Conditions nécessaires et suffisantes en contrôle optimal et applications

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If, 1895, Rudyard Kipling, adaptation d'André Maurois

Si tu peux voir détruit l'ouvrage de ta vie
Et sans dire un seul mot te mettre à rebâtir,
Ou perdre en un seul coup le gain de cent parties
Sans un geste et sans un soupir ;

Si tu peux être amant sans être fou d'amour,
Si tu peux être fort sans cesser d'être tendre,
Et, te sentant haï, sans haïr à ton tour,
Pourtant lutter et te défendre ;

Si tu peux supporter d'entendre tes paroles
Travesties par des gueux pour exciter des sots,
Et d'entendre mentir sur toi leurs bouches folles
Sans mentir toi-même d'un mot ;

Si tu peux rester digne en étant populaire,
Si tu peux rester peuple en conseillant les rois,
Et si tu peux aimer tous tes amis en frère,
Sans qu'aucun d'eux soit tout pour toi ;

Si tu sais méditer, observer et connaitre,
Sans jamais devenir sceptique ou destructeur,
Rêver, mais sans laisser ton rêve être ton maitre,
Penser sans n'être qu'un penseur ;

Si tu peux être dur sans jamais être en rage,
Si tu peux être brave et jamais imprudent,
Si tu sais être bon, si tu sais être sage,
Sans être moral ni pédant ;

Si tu peux rencontrer Triomphe après Défaite
Et recevoir ces deux menteurs d'un même front,
Si tu peux conserver ton courage et ta tête
Quand tous les autres les perdront,

Alors les Rois, les Dieux, la Chance et la Victoire
Seront à tout jamais tes esclaves soumis,
Et, ce qui vaut mieux que les Rois et la Gloire
Tu seras un homme, mon fils.

Cette thèse est dédiée à ma mère, à mon père, à ma famille.

Part 0

Introductory Pages

Résumé

La partie 1 de cette thèse traite de la programmation dynamique en contrôle optimal. On considère un problème de Bolza non-autonome en contrôle optimal pour lequel la dynamique et le lagrangien sont continus en temps seulement presque partout (avec limites à droite et à gauche partout).
Plusieurs caractérisations (proximale, de Dini, et viscosité) de la fonction valeur du problème en tant qu'unique solution généralisée de l'équation de Hamilton-Jacobi-Bellman (HJB) correspondante sont démontrées dans la classe des fonctions semi-continues inférieurement.

Le cas où une contrainte d'état est ajoutée au problème précédent est aussi considéré. Des conditions de compatibilité *ad hoc* entre l'ensemble des contraintes A et la fonction dictant la dynamique F sont introduites, ce qui permet d'approximer dans $W^{1,1}$ les trajectoires non-faisables par des F-trajectoires faisables et par suite d'établir différentes caractérisations (proximale, de Dini, et viscosité) de la fonction valeur en tant qu'unique solution de l'équation (HJB).

La partie 2 de cette thèse traite de résultats obtenus concernant les conditions nécessaires d'optimalité en calculs de variations et la régularité des minimiseurs. Le problème considéré est celui d'un problème de Bolza non-autonome d'ordre N dans lequel le lagrangien L est seulement Borel mesurable, et peut prendre pour valeur $+\infty$.
On établit d'abord les conditions nécessaires d'optimalité sous la forme d'une équation du type Euler-Lagrange, ainsi que sous la forme d'une équation du type Erdmann – Du Bois-Reymond, sans imposer au lagrangien la convexité par rapport à sa dernière variable, ni aucune condition de croissance particulière.

En imposant en plus à L une condition de croissance plus générale que la croissance super-linéaire utilisée habituellement, les conditions nécessaires sont mises à profit afin d'établir que la dernière dérivée d'un minimiseur de ce problème est essentiellement bornée.

Mots clefs. Optimisation, contrôle optimal, équation de Hamilton-Jacobi-Bellman, fonction valeur, contrainte d'état, calcul des variations, analyse non lisse, problème de Bolza, conditions nécessaires d'optimalité, régularité des minimiseurs.

Abstract

The part 1 of this thesis focuses on dynamic programming in optimal control. We consider non autonomous Bolza problem in optimal control for which the Lagrangian L and the dynamics F are allowed to be discontinuous with respect to time on a set of full measure (with left and right limits everywhere).
Several characterizations (Dini, proximal, viscosity) of the value function of the problem as the unique solution to the corresponding Hamilton-Jacobi-Bellman equation are established in the class of lower semi-continuous functions.

The case where a state constraint is added to the previous problem is also considered. Some appropriate compatibility conditions between the state of constraints and the dynamics are introduced. They allow to establish a $W^{1,1}$ neighbouring feasible trajectories result which is then exploited to prove several characterizations (Dini, proximal, viscosity) of the value function V as the unique generalized solution to (HJB).

Part 2 of this thesis presents results concerning necessary conditions for optimality in the calculus of variations and the regularity of minimizers. The problem considered is a non autonomous high order Bolza problem in which the Lagrangian is merely Borel measurable, and is possibly extended valued.

Necessary conditions for optimality in the Euler-Lagrange form and in the Erdmann – Du Bois-Reymond form are provided, without imposing on the Lagrangian to be convex with respect to the last variable, nor to have any kind of specific growth behavior.

By adding an extra growth assumption that is more general than the usual superlinearity with respect to the last variable, the necessary conditions are exploited to establish that the last derivative of a given minimizer is essentially bounded.

Key words. Optimization, optimal control, Hamilton-Jacobi-Bellman equation, value function, state constraint, calculus of variations, nonsmooth analysis, Bolza problem, necessary conditions for optimality, regularity of minimizers.

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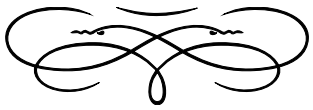

Je me considère privilégié et honoré d'avoir pu préparer mon doctorat sous la direction du professeur Piernicola Bettiol, que je remercie sincèrement pour la grande qualité de son encadrement. Durant presque quatre années, il a pris beaucoup de son temps pour s'entretenir régulièrement avec moi, me former, et partager l'impressionnante connaissance encyclopédique de son domaine de recherche, pour lequel sa passion est aussi vive que communicative. A ses qualités académiques manifestes, qu'il ne m'appartient d'ailleurs pas de juger, s'ajoutent de grandes vertus, tant humaines que pédagogiques, qui devraient être l'honneur de tout professeur d'université : humilité, bienveillance, patience, tempérance, diplomatie, optimisme et rigueur.
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1 Introduction

1.1 Optimal control theory

Control theory is a mathematical field in which a dynamical system, commonly modeled by a differential equation, is monitored by an operant. The inputs of the system, called controls, can be managed in order to modify the output of the system. A typical mathematical representation for such a system is:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), \text{ for a.e. } s \in [S, T], \\ u(s) \in U(s), \text{ for a.e. } s \in [S, T] \\ x(S) = x_0. \end{cases} \tag{1.1}$$

Here $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes the dynamics of the system, $U : [S,T] \rightsquigarrow \mathbb{R}^m$ is a multivalued function giving the admissible controls at each time t, and $u(\cdot)$ is an admissible control function chosen among the measurable selections of U. If $(x(\cdot), u(\cdot))$ satisfies (1.1) then it is called a trajectory-control pair.

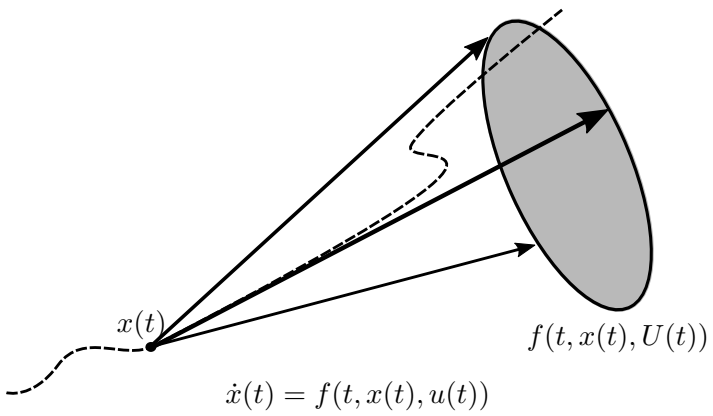

Figure 1.1: Representation of an f-trajectory and choice of the velocity vector $\dot{x}(t)$.

Control theory (together with optimal control theory) emerged to answer the need of controlling engineering devices, for example space shuttles (see [81]). Over the years, they have found many other natural applications, for instance in the car industry: auto-braking systems or parking assistants. In these two examples, the system is the car, the controller is the on-board computer, the controls are the acceleration and the angle(s) of the guide wheels with the longitudinal axis of the car, and the output of the system is the pair

trajectory/velocity of the car $(x, \dot{x})$.

When it is possible to do so, the inputs of the system are chosen to obtain a satisfactory output. A relevant criterion can, for instance, be whether or not an endpoint constraint is satisfied:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), \text{ for a.e. } s \in [S, T], \\ u(s) \in U(s), \text{ for a.e. } s \in [S, T], \\ x(S) = x_0, \\ \boxed{x(T) \in E,} \end{cases} \tag{1.2}$$

where E is the set of admissible endpoints for f-trajectories. In our examples, the satisfactory output is stopping the car before the impact or steering the car into the parking slot.

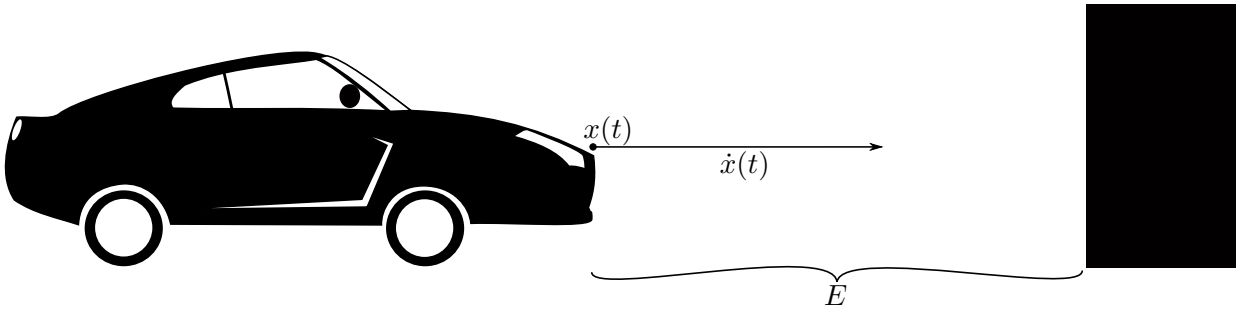

Figure 1.2: Autobraking: the car must stop in E

To decide between all the controls giving satisfying results, one can add extra requirements by introducing a cost function $J(\cdot)$ and associating a cost to each trajectory. A convenient definition for $J(\cdot)$ (and the one we will be using in this thesis) that allows to cover a large number of cases is:

$$J(x(\cdot)) = \int_S^T L(t, x(t), \dot{x}(t))\mathrm{d}t + g(x(T)),$$

where $\displaystyle\int_S^T L(t, x(t), \dot{x}(t))\mathrm{d}t$ and $g(x(T))$ are respectively called the running cost and the final cost of the trajectory $x(\cdot)$. But of course there are as many cost functions as there are control problems.
In the auto-braking system, a relevant running cost function for a trajectory/control pair $((x, \dot{x}), u)$ could be the integral of the deceleration, *i.e.* $L\left(t, (x, \dot{x}), (\dot{x}, \ddot{x})\right) = |\ddot{x}|$.

When we look, amongst all the admissible trajectories, for the ones that minimize or maximize the functional $J(\cdot)$, we enter the field of optimal control theory, which can be considered as an extension of calculus of variations with constraints on the dynamics.

$$\begin{cases} \text{Minimize } J(x) = \displaystyle\int_S^T L(s, x(s), \dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } (x(\cdot), u(\cdot)) \text{ satisfying (1.2).} \end{cases} \tag{1.3}$$

Note that it can be convenient to include the endpoint constraint $x(T) \in E$ (see (1.2)) into the final cost $g(\cdot)$ by adding the following function $\chi_E(x) = \begin{cases} 0, \text{ if } x \in E \\ +\infty, \text{ if } x \notin E \end{cases}$ (in the case of a minimization problem), which

immediately disqualifies trajectories violating the endpoint constraint. The problem (1.3) hence becomes:

$$\begin{cases} \text{Minimize } J(x) = \displaystyle\int_S^T L(s, x(s), \dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } (x(\cdot), u(\cdot)) \text{ satisfying (1.1).} \end{cases} \tag{1.4}$$

In the auto-braking system example, we can take $g = \chi_{E\times\{0\}}$ where E is the set of all the positions between the car and the obstacle at the beginning of the braking. Once it is guaranteed that the car is not getting into an accident and stops, it sounds reasonable to minimize the integral of the deceleration for the sake of passengers safety, so the cost we want to minimize in this setup is:

$$\int_S^T |\ddot{x}|(s)\mathrm{d}s + \chi_{E\times\{0\}}((x(T), \dot{x}(T))).$$

Defining a multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ by $F(t,x) := f(t,x,U(t))$, this problem can alternatively be written using a differential inclusion involving $x(\cdot)$ instead of an ordinary differential equation. We obtain a new form of the optimal control problem we started with:

$$\begin{cases} \text{Minimize } J(x) = \displaystyle\int_S^T L(s, x(s), \dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } x(\cdot) \text{ satisfying} \\ \boxed{\dot{x}(s) \in F(s, x(s)), \text{ for a.e. } s \in [S,T],} \\ x(S) = x_0, \end{cases} \tag{P_{S,x_0}}$$

where $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Lagrangian of the problem, $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a function and $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ a multivalued function. Here, we call F-trajectory on the interval $[S,T]$ any absolutely continuous arc $x(\cdot) : [S,T] \to \mathbb{R}^n$ which satisfies the reference differential inclusion $\dot{x}(s) \in F(s, x(s))$ for a.e. $s \in [S,T]$.

Another important aspect of minimization problems is the possibility to consider a state constraint on the trajectories:

$$\begin{cases} \text{Minimize } J(x) = \displaystyle\int_S^T L(s, x(s), \dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } x(\cdot) \text{ satisfying} \\ \dot{x}(s) \in F(s, x(s)), \text{ for a.e. } s \in [S,T], \\ \boxed{x(s) \in A, \text{ for all } s \in [S,T],} \\ x(S) = x_0, \end{cases} \tag{SC_{S,x_0}}$$

where A is a closed set giving all the admissible positions for the F-trajectories. If an F-trajectory takes all of its values in A, it is called a feasible F-trajectory.
In the parking assistant example, a relevant state constraint could be $(x(t), \dot{x}(t)) \in A \times [-v, v]$, where A is the side of the road along which the car is parking (the other side of the road should be reserved to traffic coming from the opposite direction) and $v \geq 0$ is a reasonable speed the car should not exceed to maneuver safely.

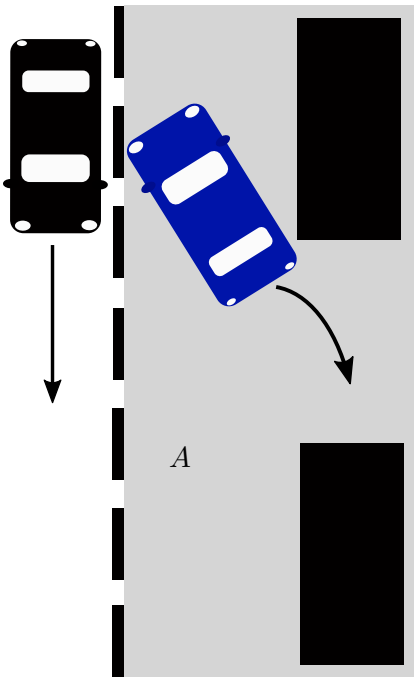

Figure 1.3: Parking assistant and state constraint: the blue car must park without rolling over the other side of the road (it must stay in A)

Once a problem that has the form (1.4), (P_{S,x_0}) or (SC_{S,x_0}) is given[1], the construction of an optimal trajectory-control pair $(x(\cdot), u(\cdot))$ or of an optimal trajectory $x(\cdot)$ naturally arises and has been a topic of research of topmost importance since the late 50s. Two different approaches were proposed: the necessary conditions (namely the maximum principle) by Lev Pontryagin initiated in 1956, *cf.* [73], and the dynamic programming principle by Richard Ernest Bellman initiated in 1957, *cf.* [26].

In this thesis, necessary conditions in optimal control theory are only used to derive regularity properties of minimizers in calculus of variations (Chapter 5), while Chapters 3 and 4 are actually devoted to dynamic programming applied to optimal control problems with discontinuous data (with respect to time).

1.2 Necessary conditions in Optimal Control

The Pontryagin maximum principle provides a set of conditions that are necessarily satisfied by a (local) minimizer of a given optimal control problem where the constraint on the dynamics was first expressed as an ordinary differential equation.

[1]Many other optimal control problems can be considered: the Mayer problem, the minimal time control problem, infinite horizon Bolza problem...

Consider first the following Mayer problem in which the data are considered as smooth as needed:

$$(P)\quad \begin{cases} \text{Minimize } g(y(T)) \\ \text{over the arcs } y \in W^{1,1}([S,T],\mathbb{R}^n) \\ \text{and measurable functions } u : [S,T] \to \mathbb{R}^m \text{ satisfying} \\ \dot{y}(s) = f(s,y(s),u(s)), \text{ for almost every } s \in [S,T], \\ u(s) \in U(s), \text{ for almost every } s \in [S,T], \\ y(S) = x_0. \end{cases}$$

Here, g is real valued and differentiable. Then if (x_*,u_*) is a $W^{1,1}$-local minimizer for (P), then there exists $p \in W^{1,1}([S,T]\times\mathbb{R}^n)$ (we note p in line $p(t) = (p_1(t),\dots,p_n(t))$ such that the following conditions are satisfied:

(i) the nontriviality condition: $p \neq 0$,

(ii) the adjoint system:

$$-\dot{p}(s) = p(s)\,\nabla_x f(s,x_*(s),u_*(s)),$$

for almost every $s \in [S,T]$,

(iii) the maximality condition:

$$p(s)f(s,x_*(s),u_*(s)) = \max_{v\in U(s)} \{p(s)f(s,x_*(s),v)\},$$

for almost every $s \in [S,T]$,

(iv) the transversality condition:

$$-p(T) = \nabla g(x_*(T)).$$

Even though other methods have been developed through years (second order necessary conditions, dynamic programming) to solve optimal control problems, necessary conditions remain an excellent source of optimal control strategies for some practical applications (*cf.* [11, 65]).

A lot of researchers have contributed to the topic, providing more and more involved versions of the maximum principle [44, 41, 62, 85]. The enhancements of the theorem previously mentioned include several aspects: extensions with less smoothness of the data, the possibility to cover minimization problems for functionals comprising a running cost, the possibility to cover involved endpoint and starting point constraints $(y(S),y(T)) \in C\dots$

Some aspects of this theory could not be dealt with using classical tools coming from smooth analysis. For instance even in the simple form we presented, how can we state the adjoint inclusion when f is not differentiable with respect to x ? or the transversality condition when g is not differentiable ?
More generally, if we want to study problems including endpoint and starting point constraints $(y(S),y(T)) \in C$, the transversality condition involves normal vectors to C, but in the case C is merely closed (with a possibly nonsmooth boundary), such vectors must be considered in a general setting.

The mathematical theory allowing to answer these questions is called nonsmooth analysis. It will be outlined in Section 1.4 and presented with more details in Chapter 2.

1.3 Dynamic programming in Optimal control

1.3.1 The value function

We proceed to make a brief introduction to the dynamic programming approach. It starts with embedding our problem (P_{S,x_0}) into a family of problems $((P_{t,x}))_{(t,x)}$ indexed by the initial data $(t,x) \in [S,T] \times \mathbb{R}^n$:

$$(P_{t,x}) \quad \begin{cases} \text{Minimize } J(y(\cdot)) = \int_t^T L(s,y(s),\dot{y}(s))\mathrm{d}s + g(y(T)) \\ \text{over the arcs } y \in W^{1,1}([t,T],\mathbb{R}^n) \text{ satisfying} \\ \dot{y}(s) \in F(s,y(s)), \text{ for almost every } s \in [t,T], \\ \boxed{y(t) = x.} \end{cases}$$

This leads to the concept of the value function $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which, for all $(t,x) \in [S,T] \times \mathbb{R}^n$, is defined taking the infimum cost for $(P_{t,x})$:

$$V(t,x) := \inf \left\{ \int_t^T L(s,y(s),\dot{y}(s))\mathrm{d}s + g(y(T)),\ y(\cdot)\ \ F\text{-trajectory on } [t,T], y(t) = x \right\}.$$

1.3.2 Hamilton-Jacobi-Bellman equation

An heuristic approach to find the partial differential equation satisfied by V can be presented invoking the principle of optimality, stating that for each $t \in [S,T]$, the function

$$\phi\ :\ s \mapsto \int_t^s L(\tau,y(\tau),\dot{y}(\tau))\mathrm{d}\tau + V(s,y(s))$$

is increasing along F-trajectories $y(\cdot)$ satisfying $y(t) = x$ and is constant if and only if $y(\cdot)$ is a minimizer for $(P_{t,x})$.

Assume that V is differentiable and that $(P_{t,x})$ has a minimizer for all $(t,x) \in [S,T] \times \mathbb{R}^n$. Fix any $v \in F(t,x)$. We admit the existence of an F-trajectory $y(\cdot)$ such that $y(t) = x$ and $\dot{y}(t) = v$. Differentiating ϕ, the principle of optimality yields:

$$\partial_t V(s,y(s)) + \partial_x V(s,y(s)) \cdot \dot{y}(s) + L(s,y(s),\dot{y}(s)) \geq 0,$$

hence if $s = t$:

$$\partial_t V(t,x) + \partial_x V(t,x) \cdot v + L(t,x,v) \geq 0.$$

Similarly, if $y_0(\cdot)$ is a minimizer for $(P_{t,x})$ and if we note $v_0 = \dot{y}_0(t) \in F(t,x)$, then

$$\partial_t V(t,x) + \partial_x V(t,x) \cdot v_0 + L(t,x,v_0) = 0.$$

This shows that V is a solution to the following Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t \varphi(t,x) + \inf_{v \in F(t,x)} \{\partial_x \varphi(t,x) \cdot v + L(t,x,v)\} = 0 \\ \varphi(T,x) = g(x). \end{cases} \tag{HJB}$$

An important feature of the dynamic programming approach is that the value function does provide the minimum cost for (P_{S,x_0}), but solving (HJB) also provides information about minimizers, supplying optimal

controls in feedback form, that are favored (over open-loop controls) in engineering applications due to their resilience to perturbations of the system. This is achieved by an analysis of the feedback map:

$$\psi \ : \ \begin{cases} [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \\ (t,x) \mapsto \Big\{ v \in F(t,x), \partial_x V(t,x) \cdot v + L(t,x,v) = \min\limits_{w \in F(t,x)} [\partial_x V(t,x) \cdot w + L(t,x,w)] \Big\}. \end{cases}$$

and of the differential inclusion:

$$\dot{x}(t) \in \psi(t, x(t)), \text{ for a.e. } t \in [S,T].$$

This type of work is called optimal synthesis and goes beyond the purpose of this thesis.

1.3.3 Non differentiability of the value function

A major issue arises when we want to characterize the value function V as the unique solution to the corresponding Hamilton-Jacobi-Bellman equation, as even for the simplest control problems, the value function is not differentiable.
We illustrate this fact by an example from this thesis. Consider the following Mayer problem:

$$\begin{cases} \text{Minimize } g(x(1)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([t_0,1],\mathbb{R}) \text{ such that} \\ \dot{x}(t) \in F(t) \quad \text{for a.e. } t \in [t_0,1], \\ x(t_0) \ = \ x_0 \ , \end{cases}$$

where $t_0 \in [0,1]$, $x_0 \in \mathbb{R}$, and $F(t) := [0,1]$, $g(x) := \begin{cases} 1, \text{if } x \neq 0, \\ 0, \text{if } x = 0. \end{cases}$
A computation of the value function $V : [0,1] \times \mathbb{R} \to \mathbb{R}$,

$$V(t,x) \ = \ \begin{cases} 0, & \text{if } x+1-t \geq 0 \text{ and } x \leq 0, \\ 1, & \text{if } x+1-t < 0 \text{ or } x > 0, \end{cases}$$

shows that it is merely lower semicontinuous, and certainly not differentiable.

To circumvent this issue, a suitable notion of generalized solution to (HJB) has to be introduced: the value function V should be the unique solution to (HJB) in this new sense. Unfortunately, the distributional derivatives theory is not suited to strongly unlinear partial differential equations such as the Hamilton-Jacobi-Bellman equation. This need for differentiating the value function led to innovative techniques. In this introduction we shall consider the approach provided by the nonsmooth analysis.

As demonstrated here, when we want to penalize or disqualify trajectories that do not respect a given endpoint constraint, we easily obtain a discontinuous value function. To cover a broad class of optimization problems, a convenient class of functions to characterize the value function as a unique solution to (HJB) would be the class of extended valued lower semicontinuous functions.

1.4 Nonsmooth analysis

Nonsmooth analysis was designed to extend notions such as derivatives and normal vectors (to a set with smooth border) to non-differentiable data. The reason for introducing such extensions was sketched in Section 1.2 and Subsection 1.3.3.

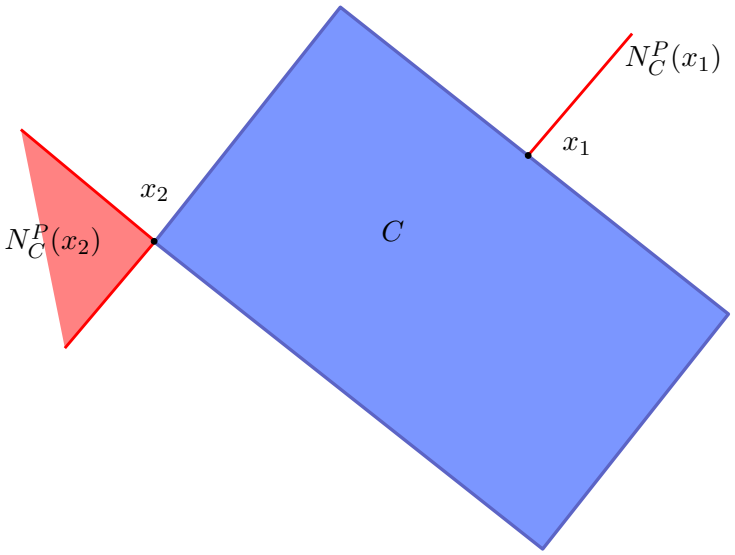

Figure 1.4: A closed set and the proximal normal cone at two different points.

In this framework, many researchers contributed to the development of nonsmooth analysis and Optimal control: for instance, see the references [4, 27, 84, 31, 66, 61].

We briefly present some important notions: the proximal normal cone and the associated proximal subdifferential, the limiting normal cone and the associated limiting subdifferential.

Let $C \subset \mathbb{R}^m$ be a closed set and $\bar{x} \in C$. The proximal normal cone to C at $\bar{x}$ is:

$$N_C^P(\bar{x}) := \left\{\eta \in \mathbb{R}^m, \exists M \geq 0, \forall x \in C,\, \eta \cdot (x - \bar{x}) \leq M|x - \bar{x}|^2\right\}.$$

The limiting normal cone $N_C(x)$ to $C \subset \mathbb{R}^m$ at $\bar{x}$ is defined as follows

$$N_C^L(\bar{x}) \;:=\; \left\{\, \eta \in \mathbb{R}^m,\ \text{there exists } x_i \xrightarrow{C} \bar{x}, \eta_i \to \eta \text{ such that } \eta_i \in N_C^P(x_i),\ \text{for all } i \in \mathbb{N}\right\}.$$

Let $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and $\bar{x} \in \mathrm{dom}(f)$. Then the epigraph of f, $\mathrm{epi}\, f$ is closed. The notions of normal cones to a closed set hence yields corresponding notions of generalized derivatives of f.
The proximal subdifferential of f at $\bar{x}$ is:

$$\partial_P f(\bar{x}) := \left\{\xi \in \mathbb{R}^m, (\xi, -1) \in N^P_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x}))\right\},$$

while the limiting subdifferential of f at $\bar{x}$ is:

$$\partial^L f(\bar{x}) := \left\{\xi \in \mathbb{R}^m, (\xi, -1) \in N^L_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x}))\right\}.$$

Note that the proximal normal cone (resp. the limiting normal cone) to the epigraph of f at $(\bar{x}, f(\bar{x}))$, is in general not spanned by $\partial_P f(\bar{x}) \times \{-1\}$ (resp. $\partial^L f(\bar{x}) \times \{-1\}$) because it may contain horizontal vectors (also called asymptotic vectors) in the form $(\xi, 0)$.

For the function $f = 2\mathbb{1}_{\mathbb{R}_+^*} - 1$, we have

$$N^P_{\mathrm{epi}\, f}(0, -1) = \{(x, y), x \geq 0, y \leq 0\},$$

so $\partial_P f(0) = [0, +\infty[$ and the asymptotic proximal subdifferential corresponding to asymptotic vectors, denoted $\partial_P^\infty f(0)$ is equal to $[0, +\infty[$.

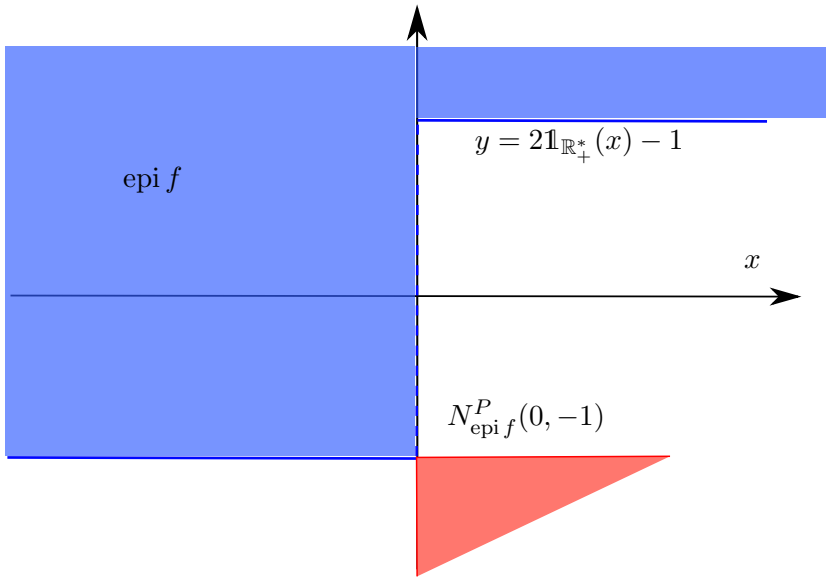

Figure 1.5: Proximal normal cone to the epigraph of $2\mathbb{1}_{\mathbb{R}_+^*} - 1$ at $(0, -1)$.

More details and references about nonsmooth analysis will be provided in Chapter 2. We proceed to show how nonsmooth analysis allows to develop both the sufficient and necessary conditions in optimal control, outlining solutions to the problems we itemised in Subsection 1.3.3 and Section 1.2.

1.4.1 Application of nonsmooth analysis to the (HJB) theory

Nonsmooth analysis can be invoked to define appropriate notions of generalized solution to (HJB). We provide an example in which we put proximal subdifferentials to good use.

In our setup, we say that an extended valued, lower semicontinuous function $W : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proximal solution to (HJB) if the following conditions are satisfied:
i) for every $(t, x) \in]S, T[\times\mathbb{R}^n) \cap \text{dom}(W)$

$$\xi^0 + \inf_{v \in F(t,x)} \left[\xi^1 \cdot v + L(t, x, v)\right] = 0, \text{ for all } (\xi^0, \xi^1) \in \partial_P W(t, x),$$

ii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x') \to (S,x) \mid t' > S\}} W(t', x') = W(S, x),$$

and

$$\liminf_{\{(t',x') \to (T,x) \mid t' < T\}} W(t', x') = W(T, x) = g(x).$$

More notions of generalized solutions to (HJB) will be detailed in section 1.8 and in this thesis.

1.4.2 Nonsmooth analysis contribution to the maximum principle

We illustrate how nonsmooth analysis intervenes even in a quite simple version of the maximum principle [37, 45, 84] for the following optimization problem:

$$(P2)\quad \begin{cases} \text{Minimize } \displaystyle\int_S^T L(s,y(s),u(s))\mathrm{d}s + g(y(S),y(T)) \\ \text{over the arcs } y \in W^{1,1}([S,T],\mathbb{R}^n) \\ \text{and measurable functions } u : [S,T] \to \mathbb{R}^m \text{ satisfying} \\ \dot{y}(s) = f(s,y(s),u(s)), \text{ for almost every } s \in [S,T], \\ u(s) \in U(s), \text{ for almost every } s \in [S,T] \\ (y(S),y(T)) \in C. \end{cases}$$

Here $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes the dynamics of the system, $U : [S,T] \rightsquigarrow \mathbb{R}^m$ is a multivalued function describing the admissible controls, $C \subset \mathbb{R}^{2n}$ is closed set. We denote by $H_\lambda : [S,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ the Hamiltonian function:

$$H_\lambda(s,x,p,u) := p \cdot f(s,x,u) - \lambda L(s,x,u).$$

Then if (x_*,u_*) is a $W^{1,1}$-local minimizer for (P2), then there exist $p \in W^{1,1}([S,T],\mathbb{R}^n)$ (the adjoint arc) and $\lambda \geq 0$ such that the following conditions are satisfied (see [45, Theorem 22.26]:

(i) the nontriviality condition: $(p,\lambda) \neq (0,0)$,

(ii) the adjoint inclusion:

$$-\dot{p}(s) \in \operatorname{co} \partial_x^L H_\lambda(s,x_*(s),p(s),u_*(s)),$$

for almost every $s \in [S,T]$,

(iii) the maximality condition:

$$H_\lambda(s,x_*(s),p(s),u_*(s)) = \sup_{u\in U(s)} H_\lambda(s,x_*(s),p(s),u),$$

for almost every $s \in [S,T]$,

(iv) the transversality condition:

$$(p(S),-p(T)) \in \lambda\partial^L g(x_*(S),x_*(T)) + N_C^L((x_*(S),x_*(T)).$$

In this theorem we resorted no less than three times to nonsmooth analysis to state the adjoint inclusion and the transversality condition.

Nonsmooth analysis also intervenes in the case we are looking for extensions to the maximum principle in the case the constraint on the dynamics is expressed as a differential inclusion $\dot{y}(s) \in F(s,y(s))$. This type of results is called extended Euler-Lagrange conditions.

Here, we consider the following control problem :

$$(P3)\quad \begin{cases} \text{Minimize } g(y(S), y(T)) \\ \text{over the arcs } y \in W^{1,1}([S,T], \mathbb{R}^n) \text{ such that} \\ \dot{y}(s) \in F(s, y(s)), \text{ for almost every } s \in [S,T] \\ (y(S), y(T)) \in C. \end{cases}$$

(here the differential equation is replaced by a differential inclusion). We can turn the problem (P3) into a Bolza problem in the calculus of variations by disqualifying trajectories violating the differential inclusion on a set of positive measure. This is achieved by adding a penalty running cost to the original cost:

$$\begin{cases} \text{Minimize } g(y(S), y(T)) + \boxed{\displaystyle\int_S^T \ell(s, y(s), \dot{y}(s))\mathrm{d}s} \\ \text{over the arcs } y \in W^{1,1}([S,T], \mathbb{R}^n) \\ \text{such that } (y(S), y(T)) \in C. \end{cases}$$

where

$$\ell(t,x,v) = \begin{cases} 0, \text{ if } v \in F(t,x), \\ +\infty, \text{ if } v \notin F(t,x). \end{cases}$$

If we could apply the **smooth version** of the Euler-Lagrange equation (in the calculus of variations) to a $W^{1,1}$ minimizer $x_*(\cdot)$ to derive the new necessary conditions, we would obtain the existence of an adjoint arc $p \in W^{1,1}([S,T], \mathbb{R}^n)$ for which the following conditions are satisfied:

(i) the adjoint inclusion:

$$(\dot{p}(s), p(s)) = \nabla_{x,v} \ell(s, x_*(s), \dot{x}_*(s)),$$

for almost every $s \in [S,T]$,

(ii) the maximality condition:

$$p(s) \cdot \dot{x}_*(s) - \ell(s, x_*(s), \dot{x}_*(s)) \geq p(s) \cdot v - \ell(s, x_*(s), v),$$

for all $v \in \mathbb{R}^n$ and for almost every $s \in [S,T]$,

(iii) the transversality condition:

$$(p(S), -p(T)) \in \nabla g(x_*(S), x_*(T)) + N_C^L((x_*(S), x_*(T)).$$

However, in the adjoint inclusion above, $\nabla_{x,v}\ell$ is ill-defined. This is yet another problem that was solved using nonsmooth analysis, since under suitable conditions, the following extended Euler-Lagrange conditions can be obtained (see [45, 84]): there exist $p \in W^{1,1}([S,T], \mathbb{R}^n)$ and $\lambda \geq 0$ such that the following conditions are satisfied

(i) the nontriviality condition: $(p, \lambda) \neq (0,0)$,

(ii) the adjoint inclusion:

$$\dot{p}(s) \in \mathrm{co}\left\{\xi \in \mathbb{R}^n, (\xi, p(s)) \in N^L_{\mathrm{Gr}(F(s,\cdot))}(x_*(s), \dot{x}_*(s))\right\},$$

for almost every $s \in [S,T]$,

(iii) the maximality condition:

$$p(s) \cdot \dot{x}_*(s) \geq p(s) \cdot v,$$

for all $v \in F(s, x_*(s))$ and for almost every $s \in [S, T]$,

(iv) the transversality condition:

$$(p(S), -p(T)) \in \lambda \nabla g(x_*(S), x_*(T)) + N_C^L((x_*(S), x_*(T)).$$

1.5 The maximum principle and the value function: sensitivity

A missing link between the maximum principle and the dynamic programming approach is given by sensitivity relations, which provide an interpretation of the adjoint arc and the Hamiltonian, evaluated along an optimal trajectory, in terms of generalized gradients of the value function.

For instance, if $(x_*(\cdot), u_*(\cdot))$ is an optimal trajectory/control pair for the Mayer problem (P), and $p(\cdot)$ is the adjoint arc given by the maximum principle, then a smooth version[2] of the sensitivity relations is :

$$\begin{cases} (H(t, x_*(t), p(t)), -p(t)) = \nabla V(t, x_*(t)), \text{ for all } t \in [t_0, T], \text{ (full sensitivity relation)} \\ -p(t) = \partial_x V(t, x_*(t)), \text{ for all } t \in [t_0, T], \text{ (partial sensitivity relation).} \end{cases}$$

A specificity of this smooth framework is that the partial sensitivity relation is trivially implied by the full sensitivity relation and that the adjoint arc $p(\cdot)$ given by the maximum principle is unique.

We mentioned many times that even in the smooth setting, the value function might be non-differentiable, which (once again) drove researchers to resort to nonsmooth analysis. For instance when the value function is merely locally Lipschitz continuous, the sensitivity relations were expressed in [30] using (Fréchet) superdifferentials (see Chapter 2). Quite remarkably in this context, the sensitivity relations help provide necessary and sufficient conditions for optimality. Note that due to the nature of generalized gradients and differentials in nonsmooth analysis, the full sensitivity relation does not imply in general the partial sensitivity relation.

When the data are not smooth or a state constrained problem is considered, the adjoint arc is not unique, which makes the proofs of sensitivity relations more elaborate. For the unconstrained case, we refer to [38] (partial sensitivity relation using Clarke's generalized gradient), [82, 84] (full sensitivity relation using limiting subdifferentials) and [32] (partial and full sensitivity relations using superdifferentials and proximal subgradients). For the state constrained case a recent reference is [20] where the existence of an adjoint arc satisfying both sensitivity relations at the same time was established. Researchs about second-order sensitivity relations have also been carried out [33].

1.6 Uniqueness of a solution to (HJB): Invariance/Viability theory

The tools allowing for a notion of generalized solution to (HJB) have been outlined in Subsection 1.4.1. Another important step of the characterization still has to be dealt with. From the humble experience of a Ph.D. student, to prove that the value function is a generalized solution to (HJB) (existence of a solution) is not nearly as challenging as proving it is the unique generalized solution (uniqueness).

[2] Here the data and the value function are assumed to be smooth.

Viability theory is a mathematical field that studies the evolution of dynamical systems under constraints on the system. In the framework of (HJB) equation, viability theory is used in two ways:

- weak invariance theorems also called viability theorems assert the existence of solutions to constrained differential inclusions that have the following form (here $y_0 \in D$):

$$\begin{cases} \dot{y}(t) \in Q(t, y(t)), \text{ for a.e. } t \in [S, T] \\ y(t) \in D, \text{ for all } t \in [S, T], \\ y(S) = y_0, \end{cases}$$

  where D is a closed set;

- strong invariance theorems assert that all trajectories starting from some $y_0 \in D$ that are solutions to the following differential inclusion

$$\begin{cases} \dot{y}(t) \in Q(t, y(t)), \text{ for a.e. } t \in [S, T] \\ y(S) = y_0 \end{cases},$$

  satisfy the constraint $y(t) \in D, \text{ for all } t \in [S, T]$.

A well-known contribution in order to establish uniqueness of solutions to (HJB) via viability theory is [51]: viability theory was exploited in order to study the behaviour of F-trajectories evolving in the epigraph of V, which yields comparison results for (HJB). As a result, the value function is the unique lower semicontinuous function satisfying (HJB) in a generalized sense which involves the lower Dini derivative, or, equivalently, the Fréchet sub/superdifferentials. This was subsequently refined in [42], by a characterization of the value function as the unique solution to (HJB) in a generalized sense which makes use of proximal subgradients. Since then, using viability theory to establish uniqueness of solution to Hamilton-Jacobi equations (coming either from calculus of variations or optimal control) has become both a useful and usual technique (see [21, 42, 50, 49, 54, 71, 72, 84] and many others).

Let us illustrate how viability can be used to derive comparison results between solutions to (HJB). For example, if $W : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proximal solution to (HJB) (as defined in Subsection 1.4.1), we sketch the proof of the fact that $V \leq W$.

We introduce a multivalued function $W : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \times \mathbb{R}$ defined by:

$$Q(\tau, x) := \Big\{(1, v, -\eta) \,|\, v \in F(\tau, x), \eta \geq L(\tau, x, v)\Big\},$$

Fix $(t_0, x_0) \in]S, T[\times\mathbb{R}^n$. Applying a weak invariance theorem to the following differential inclusion:

$$\begin{cases} (\dot{\tau}, \dot{x}, \dot{\ell})(t) \in Q(\tau(t), x(t)), \text{ for a.e. } t \in [t_0, T], \\ (\tau(t), x(t), \ell(t)) \in \text{epi } W, \text{ for all } t \in [t_0, T], \\ (\tau(t_0), x(t_0), \ell(t_0)) = (t_0, x_0, W(t_0, x_0)), \end{cases}$$

yields the desired inequality.

The fact that W is a proximal solution helps check that the relevant hypotheses of the viability theorem are satisfied, in particular the 'inward pointing condition': for every $(\tau, x) \in]S, T[$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\mathrm{epi}\, W}((\tau, x), W(\tau, x))$,

$$\min_{w \in Q(\tau,x)} (\xi^0, \xi^1, -\lambda) \cdot w \leq 0.$$

This condition somehow means that we can always find velocity vectors in $Q(\tau, x)$ pointing towards the interior of $\mathrm{epi}\, W$ (see figure 2.6 in Chapter 2).

The weak invariance theorem states that there exists $(\tau(\cdot), x(\cdot), \ell(\cdot)) \in W^{1,1}([t_0, T], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ satisfying $\tau(t) = t$ and

$$\begin{cases} (1, \dot{x}(t), \dot{\ell}(t)) \in Q(t, x(t)), \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0, \ell(t_0) = W(t_0, x_0) \\ \ell(t) \geq W(t, x(t)), \quad \text{for all } \ t \in [t_0, T] \ . \end{cases}$$

Taking into account the definition of the multivalued function Q, we deduce that $x(\cdot)$ is an F-trajectory and that $\dot{\ell}(s) \leq -L(s, x(s), \dot{x}(s))$ for a.e. $s \in [t_0, T]$. Hence we have:

$$g(x(T)) = W(T, x(T)) \leq \ell(T) = \ell(t_0) + \int_{t_0}^{T} \dot{\ell}(s) \mathrm{d}s \leq W(t_0, x_0) - \int_{t_0}^{T} L(s, x(s), \dot{x}(s)) \mathrm{d}s,$$

which implies:

$$g(x(T)) + \int_{t_0}^{T} L(s, x(s), \dot{x}(s)) \mathrm{d}s \leq W(t_0, x_0),$$

and then:

$$V(t_0, x_0) \leq W(t_0, x_0).$$

The inequality

$$V(t_0, x_0) \geq W(t_0, x_0),$$

could be obtained using a similar analysis together with a strong invariance theorem. Together with the existence result, this would show that V is the unique solution to (HJB).

1.7 Neighbouring feasible trajectories results

Neighbouring feasible trajectories results allow to approximate F-trajectories evolving possibly outside of the state constraint A by F-trajectories taking values in A. The estimates between a non feasible trajectory $\tilde{y}(\cdot)$ and its feasible approximation $y(\cdot)$ can be given using different norms: for instance $\mathbb{L}^\infty$ (these estimates are well-suited for Mayer problem) or in $W^{1,1}$ (which are more useful for Bolza problems). Moreover, the distance between these two functions should go to 0 as the violation of the constraint, namely the quantity $\max_{t \in [S,T]} \mathrm{d}_A(\tilde{y}(t))$ goes to 0. Some references are [18, 19, 22, 29, 56, 57, 75].

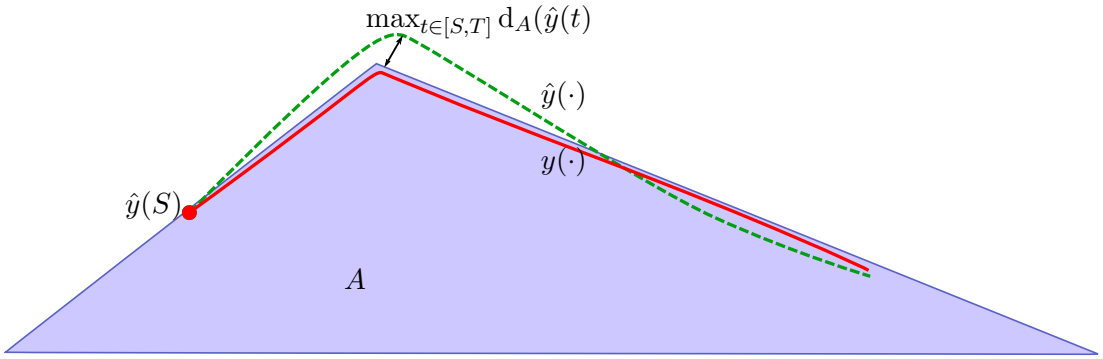

Figure 1.6: Neighbouring feasible trajectories result. The arc $\hat{y}$ is not feasible, the 'neighbour' arc y starts from the same point, but then, evolves in the interior of A.

Neighbouring feasible trajectories theorems are useful analytical tools to obtain results for state constrained problems. Roughly speaking they correspond to constraints removal and in that regard, they allow to invoke the dynamic programming approach and to develop the Hamilton-Jacobi theory (regularity of the value function and characterizations of value functions [21, 56]), as well as to deal with delicate aspects of the necessary conditions theory (see [75] for applications to abnormality) or even to establish sensitivity conditions [57].

A key hypothesis allowing to derive such neighbouring feasible trajectories results is the validity of a compatibility condition between the boundary of the constraint A and the dynamics F. Two types of compatibilities, namely inward pointing and outward pointing conditions can be considered. For instance, a smooth version of the inward pointing condition is: for all $x \in \partial A$, for all $t \in [S,T]$, there exists $v \in F(t,x)$ such that:

$$n_x \cdot v > 0,$$

where n_x is the unit inward normal to A at x.

If A does not have a smooth border, then (inward) normals to A are ill-defined, and nonsmooth analysis is helpful to generalize these conditions. For instance, using the contingent cone to A (see Chapter 2), the previous conditions can be reformulated: for all $x \in \partial A$, for all $t \in [S,T]$:

$$F(t,x) \cap \operatorname{int} T_A(x) \neq \emptyset.$$

1.8 Introduction to Part 1: Hamilton-Jacobi-Bellman equation for Optimal Control problems

1.8.1 Characterizations of value functions

It is well-known that continuous viscosity solutions for Hamilton-Jacobi equations were introduced in the viscosity theory context by the mean of test functions [47, 48]. They were named after the vanishing viscosity technique. This technique consists in adding a small friction term in a partial differential equation $(\varepsilon \Delta V)_{\varepsilon>0}$, in hopes of applying compactness results to the family of solutions $(V_\varepsilon)_{\varepsilon>0}$ as ε vanishes (a well-known ilustration of this technique can be found in [31]).

The viscosity solutions were used to study continuous value function related to Mayer's optimal control problems, which allowed to establish generalized characterization of (uniformly continuous) value function

of the Mayer problem [5, 43, 31]. Viscosity solutions also revealed themselves well-suited to characterize value functions related to optimal control problems with a state constraint (see [71] for the Mayer problem and [56]). A trade mark of viscosity solutions is that the characterization is expressed using two inequalities (whether it involves test functions, sub and superdifferentials, or even strict normals to the epigraph and hypograph), meaning that the candidate function is both a viscosity subsolution and supersolution. Slight variations of viscosity solutions have also been used to characterize continuous minimum time function for sweeping process, invoking proximal normals to the epigraph and hypograph instead of strict normals (see [46]). One can also mention [34], where the asymptotic analysis of constrained viscosity solutions related to an infinite horizon problem with a vanishing discount was carried out.

The notion of viscosity solution was also extended to deal with less regular (semicontinuous) value functions [5, 6]. For Hamilton-Jacobi-Bellman equations related to Mayer's optimal control problems it was shown that simpler notions of solutions using only subdifferentials [8, 10, 51] could be introduced. In the continuous case these solutions coincide with viscosity solutions. Similar characterization also hold true when the value function is related to a state constrained problem (see [53] for the controlled finite horizon Bolza problem and [54] for the controlled infinite horizon control problem, as well as [56]) or to characterize minimum time functions (see [25]). We can also mention characterizations for Hamilton-Jacobi equations related to Bolza problems: [58] where the characterization is given using limiting subgradients, [72] in which subdifferentials are used, and [49, 50].

Using proximal subdifferentials and lower Dini derivatives is another effective way of characterizing semicontinuous value functions. Characterizations of value functions related to Mayer's optimal control problems were first proved when the dynamics are continuous with respect to time (see [42] for a proximal characterization and [51] for a Dini characterization). In the case the dynamics are merely measurable some results were also obtained in [52] (existence and half-characterization) and [52] (full characterization) imposing additional conditions on the class of functions which are candidate to be solutions. Proximal and Dini characterizations are provided in the book [84]. Another recent reference dealing with these characterizations is [21] in the case the dynamics are continuous almost everywhere with respect to time. When it comes to value functions arising in other problems, a proximal characterization of the lower semicontinuous minimal time function was provided in [86] while Dini and proximal characterizations of value function related to a non autonomous Bolza problem in calculus of variations was achieved in [72]. See also [49, 50] for value functions of autonomous Bolza problem in the calculus of variations.

1.8.2 The stakes of the Chapter 3

Consider the non autonomous Bolza problem in optimal control:

$$(P_{S,x_0})\begin{cases}\text{Minimize } g(x(T)) + \int_S^T L(t,x(t),\dot{x}(t))\mathrm{d}t\\ \text{over arcs } x \in W^{1,1}([S,T],\mathbb{R}^n) \text{ satisfying}\\ \dot{x}(t) \in F(t,x(t)) \text{ for almost every } t \in [S,T],\\ x(S) = x_0,\end{cases}$$

in which $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ are given functions, $[S,T]$ is a given interval, $x_0 \in \mathbb{R}^n$ is a given point, and $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ a multivalued function.

Under appropriate assumptions, the value function of the problem, defined by: for all $(t,x) \in [S,T] \times \mathbb{R}^n$,

$$V(t,x) = \inf \left\{ g(y(T)) + \int_t^T L(s, y(s), \dot{y}(s)) \mathrm{d}s,\ y(\cdot)\ \ F\text{-trajectory on } [t,T], y(t) = x \right\},$$

is the unique (generalized) solution to the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t \varphi(t,x) + H(t,x,\partial_x \varphi(t,x)) = 0 \\ \varphi(T,x) = g(x), \end{cases} \tag{HJB}$$

where $H(t,x,p) = \inf_{v \in F(t,x)} \{p \cdot v + L(t,x,v)\}$ is the minimized Hamiltonian.

Characterizations of value functions related to Mayer's optimal control problems ($L = 0$) have already been widely studied. Granted that F is continuous with respect to time, the application of viability theory to characterize lower semicontinuous value functions for optimal control problems with extended valued terminal costs was first achieved in a paper by Frankowska [51] that, as previously underlined in this thesis (see page 29), had a great influence on the development of the Hamilton-Jacobi theory.
In this paper, it was established that V is the unique extended valued, lower semicontinuous function satisfying (HJB) in the following generalized sense:

(c1) for every $(t,x) \in ([S,T[\times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\inf_{v \in F(t,x)} D_\uparrow V((t,x),(1,v)) \leq 0,$$

(c2) for every $(t,x) \in (]S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\sup_{v \in F(t,x)} D_\uparrow V((t,x),(-1,-v)) \leq 0,$$

(c3) for every $x \in \mathbb{R}^n, V(T,x) = g(x)$.

Here, $D_\uparrow \varphi(x,d) := \liminf_{\substack{h \downarrow 0 \\ e \to d}} h^{-1}(\varphi(x+he) - \varphi(x))$ denotes the lower Dini derivative (see Chapter 2).
In the same paper we can also find characterizations using subdifferentials and superdifferentials that are related to viscosity solutions for continuous value functions.

With the same hypotheses, this characterization was refined using proximal subgradients [42]:

(p1) for every $(t,x) \in (]S,T[\times \mathbb{R}^n) \cap \mathrm{dom}(V)$, and every $(\xi^0, \xi^1) \in \partial_P V(t,x)$,

$$\xi^0 + \inf_{v \in F(t,x)} \xi^1 \cdot v = 0,$$

(p2) for every vector $x \in \mathbb{R}^n$, we have $\liminf_{\{(t',x') \to (S,x),\, t' > S\}} V(t',x') = V(S,x)$, and also

$$\liminf_{\{(t',x') \to (T,x),\, t' < T\}} V(t',x') = V(T,x) = g(x).$$

Note that only non asymptotic vectors of the proximal normal cone $N^P_{\mathrm{epi}\, V}((t,x), V(t,x))$ are being used in this characterization. The contribution of horizontal normals can be easily removed when F is continuous with respect to time, owing to a well-known result by Rockafellar (see Theorem 2.4.15).

In the case F is merely measurable with respect to time, the problem of the characterization of the value function as a Dini/contingent solution to (HJB) in an almost everywhere sense was first adressed in [83]. The fact that V is a solution to (HJB) was proven, as well as one comparison result in the class of functions that are uniformly absolutely continuous with respect to time.
Then a full characterization was obtained in [52], in an almost everywhere sense. More specifically, in the class of functions whose epigraph is absolutely continuous, V is the unique function satisfying: there exists a measurable $\mathcal{A} \subset [S,T]$ of zero Lebesgue measure for which

(c1') for every $(t,x) \in ([S,T[\backslash \mathcal{A} \times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\inf_{v \in F(t,x)} D_{\uparrow} V((t,x),(1,v)) \leq 0,$$

(c2') for every $(t,x) \in (]S,T] \setminus \mathcal{A} \times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\sup_{v \in F(t,x)} D_{\uparrow} V((t,x),(-1,-v)) \leq 0,$$

(c3') for every $x \in \mathbb{R}^n, V(T,x) = g(x)$.

Note that the regularity assumption is required to obtain the uniqueness of the solution, which is not achieved in the class of lower semicontinuous functions (see the introduction of [21]).

An intermediate approach was then suggested in [21]. In the case F is continuous on a set of full measure and has everywhere left and right limits with respect to time, two characterizations (using lower Dini derivatives and proximal derivatives) were established. Quite noticeably, the epigraph continuity condition could be removed and the uniqueness was obtained nonetheless, in the class of lower semicontinuous functions. However, the characterization requires to use two different inequalities involving respectively the left and right limits of F.
For the Dini/contingent characterization, V is the unique extended valued lower semicontinuous function satisfying:

(c1") for every $(t,x) \in ([S,T[\times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\inf_{v \in F(t^+,x)} D_{\uparrow} V((t,x),(1,v)) \leq 0,$$

(c2") for every $(t,x) \in (]S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(V)$,

$$\sup_{v \in F(t^-,x)} D_{\uparrow} V((t,x),(-1,-v)) \leq 0,$$

(c3") for every $x \in \mathbb{R}^n, V(T,x) = g(x)$.

On the other hand, the proximal characterization is: V is the unique extended valued lower semicontinuous function satisfying:

(p1") for every $(t,x) \in (]S,T[\times \mathbb{R}^n) \cap \mathrm{dom}(V)$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\mathrm{epi}\, V}((t,x), V(t,x))$,

$$\xi^0 + \min_{v \in F(t^+,x)} \xi^1 \cdot v \leq 0.$$

(p2") for every $(t,x) \in]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(V)$, and every $(\xi^0,\xi^1,-\lambda) \in N^P_{\mathrm{epi}\, V}((t,x),V(t,x))$,

$$\xi^0 + \min_{v\in F(t^-,x)} \xi^1 \cdot v \geq 0.$$

(p3") for every vector $x \in \mathbb{R}^n$, we have $\liminf_{\{(t',x')\to(S,x),\, t'>S\}} V(t',x') = V(S,x)$, and also

$$\liminf_{\{(t',x')\to(T,x),\, t'<T\}} V(t',x') = V(T,x) = g(x).$$

Note that this characterization involves asymptotic vectors and that, at the time of the publication of [21], it was unclear whether or not they could be removed from the characterization in this discontinuous framework.

As far as this thesis is concerned, it aims at answering the following open questions:

(Q1) In the discontinuous framework, is it possible to remove the asymptotic vectors from the proximal characterization ?

(Q2) In the case of the non autonomous Bolza problem, and in a context in which standing hypotheses don't allow to use a state augmentation technique, is it possible to achieve such characterizations of the value function as the unique lower semicontinuous function solution to (HJB), without requiring an additional regularity assumption from candidate solutions?

(Q3) In this framework, is it possible to achieve an extended-sense viscosity solution characterization of lower semicontinuous value functions, employing Fréchet subdifferentials and superdifferentials?

Chapter 3, see page 67, provides positive answers to these questions. In this introduction, only the hypotheses and the main results of the chapter are presented.

The results of this chapter have been published [15].

1.8.3 Hypotheses of Chapter 3

In Chapter 3 the following hypotheses are invoked: for every given positive number R_0, there exist functions $c_F(\cdot) \in \mathbb{L}^1([S,T],\mathbb{R}_+)$ and $k_F(\cdot) \in \mathbb{L}^1([S,T],\mathbb{R}_+)$, a modulus of continuity $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and constants $c_0 > 0$, $M_0 > 0$ such that:

(H1): i) The multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes convex, closed, nonempty values. For every $x \in \mathbb{R}^n$, $F(\cdot,x)$ is Lebesgue measurable on $[S,T]$.

ii) The function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, with nonempty domain.

(H2): i) For almost every $t \in [S,T]$ and $x \in \mathbb{R}^n$

$$F(t,x) \subset c_F(t)(1+|x|)\mathbb{B}.$$

ii) For all $(t,x) \in [S,T] \times R_0\mathbb{B}$

$$F(t,x) \subset c_0\mathbb{B}.$$

(H3): i)
$$\mathrm{d}_H(F(t,x'),F(t,x)) \leq \omega(|x-x'|), \text{ for all } x,x' \in R_0\mathbb{B} \text{ and } t \in [S,T].$$

ii)
$$F(t,x') \subset F(t,x) + k_F(t)|x-x'|\mathbb{B}, \text{ for all } x,x' \in R_0\mathbb{B} \text{ and for a.e. } t \in [S,T].$$

(H4): i) For each $x \in \mathbb{R}^n$, $s \in [S,T[$, and $t \in]S,T]$ the following limits (in the sense of Kuratowski) exist and are nonempty
$$F(s^+,x) := \lim_{s' \downarrow s} F(s',x) \text{ and } F(t^-,x) := \lim_{t' \uparrow t} F(t',x).$$

ii) For almost every $s \in [S,T[$ and $t \in]S,T]$, and every $x \in \mathbb{R}^n$ we have
$$F(s^+,x) = F(s,x) \text{ and } F(t^-,x) = F(t,x).$$

(H5): i) The Lagrangian $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable. For every $t \in [S,T]$ and $x \in \mathbb{R}^n$, $L(t,x,\cdot)$ is convex.

ii) L is locally bounded in the following sense
$$|L(t,x,v)| \leq M_0, \quad \text{for all} \quad (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_0\mathbb{B} \ .$$

(H6): i) $|L(t,x',v) - L(t,x,v)| \leq \omega(|x-x'|)$, for all $x,x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_0\mathbb{B}$.

ii) $L(t^-,x,v) := \lim_{t' \uparrow t} L(t',x,v)$ exists for every $(t,x,v) \in]S,T] \times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(t^-,x,v) = L(t,x,v)$ for a.e. $t \in]S,T]$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$.

iii) $L(s^+,x,v) := \lim_{s' \downarrow s} L(s',x,v)$ exists for every $(s,x,v) \in [S,T[\times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(s^+,x,v) = L(s,x,v)$ for a.e. $s \in [S,T[$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$.

1.8.4 Results of Chapter 3

The following subsection gathers the two main results of Chapter 3, without complementary results, interpretations, remarks or examples. More details are to be found starting from page 73.

Theorem 1.8.1 answers positively to questions (Q1) and (Q2) that were asked page 35. It gives a characterization of lower semicontinuous extended valued value function using proximal subdifferentials (not taking into account the contribution of horizontal proximal normals) and lower Dini derivatives.

Theorem 1.8.2 provides a positive answer to the second question (Q3). If we assume that g is locally bounded and satisfies some type of regularity property in terms of lower and upper semicontinuous envelopes, the value function is the unique locally bounded function solution to (HJB) in an extended viscosity sense, using the Fréchet subdifferential of V and the Fréchet superdifferential of V^*.

Theorem 1.8.1 *Assume (H1)–(H6). Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then the assertions (a), (b) and (c) below are equivalent.*

(a) The function U is the value function for $(P_{t,x})$: $U = V$.

(b) The function U is lower semicontinuous and satisfies:
i) for every $(t,x) \in ([S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$

$$\inf_{v\in F(t^+,x)} \left[D_\uparrow U((t,x),(1,v)) + L(t^+,x,v)\right] \leq 0\,;$$

ii) for every $(t,x) \in (]S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(U)$

$$\sup_{v\in F(t^-,x)} \left[D_\uparrow U((t,x),(-1,-v)) - L(t^-,x,v)\right] \leq 0\,;$$

iii) for all $x \in \mathbb{R}^n$, $U(T,x) = g(x)$.

(c) The function U is lower semicontinuous and satisfies:
i) for every $(t,x) \in (]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$,

$$\xi^0 + \min_{v\in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \leq 0\,;$$

ii) for every $(t,x) \in (]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$,

$$\xi^0 + \min_{v\in F(t^-,x)} \left[\xi^1 \cdot v + L(t^-,x,v)\right] \geq 0\,; \tag{1.5}$$

iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x),\, t'>S\}} U(t',x') = U(S,x),$$

and

$$\liminf_{\{(t',x')\to(T,x),\, t'<T\}} U(t',x') = U(T,x) = g(x).$$

Theorem 1.8.2 *Assume (H1)–(H6). Suppose, in addition, that g is locally bounded and satisfies $(g^*)_* = g$. Let $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}$ be a locally bounded function. Then, the assertions (a), (b) and (c) of Theorem 3.4.1 are equivalent to (d) below.*
(d) U is lower semicontinuous and satisfies:
i) For every $(t,x) \in]S,T[\times\mathbb{R}^n$, $(\xi^0,\xi^1) \in \partial_- U(t,x)$,

$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \leq 0\,;$$

ii) for every $(t,x) \in]S,T[\times\mathbb{R}^n$, $(\xi^0,\xi^1) \in \partial_+ U^*(t,x)$,

$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \geq 0\,; \tag{1.6}$$

iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x)\ \mid\ t'>S\}} U(t',x') = U(S,x),$$

$$U(T,x) = g(x),$$

and

$$U^*(T,x) = g^*(x).$$

1.8.5 The stakes of Chapter 4

In Chapter 4, we add a state constraint to the Bolza problem of Chapter 3:

$$(SC_{S,x_0}) \quad \begin{cases} \text{Minimize } J(x) = \displaystyle\int_S^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } x(\cdot) \text{ satisfying} \\ \dot{x}(s) \in F(s,x(s)), \text{ for a.e. } s \in [S,T], \\ x(s) \in A, \text{ for all } s \in [S,T], \\ x(S) = x_0, \end{cases}$$

in which $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multivalued function, and A is a given nonempty convex and closed set in $\mathbb{R}^n$.

Similarly to Chapter 3, under appropriate assumptions, the value function of the problem, defined by: for all $(t,x) \in [S,T] \times \mathbb{R}^n$,

$$V(t,x) = \inf \left\{ g(y(T)) + \int_t^T L(s,y(s),\dot{y}(s))\mathrm{d}s,\ y(\cdot) \text{ feasible } F\text{-trajectory on } [t,T], y(t) = x \right\},$$

is the unique (generalized) solution to the (HJB) equation. We recall that in this definition, feasible means that $y(s) \in A$, for all $s \in [t,T]$.

