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# Stabilization of some coupled systems involving (thermo-) viscoelastic/elastic transmission problems or telegraph equations in bounded domains or in networks

Alaa Hayek

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**Thèse de doctorat**  
**Pour obtenir le grade de Docteur de**  
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**Stabilisation de quelques systèmes couplés impliquant problèmes de  
transmission (thermo-) viscoélastiques/élastiques ou équations  
télégraphiques dans des domaines bornés ou sur réseaux**

**JURY**

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**Stabilization of some coupled systems involving  
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*“Solving a problem for which you know there is an answer is like climbing a mountain with a guide, along a trail someone else has laid. In mathematics, the truth is somewhere out there in a place no one knows, beyond all the beaten paths. And it’s not always at the top of the mountain. It might be in a crack on the smoothest cliff or somewhere deep in the valley. ”*

YOKO OGAWA.



**Stabilization of some coupled systems involving  
(thermo-)viscoelastic/elastic transmission problems or telegraph  
equations in bounded domains or in networks**

by Alaa HAYEK

## Abstract

This thesis is concerned with the stabilization of some coupled systems that involves (thermo-) viscoelastic/elastic transmission problems and telegraph equations in bounded domains or in networks. First, we consider the stability of a system of weakly coupled wave equations with one or two local internal Kelvin-Voigt dampings located via a non-smooth coefficient in a part of the domain. First, using a unique continuation theorem, we obtain a new uniqueness result and therefore, we show that our system is strongly stable. Next, we show that the system is not always exponentially stable. However, using a frequency domain approach combined with a multiplier technique, we establish different polynomial stability results in any space dimension by assuming that the damping region satisfies some geometric conditions. Second, we study the existence, uniqueness and stability of a generalized telegraph equation set on a one-dimensional star-shaped network. On the interior common node, we consider a dynamic boundary condition (called the improved Kirchhoff condition), while on the exterior nodes of the network, a dissipative boundary condition is applied. Using frequency domain approach combined with a multiplier technique and the construction of a new multiplier satisfying some ordinary differential inequalities, we prove that the system is uniformly (exponentially) stable. Finally, we investigate the indirect stability of an elastic/thermo-elastic transmission problem on networks. On each elastic edge, we consider two conservative wave equations while, on each thermo-elastic edge, we consider two coupled wave equations such that one of them is damped via a coupling with a heat equation. Mainly, we show that the stability of the system is influenced by the thermal effect and the speed of the two propagating waves. Consequently, different results of exponential or polynomial stability are established.

\*\*\*\*\* **Keywords** \*\*\*\*\*

Exponential stability, analytic semigroup, polynomial stability, Kelvin-Voigt, thermo-elastic, telegraph, networks, improved Kirchhoff law, dynamic boundary condition.





# Stabilisation de quelques systèmes couplés impliquant problèmes de transmission (thermo-) viscoélastiques/élastiques ou équations télégraphiques dans des domaines bornés ou sur réseaux

par Alaa HAYEK

## Résumé

Cette thèse s'intéresse à la stabilisation de certains systèmes couplés impliquant des problèmes de transmission (thermo-) viscoélastique/élastique et des équations télégraphiques dans des domaines bornés ou sur réseaux. Dans un premier temps, nous considérons la stabilité d'un système d'équations d'ondes faiblement couplées avec un ou deux amortissements de type Kelvin-Voigt. L'amortissement est localisé via un coefficient singulier dans une partie du domaine. D'abord, en adaptant un théorème de continuation unique, nous obtenons un nouveau résultat d'unicité et nous montrons, par conséquence, que notre système est fortement stable. Ensuite, nous montrons que le système n'est pas exponentiellement stable. Cependant, en utilisant une méthode fréquentielle combinée avec une méthode de multiplicateur par morceaux, nous établissons différents type de stabilisation polynomiale en toute dimension d'espace dès que la région d'amortissement satisfait certaines conditions géométriques. Dans un second temps, nous étudions l'existence, l'unicité et la stabilité de la solution d'une équation télégraphique généralisée sur un réseau en forme d'étoile mono-dimensionnel. Sur le nœud commun intérieur, nous considérons une condition aux limites dynamique (appelée condition de Kirchhoff améliorée), tandis que sur les nœuds extérieurs du réseau, une condition aux limites dissipative est appliquée. En utilisant une méthode fréquentielle combinée avec une nouvelle technique de multiplicateur, nous montrons que le système est uniformément (exponentiellement) stable. Finalement, nous étudions la stabilité indirecte d'un problème de transmission élastique/thermo-élastique sur réseaux. Sur chaque bord élastique, on considère deux équations d'onde conservatives tandis que, sur chaque bord thermo-élastique, on considère deux équations d'onde couplées avec un seul amortissement thermique. Nous montrons que la stabilité du système est influencée par l'effet thermique et la vitesse de propagation des deux ondes. Différents résultats de stabilité exponentielle ou polynomiale sont établis.

\*\*\*\*\* **Mots clés** \*\*\*\*\*

Stabilité exponentielle, semi-groupe analytique, stabilité polynomiale, Kelvin-Voigt, thermo-élastique, télégraphe, réseaux, loi de Kirchhoff améliorée, condition aux limites dynamique.



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*“Progress is never permanent, will always be threatened, must be redoubled, restated and reimagined if it is to survive.”*

ZADI SMITH



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# List of Symbols

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^*$	The set of non-zero real numbers.
$\mathbb{R}_+^*$	The set of strictly positive real numbers.
$\mathbb{N}^*$	The set of non-zero natural numbers.
$\mathbb{C}$	The set of complex numbers.
$\Im$ or $\text{Im}$	The imaginary part of a complex number.
$\Re$ or $\text{Re}$	The real part of a complex number.
$\mathcal{D}(\Omega)$	The set of smooth functions in $\Omega$ .
$\mathcal{D}'(\Omega)$	The space of distributions in $\Omega$ .
$\langle \cdot, \cdot \rangle_X$	The inner product in $X$ .
$\ \cdot\ _X$	The norm in $X$ .
$f_y = \frac{\partial f}{\partial y}$	The first-order partial derivative of $f$ with respect to $y$ .
$f_{yy} = \frac{\partial^2 f}{\partial y^2}$	The second-order partial derivative of $f$ with respect to $y$ .
$\partial_t f = f'$	The first-order partial derivative of $f$ with respect to time $t$ .

## General notations on graphs

Let  $\mathcal{N}$  be a finite connected network embedded in the Euclidean space  $\mathbb{R}^m, m \in \mathbb{N}^*$  with  $n$  vertices  $\mathcal{V} = \{a_0, a_2, \dots, a_{n-1}\}$  and  $N$  edges  $E = \{e_1, \dots, e_N\}$ , with  $\mathcal{I}(\mathcal{N}) = \{1, \dots, N\}$ , the set of indices of edges. Each edge  $e_j$  is a curve, parametrized by

$$\pi_j : [0, \ell_j] \rightarrow e_j : x_j \rightarrow \pi_j(x_j). \quad (0.0.1)$$

The degree of a vertex is the number of incident edges at the vertex. A vertex with degree 1 is called an exterior vertex. On the other hand, a vertex with degree greater than 1 is called an interior vertex. We further denote by

$V_{\text{ext}}$  The set of exterior vertices of  $\mathcal{N}$ .

$V_{\text{int}}$  The set of interior vertices of  $\mathcal{N}$ .

$\mathcal{I}(a_k)$  The set of indices of edges incident to  $a_k$ .

$\mathcal{I}_{\text{ext}}$  The set of indices of edges adjacent to an exterior vertex of  $\mathcal{N}$ .

The incidence matrix  $D = (d_{kj})_{n \times N}$  of  $\mathcal{N}$  is defined by

$$d_{kj} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = a_k, \\ -1 & \text{if } \pi_j(0) = a_k, \\ 0 & \text{otherwise,} \end{cases}$$

and for a function  $f : \mathcal{N} \rightarrow \mathbb{C}$ , a complex valued function on  $\mathcal{N}$ , we set  $f^j = f \circ \pi_j : [0, \ell_j] \rightarrow \mathbb{C}$  its restriction to the edge  $e_j$ . For simplicity, we write  $f = (f^1, \dots, f^N)$  and we denote  $f^j(x) = f^j(\pi_j(x))$  for any  $x$  in  $(0, \ell_j)$ , refer to Valein and Zuazua, 2009; Abdallah and Shel, 2012 for more details.

***This thesis is dedicated to my parents.  
For their endless love, support and  
encouragement.***

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# Part 1

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◇ **General Introduction** ◇

**Main Introduction**

**&**

**Thesis Overview**





---

*“It’s the question we can’t answer that  
teach us the most. They teach us to think.  
If you give a man an answer, all he gains  
is a little fact. But give him a question  
and he’ll look for his own answers. ”*

Patrick Rothfuss

---

In this part, we start by a general introduction that will help the reader to be familiar with the systems of partial differential equations and the types of dampings considered in this thesis, in addition to their contribution with real life applications. Then, we give the main introduction in which we outline the thesis chapter by chapter and state the main problems considered with the main results obtained. Finally, we present a thesis overview in which we give a brief mathematical explanation of the problem investigated in each chapter.



---

# Introduction

---

Mathematical models play an important role in designing, understanding and analyzing many physical systems. For instance, the mathematical modeling of the vibration of flexible structures, the regulation of blood flow in cardiovascular networks, the flow of gas in pipes and electromagnetism was set up over years. Recently, modeling has progressed tremendously which provoked the mathematicians' curiosity in the stabilization of simple and coupled systems of partial differential equations arising in several mathematical models.

In general, the stability of a coupled system of partial differential equations is influenced by several properties. Among these properties we have: the nature of the partial differential equations, the set of boundary conditions, the type of the damping, the kind of the medium and the type of the coupling. In this PhD thesis we study the stabilization of three coupled systems that arise in three different mathematical models. What characterizes the problems studied in this thesis is the diversity in the properties of the considered coupled systems:

- Partial differential equations: wave equation, heat equation, telegraph equation.
- Boundary conditions: Dirichlet, Neumann, standard Kirchhoff condition, dynamic boundary conditions.
- Dissipation mechanisms: visco-elastic damping, thermo-elastic damping, boundary damping.
- Mediums: bounded domains, networks.
- Coupling mechanisms: by displacement, by velocity, other..

