}{c^5 r} C x_{\langle i} v_{j \rangle} \right) + \mathcal{O} \left(\frac{1}{c^9} \right), \quad (6.52)$$

²There are some missing terms in the equations (C3) of [131]; the correct expression, also taking into account the final determination of the ambiguity parameters [133, 38], is given in Eqs. (E.2)–(E.3) below taken from [9].

where the factors are explicitly given up to the 4PN order by³

$$\begin{aligned}
A = & 1 + \gamma \left(-\frac{1}{42} - \frac{13}{14} \nu \right) + \gamma^2 \left(-\frac{461}{1512} - \frac{18395}{1512} \nu - \frac{241}{1512} \nu^2 \right) \\
& + \gamma^3 \left(\frac{395899}{13200} - \frac{428}{105} \ln \left(\frac{r}{r_0} \right) + \left[\frac{3304319}{166320} - \frac{44}{3} \ln \left(\frac{r}{r'_0} \right) \right] \nu + \frac{162539}{16632} \nu^2 + \frac{2351}{33264} \nu^3 \right) \\
& + \gamma^4 \left(-\frac{1023844001989}{12713500800} + \frac{31886}{2205} \ln \left(\frac{r}{r_0} \right) + \left[-\frac{18862022737}{470870400} - \frac{2783}{1792} \pi^2 \right. \right. \\
& \quad \left. \left. - \frac{24326}{735} \ln \left(\frac{r}{r_0} \right) + \frac{8495}{63} \ln \left(\frac{r}{r'_0} \right) \right] \nu + \left[\frac{171906563}{4484480} + \frac{44909}{2688} \pi^2 - \frac{4897}{21} \ln \left(\frac{r}{r'_0} \right) \right] \nu^2 \right. \\
& \quad \left. - \frac{22063949}{5189184} \nu^3 + \frac{71131}{314496} \nu^4 \right), \tag{6.53a}
\end{aligned}$$

$$\begin{aligned}
B = & \frac{11}{21} - \frac{11}{7} \nu + \gamma \left(\frac{1607}{378} - \frac{1681}{378} \nu + \frac{229}{378} \nu^2 \right) \\
& + \gamma^2 \left(-\frac{357761}{19800} + \frac{428}{105} \ln \left(\frac{r}{r_0} \right) - \frac{92339}{5544} \nu + \frac{35759}{924} \nu^2 + \frac{457}{5544} \nu^3 \right) \\
& + \gamma^3 \left(\frac{17607264287}{1589187600} - \frac{4922}{2205} \ln \left(\frac{r}{r_0} \right) + \left[\frac{5456382809}{529729200} + \frac{143}{192} \pi^2 - \frac{1714}{49} \ln \left(\frac{r}{r_0} \right) - \frac{968}{63} \ln \left(\frac{r}{r'_0} \right) \right] \nu \right. \\
& \quad \left. + \left[\frac{351838141}{5045040} - \frac{41}{24} \pi^2 + \frac{968}{21} \ln \left(\frac{r}{r'_0} \right) \right] \nu^2 - \frac{1774615}{81081} \nu^3 - \frac{3053}{432432} \nu^4 \right), \tag{6.53b}
\end{aligned}$$

$$C = \frac{48}{7} + \gamma \left(-\frac{4096}{315} - \frac{24512}{945} \nu \right). \tag{6.53c}$$

Remark that the values of A and B are different for two coefficients therein from the one we have published in [9]. Indeed, we have forgotten to include the contribution of the UV d -dimensional regularisation of the potential \hat{R}_i in two of the compact terms (in SIC_{3PN} and VIC_{3PN} in App. E.2). We recall the already introduced notations: $\gamma = \frac{Gm}{rc^2}$, $r = |y_1^i - y_2^i|$, $v^i = v_1^i - v_2^i$, $x^i = y_1^i - y_2^i$, $m = m_1 + m_2$ and $\nu = \mu/m = m_1 m_2 / (m_1 + m_2)^2$. Two constants parametrize the logarithmic terms of (6.53). The constant r_0 was introduced in the Hadamard regularisation for the IR (see Eq. (3.31a)). Then, there appears the constant r'_0 associated with the UV regularisation and which has been introduced through the shift $\xi_{1,2}$ in Eqs. (E.3).⁴

The result (6.52)–(6.53) extends to the 4PN order the expression of the mass quadrupole moment that was known at the 3.5PN order [142, 143, 147, 43]. We are now in position to compute the correct dimensional-regularised mass quadrupole.

³The terms C represent the time-odd 2.5PN and 3.5PN contributions provided here for completeness.

⁴In previous works on the 3PN/4PN equations of motion in harmonic coordinates, two gauge constants r'_1 and r'_2 in the logarithms were considered for the UV divergences instead of one [154, 110, 131]. In the CM frame this yielded the two convenient combinations (with $X_{1,2} = m_{1,2}/m$)

$$\begin{aligned}
\ln r'_0 &= X_1 \ln r'_1 + X_2 \ln r'_2, \\
\ln r''_0 &= \frac{X_1^2 \ln r'_1 - X_2^2 \ln r'_2}{X_1 - X_2},
\end{aligned}$$

with r''_0 entering specifically the expression of the particle's positions in the CM frame at the 3PN order [110]. Because of the factor $(X_1 - X_2)^{-1}$ in $\ln r''_0$ there is an apparent divergence when the two masses are equal, but of course it is compensated by a factor $X_1 - X_2$ in the CM relations. There, we make the choice $r'_1 = r'_2$, which avoids such spurious divergence and has the advantage that the particle's positions are exactly $y_1^i = -y_2^i$ when the masses are equal. Hence we have only one UV constant $r'_0 = r''_0 = r'_1 = r'_2$.

6.4 The properly IR regularised quadrupole

Let us recall that for the computation of the mass quadrupole at 3PN, DR was used only in the UV and not in the IR. The reason for such a non-trivial statement is that DR in the UV was sufficient to remove the ambiguities in the conservative sector (see discussion in Sec. 3.2.2). When we started investigating DR in the IR for the computation of the source multipoles, it was supposed that Hadamard regularisation was sufficient at 3PN. However, it had never been proved that Hadamard was equivalent to DR. The effort we furnished to compute the DDR in the IR for the mass quadrupole at 4PN also led to confirm that the statement was indeed correct not only for the mass quadrupole but also for the current quadrupole and the mass octupole at 3PN (see Sec. 6.4.7 and Sec. 7.2.4).

In order to compute the DDR in the IR for the mass quadrupole, we have to list all the possible contributions where the difference between the two regularisations can appear. Of course, differences may appear in NC terms because the treatment of the regularisation at spatial infinity should in principle be different. For the same reason, we also expect the surface terms, only limited to spatial infinity, to be treated differently. However, it is non-trivial that the C terms, in which the values of the integrands are only required at the location of the bodies, should be different because they were, in principle, treated in the previous section by means of the UV DR. At first we did not think this would contribute, but we then realized that the values of the potentials that we were using were computed by a $3d$ Hadamard calculation followed by a UV difference. Therefore, these values were not the full d -dimensional regularised potentials in the whole space and it turned out that the correct IR treatment of the potentials changed their values also on the location of the bodies and not just at spatial infinity. Note that in principle, distributional terms can also arise. Since they are treated in the same way as C terms, we include their possible contribution in C terms. Furthermore, the last possible contribution in the IR DDR is the last term (proportional to B) in Eq. (6.1). Then the result for the mass quadrupole has to be shifted using the shift derived in the conservative sector, in the same way that it was done for the UV DDR. To summarize, we had to deal with the difference between Hadamard and dimensional regularisations for

- C terms
- NC terms
- Extra and Surface terms
- Shift

so that, the total IR DDR for the mass quadrupole can be written as

$$\mathcal{D}I_{ij}^{\text{IR}} = \mathcal{D}I_{ij}^{\text{C}} + \mathcal{D}I_{ij}^{\text{NC}} + \mathcal{D}I_{ij}^{\text{Surf}} + \mathcal{D}I_{ij}^{\text{extra}} + \mathcal{D}I_{ij}^{\text{shift}}. \quad (6.54)$$

The following sections aim at computing the values of the different contributions.

6.4.1 Computation of the DDR of potentials at point 1 and C terms

We want to compute the difference between the Hadamard and dimensional regularisations treatment of IR divergencies in C terms. The first step is to compute the value of the IR DDR of the potentials at point A . To do so, we need to solve the wave equation regularised at point 1, namely

$$P(\mathbf{y}_1) = (\square^{-1}S)(\mathbf{y}_1). \quad (6.55)$$

For this integration, we start PN expanding the D'Alembert operator

$$\square^{-1} = \sum_{k=0}^{+\infty} \left(\frac{1}{c} \frac{d}{dt} \right)^{2k} \Delta^{-k-1}. \quad (6.56)$$

We then need to solve the Laplace equation in order to extend its solution to the D'Alembert solution.

Solution of the Laplace equation

Consider a d -dimensional Poisson like potential sourced by a function $F^{(d)}$

$$P^{(d)}(\mathbf{x}') = -\frac{\tilde{k}}{4\pi} \mathcal{FP}_{B=0} \int_{r>\mathcal{R}} d^d \mathbf{x} \tilde{r}^B \frac{F^{(d)}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^{d-2}}. \quad (6.57)$$

evaluated any field point \mathbf{x}' . We pose $\tilde{k} = \frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}$. The constant in the Green's function of the Laplace operator is given by $\Delta(\tilde{k} r^{2-d}) = -4\pi\delta^{(d)}(\mathbf{x})$. At point 1 we have

$$P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi} \mathcal{FP}_{B=0} \int_{r>\mathcal{R}} d^d \mathbf{x} \tilde{r}^B \frac{F^{(d)}(\mathbf{x})}{r_1^{d-2}}. \quad (6.58)$$

We consider the class of functions $F^{(d)}$ which admit a generic expansion in d dimensions when $r \equiv |\mathbf{x}| \rightarrow +\infty$, such that, for any $N \in \mathbb{N}$,

$$F^{(d)}(\mathbf{x}) = \sum_{p=p_0}^N \sum_{q=q_0}^{q_1} \frac{r^{p+q\varepsilon}}{\ell_0^{q\varepsilon}} \varphi_{p,q}^{(\varepsilon)}(\mathbf{n}) + o\left(\frac{1}{r^N}\right), \quad (6.59)$$

where $p_0 \in \mathbb{N}$ indicates the maximal order of the IR divergence, and $q_0, q_1 \in \mathbb{Z}$ represent a finite range of values for q . Further more, we know that

$$\frac{1}{r_1^{d-2}} = \sum_{\ell=0}^{+\infty} \frac{2^\ell \Gamma\left(\frac{d}{2} + \ell - 1\right)}{\ell! \Gamma\left(\frac{d}{2} - 1\right)} \frac{y_1^L \hat{n}_1^L}{r^{d+\ell-2}}. \quad (6.60)$$

After inserting (6.59) and (6.60) into (6.58), we decompose the integral into a radial and angular one. After integration the integral reads

$$P^{(d)}(\mathbf{y}_1) = \frac{\tilde{k}}{4\pi} \sum_{\ell=0}^{+\infty} \frac{2^\ell \Gamma\left(\frac{d}{2} + \ell - 1\right)}{\ell! \Gamma\left(\frac{d}{2} - 1\right)} y_1^L \sum_{p,q} \frac{\ell_0^{q\varepsilon}}{r_0^B} \frac{\mathcal{R}^{B-\ell+p+2+q\varepsilon}}{B-\ell+p+2+q\varepsilon} \int d\Omega_{d-1} \hat{n}_1^L \varphi_{p,q}^{(\varepsilon)}(\mathbf{n}). \quad (6.61)$$

We do the same in 3d and compute the difference up to order $\mathcal{O}(\varepsilon)$. The constant \mathcal{R} cancels out and we get

$$\mathcal{DP}(\mathbf{y}_1) = -\sum_{\ell=0}^{+\infty} \frac{(2\ell-1)!!}{\ell!} y_1^L \mathcal{FP}_{B=0} \sum_{q \neq 0} \frac{1}{q} \left[\frac{1}{\varepsilon} - q \ln\left(\frac{r_0}{\ell_0}\right) + \sum_{k=0}^{\ell} \frac{1}{2k-1} \right] \langle \hat{n}_L \varphi_{\ell-2,-q}^{(\varepsilon)}(\mathbf{n}) \rangle. \quad (6.62)$$

where $\langle \rangle \equiv \int \frac{d\Omega_{d-1}}{\Omega_{d-1}}$ is the spherical average performed in d dimensions, while the volume of the sphere is $\Omega_{d-1} = \frac{4\pi}{(d-2)\tilde{k}}$.

Solution of the D'Alembert equation

For potentials that are required at least at 1PN, the D'Alembertian cannot be approximated as a mere Laplacian. The inverse D'Alembertian can be PN expanded in terms of the inverse Laplacian. Then, one can show that computing the IR DDR at the location of the bodies for the D'Alembert equation is equivalent to compute the IR DDR in the Laplace equation on a different source [11]. Indeed, solving (6.55) is equivalent solving

$$P(\mathbf{y}_1) = (\Delta^{-1} \hat{S})(\mathbf{y}_1), \quad (6.63)$$

using (6.62) with \hat{S} given in terms of S by

$$\hat{S}(\mathbf{x}, t) = S(\mathbf{x}, t) + \frac{r_1^2}{2(4-d)c^2} \partial_t^2 S(\mathbf{x}, t) + \frac{r_1^4}{8(4-d)(6-d)c^4} \partial_t^4 S(\mathbf{x}, t) + \mathcal{O}\left(\frac{1}{c^6}\right). \quad (6.64)$$

The prefactors come from the so-called *Matthieu* formulae that are introduced later in (6.72) when we iterate the inverse Laplacians.

The method described in this section was used to derive the IR DDR of the potentials that enter C terms. Right below we detail which potentials were required.

Computation of C terms

As we can see from Eq. (6.62), we have to perform a Taylor expansion of the source of the potentials at spatial infinity, as $r \rightarrow \infty$. The sum $\sum_{\ell \geq 0} \varphi_{-\ell+2,q}$ indicates that only the terms in the source containing $1/r^2$ and lower order ($1/r, r^0, r \dots$) will contribute. This condition highly limits the possible contributions in the given sources of the potentials. As an example, when we look at the complicated source of the potential \hat{T} in (B.4i), only the term $\propto V \partial_t^2 \hat{K}$ can contribute because the other terms are at least of the order $1/r^3$. Based on this argument, we can list the potentials that can have a non-zero IR DDR at the location of the bodies. The only potentials required for the DDR in the IR for C terms are \hat{W} at 2PN, \hat{Z} and \hat{X} at 1PN, \hat{M} and \hat{T} at 0PN.

Regarding the superpotentials, using the previous argument, they should in principle all have a non-zero IR DDR. However, we can use a parity argument to show that their IR DDR is zero. Indeed, for each ℓ , the parity of the number of n^i entering the expression of $\varphi_{-\ell+2,q}^{(\varepsilon)}$ in the super-potentials is the opposite of the parity of ℓ . Considering the fact that the angular integral of an odd number of n^i is zero, we thus see that $\langle \hat{n}_L \varphi_{-\ell+2,q}^{(\varepsilon)} \rangle$ always vanishes.

I used the method described above to compute the IR DDR of the 5 potentials $\{\hat{W}, \hat{Z}, \hat{X}, \hat{M}, \hat{T}\}$. Then, in order to obtain the corrected values of the C terms, I inserted the IR DDR of the potentials using the method described in Sec. 6.3.1. This yields a long result for $\mathcal{D}I_{ij}^C$ which is not interesting to detail here. The main feature is that it contains poles in $1/\varepsilon$ as well as the cut-off constants r_0 and ℓ_0 linked to Hadamard and DR.

6.4.2 Computation of the d -dimensional potentials at spatial infinity

The potentials that must be computed in d dimensions at spatial infinity are the ones entering the NC terms. One could think that we also need to compute the potentials entering Surface terms at spatial infinity, however it is shown in Sec. 6.4.4 that these terms vanish using DR, so that we do not need to compute them. The potentials present in the NC terms are: V, V_i, \hat{K} at 2PN, \hat{W}_{ij} at 1PN, $\hat{R}_i, \hat{X}, \hat{Z}_{ij}$ at 0PN and the 4 superpotentials $\Psi_{ij}^{\partial_{ab}V}, \Psi_{ij}^{\partial_a V_b}, \Psi_{ijk}^{\partial_a V}$ and $\Psi_{ij}^{\partial_t \partial_a V}$. Note that V, V_i, \hat{K} and the superpotentials were already known in the whole space in d dimensions so the only thing that had to be done was to expand them for $r \rightarrow \infty$ at consistent order. The source of the other relevant potentials are divided in two parts: the compact support part, involving the PN source densities σ, σ_i and σ_{ij} , and the non-compact support part which is made of other simpler potentials (see Eq. (B.4)). Thus the general form of the potentials is the following

$$\square P(\mathbf{x}, t) = S(\mathbf{x}, t) = S_C(\mathbf{x}, t) + S_{NC}(\mathbf{x}, t). \quad (6.65)$$

Furthermore, the asymptotic value of the d -dimensional potentials solution of (6.65) at infinity is given by [133]

$$\mathcal{M}(P) = \sum_{k=0}^{+\infty} \widetilde{\square_{\mathbb{R}^{-1}}} \mathcal{M}(S) - \frac{1}{4\pi} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \overline{\partial_L S_*^\ell}, \quad (6.66)$$

where $\widetilde{\square}_R^{-1}$ is the d -dimensional retarded propagator. The first term corresponds to a particular solution of (6.65) while the second term corresponds to a homogeneous solution and is fixed by the matching between the exterior zone and the near zone. This is the same matching as the one explained in Sec. 3.1.4. Then S_\star^L reads

$$S_\star^L = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \left[\mathcal{S}_L \left(t - \frac{zr}{c} \right) + \mathcal{S}_L \left(t + \frac{zr}{c} \right) \right], \quad (6.67)$$

with

$$\gamma_{\frac{1-d}{2}}(z) = \frac{2\sqrt{\pi}}{\Gamma\left(-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1+\varepsilon}{2}\right)} (z^2 - 1)^{-1-\frac{\varepsilon}{2}} \quad \text{and} \quad S_L(u) = \mathcal{F}\mathcal{P} \int d^d \mathbf{x} \tilde{r}^B x_L S(\mathbf{x}, u). \quad (6.68)$$

The tilded notation corresponds to the inverse Laplacian regularised by means of the $B\varepsilon$ procedure. After performing a PN expansion of (6.66) and integrating over z , we obtain the more explicit solution for the multipolar expansion the considered potential as

$$\mathcal{M}(P) = \sum_{k=0}^{+\infty} \left(\frac{1}{c} \frac{d}{dt} \right)^{2k} \widetilde{\Delta}^{-k-1} \mathcal{M}(S) - \frac{1}{4\pi^{\frac{d-1}{2}}} \sum_{k,\ell} \frac{(-)^\ell}{\ell!(2k)!c^{2k}} \frac{\Gamma\left(\frac{d}{2} - k - 1\right)}{\Gamma\left(\frac{1}{2} - k\right)} S_L^{(2k)}(t) \partial_L r^{2k-1-\varepsilon}. \quad (6.69)$$

The two particular solution (on the left) and the homogeneous solution (on the right) are treated separately. Furthermore, the computation of the NC part of the potentials is made of two steps: the particular and homogeneous solutions.

Treatment of the particular solution : the compact part

For the compact part of the potentials, we want to solve at spatial infinity in d dimensions equations of the form

$$\square P_C(\mathbf{x}, t) = S_C(\mathbf{x}, t) = s_1(t) \delta_1^{(d)} + 1 \leftrightarrow 2. \quad (6.70)$$

As in the previous section, we can PN expand the D'Alembertian using (6.56). In [83] is given the solution of $\Delta u = -4\pi \delta_1^{(d)}$ with $u = \tilde{k} r_1^{2-d}$. Using this result and Eqs. (6.72), we find that the solution of Eq. (6.70) reads

$$\mathcal{M}(P_C) = -\frac{1}{4\pi} \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1-\varepsilon}{2}\right)}{\pi^{\frac{1+\varepsilon}{2}}} \sum_{k=0}^{+\infty} \frac{s_1^{(2k)}(t)}{(2c)^{2k} k!} \frac{r^{2k-1-\varepsilon}}{\Gamma\left(\frac{1-\varepsilon}{2} + k\right)} + 1 \leftrightarrow 2. \quad (6.71)$$

The sum is truncated depending on the required PN order.

Treatment of the particular solution : the non-compact part

As shown in (6.69), the first step to derive $\mathcal{M}(P_{\text{NC}})$ consists in expanding S_{NC} at spatial infinity. Then we need to integrate the inverse Laplacians of $\mathcal{M}(S_{\text{NC}})$ using the very useful *Matthieu* formula (valid $\forall \alpha \in \mathbb{C} \setminus \{\ell - 2, -d - \ell\}$) [83]

$$\Delta^{-1} r^\alpha = \frac{r^{\alpha+2}}{(\alpha+2)(\alpha+d)}, \quad (6.72a)$$

$$\Delta^{-k} r^\alpha = \frac{\Gamma\left(\frac{\alpha}{2} + 1\right) \Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(\frac{\alpha}{2} + k + 1\right) \Gamma\left(\frac{\alpha+d}{2} + k\right)} \frac{r^{\alpha+2k}}{2^{2k}}, \quad (6.72b)$$

$$\Delta^{-1} (\hat{n}_L r^\alpha) = \frac{\hat{n}_L r^{\alpha+2}}{(\alpha - \ell + 2)(\alpha + \ell + d)}. \quad (6.72c)$$

Note that in the cases where $\alpha \in \{\ell - 2, -d - \ell\}$, these formulae diverge. These problems have been dealt with by introducing of a regulator η that we added to the power of r . We performed the integration and then took the limit $\eta \rightarrow 0$.

Treatment of the homogeneous solution

All requested potentials but \hat{X} have at most quadratic sources. This means that their NC source is a sum of products of derivatives of the simple C potentials V , V_i and \hat{K} (see Eq. (B.4)). In order to compute S_L that enters in (6.69), we need to derive integrals of the form

$$\int d^d \mathbf{x} x_L r_1^\alpha r_2^\beta n_1^P n_2^Q. \quad (6.73)$$

No ready-to-use closed form of this integral was known in d dimensions when we started the computation.⁵ However, (6.73) can be simplified by substituting $x^i = y_1^i + r_1 n_1^i$ and $n_2^i = (r_{12} n_{12}^i + r_1 n_1^i)/r_2$ which only requires to know the value of the integral

$$\int d^d \mathbf{x} r_1^\alpha r_2^\beta n_1^L. \quad (6.74)$$

Again, the value of this integral was only known for $L = 0$ so we used the algorithm explained in App. A.6 of [155]. The trick is to project n_1^i onto n_{12}^i

$$n_1^i = \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}} n_{12}^i + n_{1\perp}^i \quad (6.75)$$

where $n_{1\perp}^i$ is the component of n_1^i orthogonal to n_{12}^i . The angular integral over a given number of $n_{1\perp}^i$ is known and given in the same paper. In the end we can reduce all the integrals to the following one

$$\int d^d \mathbf{x} r_1^\alpha r_2^\beta = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{\alpha+d}{2}\right) \Gamma\left(\frac{\beta+d}{2}\right) \Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right) \Gamma\left(\frac{\alpha+\beta+2d}{2}\right)} r_{12}^{\alpha+\beta+d}, \quad (6.76)$$

This allowed us to compute the homogeneous solutions of (6.65) for quadratic-sourced potentials.

The particular case of the potential \hat{X}

One term in the source of \hat{X} is not quadratic, it reads $S^c = \hat{W}_{ij} \partial_{ij} V$. This term has two contributions: a NC and a distributional part.

For the NC part, the quantity S_L entering (6.69) reads $S_L = \mathcal{F}\mathcal{P} \int d^d \mathbf{x} \tilde{r}^B x_L \hat{W}_{ij} \partial_{ij} V$. We can compute this integral by defining the STF version \hat{S}_L of S_L on its L indices. The link between S_L and \hat{S}_L is very simple. This allows to use the super-potential method and derive exactly S_L :

$$\hat{S}_L = \int d^d \mathbf{x} \left(-4\pi G \left(\sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) \right) \Psi_L^{\partial_{ij} V} - \frac{d-1}{2(d-2)} \int d^d \mathbf{x} \partial_i V \partial_j V \Psi_L^{\partial_{ij} V}. \quad (6.77)$$

Since we are using DR, we can ignore the surface terms (not written here) induced by the super-potential method. After computing explicitly \hat{S}_L , we inserted its value in (6.69) and added it to the homogeneous solution of the quadratic source.

Regarding the distributional part, using (6.25) in d dimensions and (6.71), we have to add to the NC part of \hat{X} the quantity

$$\hat{X}_{\text{Distr}} = \frac{Gm_1(\hat{W})_1}{3+\varepsilon} \left(1 + \frac{\varepsilon}{2}(1 + \ln(\bar{q})) \right) + 1 \leftrightarrow 2, \quad (6.78)$$

where $\bar{q} = 4\pi e^{\gamma_E}$ and γ_E is the Euler constant.

We performed all the computations for the required potentials. The next step is to compute the IR DDR for the NC terms using the expressions of the potentials.

⁵By the time we were doing the computations, Guillaume Faye managed to find an analytical expression for this integral, see [11], but we did not use it.

6.4.3 Computation of NC terms

In this section we regularise the IR divergences of three-dimensional volume integrals of the type (6.37). We compute the difference between two IR regularisations: the so-called $B\varepsilon$ regularisation [38] which is our target, and the B regularisation in 3 dimensions. The $B\varepsilon$ regularization considers the integral (6.37) in $d = 3 + \varepsilon$ dimensions, with $F^{(d)}$ the analogue of the function F in d dimensions, and applies to it the \mathcal{FP} procedure with parameter B namely

$$I_{\mathcal{R}}^{B\varepsilon} = \mathcal{FP}_{B=0} \int_{r>\mathcal{R}} \frac{d^d \mathbf{x}}{\ell_0^\varepsilon} \left(\frac{r}{r_0}\right)^B F^{(d)}(\mathbf{x}). \quad (6.79)$$

We have inserted the characteristic length scale ℓ_0 associated with dimensional regularisation, as well as an arbitrary constant r_0 associated with the \mathcal{FP} procedure. The $B\varepsilon$ regularisation consists of applying first the \mathcal{FP} process when $B \rightarrow 0$, and second the usual DR when $\varepsilon \rightarrow 0$. On the other hand the B regularisation considers simply the integral $I_{\mathcal{R}}^+$ in 3 dimensions (6.37) and just applies the \mathcal{FP} at $B = 0$. The decomposition of $F^{(d)}$ is given in (6.59). The coefficients which can contain poles $\propto 1/\varepsilon$, *i.e.* are of the type

$$\varphi_{p,q}^{(\varepsilon)}(\mathbf{n}) = \frac{1}{\varepsilon} \psi_{p,q}^{(-1)}(\mathbf{n}) + \psi_{p,q}^{(0)}(\mathbf{n}) + \mathcal{O}(\varepsilon). \quad (6.80)$$

The coefficients of the pole $\psi_{p,q}^{(-1)}$ naturally are defined with no dependence upon ε , while the $\psi_{p,q}^{(\varepsilon)}$ are finite when $\varepsilon \rightarrow 0$. We furthermore assume that in 3 dimensions the function F is finite. Posing thus (for $\forall p$)

$$\sum_q \psi_{p,q}^{(-1)}(\mathbf{n}) = 0, \quad \sum_q \psi_{p,q}^{(0)}(\mathbf{n}) = \varphi_p(\mathbf{n}), \quad \sum_q q \psi_{p,q}^{(-1)}(\mathbf{n}) = -\varphi_p^\ell(\mathbf{n}). \quad (6.81)$$

we get the following logarithmic corresponding expansion in 3 dimensions

$$F^{(3)} = \sum_p r^p \left[\varphi_p(\mathbf{n}) + \varphi_p^\ell(\mathbf{n}) \ln \left(\frac{r}{\ell_0} \right) \right]. \quad (6.82)$$

The difference between the two regularisations is now computed in the limit where $\varepsilon \rightarrow 0$. We pose

$$\mathcal{D}I^{\text{IR}} \equiv I_{\mathcal{R}}^{B\varepsilon} - I_{\mathcal{R}}^+. \quad (6.83)$$

Since in the limit $\varepsilon \rightarrow 0$ the complementary integrals over $r < \mathcal{R}$ are finite and agree, the difference (6.83) should be independent on the cut-off scale \mathcal{R} . Using the relations (6.81) linking the d and 3 dimensional quantities, we obtain

$$\mathcal{D}I^{\text{IR}} = - \sum_{q \neq -1} \frac{1}{\varepsilon} \left[\frac{1}{q+1} + \varepsilon \ln \left(\frac{r_0}{\ell_0} \right) \right] \int d\Omega_{d-1} \varphi_{-3,-q}^{(\varepsilon)}(\mathbf{n}) - \frac{1}{2} \ln^2 \left(\frac{r_0}{\ell_0} \right) \int d\Omega \varphi_{-3}^\ell(\mathbf{n}) + \mathcal{O}(\varepsilon), \quad (6.84)$$

where $d\Omega_{d-1}$ denotes the $(d-1)$ -dimensional volume of the sphere. Indeed we find that modulo $\mathcal{O}(\varepsilon)$ -terms this difference does not depend on the cut-off scale \mathcal{R} . Note that the value $q = -1$ has to be excluded from the sum so that, unlike in the UV case, we did not have to check that $\langle \varphi_{-3,-1}^{(\varepsilon)} \rangle = 0$. A last use of (6.81) gives the compact form

$$\mathcal{D}I^{\text{IR}} = - \sum_{q \neq -1} \frac{1}{\varepsilon} \frac{1}{q+1} \left(\frac{r_0}{\ell_0} \right)^{(q+1)\varepsilon} \int d\Omega_{d-1} \varphi_{-3,-q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon). \quad (6.85)$$

For the computation of the IR DDR of NC terms, we inserted the expressions of the d -dimensional potentials derived in Sec. 6.4.2 and applied (6.85) on the source of the NC terms. In the end, as for the C terms, we find a result for $\mathcal{D}I_{ij}^{\text{NC}}$ composed of a pole in $1/\varepsilon$ and a finite part.

6.4.4 Computation of the extra term and surface terms

The reason why both extra term and surface terms are regrouped in this section is that they do not contribute with DR in the IR. In fact, they both vanish for the same reason that we are going to see below.

Extra term

Recall that the extra term in the definition of I_L is the one proportional to the regulator B and to ε . This induces a more specific constraint on the terms in the multipolar expansion of the source that can contribute using the $B\varepsilon$ procedure. Indeed, the extra term reads for $\ell = 2$ (see the last line of (6.1))

$$\mathbb{I}_{ij}^{\text{extra}} = \frac{1}{c^2} \sum_{k=0}^{+\infty} \left(\frac{1}{c} \frac{d}{dt} \right)^{2k} \mathcal{F}_{B=0} \mathcal{P} \lambda_k B (d-3) \int d^d \mathbf{x} \tilde{r}^B r^{2k-2} \hat{x}_{ija} x_b \bar{\Sigma}_{ab} \quad (6.86)$$

where λ_k is a coefficient depending on the dimension and k ⁶. We set $\ell_0 = 1$ because it is not relevant here. Once again, we can split the integral into two parts: for $r < \mathcal{R}$ and $r > \mathcal{R}$, \mathcal{R} being an arbitrary radial constant. It was shown that in the UV, so the part for $r < \mathcal{R}$, this integral did not contribute. Consequently, we can restrict ourselves to the IR, for $r > \mathcal{R}$. For \mathcal{R} sufficiently large, we can perform a multipolar decomposition of the source $\bar{\Sigma}_{ij}$ reading

$$\bar{\Sigma}_{ab} = \sum_{p \geq 0} \sum_{q=q_0}^{q_1} r^{p+q\varepsilon} \bar{\sigma}_{ab}^{(p,q)}. \quad (6.87)$$

With this decomposition in hand, we can split the integral into a radial and an angular one and integrate over r using the $B\varepsilon$ procedure. This leads to the final result for the extra term

$$\mathbb{I}_{ij}^{\text{extra}} = \frac{1}{c^2} \sum_{k=0}^{+\infty} \lambda_k \left(\frac{1}{c} \frac{d}{dt} \right)^{2k} \int d\Omega_{d-1} \hat{n}_{ija} n_b \bar{\sigma}_{ab}^{(-5-2k,-1)}. \quad (6.88)$$

This means that only the terms in the source that admit a coefficient $q = -1$ can contribute. Remind that the source is formed of products of derivatives of potentials. All the potentials admit at most $q = -1$ and their derivatives don't affect this coefficient. This means that a product of potential will always contain $q = -2$ or less. In other words, due to the structure of $\bar{\Sigma}_{ab}$, $\mathbb{I}_{ij}^{\text{extra}}$ vanishes and doesn't need to be computed. Also note that for the computation of the mass quadrupole up to 4PN, $\bar{\Sigma}_{ab}$ was only required at 2PN because it contains a factor $1/c^4$ in the third term of (6.1). However, the extra term contains a factor $1/c^2$, thus we would have had to compute it up to 3PN. At this order, more complicated potentials appear and their computation would have been much harder than the ones needed in the previous sections. In the end, since $\mathbb{I}_{ij}^{\text{extra}} = 0$, we can drop $\mathcal{D}\mathbb{I}_{ij}^{\text{extra}}$ from (6.54).

Surface terms

The reasoning for the two types surface terms, generally of the form (6.45) and (6.49)⁷, is exactly the same. The integrals are already to be considered at spatial infinity. We first perform a multipolar expansion of the source, integrate over the radial variable and find a resulting angular integral in which only the coefficients of the source containing $q = -1$ can contribute. Since their source is

⁶More exactly, its expression is

$$\lambda_k = -\frac{1}{2^{2k-1} k!} \frac{(d+2)}{d(d-2)(d+4)} \frac{\Gamma(\frac{d}{2}+3)}{\Gamma(\frac{d}{2}+3+k)}.$$

⁷In these definitions, $d^3 \mathbf{x}$ has to be replaced by $d^d \mathbf{x}$.

made of products of potentials, coefficients with $q = -1$ cannot appear and thus these integrals are exactly 0. Hence, in order to get the correct value of the dimensional-regularised mass quadrupole, we have to subtract from (6.52) the values of the Surface terms so that

$$\mathcal{D}I_{ij}^{\text{Surf}} = -I_{ij}^{\text{Surf,Had}}. \quad (6.89)$$

In Sec. 6.2.2 and Sec. 6.2.3, we explained how we transformed NC terms involving complicated potentials (such as \hat{M}_{ij}) into simpler NC terms plus C and Surface integrals. In fact, all the potentials that start parametrizing the PN metric at 3PN and 4PN, are not contained in the NC terms but in the Surface terms. The argument above showed that the Surface terms do not contribute when treated with DR. However, in [9], we used Hadamard partie finie for the IR and in this regularisation the Surface terms do not vanish. Though, the very complicated computation of these terms made by Tanguy Marchad, François Larrouturou and Sylvain Marsat was not in vain, but crucial as we will see in Sec. 6.4.6.