Quite similarly to the unconstrained case, characterizations of value function for problems involving data that are merely measurable with respect to time have been investigated. This resulted in characterizations in an 'almost everywhere' sense for infinite horizon Lagrange control problems, imposing some additional regularity to the candidate solutions (cf [9]).

However, until now, the most general known class of time-discontinuous problems allowing to provide 'everywhere in t' characterizations of value functions was introduced in [21, Theorems 4.1 and 4.2] to investigate Mayer problems ($L = 0$). In this paper, the multivalued function F merely has bounded variation with respect to the time variable, uniformly over the state variable. Two characterizations have been obtained, which differ according to whether an outward or inward constraint qualification is assumed.

If an **outward pointing condition** is assumed, namely for each $s \in [S,T[$, $t \in]S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap (-\operatorname{int} T_A(x)) \neq \emptyset \text{ and } F(s^+,x) \cap (-\operatorname{int} T_A(x)) \neq \emptyset, \tag{OPC}$$

then V can be characterized as the unique generalized solution to (HJB) in a Dini and proximal sense.

More precisely, in [21], the following Dini characterization was shown. The value function V is the unique extended valued lower semicontinuous function satisfying $V(t,x) = +\infty$ if $x \notin A$ and :

(c1) for every $(t,x) \in ([S,T[\times A) \cap \operatorname{dom}(V)$,

$$\inf_{v \in F(t^+,x)} D_\uparrow V((t,x),(1,v)) \leq 0\,;$$

(c2) for every $(t,x) \in (]S,T] \times \operatorname{int} A) \cap \operatorname{dom}(V)$,

$$\sup_{v \in F(t^-,x)} D_\uparrow V((t,x),(-1,-v)) \leq 0\,;$$

(c3) for every $x \in A, \liminf_{\{(t',x')\to(T,x),\, t'<T, x'\in \operatorname{int} A\}} V(t',x') = V(T,x) = g(x)$.

On the other hand, the proximal characterization in this context is: V is the unique extended valued lower semicontinuous function satisfying $V(t,x) = +\infty$ if $x \notin A$ and :

(p1) for every $(t,x) \in]S,T[\times A) \cap \operatorname{dom}(V)$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} V}((t,x), V(t,x))$,

$$\xi^0 + \min_{v\in F(t^+,x)} \xi^1 \cdot v \leq 0\,;$$

(p2) for every $(t,x) \in]S,T[\times \operatorname{int} A) \cap \operatorname{dom}(V)$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} V}((t,x), V(t,x))$,

$$\xi^0 + \min_{v\in F(t^-,x)} \xi^1 \cdot v \geq 0\,;$$

(p3) for every vector $x \in A$, we have $\liminf_{\{(t',x')\to(S,x),\, t'>S\}} V(t',x') = V(S,x)$, and also

$$\liminf_{\{(t',x')\to(T,x),\, t'<T, x'\in \operatorname{int} A\}} V(t',x') = V(T,x) = g(x).$$

When an **inward pointing condition** is assumed, that is to say, for each $s \in [S,T[,\ t \in]S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \operatorname{int} T_A(x) \neq \emptyset \text{ and } F(s^+,x) \cap \operatorname{int} T_A(x) \neq \emptyset, \tag{IPC}$$

then the characterizations of V as the unique generalized solution to (HJB) is obtained at the price of more requirements for the cost function $g(\cdot)$, which is assumed to be **continuous** on A. With these hypotheses, V is the unique extended valued lower semicontinuous function satisfying $V(t,x) = +\infty$ if $x \notin A$ and :

(c1') for every $(t,x) \in ([S,T[\times A) \cap \operatorname{dom}(V)$,

$$\inf_{v\in F(t^+,x)} D_\uparrow V((t,x),(1,v)) \leq 0\,;$$

(c2') for every $(t,x) \in (]S,T] \times \operatorname{int} A) \cap \operatorname{dom}(V)$,

$$\sup_{v\in F(t^-,x)} D_\uparrow V((t,x),(-1,-v)) \leq 0\,;$$

(c3') for every $x \in A, V(T,x) = g(x)$.

On the other hand, the proximal characterization is: V is the unique extended valued lower semicontinuous function satisfying $V(t,x) = +\infty$ if $x \notin A$ and :

(p1') for every $(t,x) \in]S,T[\times A) \cap \operatorname{dom}(V)$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} V}((t,x), V(t,x))$,

$$\xi^0 + \min_{v\in F(t^+,x)} \xi^1 \cdot v \leq 0\,;$$

(p2') for every $(t,x) \in]S,T[\times \operatorname{int} A) \cap \operatorname{dom}(V)$, and every $(\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} V}((t,x), V(t,x))$,

$$\xi^0 + \min_{v\in F(t^-,x)} \xi^1 \cdot v \geq 0\,;$$

(p3') for every vector $x \in A$, we have $\liminf_{\{(t',x')\to(S,x),\, t'>S\}} V(t',x') = V(S,x)$, and also $V(T,x) = g(x)$.

An important feature here, apart from those that have already been discussed in subsection 1.8.2, is that, considering the nature of Mayer problem's cost, a mere $\mathbb{L}^\infty$ 'distance estimate' (see [22]) is sufficient to establish both characterizations of the value function in terms of lower Dini derivatives and proximal subdifferentials.

As previously mentioned, if we are looking for possible extensions of [21] to the case $L \neq 0$, we need to have $W^{1,1}$ 'distance estimates' at our disposal. More precisely, if $\hat{x}(\cdot)$ is a given F-trajectory, possibly not feasible, then we want to construct a second feasible F-trajectory $x(\cdot)$ which is close to $\hat{x}(\cdot)$ with respect to the $W^{1,1}$ distance. Usually, such estimates are provided using a suitable modulus of continuity $\theta(\cdot)$ evaluated in a 'state constraint violation' parameter ρ. Though linear estimates ($\theta(\rho) = K\rho$ with $K > 0$) are the most valuable, it has been shown (cf. [16]) that they cannot generally be obtained outside of a very smooth context. In fact, a later paper [17] pointed out that estimates of the type $\theta(\rho) = K\rho|\ln(\rho)|$ with $K > 0$ are optimal in some cases (with a constraint A that is not smooth).

For discontinuous time-dependent F's (with bounded time-variation w.r.t. time), $\mathbb{L}^\infty$-distance estimates were obtained in [22]. On the other hand, the paper [29], deals with a convex compact constraint A coupled with a differential inclusion involving a Lipschitz, **time-independent** multifunction F, obtaining $\rho|\ln(\rho)|$ $W^{1,1}$ estimates. However, to extend the results of [22] and [29] to $W^{1,1}$-estimates for a class of problems in which F depends on time in a possibly discontinuous manner is far from trivial. This is one of the aspect that has been investigated in this thesis.

Another question raised by the work presentend in [21] is whether or not it is possible to obtain an equivalent viscosity characterizations (using Fréchet subdifferential and superdifferential). When it comes to the state constrained case, this type of characterization was established in [56, Theorem 3] in the class of continuous functions, but only when the data are continuous, and at the cost of a more demanding type of constraint qualification. Keeping this stronger constraint qualification, but weakening the continuity assumptions on the data, only partial results have been obtained, like one-side comparison theorems (cf. [56, Theorem 4]).

As far as this thesis is concerned, it aims at answering the following open questions:

(Q1) Provided that A and F satisfy the inward/outward compatibility conditions (IPC)/(OPC) of [21], and that F has bounded variations with respect to time (uniformly with respect to the state variable), can we obtain $W^{1,1}$ distance estimates results. If so, under which additional hypotheses?

(Q2) Once suitable $W^{1,1}$ distance estimates results are available, in the case of the constrained non autonomous Bolza problem, is it possible to achieve 'everywhere in t' characterizations of the value function as the unique lower semicontinuous function solution to (HJB), without requiring an additional regularity assumption from candidate solutions?

(Q3) Is it possible not to resort to asymptotic normals in the proximal characterization of the value function ?

(Q4) In the same framework, is it possible to achieve extended-sense viscosity characterizations of value functions, employing Fréchet subdifferentials and superdifferentials? If so, which type of constraint qualifications are required, and which additional hypothesis do we need to impose to $g(\cdot)$?

In this introduction, only the hypotheses and the main results of the chapter are presented.

1.8.6 Hypotheses in Chapter 4

In Chapter 4, the following hypotheses are invoked: for every given positive number R_0, there exist positive functions $c_F(\cdot) \in \mathbb{L}^1(S,T)$ and $k_F(\cdot) \in \mathbb{L}^\infty(S,T)$, a modulus of continuity $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and constant $c_0 > 0$, $M_0 > 0$ such that

(H1): the multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes convex, closed, non-empty values; for every $x \in \mathbb{R}^n$, $F(\cdot, x)$ is Lebesgue measurable on $[S,T]$;

(H2): $F(t,x) \subset c_F(t)(1+|x|)\,\mathbb{B}$ for all $x \in \mathbb{R}^n$ and for a.e. $t \in [S,T]$;

(H3): $F(t,x') \subset F(t,x) + k_F(t)|x-x'|\,\mathbb{B}$ for all $x, x' \in R_0\mathbb{B}$ and a.e. $t \in [S,T]$;

(H4): $F(.,x)$ has bounded variation uniformly over $x \in R_0\mathbb{B}$, in the following sense: there exists a non-decreasing bounded variation function $\eta(\cdot) : [S,T] \to [0,\infty)$ such that

(i) for every $[s,t] \subset [S,T]$ and $x \in R_0\mathbb{B}$,

$$d_H(F(s,x), F(t,x)) \leq \eta(t) - \eta(s) \ ;$$

(ii) for every $\mu > 0$ and every $[t_0,t_1] \subset [S,T]$ there exists a partition $\{t_0 =: \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \ldots < \tilde{t}_M := t_1\}$ such that for each $k = 0, 1, \ldots, M-1$ we have

$$\lim_{\epsilon \downarrow 0} \int_{\tilde{t}_k + \epsilon}^{\tilde{t}_{k+1}} \frac{\eta(\tau) - \eta(\tilde{t}_k^+)}{\tau - \tilde{t}_k} d\tau \leq \mu \ .$$

(H5): (i) the Lagrangian $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable; for every $t \in [S,T]$ and $x \in \mathbb{R}^n$, $L(t,x,\cdot)$ is convex;

(ii) L is locally bounded in the following sense:

$$|L(t,x,v)| \leq M_0, \quad \text{for all} \quad (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_0\mathbb{B} \, ;$$

(H6): (i) $|L(t,x',v) - L(t,x,v)| \leq \omega(|x-x'|)$, for all $x, x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_0\mathbb{B}$;

(ii) $L(t^-,x,v) := \lim_{t' \uparrow t} L(t',x,v)$ exists for every $(t,x,v) \in (S,T] \times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(t^-,x,v) = L(t,x,v)$ for a.e. $t \in (S,T]$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$;

(iii) $L(s^+,x,v) := \lim_{s' \downarrow s} L(s',x,v)$ exists for every $(s,x,v) \in [S,T) \times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(s^+,x,v) = L(s,x,v)$ for a.e. $s \in [S,T)$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$;

(H7): $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, with nonempty domain;

(H8): $A \subset \mathbb{R}^n$ is convex and closed;

(OPC): for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset \qquad \text{and} \qquad F(s^+,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset \, ;$$

(IPC): for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \operatorname{int} T_A(x) \neq \emptyset \qquad \text{and} \qquad F(s^+,x) \cap \operatorname{int} T_A(x) \neq \emptyset \, .$$

1.8.7 Results of Chapter 4

The following subsection gathers the five main results of Chapter 4, without complementary results, interpretations, remarks or examples. More details are to be found starting from page 101.

A very early version of these results is published in [13]. The results proposed in this chapter have been submitted in their current form.

Chapter 4 provides positive answers to the three questions that were asked page 40.

Theorem 1.8.3 provides a positive answer to (Q1). If we assume that the bounded variation function (see hypothesis (H4) below) that controls the variations of F with respect to time satisfies some type of uniform Dini's test (see hypothesis (H4) (ii) and the Remark page 108), then we obtain the desired $W^{1,1}$ distance estimates, even when F is not convex valued.

Theorems 1.8.4 and 1.8.5 give a positive answer to question (Q2) and (Q3). They provide two 'everywhere in t' characterizations of the value function V in the class of lower semicontinuous functions.
The first one holds when g is lower semicontinuous and an outward constraint qualification is satisfied, while the second deals with a function g that is continuous on A coupled with an inward constraint qualification. In both cases, the proximal characterization is expressed using only the proximal subdifferential of V, without resorting to asymptotic vectors.
Illustrative examples allow to understand that finding the right coupling between the regularity of g and the suitable constraint qualification is crucial as characterizations may fail when other hypotheses are tested.

Eventually, Theorems 1.8.6 and 1.8.7 provide extended-sense viscosity characterizations of value functions, answering positively to question (Q4).
Theorem 1.8.6 deals with the case of a lower semicontinuous final cost g whose restriction to A is locally bounded and satisfies some type of regularity property with respect to the upper/lower semicontinuous envelopes, coupled with both an inward and outward pointing constraint qualification. It provides a characterization of lower semicontinuous, locally bounded value functions, which is expressed using the Fréchet subdifferential of V and, in the interior of A, the Fréchet superdifferential of the upper semicontinous envelope of V. An example illustrates the fact that even though the outward pointing constraint qualification does not intervene to prove that V is an extended-sense viscosity solution, it is indispensable to derive uniqueness of solutions to the associated Hamilton-Jacobi equation.
Theorem 1.8.7 deals with the case of a final cost g whose restriction to A is continuous, coupled with an inward pointing constraint qualification. It provides a characterization of value functions whose restriction to A are continuous, expressed using the Fréchet subdifferential and superdifferential of V.
Quite noticeably, the theorems presented in chapter 4 are illustrated by an economics example in which the integral cost is merely continuous w.r.t. the state variable x (and not locally Lipschitz continuous). This uncommon behaviour is not obtained by an artificial construct since it is due to an inherent fractional singularity term which is introduced to interpret the production function (cf. [1]). This shows that we do not invoke hypothesis (H6) (i) for the love of abstraction alone, but also in order to cover practical and tangible problems.

Theorem 1.8.3 *Fix $r_0 > 0$. Assume that, for some positive functions $c_F(\cdot) \in \mathbb{L}^1(S,T)$ and $k_F(\cdot) \in \mathbb{L}^\infty(S,T)$ and for $R_0 := e^{\int_S^T c_F(s)\mathrm{d}s}(r_0+1)$, the following hypothesis*

(H1)′: $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, non-empty values, $F(\cdot,x)$ is $\mathcal{L}$-measurable for all $x \in \mathbb{R}^n$,

is satisfied together with (H2), (H3), (H4) and

(IPC)′: for each $s \in [S,T)$, $t \in (S,T]$ and $x \in R_0\mathbb{B} \cap \partial A$,

$$\operatorname{co} F(t^-,x) \cap \operatorname{int} T_A(x) \;\neq\; \emptyset \qquad \text{and} \qquad \operatorname{co} F(s^+,x) \cap \operatorname{int} T_A(x) \;\neq\; \emptyset\,.$$

Then, there exists a constant $K > 0$ with the following property: given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0,t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{\int_S^{t_0} c_F(s)\mathrm{d}s}(r_0+1)-1\big)\mathbb{B}$, and any $\rho > 0$ such that

$$\rho \;\geq\; \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}\,,$$

we can find an F-trajectory $x(\cdot)$ on $[t_0,t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \;\in\; \operatorname{int} A \quad \text{for all } t \in (t_0,t_1]$$

and

$$\|\hat{x} - x\|_{\mathbb{L}^\infty([t_0,t_1])} \;\leq\; K\;\rho\,, \tag{1.7}$$

$$\|\dot{\hat{x}} - \dot{x}\|_{\mathbb{L}^1([t_0,t_1])} \;\leq\; K\;\rho(1+|\ln(\rho)|)\,. \tag{1.8}$$

Theorem 1.8.4 (Characterization of lsc Value Functions - Outward-pointing Condition) *Assume (H1)–(H8) and (OPC). Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then assertions (a), (b) and (c) below are equivalent:*

(a) U is the value function for $(SC_{t,x})$, i.e. $U = V$.

(b) U is lower semicontinuous on $[S,T] \times \mathbb{R}^n$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and

(i) for all $(t,x) \in ([S,T) \times A) \cap \operatorname{dom} U$

$$\inf_{v \in F(t^+,x)} \big[D_\uparrow U((t,x),(1,v)) + L(t^+,x,v)\big] \leq 0\,;$$

(ii) for all $(t,x) \in ((S,T] \times \operatorname{int} A) \cap \operatorname{dom} U$

$$\sup_{v \in F(t^-,x)} \big[D_\uparrow U((t,x),(-1,-v)) - L(t^-,x,v)\big] \leq 0\,;$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (T,x) \;|\; t' < T, x' \in \operatorname{int} A\}} U(t',x') = U(T,x) = g(x).$$

(c) U is lower semicontinuous on $[S,T] \times \mathbb{R}^n$ $U(t,x) = +\infty$ if $x \notin A$, and

(i) for all $(t,x) \in ((S,T) \times A) \cap \operatorname{dom} U$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$

$$\xi^0 + \min_{v \in F(t^+,x)} \big[\xi^1 \cdot v + L(t^+,x,v)\big] \leq 0\,;$$

(ii) for all $(t,x) \in ((S,T) \times \operatorname{int} A) \cap \operatorname{dom} U$, $(\xi^0, \xi^1) \in \partial_P U(t,x)$

$$\xi^0 + \min_{v \in F(t^-,x)} \left[\xi^1 \cdot v + L(t^-, x, v)\right] \geq 0\,;$$

(iii) for all $x \in A$,

$$\liminf_{\{(t',x') \to (S,x) \ \mid \ t' > S\}} U(t',x') = U(S,x) \tag{1.9}$$

and

$$\liminf_{\{(t',x') \to (T,x) \ \mid \ t' < T, \ x' \in \operatorname{int} A\}} U(t',x') = U(T,x) = g(x). \tag{1.10}$$

Theorem 1.8.5 (Characterization of lsc Value Functions - Inward-pointing Condition) *Assume that (H1)–(H8), (IPC) are satisfied and that* g *is continuous on* A. *Let* $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be an extended valued function. Then the assertions (a), (b) and (c) of Theorem 4.4.1 remain equivalent.*

Theorem 1.8.6 (Characterization of locally bounded lsc Value Functions - Inward/Outward-pointing Condition) *Assume (H1)–(H8), (OPC) and (IPC). Suppose, in addition, that* $g_{|A}$ *is locally bounded and satisfies* $((g_{|A})^*)_* = g_{|A}$. *Let* $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be an extended valued function. Then, the assertions (a), (b) and (c) of Theorem 4.4.1 are equivalent to condition (d) below:*

(d) U *is lower semicontinuous on* $[S,T] \times \mathbb{R}^n$ *and locally bounded on* $[S,T] \times A$, *satisfies* $U(t,x) = +\infty$ *whenever* $x \notin A$ *and*

(i) for all $(t,x) \in (S,T) \times A$, $(\xi^0, \xi^1) \in \partial_- U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+, x, v)\right] \leq 0\,; \tag{1.11}$$

(ii) for all $(t,x) \in (S,T) \times \operatorname{int} A$, $(\xi^0, \xi^1) \in \partial_+ U^*(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+, x, v)\right] \geq 0\,; \tag{1.12}$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (S,x) \ \mid \ t' > S\}} U(t',x') = U(S,x),$$
$$(U_{|[S,T] \times A})^*(T,x) = (g_{|A})^*(x) \quad \text{and} \quad U(T,x) = g(x).$$

Theorem 1.8.7 (Characterization of continuous Value Functions - Inward-pointing Condition) *Assume (H1)–(H8) and (IPC). Suppose, in addition, that* g *is continuous on* A. *Let* $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be an extended valued function. Then, the assertions (a), (b) and (c) of Theorem 4.4.1 are equivalent to condition (d)′ below:*

(d)′ U *is continuous on* $[S,T] \times A$, *satisfies* $U(t,x) = +\infty$ *whenever* $x \notin A$ *and*

(i) for all $(t,x) \in (S,T) \times A$, $(\xi^0, \xi^1) \in \partial_- U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+, x, v)\right] \leq 0\,; \tag{1.13}$$

(ii) for all $(t,x) \in (S,T) \times \operatorname{int} A$, $(\xi^0, \xi^1) \in \partial_+ U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+, x, v)\right] \geq 0\,; \tag{1.14}$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (S,x) \ \mid \ t' > S\}} U(t',x') = U(S,x),$$
$$\text{and} \quad U(T,x) = g(x).$$

1.9 Introduction to Part 2: Applications of Optimal control in Calculus of Variations

Consider the following classical problem of minimizing a functional:

$$\text{(CV1)} \quad \begin{cases} \text{Minimize } J(x) := \displaystyle\int_a^b L(t, x(t), \dot{x}(t))\mathrm{d}t + g(x(a), x(b)) \\ \text{over arcs } x \in W^{1,1}([a,b], \mathbb{R}^n). \end{cases}$$

The function $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is referred to as the Lagrangian of the problem. Its variables are the time variable, the state variable, and the velocity variable: (t, x, ξ). The function g is extended value, hence any kind of starting point/endpoint constraint is covered.

1.9.1 Bolza problem, existence of minimizers

Sufficient conditions on L that guarantee the existence of a minimizer to (CV1) have been provided by Tonelli's existence result (see [84]), namely:

- local boundedness,
- convexity with respect to ξ,
- uniform local Lipschitz continuity with respect to (x, ξ),
- coercivity, also referred to as superlinear growth.

However some variational problems with a Lagrangian $L(t, x, \xi)$ that is not convex in ξ may have a minimizer. Moreover, discontinuous Lagrangians in the state or in the velocity variable arise often in real life engineering problems (e.g. fuel consumption). Consequently, it is relevant to study the behavior of potential minimizers to (CV1), even when L does not satisfy the hypotheses of Tonelli's existence theorem.

1.9.2 Lavrentiev phenomenon and non Lipschitzianity of minimizer

We say that (CV1) exhibits the Lavrentiev phenomenon when the infimum of $J(\cdot)$ over absolutely continuous functions is strictly lower than the infimum over Lipschitz continuous functions:

$$\inf\left\{J(x), x \in W^{1,1}([a,b], \mathbb{R}^n)\right\} < \inf\left\{J(x), x \in W^{1,1}([a,b], \mathbb{R}^n), \dot{x} \in \mathbb{L}^\infty([a,b], \mathbb{R}^n)\right\}.$$

An historical example of a non autonomous Bolza problem exhibiting the Lavrentiev phenomenon is due to Manià (*cf.* [68]).
Take $n = 1$, $a = -1$, $b = 1$, $g = \chi_{\{(-1,1)\}}$ and $L(t, x, \xi) = \xi^6[x^3 - t]^2$. Then, since L is non negative and $J(t \mapsto \sqrt[3]{t}) = 0$, the function $t \mapsto \sqrt[3]{t}$ is a $W^{1,1}$-minimizer and is not Lipschitz continuous, while it can be shown that any Lipschitz continuous function $x(\cdot)$ satisfies $J(x) > 0$. On top of that, the Lavrentiev phenomenon persists even if we slightly enlarge the set of endpoints constraints $\{(-1,1)\}$ or add a small perturbations to L (see. [67]).

When L is autonomous, and even when it is merely Borel measurable, the Lavrentiev phenomenon cannot occur [3]. However, even in that case, some of the minimizers may not be Lipschitz continuous.

1.9.3 Necessity for establishing the regularity of minimizers

When the Lavrentiev phenomenon occurs, or when some of the minimizers are not Lipschitz, the implementation of numerical techniques such as finite-element method to solve the problem (CV1) is compromised: the computation of the infimum for J might be false or some minimizers might never be detected.

Finding minimal conditions for L under which a potential minimizer for (CV1) is necessarily Lipschitz continuous it therefore essential.

Many researchers have contributed to weakening the required hypotheses in the autonomous case. We provide a non exhaustive list: Clarke and Vinter [39], Ambrosio, Ascenzi and Buttazo [2] and finally Dal Maso and Frankowska (*cf.* [50]). This process led to a very simple and appreciable result: assuming that L is Borel measurable, locally bounded and superlinear, then the minimizers of (CV1) are Lipschitz continuous.

In the non autonomous case, fewer and less satisfying[3] results were proposed until a recent collaboration between Bettiol and Mariconda [23, 24]. A more comprehensive presentation of these results will be made in Subsection 1.10.1.

1.9.4 Erdmann – Du Bois-Reymond, a tool do derive the regularity of minimizers

An efficient way to prove that a given minimizer $x_*(\cdot)$ for (CV1) is Lipschitz continuous can be produced when an Erdmann – Du Bois-Reymond (EDBR) condition is satisfied by $x_*(\cdot)$ and L satisfies a suitable growth condition that somehow quantifies that the (EDBR) condition is violated when $|\xi|$ goes to $+\infty$.

To illustrate this fact, assume that L is smooth and that the smooth Erdmann – Du Bois-Reymond equation is satisfied, *i.e.* the arc $p : [a, b] \to \mathbb{R}^n$, defined by

$$p(t) := L(t, x_*(t), \dot{x}_*(t)) - \dot{x}_*(t) \cdot D_\xi L(t, x_*(t), \dot{x}_*(t)), \text{ for all } t \in [a, b],$$

is absolutely continuous and

$$\dot{p}(t) = D_t L(t, x_*(t), \dot{x}_*(t)), \text{ for almost every } t \in [a, b].$$

Assume that the following growth condition is satisfied :

$$\lim_{|\xi| \to +\infty} |L(t, x_*(t), \xi) - \xi \cdot D_\xi L(t, x_*(t), \xi)| = +\infty, \text{ uniformly for almost every } t \in [a, b].$$

Then there exists $R > 0$ and a set of full measure $E \subset [a, b]$ such that:

$$|L(t, x_*(t), \xi) - \xi \cdot D_\xi L(t, x_*(t), \xi)| \geq \|p\|_{\mathbb{L}^\infty} + 1, \text{ for all } t \in E \text{ and } |\xi| \geq R.$$

Necessarily, $|\dot{x}_*(t)| \leq R$ for all $t \in E$, since otherwise, for some $t_0 \in E$, we would have:

$$\|p\|_{\mathbb{L}^\infty} + 1 \leq |p(t_0)| \leq \|p\|_{\mathbb{L}^\infty}.$$

We just proved the essential boundedness of $\dot{x}_*(\cdot)$ (which yields the Lipschitz continuity of $x_*(\cdot)$) in a very concise and elegant way. Unfortunately for the need of this proof, we presupposed that we had an Erdmann – Du Bois-Reymond equation at our disposal, while such equations are not easily found nor expressed for non-autonomous Lagrangians in the nonsmooth setting.

[3] Less satisfying in the sense that the additional conditions imposed on L in the non autonomous case were not empty for autonomous Lagrangians.

1.10 Higher order Bolza problem

Consider the following non autonomous, N-th order Bolza problem:

$$\begin{cases} \text{Minimize } J(x) := \displaystyle\int_a^b L(s, x(s), x^{(1)}(s), x^{(2)}(s), \dots, x^{(N)}(s))\mathrm{d}s \\ \qquad\qquad\qquad\qquad + g\left((x, x^{(1)}, \dots, x^{(N-1)})(a), (x, x^{(1)}, \dots, x^{(N-1)})(b)\right), \\ \text{over arcs } x \in W^{N,m}([a,b], \mathbb{R}), \end{cases} \tag{CV}$$

where $N \geq 1$ is an integer, $m \geq 1$ is a real number, $L : [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R} \cup \{+\infty\}$ is a given Borel measurable function and $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a given extended valued function non identically equal to $+\infty$.

Standard hypotheses for existence of minimizers to (CV) are quite similar to those used in the case $N = 1$ (*cf.* [76, 77]), namely:

1. $(t, (x_0, x_1, \dots, x_N)) \mapsto L(t, x_0, x_1, \dots, x_N)$ is $\mathcal{L} \times \mathcal{B}$-measurable,
2. $(x_0, x_1, \dots, x_N) \mapsto L(t, x_0, x_1, \dots, x_N)$ is lower semicontinuous for each t,
3. $L(t, x_0, x_1, \dots, x_{N-1}, \cdot\,)$ is convex for each $(x_0, x_1, \dots, x_{N-1})$,
4. L is uniformly coercive (superlinear) in the sense that there exists $\beta > 0$ and a positive, scalar valued, monotone function $\theta :]0, +\infty[\to]0, +\infty[$ such that $\frac{\theta(r)}{r} \xrightarrow[r \to +\infty]{} +\infty$ and

$$L(t, x_0, x_1, \dots, x_N) \geq -\beta |x_N| + \theta(|x_N|), \text{ for all } (t, x_0, \dots, x_N).$$

However there are problems in the same form as (CV) that admit minimizers even when the existence assumptions are not satisfied. Moreover, in this higher order framework, it is quite noticeable that the Lavrentiev phenomenon, *i.e.*

$$\inf\left\{J(x), x \in W^{N,m}([a,b], \mathbb{R}^n)\right\} < \inf\left\{J(x), x \in W^{N,\infty}([a,b], \mathbb{R}^n)\right\}$$

can occur even for autonomous Lagragians.
For instance, if $a = 0$, $b = 1$,

$$L(t, x_0, x_1, x_2) = |x_2|^7 \left[3x_0 - 3|x_1 - 1|^2 - 2|x_1 - 1|^3\right]^2,$$

and

$$g = \chi_{\{(0, \frac{5}{3})\} \times \{(1,2)\}}$$

then (CV) exhibits the Lavrentiev phenomenon [79].

Finding minimal conditions for L under which minimizers are necessarily in $W^{N,\infty}$ for higher order Bolza problems is as important if not more important than for the case $N = 1$.

A few references dealing with this higher order problem are [40, 60, 59]. Here, we just detail the results of the most recent paper [60] to underline the points that have been improved in this thesis.

1.10.1 The stakes of Chapter 5

A result about higher order Bolza problems

Theorem 1.10.1 *[Gavriel, Lopes, Vinter, 2011] Let $x_*(\cdot)$ be a minimizer for (CV). Assume that*

(H1): $(t,(x_0,x_1,\dots,x_N)) \mapsto L(t,x_0,x_1,\dots,x_N)$ is $\mathcal{L}\times\mathcal{B}$-measurable, and L is bounded on bounded sets,

(H2): $L(t,x_0,x_1,\dots,x_N)$ is uniformly coercive/superlinear,

(H3): $L(t,x_0,x_1,\dots,x_{N-1},\cdot\,)$ is convex for each $(x_0,x_1,\dots,x_{N-1})$,

(H4): there exist $\varepsilon_*>0,\ \sigma_*>0$ and a Borel measurable function $k(\cdot,\cdot)$ such that:

$$t \mapsto k(t,0) \in \mathbb{L}^1([a,b],\mathbb{R}_+),$$

and, for a.e. $t\in[a,b]$, for all $\sigma\in[-\sigma_*,\sigma_*]$, the map:

$$(s,x_0,\dots,x_{N-2}) \mapsto L(s,x_0,\dots,x_{N-2},x_*^{(N-1)}(t),(1+\sigma)x_*^{(N)}(t)) \tag{1.15}$$

is Lipschitz continuous on $\mathrm{B}((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*)\cap([a,b]\times\mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Then $x_*^{(N)}(\cdot)$ is essentially bounded.

Condition (H4) can be interpreted as a condition on partial subdifferentials involving up to the x_{N-2} variable. It means that the partial subdifferentials (proximal and limiting) of L are integrally bounded when evaluated along the minimizer $x_*(\cdot)$.

To establish the N-th derivative essential boundedness of a reference minimizer $x_*(\cdot)$ two techniques were used

1. an analysis of the the set of points $t\in[a,b]$ such that $x_*^{(N)}(\cdot)$ is unbounded near t, called Tonelli set associated with $x_*(\cdot)$,
2. the time reparameterization technique that consists in studying an auxiliary Lagrange problem in optimal control in which the control is the derivative of the time variable, then applying a suitable version of the maximum principle.

An inspirational result

We now proceed to give insight of Bettiol and Mariconda recent work in the case $N=1$ (see [23, 24]). Indeed, the same approach will be used in this thesis to deal with the higher order Bolza problem.
In the case $L:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is real valued and Borel measurable, and $x_*(\cdot)$ is a given $W^{1,m}([a,b],\mathbb{R}^n)$ local minimum for (CV1), the following hypotheses (S_{x_*}) was invoked:

Hypothesis (S_{x_*}).
There exists $\varepsilon_*>0$ and a Lebesgue Borel measurable map $k:[a,b]\times]0,+\infty[\to\mathbb{R}$ such that

$$k(t,1)\in\mathbb{L}^1([a,b],\mathbb{R}_+)$$

and for a.e. $t\in[a,b]$, for all $\sigma>0$

$$|L(t_2,x_*(t),\sigma\dot{x}_*(t))-L(t_1,x_*(t),\sigma\dot{x}_*(t))|\le k(t,\sigma)|t_2-t_1|,$$

whenever $t_1, t_2 \in [t-\varepsilon_*, t+\varepsilon_*] \cap [a,b]$.

This condition is a Lipschitzianity condition of $t \mapsto L(t, x_*(t), \sigma x_*(t))$ in a neighborhood of $x_*(\cdot)$ for all $\sigma > 0$. In particular, the range of σ is not limited in this condition.

In the case $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is extended valued and Borel measurable, and $x_*(\cdot)$ is a given $W^{1,m}([a,b], \mathbb{R}^n)$ local minimum for (CV1), the following hypotheses $(S^\infty_{x_*})$ was invoked:

Hypothesis $(S^\infty_{x_*})$.

(i) the map $(s, \xi) \mapsto L(s, x_*(t), \xi)$ is lower semicontinuous for each $t \in [a,b]$,

(ii) there exists a non negligible subset E of $[a,b]$ such that for all $t \in E$, there are $0 < \sigma_1 < 1 < \sigma_2$ such that

$$L(t, x_*(t), \sigma_1 \dot{x}_*(t)) < +\infty \quad \text{and} \quad L(t, x_*(t), \sigma_2 \dot{x}_*(t)) < +\infty,$$

(iii) there exist $\beta > 0$, $A \geq 0$ and a positive function $\gamma \in \mathbb{L}^1([a,b], \mathbb{R}_+)$ such that, for a.e. $t \in [a,b]$:

$$L(\tau, x_*(t), \sigma x_*(t)) + A\sigma|\dot{x}_*(t)| + \gamma(t) \geq 0,$$

$$|\partial_{P,\tau} L(\tau, x_*(t), \sigma \dot{x}_*(t))| \leq \beta \left(L(\tau, x_*(t), \dot{x}_*(t)) + A\sigma|\dot{x}_*(t)| + \gamma(t) \right),$$

for all $\tau \in [a,b]$ and $\sigma > 0$ with $L(\tau, x_*(t), \sigma \dot{x}_*(t))$.

Using (S_{x_*}) or $(S^\infty_{x_*})$, a directional Weierstrass type condition is obtained (see [24, Theorem 1]), and subsequently, the corresponding Erdmann – Du Bois-Reymond type condition is deduced. We give the complete statement of this theorem:

Theorem 1.10.2 *Let $x_*(\cdot)$ be a $W^{1,m}([a,b], \mathbb{R}^n)$ a local minimum of (CV1). Assume that L satisfies (S_{x_*}) (resp. $(S^\infty_{x_*})$). Then, there exists an absolutely continuous function $p \in W^{1,1}([a,b], \mathbb{R}^n)$ satisfying*

$$L(t, x_*(t), r\dot{x}_*(t)) - L(t, x_*(t), \dot{x}_*(t)) \geq (r-1) \left(L(t, x_*(t), \dot{x}_*(t)) - p(t) \right), \text{ for a.e. } t \in [a,b] \text{ and all } r > 0. \tag{W_r}$$

Moreover, for a.e. $t \in [a,b]$, if (S_{x_*}) holds then

$$\dot{p}(t) \in \partial^C_t L(t, x_*(t), \dot{x}_*(t)), \tag{1.16}$$

while if $(S^\infty_{x_*})$ holds then

$$\dot{p}(t) \in \operatorname{co} \left\{ \omega, (\omega, p(t)) \in \partial^L_{(s,v)} \left(L\left(s, x_*(t), \frac{\dot{x}_*(t)}{v}\right) v \right)_{s=t, v=1} \right\}. \tag{1.17}$$

If $(t, x, \xi) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$ is such that $L(t, x, \xi) < +\infty$, we denote by $\partial_r L(t, x, r\xi)_{r=1}$ the convex subdifferential of the function $0 < r \mapsto L(t, x, r\xi)$ at $r = 1$. Then (W_r) can be reformulated:

$$L(t, x_*(t), \dot{x}_*(t)) - p(t) \in \partial_r L(t, x_*(t), r\dot{x}_*(t))_{r=1}, \text{ for a.e. } t \in [a,b],$$

and combining this with equation (1.16) or (1.17), we retrieve the Erdmann – Du Bois-Reymond equation. Another important feature of this result is that it shows that $\partial_r L(t, x_*(t), r\dot{x}_*(t))_{r=1}$ is non empty almost everywhere. Therefore, this result can be interpreted as a relaxation result, stating that the Lagrangian L

is convex in the direction of the minimizer $x_*(\cdot)$. This explains how the convexity assumption might be lift to obtain regularity results.

Once the Erdmann – Du Bois-Reymond type condition has been established, to prove the essential boundedness of $\dot{x}_*(\cdot)$, it is sufficient to introduce a suitable growth condition that quantifies the violation of the Erdmann – Du Bois-Reymond equation.

Hypothesis (G_{x_*}). Let $x_*(\cdot)$ be an absolutely continuous arc on $[a,b]$. We say that L satisfies (G_{x_*}) if, for every selection $Q(t,\xi)$ of the set-valued map $\partial_r L(t,x_*(t),\dot{x}_*(t))_{r=1}$,

$$\lim_{\substack{|\xi|\to+\infty \\ \partial_r L(t,x_*(t),r\xi)_{r=1}\neq\emptyset}} |L(t,x_*(t),\xi)-Q(t,\xi)| = +\infty,$$

uniformly for a.e. $t\in[a,b]$.

This growth condition is more general than superlinearity since the class of functions satisfying condition $(G^{\infty}_{x_*})$ contains some functions with linear growth as, for instance, $L(\xi)=|\xi|-\sqrt{|\xi|}$. Another involved notion of generalized growth condition $(H^{\infty}_{x_*})$ was also introduced in [24], but since it has not been generalized for higher order problems, we do not give more details about this last condition here.

Theorem 1.10.3 *Let $x_*(\cdot)$ be a $W^{1,m}$ local minimizer for (CV1). Assume that L satisfies either (S_{x_*}) or $(S^{\infty}_{x_*})$. If (G_{x_*}) is satisfied. Then $\dot{x}_*(\cdot)$ is essentially bounded.*

Remaining questions

Our analysis of the papers [23, 24, 60] leaves the following questions open:

(Q1) Is it possible to obtain a directional Weierstrass type condition and the corresponding Erdmann – Du Bois-Reymond type condition for higher order Bolza problems?

(Q2) In order to establish regularity of minimizers of higher order Bolza problems, is it possible (as it was done in [23, 24] in the case $N=1$) to weaken the growth assumptions (local boundedness together with the superlinearity) used in [60]?

(Q3) Is it possible to simplify the structure of the proof of the regularity, using the time reparameterization technique alone, without studying the Tonelli set associated with $x_*(\cdot)$?

(Q4) Is it possible to drop the convexity assumption (with respect to the last variable) to obtain the essential boundedness of $x_*^{(N)}(\cdot)$?

(Q5) Is it possible to find an extension of these results to the case L is extended valued?

Chapter 5 provides answers to these questions. In the introduction of this thesis, only the hypotheses and the main results of the chapter are presented.

These results have been published in [14].

1.10.2 Hypotheses in Chapter 5

In Chapter 5, two different sets of hypotheses on L for a given local $W^{N,m}$ local minimizer $x_*(\cdot)$ for (CV) are considered: (S_{x_*}) and $(\mathrm{S}^{\infty}_{x_*})$.

Hypothesis $(\mathbf{S}_{x_*})$ The function

$$L : (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto L(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R}$ and is $\mathcal{B}_{N+2}$-measurable.
There exists $\varepsilon_* > 0$ and an $\mathcal{L} \times \mathcal{B}_1$-measurable function $k : [a,b]\times]0, +\infty[\to \mathbb{R}_+$ such that:

$$t \mapsto k(t,1) \in \mathbb{L}^1([a,b], \mathbb{R}_+),$$

and, for a.e. $t \in [a,b]$, for all $\sigma \in]0, +\infty[$, the map:

$$\begin{cases} [a,b] \times \mathbb{R}^{N-1} \to \mathbb{R}, \\ (s, x_0, \dots, x_{N-2}) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)), \end{cases} \tag{1.18}$$

is Lipschitz continuous on $\mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_*) \cap ([a,b] \times \mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Hypothesis $(\mathbf{S}^{\infty}_{x_*})$ The function

$$L : (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto L(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R} \cup \{+\infty\}$ and is $\mathcal{B}_{N+2}$-measurable.
There exist a measurable set $E \subset [a,b]$ of full measure, strictly positive constants ε, c and λ, functions $d,\ \beta \in \mathbb{L}^1([a,b], \mathbb{R}_+)$ such that the following conditions are satisfied:

i) the function $(s, x_0, \dots, x_{N-2}, x_N) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N)$ is lower semicontinuous for all $t \in [a,b]$,

ii) for all $t \in E$, we can find $0 < \sigma_1(t) < 1 < \sigma_2(t) < +\infty$ for which:

$$\begin{cases} L(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_1(t) x_*^{(N)}(t)) < +\infty \\ L(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_2(t) x_*^{(N)}(t)) < +\infty; \end{cases} \tag{1.19}$$

iii) for every $t \in E$, every $(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}) \in \mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon) \cap ([a,b] \times \mathbb{R}^{N-1})$, and $x_N \in \mathbb{R}$, we have

$$|\zeta| \le c \left(|(1, \bar{x}_1, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t))| + L(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N) + \lambda |x_N|) \right) + d(t) \tag{1.20}$$

for all $\zeta \in \partial_{P,(s,x_0,\dots,x_{N-2})} L(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N)$;

iv) for all $t \in E$, there exists $\varepsilon_t > 0$ such that the function

$$(s, x_0, \dots, x_{N-2}) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N),$$

is Lipschitz continuous with Lipschitz constant $\beta(t)$ on $\mathrm{B}((t, x_*(t), \dot{x}_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_t)$, uniformly with respect to $x_N \in \mathrm{B}(x_*^{(N)}(t), \varepsilon_t) \cap \mathrm{dom}(L(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot))$.

To state the following hypothesis, we introduce an auxiliary Lagrangian $L : [a,b]\times\mathbb{R}\times]0,+\infty[\to \mathbb{R}$, defined by:

$$L(t,\xi,r) := L(t, x_*(t), \dot{x}_*(t), \dots, x_*^{(N-1)}(t), r\xi), \text{ for all } (t,\xi,r) \in [a,b]\times\mathbb{R}\times]0,+\infty[\,. \tag{1.21}$$

The partial convex subdifferential of L with respect to r at (t,ξ,r_0), which is defined by:

$$\partial_r L(t,\xi,r_0) := \{p \in \mathbb{R} : L(t,\xi,r) - L(t,\xi,r_0) \geq p(r-r_0), \forall r \in]0,+\infty[\,\}\,. \tag{1.22}$$

The growth assumption (G_{x_*}). For every selection $Q(t,\xi)$ of $\partial_r L(t,\xi,1)$,

$$\lim_{\substack{|\xi|\to+\infty \\ \partial_r L(t,\xi,1)\neq\emptyset}} |L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t,\xi)| = +\infty, \text{ uniformly for a.e. } t \in [a,b], \tag{G_{x_*}}$$

which means that for any $M > 0$, we can find a set $\mathcal{E} \subset [a,b]$ of full measure, and a real $R > 0$ satisfying:

$$\forall (t,\xi) \in \mathcal{E}\times\mathbb{R},\ Q(t,\xi) \in \partial_r L(t,\xi,1),\ |\xi| \geq R \Rightarrow |L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t,\xi)| \geq M.$$

1.10.3 Results of Chapter 5

This section gathers the four main results of Chapter 5 together with an extension of [60, Theorem 2.1]. Here we do not give complementary results, nor remarks or examples. More details are to be found starting from page 148.

The expected Weierstrass type condition and the corresponding Erdmann – Du Bois-Reymond type condition for higher order Bolza problem are given by Theorem 1.10.4 (in the case L is real valued) and Corollary 1.10.6, answering positively to the question (Q1) page 50.

Using the generalized growth condition (G_{x_*}), the regularity of a given $W^{N,m}$ local minimizer for (CV) is established in Theorem 1.10.7, answering positively to (Q2). A scrutiny of the proof (see Chapter 5) will reveal that we use the time reparameterization technique alone, without studying the Tonelli set associated with $x_*(\cdot)$ (see question (Q3)).

Necessary conditions do not require the convexity assumption with respect to the last variable (Theorems 1.10.4 and 1.10.5), nor does the regularity result in the form of Theorem 1.10.7, which answers positively to (Q4). Note that if we assume the convexity of L with respect to the last variable, we can relax hypothesis (S_{x_*}) which yields a generalization of Theorem 1.10.1 from [60] in the sense that the growth condition is weaker, see Proposition 1.10.8.

The extended valued case is also covered in Chapter 5 by Theorem 1.10.5 and Corollary 1.10.6 for the necessary conditions and by Theorem 1.10.7 for the regularity of minimizers. This is a positive answer to (Q5).

Theorem 1.10.4 *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV)*. Assume that L satisfies* (S_{x_*})*. Then there are two – mutually non exclusive – cases:*

i) The function $x_*(\cdot)$ is a polynomial function whose degree is at most $N-1 \geq 1$.

ii) There exists an arc $(p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ *for which the following Weierstrass type condition is satisfied: for all* $u \in]0, +\infty[$ *and for a.e.* $t \in [a,b]$*:*

$$L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq \\ (u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big). \tag{W}$$

Moreover, for a.e. $t \in [a,b]$*:*

$$(\dot{p}_0, \dot{p}_1, \dot{p}_2 + p_1, \dots, \dot{p}_{N-2} + p_{N-3}, \dot{p}_{N-1} + p_{N-2}) \in \partial^C_{t,x_0,\dots,x_{N-2}} L(t, x_*, \dot{x}_*, \dots, x_*^{(N)}). \tag{1.23}$$

Theorem 1.10.5 *Let* $x_*(\cdot) \in W^{N,m}([a,b], \mathbb{R}^N)$ *be a minimizer for* (CV)*. Assume that* L *satisfies* $(\mathrm{S}^\infty_{x_*})$*. Then there are two – mutually non exclusive – cases:*

i) The function $x_*(\cdot)$ *is a polynomial function whose degree is at most* $N - 1 \geq 1$*.*

ii) There exists an arc $p := (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ *for which the following Weierstrass type condition is satisfied: for all* $u \in]0, +\infty[$ *and for a.e.* $t \in [a,b]$*:*

$$L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq \\ (u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big). \tag{W}$$

Moreover, for a.e. $t \in [a,b]$*,* $\dot{p}(t)$ *belongs to the set:*

$$\mathrm{co}\Big\{\omega \in \mathbb{R}^N \ : \ \Big(\omega + \gamma(t), p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big) \\ \in \Big(\partial^L_{(s,x_0,\dots,x_{N-2},u)} L\big(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_*^{(N)}(t)/u\big)u\Big)_{\substack{(s,x_0,\dots,x_{N-2})=z_*(t) \\ u=1}} \Big\}, \tag{1.24}$$

with $\gamma(t) := (0, 0, p_1(t), \dots, p_{N-2}(t))$ *and* $z_*(t) := \big(t, x_*(t), \dots, x_*^{(N-2)}(t)\big)$*.*

Erdmann – Du Bois-Reymond type conditions.

Corollary 1.10.6 *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV)*. Assume that* L *satisfies* (S_{x_*}) *(resp.* $(\mathrm{S}^\infty_{x_*})$*). Then there are two – mutually non exclusive – cases:*

i) The function $x_*(\cdot)$ *is a polynomial function whose degree is at most* $N - 1 \geq 1$*.*

ii) There exists an arc $p := (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ *for which the following equation is satisfied: for all* $r \in]0, +\infty[$ *and for a.e.* $t \in [a,b]$*:*

$$L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), r x_*^{(N)}(t)\Big) - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq \\ (r-1)\Big(L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) - (p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t))\Big), \tag{W_r}$$

where $p(\cdot)$ *satisfies (1.23) (resp. (1.24)).*

Regularity of minimizers.

Theorem 1.10.7 *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV)*.*

(i) Assume that L satisfies (S_{x_*}) and (G_{x_*}), then $x_*^{(N)} \in \mathbb{L}^\infty([a,b],\mathbb{R})$.

(ii) Assume that L satisfies $(\mathrm{S}^\infty_{x_*})$ and (G_{x_*}), then $x_*^{(N)} \in \mathbb{L}^\infty([a,b],\mathbb{R})$.

If the Lagrangian L is convex with respect to x_N, then we can relax the condition (S_{x_*}) and invoke a weaker (merely local in σ) version of it. This result provides an extension of [60, Theorem 2.1].

Proposition 1.10.8 *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for (CV), in which we assume that $L : [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R}$ is Borel measurable and*

$(\mathrm{H})'$ *$x_N \mapsto L(t,x_0,x_1,\dots,x_{N-2},x_{N-1},x_N)$ is convex for every $(t,x_0,x_1,\dots,x_{N-2},x_{N-1})$;*

$(\mathrm{S}_{x_*})'$ *There exist $\varepsilon_* > 0$, $\sigma_* \in]0,1[$ and a $\mathcal{L}\times\mathcal{B}_1$-measurable function $k : [a,b]\times[1-\sigma_*,1+\sigma_*] \to \mathbb{R}_+$ such that:*

$$t \mapsto k(t,1) \in \mathbb{L}^1([a,b],\mathbb{R}_+),$$

and, for a.e. $t \in [a,b]$, for all $\sigma \in [1-\sigma_*,1+\sigma_*]$, the map:

$$\begin{cases} [a,b]\times\mathbb{R}^{N-1} \to \mathbb{R}, \\ (s,x_0,\dots,x_{N-2}) \mapsto L(s,x_0,\dots,x_{N-2},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)), \end{cases} \tag{1.25}$$

is Lipschitz continuous on $\mathrm{B}((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*)\cap([a,b]\times\mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Then, the same conclusions of Theorem 1.10.4 are valid. If moreover, L satisfies (G_{x_*}), then $x_*^{(N)}(\cdot)$ belongs to $\mathbb{L}^\infty([a,b],\mathbb{R})$.

2

Preliminary results

2.1 Abstract of chapter 2

This chapter gathers most of the notations, definitions, theorems and propositions that will be used in the rest of this thesis. A great part of these notions are well-known and can be found in the literature but we reproduce them here for the sake of readability. Proofs of some technical lemmas that are used in other chapters are also included.
The topic covered here are nonsmooth analysis, multivalued functions and the properties of their trajectories, and viability theory.

2.2 Résumé du chapitre 2

Ce chapitre rassemble la plupart des notations, définitions, théorèmes et propositions qui seront utilisés dans le reste de cette thèse. Une grande partie de ces notions est bien connue and peut être trouvée dans la littérature, mais nous les rappelons ici pour le confort du lecteur. Les preuves de certains lemmes techniques sont aussi incluses.
Les sujets traités ici sont l'analyse non lisse, les fonctions multivaluées et les propriétés de leurs trajectoires ainsi que le théorie de la viabilité.

2.3 Some notations

We write $\mathbb{R}_+$ the set of non negative real numbers, *i.e.* $\{x \in \mathbb{R} \,|\, r \geq 0\}$, and $\mathbb{B}$ for the closed unit ball in $\mathbb{R}^n$.

We denote the Lebesgue subsets of $[S,T]$ and the Borel subsets of $\mathbb{R}^m$ by $\mathcal{L}$ and $\mathcal{B}^m$ respectively. The (associated) product σ-algebra of sets in $[S,T] \times \mathbb{R}^m$ is written $\mathcal{L} \times \mathcal{B}^m$. We denote by $\mathbb{L}^p([S,T],\mathbb{R}^n)$ the space of $\mathbb{L}^p$ functions for the Lebesgue measure, that are defined on $[S,T]$, and take values in $\mathbb{R}^n$. We write $W^{1,1}([S,T],\mathbb{R}^n)$, the space of absolutely continuous function for the Lebesgue measure endowed with the norm:

$$\|f\|_{W^{1,1}} := |f(\alpha)| + \int_\alpha^\beta |\dot{f}(s)| \mathrm{d}s, \text{ for all } f \in W^{1,1}([S,T],\mathbb{R}^n).$$

Let $D \subset \mathbb{R}^m$, we denote by $\operatorname{co} D$, $\overline{D}$ and $\overline{\operatorname{co}}\, D$ respectively the convex hull, the closure and the closed convex hull of D.

The polar cone D^* to a subset D is given by:

$$D^* := \{v \in \mathbb{R}^m \,|\, \forall w \in D, v \cdot w \leq 0\}.$$

For arbitrary nonempty closed sets in $\mathbb{R}^n$, C' and C, we denote by $\mathrm{d}_H(C, C')$ the 'Hausdorff distance' between C and C':

$$\mathrm{d}_H(C, C') := \inf\{\beta > 0 \,|\, C' \subset C + \beta\mathbb{B}\} \vee \inf\{\beta > 0 \,|\, C \subset C' + \beta\mathbb{B}\} \,.$$

Take a closed set $C \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. Then $\min_{y\in C}\{|x - y|\}$ is the distance of x from the set C and is written $\mathrm{d}_C(x)$.

If $f : C \subset \mathbb{R}^m \to \mathbb{R}$ is a locally bounded function, we denote its lower (resp. upper) semicontinous envelope by:

$$f_*(x) := \liminf_{y \overset{C}{\to} x} f(y) \quad \left(\text{resp. } f^*(x) := \limsup_{y \overset{C}{\to} x} f(y) \right), \text{ for every } x \in C.$$

The notation $y \overset{C}{\to} x$ means that we are considering convergent sequences $(y_i)_{i\in\mathbb{N}}$ such that $y_i \to x$, and each element y_i belongs to C.

An increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity if $\lim_{s\to 0} \omega(s) = 0$.

Consider an extended valued function $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$. We write $\mathrm{dom}\,(\varphi) := \{x \in \mathbb{R}^m \,|\, \varphi(x) \neq \pm\infty\}$, $\mathrm{epi}\,\varphi := \{(x, r) \in \mathbb{R}^{m+1} \,|\, r \geq \varphi(x)\}$, and $\mathrm{hyp}\,\varphi := \{(x, r) \in \mathbb{R}^{m+1} \,|\, r \leq \varphi(x)\}$.

2.4 Nonsmooth analysis

Nonsmooth analysis allows to treat situations in which differentiability of the data doesn't necessarily stand. We will use this field of mathematics as it generalizes some notions that are usually expressed with smooth data, such as the normal vector to a set with smooth border, or the gradient of a differentiable function. The necessity for dealing with nonsmooth settings has been illustrated in the introduction, section 1.2 and subsection 1.3.3.

The concepts and tools coming from nonsmooth analysis are presented with more details in the monographs [4, 31, 43, 45, 66, 84]).

Normal cones.

Definition 2.4.1 *The proximal normal cone to a closed set $C \subset \mathbb{R}^m$ at $x \in C$, denoted $N_C^P(x)$, is defined by:*

$$N_C^P(x) := \{\eta \in \mathbb{R}^m \,|\, \exists M \geq 0 \quad \textit{such that} \quad \forall y \in C, \eta \cdot (y - x) \leq M|y - x|^2\}.$$

Definition 2.4.2 *The strict normal cone $\hat{N}_C(x)$ to a closed set $C \subset \mathbb{R}^m$ at x is defined as follows*

$$\hat{N}_C(x) \;:=\; \left\{ \eta \in \mathbb{R}^m \mid \limsup_{y \overset{C}{\to} x} |y - x|^{-1}\, \eta \cdot (y - x) \;\leq 0 \right\}.$$

Definition 2.4.3 *The limiting normal cone $N_C^L(x)$ to a closed set $C \subset \mathbb{R}^m$ at x is defined as follows*

$$N_C^L(x) \;:=\; \left\{ \eta \in \mathbb{R}^m \mid \textit{ there exists } x_i \overset{C}{\to} x, \eta_i \to \eta \textit{ such that } \eta_i \in N_C^P(x_i), \textit{ for all } i \in \mathbb{N} \right\}.$$

Definition 2.4.4 *The Bouligand tangent cone, alternatively referred to as contingent cone $T_C(x)$ to a closed set $C \subset \mathbb{R}^m$ at $x \in C$ is defined by:*

$$T_C(x) := \left\{ v \in \mathbb{R}^m \mid \liminf_{h \to 0^+} \frac{\mathrm{d}_C(x+hv)}{h} = 0 \right\} = \limsup_{h \to 0^+} \frac{C-x}{h}.$$

Proposition 2.4.5 *We have*

$$\hat{N}_C(x) = [T_C(x)]^*$$

and

$$N_C^P(x) \subset \hat{N}_C(x) \subset N_C^L(x). \tag{2.1}$$

An important feature of the limiting normal cone is that it is better suited than proximal and strict normal cones to limit-taking. Indeed, we have the following proposition, that does not hold for proximal and strict normal cones.

Proposition 2.4.6 *Let $C \subset \mathbb{R}^m$ be a closed set. The set valued map $C \rightsquigarrow \mathbb{R}^m, y \mapsto N_C^L(y)$ has a closed graph.*

Lower Dini derivative.

The *lower Dini derivative* (alternatively referred to as *the contingent epiderivative* cf. [4, 31, 61]) plays an important role in chapter 4.

Definition 2.4.7 *Consider an extended valued function $\phi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$. Take $x \in \mathrm{dom}\,(\phi)$ and $d \in \mathbb{R}^m$. The lower Dini derivative of ϕ at x in the direction $d \in \mathbb{R}^m$, denoted $D_\uparrow \phi(x,d)$, is defined by:*

$$D_\uparrow \phi(x,d) := \liminf_{\substack{h \downarrow 0 \\ e \to d}} h^{-1}(\phi(x+he) - \phi(x)).$$

The following proposition gives a simplification of $D_\uparrow U$ when U has two variables and one of which is the time variable.

Proposition 2.4.8 *Let U be an extended valued function defined on $[S,T] \times \mathbb{R}^n$, taking $\epsilon \in \{1,-1\}$, then for all $(t,x) \in \mathrm{dom}\,(U)$, we have:*

$$D_\uparrow U((t,x),(\epsilon,d)) = \liminf_{\substack{h \downarrow 0 \\ e \to d}} h^{-1}(U(t+\epsilon h, x+he) - U(t,x)).$$

Subdifferential and Superdifferential.

The subdifferential and superdifferential, also referred to as Fréchet subdifferential and superdifferential are often use to characterize the solutions to the Hamilton-Jacobi-Bellman equation in the viscosity sense, without invoking test functions [51]. More details about these notions can be found in [66].

Definition 2.4.9 *Let $\phi : X \subset \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \mathrm{dom}(\phi)$. The subdifferential of ϕ at x is defined by:*

$$\partial_- \phi(x) := \left\{ p \in \mathbb{R}^m, \liminf_{y \to x} \frac{\phi(y) - \phi(x) - p \cdot (y-x)}{|y-x|} \geq 0 \right\}$$

Similarly, the superdifferential of an upper semicontinuous function ϕ at $x \in \mathrm{dom}(\phi)$ is defined by:

$$\partial_+ \phi(x) = -\partial_-(-\phi)(x).$$

Proposition 2.4.10 *Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$. Then*

$$\zeta \in \partial_- \phi(x) \Leftrightarrow (\zeta, -1) \in \hat{N}_{\mathrm{epi}\,\phi}(x, \phi(x)),$$

and

$$\zeta \in \partial_+(-\phi)(x) \Leftrightarrow (-\zeta, 1) \in \hat{N}_{\mathrm{hyp}\,-\phi}(x, -\phi(x)).$$

The proximal subdifferential and superdifferential.

Definition 2.4.11 *Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$, then the proximal subdifferential $\partial_P \phi(x)$ of ϕ at x is the set of elements $\zeta \in \mathbb{R}^m$ such that there exist $M \geq 0$ and $\eta > 0$ satisfying:*

$$\phi(y) - \phi(x) + M|y - x|^2 \geq \zeta \cdot (y - x), \text{ for all } y \in x + \eta\mathbb{B}.$$

Let $\phi : X \to \mathbb{R} \cup \{-\infty\}$ be a given extended valued upper semicontinuous function and $x \in \mathrm{dom}(\phi)$, then the proximal superdifferential $\partial^P \phi(x)$ of ϕ at x is defined by :

$$\partial^P \phi(x) = -\partial_P(-\phi)(x)$$

Proposition 2.4.12 *Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$. Then*

$$\zeta \in \partial_P \phi(x) \Leftrightarrow (\zeta, -1) \in N^P_{\mathrm{epi}\,\phi}(x, \phi(x)).$$

The limiting subdifferential.

Definition 2.4.13 *Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$, then the limiting subdifferential of ϕ at $x \in \mathrm{dom}(\phi)$ is defined by*

$$\partial^L \phi(x) := \{\zeta \in \mathbb{R}^k : \exists x_i \to x, \zeta_i \in \partial_P \phi(x_i) \text{ s.t. } \phi(x_i) \to \phi(x) \text{ and } \zeta_i \to \zeta\}.$$

Proposition 2.4.14 *Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$. Then*

$$\zeta \in \partial^L \phi(x) \Leftrightarrow (\zeta, -1) \in N^L_{\mathrm{epi}\,\phi}(x, \phi(x)).$$

We recall that the proximal normal cone to the epigraph of ϕ at $(x, \phi(x))$, is in general not spanned by $\partial_P \phi(x) \times \{-1\}$ because it may contain horizontal vectors in the form $(\xi, 0)$. The corresponding remark also holds for Fréchet sub/superdifferential. The following theorem, known as the Rockafellar Horizontal Approximation Theorem (cf. [45, Thm. 11.31], [84, Thm. 4.6.2]), will be particularly useful in Chapters 3 and 4 to deal with these so-called asymptotic vectors.

Theorem 2.4.15 *Let $\phi : X \subset \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a given extended valued lower semicontinuous function and $x \in \mathrm{dom}(\phi)$. Let $\zeta \in \mathbb{R}^m$. Then the following conditions are equivalent :*

(a) the vector ζ is such that $(\zeta, 0) \in N^L_{\mathrm{epi}\,\phi}(x, \phi(x))$,

(b) there exist sequences $(x_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}^m$, $(\lambda_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}_+$ *and* $(\zeta_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}^m$ *such that :*

$$x_i \xrightarrow[i\to+\infty]{} x, \qquad \lambda_i \downarrow 0,$$

and

$$\forall i \in \mathbb{N}, \lambda_i^{-1}\zeta_i \in \partial_P \phi(x_i).$$

The Clarke subdifferential.
An alternative approach of generalized derivatives of locally Lipschitz continuous functions was proposed by F. Clarke in the 1970s. This approach provides a new representation of subdifferentials that is used in chapter 5.

Definition 2.4.16 *Let* $\phi : \mathbb{R}^m \to \mathbb{R}$ *be a given function and assume that* ϕ *is Lipschitz continuous on a neighborhood of a point* $x \in \mathbb{R}^m$. *Take* $v \in \mathbb{R}^m$. *The generalized directional derivative of* ϕ *at* x *in the direction* v, *written* $\phi^0(x, v)$ *is defined by:*

$$\phi^0(x, v) := \limsup_{y\to x, h\downarrow 0} h^{-1}[\phi(y + tv) - \phi(y)].$$

Then the Clarke subdifferential of ϕ *at* x *is defined by:*

$$\partial^C \phi(x) := \{\zeta \in \mathbb{R}^m \,|\, \phi^0(x, v) \geq \zeta \cdot v\}.$$

Remark 2.4.17 *Since* $\phi^0(x, 0) = 0$, *the Clarke subdifferential* $\partial^C \phi(x)$ *is the subdifferential in the sense of convex analysis to the convex function* $v \mapsto \phi^0(x, v)$.

Proposition 2.4.18 *Take a lower semicontinuous function* $\phi : \mathbb{R}^m \to \mathbb{R}$ *that is Lipschitz continuous on a neighborhood of some point* $x \in \mathbb{R}^m$. *Then*

$$\partial^C \phi(x) = \operatorname{co} \partial^L \phi(x).$$

2.5 Multivalued functions and their trajectories

2.5.1 Kuratowski limits

Let us recall the notion of limit for a multivalued function in the sense of Kuratowski, which plays a role of topmost importance in chapters 3 and 4.

Definition 2.5.1 *Let* $F :]S, T[\rightsquigarrow \mathbb{R}^m$ *a multivalued function. If the sets*

$$\liminf_{s'\downarrow\alpha} F(s') := \{v \in \mathbb{R}^m \mid \limsup_{s'\downarrow S} \mathrm{d}_{F(s')}(v) = 0\},$$

$$\limsup_{s'\downarrow\alpha} F(s') := \{v \in \mathbb{R}^m \mid \liminf_{s'\downarrow S} \mathrm{d}_{F(s')}(v) = 0\}$$

are the same, we call this common limit the right Kuratowski limit of F *in* S, *denoted* $F(S^+)$.
If

$$\liminf_{s'\uparrow\beta} F(s') := \{v \in \mathbb{R}^m \mid \limsup_{s'\uparrow T} \mathrm{d}_{F(s')}(v) = 0\},$$

$$\limsup_{s'\uparrow\beta} F(s') := \{v \in \mathbb{R}^m \mid \liminf_{s'\uparrow T} \mathrm{d}_{F(s')}(v) = 0\}$$

are the same, we call the common limit the left Kuratowski limit of F *in* T, *denoted* $F(T^-)$.