In what follows, we will start by a general introduction in which we introduce briefly some of the properties of the coupled systems of partial differential equations investigated in this thesis. To do this, first we begin in section 0.1 by introducing the wave equation used in Chapter 1 (in multi-dimensional space) and the generalized telegraph equation considered in Chapter 2. Next, in section 0.2, we recall the heat equation and explain briefly the thermo-mechanical process leading us to the thermo-elastic system encountered in Chapter 3. Then, due to the fact that the damping mechanism is the soul of the stabilization process of systems of partial differential equations, we focus in section 0.3 on three types of dampings that deals with this thesis. Finally, in section 0.4, we present some types of boundary conditions used in Chapter 2 and Chapter 3.

Interested readers can also go directly to the main introduction in which we present the main results obtained in this thesis.

---

## I) GENERAL INTRODUCTION

Partial differential equations play a relevant role in mathematical physics, mainly those of the second order. These equations are useful to describe a diversity of physical phenomena that include wavelike propagation, diffusion, and transport processes in practically all branches of physics. Namely, in continuous and classical mechanics, it is common to face problems of either the hyperbolic (vibrating strings, stretched membranes) or the parabolic (heat conduction) types.



FIGURE 1: waves in sciences

### 0.1 Vibrations and waves

Vibrations and waves are extremely important phenomena in physics. In nature, oscillations are found everywhere. From the jiggling of atoms to the large oscillations of sea waves. Wave is a flow or transfer of energy in the form of oscillation through a medium – space or mass. In Mathematics, the [wave equation](#) is a second order partial differential equation which describes the propagation of oscillations with a constant velocity in some quantity  $f = f(x_1, \dots, x_n, t)$  of time variable  $t$  and spacial variables  $(x_1, \dots, x_n)$ . The wave equation is given by

$$\frac{\partial^2 f}{\partial t^2} = s^2 \Delta f, \quad (0.1.1)$$

where  $s$  is the velocity of the wave. The quantity  $f$  may be, for example, the pressure in a liquid or gas, or the displacement, along some specific direction, of the particles of a vibrating solid away from their resting positions.

There are different types of waves including electromagnetic waves and mechanical waves.

---

**Mechanical waves:**

- A mechanical wave is an oscillation of matter and therefore a transfer of the energy through a medium.
- This kind of waves needs a medium in order to propagate itself.
- Sound waves, water waves, stadium waves, and jump rope waves are examples of mechanical waves; each requires some medium in order to exist.



FIGURE 2: Sea waves

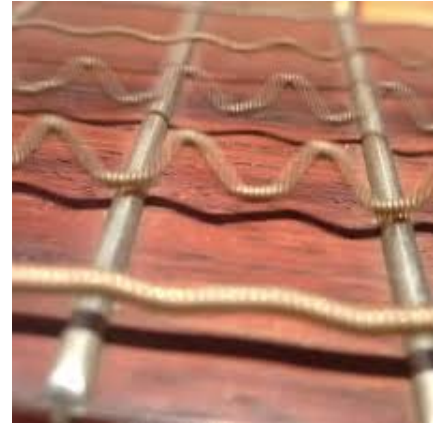


FIGURE 3: waves on the strings of a guitar

For example, the one-dimensional wave equation of a vibrating string with tension  $T$  and mass density  $m$ , is

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{m} \frac{\partial^2 u}{\partial x'^2} \quad (0.1.2)$$

where  $u$  is the displacement along the direction of the particles of the vibrating string away from their resting positions and  $\sqrt{\frac{T}{m}}$  is the speed of the propagation of the wave.

**Electromagnetic waves:**

- Electromagnetic waves are created by a fusion of electric and magnetic fields. The light you see, the colors around you are visible because of electromagnetic waves.
- Unlike mechanical waves, electromagnetic waves do not need a medium to travel.
- Microwaves, X-ray, radio waves are all examples of waves that are capable to propagate in vacuum.

The mostly used equation which models the propagation of electromagnetic waves is the telegraph equation.

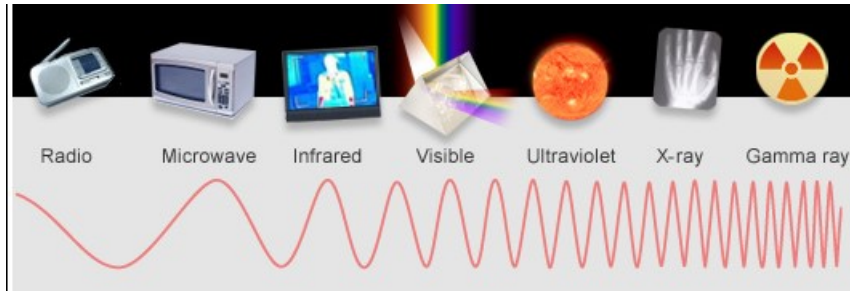


FIGURE 4: Electromagnetic waves

### 0.1.1 Telegraph equation

In many industrial applications, like microwave communication systems, transmission lines (including coaxial cables) are used to transmit signals from one point to another. In fact, these signals are transmitted along cables by electromagnetic waves which consist of variations in the magnitude and direction of the electric and magnetic fields produced by the line voltages and currents. The electromagnetic fields are the solution of the 3D Maxwell's equation. But, since a cable is a thin structure whose transverse dimensions are much smaller than the longitudinal one, one would like to use a simplified 1D model because 3D Maxwell's equations can be very complicated to solve: complex geometry due to defaults, 3D mesh for the thin cable... In such a situation, electrical engineers use the well-known [telegraph equation](#) for perfect coaxial cables (homogeneous with circular cross-section), where the electric unknowns are reduced to an electric potential  $V(x, t)$  and an electric current  $I(x, t)$ , where  $x$  denotes the abscissa along the cable,  $t$  is time and in absence of source

$$\begin{cases} C \frac{\partial V}{\partial t} + GV - \frac{\partial I}{\partial x} = 0, \\ L \frac{\partial I}{\partial t} + RI - \frac{\partial V}{\partial x} = 0, \end{cases} \quad (0.1.3)$$

where the capacitance  $C$ , the inductance  $L$ , the conductance  $G$  and the resistance  $R$  can be expressed it terms of the geometry of the cross-section.

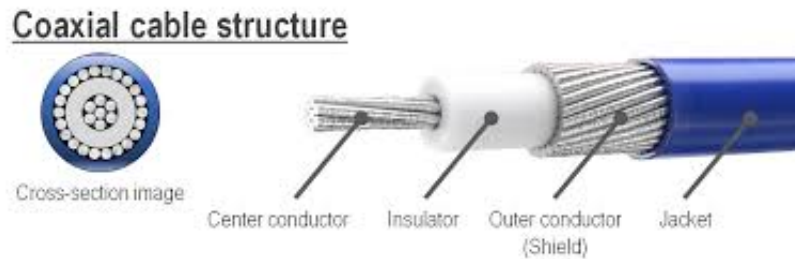


FIGURE 5: A coaxial cable

Differentiating in time (resp. space) the first (resp. second) equation in (0.1.3), we

get a partial differential equation with only one dependent variable  $V$ ,

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} + (LG + RC) \frac{\partial V}{\partial t} + RGV. \quad (0.1.4)$$

Similarly, differentiating in space (resp. time) the first (resp. second) equation in (0.1.3), we get a partial differential equation with only one dependent variable  $I$ ,

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + (LG + RC) \frac{\partial I}{\partial t} + RGI. \quad (0.1.5)$$

Further, in the case of loss-less coaxial cable (i.e.,  $R = G = 0$ ), equations (0.1.4) and (0.1.5) will be two exact wave equations

$$\begin{cases} \frac{\partial^2 V}{\partial t^2} = s^2 \frac{\partial^2 V}{\partial x^2}, \\ \frac{\partial^2 I}{\partial t^2} = s^2 \frac{\partial^2 I}{\partial x^2}, \end{cases}$$

where  $s = \frac{1}{\sqrt{LC}}$  is the propagation speed of the wave along the cable.

In the case of more general and realistic situation namely, when each cross section is heterogeneous (physical characteristics of the medium vary in the cross section) and the cable has a variable cross section and material properties along the cable, another model is derived. In fact, a 1D limit model for the propagation of electromagnetic waves in thin heterogeneous co-axial cables by means of asymptotic analysis of the original 3D model with respect to a small geometric parameter  $\delta$  (representing the transverse dimensions of the cables) is derived. The resulting model is a generalized telegraphist's model whose coefficients are determined from the solutions of 2D scalar problems in normalized cross sections. Mathematically, the [generalized telegraph equation](#) set in a real interval (model on a cable from Imperiale and Joly, 2014) given by:

$$\begin{cases} V_t + gV + aI_x + kW = 0, & \text{in } (0, L) \times (0, \infty), \\ I_t + rI + bV_x = 0, & \text{in } (0, L) \times (0, \infty), \\ W_t + cW = V, & \text{in } (0, L) \times (0, \infty). \end{cases} \quad (0.1.6)$$

This generalized telegraph equation is a coupling between the usual telegraph equation where the electric unknowns are  $V$  and  $I$  representing the electric potential and the electric current respectively with a first order differential equation of parabolic type involving an auxiliary variable  $W$  representing the non-local effects. The coefficients  $a, b, c, r, k$  and  $g$  are all non-negative functions in  $L^\infty(0, L)$  that verify some assumptions mentioned in Chapter 2.

## 0.2 Heat transfer

Heat transfer describes the flow of heat (thermal energy) due to temperature differences and the subsequent temperature distribution and changes. One of the various mechanisms of heat transfer is heat conduction. Heat conduction, also called diffusion, is the direct microscopic exchange of kinetic energy of particles through the boundary between two systems. When an object is at a different temperature from



another body or its surroundings, heat flows so that the body and the surroundings reach the same temperature, at which point they are in thermal equilibrium.



FIGURE 6: Heat conduction

Heat conduction is a diffusive process governed by a parabolic differential equation which lacks second-order derivative with respect to time, characteristic of a wave equation. In Mathematics, the [heat equation](#) is a second order partial differential equation which describes the propagation of heat with a in some quantity  $f = f(x_1, \dots, x_n, t)$  of time variable  $t$  and spacial variables  $(x_1, \dots, x_n)$ . The heat equation is given by

$$\frac{\partial f}{\partial t} = \Delta f, \quad (0.2.1)$$

where,  $f$  is the temperature deviation from the reference temperature.