6.4.5 Applying the IR shift

We recall that the shift on the bodies' trajectories were derived in the conservative sector in order to remove the poles in the EoM [134]. This shift has to be performed in the radiative sector in order to remain consistent. In the end, the contribution of the shift to the mass quadrupole is

$$\mathcal{D}I_{ij}^{\text{shift}} = 2m_1 y_1^{\langle i} \eta_1^{j \rangle} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^{10}}\right), \quad (6.90)$$

where the explicit expression of the shift is given in Eq. (E.4). Note that the shift starts at 4PN.

6.4.6 Final result for the source mass quadrupole

In the previous section, we computed all the different contributions to the IR DDR for the mass quadrupole. We have not given explicitly the expressions of $\mathcal{D}I_{ij}^{\text{C}}$ and $\mathcal{D}I_{ij}^{\text{NC}}$ because their expressions are quite long. The IR DDR reads

$$\mathcal{D}I_{ij}^{\text{IR}} = \mathcal{D}I_{ij}^{\text{C}} + \mathcal{D}I_{ij}^{\text{NC}} - I_{ij}^{\text{Surf,Had}} + \mathcal{D}I_{ij}^{\text{shift}}. \quad (6.91)$$

In the case of the UV DDR, we added the UV contributions (without shift) to the Hadamard regularised mass quadrupole, the result contained a pole in $1/\varepsilon$ that was exactly canceled by the UV shift. However, after computing the IR difference (without applying the shift), we found a pole at 3PN. This pole cannot be canceled by the IR shift because it is a 4PN quantity. In other words, a pole remained in the mass quadrupole at 3PN. After a lot of consistency checks, we found out that this pole was correct and had to be canceled later on. We investigated the non-linear effects arising from the difference between the source and radiative multipoles and we found out that the tail-of-tail effects that arise at 3PN had to be computed in d dimensions. François Larrouturou and Luc Blanchet computed the DDR of the tail-of-tail effects, that we call here $\mathcal{D}U_{ij}^{\text{tail-of-tail}}$ and this contribution canceled exactly the IR DDR at 3PN

$$\begin{aligned} \mathcal{D}I_{ij}^{\text{IR}}|_{3\text{PN}} &= -\mathcal{D}U_{ij}^{\text{tail-of-tail}}|_{3\text{PN}} \\ &= \frac{214}{105} \frac{G^2 M^2 I_{ij}^{(2)}}{2c^6} \left[\Pi_\varepsilon - \frac{246299}{22470} \right] - \frac{214}{105} \frac{G^2 M P_{\langle i} P_{j \rangle}}{c^6} \left[\Pi_\varepsilon - \frac{252599}{22470} \right] + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \quad (6.92)$$

where $P_i = m_1 v_1^i + m_2 v_2^i$ is the Newtonian linear momentum of the system and $\Pi_\varepsilon = \frac{1}{\varepsilon} - 2 \ln\left(\frac{r_0 \sqrt{q}}{\ell_0}\right)$. The linear momentum vanishes in the CoM frame, thus this cancelation is exact in any harmonic coordinate frame. In the end, we have shown that the expression of the 3PN mass quadrupole

computed with Hadamard for the IR is exactly equal to the 3PN mass quadrupole computed with DR for the IR. However, at 4PN, other types of non-linear effects [156, 11] appear and we expect to show that the poles cancel out in the end to give a finite result when $d \rightarrow 3$

$$\mathcal{D}I_{ij}^{\text{IR}}|_{\text{pole}} = -\mathcal{D}U_{ij}^{\text{non-lin}}|_{\text{pole}}. \quad (6.93)$$

Now we can wonder if we expect that the finite part should cancel. The rest of this paragraph is only a speculation and I will give my opinion. A priori, if we suppose that the computation should behave as in the UV, it shouldn't be the case because in the UV the poles vanish but a finite part remains. However, the problem is different here. The IR regularisation had to be introduced at the level of (3.14) in Sec. 3.1.4 for the first time. A priori, the general reasoning of the matching should be independent of the choice of regularisation scheme employed. Assuming this statement, we can raise the question: why did Hadamard regularisation fail in the IR? Two answers are possible: either some subtleties of this regularisation were not understood at the time of the computations, either it is the regularisation itself that, due to some bad properties, could not work. Since I did not perform the computations in the conservative sector at 4PN, I cannot form a final opinion. However, I think that *if* the finite part cancels at 4PN the same way that it did at 3PN, namely $\mathcal{D}I_{ij}^{\text{IR}} = -\mathcal{D}U_{ij}^{\text{non-lin}}$, then it would show that Hadamard regularisation would have given the same result as DR in the conservative sector. This is because the regularisation scheme has to be consistent in both sectors. This would imply that a mistake in the computations occurred in the treatment of subtleties in Hadamard that can be tricky. On the contrary, if we find that the finite part does not cancel, we could not have used another regularisation than DR and thus the computation failed due to the inconvenient properties of Hadamard. Unfortunately, as we do not know what to expect, this cannot constitute a check the way it did at 3PN because at this order we had to find the cancelation.

6.4.7 The IR DDR of the mass octupole

As mentioned in the introduction, I computed entirely the 3PN current quadrupole J_{ij} and re-computed the 3PN mass octupole I_{ijk} . The mass octupole was already published in [148] in which they used Hadamard for the treatment of IR divergencies. I confirmed the obtained value with Hadamard for the IR and I also computed for the first time its IR DDR. On the other hand, François Larrouturou computed the DDR for the tails-of-tails. After putting together our results, we have shown that, exactly as for the mass quadrupole, the contributions canceled out up to 3PN and they are of the form

$$\begin{aligned} \mathcal{D}I_{ijk}^{\text{IR}} &= -\mathcal{D}U_{ijk}^{\text{tail-of-tail}} \\ &= \frac{13}{21} \frac{G^2 M^2 I_{ijk}^{(2)}}{c^6} \left[\Pi_\epsilon - \frac{9281}{1365} \right] - \frac{26}{7} \frac{G^2 M I_{(ij}^{(1)} P_{k)}}{c^6} \left[\Pi_\epsilon - \frac{28774}{4095} \right] \\ &\quad + \frac{6}{5} \frac{G^2 M I_{(ij}^{(2)} I_{k)}}{c^6} \left[\Pi_\epsilon - \frac{63421}{3780} \right] + \frac{46}{35} \frac{G^2 I_{(i} P_j P_{k)}}{c^6} \left[\Pi_\epsilon + \frac{161597}{14490} \right] + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \quad (6.94)$$

where $I_i = m_1 y_1^i + m_2 y_2^i$ is the Newtonian mass dipole (that vanishes in the CoM). As well as for the mass quadrupole, and also current quadrupole (see Sec. 7.2.4), the cancelation shows that the Hadamard regularisation for the IR is equivalent to DR and thus that the mass octupole has been computed with the correct method.

Chapter 7

The 3PN radiative current quadrupole and the associated gravitational amplitude mode

As mentioned in the introduction of Part III, in order to get the 4PN flux, we need to derive the radiative current quadrupole at 3PN order. Even though the computation of the radiative mass quadrupole is not yet achieved, I derived the full radiative current quadrupole at 3PN order. To do so, we needed to compute the source current quadrupole. However, the generalisation in d dimensions of the source current quadrupole is more complicated than the one for the mass quadrupole. In $3d$, the STF property of the multipole moments plays a crucial role in this formalism, since it guaranties that they are linearly independent. This is the case because they are irreducible tensors, in the sense that they belong to irreducible representations of the rotation group $\text{SO}(3)$. Now, the irreducible decomposition of vector and tensor fields in a generic space of dimension d is more complicated than in 3 dimensions. Indeed, irreducible tensors in d dimensions are exactly the trace-free tensors that have Young-tableau symmetries (elements of $\text{SO}(d)$ representations). When $d = 3$, because of identities ensuing from the fact that antisymmetrization over more than three indices yields zero, any Young tableau can be written as the tensor product of STF and anti-symmetric Levi-Civita tensors. However, for generic d , notably due the impossibility of generalizing the Levi-Civita tensor, all Young tableaux must a priori be considered.

Once the current quadrupole moment was properly defined, I applied the techniques described in Chapter 6 to compute it with 3PN accuracy in the case of compact binaries without spins. The result is presented in the CoM frame and for quasi-circular orbits. Next, adding corrections coming from non-linear multipole interactions (most notably the tails-of-tails already known from [148]), I obtained the radiative current quadrupole moment measured at future null infinity. The full physical content of this moment is encoded into the gravitational amplitude mode $(\ell, m) = (2, 1)$ ¹, which is provided with 3PN relative accuracy, corresponding to 3.5PN accuracy in the full waveform. The perturbative limit or small mass-ratio limit is an important test for the mode, and found to agree with the result from black-hole perturbation theory [157, 158, 159]. The expression for this current quadrupolar mode is ready for comparison with existing numerical relativity calculations such as [160, 161].

In Sec. 7.1 is presented the d -dimensional generalisation of the source current quadrupole (3.31b) following [10]. In Sec. 7.2, are detailed the results for the computation of the current quadrupole moment J_{ij} up to 3PN order. Sec. 7.3 is devoted to the computation of the radiative current quadrupole V_{ij} defined at future null infinity, and the corresponding GW mode h^{21} , given for quasi-circular orbits by Eq. (7.28). Finally, in Sec. 7.4, we perform a partial test of our result by checking the transformation law of the quadrupole moment J_{ij} under a constant spatial shift.

¹We recall that a quantity can be decomposed in the spherical harmonics basis. Then the components in this basis is indexed by two numbers: the azimuthal ℓ and the polar m . See Sec. 7.3 for more details.

7.1 The current multipole moments in d dimensions

In [10], we presented the full d -dimensional generalisation of the expression of the linearized metric $h_{(1)}^{\mu\nu}$. As in $3d$, explained in Sec. 3.1.1, the metric can be split into a ‘‘canonical’’ metric $k_{(1)}^{\mu\nu}$ having Thorne’s form (see Eqs. (8.12) in [79]), plus a linearized gauge transformation with gauge vector $\varphi_{(1)}^\mu$

$$h_{(1)}^{\mu\nu} = k_{(1)}^{\mu\nu} + \partial^\mu \varphi_{(1)}^\nu + \partial^\nu \varphi_{(1)}^\mu - \eta^{\mu\nu} \partial_\lambda \varphi_{(1)}^\lambda. \quad (7.1)$$

In d dimensions, the canonical metric is parametrized by means of three irreducible moments I_L , $J_{i|L}$ and $K_{ij|L}$, as

$$k_{(1)}^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \tilde{I}_L, \quad (7.2a)$$

$$k_{(1)}^{0i} = \frac{4}{c^3} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\partial_{L-1} \dot{\tilde{I}}_{iL-1} + \frac{\ell}{\ell+1} \partial_L \tilde{J}_{i|L} \right], \quad (7.2b)$$

$$k_{(1)}^{ij} = -\frac{4}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\partial_{L-2} \ddot{\tilde{I}}_{ijL-2} + \frac{2\ell}{\ell+1} \partial_{L-1} \dot{\tilde{J}}_{(i|\underline{L-1}j)} + \frac{\ell-1}{\ell+1} \partial_L \tilde{K}_{ij|L} \right], \quad (7.2c)$$

where the underline notation denotes indices that are excluded from the symmetrization. The tilded quantities in the above equations are defined for an arbitrary function of time A by

$$\tilde{A}(r, t) = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) A(t - zr), \quad (7.3)$$

where $\tilde{k} = \Gamma(\frac{d}{2} - 1)/\pi^{\frac{d}{2}-1}$ and

$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)} (z^2 - 1)^{\frac{1-d}{2}}. \quad (7.4)$$

Following the d -dimensional generalisation of the reasoning presented in Sec. 3.1.4 and Sec. 3.1.5, Guillaume Faye obtained the explicit expressions of the three irreducible moments entering (7.2). Of course, the expression of the mass-type moment displayed in (6.1) was recovered. It is STF in all its indices so that its symmetry is given by a symmetric Young tableau (with multi-index $L = i_1 \cdots i_\ell$)

$$I_L = \boxed{i_\ell \cdots i_1}. \quad (7.5)$$

Then, the proper generalization of the current-type moments in d dimensions, which we want to compute, was found to be

$$\begin{aligned} J_{i|L} = \mathcal{A}_{ii_\ell} \mathcal{F} \mathcal{P} \int_{B=0} d^d \mathbf{x} \tilde{r}^B \left\{ -2 \left[\hat{x}^L \bar{\Sigma}_{[\ell]}^i - \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta^{i\langle i_\ell} \hat{x}^{L-1 \rangle a} \bar{\Sigma}_{[\ell]}^a \right] \right. \\ \left. + \frac{2(2\ell + d - 2)}{c^2(\ell + d - 1)(2\ell + d)} \left[\hat{x}^{aL} \dot{\Sigma}_{[\ell+1]}^{ia} - \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta^{i\langle i_\ell} \hat{x}^{L-1 \rangle ab} \dot{\Sigma}_{[\ell+1]}^{ab} \right] \right\}. \end{aligned} \quad (7.6)$$

Here, \mathcal{A}_{ii_ℓ} represents the anti-symmetrization with respect to the pair of indices ii_ℓ (with the factor $\frac{1}{2}$ included). Remark that the second terms inside the square brackets correspond to the d -dimensional trace of the first terms. The sources Σ , Σ^i and Σ^{ij} are defined in Eq. (6.2). Note that, in three dimensions, the quantity $J_{i|L}$ is not the usual current multipole that enters the definition of the flux. In fact, $J_{i|L}$ is the dual of this quantity and they are linked by the relation (7.9).

The tensor $K_{ij|L}$ appearing in $k_{(1)}^{ij}$ vanishes in 3 dimensions. This can be checked by counting the number of independent components of $K_{ij|L}$, which follows from the King’s rule [162] and is given by

Eq. (A6b) in the App. A of [10]. As it is proportional to $d-3$, there is no independent component in 3 dimensions. This is similar to what happens for the Weyl tensor, which does not exist either in 3 dimensions. In fact, the tensor $K_{ij|L}$ for $\ell = 2$ has exactly the same trace-free property, symmetries, and number of independent components as the Weyl tensor. Thus, we do not need to compute it from its explicit expression given in (2.34) of [10].

The symmetries of the moments $J_{i|L}$ and $K_{ij|L}$ are given by the mixed Young tableaux [163, 164, 165]

$$J_{i|L} = \begin{array}{|c|c|c|c|} \hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & & & \\ \hline \end{array}, \quad K_{ij|L} = \begin{array}{|c|c|c|c|} \hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & & & \\ \hline \end{array}, \quad (7.7)$$

respectively, with the convention that the indices are symmetrized over lines *before* being antisymmetrized over columns. It is convenient to introduce the following specific notation that allows reconstructing the symmetries of $J_{i|L}$ given $\ell + 1$ indices:

$$\text{Sym}_{i|L} \equiv \mathcal{A}_{i i_\ell} \text{TF}_{iL} \text{STF}_L, \quad (7.8)$$

where TF is the operator taking the trace-free part of a tensor. As an example, the first line of (7.6) can be constructed, starting from $x^L \overline{\Sigma}_{[\ell]}^i$, by taking its STF part on the indices L , then removing the traces of the resulting object in d dimensions and, finally, anti-symmetrizing on $\{i, i_\ell\}$. Thus, $J_{i|L}$ is trace-free, STF with respect to $L-1 = i_1 \cdots i_{\ell-1}$ and anti-symmetric with respect to the pair $\{i_\ell, i\}$; on the other hand, $K_{ij|L}$ is trace-free, STF with respect to $L-2 = i_1 \cdots i_{\ell-2}$ and anti-symmetric with respect to both pairs $\{i_\ell, j\}$ and $\{i_{\ell-1}, i\}$.

The ordinary STF mass-type moment in 3 dimensions, say $I_L^{[3]}$, is simply recovered as the limit $I_L^{[3]} = \lim_{d \rightarrow 3} I_L$. For the ordinary STF current-type moment in 3 dimensions, $J_L^{[3]}$, we have

$$\lim_{d \rightarrow 3} J_{i|L} = \varepsilon_{i i_\ell a} J_{aL-1}^{[3]} \iff J_L^{[3]} = \frac{1}{2} \varepsilon_{ab(i_\ell} \lim_{d \rightarrow 3} J_{a|bL-1}, \quad (7.9)$$

where ε_{abi} is the usual Levi-Civita symbol in 3 dimensions (we recall that underlined indices being excluded from symmetrization). Notice that $J_L^{[3]}$, as recovered from (7.9), not only is symmetric in its indices L but is also automatically trace-free.

Furthermore, the gauge vector $\varphi_{(1)}^\mu$ is also explicit with respect to some other moments as in Eq. (3.11). They are generalized in d dimensions but were not needed to be derived, their 3d expression is sufficient for the derivation of the 4PN phase. However, the value of the gauge vector in terms of the gauge moments is displayed in Eq. (2.37) of [10].

7.2 The current quadrupole moment of compact binaries

7.2.1 Computation of the current quadrupole

I computed $J_{k|ji}$ the expression (7.6) for $\ell = 2$ up to the 3PN order. Remember that $J_{k|ji}$ is the dual of the physical current quadrupole moment J_{ij} but remark also that it is not symmetric on $\{i, j\}$. I employed essentially the same set of techniques as for the computation of the mass quadrupole moment at the 4PN order presented in Sec. 6.3. All calculations were performed with *Mathematica* supplemented with the *xAct* library [13]. Here is a reminder of the method (also used for the mass quadrupole):

- I expressed $J_{k|ji}$ in terms of integrals of potentials by inserting the sources (6.2) into (7.6). I then simplified the integrand as much as possible using integration by part, together with a set of relations between potentials (see Sec. 6.2.3).

- The resulting expression for $J_{k|ji}$ is divided into three types of terms: compact support terms, non-compact support terms and surface terms at spatial infinity (when $r \rightarrow +\infty$ with fixed t).
- The required potentials were already known from the computation of the 4PN mass quadrupole except for the potential \hat{R}_i that needed to be known at 1PN instead of leading order.
- Compact support terms are calculated directly in d dimensions, as in Sec. 6.3.1, using the d -dimensional regularized potentials at the source points 1 and 2 derived in [9].
- Non-compact support terms are first computed in 3 dimensions; the difference between dimensional and Hadamard regularisations in the UV is obtained in a second stage using (6.43), and added to the 3-dimensional result. This is discussed in Secs. 6.3.2 and 6.3.3.
- For one particular non-compact support term (involving the non-linear potential \hat{Y}_i defined in (B.4h)), I employed the method of super-potentials explained in Sec. 6.2.2. The computation of this term is detailed in Sec. 7.2.2.
- All derivatives of potentials are understood as Schwartz distributional derivatives [152], *i.e.*, we apply the Gel'fand-Shilov formula in d dimensions [151] (see Eq. (6.27)). This gives rise to distributional terms that are treated exactly as compact terms.
- However, the IR divergences are treated using the Hadamard partie finie procedure with regulator \tilde{r}^B in 3 dimensions (see Sec. 7.2.4 for a justification of this point).
- The result for $J_{k|ji}$ after integration contains a pole in $\frac{1}{d-3}$: the last step consists in applying a shift, given in Eq. (E.1), on the bodies' positions; when substituting the renormalized position variables to the bare ones, the pole in $J_{k|ji}$ cancels out.
- Finally, we can take the limit $d \rightarrow 3$ using (7.9) in order to get the renormalized current quadrupole moment J_{ij} displayed in Eq. (7.15).

The investigation of the current moment in d dimensions in Sec. 7.1 is essential to treat the UV divergences with DR, which incidentally is the only way to give a full meaning to distributional derivatives. However, the calculation of the current quadrupole limited to 3PN order does not require corrections from DR for the IR divergences (see Sec. 7.2.4). We thus treat those in the source moment with the Hadamard partie finie regularisation and add the non-linear corrections due to tails and tails-of-tails as computed in 3 dimensions. We have shown that, as for the 3PN mass quadrupole and octupole, the IR corrections in the source moment computed in pure DR are cancelled by corresponding UV corrections coming from the tails-of-tails in d dimensions, yielding the equivalence of Hadamard partie finie and DR for the IR divergencies at 3PN order.

7.2.2 Example of the super-potential method

Let us illustrate one computational aspect concerning the use of “super-potentials” already developed in Sec. 6.2.2. One non-compact support term to be computed in the current quadrupole moment $J_{k|ji}$ reads

$$J_{k|ji}^{\hat{Y}} \equiv -\frac{4}{\pi G c^6} \frac{d-1}{d-2} \text{Sym}_{k|ji} \mathcal{F}_{\mathcal{P}} \int_{B=0} d^d \mathbf{x} \tilde{r}^B x^{ij} \partial_k \hat{Y}_a \partial_a V, \quad (7.10)$$

where Sym is defined in (7.8). It involves the difficult non-linear potential \hat{Y}_a obeying a Poisson equation $\Delta \hat{Y}_a = \hat{S}_a$ at leading order (see (B.4h) for the full definition). While the source term \hat{S}_a is known and relatively easy to manage, no closed-form expression is available for the potential \hat{Y}_a itself, even in 3 dimensions. On the other hand, V is a simple linear potential with compact support:

$\Delta V = -4\pi G\sigma$. By means of an integration by parts taking advantage of the symmetries of $J_{k|ji}$ (the all-integrated term being zero by analytic continuation in B), we transform this term into

$$J_{k|ji}^{\hat{Y}} = \frac{4}{\pi G c^6} \frac{d-1}{d-2} \text{Sym}_{k|ji} \mathcal{FP}_{B=0} \int d^d \mathbf{x} \tilde{r}^B \hat{x}^{ij} \hat{Y}_a \partial_{ka} V. \quad (7.11)$$

The idea behind introducing super-potentials is that the solution of the equation $\Delta \Psi_{ij}^{\partial_{ka} V} = \hat{x}_{ij} \partial_{ka} V$ (where $\hat{x}_{ij} \equiv x_{\langle ij \rangle}$ is STF) is known in analytic closed-form in terms of the super-potentials of V , namely the Poisson-like potentials V_{2k} satisfying the hierarchy of equations $\Delta V_{2k+2} = V_{2k}$, together with $V_0 \equiv V$. This solution,

$$\Psi_{ij}^{\partial_{ka} V} = x_{\langle ij \rangle} \partial_{ka} V_2 - 4x_{\langle i} \partial_{j \rangle ka} V_4 + 8\partial_{\langle ij \rangle ka} V_6, \quad (7.12)$$

is valid in the sense of distributions and its expression holds in any d dimensions (see Eq. (6.17)). The value of this superpotential was already known from the computation of the mass quadrupole. Then, we can transform the term (7.11) using (6.18) into the more tractable form

$$J_{k|ji}^{\hat{Y}} = \frac{4}{\pi G c^6} \frac{d-1}{d-2} \text{Sym}_{k|ji} \mathcal{FP}_{B=0} \int d^d \mathbf{x} \tilde{r}^B \left[\Psi_{ij}^{\partial_{ka} V} \hat{S}_a + \partial_b \left(\partial_b \Psi_{ij}^{\partial_{ka} V} \hat{Y}_a - \Psi_{ij}^{\partial_{ka} V} \partial_b \hat{Y}_a \right) \right]. \quad (7.13)$$

Here, the first term is straightforwardly computed because we know the source \hat{S}_a of the potential \hat{Y}_a . As for the second term, which is a surface term, it can be computed directly in 3 dimensions since it does not involve UV divergences. It depends only on the expansions of \hat{Y}_a and the superpotential at spatial infinity, when $r \rightarrow +\infty$. The expansion of \hat{Y}_a can be determined directly from the source term \hat{S}_a , without having to control \hat{Y}_a all-over the space. As a check of the result, I have also calculated $J_{k|ji}^{\hat{Y}}$ using an alternative form in which I did not perform the integration by part of (7.11). It reads

$$J_{k|ji}^{\hat{Y}} = -\frac{4}{\pi G c^6} \frac{d-1}{d-2} \text{Sym}_{k|ji} \mathcal{FP}_{B=0} \int d^d \mathbf{x} \tilde{r}^B \left\{ 4x^i \partial_a V_4 \partial_{jk} \hat{S}_a + x^{ij} \partial_a V_2 \partial_k \hat{S}_a \right. \\ \left. + \partial_l \left[x^{ij} \partial_{al} V_2 \partial_k \hat{Y}_a - \partial_a V_2 \partial_l (x^{ij} \partial_k \hat{Y}_a) + 4x^i \partial_{al} V_4 \partial_{kj} \hat{Y}_a - 4\partial_a V_4 \partial_l (x^i \partial_{kj} \hat{Y}_a) \right] \right\}, \quad (7.14)$$

which is derived directly by substituting $\Delta \hat{Y}_a$ to \hat{S}_a , expanding the derivatives by means of the Leibniz rule, and resorting again to the properties of the symmetry operator Sym . We see that the quantity V_6 does not appear in (7.14), thus the check on the value of this integral is robust. The result of this integral is quite long and is not relevant to be displayed here.

7.2.3 Final result

The final expression of the 3PN current quadrupole in a general frame is obtained, as mentioned above, by applying the UV shift (E.1), taking the 3 dimensional limit and, finally, using (7.9) to come back to the usual current moment J_{ij} . I have checked that the symmetry operations (7.8) commute with the computation of the difference between the dimensional and Hadamard regularisations. The expression of the final current quadrupole is quite long; so it is presented only in the CoM frame.² For general orbits, we obtain

$$J_{ij} = -\nu m \Delta \left[A L^{\langle i} x^{j \rangle} + B \frac{(vx)}{c^2} L^{\langle i} v^{j \rangle} + C \frac{Gm}{c^3} L^{\langle i} v^{j \rangle} \right] + \mathcal{O} \left(\frac{1}{c^7} \right). \quad (7.15)$$

²Notation is as follows: The relative position and velocity of the two particles (in harmonic coordinates) are $x^i = y_1^i - y_2^i$, $v^i = dx^i/dt = v_1^i - v_2^i$; the distance between the two particles $r = |\mathbf{x}|$; the masses m_1 and m_2 , the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = m_1 m_2 / m^2$ and the mass difference ratio $\Delta = (m_1 - m_2) / m$; finally, we pose $L^i \equiv \varepsilon^{iab} x^a v^b$ and $(vx) = v^i x^i$.

The coefficients A and B describe the conservative effects up to the 3PN order; A also includes a 2.5PN dissipative term, while the coefficient C is purely dissipative. We have

$$\begin{aligned}
A = & 1 + \frac{1}{c^2} \left[\left(\frac{13}{28} - \frac{17}{7} \nu \right) v^2 + \frac{Gm}{r} \left(\frac{27}{14} + \frac{15}{7} \nu \right) \right] \\
& + \frac{1}{c^4} \left[\left(\frac{29}{84} - \frac{11}{3} \nu + \frac{505}{56} \nu^2 \right) v^4 + \frac{Gm}{r^3} \left(-\frac{5}{252} - \frac{241}{252} \nu - \frac{335}{84} \nu^2 \right) (vx)^2 \right. \\
& \quad \left. + \frac{Gm}{r} \left(\frac{671}{252} - \frac{1297}{126} \nu - \frac{121}{12} \nu^2 \right) v^2 + \frac{G^2 m^2}{r^2} \left(-\frac{43}{252} - \frac{1543}{126} \nu + \frac{293}{84} \nu^2 \right) \right] \\
& + \frac{1}{c^5} \left[-\frac{62}{7} \frac{G^2 m^2}{r^3} \nu (vx) \right] \\
& + \frac{1}{c^6} \left[\frac{Gm}{r^3} \left(\frac{109}{11088} - \frac{2687}{1232} \nu - \frac{13507}{11088} \nu^2 + \frac{31387}{924} \nu^3 \right) (vx)^2 v^2 \right. \\
& \quad + \frac{G^2 m^2}{r^2} \left(\frac{166931}{27720} - \frac{273443}{5544} \nu + \frac{9283}{264} \nu^2 - \frac{145669}{5544} \nu^3 \right) v^2 \\
& \quad + \left(\frac{709}{2464} - \frac{35059}{7392} \nu + \frac{179615}{7392} \nu^2 - \frac{73517}{1848} \nu^3 \right) v^6 \\
& \quad + \frac{Gm}{r^5} \left(-\frac{67}{2464} + \frac{1667}{2464} \nu + \frac{7813}{7392} \nu^2 - \frac{3137}{462} \nu^3 \right) (vx)^4 \\
& \quad + \frac{Gm}{r} \left(\frac{74083}{22176} - \frac{201835}{7392} \nu + \frac{803783}{22176} \nu^2 + \frac{25621}{462} \nu^3 \right) v^4 \\
& \quad + \frac{G^2 m^2}{r^4} \left(-\frac{191}{560} - \frac{1909}{336} \nu - \frac{9073}{144} \nu^2 - \frac{13169}{504} \nu^3 \right) (vx)^2 \\
& \quad + \frac{G^3 m^3}{r^3} \left(\frac{23443}{7425} + \frac{79067}{9240} \nu + \frac{123}{128} \pi^2 \nu - \frac{120185}{5544} \nu^2 + \frac{85031}{16632} \nu^3 \right. \\
& \quad \left. - \frac{214}{105} \ln\left(\frac{r}{r_0}\right) - 22\nu \ln\left(\frac{r}{r'_0}\right) \right) \right], \tag{7.16a}
\end{aligned}$$

$$\begin{aligned}
B = & \frac{5}{28} - \frac{5}{14} \nu \\
& + \frac{1}{c^2} \left[\left(\frac{25}{168} - \frac{25}{24} \nu + \frac{25}{14} \nu^2 \right) v^2 + \frac{Gm}{r} \left(\frac{103}{63} + \frac{337}{126} \nu - \frac{173}{84} \nu^2 \right) \right] \\
& + \frac{1}{c^4} \left[\left(\frac{305}{2464} - \frac{11815}{7392} \nu + \frac{49025}{7392} \nu^2 - \frac{32905}{3696} \nu^3 \right) v^4 \right. \\
& \quad + \frac{Gm}{r^3} \left(\frac{1}{396} - \frac{3053}{1848} \nu + \frac{14605}{5544} \nu^2 + \frac{967}{462} \nu^3 \right) (vx)^2 \\
& \quad + \frac{Gm}{r} \left(\frac{193}{99} - \frac{6095}{1848} \nu - \frac{17783}{1386} \nu^2 + \frac{941}{66} \nu^3 \right) v^2 \\
& \quad \left. + \frac{G^2 m^2}{r^2} \left(\frac{164023}{18480} + \frac{98485}{3696} \nu + \frac{194041}{11088} \nu^2 - \frac{41215}{5544} \nu^3 \right) \right], \tag{7.16b}
\end{aligned}$$

$$C = \frac{4\nu}{5c^2} \left(\frac{54}{7} \frac{Gm}{r} - v^2 \right). \tag{7.16c}$$

Note the dependence on two regularisation scales r_0 and r'_0 through logarithms in the coefficient A . The scale r_0 , which enters a logarithm with numerical prefactor coefficient $\beta_J = -\frac{214}{105}$, is due to the IR divergences of the multipole moment. This constant will be found to properly cancel out in the final expression of the GW mode h^{21} in Sec. 7.3. In the traditional PN approach adopted here, the constant r_0 is canceled by the same constant that is present in the tail-of-tail correction

of the radiative current quadrupole (see Eq. (7.20)). As for r'_0 , it is a UV scale and it also properly cancels out when we express the orbital separation r of the particles in terms of the invariant orbital frequency ω or, equivalently, the PN parameter x (see Eq. (7.28)). Following the discussion in Sec. VI (footnote 10) of [9], we identify for simplicity the two *a priori* independent UV scales associated with the two particles, respectively, *i.e.*, we set $r'_0 \equiv r'_1 = r'_2$.

In the case of quasi-circular orbits, we get (with $\gamma \equiv Gm/(rc^2)$)

$$\mathbf{J}_{ij} = -\nu m \Delta \left[A_{\text{circ}} L^{(i} x^{j)} + C_{\text{circ}} \frac{Gm}{c^3} L^{(i} v^{j)} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (7.17a)$$

$$\begin{aligned} A_{\text{circ}} = & 1 + \gamma \left(\frac{67}{28} - \frac{2}{7}\nu \right) + \gamma^2 \left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{\nu^2}{168} \right) \\ & + \gamma^3 \left(\frac{2301023}{415800} - \frac{214}{105} \ln\left(\frac{r}{r_0}\right) \right. \\ & \left. + \left[-\frac{243853}{9240} + \frac{123}{128}\pi^2 - 22 \ln\left(\frac{r}{r'_0}\right) \right] \nu + \frac{44995}{5544}\nu^2 + \frac{599}{16632}\nu^3 \right), \end{aligned} \quad (7.17b)$$

$$C_{\text{circ}} = \frac{188}{35}\nu\gamma. \quad (7.17c)$$

7.2.4 The IR DDR for the current quadrupole

As for the mass quadrupole and octupole moments, we wanted to confirm that using the Hadamard regularisation for the IR was equivalent to DR. I had all the required potentials developed at the correct order except for the potential \hat{R}_i that was only known at Newtonian order while it was required at 1PN for the current quadrupole. After computing the extra potential, I performed the computation of the IR DDR for the source current quadrupole following the methods detailed in 6.4. On the other hand, François Larrouturou computed the DDR for the tails-of-tails and we showed that the two effects cancel out:

$$\begin{aligned} \mathcal{D}J_{ijk}^{\text{IR}} &= -\mathcal{D}V_{ijk}^{\text{tail-of-tail}} \\ &= \text{Sym}_{ijk} \left\{ \frac{107 G^2 M^2 J_{ijk}^{(2)}}{105 c^6} \left[\Pi_\varepsilon - \frac{54989}{22470} \right] + \frac{428 G^2 M I_{ij}^{(2)} P_k}{105 c^6} \left[\Pi_\varepsilon - \frac{229289}{22470} \right] \right. \\ &\quad \left. + \frac{106 G^2 M I_{ij}^{(3)} I_k}{7 c^6} \right\} + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \quad (7.18)$$

where we remind $\Pi_\varepsilon = \frac{1}{\varepsilon} - 2 \ln\left(\frac{r_0 \sqrt{q}}{\ell_0}\right)$. Even though the IR DDR computation I performed has not been double checked, the exact cancelation with the result of François Larrouturou suggests our computations are correct. Furthermore, the cancelation, which is highly non-trivial, shows that the method used to compute (7.15) is correct (in the sense that it is equivalent to DR) and that we were allowed to use the Hadamard regularisation for the IR.