Examples.
– Define a multivalued function $F :]0,1[\rightsquigarrow \mathbb{R}$ by, for all $t \in \mathbb{R}_+^*$, $F(t) = (1+t)\mathbb{B}$. Then F has a limit in 0 and

$$F(0^+) = \mathbb{B}.$$

– Define a multivalued function $F :]0,1[\rightsquigarrow \mathbb{R}$ by, for all $t \in]0,1[$, $F(t) = (2\mathbb{1}_{\mathbb{Q}}(t) - 1)[0,1/2]$. Then F does not have a limit in 0:

$$\{0\} = \liminf_{s' \downarrow 0} F(s') \neq \limsup_{s' \downarrow 0} F(s') = [-1/2, 1/2].$$

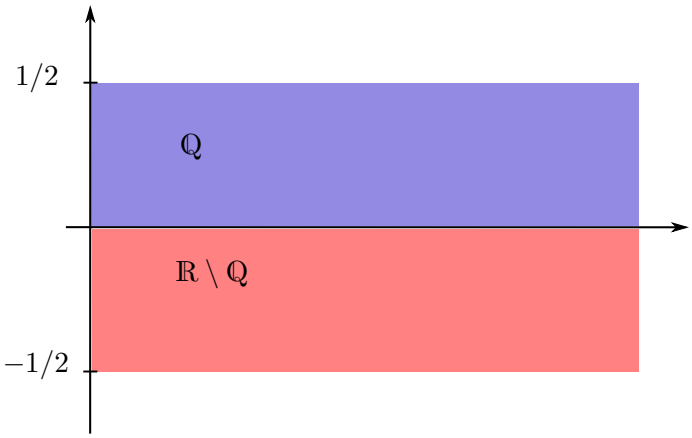

Figure 2.1: Graph of the function $F = (2\mathbb{1}_{\mathbb{Q}}(\cdot) - 1)[0,1/2]$.

2.5.2 Boundedness of F-trajectories and continuity of the support function of F

We now state and prove some useful results about multivalued functions that will be used in chapters 3 and 4. Let us first recall some of the hypotheses that are being used in these chapters.

(H1): i) The multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes convex, closed, nonempty values. For every $x \in \mathbb{R}^n$, $F(\cdot, x)$ is Lebesgue measurable on $[S,T]$.

(H2): i) For almost every $t \in [S,T]$ and $x \in \mathbb{R}^n$

$$F(t,x) \subset c_F(t)(1+|x|)\mathbb{B}.$$

ii) For all $(t,x) \in [S,T] \times R_0\mathbb{B}$

$$F(t,x) \subset c_0\mathbb{B}.$$

(H3): i)

$$\mathrm{d}_H(F(t,x'), F(t,x)) \leq \omega(|x-x'|), \text{ for all } x, x' \in R_0\mathbb{B} \text{ and } t \in [S,T].$$

ii)

$$F(t,x') \subset F(t,x) + k_F(t)|x-x'|\mathbb{B}, \text{ for all } x, x' \in R_0\mathbb{B} \text{ and for a.e. } t \in [S,T].$$

(H4): i) For each $x \in \mathbb{R}^n$, $s \in [S,T[$, and $t \in]S,T]$ the following limits (in the sense of Kuratowski) exist and are nonempty

$$F(s^+,x) := \lim_{s' \downarrow s} F(s',x) \text{ and } F(t^-,x) := \lim_{t' \uparrow t} F(t',x).$$

ii) For almost every $s \in [S,T[$ and $t \in]S,T]$, and every $x \in \mathbb{R}^n$ we have

$$F(s^+,x) = F(s,x) \text{ and } F(t^-,x) = F(t,x).$$

***A priori* uniform boundedness of F-trajectories**

We recall Grönwall's inequality (see [84]).

Proposition 2.5.2 *Take any $z \in W^{1,1}([S,T],\mathbb{R}^m)$. Assume that there exist $k(\cdot)$ and $v(\cdot)$ in $\mathbb{L}^1([S,T],\mathbb{R}_+)$ such that for almost every $t \in [S,T]$,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right| \leq k(t)|z(t)| + v(t).$$

Then for all $t \in [S,T]$:

$$|z(t)| \leq \exp\left(\int_S^t k(\sigma)\mathrm{d}\sigma\right)\left[|z(S)| + \int_S^t \exp\left(-\int_S^\tau k(\sigma)\mathrm{d}\sigma\right)v(\tau)\mathrm{d}\tau\right].$$

Grönwall's inequality together with hypothesis (H2) guarantee a well-known *a priori* uniform boundedness property for the F-trajectories that is consistently being used in chapters 3 and 4.

Lemma 2.5.3 *Choose any $(t,x) \in [S,T] \times \mathbb{R}^n$ and an F-trajectory $y \in W^{1,1}([t,T],\mathbb{R}^n)$ such that $y(t) = x$. Define $R_0 := (1+|x|)\exp\left(\int_S^T c_F(s)\mathrm{d}s\right)$ and let c_1 be any strictly positive constant greater than the constant c_0 associated with R_0 given by hypothesis (H2) ii). Then for every $s \in [t,T], y(s) \in R_0\mathbb{B}$, and for almost every $s \in [t,T]$, $\dot{y}(s) \in c_1\mathbb{B}$.*

Proof. Fix any $(t,x) \in [S,T[\times\mathbb{R}^n$ and let $y \in W^{1,1}([t,T],\mathbb{R}^n)$ be an F-trajectory such that $y(t) = x$.
From $(H2)$ $i)$, we can pick $c_F \in \mathbb{L}^1([t,T],\mathbb{R}_+)$ such that, for almost every $s \in [t,T]$ and every $\mathbf{y} \in \mathbb{R}^n$:

$$F(s,x) \subset c_F(s)(1+|\mathbf{y}|)\mathbb{B}.$$

Since $y(\cdot)$ is an F-trajectory, it follows:

$$|\dot{y}(s)| \leq c_F(s)\,(1+|y(s)|), \text{ for almost } s \in [t,T].$$

Hence, from Grönwall's inequality, for every $s \in [t,T]$:

$$|y(s)| \leq (1+|x|)\exp\left(\int_t^T c_F(s)\mathrm{d}s\right) - 1 \leq (1+|x|)\exp\left(\int_S^T c_F(s)\mathrm{d}s\right).$$

We deduce that $y(s) \in R_0\mathbb{B}$ for every $s \in [t,T]$.
From $(H2)$ $ii)$, there exists $c_0 \in \mathbb{R}_+^*$, associated with R_0, such that:

$$F(t,\mathbf{y}) \subset c_0\mathbb{B}, \text{ for every } (t,\mathbf{y}) \in [S,T] \times R_0\mathbb{B}.$$

Hence for almost every $s \in [t,T]$, $\dot{y}(s) \in F(s,y(s)) \subset c_0\mathbb{B}$. □

Continuity of $s \mapsto \max_{v\in F(s^-,y(s))} p\cdot v$ and $s \mapsto \max_{v\in F(s^-,y(s))} p\cdot v$

In the following lemma, we show continuity properties for the functions $s \mapsto \max_{v\in F(s^-,y(s))} p\cdot v$ and $s \mapsto \max_{v\in F(s^-,y(s))} p\cdot v$, where $y(\cdot)$ is an F-trajectory.

Lemma 2.5.4 *Assume (H1), (H3) i) and (H4). Take any $(t,x) \in [S,T[\times\mathbb{R}^n$. Let $y \in W^{1,1}([t,T],\mathbb{R}^n)$ be an F-trajectory such that $y(t) = x$. Then for every $\varepsilon \in \mathbb{R}_+^*$, there exists $\delta \in \mathbb{R}_+^*$ such that:*

$$F(s^+, y(s)) \subset F(t^+, y(t)) + 3\varepsilon\mathbb{B}, \text{ for every } s \in [t, t+\delta]$$

and for every $p \in \mathbb{R}^n$, the function $s \longmapsto \max_{v\in F(s^+,y(s))} p\cdot v$ is right continuous at $s = t$.

Similarly, if $y \in W^{1,1}([S,T],\mathbb{R}^n)$ is an F-trajectory and $t \in]S,T]$, then for every $p \in \mathbb{R}^n$, the function $s \longmapsto \max_{v\in F(s^-,y(s))} p\cdot v$ is left continuous at $s = t$.

Proof. Let us assume there exist $\eta \in \mathbb{R}_+^*$, a strictly decreasing sequence $(s_j)_{j\in\mathbb{N}}$ in $[t,T]$ that converges to t and a sequence $(v_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ such that: $\forall j \in \mathbb{N}, v_j \in F(s_j^+, y(s_j))$, but $v_j \notin F(t^+, y(t)) + 3\eta\mathbb{B}$.
From the definition of $F(s_j^+, y(s_j))$, for every integer $j \in \mathbb{N}$, there exists a strictly decreasing sequence $(s_{j,i})_{i\in\mathbb{N}}$ that converges to s_j, a sequence $(v_{j,i})_{i\in\mathbb{N}}$ such that:

$$(v_{j,i}) \in F(s_{j,i}, y(s_j)), \text{ and } v_j = \lim_{i\to+\infty} v_{j,i}.$$

This last convergence allows us to extract a sequence $(i_j)_{j\in\mathbb{N}}$ such that for every $j \in \mathbb{N}$, and every integer $i \geq i_j$:

$$v_{j,i} \subset v_j + \eta\mathbb{B}.$$

Let us denote $\tilde{v}_j = v_{j,i_j}$ for every $j \in \mathbb{N}$.
From the boundedness of the F-trajectories, we pick $c_0 \in \mathbb{R}_+^*$ such that for every $j \in \mathbb{N}$, $F(s_{j,i_j}, y(s_j)) \subset c_0\mathbb{B}$.
We can extract a subsequence $(\tilde{v}_{j_k})_{k\in\mathbb{N}}$ converging to a vector $\tilde{v}$. Hence, we can pick $k_0 \in \mathbb{N}$ such that for every integer $k \geq k_0$,

$$\tilde{v} \in \tilde{v}_{j_k} + \eta\mathbb{B}.$$

Thus, for every $k \in \mathbb{N}$, large enough:

$$\tilde{v} \in \tilde{v}_{j_k} + \eta\mathbb{B} \subset v_{j_k} + 2\eta\mathbb{B}.$$

But we have assumed $v_{j_k} + 3\eta\mathbb{B} \cap F(t^+, y(t)) = \emptyset$, which means $\tilde{v} \notin F(t^+, y(t))$.

We will prove that the converse is also true.
We pick any $\varepsilon \in \mathbb{R}_+^*$. From the boundedness of the F-trajectories, we can pick $R_0 \in \mathbb{R}_+^*$ such that for every $s \in [t,T]$, $y(s) \in R_0\mathbb{B}$.
From $(H3)$ $i)$, there exists a modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every integer $k \geq 0$:

$$\mathrm{d}_H(F(s_{j_k}, y(s_{j_k})), F(s_{j_k}, y(t))) \leq \omega(|y(s_{j_k}) - y(t)|).$$

From the continuity of $y(\cdot)$ at t, for every $k \in \mathbb{N}$, large enough, we have:

$$F(s_{j_k}, y(s_{j_k})) \subset F(s_{j_k}, y(t)) + \varepsilon\mathbb{B}.$$

Let us denote w_k the projection of v_{j_k} upon the closed convex set $F(s_{j_k}, y(t))$. Since we have $\lim_{k\to+\infty} \tilde{v}_{j_k} = \tilde{v}$, for every $k \in \mathbb{N}$, large enough, we have:

$$|\tilde{v} - w_k| \leq |\tilde{v} - v_{j_k}| + |v_{j_k} - w_k| \leq 2\varepsilon.$$

This shows that $\tilde{v} \in F(t^+, y(t))$.

Let $p \in \mathbb{R}^n$. We can assume $p \neq 0$ otherwise the right continuity is trivial. Let $\varepsilon \in \mathbb{R}_+^*$. From the first point of the proof, we pick $\delta \in \mathbb{R}_+^*$ such that

$$\forall s \in [t, t+\delta], F(s^+, y(s)) \subset F(t^+, y(t)) + \frac{\varepsilon}{|p|}\mathbb{B}.$$

Let $v \in F(s^+, y(s))$. We write w the projection of v upon the closed convex set $F(t^+, y(t))$. Using Cauchy-Schwarz inequality we have:

$$p \cdot v = p \cdot w + p \cdot (v - w) \leq \max_{\mathbf{v} \in F(t^+, y(t))} p \cdot \mathbf{v} + |p| \frac{\varepsilon}{|p|} \leq \max_{\mathbf{v} \in F(t^+, y(t))} p \cdot \mathbf{v} + \varepsilon.$$

From that equation we deduce: $\max_{\mathbf{v} \in F(s^+, y(s))} p \cdot \mathbf{v} \leq \max_{\mathbf{v} \in F(t^+, y(t))} p \cdot \mathbf{v} + \varepsilon$.
Switching the roles of $F(t^+, y(t))$ and $F(s^+, y(s))$, we obtain:

$$\left| \max_{v \in F(s^+, y(s))} p \cdot v - \max_{v \in F(t^+, y(t))} p \cdot v \right| \leq \varepsilon,$$

which ends the proof of the right continuity. The left continuity can be proved in a similar fashion. □

2.6 Viability/Invariance Theorems

In this section we recall results from viability theory, referred to as weak invariance/global viability theorems (cf. [4] or [84]). As stated in the introduction, usually the Hamilton-Jacobi-Bellman theory requires the use of both a weak and a strong invariance theorems. However the hypotheses in force in chapters 3 and 4 made this type of theorem unusable. To circumvent this issue, a Carathéodory's parametrization argument is invoked, making it then possible to use a weak invariance theorem alone.

Theorem 2.6.1 (Weak Invariance Theorem) *Take a multifunction $\Gamma : \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$, an interval $[S, T]$ and a closed set $D \subset \mathbb{R}^k$. Assume:*

i) The graph of Γ is closed and $\Gamma(x)$ is a nonempty, convex set for each $x \in \mathbb{R}^k$;

ii) there exists $c > 0$ such that

$$\Gamma(x) \ \subset \ c(1 + |x|)\mathbb{B} \quad \text{for all } x \in \mathbb{R}^k;$$

iii) for every $x \in D$ we have

$$\min_{v \in \Gamma(x)} \zeta \cdot v \ \leq \ 0 \quad \text{for all } \zeta \in N_D^P(x).$$

Then, given any $x_0 \in D$, there exists an absolutely continuous function $x(\cdot)$ satisfying

$$\begin{cases} \dot{x}(t) \ \in \ \Gamma(x(t)) & \text{for a.e. } t \in [S, T], \\ x(S) \ = \ x_0, & \\ x(t) \ \in \ D & \text{for all } t \in [S, T]. \end{cases}$$

Condition iii) is called inward pointing condition. This condition is crucial when constructing the arc $x(\cdot)$. It somehow means that we can always find velocity vectors in $\Gamma(x)$ pointing towards the interior of D.

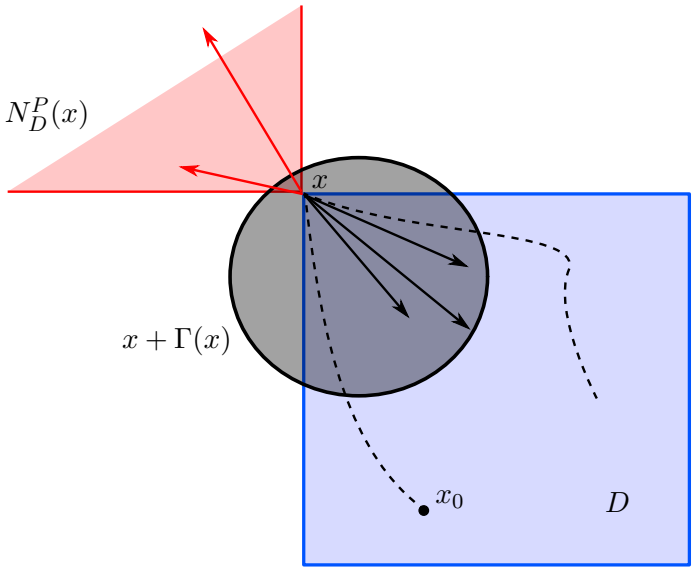

Figure 2.2: Inward pointing condition

A key role in many proofs of Chapter 5 is played by the following Local Weak Invariance result ([21]):

Theorem 2.6.2 (Local Weak Invariance Theorem) *Take an interval $[S,T]$, a multifunction $\Gamma : \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$, a closed set $D \subset \mathbb{R}^k$, a number $\varepsilon > 0$ and a Γ-trajectory $\bar{x}$ on $[S,T]$. Assume that:*

(i) the graph of Γ is closed and $\Gamma(x)$ is a non-empty, convex set for each $x \in \bar{x}([S,T]) + \varepsilon\mathbb{B}$;

(ii) there exists a real $c > 0$ such that

$$\Gamma(x) \subset c(1 + |x|)\mathbb{B}, \text{ for all } x \in \bar{x}([S,T]) + \varepsilon\mathbb{B}\,;$$

(iii) for each $[s,t] \subset [S,T]$, the restriction of $\bar{x}$ to $[s,t]$ is the unique Γ-trajectory on $[s,t]$ with initial condition $\bar{x}(s)$;

(iv) $\bar{x}(S) \in D$ and for every $t \in [S,T]$ and $x \in D \cap (\bar{x}(t) + \varepsilon\mathbb{B})$, we have:

$$\min_{w \in \Gamma(x)} \xi \cdot w \leq 0, \text{ for all } \xi \in N_D^P(x).$$

Then

$$\bar{x}(t) \in D, \text{ for all } t \in [S,T].$$

Part 1

Hamilton-Jacobi-Bellman equation for Optimal Control problems

3

Solutions to the Hamilton-Jacobi equation for Bolza problems with discontinuous time dependent data

The results of this chapter have been published [15].

3.1 Abstract of chapter 3

The results of chapter 3 present characterizations of the value function V of a non autonomous Bolza problem as the unique solution to (HJB) in three suitable generalized sense : Dini, proximal and viscosity solutions. There are two aspects by which these results stand out from previous works.
First of all, the Larangian L and the dynamic F are allowed to be discontinuous on a set of full measure. So far, the majority of previous works dealing with this topic imposed stronger regularity for L and F. For the few others using weaker regularity for the time variable (e.g. merely measurable with respect to time *cf.* [52]), the characterization of V as the solution to (HJB) becomes restrictive, since it is then necessary to remove points from the characterization. Moreover, to obtain the uniqueness of the solution to (HJB), it appears necessary to define generalized solution in a smaller class of functions than usual, losing information in the process.
On the contrary, our hypotheses allow to give a much more natural characterization of V as the unique solution to (HJB), that does not only take all the points into account, but is also given in the class of lower semicontinuous functions to which V commonly belongs.

Then, these result improve a previous work in which the regularity with respect to time was exactly the same (*cf.* [21]). There, only the case of the Mayer problem (corresponding to the case $L = 0$ in our setup) was dealt with. Henceforth, our contribution allows to deal also with the case $L \neq 0$. It must be said that because of a lack of regularity of L with respect to the state variable x, these results could not have been deduced from those proved in [21] together with the help of a state augmentation technique. In addition, the results of this chapter shows that once a suitable definition of extended-sense viscosity solution has been given, V is the unique viscosity solution to (HJB), while such characterization had not been presented in [21].

3.2 Résumé du chapitre 3

Les travaux du chapitre 3 présentent des caractérisations de la fonction valeur V d'un problème de Bolza non-autonome en tant qu'unique solution de l'équation (HJB) en trois sens généralisés adéquats utlisant : le gradient proximal, les dérivées de Dini et une notion de solution de viscosité. Ces résultats se distinguent de ceux ayant été obtenus jusque là de deux manières.

Tout d'abord, le lagrangien L, et la dynamique F sont continues seulement presque partout par rapport à la variable de temps t. Jusque là, la majorité des travaux antérieurs concernant ce sujet imposaient une régularité plus forte en la variable t. Pour les rares autres traitant du cas d'une régularité plus faible en la variable t (par exemple simplement measurable par rapport à t *cf.* [52]), la caractérisation de V en tant que solution de l'équation (HJB) est alors limitative puisqu'il est nécessaire de restreindre l'ensemble des points considérés dans la caractérisation. Par ailleurs, afin d'obtenir l'unicité de la solution de l'équation (HJB) dans une telle configuration, il apparaît nécessaire de restreindre la classe de fonctions dans laquelle les notions de solutions généralisées sont obtenues.
Au contraire, notre jeu d'hypothèses permet de fournir une caractérisation plus naturelle, dans laquelle non seulement aucun point n'est omis et qui d'autre part permet d'obtenir l'unicité de la solution de (HJB) dans la classe naturelle de fonctions à laquelle appartient V : les fonctions semi-continues inférieurement.

Ensuite, nos résultats viennent préciser et améliorer un résultat antérieur qui utilisait exactement la même régularité en temps des données (*cf.* [21]). Dans cet article, seul le problème de Mayer correspondant chez nous au cas où $L = 0$ était traité. Notre contribution permet désormais de traiter aussi le cas où L n'est pas nécessairement nul. Il est important de préciser que pour des raisons de régularité de L par rapport à la variable d'espace x, ces résultats n'auraient pas pu être déduits de ceux présentés dans [21] à l'aide d'une technique d'augmentation du nombre de variables d'états.
Par ailleurs, les travaux de cette thèse démontrent que – moyennant une définition adaptée – V est l'unique solution de viscosité de (HJB) : caractérisation qui n'avait pas été établie dans [21], même dans le cas où $L = 0$.

3.3 Introduction

Consider the non autonomous Bolza problem:

$$(P_{S,x_0}) \begin{cases} \text{Minimize } \int_S^T L(t,x(t),\dot{x}(t))\mathrm{d}t + g(x(T)) \\ \text{over arcs } x \in W^{1,1}([S,T],\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \text{ for almost every } t \in [S,T], \\ x(S) = x_0, \end{cases}$$

in which $[S,T]$ is a given interval, $x_0 \in \mathbb{R}^n$ is a given initial datum, $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given functions, and $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multivalued function. The reference problem (P_{S,x_0}) can be embedded in a family of problems $(P_{t,x})$ parametrized by pairs of initial data $(t,x) \in [S,T] \times \mathbb{R}^n$. This leads to the concept of the value function for $(P_{t,x})$, $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which, for all $(t,x) \in [S,T] \times \mathbb{R}^n$, is defined taking the infimum cost for $(P_{t,x})$:

$$V(t,x) := \inf \left\{ \int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \,\middle|\, x(\cdot) \;\; F\text{-trajectory on } [t,T], x(t) = x \right\}.$$

Here, an F-trajectory on the interval $[s,t] \subset [S,T]$ is an absolutely continuous arc $x(\cdot) : [s,t] \to \mathbb{R}^n$ which satisfies the reference differential inclusion $\dot{x}(\sigma) \in F(\sigma, x(\sigma))$ for a.e. $\sigma \in [s,t]$. We shall consider characterizations of $V(\cdot,\cdot)$ as the unique solution – in a suitable generalized sense – to the Hamilton-Jacobi equation:

$$\begin{cases} \partial_t \varphi(t,x) + \inf_{v \in F(t,x)} \{\partial_x \varphi(t,x) \cdot v + L(t,x,v)\} = 0 \\ \varphi(T,x) = g(x), \end{cases} \tag{HJB}$$

when we may have a discontinuous behavior of F and L w.r.t. the time variable t. Many techniques have been employed to characterize the value function as solution to (HJB), mainly coming from viscosity solutions theory and viability theory. In both contexts a lot of work has been done including the case of discontinuous time dependence problems (see for instance the monographs [5, 43, 31, 84] and the references therein). In this chapter we employ nonsmooth analysis tools and a viability approach to provide value function characterizations involving the notions of lower Dini derivative (also called contingent epiderivative), proximal subdifferential, and Fréchet subdifferential and superdifferential. An important feature is that we allow the final cost function g to be a lower semicontinuous function, possibly extended valued, incorporating implicit terminal constraints. As a consequence the natural class of functions in which we study the value function is the set of lower semicontinuous functions.

In presence of extended terminal costs, the first result, using viability theory, characterizing lower semicontinuous value functions as solutions to (HJB) in a generalized sense which involves the lower Dini derivative, is obtained in [51]. In the same paper we can find also characterizations using (Fréchet) subdifferentials, and eventually both subdifferentials and superdifferentials leading to a comparison with viscosity solutions for continuous value functions. These results have been achieved for the Mayer problem (i.e. for $L = 0$) assuming velocity sets F which are continuous in (t,x). A further significant contribution is [43], in which appropriate invariance theorems allow to characterize the value function also considering proximal subdifferentials.

Passing to discontinuous time-dependent optimal control problems, the relevance of the role of lower Dini derivatives to deal with measurable time-dependence was highlighted by [83]. Simple examples illustrate that the value function might not be the unique lower semicontinuous generalized (according to the concepts above-mentioned) solution to (HJB) in an 'almost everywhere' sense (cf. the discussion in [21]). However, uniqueness properties of the solution can be derived for the mere measurable time dependent case imposing additional conditions on the class of functions which are candidate to be solutions, such as the epigraph of the candidate solution is absolutely continuous w.r.t. t, see [52].

A different perspective has been recently suggested in [21] for the intermediate case (between the continuous one and the merely measurable one) when the multifunction $t \rightsquigarrow F(t,x)$ has everywhere one-sided limits, for all x, and is continuous on the complement of a zero-measure subset of $[S,T]$ (without necessarily imposing further *a priori* regularity conditions such as the absolute continuity of the epigraph of the candidate solutions). In this context, considering optimal control problems with a final cost term (i.e. $L = 0$), the value function turns out to be the unique lower semicontinuous solution to (HJB) taking into account '*everywhere in t*' characterizations which involve the concepts of lower Dini derivative and the proximal subdifferential. Further important features of the results obtained in [21] are: the presence of left and right limits $F(t^+,x)$ and $F(t^-,x)$ (the role of which cannot be exchanged) in the characterizing conditions and the presence of

the horizontal proximal subdifferentials in the concept of the proximal solution.

The main objective of this chapter is to explore lower semicontinuous characterizations of the value function in the context of non-autonomous Bolza problems, in which the velocity set F satisfies the same assumptions as in [21]. The Lagrangian L is assumed to have the same behavior in t ($L(\cdot, x, v)$ is continuous on a set of full measure and has everywhere left and right limits), but is just continuous w.r.t. x. In addition, L satisfies conditions in v (such as convexity and boundedness on bounded sets). We observe that it would be natural to invoke a well-known augmentation technique and rewrite the reference Bolza problem (P_{S,x_0}) in a Mayer form:

$$\begin{cases} \text{Minimize } g(x(T)) + y(T) \\ \text{over arcs } (x, y) \in W^{1,1}([S, T], \mathbb{R}^{n+1}) \text{ satisfying} \\ (\dot{x}(t), \dot{y}(t)) \in G(t, x(t)) \text{ for almost every } t \in [S, T], \\ x(S) = x_0, \quad y(S) = 0 \end{cases}$$

where $G(t, x) := \{(v, w) \mid v \in F(t, x), \quad w \geq L(t, x, v)\}$. Even if this method provides a good insight of a correct outcome, previous results on the Mayer problem are not applicable in our case. On the other hand, keeping the Bolza formulation of the reference problem allows us, for instance, to impose weaker assumptions on the Lagrangian L, avoiding additional (and more restrictive) Lipschitz continuity conditions of L w.r.t. the state variable x, that would be otherwise necessary to impose if we passed to the Mayer form, and which is typically required in previous work for the Mayer problem (cf. [21], [51], [52] and [56]). Therefore, the mere state augmentation technique does not simplify the task: we would add a step in the analysis and eventually end up with a (possibly more involved) problem, with exactly the same difficulties as we left the reference optimal control problem in the Bolza form.

Our first main result (see Theorem 3.4.1 below) provides a characterization of lower semicontinuous extended valued value functions involving both the notions of generalized solution in terms of lower Dini derivatives and in terms of proximal subdifferentials (confirming that a result consistent with [21, Theorem 2.2] can be obtained also for the class of Bolza problems considered here). The second main result of this chapter gives a positive answer to an important question (highlighted in [21]): it was not known whether to achieve an extended-sense viscosity solution characterization of lower semicontinuous value functions would require employing horizontal Fréchet subderivatives (and superderivatives). Theorem 3.4.2 below gives (together with the examples in Section 1.4) an answer to this issue and represents, at the same time, an extension to earlier viscosity solutions characterizations such as in [51], [52] (and [56] for the state constraints free case), to locally bounded lower semicontinuous value functions for Bolza problems with F and L discontinuous in t and a discontinuous final cost term g.

To complete the huge picture of this strand, we recall that the viability approach is applicable also to characterize value functions for state constrained optimal control problems (cf. [55], [84], [56] and [21]). In this case, the analysis requires some compatibility conditions of the velocity sets F with the state constraint (called 'existence of inward/outward pointing conditions'), which conveys more restrictive assumptions on F and is based on some distance estimates results. The discussion on these technical aspects together with the appropriate assumptions which allow to revisit our results in the context of the state constrained Bolza problems can be found in chapter 4.

The chapter is organized as follows. In Section 3.4 we display the invoked assumptions (together with an hypotheses reduction technique), our main results (Theorem 3.4.1 and Theorem 3.4.2) accompanied by some refinements and a discussion based on three illustrative examples. Section 3.5 is dedicated to some preliminary results. Section 3.6 provides the proof of Theorem 3.4.1, which is split into three main steps. The proof of Theorem 3.4.2 is provided in Section 3.7.

3.4 Main results

3.4.1 Hypotheses

In this chapter we shall invoke the following hypotheses: for every given positive number R_0, there exist functions $c_F(\cdot) \in \mathbb{L}^1([S,T],\mathbb{R}_+)$ and $k_F(\cdot) \in \mathbb{L}^1([S,T],\mathbb{R}_+)$, a modulus of continuity $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and constants $c_0 > 0$, $M_0 > 0$ such that:

(H1): i) The multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes convex, closed, nonempty values. For every $x \in \mathbb{R}^n$, $F(\cdot,x)$ is Lebesgue measurable on $[S,T]$.

ii) The function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, with nonempty domain.

(H2): i) For almost every $t \in [S,T]$ and $x \in \mathbb{R}^n$

$$F(t,x) \subset c_F(t)(1+|x|)\mathbb{B}.$$

ii) For all $(t,x) \in [S,T] \times R_0\mathbb{B}$

$$F(t,x) \subset c_0\mathbb{B}.$$

(H3): i)

$$\mathrm{d}_H(F(t,x'),F(t,x)) \leq \omega(|x-x'|), \text{ for all } x,x' \in R_0\mathbb{B} \text{ and } t \in [S,T].$$

ii)

$$F(t,x') \subset F(t,x) + k_F(t)|x-x'|\mathbb{B}, \text{ for all } x,x' \in R_0\mathbb{B} \text{ and for a.e. } t \in [S,T].$$

(H4): i) For each $x \in \mathbb{R}^n$, $s \in [S,T[$, and $t \in]S,T]$ the following limits (in the sense of Kuratowski) exist and are nonempty

$$F(s^+,x) := \lim_{s' \downarrow s} F(s',x) \text{ and } F(t^-,x) := \lim_{t' \uparrow t} F(t',x).$$

ii) For almost every $s \in [S,T[$ and $t \in]S,T]$, and every $x \in \mathbb{R}^n$ we have

$$F(s^+,x) = F(s,x) \text{ and } F(t^-,x) = F(t,x).$$

(H5): i) The Lagrangian $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable. For every $t \in [S,T]$ and $x \in \mathbb{R}^n$, $L(t,x,\cdot)$ is convex.

ii) L is locally bounded in the following sense

$$|L(t,x,v)| \leq M_0, \quad \text{for all} \quad (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_0\mathbb{B} .$$

(H6): i) $|L(t,x',v) - L(t,x,v)| \leq \omega(|x-x'|)$, for all $x, x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_0\mathbb{B}$.

ii) $L(t^-,x,v) := \lim_{t' \uparrow t} L(t',x,v)$ exists for every $(t,x,v) \in]S,T] \times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(t^-,x,v) = L(t,x,v)$ for a.e. $t \in]S,T]$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$.

iii) $L(s^+,x,v) := \lim_{s' \downarrow s} L(s',x,v)$ exists for every $(s,x,v) \in [S,T[\times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(s^+,x,v) = L(s,x,v)$ for a.e. $s \in [S,T[$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$.

A priori boundedness and hypotheses reduction technique.
We observe that condition (H2) guarantees a well-known *a priori* uniform boundedness property for the F-trajectories. More precisely, if we take initial data $(t,x) \in [S,T] \times \mathbb{R}^n$ and an F-trajectory $y \in W^{1,1}([t,T],\mathbb{R}^n)$ such that $y(t) = x$, then for every $s \in [t,T]$, $y(s) \in (1+|x|)\exp\left(\int_S^T c_F(s)\mathrm{d}s\right)\mathbb{B}$. Set $R_0 := (1+|x|)\exp\left(\int_S^T c_F(s)\mathrm{d}s\right)$, then, owing to (H2) ii), for almost every $s \in [t,T]$, $\dot{y}(s) \in c_0\mathbb{B}$. As a consequence, once we fix the initial data (t,x), using a standard hypotheses reduction argument (cf. [21] or [84]), when we are interested in studying the behavior of the value function at (t,x), we can impose much stronger assumptions. More precisely, we introduce the multifunction $\widehat{F} : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$

$$\widehat{F}(s,y) := \begin{cases} F(s,y) & \text{if } |y| \leq R_0 \\ F(s, R_0 y/|y|) & \text{if } |y| > R_0\,, \end{cases}$$

and the function $\tilde{L} : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\widehat{L}(s,y,v) := \begin{cases} L(s,y,v) & \text{if } |y| \leq R_0 \\ L(s, R_0 y/|y|, v) & \text{if } |y| > R_0\,, \end{cases}$$

The multifunction $\widehat{F}(\cdot,\cdot)$ and the function $\widehat{L}(\cdot,\cdot,\cdot)$ satisfy hypotheses $(H1)$, $(H2)^*$, $(H3)^*$, $(H4)$, $(H5)^*$ and $(H6)^*$, where we denote by $(H2)^*$, $(H3)^*$, $(H5)^*$ and $(H6)^*$ the global (stronger) version of conditions (H2), (H3), (H5) and (H6), in which we have removed the constant R_0.
The data of the problem $(P_{t,x})$ involving either (F,L) or $(\widehat{F},\widehat{L})$ do coincide in a neighborhood of the reference point (t,x). It follows that in the forthcoming analysis we can invoke the more restrictive version of conditions (H1)-(H6) without loss of generality.

3.4.2 Characterizations of lower semicontinuous value functions

We consider the following family of minimization problems indexed by initial data $(t,x) \in [S,T] \times \mathbb{R}^n$:

$$(P_{t,x}) \begin{cases} \text{Minimize } \int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over the arcs } x \in W^{1,1}([t,T],\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(s) \in F(s,x(s)), \text{ for almost every } s \in [t,T], \\ x(t) = x. \end{cases}$$

We recall that the value function $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by the infimum cost for $(P_{t,x})$:

$$V(t,x) = \inf(P_{t,x}), \quad \text{for all } (t,x) \in [S,T] \times \mathbb{R}^n.$$

The first result provides a characterization of lower semicontinuous extended valued value functions in a generalized sense involving the concepts of Dini derivative and proximal normal (to the epigraph); these are sometimes referred to as 'lower Dini solutions' and 'proximal solutions' (cf. [84, 45]).

Theorem 3.4.1 (Characterization of Lower Semicontinuous extended valued Value Functions) *Assume (H1)–(H6). Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then the assertions (a), (b) and (c) below are equivalent.*

(a) The function U is the value function for $(P_{t,x})$: $U = V$.

(b) The function U is lower semicontinuous and satisfies:
i) for every $(t,x) \in ([S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$

$$\inf_{v \in F(t^+,x)} \left[D_\uparrow U((t,x),(1,v)) + L(t^+,x,v) \right] \leq 0\,;$$

ii) for every $(t,x) \in (]S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(U)$

$$\sup_{v \in F(t^-,x)} \left[D_\uparrow U((t,x),(-1,-v)) - L(t^-,x,v) \right] \leq 0\,; \tag{3.1}$$

iii) For all $x \in \mathbb{R}^n$, $U(T,x) = g(x)$.

(c) The function U is lower semicontinuous and satisfies:
i) for every $(t,x) \in (]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$,

$$\xi^0 + \min_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v) \right] \leq 0\,;$$

ii) for every $(t,x) \in (]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$,

$$\xi^0 + \min_{v \in F(t^-,x)} \left[\xi^1 \cdot v + L(t^-,x,v) \right] \geq 0\,; \tag{3.2}$$

iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x') \to (S,x),\, t' > S\}} U(t',x') = U(S,x),$$

and

$$\liminf_{\{(t',x') \to (T,x),\, t' < T\}} U(t',x') = U(T,x) = g(x).$$

We consider now the case when the final cost is lower semicontinuous and locally bounded. In this case it is immediate to see that the value function acquires the same properties. In presence of a locally bounded candidate U to be a solution to an Hamilton-Jacobi equation, a well-known approach in viscosity solutions theory suggests to consider its lower and upper semicontinuous envelopes and check whether the properties of supersolution and subsolution in the viscosity sense are satisfied (cf. [5]). From the perspective developed here, this idea leads to a notion of viscosity solution using the Fréchet subdifferential and superdifferential of the candidate solution U.

Theorem 3.4.2 (Characterization of Lower Semicontinuous locally bounded Value Functions) *Assume (H1)–(H6). Suppose, in addition, that g is locally bounded and satisfies $(g^*)_* = g$. Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R}$ be a locally bounded function. Then, the assertions (a), (b) and (c) of Theorem 3.4.1 are equivalent to (d) below.*

(d) U is lower semicontinuous and satisfies:
i) For every $(t,x) \in]S,T[\times\mathbb{R}^n$, $(\xi^0,\xi^1) \in \partial_- U(t,x)$,

$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \leq 0\,;$$

ii) for every $(t,x) \in]S,T[\times\mathbb{R}^n$, $(\xi^0,\xi^1) \in \partial_+ U^*(t,x)$,

$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \geq 0\,; \tag{3.3}$$

iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x)\ |\ t'>S\}} U(t',x') = U(S,x),$$
$$U(T,x) = g(x),$$

and

$$U^*(T,x) = g^*(x).$$

If the condition $(g^*)_* = g$ in Theorem 3.4.2 is removed, then the implication is valid only in one sense:

Proposition 3.4.3 *Assume (H1)–(H6) are satisfied and that g is locally bounded. Let $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}$ be a locally bounded function. Then, the assertions (a), (b) or (c) of Theorem 3.4.1 imply (d) of Theorem 3.4.2.*

Theorem 3.4.2 can be restated using test functions.

Proposition 3.4.4 *Assume (H1)–(H6). Suppose, in addition, that g is locally bounded and satisfies $(g^*)_* = g$. Let $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}$ be a locally bounded function. Then, the assertions (a), (b) and (c) of Theorem 3.4.1 are equivalent to (d') below.*
(d') U is lower semicontinuous and satisfies:
i) for every $(t,x) \in]S,T[\times\mathbb{R}^n$, for every $\varphi \in \mathcal{C}^1(]S,T[\times\mathbb{R}^n,\mathbb{R})$ such that $U-\varphi$ has a local minimum at (t,x),

$$\partial_t\varphi(t,x) + \inf_{v\in F(t^+,x)} \left[\nabla_x\varphi(t,x)\cdot v + L(t^+,x,v)\right] \leq 0\,;$$

ii) for every $(t,x) \in]S,T[\times\mathbb{R}^n$, for every $\varphi \in \mathcal{C}^1(]S,T[\times\mathbb{R}^n,\mathbb{R})$ such that $U^*-\varphi$ has a local maximum at (t,x),

$$\partial_t\varphi(t,x) + \inf_{v\in F(t^+,x)} \left[\nabla_x\varphi(t,x)\cdot v + L(t^+,x,v)\right] \geq 0\,;$$

iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x)\ |\ t'>S\}} U(t',x') = U(S,x),$$
$$U(T,x) = g(x),$$

and

$$U^*(T,x) = g^*(x).$$

Imposing the lower semicontinuity of L w.r.t. t, we obtain the following result.

Proposition 3.4.5 *Assume (H1)–(H6). If, in addition, we suppose that $L(\,\cdot\,,x,v)$ is a lower semicontinuous function for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$, then the assertions of Theorem 3.4.1, Theorem 3.4.2 and Proposition 3.4.3 remain valid when we replace $L(t^+,x,v)$ by $L(t,x,v)$ in conditions (b) i), (c) i) and (d) i).*

Remark 3.4.6 *(i) The characterizations (c) and (d) of the value function $V(\cdot,\cdot)$ are expressed in terms of proximal subdifferentials, and (Fréchet) subdifferentials and superdifferentials of $V(\cdot,\cdot)$ at points (t,x) belonging to the domain of $V(\cdot,\cdot)$. In [21], where the velocity set F has the same discontinuous behavior, characterizations of the values function are provided by conditions involving the proximal normal cone, which includes both horizontal and non-horizontal proximal subdifferentials. The contribution of horizontal normals can be easily removed when 'F is continuous' (cf. [51, 84]) owing to the well-known (Rockafellar) horizontal approximation theorem (cf. [43]), and it was not clear whether this simplification procedure would be in general applicable in the discontinuous context (cf. the issue raised in [21, Remark 2.2-(d)]). In this manuscript, we establish that we can still use this technique, at the cost of some adjustments in the proof.*

(ii) Conditions in (b), (c) and (d) are formulated taking into account particular left and right limits w.r.t. t of F and L. For the Mayer problem, in [21] it is shown that the role of the left/right limits is crucial to characterize the value function, and assertions (b) and (c) become in general false if we try to exchange the role of those limits. As one may expect, our results for Bolza problems are consistent with [21]. We underline the fact that also for the viscosity solutions characterization (d) the role of the right limit is crucial as illustrated by Example 1 below. Finally we observe that, in (b) i), (c) i) and (d) i) of Theorems 3.4.1 and 3.4.2 we can avoid consideration of the limits of L w.r.t. t, owing to the lower semicontinuity of L (in t).

(iii) The characterization (d) provided by Theorem 3.4.2 concerns lower semicontinuous value functions for optimal control problems having a terminal cost g which is locally bounded and satisfies the condition $(g^*)_* = g$. A natural question would be:

> Is that possible to characterize $V(\cdot,\cdot)$ in the sense of Theorem 3.4.2 for optimal control problems removing the conditions 'g is locally bounded or $(g^*)_* = g$'?

If g is a lower semicontinuous extended valued function (taking the value $+\infty$ at some points), the issue of interpreting the viscosity subsolution replacing the condition (d)-ii) immediately arises and it is not clear how we have to interpret U^*. Taking the lim sup *operator we would lose crucial information on the boundary of* $\mathrm{dom}(V)$ *and the viability approach would not be applicable or give the desired information. On the other hand, if we consider the smaller (extended valued) upper semicontinuous function bigger than V on the domain of V, under some circumstances (such as V is continuous on its domain and* $\mathrm{dom}(V)$ *is a closed set) we would be induced to end up with the function V^- which coincides with V on* $\mathrm{dom}(V)$ *and takes the value $-\infty$ on $[S,T] \times \mathbb{R}^n \setminus \mathrm{dom}(V)$. The latter technique would not help either, as clarified by Example 2. Condition $(g^*)_* = g$ can be removed if we are interested in proving that the value function is a viscosity solution in the sense of (d) of Theorem 3.4.2 (as established by Proposition 3.4.3). However, condition $(g^*)_* = g$ becomes far from being just a technical hypothesis and emerges as crucial if we want a characterization (comparison result) for the value function. This point is illustrated in Example 3. Condition '$(g^*)_* = g$' (or '$(g^*)_* = g$' when g is not necessarily lower semicontinuous) is well-known in the viscosity theory context (cf. [7]).*

(iv) The results above are still valid if we start from a slightly more general context in which the Lagrangian in now extended valued $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ and assumption (H5)-ii) is replaced by a 'local boundedness a.e. in t' in the following sense: there exists a set of full measure $E \subset [S,T]$ such that

$$|L(t,x,v)| \leq M_0, \quad \textit{for all} \quad (t,x,v) \in E \times R_0\mathbb{B} \times 2c_0\mathbb{B} \ .$$

Indeed, using the lower semicontinuity of L we can reduce attention to the case in which L is locally bounded in the sense of (H5), and, then, the analysis remains the same.

(v) Assertions (c) and (d) of Theorems 3.4.1 and 3.4.2 can be easily reformulated in terms of an Hamiltonian function

$$H_\lambda(t,x,p) := \inf_{v \in F(t,x)} [p \cdot v + \lambda L(t,x,v)] \ .$$

Observe that under our assumptions $H_\lambda(\cdot,x,p)$ turns out to be continuous on the complement of a zero-measure subset of $[S,T]$ and has everywhere one-sided limits $H_\lambda(t^+,x,p)$ and $H_\lambda(t^-,x,p)$.

3.4.3 Examples

Example 1. Consider the optimal control problem

$$(P_{t_0,x_0}) \begin{cases} \text{Minimize } g(x(1)) + \int_0^1 L(t,x(t),\dot{x}(t))\mathrm{d}t \\ \text{over arcs } x(\cdot) \in W^{1,1}([t_0,1],\mathbb{R}) \text{ such that} \\ \dot{x}(t) \ \in \ F(t) \quad \text{for a.e. } t \in [t_0,1], \\ x(t_0) \ = \ x_0 \ , \end{cases}$$

where $t_0 \in [0,1]$, $x_0 \in \mathbb{R}$,

$$F(t) \ := \begin{cases} [-1,1], & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ [-\frac{1}{2},\frac{1}{2}], & \text{if} \quad \frac{1}{2} < t \leq 1, \end{cases}$$

$$g(x) \ := \begin{cases} 1+x, & \text{if} \quad x > 0, \\ x, & \text{if} \quad x \leq 0, \end{cases}$$

and

$$L(t,x,v) \ := \begin{cases} (v+1)^2, & \text{if} \quad 0 < t \leq \frac{1}{2}, \\ (v+\frac{1}{2})^2 + 2, & \text{if} \quad \frac{1}{2} < t \leq 1. \end{cases}$$

The value function $V : [0,1] \times \mathbb{R} \to \mathbb{R}$ is

$$V(t,x) \ = \ \begin{cases} x+t+\frac{5}{4}, & \text{if} \quad 0 \leq t \leq \frac{1}{2} \quad \text{and} \quad x+t-\frac{3}{4} > 0, \\ x+t+\frac{1}{4}, & \text{if} \quad 0 \leq t \leq \frac{1}{2} \quad \text{and} \quad x+t-\frac{3}{4} \leq 0, \\ x-\frac{3t}{2}+\frac{5}{2}, & \text{if} \quad \frac{1}{2} < t \leq 1 \quad \text{and} \quad x+\frac{t}{2}-\frac{1}{2} > 0, \\ x-\frac{3t}{2}+\frac{3}{2}, & \text{if} \quad \frac{1}{2} < t \leq 1 \quad \text{and} \quad x+\frac{t}{2}-\frac{1}{2} \leq 0. \end{cases}$$

As a result of a routine analysis, one can see that conditions (b)-(c) of Theorem 3.4.1 and condition (d) of Theorem 3.4.2 are satisfied by V. Here, we only display some calculations at the point $(t_0,x_0) = (\frac{1}{2},\frac{1}{4})$, which is of particular interest since it carries information about the discontinuous behavior of the data F, L and g at the same time. Consider, for instance, (d) ii) of Theorem 3.4.2. We have:

$$\partial_+ V^* \left(\frac{1}{2},\frac{1}{4}\right) = \left\{(\xi^0,\xi^1) \,|\, \xi^1 \geq 1, \xi^1 \geq \xi^0, -\xi^0 + \frac{\xi^1}{2} - 2 \leq 0\right\}.$$

Consistently with condition (d) ii) in Theorem 3.4.2, the value function satisfies:

$$\xi^0 + \min_{v\in[-\frac{1}{2},\frac{1}{2}]} \left\{ \xi^1 v + \left[\left(v + \frac{1}{2} \right)^2 + 2 \right] \right\} = \xi^0 - \frac{\xi^1}{2} + 2 \geq 0,$$

for every $(\xi^0, \xi^1) \in \partial_+ V^* \left(\frac{1}{2}, \frac{1}{4}\right)$.
On the other hand, switching the roles of $F\left(\frac{1}{2}^+\right)$ and $F\left(\frac{1}{2}^-\right)$ in the analysis above, we would not obtain the validity of condition (d) ii) since, taking the vector $(-\frac{3}{2}, 1) \in \partial_+ V^* \left(\frac{1}{2}, \frac{1}{4}\right)$, we obtain:

$$-\frac{3}{2} + \min_{v\in[-1,1]} \left\{ v + \left[\left(v + \frac{1}{2} \right)^2 + 2 \right] \right\} = -\frac{3}{2} + \frac{5}{4} = -\frac{1}{4} < 0.$$

Similarly, switching the roles of $L\left(\frac{1}{2}, x_0, v\right)$ and $L\left((\frac{1}{2})^+, x_0, v\right)$ for the same normal vector, we would not obtain condition (d) ii) either:

$$-\frac{3}{2} + \min_{v\in[-\frac{1}{2},\frac{1}{2}]} \left\{ v + (v+1)^2 \right\} = \frac{3}{2} - \frac{1}{4} = -\frac{7}{4} < 0.$$

Even if we switched limits for both L and F, condition (d) ii) would not be satisfied since we have:

$$-\frac{3}{2} + \min_{v\in[-1,1]} \left\{ v + (v+1)^2 \right\} = -\frac{3}{2} - 1 = -\frac{5}{2} < 0.$$

This example shows that condition (d) ii) must involve the right limits $F(t^+, x)$ and $L(t^+, x, v)$, for, if the limits were taken from the other side, the assertion would be false in general. Similar considerations show the fundamental significance of the right limits also in condition (d) i) of Theorem 3.4.2.

Theorem 3.4.2 provides a characterization for the class of lower semicontinuous functions which are also locally bounded. One might wonder whether this result can be generalized to the class of lower semicontinuous extended valued functions, like for the characterization provided by Theorem 3.4.1. The major difficulty comes from interpreting the concept of viscosity subsolution (which would correspond to condition (d) ii)) on the boundary of the domain of the candidate to be value function. The notion of viscosity subsolution used in this chapter involves consideration of the upper semicontinuous envelope V^*, which has a clear meaning if V is locally bounded. On the other hand, if V were extended valued (with a closed nonempty domain), one might be tempted to take into account the upper semicontinuous extended valued function V^-:

$$V^-(t,x) := \begin{cases} V(t,x), & \text{if } (t,x) \in \text{dom}(V), \\ -\infty, & \text{elsewhere.} \end{cases}$$

The following simple example shows that this would not provide the desired effect, even if F is continuous and the value function V is continuous on $\text{dom}(V)$.

Example 2. Consider the optimal control problem:

$$\begin{cases} \text{Minimize } g(x(1)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([t_0,1],\mathbb{R}) \text{ such that} \\ \dot{x}(t) \in F(t) \quad \text{for a.e. } t \in [t_0,1], \\ x(t_0) = x_0 \,, \end{cases}$$

where $t_0 \in [0,1]$, $x_0 \in \mathbb{R}$,

$$g(x) \;:=\; \begin{cases} +\infty, & \text{if } x > 0, \\ x, & \text{if } x \le 0, \end{cases}$$

and for all $(t, x, v) \in [0,1] \times \mathbb{R} \times \mathbb{R}$,

$$F(t) \;:= [-1, 1].$$

The value function $V : [0,1] \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is:

$$V(t,x) \;=\; \begin{cases} +\infty, & \text{if } x + t - 1 > 0, \\ x + t - 1, & \text{if } x + t - 1 \le 0. \end{cases}$$

Then

$$V^-(t,x) \;=\; \begin{cases} -\infty, & \text{if } x + t - 1 > 0, \\ x + t - 1, & \text{if } x + t - 1 \le 0. \end{cases}$$

Let us consider $(t_0, x_0) \in]0,1[\times\mathbb{R}$ such that $x_0 + t_0 + 1 = 0$. We have:

$$\partial_+ V^-(t_0, x_0) = \left\{ (\xi^0, \xi^0) \,|\, \lambda \in \mathbb{R}_+,\ \xi^0 \le 1 \right\}.$$

However if we use $(-1,-1) \in \partial_+ V^-(t_0, x_0)$, then condition (d) ii) is violated since:

$$-1 + \min_{v \in [-1,1]} \{-v\} = -2 < 0.$$

Example 3. Consider the Mayer problem:

$$\begin{cases} \text{Minimize } g(x(1)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([t_0, 1], \mathbb{R}) \text{ such that} \\ \dot{x}(t) \;\in\; F(t) \quad \text{for a.e. } t \in [t_0, 1], \\ x(t_0) \;=\; x_0\ , \end{cases}$$

where $t_0 \in [0,1]$, $x_0 \in \mathbb{R}$,

$$g(x) \;:=\; \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and for all $(t, x) \in [0,1] \times \mathbb{R}$,

$$F(t) \;:= [0, 1].$$

The value function $V : [0,1] \times \mathbb{R} \to \mathbb{R}$ is:

$$V(t,x) \;=\; \begin{cases} 0, & \text{if } x + 1 - t \ge 0 \text{ and } x \le 0, \\ 1, & \text{if } x + 1 - t < 0 \text{ or } x > 0. \end{cases}$$

One can easily check that V is a viscosity solution, *i.e.* satisfies (d) i)–iii). Consider the function $U : [0,1] \times \mathbb{R} \to \mathbb{R}$:

$$U(t,x) \;:=\; \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x \neq 0, x + 1 - t \ge 0 \text{ and } x \le 0, \\ 1, & \text{if } x + 1 - t < 0 \text{ or } x > 0. \end{cases}$$

Then U is also a viscosity solution in the sense of condition (d). This shows that, if we do not have the property $(g^*)_* = g$, we do not obtain the uniqueness of the viscosity solution in the sense of (d).

3.5 Preliminary results

We observe that, under our reference assumptions (H1)-(H6) (or under their more restrictive form provided by the a priori boundedness argument), for every $(t,x) \in [S,T] \times \mathbb{R}^n$, the problem

$$\inf_{x(\cdot)\, F\text{-trajectory on } [t,T],\ x(t)=x} J(x(\cdot)) := \int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T))$$

has a minimizer. This is due to the fact that, with respect to the $W^{1,1}$ topology, the set of F-trajectories $\{x(\cdot)\, F\text{-trajectory on } [t,T],\ x(t)=x\}$ is compact (cf. [45, Theorem 6.39] or [84, Theorem 2.5.3]) and the functional $J(\cdot)$ is lower semicontinuous.
Taking into account (H5)*, we can state a local Lipschitz regularity lemma for the function $L(s,y,\cdot)$ (locally uniformly with respect to (s,y)), the proof of which is based on standard arguments on convex functions. We observe that the role of the number $2c_0$ (instead of the simpler c_0) allows to deduce the Lipschitz regularity of L in v in a ball with the smaller radius c_0.

Lemma 3.5.1 *Assume (H5)*. Then, there exists a positive constant k_L such that for every $(s,y) \in [S,T] \times \mathbb{R}^n$, and $v, v' \in c_0\mathbb{B}$:*

$$|L(s,y,v) - L(s,y,v')| \leq k_L |v - v'|. \tag{3.4}$$

Proof. There exists $M_0 \in \mathbb{R}_+^*$ such that:

$$|L(t,x,v)| \leq M_0, \text{ for all } (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2R\mathbb{B}.$$

Take v_1 and v_2 in $R\mathbb{B}$ such that $v_1 \neq v_2$, and set:

$$v_3 = v_2 + \frac{R}{|v_2 - v_1|}(v_2 - v_1).$$

Observe that $v_3 \in 2R\mathbb{B}$ and:

$$v_2 = \frac{R}{|v_2 - v_1| + R} v_1 + \frac{|v_2 - v_1|}{|v_2 - v_1| + R} v_3.$$

Therefore, by the convexity of $L(t,x,\cdot)$:

$$L(t,x,v_2) \leq \frac{R}{|v_2 - v_1| + R} L(t,x,v_1) + \frac{|v_2 - v_1|}{|v_2 - v_1| + R} L(t,x,v_3).$$

And so, if (t,x) is in $[S,T] \times r\mathbb{B}$, we obtain:

$$\begin{aligned} L(t,x,v_2) - L(t,x,v_1) &\leq \frac{|v_2 - v_1|}{|v_2 - v_1| + R} \left(L(t,x,v_3) - L(t,x,v_1)\right) \\ &\leq \frac{2M_0}{R} |v_2 - v_1|. \end{aligned}$$

Exchanging the roles of v_1 and v_2, we obtain that $L(t,x,\cdot)$ is $\frac{2M_0}{R}$-Lipschitz on $R\mathbb{B}$, uniformly with respect to $(t,x) \in [S,T] \times r\mathbb{B}$. □

In the following lemma we establish a further (uniform) regularity property of the Lagrangian, which we will invoke several times in our analysis.

Lemma 3.5.2 *i) Assume that L satisfies (H5)* i), (H6)* i) and (H6)* iii) . Let $t \in [S,T[$ and $x \in \mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $y \in x + \delta\mathbb{B}$, for every real $s \in]t, t+\delta] \cap [S,T]$, and every $u \in c_0\mathbb{B}$:*

$$L(s,y,u) \geq L(t^+,x,u) - \varepsilon. \tag{3.5}$$

ii) Assume that L satisfies (H5) i) and (H6)* i)-ii). Let $t \in]S,T]$ and $x \in \mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $y \in x + \delta\mathbb{B}$, for every $s \in [t-\delta, t[\cap [S,T]$, and every $u \in c_0\mathbb{B}$:*

$$L(s,y,u) \geq L(t^-,x,u) - \varepsilon. \tag{3.6}$$

iii) Assume that L satisfies (H5) and (H6)* i), and that $L(\,\cdot\,,x,v)$ is lower semicontinuous for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$. Let $t \in [S,T[$ and $x \in \mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $y \in x + \delta\mathbb{B}$, for every $s \in [t-\delta, t+\delta] \cap [S,T]$, and for every $u \in c_0\mathbb{B}$:*

$$L(s,y,u) \geq L(t,x,u) - \varepsilon. \tag{3.7}$$

Proof. We start proving i). Fix any $\varepsilon > 0$. Take any $v \in c_0\mathbb{B}$. Invoking (H6)* iii), there exists $0 < \delta_1(v,\varepsilon) < 1$ such that, for all $s \in]t, t+\delta_1(v,\varepsilon)] \cap [S,T]$, we have

$$L(s,x,v) \geq L(t^+,x,v) - \frac{\varepsilon}{4}. \tag{3.8}$$

Invoking Lemma 3.5.1 we also know that, for all $\tau \in [S,T]$ and $v,u \in c_0\mathbb{B}$,

$$|L(\tau,x,v) - L(\tau,x,u)| \leq k_L|u-v| \ . \tag{3.9}$$

Set $\delta_2(v,\varepsilon) := \min\{\delta_1(v,\varepsilon); \frac{\varepsilon}{4k_L}\} (> 0)$. Then, combining inequalities (3.8) and (3.9) (this is used twice, i.e. for $\tau = t^+$ and $\tau = s$) yields: for every $s \in]t, t+\delta_2(v,\varepsilon)] \cap [S,T]$, and every $u \in (v + \delta_2(v,\varepsilon)\mathbb{B}) \cap c_0\mathbb{B}$ we have

$$L(s,x,u) \geq L(t^+,x,u) - \frac{3}{4}\varepsilon. \tag{3.10}$$

Using the compactness of $c_0\mathbb{B}$, from the open cover of the set $c_0\mathbb{B} \subset \bigcup_{v \in c_0\mathbb{B}} v + \delta_2(v,\varepsilon)\mathring{\mathbb{B}}$ ($\mathring{\mathbb{B}}$ is the open unit ball) we can extract a finite subcover:

$$c_0\mathbb{B} \subset \bigcup_{j=1}^{N} v_j + \delta_2(v_j,\varepsilon)\mathring{\mathbb{B}}.$$

Define $\delta_3 := \min_{j=1,\ldots,N} \delta_2(v_j,\varepsilon)$. We obtain:

$$L(s,x,u) \geq L(t^+,x,u) - \frac{3}{4}\varepsilon, \tag{3.11}$$

for all $s \in]t, t+\delta_3] \cap [S,T]$, and every $u \in c_0\mathbb{B}$. From (H6)* i), we know that there exists $0 < \delta \leq \delta_3$ such that $\omega(\delta) \leq \frac{1}{4}\varepsilon$, and so

$$|L(s,y,u) - L(s,x,u)| \leq \frac{1}{4}\varepsilon, \text{ for all } y \in x + \delta\mathbb{B}.$$

As a consequence, from this inequality and from (3.11), we deduce the validity of (3.5). The proofs of ii) and iii) follow along the same lines. Indeed, in the first step of the proof, we can use respectively (H6)* ii) and the lower semicontinuity of $L(\,\cdot\,,x,v)$ instead of (H6)* iii) to obtain (3.8) on the suitable time interval.

□

We now introduce the auxiliary Lagrangian L^- which will be used as a technical tool in the characterization of solutions to the Hamilton-Jacobi equation. Take any $(t,x) \in]S,T] \times \mathbb{R}^n$. We consider the following modulus of continuity of F with respect to time (from the left) $\theta_t^- : [0, t-S] \to \mathbb{R}_+$, defined by: for every $h \in [0, t-S]$,

$$\theta_t^-(h) := \begin{cases} \sup_{\substack{0<t-s\leq h \\ |x-y|\leq c_0 h}} \mathrm{d}_H(F(s,y), F(t^-,x)), & \text{if } h \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Write $K := \exp\left(\int_S^T k_F(s)\mathrm{d}s\right)$. If we take also a vector $v \in F(t^-,x)$, we define the following set:

$$\begin{aligned} Z(t,x,v) := \{z(\cdot) \ F\text{-trajectory on } [S,t] \mid z(t) = x \\ \text{and } \|x + (\cdot - t)v - z(\cdot)\|_{\mathbb{L}^\infty([t-h,t],\mathbb{R}^n)} \leq K\theta_t^-(h)h, \text{ for all } h \in [0, t-S]\}. \end{aligned} \tag{3.13}$$

The Lagrangian $L^- :]S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as follows: for every $(t,x,v) \in]S,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$L^-(t,x,v) := \begin{cases} \liminf_{h \downarrow 0} h^{-1} \inf\left\{\int_{t-h}^t L(s,z(s),\dot{z}(s))\mathrm{d}s \,\middle|\, z \in Z(t,x,v)\right\}, & \text{if } v \in F(t^-,x), \\ L(t^-,x,v), \text{ otherwise .} \end{cases} \tag{3.14}$$

The map L^- arises in a somehow natural way in some crucial steps of our analysis (cf. the proofs of Proposition 3.5.3 and Theorem 3.4.1 below). A similar auxiliary Lagrangian function was introduced in [49, 50] to investigate characterization of solutions to Hamilton-Jacobi equations in the context of calculus of variations. In our framework the expression of L^- is more involved since we have to take account of the velocity constraint given by the differential inclusion $\dot{z}(s) \in F(s,z(s))$ and the possible different (from the left and from the right) limit behavior of F w.r.t. t.

Proposition 3.5.3 *Suppose that $(H1)$, $(H2)^*$, $(H3)^*$, $(H4)$ and $(H5)^*$ are satisfied.*
i) Then, for all $(t,x,v) \in]S,T] \times \mathbb{R}^n \times \mathbb{R}^n$, we have $L^-(t,x,v) \in \mathbb{R}$ and

$$L(t^-,x,v) \leq L^-(t,x,v). \tag{3.15}$$

ii) If, in addition, L satisfies $(H6)^*$ i)-ii), then for every $(t,x,v) \in]S,T] \times \mathbb{R}^n \times \mathbb{R}^n$ we also obtain

$$L^-(t,x,v) \leq L(t^-,x,v). \tag{3.16}$$

Proof. i) Consider $(t,x) \in]S,T] \times \mathbb{R}^n$. We can assume that $v \in F(t^-,x)$, since otherwise the stated inequality immediately follows from the definition of L^-. Using Filippov existence theorem (cf. [84, Theorem 2.4.3]), we have $Z(t,x,v) \neq \emptyset$. As a consequence, we obtain $\inf\left\{\int_{t-h}^t L(s,z(s),\dot{z}(s))\mathrm{d}s \,|\, z \in Z(t,x,v)\right\} \neq +\infty$, for all $h \in]0, t-S]$.
Invoking the *a priori* uniform boundedness of the F-trajectories, it is straightforward to see that all the arcs in $Z(t,x,v)$ are uniformly bounded and uniformly Lipschitz continuous. Since L is bounded in the sense of condition $(H5)^*$, we deduce that there exists a constant $M_0 > 0$ such that, for every $z(\cdot) \in Z(t,x,v)$,

$$|L(s,z(s),\dot{z}(s))| \leq M_0, \text{ for almost every } s \in [S,t].$$

It follows that, for all $h \in]0, t-S]$: $\left|\inf\left\{\int_{t-h}^t L(s,z(s),\dot{z}(s))\mathrm{d}s \,|\, z \in Z(t,x,v)\right\}\right| \leq hM_0$, which implies that $|L^-(t,x,v)| \leq M_0$, and therefore $L^-(t,x,v) \in \mathbb{R}$.