### 0.2.1 Combination between mechanical and thermal processes

Mechanical equipment are affected by various interactions during their operation, the most significant being the mechanical and thermal effects. Mechanical and thermal loads usually occur simultaneously and as a result, the displacement and temperature fields are created in close connection with each other. In fact, Changes in temperatures causes thermal effects on materials. Some of these thermal effects include thermal stress, strain, and deformation. Thermal deformation simply means that as the "thermal" energy (and temperature) of a material increases, so does the vibration of its atoms/molecules and this increased vibration results in what can be considered a stretching of the molecular bonds - which causes the material to expand. In the same manner, if the thermal energy (and temperature) of a material decreases, the material will shrink or contract. Thus, thermo-elasticity is based on temperature changes induced by expansion and compression of the test part. Therefore, both mechanical and thermal fields have to be defined simultaneously taking the relationship between them into account which in practice proves to be rather difficult. Thermo-mechanical processes are described by the basic equations of continuum mechanics and thermodynamics. Mathematically, a linear one-dimensional [thermo-elastic system](#) satisfied by a thermoelastic bar  $(0, L)$  is represented by the following two equations:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = 0, & \text{in } (0, L) \times (0, \infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} = 0, & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (0.2.2)$$

where,  $u$  is the displacement,  $\theta$  is the temperature deviation from the reference temperature and the mechanical-thermal coupling  $\alpha$  is a positive constant.

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Recently, the study of the control and stabilization of thermo-elastic systems constitutes the framework of many dynamical developing areas of industry. For instance, the production of high-speed airplanes, the design of space vehicles, rocket and jet engines, the technology of large turbines and the design of nuclear reactors. This is due to the need of designing equipment that can operate at very high temperatures without being destroyed.

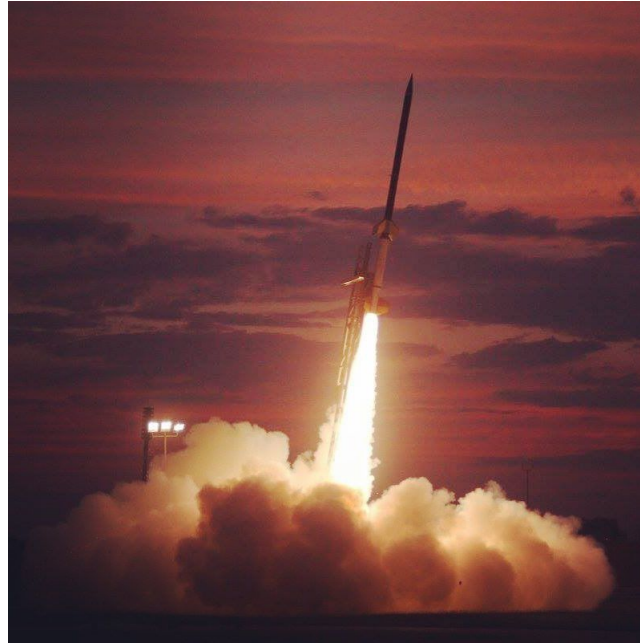


FIGURE 7: A launched rocket

In general, during wave propagation, the wave eventually reaches an end of the material. Some of the wave will reflect back to its source. If the reflection reaches the source at the same time a new wave is generated, the two waves will combine and be synchronized in phase. Later, if those two waves reflections return to the source at the same time the next new wave is generated, all three waves will combine. This will continue for as long as waves are generated under these conditions, and the resultant wave will increase in amplitude, theoretically to infinity. This is called resonance. If this wave reinforcement is allowed to continue, the system eventually will either self-destruct or fatigue to failure.

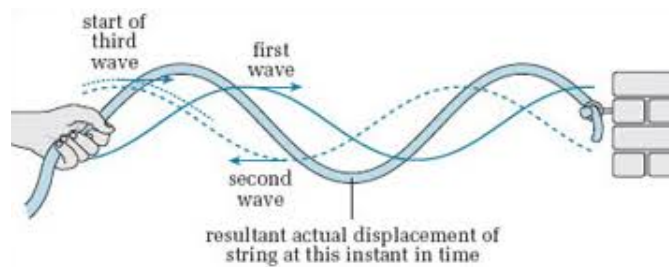


FIGURE 8: The generation of several waves

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Hence, in order to prevent the resonance energy from reaching an infinite value and thus forbidding the structure failure, a dissipation mechanism via a damping is invoked.

### 0.3 Damping

**Damping** is a phenomenon that dissipates the energy of every vibrating system. Depending on its spatial origin, damping can be classified as material damping, boundary damping and damping due to fluid-structure interactions. Materials with high damping coefficients are used in applications of shock absorption, vibration control, noise reduction, and dissipation of increased heat. Engineers use damping coefficients to compare materials to see which will be the best one for the application. Within this section, we will focus on three types of dampings which are: viscous damping, visco-elastic damping and thermo-elastic damping.

**Viscous** damping usually models external friction forces such as air resistance acting on the vibrating structures and is thus called "external damping". Most often they damp motion by providing a force or torque opposing motion proportional to the velocity.



FIGURE 9: viscous fluids

**Visco-elastic damping** originates from the internal friction of the material of the vibrating structures and thus called "internal damping". Visco-elastic materials, as their name suggests, combine two different properties which are elasticity and viscosity. The term "viscous" implies that they deform slowly when exposed to an external force. The term "elastic" implies that once a deforming force has been removed the material will return to its original configuration. So, visco-elastic materials exhibit mechanical properties intermediate between those of viscous liquid and those of elastic solid. When a visco-elastic material is subjected to a stress, the response is composed by an elastic deformation (which stores energy) and a viscous flow (which dissipates energy). For example, tall buildings vibrate when dynamically loaded by wind or earthquakes. Visco-elastic materials have the property of absorbing such vibrational energy – damping the vibrations. Visco-elastic dampers are used in some tall buildings, for example in the Columbia Center in Seattle, in which the dampers consist of steel plates coated with a visco-elastic polymer compound - the dampers are fixed to some of the diagonal bracing members. Different models can be used to describe the visco-elastic behavior among them, Maxwell model (i.e., spring and shock absorber in series) and **Kelvin-Voigt** model (spring and shock absorber in parallel) describe the simpler ones.



FIGURE 10: A group of visco-elastic materials

**Thermo-elastic** damping is a source of intrinsic material damping due to thermo-elasticity present in almost all materials. As the name thermo-elastic suggests, it describes the coupling between the elastic field in the structure (caused by deformation) and the temperature field. The effect of including thermo-elastic damping is that a heat source term, proportional to the rate of stress change, is added to the heat balance equations, see (0.2.2).

In any vibrating structure, the strain field causes a change in the internal energy such that compressed region becomes hotter (assuming a positive coefficient of thermal expansion) and extended region becomes cooler. The mechanism responsible for thermo-elastic damping is the resulting lack of thermal equilibrium between various parts of the vibrating structure. Energy is dissipated when irreversible heat flow which is driven by the temperature gradient causes vibrational energy to be dissipated. So, thermo-elastic damping refers to the process in which part of the vibration energy of a mechanical resonator is dissipated into thermal energy, through irreversible heat conduction accompanying elastic vibrations in the resonator.

In practice, from the research laboratory to the production line, scientists study the influence of such dampings (viscous, visco-elastic or thermo-elastic) on the behavior of the considered physical system. Hence, in this context many questions arise. Among the most frequently asked questions we mention:

1) Is the damping involved in a system of partial differential equations set in a certain medium (bounded domain or a network) strong enough to prove the strong stability of the system? In other words, is the damping able to dissipate the waves propagating in this medium and therefore leading the energy of the solution of the system to converge to zero?

2) How is the decay of the energy of the solution of the system varying with respect to the different localizations of the damping?

3) What is the optimal decay rate obtained and in which geometric situations?

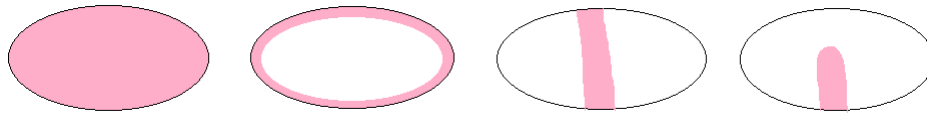


FIGURE 11: Different localizations of the damping. The pink color represents the damping region.

On the other hand, in addition to the significant role of the damping, the type of the medium and the type of the boundary conditions also contribute in the stabilization process of a physical system. For this aim, we will discuss briefly in the following section some boundary conditions used in this thesis.

## 0.4 Boundary conditions on networks

From the theoretical point of view the most challenging current problems are probably in the control questions for flows on [networks](#). Probably, this is due to the fact that the recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks, computer networks, chemical and biological networks, all involve systems of partial differential equations set on networks or graphs. Typical examples are the telegrapher equations for electrical lines, the shallow water (Saint-Venant) equations for open channels, the Euler equations for gas flow in pipelines or the Aw-Rascle equations for road traffic (see Bastin and Coron, 2016; Gugat, Dick, and Leugering, 2011; Leugering, Gugat, and Dick, 2010; Valein and Zuazua, 2009; Zhang and Xu, 2012; Kramar, Mugnolo, and Nicaise, 2020b).