7.3 The gravitational-wave mode h_{21} at 3PN order

Let us now review the expression of the radiative current quadrupole moment V_{ij} in terms of the source moment J_{ij} . The radiative moments parametrize the asymptotic waveform to leading order $1/R$ in the distance, within the class of radiative coordinates (T, R) such that $u \equiv T - R/c$ is a null coordinate, or becomes asymptotically null in the limit $R \rightarrow +\infty$. In terms of the harmonic coordinates (t, r) , we have

$$u = t - \frac{r}{c} - \frac{2GM}{c^3} \ln\left(\frac{r}{cb}\right) + \mathcal{O}\left(\frac{1}{r}\right), \quad (7.19)$$

where M is the ADM mass of the source and b an arbitrary constant time scale (independent from r_0). Here, we are interested in the radiative current quadrupole moment, linking to the source moments through a series of non-linear corrections. At 3PN order, we have

$$\begin{aligned}
V_{ij}(u) &= J_{ij}^{(2)}(u) \\
&+ \frac{GM}{c^3} \int_{-\infty}^u d\tau \left[2 \ln \left(\frac{u-\tau}{2b} \right) + \frac{7}{3} \right] J_{ij}^{(4)}(\tau) \\
&+ \frac{G}{7c^5} \left\{ 4J_{a\langle i}^{(2)} I_{j\rangle a}^{(3)} + 8I_{a\langle i}^{(2)} J_{j\rangle a}^{(3)} + 17J_{a\langle i}^{(1)} I_{j\rangle a}^{(4)} - 3I_{a\langle i}^{(1)} J_{j\rangle a}^{(4)} + 9J_{a\langle i} I_{j\rangle a}^{(5)} \right. \\
&\quad - 3I_{a\langle i} J_{j\rangle a}^{(5)} - \frac{1}{4} J_a I_{ija}^{(5)} - 7\varepsilon_{ab\langle i} J_a J_{j\rangle b}^{(4)} + \frac{1}{2} \varepsilon_{ac\langle i} \left[3I_{ab}^{(3)} I_{j\rangle bc}^{(3)} + \frac{353}{24} I_{j\rangle bc}^{(2)} I_{ab}^{(4)} \right. \\
&\quad \left. \left. - \frac{5}{12} I_{ab}^{(2)} I_{j\rangle bc}^{(4)} + \frac{113}{8} I_{j\rangle bc}^{(1)} I_{ab}^{(5)} - \frac{3}{8} I_{ab}^{(1)} I_{j\rangle bc}^{(5)} + \frac{15}{4} I_{j\rangle bc} I_{ab}^{(6)} + \frac{3}{8} I_{ab} I_{j\rangle bc}^{(6)} \right] \right\} \\
&+ \frac{2G}{c^5} \frac{d^2}{dt^2} \left[\varepsilon_{ab\langle i} \left(-I_{j\rangle b}^{(3)} W_a - 2I_{j\rangle b} Y_a^{(2)} + I_{j\rangle b}^{(1)} Y_a^{(1)} \right) + 3J_{\langle i} Y_{j\rangle}^{(1)} - 2J_{ij}^{(1)} W^{(1)} \right] \\
&+ \frac{G^2 M^2}{c^6} \int_{-\infty}^u d\tau \left[2 \ln^2 \left(\frac{u-\tau}{2b} \right) \right. \\
&\quad \left. + \frac{14}{3} \ln \left(\frac{u-\tau}{2b} \right) - \frac{214}{105} \ln \left(\frac{c(u-\tau)}{2r_0} \right) - \frac{26254}{11025} \right] J_{ij}^{(5)}(\tau) + \mathcal{O} \left(\frac{1}{c^7} \right). \tag{7.20}
\end{aligned}$$

At this accuracy level, the right-hand side contains the dominant quadratic tail term at 1.5PN order [74], a number of instantaneous corrections at 2.5PN order involving the physical moments I_L , J_L but also the gauge moments W_L , Y_L [147], and the cubic tail-of-tail term at 3PN order [148]. Note the presence of the IR scale r_0 in one of the logarithms of the tail-of-tail term in (7.20), with the same coefficient $\beta_J = -\frac{214}{105}$ as in the source moment (7.15), which shows that the constant r_0 will finally drop from the radiative moment. The method used to compute the tail terms is described in Sec. 5.4.1 and the integral that is needed for the tail-of-tail term is very similar to Eq. (5.34), see for example (7.10) and (7.11) of [147].

With our knowledge of the radiative current moment V_{ij} , we are in the position to compute the mode h^{21} at the 3PN order. The modes are defined from the $+$ and \times polarisation waveforms as

$$h \equiv h_+ - ih_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi), \tag{7.21}$$

where the spin-weighted spherical harmonics of weight -2 are functions of the spherical angles (Θ, Φ) defining the direction of propagation.³ For planar binaries, which are either non-spinning or with spins aligned or anti-aligned with the orbital angular momentum, there is a clean separation of modes between mass-type and current-type contributions (see [166] and Sec. III B in [43]). In particular, the ‘‘current’’ modes are entirely determined by the current radiative moments when $\ell + m$ is an odd integer,

$$h^{\ell m} = \frac{i}{\sqrt{2}} \frac{G}{R c^{\ell+3}} V^{\ell m} \quad (\text{for } \ell + m \text{ odd}), \tag{7.22}$$

hence the only relevant mode for the current quadrupole is h^{21} (recall that $h^{2,-1} = \overline{h^{21}}$ in the planar case). The current moment in non-STF spin-weighted guise is given in terms of the STF version V_L

³We adopt the same conventions as in Refs. [147, 43]. The spin-weighted spherical harmonics are given by (2.4) in [147], and Fig. 1 in [43] specifies our convention for the polarization vectors.

by

$$V^{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_L^{\ell m} V_L, \quad (7.23)$$

where the STF tensorial coefficient $\alpha_L^{\ell m} \equiv \int d\Omega \hat{N}_L \bar{Y}^{\ell m}$ is defined from the ordinary spherical harmonics $Y^{\ell m}$ (or in fact its complex conjugate $\bar{Y}^{\ell m}$). In practice, I used the convenient orthonormal triad $(\mathbf{n}, \boldsymbol{\lambda}, \mathbf{l})$ where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, $\mathbf{l} = \mathbf{L}/|\mathbf{L}|$, while $\boldsymbol{\lambda}$ completes the triad with right-handed orientation. We can also define $\mathbf{m} = (\mathbf{n} + i\boldsymbol{\lambda})/\sqrt{2}$, as well as its value \mathbf{m}_0 at some reference time t_0 , to obtain the explicit expression of $\alpha_L^{\ell m}$:⁴

$$\alpha_L^{\ell m} = \frac{\sqrt{4\pi}(-\sqrt{2})^m \ell!}{\sqrt{(2\ell+1)(\ell+m)!(\ell-m)!}} \bar{\mathbf{m}}_0^{\langle M} \mathbf{l}^{L-M\rangle}. \quad (7.24)$$

In the tail and tail-of-tail terms at 1.5PN and 3PN orders, there appears the total ADM mass M , which therefore needs to be computed with 1PN precision. It is convenient, following [167, 168], to perform a change of phase variable, from the actual orbital phase ϕ to the new variable

$$\psi \equiv \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right). \quad (7.25)$$

The constant ω_0 , equivalent to b in (7.19), is conveniently defined by $\omega_0 = \frac{1}{4b} \exp[\frac{11}{12} - \gamma_E]$ with γ_E the Euler constant. The main advantage of the new phase variable (7.25) is that it minimizes the occurrence of logarithms due to tails in the waveform and modes. Notice, however, that the use of either ϕ or ψ is equivalent for the present 3PN level of accuracy. Indeed, the correction term in (7.25) is of order 1.5PN, which means that the effect as seen as a correction to the phase evolution is actually of order 4PN with respect to the leading order in the phase provided by the usual quadrupole formula. A similar variable is also introduced in black-hole perturbation theory [157, 158, 159].

The modes are thus defined with respect to the phase variable (7.25) as

$$h^{\ell m} = \frac{2G m \nu x}{R c^2} \sqrt{\frac{16\pi}{5}} \hat{H}^{\ell m} e^{-im\psi}. \quad (7.26)$$

Beware that the symbol m in the first factor of the right-hand side denotes the total mass $m = m_1 + m_2$, whereas its two other occurrences refer to an integral label. The final result for the 3PN mode (2,1) (corresponding in fact to the 3.5PN accurate waveform) is expressed with the usual gauge invariant PN parameter (with $\omega = \dot{\phi}$)

$$x \equiv \left(\frac{Gm\omega}{c^3}\right)^{2/3}. \quad (7.27)$$

As already announced, the regularisation constants r_0 , r'_0 and the gauge constant b disappear from the end result, and we find, extending (9.4b) in [147]:

$$\begin{aligned} \hat{H}^{21} = & \frac{i}{3} \Delta \left[x^{1/2} + x^{3/2} \left(-\frac{17}{28} + \frac{5\nu}{7} \right) + x^2 \left(\pi + i \left[-\frac{1}{2} - 2 \ln 2 \right] \right) \right. \\ & + x^{5/2} \left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168} \right) \\ & \left. + x^3 \left(\pi \left[-\frac{17}{28} + \frac{3\nu}{14} \right] + i \left[\frac{17}{56} + \nu \left(-\frac{353}{28} - \frac{3}{7} \ln 2 \right) + \frac{17}{14} \ln 2 \right] \right) \right] \end{aligned} \quad (7.28)$$

⁴We have $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!} \bar{\alpha}_L^{\ell m}$ in the alternative definition used in [79, 166].

$$\begin{aligned}
& + x^{7/2} \left(\frac{15223771}{1455300} + \frac{\pi^2}{6} - \frac{214}{105} \gamma_E - \frac{107}{105} \ln(4x) - \ln 2 - 2(\ln 2)^2 \right. \\
& \left. + \nu \left[-\frac{102119}{2376} + \frac{205}{128} \pi^2 \right] - \frac{4211}{8316} \nu^2 + \frac{2263}{8316} \nu^3 + i\pi \left[\frac{109}{210} - 2 \ln 2 \right] \right) + \mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned}$$

The perturbative limit $\nu \rightarrow 0$ is in perfect agreement with the result of black-hole perturbation theory [157, 158, 159], as one may check with the mode provided in this limit in the Appendix B in [158].⁵ Interestingly, our result (7.28) can be compared directly with accurate numerical relativity calculations, such as those in [160, 161].

7.4 Test of the current quadrupole with a constant shift

This section aims at testing the final expression of the 3PN source current quadrupole (7.15) with the test of constant shifts. By this, we mean that when the two trajectories of the particles are shifted by $y_1^i \rightarrow y_1^i + \epsilon^i$ and $y_2^i \rightarrow y_2^i + \epsilon^i$, where ϵ^i denotes an infinitesimal constant purely spatial vector, the variation of the moment obeys the expected law of transformation of the moment under the shift to first order in that shift.

We consider the case of the moments of a general isolated system made of an extended smooth matter distribution. There is then no need of UV regularisation and we may focus on the moments I_L and J_L in 3 dimensions, given by Eqs. (3.31). The laws of transformation of such moments to first order in the shift have been found in *linearized* gravity by Damour and Iyer [145, 169] (see also [149]). In this case, the pseudo-tensor $\tau^{\mu\nu}$ reduces to the matter stress-energy tensor $T^{\mu\nu}$ with compact support, so that there is no necessity to resort to the IR regularisation with regulator \tilde{r}^B and finite part when $B = 0$. In this situation, the transformation laws read (see Appendix B in [145])⁶

$$\delta_\epsilon^{\text{lin}} I_L = \ell \epsilon_{\langle i_\ell} I_{L-1 \rangle} - \frac{4\ell}{c^2(\ell+1)^2} \epsilon_a J_{b\langle L-1} \epsilon_{i_\ell \rangle ab} + \frac{(\ell-1)(\ell+3)}{c^2(\ell+1)^2(2\ell+3)} \epsilon_a I_{aL}^{(2)}, \quad (7.29a)$$

$$\delta_\epsilon^{\text{lin}} J_L = \frac{(\ell-1)(\ell+1)}{\ell} \epsilon_{\langle i_\ell} J_{L-1 \rangle} + \frac{1}{\ell} \epsilon_a I_{b\langle L-1} \epsilon_{i_\ell \rangle ab} + \frac{(\ell-1)(\ell+3)}{c^2\ell(\ell+2)(2\ell+3)} \epsilon_a J_{aL}^{(2)}. \quad (7.29b)$$

Such laws follow from the irreducible decomposition of the metric and from the fact that the components of the matter tensor behave under the spatial constant shifts like scalars, *i.e.*, $T^{\mu\nu}(x') = T^{\mu\nu}(x)$, considering now the action of the shifts as a passive coordinate transformation $x'^\mu = x^\mu + \epsilon^\mu$ with $\epsilon^\mu = (0, \epsilon^i)$.

For the full non-linear theory, driven by Eqs. (3.3)–(3.4), one must take into account the non-linear gravitational source term $\Lambda^{\mu\nu}$. Now, even though the pseudo-tensor $\tau^{\mu\nu}$ behaves in the same way as the matter tensor under constant spatial shifts, *i.e.*, $\tau^{\mu\nu}(x') = \tau^{\mu\nu}(x)$, the transformation laws (7.29) obeyed by the moments I_L and J_L are then expected to be modified, notably because those involve the regularisation factor \tilde{r}^B dealing with the fact that the pseudo-tensor is no longer with compact support. This *a priori* implies that the linear transformation laws (7.29) must be augmented by certain non-linear corrections, which may be referred to as $\delta_\epsilon^{\text{nonlin}} I_L$ and $\delta_\epsilon^{\text{nonlin}} J_L$.

Ignoring all odd powers of $1/c$, we have found that the mass quadrupole moment I_{ij} at 3PN order does satisfy the linear transformation law (7.29a), which means that the non-linear correction in this case actually happens to start only at the 4PN order:

$$\delta_\epsilon^{\text{nonlin}} I_{ij} = \mathcal{O}\left(\frac{1}{c^8}\right). \quad (7.30)$$

⁵For the comparison, note that the phase variable used in the Appendix B of [158] is related to ours by $\psi_{\text{TS}} = \psi + \pi/2 + 2x^{3/2}[\ln 2 - 17/12]$.

⁶Here, the shift vector is denoted $\epsilon_i = \epsilon^i$, which contrasts with the usual Levi-Civita symbol $\epsilon_{iab} = \epsilon^{iab}$.

However, in the case of the current quadrupole moment J_{ij} (neglecting again possible odd-type contributions at 2.5PN order), we find that the non-linear contribution arises precisely at the 3PN order. Looking at the only possible contribution at that order, we infer from dimensional analysis that it is necessarily of the type

$$\delta_\epsilon^{\text{nonlin}} J_{ij} = \eta \frac{G^2 M^2}{c^6} \epsilon^a \varepsilon_{ab\langle i} I_{j\rangle b}^{(3)} + \mathcal{O}\left(\frac{1}{c^8}\right), \quad (7.31)$$

where I_{ij} is the (Newtonian here) quadrupole moment and the other factors comprise two masses M so that the interaction is cubic. We have introduced an unknown numerical coefficient η in front. Now, as an important though partial check of our final result for the 3PN current-type quadrupole J_{ij} , we have verified that it satisfies the law of transformation under the shift if and only if the numerical coefficient in the non-linear term (7.31) is $\eta = \frac{58}{105}$.

Although the coefficient η was not determined from scratch, the latter verification is enough for our purpose. Indeed, it is straightforward to see that any offending term in the quadrupole moment itself that is not checked by the fact that we have not determined the coefficient η is necessarily of the type

$$\Delta J_{ij} = \eta' \frac{G^2 M^2}{c^6} J_{ij}^{(2)} \quad \implies \quad \delta_\epsilon \Delta J_{ij} = \frac{\eta'}{2} \frac{G^2 M^2}{c^6} \epsilon^a \varepsilon_{ab\langle i} I_{j\rangle b}^{(3)}. \quad (7.32)$$

Now, such term ΔJ_{ij} does not vanish in the test-mass limit $\nu \rightarrow 0$ and, therefore, its coefficient η' has already been verified by the correct perturbative limit, which we have checked independently from black-hole perturbation theory [158]. We thus conclude that the above partial test of the constant shifts together with the perturbative limit grant us with a satisfying level of confidence in our result.

7.5 Conclusion of Part III

The goal of this project is to compute the 4PN phase for non-spinning point particles. In order to do so, we need to know both sides of the balance equation (3.42). The 4PN EoM and conserved quantities, especially the 4PN energy, were already computed from several groups with different approaches. However, the 4PN flux is still not known. As it is detailed in Sec. 3.3, to get the energy flux, we need to know all the source multipoles moments at consistent PN order. We have shown in Part III that the computations of the source multipoles are now finished. Indeed, in Chapter 6 we presented the computation of the 4PN source mass quadrupole and in Chapter 7 we displayed the 3PN current quadrupole. Furthermore, we know all the required radiative multipoles except the 4PN mass radiative quadrupole. Two steps are yet to be achieved, the first one is to compute all the non-linear effects, such as tails, tails-of-tails, memory or tails-of-memory in $3d$. The second is to compute the DDR for these effects, which is currently investigated and soon to be published in [11]. Then, once we know the radiative mass quadrupole, we will have all the required expressions to derive the 4PN flux. Once the expression of the CoM flux for quasi-circular orbits is known, we will insert it in the balance equation to derive the GW phase.

Conclusion

So far, all direct detections of GWs came from binary systems of compact objects. The data analysis requires accurate templates in order to extract the signal from the noise and give better parameter estimations. Such templates are built upon numerical and analytical solutions of the Einstein equations. Among the analytical methods, the PN formalism, which allows to describe the inspiral phase of compact binaries, plays a crucial role in template buildings. This is why extending the already known PN results will be useful notably for next generation detectors such as LISA or Einstein telescope. The work performed during this PhD used this approach and was made of two main projects:

1. The computation of finite-size effects in the GW phase up to 7.5PN order for spinless compact objects. Finite-size effects in binary systems occur through the tidal interaction of the bodies. In the PN expansion, the leading order of these effects arise at 5PN. Although it is a high PN order, the numerical value of the coefficient, related to the internal structure of the bodies, can be quite large so that such effects can be measurable. Tidal effects start to be quantitatively distinguishable from the point-particle approximation in the late inspiral phase. Thus, extending the precision on the waveform for these effects will be useful for Einstein telescope. The action describing tidal effects is an effective action coming from an EFT approach. Then, the computation of the EoM and conserved quantities was performed by means of a Fokker action and the results are in complete agreement with the literature. Regarding the radiative sector, the derivation of the GW flux has been implemented using the PN-MPM formalism. This allowed to compute the GW phase including tidal effects up to 2.5PN relative order (or 7.5PN). The results are in agreement with the literature up to 2PN relative order and new coefficients were provided at 2PN. This can be used to reduce biases in the EOB templates. However, a disagreement has been found in the 2.5PN term and it is currently under investigation. This project, complete and published, is displayed in Part II.

We discussed in Sec. 5.6 some ways to improve the model by: relaxing the quasi-circular orbit approximation, considering dynamical tides instead of adiabatic tides and considering the coupling to spins up to 7.5PN order.

2. The computation of the 4PN phase for spinless point particles in the quasi-circular orbits approximation. It is computed by combining the 4PN conserved energy of the system with the 4PN radiated flux of GW through the balance equation. The 4PN conserved energy, as well as the EoM, are known and have been derived by several groups using different methods. All the results are in agreement. However, the 4PN flux is not yet known. To tackle this problem, we use the PN-MPM formalism that allows to compute the energy flux expressed in terms of radiative multipole moments. These moments are related to the source multipole moments, themselves expressed in terms of the stress-energy tensor and the potentials parametrizing the PN metric. At high PN orders, many subtleties arise notably due to the regularisation scheme employed. Moreover, the proliferation of terms in the required quantities at 4PN further complicates the computations. A core part of this project was to optimize the codes in order to reduce the computational time. Now, all the source multipole moments are known. We also know all the radiative multipole moments except the mass quadrupole, for which more work is required to be done. Indeed, the non-linear effects have to

be computed in 3 dimensions using the Hadamard partie finie regularisation. Then, we need to compute the difference between Hadamard and dimensional regularisations to obtain the radiative mass quadrupole fully regularised with dimensional regularisation. Once this is done, we will be able to compute the flux up to 4PN. Thus, we will be able to provide the full 4PN phase for spinless point particles in the quasi-circular orbits approximation by combining the 4PN energy with the 4PN flux through the energy balance equation.

Appendices

Appendix A

Conventions and some technical aspects of General Relativity

A.1 General relativity

We use the Einstein summation convention. The signature is $(-, +, +, +)$. Greek indices correspond to 4-dimensional components. They are raised and lowered with the metric $g_{\mu\nu}$. Latin indices correspond to spatial components of the tensors and are raised and lowered using the flat metric δ_{ij} . Furthermore, a tensor of the form A_{ii} has to be understood as $A^{ij}\delta_{ij}$. The convention on the Riemann tensor is the following

$$\left(\nabla_{\mu}\nabla_{\nu}-\nabla_{\nu}\nabla_{\mu}\right)v^{\alpha}=R^{\alpha}{}_{\lambda\mu\nu}v^{\lambda}. \quad (\text{A.1})$$

The symmetry operators denoted (\dots) are given by

$$T_{(a_1\dots a_k)}=\frac{1}{k!}\sum_{\pi}T_{a_{\pi(1)}}\dots T_{a_{\pi(k)}} \quad (\text{A.2})$$

where the sum is taken over all the permutations π . The antisymmetry is given by

$$T_{[a_1\dots a_k]}=\frac{1}{k!}\sum_{\pi}\delta_{\pi}T_{a_{\pi(1)}}\dots T_{a_{\pi(k)}}, \quad (\text{A.3})$$

where δ_{π} is +1 for even permutations and -1 for odd permutations.

The multi-indices notation is the following: $L = i_1 \cdots i_{\ell}$ for a multi-index composed of ℓ multipolar indices i_1, \dots, i_{ℓ} ; $iL = ii_1 \cdots i_{\ell}$ for a multi-index with $\ell + 1$ indices; $x_L = x_{i_1} \cdots x_{i_{\ell}}$ for the product of ℓ spatial vectors $x^i = x_i$. The symmetric-trace-free (STF) projection is denoted by $\hat{x}_L = \text{STF}(x_{i_1} \cdots x_{i_{\ell}})$, or sometimes using brackets surrounding the indices, for instance $x_{\langle L\nu P \rangle}$. Similarly, $\partial_L = \partial_{i_1} \cdots \partial_{i_{\ell}}$ for the product of ℓ partial derivatives $\partial_i = \partial/\partial x^i$, and $\hat{\partial}_L = \text{STF}(\partial_{i_1} \cdots \partial_{i_{\ell}})$. In the case of summed-up (dummy) multi-indices L , we do not write the ℓ summations from 1 to 3 over their indices. The superscript (n) denotes n time derivatives, and an overbar indicates a PN-expanded quantity.

A.2 Notations and conventions in the PN approach

The notations in a general frame the notations are given following Fig. [A.1](#).

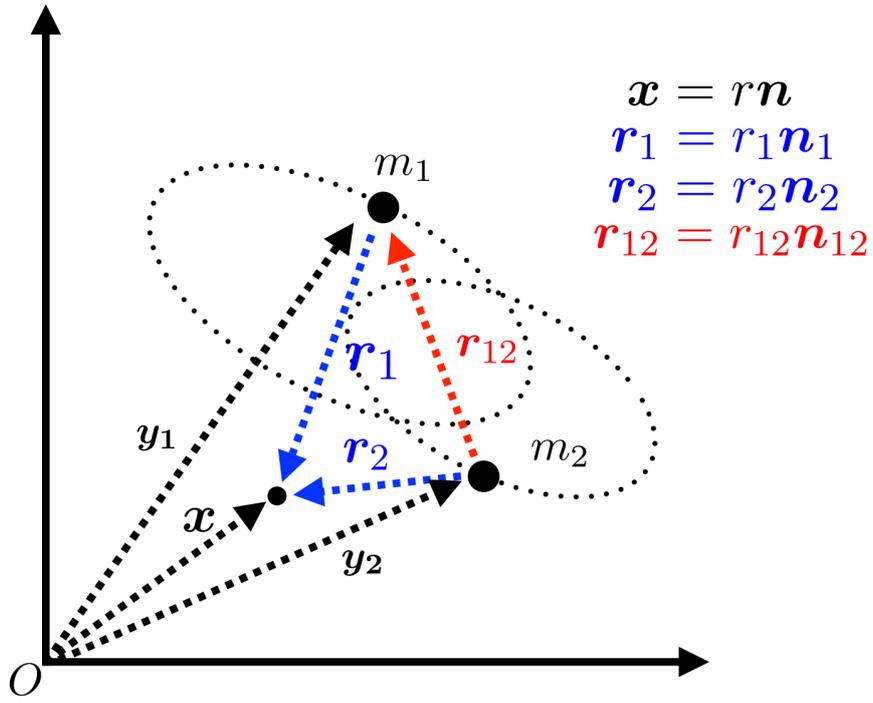


Figure A.1: Notations used throughout this thesis for the relevant parameters in the problem of binary systems.

Only in the center of mass frame, we pose $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt$; $r = |\mathbf{x}| = r_{12}$ denotes the separation, $n^i = x^i/r$ the unit direction, and we have $\dot{r} = (nv) = \mathbf{n} \cdot \mathbf{v}$; mass parameters are: the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = m_1 m_2 / m^2 = X_1 X_2$ and the mass difference $\Delta = X_1 - X_2$, with $X_A = m_A / m$.

Appendix B

The d -dimensional 4PN metric as function of potentials

We present here the complete 4PN metric with our choice of parametrization, in d dimensions since the metric is also used to compute the corrections due to DR. For the usual covariant components $g_{\mu\nu}$:

$$\begin{aligned}
g_{00} = & -1 + 2\frac{V}{c^2} + \frac{1}{c^4} \left[-\frac{4(d-3)K}{d-2} - 2V^2 \right] + \frac{1}{c^6} \left[\frac{8(d-3)KV}{d-2} + \frac{4}{3}V^3 + 8V_a V^a + 8\hat{X} \right] \\
& + \frac{1}{c^8} \left[-\frac{8(d-3)^2 K^2}{(d-2)^2} + 32\hat{T} - \frac{8(d-3)KV^2}{d-2} - \frac{2}{3}V^4 + 32\hat{R}_a V_a + V(-16V_a V_a - 16\hat{X}) \right] \\
& + \frac{1}{c^{10}} \left(64\hat{P} + 32\hat{R}_a \hat{R}_a + \frac{16(d-3)^2 K^2 V}{(d-2)^2} - 64\hat{T}V + \frac{4}{15}V^5 - \frac{32(d-3)\hat{R}_a V V_a}{d-2} \right. \\
& \quad \left. + V^2 \left[\frac{8(9-10d+3d^2)V_a V_a}{(d-2)^2} + 16\hat{X} \right] + K \left[\frac{16(d-3)V^3}{3(d-2)} \right. \right. \\
& \quad \left. \left. + \frac{16(d-3)^2 V_a V_a}{(d-2)^2} + \frac{32(d-3)\hat{X}}{d-2} \right] + 64V_a \hat{Y}_a \right) + \mathcal{O}\left(\frac{1}{c^{12}}\right), \tag{B.1a}
\end{aligned}$$

$$\begin{aligned}
g_{0i} = & -4\frac{V_i}{c^3} + \frac{1}{c^5} \left[-8\hat{R}_i + \frac{4(d-3)V V_i}{d-2} \right] \\
& + \frac{1}{c^7} \left[\frac{8(d-3)\hat{R}_i V}{d-2} - \frac{8(d-3)^2 K V_i}{(d-2)^2} - \frac{4(5-4d+d^2)V^2 V_i}{(d-2)^2} - 8V_a \hat{W}_{ia} - 16\hat{Y}^i \right] \\
& + \frac{1}{c^9} \left(-32\hat{N}_i + \hat{R}_i \left[-\frac{16(d-3)^2 K}{(d-2)^2} - \frac{4(d-3)^2 V^2}{(d-2)^2} \right] + \frac{8(d-3)^3 K V V^i}{(d-2)^3} \right. \\
& \quad \left. + \frac{8(d-3)(3-3d+d^2)V^3 V_i}{3(d-2)^3} - 16\hat{R}_a \hat{W}_{ia} + V_i \left[\frac{16(-5+2d)V_a V_a}{d-2} \right. \right. \\
& \quad \left. \left. + \frac{16(d-3)\hat{X}}{d-2} \right] + V \left[\frac{8(d-3)V_a \hat{W}_{ia}}{d-2} + \frac{16(d-3)\hat{Y}_i}{d-2} \right] - 32V_a \hat{Z}_{ia} \right) + \mathcal{O}\left(\frac{1}{c^{11}}\right), \tag{B.1b}
\end{aligned}$$

$$\begin{aligned}
g_{ij} = & \delta_{ij} + \frac{1}{c^2} \frac{2V}{d-2} \delta_{ij} + \frac{1}{c^4} \left(\delta_{ij} \left[-\frac{4(d-3)K}{(d-2)^2} + \frac{2V^2}{(d-2)^2} \right] + 4\hat{W}_{ij} \right) \\
& + \frac{1}{c^6} \left(-16V_i V_j + \frac{8V \hat{W}_{ij}}{d-2} + \delta_{ij} \left[-\frac{8(d-3)KV}{(d-2)^3} + \frac{4V^3}{3(d-2)^3} + \frac{8V_a V_a}{d-2} + \frac{8\hat{X}}{d-2} \right] + 16\hat{Z}_{ij} \right) \\
& + \frac{1}{c^8} \left[32\hat{M}_{ij} - 32\hat{R}_{(i} V_{j)} - \frac{16(d-3)K \hat{W}_{ij}}{(d-2)^2} + \frac{8V^2 \hat{W}_{ij}}{(d-2)^2} + 8\hat{W}_{ia} \hat{W}_{ja} + \delta_{ij} \left(-\frac{4(d-3)^3 K^2}{(d-2)^4} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{32\hat{M}}{d-2} + \frac{32\hat{T}}{d-2} - \frac{8(d-3)KV^2}{(d-2)^4} + \frac{2V^4}{3(d-2)^4} + \frac{32\hat{R}_aV_a}{d-2} - \frac{4\hat{W}_{ab}\hat{W}_{ab}}{d-2} \\
& + V\left[\frac{16V_aV_a}{(d-2)^2} - \frac{8(d-3)\hat{X}}{(d-2)^2}\right] + V\left(-\frac{32V_iV_j}{d-2} + \frac{32\hat{Z}_{ij}}{d-2}\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right). \tag{B.1c}
\end{aligned}$$

Up to 3PN order there are nine potentials V , V_i , K , \hat{W}_{ij} , \hat{R}_i , \hat{X} , \hat{Z}_{ij} , \hat{Y}_i , \hat{T} . The new generation of potentials at the 4PN order are denoted \hat{P} , \hat{N}_i and \hat{M}_{ij} . Since we need for the 4PN quadrupole moment the metric components in gothic form $h^{\mu\nu}$ up to order c^{-8} , c^{-7} and c^{-8} , we also present them here:

$$\begin{aligned}
h^{00} = & -\frac{1}{c^2} \frac{2(d-1)V}{d-2} + \frac{1}{c^4} \left[\frac{4(d-3)(d-1)K}{(d-2)^2} - \frac{2(d-1)^2V^2}{(d-2)^2} - 2\hat{W} \right] \\
& + \frac{1}{c^6} \left[-\frac{4(d-1)^3V^3}{3(d-2)^3} + \frac{8(d-3)V_aV_a}{d-2} + V\left(\frac{8(d-3)(d-1)^2K}{(d-2)^3} \right. \right. \\
& \left. \left. - \frac{4(d-1)\hat{W}}{d-2} - \frac{8(d-1)\hat{X}}{d-2} - 8\hat{Z} \right) \right] \\
& + \frac{1}{c^8} \left[\frac{2d\hat{W}_{ab}\hat{W}_{ab}}{d-2} - \frac{2(d-3)^2(d-1)(-4+3d)K^2}{(d-2)^4} + \frac{32\hat{M}}{d-2} - \frac{32(d-1)\hat{T}}{d-2} \right. \\
& - \frac{2(d-1)^4V^4}{3(d-2)^4} + \frac{16(d-4)\hat{R}_aV_a}{d-2} + \frac{8(d-3)(d-1)K\hat{W}}{(d-2)^2} - 2\hat{W}^2 \\
& + \frac{8(d-3)(d-1)^3KV^2}{(d-2)^4} - \frac{4(d-1)^2\hat{W}V^2}{(d-2)^2} + \frac{16(d-3)(d-1)VV_aV_a}{(d-2)^2} \\
& \left. - \frac{4(d-1)(-4+3d)\hat{X}V}{(d-2)^2} - \frac{16(d-1)\hat{Z}V}{d-2} \right] + \mathcal{O}\left(\frac{1}{c^{10}}\right), \tag{B.2a}
\end{aligned}$$

$$\begin{aligned}
h^{0i} = & -\frac{4}{c^3}V_i + \frac{1}{c^5} \left[-8\hat{R}_i - \frac{4(d-1)V_iV}{d-2} \right] \\
& + \frac{1}{c^7} \left[-16\hat{Y}_i + \frac{8(d-3)(d-1)V_iK}{(d-2)^2} - \frac{8(d-1)\hat{R}_iV}{d-2} - \frac{4(d-1)^2V_iV^2}{(d-2)^2} \right. \\
& \left. + 8\hat{W}_{ia}V_a - 8V_i\hat{W} \right] + \mathcal{O}\left(\frac{1}{c^9}\right), \tag{B.2b}
\end{aligned}$$

$$\begin{aligned}
h^{ij} = & \frac{1}{c^4} \left[-4\hat{W}_{ij} + 2\delta_{ij}\hat{W} \right] + \frac{1}{c^6} \left[-16\hat{Z}_{ij} + 8\delta_{ij}\hat{Z} \right] \\
& + \frac{1}{c^8} \left[-32\hat{M}_{ij} - 16V_i\hat{R}_j - 16\hat{R}_iV_j - 8\hat{W}_{ij}\hat{W} + 8\hat{W}_{ia}\hat{W}_{ja} \right. \\
& \left. + \delta_{ij} \left\{ -\frac{2(d-3)^2(d-1)K^2}{(d-2)^3} + 16\hat{R}_aV_a + 2\hat{W}^2 - 2\hat{W}_{ab}\hat{W}_{ab} - \frac{4(d-1)V\hat{X}}{d-2} \right\} \right] \\
& + \mathcal{O}\left(\frac{1}{c^{10}}\right). \tag{B.2c}
\end{aligned}$$

The compact-support parts of the potentials are generated by the matter stress energy-tensor $T^{\mu\nu}$ through the definitions

$$\sigma = 2\frac{(d-2)T^{00} + T^{ii}}{(d-1)c^2}, \quad \sigma_i = \frac{T^{0i}}{c}, \quad \sigma_{ij} = T^{ij}. \tag{B.3}$$