We now establish (3.15). Let $\varepsilon > 0$. For every $h \in]0, t-S]$, small enough, we choose $z_h \in W^{1,1}([S,t],\mathbb{R}^n)$ an εh-minimizer for the following Lagrange problem:

$$\inf\left\{\int_{t-h}^{t} L(s,z(s),\dot{z}(s))\mathrm{d}s \;\middle|\; z \in Z(t,x,v)\right\}.$$

Invoking again the *a priori* boundedness of F-trajectories, the family $(\dot{z}_h)_{h\in]0,t-S]}$ is bounded in $\mathbb{L}^\infty$ by c_0. Using (3.6) of Lemma 3.5.2 and Jensen's inequality, we obtain for all $h \in]0, t-S]$:

$$\frac{1}{h}\int_{t-h}^{t} L(s,z_h(s),\dot{z}_h(s))\mathrm{d}s \geq \frac{1}{h}\int_{t-h}^{t} L(t^-,x,\dot{z}_h(s))\mathrm{d}s - \varepsilon \geq L\left(t^-,x,\frac{1}{h}\int_{t-h}^{t}\dot{z}_h(s)\mathrm{d}s\right) - \varepsilon.$$

From standard analysis, we also know that $\lim_{h\downarrow 0}\frac{1}{h}\int_{t-h}^{t}\dot{z}_h(s)\mathrm{d}s = v$. Passing to the limit inferior in the last equation, we have:

$$\varepsilon + L^-(t,x,v) \geq L(t^-,x,v) - \varepsilon.$$

which confirms (3.15) since ε is arbitrary.

ii) Consider $(t,x) \in]S,T] \times \mathbb{R}^n$. Again, we can restrict attention to the case $v \in F(t^-,x)$, since otherwise the assertion easily follows from the definition of L^-, and claim that

$$L^-(t,x,v) \leq \lim_{h\downarrow 0}\sup_{\substack{0<t-s\leq h\\ |x-y|\leq c_0 h}} L(s,y,v). \tag{3.17}$$

Indeed, using Filippov existence theorem, we can find an F-trajectory $z \in W^{1,1}([S,t],\mathbb{R}^n)$, such that $z(t) = x$ and for every $h \in [0, t-S]$:

$$\|z - (x + (\cdot - t)v)\|_{\mathbb{L}^\infty([t-h,t],\mathbb{R}^n)} \leq \int_{t-h}^{t}|\dot{z}(s) - v|\mathrm{d}s \leq K\theta_t^-(h)h, \tag{3.18}$$

where $K = \exp\left(\int_S^T k_F(s)\mathrm{d}s\right)$. From Lemma 3.5.1, there exists $k_L > 0$ such that for every $(t',x') \in [S,T]\times\mathbb{R}^n$, $v, v' \in c_0\mathbb{B}$:

$$|L(t',x',v) - L(t',x',v')| \leq k_L|v - v'|.$$

As a consequence, for every $h \in]0, t-S]$, we have:

$$\begin{aligned}
\inf\left\{\int_{t-h}^{t} L(s,z(s),\dot{z}(s))\mathrm{d}s \,|\, z \in Z(t,h,v)\right\} &\leq \int_{t-h}^{t} L(s,z(s),\dot{z}(s))\mathrm{d}s,\\
&\leq \int_{t-h}^{t} L(s,z(s),v)\mathrm{d}s + \int_{t-h}^{t} k_L|\dot{z}(s) - v|\mathrm{d}s\\
&\leq h\sup_{\substack{0<t-s\leq h\\ |x-y|\leq c_0 h}} L(s,y,v) + h\,k_L\,K\theta_t^-(h).
\end{aligned}$$

Dividing across by h, passing to the limit inferior as h goes to 0, yields (3.17). If L satisfies satisfies also (H6)* i)-ii), then:

$$L^-(t,x,v) \leq \lim_{h\downarrow 0}\sup_{\substack{0<t-s\leq h\\ |x-y|\leq c_0 h}} L(s,y,v) = L(t^-,x,v),$$

which confirms (3.16).

□

3.6 Proof of Theorem 3.4.1

The proof has the following structure: we first show that the value function satisfies property (b) of Theorem 3.4.1. We subsequently prove that condition (b) implies condition (c). Finally, if a lower semicontinuous function U satisfies (c) then we show that it coincides with the value function. Each step is highlighted by a proposition or a theorem statement.

3.6.1 The value function satisfies (b) of Theorem 3.4.1

Proposition 3.6.1 *Assume (H1)–(H6). Let $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the value function of the problem. Then V satisfies (b) i)–iii) of Theorem 3.4.1.*

Proof. From the definition of V it immediately follows that $V(T,\cdot) = g(\cdot)$ confirming (b) iii). The lower semicontinuity of V can be deduced by standard arguments (see for instance [64, Theorem 1.1]). We have to prove that V satisfies (b) i) and (b) ii) of Theorem 3.4.1.

Step 1 The first part of this step is somewhat standard (cf. [84, 21]). We briefly reproduce this analysis since, in the second part of this step, it has to be properly combined with suitable properties on the Lagrangian L, mainly described by Lemma 3.5.2.

Take $(t,x) \in ([S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(V)$. Let $y \in W^{1,1}([t,T],\mathbb{R}^n)$ be a minimizing F-trajectory for $(P_{t,x})$, whose existence is guaranteed by our assumptions on F and L (see Section 3.5). Using the principle of optimality, for every $\delta \in]0,T-t]$, we have:

$$V(t+\delta, y(t+\delta)) - V(t,x) = \int_{t+\delta}^{t} L(s,y(s),\dot{y}(s))\mathrm{d}s.$$

From the fundamental theorem of calculus we also have for every $\delta \in]0,T-t]$:

$$\delta^{-1}(y(t+\delta) - y(t)) = \delta^{-1}\int_{t}^{t+\delta} \dot{y}(s)\mathrm{d}s.$$

Let $(\delta_i)_{i\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers that converges to 0. For every integer $i \in \mathbb{N}$, let us define $v_i \in \mathbb{R}^n$ by:

$$v_i := \delta_i^{-1}\int_{t}^{t+\delta_i} \dot{y}(s)\mathrm{d}s.$$

From the *a priori* boundedness of the F-trajectories guaranteed by the *hypotheses reduction* of Section 3.4, $|\dot{y}(s)| \le c_0$ for almost every $s \in [t,T]$. From this inequality, we deduce that the sequence $(v_i)_{i\in\mathbb{N}}$ is bounded by c_0. Then, there exists a vector $\bar{v} \in c_0\mathbb{B}$ such that, up to a subsequence, $v_i \xrightarrow[i\to+\infty]{} \bar{v}$.

Take any $p \in \mathbb{R}^n$ and $i \in \mathbb{N}$. Since $F(s^+, y(s)) = F(s, y(s))$ almost everywhere for each $i \in \mathbb{N}$, we have:

$$p\cdot v_i = \delta_i^{-1}\int_{t}^{t+\delta_i} p\cdot\dot{y}(s)\mathrm{d}s \le \delta_i^{-1}\int_{t}^{t+\delta_i} \max_{v\in F(s^+,y(s))} p\cdot v\,\mathrm{d}s. \tag{3.19}$$

But since the function $s \longmapsto \max_{v\in F(s^+,y(s))} p\cdot v$ is right continuous at $s=t$, letting i go to $+\infty$ in equation (3.19), we have:

$$p\cdot\bar{v} \le \max_{v\in F(t^+,y(t))} p\cdot v.$$

Employing the characterization of the closed convex hull of a set by the support function [78, Thm 13.1], we deduce that $\bar{v} \in F(t^+, y(t)) = F(t^+, x)$.

Using the definition of $\mathrm{D}_\uparrow V((t,x),(1,\bar{v}))$ and the principle of optimality, we obtain:

$$\begin{aligned}\mathrm{D}_\uparrow V((t,x),(1,\bar{v})) + L(t^+,x,\bar{v}) &\leq \liminf_{i\to\infty} \delta_i^{-1}(V(t+\delta_i, y(t+\delta_i)) - V(t,x)) + L(t^+,x,\bar{v}),\\ &= \liminf_{i\to\infty} \delta_i^{-1} \int_{t+\delta_i}^{t} L(s,y(s),\dot{y}(s))\mathrm{d}s + L(t^+,x,\bar{v}),\\ &= \liminf_{i\to\infty} -\delta_i^{-1} \int_{t}^{t+\delta_i} L(s,y(s),\dot{y}(s))\mathrm{d}s + L(t^+,x,\bar{v}). \end{aligned} \tag{3.20}$$

Fix any $\varepsilon > 0$. We consider the constant $\delta_\varepsilon > 0$ given by Lemma 3.5.2 i) for the reference pair $(t,x) \in [S,T[\times\mathbb{R}^n$. Since $v_i \xrightarrow[i\to+\infty]{} \bar{v}$ and $L(t^+,x,\cdot)$ is continuous, there exists $N_0 \in \mathbb{N}$ such that:

$$|L(t^+,x,v_i) - L(t^+,x,\bar{v})| \leq \varepsilon, \text{ for all } i \geq N_0,$$

and from the continuity of the F-trajectory $y(\cdot)$, we can choose an integer $N \geq N_0$ such that for every integer $i \geq N$: $\delta_i < \delta_\varepsilon$, and for all $s \in [t, t+\delta_i]$, $|y(s) - x| \leq \delta_\varepsilon$. Since for almost every $s \in [t,T]$, $|\dot{y}(s)| \leq c_0$, Lemma 3.5.2 guarantees that for almost every $s \in [t, t+\delta_\varepsilon] \cap [t,T]$,

$$L(s,y(s),\dot{y}(s)) \geq L(t^+,x,\dot{y}(s)) - \varepsilon.$$

Thus, for any integer $i \geq N$:

$$\delta_i^{-1} \int_t^{t+\delta_i} L(s,y(s),\dot{y}(s))\mathrm{d}s \geq \delta_i^{-1} \int_t^{t+\delta_i} L(t^+,x,\dot{y}(s))\mathrm{d}s - \varepsilon.$$

Applying Jensen's inequality to the convex function $L(t,x,\cdot)$, we also obtain:

$$\delta_i^{-1} \int_t^{t+\delta_i} L(t^+,x,\dot{y}(s))\mathrm{d}s \geq L\left(t^+, x, \delta_i^{-1}\int_t^{t+\delta_i} \dot{y}(s)\mathrm{d}s\right) - \varepsilon = L\left(t^+,x,v_i\right) - \varepsilon.$$

We deduce that $-\delta_i^{-1}\int_t^{t+\delta_i} L(s,y(s),\dot{y}(s))\mathrm{d}s + L(t^+,x,\bar{v}) \leq 2\varepsilon$ for every integer $i \geq N$, and so, from (3.20) we obtain:

$$\mathrm{D}_\uparrow V((t,x),(1,\bar{v})) + L(t^+,x,\bar{v}) \leq 2\varepsilon.$$

Since ε is arbitrary, this confirms (b)-i).

Step 2 Let $(t,x) \in \mathrm{dom}(V) \cap]S,T] \times \mathbb{R}^n$. Let $\tilde{v} \in F(t^-,x)$. For every $s \in [S,t]$, set $y(s) = x + (s-t)\tilde{v}$. Hypotheses on the multifunction F allow us to use the Filippov existence theorem: there exists an F-trajectory $\tilde{z}(\cdot)$ that satisfies $\tilde{z}(t) = x$, such that for every $h \in]0, t-S]$,

$$\|\tilde{z} - y\|_{\mathbb{L}^\infty([t-h,t],\mathbb{R}^n)} \leq K\left(\int_{t-h}^{t} \mathrm{d}_{F(s,y(s))}(\tilde{v})\mathrm{d}s\right) \leq K\theta_t^-(h)h,$$

where $K = \exp\left(\int_S^T k_F(s)\mathrm{d}s\right)$ and θ_t^- is the modulus of continuity defined in (3.12). Recalling the *hypotheses reduction* and definition of $Z(t,x,\tilde{v})$ given in (3.13) (see Section 3.4), it follows that $\tilde{z}(\cdot) \in Z(t,x,\tilde{v}) \neq \emptyset$. For any $h \in]0, t-S]$, there exists an h^2 minimizer $z_h(\cdot) \in Z(t,x,\tilde{v})$ of the Lagrange problem:

$$\inf\left\{\int_{t-h}^{t} L(s,z(s),\dot{z}(s))\mathrm{d}s \,\middle|\, z \in Z(t,x,\tilde{v})\right\}$$

For any $h \in]0, t-S]$, we write $v_h := h^{-1}(x - z_h(t-h))$. We obtain:

$$|\tilde{v} - v_h| = h^{-1}|z_h(t-h) - y(t-h)| \leq K\theta_t^-(h).$$

We deduce $v_h \underset{h\downarrow 0}{\longrightarrow} \tilde{v}$.

Using the principle of optimality applied to the F-trajectories $z_h(\cdot)$, we also have:

$$V(t-h, x-hv_h) - V(t,x) \leq \int_{t-h}^{t} L(s, z_h(s), \dot{z}_h(s))\mathrm{d}s, \text{ for every } h \in]0, t-S].$$

It follows that for every $h \in]0, t-S[$,

$$h^{-1}(V(t-h,x-hv_h) - V(t,x)) \leq h^{-1} \inf \left\{ \int_{t-h}^{t} L(s, z(s), \dot{z}(s))\mathrm{d}s \,\middle|\, z \in Z(t,x,\tilde{v}) \right\} + h.$$

Hence, passing to the limit inferior when h goes to 0 and recalling the definition of L^- in (3.14),

$$\mathrm{D}_\uparrow V((t,x),(-1,-\tilde{v})) \leq \liminf_{h\to 0} h^{-1}\left(V(t-h, x-hv_h) - V(t,x)\right) \leq L^-(t,x,\tilde{v}).$$

As a consequence, owing to ii) of Proposition 3.5.3, we obtain:

$$\mathrm{D}_\uparrow V((t,x),(-1,-\tilde{v})) \leq L(t^-,x,\tilde{v}),$$

which establishes the validity of (b)-ii), concluding the proof of Proposition 3.6.1.

□

3.6.2 The value function is a proximal solution

In this subsection we prove that any lower semicontinous function $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfying condition (b) from Theorem 3.4.1 also verifies condition (c) from Theorem 3.4.1, *i.e.* is a proximal solution.

Proposition 3.6.2 *Assume (H1)–(H6) and let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function satisfying (b) i)–iii) from Theorem 3.4.1. Then U is a proximal solution to (HJB), i.e. satisfies (c) i)–iii).*

We shall make use of two technical lemmas, which provide consequences of properties (b) i) and (b) ii) of Theorem 3.4.1.

Lemma 3.6.3 *Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinous function. Take any $(t,x) \in ([S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$. Then, there exists $v \in F(t^+,x)$, a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v, and a strictly decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$, converging to 0 as i goes to $+\infty$, such that:*

$$\lim_{i\to+\infty} h_i^{-1}(U(t+h_i, x+h_iv_i) - U(t,x)) = \left(\inf_{w\in F(t^+,x)} D_\uparrow U((t,x),(1,w)) + L(t^+,x,w)\right) - L(t^+,x,v).$$

Assume, in addition, that U satisfies (b) i). Then we have:

$$\lim_{i\to+\infty} h_i^{-1}(U(t+h_i, x+h_iv_i) - U(t,x)) \leq -L(t^+,x,v). \tag{3.21}$$

Proof. Fix any $(t,x) \in ([S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$. Write $\Delta := \inf_{w\in F(t^+,x)} D_\uparrow U((t,x),(1,w)) + L(t^+,x,w)$. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ in $\mathbb{R}_+$ be a strictly decreasing sequence that converges to 0. For any $j \in \mathbb{N}$, there exists a vector $v_j \in F(t^+,x)$ such that:

$$\Delta \leq D_\uparrow U((t,x),(1,v_j)) + L(t^+,x,v_j) \leq \Delta + \varepsilon_j.$$

Since $F(t^+,x)$ is compact, there exists $\tilde{v}$ in $F(t^+,x)$ for which, up to a subsequence, $(v_j)_{j\in\mathbb{N}}$ converges to $\tilde{v}$. By definition of the limit inferior, for each $j \in \mathbb{N}$, there exists a sequence $(v_{j,i})_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v_j and a strictly decreasing sequence $(h_{j,i})_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ converging to 0 such that:

$$\lim_{i\to+\infty} h_{j,i}^{-1}(U(t+h_{j,i},\, x+h_{j,i}v_{j,i}) - U(t,x)) = \liminf_{v'\to v_j, h'\downarrow 0} h'^{-1}(U(t+h', x+h'v') - U(t,x)).$$

It follows that we can construct a sequence $(\varphi(j))_{j\in\mathbb{N}}$ for which the subsequence $(h_{j,\varphi(j)})_{j\in\mathbb{N}}$ is strictly decreasing, converges to 0, and such that for every $j \in \mathbb{N}^*$:

$$|v_j - v_{j,\varphi(j)}| \leq \varepsilon_j,$$

$$h_{j,\varphi(j)}^{-1}(U(t+h_{j,\varphi(j)}, x+h_{j,\varphi(j)}v_{j,\varphi(j)}) - U(t,x)) \in [\Delta - \varepsilon_j - L(t^+,x,v_j), \Delta + 2\varepsilon_j - L(t^+,x,v_j)]. \quad (3.22)$$

Write $\tilde{h}_j := h_{j,\varphi(j)}$ and $\tilde{v}_j := v_{j,\varphi(j)}$ for each $j \in \mathbb{N}^*$. As a consequence, we have $\lim_{j\to+\infty} \tilde{v}_j = \tilde{v}$ and $\lim_{j\to+\infty} \tilde{h}_j = 0$. Moreover, using the continuity of $L(t^+,x,\cdot)$, we obtain $\lim_{j\to+\infty} L(t^+,x,v_j) = L(t^+,x,\tilde{v})$. Therefore, letting j go to $+\infty$ in (3.22) yields:

$$\lim_{j\to+\infty} \tilde{h}_j^{-1}(U(t+\tilde{h}_j, x+\tilde{h}_j\tilde{v}_j) - U(t,x)) = \Delta - L(t^+,x,\tilde{v}).$$

If U satisfies (b) i), we have $\Delta \leq 0$, which implies

$$\lim_{j\to+\infty} \tilde{h}_j^{-1}(U(t+\tilde{h}_j, x+\tilde{h}_j\tilde{v}_j) - U(t,x)) \leq -L(t^+,x,\tilde{v}),$$

and concludes the proof of the lemma.

□

Lemma 3.6.4 *Let $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinous function. Assume that U satisfies (b) ii). Let $(t,x) \in (]S,T]\times\mathbb{R}^n) \cap \mathrm{dom}(U)$. Then for every $v \in F(t^-,x)$, there exists a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v and a decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ which converges to 0, such that:*

$$\lim_{i\to+\infty} h_i^{-1}(U(t-h_i, x-h_iv_i) - U(t,x)) \leq L(t^-,x,v). \quad (3.23)$$

Proof. Consider any $(t,x) \in (]S,T]\times\mathbb{R}^n) \cap \mathrm{dom}(U)$ and $v \in F(t^-,x)$. We have:

$$D_\uparrow U((t,x),(-1,-v)) \leq L(t^-,x,v).$$

Using the definition of $D_\uparrow U$, there exists a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v and a decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$, converging to 0, such that:

$$\liminf_{h\downarrow 0, v_h\to v} h^{-1}(U(t-h, t-hv_h) - U(t,x)) = \lim_{i\to+\infty} h_i^{-1}(U(t-h_i, t-h_iv_i) - U(t,x)) \leq L(t^-,x,v),$$

which concludes the proof.

□

We are now ready to prove Proposition 3.6.2. The proof is split into three steps.

Step 1 We first claim that (c) i) from Theorem 3.4.1 holds: for every $(t,x) \in]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$, we have

$$\xi^0 + \min_{v\in F(t^+,x)} \xi^1 \cdot v + L(t^+,x,v) \leq 0.$$

Take any $(t,x) \in]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$ and $(\xi^0,\xi^1) \in \partial_P U(t,x)$. From the definition of the proximal subdifferential, there exist $M>0$ and $\epsilon>0$ such that

$$\xi^0(t'-t) + \xi^1 \cdot (x'-x) \;\leq\; U(t',x') - U(t,x) + M(|t'-t|^2 + |x'-x|^2) \qquad (3.24)$$
$$\text{for all } (t',x') \in (t,x) + \epsilon\mathbb{B}.$$

From Lemma 3.6.3, there exists $v \in F(t^+,x)$, a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ that converges to v and a strictly decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$, converging to 0, such that:

$$\lim_{i\to+\infty} h_i^{-1}(U(t+h_i,x+h_iv_i) - U(t,x)) \leq -L(t^+,x,v).$$

Taking the particular values $(t+h_i,x+h_iv_i)$ for (t',x') in (3.24), and dividing across by h_i, for every $i\in\mathbb{N}$ sufficiently large, we obtain:

$$\xi^0 + \xi^1\cdot v_i \leq h_i^{-1}(U(t+h_i,x+h_iv_i) - U(t,x)) + Mh_i(1+|v_i|^2).$$

Letting the integer i go to $+\infty$, we have: $\xi^0 + \xi^1\cdot v = \lim_{i\to+\infty}(\xi^0 + \xi^1\cdot v_i) \leq -L(t^+,x,v)$, and thus we obtain: $\xi^0 + \min_{v\in F(t^+,x)} \left[\xi^1\cdot v + L(t^+,x,v)\right] \leq 0$, which confirms the claim of step 1.

Step 2 We now prove that (c) ii) is satisfied: for every $(t,x) \in]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$, for every proximal vector $(\xi^0,\xi^1) \in \partial_P U(t,x)$, we have

$$\xi^0 + \min_{v\in F(t^-,x)} \left[\xi^1\cdot v + L(t^-,x,v)\right] \geq 0.$$

Consider any $(t,x) \in]S,T[\times\mathbb{R}^n) \cap \mathrm{dom}(U)$ and $(\xi^0,\xi^1) \in \partial_P U(t,x)$.
Take any $v \in F(t^-,x)$. Owing to Lemma 3.6.4, we can find two sequences $(v_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ satisfying (3.23). Employing the same arguments used in the first step, there exist $M>0$, $\varepsilon>0$ such that for every $i\in\mathbb{N}$, sufficiently large:

$$-(\xi^0 + \xi^1\cdot v_i) \leq h_i^{-1}(U(t-h_i,x-h_iv_i) - U(t,x)) + Mh_i(1+|v_i|^2).$$

Bearing in mind (3.23), letting i go to $+\infty$, we obtain:

$$-(\xi^0 + \xi^1\cdot v) \leq L(t^-,x,v).$$

Thus we have $\xi^0 + \xi^1\cdot v + L(t^-,x,v) \geq 0$ and consequently:

$$\xi^0 + \inf_{w\in F(t^-,x)} \left[\xi^1\cdot w + \lambda L(t^-,x,w)\right] \geq 0.$$

Step 3 To conclude the proof we have to consider the boundary conditions (c) iii). Take any $x\in\mathbb{R}^n$. Using the lower semicontinuity of U, we have:

$$\liminf_{\{(t',x')\to(S,x)\ |\ t'>S\}} U(t',x') \geq U(S,x), \text{ and } \liminf_{\{(t',x')\to(T,x)\ |\ t'<T\}} U(t',x') \geq U(T,x).$$

If $(S,x) \notin \mathrm{dom}(U)$, then we immediately obtain:

$$\liminf_{\{(t',x')\to(S,x) \mid t'>S\}} U(t',x') \leq U(S,x) = +\infty.$$

If $(S,x) \in \mathrm{dom}(U)$, then using Lemma 3.6.3, we can find $v \in F(S^+,x)$, a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v, and a strictly decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$ converging to 0, such that (3.21) holds at $t = S$.
As a consequence $\limsup_{i\to+\infty} U(S+h_i, x+h_iv_i) \leq U(S,x)$. Thus we have:

$$U(S,x) \leq \liminf_{\{(t',x')\to(S,x) \mid t'>S\}} U(t',x') \leq \liminf_{i\to+\infty} U(S+h_i, x+h_iv_i) \leq U(S,x),$$

which gives the first equality in (c) iii) from Theorem 3.4.1.

If $(T,x) \notin \mathrm{dom}(U)$, clearly we have: $\liminf_{\{(t',x')\to(T,x) \mid t'<T\}} U(t',x') \leq U(T,x)$, so we consider the case when $(T,x) \in \mathrm{dom}(U)$. Then fix any $v \in F(T^-,x)$. Using Lemma 3.6.4, we deduce the existence of a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to v and a decreasing sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_+$, that converges to 0, such that (3.23) holds at $t = T$.
Since $L(t^-,x,v)$ is finite, from (3.23) we deduce that $\limsup_{i\to+\infty} U(T-h_i, x-h_iv_i) \leq U(T,x)$. It follows that

$$U(T,x) \leq \liminf_{\{(t',x')\to(T,x) \mid t'<T\}} U(t',x') \leq \liminf_{i\to+\infty} U(T-h_i, x-h_iv_i) \leq U(T,x).$$

Using the relation $U(T,x) = g(x)$, given by the fact U satisfies (b) iii), we obtain the last desired boundary condition at $t = T$. □

3.6.3 A proximal solution coincides with the value function: comparison results

We display the last part of the proof which consists in showing that if a lower semicontinuous function $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfies (c) of Theorem 3.4.1, then it coincides with the value function for $(P_{t,x})$. We observe that for the inequality $V(t,x) \leq U(t,x)$ conditions (H6) ii) and iii) are not necessary, but they are required for the opposite inequality. More precisely we will prove the following result.

Theorem 3.6.5 *Assume (H1)–(H5) and (H6) i). Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinous function.*

i) Suppose that U satisfies (c) i), and that, for all $x \in \mathbb{R}^n$,

$$U(T,x) = g(x) \quad \textit{and} \quad \liminf_{\{(t',x')\to(S,x),\, t'>S\}} U(t',x') = U(S,x).$$

Then $V(t,x) \leq U(t,x)$ for any $(t,x) \in ([S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(U)$.

ii) Assume, in addition, that L satisfies (H6) ii) and iii). Suppose that U satisfies (c) ii), and for all $x \in \mathbb{R}^n$

$$\liminf_{\{(t',x')\to(T,x),\, t'<T\}} U(t',x') = U(T,x) = g(x).$$

Then $V(t,x) \geq U(t,x)$ for any $(t,x) \in ([S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(U)$.

Theorem 3.6.5 above contains two 'comparison results' establishing the last part of the proof of Theorem 3.4.1 with the implication '(c) ⇒ (a)'. Combined with Propositions 3.6.1 and 3.6.2, it provides uniqueness result for the characterization of the value function in the class of lower semicontinuous functions, as summarized in the Corollary below.

Corollary 3.6.6 *Assume that (H1)–(H6) are satisfied. Then the value function V is the unique lower semicontinuous function solution to (HJB) in the sense of (b)-(c) of Theorem 3.4.1.*

Proof of Theorem 3.6.5 i). In order to establish the first comparison result, bearing in mind the *hypotheses reduction* of Section 3.4, we introduce an auxiliary multivalued function: $Q : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \times \mathbb{R}$ defined by:

$$Q(\tau,x) \;:=\; \begin{cases} \Big\{(v,-\eta)\,|\,v \in F(S^+,x), M_0 \geq \eta \geq L(S^+,x,v)\Big\}, & \text{if} \quad \tau = S, \\ \text{co}\Big\{(v,-\eta)\,|\,v \in \{F(\tau^-,x) \cup F(\tau^+,x)\}\,, M_0 \geq \eta \geq \widetilde{L}(\tau,x,v)\Big\}, & \text{if} \quad \tau \in]S,T[, \\ \Big\{(v,-\eta)\,|\,v \in F(T^-,x), M_0 \geq \eta \geq L(T^-,x,v)\Big\}, & \text{if} \quad \tau = T\ , \end{cases}$$

where $\widetilde{L}(\tau,x,v) := \min\{L(\tau^+,x,v), L(\tau^-,x,v)\}$. A routine analysis allows to verify that the multifunction Q takes as values nonempty convex sets with elements which are (uniformly) bounded by $c := \sqrt{c_0^2 + M_0^2}$; moreover the graph of Q is closed.
Take any $(t_0,x_0) \in (]S,T[\times\mathbb{R}^n) \cap \text{dom}(U)$. The crucial point of Theorem 3.6.5 i) is establishing the applicability of the Weak Invariance Theorem 2.6.1 for the following differential inclusion:

$$\begin{cases} (\dot\tau,\dot x,\dot\ell)(t) \in \Gamma(\tau(t),x(t),\ell(t)), \quad \text{for a.e. } t \in [t_0,T], \\ (\tau(t),x(t),\ell(t)) \in \text{epi } U, \quad \text{for all } t \in [t_0,T], \\ (\tau(t_0),x(t_0),\ell(t_0)) = (t_0,x_0,U(t_0,x_0)). \end{cases}$$

where $\Gamma : [S,T] \times \mathbb{R}^{n+1} \rightsquigarrow \mathbb{R}^{n+2}$ is defined by

$$\Gamma(\tau,x,\ell) \;:=\; \begin{cases} \text{co}\ (\{(0,0,0)\} \cup (\{1\} \times Q(S,x))), & \text{if} \quad \tau = S, \\ \{1\} \times Q(\tau,x), & \text{if} \quad \tau \in]S,T[, \\ \text{co}\ (\{(0,0,0)\} \cup (\{1\} \times Q(T,x))), & \text{if} \quad \tau = T\ . \end{cases}$$

Clearly the multifunction Γ inherits the following properties from Q: the graph of Γ is closed, for all $(\tau,x,\ell) \in [S,T] \times \mathbb{R}^{n+1}$, $\Gamma(\tau,x,\ell)$ is nonempty convex set and $\Gamma(\tau,x,\ell) \subset (c+1)\mathbb{B}$.

It remains to check that 'inward pointing condition' of the Weak Invariance Theorem (Theorem 2.6.1) is also satisfied: take any $(\tau,x,\ell) \in \text{epi } U$ and any $(\xi^0,\xi^1,-\lambda) \in N^P_{\text{epi } U}(\tau,x,\ell)$, we must show

$$\min_{w \in \Gamma(\tau,x,\ell)} (\xi^0,\xi^1,-\lambda) \cdot w \;\leq\; 0\ . \tag{3.25}$$

If $\tau = S,T$, then it is immediately verified (taking $w = 0$ that belongs to both $\Gamma(S,x,\ell)$ and $\Gamma(T,x,\ell)$).
Suppose then that $S < \tau < T$. Observe that, by the nature of proximal normals to epigraph sets, we know that $\lambda \geq 0$ and we need to check (3.25) only when $\ell = U(\tau,x)$. We shall show

$$\xi^0 + \min_{v \in F(\tau^-,x) \cup F(\tau^+,x)} \xi^1 \cdot v + \lambda L(\tau^+,x,v) \;\leq\; 0\ . \tag{3.26}$$

This will confirm the inward pointing condition, indeed it implies

$$\min_{w \in \Gamma(\tau,x,\ell)} (\xi^0,\xi^1,-\lambda) \cdot w \;\leq\; \xi^0 + \min_{v \in \text{co}\{F(\tau^-,x) \cup F(\tau^+,x)\}} \xi^1 \cdot v + \lambda L(\tau^+,x,v) \;\leq\; 0\ .$$

To check (3.26) we need to consider two cases.

Case 1: $\lambda > 0$. In this case $((1/\lambda)\xi^0, (1/\lambda)\xi^1, -1) \in N^P_{\text{epi } U}((\tau, x), U(\tau, x))$. It follows that

$$((1/\lambda)\xi^0, (1/\lambda)\xi^1) \ \in \ \partial_P U(\tau, x).$$

But then, by (c)(i), $(1/\lambda)\left(\xi^0 + \min_{v \in F(\tau^+, x)}[\xi^1 \cdot v + \lambda L(\tau^+, x, v)]\right) \ \leq \ 0$. This implies (3.26).

Case 2: $\lambda = 0$. In this case, we know from the Rockafellar Horizontal Approximation Theorem (see Theorem 2.4.15) that there exist $(\xi^0_i, \xi^1_i) \to (\xi^0, \xi^1)$, $\lambda_i \downarrow 0$ and $(t_i, x_i) \to (\tau, x)$ such that, for each i,

$$(\lambda_i^{-1}\xi^0_i, \lambda_i^{-1}\xi_i) \in \partial_P U(t_i, x_i)\,.$$

But then, by condition (c)(i), there exists $v_i \in F(t_i^+, x_i)$ such that

$$\xi^0_i + \xi^1_i \cdot v_i + \lambda_i L(\tau^+, x, v_i) \leq 0\,.$$

But $(v_i)_{i \in \mathbb{N}}$ is a bounded sequence. We can therefore arrange, by extracting a subsequence, that $v_i \to \bar{v}$, for some $\bar{v} \in \mathbb{R}^n$. Since $(t, x) \rightsquigarrow F(t^-, x) \cup F(t^+, x)$ is an upper semi-continuous multifunction, it follows that $\bar{v} \in F(\tau^-, x) \cup F(\tau^+, x)$. So, recalling also that L is bounded on bounded sets, in the limit as $i \to \infty$, we obtain

$$0 \geq \xi^0 + \xi^1 \cdot \bar{v} + 0 \times L(\tau^+, x, \bar{v}) \geq \xi^0 + \min_{v \in F(\tau^-, x) \cup F(\tau^+, x)} \xi^1 \cdot v.$$

We have confirmed (3.26) in this case too.

Then, the Weak Invariance Theorem 2.6.1 is applicable and there exists $(\tau(\cdot), x(\cdot), \ell(\cdot)) \in W^{1,1}([t_0, T], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ satisfying $\tau(t) = t$ and

$$\begin{cases} (\dot{x}(t), \dot{\ell}(t)) \in Q(t, x(t)), \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0, \ell(t_0) = U(t_0, x_0) \\ \ell(t) \geq U(t, x(t)) \quad \text{for all } \ t \in [t_0, T]\ . \end{cases}$$

Taking into account the definition of the multivalued function Q and the hypotheses on both F and L, we deduce that $x(\cdot)$ is an F-trajectory and that $\dot{\ell}(s) \leq -L(s, x(s), \dot{x}(s))$ for a.e. $s \in [t_0, T]$. Hence we have:

$$g(x(T)) = U(T, x(T)) \leq \ell(T) = \ell(t_0) + \int_{t_0}^T \dot{\ell}(s)\mathrm{d}s \leq U(t_0, x_0) - \int_{t_0}^T L(s, x(s), \dot{x}(s))\mathrm{d}s,$$

which implies:

$$g(x(T)) + \int_{t_0}^T L(s, x(s), \dot{x}(s))\mathrm{d}s \leq U(t_0, x_0).$$

Thus we obtain:

$$V(t_0, x_0) \leq U(t_0, x_0).$$

If (S, x_0) belongs to $\mathrm{dom}(U)$, we pick a decreasing sequence $(h_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_+$ that converges to 0 and a sequence $(y_i)_{i \in \mathbb{N}}$ in $\mathbb{R}^n$ that converges to x_0 such that:

$$\lim_{i \to +\infty} U(S + h_i, y_i) = \liminf_{\{(t', x') \to (S, x) \ | \ t' > S\}} U(t', x') = U(S, x_0).$$

From what precedes, for every integer $i \in \mathbb{N}$, we have:

$$V(S + h_i, y_i) \leq U(S + h_i, y_i).$$

Passing to the limit inferior in that last equation yields:

$$V(S, x_0) = \liminf_{\{(t', x') \to (S, x) | t' > S\}} V(t', x') \leq \liminf_{i \to +\infty} V(S + h_i, y_i) \leq \lim_{i \to +\infty} U(S + h_i, y_i) = U(S, x_0).$$

Note that we also have $g(x_0) = V(T, x_0) \leq U(T, x_0) = g(x_0)$.

□

Proof of Theorem 3.6.5 ii). Pick $(\bar{t},\bar{x}) \in ([S,T]\times\mathbb{R}^n)\cap \mathrm{dom}(U)$ and let $x \in W^{1,1}([\bar{t},T],\mathbb{R}^n)$ be an F-trajectory such that $x(\bar{t}\,)=\bar{x}$. We want to prove:

$$U(\bar{t},\bar{x}) \leq g(x(T)) + \int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s.$$

We can assume that $g(x(T)) < +\infty$, otherwise we automatically have the desired inequality. Using the fact that $\liminf_{\{(t',x')\to(T,x)\ |\ t'<T\}} U(t',x') = U(T,x) = g(x)$, we can find a sequence of points (T_i,y_i) in $]\bar{t},T[\times\mathbb{R}^n$ such that $\lim_{i\to+\infty}(T_i,y_i)=(T,x(T))$ and $\lim_{i\to+\infty} U(T_i,y_i) = U(T,x(T))$. Invoking Filippov's Existence Theorem and arguing as in [21], we obtain a subsequence of F-trajectories $x_i(\cdot)$ on $[\bar{t},T]$ such that $x_i(T_i)=y_i$, for all i, and $\|x_i(\cdot)-x(\cdot)\|_{W^{1,1}}\to 0$ as $i\to+\infty$.
The multivalued function F satisfies the assumptions which allow to apply Carathéodory's parametrization theorems [4, Theorems 9.6.2 and 9.7.2]. Hence there exists a measurable function:

$$f\ :\ [S,T]\times\mathbb{R}^n\times\mathbb{B}\to\mathbb{R}^n,$$

such that:

$$\begin{cases} \text{For every } (t,x)\in[S,T]\times\mathbb{R}^n, F(t,x)=f(t,x,\mathbb{B})\,; \\ \text{For every } (x,u)\in\mathbb{R}^n\times\mathbb{B}, f(\cdot,x,u) \text{ is measurable}\,; \\ \text{For every } (t,u)\in[S,T]\times\mathbb{B}, f(t,\cdot,u) \text{ is } 10nk_F(t)\text{-Lipschitz}\,; \\ \text{For every } (t,x)\in[S,T]\times\mathbb{R}^n, (u,u')\in\mathbb{B}^2, |f(t,x,u)-f(t,x,u')|\leq 5n\max_{v\in F(t,x)}|v||u-u'|. \end{cases} \tag{3.27}$$

Under our hypotheses, for all $(s,x,u)\in]S,T[\times\mathbb{R}^n\times\mathbb{B}$, we have (cf. [21]):

$$\lim_{s'\downarrow s,x'\to x,u'\to u} f(s',x',u')\in F(s^-,x) \text{ and } \lim_{s'\uparrow s,x'\to x,u'\to u} f(s',x',u')\in F(s^+,x).$$

Fix $i\geq 0$. Since $x_i(\cdot)$ is an F-trajectory, for almost every $t\in[\bar{t},T]$:

$$\dot{x}_i(t)\in f(t,x_i(t),\mathbb{B}).$$

and, using Filippov's selection theorem (cf. [84, Theorem 2.3.13]), there exists a measurable selection $u_i : [\bar{t},T]\to\mathbb{B}$ such that:

$$\dot{x}_i(t) = f(t,x_i(t),u_i(t)),\ \text{for almost every } t\in[\bar{t},T].$$

Let $\varepsilon>0$. Lusin's theorem (cf. [45, Proposition 6.14]) allows us to find a pair of functions $(x_i^\varepsilon,u_i^\varepsilon)$ defined on $[\bar{t},T]$ such that $x_i^\varepsilon(T_i)=x_i(T_i)$ and

$$\begin{cases} \dot{x}_i^\varepsilon(t)=f(t,x_i^\varepsilon(t),u_i^\varepsilon(t)) \text{ for almost every } t\in[\bar{t},T]\,; \\ \text{the control } u_i^\varepsilon \text{ is continuous}\,; \\ \|x_i-x_i^\varepsilon\|_{\mathbb{L}^\infty([\bar{t},T],\mathbb{R}^n)}\leq\varepsilon\,; \\ \mathrm{meas}\,(\{t\in[\bar{t},T]\,|\,u_i(t)-u_i^\varepsilon(t)\neq 0\})\leq\varepsilon. \end{cases} \tag{3.28}$$

For every $(t,x)\in[\bar{t},T]\times\mathbb{R}^n$, we write $v^+(t,x) := f(t^+,x,u_i^\varepsilon(t))$, $v^-(t,x) := f(t^-,x,u_i^\varepsilon(t))$. We define two multivalued functions $F_i^\varepsilon : [\bar{t},T_i]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $\Lambda_i^\varepsilon : [\bar{t},T_i]\times\mathbb{R}^n\rightsquigarrow\mathbb{R}$ by the relations: for every $(t,x)\in[\bar{t},T_i]\times\mathbb{R}^n$:

$$F_i^\varepsilon(t,x) := \mathrm{co}\,\{v^-(t,x),v^+(t,x)\},$$

$$\Lambda_i^\varepsilon(t,x) := \mathrm{co}\,\{L(t^-,x,v^-(t,x)), L(t^+,x,v^+(t,x))\}.$$

Then we set a new multifunction $\Gamma_i^\varepsilon : [0, T_i - \bar{t}\,] \times \mathbb{R}^n \times \mathbb{R} \rightsquigarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ defined for every $(t,x,\ell) \in [0, T_i - \bar{t}\,] \times \mathbb{R}^n \times \mathbb{R}$:

$$\Gamma_i^\varepsilon(t,x,\ell) = \begin{cases} \mathrm{co}\,\{(0,0,0) \cup \{1\} \times -F_i^\varepsilon(\bar{t},x) \times \Lambda_i^\varepsilon(\bar{t},x)\}, & \text{if } t = T_i - \bar{t}, \\ \{1\} \times -F_i^\varepsilon(T_i - t, x) \times \Lambda_i^\varepsilon(T_i - t, x), & \text{if } t \in [0, T_i - \bar{t}\,[\,. \end{cases}$$

The multivalued function Γ_i^ε is convex, compact valued and has closed graph. We consider the following differential inclusion:

$$\begin{cases} (\dot{\tau}(t), \dot{y}(t), \dot{\ell}(t)) \in \Gamma_i^\varepsilon(\tau(t), y(t), \ell(t)), \text{ for a.e. } t \in [0, T_i - \bar{t}\,], \\ \tau(0) = 0, y(0) = x_i^\varepsilon(T_i) = x_i(T_i), \ell(0) = U(T_i, x(T_i)), \\ \ell(t) \geq U(T_i - \tau(t), y(t)), \text{ for all } t \in [0, T_i - \bar{t}\,]. \end{cases} \tag{3.29}$$

We define the arc

$$t \mapsto (\tau_i^\varepsilon(t), y_i^\varepsilon(t), \ell_i^\varepsilon(t)) := \left(t, x_i^\varepsilon(T_i - t), U(T_i, x_i(T_i)) + \int_0^t L(T_i - s, x_i^\varepsilon(T_i - s), \dot{x}_i^\varepsilon(T_i - s))\mathrm{d}s\right).$$

The arc $(\tau_i^\varepsilon, y_i^\varepsilon, \ell_i^\varepsilon)$ is the unique Γ_i^ε-trajectory with initial condition $(0, x(T_i), U(T_i, x_i(T_i)))$. Owing to the 'hypotheses reduction' argument of Section 3.4, we deduce that there exists a constant $c := \sqrt{c_0^2 + M_0^2}$ such that $\Gamma_i^\varepsilon(t,x,\ell) \subset (c+1)\mathbb{B}$, for every (t,x,ℓ).

For every $(\tau, x) \in [0, T_i - \bar{t}\,] \times \mathbb{R}^n$, we set $\widetilde{U}(\tau, x) := U(T_i - \tau, x)$. Therefore the last condition in (3.29) can be interpreted as the inclusion $(\tau(t), y(t), \ell(t)) \in \mathrm{epi}\,\widetilde{U}$ for all $t \in [0, T_i - \bar{t}\,]$. We claim that the Weak Invariance Theorem 2.6.1 is applicable to the differential inclusion (3.29). We have already observed that the assumptions i) and ii) of this theorem are satisfied. We show now that Γ_i^ε also satisfies the last ('inward pointing') condition iii). That is, for every pair $(\tau, x) \in ([0, T_i - \bar{t}\,[\times \mathbb{R}^n) \cap \mathrm{dom}(\widetilde{U})$ and every $\ell \geq \widetilde{U}(\tau, x)$:

$$\min_{w \in \Gamma_i^\varepsilon(\tau,x,\ell)} (\xi^0, \xi^1, -\lambda) \cdot w \leq 0, \text{ for all } (\xi^0, \xi^1, -\lambda) \in N^P_{\mathrm{epi}\,\widetilde{U}}((\tau, x), \ell). \tag{3.30}$$

Indeed, let $(\tau, x) \in ([0, T_i - \bar{t}\,[\times \mathbb{R}^n) \cap \mathrm{dom}(\widetilde{U})$ and $(\xi^0, \xi^1, -\lambda) \in N^P_{\mathrm{epi}\,\widetilde{U}}((\tau, x), \widetilde{U}(\tau, x))$ (we recall that we can always reduce to the case $\ell = \widetilde{U}(\tau, x)$), which is equivalent to say:

$$(-\xi^0, \xi^1, -\lambda) \in N^P_{\mathrm{epi}\,U}((T_i - \tau, x), U(T_i - \tau, x)).$$

We consider two different cases.

Case 1 $\lambda > 0$**.** Then $\lambda^{-1}(-\xi^0, \xi^1) \in \partial_P U((T_i - \tau, x))$. Hence, since the vector $v^-(T_i - \tau, x) \in F_i^\varepsilon(T_i - \tau, x) \cap F((T_i - \tau)^-, x)$, condition (c) ii) implies:

$$\lambda^{-1}\left(-\xi^0 + \xi^1 \cdot v^-(T_i - \tau, x) + \lambda L((T_i - \tau)^-, x, v^-(T_i - \tau, x))\right) \geq 0,$$

which gives:

$$-\xi^0 + \xi^1 \cdot v^-(T_i - \tau, x) + \lambda L((T_i - \tau)^-, x, v^-(T_i - \tau, x)) \geq 0.$$

Hence we can confirm (3.30) by choosing $w = (1, -v^-(T_i - \tau, x), L((T_i - \tau)^-, x, v^-(T_i - \tau, x)))$.

Case 2 $\lambda = 0$**.** Then invoking the Rockafellar Horizontal Approximation Theorem (see Theorem 2.4.15), there exist sequences $(-\xi_k^0, \xi_k^1) \xrightarrow[k \to +\infty]{} (-\xi^0, \xi^1)$, $\lambda_k \downarrow 0$ and $(s_k, x_k) \xrightarrow[k \to +\infty]{} (T_i - \tau, x)$ such that, for each k:

$$\left(-\lambda_k^{-1}\xi_k^0, \lambda_k^{-1}\xi_k^1\right) \in \partial_P U(s_k, x_k).$$

But then, by condition (c) ii), for each k, we have:

$$-\xi_k^0 + \xi_k^1 \cdot v^-(s_k, x_k) + \lambda_k L(s_k^-, x_k, v^-(s_k, x_k)) \geq 0.$$

By extracting a subsequence we can arrange that, either $s_k \leq T_i - \tau$ for all k, or $s_k > T_i - \tau$ for all k. If $s_k \leq T_i - \tau$ for all k, then since $(s_k, x_k) \xrightarrow[i\to+\infty]{} (T_i - \tau, x)$, $v^-(s_k, x_k) \xrightarrow[i\to+\infty]{} v^-(T_i - \tau, x)$, and, in consequence of the hypotheses (H3), (H4)(i), (H5) and (H6), we can pass to the limit as $k \to +\infty$ in the preceding relation to obtain

$$-\xi^0 + \xi^1 \cdot v^-(T_i - \tau, x) \geq 0.$$

If $s_k > T_i - \tau$ for all k, this time, the passage to the limit gives $v^-(s_k, x_k) \to v^+(T_i - \tau, x)$ and

$$-\xi^0 + \xi^1 \cdot v^+(T_i - \tau, x) \geq 0.$$

In any case, we are able to confirm (3.30).

As a consequence we can apply the Weak Invariance Theorem obtaining that the arc $(\tau_i^\varepsilon(\cdot), y_i^\varepsilon(\cdot), \ell_i^\varepsilon(\cdot))$ is the solution to (3.29). For $t = T_i - \bar{t}$, by a change of variable, we have:

$$U(T_i, x_i(T_i)) + \int_{\bar{t}}^{T_i} L(s, x_i^\varepsilon(s), \dot{x}_i^\varepsilon(s))\mathrm{d}s \geq U(\bar{t}, x_i^\varepsilon(\bar{t})), \text{ for all } \varepsilon. \tag{3.31}$$

Invoking condition (H2)*, for all $t \in [\bar{t}, T]$, $\max_{v\in F(t,x_i(t))} |v| \leq c_0$. So, for almost every $s \in [\bar{t}, T]$ we have:

$$\begin{aligned}
|\dot{x}_i^\varepsilon(s) - \dot{x}_i(s)| &= |f(s, x_i^\varepsilon(s), u_i^\varepsilon(s)) - f(s, x_i(s), u_i(s))| \\
&\leq |f(s, x_i^\varepsilon(s), u_i^\varepsilon(s)) - f(s, x_i(s), u_i^\varepsilon(s))| + |f(s, x_i(s), u_i^\varepsilon(s)) - f(s, x_i(s), u_i(s))| \\
&\leq 10nk_F(s)|x_i^\varepsilon(s) - x_i(s)| + 5n \max_{v\in F(s,x_i(s))} |v||u_i^\varepsilon(s) - u_i(s)| \\
&\leq 10nk_F(s)\varepsilon + 5nc_0|u_i^\varepsilon(s) - u_i(s)|.
\end{aligned}$$

Since $\mathrm{meas}\,(\{t \in [\bar{t}, T] \,|\, u_i(t) - u_i^\varepsilon(t) \neq 0\}) \leq \varepsilon$ and $\|u_i - u_i^\varepsilon\|_{\mathbb{L}^\infty} \leq 2$, this implies that

$$\dot{x}_i^\varepsilon \xrightarrow[\varepsilon\to 0]{\mathbb{L}^1([\bar{t},T],\mathbb{R}^n)} \dot{x}_i.$$

As a consequence, up to a subsequence, $\dot{x}_i^\varepsilon(\cdot)$ converges to $\dot{x}_i(\cdot)$ almost everywhere in $[\bar{t}, T]$. Using (H6)* and (3.4), we can apply Lebesgue's dominated convergence theorem, and we obtain:

$$\int_{\bar{t}}^{T_i} L(s, x_i^\varepsilon(s), \dot{x}_i^\varepsilon(s))\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \int_{\bar{t}}^{T_i} L(s, x_i(s), \dot{x}_i(s))\mathrm{d}s.$$

Passing to the limit inferior in (3.31), bearing in mind that $x_i^\varepsilon(\bar{t}) \xrightarrow[\varepsilon\to 0]{} x_i(\bar{t})$, and U is lower semicontinuous, we obtain

$$U(T_i, x_i(T_i)) + \int_{\bar{t}}^{T_i} L(s, x_i(s), \dot{x}_i(s))\mathrm{d}s \geq U(\bar{t}, x_i(\bar{t})), \text{ for all } i.$$

Then, as $i \to +\infty$,

$$g(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s \geq U(\bar{t}, \bar{x}).$$

Since $x(\cdot)$ was an arbitrary F-trajectory satisfying $x(\bar{t}) = \bar{x}$, we deduce that

$$V(\bar{t}, \bar{x}) \geq U(\bar{t}, \bar{x}).$$

This concludes the proof.

□

3.7 Proofs of Theorem 3.4.2, and Propositions 3.4.3 and 3.4.5

Proof of Theorem 3.4.2. The proof is organized as follows: in Step 1 we show that, assuming hypotheses (H1)-(H6), the value function V is a viscosity solution in the sense of condition (d) of Theorem 3.4.2. In Step 2, we prove that if a lower semicontinuous function U satisfies (d) i) and (d) iii) of Theorem 3.4.2, then $V \leq U$. In Step 3 we prove that if we impose the additional assumption $(g^*)_* = g$, then any lower semicontinuous function U satisfying (d) ii) and (d) iii) satisfies $U \leq V$.

Step 1: The value function V satisfies (d) i)–iii).
Assume that hypotheses (H1)-(H6) are satisfied. We first observe that, from the *a priori* boundedness of the F-trajectories, and the local boundedness of g and L, it immediately follows that V is locally bounded. Let $(t,x) \in]S,T[\times\mathbb{R}^n$. Take any $(\xi^0,\xi^1) \in \partial_- V(t,x)$. Then,

$$\xi^0(t'-t) + \xi^1 \cdot (x'-x) \;\leq\; V(t',x') - V(t,x) + \mathrm{o}(|(t'-t,x'-x)|) \quad \text{for all } (t',x') \tag{3.32}$$

in which $o(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying $o(\epsilon)/\epsilon \xrightarrow[\epsilon\to 0]{} 0$. From Lemma 3.6.3, we can find sequences $h_i \downarrow 0$ and $v_i \to \bar{v}$, for some $\bar{v} \in F(t^+,x)$, such that

$$\lim_{i\to+\infty} h_i^{-1}(V(t+h_i, x+h_i v_i) - V(t,x)) \;\leq\; -L(t^+,x,\bar{v}).$$

Taking $(t',x') = (t+h_i, x+h_i v_i)$ in (3.32), and dividing across by h_i, we obtain, for i sufficiently large,

$$\xi^0 + \xi^1 \cdot v_i \leq h_i^{-1}(V(t+h_i,x+h_i v_i) - V(t,x)) + h_i^{-1}\,\mathrm{o}\left(h_i\sqrt{1+|v_i|^2}\right).$$

Therefore, in the limit as $i \to \infty$, it follows that

$$\xi^0 + \inf_{v\in F(t^+,x)} [\xi^1 \cdot v + L(t^+,x,v)] \;\leq\; \xi^0 + \xi^1 \cdot \bar{v} + L(t^+,x,\bar{v}) \;\leq\; 0,$$

which confirms (d) i).

Let $(t,x) \in]S,T[\times\mathbb{R}^n$ and $\tilde{v} \in F(t^+,x)$. There exists a sequence $(t_i,x_i)_{i\in\mathbb{N}}$ in $]S,T[\times\mathbb{R}^n \setminus \{(t,x)\}$ that converges to (t,x) such that:

$$\lim_{i\to+\infty} V(t_i,x_i) = V^*(t,x).$$

We claim that we can extract a subsequence such that $t_i > t$ for all $i \in \mathbb{N}$. Let us assume that $t_i \leq t$ for every $i \in \mathbb{N}$ and take a strictly decreasing sequence $(\tau_i)_{i\in\mathbb{N}}$ in $]t,T]$ that converges to t. Fix any $i \in \mathbb{N}$, and take an F-trajectory $x_i(\cdot) \in W^{1,1}([t_i,T],\mathbb{R}^n)$ such that $x_i(t_i) = x_i$. Using the principle of optimality, we obtain:

$$V(t_i,x_i) - \int_{t_i}^{\tau_i} L(s,x_i(s),\dot{x}_i(s))\mathrm{d}s \leq V(\tau_i,x_i(\tau_i)).$$

Using the local boundedness of L given by condition (H5)*, there exists $M_0 > 0$ such that for every $i \in \mathbb{N}$:

$$V(t_i,x_i) - M_0|\tau_i - t_i| \leq V(\tau_i,x_i(\tau_i)).$$

Passing to the limit superior and using the upper semicontinuity of V^*, we obtain:

$$V^*(t,x) = \limsup_{i\to+\infty} V(t_i,x_i) \leq \limsup_{i\to+\infty} V(\tau_i,x_i(\tau_i)) \leq \limsup_{i\to+\infty} V^*(\tau_i,x_i(\tau_i)) \leq V^*(t,x).$$

Hence $\limsup_{i\to+\infty} V(\tau_i, x_i(\tau_i)) = V^*(t,x)$ and there exists a subsequence $(i_k)_{k\in\mathbb{N}}$ for which:

$$V(\tau_{i_k}, x_{i_k}(\tau_{i_k})) \xrightarrow[k\to+\infty]{} V^*(t,x).$$

Fix any $\tilde{v} \in F(t^+, x)$. Then for every $i \in \mathbb{N}$, there exists $v_i \in F(t_i^+, x_i)$ such that $\lim_{i\to+\infty} v_i = \tilde{v}$. For every $i \in \mathbb{N}$, we consider the arc

$$y_i(s) := x_i + (s - t_i)v_i, \text{ for all } s \in [t_i, T].$$

Using the Filippov existence theorem, for every $i \in \mathbb{N}$ there exists an F-trajectory $z_i(\cdot)$ that satisfies $z_i(t_i) = x_i$ and such that for every $h \in]0, T - t_i]$

$$\|z_i - y_i\|_{\mathbb{L}^\infty([t_i, t_i+h],\mathbb{R}^n)} \leq K \left(\int_{t_i}^{t_i+h} \mathrm{d}_{F(s,y_i(s))}(v_i)\mathrm{d}s \right),$$

where $K = \exp\left(\int_S^T k_F(s)\mathrm{d}s\right)$. From the *a priori* boundedness of F-trajectories, we can pick $R_0 > 0$ such that, for every $i \in \mathbb{N}$, $|y_i(s)| \leq R_0$ for every $s \in [t_i, T]$. Observe also that $|\dot{y}_i(s)| \leq c_0$, for any $i \in \mathbb{N}$ and for almost every $s \in [t_i, T]$.
For every $i \in \mathbb{N}$, we define $\delta_i = \max\{|V(t_i, x_i) - V^*(t,x)|, |x_i - x|, |t_i - t|\}$. Take a strictly decreasing sequence $(h_i)_{i\in\mathbb{N}}$ that converges to 0 such that $h_i \geq \sqrt{\delta_i}$.

Fix any $i \in \mathbb{N}$, and set $w_i = \frac{1}{h_i}\int_{t_i}^{t_i+h_i} \dot{z}_i(s)\mathrm{d}s$. Note that we have:

$$|v_i - w_i| \leq K \left(\int_{t_i}^{t_i+h_i} \mathrm{d}_{F(s,y_i(s))}(v_i)\mathrm{d}s \right) \leq K\theta_i(h_i),$$

where

$$\theta_i(h) := \begin{cases} \sup\limits_{\substack{0<s-t_i\leq h \\ |x_i-y|\leq c_0 h}} \mathrm{d}_H(F(s,y), F(t_i^+, x_i)), \text{ if } h \neq 0, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

There exists $\tau \in [t_i, t_i + h_i]$ and $z \in \mathbb{R}^n$ verifying $|x_i - z| \leq c_0 h_i$ such that:

$$\theta_i(h_i) \leq \mathrm{d}_H(F(\tau, z), F(t_i^+, x_i)) + \frac{1}{i+1}.$$

Hence we obtain:

$$\theta_i(h_i) \leq \mathrm{d}_H(F(\tau, z), F(t^+, x)) + \mathrm{d}_H(F(t^+, x), F(t_i^+, x_i)) + \frac{1}{i+1}.$$

We notice that:

$$\tau - t \leq (\tau - t_i) + (t_i - t) \leq h_i + h_i^2 \text{ and } |x - z| \leq |z - x_i| + |x_i - x| \leq h_i c_0 + h_i^2.$$

This yields:

$$\theta_i(h_i) \leq \sup_{\substack{0<s-t\leq h_i+h_i^2 \\ |y-x|\leq c_0 h_i + h_i^2}} \mathrm{d}_H(F(s,y), F(t^+, x)) + \sup_{\substack{0<s-t\leq h_i^2 \\ |y-x|\leq h_i^2}} \mathrm{d}_H(F(t^+, x), F(s^+, y)) + \frac{1}{1+i},$$

which implies that $\theta_i(h_i) \xrightarrow[i\to+\infty]{} 0$. Recalling that for every $i \in \mathbb{N}$, $|\tilde{v} - w_i| \leq \theta_i(h_i) + |v_i - \tilde{v}|$, we obtain:

$$w_i \xrightarrow[i\to+\infty]{} \tilde{v}.$$

For every $i \in \mathbb{N}$, we define: $e_i := 1 - \frac{t-t_i}{h_i}$ and $\tilde{w}_i := w_i - \frac{x-x_i}{h_i}$, and immediately notice that $\lim_{i\to+\infty}(e_i, \tilde{w}_i) = (1, \tilde{v})$. This yields:

$$\begin{aligned}\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] &\geq \limsup_{i\to+\infty} h_i^{-1}\left[V^*(t_i+h_i, x_i+h_iw_i) - V^*(t,x)\right], \\ &= \limsup_{i\to+\infty} h_i^{-1}\left[V(t_i+h_i, x_i+h_iw_i) - V^*(t,x)\right].\end{aligned}$$

Fix $i \in \mathbb{N}$. We have:

$$V(t_i+h_i, x_i+h_iw_i) - V^*(t,x) \geq V(t_i+h_i, x_i+h_iw_i) - V(t_i,x_i) - \delta_i. \tag{3.33}$$

Using the principle of optimality, we obtain:

$$V(t_i+h_i, x_i+h_iw_i) - V(t_i,x_i) \geq -\int_{t_i}^{t_i+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s.$$

Hence, dividing across equation (3.33) by h_i, passing to the limit superior in this inequality while recalling $\frac{\delta_i}{h_i} \leq \sqrt{\delta_i}$, we obtain:

$$\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] \geq -\liminf_{i\to+\infty} \frac{1}{h_i}\int_{t_i}^{t+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s. \tag{3.34}$$

We recall that from Lemma 3.5.1, there exists $k_L > 0$ such that for every $(t', x') \in [S,T]\times\mathbb{R}^n$, and $v, v' \in c_0\mathbb{B}$:

$$|L(t',x',v) - L(t',x',v')| \leq k_L|v-v'|.$$

As a consequence, for every $i \in \mathbb{N}$ we have:

$$\begin{aligned}\int_{t_i}^{t_i+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s &\leq \int_{t_i}^{t_i+h_i} L(s, z_i(s), \tilde{v})\mathrm{d}s + \int_{t_i}^{t_i+h_i} k_L|\dot{z}_i(s) - \tilde{v}|\mathrm{d}s, \\ &\leq h_i \sup_{\substack{|z-x|\leq c_0h_i+h_i^2 \\ 0<s-t\leq h_i+h_i^2}} L(s,z,\tilde{v}) + h_i\, k_L(K\theta_i(h_i) + |v_i - \tilde{v}|).\end{aligned}$$

Dividing across by h_i, passing to the limit inferior as i goes to $+\infty$ gives:

$$\liminf_{i\to+\infty} \frac{1}{h_i}\int_{t_i}^{t+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s \leq L(t^+, x, \tilde{v}). \tag{3.35}$$

Combining (3.34) and (3.35) we obtain:

$$\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] \geq -L(t^+, x, \tilde{v}).$$

Now, take any $(\xi^0, \xi^1) \in \partial_+ V^*(t,x)$. Then,

$$V^*(t',x') - V^*(t,x) - \xi^0(t'-t) - \xi^1\cdot(x'-x) \;\leq\; \mathrm{o}(|(t'-t, x'-x)|) \quad \text{for all } (t',x') \tag{3.36}$$

where $o(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\mathrm{o}(\epsilon)/\epsilon \xrightarrow[\varepsilon\to 0]{} 0$. Setting $(t',x') = (t+h_ie_i, x+h_i\tilde{w}_i)$ in (3.36), and dividing across by h_i, we have

$$h_i^{-1}(V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)) - \xi^0 e_i - \xi^1\cdot\tilde{w}_i \leq h_i^{-1}\mathrm{o}(h_i|(e_i,\tilde{w}_i)|).$$

From these relations, in the limit as $i \to \infty$, it follows that

$$-L(t^+, x, \tilde{v}) - \xi^0 - \xi^1 \cdot \tilde{v} \ \leq \ 0.$$

This relation being valid for all $v \in F(t^+, x)$, we obtain:

$$\xi^0 + \min_{v \in F(t^+, x)} \ \left[\xi^1 \cdot \tilde{v} + L(t^+, x, \tilde{v})\right] \geq 0,$$

which confirms (d) ii).

To prove that V satisfies (d) iii), only the assertion $V^*(T, \cdot) = g^*(\cdot)$ remains to be proved. Since $V(T, \cdot) = g(\cdot)$, it is obvious that $V^*(T, x) \geq g^*(x)$ for every $x \in \mathbb{R}^n$. We prove that the converse inequality is also satisfied. Fix any $x \in \mathbb{R}^n$. There exists a sequence $(t_i, x_i)_{i \in \mathbb{N}}$ in $[S, T] \times \mathbb{R}^n \setminus \{(T, x)\}$ converging to (T, x) such that:

$$\lim_{i \to +\infty} V(t_i, x_i) = \limsup_{(t,y) \to (T,x)} V(t, y) = V^*(T, x).$$

For every $i \in \mathbb{N}$ there exists an F-trajectory $x_i(\cdot) \in W^{1,1}([t_i, T], \mathbb{R}^n)$ such that $x_i(t_i) = x_i$. By the principle of optimality:

$$V(t_i, x_i) - \int_{t_i}^{t} L(s, x_i(s), \dot{x}_i(s))\mathrm{d}s \leq V(t, x_i(t)), \text{ for all } t \in [t_i, T].$$

Using again condition (H5)*, we know that there exists a constant $M_0 > 0$ such that for every $i \in \mathbb{N}$:

$$V(t_i, x_i) - M_0|T - t_i| \leq V(T, x_i(T)) = g(x_i(T)).$$

Using the fact $\lim_{i \to +\infty} x_i(T) = x$, we pass to the limit superior as i tends to $+\infty$ and obtain:

$$V^*(T, x) \leq \limsup_{i \to +\infty} g(x_i(T)) \leq \limsup_{y \to x} g(y) = g^*(x),$$

which achieves to show that V satisfies (d) iii).