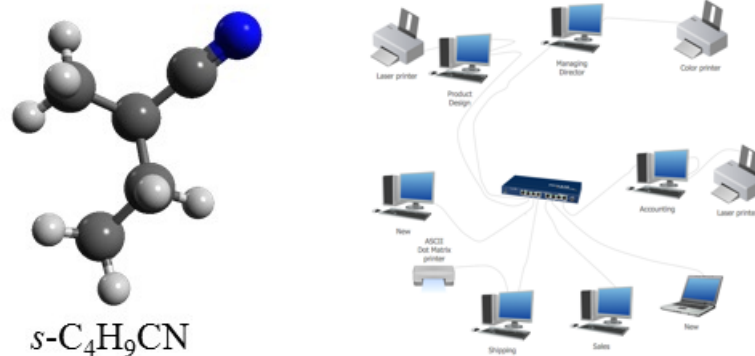


FIGURE 12: Chemical and computer networks

For a spatial network, the dynamics is typically described first at the level of individual edges, followed by a model for the junctions. So, once a reduced model is obtained, the next question is to impose the correct coupling conditions between the different branches (edges). In this case some transmission conditions that translate some physical preservation have to be imposed at the junctions. In fact, a particular choice for the coupling conditions determines the particular physics of the underlying problem. Hence, different coupling conditions give rise to different network



FIGURE 13: Electrical and arterial networks

problems. In this sense, the commonly used boundary condition at the junction is the standard Kirchhoff condition given by

$$\begin{cases} \sum_{j \in \mathcal{I}(v)} q^j(v) = 0, \\ p^j(v) = p^\ell(v); \quad j, \ell \in \mathcal{I}(v), \end{cases} \quad (0.4.1)$$

where  $v$  is the node of the junction of the connected edges adjacent to  $v$ . The first equation of (0.4.1) describes the mass conservation (resp. charge conservation) for fluid flow models (resp. electrical circuits); on the contrary the second equation means that the pressure (resp. voltage) is continuous at the junction.

But since such transmission conditions do not take into account the geometry of the 3D structure, another coupling conditions called the [improved Kirchhoff's circuit laws](#) characterizing (up to a certain extent) the 3D properties and the physical parameters of the junction was derived (see Beck, 2016 and Joly and Semin, 2008). In this sense, the interesting 1D variables used are the electrical potential and the electrical current present in the limit star-like graph made out of one knot (assume  $x=0$  is the node at the knot) and  $L + 1$  branches. Such a reduced model was derived using asymptotic analysis of Maxwell's equations. The so-called improved Kirchhoff's law is a dynamic boundary condition made of  $L + 1$  equations and given by

$$\begin{cases} \sum_{k=1}^N Z_{\ell k} I_{k,t}(0, t) = \frac{1}{\delta} (V_0(0, t) - V_\ell(0, t)), & \text{in } \mathbb{R}_+^*, \quad \forall \ell \in \{1, \dots, N\}, \\ V_{0,t}(0, t) = \frac{-1}{\delta Y} \sum_{\ell=0}^N I_\ell(0, t) & \text{in } \mathbb{R}_+^*, \end{cases} \quad (0.4.2)$$

where  $Y$  is a positive constant and  $Z = (Z_{\ell k})_{N \times N}$  is a symmetric, positive definite matrix. The two effective coefficients  $Y$  and  $Z$  integrate the complex structure of the knot area. The cable 0<sup>th</sup> plays a privileged role because it is chosen as the reference from which the electrical potential of each cable is expressed. Further, if  $\delta = 0$  then, the improved Kirchhoff's circuit rules become the usual Kirchhoff's law given in (0.4.1).

On the other hand, concerning the terminals of a network also several boundary conditions can be considered such as the well-known Dirichlet and Neumann boundary conditions. In addition to these conditions and in the presence of disturbances, such as a blockage, the response of the system is to return it to its

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equilibrium. Hence, boundary control conditions are imposed at the terminals to control the dynamic system. For example, in the network of large arteries, the process of auto-regulation can be modeled using boundary controls (at the root of the network-the heart and the peripheral nodes-the ends of the large arteries) deriving the system back to a dynamic equilibrium in a minimum time. In fact, the terminal boundary condition widely used in the modelling of arterial networks in cardio-vascular systems (in analogy with electrical circuits) is the pure resistance condition given by

$$W_b = -R_T W_f$$

where,  $W_f$  is the forward characteristics information leaving the outlet of an arterial domain,  $W_b$  is the backward characteristic information reflected by the terminal model and  $-1 \leq R_T \leq 1$  is the terminal reflection coefficient.  $R_t = 1$  corresponds to a complete reflection of the characteristic (complete blockage in the terminal site);  $R_t = 0$  means that there is no reflected characteristic at the terminal site, and  $R_T = 1$  represents an outflow with an open end (free end), see Cascaval et al., 2017.

In reality, a controlling mechanism specifies what resistance values are applied in order to regulate the desired equilibrium.

Beside the boundary conditions stated above there are still various dynamic, mixed or control boundary conditions used in the control and stabilization of dynamic systems on flow networks (see Bressan et al., 2014; Kramar, Mugnolo, and Nicaise, 2020a) but within this section we just focus on the boundary conditions that will be used within the thesis.

Now, we will introduce the main introduction followed by a thesis overview in which we give a brief explanation of the problems and the results obtained.

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## II) MAIN INTRODUCTION

In this thesis we study the stabilization of three coupled systems concerned with (thermo-) viscoelastic/elastic transmission problems and telegraph equations in bounded domains or in networks. This PhD thesis is divided into three parts. In part 1, we give a general introduction followed by a main introduction and a thesis overview in which we give a brief mathematical description of the problems and the results obtained in each chapter.

In part 2, we introduce Chapter 1, in which we consider a viscoelastic/elastic transmission problem in a bounded domain. Mainly, in this part we study the stabilization of a system of weakly coupled wave equations with one or two Kelvin-Voigt dampings and non-smooth coefficient at the interface. First, using semigroup theory, we prove the well-posedness of the system. Second, in the absence of the compactness of the resolvent and using a unique continuation result combined with a general criteria of Arendt and Batty, 1988, we show the strong stability of our system in different situations. Then, we establish a non-uniform stability result for the case when only one equation is damped with a globally distributed damping. Next, we study the energy decay rate in several distinguished cases. We prove that the corresponding semigroup is analytic when the Kelvin-Voigt dampings are globally distributed. Then, when only one non-smooth local Kelvin-Voigt damping is effective and under some assumptions (some of which are related to the decay rate of a wave equation with a frictional damping), we prove that the energy of the system decays polynomially with different decay rates according to these conditions. The results obtained are established by using a frequency domain approach combined with a multiplier technique. The method used give different quantitative results related to the asymptotic behavior of the energy depending on some geometric situations.

In part 3, we investigate the stabilization of two coupled systems involving a thermo-elastic/elastic transmission problem and telegraph equations on networks. This part is composed of two chapters, Chapter 2 and Chapter 3.

Chapter 2 concerns with the existence, uniqueness and stabilization of solutions of a generalized telegraph equation on star shaped networks. At the central interior node of the network, we consider the improved Kirchhoff conditions. In such node conditions, not only function values but also time derivatives appear. That is at the junction, the conditions considered are not system of algebraic equations but instead a system of ordinary differential equations. At the exterior nodes, we imposed a dissipative boundary condition. First, we provide a well-posedness result. Next, in the absence of the compactness of the resolvent, we prove that our system is strongly stable using a general criteria in Arendt and Batty, 1988, combined with a new uniqueness result. Finally, using a frequency domain approach combined with a multiplier technique and the construction of a new multiplier satisfying some ordinary differential inequalities, we show that the energy of the system decays exponentially to zero, under some regularity assumptions on two coefficient functions of the system.

Chapter 3 concerns with the stability of a transmission problem of a thermo-elastic system on networks. Mainly, we focus on networks composed of both elastic and thermo-elastic materials. On the thermo-elastic edges, we consider a system of



two wave equations coupled by velocity, such that one wave equation is coupled to a heat equation with a thermal effect. On the purely elastic edges, we consider only a system of two conservative wave equations. Using a general criteria of Arendt and Batty, 1988, we prove the strong stability of the system under some conditions related to the coupling coefficient and the geometry of the network. Then, we prove the exponential stability of the system under the condition that the two waves propagate with the same speed on each thermo-elastic edge. Otherwise, we establish a polynomial decay rate.

## Thesis Overview

Let us now briefly outline the content of this thesis.

**Chapter 1:** In this chapter, we consider the following system of coupled wave equations with viscoelastic dampings:

$$\begin{cases} Z_{tt} - \operatorname{div}(\nabla Z + D\nabla Z_t) + BZ = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ Z = 0, & \text{on } \Gamma \times \mathbb{R}_+^*, \\ Z(0) = Z_0, Z_t(0) = Z_1 & \text{in } \Omega, \end{cases} \quad (0.4.3)$$

where

- $Z = (u, y)$  with  $u = u(x, t)$  and  $y = y(x, t)$ ,  $Z_t = (\partial_t u, \partial_t y)$  denotes the time derivative of  $Z$ .

- $D(x) = \begin{pmatrix} b(x) & 0 \\ 0 & c(x) \end{pmatrix}$  with  $b, c : \Omega \rightarrow \mathbb{R}$  are the visco-elastic damping functions satisfying

$$b, c \in L^\infty(\Omega), \quad (0.4.4)$$

and

$$b(x) \geq 0, c(x) \geq 0 \text{ for almost all } x \in \Omega. \quad (0.4.5)$$

- $B(x) = \begin{pmatrix} 0 & \alpha(x) \\ \alpha(x) & 0 \end{pmatrix}$  with  $\alpha : \Omega \rightarrow \mathbb{R}$  is the coupling function satisfying

$$\alpha \in L^\infty(\Omega), \quad \|\alpha\|_\infty < \frac{1}{C_0}, \quad (0.4.6)$$

where  $C_0$  denotes the Poincaré constant.

In Najdi, 2016, the author considered system (0.4.3) and proved a polynomial energy decay rate of type  $t^{-1/4}$  provided that the coupling coefficient is a constant real number, the damping coefficient  $b \in C^{1,1}(\overline{\Omega})$ , and the damping region is covering the whole boundary (that represent a quite restrictive assumption). In addition, it was proved that the polynomial energy decay rate of order  $t^{-1/4}$  is optimal in some sense. Also a decay rate of the energy in  $t^{-1/2}$  and an optimality result is proved in Oquendo and Pacheco, 2017 provided that the coupling function is a constant,  $c = 0$  and the visco-elastic coefficient  $b$  is a positive constant (that means that the damping acts on one equation but on the whole domain).

In this chapter, our main interest is to improve these previous results. Mainly,

we consider system (0.4.3) with a coupling function and a non-smooth locally Kelvin-Voigt damping distributed in a part of the domain and we get a better polynomial decay rate than the one obtained in Najdi, 2016.