All the potentials obey flat space-time wave equations in d dimensions (where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$) given at 3PN order by:

$$\square V = -4\pi G \sigma, \tag{B.4a}$$

$$\square V_i = -4\pi G \sigma_i, \quad (\text{B.4b})$$

$$\square K = -4\pi G \sigma V, \quad (\text{B.4c})$$

$$\square \hat{W}_{ij} = -4\pi G \left(\sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) - \frac{1}{2} \left(\frac{d-1}{d-2} \right) \partial_i V \partial_j V, \quad (\text{B.4d})$$

$$\begin{aligned} \square \hat{R}_i &= -\frac{4\pi G}{d-2} \left(\frac{5-d}{2} V \sigma_i - \frac{d-1}{2} V_i \sigma \right) \\ &\quad - \frac{d-1}{d-2} \partial_k V \partial_i V_k - \frac{d(d-1)}{4(d-2)^2} \partial_t V \partial_i V, \end{aligned} \quad (\text{B.4e})$$

$$\begin{aligned} \square \hat{X} &= -4\pi G \left[\frac{V \sigma_{ii}}{d-2} + 2 \left(\frac{d-3}{d-1} \right) \sigma_i V_i + \left(\frac{d-3}{d-2} \right)^2 \sigma \left(\frac{V^2}{2} + K \right) \right] \\ &\quad + \hat{W}_{ij} \partial_{ij} V + 2V_i \partial_t \partial_i V + \frac{1}{2} \left(\frac{d-1}{d-2} \right) V \partial_t^2 V \\ &\quad + \frac{d(d-1)}{4(d-2)^2} (\partial_t V)^2 - 2\partial_i V_j \partial_j V_i, \end{aligned} \quad (\text{B.4f})$$

$$\begin{aligned} \square \hat{Z}_{ij} &= -\frac{4\pi G}{d-2} V \left(\sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) - \frac{d-1}{d-2} \partial_t V_{(i} \partial_{j)} V + \partial_i V_k \partial_j V_k \\ &\quad + \partial_k V_i \partial_k V_j - 2\partial_k V_{(i} \partial_{j)} V_k - \frac{\delta_{ij}}{d-2} \partial_k V_m (\partial_k V_m - \partial_m V_k) \\ &\quad - \frac{d(d-1)}{8(d-2)^3} \delta_{ij} (\partial_t V)^2 + \frac{(d-1)(d-3)}{2(d-2)^2} \partial_i V \partial_j K, \end{aligned} \quad (\text{B.4g})$$

$$\begin{aligned} \square \hat{Y}_i &= -4\pi G \left[-\frac{1}{2} \left(\frac{d-1}{d-2} \right) \sigma \hat{R}_i - \frac{(5-d)(d-1)}{4(d-2)^2} \sigma V V_i + \frac{1}{2} \sigma_k \hat{W}_{ik} + \frac{1}{2} \sigma_{ik} V_k \right. \\ &\quad \left. + \frac{1}{2(d-2)} \sigma_{kk} V_i - \frac{d-3}{(d-2)^2} \sigma_i \left(V^2 + \frac{5-d}{2} K \right) \right] \\ &\quad + \hat{W}_{kl} \partial_{kl} V_i - \frac{1}{2} \left(\frac{d-1}{d-2} \right) \partial_t \hat{W}_{ik} \partial_k V + \partial_i \hat{W}_{kl} \partial_k V_l - \partial_k \hat{W}_{il} \partial_l V_k \\ &\quad - \frac{d-1}{d-2} \partial_k V \partial_i \hat{R}_k - \frac{d(d-1)}{4(d-2)^2} V_k \partial_i V \partial_k V - \frac{d(d-1)^2}{8(d-2)^3} V \partial_t V \partial_i V \\ &\quad - \frac{1}{2} \left(\frac{d-1}{d-2} \right)^2 V \partial_k V \partial_k V_i + \frac{1}{2} \left(\frac{d-1}{d-2} \right) V \partial_t V_i + 2V_k \partial_k \partial_t V_i \\ &\quad + \frac{(d-1)(d-3)}{(d-2)^2} \partial_k K \partial_i V_k + \frac{d(d-1)(d-3)}{4(d-2)^3} (\partial_t V \partial_i K + \partial_i V \partial_t K), \end{aligned} \quad (\text{B.4h})$$

$$\begin{aligned} \square \hat{T} &= -4\pi G \left[\frac{1}{2(d-1)} \sigma_{ij} \hat{W}_{ij} + \frac{5-d}{4(d-2)^2} V^2 \sigma_{ii} + \frac{1}{d-2} \sigma V_i V_i - \frac{1}{2} \left(\frac{d-3}{d-2} \right) \sigma \hat{X} \right. \\ &\quad - \frac{1}{12} \left(\frac{d-3}{d-2} \right)^3 \sigma V^3 - \frac{1}{2} \left(\frac{d-3}{d-2} \right)^3 \sigma V K + \frac{(5-d)(d-3)}{2(d-1)(d-2)} \sigma_i V_i V \\ &\quad \left. + \frac{d-3}{d-1} \sigma_i \hat{R}_i - \frac{d-3}{2(d-2)^2} \sigma_{ii} K \right] + \hat{Z}_{ij} \partial_{ij} V + \hat{R}_i \partial_t \partial_i V \\ &\quad - 2\partial_i V_j \partial_j \hat{R}_i - \partial_i V_j \partial_t \hat{W}_{ij} + \frac{1}{2} \left(\frac{d-1}{d-2} \right) V V_i \partial_t \partial_i V + \frac{d-1}{d-2} V_i \partial_j V_i \partial_j V \\ &\quad + \frac{d(d-1)}{4(d-2)^2} V_i \partial_t V \partial_i V + \frac{1}{8} \left(\frac{d-1}{d-2} \right)^2 V^2 \partial_t^2 V + \frac{d(d-1)^2}{8(d-2)^3} V (\partial_t V)^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(\partial_t V_i)^2 - \frac{(d-1)(d-3)}{4(d-2)^2} V \partial_t^2 K - \frac{d(d-1)(d-3)}{4(d-2)^3} \partial_t V \partial_t K \\
& - \frac{(d-1)(d-3)}{4(d-2)^2} K \partial_t^2 V - \frac{d-3}{d-2} V_i \partial_t \partial_i K - \frac{1}{2} \left(\frac{d-3}{d-2} \right) \hat{W}_{ij} \partial_{ij} K, \tag{B.4i}
\end{aligned}$$

$$\begin{aligned}
\Box \hat{M}_{ij} = & 4\pi G \left[\left(-\frac{(d-1)V^i V^j}{2(d-2)} + \delta^{ij} \left[\frac{(d-3)^2(d-1)KV}{4(d-2)^3} + \frac{(d-3)^2(d-1)V^3}{16(d-2)^3} \right. \right. \right. \\
& + \left. \left. \frac{(d-1)V_a V^a}{4(d-2)} + \frac{(d-1)\hat{X}}{8(d-2)} \right] \right) \sigma + \delta^{ij} \left[-\frac{1}{2}\hat{R}^a + \frac{(d-4)V V^a}{2(d-2)} \right] \sigma_a + \left[\hat{R}^{(i} \right. \\
& - \left. \frac{(d-5)V V^{(i} \right] \sigma^{j)} + \frac{1}{8} \delta^{ij} \hat{W}^{ab} \sigma_{ab} + \frac{(d-1)\delta^{ij} V^2 \sigma^a_a}{8(d-2)^2} - \frac{\delta^{ij} \hat{W}^a_a \sigma^b_b}{8(d-2)} \\
& + \left[\frac{(d-3)K}{(d-2)^2} - \frac{V^2}{(d-2)^2} \right] \sigma^{ij} - \frac{1}{2} \hat{W}^{a(i} \sigma^{j)}_a \left. \right] + \frac{1}{2} \hat{W}^{ab} \partial_b \partial_a \hat{W}^{ij} \\
& - \frac{1}{2} \partial_a \hat{W}^i_b \partial^b \hat{W}^{ja} - \frac{(d-3)^2(d-1)\partial^i K \partial^j K}{4(d-2)^3} + \partial_t V^{(i} \left[-\partial_t V^{j)} \right. \\
& + \left. \frac{(d-3)(d-1)\partial^j K}{(d-2)^2} \right] - 2\partial_a V^{(i} \partial^j) \hat{R}^a + 2\partial^{(i} V_a \partial^j) \hat{R}^a - \frac{(d-1)\partial_t \hat{R}^{(i} \partial^j) V}{d-2} \\
& + \frac{(d-1)\hat{W}^{(i} \partial^j) V \partial^a V}{4(d-2)} - \frac{(d-1)\partial^{(i} \hat{X} \partial^j) V}{2(d-2)} + V^{(i} \left[\frac{(d-1)\partial_a V^{j)} \partial^a V}{d-2} \right. \\
& + \left. \frac{(d-1)\partial_t V \partial^j) V}{4(d-2)} \right] + V \left[\frac{(d-1)\partial_t^2 \hat{W}^{ij}}{4(d-2)} - \frac{(d-1)^2 \partial_t V^{(i} \partial^j) V}{2(d-2)^2} \right] + V^a \left[\partial_t \partial_a \hat{W}^{ij} \right. \\
& - \left. \frac{(d-1)\partial_a V^{(i} \partial^j) V}{d-2} \right] - 2\partial^a \hat{R}^{(i} \partial^j) V_a + \partial_t \hat{W}^{(i} \partial^j) (-\partial^a V^j) + \partial^j) V^a \\
& + \partial^b \hat{W}^{a(i} \partial^j) \hat{W}_{ab} - \frac{1}{4} \partial^i \hat{W}^{ab} \partial^j \hat{W}_{ab} \\
& + \delta^{ij} \left(-\frac{(d-3)(d-1)d\partial_t K \partial_t V}{8(d-2)^3} - \frac{(d-1)^2 V^2 \partial_t^2 V}{16(d-2)^2} + V^a \left[-\frac{1}{2} \partial_t \partial_a \hat{W}^b_b \right. \right. \\
& - \left. \left. \frac{(d-1)\partial_t V \partial_a V}{8(d-2)} \right] - \frac{(d-3)(d-1)\partial_t V_a \partial^a K}{2(d-2)^2} + \frac{(d-1)\partial_t \hat{R}_a \partial^a V}{2(d-2)} \right. \\
& - \left. \frac{1}{4} \hat{W}^{ab} \partial_b \partial_a \hat{W}^k_k + \partial_a V_b \partial^b \hat{R}^a - \frac{(d-1)\hat{W}_{ab} \partial^a V \partial^b V}{16(d-2)} - \frac{1}{2} \partial_t \hat{W}_{ab} \partial^b V^a \right. \\
& + V \left[\frac{(d-1)^2 d(\partial_t V)^2}{32(d-2)^3} - \frac{(d-1)V^a \partial_t \partial_a V}{4(d-2)} - \frac{(d-1)\partial_t^2 \hat{W}^a_a}{8(d-2)} \right. \\
& + \left. \frac{(d-1)^2 \partial_t V_a \partial^a V}{4(d-2)^2} - \frac{(d-1)\hat{W}^{ab} \partial_b \partial_a V}{8(d-2)} + \frac{(d-1)\partial_a V_b \partial^b V^a}{4(d-2)} \right] \\
& \left. - \frac{1}{4} \partial_b \hat{W}_{ak} \partial^k \hat{W}^{ab} \right). \tag{B.4j}
\end{aligned}$$

Finally we give the expressions of the effective masses μ_1 and $\tilde{\mu}_1$ in d dimensions which parametrize the source densities (6.7) for point particle sources. They are defined by

$$\mu_1 = \frac{1}{\sqrt{-(g)_1}} \frac{m_1}{\sqrt{-(g_{\mu\nu})_1 \frac{v_1^\mu v_1^\nu}{c^2}}}, \tag{B.5a}$$

$$\tilde{\mu}_1 = \frac{2}{d-1} \left(d-2 + \frac{v_1^2}{c^2} \right) \mu_1, \tag{B.5b}$$

where g is the determinant of the metric, and $(g)_1$ and $(g_{\mu\nu})_1$ are evaluated at $\mathbf{x} = \mathbf{y}_1$ following DR. Most importantly the expression of $\tilde{\mu}_1$ with full 4PN accuracy is required for the 4PN mass quadrupole moment. It is given in terms of the potentials by

$$\begin{aligned}
\tilde{\mu}_1 = & \frac{2m_1}{d-1} \left(d - 2 + \frac{1}{c^2} \left[\frac{d}{2} v_1^2 + (d-4)V \right] \right. \\
& + \frac{1}{c^4} \left[-4(d-2)v_1^i V_i - \frac{2(d-4)(d-3)K}{d-2} + \frac{1}{8}(-2+3d)v_1^4 \right. \\
& \quad \left. + \frac{(4-10d+3d^2)v_1^2 V}{2(d-2)} + \frac{(d-4)^2 V^2}{2(d-2)} - 2(d-2)\hat{W} \right] \\
& + \frac{1}{c^6} \left[-8(d-2)\hat{R}^i v_{1i} + 4(d-4)V_i V^i - \frac{(d-3)(4-10d+3d^2)v_1^2 K}{(d-2)^2} \right. \\
& \quad + 2(-2+d)\hat{W}_{ij} v_1^i v_1^j + \frac{1}{16}(-4+5d)v_1^6 - 4(-5+2d)v_1^i V_i V \\
& \quad - \frac{2(d-4)^2(d-3)KV}{(d-2)^2} + \frac{3}{8}(-4+5d)v_1^4 V \\
& \quad + \frac{(-40+92d-52d^2+9d^3)v_1^2 V^2}{4(d-2)^2} + \frac{(d-4)^3 V^3}{6(d-2)^2} - 2(-4+3d)v_1^2 v_1^j V_j \\
& \quad \left. - dv_1^2 \hat{W} - 2(d-4)V\hat{W} + 4(d-4)\hat{X} - 8(d-2)\hat{Z} \right] \\
& + \frac{1}{c^8} \left[-32\hat{R}^i V_i + 2d\hat{W}_{ij}^2 - 16(d-2)v_1^i \hat{Y}_i + \frac{8(d-3)(-5+2d)v_1^i V_i K}{d-2} \right. \\
& \quad + \frac{2(d-3)^2(16-9d+2d^2)K^2}{(d-2)^3} + 32\hat{M} + 16(d-4)\hat{T} + 8(d-2)\hat{Z}_{ij} v_1^i v_1^j \\
& \quad - 4(-4+3d)\hat{R}^i v_{1i} v_1^2 - \frac{3(d-3)(-4+5d)K v_1^4}{4(d-2)} + \frac{5}{128}(-6+7d)v_1^4 \\
& \quad - 8(-5+2d)\hat{R}^i v_{1i} V + \frac{4(d-4)^2 V_i V^i V}{d-2} \\
& \quad - \frac{(d-3)(-40+92d-52d^2+9d^3)v_1^2 KV}{(d-2)^3} + 6(d-2)\hat{W}_{ij} v_1^i v_1^j V \\
& \quad + \frac{(52-90d+35d^2)v_1^6 V}{16(d-2)} - \frac{2(26-22d+5d^2)v_1^i V_i V^2}{d-2} \\
& \quad - \frac{(d-4)^3(d-3)KV^2}{(d-2)^3} + \frac{3(40-68d+25d^2)v_1^4 V^2}{16(d-2)} \\
& \quad + \frac{(304-768d+624d^2-214d^3+27d^4)v_1^2 V^3}{12(d-2)^3} + \frac{(d-4)^4 V^4}{24(d-2)^3} - \frac{3}{2}(-6+5d)v_1^4 v_1^a V_a \\
& \quad + 16(d-2)v_1^i v_1^j V_i V_j - \frac{2(32-41d+12d^2)v_1^2 v_1^j V V_j}{d-2} \\
& \quad - 8(d-2)v_1^i \hat{W}_{ij} V^j + \frac{2(4-10d+3d^2)v_1^2 V_j V^j}{d-2} + 8(d-2)v_1^i V_i \hat{W} \\
& \quad + \frac{4(d-4)(d-3)K\hat{W}}{d-2} - \frac{1}{4}(-2+3d)v_1^4 \hat{W} - \frac{(4-10d+3d^2)v_1^2 V \hat{W}}{d-2} \\
& \quad \left. - \frac{(d-4)^2 V^2 \hat{W}}{d-2} + 2(d-2)\hat{W}^2 + (-4+3d)v_1^i v_1^j \hat{W}_{ij} - 4dv_1^2 \hat{Z} \right]
\end{aligned}$$

$$\left. + \frac{2(4 - 10d + 3d^2)v_1^2 \hat{X}}{d - 2} + \frac{4(16 - 9d + 2d^2)V \hat{X}}{d - 2} - 8(d - 4)V \hat{Z} \right]_1, \quad (\text{B.6})$$

where it is understood that all the potentials are evaluated at the point $\mathbf{x} = \mathbf{y}_1$ following DR. Note that we must apply the rule of “distributivity”, *e.g.* $(V \hat{Z})_1 = (V)_1(\hat{Z})_1$, see [83] for discussions. Besides the 4PN expression of $\tilde{\mu}_1$ we require also the 3PN expression of μ_1 which we provide for completeness:

$$\begin{aligned} \mu_1 = m_1 & \left(1 + \frac{1}{c^2} \left[\frac{1}{2} v_1^2 + \frac{(d-4)V}{d-2} \right] \right. \\ & + \frac{1}{c^4} \left[-4v_1^i V_i - \frac{2(d-4)(d-3)K}{(d-2)^2} + \frac{3}{8} v_1^4 + \frac{3}{2} v_1^2 V + \frac{(d-4)^2 V^2}{2(d-2)^2} - 2\hat{W} \right] \\ & + \frac{1}{c^6} \left[-8\hat{R}^i v_{1i} + \frac{4(d-4)V_i V^i}{d-2} - \frac{3(d-3)v_1^2 K}{d-2} + 2\hat{W}_{ij} v_1^i v_1^j + \frac{5}{16} v_1^6 \right. \\ & \quad - \frac{4(-5+2d)v_1^i V_i V}{d-2} - \frac{2(d-4)^2(d-3)KV}{(d-2)^3} + \frac{3(-8+5d)v_1^4 V}{8(d-2)} \\ & \quad + \frac{9}{4} v_1^2 V^2 + \frac{(d-4)^3 V^3}{6(d-2)^3} - 6v_1^2 v_1^j V_j - v_1^2 \hat{W} - \frac{2(d-4)V \hat{W}}{d-2} \\ & \quad \left. + \frac{4(d-4)\hat{X}}{d-2} - 8\hat{Z} \right]_1, \end{aligned} \quad (\text{B.7})$$

Appendix C

Proof that the Weyl tensor trace terms can be removed by a redefinition of the metric in the effective tidal action

In this section, we show that the tidal moments entering the action may be defined in terms of the Riemann tensor instead of the Weyl tensor, since the traces of the Riemann tensor do not play any role in the dynamics. This is a well known result from EFT that we explicit here. This proof is valid without considering the term in the action that fixes the harmonic gauge in (4.10). Thus, it shows that for any gauge we can equivalently choose the Weyl or the Riemann tensors in the definition of the tidal moments. Once the choice is made, we can fix later on the gauge by adding the gauge-fixing term in the action to impose the harmonic coordinates condition.

We denote $G_{\mu\nu}^{(R)}$, $G_{\lambda\mu\nu}^{(R)}$ and $H_{\mu\nu}^{(R)}$ the tidal mass-quadrupole, mass-octupole and current-quadrupole introduced in Eqs. (4.13), while $G_{\mu\nu}^{(C)}$, $G_{\lambda\mu\nu}^{(C)}$ and $H_{\mu\nu}^{(C)}$ will represent the original tidal moments built with the Weyl tensor instead of the Riemann tensor. We thus pose (setting also $G = c = 1$, and omitting particles' labels as well as explicit mention of the regularisation)

$$G_{\mu\nu}^{(R)} = -R_{\rho\mu\sigma\nu}u^\rho u^\sigma, \quad G_{\mu\nu}^{(C)} = -C_{\rho\mu\sigma\nu}u^\rho u^\sigma, \quad (\text{C.1a})$$

$$H_{\mu\nu}^{(R)} = -\varepsilon_{\rho(\mu}{}^{\eta\tau} R_{\underline{\eta}\tau\nu)\sigma}u^\rho u^\sigma, \quad H_{\mu\nu}^{(C)} = -\varepsilon_{\rho(\mu}{}^{\eta\tau} C_{\underline{\eta}\tau\nu)\sigma}u^\rho u^\sigma, \quad (\text{C.1b})$$

$$G_{\lambda\mu\nu}^{(R)} = -\nabla_{(\lambda}^\perp R_{\rho\mu\underline{\sigma}\nu)}u^\rho u^\sigma, \quad G_{\lambda\mu\nu}^{(C)} = -\nabla_{(\lambda}^\perp C_{\rho\mu\underline{\sigma}\nu)}u^\rho u^\sigma, \quad (\text{C.1c})$$

where $C_{\mu\nu\rho\sigma}$ stands for the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu} \right) + \frac{1}{3}g_{\mu[\rho}g_{\sigma]\nu}R. \quad (\text{C.2})$$

Let us now proof that using the (R)-type tidal multipole moments in the action yields the same EoM as when resorting to (C) moments. The latter formulation induces some extra terms, with respect to the former one, which contain at least one Ricci scalar or Ricci tensor. We call ‘‘double zero terms’’ those that are at least quadratic in the Ricci scalar and tensor. Varying a scalar of this form, say $\propto A^{\mu\nu\rho\sigma\dots}\nabla\dots R_{\mu\nu}\nabla\dots R_{\rho\sigma}$, leads to a sum of terms $\propto \int d^4x\sqrt{-g}\nabla\dots[(A^{\mu\nu\rho\sigma\dots} + A^{\rho\sigma\mu\nu})\nabla\dots R_{\rho\sigma}]\delta R_{\mu\nu}$, plus surface integrals at infinity whose integrands contain factors $\nabla\dots R_{\rho\sigma} = 0$ in vacuum. By virtue of Einstein's equations $R_{\mu\nu} = 8\pi[(T_{\text{pp}})_{\mu\nu} - (T_{\text{pp}})^\lambda{}_\lambda g_{\mu\nu}]/2 + \mathcal{O}(\epsilon_{\text{tidal}})$, those are necessarily proportional to (the covariant derivatives of) the stress-energy tensor of our system of particles

$$T_{\text{pp}}^{\mu\nu} = \sum_B m_B \int d\tau_B u_B^\mu u_B^\nu \frac{\delta^{(4)}[x - y_B(\tau_B)]}{\sqrt{-g}}, \quad (\text{C.3})$$

evaluated at one particle's location, *e.g.*, at $\mathbf{x} = \mathbf{y}_A$, in the sense of dimensional regularisation. If $A \neq B$, then $\delta^{(4)}[y_B(\tau_B) - y_A(\tau_A)] = 0$ because the compact objects never collide in the PN regime. If

$A = B$, the Dirac distribution reduces to $\delta^4(0)$, which is precisely zero in dimensional regularisation, as the limit of $\int d^d k e^{2\pi i 0} = 0$ when $d \rightarrow 3$. Hence $[T_{\text{pp}}^{\mu\nu}]_A$ identically vanishes, which implies that the Euler-Lagrange equations for the remains unaffected.

Apart from quadratic or higher order contributions of the Ricci scalar and tensor, the (R) tidal invariants written in terms of their Weyl counterparts involve terms that are linear in both the Riemann and the Ricci tensor:

$$(G_{\mu\nu}G^{\mu\nu})^{(C)} = (G_{\mu\nu}G^{\mu\nu})^{(R)} + R^{\mu\nu}R_{\mu\rho\nu\sigma}u^\rho u^\sigma + (\text{double zero terms}), \quad (\text{C.4a})$$

$$(H_{\mu\nu}H^{\mu\nu})^{(C)} = (H_{\mu\nu}H^{\mu\nu})^{(R)} + (\text{double zero terms}), \quad (\text{C.4b})$$

$$(G_{\mu\nu\rho}G^{\mu\nu\rho})^{(C)} = (G_{\mu\nu\rho}G^{\mu\nu\rho})^{(R)} + \frac{u^\mu u^\nu}{3} \left[\nabla^\lambda R^{\rho\sigma} (\nabla_\lambda R_{\mu\rho\nu\sigma} + 2\nabla_\sigma R_{\mu\rho\nu\lambda}) - \frac{2}{3} \nabla^\rho R \nabla^\sigma R_{\mu\rho\nu\sigma} \right. \\ \left. + u^\rho u^\sigma (\nabla_\mu R^{\kappa\lambda} \nabla_\sigma R_{\nu\kappa\rho\lambda} - 2\nabla^\kappa R_{\mu\nu} \nabla^\lambda R_{\rho\kappa\sigma\lambda} + 2\nabla_\sigma R_{\nu\kappa\rho\lambda} \nabla^\lambda R_\mu{}^\kappa) \right] + (\text{double zero terms}), \quad (\text{C.4c})$$

Those cannot be dealt with in the same way. Instead, they can be treated by making appropriate infinitesimal changes of variable on the metric, say $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$. The action with the new metric, at first order reads

$$S[g'_{\mu\nu}] = S[g_{\mu\nu}] - \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} - 8\pi T^{\mu\nu} \right) h_{\mu\nu} + \mathcal{O}(h^2), \quad (\text{C.5})$$

and we just have to choose conveniently the redefinition of the metric so as to cancel the remaining terms in (C.4). The term $R^{\mu\nu}R_{\mu\rho\nu\sigma}u^\rho u^\sigma$ in (C.4a) is cancelled in the action if we choose $h_{\mu\nu}$ to be (reinstalling the particles' label)

$$h_{\mu\nu}^{(G_{\alpha\beta})} = -4\pi \sum_A \mu_A^{(2)} \int d\tau_A G_{\mu\nu}^A \frac{\delta^{(4)}[x^\mu - y_A^\mu(\tau_A)]}{\sqrt{-g}}. \quad (\text{C.6})$$

This change of variable induces another double zero term that also has no physical relevance. The stress-energy tensor is $T^{\mu\nu} = T_{\text{pp}}^{\mu\nu} + T_{\text{tidal}}^{\mu\nu}$, but since we consider the contraction $h_{\mu\nu}T^{\mu\nu}$ in (C.5) and that $h_{\mu\nu} \propto \mu_A^{(2)}$, we only require to consider the term $h_{\mu\nu}T_{\text{pp}}^{\mu\nu}$. We have

$$\int d^4 x \sqrt{-g} h_{\mu\nu}^{(G_{\alpha\beta})} T_{\text{pp}}^{\mu\nu} \propto \sum_{A,B} \mu_A^{(2)} \int d\tau_A [R_{\rho\mu\sigma\nu}]_A u_A^\rho u_A^\sigma u_B^\mu u_B^\nu \frac{\delta^{(4)}[y_B(\tau_B) - y_A(\tau_A)]}{\sqrt{-g}}. \quad (\text{C.7})$$

If $A = B$, the term is zero due to the symmetries of the Riemann tensor, and if $A \neq B$, $\delta^{(4)}[y_B(\tau_B) - y_A(\tau_A)] = 0$ because the two compact objects never collide, so this term vanishes.

One can see in Eq. (C.4b) that we do not need to make a redefinition for the current quadrupole since only double zero terms appear. Regarding the mass octupole, by making the following change of variable

$$h_{\mu\nu}^{(G_{\alpha\beta\gamma})} = 4\pi \sum_A \frac{\mu_A^{(3)}}{3} \int d\tau_A \nabla^\lambda \left[\left(G_{\lambda\mu\nu}^{(R)} + \frac{2}{3} \nabla^\kappa R_{\kappa\rho\lambda\sigma} u^\rho u^\sigma \left(u_\mu u_\nu + \frac{2}{3} g_{\mu\nu} \right) \right)_A \frac{\delta^{(4)}[x - y_A(\tau_A)]}{\sqrt{-g}} \right]. \quad (\text{C.8})$$

one can show that the extra terms in (C.4c) are cancelled but induce boundary terms due to an integration by part. These boundary terms vanish due to the presence of the delta function $\delta^{(4)}[x^\mu - y_A^\mu(\tau_A)]$ by integrating over a sphere of radius $R_A \rightarrow +\infty$. The last term to treat due to this change of variable is $h_{\mu\nu}T_{\text{pp}}^{\mu\nu}$. As before, if $A \neq B$ this term vanishes because the bodies do not collide, but if $A = B$, we cannot use the same argument as for the mass quadrupole and we get instead $\int d^4 x \sqrt{-g} h_{\mu\nu}^{(G_{\rho\sigma\tau})} T_{\text{pp}}^{\mu\nu} \propto \delta^{(4)}(0)$, which in our regularisation is zero.

Appendix D

Lengthy expressions of the project on tidal effects

D.1 Conservative sector

The tidal part of the acceleration of body 1 is given, for general orbits in arbitrary frame by

$$\begin{aligned}
m_1(a_1^i)_{\text{tidal}} = & \frac{G^2}{r_{12}^7} \left\{ n_{12}^i \left(-9m_2^2\mu_1^{(2)} - 9m_1^2\mu_2^{(2)} \right) + \frac{1}{c^2} \left\{ n_{12}^i \left[m_2^2\mu_1^{(2)} \left(-36(n_{12}v_1)^2 + 72(n_{12}v_1)(n_{12}v_2) \right. \right. \right. \\
& - 18v_{12}^2 + 9v_1^2 \left. \right) + m_1^2\mu_2^{(2)} \left(144(n_{12}v_1)^2 - 288(n_{12}v_1)(n_{12}v_2) + 180(n_{12}v_2)^2 - \frac{81}{2}v_{12}^2 + 9v_1^2 \right) \\
& + m_2^2\sigma_1^{(2)} \left(-96(n_{12}v_{12})^2 - 48v_{12}^2 \right) + m_1^2\sigma_2^{(2)} \left(-96(n_{12}v_{12})^2 - 48v_{12}^2 \right) + \frac{Gm_1}{r_{12}} \left(\frac{159}{2}m_2^2\mu_1^{(2)} \right. \\
& + 132m_1^2\mu_2^{(2)} \left. \right) + \frac{Gm_2}{r_{12}} \left(99m_2^2\mu_1^{(2)} + 84m_1^2\mu_2^{(2)} \right) \left. \right] + v_1^i \left[m_2^2\mu_1^{(2)} \left(54(n_{12}v_1) - 45(n_{12}v_2) \right) \right. \\
& + 9m_1^2\mu_2^{(2)}(n_{12}v_1) + 144m_2^2\sigma_1^{(2)}(n_{12}v_{12}) + 144m_1^2\sigma_2^{(2)}(n_{12}v_{12}) \left. \right] + v_2^i \left[m_2^2\mu_1^{(2)} \left(-54(n_{12}v_1) \right. \right. \\
& + 45(n_{12}v_2) \left. \right) - 9m_1^2\mu_2^{(2)}(n_{12}v_1) - 144m_2^2\sigma_1^{(2)}(n_{12}v_{12}) - 144m_1^2\sigma_2^{(2)}(n_{12}v_{12}) \left. \right] \left. \right\} \\
& + \frac{1}{c^4} \left[n_{12}^i \left\{ m_2^2\mu_1^{(2)} \left(135(n_{12}v_1)^4 - 540(n_{12}v_1)^3(n_{12}v_2) + 990(n_{12}v_1)^2(n_{12}v_2)^2 \right. \right. \right. \\
& - 900(n_{12}v_1)(n_{12}v_2)^3 + 225(n_{12}v_2)^4 + 72(n_{12}v_1)(n_{12}v_2)(v_1v_2) - 18(v_1v_2)^2 - 126(n_{12}v_1)^2v_{12}^2 \\
& + 324(n_{12}v_1)(n_{12}v_2)v_{12}^2 - 90(n_{12}v_2)^2v_{12}^2 - 36(v_1v_2)v_{12}^2 - 27v_{12}^4 - 72(n_{12}v_1)(n_{12}v_2)v_1^2 \\
& + 36(v_1v_2)v_1^2 + 36v_{12}^2v_1^2 - 18v_1^4 \left. \right) + m_1^2\mu_2^{(2)} \left(-3855(n_{12}v_1)^4 + 15420(n_{12}v_1)^3(n_{12}v_2) \right. \\
& - 23850(n_{12}v_1)^2(n_{12}v_2)^2 + 16860(n_{12}v_1)(n_{12}v_2)^3 - 4665(n_{12}v_2)^4 - 288(n_{12}v_1)(n_{12}v_2)(v_1v_2) \\
& + 360(n_{12}v_2)^2(v_1v_2) - \frac{81}{2}(v_1v_2)^2 + 2598(n_{12}v_1)^2v_{12}^2 - 5484(n_{12}v_1)(n_{12}v_2)v_{12}^2 \\
& + 3084(n_{12}v_2)^2v_{12}^2 - 81(v_1v_2)v_{12}^2 - \frac{1923}{8}v_{12}^4 + 288(n_{12}v_1)(n_{12}v_2)v_1^2 - 360(n_{12}v_2)^2v_1^2 \\
& + 81(v_1v_2)v_1^2 + 81v_{12}^2v_1^2 - \frac{81}{2}v_1^4 \left. \right) + m_2^2\sigma_1^{(2)} \left(840(n_{12}v_{12})^4 - 960(n_{12}v_{12})^3(n_{12}v_1) \right. \\
& + 480(n_{12}v_{12})^2(n_{12}v_1)^2 - 192(n_{12}v_{12})^2(v_1v_2) + 192(n_{12}v_{12})(n_{12}v_1)(v_1v_2) - 48(v_1v_2)^2 \\
& - 336(n_{12}v_{12})^2v_{12}^2 - 192(n_{12}v_{12})(n_{12}v_1)v_{12}^2 + 192(n_{12}v_1)^2v_{12}^2 - 96(v_1v_2)v_{12}^2 - 72v_{12}^4 \\
& + 192(n_{12}v_{12})^2v_1^2 - 192(n_{12}v_{12})(n_{12}v_1)v_1^2 + 96(v_1v_2)v_1^2 + 96v_{12}^2v_1^2 - 48v_1^4 \left. \right) \\
& + m_1^2\sigma_2^{(2)} \left(1000(n_{12}v_{12})^4 - 960(n_{12}v_{12})^3(n_{12}v_1) + 480(n_{12}v_{12})^2(n_{12}v_1)^2 - 192(n_{12}v_{12})^2(v_1v_2) \right. \\
& + 192(n_{12}v_{12})(n_{12}v_1)(v_1v_2) - 48(v_1v_2)^2 + 64(n_{12}v_{12})^2v_{12}^2 - 192(n_{12}v_{12})(n_{12}v_1)v_{12}^2 \\
& + 192(n_{12}v_1)^2v_{12}^2 - 96(v_1v_2)v_{12}^2 - 128v_{12}^4 + 192(n_{12}v_{12})^2v_1^2 - 192(n_{12}v_{12})(n_{12}v_1)v_1^2 \left. \right) \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + 96(v_1 v_2) v_1^2 + 96 v_{12}^2 v_1^2 - 48 v_1^4) + \frac{G m_1}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(\frac{7215}{8} (n_{12} v_1)^2 - \frac{7431}{4} (n_{12} v_1) (n_{12} v_2) \right. \right. \\
& + \frac{4461}{8} (n_{12} v_2)^2 - \frac{285}{8} v_{12}^2 - \frac{159}{2} v_1^2) + m_1^2 \mu_2^{(2)} \left(-\frac{15717}{8} (n_{12} v_1)^2 + \frac{16581}{4} (n_{12} v_1) (n_{12} v_2) \right. \\
& - \frac{22521}{8} (n_{12} v_2)^2 + \frac{4597}{8} v_{12}^2 - 132 v_1^2) + m_2^2 \sigma_1^{(2)} \left(656 (n_{12} v_{12})^2 - 144 (n_{12} v_{12}) (n_{12} v_1) \right. \\
& + 200 v_{12}^2) + m_1^2 \sigma_2^{(2)} \left(1124 (n_{12} v_{12})^2 - 144 (n_{12} v_{12}) (n_{12} v_1) + 436 v_{12}^2 \right) \left. \right] \\
& + \frac{G m_2}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(252 (n_{12} v_1)^2 - 504 (n_{12} v_1) (n_{12} v_2) - \frac{387}{2} (n_{12} v_2)^2 + 162 v_{12}^2 - 99 v_1^2 \right) \right. \\
& + m_1^2 \mu_2^{(2)} \left(-2568 (n_{12} v_1)^2 + 5136 (n_{12} v_1) (n_{12} v_2) - 2946 (n_{12} v_2)^2 + 426 v_{12}^2 - 84 v_1^2 \right) \\
& + m_2^2 \sigma_1^{(2)} \left(672 (n_{12} v_{12})^2 + 336 v_{12}^2 \right) + m_1^2 \sigma_2^{(2)} \left(592 (n_{12} v_{12})^2 + 192 v_{12}^2 \right) \left. \right] + \frac{G^2 m_1^2}{r_{12}^2} \left(-\frac{2145}{7} m_2^2 \mu_1^{(2)} \right. \\
& - 1008 m_1^2 \mu_2^{(2)}) + \frac{G^2 m_1 m_2}{r_{12}^2} \left(-\frac{2581}{2} m_2^2 \mu_1^{(2)} - 1805 m_1^2 \mu_2^{(2)} \right) + \frac{G^2 m_2^2}{r_{12}^2} \left(-576 m_2^2 \mu_1^{(2)} \right. \\
& - \left. \frac{6705}{14} m_1^2 \mu_2^{(2)} \right) \left. \right\} + v_1^i \left\{ m_2^2 \mu_1^{(2)} \left(-144 (n_{12} v_1)^3 + 468 (n_{12} v_1)^2 (n_{12} v_2) - 720 (n_{12} v_1) (n_{12} v_2)^2 \right. \right. \\
& + 360 (n_{12} v_2)^3 - 342 (n_{12} v_1) (v_1 v_2) + 360 (n_{12} v_2) (v_1 v_2) + 144 (n_{12} v_1) v_1^2 - 135 (n_{12} v_2) v_1^2 \\
& + 198 (n_{12} v_1) v_2^2 - 225 (n_{12} v_2) v_2^2) + m_1^2 \mu_2^{(2)} \left(1248 (n_{12} v_1)^3 - 3888 (n_{12} v_1)^2 (n_{12} v_2) \right. \\
& + 3996 (n_{12} v_1) (n_{12} v_2)^2 - 1392 (n_{12} v_2)^3 + 9 (n_{12} v_1) (v_1 v_2) - \frac{903}{2} (n_{12} v_1) v_{12}^2 + 492 (n_{12} v_2) v_{12}^2 \\
& - 9 (n_{12} v_1) v_1^2) + m_2^2 \sigma_1^{(2)} \left(-1056 (n_{12} v_{12})^3 + 1248 (n_{12} v_{12})^2 (n_{12} v_1) - 576 (n_{12} v_{12}) (n_{12} v_1)^2 \right. \\
& - 960 (n_{12} v_{12}) (v_1 v_2) + 48 (n_{12} v_1) (v_1 v_2) + 336 (n_{12} v_{12}) v_1^2 + 48 (n_{12} v_1) v_1^2 + 624 (n_{12} v_{12}) v_2^2 \\
& - 96 (n_{12} v_1) v_2^2) + m_1^2 \sigma_2^{(2)} \left(-1664 (n_{12} v_{12})^3 + 1248 (n_{12} v_{12})^2 (n_{12} v_1) - 576 (n_{12} v_{12}) (n_{12} v_1)^2 \right. \\
& - 1168 (n_{12} v_{12}) (v_1 v_2) + 48 (n_{12} v_1) (v_1 v_2) + 440 (n_{12} v_{12}) v_1^2 + 48 (n_{12} v_1) v_1^2 + 728 (n_{12} v_{12}) v_2^2 \\
& - 96 (n_{12} v_1) v_2^2) + \frac{G m_1}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(-\frac{1209}{4} (n_{12} v_1) + \frac{1179}{4} (n_{12} v_2) \right) + m_1^2 \mu_2^{(2)} \left(\frac{241}{4} (n_{12} v_1) \right. \right. \\
& - \left. \frac{661}{4} (n_{12} v_2) \right) + m_2^2 \sigma_1^{(2)} \left(-712 (n_{12} v_1) + 856 (n_{12} v_2) \right) + m_1^2 \sigma_2^{(2)} \left(-1416 (n_{12} v_1) + 1560 (n_{12} v_2) \right) \left. \right] \\
& + \frac{G m_2}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(-378 (n_{12} v_1) + 279 (n_{12} v_2) \right) + m_1^2 \mu_2^{(2)} \left(714 (n_{12} v_1) - 798 (n_{12} v_2) \right) \right. \\
& - 1008 m_2^2 \sigma_1^{(2)} (n_{12} v_{12}) - 784 m_1^2 \sigma_2^{(2)} (n_{12} v_{12}) \left. \right] \left. \right\} + v_2^i \left\{ m_2^2 \mu_1^{(2)} \left(144 (n_{12} v_1)^3 - 468 (n_{12} v_1)^2 (n_{12} v_2) \right. \right. \\
& + 720 (n_{12} v_1) (n_{12} v_2)^2 - 360 (n_{12} v_2)^3 + 342 (n_{12} v_1) (v_1 v_2) - 360 (n_{12} v_2) (v_1 v_2) - 144 (n_{12} v_1) v_1^2 \\
& + 135 (n_{12} v_2) v_1^2 - 198 (n_{12} v_1) v_2^2 + 225 (n_{12} v_2) v_2^2) + m_1^2 \mu_2^{(2)} \left(-1248 (n_{12} v_1)^3 \right. \\
& + 3888 (n_{12} v_1)^2 (n_{12} v_2) - 3996 (n_{12} v_1) (n_{12} v_2)^2 + 1392 (n_{12} v_2)^3 - 9 (n_{12} v_1) (v_1 v_2) + \frac{903}{2} (n_{12} v_1) v_{12}^2 \\
& - 492 (n_{12} v_2) v_{12}^2 + 9 (n_{12} v_1) v_1^2) + m_2^2 \sigma_1^{(2)} \left(1056 (n_{12} v_{12})^3 - 1248 (n_{12} v_{12})^2 (n_{12} v_1) \right. \\
& + 576 (n_{12} v_{12}) (n_{12} v_1)^2 + 960 (n_{12} v_{12}) (v_1 v_2) - 48 (n_{12} v_1) (v_1 v_2) - 336 (n_{12} v_{12}) v_1^2 - 48 (n_{12} v_1) v_1^2 \\
& - 624 (n_{12} v_{12}) v_2^2 + 96 (n_{12} v_1) v_2^2) + m_1^2 \sigma_2^{(2)} \left(1664 (n_{12} v_{12})^3 - 1248 (n_{12} v_{12})^2 (n_{12} v_1) \right. \\
& + 576 (n_{12} v_{12}) (n_{12} v_1)^2 + 1168 (n_{12} v_{12}) (v_1 v_2) - 48 (n_{12} v_1) (v_1 v_2) - 440 (n_{12} v_{12}) v_1^2 - 48 (n_{12} v_1) v_1^2 \\
& - 728 (n_{12} v_{12}) v_2^2 + 96 (n_{12} v_1) v_2^2) + \frac{G m_1}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(\frac{1209}{4} (n_{12} v_1) - \frac{1179}{4} (n_{12} v_2) \right) \right. \\
& + m_1^2 \mu_2^{(2)} \left(-\frac{241}{4} (n_{12} v_1) + \frac{661}{4} (n_{12} v_2) \right) + m_2^2 \sigma_1^{(2)} \left(712 (n_{12} v_1) - 856 (n_{12} v_2) \right) \\
& + m_1^2 \sigma_2^{(2)} \left(1416 (n_{12} v_1) - 1560 (n_{12} v_2) \right) \left. \right] + \frac{G m_2}{r_{12}} \left[m_2^2 \mu_1^{(2)} \left(378 (n_{12} v_1) - 279 (n_{12} v_2) \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + m_1^2 \mu_2^{(2)} \left(-714(n_{12}v_1) + 798(n_{12}v_2) \right) + 1008m_2^2 \sigma_1^{(2)}(n_{12}v_{12}) + 784m_1^2 \sigma_2^{(2)}(n_{12}v_{12}) \Big] \Big\} \\
& + \frac{1}{r_{12}^2} n_{12}^i \left(-60m_2^2 \mu_1^{(3)} - 60m_1^2 \mu_2^{(3)} \right) \Big\} + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right). \tag{D.1}
\end{aligned}$$