Step 2 We show that if U satisfies (d) i) and (d) iii), then for every $(\bar{t}, \bar{x}) \in [S, T] \times \mathbb{R}^n$ we have $V(\bar{t}, \bar{x}) \leq U(\bar{t}, \bar{x})$.
Since $\partial_P U(t, x) \subset \partial_- U(t, x)$, it is not difficult to see that we can use the analysis employed in the proof of Theorem 3.6.5 i) to obtain the desired inequality.

Step 3 We prove that if U satisfies (d) ii) and (d) iii), then for every $(\bar{t}, \bar{x}) \in [S, T] \times \mathbb{R}^n$ we have $U(\bar{t}, \bar{x}) \leq V(\bar{t}, \bar{x})$.
Using (d) iii), we can restrict attention to the case when $(\bar{t}, \bar{x}) \in]S, T[\times \mathbb{R}^n$. Let $x \in W^{1,1}([\bar{t}, T], \mathbb{R}^n)$ be an F-trajectory such that $x(\bar{t}\,) = \bar{x}$. We want to prove prove that:

$$U(\bar{t}, \bar{x}) \leq g(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s.$$

We can find a sequence $(\xi_j)_{j \in \mathbb{N}^*}$ in $\mathbb{R}^n$, converging to $x(T)$, such that:

$$\lim_{j \to +\infty} g^*(\xi_j) = (g^*)_*(x(T)).$$

Applying Carathéodory's parametrization theorem and Filippov's selection theorem, we can find a measurable function $u(\cdot)$ such that $\dot{x}(t) = f(t, x(t), u(t))$ for almost every $t \in [\bar{t}, T]$, for a Lipschitz continuous

parametrization f of F satisfying (3.27). Applying Lusin's theorem, for every $j \in \mathbb{N}^*$ we construct a pair of functions (z_j, u_j) defined on $[\bar{t}, T]$ such that:

$$\begin{cases} \dot{z}_j(t) = f(t, z_j(t), u_j(t)) \text{ for almost every } t \in [\bar{t}, T] \text{ and } z_j(\bar{t}) = x(\bar{t})\,; \\ \text{the control } u_j \text{ is continuous}\,; \\ \|x - z_j\|_{\mathbb{L}^\infty([\bar{t},T],\mathbb{R}^n)} \leq \frac{1}{j}\,; \\ \text{meas}\left(\{t \in [S,T] \,|\, u(t) - u_j(t) \neq 0\}\right) \leq \frac{1}{j}. \end{cases} \tag{3.37}$$

For every $j \in \mathbb{N}^*$, we define $y_j(\cdot) \in W^{1,1}([0, T - \bar{t}\,], \mathbb{R}^n)$ as the solution to the following differential equation:

$$\begin{cases} \dot{y}(s) = -f(T - s, y(T - s), u_j(T - s)) \text{ for a.e. } s \in [0, T - \bar{t}\,], \\ y(0) = \xi_j. \end{cases}$$

For every $j \in \mathbb{N}^*$, we note $\bar{x}_j := y_j(T - \bar{t})$ and define $x_j(\cdot) \in W^{1,1}([\bar{t}, T], \mathbb{R}^n)$ by:

$$x_j(s) := y_j(T - s),$$

which implies that $x_j(\cdot)$ is the solution to the following differential equation:

$$\begin{cases} \dot{y}(s) = f(s, y(s), u_j(s)) \text{ for a.e. } s \in [\bar{t}, T], \\ y(0) = \bar{x}_j. \end{cases}$$

Owing to the Lipschitz continuity of f and the properties of $(u_j)_{j\in\mathbb{N}^*}$ we have:

$$\|x_j - x\|_{W^{1,1}([\bar{t},T],\mathbb{R}^n)} \xrightarrow[j\to+\infty]{} 0.$$

For every $(t, x) \in [\bar{t}, T] \times \mathbb{R}^n$, we write $v^+(t, x) := f(t^+, x, u_j(t))$, $v^-(t, x) := f(t^-, x, u_j(t))$.
We then define two multivalued functions $F_j : [\bar{t}, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $\Lambda_j : [\bar{t}, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}$ by the relations: for every $(t, x) \in [\bar{t}, T] \times \mathbb{R}^n$:

$$F_j(t, x) := \text{co}\,\{v^-(t, x), v^+(t, x)\},$$

$$\Lambda_j(t, x) := -\text{co}\,\{L(t^-, x, v^-(t, x)), L(t^+, x, v^+(t, x))\}.$$

Then we set a new multifunction $\Gamma_j : [\bar{t}, T\,] \times \mathbb{R}^n \times \mathbb{R} \rightsquigarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ defined for every $(t, x, \ell) \in [\bar{t}, T\,] \times \mathbb{R}^n \times \mathbb{R}$:

$$\Gamma_j(t, x, \ell) := \begin{cases} \{1\} \times F_j(t, x) \times \Lambda_j(t, x), & \text{if } t \in [\bar{t}, T\,[, \\ \text{co}\,\{(0, 0, 0) \cup \{1\} \times F_j(T, x) \times \Lambda_j(T, x)\}\,, & \text{if } t = T. \end{cases}$$

Observe that the multivalued function Γ_j is convex, compact valued and has closed graph.
We consider the following differential inclusion:

$$\begin{cases} (\dot{\tau}(t), \dot{y}(t), \dot{\ell}(t)) \in \Gamma_j(\tau(t), y(t), \ell(t)), \text{ for a.e. } t \in [\bar{t}, T], \\ \tau(\bar{t}) = \bar{t}, y(\bar{t}) = \bar{x}_j, \ell(\bar{t}) = U^*(\bar{t}, \bar{x}_j), \\ (\tau(t), y(t), \ell(t)) \in \text{hyp}\, U^*, \text{ for all } t \in [\bar{t}, T]. \end{cases} \tag{3.38}$$

Observe that the last condition in (3.38) means that $\ell(t) \leq U^*(\tau(t), y(t))$, for all $t \in [\bar{t}, T]$.
We define the arc on $[\bar{t}, T]$

$$t \mapsto (\tau_j(t), x_j(t), \ell_j(t)) := \left(t, x_j(t), U^*(\bar{t}, \bar{x}_j) - \int_{\bar{t}}^{t} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s\right).$$

Observe that (τ_j, x_j, ℓ_j) is the unique Γ_j-trajectory with initial condition $(\bar{t}, \bar{x}_j, U^*(\bar{t}, \bar{x}_j))$.
Assumptions i) and ii) of Weak Invariance Theorem 2.6.1 are satisfied from the discussion above and from the fact that the 'hypotheses reduction' of Section 3.4 guarantee also that $\Gamma_j(\tau, x, \ell) \subset (c+1)\mathbb{B}$, for every (τ, x, ℓ), where $c := \sqrt{c_0^2 + M_0^2}$.

The 'inward pointing' condition iii) is also satisfied, we prove the validity of the following property: for every pair $(\tau, x) \in]\bar{t}, T[\times\mathbb{R}^n$ and every $\ell \leq U^*(\tau, x)$

$$\min_{w \in \Gamma_j(\tau,x,\ell)} (-\xi^0, -\xi^1, \lambda) \cdot w \leq 0, \text{ for all } (-\xi^0, -\xi^1, \lambda) \in N^P_{\mathrm{hyp}\, U^*}((\tau, x), \ell). \tag{3.39}$$

Let $(\tau, x) \in [\bar{t}, T[\times\mathbb{R}^n$, $\ell \leq U^*(t, x)$ and $(-\xi^0, -\xi^1, \lambda) \in N^P_{\mathrm{hyp}\, U^*}((\tau, x), U^*(\tau, x))$ (we recall that we can always reduce to the case $\ell = U^*(\tau, x)$). Depending on the value of λ we can consider two distinct cases.

Case 1: $\lambda > 0$**.** Then $\lambda^{-1}(\xi^0, \xi^1) \in \partial_+ U^*(\tau, x)$. We notice that $v^+(\tau, x) \in F_j(\tau, x) \cap F(\tau^+, x)$ and, bearing in mind that U satisfies condition (d) ii), we deduce that:

$$\xi^0 + \xi^1 \cdot v^+(\tau, x) + \lambda L(\tau^+, x, v^+(\tau, x)) \geq 0,$$

and then:

$$-\xi^0 - \xi^1 \cdot v^+(\tau, x) - \lambda L(\tau^+, x, v^+(\tau, x)) \leq 0,$$

So (3.39) is confirmed since $(1, v^+(\tau, x), -L(\tau^+, x, v^+(\tau, x))) \in \Gamma_j(\tau, x, U^*(\tau, x))$.

Case 2: $\lambda > 0$**.** Then invoking the Rockafellar Horizontal Approximation Theorem (see Theorem 2.4.15), there exist sequences $(\xi^0_m, \xi^1_m) \xrightarrow[m\to+\infty]{} (\xi^0, \xi^1)$, $\lambda_m \downarrow 0$ and $(s_m, x_m) \xrightarrow[m\to+\infty]{} (\tau, x)$ such that, for each m,

$$(\lambda_m^{-1}\xi^0_m, \lambda_m^{-1}\xi^1_m) \in \partial_+ U^*(s_m, x_m)\,.$$

By condition (d) ii), we obtain, for each m:

$$\xi^0_m + \xi^1_m \cdot v^+(s_m, x_m) + \lambda_m L((s_m)^+, x_m, v^+(s_m, x_m)) \geq 0.$$

By extracting suitable subsequences and arguing as in the proof of Theorem 3.6.5 ii) we can confirm (3.39).

As a consequence, the Weak Invariance Theorem 2.6.1 is applicable to the differential inclusion (3.38), and we can conclude that the arc (τ_j, x_j, ℓ_j) is a solution to the constrained differential inclusion (3.38). It follows that at $t = T$:

$$U^*(\bar{t}, \bar{x}_j) - \int_{\bar{t}}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s \leq U^*(T, x_j(T)) = g^*(x_j(T)), \text{ for every } j. \tag{3.40}$$

Since $\|x_j - x\|_{W^{1,1}([\bar{t},T],\mathbb{R}^n)} \xrightarrow[j\to+\infty]{} 0$, by Lebesgue's dominated convergence theorem we have:

$$\int_{\bar{t}}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s \xrightarrow[j\to+\infty]{} \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s.$$

Since $U \leq U^*$, and U is lower semicontinuous, passing to the limit inferior in (3.40) yields:

$$U(\bar{t}, \bar{x}) \leq \liminf_{j\to+\infty} g^*(\xi_j) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s.$$

Recalling that $\lim_{j\to+\infty} g^*(\xi_j) = (g^*)_*(x(T))$ and $(g^*)_* = g$, we obtain:

$$U(\bar{t},\bar{x}) \le g(x(T)) + \int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s,$$

which implies

$$U(\bar{t},\bar{x}) \le V(\bar{t},\bar{x}),$$

and concludes the proof.

□

Proof of Proposition 3.4.3. The proof immediately follows from the proof of Theorem 3.4.2, observing that condition $(g^*)_* = g$ is used only in Step 3.

□

Proof of Proposition 3.4.5. The proof of Proposition 3.4.5 follows along the same lines as the proof of Theorem 3.4.1 and Theorem 3.4.2, replacing $\widetilde{L}$ with L in the definition of the the multivalued function Q (steps '(c) ⇒ (a)', '(d) ⇒ (a)', and proof of Theorem 3.6.5 i)), and taking into account that, when L is lower semicontinuous with respect to the time variable, we have $L(t,x,v) \le L(t^+,x,v)$, for all $(t,x,v) \in [S,T[\times\mathbb{R}^n\times\mathbb{R}^n$ (steps '(a) ⇒ (b)', '(b) ⇒ (c)' and '(a) ⇒ (d)').

□

4

Solutions to the Hamilton-Jacobi equation for Bolza problems with discontinuous time dependent data and state constraint

A very early version of these results is published in [13]. The results proposed in this chapter have been submitted.

4.1 Abstract of chapter 4

Chapter 4, is quite similar to chapter 3. It deals with the case of the Hamilton-Jacobi-Bellman equation that is associated to the non autonomous Bolza problem with a state constraint:

$$\begin{cases} \text{Minimize } \displaystyle\int_S^T L(s, y(s), \dot{y}(s))\mathrm{d}s + g(y(T)) \\ \text{over arcs } y \in W^{1,1}([S,T], \mathbb{R}^n) \text{ satisfying} \\ \dot{y}(s) \in F(s, y(s)), \text{ for almost every } s \in [S,T], \\ y(S) = x_0, \\ \boxed{y(t) \in A, \text{ for all } t \in [S,T].} \end{cases}$$

The Lagrangian L and the multivalued function F are allowed to be discontinuous on a set of full measure with respect to time (they still have left and right limits everywhere).

Some appropriate constraint qualifications conditions between the state of constraints A and the function controlling the dynamic F are introduced. They allow for $W^{1,1}$ continuous distance estimates, making it then possible to establish several characterizations of the value function V as the unique generalized solution to (HJB), using the lower Dini derivative, the proximal subdifferential, and the Fréchet sub/superdifferential. These results complete the ones from [21] (that applied to the Mayer problem), but they also include new features such as generalized viscosity characterizations of V.

4.2 Résumé du chapitre 4

Le chapitre 4 est similaire au chapitre 3. Il traite du cas où l'équation de Hamilton-Jacobi-Bellman est associée au problème de Bolza non-autonome avec dynamique contrôlée auquel on a cette fois ajouté une contrainte d'état :

$$\begin{cases} \text{Minimiser } \displaystyle\int_S^T L(s,y(s),\dot{y}(s))\mathrm{d}s + g(y(T)) \\ \text{parmi les arcs } y \in W^{1,1}([S,T],\mathbb{R}^n) \text{ qui satisfont} \\ \dot{y}(s) \in F(s,y(s)), \text{ pour presque tout } s \in [S,T], \\ y(S) = x_0, \\ \boxed{y(t) \in A, \text{ pour tout } t \in [S,T].} \end{cases}$$

Le lagrangien L et la fonction multivaluée F sont discontinues presque partout par rapport au temps (elles ont toutefois des limites à droite et à gauche partout).

Des conditons de compatibilité *ad hoc* entre l'ensemble des contraintes A et la fonction multivaluée de la dynamique F sont introduites, ce qui permet d'approximer dans $W^{1,1}$ les F-trajectoires violant la contrainte d'état par des F-trajectoires la respectant, et par suite, d'établir différentes caractérisations de la fonction V en tant qu'unique solution de l'équation (HJB), en ayant recours aux dérivées de Dini, à la sous-différentiel proximale et aux différentiels de Fréchet. Ces résultats viennent d'une part compléter ceux obtenus dans [21] pour le problème de Mayer, et s'en distinguent puisque nous présentons dans ce contexte des caractérisations de V au sens des solutions de viscosité qui n'avaient pas été établies dans [21].

4.3 Introduction

We consider the following state constrained Bolza problem with initial data $(t,x) \in [S,T] \times \mathbb{R}^n$:

$$(SC_{t,x}) \begin{cases} \text{Minimize } \displaystyle\int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([t,T],\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(s) \in F(s,x(s)) \quad \text{for a.e. } s \in [t,T], \\ x(s) \in A \quad \text{for all } s \in [t,T], \\ x(t) = x, \end{cases}$$

in which $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multivalued function, and A is a given nonempty closed set in $\mathbb{R}^n$. An *F-trajectory* on the interval $[t_0,t_1] \subset [S,T]$ is an absolutely continuous arc $x : [t_0,t_1] \to \mathbb{R}^n$ which satisfies the reference differential inclusion $\dot{x}(s) \in F(s,x(s))$ for a.e. $s \in [t_0,t_1]$. We say that an F-trajectory $x \in W^{1,1}([t_0,t_1],\mathbb{R}^n)$ is *feasible* on $[t_0,t_1]$ if $x(s) \in A$ for all $s \in [t_0,t_1]$. The value function $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by the infimum cost of $(SC_{t,x})$:

$$V(t,x) := \inf(SC_{t,x}),$$

interpreting '$+\infty$' the cost of F-trajectories which are not feasible (that clearly includes the case when $x \notin A$) and the cost of feasible F-trajectories $x(\cdot)$'s such that $g(x(T)) = +\infty$.

The aim of this chapter is to characterize V as the unique solution in the class of lower semicontinuous (lsc) functions, in a suitable generalized sense, to the Hamilton-Jacobi equation associated with $(SC_{t,x})$. More precisely the notions of solution, that shall be considered here, will involve the *lower Dini derivative* (also referred to as *contingent derivative*), the *proximal* subdifferentials, and the *Fréchet* subdifferentials

and superdifferentials (in which we take into account additional information provided by the 'horizontal' gradients). The characterization of the value function as a generalized solution to the Hamilton-Jacobi equation associated with a reference optimal control problem is a long-standing research topic and a lot of work has been done in this context. The present contribution is in the strand which employs nonsmooth analysis and viability theory techniques (cf. widely-known references as [51, 43, 84, 45], and the recent papers [56, 21] for an overview), rather than the tools which are typical in the viscosity solutions theory framework, cf. [7, 5].

In this chapter we shall consider 'everywhere in t' characterizations of the value function for optimal control problems in which F and L may have a discontinuous behaviour w.r.t. the time variable t. The most general known class of time-discontinuous problems, which allows to provide an 'everywhere in t' characterization, was introduced in [21] to investigate value functions for optimal control Mayer problems ($L = 0$). In this class of problems the time-dependent data are time-discontinuous in the following sense: they have everywhere one-sided limits in t and are continuous on the complement of a zero-measure subset of $[S, T]$. For this class of optimal control problems we highlight the following peculiarities of the value characterization: the role of the limits $F(t^-, x)$ and $F(t^+, x)$ (which cannot be exchanged) in the characterization conditions and the presence of the horizontal proximal subdifferentials in the proximal solution.

A crucial feature, in the context of this chapter, is the possibility to have at hand a $W^{1,1}$ 'distance estimate' which has a continuous behaviour w.r.t. a parameter which quantifies the F-trajectories 'constraint violation': this is an important analytical tool which allows to construct, from an arbitrary F-trajectory, a feasible F-trajectory having, in our case, a suitable $W^{1,1}$-distance from the reference F-trajectory. We recall that limiting attention to the case $L = 0$ (Mayer problems) an $\mathbb{L}^\infty$ distance estimate result would be enough to provide an 'everywhere in t' characterization of value function in terms of the lower Dini derivative and the proximal subdifferentials (see [21, Theorems 4.1 and 4.2]). We underline that, in this chapter, the Lagrangian L is merely continuous w.r.t. the state variable x and is not necessarily of bounded variation w.r.t. t. Moreover, we do not impose any *a priori* regularity condition on the epigraph of the candidate solutions (as in [9, 52]). As a consequence, the class of Bolza problems that we consider here is not covered by previous work; in particular a state augmentation technique would not reduce the difficulties and would not allow to employ, for instance, the results obtained in [21, 56] (which might be taken into account to derive just some parts of the results of these chapter only if L had stronger properties such as Lipschitz regularity in x and a bounded variation w.r.t. t).

In the state constraint-free case, when the data F, L and g satisfy particular continuity properties (which yield also that the value function V is continuous on $[S, T] \times \mathbb{R}^n$), it is well known that V can be characterized also in terms of the Fréchet subdifferentials and superdifferentials (also called 'strict' or 'viscosity' sub/superdifferentials), cf. [43, 45, 51]. Passing to the state constrained case, this 'equivalent' characterization was proved in [56, Thm. 3] in the class of continuous functions (as set of candidate solutions), still when the data are continuous, and imposing a stronger version of the 'standard' inward pointing conditions. Keeping this stronger constraint qualification, but weakening these continuity assumptions on the data, only partial results are known providing, for instance, just one-side comparison theorems (cf. [56]).
Therefore natural questions are:

(Q1) Is that possible to provide an 'everywhere in t' characterization of the value function $V(\cdot, \cdot)$ (in an equivalent way) in terms of lower Dini derivative, the proximal subdifferentials, and the Fréchet subdifferentials and superdifferentials for optimal control problems which may have also a time-discontinuous behaviour of F and L in the sense of [21] (*i.e.* we have everywhere left and right limits, but we allow a time-discontinuous behaviour on a zero-measure set)?

(Q2) And what happens if we merely impose the 'standard' constraint qualifications for nonsmooth sets (rather than imposing stronger versions of them)?

In this chapter, we consider 'standard' constraint qualifications as in [21], and, first, we provide an extension of [21, Theorems 4.1 and 4.2] to the context of Bolza problems: so Theorem 4.4.1 below deals with the case of a lower semicontinuous (on A) final cost function g assuming an 'outward pointing' constraint qualification; while Theorem 4.4.2 concerns the case in which g is continuous on A and an 'inward pointing' constraint qualification is satisfied. Subsequently, we provide positive answers to questions (Q1) and (Q2) above with Theorems 4.4.3 and 4.4.4 for problems involving respectively a locally bonded lower semicontinuous final cost coupled with an 'inward/outward pointing' constraint qualification, and a continuous final cost associated with a mere 'inward pointing' constraint qualification.

An important contribution of this work is that horizontal normal vectors to the epigraph can be removed to characterize the value function: this is a crucial aspect in the value characterization when the data of the problem are continuous, and can be obtained involving a suitable approximation technique of proximal normals to an epigraph set by non-horizontal proximal normals; it is shown that this approximation technique is still applicable even in presence of discontinuous time dependent data (in the sense specified above).

The notion of generalized solution used here involves (for discontinuous value functions) the concepts of lower and upper semicontinuous envelopes.
This appears to be a reasonable feature bearing in mind that a well-known approach in viscosity solutions theory suggests, in presence of a locally bounded candidate V to be a solution to an Hamilton-Jacobi equation, to consider its lower and upper semicontinuous envelopes and check whether the properties of being supersolution and subsolution in the viscosity sense are satisfied (cf. [7, 5]).
We emphasize the fact that the right coupling between 'regularity of g' and 'constraint qualification' plays a crucial role in these results and is far to be merely a matter of adding technical assumptions: if g is just lower semicontinuous, for instance, a bad constraint qualification ('inward pointing' in this case) would not yield the desired characterization; this aspect is clearly illustrated by Example 4.4.6 below, in which the value function V is not a generalized solution in the 'proximal' or 'lower Dini' sense of Theorem 4.4.1, but it is a (non-unique) solution in terms of the 'viscosity sense' of Theorem 4.4.3. An important aspect of our characterization involving Fréchet sub/super-differentials is the presence of the condition $((g_{|A})^*)_* = g_{|A}$ for the (locally bounded lsc) final cost g, that we comment now (here, $((g_{|A})^*)_*$ is the lower semicontinuous envelope of the upper semicontinuous envelope of $g_{|A}$).
As highlighted in Chapter 3, this condition becomes crucial if we want a characterization result (including a uniqueness/comparison property) for the value function and in the state constraint-free case (with lsc final cost g) adding condition '$(g^*)_* = g$' the characterization in terms of Fréchet sub/super-differentials is equivalent to the Dini and proximal solutions (see Chapter 3). A maybe surprising feature is that the same conclusion is not, in general, true for state constrained problems and a crucial role is played by the validity of both inward and outward pointing constraint qualifications.
Indeed even if, when the inward pointing constraint qualification is in force, the value V is a solution in the sense of the Fréchet sub/super-differentials (and this is a general fact, see Proposition 4.9.1 below), we have an uniqueness (comparison) result only when also an outward pointing constraint qualification is satisfied at the same time.
To clarify this point we provide a simple example (Example 4.4.7 below) in which only an inward pointing condition is satisfied and the value function is a solution to the associated Hamilton-Jacobi equation in the 'viscosity sense' described by Theorem 4.4.3, but it is not unique.

We apply our theory to an illustrative economics example in which the integral cost is merely continuous w.r.t. the state variable x: this is due an inherent fractional singularity term which is introduced to interpret the production function (cf. [1]).

A further contribution of this work is a $W^{1,1}$ distance estimate result (Theorem 4.6.1 below) which extends to the case of time-dependent multifunctions F the result obtained by [29] for the time-independent Lipschitz continuous F. We observe that the distance estimate in [29] is not applicable in our context since our aim is to investigate Bolza problems with discontinuous time-dependent data in the sense specified above. For our $W^{1,1}$ distance estimate we consider a closed convex set A (as in [29]), but a multifunction $F(t,x)$ which has a regularity similar to [21, Theorems 4.1 and 4.2] and [22] (where an $\mathbb{L}^\infty$ distance estimate is provided for merely closed domains), allowing F to have a bounded variation (possibly discontinuous) regularity w.r.t. the time variable. As far as the state constraint qualification is concerned, we consider a 'standard' inward pointing condition (as in [29, 22]) avoiding imposing stronger version of it. (For a discussion on $W^{1,1}$ and $\mathbb{L}^\infty$ distance estimate results, illustrative examples, and the possible issues arising when we pass from smooth to nonsmooth sets A we refer the reader to the papers [16, 18, 22] and the references therein.)

4.4 Characterizations of the Value function for Bolza problems

In this chapter we shall invoke the following hypotheses. For every $R_0 > 0$, there exist positive functions $c_F(\cdot) \in \mathbb{L}^1(S,T)$ and $k_F(\cdot) \in \mathbb{L}^\infty(S,T)$, a modulus of continuity $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and constant $c_0 > 0$, $M_0 > 0$ such that

(H1): the multivalued function $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes convex, closed, non-empty values; for every $x \in \mathbb{R}^n$, $F(\cdot, x)$ is Lebesgue measurable on $[S,T]$;

(H2): $F(t,x) \subset c_F(t)(1+|x|)\,\mathbb{B}$ for all $x \in \mathbb{R}^n$ and for a.e. $t \in [S,T]$;

(H3): $F(t,x') \subset F(t,x) + k_F(t)|x-x'|\,\mathbb{B}$ for all $x, x' \in R_0\mathbb{B}$ and a.e. $t \in [S,T]$;

(H4): $F(.,x)$ has bounded variation uniformly over $x \in R_0\mathbb{B}$, in the following sense: there exists a non-decreasing bounded variation function $\eta(\cdot) : [S,T] \to [0,\infty)$ such that

(i) for every $[s,t] \subset [S,T]$ and $x \in R_0\mathbb{B}$,

$$d_H(F(s,x), F(t,x)) \le \eta(t) - \eta(s) \ ;$$

(ii) for every $\mu > 0$ and every $[t_0,t_1] \subset [S,T]$ there exists a partition $\{t_0 =: \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \ldots < \tilde{t}_M := t_1\}$ such that for each $k = 0, 1, \ldots, M-1$ we have

$$\lim_{\epsilon \downarrow 0} \int_{\tilde{t}_k+\epsilon}^{\tilde{t}_{k+1}} \frac{\eta(\tau) - \eta(\tilde{t}_k^+)}{\tau - \tilde{t}_k} d\tau \le \mu \ .$$

(H5): (i) the Lagrangian $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable; for every $t \in [S,T]$ and $x \in \mathbb{R}^n$, $L(t,x,\cdot)$ is convex;

(ii) L is locally bounded in the following sense:

$$|L(t,x,v)| \le M_0, \quad \text{for all} \quad (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_0\mathbb{B};$$

(H6): (i) $|L(t,x',v) - L(t,x,v)| \le \omega(|x-x'|)$, for all $x, x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_0\mathbb{B}$;

(ii) $L(t^-,x,v) := \lim_{t'\uparrow t} L(t',x,v)$ exists for every $(t,x,v) \in]S,T] \times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(t^-,x,v) = L(t,x,v)$ for a.e. $t \in (S,T]$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$;

(iii) $L(s^+,x,v) := \lim_{s'\downarrow s} L(s',x,v)$ exists for every $(s,x,v) \in [S,T[\times R_0\mathbb{B} \times c_0\mathbb{B}$, and $L(s^+,x,v) = L(s,x,v)$ for a.e. $s \in [S,T)$ and for all $(x,v) \in R_0\mathbb{B} \times c_0\mathbb{B}$;

(H7): $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, with nonempty domain;

(H8): $A \subset \mathbb{R}^n$ is convex and closed;

(OPC): for each $s \in [S,T[$, $t \in]S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset \qquad \text{and} \qquad F(s^+,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset\,;$$

(IPC): for each $s \in [S,T[$, $t \in]S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \operatorname{int} T_A(x) \neq \emptyset \qquad \text{and} \qquad F(s^+,x) \cap \operatorname{int} T_A(x) \neq \emptyset\,.$$

Theorem 4.4.1 (Characterization of lsc Value Functions - Outward-pointing Condition) *Assume (H1)–(H8) and (OPC). Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then assertions (a), (b) and (c) below are equivalent:*

(a) U is the value function for $(SC_{t,x})$, i.e. $U = V$.

(b) U is lower semicontinuous on $[S,T] \times \mathbb{R}^n$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and

(i) for all $(t,x) \in ([S,T[\times A) \cap \operatorname{dom} U$

$$\inf_{v \in F(t^+,x)} \big[D_\uparrow U((t,x),(1,v)) + L(t^+,x,v)\big] \leq 0\,;$$

(ii) for all $(t,x) \in (\,]S,T] \times \operatorname{int} A) \cap \operatorname{dom} U$

$$\sup_{v \in F(t^-,x)} \big[D_\uparrow U((t,x),(-1,-v)) - L(t^-,x,v)\big] \leq 0\,;$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (T,x) \;\mid\; t'<T, x' \in \operatorname{int} A\}} U(t',x') = U(T,x) = g(x).$$

(c) U is lower semicontinuous on $[S,T] \times \mathbb{R}^n$ $U(t,x) = +\infty$ if $x \notin A$, and

(i) for all $(t,x) \in ((S,T) \times A) \cap \operatorname{dom} U$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$

$$\xi^0 + \min_{v \in F(t^+,x)} \big[\xi^1 \cdot v + L(t^+,x,v)\big] \leq 0\,;$$

(ii) for all $(t,x) \in]S,T[\times \operatorname{int} A) \cap \operatorname{dom} U$, $(\xi^0,\xi^1) \in \partial_P U(t,x)$

$$\xi^0 + \min_{v \in F(t^-,x)} \big[\xi^1 \cdot v + L(t^-,x,v)\big] \geq 0\,;$$

(iii) for all $x \in A$,

$$\liminf_{\{(t',x') \to (S,x) \;\mid\; t'>S\}} U(t',x') = U(S,x) \tag{4.1}$$

and

$$\liminf_{\{(t',x') \to (T,x) \;\mid\; t'<T,\; x' \in \operatorname{int} A\}} U(t',x') = U(T,x) = g(x). \tag{4.2}$$

Theorem 4.4.2 (Characterization of lsc Value Functions - Inward-pointing Condition) *Assume that (H1)–(H8), (IPC) are satisfied and that g is continuous on A. Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then the assertions (a), (b) and (c) of Theorem 4.4.1 remain equivalent.*

Theorem 4.4.3 (Characterization of locally bounded lsc Value Functions - Inward/Outward-pointing Condition) *Assume (H1)–(H8), (OPC) and (IPC). Suppose, in addition, that $g_{|A}$ is locally bounded and satisfies $((g_{|A})^*)_* = g_{|A}$. Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then, the assertions (a), (b) and (c) of Theorem 4.4.1 are equivalent to condition (d) below:*

(d) U is lower semicontinuous on $[S,T] \times \mathbb{R}^n$ and locally bounded on $[S,T] \times A$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and

(i) for all $(t,x) \in (S,T) \times A$, $(\xi^0, \xi^1) \in \partial_- U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \leq 0\,; \tag{4.3}$$

(ii) for all $(t,x) \in (S,T) \times \operatorname{int} A$, $(\xi^0, \xi^1) \in \partial_+ U^*(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \geq 0\,; \tag{4.4}$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (S,x) \ \mid \ t' > S\}} U(t',x') = U(S,x),$$

$$(U_{|[S,T]\times A})^*(T,x) = (g_{|A})^*(x) \quad \text{and} \quad U(T,x) = g(x).$$

Theorem 4.4.4 (Characterization of continuous Value Functions - Inward-pointing Condition) *Assume (H1)–(H8) and (IPC). Suppose, in addition, that g is continuous on A. Let $U : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then, the assertions (a), (b) and (c) of Theorem 4.4.1 are equivalent to condition (d)$'$ below:*

(d)$'$ U is continuous on $[S,T] \times A$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and

(i) for all $(t,x) \in (S,T) \times A$, $(\xi^0, \xi^1) \in \partial_- U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \leq 0\,; \tag{4.5}$$

(ii) for all $(t,x) \in (S,T) \times \operatorname{int} A$, $(\xi^0, \xi^1) \in \partial_+ U(t,x)$

$$\xi^0 + \inf_{v \in F(t^+,x)} \left[\xi^1 \cdot v + L(t^+,x,v)\right] \geq 0\,; \tag{4.6}$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (S,x) \ \mid \ t' > S\}} U(t',x') = U(S,x),$$

$$\text{and} \quad U(T,x) = g(x).$$

Comments and Examples. Before passing to the proof of the value function characterizations we comment our results and give two illustrative examples.

Remark 4.4.5 *1. Hypotheses (H1), (H2) and (H4)(i) implies that: we can always find $c > 0$ such that $F(t,x) \subset c\,\mathbb{B}$ for all $(t,x) \in [S,T] \times R_0\mathbb{B}$; moreover, for each $s \in [S,T)$ and $t \in (S,T]$ the following limits (in the sense of Kuratowski) exist and are nonempty*

$$F(s^+,x) := \lim_{s' \downarrow s} F(s',x) \;\; \textit{and} \;\; F(t^-,x) := \lim_{t' \uparrow t} F(t',x), \quad \textit{for all } x \in \mathbb{R}^n,$$

and for almost every $s \in [S,T[$ and $t \in]S,T]$ we have

$$F(s^+,x) = F(s,x) \;\; \textit{and} \;\; F(t^-,x) = F(t,x), \quad \textit{for all } x \in \mathbb{R}^n.$$

2. *Condition (H4)(ii) means that the function of bounded variation η satisfies a sort of uniform Dini's test. Since η has right limit at each point $s \in [S,T)$, Dini's criterion establishes that the Fourier series associated with the periodic extension of $\eta(\cdot)$ (out of $[S,T)$) converges to $\eta(s^+)$. Assumption (H4)(ii) is satisfied, for instance, when $\eta(\cdot) - \eta(s^+)$ has a uniform linear growth over $s \in [S,T)$.*

3. *Observe that the conclusions of Theorems 4.4.1, 4.4.2, 4.4.3, 4.4.4 remain valid if A is only a closed set (instead of a convex set as requested in (H8)) with nonempty interior, if $F(t,\cdot)$ is locally uniformly continuous, (H3) is satisfied with $k_F \in \mathbb{L}^1(S,T)$ (in place of $k_F \in \mathbb{L}^\infty(S,T)$), and the constraint qualifications (OPC) and (IPC) are substituted by the following conditions given in terms of the distance estimates (in place of the data of the problem):*

$(CQ)_{BW}$: *for any $r_0 > 0$, there exists a modulus of continuity $\tilde\theta(\cdot)$, such that given any interval $[s_0,s_1] \subset [0,T-S]$, any $\tilde F$-trajectory $\hat y(\cdot)$ on $[s_0,s_1]$ with $\hat y(s_0) \in A \cap \big(e^{\int_0^{s_0} c_F(s)\mathrm{d}s}(r_0+1)-1\big)\mathbb{B}$, and any $\rho > 0$ such that $\rho \geq \max\{d_A(\hat y(t)) \,|\, s \in [s_0,s_1]\}$, we can find an $\tilde F$-trajectory $y(\cdot)$ on $[s_0,s_1]$ such that $y(s_0) = \hat y(s_0)$, $y(s) \in \operatorname{int} A$ for all $s \in (s_0,s_1]$ and*

$$\|\hat y - y\|_{W^{1,1}([s_0,s_1])} \leq \tilde\theta(\rho). \tag{4.7}$$

(Here, $\tilde F(s,y) := -F(T-s,y)$ for all $s \in [0,T-S]$.)

$(CQ)_{FW}$: *for any $r_0 > 0$, there exists a modulus of continuity $\theta(\cdot)$, such that given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat x(\cdot)$ on $[t_0,t_1]$ with $\hat x(t_0) \in A \cap \big(e^{\int_S^{t_0} c_F(t)\mathrm{d}t}(r_0+1)-1\big)\mathbb{B}$, and any $\rho > 0$ such that $\rho \geq \max\{d_A(\hat x(t)) \,|\, t \in [t_0,t_1]\}$, we can find an F-trajectory $x(\cdot)$ on $[t_0,t_1]$ such that $x(t_0) = \hat x(t_0)$, $x(t) \in \operatorname{int} A$ for all $t \in (t_0,t_1]$ and*

$$\|\hat x - x\|_{W^{1,1}([s_0,s_1])} \leq \theta(\rho). \tag{4.8}$$

Observe that if hypotheses (H1)-(H4) and (H8) are in force together with (IPC) (resp. (OPC)), then owing to Theorem 4.6.1 below, $(CQ)_{FW}$ (resp. $(CQ)_{BW}$) is satisfied with $\theta(\rho) := K\rho(1+|\ln(\rho)|)$ for some constant $K > 0$ (resp. $\tilde\theta(\rho) := K\rho(1+|\ln(\rho)|)$). It is well-known that it is not possible to obtain in general linear $W^{1,1}$ distance estimates (i.e. $\theta(\rho) = K\rho$) when A is nonsmooth and the 'simple' inward pointing condition (IPC) is satisfied; and different $W^{1,1}$ distance estimate results can be obtained imposing additional assumptions, such as regularity of the state constraint or stronger inward pointing conditions (cf. [16, 19, 56]).

4. *Conditions (c), (d) and (d)′ of Theorems 4.4.1, 4.4.2, 4.4.3, 4.4.4 can be easily reformulated in terms of an Hamiltonian function*

$$H_\lambda(t,x,p) := \inf_{v \in F(t,x)} [p \cdot v + \lambda L(t,x,v)] \; .$$

5. *The examples below show that the right coupling between the 'regularity of g' and the 'constraint qualification' ((IPC) or (OPC)) plays a crucial role to obtain the characterizations provided by Theorems 4.4.1, 4.4.2, 4.4.3 and 4.4.4. In particular, we point out the following important facts: condition (OPC) cannot be replaced by (IPC) in Theorem 4.4.1; the continuity of g cannot be dropped in Theorems 4.4.2 and 4.4.4; the validity of both (IPC) and (OPC) is required for Theorem 4.4.3. We highlight also that, if conditions (H1)-(H8) and (IPC) alone are in force (without (OPC) and condition '$((g_{|A})^*)_* = g_{|A}$') then the value function does satisfy property (d) of Theorem 4.4.3 (even if it may fail to meet all the requirements of (c) or (b)): this is a general fact as stated by Proposition 4.9.1 proved in Section 4.9.*

Example 4.4.6 *Consider the case in which $n = 1$, $[S,T] = [0,1]$, $A = \{x \in \mathbb{R}, x \geq 0\}$, $F(t,x) \equiv [0,1]$, $L = 0$, and*

$$g(x) = \begin{cases} -x - 2, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

Observe that all the assumptions in Theorem 4.4.1 are satisfied except (OPC), since we have $\partial A = \{0\}$ and $T_A(0) = \mathbb{R}_+$. On the other hand, (IPC) is valid. The value function is, for all $(t,x) \in [0,1] \times \mathbb{R}$

$$V(t,x) = \begin{cases} t - x - 1, & \text{if } x > 0, \\ -2, & \text{if } x = 0, \\ +\infty, & \text{if } x < 0. \end{cases}$$

Notice that $\liminf_{\{(t',x') \to (1,0) \,|\, x' > 0\}} V(t',x') = 0 \neq V(1,0) = -2$. Therefore, the value function does not satisfy condition (iii) of (b) and (c). Observe also that all the hypotheses of Theorem 4.4.2 are satisfied, except the continuity of g on A. The consequence is that still (b) and (c) cannot be used to characterize the value function. Moreover, V satisfies (d) of Theorem 4.4.3 but it not the unique, indeed the following function W ($\neq V$) satisfies condition (d) as well:

$$W(t,x) := \begin{cases} t - x - 1, & \text{if } x > 0, \\ -3/2, & \text{if } x = 0 \text{ and } t \neq 1, \\ -2, & \text{if } (t,x) = (1,0), \\ +\infty, & \text{if } x < 0. \end{cases}$$

The data of Example 4.4.6 do not satisfy all the hypotheses of Theorem 4.4.3 in two respects: the 'outward pointing condition' and condition '$((g_{|A})^*)_* = g_{|A}$' are not valid. It is well-known that condition '$((g_{|A})^*)_* = g_{|A}$' is crucial for the uniqueness of the solution (in terms of (d)) even for the state constraint-free case (see Chapter 3). The following example shows that, even if '$((g_{|A})^*)_* = g_{|A}$' is satisfied, the absence of one constraint qualification ((OPC) in this case) might compromise the characterization of the value function in the sense of (d) of Theorem 4.4.3.

Example 4.4.7 *Take $n = 1$, $[S,T] = [0,1]$, $A = \{x \in \mathbb{R}, x \geq 0\}$, $L = 0$,*

$$F(t) = \begin{cases} [0,1], & \text{if } t \leq \frac{1}{2} \\ [1/2,1], & \text{if } t > \frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{4} \\ 1, & \text{if } x > \frac{1}{4}. \end{cases}$$

Then for all $(t,x) \in [0,1] \times \mathbb{R}$, the Value function is

$$V(t,x) = \begin{cases} 0, & \text{if } x = 0 \text{ and } t \leq \frac{1}{2}, \\ 1, & \text{if } x > 0 \text{ and } t \leq \frac{1}{2}, \\ 0, & \text{if } 0 \leq x \leq \frac{t}{2} - \frac{1}{4} \text{ and } t > \frac{1}{2}, \\ 1, & \text{if } x > \frac{t}{2} - \frac{1}{4} \text{ and } t > \frac{1}{2}, \\ +\infty, & \text{if } x < 0. \end{cases}$$

We observe that all the assumptions in Theorem 4.4.3 are satisfied except (OPC), and the value function V satisfies (d) of Theorem 4.4.3. However, also the following function satisfies (d)

$$W(t,x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0 \text{ and } t < \frac{1}{2}, \\ 1, & \text{if } x > 0 \text{ and } t < \frac{1}{2}, \\ 0, & \text{if } x \leq \frac{t}{2} - \frac{1}{4} \text{ and } t \geq \frac{1}{2}, \\ 1, & \text{if } x > \frac{t}{2} - \frac{1}{4} \text{ and } t \geq \frac{1}{2}, \\ +\infty, & \text{if } x < 0. \end{cases}$$

Consequently, V is not the unique solution to the Hamilton-Jacobi equation in the sense of (d) in Theorem 4.4.3.

4.5 Economics Example

The following 'growth versus consumption' problem arises in neo-classical macro-economics:

$$(GC) \begin{cases} \text{Maximize } \int_0^T (1-u(t))x^\alpha(t)\mathrm{d}t \\ \text{subject to} \\ \dot{x}(t) = -ax(t) + bu(t)x^\alpha(t) \quad \text{for a.e. } t \in [0,T], \\ u(t) \in [0,1] \quad \text{for a.e. } t \in [0,T], \\ x(t) \geq 0 \text{ for all } t \in [0,T], \\ x(0) = x_0. \end{cases}$$

Data: constants $a > 0$, $b > 0$, $T > 0$ and $\alpha \in (0,1)$, and initial point $x_0 \geq 0$.

It has the following interpretation: x denotes (aggregated) economic output. The (normalized) rate of financial return $r(x)$ from economic output x is modelled by the production function $r(x) = x^\alpha$. The problem is to choose the proportion u of rate of return for investment and for expenditure, over a given time horizon $[0,T]$, to maximize total expenditure over the time horizon $[0,T]$.
The cost function and underlying dynamic model in (GC) provides a (finite horizon) example of a class of utility and macro economic growth models studied in Solow's classic paper [80, 1]. The dynamic model was elaborated in subsequent decades, most notably as the Ramsey Cass Koopmans growth model [12] which incorporates a more precise description of savings behavior.
From a variational analysis point of view, an unusual feature of this optimal control problem is the presence of a fractional singularity introduced by the production function $r(x) = x^\alpha$, with $0 < \alpha < 1$. This is not an artificial construct; a singularity is inherent to such problems, where, typically, the production function is required to satisfy the Inanda conditions [1], which include an 'infinite slope' condition at the origin.

Proposition 4.5.1 *Denote by $V : [0,T]\times]0,+\infty[\to \mathbb{R}$ the value function for (GC) and let $\psi : [0,\infty[\to [0,+\infty[$ be the mapping*

$$\psi(x) := x^{1-\alpha} \text{ for all } x \in [0,+\infty[\,. \tag{4.9}$$

Then

$$V(t,x) = (W \circ (Id,\psi))(t,x), \quad \text{for all } (t,x) \in [0,T]\times]0,+\infty[\,, \tag{4.10}$$

where $W : [0,T]\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ is the unique upper semicontinuous function such that $W(t,y) = -\infty$ whenever $y < 0$ and

(i) for all $(t,y) \in (0,T)\times[0,+\infty[$, $(\xi^0,\xi^1) \in \partial^P W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \geq 0;$$

(ii) for all $(t,y) \in (0,T)\times]0,+\infty[$, $(\xi^0,\xi^1) \in \partial^P W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \leq 0;$$

(iii) for all $y \in [0,+\infty[$

$$\limsup_{\{(t',y')\to(0,y)\,|\,t'>0\}} W(t',y') = W(0,y)$$

and

$$\limsup_{\{(t',y')\to(T,x)\,|\,t'<T,\ y'>0\}} W(t',y') = W(T,y) = 0.$$

(In conditions (i) and (ii), $\partial^P W(t,y)$ *denotes the proximal* superdifferential *of* W *at* (t,y) *.)*

Proof: Problem (GC) does not immediately fit into the Hamilton Jacobi framework of this paper, but can be made to do so by means of the transformation ψ (see (4.9)) of the state variable. Notice, however, that for any $s \in [0,T]$, $x_0 > 0$ and state trajectory/control pair (x,u) on $[s,T]$, we have that $x(t) > 0$ for all $t \in [s,T]$ and the transformed state trajectory $y(t) = \psi(x(t))$ is such that $y(t) > 0$ for all $t \in [s,T]$ and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = (1-\alpha)x^{-\alpha}(t)(-ax(t)) + bx^{\alpha}(t)u(t)) = -a(1-\alpha)y(t) + (1-\alpha)bu(t)\,.$$

Furthermore, $\displaystyle\int_s^T (1-u(t))x^{\alpha}(t)\mathrm{d}t = \int_s^T (1-u(t))y^{\frac{\alpha}{1-\alpha}}(t)\mathrm{d}t$. The same final expressions for the transformed problem can be employed also when $x_0 = 0$: indeed an admissible state trajectory $x(\cdot)$ might stay on the boundary of the state constraint '$x \geq 0$', but as soon as we have $x(t') > 0$ for some $t' \in (s,T)$, then $x(t) > 0$ for all $t \in [t',T]$. Write $s_0 := \inf\{t' \mid x(t') > 0\}$. Then, the transformed trajectory/control pair is $(y,\tilde{u})$ where $(y = 0, \tilde{u} = 0)$ on $[s,s_0]$ and $(y(\cdot) = \psi(x(\cdot)), \tilde{u}(\cdot) = u(\cdot))$ on $(s_0,T]$.
Take $W : [0,T]\times\mathbb{R} \to \mathbb{R}$ the value function for the transformed optimal control problem

$$(T)\begin{cases} \text{Maximize } \int_0^T (1-u(t))y^{\frac{\alpha}{1-\alpha}}(t)\mathrm{d}t \\ \text{subject to} \\ \dot{y}(t) = -a(1-\alpha)y(t) + (1-\alpha)bu(t) \quad \text{for a.e. } t \in [0,T], \\ y(t) \geq 0, \\ u(t) \in [0,1] \quad \text{for a.e. } t. \in [0,T]\,, \\ y(0) = y_0\ , \end{cases}$$

in which $y_0 \geq 0$. It follows from the preceding remarks that the value functions of (GC) and (T) are related by (4.10). The proof is completed by noting that the data for (T) satisfies the hypotheses of Thm. 4.4.2. We deduce that the value function W is characterized as in the proposition statement. (Observe that Thm. 4.4.2 must be applied in a modified form, since (T) is a maximization problem.)

□

In a similar way, using Thm. 1.4, we can characterize the value function also in the viscosity sense.

Proposition 4.5.2 *Denote by* $V : [0,T]\times]0,+\infty[\to \mathbb{R}$ *the value function for (GC) and let* $\psi : [0,+\infty[\to [0,+\infty)$ *be the mapping defined in (4.9). Then*

$$V(t,x) = (W \circ (Id,\psi))\,(t,x), \quad \textit{for all } (t,x) \in [0,T]\times]0,+\infty[\,,$$

where $W : [0,T]\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ *is the unique upper semicontinuous function such that* W *is continuous on* $[0,T]\times[0,+\infty[$, $W(t,y) = -\infty$ *whenever* $y < 0$ *and*

(i) for all $(t,y) \in]0,T[\times[0,+\infty[$, $(\xi^0,\xi^1) \in \partial_+ W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \geq 0;$$

(ii) for all $(t,y) \in]0,T[\times]0,+\infty[$, $(\xi^0,\xi^1) \in \partial_- W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \leq 0;$$

(iii) for all $y \in [0,+\infty[$

$$\limsup_{\{(t',y')\to(0,y)\,|\,t'>0\}} W(t',y') = W(0,y)$$

and

$$W(T,y) = 0.$$

The usefulness of this exercise can be described as follows. Solutions to growth model optimal control problems akin to (GC) are typically studied in the economics literature by means of the Pontryagin Maximum Principle (PMP). (See, e.g. [1].) But the PMP, which is merely a necessary condition of optimality, provides only putative minimizers (referred to as 'extremals') for (GC). Optimality can be confirmed by constructing a field of extremals and evaluating the extremal cost function $\tilde{V}$, that is the cost of each extremal, parameterized by its initial data.
Problem (GC) was studied and, applying a non-standard verification technique, completely solved in [70] for the 'soft' state constraint '$x(t) > 0$' (with initial data $x_0 > 0$). When we consider the (full) state constrained problem (GC) (i.e. with '$x(t) \geq 0$'), our theory tells us that this procedure will be successful, if we can show that $\tilde{W} := \tilde{V} \circ (Id, \psi^{-1})$ is a proximal solution (resp. viscosity solution) to the Hamilton Jacobi equation in the sense of the Proposition 4.5.1 (resp. Proposition 4.5.2).

4.6 $W^{1,1}$ distance estimate, preliminary results and hypotheses reduction

In this chapter we shall make use of some important analytical tools the first of which is a $W^{1,1}$ distance estimate theorem. If $\hat{x}(\cdot)$ is an F-trajectory starting from $(t_0,x_0) \in [S,T] \times A$, the real number $\hat{\rho} := \max_{t\in[t_0,T]} \mathrm{d}_A(\hat{x}(t))$ can be interpreted as a measure of the 'state constraint violation' of $\hat{x}(\cdot)$. A key point in our analysis is the possibility to construct a second feasible F trajectory $x(\cdot)$ which satisfies particular properties including the fact that $x(\cdot)$ is close to $\hat{x}(\cdot)$ w.r.t. $W^{1,1}$-distance: the obtained estimate is provided in terms of a suitable modulus of continuity which depends on the 'state constraint violation' parameter. The following theorem is valid even if F is not convex (so we shall consider (H1)$'$ below in place of (H1)).

Theorem 4.6.1 *Fix* $r_0 > 0$. *Assume that, for some positive functions* $c_F(\cdot) \in \mathbb{L}^1(S,T)$ *and* $k_F(\cdot) \in \mathbb{L}^\infty([S,T])$ *and for* $R_0 := e^{\int_S^T c_F(s)\mathrm{d}s}(r_0+1)$, *the following hypothesis*

(H1)$'$: $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, non-empty values, $F(\cdot,x)$ is $\mathcal{L}$-measurable for all $x \in \mathbb{R}^n$,

is satisfied together with (H2), (H3), (H4) and

(IPC)$'$: for each $s \in [S,T[$, $t \in]S,T]$ and $x \in R_0\mathbb{B} \cap \partial A$,

$$\mathrm{co}\, F(t^-,x) \cap \mathrm{int}\, T_A(x) \neq \emptyset \qquad \textit{and} \qquad \mathrm{co}\, F(s^+,x) \cap \mathrm{int}\, T_A(x) \neq \emptyset\,.$$

Then, there exists a constant $K > 0$ with the following property: given any interval $[t_0, t_1] \subset [S, T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{\int_S^{t_0} c_F(s)\mathrm{d}s}(r_0 + 1) - 1\big)\mathbb{B}$, and any $\rho > 0$ such that

$$\rho \;\geq\; \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0, t_1]\}\;,$$

we can find an F-trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \;\in\; \mathrm{int}\, A \quad \textit{for all } t \in (t_0, t_1]$$

and

$$||\hat{x} - x||_{\mathbb{L}^\infty([t_0,t_1])} \;\leq\; K\;\rho\;, \tag{4.11}$$

$$||\dot{\hat{x}} - \dot{x}||_{\mathbb{L}^1([t_0,t_1])} \;\leq\; K\;\rho(1 + |\ln(\rho)|)\;. \tag{4.12}$$

The proof of Theorem 4.6.1 is provided in Section 4.11.

Remark 4.6.2 *Theorem 4.6.1 can be reformulated in an equivalent way (as in [29], where this result is proved for a time-independent Lipschitz continuous F), in which the reference 'violating' F-trajectory does not necessarily start from the state constraint A; this alternative formulation can be immediately deduced making use of Filippov's Existence Theorem.*

We observe that the analysis is simplified employing a standard **hypotheses reduction** argument (cf. [21] or [84]), which allows us to invoke (without loss of generality) hypotheses (H1), (H2), (H3)*, (H4)*, (H5)* and (H6)*, where by (H3)*, (H4)*, (H5)* and (H6)* we denote the global (stronger) version of conditions (H3), (H4), (H5) and (H6) in which we have removed the constant R_0. This is due to the fact that (H2), (H3) and (H4)(i) yield an a priori uniform boundedness of F-trajectories. Indeed, given an initial data $(t, x) \in [S, T] \times \mathbb{R}^n$ and an F-trajectory $y \in W^{1,1}([t, T], \mathbb{R}^n)$ such that $y(t) = x$, for every $s \in [t, T], y(s) \in (1 + |x|) \exp\Big(\int_S^T c_F(\tau)\mathrm{d}\tau\Big)\,\mathbb{B}$. Set $R_0 := (1 + |x|) \exp\Big(\int_S^T c_F(\tau)\mathrm{d}\tau\Big)$, then, owing to (H3) and (H4)(i) there exists $c_0 > 0$ such that, for almost every $s \in [t, T]$, $\dot{y}(s) \in c_0\mathbb{B}$. So we consider a new multifunction $\widehat{F} : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a new function $\tilde{L} : [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, defined by

$$\widehat{F}(s, y) := \begin{cases} F(s, y) & \text{if } |y| \leq R_0 \\ F(s, R_0 y/|y|) & \text{if } |y| > R_0\,, \end{cases} \qquad \widehat{L}(s, y, v) := \begin{cases} L(s, y, v) & \text{if } |y| \leq R_0 \\ L(s, R_0 y/|y|, v) & \text{if } |y| > R_0\,. \end{cases}$$

The data $(\widehat{F}, \widehat{L})$ satisfy hypotheses (H1), (H2), (H3)*, (H4)*, (H5)* and (H6)*. But, in a neighbourhood of the given point (t, x), the data of the problem $(P_{t,x})$ involving either (F, L) or $(\widehat{F}, \widehat{L})$ do coincide. Therefore, in the forthcoming analysis we can invoke the more restrictive version of conditions (H3)-(H6) without loss of generality.

We shall invoke an useful Carathéodory's Parametrization Result ([4, Theorems 9.6.2 and 9.7.2]):

Proposition 4.6.3 *Assume that (H1), (H2) and (H3)*. Then, there exists a measurable function: $f : [S, T] \times \mathbb{R}^n \times \mathbb{B} \to \mathbb{R}^n$, such that:*

$$\begin{cases} \textit{For every } (t, x) \in [S, T] \times \mathbb{R}^n, F(t, x) = f(t, x, \mathbb{B})\,; \\ \textit{For every } (x, u) \in \mathbb{R}^n \times \mathbb{B}, f(\cdot, x, u) \textit{ is mesurable}\,; \\ \textit{For every } (t, u) \in [S, T] \times \mathbb{B}, f(t, \cdot, u) \textit{ is } 10nk_F(t)\textit{-Lipschitz}; \\ \textit{For every } (t, x) \in [S, T] \times \mathbb{R}^n, (u, u') \in \mathbb{B}^2, |f(t, x, u) - f(t, x, u')| \leq 5n \max_{v \in F(t,x)} |v||u - u'|. \end{cases}$$

Moreover, for all $(s, x, u) \in (S, T) \times \mathbb{R}^n \times \mathbb{B}$:

$$\lim_{s' \downarrow s, x' \to x, u' \to u} f(s', x', u') \in F(s^-, x) \textit{ and } \lim_{s' \uparrow s, x' \to x, u' \to u} f(s', x', u') \in F(s^+, x). \tag{4.13}$$

(Inclusions (4.13) do not appear in [4], but are a consequence of the construction of the parametrization f of F, which is based on the Steiner selection argument used in [4], see [21] for a detailed discussion on this point.)

4.7 Proof of Theorem 4.4.1

We assume that hypotheses (H1), (H2), (H3)*, (H4)*, (H5)*, (H6)* and (H7) are satisfied together with property $(CQ)_{BW}$ (of Remark 4.4.5) which is valid owing to (H8) and Theorem 4.6.1 with $\tilde{\theta}(\rho) := K\rho(1 + |\ln(\rho)|)$ for some constant $K > 0$. We know that the value function is lower semicontinuous (cf. [84, Chapter 12]).

'(a) $\Rightarrow$ (b)'. This implication can be proved using the principle of optimality for the value function and the analysis does not change in presence of state constraints and is the same of the state constraint-free case (cf. [21] and, for the details with a Lagrangian satisfying (H6), see Chapter 3), except for condition (b)(iii), for which property $(CQ)_{BW}$ turns out to be useful. (At this stage we actually do not need a distance estimate involving necessarily a $W^{1,1}$ norm: an L^∞ distance estimate is enough and the proof is as in [21].)

'(b) $\Rightarrow$ (c)'. Consider a lower semicontinuous function U satisfying (b). Then condition (c)(iii) is an easy consequence of the definition of $D_\uparrow U$ (cf. [84, Prop. 12.3.4]). Now, take $(t, x) \in ((S, T) \times \mathbb{R}^n) \cap \operatorname{dom} U$ and $(\xi^0, \xi^1) \in \partial_P U(t, x)$. Then, there exist $M > 0$ and $\epsilon > 0$ such that

$$\xi^0(t' - t) + \xi^1 \cdot (x' - x) \ \le\ U(t', x') - U(t, x) + M(|t' - t|^2 + |x' - x|^2) \qquad \text{for all } (t', x') \in (t, x) + \epsilon\mathbb{B}. \tag{4.14}$$

Since U satisfies (b) and $F(t^+, x)$ is compact, there exist sequences $h_i \downarrow 0$ and $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $v_i \xrightarrow[i\to+\infty]{} \bar{v}$ for some $\bar{v} \in F(t^+, x)$ and (cf. [15, Lemma 3.3] for Bolza problems with L satisfying (H6))

$$\lim_{i\to+\infty} h_i^{-1}(U(t + h_i, x + h_i v_i) - U(t, x)) \ \le\ -L(t^+, x, \bar{v}).$$

Setting $(t', x') = (t + h_i, x + h_i v_i)$, and dividing across by h_i, we obtain that, for i sufficiently large,

$$\xi^0 + \xi^1 \cdot v_i \ \le\ h_i^{-1}(U(t + h_i, x + h_i v_i) - U(t, x)) \ +\ M h_i(1 + |v_i|^2).$$

Since $v_i \xrightarrow[i\to+\infty]{} \bar{v}$ and $\bar{v} \in F(t^+, x)$, we have, in the limit as $i \to \infty$, that

$$\xi^0 + \inf_{v\in F(t^+,x)} \xi^1 \cdot v \ \le\ \xi^0 + \xi^1 \cdot \bar{v} \ =\ \xi^0 + \lim_{i\to+\infty} \xi^1 \cdot v_i \ \le\ L(t^+, x, \bar{v}).$$

On the other hand, for any given $v \in F(t^-, x)$, since U satisfies (b), there exist sequences $h_i \downarrow 0$ and $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$, such that $v_i \xrightarrow[i\to+\infty]{} v$ and

$$\lim_{i\to+\infty} h_i^{-1}(U(t - h_i, x - h_i v_i) - U(t, x)) \ \le\ L(t^-, x, v).$$

According to the notion of $\partial_P U(t, x)$, setting $(t', x') = (t - h_i, x - h_i v_i)$, we obtain

$$-(\xi^0 + \xi^1 \cdot v_i) \ \le\ h_i^{-1}(U(t - h_i, x - h_i v_i) - U(t, x)) \ +\ M h_i(1 + |v_i|^2),$$

for i sufficiently large. Since $v_i \xrightarrow[i\to+\infty]{} v$, we deduce from these relations that

$$\xi^0 + \xi^1 \cdot v \ \ge\ -L(t^-, x, v).$$

Since v was an arbitrary element in $F(t^-, x)$, we have shown that

$$\inf_{v\in F(t^-,x)} [\ \xi^0 + \xi^1 \cdot v + L(t^-, x, v)] \ \ge\ 0.$$

We deduce that U satisfies (c).

'(c) $\Rightarrow$ (a)'. A first step consists in showing that if U satisfies (c)(i) and (c)(iii), then $V(t,x) \leq U(t,x)$ for any $(t,x) \in [S,T] \times A$; more precisely one can show that $\int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \leq U(t,x)$ for some feasible F-trajectory $x(\cdot)$ on $[t,T]$ such that $x(t) = x$. This can be derived invoking well-known weak invariance results, and the analysis does not differ from the state constraint-free case used in Chapter 3. However, we show here that, as for the continuous time dependent case, we can eliminate the horizontal subdifferentials in the characterization of the value function. We introduce an auxiliary multivalued function: $Q : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \times \mathbb{R}$ defined by:

$$Q(\tau,x) \;:=\; \begin{cases} \Big\{(v,-\eta)\,|\,v \in F(S^+,x), M_0 \geq \eta \geq L(S^+,x,v)\Big\}, & \text{if} \quad \tau = S, \\ \mathrm{co}\Big\{(v,-\eta)\,|\,v \in \{F(\tau^-,x) \cup F(\tau^+,x)\}, M_0 \geq \eta \geq \widetilde{L}(\tau,x,v)\Big\}, & \text{if} \quad \tau \in]S,T[, \\ \Big\{(v,-\eta)\,|\,v \in F(T^-,x), M_0 \geq \eta \geq L(T^-,x,v)\Big\}, & \text{if} \quad \tau = T\ , \end{cases}$$

where $\widetilde{L}(\tau,x,v) := \min\{L(\tau^+,x,v), L(\tau^-,x,v)\}$. A routine analysis allows to verify that the multifunction Q takes as values non-empty convex sets with elements which are (uniformly) bounded by $c := \sqrt{c_0^2 + M_0^2}$; moreover the graph of Q is closed.
Take any $(\bar{t},\bar{x}) \in ((S,T) \times A) \cap \mathrm{dom}(U)$. Our aim is to apply the Viability/Weak Invariance Theorem (cf. Theorem 2.6.1) to the following differential inclusion:

$$\begin{cases} (\dot{\tau},\dot{x},\dot{\ell})(t) & \in & \Gamma(\tau(t),x(t),\ell(t)), \quad \text{for a.e. } t \in [\bar{t},T], \\ (\tau(t),x(t),\ell(t)) & \in & \mathrm{epi}\ U, \quad \text{for all } t \in [\bar{t},T], \\ (\tau(\bar{t}),x(\bar{t}),\ell(\bar{t})) & = & (\bar{t},\bar{x},U(\bar{t},\bar{x})). \end{cases} \tag{4.15}$$

where $\Gamma : [S,T] \times \mathbb{R}^{n+1} \rightsquigarrow \mathbb{R}^{n+2}$ is defined by

$$\Gamma(\tau,x,\ell) \;:=\; \begin{cases} \mathrm{co}\ (\{(0,0,0)\} \cup (\{1\} \times Q(S,x))), & \text{if} \quad \tau = S, \\ \{1\} \times Q(\tau,x), & \text{if} \quad \tau \in]S,T[, \\ \mathrm{co}\ (\{(0,0,0)\} \cup (\{1\} \times Q(T,x))), & \text{if} \quad \tau = T\ . \end{cases}$$

Clearly the multifunction Γ inherits the following properties from Q: the graph of Γ is closed, for all $(\tau,x,\ell) \in [S,T] \times \mathbb{R}^{n+1}$, $\Gamma(\tau,x,\ell)$ is nonempty convex set and $\Gamma(\tau,x,\ell) \subset (c+1)\mathbb{B}$. It remains to check that 'inward pointing condition' of the Weak Invariance Theorem is also satisfied: take any $(\tau,x,\ell) \in \mathrm{epi}\ U$ and any $(\xi^0,\xi^1,-\lambda) \in N^P_{\mathrm{epi}\ U}(\tau,x,\ell)$, we must show

$$\min_{w \in \Gamma(\tau,x,\ell)} (\xi^0,\xi^1,-\lambda) \cdot w \;\leq\; 0\ . \tag{4.16}$$

If $\tau = S,T$, then it is immediately verified (taking $w = 0$ that belongs to both $\Gamma(S,x,\ell)$ and $\Gamma(T,x,\ell)$). Suppose then that $S < \tau < T$. Observe that, by the nature of proximal normals to epigraph sets, we know that $\lambda \geq 0$ and we need to check (4.16) only when $\ell = U(\tau,x)$. We shall show

$$\xi^0 + \min_{v \in F(\tau^-,x) \cup F(\tau^+,x)} \xi^1 \cdot v + \lambda L(\tau^+,x,v) \;\leq\; 0\ . \tag{4.17}$$

This will confirm the inward pointing condition, indeed it implies

$$\min_{w \in \Gamma(\tau,x,\ell)} (\xi^0,\xi^1,-\lambda) \cdot w \;\leq\; \xi^0 + \min_{v \in \mathrm{co}\{F(\tau^-,x) \cup F(\tau^+,x)\}} \xi^1 \cdot v + \lambda L(\tau^+,x,v) \;\leq\; 0\ .$$

To check (4.17) we need to consider two cases.