We assume that there exist three constants  $b_0, c_0$  and  $\alpha_0$  and three open sets  $\omega_b, \omega_\alpha$  and  $\omega_c$  contained in  $\Omega$  such that

$$b(x) \geq b_0 > 0, \quad \forall x \in \omega_b, \quad (0.4.7)$$

$$\alpha(x) \geq \alpha_0 > 0, \quad \forall x \in \omega_\alpha, \quad (0.4.8)$$

and

$$c(x) \geq c_0 > 0, \quad \forall x \in \omega_c. \quad (0.4.9)$$

First, we study the existence, uniqueness and regularity of the solution of system (0.4.3) using a semigroup approach. We start by defining the energy space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \quad (0.4.10)$$

equipped with the following inner product:

$$\begin{aligned} (U, V)_{\mathcal{H}} = & \int_{\Omega} (\nabla u \cdot \nabla \bar{\hat{u}} + \nabla y \cdot \nabla \bar{\hat{y}}) dx + \int_{\Omega} (v \bar{\hat{v}} + z \bar{\hat{z}}) dx \\ & + \operatorname{Re} \int_{\Omega} \alpha(x) (u \bar{\hat{y}} + y \bar{\hat{u}}) dx, \end{aligned} \quad (0.4.11)$$

for all  $U = (u, v, y, z)$ ,  $V = (\hat{u}, \hat{v}, \hat{y}, \hat{z}) \in \mathcal{H}$ .

Then, we define the unbounded linear operator  $\mathcal{A}$  by:

$$D(\mathcal{A}) = \left\{ (u, v, y, z) \in \mathcal{H} : \operatorname{div}(\nabla u + b \nabla v) \in L^2(\Omega), \operatorname{div}(\nabla y + c \nabla z) \in L^2(\Omega) \right. \\ \left. \text{and } v, z \in H_0^1(\Omega) \right\},$$

$$\mathcal{A}U = (v, \operatorname{div}(\nabla u + b(x) \nabla v) - \alpha y, z, \operatorname{div}(\nabla y + c(x) \nabla z) - \alpha u), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}).$$

The energy of the system is given by

$$E(t) = \frac{1}{2} \left[ \int_{\Omega} (|u_t|^2 + |\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx + 2 \operatorname{Re} \int_{\Omega} \alpha u \bar{y} dx \right]. \quad (0.4.12)$$

For a smooth solution, a straightforward computation leads to

$$E'(t) = - \int_{\Omega} (b |\nabla v|^2 + c |\nabla z|^2) dx \leq 0. \quad (0.4.13)$$

Then, system (0.4.3) is dissipative in the sense that its energy is non-increasing with respect to time  $t$ .

By putting  $v = u_t, z = y_t$  and if  $U = (u, u_t, y, y_t)$  is a regular solution of system (0.4.3), we can rewrite this system as the following evolution equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (0.4.14)$$

where  $U_0 = (u_0, u_1, y_0, y_1)$ .

We easily prove that the operator  $\mathcal{A}$  is maximal dissipative. Then, using Lumer-Phillips Theorem (see Pazy, 1983) we conclude that  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ . Hence, semigroup theory leads us to an existence and uniqueness result stated in Chapter 1, Theorem 1.2.2.

Second, we study the strong stability of the system in different geometrical situations. More precisely, we prove that for an initial data in  $\mathcal{H}$ , the energy converges to zero as  $t$  tends to infinity if one of the following geometric situations hold.

(H1) If  $\alpha = \alpha_0$  is a positive constant, then suppose that either  $\omega_b$  or  $\omega_c$  is a non-empty open subset of  $\Omega$ .

(H2) If  $\alpha$  is a non-zero function, then suppose that  $\omega_b$  or  $\omega_c$  is a non-empty open subset of  $\Omega$ . Further, assume that either  $\omega_1 = \omega_b \cap \omega_\alpha$  satisfies  $\text{meas}(\overline{\omega_1} \cap \Gamma) > 0$  or  $\omega_2 = \omega_c \cap \omega_\alpha$  satisfies  $\text{meas}(\overline{\omega_2} \cap \Gamma) > 0$ .

(H3)  $\omega_3 = \omega_b \cap \omega_c$  is a non-empty open subset of  $\Omega$ .

To prove the system strongly stable, we first use a unique continuation result based on a Carleman estimate to derive a new uniqueness result. Then, we use the obtained uniqueness result combined with a general criteria of Arendt and Batty, 1988 to achieve our goal. Later on, we show that strongly stable semigroup associated to system (0.4.3) is analytic when the Kelvin-Voigt dampings are globally distributed. Then, we show that system (0.4.3) is not uniformly stable for the case when the coupling coefficient is a positive constant and only one equation is damped with a globally distributed damping. For this desire, we study the asymptotic behavior of the eigenvalues of  $\mathcal{A}$  near the imaginary axis. In fact, we show that there exists  $k_0 \in \mathbb{N}^*$  sufficiently large and two sequences  $(\lambda_k^+)$  and  $(\lambda_k^-)$  satisfying the following asymptotic behavior

$$\lambda_k^+ = i\mu_k - \frac{\alpha^2}{2\mu_k^4} + O\left(\frac{1}{\mu_k^9}\right), \quad \forall |k| \geq k_0 \quad (0.4.15)$$

and

$$\lambda_k^- = -i\mu_k - \frac{\alpha^2}{2\mu_k^4} + O\left(\frac{1}{\mu_k^9}\right), \quad \forall |k| \geq k_0, \quad (0.4.16)$$

where  $\{\mu_k^2\}_{k=1}^\infty$  is the sequence of eigenvalues of the Laplace operator  $\Delta$ .

Due to the fact that the sequences of eigenvalues of  $\mathcal{A}$  ((0.4.15) and (0.4.16)) are close to the imaginary axis as  $k$  tends to infinity, a polynomial decay is hoped. This leads us to our main result in this chapter. Mainly, when the system (0.4.3) is partially damped by one locally Kelvin-Voigt damping distributed in a flexible geometry that covers several situations and under conditions (0.4.7) and (0.4.8), we confirm that there exists a constant  $C > 0$  such that for every initial data

$U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ , the energy of system (0.4.3) verifies the following estimate:

$$E(t) \leq C \frac{1}{\sqrt{t}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (0.4.17)$$

This result is achieved by using a frequency domain approach combined with a multiplier technique and using the exponential stability of a wave equation with a frictional damping in  $\omega_1 = \omega_b \cap \omega_\alpha$ , namely

$$\begin{cases} \varphi_{tt} - \Delta \varphi + \mathbb{1}_{\omega_1} \varphi_t = 0 & \text{in } \Omega \times (0, \infty), \\ \varphi = 0 & \text{on } \Gamma \times (0, \infty), \\ \varphi(t=0) = \varphi_0, \varphi_t(t=0) = \varphi_1. \end{cases} \quad (0.4.18)$$

After that and under an additional condition on  $\omega_\alpha$  and by using the exponential or polynomial decay of the energy of system (0.4.18), we establish different quantitative results relating the decay rate of the energy of system (0.4.18) and the decay rate of the energy of system (0.4.3). Then, we prove a better polynomial decay rate in the case of two active dampings. Finally, we prove the optimality of the polynomial decay rate  $t^{-1/2}$  in the case where only one of the damping coefficients is effective and is globally distributed.

In the next chapter, we focus on a coupled system involving the telegraph equation on a network.

Let  $\mathcal{S}$  be a star shaped network made of  $N + 1$  cables where,  $N$  is a positive integer. Without loss of generality, we assume that all the  $N + 1$  cables are of equal length  $L > 0$ . Also, fix different real valued and non-negative functions  $\mathbf{a} = (a_\ell)_{\ell=0}^N$ ,  $\mathbf{b} = (b_\ell)_{\ell=0}^N$ ,  $\mathbf{c} = (c_\ell)_{\ell=0}^N$ ,  $\mathbf{k} = (k_\ell)_{\ell=0}^N$ ,  $\mathbf{r} = (r_\ell)_{\ell=0}^N$  and  $\mathbf{g} = (g_\ell)_{\ell=0}^N$  in  $(L^\infty(0, L))^{N+1}$  satisfying the following assumption

$$a_\ell \geq \mu, b_\ell \geq \mu, c_\ell \geq \mu, k_\ell + g_\ell \geq \mu \quad a.e \text{ in } (0, L), \quad \forall \ell = 0, \dots, N, \quad (0.4.19)$$

where  $\mu > 0$  is a positive real number.

**Chapter 2:** In this chapter, we consider the generalized telegraph equation on a star shaped network  $\mathcal{S}$ :

$$\begin{cases} V_{\ell,t} + g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell & = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ I_{\ell,t} + r_\ell I_\ell + b_\ell V_{\ell,x} & = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ W_{\ell,t} + c_\ell W_\ell & = V_\ell, & (x, t) \in (0, L) \times \mathbb{R}_+^*. \end{cases} \quad (0.4.20)$$

Previously, Nicaise, 2015, has considered the stabilization of the generalized telegraph equation set in a real interval  $(0, L)$  with  $L > 0$  (model on a cable from Imperiale and Joly, 2014) with the Dirichlet boundary condition at  $x = 0, L$ . He proved that under the condition  $r \neq 0$ , system (0.4.20) is strongly stable in the energy Hilbert space. Then, under the additional conditions that are  $r \in W^{1,\infty}(0, L)$  and  $r + g > 0$  in  $(0, L)$ , an exponential energy decay rate was established. Otherwise, a polynomial energy decay rate of type  $t^{-1}$  for smooth initial data was attained. Moreover, he showed that the obtained polynomial decay rate is optimal in the case  $r = g = 0$ .

Our main interest is to extend the analysis to a networked system. This generalization is very well-motivated by engineering applications since electric power grids are complex networked systems.