The tidal part of the relative acceleration in the CoM frame, deriving from the CoM Lagrangian whose tidal part is shown in (4.29), reads

$$\begin{aligned}
(a^i)_{\text{tidal}} = & -18 \frac{G^2 m}{r^7} \mu_+^{(2)} n^i \\
& + \frac{1}{c^2} \left\{ \frac{G^2 m}{r^7} \left[\left((108 + 72\nu) \mu_+^{(2)} + 180\Delta \mu_-^{(2)} - 192\sigma_+^{(2)} \right) \dot{r}^2 n^i + \left(\left(-\frac{81}{2} - 54\nu \right) \mu_+^{(2)} - \frac{45}{2} \Delta \mu_-^{(2)} \right. \right. \right. \\
& \left. \left. \left. - 96\sigma_+^{(2)} \right) v^2 n^i + \left((63 - 36\nu) \mu_+^{(2)} - 45\Delta \mu_-^{(2)} + 288\sigma_+^{(2)} \right) \dot{r} v^i \right] + \frac{G^3 m^2}{r^8} \left[(183 + 57\nu) \mu_+^{(2)} - 15\Delta \mu_-^{(2)} \right] n^i \right\} \\
& + \frac{1}{c^4} \left\{ \frac{G^2 m}{r^7} \left[\left((-3720 - 720\nu + 540\nu^2) \mu_+^{(2)} + (-3990 + 900\nu) \Delta \mu_-^{(2)} + (880 + 960\nu) \sigma_+^{(2)} + 160\Delta \sigma_-^{(2)} \right) \dot{r}^4 n^i \right. \right. \\
& \left. \left. + \left((2472 + 522\nu - 288\nu^2) \mu_+^{(2)} + (2724 - 450\nu) \Delta \mu_-^{(2)} - 272\sigma_+^{(2)} + 400\Delta \sigma_-^{(2)} \right) \dot{r}^2 v^2 n^i \right. \right. \\
& \left. \left. + \left(\left(-\frac{1671}{8} - \frac{153}{2} \nu + 72\nu^2 \right) \mu_+^{(2)} + \left(-\frac{1527}{8} + \frac{45}{2} \nu \right) \Delta \mu_-^{(2)} + (-104 - 96\nu) \sigma_+^{(2)} - 56\Delta \sigma_-^{(2)} \right) v^4 n^i \right. \right. \\
& \left. \left. + \left((1104 + 36\nu - 144\nu^2) \mu_+^{(2)} + (1392 - 180\nu) \Delta \mu_-^{(2)} + (-1376 - 1536\nu) \sigma_+^{(2)} - 608\Delta \sigma_-^{(2)} \right) \dot{r}^3 v^i \right. \right. \\
& \left. \left. + \left(\left(-\frac{633}{2} + 63\nu + 36\nu^2 \right) \mu_+^{(2)} + \left(-\frac{1209}{2} + 45\nu \right) \Delta \mu_-^{(2)} + (872 + 672\nu) \sigma_+^{(2)} + 104\Delta \sigma_-^{(2)} \right) \dot{r} v^2 v^i \right] \right. \\
& \left. + \frac{G^3 m^2}{r^8} \left[\left(\left(-2316 + \frac{5391}{4} \nu + \frac{549}{2} \nu^2 \right) \mu_+^{(2)} + \left(-2820 + \frac{1665}{4} \nu \right) \Delta \mu_-^{(2)} + (1264 + 744\nu) \sigma_+^{(2)} - 80\Delta \sigma_-^{(2)} \right) \dot{r}^2 n^i \right. \right. \\
& \left. \left. + \left(\left(405 + \frac{887}{2} \nu - 27\nu^2 \right) \mu_+^{(2)} + \left(279 - \frac{135}{2} \nu \right) \Delta \mu_-^{(2)} + (528 + 216\nu) \sigma_+^{(2)} - 144\Delta \sigma_-^{(2)} \right) v^2 n^i \right. \right. \\
& \left. \left. + \left((336 - 832\nu - 114\nu^2) \mu_+^{(2)} + (1092 - 150\nu) \Delta \mu_-^{(2)} + (-1792 - 960\nu) \sigma_+^{(2)} + 224\Delta \sigma_-^{(2)} \right) \dot{r} v^i \right] \right. \\
& \left. + \frac{G^4 m^3}{r^9} \left[\left(-\frac{14769}{14} - \frac{8716}{7} \nu \right) \mu_+^{(2)} + \left(\frac{1359}{14} + 90\nu \right) \Delta \mu_-^{(2)} \right] n^i \right\} - 120 \frac{G^2 m}{r^9} \mu_+^{(3)} n^i + \mathcal{O}\left(\frac{\epsilon_{\text{tidal}}}{c^6}\right). \tag{D.2}
\end{aligned}$$

The Hamiltonian deriving from the NNL/2PN generalized Fokker Lagrangian in harmonic coordinates reads for the point-particle part

$$\begin{aligned}
H'_{\text{pp}} = & \frac{P_1^2}{2m_1} - \frac{Gm_1 m_2}{2R_{12}} \\
& + \frac{1}{c^2} \left\{ -\frac{P_1^4}{8m_1^3} + \frac{G^2 m_1^2 m_2}{2R_{12}^2} + \frac{Gm_1 m_2}{R_{12}} \left(\frac{1}{4} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2} - \frac{3}{2} \frac{P_1^2}{m_1^2} + \frac{7}{4} \frac{(P_1 P_2)}{m_1 m_2} \right) \right\} \\
& + \frac{1}{c^4} \left\{ \frac{P_1^6}{16m_1^5} - \frac{G^3 m_1^3 m_2}{4R_{12}^3} - \frac{5G^3 m_1^2 m_2^2}{8R_{12}^3} \right. \\
& \left. + \frac{G^2 m_1^2 m_2}{R_{12}^2} \left(-\frac{3}{2} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2} + \frac{19}{4} \frac{P_1^2}{m_1^2} - \frac{27}{4} \frac{(P_1 P_2)}{m_1 m_2} + \frac{5P_2^2}{2m_2^2} \right) \right. \\
& \left. + \frac{Gm_1 m_2}{R_{12}} \left(-\frac{3}{16} \frac{(N_{12}P_1)^2 (N_{12}P_2)^2}{m_1^2 m_2^2} + \frac{5}{8} \frac{(N_{12}P_2)^2 P_1^2}{m_1^2 m_2^2} \right. \right. \\
& \left. \left. + \frac{5}{8} \frac{P_1^4}{m_1^4} - \frac{3}{4} \frac{(N_{12}P_1)(N_{12}P_2)(P_1 P_2)}{m_1^2 m_2^2} - \frac{1}{8} \frac{(P_1 P_2)^2}{m_1^2 m_2^2} - \frac{11}{16} \frac{P_1^2 P_2^2}{m_1^2 m_2^2} \right) \right\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{D.3}
\end{aligned}$$

and where the tidal part, accurate up to NNL/2PN order, reads

$$H'_{\text{tidal}} = \frac{G^2 m_2^2}{R_{12}^6} \left\{ -\frac{3}{2} \mu_1^{(2)} \right.$$

$$\begin{aligned}
& + \frac{1}{c^2} \left[-\frac{12\sigma_1^{(2)} P_2^2}{m_2^2} + \frac{(N_{12}P_2)^2}{m_2^2} \left(-18\mu_1^{(2)} + 12\sigma_1^{(2)} \right) + \frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2} \left(18\mu_1^{(2)} - 24\sigma_1^{(2)} \right) \right. \\
& + \frac{(P_1 P_2)}{m_1 m_2} \left(\frac{9}{2}\mu_1^{(2)} + 24\sigma_1^{(2)} \right) + \frac{(N_{12}P_1)^2}{m_1^2} \left(\frac{9}{2}\mu_1^{(2)} + 12\sigma_1^{(2)} \right) + \frac{P_1^2}{m_1^2} \left(-\frac{15}{4}\mu_1^{(2)} - 12\sigma_1^{(2)} \right) \\
& \left. + \frac{G}{R_{12}} \left(3m_1\mu_1^{(2)} + \frac{21}{2}m_2\mu_1^{(2)} \right) \right] \\
& + \frac{1}{c^4} \left[\frac{(N_{12}P_2)^4}{m_2^4} \left(-\frac{63}{2}\mu_1^{(2)} - 60\sigma_1^{(2)} \right) + \frac{P_2^4}{m_2^4} \left(-9\mu_1^{(2)} - 12\sigma_1^{(2)} \right) + \frac{(P_1 P_2)P_2^2}{m_1 m_2^3} \left(\frac{99}{4}\mu_1^{(2)} + 60\sigma_1^{(2)} \right) \right. \\
& + \frac{(N_{12}P_2)^2}{m_2^2} \left[\frac{P_2^2}{m_2^2} \left(54\mu_1^{(2)} + 72\sigma_1^{(2)} \right) + \frac{(P_1 P_2)}{m_1 m_2} \left(-54\mu_1^{(2)} - 144\sigma_1^{(2)} \right) \right] \\
& + \frac{(P_1 P_2)^2}{m_1^2 m_2^2} \left(-\frac{45}{2}\mu_1^{(2)} - 60\sigma_1^{(2)} \right) + \frac{(N_{12}P_1)^3 (N_{12}P_2)}{m_1^3 m_2} \left(18\mu_1^{(2)} + 48\sigma_1^{(2)} \right) \\
& + \frac{P_1^2}{m_1^2} \left[\frac{(N_{12}P_2)^2}{m_2^2} \left(\frac{45}{2}\mu_1^{(2)} + 66\sigma_1^{(2)} \right) + \frac{P_2^2}{m_2^2} \left(-\frac{45}{4}\mu_1^{(2)} - 30\sigma_1^{(2)} \right) + \frac{(P_1 P_2)}{m_1 m_2} \left(\frac{81}{4}\mu_1^{(2)} + 48\sigma_1^{(2)} \right) \right] \\
& + \frac{(N_{12}P_1)}{m_1} \left(\frac{(N_{12}P_2)^3}{m_2^3} \left(54\mu_1^{(2)} + 144\sigma_1^{(2)} \right) + \frac{(N_{12}P_2)P_1^2}{m_1^2 m_2} \left(-\frac{63}{2}\mu_1^{(2)} - 48\sigma_1^{(2)} \right) \right. \\
& \left. + \frac{(N_{12}P_2)}{m_2} \left[\frac{P_2^2}{m_2^2} \left(-36\mu_1^{(2)} - 60\sigma_1^{(2)} \right) + \frac{(P_1 P_2)}{m_1 m_2} \left(45\mu_1^{(2)} + 120\sigma_1^{(2)} \right) \right] \right) \\
& + \frac{(N_{12}P_1)^4}{m_1^4} \left(-\frac{9}{2}\mu_1^{(2)} - 12\sigma_1^{(2)} \right) + \frac{P_1^4}{m_1^4} \left(-\frac{45}{16}\mu_1^{(2)} - 6\sigma_1^{(2)} \right) \\
& + \frac{(N_{12}P_1)^2}{m_1^2} \left[\frac{(N_{12}P_2)^2}{m_2^2} \left(-45\mu_1^{(2)} - 120\sigma_1^{(2)} \right) + \frac{P_2^2}{m_2^2} \left(9\mu_1^{(2)} + 24\sigma_1^{(2)} \right) \right. \\
& \left. + \frac{(P_1 P_2)}{m_1 m_2} \left(-18\mu_1^{(2)} - 48\sigma_1^{(2)} \right) + \frac{P_1^2}{m_1^2} \left(\frac{27}{4}\mu_1^{(2)} + 18\sigma_1^{(2)} \right) \right] \\
& + \frac{G}{R_{12}} \left(m_1 \left[\frac{(N_{12}P_2)^2}{m_2^2} \left(207\mu_1^{(2)} - 80\sigma_1^{(2)} \right) + \frac{P_2^2}{m_2^2} \left(-\frac{45}{2}\mu_1^{(2)} + 80\sigma_1^{(2)} \right) \right. \right. \\
& + \frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2} \left(-\frac{1341}{8}\mu_1^{(2)} + 172\sigma_1^{(2)} \right) + \frac{(P_1 P_2)}{m_1 m_2} \left(\frac{3}{8}\mu_1^{(2)} - 172\sigma_1^{(2)} \right) \\
& + \frac{(N_{12}P_1)^2}{m_1^2} \left(-\frac{183}{2}\mu_1^{(2)} - 92\sigma_1^{(2)} \right) + \frac{P_1^2}{m_1^2} \left(\frac{123}{4}\mu_1^{(2)} + 92\sigma_1^{(2)} \right) \left. \right] \\
& + m_2 \left[\frac{(N_{12}P_2)^2}{m_2^2} \left(\frac{331}{2}\mu_1^{(2)} - 120\sigma_1^{(2)} \right) + \frac{P_2^2}{m_2^2} \left(\frac{61}{4}\mu_1^{(2)} + 120\sigma_1^{(2)} \right) \right. \\
& + \frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2} \left(-\frac{1189}{8}\mu_1^{(2)} + 228\sigma_1^{(2)} \right) + \frac{(P_1 P_2)}{m_1 m_2} \left(-\frac{401}{8}\mu_1^{(2)} - 228\sigma_1^{(2)} \right) \\
& \left. + \frac{(N_{12}P_1)^2}{m_1^2} \left(-\frac{81}{2}\mu_1^{(2)} - 108\sigma_1^{(2)} \right) + \frac{P_1^2}{m_1^2} \left(\frac{135}{4}\mu_1^{(2)} + 108\sigma_1^{(2)} \right) \right] \left. \right) \\
& + \frac{G^2}{R_{12}^2} \left(\frac{303}{28}m_1^2\mu_1^{(2)} - \frac{455}{8}m_1 m_2\mu_1^{(2)} - 39m_2^2\mu_1^{(2)} \right) \left. \right] - \frac{15\mu_1^{(3)}}{2R_{12}^2} \left. \right\} + 1 \leftrightarrow 2 + \mathcal{O} \left(\frac{\epsilon_{\text{tidal}}}{c^6} \right). \tag{D.4}
\end{aligned}$$

D.2 Radiative sector

$$\begin{aligned}
\sigma_{\text{tidal}} = & -\frac{1}{\sqrt{-g}}\partial_{ab}\left\{\delta_1\left(\mu_1^{(2)}\left[-\frac{1}{2}\hat{G}_{1ab} + \frac{1}{c^2}\left(-\frac{3}{4}\hat{G}_{1ab}v_1^2 + \frac{3}{2}\hat{G}_{1ai}v_1^bv_1^i + \frac{1}{2}\hat{G}_{1ab}V\right)\right.\right.\right. \\
& + \frac{1}{c^4}\left(-\frac{7}{16}\hat{G}_{1ab}v_1^4 - \frac{1}{8}(\hat{G}_{1ij}v_1^iv_1^j)v_1^av_1^b + \frac{7}{8}\hat{G}_{1ai}v_1^2v_1^bv_1^i - \frac{1}{4}\hat{G}_{1ab}v_1^2V + \frac{1}{2}\hat{G}_{1ai}Vv_1^bv_1^i - \frac{1}{4}\hat{G}_{1ab}V^2\right. \\
& \left.\left.\left.+ 2\hat{G}_{1ab}(v_1^iV_i) - 2\hat{G}_{1ai}v_1^iV_b - 2\hat{G}_{1ai}v_1^bV_i + \hat{G}_{1bi}\hat{W}_{ai} + \hat{G}_{1ai}\hat{W}_{bi}\right)\right]\right\} \\
& + \sigma_1^{(2)}\left(-\frac{4\varepsilon_{aij}\hat{H}_{1bj}v_1^i}{3c^2} + \frac{1}{c^4}\left(-\frac{2}{3}\varepsilon_{aij}\hat{H}_{1bj}v_1^2v_1^i + \frac{2}{3}\varepsilon_{ajk}\hat{H}_{1ik}v_1^bv_1^iv_1^j + \frac{4}{3}\varepsilon_{aij}\hat{H}_{1bj}Vv_1^i + \frac{8}{3}\varepsilon_{aij}\hat{H}_{1bj}V_i\right)\right)\left.\right\} \\
& - \frac{1}{\sqrt{-g}}\left(\partial_t\partial_a\left\{\mu_1^{(2)}\delta_1\left[\frac{\hat{G}_{1ab}v_1^b}{c^2} + \frac{1}{c^4}\left(\frac{1}{2}(\hat{G}_{1ij}v_1^iv_1^j)v_1^a - \hat{G}_{1ab}Vv_1^b\right)\right]\right\}\right) \\
& + \partial_t\left\{\frac{\mu_1^{(2)}}{c^4}\delta_1\left((\hat{G}_{1ab}v_1^a\partial_bV) + 2(\hat{G}_{1ab}\partial_bV_a)\right)\right\} \\
& - \frac{1}{\sqrt{-g}}\partial_a\left\{\delta_1\left(\mu_1^{(2)}\left[-\frac{\hat{G}_{1ab}\partial_bV}{c^2} + \frac{1}{c^4}\left(\hat{G}_{1ab}v_1^b\partial_tV + \frac{7}{2}(\hat{G}_{1ij}v_1^i\partial_jV)v_1^a + \frac{7}{2}\hat{G}_{1ab}(v_1^i\partial_iV)v_1^b\right.\right.\right.\right. \\
& \left.\left.\left.- 2(\hat{G}_{1ij}v_1^iv_1^j)\partial_aV - \frac{7}{2}\hat{G}_{1ab}v_1^2\partial_bV - 4\hat{G}_{1ab}\partial_tV_b + 5\hat{G}_{1ab}V\partial_bV + 4(\hat{G}_{1ij}\partial_jV_i)v_1^a + 2\hat{G}_{1bi}v_1^b\partial_aV_i\right.\right.\right. \\
& \left.\left.\left.- 8\hat{G}_{1ai}v_1^b\partial_bV_i - 2\hat{G}_{1bi}v_1^b\partial_iV_a + 4\hat{G}_{1ai}v_1^b\partial_iV_b + \hat{G}_{1bi}\partial_a\hat{W}_{bi} - 2\hat{G}_{1bi}\partial_i\hat{W}_{ab}\right)\right]\right\} \\
& + \frac{\sigma_1^{(2)}}{c^4}\left(\frac{8}{3}\varepsilon_{bij}\hat{H}_{1aj}v_1^b\partial_iV - \frac{8}{3}\varepsilon_{abj}\hat{H}_{1ij}v_1^b\partial_iV - \frac{8}{3}\varepsilon_{bij}\hat{H}_{1aj}\partial_iV_b - \frac{8}{3}\varepsilon_{aij}\hat{H}_{1bj}\partial_iV_b\right)\left.\right\} \\
& + \delta_1\left(\mu_1^{(2)}\left[\frac{1}{c^2}\left(-(\hat{G}_{1ab}\partial_{ab}V) + \frac{3}{4}(\hat{G}_{1ab}\hat{G}_{1ab})\right) + \frac{1}{c^4}\left(2(\hat{G}_{1ab}v_1^a\partial_t\partial_bV) + (\hat{G}_{1ai}v_1^av_1^b\partial_{ib}V)\right.\right.\right. \\
& \left.\left.\left.- \frac{3}{2}(\hat{G}_{1ab}\partial_{ab}V)v_1^2 - 4(\hat{G}_{1ab}\partial_t\partial_bV_a) + 6(\hat{G}_{1ab}\partial_aV\partial_bV) + 7(\hat{G}_{1ab}\partial_{ab}V)V - 4(\hat{G}_{1bi}v_1^a\partial_{ia}V_b)\right.\right.\right. \\
& \left.\left.\left.+ 4(\hat{G}_{1bi}v_1^a\partial_{ib}V_a) + \frac{9}{8}(\hat{G}_{1ab}\hat{G}_{1ab})v_1^2 - \frac{3}{4}(\hat{G}_{1ab}\hat{G}_{1ab})V\right)\right]\right) \\
& + \frac{\sigma_1^{(2)}}{c^4}\left(-\frac{8}{3}(\varepsilon_{aij}\hat{H}_{1bi}v_1^a\partial_{jb}V) - \frac{16}{3}(\varepsilon_{bij}\hat{H}_{1ab}\partial_{ja}V_i) + \frac{1}{2}(\hat{H}_{1ab}\hat{H}_{1ab})\right) - \frac{1}{\sqrt{-g}}\partial_t^2\left\{\frac{\mu_1^{(2)}\delta_1(\hat{G}_{1ab}v_1^av_1^b)}{2c^4}\right\} \\
& - \frac{1}{\sqrt{-g}}\partial_{abi}\left\{\frac{1}{6}\mu_1^{(3)}\delta_1\hat{G}_{1abi}\right\} + 1 \leftrightarrow 2, \tag{D.5a}
\end{aligned}$$

$$\begin{aligned}
(\sigma_i)_{\text{tidal}} = & -\frac{1}{\sqrt{-g}}\left(\partial_t\partial_a\left\{\delta_1\left[\mu_1^{(2)}\left(\frac{1}{2}\hat{G}_{1ai} + \frac{1}{c^2}\left(\frac{1}{4}\hat{G}_{1ai}v_1^2 - \frac{1}{4}\hat{G}_{1ib}v_1^av_1^b - \frac{1}{4}\hat{G}_{1ab}v_1^bv_1^i - \frac{1}{2}\hat{G}_{1ai}V\right)\right)\right.\right.\right. \\
& \left.\left.\left.+ \frac{\sigma_1^{(2)}}{c^2}\left(\frac{1}{3}\varepsilon_{ibj}\hat{H}_{1aj}v_1^b + \frac{1}{3}\varepsilon_{abj}\hat{H}_{1ij}v_1^b\right)\right]\right\}\right) + \partial_t\left\{\frac{\mu_1^{(2)}\delta_1\hat{G}_{1ia}\partial_aV}{c^2}\right\} \\
& - \frac{1}{\sqrt{-g}}\partial_{ab}\left\{\delta_1\left[\mu_1^{(2)}\left(\frac{1}{2}\hat{G}_{1ai}v_1^b - \frac{1}{2}\hat{G}_{1ab}v_1^i + \frac{1}{c^2}\left(\frac{1}{4}\hat{G}_{1ai}v_1^2v_1^b - \frac{1}{4}\hat{G}_{1ab}v_1^2v_1^i - \frac{1}{4}\hat{G}_{1ij}v_1^av_1^bv_1^j + \frac{1}{4}\hat{G}_{1aj}v_1^bv_1^iv_1^j\right.\right.\right.\right. \\
& \left.\left.\left.- \frac{1}{2}\hat{G}_{1ai}Vv_1^b + \frac{1}{2}\hat{G}_{1ab}Vv_1^i\right)\right)\right] \\
& + \sigma_1^{(2)}\left(-\frac{1}{3}\varepsilon_{aij}\hat{H}_{1bj} + \frac{1}{c^2}\left(-\frac{1}{6}\varepsilon_{aij}\hat{H}_{1bj}v_1^2 + \frac{1}{3}\varepsilon_{ajk}\hat{H}_{1ik}v_1^bv_1^j + \frac{1}{6}\varepsilon_{aik}\hat{H}_{1jk}v_1^bv_1^j - \frac{1}{3}\varepsilon_{ajk}\hat{H}_{1bk}v_1^iv_1^j\right.\right. \\
& \left.\left.+ \frac{1}{6}\varepsilon_{aik}\hat{H}_{1bj}v_1^jv_1^k + \varepsilon_{aij}\hat{H}_{1bj}V\right)\right)\left.\right\} \\
& + \delta_1\left[\frac{2\sigma_1^{(2)}\varepsilon_{ibj}\hat{H}_{1ab}\partial_{ja}V}{3c^2} + \frac{\mu_1^{(2)}}{c^2}\left(-\frac{1}{2}\hat{G}_{1ib}v_1^a\partial_{ab}V - \frac{1}{2}\hat{G}_{1ab}v_1^a\partial_{ib}V + \frac{3}{4}(\hat{G}_{1ab}\hat{G}_{1ab})v_1^i\right)\right] \\
& - \frac{1}{\sqrt{-g}}\partial_a\left\{\delta_1\left[\frac{\sigma_1^{(2)}}{c^2}\left(-\frac{2}{3}\varepsilon_{aij}\hat{H}_{1bj}\partial_bV - \frac{2}{3}\varepsilon_{abj}\hat{H}_{1ij}\partial_bV\right)\right.\right. \\
& \left.\left.+ \frac{\mu_1^{(2)}}{c^2}\left(-\frac{1}{2}\hat{G}_{1ai}\partial_tV - \hat{G}_{1ai}(v_1^j\partial_jV) - \frac{1}{2}\hat{G}_{1ib}v_1^b\partial_aV + 2\hat{G}_{1ib}v_1^a\partial_bV - \hat{G}_{1ab}v_1^i\partial_bV + \frac{1}{2}\hat{G}_{1ab}v_1^b\partial_iV\right.\right.\right.
\end{aligned}$$

$$+ \hat{G}_{1ib} \partial_a V_b - \hat{G}_{1ib} \partial_b V_a + \hat{G}_{1ab} \partial_b V_i - \hat{G}_{1ab} \partial_i V_b \Big] \Big\} + 1 \leftrightarrow 2, \quad (\text{D.5b})$$

$$\begin{aligned} (\sigma_{ij})_{\text{tidal}} = & -\frac{1}{\sqrt{-g}} \partial_t^2 \left\{ -\frac{1}{2} \mu_+^{(2)} \delta_1 \hat{G}_{1ij} \right\} - \mu_+^{(2)} \delta_1 \hat{G}_{1a(i} \partial_{j)a} V - \frac{1}{\sqrt{-g}} \partial_a \left\{ \mu_+^{(2)} \delta_1 \left(\frac{1}{2} \hat{G}_{1ij} \partial_a V - \hat{G}_{1a(i} \partial_{j)V} \right) \right\} \\ & - \frac{1}{\sqrt{-g}} \partial_t \partial_a \left\{ \delta_1 \left(\mu_+^{(2)} (-\hat{G}_{1ij} v_1^a + \hat{G}_{1a(i} v_1^{j)}) - \frac{2\sigma_+^{(2)} \varepsilon_{ab(i} \hat{H}_{1j)b}}{3} \right) \right\} \\ & - \frac{1}{\sqrt{-g}} \partial_{ab} \left\{ \delta_1 \left(\mu_+^{(2)} \left(-\frac{1}{2} \hat{G}_{1ij} v_1^a v_1^b + \hat{G}_{1a(i} v_1^{j)} v_1^b - \frac{1}{2} \hat{G}_{1ab} v_1^i v_1^j \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(\frac{2}{3} \varepsilon_{ka(i} \hat{H}_{1j)k} v_1^b - \frac{2}{3} \varepsilon_{ka(i} v_1^{j)} \hat{H}_{1bk} \right) \right) \right\} + 1 \leftrightarrow 2. \quad (\text{D.5c}) \end{aligned}$$