Case 1: $\lambda > 0$**.** In this case $((1/\lambda)\xi^0, (1/\lambda)\xi^1, -1) \in N^P_{\text{epi } U}((\tau, x), U(\tau, x))$. It follows that

$$((1/\lambda)\xi^0, (1/\lambda)\xi^1) \ \in \ \partial_P U(\tau, x) \ .$$

But then, by (c)(i), $(1/\lambda)\left(\xi^0 + \min_{v\in F(\tau^+,x)}[\xi^1 \cdot v + \lambda L(\tau^+, x, v)]\right) \ \leq \ 0$. This implies (4.17).

Case 2: $\lambda = 0$**.** In this case, we know from the Rockafellar Horizontal Approximation Theorem (cf. Theorem 2.4.15) that there exist $(\xi^0_i, \xi^1_i) \to (\xi^0, \xi^1)$, $\lambda_i \downarrow 0$ and $(t_i, x_i) \to (\tau, x)$ such that, for each i,

$$(\lambda_i^{-1}\xi^0_i, \lambda_i^{-1}\xi_i) \in \partial_P U(t_i, x_i)\,.$$

But then, by condition (c)(i), there exists $v_i \in F(t_i^+, x_i)$ such that

$$\xi^0_i + \xi^1_i \cdot v_i + \lambda_i L(\tau^+, x, v_i) \leq 0\,.$$

But $(v_i)_{i\in\mathbb{N}}$ is a bounded sequence. We can therefore arrange, by extracting a subsequence, that $v_i \xrightarrow[i\to+\infty]{} \bar{v}$, for some $\bar{v} \in \mathbb{R}^n$. Since $(t, x) \rightsquigarrow F(t^-, x) \cup F(t^+, x)$ is an upper semi-continuous multifunction, it follows that $\bar{v} \in F(\tau^-, x) \cup F(\tau^+, x)$. So, recalling also that L is bounded on bounded sets, in the limit as $i \to \infty$, we obtain

$$0 \geq \xi^0 + \xi^1 \cdot \bar{v} + 0 \times L(\tau^+, x, \bar{v}) \geq \xi^0 + \inf_{v\in F(\tau^-,x)\cup F(\tau^+,x)} \xi^1 \cdot v.$$

We have confirmed (4.17) in this case too.

Then, the Weak Invariance Theorem is applicable to system (4.15). We deduce that there exists $(\tau(\cdot), x(\cdot), \ell(\cdot)) \in W^{1,1}([\bar{t}, T], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ satisfying $\tau(t) = t$ and

$$\begin{cases} (\dot{x}(t), \dot{\ell}(t)) \in Q(t, x(t)), \text{for a.e. } t \in [\bar{t}, T] \\ x(\bar{t}) = \bar{x}, \ell(\bar{t}) = U(\bar{t}, \bar{x}) \\ \ell(t) \geq U(t, x(t)) \quad \text{for all } \ t \in [\bar{t}, T] \ . \end{cases}$$

Taking into account the definition of the multivalued function Q and the hypotheses on both F and L, we it follows that $x(\cdot)$ is an F-trajectory and that $\dot{\ell}(s) \leq -L(s, x(s), \dot{x}(s))$ for a.e. $s \in [\bar{t}, T]$. Hence we have:

$$g(x(T)) = U(T, x(T)) \leq \ell(T) = \ell(\bar{t}) + \int_{\bar{t}}^{T} \dot{\ell}(s)\mathrm{d}s \leq U(\bar{t}, \bar{x}) - \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s,$$

which implies:

$$g(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s \leq U(\bar{t}, \bar{x}).$$

Thus we obtain:

$$V(\bar{t}, \bar{x}) \leq U(\bar{t}, \bar{x}).$$

It remains to investigate the case in which $\bar{t} = S$: if $(S, \bar{x})$ belongs to $\mathrm{dom}(U)$, then from (c)(iii) we can take a sequence $(S_i, x_i) \to (S, \bar{x})$ as $i \to \infty$ such that $S_i \downarrow S$ and $\lim_{i\to\infty} U(S_i, x_i) = U(S, \bar{x})$. For each $i \geq 0$, from the previous argument we know that there exists an F-trajectory $y_i(\cdot)$ such that

$$U(S_i, x_i) \geq g(y_i(T)) \ . \tag{4.18}$$

Extending $y_i(\cdot)$ to all $[S, T]$ by constant extrapolation from the right on $[S, S_i]$, with the help of the Compactness of Trajectories Theorem (cf. [84, Thm. 2.5.3]) we can arrange, by extracting a suitable subsequence,

that $y_i(\cdot) \xrightarrow[i\to+\infty]{} \bar{y}(\cdot)$ uniformly, for some F-trajectory $\bar{y}(\cdot)$ such that $\bar{y}(\bar{t}\,) = \bar{x}$. Therefore, since g is lower semicontinuous, in the limit, as $i \to \infty$ from (4.18) we deduce that

$$U(S,\bar{x}) = \liminf_{i\to\infty} U(S_i,x_i) \geq \liminf_{i\to\infty} g(y_i(T)) \geq g(\bar{y}(T))\ .$$

This completes the first step.

We consider now the second step of the implication '(c) $\Rightarrow$ (a)', where a $W^{1,1}$ distance estimate is required: if U satisfies (c)(ii) and (c)(iii), then $V(t,x) \geq U(t,x)$ for any $(t,x) \in [S,T]\times\mathbb{R}^n$. This can be achieved showing that, given any arbitrary point $(\bar{t},\bar{x}) \in [S,T]\times A$, and any feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$, we have $\int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \leq U(\bar{t},\bar{x})$. We can assume that $g(x(T)) < +\infty$, otherwise we automatically have the desired inequality.
Since U satisfies (4.2), we can pick sequences $(t_j)_{j\geq 1}$ in $(\bar{t},T)$ and $(\xi_j)_{j\geq 1}$ in $\operatorname{int} A$ such that $(t_j,\xi_j) \in \operatorname{dom}(U)$ for every $j \geq 1$, $t_j \uparrow T$ and $\xi_j \to x(T)$ as $j \to +\infty$, and

$$\lim_{j\to+\infty} U(t_j,\xi_j) = \liminf_{\{(t',x')\to(T,x(T))\,|\,t'<T,x'\in\operatorname{int} A\}} U(t',x') = U(T,x(T)) = g(x(T)). \tag{4.19}$$

We write $s_j := T - t_j$ and $\rho_j := \exp(k_F|T-S|)|x(t_j) - \xi_j|$; clearly $\lim_{j\to+\infty} s_j = 0$ and $\lim_{j\to+\infty}\rho_j = 0$. Using Filippov's Existence Theorem, for each $j \geq 1$, we can find an $\tilde{F}$-trajectory $\tilde{y}_j(\cdot)$ on $[s_j, T-\bar{t}\,]$ such that $\tilde{y}_j(s_j) = \xi_j$ and:

$$\|\tilde{y}_j(\cdot) - x(T-\cdot)\|_{W^{1,1}([s_j,T-\bar{t}\,])} \leq \rho_j.$$

($\tilde{F} : [0,T-S]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is the multivalued function defined by $\tilde{F}(s,y) := -F(T-s,y)$.) Observe that, for every $j \geq 1$, since $x(\cdot)$ is feasible we have $\max\{\mathrm{d}_A(\tilde{y}_j(s))\,|\,s\in[s_j,T-\bar{t}\,]\} \leq \rho_j$. Now, we can use property $(CQ)_{BW}$ (with $\tilde{\theta}(\rho) := K\rho(1+|\ln(\rho)|)$ owing to Theorem 4.6.1) and deduce that, for any $j \geq 1$, there exists a feasible $\tilde{F}$-trajectory $z_j(\cdot)$ on $[s_j,T-\bar{t}]$, such that $z_j(s_j) = \xi_j$, $z_j(s) \in \operatorname{int} A$ for all $s \in (s_j, T-\bar{t}\,]$ and

$$\|\tilde{y}_j - z_j\|_{W^{1,1}([s_j,T-\bar{t}\,])} \leq \tilde{\theta}\,(\rho_j)\,.$$

This implies that

$$\|x(T-\cdot) - z_j(\cdot)\|_{W^{1,1}([s_j,T-\bar{t}\,])} \leq \tilde{\theta}\,(\rho_j) + \rho_j.$$

For each $j \geq 1$ consider the arc $x_j(t) := z_j(T-t)$, for $t \in [\bar{t},t_j]$. Then each $x_j(\cdot)$ is an F-trajectory on $[\bar{t},t_j]$ that can be extended to $[\bar{t},T]$ using Filippov's Existence Theorem in such a manner that each extension, still written $x_j(\cdot)$, satisfies the estimate

$$\|x - x_j\|_{W^{1,1}([\bar{t},T])} \leq \tilde{\theta}\,(\rho_j) + \rho_j. \tag{4.20}$$

Using the fact $z_j(s) \in \operatorname{int} A$ for all $s \in]s_j, T-\bar{t}\,]$, we can find sequences $(\tau_j)_{j\geq 1}$ in (t_j,T) and $(\delta_j)_{j\geq 1}$ such that $\tau_j \uparrow T$ and $\delta_j \downarrow 0$ as $j \to +\infty$ and

$$x_j(t) + \delta_j\mathbb{B} \subset \operatorname{int} A,\ \text{for all } t \in [\bar{t},\tau_j].$$

Let $f : [S,T]\times\mathbb{R}^n\times\mathbb{B} \to \mathbb{R}^n$ be the parametrization of F given by Proposition 4.6.3. Using Filippov's Selection Theorem (cf. [84, Theorem 2.3.13]), for every $j \geq 1$, there exists a measurable selection $u_j : [\bar{t},t_j] \to \mathbb{B}$ such that $\dot{x}_j(t) = f(t,x_j(t),u_j(t))$. for a.e. $t \in [\bar{t},t_j]$.

Fix any $j \geq 1$. Take any $i \in \mathbb{N}$ such that $i \geq \frac{2}{\delta_j}$. Lusin's Theorem guarantees the existence of a continuous function $u_j^i : [\bar{t},t_j] \to \mathbb{B}$ such that

$$\operatorname{meas}\big(\{t \in [\bar{t},t_j]\,|\,u_j(t) - u_j^i(t) \neq 0\}\big) \leq \frac{1}{i} \tag{4.21}$$

and the corresponding solution $x_j^i(\cdot)$ to system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u_j^i(t)), \text{ for a.e. } t \in [\bar{t}, t_j] \\ x(t_j) = \xi_j \, (= x_j(t_j)) \end{cases}$$

satisfies

$$\|x_j - x_j^i\|_{\mathbb{L}^\infty([\bar{t}, t_j])} \leq \frac{1}{i}. \tag{4.22}$$

For every $(t, x) \in [\bar{t}, t_j] \times \mathbb{R}^n$, we write $v^+(t, x) := f(t^+, x, u_j^i(t))$, $v^-(t, x) := f(t^-, x, u_j^i(t))$. We define two multivalued functions $F_{j,i} : [\bar{t}, t_j] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and $\Lambda_{j,i} : [\bar{t}, t_j] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}$ as follows: for every $(t, x) \in [\bar{t}, t_j] \times \mathbb{R}^n$

$$F_{j,i}(t, x) := \operatorname{co} \{v^-(t, x), v^+(t, x)\},$$

$$\Lambda_{j,i}(t, x) := \operatorname{co} \{L(t^-, x, v^-(t, x)), L(t^+, x, v^+(t, x))\}.$$

We define the arc $(\tilde{\tau}, \tilde{z}, \tilde{\ell})$ on $[0, t_j - \bar{t}\,]$:

$$s \mapsto (\tilde{\tau}(s), \tilde{z}(s), \tilde{\ell}(s)) := \left(s, x_j^i(t_j - s), U(t_j, \xi_j) + \int_0^s L(t_j - r, x_j^i(t_j - r), \dot{x}_j^i(t_j - r)) \mathrm{d}r \right),$$

and the function $\widetilde{U}_j$ defined by $\widetilde{U}_j(s, y) := U(t_j - s, y)$, for any $s \in [0, t_j - \bar{t}\,]$ and $y \in \mathbb{R}^n$.

We introduce also the multifunction $\Gamma_{j,i} : [0, t_j - \bar{t}\,] \times \mathbb{R}^n \times \mathbb{R} \rightsquigarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ defined as

$$\Gamma_{j,i}(\tau, x, \ell) := \begin{cases} \operatorname{co} \{(0, 0, 0) \cup \{1\} \times -F_{j,i}(\bar{t}, x) \times \Lambda_{j,i}(\bar{t}, x)\}, & \text{if } \tau = t_j - \bar{t}, \\ \{1\} \times -F_{j,i}(t_j - \tau, x) \times \Lambda_{j,i}(t_j - \tau, x), & \text{if } \tau \in [0, t_j - \bar{t}\,). \end{cases}$$

Clearly, $\Gamma_{j,i}$ takes convex, nonempty values, has a closed graph, and owing to our 'hypotheses reduction' (of Section 4.6), $\Gamma_{j,i}(t, x, \ell) \subset (1 + |c|)\mathbb{B}$ where $c := \sqrt{c_0^2 + M_0^2}$ for every $(t, x, \ell) \in [0, t_j - \bar{t}\,] \times \mathbb{R}^n \times \mathbb{R}$. Observe that the arc $(\tilde{\tau}, \tilde{z}, \tilde{\ell})$ is a $\Gamma_{j,i}$-trajectory such that $(\tilde{\tau}, \tilde{z}, \tilde{\ell})(0) = (0, \xi_j, U(t_j, \xi_j))$, and for every interval $[s_1, s_2] \subset [0, t_j - \bar{t}\,]$, the restriction of $(\tilde{\tau}, \tilde{z}, \tilde{\ell})$ to $[s_1, s_2]$ is the unique $\Gamma_{j,i}$-trajectory with initial condition $(\tilde{\tau}(s_1), \tilde{z}(s_1), \tilde{\ell}(s_1))$. Consider the state constrained differential inclusion:

$$\begin{cases} (\dot{\tau}(s), \dot{y}(s), \dot{\ell}(s)) \in \Gamma_{j,i}(\tau(s), y(s), \ell(s)) \text{ for a.e. } s \in [0, t_j - \bar{t}\,], \\ \tau(0) = 0, y(0) = \xi_j, \ell(0) = U(t_j, \xi_j) \, (= \widetilde{U}_j(0, \xi_j)), \\ (\tau(s), y(s), \ell(s)) \in \operatorname{epi} \widetilde{U}_j \text{ for all } s \in [0, t_j - \bar{t}\,]. \end{cases} \tag{4.23}$$

We claim that $(\tilde{\tau}, \tilde{z}, \tilde{\ell})$ is a solution to (4.23), *i.e.* the constraint $(\tilde{\tau}, \tilde{z}, \tilde{\ell}) \in \operatorname{epi} \widetilde{U}_j$ is also satisfied. Indeed, all the necessary conditions for the applicability of the Local Weak Invariance Theorem (Theorem 2.6.2) have already been discussed above, except the local inward pointing condition (*(iv)* of Theorem 2.6.2) which in our case takes the form: for any $(\tau, x, \ell) \in \operatorname{epi} \widetilde{U}_j \cap ((\tilde{\tau}, \tilde{z}, \tilde{\ell})([0, t_j - \bar{t}\,]) + \frac{1}{i}\mathbb{B})$:

$$\min_{w \in \Gamma_{j,i}(\tau, x, \ell)} w \cdot (\xi^0, \xi^1, -\lambda) \leq 0, \text{ for all } (\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} \widetilde{U}_j}(\tau, x, \ell). \tag{4.24}$$

Take any $(\tau, x, \ell) \in \operatorname{epi} \widetilde{U}_j \cap ((\tilde{\tau}, \tilde{z}, \tilde{\ell})([0, t_j - \bar{t}\,]) + \frac{1}{i}\mathbb{B})$ and any normal vector $(\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} \widetilde{U}_j}((\tau, x), \ell)$. By construction of $\Gamma_{j,i}$, the inward pointing condition is easily verified if $\tau = t_j - \bar{t}$, so we assume that $\tau < t_j - \bar{t}$. Notice that from well-known properties of the proximal cone to the epigraph, we can restrict attention to the case $\ell = \widetilde{U}_j(\tau, x)$, and we know that $\lambda \geq 0$ and

$$(-\xi^0, \xi^1, -\lambda) \in N^P_{\operatorname{epi} U}((t_j - \tau, x), U(t_j - \tau, x)).$$

We also have $v^-(t_j-\tau,x)\in F_{j,i}(t_j-\tau,x)\cap F((t_j-\tau)^-,x)$ and if $x\in\tilde z(\tau)+\frac{1}{i}\mathbb{B}$ then $x\in\tilde z(\tau)+\frac{\delta_j}{2}\mathbb{B}\subset\operatorname{int}A$. We continue considering two possible situations. In the first case $\lambda>0$. Then, $\lambda^{-1}(-\xi^0,\xi^1)\in\partial_P U(t_j-\tau,x)$ and so, condition (c)(ii) implies

$$-\xi^0+\xi^1\cdot v^-(t_j-\tau,x)+\lambda L((t_j-\tau)^-,x,v^-(t_j-\tau,x))\geq 0,$$

which confirms (4.24) taking $w=(1,-v^-(t_j-\tau,x),L((t_j-\tau)^-,x,v^-(t_j-\tau,x)))$. On the other hand, if $\lambda=0$, then there exist sequences $(-\xi^0_k,\xi^1_k)\to(-\xi^0,\xi^1)$, $\lambda_k\downarrow 0$ and $(s_k,x_k)\to(t_j-\tau,x)$ such that, for each k,

$$(-\lambda_k^{-1}\xi^0_k,\lambda_k^{-1}\xi^1_k)\in\partial_P U(s_k,x_k)\,.$$

But then, by condition (c)(ii),

$$-\xi^0_k+\xi^1_k\cdot v^-(s_k,x_k)+\lambda_k L((s_k)^-,x_k,v^-(s_k,x_k))\geq -\xi^0_k+\inf_{v\in F(s_k^-,x_k)}[\xi^1_k\cdot v+\lambda_k L((s_k)^-,x_k,v)\geq 0. \quad (4.25)$$

By extracting a subsequence we can arrange that, either $s_k\leq t_j-\tau$ for all k, or $s_k>t_j-\tau$ for all k. If $s_k\leq t_j-\tau$ for all k, then since $(s_k,x_k)\to(t_j-\tau,x)$, $v^-(s_k,x_k)\to v^-(t_j-\tau,x)$, and, in consequence of the hypotheses (H3), (H4)(i), (H5) and (H6), we can pass to the limit as $k\to\infty$ in the preceding relation to obtain

$$-\xi^0+\xi^1\cdot v^-(t_j-\tau,x)\geq 0.$$

If $s_k>t_j-\tau$ for all k, passage to the limit gives $v^-(s_k,x_k)\to v^+(t_j-\tau,x)$ and

$$-\xi^0+\xi^1\cdot v^+(t_j-\tau,x)\geq 0.$$

In either case then, (4.24) is verified.
Since all the hypotheses of the Local Weak Invariance Theorem (Theorem 2.6.2) are verified, we conclude that for every $t\in[0,t_j-\bar t]$: $(\tilde\tau,\tilde z,\tilde\ell)(t)\in\operatorname{epi}\widetilde U_j$. Our claim is confirmed.

Therefore, bearing in mind the definition of the arc $(\tilde\tau,\tilde z,\tilde\ell)$, we deduce that for that for any $j\geq 1$ and any $i\in\mathbb{N}$ such that $i\geq\frac{2}{\delta_j}$:

$$U(\bar t,x^i_j(\bar t))\leq U(t_j,\xi_j)+\int_{\bar t}^{t_j}L(s,x^i_j(s),\dot x^i_j(s))\mathrm{d}s. \quad (4.26)$$

Since U is lower semicontinuous and $x^i_j\xrightarrow[i\to+\infty]{}x_j(\bar t)$, in the limit as $i\to+\infty$, we obtain:

$$U(\bar t,x_j(\bar t))\leq U(t_j,\xi_j)+\liminf_{i\to+\infty}\int_{\bar t}^{t_j}L(s,x^i_j(s),\dot x^i_j(s))\mathrm{d}s,\ \text{for all } j\geq 1.$$

From condition (H2)*, for every integer $j\geq 1$ and every $t\in[\bar t,T]$, we have $\max_{v\in F(t,x_j(t))}|v|\leq c_0$, and so, using (4.22) and the regularity properties of f (Proposition 4.6.3), we deduce that, for a.e. $s\in[\bar t,t_j]$,

$$\begin{aligned}|\dot x^i_j(s)-\dot x_j(s)|&=|f(s,x^i_j(s),u^i_j(s))-f(s,x_j(s),u_j(s))|\\&\leq|f(s,x^i_j(s),u^i_j(s))-f(s,x_j(s),u^i_j(s))|+|f(s,x_j(s),u^i_j(s))-f(s,x_j(s),u_j(s))|\\&\leq 10nk_F|x^i_j(s)-x_j(s)|+5n\max_{v\in F(s,x_j(s))}|v||u^i_j(s)-u_j(s)|\\&\leq\frac{10n}{i}k_F+5nc_0|u^i_j(s)-u_j(s)|.\end{aligned} \quad (4.27)$$

We also know that $\|u_j-u^i_j\|_{\mathbb{L}^\infty(\bar t,t_j)}\leq 2$, then from (4.21) and (4.27) it follows that

$$\dot x^i_j\xrightarrow[i\to+\infty]{\mathbb{L}^1(\bar t,t_j)}\dot x_j. \quad (4.28)$$

Moreover, from (H5)* we can easily derive the local Lipschitz regularity of $L(s,y,\cdot)$, uniformly with respect to (s,y). Hence using (H6)(i), (4.22), (4.28), and the Lebesgue Dominated Convergence Theorem (possibly taking a subsequence), we obtain:

$$U(\bar{t},x_j(\bar{t})) \leq U(t_j,\xi_j) + \int_{\bar{t}}^{t_j} L(s,x_j(s),\dot{x}_j(s))\mathrm{d}s, \text{ for all } j \geq 1.$$

Invoking now (4.20), (4.19), and the lower semicontinuity of U, in the limit as $j \to +\infty$ we have

$$U(\bar{t},\bar{x}) = U(\bar{t},x(\bar{t})) \leq U(T,x(T)) + \liminf_{j\to+\infty} \int_{\bar{t}}^{t_j} L(s,x_j(s),\dot{x}_j(s))\mathrm{d}s.$$

We once again make use of (4.20) and of the Lebesgue's Dominated Convergence Theorem (possibly taking a subsequence), deducing:

$$U(\bar{t},\bar{x}) \leq g(x(T)) + \int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s.$$

□

4.8 Proof of Theorem 4.4.2

The implications '(a) ⇒ (b)' and '(b) ⇒ (c)' and the relation '$V(t,x) \leq U(t,x)$' of the implication '(c) ⇒ (a)', can be proved arguing as in the proof of Theorem 4.4.1 except for the boundary condition which now immediately follows from the continuity of the value function, which, making use of the distance estimate $(CQ)_{FW}$ (with $\theta(\rho) := K\rho(1+|\ln(\rho)|)$, owing to Theorem 4.6.1) can be deduced from the continuity of g.

It remains to show that, if U satisfies (c)(ii) and (c)(iii) of Theorem 4.4.2, then $V(\bar{t},\bar{x}) \geq U(\bar{t},\bar{x})$ for all $(\bar{t},\bar{x}) \in [S,T] \times A$. Take any feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$. (We can assume that $g(x(T)) < +\infty$). Using property $(CQ)_{FW}$ (with $\theta(\rho) = K\rho(1+|\ln(\rho)|)$), there exists a sequence of feasible F-trajectories $(x_j)_{j\geq 1}$ on $[\bar{t},T]$ such that, for every $j \geq 1$, $x_j(\bar{t}) = \bar{x}$, $x_j(t) \in \operatorname{int} A$ for all $t \in (\bar{t},T]$, and:

$$\|x - x_j\|_{W^{1,1}([\bar{t},T])} \leq \frac{1}{j}. \tag{4.29}$$

Take a strictly decreasing sequence $(t_j)_{j\geq 1}$ in $]\bar{t},T[$ that converges to $\bar{t}$. For every $j \geq 1$, there exists $\delta_j > 0$ such that:

$$x_j(t) + \delta_j \mathbb{B} \subset \operatorname{int} A, \text{ for all } t \in [t_j,T].$$

Let $f : [S,T] \times \mathbb{R}^n \times \mathbb{B} \to \mathbb{R}^n$ be the Carathéodory parametrization given by Proposition 4.6.3. Applying Filippov's Selection Theorem for every $j \geq 1$, we can find a measurable control $u_j : [t_j,T] \to \mathbb{B}$ such that $\dot{x}_j(t) = f(t,x_j(t),u_j(t))$, for a.e. $t \in [t_j,T]$.
Fix any $j \geq 1$. From the first relation of condition (c)(iii) we know that there exist a sequence $((T_j^i,\xi_j^i))_{i\geq 1}$ such that $|(T_j^i,\xi_j^i) - (T,x_j(T))| \leq 1/i$ for all $i \geq 1$ and $U(T_j^i,\xi_j^i) \to U(T,x_j(T))$ $(= g(x_j(T)))$ as $i \to +\infty$. Now, invoking Lusin's Theorem and employing arguments (including the application of the Local Weak Invariance Theorem 2.6.2) similar to those of the proof of Theorem 4.4.1 we arrive at a sequence of feasible F-trajectories $(x_j^i)_{i\in\mathbb{N}}$ on $[t_j,T]$ such that $x_j^i(T_j^i) = \xi_j^i$, $\|x_j - x_j^i\|_{\mathbb{L}^\infty([t_j,T])} \leq \frac{1}{i}$ and

$$U(t_j,x_j^i(t_j)) \leq U(T_j^i,\xi_j^i) + \int_{t_j}^{T_j^i} L(s,x_j^i(s),\dot{x}_j^i(s))\mathrm{d}s.$$

Since U is lower semicontinuous, $x_j^i(t_j) \to x_j(t_j)$ and $U(T_j^i, \xi_j^i) \to U(T, x_j(T))$ $(= g(x_j(T)))$ as $i \to +\infty$, arguing as in the proof of Theorem 4.4.1, in the limit as $i \to +\infty$, for each $j \geq 1$, we obtain:

$$U(t_j, x_j(t_j)) \leq g(x_j(T)) + \int_{t_j}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s.$$

Since $x_j(T) \xrightarrow[j\to+\infty]{A} x(T)$, g is continuous on A and U is lower semicontinuous, recalling (4.29) and passing to the limit as $j \to +\infty$, with the help again of Lebesgue's Dominated Convergence Theorem we deduce (for a subsequence if necessary)

$$U(\bar{t}, \bar{x}) = U(\bar{t}, x(\bar{t})) \leq \liminf_{j\to+\infty} U(t_j, x_j(t_j)) \leq g(x(T)) + \liminf_{j\to+\infty} \int_{t_j}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s.$$

and, owing to the lower semicontinuity of U, we deduce:

$$\begin{aligned} U(\bar{t}, \bar{x}) &= U(\bar{t}, x(\bar{t})) \leq g(x(T)) + \liminf_{j\to+\infty} \int_{t_j}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s & (4.30)\\ &\leq g(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s. & (4.31) \end{aligned}$$

Since $x(\cdot)$ was an arbitrary feasible F-trajectory such that $x(\bar{t}) = \bar{t}$, we conclude that:

$$U(\bar{t}, \bar{x}) \leq V(\bar{t}, \bar{x}).$$

□

4.9 Proof of Theorem 4.4.3

Since from Theorem 4.4.1 we already know that conditions (a), (b) and (c) are equivalent, we proceed to show the implications '(a) ⇒ (d)' and '(d) ⇒ (a)'.
The first relation '(a) ⇒ (d)' is actually valid even if the outward pointing constraint qualification and condition '$((g_{|A})^*)_* = g_{|A}$' are not in force. This is established by the following proposition.

Proposition 4.9.1 *Assume (H1)–(H8) and (IPC). Suppose, in addition, that $g_{|A}(\cdot)$ is locally bounded. Then the value function V satisfies (d) of Theorem 4.4.3.*

Proof. We observe that, from the *a priori* boundedness of the F-trajectories, and the local boundedness of $g_{|A}(\cdot)$ and L, it immediately follows that $V_{|[S,T]\times A}$ is locally bounded. Let $(t, x) \in (S, T) \times A$.

Take any $(\xi^0, \xi^1) \in \partial_- V(t, x)$. Then,

$$\xi^0(t' - t) + \xi^1 \cdot (x' - x) \ \leq \ V(t', x') - V(t, x) + \mathrm{o}(|(t' - t, x' - x)|) \quad \text{for all } (t', x') \tag{4.32}$$

in which $\mathrm{o}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying $\mathrm{o}(\epsilon)/\epsilon \to 0$ as $\epsilon \downarrow 0$. Arguing as in the proof of Theorem 4.4.1, we can find sequences $h_i \downarrow 0$ and $v_i \xrightarrow[i\to+\infty]{} \bar{v}$, for some $\bar{v} \in F(t^+, x)$, such that

$$\lim_{i\to+\infty} h_i^{-1}(V(t + h_i, x + h_i v_i) - V(t, x)) \ \leq \ -L(t^+, x, \bar{v}).$$

Taking $(t', x') = (t + h_i, x + h_i v_i)$ in (4.32), and dividing across by h_i, we obtain, for i sufficiently large,

$$\xi^0 + \xi^1 \cdot v_i \leq h_i^{-1}(V(t + h_i, x + h_i v_i) - V(t, x)) + h_i^{-1}\mathrm{o}(h_i\sqrt{1 + |v_i|^2}).$$

Therefore, in the limit as $i \to \infty$, it follows that

$$\xi^0 + \inf_{v \in F(t^+,x)} [\xi^1 \cdot v + L(t^+, x, v)] \ \leq \ \xi^0 + \xi^1 \cdot \bar{v} + L(t^+, x, \bar{v}) \ \leq \ 0.$$

Let $(t, x) \in]S, T[\times \operatorname{int} A$ and take $\varepsilon > 0$ such that $x + \varepsilon\mathbb{B} \subset \operatorname{int} A$. There exists a sequence $((t_i, x_i))_{i\in\mathbb{N}}$ in $(S, T) \times \operatorname{int} A \setminus \{(t, x)\}$ that converges to (t, x) such that

$$\lim_{i\to+\infty} V(t_i, x_i) = (V_{|[S,T]\times A})^*(t, x) \ \ (= V^*(t, x)).$$

We claim that, extracting a subsequence if necessary (we do not relabel), we have $t_i \downarrow t$ as $i \to +\infty$. Indeed, if $t_i < t$ for every $i \geq i_0$ for some $i_0 \in \mathbb{N}$, then we proceed as follows. We take a strictly decreasing sequence $(\tau_i)_{i\geq i_0}$ in $]t, T]$ that converges to t as $i \to +\infty$. We can assume that for all $i \geq i_0$, $x_i \in x + \frac{\varepsilon}{3}\mathbb{B}$ and $\tau_i - t_i \leq \frac{\varepsilon}{3c_0}$. Fix any $i \geq i_0$, and take a feasible F-trajectory $\tilde{y}_i(\cdot)$ on $[t_i, \tau_i]$ such that $\tilde{y}_i(t_i) = x_i$. Using the principle of optimality, we obtain:

$$V(t_i, x_i) - \int_{t_i}^{\tau_i} L(s, \tilde{y}_i(s), \dot{\tilde{y}}_i(s))\mathrm{d}s \leq V(\tau_i, \tilde{y}_i(\tau_i)).$$

Using the local boundedness of L (*cf.* condition (H5)*), there exists $M_0 > 0$ such that for every $i \geq i_0$:

$$V(t_i, x_i) - M_0|\tau_i - t_i| \leq V(\tau_i, \tilde{y}_i(\tau_i)).$$

Passing to the limit to both sides and using the upper semicontinuity of $(V_{|[S,T]\times A})^*$, we obtain:

$$\begin{aligned}(V_{|[S,T]\times A})^*(t, x) = \limsup_{i\to+\infty} V(t_i, x_i) &\leq \limsup_{i\to+\infty} V(\tau_i, \tilde{y}_i(\tau_i)) + \lim_{i\to+\infty} M_0|t_i - \tau_i| \\ &\leq \limsup_{i\to+\infty} (V_{|[S,T]\times A})^*(\tau_i, \tilde{y}_i(\tau_i)) \leq (V_{|[S,T]\times A})^*(t, x).\end{aligned}$$

Thus $\limsup_{i\to+\infty} V(\tau_i, \tilde{y}_i(\tau_i)) = (V_{|[S,T]\times A})^*(t, x)$ and there exists a subsequence $(i_k)_{k\in\mathbb{N}}$ for which

$$V(\tau_{i_k}, \tilde{y}_{i_k}(\tau_{i_k})) \xrightarrow[k\to+\infty]{} (V_{|[S,T]\times A})^*(t, x).$$

This confirms our claim.

Now, fix any $\tilde{v} \in F(t^+, x)$. Then, there exists a sequence of vectors $(v_i)_{i\in\mathbb{N}}$ such that $v_i \in F(t_i^+, x_i)$ for all i, and $\lim_{i\to+\infty} v_i = \tilde{v}$. For each i, we consider the arc

$$y_i(s) := x_i + (s - t_i)v_i, \ \text{for all } s \in [t_i, T].$$

Using Filippov's Existence Theorem, there exists an F-trajectory $z_i(\cdot)$ that satisfies $z_i(t_i) = x_i$ and such that for every $h \in (0, T - t_i]$

$$\|z_i - y_i\|_{\mathbb{L}^\infty(t_i,t_i+h)} \leq K \left(\int_{t_i}^{t_i+h} \mathrm{d}_{F(s,y_i(s))}(v_i)\mathrm{d}s \right),$$

where $K = \exp(k_F|T - S|)$. From the *a priori* boundedness of F-trajectories, we can pick $R_0 > 0$ such that, for each i, $|z_i(s)| \leq R_0$ for every $s \in [t_i, T]$ and $|\dot{z}_i(s)| \leq c_0$, for almost every $s \in [t_i, T]$. Observe that we can find $\bar{h} > 0$ such that, for all $i \geq 1$, we have $t_i + \bar{h} < T$ and $z_i(s) \in x + \varepsilon\mathbb{B} \subset \operatorname{int} A$ for every $s \in [t_i, t_i + \bar{h}]$. Moreover, since property $(CQ)_{FW}$ is valid (owing to Theorem 4.6.1) guarantees that each $z_i(\cdot)$ can be extended to a feasible F-trajectory on $[t_i, T]$. For every $i \in \mathbb{N}$, define $\delta_i := \max\{|V(t_i, x_i) - V^*(t, x)|, |x_i - x|, |t_i - t|\}$. Extracting a subsequence, we can find a strictly decreasing sequence $(h_i)_{i\in\mathbb{N}}$ that converges to 0 such that

$h_i \in [\sqrt{\delta_i}, \bar{h}\,]$.

For any $i \in \mathbb{N}$ we define $w_i := \frac{1}{h_i}\int_{t_i}^{t_i+h_i} \dot{z}_i(s)\mathrm{d}s$, $e_i := 1 - \frac{t-t_i}{h_i}$ and $\tilde{w}_i := w_i - \frac{x-x_i}{h_i}$. One can easily prove that $w_i \xrightarrow[i\to+\infty]{} \tilde{v}$. Observe also that $(t+h_ie_i, x+h_i\tilde{w}_i) \in (S,T)\times \operatorname{int} A$ and $\lim_{i\to+\infty}(e_i,\tilde{w}_i) = (1,\tilde{v})$. Therefore recalling also that $(t,x) \in [S,T]\times \operatorname{int} A$, we deduce that

$$\begin{aligned}\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] &= \limsup_{i\to+\infty} h_i^{-1}\left[V^*(t_i+h_i, x_i+h_iw_i) - V^*(t,x)\right]\\ &\geq \limsup_{i\to+\infty} h_i^{-1}\left[V(t_i+h_i, x_i+h_iw_i) - V^*(t,x)\right].\end{aligned}$$

For all $i \in \mathbb{N}$, we have:

$$V(t_i+h_i, x_i+h_iw_i) - V^*(t,x) \geq V(t_i+h_i, x_i+h_iw_i) - V(t_i,x_i) - \delta_i. \tag{4.33}$$

We have arranged $z_i(\cdot)$ to be feasible for all $i \in \mathbb{N}$, so the principle of optimality yields:

$$V(t_i+h_i, x_i+h_iw_i) - V(t_i,x_i) \geq -\int_{t_i}^{t_i+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s, \text{ for all } i \in \mathbb{N}.$$

Hence, dividing across the inequality (3.33) by h_i, passing to the limit superior and recalling that $\frac{\delta_i}{h_i} \leq \sqrt{\delta_i}$, we deduce:

$$\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] \geq -\liminf_{i\to+\infty} \frac{1}{h_i}\int_{t_i}^{t_i+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s. \tag{4.34}$$

From (H5)* one can easily show that there exists $k_L > 0$ such that $v \mapsto L(s,z,v)$ is k_L-Lipschitz continuous on $c_0\mathbb{B}$ (uniformly w.r.t. s and z), and so, for every i, we have:

$$\begin{aligned}\int_{t_i}^{t_i+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s &\leq \int_{t_i}^{t_i+h_i} L(s, z_i(s), \tilde{v})\mathrm{d}s + \int_{t_i}^{t_i+h_i} k_L|\dot{z}_i(s) - \tilde{v}|\mathrm{d}s\\ &\leq h_i \sup_{\substack{|z-x|\leq c_0h_i+h_i^2\\ 0<s-t\leq h_i+h_i^2}} L(s,z,\tilde{v}) + h_i\, k_L(K\theta_i(h_i) + |v_i - \tilde{v}|),\end{aligned} \tag{4.35}$$

where θ_i is defined by

$$\theta_i(h) := \begin{cases}\sup\limits_{\substack{0<s-t_i\leq h\\ |x_i-y|\leq c_0h}} \mathrm{d}_H(F(s,y), F(t_i^+, x_i)), & \text{if } h \neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and satisfies $\theta_i(h_i) \xrightarrow[i\to+\infty]{} 0$ (*cf.* [15]). Dividing across by h_i in (4.35) and passing to the limit as $i \to +\infty$, it follows that:

$$\liminf_{i\to+\infty} \frac{1}{h_i}\int_{t_i}^{t+h_i} L(s, z_i(s), \dot{z}_i(s))\mathrm{d}s \leq L(t^+, x, \tilde{v}). \tag{4.36}$$

Combining (4.34) with (4.36) we obtain:

$$\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] \geq -L(t^+, x, \tilde{v}).$$

Now, take any $(\xi^0, \xi^1) \in \partial_+ V^*(t,x)$. Then,

$$V^*(t',x') - V^*(t,x) - \xi^0(t'-t) - \xi^1\cdot(x'-x) \;\leq\; \mathrm{o}(|(t'-t, x'-x)|) \quad \text{for all } (t',x') \tag{4.37}$$

where $\mathrm{o}(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ is a function such that $\mathrm{o}(\epsilon)/\epsilon\to 0$ as $\epsilon\downarrow 0$. Setting $(t',x')=(t+h_ie_i,x+h_i\tilde{w}_i)$ in (4.37), and dividing across by h_i, we have

$$h_i^{-1}(V^*(t+h_ie_i,x+h_i\tilde{w}_i)-V^*(t,x))-\xi^0e_i-\xi^1\cdot\tilde{w}_i\leq h_i^{-1}\mathrm{o}(h_i|(e_i,\tilde{w}_i)|).$$

From these relations, in the limit as $i\to\infty$, it follows that

$$-L(t^+,x,\tilde{v})-\xi^0-\xi^1\cdot\tilde{v}\ \leq\ 0.$$

This relation being valid for all $\tilde{v}\in F(t^+,x)$, we obtain (d)(ii).

Concerning condition (d)(iii), the first condition, $\liminf_{\{(t',x')\to(S,x),\,t'>S\}}V(t',x')=V(S,x)$, as in the proof of Theorem 4.4.1, can be easily deduced from the fact that V satisfies (b)(i).

It remains to prove that $V^*_{|[S,T]\times A}(T,\cdot)=g^*_{|A}(\cdot)$. Since $V_{|[S,T]\times A}(T,\cdot)=g_{|A}(\cdot)$, we immediately deduce that $(V_{|[S,T]\times A})^*(T,x)\geq(g_{|A})^*(x)$ for all $x\in A$. We prove that the converse inequality is also satisfied. Fix any $x\in A$. There exists a sequence $((t_i,x_i))_{i\in\mathbb{N}}$ in $[S,T]\times A\setminus\{(T,x)\}$ converging to (T,x) such that:

$$\lim_{i\to+\infty}V(t_i,x_i)=\limsup_{\{(t,y)\to(T,x),\,y\in A\}}V(t,y)=V^*_{|[S,T]\times A}(T,x).$$

Using $(CQ)_{FW}$, we know that, for every i, there exists a feasible F-trajectory $x_i(\cdot)$ on $[t_i,T]$ such that $x_i(t_i)=x_i$. By the principle of optimality:

$$V(t_i,x_i)-\int_{t_i}^{T}L(s,x_i(s),\dot{x}_i(s))\mathrm{d}s\leq V(T,x_i(T)),\ \text{for all } i\in\mathbb{N}.$$

Using condition (H5)*, we know that there exists a constant $M_0>0$ such that for every $i\in\mathbb{N}$:

$$V(t_i,x_i)-M_0|T-t_i|\leq V(T,x_i(T))=g(x_i(T)).$$

Using the fact $\lim_{i\to+\infty}x_i(T)=x$, passing to the limit as $i\to+\infty$, we obtain:

$$(V_{|[S,T]\times A})^*(T,x)\leq\limsup_{i\to+\infty}g(x_i(T))\leq\limsup_{\substack{A\\ y\to x}}g(y)=(g_{|A})^*(x).$$

Therefore V satisfies (d)(iii).

□

Since $\partial_PU(t,x)\subset\partial_-U(t,x)$, it is not difficult to see that the proof of (A) can be reduced to the analysis employed in the proof of Theorem 4.4.1. On the other hand, the proof of (B) requires a different construction of several sequences of arcs, taking into account the new conditions involved, and the notions of lower/upper semicontinuous envelopes.
So here we provide details of the proof of (B). Take any point $(\bar{t},\bar{x})\in[S,T]\times A$ and any feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t})=\bar{x}$. Since $((g_{|A})^*)_*(x(T))=g_{|A}(x(T))$, there exists a sequence $(\xi_j)_{j\in\mathbb{N}}$ in A such that $\lim_{j\to+\infty}\xi_j=x(T)$ and

$$\lim_{j\to+\infty}(g_{|A})^*(\xi_j)=g_{|A}(x(T)).\tag{4.38}$$

Consider a decreasing sequence $(\bar{t}_j)_{j\in\mathbb{N}}$ in $]\bar{t},T]$ that converges to $\bar{t}$. Since we are assuming that the outward pointing constraint qualification is satisfied, we can employ the same construction in the proof of Theorem

4.4.1 obtaining, for each $j \geq 1$, the existence of a feasible F-trajectory $x_j(\cdot)$ on $[\bar{t}_j, T]$ such that $x_j(T) = \xi_j$, $x_j(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j, T[$ and:

$$\|x_j(\cdot) - x(\cdot)\|_{W^{1,1}([\bar{t}_j,T])} \leq \tilde{\theta}(\rho_j) + \rho_j,$$

where $\rho_j := \exp(k_F|T-S|)|\xi_j - x(T)|$ and $\tilde{\theta}(\rho) = K\rho(1 + |\ln(\rho)|)$ for some constant $K > 0$.
Fix any $j \in \mathbb{N}$. We can take an increasing sequence $(t_{j,i})_{i\geq 1}$ in $]\bar{t}_j, T[$, converging to T, and a sequence $(\omega_{j,i})_{i\geq 1}$ in $\operatorname{int} A$, converging to ξ_j such that:

$$\lim_{i\to+\infty} (U_{|[S,T]\times A})^*(t_{j,i}, \omega_{j,i}) = \liminf_{\{(t',x')\to(T,\xi_j)\,|\,t'<T, x'\in\operatorname{int} A\}} (U_{|[S,T]\times A})^*(t', x'). \tag{4.39}$$

Invoking the same argument above, now with the base points $(t_{j,i}, \omega_{j,i})$, for each integer $i \geq 1$, we can find a feasible F-trajectory $x_{j,i}(\cdot)$ on $[\bar{t}_j, t_{j,i}]$ such that $x_{j,i}(t_{j,i}) = \omega_{j,i}$, $x_{j,i}(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j, t_{j,i}]$ and:

$$\|x_{j,i}(\cdot) - x_j(\cdot)\|_{W^{1,1}([\bar{t}_j,t_{j,i}])} \leq \tilde{\theta}(\rho_{j,i}) + \rho_{j,i},$$

where $\rho_{j,i} := \exp(k_F|T-S|)|\xi_j - \omega_{j,i}|$. It is not difficult to see that, using Filippov's Existence Theorem, each $x_{j,i}(\cdot)$ can be extended on $[\bar{t}_j, T]$ obtaining the estimate (we do not relabel)

$$\|x_{j,i}(\cdot) - x_j(\cdot)\|_{W^{1,1}([\bar{t}_j,T])} \leq \tilde{\theta}(\rho_{j,i}) + \rho_{j,i}.$$

Since $x_{j,i}(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j, t_{j,i}]$, there exist sequences $(\delta_{j,i})_{i\geq 1}$ and $(\tau_{j,i})_{i\geq 1}$ with $\tau_{j,i} \in]t_{j,i}, T[$ such that $\delta_{j,i} \downarrow 0$ and $\tau_{j,i} \uparrow T$ as $i \to +\infty$, and

$$x_{j,i}(s) + \delta_{j,i}\mathbb{B} \subset \operatorname{int} A, \text{ for all } s \in [\bar{t}_j, \tau_{j,i}].$$

Let $f : [S,T] \times \mathbb{R}^n \times \mathbb{B} \to \mathbb{R}^n$, be the Carathéodory parametrization (see Proposition 3.27). Using Filippov's Selection Theorem, for each $i \geq 1$, there exists a measurable selection $u_{j,i} : [\bar{t}_j, T] \to \mathbb{B}$ such that $\dot{x}_{j,i}(t) = f(t, x_{j,i}(t), u_{j,i}(t))$ for a.e. $t \in [\bar{t}_j, T]$.
Fix any $i \geq 1$ and any $j \geq 1$. Take $k \in \mathbb{N}$ such that $k \geq \frac{2}{\delta_{j,i}}$. Invoking a now familiar argument based on Lusin's Theorem, we can find a continuous control $u^k_{j,i} : [\bar{t}_j, T] \to \mathbb{B}$ and an arc $x^k_{j,i} : [\bar{t}_j, t_{j,i}] \to \mathbb{R}^n$ such that:

$$\begin{cases} \dot{x}^k_{j,i}(t) = f(t, x^k_{j,i}(t), u^k_{j,i}(t)), \text{ for a.e. } t \in [\bar{t}_j, T]\,, \\ \|x_{j,i} - x^k_{j,i}\|_{\mathbb{L}^\infty([\bar{t}_j,T])} \leq \frac{1}{k} \text{ and } x^k_{j,i}(t_{j,i}) = x_{j,i}(t_{j,i}) = \omega_{j,i}\,, \\ \operatorname{meas}\Big(\{t \in [\bar{t}_j, T]\,|\,u_{j,i}(t) - u^k_{j,i}(t) \neq 0\}\Big) \leq \frac{1}{k}. \end{cases}$$

For every $(t,x) \in [\bar{t}_j, T] \times \mathbb{R}^n$, we write $v^+(t,x) := f(t^+, x, u^k_{j,i}(t))$, $v^-(t,x) := f(t^-, x, u^k_{j,i}(t))$.
We then define the multivalued functions $F^k_{j,i} : [\bar{t}_j, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $\Lambda^k_{j,i} : [\bar{t}_j, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}$ and $\Gamma^k_{j,i} : [\bar{t}_j, T\,] \times \mathbb{R}^n \times \mathbb{R} \rightsquigarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ by the relations:

$$F^k_{j,i}(t,x) := \operatorname{co}\{v^-(t,x), v^+(t,x)\},$$

$$\Lambda^k_{j,i}(t,x) := -\operatorname{co}\{L(t^-, x, v^-(t,x)), L(t^+, x, v^+(t,x))\},$$

$$\Gamma^k_{j,i}(t,x,\ell) := \begin{cases} \{1\} \times F^k_{j,i}(t,x) \times \Lambda^k_{j,i}(t,x), & \text{if } t \in [\bar{t}_j, t_{j,i}\,), \\ \operatorname{co}\Big\{(0,0,0) \cup \{1\} \times F^k_{j,i}(t_{j,i}, x) \times \Lambda^k_{j,i}(t_{j,i}, x)\Big\}, & \text{if } t = t_{j,i}. \end{cases}$$

Consider the arc $(\tau^k_{j,i}, x^k_{j,i}, \ell^k_{j,i})$ on $[\bar{t}_j, T]$ defined by

$$t \mapsto (\tau^k_{j,i}(t), x^k_{j,i}(t), \ell^k_{j,i}(t)) := \left(t, x^k_{j,i}(t), (U_{|[S,T]\times A})^*(\bar{t}_j, x^k_{j,i}(\bar{t}_j)) - \int_{\bar{t}_j}^t L(s, x^k_{j,i}(s), \dot{x}^k_{j,i}(s))\mathrm{d}s\right).$$

Observe that $[\bar{t}_j, t_{j,i}] \subset]S,T[$ and

$$x^k_{j,i}(t) + \frac{\delta_{j,i}}{2}\mathbb{B} \subset x_{j,i}(t) + \delta_{j,i}\mathbb{B} \subset \operatorname{int} A, \text{ for all } t \in [\bar{t}_j, t_{j,i}].$$

Now, our aim is to apply the Local Weak Invariance (Theorem 2.6.2) to the constrained differential inclusion:

$$\begin{cases} (\dot{\tau}(t), \dot{y}(t), \dot{\ell}(t)) \in \Gamma^k_{j,i}(\tau(t), y(t), \ell(t)), \text{ for a.e. } t \in [\bar{t}_j, t_{j,i}], \\ \tau(\bar{t}_j) = \bar{t}_j, y(\bar{t}_j) = x^k_{j,i}(\bar{t}_j), \ell(\bar{t}_j) = (U_{|[S,T]\times A})^*(\bar{t}_j, x^k_{j,i}(\bar{t}_j)), \\ (\tau(t), y(t), \ell(t)) \in \operatorname{hyp}(U_{|[S,T]\times A})^*, \text{ for all } t \in [\bar{t}_j, t_{j,i}], \end{cases} \tag{4.40}$$

with the reference arc $(\tau^k_{j,i}, x^k_{j,i}, \ell^k_{j,i})$. We discuss here only the validity of the inward pointing condition (iv) of Theorem 2.6.2 since it is easy to see that, as in the proof of Theorem 4.4.1, all the assumptions are satisfied. Observe that along a '$\frac{\delta_{j,i}}{2}$-tube' around the arc $(\tau^k_{j,i}, x^k_{j,i}, \ell^k_{j,i})$ we have $(U_{|[S,T]\times A})^* = U^*$. Therefore the required inward pointing condition to check is: for any $(\tau, x, \ell) \in \operatorname{hyp} U^* \cap ((\tau^k_{j,i}, x^k_{j,i}, \ell^k_{j,i})([\bar{t}_j, t_{j,i}]) + \frac{\delta_{j,i}}{2}\mathbb{B})$ we have

$$\min_{w \in \Gamma^k_{j,i}(\tau,x,\ell)} w \cdot (-\xi^0, -\xi^1, \lambda) \leq 0, \text{ for all } (-\xi^0, -\xi^1, \lambda) \in N^P_{\operatorname{hyp} U^*}(\tau, x, \ell). \tag{4.41}$$

Take any $(\tau, x, \ell) \in \operatorname{hyp} U^* \cap ((\tau^k_{j,i}, x^k_{j,i}, \ell^k_{j,i})([\bar{t}_j, t_{j,i}]) + \frac{\delta_{j,i}}{2}\mathbb{B})$ and any $(-\xi^0, -\xi^1, \lambda) \in N^P_{\operatorname{hyp} U^*}(\tau, x, \ell)$ $(\subset \hat{N}_{\operatorname{hyp} U^*}(\tau, x, \ell))$. We have $\lambda \geq 0$. Since (4.41) is clearly verified when $\tau = t_{j,i}$, we assume that $\tau \in [\bar{t}_j, t_{j,i}[$. We can also restrict attention to the case $\ell = U^*(\tau, x)$.
Depending on the value of λ we can consider two distinct cases.

If $\lambda > 0$, then $\lambda^{-1}(\xi^0, \xi^1) \in \partial_+ U^*(\tau, x)$ and so, from condition (d)(ii), we deduce that

$$\xi^0 + \xi^1 \cdot v^+(\tau, x) + \lambda L(\tau^+, x, v^+(\tau, x)) \geq 0.$$

This confirms (4.41) with $w = (1, v^+(\tau, x), -L(\tau^+, x, v^+(\tau, x)))$.

If $\lambda = 0$, then invoking Theorem 2.4.15, there exist sequences $(\xi^0_m, \xi^1_m) \xrightarrow[m\to+\infty]{} (\xi^0, \xi^1)$, $\lambda_m \downarrow 0$ and $(s_m, x_m) \xrightarrow[m\to+\infty]{} (\tau, x)$ such that, for each m,

$$(\lambda_m^{-1}\xi^0_m, \lambda_m^{-1}\xi^1_m) \in \partial_+ U^*(s_m, x_m)\,.$$

By (d)(ii), we obtain

$$\xi^0_m + \xi^1_m \cdot v^+(s_m, x_m) + \lambda_m L((s_m)^+, x_m, v^+(s_m, x_m)) \geq 0.$$

By extracting suitable subsequences and arguing as in the proof of Theorem 4.4.1 we deduce that (4.41) is verified.

Observe that the last condition in (4.40) means that $\ell^k_{j,i}(t) \leq (U_{|[S,T]\times A})^*(\tau^k_{j,i}(t), x^k_{j,i}(t))$, for all $t \in [\bar{t}_j, t_{j,i}]$. It follows that at $t = t_{j,i}$:

$$(U_{|[S,T]\times A})^*(\bar{t}_j, x^k_{j,i}(\bar{t}_j)) - \int_{\bar{t}_j}^{t_{j,i}} L(s, x^k_{j,i}(s), \dot{x}^k_{j,i}(s))\mathrm{d}s \leq (U_{|[S,T]\times A})^*(t_{j,i}, \omega_{j,i}), \text{ for every } k, i \text{ and } j.$$

Applying Lebesgue's dominated convergence, taking a subsequence if necessary and without relabeling, in the limit as $k \to +\infty$ we obtain

$$U(\bar{t}_j, x_{j,i}(\bar{t}_j)) \le \liminf_{k\to+\infty} U(\bar{t}_j, x^k_{j,i}(\bar{t}_j)) \le \liminf_{k\to+\infty} (U_{|[S,T]\times A})^*(\bar{t}_j, x^k_{j,i}(\bar{t}_j))$$
$$\le (U_{|[S,T]\times A})^*(t_{j,i}, \omega_{j,i}) + \int_{\bar{t}_j}^{t_{j,i}} L(s, x_{j,i}(s), \dot{x}_{j,i}(s))\mathrm{d}s, \text{ for every } i \text{ and } j.$$

Then, taking the limit as $i \to +\infty$, we have

$$\begin{aligned} U(\bar{t}_j, x_j(\bar{t}_j)) &\le \liminf_{i\to+\infty} U(\bar{t}_j, x_{j,i}(\bar{t}_j)) \\ &\le \lim_{i\to+\infty} U^*_{|[S,T]\times A}(t_{j,i}, \omega_{j,i}) + \int_{\bar{t}_j}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s, \text{ for every } j \ge 1. \end{aligned} \tag{4.42}$$

From (4.39) and the upper semicontinuity of $(U_{|[S,T]\times A})^*$ it follows that

$$\liminf_{\{(t',x')\to(T,\xi_j)\,|\,t'<T, x'\in\mathrm{int}\,A\}} (U_{|[S,T]\times A})^*(t',x') \le \limsup_{\{(t',x')\to(T,\xi_j), x'\in A\}} (U_{|[S,T]\times A})^*(t',x') \le (U_{|[S,T]\times A})^*(T,\xi_j).$$

Using these relations and the fact that $(U_{|[S,T]\times A})^*(T,\cdot) = (g_{|A})^*(\cdot)$ (from (4.42)), we deduce that

$$U(\bar{t}_j, x_j(\bar{t}_j)) \le (g_{|A})^*(\xi_j) + \int_{\bar{t}_j}^{T} L(s, x_j(s), \dot{x}_j(s))\mathrm{d}s, \text{ for every } j \ge 1.$$

Recalling that the sequence $(\xi_j)_{j\in\mathbb{N}}$ satisfies (4.38) and that U is lower semicontinuous, passing to the limit as $j \to +\infty$ we obtain:

$$U(\bar{t}, x(\bar{t})) \le \liminf_{j\to+\infty} U(\bar{t}_j, x_j(\bar{t}_j)) \le g_{|A}(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s,$$

which concludes the proof since $x(\cdot)$ was an arbitrary feasible F-trajectory such that $x(\bar{t}) = \bar{t}$.

□

4.10 Proof of Theorem 4.4.4

We observe that, if the inward pointing constraint qualification is in force and g is continuous on A, we obtain that $V_{|[S,T]\times A}$ is continuous too, and so, in view of Theorem 4.4.2 and Proposition 4.9.1 we obtain that (a), (b) and (c) are equivalent and '(a) $\Rightarrow$ (d)'' ((d)$'$ is the simplified version of (d) which appears in Theorem 4.4.4 statement). As a consequence, we can restrict attention to the implication '(d)$'$ $\Rightarrow$ (a)'. To this aim, we assume (H1)-(H7) and take an extended valued function $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ such that $U_{|[S,T]\times A}$ is continuous and $U(t,x) = +\infty$ when $x \notin A$, and employing a consolidated approach we divide the proof of the desired property into two steps:

(A)$'$ if U satisfies (d)$'$(i) and $U(T,x) = g(x)$ for all $x \in A$, then $V(t,x) \le U(t,x)$ for any $(t,x) \in [S,T]\times A$.

(B)$'$ if, in addition, (H8) and (OPC) hold, and U satisfies (d)$'$(ii) and (d)$'$(iii), then $V(t,x) \ge U(t,x)$ for any $(t,x) \in [S,T]\times A$.

Since (A)$'$ can be easily deduced as in step (A) of the proof of Theorem 4.4.3, consider here only (B)$'$. Take $(\bar{t},\bar{x}) \in [S,T] \times A$, and a feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$. With the help of property $(CQ)_{FW}$ (with $\theta(\rho) = K\rho(1 + |\ln(\rho)|)$), we can find a sequence of feasible F-trajectories $(x_j)_{j\geq 1}$ on $[\bar{t},T]$ such that, for every $j \geq 1$, $x_j(\bar{t}) = \bar{x}$, $x_j(t) \in \operatorname{int} A$ for all $t \in]\bar{t},T]$, and:

$$\|x - x_j\|_{W^{1,1}([\bar{t},T])} \leq \frac{1}{j}\ . \tag{4.43}$$

Take a sequence $(t_j)_{j\geq 1}$ in $]\bar{t},T[$ such that $t_j \downarrow \bar{t}$. A now routine analysis reveals that, for each j, we can find a sequence of F-trajectories $(x_j^i)_{i\in\mathbb{N}}$ on $[t_j,T]$ such that

$$\|x_j - x_j^i\|_{W^{1,1}([t_j,T])} \leq \frac{1}{i} \text{ and } x_j^i(t_j) = x_j(t_j), \tag{4.44}$$

and the arc $(\tilde{\tau},\tilde{x},\tilde{\ell})$ on $[t_j,T]$ defined by

$$t \mapsto (\tilde{\tau},\tilde{x},\tilde{\ell})(t) := \left(t, x_j^i(t), U(t_j,x_j^i(t_j)) - \int_{t_j}^{t} L(s,x_j^i(s),\dot{x}_j^i(s))\mathrm{d}s\right),$$

is such that

$$\begin{cases} \tau(t_j) = t_j, y(t_j) = x_j^i(t_j), \ell(t_j) = U(t_j,x_j^i(t_j)), \\ (\tau(t),y(t),\ell(t)) \in \operatorname{hyp} U, \text{ for all } t \in [t_j,T]. \end{cases} \tag{4.45}$$

The last condition in (4.45) yields $\tilde{\ell}(t) \leq U(t,x_j^i(t))$, for all $t \in [t_j,T]$. It follows that for $t = T$:

$$U(t_j,x_j^i(t_j)) - \int_{t_j}^{T} L(s,x_j^i(s),\dot{x}_j^i(s))\mathrm{d}s \leq U(T,x_j^i(T)). \tag{4.46}$$

Bearing in mind (4.44), and passing to the limit in (4.46) as $i \to +\infty$, we obtain

$$U(t_j,x_j(t_j)) \leq U(T,x_j(T)) + \int_{t_j}^{T} L(s,x_j(s),\dot{x}_j(s))\mathrm{d}s, \text{ for every } j \geq 1.$$

Now invoking (4.43), the feasibility of $x_j(\cdot)$ and the continuity of U on $[S,T] \times A$, and condition (d)$'$(iii), we take the limit as $j \to +\infty$ in the previous inequality deducing:

$$U(\bar{t},\bar{x}) \leq \lim_{j\to+\infty} U(t_j,x_j(t_j)) \leq g(x(T)) + \int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s.$$

This shows that:

$$U(\bar{t},\bar{x}) \leq V(\bar{t},\bar{x}).$$

□

4.11 Proof of Theorem 4.6.1

The proof of Theorem 4.6.1 is inspired by techniques proposed in two papers: [22] and [29]. The first one provides linear $\mathbb{L}^\infty$ distance estimates for general closed sets A and differential inclusions with bounded time variation. On the other hand, [29] deals with convex compact sets A coupled with differential inclusions involving Lipschitz (time-independent) multifunctions F obtaining $\rho|\ln(\rho)|$-$W^{1,1}$ estimates. However, our proof gives a $\rho|\ln(\rho)|$-$W^{1,1}$ estimate for convex sets A with bounded time variation multifunction F, necessarily differs in many points from [22] and [29] and, for this reason, we provide it in detail, referring to

previous work only for some technical lemmas.

We start providing some technical lemmas, which we shall employ in our analysis. The first one (a proof of which can be found in [22, Lemma 5]) summarises implications of the inward pointing condition (IPC)′.

Lemma 4.11.1 *Suppose the multifunction $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and the closed set A satisfy hypotheses (H1)′, (H2), (H3), (H4)(i) and assumption (IPC)′ (for some $R_0 \geq 0$). Then there exist $M > 0$, $\epsilon > 0$, $\bar{\eta} > 0$ and a finite time set $\{\bar{\tau}_j\}_{j\in J} \subset [S,T]$ with the following property: for any $(t,x) \in [S,T]\times\big((\partial A+\bar{\eta}\mathbb{B})\cap R_0\mathbb{B}\cap A\big)$, there exists*

$$v \in \begin{cases} \operatorname{co} F(t,x) \cap M\mathbb{B} & \text{if } t \notin \{\bar{\tau}_j\}_{j\in J} \\ \operatorname{co} F(t^+,x) \cap M\mathbb{B} & \text{if } t \in \{\bar{\tau}_j\}_{j\in J}\ , \end{cases}$$

such that

$$y + [0,\epsilon](v + \epsilon\mathbb{B}) \ \subset \ A$$

for all $y \in (x + \epsilon\mathbb{B}) \cap A$.

The next lemma represents an useful 'hypotheses reduction' result, a proof of which can be obtained following the argument of [22, Lemma 4], the statement of which differs from Lemma4.11.2 below in three respects: it concerns merely $\mathbb{L}^\infty$-estimates, it reduces attention to the case in which '$F(t,x)$ is convex for all (t,x)', and condition (iv) of Lemma4.11.2 is not considered. We observe that the same analysis of the proof of [22, Lemma 4] can be employed here to derive $W^{1,1}$-estimates (which is what we need to prove Theorem 4.6.1), except the fact that, at this stage, we cannot reduce to the situation where F is convex valued. Condition (iv) of Lemma 4.11.2 does not appear in [22, Lemma 4], but can be easily deduced for multifunctions F satisfying (H1)′, (H2), (H3) and in particular the bounded time variation hypothesis (H4)(i) (cf. discussion on [21, 22]).