We considered system (0.4.20) with the following dissipative boundary condition at the exterior vertices

$$V_\ell(L, t) - \alpha_\ell I_\ell(L, t) = 0, \quad \text{in } \mathbb{R}_+^*, \quad \alpha_\ell > 0, \quad \forall \ell \in \{0, \dots, N\}, \quad (0.4.21)$$

and the transmission conditions from Beck, 2016, §8.2 or Joly and Semin, 2008 (called by these authors the improved Kirchhoff conditions) at the interior common vertex

$$\begin{cases} \sum_{k=1}^N Z_{\ell k} I_{k,t}(0, t) &= \frac{1}{\delta} (V_0(0, t) - V_\ell(0, t)), \quad \text{in } \mathbb{R}_+^*, \quad \forall \ell \in \{1, \dots, N\}, \\ V_{0,t}(0, t) &= \frac{-1}{\delta Y} \sum_{\ell=0}^N I_\ell(0, t) \quad \text{in } \mathbb{R}_+^*, \end{cases} \quad (0.4.22)$$

where,  $Y$  and  $\delta$  are two positive constants and  $Z = (Z_{\ell k})_{N \times N}$  is a symmetric, positive definite matrix. These boundary conditions are derived in Beck, 2016, §8.2 or Joly and Semin, 2008. In fact, these junction nodes are improved in the sense that instead of zero-order Taylor approximation, a first order Taylor approximation is considered. As a consequence, in the node conditions not only functions appear but also time derivative appears and thus leading to a system of ordinary differential equations. The geometry of the joint can be described by the angles between the joined edges. This angle enters in the node conditions as a parameter. In addition, the coefficient  $Y$  and the matrix  $Z$  are defined from the material properties of the medium and from 3D potentials defined in the reference domain.

By setting  $v(t) = V_0(0, t)$  and  $\boldsymbol{\eta}(t) = (\eta_\ell(t))_{\ell=1}^N$  with  $\eta_\ell(t) = \sum_{k=1}^N Z_{\ell k} I_k(0, t)$ , system (0.4.20)-(0.4.22) is completed with the following initial conditions

$$\begin{cases} V(x, 0) = V^0(x), \quad I(x, 0) = I^0(x), \quad W(x, 0) = W^0(x), \quad x \in (0, L), \\ v(0) = v^0, \quad \boldsymbol{\eta}(0) = \boldsymbol{\eta}^0. \end{cases} \quad (0.4.23)$$

Our main desire within this chapter is to find the sufficient conditions on the coefficients of the system, i.e., on the functions  $a_\ell, b_\ell, c_\ell, k_\ell, r_\ell$ , and  $g_\ell$  that guarantee that system (0.4.20)-(0.4.23) is strongly stable and then exponentially stable.

For all  $p \in [1, \infty]$ , denote by  $\mathbb{L}^p(\mathcal{S}) = L^p(0, L)^{N+1}$ ,  $1 \leq p \leq \infty$  and  $\mathbb{H}^1(\mathcal{S}) = H^1(0, L)^{N+1}$ .

First, we study the existence, uniqueness and regularity of the solution of system (0.4.20) using a semigroup approach. We start by defining the energy space

$$\mathcal{H} = (\mathbb{L}^2(\mathcal{S}))^3 \times \mathbb{C}^{N+1},$$

that is a Hilbert space equipped with the following inner product:

$$\begin{aligned} \langle (\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})^\top, (\mathbf{V}^*, \mathbf{I}^*, \mathbf{W}^*, v^*, \boldsymbol{\eta}^*)^\top \rangle_{\mathcal{H}} &= \sum_{\ell=0}^N \int_0^L (\theta_\ell V_\ell \overline{V_\ell^*} + \beta_\ell I_\ell \overline{I_\ell^*} + \gamma_\ell W_\ell \overline{W_\ell^*}) dx \\ &\quad + \delta Y v \overline{v^*} + \delta (\boldsymbol{\eta}, Z^{-1} \boldsymbol{\eta}^*)_{\mathbb{C}^N} \end{aligned}$$

where  $\boldsymbol{\theta} = (\theta_\ell)_{\ell=0}^N$ ,  $\boldsymbol{\beta} = (\beta_\ell)_{\ell=0}^N$ ,  $\boldsymbol{\gamma} = (\gamma_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$  will be fixed later but are such that

$$\theta_\ell \geq \mu_0, \quad \beta_\ell \geq \mu_0, \quad \gamma_\ell \geq \mu_0 \quad \text{a.e in } \Omega, \quad \ell = 0, \dots, N, \quad (0.4.24)$$

for some  $\mu_0 > 0$ .

The energy of the system is given by

$$E(t) = \frac{1}{2} \|(\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})\|_{\mathcal{H}}^2.$$

For a strong solution, technical computations gives

$$E'(t) \leq -\frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \leq 0.$$

This means that our system is dissipative in the sense that its energy is non-increasing with respect to  $t$ .

Next, we define the unbounded linear operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$  by

$$\begin{aligned} D(\mathcal{A}) &= \left\{ (\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta}) \in \mathcal{H} : \mathbf{V} \in \mathbb{H}^1(\mathcal{S}), \mathbf{I} \in \mathbb{H}^1(\mathcal{S}), V_\ell(L) = \alpha_\ell I_\ell(L), \forall \ell \in \{0, \dots, N\}, \right. \\ &\quad \left. v = V_0(0), \text{ and } \boldsymbol{\eta} = (\eta_\ell)_{\ell=1}^N \text{ with } \eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) \right\}, \end{aligned}$$

and

$$\mathcal{A} \begin{pmatrix} \mathbf{V} \\ \mathbf{I} \\ \mathbf{W} \\ v \\ \boldsymbol{\eta} \end{pmatrix} = - \left( \mathbf{g} \otimes \mathbf{V} + \mathbf{a} \otimes \mathbf{I}_x + \mathbf{k} \otimes \mathbf{W}, \mathbf{r} \otimes \mathbf{I} + \mathbf{b} \otimes \mathbf{V}_x, \mathbf{c} \otimes \mathbf{W} - \mathbf{V}, \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0), \frac{1}{\delta} (V_\ell(0) - v)_{\ell=1}^N \right)^\top,$$

where for two vector functions  $\mathbf{P} = (p_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$ , and  $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{L}^2(\mathcal{S})$ , we set

$$\mathbf{P} \otimes \mathbf{Q} = (p_\ell q_\ell)_{\ell=0}^N,$$

while for  $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{H}^1(\mathcal{S})$ ,

$$\mathbf{Q}_x = (q_{\ell,x})_{\ell=0}^N.$$

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Then, setting  $\mathbf{U}(t) = (\mathbf{V}(\cdot, t), \mathbf{I}(\cdot, t), \mathbf{W}(\cdot, t), \nu(t), \boldsymbol{\eta}(t))^\top$ , System (0.4.20)-(0.4.22) can be formally written as a linear evolution equation in the space  $\mathcal{H}$

$$\begin{cases} \mathbf{U}_t = \mathcal{A}\mathbf{U}, & t \in \mathbb{R}_+^*, \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \quad (0.4.25)$$

where  $\mathbf{U}_0 = (\mathbf{V}_0, \mathbf{I}_0, \mathbf{W}_0, \nu_0, \boldsymbol{\eta}_0)^\top$ .

First, we prove that the operator  $\mathcal{A}$  is maximal dissipative. Then, using Lumer-Phillips Theorem (see Pazy, 1983) we conclude that  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ . Hence, semigroup theory leads us to an existence and uniqueness result stated in Chapter 2, Theorem 2.2.2.

Second, we study the strong stability of the system in the absence of the compactness of the resolvent and without any additional conditions on the coefficients of the system. Within this chapter, the reader will realize that analyzing the stability of our system is not immediate. The result is given in the following theorem.

**Theorem 0.4.1.** *Assume that assumptions (0.4.19) and (0.4.24) hold. Then, the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable on the energy space  $\mathcal{H}$ , i.e., for any  $\mathbf{U}_0 \in \mathcal{H}$ , we have*

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}}\mathbf{U}_0\|_{\mathcal{H}} = 0.$$

The proof of this theorem is based on the following lemmas.

**Lemma 0.4.2.** *Under the assumptions of Theorem 0.4.1, we have*

$$\ker(i\lambda - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

**Lemma 0.4.3.** *Under the assumptions of Theorem 0.4.1, we have*

$$\mathcal{R}(i\lambda - \mathcal{A}) = \mathcal{H}, \quad \forall \lambda \in \mathbb{R}.$$

The first result is proved by using the dissipativeness of  $\mathcal{A}$  and our boundary and transmission conditions, while the second one is based on a compact perturbation argument in addition to a new uniqueness result and Lemma 0.4.2.

These two Lemmas guarantee that the resolvent set  $\rho(\mathcal{A})$  of  $\mathcal{A}$  contains the whole imaginary axis. Hence, by a general criteria of Arendt-Batty (see Arendt and Batty, 1988), we deduce that the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable, i.e., the energy of system (0.4.20)-(0.4.23) converges to zero as time  $t$  tends to infinity.

Later on, under the condition that  $a_\ell, b_\ell \in W^{1,\infty}(0, L)$ , for all  $\ell \in \{0, \dots, N\}$  we prove that the semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is exponentially stable, i.e. there exist two positive constants  $M$  and  $\epsilon > 0$  such that

$$\|e^{t\mathcal{A}}\mathbf{U}_0\|_{\mathcal{H}} \leq M e^{-\epsilon t} \|\mathbf{U}_0\|_{\mathcal{H}}, \quad \forall t \geq 0, \quad \forall \mathbf{U}_0 \in \mathcal{H}.$$

Here contrary to Nicaise, 2015, we do not require that  $r_\ell + g_\ell$  is uniformly bounded from below for each  $\ell$  (see the assumption (25) in Nicaise, 2015) to get exponential decay of the energy but rather exploit the dissipative boundary conditions (0.4.21) at the exterior nodes. To prove our result we combine the frequency domain approach with an ad-hoc multiplier method given in the following lemma.

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**Lemma 0.4.4.** Let  $\ell \in \{0, \dots, N\}$  be fixed and assume that the functions  $a_\ell, b_\ell \in W^{1,\infty}(0, L)$ . Then, there exists  $h_\ell \in W^{1,\infty}(0, L)$  and a positive real number  $\rho_\ell$  such that

$$h_\ell(0) = 0, \quad (0.4.26a)$$

$$h_{\ell,x}(x) \geq \rho_\ell > 0, \text{ for a.a. } x \in (0, L), \quad (0.4.26b)$$

$$h_{\ell,x}(x) \left( \frac{a_\ell(x)}{b_\ell(x)} \right) + h_\ell(x) \left( \frac{a_\ell(x)}{b_\ell(x)} \right)_x \geq 0, \text{ for a.a. } x \in (0, L). \quad (0.4.26c)$$

Finally, we discuss the stability of the system (0.4.20) with some extensions, either by considering a general dynamic boundary condition or by considering general networks.