The source multipole moments in the CoM for general orbits required for the computation of the 2PN radiated energy flux read

$$\begin{aligned} \Gamma_{ij}^{\text{tidal}} = & \frac{Gm}{r^3} \left\{ n^{(i} n^{j)} \left(3\mu_+^{(2)} + 3\Delta\mu_-^{(2)} \right) + \frac{1}{c^2} \left[n^{(i} n^{j)} \left\{ \left[\mu_+^{(2)} \left(-\frac{15}{2} + \frac{250}{7} \nu \right) + \Delta\mu_-^{(2)} \left(-\frac{15}{2} - \frac{40}{7} \nu \right) + \frac{160}{3} \nu \sigma_+^{(2)} \right] \dot{r}^2 \right. \right. \\ & + \left[\mu_+^{(2)} \left(6 + \frac{13}{7} \nu \right) + \Delta\mu_-^{(2)} \left(6 - \frac{13}{7} \nu \right) + \frac{160}{3} \nu \sigma_+^{(2)} \right] v^2 + \frac{Gm}{r} \left[\mu_+^{(2)} \left(-\frac{15}{2} - \frac{99}{7} \nu + \frac{39}{7} \nu^2 \right) \right. \\ & \left. \left. + \Delta\mu_-^{(2)} \left(-\frac{15}{2} - \frac{54}{7} \nu \right) \right] \right\} + n^{(i} v^{j)} \left[\mu_+^{(2)} \left(-6 - \frac{214}{7} \nu \right) + \Delta\mu_-^{(2)} \left(-6 + \frac{130}{7} \nu \right) - \frac{448}{3} \nu \sigma_+^{(2)} \right] \dot{r} \\ & + v^{(i} v^{j)} \left[\mu_+^{(2)} \left(3 + \frac{38}{7} \nu \right) + \Delta\mu_-^{(2)} \left(3 - \frac{38}{7} \nu \right) + \frac{128}{3} \nu \sigma_+^{(2)} \right] \\ & + \frac{1}{c^4} \left\{ n^{(i} n^{j)} \left[\left[\mu_+^{(2)} \left(\frac{105}{8} - 20\nu - 85\nu^2 \right) + \Delta\mu_-^{(2)} \left(\frac{105}{8} - 85\nu - 15\nu^2 \right) + \sigma_+^{(2)} \left(\frac{80}{3} \nu - \frac{160}{3} \nu^2 \right) - \frac{640}{3} \Delta\sigma_-^{(2)} \nu \right] \dot{r}^4 \right. \right. \\ & + \left[\mu_+^{(2)} \left(-\frac{45}{2} + 120\nu - \frac{515}{14} \nu^2 \right) + \Delta\mu_-^{(2)} \left(-\frac{45}{2} + 15\nu + \frac{205}{14} \nu^2 \right) + \sigma_+^{(2)} \left(\frac{600}{7} \nu - \frac{640}{7} \nu^2 \right) - \frac{600}{7} \Delta\sigma_-^{(2)} \nu \right] \dot{r}^2 v^2 \\ & + \left[\mu_+^{(2)} \left(6 - \frac{72}{7} \nu - \frac{47}{14} \nu^2 \right) + \Delta\mu_-^{(2)} \left(6 - \frac{96}{7} \nu + \frac{61}{14} \nu^2 \right) + \sigma_+^{(2)} \left(\frac{424}{7} \nu - \frac{736}{7} \nu^2 \right) + \frac{136}{7} \Delta\sigma_-^{(2)} \nu \right] v^4 \\ & + \frac{Gm}{r} \left\{ \left[\mu_+^{(2)} \left(-\frac{639}{8} + \frac{22885}{56} \nu + \frac{3539}{56} \nu^2 - \frac{213}{7} \nu^3 \right) + \Delta\mu_-^{(2)} \left(-\frac{639}{8} + \frac{9565}{56} \nu - \frac{4909}{56} \nu^2 \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(-\frac{164}{7} \nu + \frac{2600}{21} \nu^2 \right) - \frac{340}{21} \Delta\sigma_-^{(2)} \nu \right] \dot{r}^2 + \left[\mu_+^{(2)} \left(\frac{139}{8} - \frac{14405}{168} \nu - \frac{1537}{168} \nu^2 - \frac{953}{42} \nu^3 \right) \right. \right. \\ & \left. \left. + \Delta\mu_-^{(2)} \left(\frac{139}{8} - \frac{3211}{56} \nu + \frac{709}{24} \nu^2 \right) + \sigma_+^{(2)} \left(\frac{4804}{63} \nu - \frac{2248}{21} \nu^2 \right) + \frac{508}{63} \Delta\sigma_-^{(2)} \nu \right] v^2 \right\} \\ & + \frac{G^2 m^2}{r^2} \left[\mu_+^{(2)} \left(\frac{285}{28} + \frac{8885}{84} \nu - \frac{3147}{28} \nu^2 + \frac{27}{2} \nu^3 \right) + \Delta\mu_-^{(2)} \left(\frac{285}{28} + \frac{8047}{84} \nu + \frac{233}{84} \nu^2 \right) \right] \\ & + n^{(i} v^{j)} \left\{ \left[\mu_+^{(2)} \left(15 - \frac{1265}{7} \nu + \frac{1810}{7} \nu^2 \right) + \Delta\mu_-^{(2)} \left(15 + \frac{1055}{7} \nu + \frac{110}{7} \nu^2 \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(-\frac{560}{3} \nu + \frac{640}{3} \nu^2 \right) + \frac{1840}{3} \Delta\sigma_-^{(2)} \nu \right] \dot{r}^3 + \left[\mu_+^{(2)} \left(-6 + \frac{15}{7} \nu - \frac{346}{7} \nu^2 \right) \right. \right. \\ & \left. \left. + \Delta\mu_-^{(2)} \left(-6 + \frac{27}{7} \nu - \frac{286}{7} \nu^2 \right) + \sigma_+^{(2)} \left(-\frac{3392}{21} \nu + \frac{5440}{21} \nu^2 \right) - \frac{2432}{21} \Delta\sigma_-^{(2)} \nu \right] \dot{r} v^2 \right. \\ & \left. + \frac{Gm}{r} \left[\mu_+^{(2)} \left(\frac{105}{2} - \frac{10601}{42} \nu - \frac{569}{14} \nu^2 + \frac{16}{7} \nu^3 \right) + \Delta\mu_-^{(2)} \left(\frac{105}{2} - \frac{2329}{42} \nu - \frac{731}{42} \nu^2 \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(-96\nu + \frac{1312}{21} \nu^2 \right) + \frac{256}{21} \Delta\sigma_-^{(2)} \nu \right] \dot{r} \right\} \\ & + v^{(i} v^{j)} \left\{ \left[\mu_+^{(2)} \left(-\frac{15}{2} + \frac{687}{7} \nu - \frac{760}{7} \nu^2 \right) + \Delta\mu_-^{(2)} \left(-\frac{15}{2} - \frac{477}{7} \nu + \frac{20}{7} \nu^2 \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(\frac{3104}{21} \nu - \frac{3520}{21} \nu^2 \right) - \frac{5344}{21} \Delta\sigma_-^{(2)} \nu \right] \dot{r}^2 + \left[\mu_+^{(2)} \left(3 - 11\nu + \frac{109}{7} \nu^2 \right) + \Delta\mu_-^{(2)} \left(3 - \nu + \frac{65}{7} \nu^2 \right) \right. \right. \\ & \left. \left. + \sigma_+^{(2)} \left(\frac{192}{7} \nu - \frac{384}{7} \nu^2 \right) + \frac{256}{7} \Delta\sigma_-^{(2)} \nu \right] v^2 \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{Gm}{r} \left[\mu_+^{(2)} \left(-\frac{7}{2} + \frac{458}{21}\nu + \frac{101}{21}\nu^2 - \frac{485}{21}\nu^3 \right) + \Delta\mu_-^{(2)} \left(-\frac{7}{2} - \frac{151}{21}\nu + \frac{332}{21}\nu^2 \right) \right. \\
& \quad \left. + \sigma_+^{(2)} \left(\frac{2720}{63}\nu - \frac{1664}{21}\nu^2 \right) - \frac{256}{63}\Delta\sigma_-^{(2)}\nu \right] \Bigg\} \Bigg\}, \tag{D.6a}
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_{ijk}^{\text{tidal}} &= \frac{Gm\nu}{r^2} \left\{ 18\mu_-^{(2)} n^{(i} n^j n^k) + \frac{1}{c^2} \left[n^{(i} n^j n^k) \left\{ \left[-\frac{105}{2}\Delta\mu_+^{(2)} + \mu_-^{(2)} \left(\frac{15}{2} - 15\nu \right) - 60\Delta\sigma_+^{(2)} + 60\sigma_-^{(2)} \right] \dot{r}^2 \right. \right. \right. \\
& \quad \left. \left. + \left[-3\Delta\mu_+^{(2)} + \mu_-^{(2)} (48 - 33\nu) - 84\Delta\sigma_+^{(2)} + 84\sigma_-^{(2)} \right] v^2 \right. \right. \\
& \quad \left. \left. + \frac{Gm}{r} \left[\Delta\mu_+^{(2)} \left(\frac{33}{2} - 9\nu \right) + \mu_-^{(2)} \left(-\frac{135}{2} - 27\nu \right) \right] \right\} \right\} \\
& \quad + n^{(i} n^j v^k) \left[72\Delta\mu_+^{(2)} + \mu_-^{(2)} (-108 + 108\nu) + 216\Delta\sigma_+^{(2)} - 216\sigma_-^{(2)} \right] \dot{r} \\
& \quad \left. + n^{(i} v^j v^k) \left[-21\Delta\mu_+^{(2)} + \mu_-^{(2)} (39 - 42\nu) - 72\Delta\sigma_+^{(2)} + 72\sigma_-^{(2)} \right] \right] \Bigg\}, \tag{D.6b}
\end{aligned}$$

$$\mathbb{J}_{ijkl}^{\text{tidal}} = \frac{Gm\nu}{r} n^{(i} n^j n^k n^l) (18\mu_+^{(2)} - 18\Delta\mu_-^{(2)}), \tag{D.6c}$$

$$\begin{aligned}
\mathbb{J}_{ij}^{\text{tidal}} &= \frac{Gm}{r^3} \left\{ n^{(i} (n \times v)^{j)} \left(9\nu\mu_-^{(2)} + 12\Delta\sigma_+^{(2)} + 12\sigma_-^{(2)} \right) \right. \\
& \quad + \frac{1}{c^2} \left[n^{(i} (n \times v)^{j)} \left\{ \left[-\frac{240}{7}\Delta\mu_+^{(2)}\nu + \mu_-^{(2)} \left(\frac{375}{14}\nu - \frac{60}{7}\nu^2 \right) + \Delta\sigma_+^{(2)} \left(-30 - \frac{360}{7}\nu \right) + \sigma_-^{(2)} \left(-30 + \frac{1200}{7}\nu \right) \right] \dot{r}^2 \right. \right. \\
& \quad \left. \left. + \left[\frac{129}{28}\Delta\mu_+^{(2)}\nu + \mu_-^{(2)} \left(\frac{417}{28}\nu - \frac{51}{7}\nu^2 \right) + \Delta\sigma_+^{(2)} \left(12 - \frac{96}{7}\nu \right) + \sigma_-^{(2)} \left(12 - \frac{72}{7}\nu \right) \right] v^2 \right. \right. \\
& \quad \left. \left. + \frac{Gm}{r} \left[\Delta\mu_+^{(2)} \left(\frac{3}{14}\nu - \frac{60}{7}\nu^2 \right) + \mu_-^{(2)} \left(-\frac{111}{14}\nu - \frac{267}{7}\nu^2 \right) + \Delta\sigma_+^{(2)} \left(-4 - \frac{300}{7}\nu \right) + \sigma_-^{(2)} \left(-4 - \frac{316}{7}\nu \right) \right] \right\} \right\} \\
& \quad + (n \times v)^{(i} v^{j)} \left[\frac{213}{14}\Delta\mu_+^{(2)}\nu + \mu_-^{(2)} \left(-\frac{507}{14}\nu + \frac{129}{7}\nu^2 \right) + \frac{396}{7}\Delta\sigma_+^{(2)}\nu - \frac{564}{7}\nu\sigma_-^{(2)} \right] \dot{r} \Bigg\}, \tag{D.6d}
\end{aligned}$$

$$\mathbb{J}_{ijk}^{\text{tidal}} = \frac{Gm\nu}{r^2} n^{(j} n^k (n \times v)^i) (12\mu_+^{(2)} - 12\Delta\mu_-^{(2)} + 64\sigma_+^{(2)}). \tag{D.6e}$$

Appendix E

4PN

E.1 The shifts applied in the 4PN equations of motion

In the recent work on the 4PN equations of motion [131, 132, 133, 38, 134], which succeeded in computing from first principles all the “ambiguities” in the problem, we applied to our “brute” calculation a series of shifts of the trajectories so as to remove UV- and IR-types poles and simplify the end result. Now, in order to be consistent we have to apply after our “brute” derivation of the 4PN quadrupole moment the *same* series of shifts. The total shift is composed of three pieces. A first shift $\xi_{1,2}$ removed the poles $1/\varepsilon$ corresponding to UV divergences and led to the Lagrangian provided in [132, 133]; a second shift $\chi_{1,2}$, used in [133], dealt with the $1/\varepsilon$ poles due to IR divergences; and a third shift $\eta_{1,2}$ presented in [134] was mainly used for convenience. We provide here the shifts $\xi_{1,2}$ and $\eta_{1,2}$ *in extenso*.

The shift ξ_1 is composed of 3PN and 4PN contributions and is given by

$$\xi_1^i = \frac{11}{3} \frac{G^2 m_1^2}{c^6} \left[\frac{1}{\varepsilon} - 2 \ln \left(\frac{\bar{q}^{1/2} r'_0}{\ell_0} \right) - \frac{327}{1540} \right] a_1^i + \frac{1}{c^8} \xi_{1,4\text{PN}}^i. \quad (\text{E.1})$$

The first term represents the 3PN contribution determined in [83]. Here $\bar{q} = 4\pi e^{\gamma_E}$ where γ_E is the Euler constant, and \mathbf{a}_1 represents the Newtonian acceleration of 1 in d dimensions. The 4PN contributions have already been shown in the Appendix C of [131], however the terms proportional to y_1^i (see Eq. (E.3b) below) were inadvertently missing there. Furthermore we have adjusted one coefficient in the expression of the shift to take into account the subsequent determination of the ambiguity parameters in [133, 38]. The 4PN terms in the shift are conveniently written in the form

$$\xi_{1,4\text{PN}}^i = \frac{1}{\varepsilon} \xi_{1,4\text{PN}}^{i(-1)} + \xi_{1,4\text{PN}}^{(0,y_1)} y_1^i + \xi_{1,4\text{PN}}^{(0,n_{12})} n_{12}^i + \xi_{1,4\text{PN}}^{(0,v_1)} v_1^i + \xi_{1,4\text{PN}}^{(0,v_{12})} v_{12}^i, \quad (\text{E.2})$$

with $1/\varepsilon$ being the UV pole and $v_{12}^i = v_1^i - v_2^i$. We have

$$\begin{aligned} \xi_{1,4\text{PN}}^{i(-1)} = & \frac{G^3 m_1^2 m_2 v_{12}^i}{r_{12}^2} \left(11(n_{12} v_{12}) + \frac{11}{3}(n_{12} v_1) \right) + n_{12}^i \left[\frac{G^4}{r_{12}^3} \left(\frac{55}{3} m_1^3 m_2 + \frac{22}{3} m_1^2 m_2^2 + 4 m_1 m_2^3 \right) \right. \\ & \left. + \frac{G^3 m_1^2 m_2}{r_{12}^2} \left(\frac{11}{2} (n_{12} v_{12})^2 - 11(n_{12} v_{12})(n_{12} v_1) + \frac{11}{2} (n_{12} v_1)^2 - \frac{22}{3} v_{12}^2 \right) \right], \end{aligned} \quad (\text{E.3a})$$

$$\begin{aligned} \xi_{1,4\text{PN}}^{(0,y_1)} = & G^3 (m_1 + m_2) \frac{m_1 m_2}{15 r_{12}^3} \left(\left[-67 + 48 \ln \left(\frac{r_{12}}{r_0} \right) \right] (n_{12} v_{12})^2 + \left[17 - 16 \ln \left(\frac{r_{12}}{r_0} \right) \right] v_{12}^2 \right) \\ & + G^4 (m_1 + m_2)^2 \frac{m_1 m_2}{15 r_{12}^4} \left[-17 + 16 \ln \left(\frac{r_{12}}{r_0} \right) \right], \end{aligned} \quad (\text{E.3b})$$

$$\xi_{1,4\text{PN}}^{(0,n_{12})} = G^3 m_1 m_2^2 \left[\frac{1}{r_{12}^2} \left(\left[-\frac{1753}{80} + \frac{72}{5} \ln \left(\frac{r_{12}}{r_0} \right) \right] (n_{12} v_{12})^2 + \left[\frac{1753}{40} + \frac{96}{5} \ln \left(\frac{r_{12}}{r_0} \right) \right] (n_{12} v_{12})(n_{12} v_1) \right) \right]$$

$$\begin{aligned}
& + \left[-\frac{1753}{60} - \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (v_{12}v_1) + \left[\frac{1753}{120} - \frac{32}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] v_{12}^2 \\
& + \frac{1}{r_{12}^3} \left(\left[-\frac{431}{10} + 48 \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})^2 (n_{12}y_1) + \left[\frac{53}{2} - \frac{144}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})(v_{12}y_1) \right. \\
& \quad \left. + \left[\frac{67}{10} - \frac{48}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}y_1)v_{12}^2 \right) \\
& + G^3 m_1^2 m_2 \left[\frac{1}{r_{12}^2} \left(\left[-\frac{2761}{168} - \frac{33}{2} \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) - \frac{11}{2} \ln\left(\frac{r_{12}}{r'_0}\right) \right] (n_{12}v_{12})^2 \right. \right. \\
& \quad \left. \left. + \left[\frac{4367}{168} + 33 \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) + \frac{96}{5} \ln\left(\frac{r_{12}}{r_0}\right) + 11 \ln\left(\frac{r_{12}}{r'_0}\right) \right] (n_{12}v_{12})(n_{12}v_1) \right. \right. \\
& \quad \left. \left. + \left[\frac{7489}{840} - \frac{33}{2} \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) - \frac{11}{2} \ln\left(\frac{r_{12}}{r'_0}\right) \right] (n_{12}v_1)^2 + \left[-\frac{1753}{60} - \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (v_{12}v_1) \right. \right. \\
& \quad \left. \left. + \left[\frac{1753}{120} + 22 \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) + \frac{16}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] v_{12}^2 \right) \right. \\
& + \frac{1}{r_{12}^3} \left(\left[-\frac{431}{10} + 48 \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})^2 (n_{12}y_1) + \left[\frac{53}{2} - \frac{144}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})(v_{12}y_1) \right. \\
& \quad \left. + \left[\frac{67}{10} - \frac{48}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}y_1)v_{12}^2 \right) \\
& + G^4 m_1^2 m_2^2 \left(\frac{1}{r_{12}^3} \left[-\frac{88}{3} \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) - \frac{16}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] + \frac{1}{r_{12}^4} \left[\frac{67}{15} - \frac{32}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}y_1) \right) \\
& + G^4 m_1^3 m_2 \left(\frac{1}{r_{12}^3} \left[-\frac{220}{3} \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) - \frac{32}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] + \frac{1}{r_{12}^4} \left[\frac{67}{30} - \frac{16}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}y_1) \right) \\
& + G^4 m_1 m_2^3 \left(\frac{1}{r_{12}^3} \left[-16 \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) + \frac{16}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] + \frac{1}{r_{12}^4} \left[\frac{67}{30} - \frac{16}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}y_1) \right), \tag{E.3c}
\end{aligned}$$

$$\xi_{1,4\text{PN}}^{(0,v_1)} = G^3 \left(\frac{m_1^2 m_2}{r_{12}^2} \left[-\frac{269}{120} + \frac{32}{15} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12}) + \frac{m_1 m_2^2}{r_{12}^2} \left[-\frac{269}{120} + \frac{32}{15} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12}) \right), \tag{E.3d}$$

$$\begin{aligned}
\xi_{1,4\text{PN}}^{(0,v_{12})} & = G^3 m_1 m_2^2 \left[\frac{1}{r_{12}^2} \left(\left[\frac{755}{48} - 8 \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12}) + \left[-\frac{1753}{60} - \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_1) \right) \right. \\
& \quad \left. + \frac{1}{r_{12}^3} \left(\left[\frac{53}{2} - \frac{144}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})(n_{12}y_1) + \left[-\frac{134}{15} + \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (v_{12}y_1) \right) \right] \\
& + G^3 m_1^2 m_2 \left[\frac{1}{r_{12}^2} \left(\left[\frac{23113}{1680} - 33 \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) + \frac{11}{3} \ln\left(\frac{r_{12}}{r'_0}\right) \right] (n_{12}v_{12}) \right. \right. \\
& \quad \left. \left. + \left[-\frac{2572}{105} - 11 \ln\left(\frac{\bar{q}^{1/2} r'_0}{\ell_0}\right) - \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) - \frac{11}{3} \ln\left(\frac{r_{12}}{r'_0}\right) \right] (n_{12}v_1) \right) \right. \\
& \quad \left. + \frac{1}{r_{12}^3} \left(\left[\frac{53}{2} - \frac{144}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (n_{12}v_{12})(n_{12}y_1) + \left[-\frac{134}{15} + \frac{64}{5} \ln\left(\frac{r_{12}}{r_0}\right) \right] (v_{12}y_1) \right) \right]. \tag{E.3e}
\end{aligned}$$

On the other hand the shift $\boldsymbol{\eta}_1$ is quite simple. It starts only at the 4PN order and it is made

only of G^3 and G^4 terms,

$$\boldsymbol{\eta}_1 = \frac{G^3}{c^8} \boldsymbol{\eta}_{1,4\text{PN}}^{(3)} + \frac{G^4}{c^8} \boldsymbol{\eta}_{1,4\text{PN}}^{(4)}. \quad (\text{E.4})$$

where

$$\boldsymbol{\eta}_{1,4\text{PN}}^{(3)} = \frac{\mathbf{v}_{12}}{r_{12}^2} \left(\frac{769}{24} m_1^2 m_2 (n_{12} v_{12}) + \frac{561}{35} m_1 m_2^2 (n_{12} v_{12}) \right) \quad (\text{E.5a})$$

$$+ \frac{\mathbf{n}_{12}}{r_{12}^2} \left[m_1 m_2^2 \left(\frac{21719}{1400} (n_{12} v_{12})^2 - \frac{2096}{175} v_{12}^2 \right) + m_1^2 m_2 \left(-\frac{2119}{50} (n_{12} v_{12})^2 + \frac{58769}{2100} v_{12}^2 \right) \right],$$

$$\boldsymbol{\eta}_{1,4\text{PN}}^{(4)} = \left(\frac{8861}{2100} m_1^3 m_2 + \frac{613}{350} m_1^2 m_2^2 - \frac{5183}{2100} m_1 m_2^3 \right) \frac{\mathbf{n}_{12}}{r_{12}^3}. \quad (\text{E.5b})$$

E.2 The 4PN mass quadrupole as a function of the potentials

We indicate the different PN pieces with the index $n\text{PN}$, it being understood that the term is to be multiplied by the appropriate factor $1/c^{2n}$. For simplicity we omit writing the finite part integral $\mathcal{FP}_{B=0} \int d^d \mathbf{x} \tilde{r}^B$ in front of each terms, and we also do not write the necessary time derivatives $(d/dt)^{2k}$ present in the series expansion, see Eq. (6.5). Also we do not write the terms that are equal to zero. All the non-compact and surface terms shown below have to be divided by a factor $G\pi$. Remind also our notation (6.17) for the terms involving super-potentials, for instance $\Psi_{ij}^{\partial_{ab}V}$ or $\Psi_{ijk}^{\partial_a V}$. We exhaustively provide all the terms composing the 4PN quadrupole moment following this nomenclature and conventions. We start with the list of compact (C) terms.

$$\begin{aligned} \text{SIC}_{0\text{PN}} &= \frac{(d-1)}{2(d-2)} \hat{x}_{ij} \sigma, \\ \text{SIC}_{1\text{PN}} &= -\frac{(d-3)(d-1)}{(d-2)^2} V \hat{x}_{ij} \sigma, \\ \text{SIC}_{2\text{PN}} &= \left(\frac{(d-3)^2(d-1)}{(d-2)^3} V^2 \sigma + \frac{4(d-3)}{d-2} V_a \sigma_a + \frac{2(d-1)}{(d-2)^2} V \sigma_{aa} + \frac{2(d-3)^2(d-1)}{(d-2)^3} K \sigma \right) \hat{x}_{ij} \\ &\quad + \frac{2(d-1)}{d-2} \sigma_{ab} \Psi_{ij}^{\partial_{ab}V} - \frac{2(d-1)}{(d-2)^2} \sigma_{bb} \Psi_{ij}^{\partial_{aa}V}, \\ \text{SIC}_{3\text{PN}} &= \left(-\frac{2(d-3)^3(d-1)}{3(d-2)^4} V^3 \sigma + \frac{8(d-1)}{(d-2)^2} V_a V_a \sigma - \frac{4(d-3)(d-1)}{(d-2)^2} \hat{X} \sigma - \frac{4(d-5)(d-3)}{(d-2)^2} V V_a \sigma_a \right. \\ &\quad + \frac{8(d-3)}{d-2} \hat{R}_a \sigma_a - \frac{2(d-5)(d-1)}{(d-2)^3} V^2 \sigma_{aa} + \frac{4}{d-2} \hat{W}_{ab} \sigma_{ab} - \frac{4(d-3)^3(d-1)}{(d-2)^4} K V \sigma \\ &\quad - \frac{4(d-3)(d-1)}{(d-2)^3} K \sigma_{aa} \left. \right) \hat{x}_{ij} + \frac{8(d-1)^2}{(d-2)^2} \sigma \Psi_{ij}^{\partial_a V_b} \partial_b V_a + \frac{8(d-1)^2}{(d-2)^2} V_a \Psi_{ij}^{\partial_a V_b} \partial_b \sigma \\ &\quad - \frac{4(d-1)^2}{(d-2)^2} V_a \sigma \Psi_{ij}^{\partial_{ta}V} + \frac{8(d-5)(d-1)}{(d-2)^2} V \Psi_{ij}^{\partial_a V_b} \partial_b \sigma_a + \frac{8(d-5)(d-1)}{(d-2)^2} \sigma_a \Psi_{ij}^{\partial_a V_b} \partial_b V \\ &\quad - \frac{4(d-5)(d-1)}{(d-2)^2} V \sigma_a \Psi_{ij}^{\partial_{ta}V} + \frac{8(d-1)}{(d-2)^2} V \sigma_{ab} \Psi_{ij}^{\partial_{ab}V} - \frac{8(d-1)}{(d-2)^3} V \sigma_{bb} \Psi_{ij}^{\partial_{aa}V}, \\ \text{SIC}_{4\text{PN}} &= \left(\frac{(d-3)(d-1)(3d^3 - 23d^2 + 55d - 43)}{3(d-2)^5} V^4 \sigma + \frac{4(d-1)(2d^2 - 13d + 19)}{(d-2)^3} V V_a V_a \sigma \right. \\ &\quad - \frac{4(d-3)(d-1)^2}{(d-2)^3} V^2 \hat{W} \sigma - \frac{(d-1)(d+2)}{(d-2)^2} \hat{W}_{ab} \hat{W}_{ab} \sigma - \frac{8(d-8)(d-1)}{(d-2)^2} \hat{R}_a V_a \sigma \\ &\quad + \frac{4(d-3)(d-1)(3d-7)}{(d-2)^3} V \hat{X} \sigma + \frac{8(d-1)^2}{(d-2)^2} V \hat{Z} \sigma - \frac{16(d-3)(d-1)}{(d-2)^2} \hat{T} \sigma - \frac{16(d-1)}{(d-2)^2} \hat{M} \sigma \\ &\quad + \frac{4(3d^3 - 25d^2 + 71d - 65)}{(d-2)^3} V^2 V_a \sigma_a - \frac{8(d-3)}{d-2} V_a \hat{W} \sigma_a + \frac{8(d-3)}{d-2} V_a \hat{W}_{ab} \sigma_b \\ &\quad \left. - \frac{8(2d^2 - 13d + 19)}{(d-2)^2} \hat{R}_a V \sigma_a + \frac{16(d-3)}{d-2} \hat{Y}_a \sigma_a + \frac{2(d-1)(5d^2 - 37d + 56)}{3(d-2)^4} V^3 \sigma_{aa} + \frac{8(d-3)}{d-2} V_a V_b \sigma_{ab} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{8(d-3)}{(d-2)^2} V_a V_a \sigma_{bb} - \frac{2(d^2-d-8)}{(d-2)^2} V \hat{W}_{ab} \sigma_{ab} + \frac{2d(d-1)}{(d-2)^3} V \hat{W} \sigma_{aa} + \frac{8(d-1)}{(d-2)^2} \hat{X} \sigma_{aa} + \frac{16}{d-2} \hat{Z}_{ab} \sigma_{ab} \\
& + \frac{(d-3)(d-1)(11d^3-76d^2+175d-142)}{(d-2)^5} K V^2 \sigma + \frac{(d-3)^2(d-1)(7d^2-33d+42)}{(d-2)^5} K^2 \sigma \\
& - \frac{4(d-3)(d-1)^2}{(d-2)^3} K \hat{W} \sigma + \frac{16(d-3)(d^2-5d+8)}{(d-2)^3} K V_a \sigma_a + \frac{16(d-3)^2(d-1)}{(d-2)^4} K V \sigma_{aa} \Big) \hat{x}_{ij} \\
& - \frac{8(d-5)(d-1)^2}{(d-2)^3} V \sigma \Psi_{ij}^{\partial_a V_b} \partial_b V_a - \frac{8(d-5)(d-1)^2}{(d-2)^3} V_a \sigma \Psi_{ij}^{\partial_a V_b} \partial_b V - \frac{8(d-5)(d-1)^2}{(d-2)^3} V V_a \Psi_{ij}^{\partial_a V_b} \partial_b \sigma \\
& + \frac{16(d-1)^2}{(d-2)^2} \sigma \Psi_{ij}^{\partial_a V_b} \partial_b \hat{R}_a + \frac{16(d-1)^2}{(d-2)^2} \hat{R}_a \Psi_{ij}^{\partial_a V_b} \partial_b \sigma + \frac{4(d-5)(d-1)^2}{(d-2)^3} V V_a \sigma \Psi_{ij}^{\partial_a V} \\
& - \frac{8(d-1)^2}{(d-2)^2} \hat{R}_a \sigma \Psi_{ij}^{\partial_a V} - \frac{(d-3)^2(d-1)^2}{(d-2)^4} V^3 \sigma \Psi_{ij}^{\partial_a V} + \frac{8(d-1)^2}{(d-2)^2} V_a V_b \sigma \Psi_{ij}^{\partial_a V} \\
& - \frac{4(d-1)^2}{(d-2)^2} V_b V_b \sigma \Psi_{ij}^{\partial_a V} - \frac{2(d-1)^2}{(d-2)^2} \hat{X} \sigma \Psi_{ij}^{\partial_a V} + \frac{32(d-3)(d-1)}{(d-2)^3} V^2 \Psi_{ij}^{\partial_a V_b} \partial_b \sigma_a \\
& + \frac{64(d-3)(d-1)}{(d-2)^3} V \sigma_a \Psi_{ij}^{\partial_a V_b} \partial_b V - \frac{16(d-1)}{d-2} \hat{W}_{ak} \Psi_{ij}^{\partial_k V_b} \partial_b \sigma_a - \frac{16(d-1)}{d-2} \sigma_a \Psi_{ij}^{\partial_b V_k} \partial_k \hat{W}_{ab} \\
& - \frac{16(d-3)(d-1)}{(d-2)^3} V^2 \sigma_a \Psi_{ij}^{\partial_a V} + \frac{8(d-1)}{d-2} \hat{W}_{ab} \sigma_a \Psi_{ij}^{\partial_b V} + \frac{8(d-5)(d-1)}{(d-2)^2} V V_a \sigma_b \Psi_{ij}^{\partial_a V} \\
& - \frac{8(d-4)(d-1)}{(d-2)^2} V V_b \sigma_b \Psi_{ij}^{\partial_a V} - \frac{16(d-1)}{d-2} \hat{R}_a \sigma_b \Psi_{ij}^{\partial_a V} + \frac{8(d-1)}{d-2} \hat{R}_b \sigma_b \Psi_{ij}^{\partial_a V} \\
& - \frac{16(d-1)}{(d-2)^2} \sigma_{kk} \Psi_{ij}^{\partial_a V_b} \partial_b V_a - \frac{16(d-1)}{d-2} \sigma_{ak} \Psi_{ij}^{\partial_k V_b} \partial_b V_a - \frac{16(d-1)}{d-2} V_a \Psi_{ij}^{\partial_b V_k} \partial_k \sigma_{ab} \\
& - \frac{16(d-1)}{(d-2)^2} V_a \Psi_{ij}^{\partial_a V_b} \partial_b \sigma_{kk} + \frac{8(d-1)}{(d-2)^2} V_a \sigma_{bb} \Psi_{ij}^{\partial_a V} + \frac{8(d-1)}{d-2} V_a \sigma_{ab} \Psi_{ij}^{\partial_b V} + \frac{16(d-1)}{(d-2)^3} V^2 \sigma_{ab} \Psi_{ij}^{\partial_a V} \\
& - \frac{2(d-1)^2}{(d-2)^3} V^2 \sigma_{bb} \Psi_{ij}^{\partial_a V} + \frac{8(d-1)}{d-2} \hat{W}_{ab} \sigma_{ak} \Psi_{ij}^{\partial_b V} - \frac{2(d-1)}{d-2} \hat{W}_{ka} \sigma_{ka} \Psi_{ij}^{\partial_b V} + \frac{2(d-1)}{(d-2)^2} \hat{W} \sigma_{kk} \Psi_{ij}^{\partial_a V} \\
& - \frac{4(d-3)^2(d-1)^2}{(d-2)^4} K V \sigma \Psi_{ij}^{\partial_a V} - \frac{16(d-5)(d-3)(d-1)}{(d-2)^3} K \Psi_{ij}^{\partial_a V_b} \partial_b \sigma_a \\
& - \frac{16(d-5)(d-3)(d-1)}{(d-2)^3} \sigma_a \Psi_{ij}^{\partial_a V_b} \partial_b K + \frac{8(d-5)(d-3)(d-1)}{(d-2)^3} K \sigma_a \Psi_{ij}^{\partial_a V} \\
& - \frac{16(d-3)(d-1)}{(d-2)^3} K \sigma_{ab} \Psi_{ij}^{\partial_a V},
\end{aligned}$$

$$SIIC_{1PN} = \frac{(d-1)}{4(d-2)(d+4)} r^2 \hat{x}_{ij} \sigma,$$

$$SIIC_{2PN} = \frac{(d-1)}{(d-2)^2(d+4)} r^2 V \hat{x}_{ij} \sigma,$$

$$SIIC_{3PN} = \left(\frac{2(d-1)}{(d-2)^3(d+4)} V^2 + \frac{(d-1)}{2(d-2)(d+4)} \hat{W} - \frac{2(d-3)(d-1)}{(d-2)^3(d+4)} K \right) r^2 \hat{x}_{ij} \sigma,$$

$$\begin{aligned}
SIIC_{4PN} = & \left(\frac{8(d-1)}{3(d-2)^4(d+4)} V^3 + \frac{4(d-1)}{(d-2)^2(d+4)} V_a V_a + \frac{2(d-1)}{(d-2)^2(d+4)} V \hat{W} + \frac{4(d-1)}{(d-2)^2(d+4)} \hat{X} \right. \\
& \left. + \frac{2(d-1)}{(d-2)(d+4)} \hat{Z} - \frac{8(d-3)(d-1)}{(d-2)^4(d+4)} K V \right) r^2 \hat{x}_{ij} \sigma,
\end{aligned}$$