Lemma 4.11.2 *Assume that, for $\delta > 0$, $\bar{\rho} > 0$ and $\gamma > 0$ sufficiently small, the assertions of Theorem 4.6.1 are valid under hypotheses (H1)′, (H2), (H3), (H4) and (IPC), and when the following conditions are imposed on the reference F-trajectory $\hat{x}(\cdot) : [t_0,t_1] \to \mathbb{R}^n$, with $\hat{x}(t_0) \in A \cap \big(e^{\int_S^{t_0} c_F(s)\mathrm{d}s}(r_0+1) - 1\big)\mathbb{B}$, and the positive number $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$:*

(i): $\rho \ \leq \ \bar{\rho}$;

(ii): $t_1 - t_0 \leq \delta$;

(iii): $\eta(t_0) - \eta(t_1) \ \leq \ \gamma$;

(iv): there exist $c_0 > 0$ and $k_F > 0$ such that

$$F(t_0,x) = F(t_0^+,x) \quad \text{for all } x \in R_0\mathbb{B},$$

$$F(t,x) \ \subset \ c_0\ \mathbb{B} \quad \text{for all } (t,x) \in [S,T] \times R_0\mathbb{B}, \text{ and}$$

$$F(t,x') \ \subset \ F(t,x) + k_F|x - x'|\ \mathbb{B} \quad \text{for all } x, x' \in R_0\mathbb{B} \text{ and } t \in [S,T],$$

where $R_0 = e^{\int_S^T c_F(s)\mathrm{d}s}(r_0+1)$.

Then the assertions are valid under (H1)′, (H2), (H3), (H4) and (IPC) alone, and without conditions (i) – (iv).

The third lemma guarantees the existence of solutions to a differential inclusion $\dot{x} \in G(t,x)$, where the velocity set G is obtained modifying the reference multivalued function F intersecting it with a ball with a (t,x)-dependent radius and centred at $v(t)$ where v is a measurable selection of a perturbation of F evaluated along a given continuous arc ϕ. Lemma 4.11.3 below extends [29, Lemma 3] (which deals with time-independent Lipschitz continuous F) providing an existence result when bounded time variation multifunctions F are involved. It will be coupled with a 'convexification argument' which is based on a Lyapunov type Theorem. (We refer the reader to [36, Chapter 16] for a proof and for a discussion on Lyapunov type Theorems; the application of these results to derive distance estimates results was first suggested in [29].)

Lemma 4.11.3 *Consider a multifunction F satisfying (H1)$'$, (H2), (H3), (H4)(i) and condition (iv) of Lemma 4.11.2. Given an interval $[t_0,t_1] \subset [S,T]$ and measurable functions $w,\ \phi : [t_0,t_1] \to \mathbb{R}^n$, we consider a multifunction $G : [t_0,t_1] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ defined as follows:*

$$G(t,x) := \begin{cases} \overline{(w(t) + [2k_F|x-\phi(t)| + \alpha(t)]\mathrm{int}\,\mathbb{B}) \cap F(t,x)} & \text{if } |x-\phi(t)| + \alpha(t) > 0 \\ \{w(t)\} & \text{if } |x-\phi(t)| + \alpha(t) = 0, \end{cases} \tag{4.47}$$

where $\alpha : [t_0,t_1] \to \mathbb{R}_+$ is a measurable function such that $(w(t) + [2k_0|x-\phi(t)| + \alpha(t)]\mathrm{int}\,\mathbb{B}) \cap F(t,x) \neq \emptyset$ whenever $|x-\phi(t)| + \alpha(t) > 0$. Then

(i) G takes non-empty compact values in $c_0\mathbb{B}$ and is 'Scorza-Dragoni lower semicontinuous' on $[t_0,t_1]\times\mathbb{R}^n$, and

(ii) for every $x_0 \in \mathbb{R}^n$, the differential inclusion

$$\begin{cases} \dot{x}(t) \in G(t,x(t)), \text{ for all } t \in [t_0,t_1], \\ x(t_0) = x_0, \end{cases} \tag{4.48}$$

admits at least a solution.

Proof. Making use of an existence result by Bressan [28] on solutions to differential inclusions involving merely 'Scorza-Dragoni lower semicontinuous' multifunctions, property (ii) becomes an immediate consequence of (i). So we prove (i) recalling that the multifunction $G : [t_0,t_1] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is Scorza-Dragoni lower semicontinuous if for every $\varepsilon > 0$ there exists a compact set $J_\varepsilon \subset [t_0,t_1]$ with meas $(J_\varepsilon) > t_1 - t_0 - \varepsilon$ such that G is lower semicontinuous when restricted to $J_\varepsilon \times \mathbb{R}^n$.
Fix $\varepsilon > 0$. From Lusin's Theorem, we can find a compact set $J_\varepsilon \subset [t_0,t_1]$ such that meas$(J_\varepsilon) > t_1 - t_0 - \varepsilon$ and such that $w(\cdot)$, $\phi(\cdot)$, $\alpha(\cdot)$ and $\eta(\cdot)$ are continuous on J_ε. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. Our aim is to show that the set $G^{-1}(\mathcal{O}) \cap (J_\varepsilon \times \mathbb{R}^n) = \{(t,x) \in J_\varepsilon \times \mathbb{R}^n \mid G(t,x) \cap \mathcal{O} \neq \emptyset\}$ is open in $J_\varepsilon \times \mathbb{R}^n$. Take any $(\bar{t},\bar{x}) \in G^{-1}(\mathcal{O}) \cap (J_\varepsilon \times \mathbb{R}^n)$. Then there exists $\bar{w} \in G(\bar{t},\bar{x}) \cap \mathcal{O}$. We distinguish two situations.

(a): $|\bar{x} - \phi(\bar{t})| + \alpha(\bar{t}) = 0$. It means that $\bar{w} = w(\bar{t}) \in \mathcal{O}$. Since $\mathcal{O}$ is open, there exist $r > 0$ and $\bar{\delta} \in (0, \frac{r}{8k_F})$ such that $w(\bar{t}) + r\mathbb{B} \subset \mathcal{O}$ and

$$|\eta(t) - \eta(\bar{t})| \leq \frac{r}{8}, \quad |w(t) - w(\bar{t})| \leq \frac{r}{8}, \quad |\alpha(t) - \alpha(\bar{t})| \leq \frac{r}{8}, \quad |\phi(t) - \phi(\bar{t})| \leq \frac{r}{8k_F}, \tag{4.49}$$

for all $t \in J_\varepsilon$ with $|\bar{t} - t| \leq \bar{\delta}$. Then it is easy to see that for all $(t,x) \in J_\varepsilon \times \mathbb{R}^n$ with $|\bar{t} - t| \leq \bar{\delta}$ and $|\bar{x} - x| \leq \bar{\delta}$, we obtain that

$$G(t,x) \subset w(\bar{t}) + \frac{3}{4} r\ \mathbb{B} \subset \mathcal{O}\ .$$

(b): $|\bar{x} - \phi(\bar{t})| + \alpha(\bar{t}) > 0$. In this case we observe that $\bar{w} \in F(\bar{t}, \bar{x})$ is such that it belongs also to the open set $(w(\bar{t}) + [2k_F|\bar{x} - \phi(\bar{t})| + \alpha(\bar{t})]\text{int}\,\mathbb{B}) \cap \mathcal{O}$. Then, again we can find $r > 0$ and $\bar{\delta} \in (0, \frac{r}{8k_F})$ such that (4.49) is satisfied and

$$\bar{w} + 2r\mathbb{B} \subset (w(\bar{t}) + [2k_F|\bar{x} - \phi(\bar{t})| + \alpha(\bar{t})]\text{int}\,\mathbb{B}) \cap \mathcal{O}.$$

It follows that for all $(t, x) \in J_\varepsilon \times \mathbb{R}^n$ with $|\bar{t} - t| \leq \bar{\delta}$ and $|\bar{x} - x| \leq \bar{\delta}$, we have

$$(\bar{w} + r\mathbb{B}) \cap F(t, x) \neq \emptyset \quad \text{and} \quad \bar{w} + r\mathbb{B} \subset (w(t) + [2k_F|x - \phi(t)| + \alpha(t)]\text{int}\,\mathbb{B}) \cap \mathcal{O}.$$

Then in both situations $(\bar{t}, \bar{x})$ belongs to the relative interior of $G^{-1}(\mathcal{O}) \cap (J_\varepsilon \times \mathbb{R}^n)$.

□

Proof of Theorem 4.6.1 Fix $r_0 > 0$. Assume that the multifunction F and the set A in the theorem statement satisfy (H1)′, (H2), (H3), (H4) and (IPC) with functions $c_F \in \mathbb{L}^1(S, T)$ and $k_F \in \mathbb{L}^\infty(S, T)$, for $R_0 := e^{\int_S^T c(s)\mathrm{d}s}(1 + r_0)$. (Observe that the constant R_0 bounds the magnitude of the F-trajectories $x(\cdot)$ on subintervals of $[S, T]$ originating in $r_0\mathbb{B}$.)
Let $\eta(\cdot)$ be modulus of variation appearing in (H4). Recall that Lemma 4.11.1 establishes the existence of constants $M > 0$, $\epsilon > 0$, $\bar{\eta} > 0$ and a finite time set $\{\bar{\tau}_j\}_{j\in J} \subset [S, T]$ such that, given any $(t, x) \in [S, T] \times \big((\partial A + \bar{\eta}\mathbb{B}) \cap R_0\mathbb{B} \cap A\big)$, we can find

$$v \in \begin{cases} \text{co}\,F(t, x) \cap M\mathbb{B} & \text{if } t \notin \{\bar{\tau}_j\}_{j\in J} \\ \\ \text{co}\,F(t^+, x) \cap M\mathbb{B} & \text{if } t \in \{\bar{\tau}_j\}_{j\in J}\,, \end{cases}$$

with the following property:

$$x' + [0, \epsilon](v + \epsilon\mathbb{B}) \ \subset \ A \tag{4.50}$$

for all $x' \in (x + \epsilon\mathbb{B}) \cap A$.

To validate the theorem statement we will show that there exists a constant $K > 0$ such that, for any interval $[t_0, t_1] \subset [S, T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{\int_S^{t_0} c_F(s)\mathrm{d}s}(r_0 + 1) - 1\big)\mathbb{B}$, and any $\rho > 0$ satisfying $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0, t_1]\}$, we can find a feasible F-trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \ \in \ \text{int}\,A \quad \text{for all } t \in (t_0, t_1] \tag{4.51}$$

$$\|\hat{x} - x\|_{\mathbb{L}^\infty([t_0,t_1])} \ \leq \ K\,\rho\,, \tag{4.52}$$

and

$$\|\dot{\hat{x}} - \dot{x}\|_{\mathbb{L}^1([t_0,t_1])} \ \leq \ K\,\rho(1 + |\ln(\rho)|)\,. \tag{4.53}$$

Lemma 4.11.2 allows us to restrict attention, without loss of generality, to the case when

(i): $\rho \leq \bar{\rho}$;

(ii): $t_1 - t_0 \leq \delta$ and, if $\bar{\tau}_j \in [t_0, t_1]$ for some $j \in J$, then either $\bar{\tau}_j = t_0$ or $\bar{\tau}_j = t_1$;

(iii): $\eta(t_1) - \eta(t_0) \leq \gamma$;

(iv):

$$F(t_0,x) = F(t_0^+,x) \quad \text{for all } x \in R_0\mathbb{B}\,,$$

$$F(t,x) \,\subset\, c_0\,\mathbb{B} \quad \text{for all } (t,x) \in [S,T]\times R_0\mathbb{B}\,,\ \text{and}$$

$$F(t,x') \,\subset\, F(t,x) + k_F|x-x'|\,\mathbb{B} \quad \text{for all } \ x,x' \in R_0\mathbb{B} \ \text{and } t\in[S,T]$$

for $\delta>0$, $\bar\rho>0$ and $\gamma>0$ sufficiently small, and $c_0>0$ and $k_F>0$ sufficiently large. Since we can assume that $F(t_0,x)=F(t_0^+,x)$ for all $x\in R_0\mathbb{B}$, we can also replace $\eta(\cdot)$ by the non-decreasing bounded variation function $\eta_0:[S,T]\to[0,\infty)$ defined by the relation $\eta_0(t) := \left\{\begin{array}{ll} 0 & \text{if} \quad t=t_0 \\ \eta(t)-\eta(t_0^+) & \text{if} \quad t\in\,]t_0,t_1]\end{array}\right.$.
Observe that, increasing the size of $c_0>0$ and, then, reducing the size of $\delta>0$, $\bar\rho>0$ and $\gamma>0$, if necessary, we can also ensure that

$$c_0 \geq M(\geq 1), \quad \epsilon<1, \quad \delta\leq\epsilon, \quad k_Fc_0\delta+\gamma\leq\frac{\epsilon}{16}, \quad \bar\rho+2c_0\delta<\epsilon, \quad \bar\rho<\frac{\delta\epsilon}{4}, \quad \bar\rho\leq\bar\eta, \tag{4.54}$$

and

$$4\delta c_0\leq\bar\eta, \quad e^{2k_F\delta}\Big[1/4+3k_F(1+4c_0/\epsilon)\delta+\delta/4+4\mu/\epsilon+(\bar\rho/\epsilon)\times|\ln(4\bar\rho/\epsilon)|\Big]\leq\frac{3}{4}\,. \tag{4.55}$$

Here $\mu>0$ is the number appearing in assumption (H4)(ii).
Let $\{t_0=:\tilde t_0<\tilde t_1<\tilde t_2<\ldots<\tilde t_M:=t_1\}$ be a partition of $[t_0,t_1]$ such that for each $k=0,1,\ldots,M-1$ we have

$$\lim_{\epsilon\downarrow0}\int_{\tilde t_k+\epsilon}^{\tilde t_{k+1}}\frac{\eta(\tau)-\eta(\tilde t_k^+)}{\tau-\tilde t_k}d\tau\leq\mu\ .$$

Observe that, employing a standard argument which allows to concatenate a finite number of intervals, it is not restrictive to assume the case when $\tilde t_0=t_0$ and $\tilde t_1=t_1$.
Set $k:=\frac{4}{\epsilon}$.
Notice that, if $\hat x(t_0)\in\big(A\cap\big(e^{\int_S^{t_0}c_F(s)\mathrm{d}s}(r_0+1)-1\big)\mathbb{B}\big)\setminus(\partial A+\frac{\bar\eta}{2}\mathbb{B})$, then from the first condition in (4.55) and (ii) above we deduce that $x(\cdot)=\hat x(\cdot)$ is a feasible F-trajectory having the required properties. Therefore, it is not restrictive to impose also that $\hat x(t_0)\in(\partial A+\frac{\bar\eta}{2}\mathbb{B})\cap A\cap\big(e^{\int_S^{t_0}c_F(s)\mathrm{d}s}(r_0+1)-1\big)\mathbb{B}$. From the definition of R_0 is follows that

$$\hat x(t_0)\in(\partial A+\frac{\bar\eta}{2}\mathbb{B})\cap R_0\mathbb{B}\cap A.$$

Owing to Lemma 4.11.1 and (iv) above we can take a vector $v\in\operatorname{co}F(t_0,\hat x(t_0))=\operatorname{co}F(t_0^+,\hat x(t_0))$ such that $|v|\leq c_0$ and property (4.50) holds for $(t,x)=(t_0,\hat x(t_0))$. Then, we consider the arc $y(\cdot):[t_0,t_1]\to\mathbb{R}^n$ such that $y(t_0)=\hat x(t_0)$ and

$$\dot y(t) \ = \ \left\{\begin{array}{ll} v & \text{if } t\in[t_0,(t_0+k\rho)\wedge t_1] \\ \frac{k\rho}{t-t_0}\frac{\hat x(t)-\hat x(t_0)}{t-t_0}+\Big(1-\frac{k\rho}{t-t_0}\Big)\dot{\hat x}(t) & \text{if } t\in\,]t_0+k\rho,t_1]\,.\end{array}\right. \tag{4.56}$$

Observe that

$$y(t) \ = \ \left\{\begin{array}{ll} \hat x(t_0)+(t-t_0)v & \text{if } t\in[t_0,(t_0+k\rho)\wedge t_1] \\ \hat x(t_0)+k\rho v+\Big(1-\frac{k\rho}{t-t_0}\Big)\,(\hat x(t)-\hat x(t_0)) & \text{if } t\in\,]t_0+k\rho,t_1]\,.\end{array}\right. \tag{4.57}$$

Therefore, for all $t\in(t_0,(t_0+k\rho)\wedge t_1]$, since $t-t_0\leq\delta\leq\epsilon$, it follows from (4.50) that

$$y(t)+(t-t_0)\epsilon\mathbb{B} \ = \ \hat x(t_0)+(t-t_0)(v+\epsilon\mathbb{B}) \ \subset \ A\,.$$

Moreover, whenever $t_0 + k\rho < t_1$, for all $t \in [t_0 + k\rho, t_1]$, from (4.57) we deduce that

$$|y(t) - \hat{x}(t)| \leq c_0 k\rho + k\rho \frac{\hat{x}(t) - \hat{x}(t_0)}{t - t_0} \leq 2c_0 k\rho, \tag{4.58}$$

and

$$y(t) = \frac{k\rho}{t - t_0}\hat{x}(t_0) + \left(1 - \frac{k\rho}{t - t_0}\right)\hat{x}(t) + k\rho v \in A + k\rho v + \hat{\rho}\mathbb{B}, \tag{4.59}$$

where $\hat{\rho} := \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0, t_1]\}$. Write $\tilde{y}(t)$ the projection on A of $t \mapsto y(t)$. As a consequence, from (4.50), (4.54) and (4.59) and bearing in mind that $\rho \geq \hat{\rho}$ and that $k = 4/\epsilon$, for all $t \in [t_0 + k\rho, t_1]$ we have

$$y(t) + 2\rho\mathbb{B} \;\subset\; \tilde{y}(t) + \frac{4\rho}{\epsilon}\left(v + \frac{3}{4}\epsilon\mathbb{B}\right) \;\subset\; \tilde{y}(t) + \epsilon(v + \epsilon\mathbb{B}) \;\subset\; A\,. \tag{4.60}$$

From (4.56) and condition (iv) it follows that (it is not restrictive to suppose that $t_0 + k\rho < t_1$):

$$\begin{aligned}
\|\dot{y} - \dot{\hat{x}}\|_{\mathbb{L}^1(t_0,t_1)} &\leq \int_{t_0}^{t_0+k\rho} |\dot{y}(t) - \dot{\hat{x}}(t)|\mathrm{d}t + \int_{t_0+k\rho}^{t_1} \left(\frac{k\rho}{t - t_0}\frac{\hat{x}(t) - \hat{x}(t_0)}{t - t_0} + \frac{k\rho}{t - t_0}|\dot{\hat{x}}(t)|\right)\mathrm{d}t \\
&\leq 2c_0 k\rho + 2c_0 k\rho \int_{t_0+k\rho}^{t_1} \frac{1}{t - t_0}\mathrm{d}t \\
&\leq K_0\rho \times (1 + |\ln\rho|),
\end{aligned} \tag{4.61}$$

where $K_0 := 2c_0 k(1 + \ln k)$. On the other hand, from (4.57) and (4.58) we easily arrive at

$$\|y - \hat{x}\|_{\mathbb{L}^\infty([t_0,t_1])} \leq c_0 k\rho \leq K_0\rho. \tag{4.62}$$

Let $\{\tau_k\}_{k\geq 1}$ be a strictly decreasing sequence such that $\tau_1 := (t_0 + k\rho) \wedge t_1$, $\lim_{k\to+\infty} \tau_k = t_0$ and

$$\tau_k - \tau_{k+1} < \frac{\tau_k - t_0}{4c_0}\epsilon. \tag{4.63}$$

Since $v \in \operatorname{co} F(t_0, \hat{x}(t_0))$, owing to Carathéodory's Theorem, we can find vectors $v_1, \ldots, v_{n+1} \in F(t_0, \hat{x}(t_0))$ and numbers $\lambda_1, \ldots, \lambda_{n+1} \in [0, 1]$ such that $\sum_{i=1}^{n+1} \lambda_i = 1$ and $v = \sum_{i=1}^{n+1} \lambda_i v_i$. For each integer $k \geq 1$, we consider a partition of the interval $[\tau_{k+1}, \tau_k](\subset [t_0, (t_0 + k\rho) \wedge t_1])$ into $n + 1$ subintervals $I_{k,i}$ such that $\mathrm{meas}(I_{k,i}) = \lambda_i(\tau_{k+1} - \tau_k)$, for all $i = 1, \ldots, n + 1$. Set $I_i := \bigcup_{k=1}^{+\infty} I_{k,i}$ (which is a measurable set) and $w(t) := \sum_{i=1}^{n+1} v_i \chi_{I_i}(t)$($\chi_{I_i}(\cdot)$ is the indicator function of I_i.) Let $z(\cdot)$ be a solution to the following differential equation

$$\begin{cases} \dot{z}(t) = w(t) \quad \text{a.e. } t \in [t_0, \tau_1], \\ z(t_0) = \hat{x}(t_0). \end{cases}$$

Consider the multivalued function $G_1 : [t_0, \tau_1] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ defined as

$$G_1(t, x) := \begin{cases} \overline{(w(t) + [2k_F|x - \hat{x}(t_0)| + \eta_0(t)]\mathrm{int}\,\mathbb{B}) \cap F(t, x)} & \text{if } |x - \hat{x}(t_0)| + \eta_0(t) > 0 \\ \{w(t)\} & \text{if } |x - \hat{x}(t_0)| + \eta_0(t) = 0. \end{cases} \tag{4.64}$$

and the associated differential inclusion

$$\begin{cases} \dot{x}(t) \in G_1(t, x(t)) \quad \text{for a.e. } t \in [t_0, \tau_1], \\ x(t_0) = \hat{x}(t_0). \end{cases} \tag{4.65}$$

Since $w(\cdot)$ is measurable and $t \mapsto \eta_0(t)$ is a measurable (increasing) function such that $(w(t) + [2k_F|x - \hat{x}(t_0)| + \eta_0(t)]\mathrm{int}\,\mathbb{B}) \cap F(t, x) \neq \emptyset$, when $|x - \hat{x}(t_0)| + \eta_0(t) > 0$, we can apply Lemma 4.11.3 (with $\phi = \hat{x}(t_0)$

and $\alpha(\cdot) = \eta_0(\cdot)$), obtaining the existence of a solution $x(\cdot)$ to (4.65). Observe that for all $t \in [t_0, \tau_1]$ we have:

$$|\dot{x}(t) - \dot{z}(t)| \leq 2k_F|x(t) - x_0| + \eta_0(t) \leq 2k_F c_0(t - t_0) + \eta_0(t).$$

From Gronwall's Inequality and (4.54) it follows that, for all $t \in [t_0, \tau_1]$,

$$\begin{aligned} |x(t) - z(t)| &\leq k_F c_0(t - t_0)^2 + \int_{t_0}^{t} \eta_0(s)\mathrm{d}s \leq k_F c_0(t - t_0)^2 + \gamma(t - t_0) \\ &\leq (t - t_0)[k_F c_0\delta + \gamma] \leq \frac{\epsilon}{16}k\rho \ (= \frac{\rho}{4}). \end{aligned} \tag{4.66}$$

Recalling that, for all $k \geq 1$, we have $z(\tau_k) = y(\tau_k)$, using (4.63) we deduce that , for all $t \in [\tau_{k+1}, \tau_k]$,

$$|z(t) - y(t)| \leq 2c_0|\tau_k - \tau_{k+1}| \leq \frac{\epsilon}{2}(\tau_k - t_0) \leq \frac{\epsilon}{16}k\rho \ (= \frac{\rho}{4}). \tag{4.67}$$

Moreover, we also obtain that $z(\tau_k) = \hat{x}(t_0) + (\tau_k - t_0)v$ and, bearing in mind (4.66) and (4.63), that for all $t \in [\tau_{k+1}, \tau_k]$

$$\begin{aligned} x(t) \in x(\tau_k) + c_0|\tau_k - \tau_{k+1}|\mathbb{B} \ &\subset \ z(\tau_k) + (\tau_k - t_0)[k_F c_0\delta + \gamma]\mathbb{B} + c_0|\tau_k - \tau_{k+1}|\mathbb{B} \\ &\subset \hat{x}(t_0) + (\tau_k - t_0)(v + [k_F c_0\delta + \gamma + \epsilon/4]\mathbb{B}) \\ &\subset \hat{x}(t_0) + \epsilon(v + \frac{1}{2}\epsilon\mathbb{B}). \end{aligned}$$

As a consequence, owing to (4.50), it follows that

$$x(t) \in \operatorname{int} A \text{ for all } t \in (t_0, t_0 + k\rho]. \tag{4.68}$$

Recalling (4.62), (4.66) and (4.67), we obtain:

$$\begin{aligned} \|x - \hat{x}\|_{\mathbb{L}^\infty([t_0,\tau_1])} &\leq \|x - z\|_{\mathbb{L}^\infty([t_0,\tau_1])} + \|z - y\|_{\mathbb{L}^\infty([t_0,\tau_1])} + \|y - \hat{x}\|_{\mathbb{L}^\infty([t_0,\tau_1])}, \\ &\leq \frac{\rho}{4} + \frac{\rho}{4} + K_0\rho, \\ &< (1 + K_0)\rho. \end{aligned} \tag{4.69}$$

On the other hand it is straightforward to see that

$$\|\dot{x} - \dot{\hat{x}}\|_{\mathbb{L}^1([t_0,\tau_1])} \leq 2c_0 k\rho \leq K_0\rho. \tag{4.70}$$

If $t_0 + k\rho \geq t_1$ (that is $\tau_1 = t_1$), then the F-trajectory $x(.)$ satisfies all the required properties (4.51), (4.52) and (4.53). Then we continue our analysis assuming that $t_0 + k\rho < t_1$. Set $\bar{x}_0 := x(t_0 + k\rho)$.

Observe that the function $\eta_0(\cdot)$ can be decomposed into the sum of a continuous functions $\eta_0^c(\cdot)$ on $[t_0, t_1]$ and a countable family of step functions $\{\psi_j(\cdot)\}$ satisfying, for each $j \geq 1$,

$$\psi_j(t) \ = a_j \times \begin{cases} 1 & \text{if } t > \sigma_j \\ 0 & \text{if } t < \sigma_j \,, \end{cases}$$

in which $\{\sigma_j\}$ is a sequence of distinct points in $[t_0, t_1]$ and $\{a_j\}$ is a sequence of non-negative numbers. (In the analysis to follow, we do not have to take account of the value of $\psi_j(\cdot)$ at its 'jump' time σ_j.) Take $\delta_0 \in (0, \delta \wedge \frac{\rho}{4\delta})$ such that we can ensure that

$$\eta_0^c(t') - \eta_0^c(s') < \frac{\rho}{32c_0k_F}$$

for all subinterval $[s',t'] \subset [t_0,t_1]$ such that $t'-s' \leq \delta_0$ We also know that the sequence $\{a_j\}$ is necessarily such that $\sum_j a_j < \infty$. Then, there exists a finite index set $J_0 \subset \{1,2,\ldots\}$ such that

$$\sum_{j \notin J_0} a_j < \frac{\rho}{32 c_0 k_F}.$$

Take $\beta \in \left]0, \frac{k^2\rho^3}{12(2c_0+|\hat{x}(t_0)|)}\right]$. Notice that there exist N positive numbers $b_0, b_1, \ldots, b_{N-1}$ and a partition of $[t_0+k\rho, t_1]$, $\mathcal{S} := \{t_0+k\rho =: s_0 < s_1 < s_2 < \ldots < s_N := t_1\}$, such that $\mathcal{S} \supset \{\sigma_j \mid \sigma_j \in [t_0+k\rho, t_1] \text{ and } j \in J_0\}$, $s_{k+1}-s_k \leq \beta$ for all $k=0,1,\ldots,N-1$, and, for each $k=0,1,\ldots,N-1$,

$$|(c_0 k_F(s-t_0)+\eta_0(s))-b_k| \leq \frac{\rho}{8}, \quad \text{for all } s \in (s_k, s_{k+1}), \tag{4.71}$$

which yields

$$\sum_{k=0}^{N-1} \int_{s_k}^{s_{k+1}} |(c_0 k_F(s-t_0)+\eta_0(s))-b_k| \mathrm{d}s \leq \frac{\rho}{8}(t_1-t_0). \tag{4.72}$$

Observe that for every $t \in [t_0+k\rho, t_1]$ we have:

$$\frac{\hat{x}(t)-\hat{x}(t_0)}{t-t_0} = \frac{1}{t-t_0}\int_{t_0}^{t} \dot{\hat{x}}(s)\mathrm{d}s \in \mathrm{co}\big[F(t_0^+, \hat{x}(t_0)) + (k_F c_0(t-t_0)+\eta_0(t))\mathbb{B}\big].$$

From Carathéodory's Theorem, for each $k=0,\ldots,N-1$, we can choose vectors

$$v_{k,1},\ldots,v_{k,n+1} \in F(t_0^+,\hat{x}(t_0)) + (k_F c_0(s_k-t_0)+\eta_0(s_k))\mathbb{B} = F(t_0,\hat{x}(t_0)) + (k_F c_0(s_k-t_0)+\eta_0(s_k))\mathbb{B}$$

and numbers $\lambda_{k,1},\ldots,\lambda_{k,n+1}$ in $[0,1]$ such that:

$$\frac{\hat{x}(s_k)-\hat{x}(t_0)}{s_k-t_0} = \sum_{i=1}^{n+1} \lambda_{k,i} v_{k,i} \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_{k,i} = 1.$$

Fix $k \in \{0,\ldots,N-1\}$. We apply the Lyapunov Theorem (cf. [36, Theorem 16.1.v]) on the reference interval $[s_k, s_{k+1}]$ with the $n+2$ integrable vector functions $f_{k,0}(\cdot) = \dot{\hat{x}}(\cdot)$, $f_{k,1}(\cdot) \equiv v_{k,1}, \ldots, f_{k,n+1}(\cdot) \equiv v_{k,n+1}$, and the $n+2$ weight functions $p_{k,0}(s) := 1 - \frac{k\rho}{s-t_0}$, $p_{k,1}(s) := \frac{k\rho}{s-t_0}\lambda_{k,1}, \ldots, p_{k,n+1}(s) := \frac{k\rho}{s-t_0}\lambda_{k,n+1}$. We obtain a decomposition of $[s_k, s_{k+1}]$ into disjoint measurable subsets $E_{k,0}$, $E_{k,1},\ldots,E_{k,n+1} \subset [s_k, s_{k+1}]$, satisfying

$$\mathrm{meas}(E_{k,i}) = \int_{s_k}^{s_{k+1}} p_{k,i}(s)\mathrm{d}s, \quad \text{for each } i=0,1,\ldots,n+1, \tag{4.73}$$

and

$$\sum_{i=0}^{n+1} \int_{s_k}^{s_{k+1}} f_{k,i}(s) p_{k,i}(s)\mathrm{d}s = \sum_{i=0}^{n+1} \int_{E_{k,i}} f_{k,i}(s)\mathrm{d}s. \tag{4.74}$$

From these relations it easily follows that

$$\int_{E_{k,0}} \dot{\hat{x}}(s)ds = \int_{s_k}^{s_{k+1}} \left(1-\frac{k\rho}{s-t_0}\right)\dot{\hat{x}}(s)\mathrm{d}s. \tag{4.75}$$

Now we provide an extension of the F trajectory $x(\cdot)$ above on $[t_0+k\rho, t_1]$ considering the following differential inclusion

$$\begin{cases} \dot{x}(t) \in G_2(t,x(t)) \quad a.e.\ t \in [t_0+k\rho, t_1], \\ x(t_0+k\rho) = \bar{x}_0 \end{cases} \tag{4.76}$$

where the multivalued function $G_2 : [t_0 + k\rho, t_1] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is defined as

$$G_2(t,x) := \begin{cases} \overline{(w_2(t) + [2k_F|x - \hat{x}(t)| + \alpha_2(t)]\mathrm{int}\,\mathbb{B}) \cap F(t,x)} & \text{if } |x - \hat{x}(t_0)| + \eta_0(t) > 0 \\ \{w_2(t)\} & \text{if } |x - \hat{x}(t_0)| + \eta_0(t) = 0. \end{cases} \tag{4.77}$$

in which $w_2(t) := \dot{\hat{x}}(t)\chi_{E_0}(t) + \sum_{k=0}^{N-1}\sum_{i=1}^{n+1} v_{k,i}\chi_{E_{k,i}}(t)$, $E_0 := \cup_{k=0}^{N-1} E_{k,0}$, and $\alpha_2(t) := 2[k_F c_0(t - t_0) + \eta_0(t)]\chi_{\cup_{i=1}^{n+1} E_{k,i}}(t)$. All the hypotheses of Lemma 4.11.3 are satisfied, then there exists a solution to (4.76), which is also an F-trajectory extending the F-trajectory $x(\cdot)$ previously constructed on $[t_0, t_0 + k\rho]$: we write $x(\cdot)$ the obtained F-trajectory on all $[t_0, t_1]$.
Consider the arc $\hat{y} : [t_0 + k\rho, t_1] \to \mathbb{R}^n$ such that

$$\begin{cases} \dot{\hat{y}}(t) = w_2(t) \quad \text{for a.e. } t \in [t_0 + k\rho, t_1] \\ \hat{y}(t_0 + k\rho) = y(t_0 + k\rho) = \hat{x}(t_0) + k\rho v\ . \end{cases}$$

Since $y(t_0 + k\rho) = \hat{y}(t_0 + k\rho)$, using (4.73) and (4.75), for all $k' \in \{0, 1, \ldots, N-1\}$, we have

$$\begin{aligned} |y(s_{k'}) - \hat{y}(s_{k'})| &\leq \sum_{k=0}^{k'-1} \left| \int_{s_k}^{s_{k+1}} (\dot{y}(s) - \dot{\hat{y}}(s))\mathrm{d}s \right| \\ &= \sum_{k=0}^{k'-1} \left| \int_{s_k}^{s_{k+1}} \frac{k\rho}{s - t_0} \Big(\frac{\hat{x}(s) - \hat{x}(t_0)}{s - t_0} - \frac{\hat{x}(s_k) - \hat{x}(t_0)}{s_k - t_0} \Big) \mathrm{d}s \right| \\ &\leq \frac{2(|\hat{x}(t_0)| + \delta c_0)}{(k\rho)^2}\beta. \end{aligned}$$

Therefore, taking into account also the fact that, for all $k \in \{0, 1, \ldots, N-1\}$ and for all $s \in [s_k, s_{k+1}]$

$$|y(s) - y(s_k)| \leq \frac{|\hat{x}(t_0)| + 2\delta c_0}{k\rho}\beta \quad \text{and} \quad |\hat{y}(s) - \hat{y}(s_k)| \leq c_0\beta,$$

for our choice of β it follows that

$$\|y - \hat{y}\|_{\mathbb{L}^\infty(t_0 + k\rho, t_1)} \leq \frac{\rho}{4}. \tag{4.78}$$

On the other hand, from (4.62) and (4.78) we also know that for a.e. $t \in [t_0 + k\rho, t_1]$

$$\begin{aligned} |\dot{x}(t) - \dot{\hat{y}}(t)| &\leq 2k_F|x(t) - \hat{x}(t)| + \alpha_2(t) \\ &\leq 2k_F(|x(t) - \hat{y}(t)| + |\hat{y}(t) - y(t)| + |y(t) - \hat{x}(t)|) + \alpha_2(t) \\ &\leq 2k_F|x(t) - \hat{y}(t)| + 2k_F(1/4 + c_0 k)\rho + \alpha_2(t). \end{aligned}$$

Then, recalling that $\hat{y}(t_0 + k\rho) = z(t_0 + k\rho)$ from (4.66) we have $|x(t_0 + k\rho) - \hat{y}(t_0 + k\rho)| \leq \rho/4$, and so making use of Gronwall's Inequality, and bearing in mind (4.54), (4.71) and (4.72), we deduce that for all $t \in [t_0 + k\rho, t_1]$

$$\begin{aligned} |x(t) - \hat{y}(t)| &\leq e^{2k_F\delta}\Big[\rho/4 + 2k_F(1/4 + c_0 k)\delta\rho + 2\sum_{k=0}^{N-1}\int_{s_k}^{s_{k+1}} [k_F c_0(s - t_0) + \eta_0(s)]\chi_{\cup_{i=1}^{n+1} E_{k,i}}(s)\mathrm{d}s\Big] \\ &\leq e^{2k_F\delta}\Big[\rho/4 + 2k_F(1/4 + c_0 k)\delta\rho + \delta\rho/4 + c_0 k_F k\delta\rho \\ &\qquad + k\rho\int_{t_0 + k\rho}^{t_1} \frac{\eta(s) - \eta(t_0^+)}{s - t_0}\mathrm{d}s + k\rho^2/8|\ln(k\rho)|\Big] \\ &< e^{2k_F\delta}\left[1/4 + 3k_F(1 + 4c_0/\epsilon)\delta + \delta/4 + 4\mu/\epsilon + (\rho/\epsilon) \times |\ln(4\rho/\epsilon)|\right]\rho \\ &\leq \frac{3}{4}\rho. \end{aligned} \tag{4.79}$$

From (4.78) and (4.79), it follows that

$$\|y - x\|_{\mathbb{L}^\infty([t_0+k\rho,t_1])} \leq \rho, \tag{4.80}$$

and, recalling also (4.60), we deduce that

$$x(t) + \rho\mathbb{B} \subset A, \text{ for all } t \in [t_0 + k\rho, t_1],$$

which yields, with (4.68), (4.51). A further consequence of (4.80), together with (4.62), is

$$\|\hat{x} - x\|_{\mathbb{L}^\infty([t_0+k\rho,t_1])} \leq (K_0 + 1)\rho. \tag{4.81}$$

It remains to derive an estimate for $\|\dot{x} - \dot{\hat{x}}\|_{\mathbb{L}^1([t_0+k\rho,t_1])}$. To see this we consider

$$\begin{aligned}\int_{t_0+k\rho}^{t_1} |\dot{x}(s) - \dot{\hat{x}}(s)|\mathrm{d}s &= \int_{E_0} |\dot{x}(s) - \dot{\hat{x}}(s)|\mathrm{d}s + \int_{[t_0+k\rho,t_1]\setminus E_0} |\dot{x}(s) - \dot{\hat{x}}(s)|\mathrm{d}s, \\ &\leq \int_{E_0} 2k_F|x(s) - \hat{x}(s)|ds + \int_{[t_0+k\rho,t_1]\setminus E_0} 2c_0\mathrm{d}s.\end{aligned}$$

Thus, using also (4.54), (4.73) and (4.81), we obtain

$$\begin{aligned}\|\dot{x} - \dot{\hat{x}}\|_{\mathbb{L}^1([t_0+k\rho,t_1])} &\leq 2k_F(K_0 + 1)\delta\rho + 2c_0 \int_{t_0+k\rho}^{t_1} \frac{k\rho}{s - t_0}\mathrm{d}s, \\ &\leq (K_0 + 1)\rho + K_0\rho(1 + |\ln(\rho)|).\end{aligned}$$

This estimate and (4.70) confirm (4.53) taking $K := 2(K_0 + 1)$. From (4.69) and (4.69), clearly also (4.52) is validated with the same choice for K.

□

Part 2

Applications of Optimal control in Calculus of Variations

5

Higher order problems in the Calculus of Variations : Du Bois-Reymond inclusion and regularity of minimizers

The results of this chapter have been published [14].

5.1 Abstract of chapter 5

The problem considered in chapter 5 is a non autonomous N-order Bolza problem

$$\begin{cases} \text{Minimize } I(x) := \displaystyle\int_a^b L(s, x(s), x^{(1)}(s), x^{(2)}(s), \dots, x^{(N)}(s))\mathrm{d}s \\ \qquad\qquad + \Psi\left((x, x^{(1)}, \dots, x^{(N-1)})(a), (x, x^{(1)}, \dots, x^{(N-1)})(b)\right), \\ \text{over arcs } x \in W^{N,m}([a,b], \mathbb{R}), \end{cases} \tag{CV}$$

where L is merely Borel measurable, and is possibly extended valued. Two different types of result are presented.

Imposing the integrable boundedness of the partial proximal subgradients (up to the $(N-2)$-order variable) along a neighborhood of a local minimizer $x_*(\cdot)$, necessary conditions for optimality in the Euler-Lagrange form and, for the first time for higher order problems, in the Erdmann – Du Bois-Reymond form are provided. Quite noticeably, these results are obtained without imposing on L to be convex with respect to the last variable, nor to have any kind of specific growth behavior.

By adding an extra growth assumption that is more general than the usual superlinearity with respect to the last variable, the necessary conditions we obtained are exploited to establish that the last derivative of the minimizer $x_*(\cdot)$, that is to say $x_*^{(N)}(\cdot)$, is essentially bounded.

The important contributions of this work compared with the most recent article dealing with this problem [60] are the following: the possibility to consider an extended-value Lagrangian, a more general growth condition for L, necessary conditions expressed in the Erdmann – Du Bois-Reymond form, as well as a

simplification of the proof of the regularity of the minimizers, invoking a time reparameterization argument alone instead of combining it with the study of the Tonelli set of $x_*(\cdot)$.

5.2 Résumé du chapitre 5

Le problème considéré dans le chapitre 5 est celui d'un problème de Bolza non-autonome d'ordre N :

$$\begin{cases} \text{Minimiser } I(x) := \displaystyle\int_a^b L(s, x(s), x^{(1)}(s), x^{(2)}(s), \dots, x^{(N)}(s))\mathrm{d}s \\ \qquad\qquad\qquad\qquad + \Psi\left((x, x^{(1)}, \dots, x^{(N-1)})(a), (x, x^{(1)}, \dots, x^{(N-1)})(b)\right), \\ \text{parmi les arcs } x \in W^{N,m}([a,b], \mathbb{R}), \end{cases} \tag{CV}$$

dans lequel le lagrangien L est seulement Borel mesurable, et peut prendre pour valeur $+\infty$. Les travaux présentés dans le chapitre 5, sont de deux natures.

En imposant que le gradient proximal de L par rapport à ses $(N-2)$ premières variables soit borné dans $\mathbb{L}^1$ au voisinage d'un minimiseur local $x_*(\cdot)$, on établit les conditions nécessaires d'optimalité sous la forme d'une équation du type Euler-Lagrange, ainsi que sous la forme d'une équation du type Erdmann – Du Bois-Reymond, ce qui est une contribution nouvelle pour les problèmes d'ordre N à la connaissance des auteurs. Il est notable que ces résultats soient obtenus sans imposer à L la convexité par rapport à sa dernière variable, ni aucune condition de croissance particulière.

En imposant en plus à L une condition de croissance plus générale que la croissance super-linéaire utilisée habituellement dans les autres articles concernant ce sujet, les conditions nécessaires obtenues sont mises à profit afin d'établir que la dernière dérivée du minimiseur $x_*(\cdot)$, c'est-à-dire $x_*^{(N)}(\cdot)$, est essentiellement bornée.
Les contributions essentielles de ce travail par rapport à l'article le plus récent concernant ce sujet [60] sont les suivantes : la possibilité de considérer un lagrangien prenant pour valeur $+\infty$, une hypothèse de croissance sur L plus générale, une condition nécessaire d'optimalité exprimée sous la forme d'une équation du type Erdmann – Du Bois-Reymond, ainsi que la simplification de la preuve de la régularité des minimiseurs qui fait seulement appel à un argument de reparamétrisation du temps plutôt que de combiner cet argument avec l'étude de l'ensemble de Tonelli de $x_*(\cdot)$.

5.3 Introduction

In this chapter, we consider the following calculus of variations problem:

$$\begin{cases} \text{Minimize } I(x) := \displaystyle\int_a^b L(s, x(s), x^{(1)}(s), x^{(2)}(s), \dots, x^{(N)}(s))\mathrm{d}s \\ \qquad\qquad\qquad\qquad + \Psi\left((x, x^{(1)}, \dots, x^{(N-1)})(a), (x, x^{(1)}, \dots, x^{(N-1)})(b)\right), \\ \text{over arcs } x \in W^{N,m}([a,b], \mathbb{R}), \end{cases} \tag{CV}$$

where $N \geq 1$ is an integer, $m \geq 1$ is a real number, $L : [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R} \cup \{+\infty\}$ is a given Borel measurable function and $\Psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a given extended valued function non identically equal to $+\infty$. Here, $x^{(k)}(\cdot)$ is the k-th derivative of the function $x \in W^{N,1}([a,b], \mathbb{R})$ (interpreting $x^{(0)}(\cdot) = x(\cdot)$), and we

sometimes write $\dot{x}(\cdot)$ or $\frac{\mathrm{d}}{\mathrm{d}s}x(\cdot)$ for the first derivative $x^{(1)}(\cdot)$ to simplify notation.

It is well known that the problem (CV) has a solution if $(x_0, x_1, \ldots, x_N) \mapsto L(t, x_0, x_1, \ldots, x_N)$ is lower semicontinuous, $x_N \mapsto L(t, x_0, x_1, \ldots, x_{N-1}, x_N)$ is convex and uniformly coercive (cf. [40]). A classical issue in this context concerns the possibility to establish the conditions needed, in addition to the existence hypotheses above, to obtain the essential bounded N-th order derivative of a reference minimizer. The significance of a positive answer to this question is explained by the fact that the N-th order derivative essential boundedness allows to derive first order necessary conditions and to use numerical methods to detect minimizers, which in general would not be valid if the mere existence hypotheses are in force.

The case when $N = 1$ corresponds to establish Lipschitz regularity of minimizers and has been extensively studied in the literature for a broad class of problems involving vector valued arcs $x(\cdot)$, covering even situations in which L is not necessarily convex or coercive in $\dot{x}$, cf. [23, 35, 41, 39, 44, 45, 69, 84] and the references therein (for an advanced result in the theory of necessary optimality conditions, we also refer to the recent paper by Ioffe [63]). For higher ($N > 1$) order problems the N-th derivative essential boundedness of a reference minimizer $x_*(\cdot)$ was demonstrated in [40] analyzing the '*Tonelli set*' associated with $x_*(\cdot)$ (*i.e.* the set of points $t \in [a, b]$ such that $x_*^{(N)}(\cdot)$ is unbounded near t), when, in addition to the existence hypotheses, the Lagrangian is real valued and satisfies the following assumptions:

(A1) L is locally bounded, $(x_0, x_1, \ldots, x_N) \mapsto L(t, x_0, x_1, \ldots, x_N)$ is locally Lipschitz continuous (uniformly in t),

(A2) The partial limiting subdifferential $\partial^L_{(x_0,\ldots,x_{N-1})}L$ is integrally bounded when evaluated along the minimizer.

This result remains true when $N = 2$ for autonomous Lagrangians when we replace $(A2)$ by a less restrictive condition, see [59]:

$(\mathrm{A2})'$ The partial limiting subdifferential $\partial^L_{x_0}L$ is integrally bounded when evaluated along the minimizer.

(Observe that $0 = N - 2$ in this case, and it is not necessary to evaluate the limiting subdifferential of L also w.r.t. the x_{N-1} variable as in (A2).)

The question whether a condition on partial subdifferentials involving only up to the x_{N-2} variable could take the place of $(A2)$ also for general N (including the case $N > 2$) was investigated in [60], substituting $(A2)$ with

$(\mathrm{A2})''$ The partial subdifferential $\partial^L_{(t,x_0,\ldots,x_{N-2})}L$ is integrally bounded when evaluated along the minimizer.

The higher order regularity result of [60] was obtained for problems involving real valued arcs $x(\cdot)$, combining two main approaches used for regularity analysis: the Tonelli set theory (mentioned above) and a time reparameterization.

The major contribution of this work is to show that higher order regularity results can be derived employing the time reparameterization alone, and for a wide class of Lagrangians, including possibly extended valued L's. The two main sets of hypotheses that we consider can be summarized as follows:

(a) *The finite case*: L is a Borel measurable real valued function and satisfies a (generalized) growth condition, the partial proximal subdifferential[1] $\partial^P_{(t,x_0,\dots,x_{N-2})}L$ is integrally bounded in a neighborhood of the reference minimizer, uniformly on x_N;

(b) *The extended valued case*: L is a lower semicontinuous (w.r.t. all variables except possibly x_{N-1}) and satisfy a (generalized) growth condition, the partial proximal subdifferential $\partial^P_{(t,x_0,\dots,x_{N-2})}L$ is integrally bounded (uniformly on x_N).

Another important feature is that we provide not only first order necessary conditions in the Euler-Lagrange form together with a Weierstrass type condition, but also, without requiring any kind of growth condition nor convexity, an Erdmann – Du Bois-Reymond condition which can be expressed in terms of a (partial) convex subdifferential. It turns out, in particular, that $L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot\Big)$ is convex in the direction $x_*^{(N)}(t)$. These are an extension to $N \geq 2$ (for scalar problems) of the results obtained in [23, 24] established there for $N = 1$ (for vectorial problems).
The generalized growth condition considered in this chapter is more general than the superlinearity of $x_N \mapsto L(t, x_0, \dots, x_N)$ and represents a sort of violation of the Erdmann – Du Bois-Reymond condition when $|x_N| \to +\infty$.

Notation.
Given an extended valued function $\phi(\cdot,\cdot)$ of two vector variables (x, y) and a point $(\bar{x}, \bar{y}) \in \mathrm{dom}(\phi)$, we denote the proximal (resp. limiting, Clarke) partial subdifferential of $\phi(\cdot, \bar{y})$ at $\bar{x}$ by $\partial^P_x \phi(\bar{x}, \bar{y})$ (resp. $\partial^L_x \phi(\bar{x}, \bar{y})$, $\partial^C_x \phi(\bar{x}, \bar{y})$).

For a given minimizer $x_*(\cdot)$ of (CV), we introduce an auxiliary Lagrangian $\Lambda : [a, b] \times \mathbb{R} \times]0, +\infty[\to \mathbb{R}$, defined by:

$$\Lambda(t, \xi, r) := L(t, x_*(t), \dot{x}_*(t), \dots, x_*^{(N-1)}(t), r\xi), \text{ for all } (t, \xi, r) \in [a, b] \times \mathbb{R} \times]0, +\infty[\,. \tag{5.1}$$

We shall make use of **partial convex subdifferential** of Λ with respect to r at (t, ξ, r_0), which is defined by:

$$\partial_r \Lambda(t, \xi, r_0) := \{p \in \mathbb{R} : \Lambda(t, \xi, r) - \Lambda(t, \xi, r_0) \geq p(r - r_0), \forall r \in]0, +\infty[\,\}\,. \tag{5.2}$$

The space of absolutely continuous functions defined on $[a, b]$, taking values in $\mathbb{R}^k$, with derivative in $\mathbb{L}^m([a, b], \mathbb{R}^k)$ is written $W^{1,m}([a, b], \mathbb{R}^k)$ and endowed with the norm:

$$\|f\|_{W^{1,m}} := \|f\|_\infty + \|\dot{f}\|_{\mathbb{L}^m}.$$

We denote by $W^{N,m}([a, b], \mathbb{R}^k)$ the space of functions defined on $[a, b]$, taking values in $\mathbb{R}^k$ which are $N - 1$ times continuously differentiable and whose $(N-1)$-th derivative belongs to $W^{1,m}([a, b], \mathbb{R}^k)$. We endow this space with the following norm:

$$\|f\|_{W^{N,m}} := \sum_{i=0}^{N-2} \|f^{(i)}\|_\infty + \|f^{(N-1)}\|_{W^{1,m}}.$$

We shall often write $L\Big(t, x_*, \dots, x_*^{(N-1)}, x_*^{(N)}\Big)$ instead of $L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_*^{(N)}(t)\Big)$ to simplify notation.

[1] Note that the notation for the convex subdifferential is different in Part 2 of this thesis (the P went from subscript to superscript). Even though ∂^P denotes the proximal superdifferential, it is much more convenient to reserve the subscript for the variables with respect to which we differentiate the function.

5.4 Hypotheses

We shall consider two different sets of hypotheses on L for a given local $W^{N,m}$ local minimizer $x_*(\cdot)$ for (CV): (S_{x_*}) and ($\mathrm{S}^{\infty}_{x_*}$).

Hypothesis ($\mathbf{S}_{x_*}$) The function

$$L : (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto L(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R}$ and is $\mathcal{B}_{N+2}$-measurable.
There exists $\varepsilon_* > 0$ and an $\mathcal{L} \times \mathcal{B}_1$-measurable function $k : [a,b]\times]0, +\infty[\to \mathbb{R}_+$ such that:

$$t \mapsto k(t,1) \in \mathbb{L}^1([a,b], \mathbb{R}_+),$$

and, for a.e. $t \in [a,b]$, for all $\sigma \in]0, +\infty[$, the map:

$$\begin{cases} [a,b] \times \mathbb{R}^{N-1} \to \mathbb{R}, \\ (s, x_0, \dots, x_{N-2}) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)), \end{cases} \tag{5.3}$$

is Lipschitz continuous on $\mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_*) \cap ([a,b] \times \mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Remark 5.4.1 *Making use of hypothesis (S_{x_*}), we deduce that for a.e. $t \in [a,b]$, if ζ is a vector in $\partial^C_{(t,x_0,\dots,x_{N-2})} L(t, x_*(t), \dots, x_*^{(N)}(t))$, then $|\zeta| \leq k(t,1)$. Notice also that (S_{x_*}) is satisfied whenever L depends only on x_{N-1} and x_N.*

Hypothesis ($\mathbf{S}^{\infty}_{x_*}$) The function

$$L : (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto L(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R} \cup \{+\infty\}$ and is $\mathcal{B}_{N+2}$-measurable.
There exist a measurable set $E \subset [a,b]$ of full measure, strictly positive constants ε, c and λ, functions d, $\beta \in \mathbb{L}^1([a,b], \mathbb{R}_+)$ such that the following conditions are satisfied:

i) the function $(s, x_0, \dots, x_{N-2}, x_N) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N)$ is lower semicontinuous for all $t \in [a,b]$,

ii) for all $t \in E$, we can find $0 < \sigma_1(t) < 1 < \sigma_2(t) < +\infty$ for which:

$$\begin{cases} L(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_1(t) x_*^{(N)}(t)) < +\infty \\ L(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_2(t) x_*^{(N)}(t)) < +\infty; \end{cases} \tag{5.4}$$

iii) for every $t \in E$, every $(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}) \in \mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon) \cap ([a,b] \times \mathbb{R}^{N-1})$, and $x_N \in \mathbb{R}$, we have

$$|\zeta| \leq c \left(|(1, \bar{x}_1, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t))| + L(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N) + \lambda |x_N|) \right) + d(t) \tag{5.5}$$

for all $\zeta \in \partial^P_{(s,x_0,\dots,x_{N-2})} L(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N)$;

iv) for all $t \in E$, there exists $\varepsilon_t > 0$ such that the function

$$(s, x_0, \dots, x_{N-2}) \mapsto L(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N),$$

is Lipschitz continuous with Lipschitz constant $\beta(t)$ on $\mathrm{B}((t, x_*(t), \dot{x}_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_t)$, uniformly with respect to $x_N \in \mathrm{B}(x_*^{(N)}(t), \varepsilon_t) \cap \mathrm{dom}(L(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot))$.

The growth assumption (G_{x_*}). For every selection $Q(t,\xi)$ of $\partial_r \Lambda(t, \xi, 1)$,

$$\lim_{\substack{|\xi| \to +\infty \\ \partial_r \Lambda(t,\xi,1) \neq \emptyset}} |L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t,\xi)| = +\infty, \text{ uniformly for a.e. } t \in [a,b], \qquad (\mathrm{G}_{x_*})$$

which means that for any $M > 0$, we can find a set $\mathcal{E} \subset [a,b]$ of full measure, and a real $R > 0$ satisfying:

$$\forall (t,\xi) \in \mathcal{E} \times \mathbb{R},\ Q(t,\xi) \in \partial_r \Lambda(t,\xi,1),\ |\xi| \geq R \Rightarrow |L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t,\xi)| \geq M.$$

Observe that condition (G_{x_*}) is satisfied independently of a minimizer $x_*(\cdot)$ when for every selection $Q(t, x_0, \dots, x_{N-1}, \xi)$ of $(\partial_r L(t, x_0, \dots, x_{N-1}, r\xi))_{r=1}$ and for every compact set $K \subset \mathbb{R}^N$, we have:

$$\lim_{\substack{|\xi| \to +\infty \\ (\partial_r L(t,x_0,\dots,x_{N-1},r\xi))_{r=1} \neq \emptyset}} |L(t, x_0, \dots, x_{N-1}, \xi) - Q(t, x_0, \dots, x_{N-1}, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a,b]$ and for all $(x_0, \dots, x_{N-1}) \in K$.

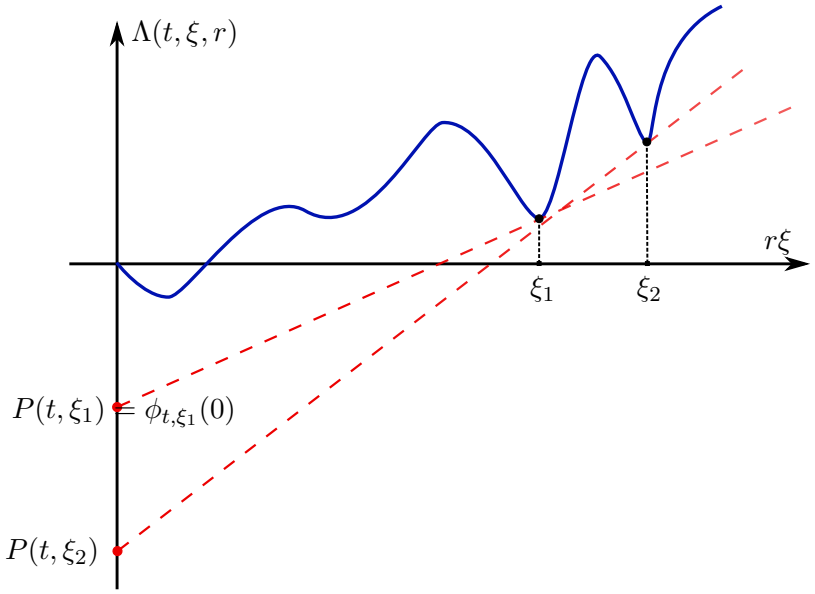

Figure 5.1: Condition (G_{x_*})

Remark 5.4.2 (Interpretation of (G_{x_*})) *Assume that* $L(t, x_*(t), ..., x_*^{(N-1)}(t), \xi) < +\infty$ *and let* $Q(t,\xi) \in \partial_r \Lambda(t,\xi,1)$. *Then*

$$\Lambda(t,\xi,r)) \geq \phi(r) := \Lambda(t,\xi,r) + Q(t,\xi)(r-1), \text{ for all } r > 0$$

and $P(t,\xi) := \phi(0) = \Lambda(t,\xi,1) - Q(t,\xi)$ *is the intersection with the* z *axis of the 'tangent' line* $z = \phi(r)$ *to* $0 < r \mapsto \Lambda(t,\xi,r)$ *at* $r = 1$.
Condition (G_{x_*}) *thus means that the ordinate* $P(t,\xi)$ *of the above intersection point goes to* ∞ *as* $|\xi|$ *goes to* ∞, *for those points* ξ *where* $0 < r \mapsto \Lambda(t,\xi,r)$ *has a nonempty convex subdifferential at* $r = 1$.

Remark 5.4.3 *1) If L is smooth in the last variable, (G_{x_*}) becomes*

$$\lim_{|\xi|\to+\infty} |L\big(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi\big) \ - \ \xi \cdot \partial_{x_N} L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$.
2) If L is convex in the last variable, (G_{x_*}) is satisfied whenever for every selection $\varphi(t, \xi)$ of the convex subdifferential $\partial_\xi L\big(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi\big)$, we have

$$\lim_{|\xi|\to+\infty} |L(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) \ - \ \xi \cdot \varphi(t, \xi)| = +\infty, \ \textit{uniformly for a.e. } t \in [a, b].$$

Condition (G_{x_*}), which was considered in the case $N = 1$ in [24], extends analogous conditions considered in [35, 69] in the autonomous case. This growth condition (G_{x_*}) is satisfied in the superlinear case. More precisely, assume that L satisfies both conditions below:
a) $L(s, x_0, \dots, x_N)$ **is bounded on an annulus along** $x_*(\cdot)$: there exist $\rho > 0$, and $M > 0$ such that, for almost every $t \in [a, b]$ we have:

$$|x_N| = \rho \Rightarrow L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_N\Big) \le M. \tag{B_{x_*}}$$

b) L **is uniformly coercive along** $x_*(\cdot)$ w.r.t. the last variable: there exists a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim\limits_{r\to+\infty} \dfrac{\theta(r)}{r} = +\infty$, such that for a.e. $t \in [a, b]$ and every $x_N \in \mathbb{R}$:

$$L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_N\Big) \ge \theta(|x_N|). \tag{C_{x_*}}$$

(This also covers hypothesis (H3) used in [60], where $\theta(\cdot)$ is taken *positive valued* and L satisfies the following estimation $L(t, x_*(t), \dots, x_*^{(N-1)}(t), x_N) \ge \theta(|x_N|) - \beta|x_N|$, where $\beta > 0$.) Then it may be shown as in [24, Proposition 2] that (G_{x_*}) is valid.
Notice however that there are Lagrangians that have just a linear growth with respect to x_N but nonetheless satisfy (G_{x_*}), for example $L(t, x_0 \dots, x_N) = |x_N| - \sqrt{|x_N|}$.

Remark 5.4.4 *Assume that L is bounded on bounded sets in the following sense: For every bounded set $K \subset \mathbb{R}^N$, the following property is satisfied: there exist $\rho > 0$ and $M_K > 0$ such that, for almost every $t \in [a, b]$, every $(x_0, \dots, x_{N-1}) \in K$ and any $x_N \in \mathbb{R}$:*

$$|x_N| = \rho \Rightarrow L(t, x_0, \dots, x_{N-1}, x_N) \le M_K.$$

Assume additionally that L is uniformly coercive in the following sense: there exist an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim\limits_{r\to+\infty} \dfrac{\theta(r)}{r} = +\infty$, a function $h : \mathbb{R}^N \to \mathbb{R}$ that is bounded on bounded sets, and a constant $\alpha > 0$, for which the following property holds: for all $(t, x_0, \dots, x_N) \in [a, b] \times \mathbb{R}^{N+1}$ satisfying $|x_N| \ge \alpha|(x_0, \dots, x_{N-1})|$:

$$L(t, x_0, \dots, x_{N-1}, x_N) \ge \theta(|x_N| - \alpha|(x_0, \dots, x_{N-1})|) - h(x_0, \dots, x_{N-1}).$$

Then L satisfies the conditions (B_{x_*}) and (C_{x_*}) above for any function $x_*(\cdot) \in W^{N,1}([a, b], \mathbb{R})$, and it yields that the growth condition (G_{x_*}) is also valid for any function $x_*(\cdot) \in W^{N,1}([a, b], \mathbb{R})$.

5.5 Main results

We establish here a new necessary condition and a subsequent regularity result for minimizers of (CV).

5.5.1 Necessary conditions

Weierstrass type conditions.

Theorem 5.5.1 *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV). *Assume that L satisfies* (S_{x_*}). *Then there are two – mutually non exclusive – cases:*

i) The function x_* is a polynomial function whose degree is at most $N-1 \geq 1$.

ii) There exists an arc $(p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ for which the following Weierstrass type condition is satisfied: for all $u \in]0, +\infty[$ and for a.e. $t \in [a,b]$:

$$L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq (u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big). \tag{W}$$

Moreover, for a.e. $t \in [a,b]$:

$$(\dot{p}_0, \dot{p}_1, \dot{p}_2 + p_1, \dots, \dot{p}_{N-2} + p_{N-3}, \dot{p}_{N-1} + p_{N-2}) \in \partial^C_{t,x_0,\dots,x_{N-2}} L(t, x_*, \dot{x}_*, \dots, x_*^{(N)}). \tag{5.6}$$

Theorem 5.5.2 *Let $x_*(\cdot) \in W^{N,m}([a,b], \mathbb{R}^N)$ be a minimizer for* (CV). *Assume that L satisfies* ($\mathrm{S}^\infty_{x_*}$). *Then there are two – mutually non exclusive – cases:*

i) The function x_* is a polynomial function whose degree is at most $N-1 \geq 1$.

ii) There exists an arc $p := (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ for which the following Weierstrass type condition is satisfied: for all $u \in]0, +\infty[$ and for a.e. $t \in [a,b]$:

$$L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq (u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big). \tag{W}$$

Moreover, for a.e. $t \in [a,b]$, $\dot{p}(t)$ belongs to the set:

$$\begin{aligned} &\mathrm{co}\Big\{\omega \in \mathbb{R}^N \,:\, \Big(\omega + \gamma(t), p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t)x_*^{(N-1)}(t)\Big) \\ &\quad \in \Big(\partial^L_{(s,x_0,\dots,x_{N-2},u)} L\big(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_*^{(N)}(t)/u\big)u\Big)_{\substack{(s,x_0,\dots,x_{N-2}) = z_*(t) \\ u=1}}\Big\}, \end{aligned} \tag{5.7}$$

with $\gamma(t) := (0, 0, p_1(t), \dots, p_{N-2}(t))$ and $z_*(t) := \big(t, x_*(t), \dots, x_*^{(N-2)}(t)\big)$.

Remark 5.5.3 *1) We observe that the inequality (W) of Theorems 5.5.1 and 5.5.2 is a Weierstrass type condition which is an extension (to the case $N \geq 2$) of the results [23, Theorem 4.1] and [23, Theorem 4.3]. 2) Assume that L is of class $\mathcal{C}^2$ with respect to $t, x_0, \dots, x_{N-2}$. Then we have*

$$\partial^C_{t,x_0,\dots,x_{N-2}} L\Big(t, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big) = \Big\{\nabla_{t,x_0,\dots,x_{N-2}} L\Big(t, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\Big\},$$

hence the arc $p := (p_0, \dots, p_{N-1})$ satisfies the following equations: for all $s \in [a,b]$

$$\begin{cases} p_0(s) = p_0(a) + \displaystyle\int_a^s \partial_t L\Big(\tau, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\mathrm{d}\tau, \\ \\ p_1(s) = p_1(a) + \displaystyle\int_a^s \partial_{x_0} L\Big(\tau, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\mathrm{d}\tau, \end{cases}$$

and for a.e. $s \in [a,b]$, *for all* $i = 2, \dots, N-1$,

$$\dot{p}_i(s) = -p_{i-1}(s) + \partial_{x_{i-1}} L\Big(s, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big).$$

Erdmann – Du Bois-Reymond type conditions. The change of variable $r = 1/u$ in (W) yields the following equivalent version of Theorems 5.5.1 and 5.5.2.