In the last chapter, we consider a coupled system involving a thermo-elastic/elastic transmission problem on networks (see "**General notations on graphs**" page xiii).

**Chapter 3:** Let  $\mathcal{N}$  be a network of elastic and thermo-elastic materials. The set of edges  $\mathcal{I}(\mathcal{N})$  is split up into  $\mathcal{I}(\mathcal{N}) = \mathcal{I}_{te} \cup \mathcal{I}_e$ , with  $\mathcal{I}_e \cap \mathcal{I}_{te} = \emptyset$ , in other words,  $\mathcal{I}_{te}$  (resp.  $\mathcal{I}_e$ ) is the set of thermo-elastic (resp. elastic) edges. Assume that  $\mathcal{N}$  contains at least one thermoelastic edge, that  $V_{ext} \neq \emptyset$ , that every maximal subgraph of elastic edges is a tree whose all of its exterior vertices except one are attached to thermo-elastic edges and that every subgraph of thermo-elastic edges is not a circuit.

In this chapter, we investigate the stabilization of the following thermo-elastic/elastic transmission problem on  $\mathcal{N}$ .

- On every thermo-elastic edge of  $\mathcal{N}$  ( $j \in \mathcal{I}_{te}$ ) the following equations hold:

$$\begin{cases} u_{tt}^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j y_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ \theta_t^j - \kappa_j \theta_{xx}^j + \alpha_j u_{tx}^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (0.4.27)$$

where  $\alpha_j, \rho_j, \kappa_j$  and  $\beta_j$  are positive constants.

- On every elastic edge of  $\mathcal{N}$  ( $j \in \mathcal{I}_e$ ) one has:

$$\begin{cases} u_{tt}^j - u_{xx}^j - \beta_j y_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (0.4.28)$$

where  $\beta_j$  and  $\rho_j$  are positive constants.

We assume that the initial data on the network  $\mathcal{N}$  are

$$\begin{aligned} u^j(x, 0) &= u_0^j(x), \quad u_t^j(x, 0) = u_1^j(x), \\ y^j(x, 0) &= y_0^j(x), \quad y_t^j(x, 0) = y_1^j(x), \quad \theta^j(x, 0) = \theta_0^j(x), \quad \forall j \in \mathcal{I}(\mathcal{N}). \end{aligned} \quad (0.4.29)$$

Previously, several authors considered the linear one-dimensional thermo-elastic system on a thermo-elastic rod  $(0, L)$  given by:

$$\begin{cases} u_{tt} - u_{xx} + \alpha \theta_x & = 0, & \text{in } (0, L) \times (0, \infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} & = 0, & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (0.4.30)$$



with the initial conditions

$$u(x, 0) = u_0, u_t(x, 0) = u_1, \theta(x, 0) = \theta_0, \quad x \in (0, L), \quad (0.4.31)$$

where  $u$  is the displacement,  $\theta$  is the temperature deviation from the reference temperature and  $\alpha$  is a positive constant.

Both Hansen, 1992 and Liu and Zheng, 1993 considered the stabilization of system (0.4.30)-(0.4.31) on a thermo-elastic rod (see Figure 14). Hansen, 1992 considered (0.4.30)-(0.4.31) with  $u$  and  $\theta$  satisfying the Dirichlet and Neumann condition respectively (or vice versa) while, Liu and Zheng, 1993 considered (0.4.30)-(0.4.31) with  $u$  and  $\theta$  satisfying both the Dirichlet condition. Both authors succeeded in proving the exponential stability of the system.

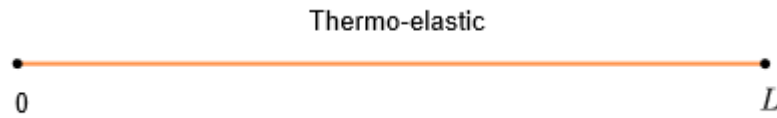
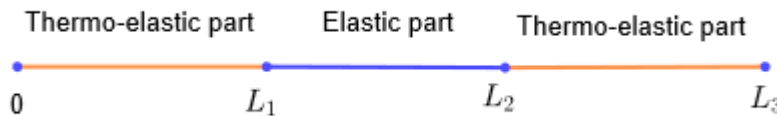


FIGURE 14: A thermoelastic rod

Later on, Marzocchi, Rivera, and Naso, 2002 considered the stabilization of system (0.4.30)-(0.4.31) on a one dimensional body which is configured in  $[0, L_3] \subset \mathbb{R}$  and for a given  $L_1, L_2 \in ]0, L_3[$ , they assumed that the material is thermo-elastic over  $]0, L_1[ \cup ]L_2, L_3[$  and elastic over  $]L_1, L_2[$ . The authors proved that the whole system is exponentially stable.



Then, Fatori, Lueders, and Rivera, 2003, considered the stabilization of a transmission problem for the thermo-elastic system with local thermal effect which is effective only over the interval  $[0, L_0], L_0 \in [0, L]$ . Similarly, the authors proved that the thermal damping are strong enough to prove the exponential decay of the energy to zero.

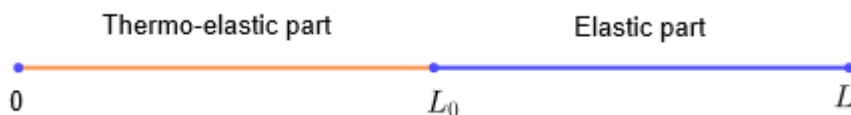


FIGURE 15: An elastic/thermo-elastic transmission problem

In Shel, 2012, the author studied the stability problem of a thermoelastic system

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on particular cases of networks of elastic and thermoelastic materials. In the first case, they suppose that two elastic edges cannot be adjacent. In the second one, they consider a tree of elastic materials, the leaves of which thermoelastic materials are added as follows: the thermoelastic body is related to only one leaf by an end, and the second is free or connects two leaves, with the condition that each leaf is connected to only one thermoelastic body, see Figure 16. Under the continuity condition of the displacement, the Neumann condition for the temperature at the internal nodes, and the balance condition, an exponential stability was proved.

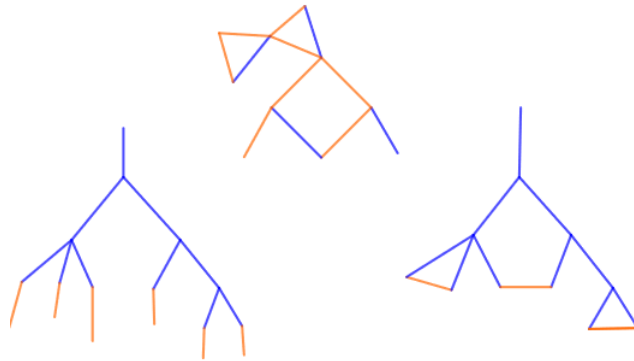


FIGURE 16: Elastic/therm-elastic networks

Later on Han and Zuazua, 2017 discussed the asymptotic behaviour of a transmission problem of the thermo-elastic system on star shaped networks of elastic and thermo-elastic rods, see Figure 17.

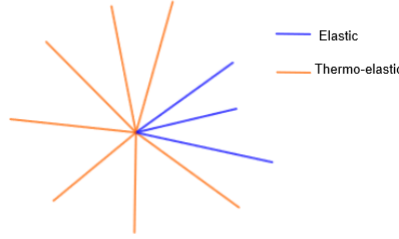


FIGURE 17: A star shaped network of elastic/ thermo-elastic rods

When only one purely elastic edge is present the uniform exponential decay rate was proved by a frequency domain analysis. Otherwise, a polynomial decay rate was deduced under a suitable irrationality condition on the lengths of the rods when more than one purely elastic edge is involved.

In this chapter, our main interest is to generalize the previous results by considering more general networks and by studying a more complicated thermo-elastic system such that on each thermo-elastic edge of the considered networks, we have a system of two wave equations coupled by velocity, such that one wave equation is coupled to a heat equation with a thermal effect while, on each purely undamped elastic edges, we have only a system of two conservative wave equations coupled by velocity, namely we consider system (0.4.27)-(0.4.29).

We prove that the dissipation due to the thermal effect is also strong enough to prove the exponential stability of the energy of the whole system when the two wave equation propagate with the same speed at each thermo-elastic edge of  $\mathcal{N}$ . Otherwise, a polynomial decay is established.

To complete formulating our system, we need to indicate the boundary conditions considered with system (0.4.27)-(0.4.29). Denote by  $\mathcal{I}_{te}(a_k)$  (resp.  $\mathcal{I}_e(a_k)$ ) the set of indices of thermo-elastic (resp. elastic) edges adjacent to  $a_k$  and denote by  $V'_{ext}$  (resp.  $V'_{int}$ ) the set of exterior (resp. interior) nodes of maximal subgraph of thermo-elastic edges. Then the boundary condition on  $\mathcal{N}$  are described as follows:

The displacement and temperature satisfies the Dirichlet boundary condition,

$$\begin{cases} u^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{ext}, \\ y^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{ext}, \\ \theta^j(a_k, t) = 0, & j \in \mathcal{I}_{te}(a_k), a_k \in V'_{ext}. \end{cases} \quad (0.4.32)$$

The displacement and temperature are continuous,

$$\begin{cases} u^j(a_k, t) = u^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{int}, \\ y^j(a_k, t) = y^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{int}, \\ \theta^j(a_k, t) = \theta^\ell(a_k, t), & j, \ell \in \mathcal{I}_{te}(a_k), a_k \in V'_{int}. \end{cases} \quad (0.4.33)$$

The system satisfies the balance condition on  $y$  at every interior node,

$$\sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, \quad a_k \in V_{\text{int}}. \quad (0.4.34)$$

The system satisfies the following balance conditions on  $u$  and  $\theta$ ,

$$\begin{cases} \sum_{j \in \mathcal{I}_{\text{te}}(a_k)} d_{kj} \kappa_j \theta_x^j(a_k, t) = 0, & a_k \in V'_{\text{int}}, \\ \sum_{j \in \mathcal{I}_{\text{te}}(a_k)} d_{kj} (u_x^j(a_k) - \alpha_j \theta^j(a_k)) + \sum_{j \in \mathcal{I}_e(a_k)} d_{kj} u_x^j(a_k) = 0, & a_k \in V_{\text{int}}. \end{cases} \quad (0.4.35)$$

First, we prove that system (0.4.27)-(0.4.29) with the boundary conditions (0.4.32)-(0.4.35) admits a unique solution in an appropriate Hilbert space  $\mathcal{H}$  using semigroup theory. Then, we give sufficient conditions that guarantee the strong stability of the system. In fact, we prove that the energy of the system converges to zero as  $t$  tends to infinity, if one of the following conditions holds,

- 1) Each maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to  $V_{\text{ext}}$ .
- 2) There exists a maximal subgraph of thermo-elastic edges with no exterior vertices that belong to  $V_{\text{ext}}$  and  $\beta_j = \beta$ , for all  $j \in \mathcal{I}(\mathcal{N})$ .