$$SIIC_{2PN} = \frac{(d-1)}{16(d-2)(d+4)(d+6)} r^4 \hat{x}_{ij} \sigma,$$

$$SIIC_{3PN} = \frac{(d-1)}{4(d-2)^2(d+4)(d+6)} r^4 V \hat{x}_{ij} \sigma,$$

$$SIIC_{4PN} = \left(\frac{(d-1)}{2(d-2)^3(d+4)(d+6)} V^2 + \frac{(d-1)}{8(d-2)(d+4)(d+6)} \hat{W} - \frac{(d-3)(d-1)}{2(d-2)^3(d+4)(d+6)} K \right) r^4 \hat{x}_{ij} \sigma,$$

$$SIVNC_{4PN} = - \frac{(d-1)^2}{192(d-2)^2(d+4)(d+6)(d+8)} r^6 \hat{x}_{ij} \partial_a V \partial_a V$$

$$SVC_{4PN} = \frac{(d-1)}{768(d-2)(d+4)(d+6)(d+8)(d+10)} r^8 \hat{x}_{ij} \sigma,$$

$$\begin{aligned}
\text{VIC}_{1\text{PN}} &= -\frac{2(d-1)(d+2)}{d(d-2)(d+4)}\hat{x}_{ija}\sigma_a, \\
\text{VIC}_{2\text{PN}} &= \left(\frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_a\sigma + \frac{2(d-5)(d-1)(d+2)}{d(d-2)^2(d+4)}V\sigma_a\right)\hat{x}_{ija}, \\
\text{VIC}_{3\text{PN}} &= \left(-\frac{4(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}VV_a\sigma + \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{R}_a\sigma - \frac{(d-5)^2(d-1)(d+2)}{d(d-2)^3(d+4)}V^2\sigma_a\right. \\
&\quad - \frac{4(d-1)(d+2)}{d(d-2)(d+4)}\hat{W}_{ab}\sigma_b - \frac{4(d-1)(d+2)}{d(d-2)(d+4)}V_b\sigma_{ab} - \frac{4(d-1)(d+2)}{d(d-2)^2(d+4)}V_a\sigma_{bb} \\
&\quad - \frac{4(d-5)(d-3)(d-1)(d+2)}{d(d-2)^3(d+4)}K\sigma_a\left.\right)\hat{x}_{ija} - \left(\frac{4(d-1)^3(d+2)}{d(d-2)^3(d+4)}\sigma\partial_bV_a + \frac{4(d-1)^3(d+2)}{d(d-2)^3(d+4)}V_a\partial_b\sigma\right. \\
&\quad \left.+ \frac{4(d-5)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\sigma_a\partial_bV + \frac{4(d-5)(d-1)^2(d+2)}{d(d-2)^3(d+4)}V\partial_b\sigma_a\right)\Psi_{ijb}^{\partial_aV}, \\
\text{VIC}_{4\text{PN}} &= \left(-\frac{16(d-1)^2(d+2)}{d(d-2)^3(d+4)}V^2V_a\sigma - \frac{4(d-5)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\hat{R}_aV\sigma + \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{Y}_a\sigma\right. \\
&\quad - \frac{2(d-1)(d+2)(3d^3-9d^2-15d+53)}{3d(d-2)^4(d+4)}V^3\sigma_a + \frac{24(d-1)(d+2)}{d(d-2)(d+4)}V_aV_b\sigma_b \\
&\quad + \frac{4(d-8)(d-1)(d+2)}{d(d-2)^2(d+4)}V_bV_b\sigma_a + \frac{4(d-5)(d-1)(d+2)}{d(d-2)^2(d+4)}V\hat{W}_{ab}\sigma_b + \frac{8(d-5)(d-1)(d+2)}{d(d-2)^2(d+4)}\hat{X}\sigma_a \\
&\quad - \frac{16(d-1)(d+2)}{d(d-2)(d+4)}\hat{Z}_{ab}\sigma_b - \frac{16(d-1)(d+2)}{d(d-2)^2(d+4)}VV_b\sigma_{ab} + \frac{8(d-3)(d-1)(d+2)}{d(d-2)^3(d+4)}VV_a\sigma_{bb} \\
&\quad - \frac{8(d-1)(d+2)}{d(d-2)(d+4)}\hat{R}_b\sigma_{ab} - \frac{8(d-1)(d+2)}{d(d-2)^2(d+4)}\hat{R}_a\sigma_{bb} + \frac{8(d-3)^2(d-1)^2(d+2)}{d(d-2)^4(d+4)}KV_a\sigma \\
&\quad + \frac{4(d-5)^2(d-3)(d-1)(d+2)}{d(d-2)^4(d+4)}KV\sigma_a\left.\right)\hat{x}_{ija} + \frac{4(d-5)(d-1)^3(d+2)}{d(d-2)^4(d+4)}V_a\sigma\Psi_{ijb}^{\partial_aV}\partial_bV \\
&\quad + \frac{4(d-5)(d-1)^3(d+2)}{d(d-2)^4(d+4)}V\sigma\Psi_{ijb}^{\partial_aV}\partial_bV_a + \frac{4(d-5)(d-1)^3(d+2)}{d(d-2)^4(d+4)}VV_a\Psi_{ijb}^{\partial_aV}\partial_b\sigma \\
&\quad - \frac{8(d-1)^3(d+2)}{d(d-2)^3(d+4)}\sigma\Psi_{ijb}^{\partial_aV}\partial_b\hat{R}_a - \frac{8(d-1)^3(d+2)}{d(d-2)^3(d+4)}\hat{R}_a\Psi_{ijb}^{\partial_aV}\partial_b\sigma \\
&\quad - \frac{32(d-3)(d-1)^2(d+2)}{d(d-2)^4(d+4)}V\sigma_a\Psi_{ijb}^{\partial_aV}\partial_bV - \frac{16(d-3)(d-1)^2(d+2)}{d(d-2)^4(d+4)}V^2\Psi_{ijb}^{\partial_aV}\partial_b\sigma_a \\
&\quad + \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{W}_{ak}\Psi_{ijb}^{\partial_kV}\partial_b\sigma_a + \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}\sigma_a\Psi_{ijk}^{\partial_bV}\partial_k\hat{W}_{ab} \\
&\quad + \frac{8(d-1)^2(d+2)}{d(d-2)^3(d+4)}\sigma_{kk}\Psi_{ijb}^{\partial_aV}\partial_bV_a + \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}\sigma_{ak}\Psi_{ijb}^{\partial_kV}\partial_bV_a + \frac{8(d-1)^2(d+2)}{d(d-2)^3(d+4)}V_a\Psi_{ijk}^{\partial_aV}\partial_k\sigma_{bb} \\
&\quad + \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_a\Psi_{ijk}^{\partial_bV}\partial_k\sigma_{ab} + \frac{8(d-5)(d-3)(d-1)^2(d+2)}{d(d-2)^4(d+4)}\sigma_a\Psi_{ijb}^{\partial_aV}\partial_bK \\
&\quad \left.+ \frac{8(d-5)(d-3)(d-1)^2(d+2)}{d(d-2)^4(d+4)}K\Psi_{ijb}^{\partial_aV}\partial_b\sigma_a,\right. \\
\text{VIIC}_{2\text{PN}} &= -\frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)}r^2\hat{x}_{ija}\sigma_a, \\
\text{VIIC}_{3\text{PN}} &= -\frac{4(d-1)(d+2)}{d(d-2)^2(d+4)(d+6)}r^2V\hat{x}_{ija}\sigma_a, \\
\text{VIIC}_{4\text{PN}} &= \left(-\frac{8(d-1)(d+2)}{d(d-2)^3(d+4)(d+6)}V^2 - \frac{2(d-1)(d+2)}{d(d-2)(d+4)(d+6)}\hat{W} + \frac{8(d-3)(d-1)(d+2)}{d(d-2)^3(d+4)(d+6)}K\right)r^2\hat{x}_{ija}\sigma_a, \\
\text{VIIC}_{3\text{PN}} &= -\frac{(d-1)(d+2)}{4d(d-2)(d+4)(d+6)(d+8)}r^4\hat{x}_{ija}\sigma_a, \\
\text{VIIC}_{4\text{PN}} &= -\frac{(d-1)(d+2)}{d(d-2)^2(d+4)(d+6)(d+8)}r^4V\hat{x}_{ija}\sigma_a, \\
\text{VIVC}_{4\text{PN}} &= -\frac{(d-1)(d+2)}{24d(d-2)(d+4)(d+6)(d+8)(d+10)}r^6\hat{x}_{ija}\sigma_a, \\
\text{TIC}_{2\text{PN}} &= \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}\hat{x}_{ijab}\sigma_{ab},
\end{aligned}$$

$$\begin{aligned}
\text{TIC}_{3\text{PN}} &= \frac{4(d-1)(d+2)}{d(d-2)^2(d+1)(d+6)} V \hat{x}_{ijab} \sigma_{ab}, \\
\text{TIC}_{4\text{PN}} &= \left(\frac{8(d-1)(d+2)}{d(d-2)^3(d+1)(d+6)} V^2 - \frac{8(d-3)(d-1)(d+2)}{d(d-2)^3(d+1)(d+6)} K + \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)} \hat{W} \right. \\
&\quad \left. - \frac{8(d-3)(d-1)(d+2)}{d(d-2)^3(d+1)(d+6)} K \right) \hat{x}_{ijab} \sigma_{ab}, \\
\text{THIC}_{3\text{PN}} &= \frac{(d-1)(d+2)}{2d(d-2)(d+1)(d+6)(d+8)} r^2 \hat{x}_{ijab} \sigma_{ab}, \\
\text{THIC}_{4\text{PN}} &= \frac{2(d-1)(d+2)}{d(d-2)^2(d+1)(d+6)(d+8)} r^2 V \hat{x}_{ijab} \sigma_{ab}, \\
\text{TTHIC}_{4\text{PN}} &= \frac{(d-1)(d+2)}{8d(d-2)(d+1)(d+6)(d+8)(d+10)} r^4 \hat{x}_{ijab} \sigma_{ab}.
\end{aligned}$$

Next we present the long list of the non-compact (NC) support terms. Remind that all the terms have to be divided by a factor $G\pi$.

$$\begin{aligned}
\text{SINC}_{2\text{PN}} &= \left(\frac{(d-4)(d-1)^2}{8(d-2)^3} (\partial_t V)^2 - \frac{(d-1)}{d-2} V_a \partial_t \partial_a V + \frac{(d-1)}{d-2} \partial_a V_b \partial_b V_a \right) \hat{x}_{ij} + \frac{(d-1)^2}{4(d-2)^2} \Psi_{ij}^{\partial_{ab} V} \partial_a V \partial_b V, \\
\text{SINC}_{3\text{PN}} &= \left(\frac{(d-4)(d-1)^3}{4(d-2)^4} (\partial_t V)^2 V - \frac{(d-1)^2}{(d-2)^2} V V_a \partial_t \partial_a V - \frac{d(d-1)^2}{2(d-2)^3} V_a \partial_t V \partial_a V + \frac{2}{d-2} \partial_t V_a \partial_t V_a \right. \\
&\quad - \frac{(d-3)}{d-2} V_a \partial_t^2 V_a - \frac{2(d-1)^2}{(d-2)^2} V_a \partial_b V_a \partial_b V + \frac{(d-1)}{2(d-2)} \partial_t V \partial_t \hat{W} + \frac{(d-1)}{4(d-2)} \hat{W} \partial_t^2 V + \frac{(d-1)}{4(d-2)} V \partial_t^2 \hat{W} \\
&\quad + \frac{2(d-1)}{d-2} \partial_t \hat{W}_{ab} \partial_b V_a - \frac{(d-4)(d-3)(d-1)^2}{2(d-2)^4} \partial_t K \partial_t V + \frac{2(d-3)(d-1)}{(d-2)^2} V_a \partial_t \partial_a K \\
&\quad + \frac{(d-3)(d-1)}{(d-2)^2} \hat{W}_{ab} \partial_{ba} K \left. \right) \hat{x}_{ij} - \frac{d(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_a V_b} \partial_t \partial_b V \partial_a V - \frac{d(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_a V_b} \partial_t V \partial_{ba} V \\
&\quad - \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_k} \partial_b V_a \partial_{ka} V - \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_k} \partial_a V \partial_{kb} V_a + \frac{d(d-1)^2}{2(d-2)^3} \Psi_{ij}^{\partial_{ta} V} \partial_t V \partial_a V \\
&\quad + \frac{2(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{tb} V} \partial_a V \partial_b V_a + \frac{d(d-1)^2}{4(d-2)^4} \Psi_{ij}^{\partial_{aa} V} (\partial_t V)^2 + \frac{2(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{ab} V} \partial_t V_b \partial_a V \\
&\quad - \frac{2(d-1)}{(d-2)^2} \Psi_{ij}^{\partial_{bb} V} \partial_a V_k \partial_k V_a + \frac{2(d-1)}{(d-2)^2} \Psi_{ij}^{\partial_{bb} V} \partial_a V_k \partial_a V_k - \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{ak} V} \partial_b V_k \partial_b V_a \\
&\quad - \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{bk} V} \partial_b V_a \partial_k V_a + \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{ak} V} \partial_b V_a \partial_k V_b - \frac{(d-1)}{2(d-2)} \Psi_{ij}^{\partial_{ab} V} \partial_t^2 \hat{W}_{ab} \\
&\quad - \frac{(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_{ab} V} \partial_a K \partial_b V, \\
\text{SINC}_{4\text{PN}} &= \left(\frac{(d-1)^3}{4(d-2)^3} (\partial_t V)^2 V^2 - \frac{(d-1)^3}{4(d-2)^3} V^3 \partial_t^2 V - \frac{(d-1)^2(2d-3)}{(d-2)^3} V^2 V_a \partial_t \partial_a V \right. \\
&\quad - \frac{d(d-1)^2(2d-5)}{2(d-2)^4} V V_a \partial_t V \partial_a V + \frac{(d-1)^3}{(d-2)^3} V^2 \partial_t V_a \partial_a V - \frac{2(d-1)(d^2-5d+8)}{(d-2)^3} V_a \partial_t V \partial_t V_a \\
&\quad + \frac{4(d-1)}{(d-2)^2} V \partial_t V_a \partial_t V_a - \frac{2(d-3)(d-1)}{(d-2)^2} V V_a \partial_t^2 V_a - \frac{(d-1)^2}{(d-2)^3} V_a V_b \partial_a V \partial_b V \\
&\quad + \frac{6(d-1)^2}{(d-2)^3} V V_a \partial_a V_b \partial_b V - \frac{(d-3)(d-1)}{2(d-2)^2} V_a V_a \partial_b V \partial_b V + \frac{(d-1)^2(3d-7)}{(d-2)^3} V^2 \partial_a V_b \partial_b V_a \\
&\quad - \frac{(d-4)(d-1)^2}{(d-2)^3} V^2 \partial_b V_a \partial_b V_a - \frac{8(d-1)}{d-2} V_a \partial_t V_b \partial_b V_a + \frac{(d-1)^2}{2(d-2)^2} (\partial_t V)^2 \hat{W} + \frac{(d-1)^2}{(d-2)^2} V \partial_t V \partial_t \hat{W} \\
&\quad + \frac{(d-1)^2}{2(d-2)^2} V \hat{W} \partial_t^2 V + \frac{(d-1)^2}{4(d-2)^2} V^2 \partial_t^2 \hat{W} - \frac{(d-1)^2}{2(d-2)^2} V^2 \hat{W}_{ab} \partial_{ba} V - \frac{d(d-1)^2}{4(d-2)^3} V \hat{W}_{ab} \partial_a V \partial_b V \\
&\quad + \frac{2(d-1)}{d-2} V_a \hat{W}_{ab} \partial_t \partial_b V + \frac{2(d-1)^2}{(d-2)^2} V \partial_t \hat{W}_{ab} \partial_b V_a + \frac{4(d-3)}{d-2} V_a \partial_b \hat{W} \partial_b V_a + \frac{2(d-3)}{d-2} \hat{W} \partial_b V_a \partial_b V_a \\
&\quad - \frac{4(d-1)}{d-2} \hat{W}_{ai} \partial_b V_a \partial_i V_b + \frac{4(d-1)}{d-2} V_a \partial_a \hat{W}_{bi} \partial_i V_b - \frac{4(d-1)}{d-2} V_a \partial_i \hat{W}_{ab} \partial_i V_b + \frac{(d-3)}{2(d-2)} \partial_t \hat{W}_{ab} \partial_t \hat{W}_{ab} \\
&\quad + \frac{1}{2(d-2)} (\partial_t \hat{W})^2 - \frac{1}{d-2} \hat{W}_{ab} \partial_t^2 \hat{W}_{ab} + \frac{1}{2(d-2)} \hat{W} \partial_t^2 \hat{W} + \frac{d(d-1)}{(d-2)^2} \hat{W}_{bi} \partial_a \hat{W}_{bi} \partial_a V
\end{aligned}$$

$$\begin{aligned}
& + \frac{(d-1)}{d-2} \hat{W}_{ai} \hat{W}_{ab} \partial_{ib} V + \frac{d(d-1)}{2(d-2)^2} V \partial_i \hat{W}_{ab} \partial_i \hat{W}_{ab} - \frac{2(d-1)^2}{(d-2)^2} \hat{R}_a V \partial_t \partial_a V - \frac{d(d-1)^2}{(d-2)^3} \hat{R}_a \partial_t V \partial_a V \\
& + \frac{8}{d-2} \partial_t \hat{R}_a \partial_t V_a - \frac{2(d-3)}{d-2} V_a \partial_t^2 \hat{R}_a - \frac{2(d-3)}{d-2} \hat{R}_a \partial_t^2 V_a - \frac{4(d-1)}{d-2} \hat{R}_a V_b \partial_{ba} V \\
& - \frac{12(d-1)}{(d-2)^2} V_a \partial_b V \partial_b \hat{R}_a + \frac{4(d-4)(d-1)}{(d-2)^2} V \partial_b V_a \partial_b \hat{R}_a - \frac{12(d-1)}{(d-2)^2} \hat{R}_a \partial_b V_a \partial_b V \\
& + \frac{4(d-1)}{d-2} \partial_t \hat{W}_{ab} \partial_b \hat{R}_a + \frac{4(d-1)}{d-2} \partial_a \hat{R}_b \partial_b \hat{R}_a + \frac{(d-4)(d-1)^2}{(d-2)^3} \partial_t V \partial_t \hat{X} - \frac{(d-1)^2}{(d-2)^2} \hat{X} \partial_a V \partial_a V \\
& - \frac{2(d-1)^2}{(d-2)^2} V \partial_a \hat{X} \partial_a V - \frac{4(d-1)}{d-2} V_a \partial_t \partial_a \hat{X} - \frac{2(d-1)}{d-2} \hat{W}_{ab} \partial_{ba} \hat{X} + \frac{2(d-1)}{d-2} \partial_t V \partial_t \hat{Z} \\
& + \frac{(d-1)}{d-2} \hat{Z} \partial_t^2 V + \frac{(d-1)}{d-2} V \partial_t^2 \hat{Z} - \frac{2(d-1)^2}{(d-2)^2} \hat{Z} \partial_a V \partial_a V - \frac{4(d-1)^2}{(d-2)^2} V \partial_a \hat{Z} \partial_a V \\
& + \frac{8(d-1)}{d-2} \partial_t \hat{Z}_{ab} \partial_b V_a - \frac{(d-3)^2 (d-1)^2 (3d-4)}{2(d-2)^5} V \partial_a K \partial_a K - \frac{(d-3)^2 (d-1)^2 (3d-4)}{(d-2)^5} K \partial_a V \partial_a K \\
& - \frac{(d-4)(d-3)(d-1)^3}{2(d-2)^5} K (\partial_t V)^2 - \frac{(d-4)(d-3)(d-1)^3}{(d-2)^5} V \partial_t K \partial_t V \\
& + \frac{(d-4)(d-3)^2 (d-1)^2}{2(d-2)^5} (\partial_t K)^2 + \frac{2(d-3)(d-1)^2}{(d-2)^3} V V_a \partial_t \partial_a K + \frac{2(d-3)(d-1)^2}{(d-2)^3} K V_a \partial_t \partial_a V \\
& + \frac{d(d-3)(d-1)^2}{(d-2)^4} V_a \partial_t V \partial_a K + \frac{d(d-3)(d-1)^2}{(d-2)^4} V_a \partial_t K \partial_a V - \frac{2(d-3)(d-1)^2}{(d-2)^3} K \partial_b V_a \partial_b V_a \\
& - \frac{(d-3)(d-1)}{(d-2)^2} \partial_t K \partial_t \hat{W} - \frac{(d-3)(d-1)}{2(d-2)^2} \hat{W} \partial_t^2 K - \frac{(d-3)(d-1)}{2(d-2)^2} K \partial_t^2 \hat{W} \\
& + \frac{2(d-3)(d-1)^2}{(d-2)^3} \hat{W} \partial_a V \partial_a K + \frac{2(d-3)(d-1)^2}{(d-2)^3} V \partial_a \hat{W} \partial_a K + \frac{2(d-3)(d-1)^2}{(d-2)^3} K \partial_a \hat{W} \partial_a V \\
& + \frac{4(d-3)(d-1)}{(d-2)^2} \hat{R}_a \partial_t \partial_a K + \frac{4(d-3)(d-1)}{(d-2)^2} \hat{Z}_{ab} \partial_{ba} K \hat{x}_{ij} - \frac{d(d-1)^3}{(d-2)^4} V \Psi_{ij}^{\partial_a V_b} \partial_t \partial_b V \partial_a V \\
& - \frac{d(d-1)^3}{(d-2)^4} \Psi_{ij}^{\partial_a V_b} \partial_t V \partial_a V \partial_b V - \frac{d(d-1)^3}{(d-2)^4} V \Psi_{ij}^{\partial_a V_b} \partial_t V \partial_{ba} V + \frac{4(d-1)^2}{(d-2)^2} V \Psi_{ij}^{\partial_a V_b} \partial_t^2 \partial_b V_a \\
& + \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_a} \partial_t^2 V_b \partial_a V - \frac{2d(d-1)^2}{(d-2)^3} V_a \Psi_{ij}^{\partial_b V_k} \partial_b V \partial_{ka} V - \frac{2d(d-1)^2}{(d-2)^3} V_b \Psi_{ij}^{\partial_k V_a} \partial_b V \partial_{ka} V \\
& - \frac{2d(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_b V_k} \partial_a V \partial_b V \partial_k V_a - \frac{4(d-1)^3}{(d-2)^3} \Psi_{ij}^{\partial_k V_b} \partial_a V_k \partial_a V \partial_b V - \frac{4(d-1)^3}{(d-2)^3} V \Psi_{ij}^{\partial_b V_k} \partial_a V \partial_{ka} V_b \\
& - \frac{4(d-1)^3}{(d-2)^3} V \Psi_{ij}^{\partial_a V_k} \partial_b V_a \partial_{kb} V + \frac{16(d-1)}{d-2} V_a \Psi_{ij}^{\partial_b V_k} \partial_t \partial_{ka} V_b + \frac{16(d-1)}{d-2} \Psi_{ij}^{\partial_k V_b} \partial_t \partial_a V_k \partial_b V_a \\
& - \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_k} \partial_t \partial_k \hat{W}_{ab} \partial_a V - \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_a V_b} \partial_t \hat{W}_{ak} \partial_{kb} V + \frac{8(d-1)}{d-2} \hat{W}_{ab} \Psi_{ij}^{\partial_k V_l} \partial_{lb} V_k \\
& + \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_k V_l} \partial_b V_a \partial_{lk} \hat{W}_{ab} - \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_k V_l} \partial_b V_a \partial_{la} \hat{W}_{bk} + \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_a V_b} \partial_a \hat{W}_{kl} \partial_{lb} V_k \\
& - \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_a V_b} \partial_k \hat{W}_{al} \partial_{lb} V_k + \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_a V_b} \partial_b \hat{W}_{kl} \partial_{lk} V_a + \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_a V_b} \partial_t^2 \partial_b \hat{R}_a \\
& - \frac{8(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_k} \partial_b \hat{R}_a \partial_{ka} V - \frac{8(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_b V_k} \partial_a V \partial_{kb} \hat{R}_a + \frac{d(d-1)^3}{2(d-2)^4} V \Psi_{ij}^{\partial_{ta} V} \partial_t V \partial_a V \\
& - \frac{2(d-1)^2}{(d-2)^2} V \Psi_{ij}^{\partial_{ta} V} \partial_t^2 V_a + \frac{d(d-1)^2}{(d-2)^3} V_b \Psi_{ij}^{\partial_{ta} V} \partial_a V \partial_b V + \frac{2(d-1)^3}{(d-2)^3} V \Psi_{ij}^{\partial_{tb} V} \partial_a V_b \partial_a V \\
& - \frac{8(d-1)}{d-2} V_a \Psi_{ij}^{\partial_{tb} V} \partial_t \partial_a V_b + \frac{2(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{tb} V} \partial_t \hat{W}_{ab} \partial_a V - \frac{4(d-1)}{d-2} \hat{W}_{ab} \Psi_{ij}^{\partial_{tk} V} \partial_{ba} V_k \\
& - \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{tk} V} \partial_b V_a \partial_k \hat{W}_{ab} + \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{tk} V} \partial_a \hat{W}_{bk} \partial_b V_a - \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{ta} V} \partial_t^2 \hat{R}_a \\
& + \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{tb} V} \partial_a V \partial_b \hat{R}_a - \frac{d(d-1)^3}{8(d-2)^4} V \Psi_{ij}^{\partial_{aa} V} (\partial_t V)^2 + \frac{(d-1)^3}{4(d-2)^3} V^2 \Psi_{ij}^{\partial_{aa} V} \partial_t^2 V \\
& + \frac{(d-1)^2}{(d-2)^2} V V_b \Psi_{ij}^{\partial_{aa} V} \partial_t \partial_b V + \frac{(d-1)^2}{2(d-2)^2} V_b \Psi_{ij}^{\partial_{aa} V} \partial_t V \partial_b V - \frac{(d-1)^3}{(d-2)^3} V \Psi_{ij}^{\partial_{aa} V} \partial_t V_b \partial_b V
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(d-1)^3}{(d-2)^3} V \Psi_{ij}^{\partial_{ab}V} \partial_t V_b \partial_a V - \frac{(d-1)^2}{(d-2)^2} V_a \Psi_{ij}^{\partial_{ab}V} \partial_t V \partial_b V + \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{ab}V} \partial_t V_a \partial_t V_b \\
& + \frac{4(d-1)^2}{(d-2)^2} V_a \Psi_{ij}^{\partial_{bk}V} \partial_a V_k \partial_b V - \frac{4(d-1)^2}{(d-2)^2} V_a \Psi_{ij}^{\partial_{ak}V} \partial_b V_k \partial_b V - \frac{(d-1)^2}{(d-2)^2} V \Psi_{ij}^{\partial_{bk}V} \partial_a V_k \partial_k V_a \\
& - \frac{(d-1)^2}{(d-2)^2} V \Psi_{ij}^{\partial_{ab}V} \partial_t^2 \hat{W}_{ab} + \frac{(d-1)^2}{2(d-2)^2} V \Psi_{ij}^{\partial_{aa}V} \partial_t^2 \hat{W} + \frac{(d-1)^2}{2(d-2)^2} V \hat{W}_{ka} \Psi_{ij}^{\partial_{bb}V} \partial_{ka} V \\
& - \frac{(d-1)^2}{(d-2)^2} \hat{W}_{ak} \Psi_{ij}^{\partial_{bk}V} \partial_a V \partial_b V + \frac{(d-1)^2}{4(d-2)^2} \hat{W}_{ka} \Psi_{ij}^{\partial_{bb}V} \partial_a V \partial_k V - \frac{4(d-1)}{d-2} V_a \Psi_{ij}^{\partial_{bk}V} \partial_t \partial_a \hat{W}_{bk} \\
& + \frac{2(d-1)}{d-2} V_b \Psi_{ij}^{\partial_{aa}V} \partial_t \partial_b \hat{W} + \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{bb}V} \partial_t \hat{W}_{ka} \partial_a V_k - \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{bk}V} \partial_t \hat{W}_{ak} \partial_b V_a \\
& + \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{ak}V} \partial_t \hat{W}_{bk} \partial_b V_a + \frac{(d-1)}{d-2} \Psi_{ij}^{\partial_{kl}V} \partial_k \hat{W}_{ab} \partial_l \hat{W}_{ab} - \frac{2(d-1)}{d-2} \hat{W}_{ab} \Psi_{ij}^{\partial_{kl}V} \partial_{ba} \hat{W}_{kl} \\
& + \frac{(d-1)}{d-2} \hat{W}_{ka} \Psi_{ij}^{\partial_{bb}V} \partial_{ka} \hat{W} - \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{bl}V} \partial_k \hat{W}_{ab} \partial_l \hat{W}_{ak} + \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{bl}V} \partial_a \hat{W}_{kl} \partial_k \hat{W}_{ab} \\
& + \frac{(d-1)}{d-2} \Psi_{ij}^{\partial_{il}V} \partial_b \hat{W}_{ak} \partial_k \hat{W}_{ab} - \frac{2(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{aa}V} \partial_t \hat{R}_b \partial_b V + \frac{4(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{ab}V} \partial_t \hat{R}_b \partial_a V \\
& - \frac{4(d-1)}{d-2} \Psi_{ij}^{\partial_{bb}V} \partial_a \hat{R}_k \partial_k V_a + \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_{bk}V} \partial_a V_k \partial_b \hat{R}_a - \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_{bk}V} \partial_b \hat{R}_a \partial_k V_a \\
& + \frac{8(d-1)}{d-2} \Psi_{ij}^{\partial_{ak}V} \partial_b \hat{R}_a \partial_k V_b + \frac{2(d-1)^2}{(d-2)^2} \Psi_{ij}^{\partial_{ab}V} \partial_a V \partial_b \hat{X} - \frac{2(d-1)}{d-2} \Psi_{ij}^{\partial_{ab}V} \partial_t^2 \hat{Z}_{ab} \\
& + \frac{2d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_a V_b} \partial_t \partial_b V \partial_a K + \frac{2d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_a V_b} \partial_t \partial_b K \partial_a V \\
& + \frac{2d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_a V_b} \partial_t V \partial_{ba} K + \frac{2d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_a V_b} \partial_t K \partial_{ba} V \\
& + \frac{8(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_b V_k} \partial_b V_a \partial_{ka} K + \frac{8(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_b V_k} \partial_a K \partial_{kb} V_a \\
& - \frac{d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_{ta}V} \partial_t V \partial_a K - \frac{d(d-3)(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_{ta}V} \partial_t K \partial_a V \\
& - \frac{4(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_{tb}V} \partial_a K \partial_b V_a + \frac{(d-3)^2(d-1)^2}{(d-2)^4} \Psi_{ij}^{\partial_{ab}V} \partial_a K \partial_b K \\
& + \frac{d(d-3)(d-1)^2}{2(d-2)^4} \Psi_{ij}^{\partial_{aa}V} \partial_t K \partial_t V + \frac{2(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_{aa}V} \partial_t V_b \partial_b K \\
& - \frac{4(d-3)(d-1)^2}{(d-2)^3} \Psi_{ij}^{\partial_{ab}V} \partial_t V_b \partial_a K,
\end{aligned}$$

$$SIINC_{2PN} = -\frac{(d-1)^2}{8(d-2)^2(d+4)} r^2 \hat{x}_{ij} \partial_a V \partial_a V,$$

$$\begin{aligned}
SIINC_{3PN} = & \left(\frac{(d-4)(d-1)^2}{16(d-2)^3(d+4)} (\partial_t V)^2 - \frac{(d-1)^2}{8(d-2)^2(d+4)} V \partial_t^2 V - \frac{(d-1)^2(3d-2)}{16(d-2)^3(d+4)} V \partial_a V \partial_a V \right. \\
& - \frac{(d-1)}{2(d-2)(d+4)} V_a \partial_t \partial_a V + \frac{(d-1)}{2(d-2)(d+4)} \partial_a V_b \partial_b V_a + \frac{(d-3)}{2(d-2)(d+4)} \partial_b V_a \partial_b V_a \\
& \left. - \frac{(d-1)}{4(d-2)(d+4)} \partial_a \hat{W} \partial_a V - \frac{(d-1)}{4(d-2)(d+4)} \hat{W}_{ab} \partial_{ba} V + \frac{(d-3)(d-1)^2}{2(d-2)^3(d+4)} \partial_a V \partial_a K \right) r^2 \hat{x}_{ij},
\end{aligned}$$