Corollary 5.5.4 *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV). *Assume that* L *satisfies* (S_{x_*}) *(resp.* $(\mathrm{S}^\infty_{x_*})$*). Then there are two – mutually non exclusive – cases:*

i) The function x_* *is a polynomial function whose degree is at most* $N-1 \geq 1$.

ii) There exists an arc $p := (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ *for which the following equation is satisfied: for all* $r \in]0, +\infty[$ *and for a.e.* $t \in [a,b]$*:*

$$\begin{aligned} &L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), r x_*^{(N)}(t)\Big) - L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) \geq \\ &(r-1)\Big(L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) - (p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t) x_*^{(N-1)}(t))\Big), \end{aligned} \tag{W_r}$$

where p *satisfies (5.6) (resp. (5.7)).*

Remark 5.5.5 *Condition* (W_r) *is a sort of variational form of an Erdmann – Du Bois-Reymond equation. Indeed, if* L *is smooth and satisfies* (S_{x_*})*, Corollary 5.5.4 implies that:*

$$\begin{aligned} x_*^{(N)}(t) \cdot \partial_{x_N} L\left(t, x_*, \dots, x_*^{(N)}\right) = \\ L\left(t, x_*, \dots, x_*^{(N)}\right) - (p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t) x_*^{(N-1)}(t)), \end{aligned}$$

where p *satisfies the conditions expressed in Remark 5.5.3 2). Whereas, under the (nonsmooth) more general hypotheses of Corollary 5.5.4, we obtain that for a.e.* $t \in [a,b]$

$$L\Big(t, x_*(t), \dots, x_*^{(N)}(t)\Big) - (p_0(t) + p_1(t)\dot{x}_*(t) + \dots + p_{N-1}(t) x_*^{(N-1)}(t)) \in \partial_r \Lambda(t, x_*^{(N)}(t), 1),$$

where Λ *and* $\partial_r \Lambda$ *are defined in (5.1) and (5.2).*
Condition (W_r) *is also a relaxation type result, namely the convexity of* $L(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot\,)$ *along the direction* $x_*^{(N)}(t)$.

5.5.2 Regularity Results

Here, the additional growth conditions (G_{x_*}) and (C_{x_*}) play a central role.

Theorem 5.5.6 *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV).

(i) Assume that L *satisfies* (S_{x_*}) *and* (G_{x_*})*, then* $x_*^{(N)} \in \mathbb{L}^\infty([a,b], \mathbb{R})$.

(ii) Assume that L *satisfies* $(\mathrm{S}^\infty_{x_*})$ *and* (G_{x_*})*, then* $x_*^{(N)} \in \mathbb{L}^\infty([a,b], \mathbb{R})$.

An immediate consequence of Theorem 5.5.6 and the discussion about the above-mentioned conditions (B_{x_*}) and (C_{x_*}) is the following corollary.

Corollary 5.5.7 *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV).

(i) Assume that L *satisfies* (S_{x_*})*,* (C_{x_*}) *and* (B_{x_*})*, then* $x_*^{(N)} \in \mathbb{L}^\infty([a,b], \mathbb{R})$.

(ii) Assume that L satisfies $(\mathrm{S}^{\infty}_{x_*})$, (C_{x_*}) and (B_{x_*}), then $x_*^{(N)} \in \mathbb{L}^{\infty}([a,b],\mathbb{R})$.

Next proposition shows that, if the Lagrangian L is convex w.r.t. x_N, then we can relax the condition (S_{x_*}) and invoke a weaker (merely local in σ) version of it. This result provides an extension of [60, Theorem 2.1].

Proposition 5.5.8 *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for (CV), in which we assume that $L : [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R}$ is Borel measurable and*

$(\mathrm{H})'$ *$x_N \mapsto L(t,x_0,x_1,\dots,x_{N-2},x_{N-1},x_N)$ is convex for every $(t,x_0,x_1,\dots,x_{N-2},x_{N-1})$;*

$(\mathrm{S}_{x_*})'$ *There exist $\varepsilon_* > 0$, $\sigma_* \in]0,1[$ and a $\mathcal{L}\times\mathcal{B}_1$-measurable function $k : [a,b]\times[1-\sigma_*,1+\sigma_*] \to \mathbb{R}_+$ such that:*

$$t \mapsto k(t,1) \in \mathbb{L}^1([a,b],\mathbb{R}_+),$$

and, for a.e. $t \in [a,b]$, for all $\sigma \in [1-\sigma_*,1+\sigma_*]$, the map:

$$\begin{cases} [a,b]\times\mathbb{R}^{N-1} \to \mathbb{R}, \\ (s,x_0,\dots,x_{N-2}) \mapsto L(s,x_0,\dots,x_{N-2},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)), \end{cases} \tag{5.8}$$

is Lipschitz continuous on $\mathrm{B}((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*)\cap([a,b]\times\mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Then, the same conclusions of Theorem 5.5.1 are valid. If moreover, L satisfies (G_{x_*}), then $x_*^{(N)}(\cdot)$ belongs to $\mathbb{L}^{\infty}([a,b],\mathbb{R})$.

5.6 Proofs of Theorem 5.5.1 and Proposition 5.5.8

We shall make use of the following technical lemma, which has been used and proved in [23, Lemma 7.1].

Lemma 5.6.1 *Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of invertible functions in $W^{1,1}([a,b],\mathbb{R})$ that satisfies the following properties:*

a) for all $k \in \mathbb{N}$, $z_k(a) = a$ and $z_k(b) = b$,

b) there exists $\alpha > 0$ such that $\dot{z}_k(t) \geq \alpha$, for all $k \in \mathbb{N}$ and for a.e. $t \in [a,b]$,

c) the sequence $(z_k)_{k\in\mathbb{N}}$ converges to Id in $W^{1,1}([a,b],\mathbb{R})$, where $\mathrm{Id} : t \mapsto t$.

Then for all $x \in W^{1,m}([a,b],\mathbb{R})$, there exists a subsequence of $(x\circ z_k^{-1})_{k\in\mathbb{N}}$ that converges to x in $W^{1,m}([a,b],\mathbb{R})$ as k goes to $+\infty$.

5.6.1 An auxiliary control problem (CP)

We consider the following extension $\widetilde{L}$ of L to the whole space $\mathbb{R}^{N+2}$: for all $(t,x_0,\dots,x_N) \in \mathbb{R}^{N+2}$,

$$\widetilde{L}(t,x_0,\dots,x_N) := \begin{cases} L(a,x_0,\dots,x_N), \text{ if } t \leq a, \\ L(t,x_0,\dots,x_N), \text{ if } t \in [a,b], \\ L(b,x_0,\dots,x_N), \text{ if } t \geq b. \end{cases} \tag{5.9}$$

We introduce also the auxiliary Lagrangian $\ell : [a,b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which is defined by: for all $(t,z,u) \in [a,b] \times \mathbb{R}^N \times \mathbb{R}$

$$\ell(t,z,u) := \begin{cases} \widetilde{L}\left(z, x_*^{(N-1)}(t), \frac{x_*^{(N)}(t)}{u}\right) u, \text{ if } x_*^{(N)}(t) \text{ is defined and } u > 0, \\ 0, \text{ otherwise.} \end{cases}$$

where $z = (z_0, z_1, \ldots, z_{N-1})$.
Fix any integer $j \geq 2$. We shall consider the following control problem:

$$\begin{cases} \text{Minimize } J(z,u) := \displaystyle\int_a^b \ell(s, z(s), u(s)) \mathrm{d}s, \\ \text{over the arcs } z \in W^{1,1}([a,b], \mathbb{R}^N) \\ \text{and } \mathcal{L} - \text{measurable controls } u : [a,b] \to \mathbb{R} \text{ such that:} \\ \dot{z}(s) = f(s, z(s), u(s)), \text{ for a.e. } s \in [a,b], \\ u(s) \in \left[\frac{1}{j}, j\right], \text{ for a.e. } s \in [a,b], \\ z(a) = \left(a, x_*(a), \dot{x}_*(a), \ldots, x_*^{(N-2)}(a)\right), \\ z(b) = \left(b, x_*(b), \dot{x}_*(b), \ldots, x_*^{(N-2)}(b)\right), \end{cases} \tag{CP}$$

where $f : [a,b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is defined by:

$$f(s,z,u) := uA_N z + ub_N(s), \text{ for } (s,z,u) \in [a,b] \times \mathbb{R}^N \times \mathbb{R},$$

with $b_N(s) := 1$ if $N = 1$, $b_N(s) := (1, 0, \ldots, 0, x_*^{(N-1)}(s))$ if $N > 1$, and $A_N := 0$ if $N = 1, 2$ and

$$A_N := \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \text{ if } N > 2.$$

We say that a trajectory/control pair (z,u) is admissible for the problem (CP) whenever $\dot{z}(s) = f(s, z(s), u(s))$ and $u(s) \in \left[\frac{1}{j}, j\right]$ for a.e. $s \in [a,b]$, together with $z(a) = \left(a, x_*(a), \dot{x}_*(a), \ldots, x_*^{(N-2)}(a)\right)$ and also $z(b) = \left(b, x_*(b), \dot{x}_*(b), \ldots, x_*^{(N-2)}(b)\right)$.
Observe that the differential equation $\dot{z}(s) = f(s, z(s), u(s))$, can be rewritten in an extended form (in the case $N > 2$): for a.e. $s \in [a,b]$

$$\begin{cases} \dot{z}_0(s) = u(s), \\ \dot{z}_1(s) = u(s) z_2(s), \\ \dot{z}_2(s) = u(s) z_3(s), \\ \qquad \vdots \\ \dot{z}_{N-2}(s) = u(s) z_{N-1}(s), \\ \dot{z}_{N-1}(s) = u(s) x_*^{(N-1)}(s). \end{cases} \tag{5.10}$$

Moreover any solution (z,u) to the control system in (CP) satisfies $\dot{z}_0 = u \geq \frac{1}{j}$ a.e., hence z_0^{-1} exists and is Lipschitz continuous with Lipschitz constant bounded above by j.
Using the fact that x_* is a minimizer for the problem (CV), we can deduce that a natural minimizer to the control problem (CP) is the trajectory/control pair (z_*, u_*) defined by:

$$u_*(s) := 1 \text{ and } z_*(s) := \left(s, x_*(s), \dot{x}_*(s), \ldots, x_*^{(N-2)}(s)\right), \text{ for all } s \in [a,b].$$

Lemma 5.6.2 *For all $\varepsilon > 0$, there exists $\rho > 0$ such that, for any admissible pair $(z, u) \in W^{1,1}([a,b], \mathbb{R}^N) \times \mathcal{L}$ for* (CP) *we have:*

$$\|z - z_*\|_{W^{1,1}} \leq \rho \Rightarrow \|z_1 \circ z_0^{-1} - x_*\|_{W^{N,m}} \leq \varepsilon, \tag{5.11}$$

in which $z_1 = x_*$ if $N = 1$.

Proof. The case $N = 1$ is an immediate consequence of Lemma 5.6.1, so we continue considering $N \geq 2$. Assuming that (5.11) is not satisfied, then we can find $\varepsilon_0 > 0$ and a sequence of admissible pairs for the control system in (CP), say $(z^k, u^k)_{k\in\mathbb{N}}$, $z^k := (z_0^k, \ldots, z_{N-1}^k)$ such that:

$$\|z^k - z_*\|_{W^{1,1}} \leq \frac{1}{k+1} \quad \text{and} \quad \|z_1^k \circ (z_0^k)^{-1} - x_*\|_{W^{N,m}} > \varepsilon_0. \tag{5.12}$$

We define $y^k := z^k \circ (z_0^k)^{-1}$ and we write $y^k = (y_0^k, y_1^k, \ldots, y_{N-1}^k)$.
For each $k \in \mathbb{N}$, we have $(y_1^k)^{(i)} = z_{i+1}^k \circ (z_0^k)^{-1}$ for all $i = 0, \ldots, N-2$, and $(y_1^k)^{(N-1)} = x_*^{(N-1)} \circ (z_0^k)^{-1}$. As a consequence if $i \leq N-3$, we obtain:

$$\begin{aligned}
\|(y_1^k)^{(i)} - x_*^{(i)}\|_\infty &\leq \|z_{i+1}^k \circ (z_0^k)^{-1} - z_{i+1}^k\|_\infty + \|z_{i+1}^k - x_*^{(i)}\|_\infty, \\
&\leq \sup_{|t-t'|\leq \frac{1}{k+1}} |z_{i+1}^k(t) - z_{i+1}^k(t')| + \frac{1}{k+1}, \\
&\leq \frac{M+1}{k+1},
\end{aligned} \tag{5.13}$$

where $M := \max\{\|x_*^{(i)}\|_\infty + 1, i = 1, \ldots, N-2\}$.

On the other hand, for $i = N-2$, we have:

$$\begin{aligned}
\|(y_1^k)^{(N-2)} - x_*^{(N-2)}\|_\infty &= \|z_{N-1}^k \circ (z_0^k)^{-1} - x_*^{(N-2)}\|_\infty, \\
&\leq \sup_{t\in[a,b]} \int_a^t |x_*^{(N-1)}((z_0^k)^{-1}(s)) - x_*^{(N-1)}(s)| \mathrm{d}s, \\
&\leq |b-a| \sup_{|t-t'|\leq \frac{1}{k+1}} |x_*^{(N-1)}(t) - x_*^{(N-1)}(t')|.
\end{aligned}$$

Therefore by uniform continuity of $x_*^{(N-1)}(\cdot)$ we deduce that:

$$\|(y_1^k)^{(N-2)} - x_*^{(N-2)}\|_\infty \xrightarrow[k\to+\infty]{} 0. \tag{5.14}$$

We now claim that

$$\|(y_1^k)^{(N-1)} - x_*^{(N-1)}\|_{W^{1,m}} \xrightarrow[k\to+\infty]{} 0, \tag{5.15}$$

(possibly for a subsequence we do not relabel). Since $(y_1^k)^{(N-1)} = x_*^{(N-1)} \circ (z_0^k)^{-1}$, this is equivalent to prove that:

$$\|x_*^{(N-1)} \circ (z_0^k)^{-1} - x_*^{(N-1)}\|_{W^{1,m}} \xrightarrow[k\to+\infty]{} 0.$$

The sequence $(z_0^k)_{k\in\mathbb{N}}$ satisfies all the hypotheses of Lemma 5.6.1. Applying it to the reference arc $x = x_*^{(N-1)}$ confirms the claim.
From (5.13), (5.14) and (5.15), we deduce that for all $k \in \mathbb{N}$:

$$\|z_1^k \circ (z_0^k)^{-1} - x_*\|_{W^{N,m}} = \|y_1^k - x_*\|_{W^{N,m}} \xrightarrow[k\to+\infty]{} 0,$$

which contradicts (5.12). □

Proposition 5.6.3 *The trajectory/control pair (z_*, u_*) is a local $W^{1,1}$-minimizer for* (CP)*, i.e. there exists $\rho > 0$, such that, for any admissible pair (z, u) for* (CP)*, we have:*

$$\|z - z_*\|_{W^{1,1}} \leq \rho \Rightarrow J(z, u) \geq J(z_*, u_*).$$

Proof. Let $\varepsilon > 0$ such that $x_*(\cdot)$ is an ε-minimizer for the $W^{N,m}$ topology. We invoke Lemma 5.6.2 to obtain a real $\rho > 0$ such that (5.11) is satisfied.

Take any admissible pair (z, u) for (CP) such that $\|z - z_*\|_{W^{1,1}} \leq \rho$, and define $y \in W^{1,1}([a, b], \mathbb{R}^N)$ by $y = z \circ z_0^{-1}$.

We claim that $y_1 \in W^{N,m}([a, b], \mathbb{R})$ is a solution of the reference problem (CV). Indeed, we have $y_1^{(N-1)} = x_*^{(N-1)} \circ z_0^{-1}$ and $y_1^{(i)} = y_{i+1} = z_{i+1} \circ z_0^{-1}$ for all $i = 1, \ldots, N-2$. Recalling the conditions satisfied by z at a and b, in particular $z_0^{-1}(a) = a$, $z_0^{-1}(b) = b$, we obtain:

$$\left(y_1, \ldots, y_1^{(N-1)}\right)(a) = \left(x_*, \ldots, x_*^{(N-1)}\right)(a), \quad \left(y_1, \ldots, y_1^{(N-1)}\right)(b) = \left(x_*, \ldots, x_*^{(N-1)}\right)(b). \tag{5.16}$$

Using the change of variable $t = z_0(s)$, we obtain:

$$\begin{aligned}
\int_a^b \ell(s, z(s), u(s))\mathrm{d}s &= \int_a^b \widetilde{L}\left(z(s), x_*^{(N-1)}(s), \frac{x_*^{(N)}(s)}{u(s)}\right) u(s)\mathrm{d}s \\
&= \int_a^b \widetilde{L}\left(y(t), x_*^{(N-1)}(z_0^{-1}(t)),\, \dot{z}_0^{-1}(t)\, x_*^{(N)}(z_0^{-1}(t))\right) \mathrm{d}t \\
&= \int_a^b L\left(t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(N-1)}(t), y_1^{(N)}(t)\right) \mathrm{d}t.
\end{aligned}$$

We recall that, from (5.11), we have $\|y_1 - x_*\|_{W^{N,m}} \leq \varepsilon$. Since $x_*(\cdot)$ in a ε-minimizer for the problem (CV). It follows from (5.16) that:

$$\int_a^b L\left(t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(N)}(t)\right) \mathrm{d}t \geq \int_a^b L\left(t, x_*(t), \dot{x}_*(t), \ldots, x_*^{(N)}(t)\right) \mathrm{d}t.$$

We deduce that:

$$J(z, u) = \int_a^b \ell(s, z(s), u(s))\mathrm{d}s \geq \int_a^b \ell(s, z_*(s), u_*(s))\mathrm{d}s = J(z_*, u_*),$$

which concludes the proof.

□

5.6.2 Application of the maximum principle to (CP)

We shall employ the maximum principle [45, Theorem 22.26] to the optimal control problem (CP) and the reference minimizer (z_*, u_*). In our case it is easy to see that all the assumptions of [45, Theorem 22.26] are satisfied and the only detail which requires particular attention is to prove the appropriate Lipschitz regularity of ℓ.

Lemma 5.6.4 *If $\varepsilon_* > 0$ and $k : [a,b]\times]0,+\infty[\to \mathbb{R}_+$ are given by hypothesis (S_{x_*}), then for a.e. $t \in [a,b]$ and for all $\sigma \in]0,+\infty[$, the application:*

$$\begin{cases} \mathbb{R}\times\mathbb{R}^{N-1} \to \mathbb{R}, \\ (s,x_0,\dots,x_{N-2}) \mapsto \widetilde{L}(s,x_0,\dots,x_{N-2},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)), \end{cases} \tag{5.17}$$

is Lipschitz continuous on $\mathrm{B}\Big((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*\Big)$ *with Lipschitz constant* $k(t,\sigma)$.

Proof. Take $\sigma \in]0,+\infty[$ and any $t \in [a,b]$ such that (S_{x_*}) is satisfied and two vectors z, w in the ball $\mathrm{B}((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*)$. We can always assume that $z_0 \le w_0$.
Using (S_{x_*}), if both z_0 and w_0 are in $[a,b]$, the inequality is easily verified.
If $z_0 \le w_0 \le a$. We have:

$$\begin{aligned} &\Big|\widetilde{L}\Big(z,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - \widetilde{L}\Big(w,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)\Big|, \\ &\quad = \Big|L\Big(a,z_1\dots,z_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - L\Big(a,w_1,\dots,w_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)\Big|, \\ &\quad \le k(t,\sigma)|(a,z_1,\dots,z_{N-1}) - (a,w_1,\dots,w_{N-1})|, \\ &\quad \le k(t,\sigma)|z-w|. \end{aligned}$$

If $z_0 \le a \le w_0 \le b$, we have:

$$\begin{aligned} &\Big|\widetilde{L}\Big(z,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - \widetilde{L}\Big(w,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)\Big|, \\ &\quad = \Big|L\Big(a,z_1\dots,z_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - L\Big(w_0,w_1,\dots,w_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)\Big|, \\ &\quad \le k(t,\sigma)|(a,z_1,\dots,z_{N-1}) - (w_0,w_1,\dots,w_{N-1})|, \\ &\quad \le k(t,\sigma)|z-w|. \end{aligned}$$

If $z_0 \le a < b \le w_0$, we have:

$$\begin{aligned} &\Big|\widetilde{L}\Big(z,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - \widetilde{L}\Big(w,x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)|, \\ &\quad = \Big|L\Big(a,z_1\dots,z_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big) - L\Big(b,w_1,\dots,w_{N-1},x_*^{(N-1)}(t),\sigma x_*^{(N)}(t)\Big)\Big|, \\ &\quad \le k(t,\sigma)|(a,z_1,\dots,z_{N-1}) - (b,w_1,\dots,w_{N-1})|, \\ &\quad \le k(t,\sigma)|z-w|. \end{aligned}$$

The cases $a \le z_0 \le b \le w_0$ and $a < b \le z_0 \le w_0$ can be proved in a similar way.

□

We are now ready to show the required Lipschitz regularity of ℓ.

Lemma 5.6.5 *There exists a $\mathcal{L}\times\mathcal{B}_1$-measurable function $\widetilde{k} : [a,b]\times\left[\frac{1}{j},j\right]$ such that $t \mapsto \widetilde{k}(t,1) \in \mathbb{L}^1([a,b],\mathbb{R}_+)$, and for almost every $t \in [a,b]$, we have:*

$$z_1,z_2 \in \mathrm{B}(z_*(t),\varepsilon_*), u \in \mathbb{R} \Rightarrow |\ell(t,z_2,u) - \ell(t,z_1,u)| \le \widetilde{k}(t,u)|z_2 - z_1|.$$

Proof. Define:

$$\widetilde{k}(t,u) := k\Big(t,\frac{1}{u}\Big)\,u.$$

The function $\widetilde{k}$ is $\mathcal{L}\times\mathcal{B}_1$-measurable and $t\mapsto\widetilde{k}(t,1)=k\,(t,1)$ is in $\mathbb{L}^1([a,b],\mathbb{R}_+)$ by hypothesis (S_{x_*}). Take any $z_1,\ z_2$ in $\mathrm{B}(z_*(t),\varepsilon_*)=\mathrm{B}\Big((t,x_*(t),\dots,x_*^{(N-2)}(t)),\varepsilon_*\Big)$ and $u\in\left[\frac{1}{j},j\right]$. Pick any $t\in[a,b]$ at which the Lipschitz continuity of (S_{x_*}) holds. From Lemma 5.6.4, we have:

$$\begin{aligned}|\ell(t,z_2,u)-\ell(t,z_1,u)|&\leq u\left|\widetilde{L}\left(z_2,x^{(N-1)}(t),\frac{x^{(N)}(t)}{u}\right)-\widetilde{L}\left(z_1,x^{(N-1)}(t),\frac{x^{(N)}(t)}{u}\right)\right|\\&\leq k\Big(t,\frac{1}{u}\Big)u\ \ |z_1-z_2|\leq\widetilde{k}(t,u)\,|z_2-z_1|.\end{aligned}$$

□

For $\eta\geq 0$, we define the Hamiltonian of the problem (CP):

$$\begin{aligned}H^\eta(t,z,p,u)&=p\cdot f(t,z,u)-\eta\ell(t,z,u)\\&=u\Big(p_0+p_1z_2+p_2z_3+\dots p_{N-2}z_{N-1}+p_{N-1}x_*^{(N-1)}(t)\Big)-\eta\ell(t,z,u).\end{aligned}$$

Applying [45, Theorem 22.26] for each integer $j\geq 2$, there exist an arc $p^j=(p_0^j,\dots,p_{N-1}^j)\in W^{1,1}([a,b],\mathbb{R}^N)$, a scalar $\eta^j\in\{0,1\}$ and a set of full measure $E_j\subset[a,b]$ such that the following properties are satisfied:

i) The nontriviality condition: $(\eta^j,p^j(t))\neq 0,$ for all $t\in[a,b]$,

ii) The adjoint inclusion:

$$-\dot{p}^j(t)\in\partial_z^C H^\eta(t,\cdot,p^j(t),u_*(t))_{|z=z_*(t)},\ \text{for all } t\in E_j,\tag{5.18}$$

iii) The maximality condition:

$$H^{\eta^j}(t,z_*(t),p^j(t),u_*(t))=\sup_{u\in\left[\frac{1}{j},j\right]}H^{\eta^j}(t,z_*(t),p^j(t),u),\ \text{for all } t\in E_j.\tag{5.19}$$

(Note that for this problem, the transversality condition $(p^j(a),-p^j(b))\in\mathbb{R}^{2N}$ does not provide useful information.)
From the maximality condition (5.19), for all $u\in\left[\frac{1}{j},j\right]$ and every $t\in E_j$ we have:

$$\begin{aligned}u(p_0^j(t)+p_1^j(t)\dot{x}_*(t)+\dots+p_{N-1}^j(t)x_*^{(N-1)}(t))-\eta^j\ell\left(t,z_*(t),\frac{x_*^{(N)}(t)}{u}\right)\leq\\p_0^j(t)+p_1^j(t)\dot{x}_*(t)+\dots+p_{N-1}^j(t)x_*^{(N-1)}(t)-\eta^j\ell\left(t,z_*(t),x_*^{(N)}(t)\right),\end{aligned}$$

which implies that for any $u\in\left[\frac{1}{j},j\right]$ and every $t\in E_j$:

$$\begin{aligned}\eta^jL\left(t,x_*(t),\dots,\frac{x_*^{(N)}(t)}{u}\right)u-\eta^jL\left(t,x_*(t),\dots,x_*^{(N)}(t)\right)\geq\\(u-1)(p_0^j(t)+p_1^j(t)\dot{x}_*(t)+\dots+p_{N-1}^j(t)x_*^{(N-1)}(t)).\end{aligned}\tag{5.20}$$

Proof of Theorem 5.5.1.
We need a lemma which allows us to handle the abnormality phenomenon ($\eta^j=0$) that can arise when we apply the maximum principle.

Lemma 5.6.6 *Assume that there exists an arc $p = (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ such that:*

$$p(t) \neq 0, \text{ for all } t \in [a,b], \tag{5.21}$$

$$p(t) \cdot f(t, z_*(t), 1) = 0, \text{ for a.e. } t \in [a,b], \tag{5.22}$$

$$-\dot{p}(t) = A_N^T p(t), \text{ for a.e. } t \in [a,b], \tag{5.23}$$

where A_N^T is the transpose of the matrix A_N. Then $N \geq 2$, and $x_*(\cdot)$ is a polynomial function with degree at most $N-1$.

Proof. We introduce the following control system, in which $\nu(\cdot)$ is a control function in $\mathbb{L}^1([a,b], \mathbb{R})$:

$$\begin{cases} \dot{w}(s) = A_N w(s) + \nu(s) f(s, z_*(s), 1) \text{ for a.e. } s \in [a,b], \\ w(a) = 0. \end{cases} \tag{5.24}$$

Take any solution (w_ν, ν) to (5.24). For almost every $s \in [a,b]$, we have:

$$\dot{w}_\nu(s) \cdot p(s) = A_N w_\nu(s) \cdot p(s) + \nu(s) f(s, z_*(s), 1) \cdot p(s),$$

Using successively (5.23) and (5.22) gives us:

$$\dot{w}_\nu(s) \cdot p(s) = A_N w_\nu(s) \cdot p(s) = -w_\nu(s) \cdot \dot{p}(s), \text{ for a.e. } s \in [a,b].$$

This implies that $\frac{\mathrm{d}}{\mathrm{d}t}(p \cdot w_\nu) = 0$ a.e. and since $w_\nu(a) = 0$, $p \cdot w_\nu = 0$ in $[a,b]$. Since $p(s) \neq 0$ for all $s \in [a,b]$, for any $\nu \in \mathbb{L}^1([a,b], \mathbb{R})$ the arc $w_\nu(\cdot)$ remains in the hyperplane $\{w \in \mathbb{R}^N \,:\, w \cdot p(s) = 0\}$, for all $s \in [a,b]$. Therefore system (5.24) is not reachable at any time $s \in [a,b]$.
Solving system (5.24), the reachable set at time b is:

$$\mathcal{R}(b) := \left\{ \int_a^b \nu(s)\, \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) \mathrm{d}s \,:\, \nu \in \mathbb{L}^1([a,b], \mathbb{R}) \right\}.$$

From what precedes, $p(b) \neq 0$ and for any $\nu \in \mathbb{L}^1([a,b], \mathbb{R})$,

$$\int_a^b \nu(s)\, p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) \mathrm{d}s = 0.$$

In particular, choosing $\nu \,:\, s \mapsto p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1)$ yields:

$$p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) = 0, \text{ for all } s \in [a,b]. \tag{5.25}$$

If $N = 1$, by definition, we have $\mathrm{e}^{(b-a)A_1} f(s, z_*(s), 1) = 1$ and (5.25) gives that $p(b) = 0$, which is a contradiction. If $N = 2$, by definition, $\mathrm{e}^{(b-a)A_2} f(s, z_*(s), 1) = (1, \dot{x}_*(s))$ for all $s \in [a,b]$. From (5.25) we have that:

$$p_0(b) + p_1(b)\dot{x}_*(s) = 0, \text{ for all } s \in [a,b].$$

Since $p(b) \neq 0$, this implies that $p_1(b) \neq 0$. We obtain $\dot{x}_*(s) = -\frac{p_0(b)}{p_1(b)}$ for all $s \in [a,b]$, which implies that $x_*(\cdot)$ is a polynomial function with degree 0 or 1.
Assume now that $N > 2$. A standard development of the exponential function $\mathrm{e}^{(b-s)A_N}$ shows that for every $s \in [a,b]$:

$$\mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) = \left(1, \sum_{k=0}^{N-2} \frac{(b-s)^k}{k!} x_*^{(k+1)}(s), \sum_{k=0}^{N-3} \frac{(b-s)^k}{k!} x_*^{(k+2)}(s), \dots, x_*^{(N-1)}(s) \right).$$

From equation (5.25), we first deduce that $(p_1(b), \ldots, p_{N-1}(b)) \neq 0$ since $p(b) \neq 0$. Moreover, differentiating both sides in (5.25), we obtain that for almost every $s \in [a, b]$:

$$\left(p_1(b)\frac{(b-s)^{N-2}}{(N-2)!} + p_2(b)\frac{(b-s)^{N-3}}{(N-3)!} + \ldots + p_{N-2}(b)(b-s) + p_{N-1}(b) \right) x_*^{(N)}(s) = 0.$$

Observe that the term that multiplies $x_*^{(N)}(s)$ is a linear combination of linearly independent polynomials with coefficients which cannot be all simultaneously zero. This implies that $x_*^{(N)}(s) = 0$ for almost every $s \in [a, b]$, hence $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N - 1$.

□

Assume first that, for some $j_0 \geq 2$, $\eta^{j_0} = 0$. Then by nontriviality, $p^{j_0}(t) \neq 0$ for all $t \in [a, b]$. Using (5.18), we have that:

$$-\dot{p}^{j_0}(t) = A_N^T p^{j_0}(t) \text{ for all } t \in E_{j_0}.$$

The maximality condition (5.20) in the abnormal case combined with the continuity of the functions $p^{j_0}(\cdot), \dot{x}_*(\cdot), \ldots, x_*^{(N-1)}(\cdot)$ gives:

$$p^{j_0}(t) \cdot f(t, z_*(t), 1) = 0, \text{ for all } t \in [a, b].$$

Then invoking Lemma 5.6.6, we deduce that $x_*(\cdot)$ is a polynomial function whose degree is at most $N - 1$.

We now assume that $\eta^j = 1$ for all $j \geq 2$ and to complete the proof of Theorem 5.5.1 we employ a compactness argument. For every integer $j \geq 2$, we denote $\alpha_j := (\|p^j\|_\infty + 1)$ and define $(\tilde{p}^j, \tilde{\eta}^j) := \alpha_j^{-1}(p^j, \eta^j)$. From Remark 5.4.1, we deduce that for a.e. $t \in [a, b]$:

$$|\dot{\tilde{p}}^j(t)| \leq |A_N^T \tilde{p}^j(t)| + \alpha_j^{-1} k(t, 1) \leq \|A_N\| + k(t, 1), \text{ for all } j \geq 2, \tag{5.26}$$

where $\|A_N\|$ is the matrix norm induced by the vector norm $\|(p_0, \ldots, p_{N-1})\| = \max_{i=0,\ldots,N-1} |p_i|$.

Estimate (5.26) shows that the sequence $(\dot{\tilde{p}}^j)_{j\geq 2}$ is equi-integrable. Then there exists $(\tilde{p}, \tilde{\eta}) \in W^{1,1}([a, b], \mathbb{R}^N) \times [0, 1]$ such that, for a subsequence we do not relabel, $(\tilde{p}^j)_{j\geq 2}$ converges to $\tilde{p}$ in $\mathbb{L}^\infty([a, b], \mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{\tilde{p}}$ weakly in $\mathbb{L}^1([a, b], \mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to $\tilde{\eta}$. We define $(p, \eta) := (\tilde{p}, 0)$ if $\tilde{\eta} = 0$ and $(p, \eta) := (\tilde{\eta}^{-1}\tilde{p}, 1)$ if $\tilde{\eta} > 0$. We also define $\tilde{\mathcal{E}} := \bigcap_{j\geq 2} E_j$, which is a set of full measure as an intersection of such sets.

Fix any $t \in \tilde{\mathcal{E}}$ and $u \in]0, +\infty[$. There exists $j_0 \geq 2$ such that, for all $j \geq j_0$, $u \in [1/j, j]$. Then for all $j \geq j_0$:

$$\eta^j L\left(t, x_*(t), \ldots, \frac{x_*^{(N)}(t)}{u}\right) u - \eta^j L\left(t, x_*(t), \ldots, x_*^{(N)}(t)\right) \geq$$
$$(u-1)(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \ldots + p_{N-1}^j(t)x_*^{(N-1)}(t)).$$

Multiplying both sides of the inequality by α_j^{-1} and passing to the limit as j goes to $+\infty$, we deduce that:

$$\tilde{\eta} L\left(t, x_*(t), \ldots, \frac{x_*^{(N)}(t)}{u}\right) u - \tilde{\eta} L\left(t, x_*(t), \ldots, x_*^{(N)}(t)\right) \geq$$
$$(u-1)(\tilde{p}_0(t) + \tilde{p}_1(t)\dot{x}_*(t) + \ldots + \tilde{p}_{N-1}(t)x_*^{(N-1)}(t)). \tag{5.27}$$

If $\tilde{\eta} = 0$, (5.27) gives

$$\tilde{p}(t) \cdot f(t, z_*(t), 1) = 0. \tag{5.28}$$

If $\tilde{\eta} \neq 0$, dividing both terms in (5.27) by $\tilde{\eta}$, we deduce that (p, η) satisfies the maximality condition (5.20) for all $t \in \tilde{\mathcal{E}}$ and every $u \in]0, +\infty[$.
It remains to prove that the adjoint inclusion is also satisfied by (p, η). Define the function $r_j(t) := |\alpha_j^{-1} - \tilde{\eta}|k(t, 1)$, and note that (5.18) gives:

$$\dot{\tilde{p}}^j(t) \in -\partial_z^C H^{\tilde{\eta}}(t, z_*(t), \tilde{p}^j(t), 1) + \mathrm{B}(0, r_j(t)), \text{ for a.e. } t \in [a, b],$$

with $\|r_j\|_{\mathbb{L}^1} \xrightarrow[j\to+\infty]{} 0$. Invoking [84, Thm 2.5.3], we deduce that $\dot{\tilde{p}}(t) \in -\partial_z^C H^{\tilde{\eta}}(t, z_*(t), \tilde{p}(t), 1)$ for a.e. $t \in [a, b]$, implying that p satisfies (5.6). If $\tilde{\eta} \neq 0$, we divide this differential inclusion by $\tilde{\eta}$ and we obtain that (5.18) is satisfied by the pair (p, η), for a.e. $t \in [a, b]$. Dividing by $\tilde{\eta}$ both terms in (5.27), we conclude that (W) is satisfied for all $u \in]0, +\infty[$ and a.e. $t \in [a, b]$.

If $\tilde{\eta} = 0$, then $\tilde{p}$ satisfies $\dot{\tilde{p}}(s) = -A_N^T \tilde{p}(s)$ for a.e. $s \in [a, b]$ and $\|\tilde{p}\|_\infty = 1$, which implies that $p(s) = \tilde{p}(s) \neq 0$ for all $s \in [a, b]$. Recalling (5.28), we invoke Lemma 5.6.6, and deduce that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N - 1$.

□

Proof of Proposition 5.5.8.
The proof of Proposition 5.5.8 follows along the same lines of the proof of Theorem 5.5.1, except that, when we apply the maximum principle to the auxiliary optimal control problem, the maximality condition is valid only for $u \in [\frac{1}{1+\sigma_*}, \frac{1}{1-\sigma_*}]$ (from $(\mathrm{S}_{x_*})'$). The extension of this property to the set $\{u \in]0, +\infty[\}$ is a consequence of (H)′ invoking a well known convexity argument (cf. [60, 84]).

□

5.7 Proof of Theorem 5.5.2

5.7.1 An auxiliary control problem (CP2)

Employing a standard 'truncation argument' which allows to extend local properties of a given function to global ones (cf. [84]), we introduce the Lagrangian $\widehat{L} : [a, b] \times \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$\widehat{L}(t, (s, x_0, \dots, x_{N-2}), x_N) := \begin{cases} \widetilde{L}(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N), \\ \qquad\qquad \text{if } |(s, x_0, \dots, x_{N-2}) - (t, x_*(t), \dots, x_*^{(N-2)}(t))| \leq \varepsilon, \\ \widetilde{L}\Big(\pi(s, x_0, \dots, x_{N-2}), x_*^{(N-1)}(t), x_N\Big), \text{ otherwise,} \end{cases}$$

where $\pi(s, x_0, \dots, x_{N-2}) := (t, x_*(t), \dots, x_*^{(N-2)}(t)) + \varepsilon \frac{(s, x_0, \dots, x_{N-2}) - (t, x_*(t), \dots, x_*^{(N-2)}(t))}{|(s, x_0, \dots, x_{N-2}) - (t, x_*(t), \dots, x_*^{(N-2)}(t))|}$ is the projection of $(s, x_0, \dots, x_{N-2})$ over the sphere of center $(t, x_*(t), \dots, x_*^{(N-2)}(t))$ with radius ε, and $\widetilde{L}$ is the extension of the Lagrangian L to $\mathbb{R}^{N+2}$ defined as in (5.9). The function $\widehat{L}$ is clearly Borel measurable, $(s, x_0, \dots, x_{N-2}, x_N) \mapsto \widehat{L}(t, (s, x_0, \dots, x_{N-2}), x_N)$ is lower semicontinuous for all $t \in [a, b]$. Moreover $\widehat{L}$ satisfies a global (stronger) version of condition $(\mathrm{S}_{x_*}^\infty)$ iii). More precisely, for every $(t, (\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}), x_N) \in E \times \mathbb{R}^N \times \mathbb{R}$, we have

$$|\zeta| \leq c\Big(|(1, \bar{x}_1, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t))| + \widehat{L}(t, (\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}), x_N) + \lambda|x_N|)\Big) + d(t) \tag{5.29}$$

for all $\zeta \in \partial^P_{(s,x_0,\dots,x_{N-2})}\widehat{L}(t,(\bar{s},\bar{x}_0,\dots,\bar{x}_{N-2}),x_N)$.
Fix any integer $j \geq 2$. We set $\widehat{\ell} : [a,b]\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by: for all $(t,z,u)\in[a,b]\times\mathbb{R}^N\times\mathbb{R}$,

$$\widehat{\ell}(t,z,u) := \begin{cases} \widehat{L}\left(t,z,\frac{x_*^{(N)}(t)}{u}\right)u, \text{ if } x_*^{(N)}(t) \text{ is defined and } u\in[1/j,j],\\ +\infty, \text{ otherwise.}\end{cases}$$

We also consider the following control problem, which differs from (CP) since it allows to consider extended valued Lagrangians and incorporates the 'control constraint' in the integral term:

$$\begin{cases} \text{Minimize } J(z,u) := \displaystyle\int_a^b \widehat{\ell}(s,z(s),u(s))\mathrm{d}s,\\ \text{over arcs } z\in W^{1,1}([a,b],\mathbb{R}^N)\\ \text{and } \mathcal{L}\text{-measurable functions } u:[a,b]\to\mathbb{R} \text{ such that:}\\ \dot{z}(s) = f(s,z(s),u(s)), \text{ for a.e. } s\in[a,b],\\ z(a) = (a,x_*(a),\dot{x}_*(a),\dots,x_*^{(N-2)}(a)),\\ z(b) = (b,x_*(b),\dot{x}_*(b),\dots,x_*^{(N-2)}(b)),\end{cases} \tag{CP2}$$

Observe that $\widehat{\ell}$ is an extended valued function (the value $+\infty$ might arise for some $t\in[a,b]$, even if $u\in[1/j,j]$), so we cannot invoke [45, Theorem 22.26]. However, the structure of the function f allows us to employ the hybrid maximum principle [44, Theorem 5.3.1].
The definition of $\widehat{\ell}$ has the following consequence: any admissible trajectory/control pair (z,u) to (CP2) with a finite cost is also an admissible trajectory/control pair for (CP). This gives the same minimizing property to the pair (z_*,u_*) for the problem (CP2).

Lemma 5.7.1 *The pair (z_*,u_*) is a local $W^{1,1}$-minimizer for* (CP2).

We check that all the relevant hypotheses of [44, Theorem 5.3.1] are satisfied. First of all, we observe that the function f is Lebesgue measurable in the time variable t, and continuously differentiable with respect to (z,u).

We claim that $\widehat{\ell}(t,\cdot,\cdot)$ is lower semicontinuous for all $t\in[a,b]$.
Fix any $t\in[a,b]$. We can assume that $x_*^{(N)}(t)$ exists otherwise the lower semicontinuity is an immediate consequence of the definition of $\widehat{\ell}$. Take any $z=(z_0,\dots,z_{N-1})\in\mathbb{R}^N$ and $u\in\mathbb{R}$. Assume first that $u\in]-\infty,1/j[$ or $u\in]j,+\infty[$. Since $]-\infty,1/j[$ and $]j,+\infty[$ are open subsets, the definition of $\widehat{\ell}$ yields:

$$+\infty = \liminf_{(z',u')\to(z,u)}\widehat{\ell}(t,z',u') \geq \widehat{\ell}(t,z,u) = +\infty.$$

Now assume that $u\in[1/j,j]$ and $z_0\in]a,b[$. Recalling that $\widehat{L}$ satisfies $(\mathrm{S}^\infty_{x_*})$ i), we have:

$$\begin{aligned}\liminf_{(z',u')\to(z,u)}\widehat{\ell}(t,z',u') &\geq \liminf_{\{(z',u')\to(z,u)\,:\,u'\in[1/j,j]\}}\widehat{\ell}(t,z',u'),\\ &= \liminf_{\{(z',u')\to(z,u)\,:\,u'\in[1/j,j]\}}\widehat{L}\left(t,z',\frac{x_*^{(N)}(t)}{u'}\right)u',\\ &\geq \widehat{L}\left(t,z,\frac{x_*^{(N)}(t)}{u}\right)u,\\ &= \widehat{\ell}(t,z,u).\end{aligned}$$

On the other hand, if $u \in [1/j, j]$ and $z_0 \leq a$ or $z_0 \geq b$, then using the definition of $\widehat{L}$ and $(\mathrm{S}^{\infty}_{x_*})$ i), we once again obtain that:

$$\liminf_{(z',u')\to(z,u)} \widehat{\ell}(t,z',u') \geq \widehat{\ell}(t,z,u),$$

confirming the claim.

We proceed to check that the appropriate growth conditions are satisfied by f and $\widehat{\ell}$. From the global version of $(\mathrm{S}^{\infty}_{x_*})$ iii) (see (5.29)), there exist strictly positive constants c, λ and $d \in \mathbb{L}^1([a,b],\mathbb{R}_+)$ such that, for all $(t,(\bar{s},\bar{x}_0,\ldots,\bar{x}_{N-2}),x_N) \in E \times \mathbb{R}^N \times \mathbb{R}$,

$$|\zeta| \leq c\Big(|(1,\bar{x}_1,\ldots,\bar{x}_{N-2},x_*^{(N-1)}(t))| + \widehat{L}(t,(\bar{s},\bar{x}_0,\ldots,\bar{x}_{N-2}),x_N) + \lambda|x_N|\Big) + d(t), \tag{5.30}$$

for any $\zeta \in \partial^P_{s,x_0,\ldots,x_{N-2}} \widehat{L}(t,(\bar{s},\bar{x}_0,\ldots,\bar{x}_{N-2}),x_N)$.
Take a bounded subset K of $\mathbb{R}^N$. Let $(t,z,u) \in [a,b] \times K \times \mathbb{R}$ such that $\widehat{\ell}(t,z,u) < +\infty$. We have:

$$\|\nabla_z f(t,z,u)\| = u\|A_N\| \leq j\|A_N\|.$$

We claim that there exist $c_K > 0$ and $d_K \in \mathbb{L}^1([a,b],\mathbb{R})$, such that, for all $(\nu,\psi) \in \partial^P_{(z,u)} \widehat{\ell}(t,z,u)$, we have:

$$\frac{|\nu|(1+\|\nabla_u f(t,z,u)\|)}{1+|\psi|} \leq c_K\left(|f(t,z,u)| + \widehat{\ell}(t,z,u) + d_K(t)\right).$$

Observe that it is enough to prove that for all $\nu \in \partial^P_z \widehat{\ell}(t,z,u)$:

$$|\nu|(1+\|\nabla_u f(t,z,u)\|) \leq c_K\left(|f(t,z,u)| + \widehat{\ell}(t,z,u) + d_K(t)\right). \tag{5.31}$$

Fix any $\nu \in \partial^P_z \widehat{\ell}(t,z,u)$. We can find $\zeta \in \partial^P_{s,x_0,\ldots,x_{N-2}} \widehat{L}\Big(t,z,x_*^{(N)}(t)/u\Big)$ such that $\nu = u\zeta$. Moreover, from (5.30) we obtain (recall that $z = (z_0,\ldots,z_{N-1})$):

$$\begin{aligned} |\zeta| \leq c\Big(\Big|\Big(1,z_2,\ldots,z_{N-1},x_*^{(N-1)}(t)\Big)\Big| + \\ \widehat{L}\Big(t,(z_0,\ldots,z_{N-1}),x_*^{(N)}(t)/u\Big) + \frac{\lambda}{u}|x_*^{(N)}(t)|\Big) + d(t). \end{aligned} \tag{5.32}$$

Note that since $z \in K$ and $x_*^{(N-1)}(\cdot)$ is bounded on $[a,b]$, for some constant $\widetilde{c} > 0$, we also have:

$$1 + \|\nabla_u f(t,z,u)\| = 1 + |A_N z + b(t)| \leq 1 + |(1,z_2,\ldots,z_{N-1},x_*^{(N-1)}(t))| \leq \widetilde{c}.$$

Hence from (5.32) we have:

$$|\nu|(1+\|\nabla_u f(t,z,u)\|) \leq \widetilde{c}c\left(|f(t,z,u)| + \widehat{\ell}(t,z,u) + \lambda|x_*^{(N)}(t)| + \frac{j}{c}d(t)\right).$$

Recalling that $t \mapsto \lambda|x_*^{(N)}(t)|$ is in $\mathbb{L}^1([a,b],\mathbb{R})$ since $x_* \in W^{N,m}([a,b],\mathbb{R})$, we define $d_K(\cdot) := \frac{j}{c}d(\cdot) + \lambda|x_*^{(N)}(\cdot)|$ and $c_K := \widetilde{c}c$, confirming (5.31).

To better handle the abnormal case that can arise from the maximum principle, we need some information about the first coordinate of $\partial^{\infty}_{(z,u)} \widehat{\ell}(t,z_*(t),1)$ and $\partial^{L}_{(z,u)} \widehat{\ell}(t,z_*(t),1)$, which are provided by the following lemma.

Lemma 5.7.2 *1) For a.e. $t \in]a,b[$, if $(\nu,\psi) \in \partial^{\infty}_{(z,u)} \widehat{\ell}(t,z_*(t),1)$ then $\nu = 0$.*
2) For a.e. $t \in]a,b[$, if $(\nu,\psi) \in \partial^{L}_{(z,u)} \widehat{\ell}(t,z_*(t),1)$ then $|\nu| \leq \beta(t)$.

Proof. 1) Recalling the characterization [84, Theorem 4.6.2] of asymptotic limiting subgradients, there exist a sequence $(z^i,u_i)_{i\in\mathbb{N}}$ such that $\widehat{\ell}(t,z^i,u_i)<+\infty$ for all $i\in\mathbb{N}$ and $(z^i,u_i)\xrightarrow[i\to+\infty]{}(z_*(t),1)$, a sequence $(h_i)_{i\in\mathbb{N}}$ of positive real numbers such that $h_i\downarrow 0$, and a sequence $(\nu_i,\psi_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ for which the following property is satisfied:

$$\forall i\in\mathbb{N}, h_i^{-1}(\nu_i,\psi_i)\in\partial^P_{(z,u)}\widehat{\ell}(t,z^i,u_i)\ \text{ and }\ (\nu_i,\psi_i)\xrightarrow[i\to+\infty]{}(\nu,\psi).$$

For each $i\in\mathbb{N}$, there exists a vector $\zeta_i\in\partial^P_{(s,x_0,\dots,x_{N-2})}\widehat{L}\left(t,z^i,\frac{x_*^{(N)}(t)}{u_i}\right)$ such that $h_i^{-1}\nu_i=u_i\zeta_i$. Also, from the definition of $\widehat{L}$ and hypothesis $(\mathrm{S}^\infty_{x_*})$ iv) for L, there exists $i_t\in\mathbb{N}$ for which:

$$|\zeta_i|\le\beta(t),\ \text{for all } i\ge i_t.$$

Hence we obtain:

$$|\nu|\le\limsup_{i\to+\infty}u_ih_i|\zeta_i|\le\limsup_{i\to+\infty}u_ih_i\beta(t)=0,$$

which implies $\nu=0$.

2) There exist a sequence $(z^i,u_i)_{i\in\mathbb{N}}$ such that $(z^i,u_i)\xrightarrow[i\to+\infty]{}(z_*(t),1)$ and $\widehat{\ell}(t,z^i,u_i)\xrightarrow[i\to+\infty]{}\widehat{\ell}(t,z_*(t),1)$ and a sequence $(\nu_i,\psi_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ satisfying:

$$(\nu_i,\psi_i)\xrightarrow[i\to+\infty]{}(\nu,\psi)\text{ and }(\nu_i,\psi_i)\in\partial^P_{(z,u)}\widehat{\ell}(t,z^i,u_i),\ \text{for all } i\in\mathbb{N}.$$

As in the proof of 1), for all $i\in\mathbb{N}$, there exists $\zeta_i\in\partial^P_{(s,x_0,\dots,x_{N-2})}\widehat{L}\left(t,z^i,\frac{x_*^{(N)}(t)}{u_i}\right)$ such that $\nu_i=u_i\zeta_i$. From $(\mathrm{S}^\infty_{x_*})$ iv), there exists $i_t\in\mathbb{N}$ for which:

$$|\zeta_i|\le\beta(t),\ \text{for all } i\ge i_t.$$

Hence we obtain:

$$|\nu|=\lim_{i\to+\infty}u_i|\zeta_i|\le\beta(t),$$

which concludes the proof of Lemma 5.7.2.

□

For $\eta\ge 0$, we define the Hamiltonian of the problem (CP2):

$$\begin{aligned}\widehat{H}^\eta(t,z,p,u)&=p\cdot f(t,z,u)-\eta\widehat{\ell}(t,z,u)\\&=u\Big(p_0+p_1z_2+p_2z_3+\dots p_{N-2}z_{N-1}+p_{N-1}x_*^{(N-1)}(t)\Big)-\eta\widehat{\ell}(t,z,u).\end{aligned}$$

Applying [44, Theorem 5.3.1] to (CP2) for each $j\ge 2$, there exist an arc $p^j=(p_0^j,\dots,p_{N-1}^j)\in W^{1,1}([a,b],\mathbb{R}^N)$, a scalar $\eta^j\in\{0,1\}$ and a set of full measure $E_j\subset[a,b]$ such that the following conditions hold:

i) The nontriviality condition: $(\eta^j,p^j(t))\neq 0,\ \text{for all } t\in[a,b]$,

ii) The maximality condition: for all $u\in[1/j,j]$ s.t. $L(t,x_*(t),\dots,x_*^{(N-1)}(t),x_*^{(N)}(t)/u)<+\infty$,

$$\widehat{H}^{\eta^j}(t,z_*(t),p^j(t),u_*(t))\le\widehat{H}^{\eta^j}(t,z_*(t),p^j(t),u),\ \text{for all } t\in E_j,\tag{5.33}$$

iii) The adjoint inclusion: for all $t \in E_j$,

$$\dot{p}^j(t) \in \text{co}\left\{\omega \in \mathbb{R}^N \,:\, \Big(\omega + A_N^T p^j(t), f(t, z_*(t), 1)\cdot p^j(t)\Big)\right) \in \partial^{L,\eta^j}_{(z,u)}\widehat{\ell}(t, z_*(t), 1)\Big\}, \tag{5.34}$$

where $\partial^{L,\eta^j}_{(z,u)}\widehat{\ell}(t, z_*(t), 1) = \partial^{\infty}_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$ if $\eta^j = 0$, $\partial^{L}_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$ if $\eta^j = 1$.

In particular from (5.33), we obtain the following equation: for a.e. $t \in E_j$ and every $u \in [1/j, j]$ such that $L\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_*^{(N)}(t)/u\Big) < +\infty$,

$$\begin{aligned}\eta^j L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - \eta^j L\left(t, x_*(t), \dots, x_*^{(N)}(t)\right) \geq \\ (u-1)\big(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \ldots + p_{N-1}^j(t)x_*^{(N-1)}(t)\big).\end{aligned} \tag{5.35}$$

Observe that condition (5.35) can be expressed using L instead of $\widehat{L}$ since, from the definition of $\widehat{L}$, we have that $\widehat{L}\Big(t, \big(t, x_*(t), \dots, x_*^{(N-2)}(t)\big), \frac{x_*^{(N)}(t)}{u}\Big) = L\left(t, x_*(t), \dots, x_*^{(N-1)}(t), \frac{x_*^{(N)}(t)}{u}\right)$.

5.7.2 Compactness argument

Let I be the set $I := \{j \geq 2, \eta^j = 0\}$. Two cases may occur: I is either infinite or finite.

Assume first that I is infinite. Then we can extract a subsequence (we do not relabel) such that $(p^j, \eta^j)_{j\geq 2}$ satisfies $\eta_j = 0$ for all $j \geq 2$. Then by nontriviality, $p^j(t) \neq 0$ for all $j \geq 2$ and all $t \in [a, b]$. For all $j \geq 2$, we define $\tilde{p}^j(t) := \frac{p^j(t)}{\|p^j\|_\infty}$. Using (5.34) and property 1) of Lemma 5.7.2, we have that:

$$|\dot{\tilde{p}}^j(t)| \leq |A_N^T \tilde{p}^j(t)| \leq \|A_N\| \text{ for all } j \geq 2 \text{ and for all } t \in \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}} := \bigcap_{j\geq 2} E_j$ is a set of full measure. This implies that the sequence $(\dot{\tilde{p}}^j)_{j\geq 2}$ is equi-integrable. Then there exists $(\tilde{p}, \tilde{\eta}) \in W^{1,1}([a, b], \mathbb{R}^N) \times \{0\}$ such that, for a subsequence we do not relabel, $(\tilde{p}^j)_{j\geq 2}$ converges to p in $\mathbb{L}^\infty([a, b], \mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{p}$ weakly in $\mathbb{L}^1([a, b], \mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to 0. Since $\dot{\tilde{p}}^j(t) = -A_N^T \tilde{p}^j(t)$ for all $j \geq 2$ and a.e. $t \in [a, b]$, we invoke [84, Thm 2.5.3] and obtain that $\dot{\tilde{p}}(t) = -A_N^T \tilde{p}(t)$ for a.e. $t \in [a, b]$. Recalling that $\|\tilde{p}\|_\infty = 1$, this implies that $\tilde{p}(t) \neq 0$ for all $t \in [a, b]$.
Passing to the limit in (5.35), we have that for almost every $t \in [a, b]$,

$$(u-1)(p(t)\cdot f(t, z_*(t), 1)) \leq 0, \text{ for all } u \in]0, +\infty[\text{ s.t. } L\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) < +\infty.$$

Then, invoking hypothesis ($\mathrm{S}^\infty_{x_*}$) ii), we obtain $p(t)\cdot f(t, z_*(t), 1) = 0$ for a.e. $t \in [a, b]$, and by continuity of $p(\cdot), x_*(\cdot), \dots, x_*^{(N-1)}(\cdot)$, we derive that $p(t)\cdot f(t, z_*(t), 1) = 0$ for all $t \in [a, b]$. Using Lemma 5.6.6, we deduce that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N - 1$.

Assume now that I is finite. Extracting a subsequence if so needed, we can assume that $\eta_j = 1$ for all $j \geq 2$. We define $\alpha_j := \|p^j\|_\infty + 1$ for all $j \geq 2$ and $(\tilde{p}^j(t), \tilde{\eta}^j) := \alpha_j^{-1}(p^j(t), \eta^j)$. We obtain that, for a.e. $t \in [a, b]$

$$\dot{\tilde{p}}^j(t) \in \text{co}\left\{\omega \in \mathbb{R}^N \,:\, \Big(\omega + A_N^T \tilde{p}^j(t), f(t, z_*(t), 1)\cdot \tilde{p}^j(t)\Big)\right) \in \alpha_j^{-1}\partial^{L}_{(z,u)}\widehat{\ell}(t, z_*(t), 1)\Big\}.$$

As a consequence of property 2) of Lemma 5.7.2, we deduce that:

$$|\dot{\tilde{p}}^j(t)| \leq \|A_N\| + \alpha_j^{-1}\beta(t) \leq \|A_N\| + \beta(t), \text{ for a.e. } t \in [a, b].$$

This implies that the sequence $(\dot{\tilde{p}}^j)_{j\geq 2}$ is equi-integrable. Then there exists a pair $(\tilde{p},\tilde{\eta})\in W^{1,1}([a,b],\mathbb{R}^N)\times[0,1]$ such that, for a subsequence (we do not relabel), $(\tilde{p}^j)_{j\geq 2}$ converges to $\tilde{p}$ in $\mathbb{L}^\infty([a,b],\mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{\tilde{p}}$ weakly in $\mathbb{L}^1([a,b],\mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to $\tilde{\eta}$.
Employing a standard argument (cf. [84, pages 250-251]), we obtain that for a.e. $t\in[a,b]$:

$$\dot{\tilde{p}}(t)\in \mathrm{co}\left\{\omega\in\mathbb{R}^N\,:\,\Big(\omega+A_N^T\tilde{p}(t),f(t,z_*(t),1)\cdot\tilde{p}(t)\Big)\right)\in\tilde{\eta}\partial^L_{(z,u)}\widehat{\ell}(t,z_*(t),1)\Big\}.$$

If $\tilde{\eta}=0$, we proceed as in the first case, and conclude by Lemma 5.6.6 that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N-1$. If $\tilde{\eta}>0$, then $p:=\tilde{\eta}^{-1}\tilde{p}$ satisfies (5.7) and (W) of the theorem.

5.8 Proof of Theorem 5.5.6

Regularity of the minimizer. From Theorem 5.5.1 or 5.5.2, we deduce that $x_*(\cdot)$ is a polynomial function or that the Weierstrass condition (W) is satisfied. If $x_*(\cdot)$ is a polynomial function, then $x_*^{(N)}(\cdot)$ is obviously essentially bounded on $[a,b]$. We therefore assume without restriction that condition (W) is valid for an arc $(p_0,\ldots,p_{N-1})\in W^{1,1}([a,b],\mathbb{R}^N)$. From Corollary 5.5.4, condition (W_r) is satisfied for the same arc $(p_0,\ldots,p_{N-1})$.

Recalling the definition of Λ and $\partial_r\Lambda$, (W_r) implies that for a.e. $t\in[a,b]$:

$$L\Big(t,x_*(t),\ldots,x_*^{(N)}(t)\Big)-\big(p_0(t)+p_1(t)\dot{x}_*(t)+\ldots+p_{N-1}(t)x_*^{(N-1)}(t)\big)\in\partial_r\Lambda(t,x_*^{(N)}(t),1).\tag{5.36}$$

Let $Q:[a,b]\times\mathbb{R}\to\mathbb{R}$ be a map such that for a.e. $t\in[a,b]$

$$Q(t,x_*^{(N)}(t))=L\Big(t,x_*(t),\ldots,x_*^{(N)}(t)\Big)-\big(p_0(t)+p_1(t)\dot{x}_*(t)+\ldots+p_{N-1}(t)x_*^{(N-1)}(t)\big).$$

We set $M:=1+\|p_0+p_1\dot{x}_*+\ldots+p_{N-1}x_*^{(N-1)}\|_\infty$. From the growth condition (G_{x_*}), we can find a set of full measure $\mathcal{E}\subset[a,b]$, and a constant $R>0$ satisfying:

$$\forall(t,\xi)\in\mathcal{E}\times\mathbb{R},Q(t,\xi)\in\partial_r\Lambda(t,\xi,1),\ |\xi|\geq R\Rightarrow|L(t,x_*(t),\ldots,\xi)-Q(t,\xi)|\geq M,$$

that is to say:

$$\forall(t,\xi)\in\mathcal{E}\times\mathbb{R},Q(t,\xi)\in\partial_r\Lambda(t,\xi,1),\ |\xi|\geq R\Rightarrow|p_0(t)+p_1(t)\dot{x}_*(t)+\ldots+p_{N-1}(t)x_*^{(N-1)}(t)|\geq M.$$

From the definition of M, we immediately deduce that $|x_*^{(N)}(t)|\leq R$ for a.e. $t\in[a,b]$, which concludes the proof of the theorem.

□

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DOCTORAT BRETAGNE LOIRE / MATHSTIC

Université de Bretagne Occidentale

Titre : Conditions Nécessaires et Suffisantes en Contrôle Optimal et Applications

Mots-clés : Contrôle optimal, équation de Hamilton-Jacobi-Bellman, fonction valeur, conditions nécessaires d'optimalité, régularité des minimiseurs.

Résumé : La partie 1 de cette thèse traite d'un problème de Bolza non-autonome en contrôle optimal pour lequel la dynamique et le lagrangien sont continus en temps seulement presque partout.

Plusieurs caractérisations (proximale, contingente et viscosité) de la fonction valeur du problème en tant qu'unique solution généralisée de l'équation de Hamilton-Jacobi-Bellman correspondante sont démontrées dans la classe des fonctions semi-continues inférieurement.

Le cas d'un problème avec contrainte d'état est aussi considéré. Des conditions de compatibilité *ad hoc* entre l'ensemble des contraintes et la dynamique sont introduites, ce qui permet d'approximer les trajectoires violant la contrainte d'état par des trajectoires faisables et par suite d'établir des caractérisations (proximale, contingente et viscosité) de la fonction valeur en tant qu'unique solution de l'équation de Hamilton-Jacobi-Bellman.

La partie 2 de cette thèse traite d'un problème de Bolza non-autonome d'ordre N dans lequel le lagrangien est seulement Borel mesurable, et peut prendre pour valeur $+\infty$.
On établit d'abord les conditions nécessaires d'optimalité sous la forme d'une équation du type Euler-Lagrange et d'une équation du type Erdmann – Du Bois-Reymond, sans imposer au lagrangien la convexité par rapport à sa dernière variable, ni aucune condition de croissance particulière.
En imposant en plus au lagrangien une condition de croissance plus générale que la croissance super-linéaire utilisée habituellement, les conditions nécessaires sont mises à profit afin d'établir que la dernière dérivée d'un minimiseur de ce problème est essentiellement bornée.

Title: Necessary and Sufficient Conditions in Optimal control and Applications

Keywords: Optimal control, Hamilton-Jacobi-Bellman equation, value function, necessary conditions for optimality, regularity of minimizers.

Abstract: The part 1 of this thesis focuses on a non autonomous Bolza problem in optimal control for which the Lagrangian L and the dynamics F are allowed to be discontinuous with respect to time on a set of full measure (with left and right limits everywhere). Several characterizations (contingent, proximal, viscosity) of the value function of the problem as the unique solution to the corresponding Hamilton-Jacobi-Bellman equation are established in the class of lower semicontinuous functions.
The state-constrained case is also considered. Some appropriate compatibility conditions between the state of constraints and the dynamics are introduced. They allow to establish a $W^{1,1}$ neighbouring feasible trajectories result which is then exploited to prove several characterizations (contingent, proximal, viscosity) of the value function V as the unique generalized solution to the Hamilton-Jacobi equation.

Part 2 of this thesis presents results concerning a non autonomous high order Bolza problem in which the Lagrangian is merely Borel measurable, and is possibly extended valued.
Necessary conditions for optimality in the Euler-Lagrange form and in the Erdmann – Du Bois-Reymond form are provided, without imposing on the Lagrangian to be convex with respect to the last variable, nor to have any kind of specific growth behavior.
By adding an extra growth assumption that is more general than the usual superlinearity with respect to the last variable, the necessary conditions are exploited to establish that the last derivative of a given minimizer is essentially bounded.