$$\begin{aligned}
SIINC_{4PN} = & \left(\frac{(d-1)^2(d^2-8d+8)}{16(d-2)^4(d+4)} (\partial_t V)^2 V - \frac{3(d-1)^3}{8(d-2)^3(d+4)} V^2 \partial_t^2 V - \frac{(d-1)^3(3d-2)}{16(d-2)^4(d+4)} V^2 \partial_a V \partial_a V \right. \\
& - \frac{3(d-1)^2}{2(d-2)^2(d+4)} V V_a \partial_t \partial_a V - \frac{d(d-1)}{2(d-2)^2(d+4)} V_a \partial_t V \partial_a V + \frac{(d-1)^2}{2(d-2)^2(d+4)} V \partial_t V_a \partial_a V \\
& + \frac{1}{(d-2)(d+4)} \partial_t V_a \partial_t V_a - \frac{(d-3)(d-1)}{(d-2)^2(d+4)} V_a \partial_a V_b \partial_b V + \frac{(d-5)(d-1)}{(d-2)^2(d+4)} V_a \partial_b V_a \partial_b V \\
& + \frac{3(d-1)}{2(d-2)(d+4)} V \partial_a V_b \partial_b V_a + \frac{(d-1)}{2(d-2)(d+4)} V \partial_b V_a \partial_b V_a + \frac{(d-1)}{4(d-2)(d+4)} \partial_t V \partial_t \hat{W} \\
& \left. - \frac{(d-1)(2d-3)}{8(d-2)^2(d+4)} \hat{W} \partial_a V \partial_a V - \frac{(d-1)^2}{2(d-2)^2(d+4)} V \partial_a \hat{W} \partial_a V - \frac{(d-1)^2}{2(d-2)^2(d+4)} V \hat{W}_{ab} \partial_{ba} V \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{(d-1)}{4(d-2)^2(d+4)}\hat{W}_{ab}\partial_a V\partial_b V + \frac{(d-1)}{(d-2)(d+4)}\partial_t\hat{W}_{ab}\partial_b V_a - \frac{1}{4(d-2)(d+4)}\partial_a\hat{W}\partial_a\hat{W} \\
& + \frac{1}{2(d-2)(d+4)}\partial_i\hat{W}_{ab}\partial_i\hat{W}_{ab} - \frac{(d-1)}{(d-2)(d+4)}\hat{R}_a\partial_t\partial_a V + \frac{2(d-1)}{(d-2)(d+4)}\partial_a V_b\partial_b\hat{R}_a \\
& + \frac{2(d-3)}{(d-2)(d+4)}\partial_b V_a\partial_b\hat{R}_a - \frac{(d-1)^2}{(d-2)^2(d+4)}\partial_a\hat{X}\partial_a V - \frac{(d-1)}{(d-2)(d+4)}\partial_a\hat{Z}\partial_a V \\
& - \frac{(d-1)}{(d-2)(d+4)}\hat{Z}_{ab}\partial_{ba}V - \frac{(d-4)(d-3)(d-1)^2}{4(d-2)^4(d+4)}\partial_t K\partial_t V + \frac{(d-3)(d-1)^2}{4(d-2)^3(d+4)}V\partial_t^2 K \\
& + \frac{(d-3)(d-1)^2}{4(d-2)^3(d+4)}K\partial_t^2 V + \frac{(d-3)(d-1)^2(3d-2)}{4(d-2)^4(d+4)}V\partial_a V\partial_a K \\
& + \frac{(d-3)(d-1)^2(3d-2)}{8(d-2)^4(d+4)}K\partial_a V\partial_a V - \frac{(d-3)^2(d-1)^2}{2(d-2)^4(d+4)}\partial_a K\partial_a K + \frac{(d-3)(d-1)}{(d-2)^2(d+4)}V_a\partial_t\partial_a K \\
& + \frac{(d-3)(d-1)}{2(d-2)^2(d+4)}\partial_a\hat{W}\partial_a K + \frac{(d-3)(d-1)}{2(d-2)^2(d+4)}\hat{W}_{ab}\partial_{ba}K)r^2\hat{x}_{ij}, \\
\text{SHINC}_{3\text{PN}} = & -\frac{(d-1)^2}{32(d-2)^2(d+4)(d+6)}r^4\hat{x}_{ij}\partial_a V\partial_a V, \\
\text{SHINC}_{4\text{PN}} = & \left(\frac{(d-4)(d-1)^2}{64(d-2)^3(d+4)(d+6)}(\partial_t V)^2 - \frac{(d-1)^2}{32(d-2)^2(d+4)(d+6)}V\partial_t^2 V \right. \\
& - \frac{(d-1)^2(3d-2)}{64(d-2)^3(d+4)(d+6)}V\partial_a V\partial_a V - \frac{(d-1)}{8(d-2)(d+4)(d+6)}V_a\partial_t\partial_a V \\
& + \frac{(d-1)}{8(d-2)(d+4)(d+6)}\partial_a V_b\partial_b V_a + \frac{(d-3)}{8(d-2)(d+4)(d+6)}\partial_b V_a\partial_b V_a \\
& - \frac{(d-1)}{16(d-2)(d+4)(d+6)}\partial_a\hat{W}\partial_a V - \frac{(d-1)}{16(d-2)(d+4)(d+6)}\hat{W}_{ab}\partial_{ba}V \\
& \left. + \frac{(d-3)(d-1)^2}{8(d-2)^3(d+4)(d+6)}\partial_a V\partial_a K\right)r^4\hat{x}_{ij}, \\
\text{SIVNC}_{4\text{PN}} = & -\frac{(d-1)^2}{192(d-2)^2(d+4)(d+6)(d+8)}r^6\hat{x}_{ij}\partial_a V\partial_a V, \\
\text{VINC}_{2\text{PN}} = & -\left(\frac{(d-1)^2(d+2)}{4(d-2)^3(d+4)}\partial_t V\partial_a V + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_a V_b\partial_b V\right)\hat{x}_{ija}, \\
\text{VINC}_{3\text{PN}} = & \left(-\frac{(d-1)^3(d+2)}{4(d-2)^4(d+4)}V\partial_t V\partial_a V - \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_t V\partial_t V_a - \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)}V_a\partial_t^2 V \right. \\
& + \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)}V\partial_t^2 V_a - \frac{(d-1)^2(d+2)}{2(d-2)^3(d+4)}V_b\partial_a V\partial_b V + \frac{(d-1)^3(d+2)}{2d(d-2)^3(d+4)}V_a\partial_b V\partial_b V \\
& + \frac{4(d-1)(d+2)}{d(d-2)(d+4)}V_b\partial_t\partial_b V_a - \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_t\hat{W}_{ab}\partial_b V + \frac{2(d-1)(d+2)}{d(d-2)(d+4)}\hat{W}_{bi}\partial_{ib}V_a \\
& + \frac{2(d-1)(d+2)}{d(d-2)(d+4)}\partial_a\hat{W}_{bi}\partial_i V_b - \frac{2(d-1)(d+2)}{d(d-2)(d+4)}\partial_b\hat{W}_{ai}\partial_i V_b + \frac{(d-3)(d-1)^2(d+2)}{2(d-2)^4(d+4)}\partial_t V\partial_a K \\
& + \frac{(d-3)(d-1)^2(d+2)}{2(d-2)^4(d+4)}\partial_t K\partial_a V + \frac{2(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\partial_a V_b\partial_b K\left.\right)\hat{x}_{ija} \\
& + \frac{(d-1)^3(d+2)}{2(d-2)^4(d+4)}\Psi_{ijb}^{\partial_a V}\partial_t\partial_b V\partial_a V + \frac{(d-1)^3(d+2)}{2(d-2)^4(d+4)}\Psi_{ija}^{\partial_b V}\partial_t V\partial_{ba}V \\
& + \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}\Psi_{ijk}^{\partial_b V}\partial_b V_a\partial_{ka}V + \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}\Psi_{ijb}^{\partial_k V}\partial_a V\partial_{kb}V_a, \\
\text{VINC}_{4\text{PN}} = & \left(-\frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)}(\partial_t V)^2 V_a - \frac{(d-1)^3(d+2)}{d(d-2)^3(d+4)}V\partial_t V\partial_t V_a - \frac{(d-1)^3(d+2)}{2d(d-2)^3(d+4)}VV_a\partial_t^2 V \right. \\
& + \frac{3(d-1)^3(d+2)}{4d(d-2)^3(d+4)}V^2\partial_t^2 V_a - \frac{(d-1)^4(d+2)}{d(d-2)^4(d+4)}VV_b\partial_a V\partial_b V + \frac{2(d-1)^4(d+2)}{d(d-2)^4(d+4)}VV_a\partial_b V\partial_b V \\
& + \frac{2(d-1)^4(d+2)}{d(d-2)^4(d+4)}V^2\partial_b V_a\partial_b V + \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}VV_b\partial_t\partial_b V_a + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_b\partial_t V_a\partial_b V \\
& - \frac{4(d-1)^2(d+2)}{d(d-2)^3(d+4)}V_b\partial_t V_b\partial_a V + \frac{2(d-4)(d-1)^2(d+2)}{d(d-2)^3(d+4)}V_b\partial_t V\partial_a V_b + \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_a\partial_t V_b\partial_b V
\end{aligned}$$

$$\begin{aligned}
& -\frac{4(d-1)(d+2)}{d(d-2)(d+4)}V_b\partial_aV_i\partial_bV_i - \frac{4(d-1)(d+2)}{(d-2)^2(d+4)}V_b\partial_aV_i\partial_iV_b + \frac{4(d-1)(d+2)}{d(d-2)(d+4)}V_b\partial_bV_i\partial_iV_a \\
& + \frac{4(d-1)(d+2)}{d(d-2)(d+4)}V_a\partial_bV_i\partial_iV_b - \frac{2(d-1)(d+2)}{d(d-2)(d+4)}V_a\partial_iV_b\partial_iV_b + \frac{(d-1)^2(d+2)}{2(d-2)^3(d+4)}\hat{W}_{ab}\partial_tV\partial_bV \\
& - \frac{(d-1)^3(d+2)}{d(d-2)^3(d+4)}V\partial_t\hat{W}_{ab}\partial_bV - \frac{2(d-1)(d+2)}{d(d-2)(d+4)}\partial_tV_a\partial_t\hat{W} - \frac{(d-1)(d+2)}{d(d-2)(d+4)}\hat{W}\partial_t^2V_a \\
& + \frac{(d-1)(d+2)}{d(d-2)(d+4)}\hat{W}_{ab}\partial_t^2V_b + \frac{(d-1)(d+2)}{d(d-2)(d+4)}V_b\partial_t^2\hat{W}_{ab} - \frac{(d-1)(d+2)}{d(d-2)(d+4)}V_a\partial_t^2\hat{W} \\
& + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{W}_{bi}\partial_aV_i\partial_bV - \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_b\partial_b\hat{W}_{ai}\partial_iV + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_b\partial_i\hat{W}_{ab}\partial_iV \\
& + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{W}_{ai}\partial_bV\partial_iV_b + \frac{(d-1)(d+2)}{d(d-2)(d+4)}\partial_t\hat{W}_{bi}\partial_a\hat{W}_{bi} - \frac{2(d-1)(d+2)}{d(d-2)(d+4)}\partial_t\hat{W}_{bi}\partial_i\hat{W}_{ab} \\
& - \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_t\hat{R}_a\partial_tV + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)}V\partial_t^2\hat{R}_a - \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{R}_a\partial_t^2V \\
& - \frac{(d-1)^2(d+2)}{(d-2)^3(d+4)}\hat{R}_b\partial_aV\partial_bV - \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}V\partial_bV\partial_b\hat{R}_a + \frac{8(d-1)(d+2)}{d(d-2)(d+4)}V_b\partial_t\partial_b\hat{R}_a \\
& + \frac{8(d-1)(d+2)}{d(d-2)(d+4)}\hat{R}_b\partial_t\partial_bV_a + \frac{4(d-1)(d+2)}{d(d-2)(d+4)}\hat{W}_{bi}\partial_{ib}\hat{R}_a + \frac{4(d-1)(d+2)}{d(d-2)(d+4)}\partial_a\hat{W}_{bi}\partial_i\hat{R}_b \\
& - \frac{4(d-1)(d+2)}{d(d-2)(d+4)}\partial_b\hat{W}_{ai}\partial_i\hat{R}_b - \frac{(d-1)^2(d+2)}{(d-2)^3(d+4)}\partial_t\hat{X}\partial_aV - \frac{(d-1)^2(d+2)}{(d-2)^3(d+4)}\partial_tV\partial_a\hat{X} \\
& - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_aV_b\partial_b\hat{X} - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}\partial_t\hat{Z}_{ab}\partial_bV + \frac{8(d-1)(d+2)}{d(d-2)(d+4)}\hat{Z}_{bi}\partial_{ib}V_a \\
& + \frac{8(d-1)(d+2)}{d(d-2)(d+4)}\partial_a\hat{Z}_{bi}\partial_iV_b - \frac{8(d-1)(d+2)}{d(d-2)(d+4)}\partial_b\hat{Z}_{ai}\partial_iV_b + \frac{(d-3)(d-1)^3(d+2)}{2(d-2)^5(d+4)}V\partial_tV\partial_aK \\
& + \frac{(d-3)(d-1)^3(d+2)}{2(d-2)^5(d+4)}V\partial_tK\partial_aV + \frac{(d-3)(d-1)^3(d+2)}{2(d-2)^5(d+4)}K\partial_tV\partial_aV \\
& - \frac{(d-3)^2(d-1)^2(d+2)}{(d-2)^5(d+4)}\partial_tK\partial_aK + \frac{2(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\partial_tK\partial_iV_a \\
& + \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}V_a\partial_t^2K - \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}K\partial_t^2V_a \\
& + \frac{(d-3)(d-1)^2(d+2)}{(d-2)^4(d+4)}V_b\partial_aK\partial_bV + \frac{(d-3)(d-1)^2(d+2)}{(d-2)^4(d+4)}V_b\partial_aV\partial_bK \\
& - \frac{2(d-3)(d-1)^3(d+2)}{d(d-2)^4(d+4)}V_a\partial_bV\partial_bK + \frac{2(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\partial_t\hat{W}_{ab}\partial_bK \\
& + \frac{4(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)}\partial_a\hat{R}_b\partial_bK\Big)\hat{x}_{ija} + \frac{(d-1)^4(d+2)}{2(d-2)^5(d+4)}V\Psi_{ijb}^{\partial_aV}\partial_t\partial_bV\partial_aV \\
& + \frac{(d-1)^4(d+2)}{2(d-2)^5(d+4)}\Psi_{ija}^{\partial_bV}\partial_tV\partial_aV\partial_bV + \frac{(d-1)^4(d+2)}{2(d-2)^5(d+4)}V\Psi_{ija}^{\partial_bV}\partial_tV\partial_baV \\
& - \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}V\Psi_{ijb}^{\partial_aV}\partial_t^2\partial_bV_a - \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}\Psi_{ija}^{\partial_bV}\partial_t^2V_b\partial_aV \\
& + \frac{(d-1)^3(d+2)}{(d-2)^4(d+4)}V_a\Psi_{ijk}^{\partial_bV}\partial_bV\partial_{ka}V + \frac{(d-1)^3(d+2)}{(d-2)^4(d+4)}V_b\Psi_{ijk}^{\partial_aV}\partial_bV\partial_{ka}V \\
& + \frac{(d-1)^3(d+2)}{(d-2)^4(d+4)}\Psi_{ijk}^{\partial_bV}\partial_aV\partial_bV\partial_kV_a + \frac{2(d-1)^4(d+2)}{d(d-2)^4(d+4)}\Psi_{ijb}^{\partial_kV}\partial_aV_k\partial_aV\partial_bV \\
& + \frac{2(d-1)^4(d+2)}{d(d-2)^4(d+4)}V\Psi_{ijk}^{\partial_bV}\partial_aV\partial_{ka}V_b + \frac{2(d-1)^4(d+2)}{d(d-2)^4(d+4)}V\Psi_{ijk}^{\partial_aV}\partial_bV_a\partial_{kb}V \\
& - \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}V_a\Psi_{ijk}^{\partial_bV}\partial_t\partial_{ka}V_b - \frac{8(d-1)^2(d+2)}{d(d-2)^2(d+4)}\Psi_{ijb}^{\partial_kV}\partial_t\partial_aV_k\partial_bV_a \\
& + \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}\Psi_{ijk}^{\partial_bV}\partial_t\partial_k\hat{W}_{ab}\partial_aV + \frac{2(d-1)^3(d+2)}{d(d-2)^3(d+4)}\Psi_{ijb}^{\partial_kV}\partial_t\hat{W}_{ak}\partial_baV \\
& - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}\hat{W}_{ab}\Psi_{ijk}^{\partial_iV}\partial_{kba}V_l - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)}\Psi_{ijk}^{\partial_iV}\partial_bV_a\partial_{lk}\hat{W}_{ab}
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)} \Psi_{ijl}^{\partial_k V} \partial_b V_a \partial_{la} \hat{W}_{bk} - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)} \Psi_{ijk}^{\partial_l V} \partial_{kb} V_a \partial_l \hat{W}_{ab} \\
& + \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)} \Psi_{ijk}^{\partial_l V} \partial_a \hat{W}_{bl} \partial_{kb} V_a - \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)} \Psi_{ijl}^{\partial_a V} \partial_{kb} V_a \partial_l \hat{W}_{bk} \\
& - \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)} \Psi_{ijb}^{\partial_a V} \partial_t^2 \partial_b \hat{R}_a + \frac{4(d-1)^3(d+2)}{d(d-2)^3(d+4)} \Psi_{ijk}^{\partial_b V} \partial_b \hat{R}_a \partial_{ka} V \\
& + \frac{4(d-1)^3(d+2)}{d(d-2)^3(d+4)} \Psi_{ijb}^{\partial_k V} \partial_a V \partial_{kb} \hat{R}_a - \frac{(d-3)(d-1)^3(d+2)}{(d-2)^5(d+4)} \Psi_{ijb}^{\partial_a V} \partial_t \partial_b V \partial_a K \\
& - \frac{(d-3)(d-1)^3(d+2)}{(d-2)^5(d+4)} \Psi_{ijb}^{\partial_a V} \partial_t \partial_b K \partial_a V - \frac{(d-3)(d-1)^3(d+2)}{(d-2)^5(d+4)} \Psi_{ija}^{\partial_b V} \partial_t V \partial_{ba} K \\
& - \frac{(d-3)(d-1)^3(d+2)}{(d-2)^5(d+4)} \Psi_{ija}^{\partial_b V} \partial_t K \partial_{ba} V - \frac{4(d-3)(d-1)^3(d+2)}{d(d-2)^4(d+4)} \Psi_{ijk}^{\partial_b V} \partial_b V_a \partial_{ka} K \\
& - \frac{4(d-3)(d-1)^3(d+2)}{d(d-2)^4(d+4)} \Psi_{ijb}^{\partial_k V} \partial_a K \partial_{kb} V_a, \\
\text{SVIINC}_{3\text{PN}} = & \left(-\frac{(d-1)^2(d+2)}{8(d-2)^3(d+4)(d+6)} \partial_t V \partial_a V - \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)(d+6)} \partial_a V_b \partial_b V \right. \\
& \left. + \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)(d+6)} \partial_b V_a \partial_b V \right) r^2 \hat{x}_{ija}, \\
\text{SVIINC}_{4\text{PN}} = & \left(-\frac{(d-1)^3(d+2)}{4(d-2)^4(d+4)(d+6)} V \partial_t V \partial_a V - \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)(d+6)} \partial_t V \partial_t V_a \right. \\
& + \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)(d+6)} V \partial_t^2 V_a - \frac{(d-1)^2(d+2)}{2d(d-2)^3(d+4)(d+6)} V_b \partial_a V \partial_b V \\
& - \frac{(d-1)^3(d+2)}{2d(d-2)^3(d+4)(d+6)} V \partial_a V_b \partial_b V + \frac{(d-1)^2(d+2)}{4(d-2)^3(d+4)(d+6)} V_a \partial_b V \partial_b V \\
& + \frac{(d-1)^3(d+2)}{2d(d-2)^3(d+4)(d+6)} V \partial_b V_a \partial_b V + \frac{2(d-1)(d+2)}{d(d-2)(d+4)(d+6)} V_b \partial_t \partial_b V_a \\
& - \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)(d+6)} \partial_t \hat{W}_{ab} \partial_b V + \frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)} \partial_b \hat{W} \partial_b V_a \\
& + \frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)} \hat{W}_{bi} \partial_{ib} V_a + \frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)} \partial_a \hat{W}_{bi} \partial_i V_b \\
& - \frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)} \partial_b \hat{W}_{ai} \partial_i V_b - \frac{(d-1)(d+2)}{d(d-2)(d+4)(d+6)} \partial_i \hat{W}_{ab} \partial_i V_b \\
& - \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)(d+6)} \partial_a \hat{R}_b \partial_b V + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)(d+6)} \partial_b V \partial_b \hat{R}_a \\
& + \frac{(d-3)(d-1)^2(d+2)}{4(d-2)^4(d+4)(d+6)} \partial_t V \partial_a K + \frac{(d-3)(d-1)^2(d+2)}{4(d-2)^4(d+4)(d+6)} \partial_t K \partial_a V \\
& \left. + \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)(d+6)} \partial_a V_b \partial_b K - \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)(d+6)} \partial_b V_a \partial_b K \right) r^2 \hat{x}_{ija}, \\
\text{SVIINC}_{4\text{PN}} = & \left(-\frac{(d-1)^2(d+2)}{32(d-2)^3(d+4)(d+6)(d+8)} \partial_t V \partial_a V - \frac{(d-1)^2(d+2)}{8d(d-2)^2(d+4)(d+6)(d+8)} \partial_a V_b \partial_b V \right. \\
& \left. + \frac{(d-1)^2(d+2)}{8d(d-2)^2(d+4)(d+6)(d+8)} \partial_b V_a \partial_b V \right) r^4 \hat{x}_{ija}, \\
\text{VIVC}_{4\text{PN}} = & -\frac{(d-1)(d+2)}{24d(d-2)(d+4)(d+6)(d+8)(d+10)} r^6 \hat{x}_{ija} \sigma_a, \\
\text{TINC}_{2\text{PN}} = & \frac{(d-1)^2(d+2)}{8d(d-2)^2(d+1)(d+6)} \hat{x}_{ijab} \partial_a V \partial_b V, \\
\text{TINC}_{3\text{PN}} = & \left(\frac{(d-1)^2(d+2)}{2d(d-2)^2(d+1)(d+6)} (\partial_t V_a \partial_b V + \partial_t V_b \partial_a V) - \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)} \partial_i V_b \partial_i V_a \right. \\
& - \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)} \partial_a V_i \partial_b V_i + \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)} (\partial_a V_i \partial_i V_b + \partial_b V_i \partial_i V_a) \\
& \left. - \frac{(d-3)(d-1)^2(d+2)}{4d(d-2)^3(d+1)(d+6)} (\partial_a K \partial_b V + \partial_a V \partial_b K) \right) \hat{x}_{ijab},
\end{aligned}$$

$$\begin{aligned}
TINC_{4PN} = & \frac{(d-1)^3(d+2)}{2d(d-2)^3(d+1)(d+6)}(\partial_t V_a \partial_b V + \partial_t V_b \partial_a V)V \hat{x}_{ijab} + \frac{(-1+d)^2(2+d)}{2(-2+d)^3 d(1+d)(6+d)}V_a \hat{x}_{ijab} \partial_t V \partial_b V \\
& + \frac{(-1+d)^2(2+d)}{2(-2+d)^3 d(1+d)(6+d)}V_b \hat{x}_{ijab} \partial_t V \partial_a V + \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)}\hat{x}_{ijab} \partial_t V_a \partial_t V_b \\
& + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+1)(d+6)}(\partial_a V \partial_i V_b + \partial_b V \partial_i V_a)V_i \hat{x}_{ijab} - \frac{(-1+d)^2(2+d)}{(-2+d)^2 d(1+d)(6+d)}V_a \hat{x}_{ijab} \partial_i V_b \partial_i V \\
& - \frac{(-1+d)^2(2+d)}{(-2+d)^2 d(1+d)(6+d)}V_b \hat{x}_{ijab} \partial_i V_a \partial_i V + \frac{(-1+d)^2(2+d)}{(-2+d)^2 d(1+d)(6+d)}V_a \hat{x}_{ijab} \partial_b V_i \partial_i V \\
& + \frac{(-1+d)^2(2+d)}{(-2+d)^2 d(1+d)(6+d)}V_b \hat{x}_{ijab} \partial_a V_i \partial_i V + \frac{(d-1)^2(d+2)}{4d(d-2)^2(d+1)(d+6)}\hat{W} \hat{x}_{ijab} \partial_a V \partial_b V \\
& - \frac{(-1+d)^2(2+d)}{2(-2+d)^2 d(1+d)(6+d)}\hat{W}_{ai} \hat{x}_{ijab} \partial_b V \partial_i V - \frac{(-1+d)^2(2+d)}{2(-2+d)^2 d(1+d)(6+d)}\hat{W}_{bi} \hat{x}_{ijab} \partial_a V \partial_i V \\
& - \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_t \hat{W}_{ai} \partial_b V_i + \partial_t \hat{W}_{bi} \partial_a V_i)\hat{x}_{ijab} + \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_t \hat{W}_{ai} \partial_i V_b \\
& + \partial_t \hat{W}_{bi} \partial_i V_a)\hat{x}_{ijab} + \frac{(d-1)(d+2)}{2d(d-2)(d+1)(d+6)}\hat{x}_{ijab} \partial_a \hat{W}_{ij} \partial_b \hat{W}_{ij} \\
& - \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_a \hat{W}_{ij} \partial_j \hat{W}_{bi} + \partial_b \hat{W}_{ij} \partial_j \hat{W}_{ai})\hat{x}_{ijab} \\
& + \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}\hat{x}_{ijab} \partial_i \hat{W}_{bj} \partial_j \hat{W}_{ai} + \frac{(d-1)(d+2)}{d(d-2)(d+1)(d+6)}\hat{x}_{ijab} \partial_j \hat{W}_{bi} \partial_j \hat{W}_{ai} \\
& + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+1)(d+6)}(\partial_t \hat{R}_a \partial_b V + \partial_t \hat{R}_b \partial_a V)\hat{x}_{ijab} + \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_a \hat{R}_i \partial_i V_b \\
& + \partial_b \hat{R}_i \partial_i V_a)\hat{x}_{ijab} - \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_i V_a \partial_i \hat{R}_b + \partial_i V_b \partial_i \hat{R}_a)\hat{x}_{ijab} \\
& - \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_a \hat{R}_i \partial_b V_i + \partial_a V_i \partial_b \hat{R}_i)\hat{x}_{ijab} + \frac{2(d-1)(d+2)}{d(d-2)(d+1)(d+6)}(\partial_a V_i \partial_i \hat{R}_b \\
& + \partial_b V_i \partial_i \hat{R}_a)\hat{x}_{ijab} + \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+1)(d+6)}(\partial_a V \partial_b \hat{X} + \partial_a \hat{X} \partial_b V)\hat{x}_{ijab} \\
& + \frac{(d-3)^2(d-1)^2(d+2)}{2d(d-2)^4(d+1)(d+6)}\hat{x}_{ijab} \partial_a K \partial_b K - \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+1)(d+6)}(\partial_t V_a \partial_b K + \partial_t V_b \partial_a K)\hat{x}_{ijab}, \\
THINC_{3PN} = & \frac{(d-1)^2(d+2)}{16d(d-2)^2(d+1)(d+6)(d+8)}r^2 \hat{x}_{ijab} \partial_a V \partial_b V, \\
THINC_{4PN} = & \left(-\frac{(d-3)(d-1)^2(d+2)}{8d(d-2)^3(d+1)(d+6)(d+8)}(\partial_a K \partial_b V + \partial_a V \partial_b K) \right. \\
& + \frac{(d-1)^2(d+2)}{4d(d-2)^2(d+1)(d+6)(d+8)}(\partial_t V_a \partial_b V + \partial_t V_b \partial_a V) \\
& - \frac{(d-1)(d+2)}{2d(d-2)(d+1)(d+6)(d+8)}\partial_i V_b \partial_i V_a - \frac{(d-1)(d+2)}{2d(d-2)(d+1)(d+6)(d+8)}\partial_a V_i \partial_b V_i \\
& \left. + \frac{(d-1)(d+2)}{2d(d-2)(d+1)(d+6)(d+8)}(\partial_a V_i \partial_i V_b + \partial_b V_i \partial_i V_a) \right) r^2 \hat{x}_{ijab}, \\
THINC_{4PN} = & \frac{(d-1)^2(d+2)}{64d(d-2)^2(d+1)(d+6)(d+8)(d+10)}r^4 \hat{x}_{ijab} \partial_a V \partial_b V.
\end{aligned}$$

Finally we present the surface terms. The terms of the Laplacian type should be multiplied by the operator $\hat{x}_{ij}\Delta$ for the scalar (S) terms, and by $\hat{x}_{ija}\Delta$ for the vector (V) terms. The terms of the divergence type should be multiplied by a spatial derivative ∂_a .

$$\begin{aligned}
SISL_{1PN} = & -\frac{(d-1)^2}{8(d-2)^2}V^2, \\
SISL_{2PN} = & -\frac{(d-1)^3}{12(d-2)^3}V^3 + \frac{(d-3)}{2(d-2)}V_a V_a - \frac{(d-1)}{4(d-2)}V\hat{W} + \frac{(d-3)(d-1)^2}{2(d-2)^3}KV, \\
SISL_{3PN} = & -\frac{(d-1)^4}{24(d-2)^4}V^4 + \frac{(d-3)(d-1)}{(d-2)^2}VV_a V_a - \frac{(d-1)^2}{4(d-2)^2}V^2\hat{W} - \frac{1}{4(d-2)}\hat{W}^2 + \frac{1}{2(d-2)}\hat{W}_{ab}\hat{W}_{ab}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(d-3)}{d-2} \hat{R}_a V_a - \frac{(d-1)^2}{(d-2)^2} V \hat{X} - \frac{(d-1)}{d-2} V \hat{Z} + \frac{(d-3)(d-1)^3}{2(d-2)^4} K V^2 - \frac{(d-3)^2(d-1)^2}{2(d-2)^4} K^2 \\
& + \frac{(d-3)(d-1)}{2(d-2)^2} K \hat{W}, \\
SISL_{4PN} = & - \frac{(d-1)^5}{60(d-2)^5} V^5 + \frac{3(d-3)(d-1)^2}{2(d-2)^3} V^2 V_a V_a - \frac{(d-1)}{4(d-2)} V \hat{W}^2 - \frac{(d-1)^3}{6(d-2)^3} V^3 \hat{W} + \frac{2(d-3)}{d-2} \hat{R}_a \hat{R}_a \\
& - \frac{(d-1)}{d-2} \hat{W} \hat{X} - \frac{2}{d-2} \hat{W} \hat{Z} + \frac{4}{d-2} \hat{W}_{ab} \hat{Z}_{ab} + \frac{4(d-3)}{d-2} V_a \hat{Y}_a - \frac{4(d-1)^2}{(d-2)^2} \hat{T} V + \frac{4(d-1)}{(d-2)^2} \hat{M} V \\
& + \frac{(d-3)(d-1)^3(3d-2)}{12(d-2)^5} K V^3 - \frac{(d-5)(d-3)(d-1)}{(d-2)^3} K V_a V_a + \frac{2(d-3)(d-1)^2}{(d-2)^3} K \hat{X} \\
& + \frac{2(d-3)(d-1)}{(d-2)^2} K \hat{Z}, \\
VISL_{2PN} = & \frac{(d-1)^2(d+2)}{2d(d-2)^2(d+4)} V V_a, \\
VISL_{3PN} = & \frac{(d-1)^3(d+2)}{4d(d-2)^3(d+4)} V^2 V_a + \frac{(d-1)(d+2)}{d(d-2)(d+4)} V_a \hat{W} - \frac{(d-1)(d+2)}{d(d-2)(d+4)} V_b \hat{W}_{ab} + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)} \hat{R}_a V \\
& - \frac{(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)} K V_a, \\
VISL_{4PN} = & \frac{(d-1)(d+2)}{(d-2)^2(d+4)} V_a V_b V_b + \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)} V V_a \hat{W} - \frac{(d-1)^2(d+2)}{d(d-2)^2(d+4)} V V_b \hat{W}_{ab} \\
& + \frac{(d-1)^3(d+2)}{d(d-2)^3(d+4)} \hat{R}_a V^2 + \frac{2(d-1)(d+2)}{d(d-2)(d+4)} \hat{R}_a \hat{W} - \frac{2(d-1)(d+2)}{d(d-2)(d+4)} \hat{R}_b \hat{W}_{ab} + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)} V_a \hat{X} \\
& + \frac{4(d-1)(d+2)}{d(d-2)(d+4)} V_a \hat{Z} - \frac{4(d-1)(d+2)}{d(d-2)(d+4)} V_b \hat{Z}_{ab} + \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)} V \hat{Y}_a \\
& - \frac{(d-3)(d-1)^3(d+2)}{d(d-2)^4(d+4)} K V V_a - \frac{2(d-3)(d-1)^2(d+2)}{d(d-2)^3(d+4)} K \hat{R}_a, \\
SISD_{2PN} = & \frac{(d-1)}{2(d-2)} (\Psi_{ai}^{\partial_{bk} V} \partial_j \hat{W}_{bk} - \hat{W}_{bk} \partial_a \Psi_{ij}^{\partial_{bk} V}), \\
SISD_{3PN} = & - \frac{4(d-1)}{d-2} \Psi_{ai}^{\partial_b V_k} \partial_{kj} \hat{R}_b + \frac{4(d-1)}{d-2} \partial_a \Psi_{ij}^{\partial_b V_k} \partial_k \hat{R}_b + \frac{2(d-1)}{d-2} \Psi_{ai}^{\partial_{tb} V} \partial_j \hat{R}_b - \frac{2(d-1)}{d-2} \hat{R}_b \partial_a \Psi_{ij}^{\partial_{tb} V} \\
& + \frac{2(d-1)}{d-2} \Psi_{ai}^{\partial_{bk} V} \partial_j \hat{Z}_{bk} - \frac{2(d-1)}{d-2} \hat{Z}_{bk} \partial_a \Psi_{ij}^{\partial_{bk} V}, \\
SISD_{4PN} = & - \frac{8(d-1)}{d-2} \Psi_{ai}^{\partial_b V_k} \partial_{kj} \hat{Y}_b + \frac{8(d-1)}{d-2} \partial_a \Psi_{ij}^{\partial_b V_k} \partial_k \hat{Y}_b + \frac{4(d-1)}{d-2} \Psi_{ai}^{\partial_{tb} V} \partial_j \hat{Y}_b - \frac{4(d-1)}{d-2} \hat{Y}_b \partial_a \Psi_{ij}^{\partial_{tb} V} \\
& + \frac{4(d-1)}{d-2} \Psi_{ai}^{\partial_{bk} V} \partial_j \hat{M}_{bk} - \frac{4(d-1)}{d-2} \hat{M}_{bk} \partial_a \Psi_{ij}^{\partial_{bk} V}, \\
VISD_{3PN} = & \frac{2(d-1)^2(d+2)}{d(d-2)^2(d+4)} (\Psi_{aib}^{\partial_k V} \partial_{kj} \hat{R}_b - \partial_a \Psi_{ijb}^{\partial_k V} \partial_k \hat{R}_b), \\
VISD_{4PN} = & \frac{4(d-1)^2(d+2)}{d(d-2)^2(d+4)} (\Psi_{aib}^{\partial_k V} \partial_{kj} \hat{Y}_b - \partial_a \Psi_{ijb}^{\partial_k V} \partial_k \hat{Y}_b).
\end{aligned}$$

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Étude des ondes gravitationnelles émises par un système binaire d'objets compacts dans la phase spirale dans l'approximation post-newtonienne.

Résumé : En 2015, une nouvelle ère de l'astronomie multimessager s'est ouverte avec la première détection directe d'ondes gravitationnelles. Cette onde, ainsi que toutes les autres qui ont été détectées depuis, a été générée lors du fusionnement de deux objets compacts. Lorsque deux de ces objets orbitent l'un autour de l'autre, le système rayonne de l'énergie sous forme d'ondes gravitationnelles. De ce fait, les deux corps se rapprochent jusqu'à leur fusionnement. Pour réussir à détecter ces ondes, il est crucial de connaître leur forme théorique en résolvant les équations d'Einstein. Le travail de cette thèse consiste à améliorer la précision de la forme d'onde théorique lors de la phase spirale, c'est à dire avant le fusionnement, en utilisant le formalisme post-newtonien (PN). Il est articulé en deux projets principaux: l'amélioration de la précision de l'impact de la déformabilité des étoiles à neutrons sur l'onde gravitationnelle émise; et des calculs de haute précision sur la forme d'onde sans tenir compte de la déformabilité des corps. Plus spécifiquement, j'ai calculé le quadrupôle de masse jusqu'au quatrième ordre PN et le quadrupôle de courant jusqu'au troisième ordre PN nécessaires pour l'obtention de la phase d'onde gravitationnelle au quatrième ordre PN. J'ai aussi calculé la phase d'onde gravitationnelle associée aux effets de marée jusqu'au deuxième et demi ordre PN.

Mots clés : relativité générale, ondes gravitationnelles, théorie post-newtonienne, effets de marée

Study of gravitational waves emitted by a binary system of compact objects in the inspiral phase in the post-Newtonian approximation.

Abstract: In 2015, a new era of multimessenger astronomy began with the first ever direct detection of gravitational waves (GW). This GW, as well as all the others that have been detected since, was generated when two compact objects merged. When two such objects orbit around each other, the system radiates energy in the form of GWs. As a result, the two bodies get closer with time until they merge. To detect these waves, it is crucial to know their theoretical form by solving Einstein's equations. The work performed in this PhD consists in improving the accuracy of the theoretical waveform during the spiral phase, i.e. before the merger, by using the post-Newtonian (PN) formalism. It is divided in two main projects: the improvement of the accuracy of the impact of the deformability of neutron stars on the GW emitted; and high accuracy calculations of the waveform without taking into account the deformability of the bodies. More specifically, I computed the mass quadrupole up to the fourth PN order and the current quadrupole up to the third PN order required to obtain the GW phase up to the fourth PN order. I also computed the GW phase associated with tidal effects up to the second and a half relative PN order.

Keywords : general relativity, gravitational waves, post-Newtonian theory, tidal effects