We rely on the Theorem of Arendt and Batty, 1988 combined with a unique continuation result derived in Hayek et al., 2020 and an iteration method to prove that our system is strongly stable. Later on, we distinguish between two different energy decay rates. More precisely, our main results are given in the following theorems.

**Theorem 0.4.5.** *Let  $\mathcal{N}$  be an arbitrary network for which system  $\{(0.4.27) - (0.4.29)\}$  with the boundary conditions  $\{(0.4.32) - (0.4.35)\}$  is stable. If  $\rho_j = 1$ , for all  $j \in \mathcal{I}_{\text{te}}$ , then the energy of the system decays exponentially in  $\mathcal{H}$ . In other words, there exist two positive constants  $M$  and  $\epsilon$  such that*

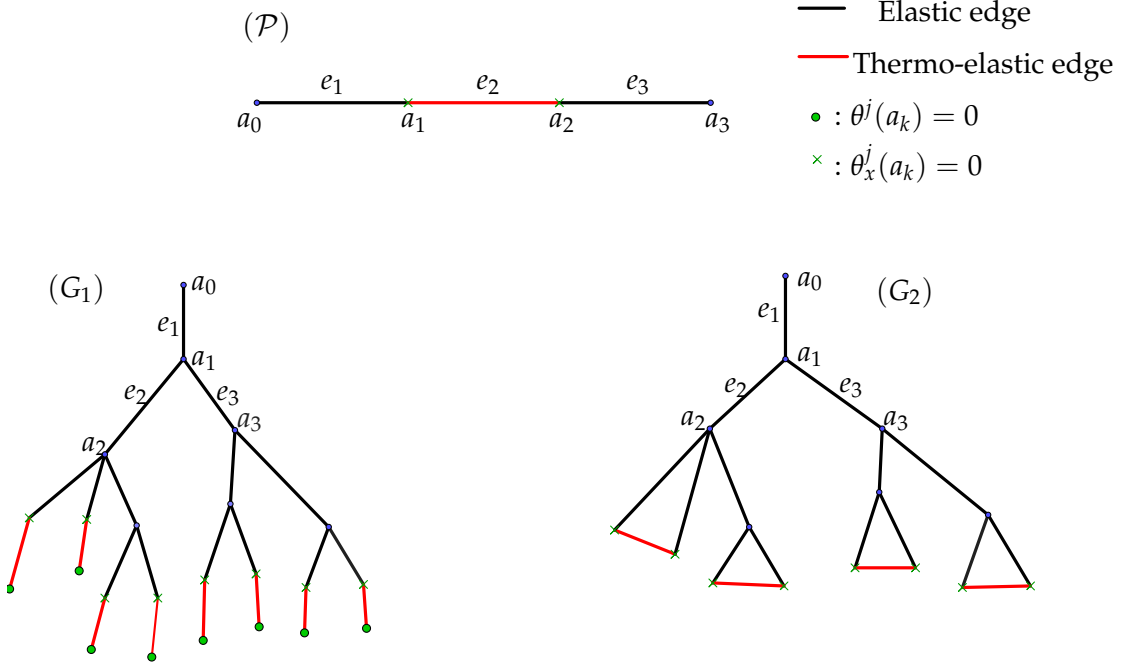
$$\|e^{tA} x_0\|_{\mathcal{H}} \leq M e^{-\epsilon t} \|x_0\|_{\mathcal{H}}, \quad \forall t > 0, \forall x_0 \in \mathcal{H}.$$

Let  $\mathcal{I}'_{\text{ext}}$  denotes the set of indices of edges adjacent to a vertex in  $V'_{\text{ext}}$ .

**Theorem 0.4.6.** *Let  $\mathcal{N}$  be an arbitrary network for which system (0.4.27) – (0.4.29) with the boundary conditions (0.4.32) – (0.4.35) is stable. Assume that there exists  $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$  such that  $\rho_j \neq 1$ . Then, the energy of the system decays polynomially in  $\mathcal{H}$ . More precisely, there exists  $C > 0$  such that for all  $t > 0$  we have*

$$E(t) \leq \frac{C}{t^{1/3}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall U_0 \in D(\mathcal{A}). \quad (0.4.36)$$

Finally, we present some other boundary conditions (in particular the Neumann boundary conditions) and we study the stabilization of system (0.4.27)-(0.4.29) on some particular networks similar to the networks considered in Shel, 2012. The networks considered in this case are shown in the following figure.



The boundary conditions of system (0.4.27)-(0.4.29) on the considered networks  $\mathcal{P}$ ,  $G_1$  and  $G_2$  will be as follows.

The system satisfies the Dirichlet boundary condition for the displacement and temperature at the exterior nodes,

$$\begin{cases} u^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{\text{ext}}, \\ y^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{\text{ext}}, \\ \theta^j(a_k, t) = 0, & j \in \mathcal{I}_{\text{te}}(a_k), a_k \in V_{\text{ext}}. \end{cases} \quad (0.4.37)$$

The displacement is continuous at every interior node,

$$\begin{cases} u^j(a_k, t) = u^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{\text{int}}, \\ y^j(a_k, t) = y^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{\text{int}}. \end{cases} \quad (0.4.38)$$

The temperature satisfies the Neumann condition at the interior nodes,

$$\theta_x^j(a_k, t) = 0, \quad j \in \mathcal{I}_{\text{te}}(a_k), a_k \in V_{\text{int}}. \quad (0.4.39)$$

The system satisfies the balance condition at every interior node,

$$\begin{cases} \sum_{j \in \mathcal{I}_{\text{te}}(a_k)} d_{kj} (u_x^j(a_k) - \alpha_j \theta^j(a_k)) + \sum_{j \in \mathcal{I}_e(a_k)} d_{kj} u_x^j(a_k) = 0, & a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, & a_k \in V_{\text{int}}. \end{cases} \quad (0.4.40)$$

Using semigroup theory, we easily prove the well-posedness of the system. Then, we discuss its strong stability and we show that under some sufficient conditions on the lengths of the purely elastic edges attached to the thermo-elastic ones, the

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system (0.4.27)-(0.4.29) with the boundary conditions (0.4.37)-(0.4.40) is strongly stable and then exponentially stable on the above described networks. More precisely, we show that the energy of the solution of the system (0.4.27)-(0.4.29) with the boundary conditions (0.4.37)-(0.4.40) converges to zero as  $t$  tends to infinity if one of the following conditions hold.

1)  $\mathcal{N}$  is the graph  $G_1$ ,

2)  $\mathcal{N}$  is the graph  $\mathcal{P}$ ,  $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(\mathcal{P})$ , and there exists  $j \in \{1, 3\}$  such that

$$\ell_j \neq \frac{m\pi}{\sqrt{2}\beta_j}, \quad \forall m \in \mathbb{N}^*, \quad (0.4.41)$$

3)  $\mathcal{N}$  is the graph  $G_2$ ,  $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(G_2)$ , and in every circuit  $C$ , for the unique  $j, k \in \Omega$  such that  $e_j$  and  $e_k$  are edges of  $C$ , we have

$$\sin(\sqrt{2}\beta_j\ell_j) + \sin(\sqrt{2}\beta_k\ell_k) \neq 0. \quad (0.4.42)$$

Finally, we again guarantee that the system (0.4.27)-(0.4.29) with the boundary conditions (0.4.37)-(0.4.40) is also exponentially stable whenever it is strongly stable.



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## Part 2

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◇ A viscoelastic/elastic transmission  
problem in a bounded domain of  $\mathbb{R}^N$  ◇





## Chapter 1

# A transmission problem of a system of weakly coupled wave equations with Kelvin-Voigt dampings and non-smooth coefficient at the interface

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### Abstract

The purpose of this chapter is to investigate the stabilization of a system of weakly coupled wave equations with one or two locally internal Kelvin-Voigt damping and non-smooth coefficient at the interface. The main novelty in this work is that the considered system is a coupled system and that the geometrical situations covered (see Remark 1.5.6, 1.5.12) are richer than all previous results, even for simple wave equation with Kelvin-Voigt damping. First, using a unique continuation result, we prove that the system is strongly stable. Second, we show that the system is not always exponentially stable, instead, we establish some polynomial energy decay estimates. Further, we prove that a polynomial energy decay rate of order  $t^{-1/2}$  is optimal in some sense.

### 1.1 Introduction

Let  $\Omega \subseteq \mathbb{R}^N$ ,  $N \geq 1$  be a bounded open set with a Lipschitz boundary  $\Gamma$ . We consider the following system of coupled wave equations with viscoelastic dampings:

$$\begin{cases} Z_{tt} - \operatorname{div}(\nabla Z + D\nabla Z_t) + BZ = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ Z = 0, & \text{on } \Gamma \times \mathbb{R}_+^*, \\ Z(0) = Z_0, Z_t(0) = Z_1 & \text{in } \Omega, \end{cases} \quad (1.1.1)$$

where

- $Z = (u, y)$  with  $u = u(x, t)$  and  $y = y(x, t)$ ,  $Z_t = (\partial_t u, \partial_t y)$  denotes the time derivative of  $Z$ .

- $D(x) = \begin{pmatrix} b(x) & 0 \\ 0 & c(x) \end{pmatrix}$  with  $b, c : \Omega \rightarrow \mathbb{R}$  are the visco-elastic damping functions satisfying

$$b, c \in L^\infty(\Omega), \quad (1.1.2)$$

and

$$b(x) \geq 0, c(x) \geq 0 \text{ for almost all } x \in \Omega. \quad (1.1.3)$$

- $B(x) = \begin{pmatrix} 0 & \alpha(x) \\ \alpha(x) & 0 \end{pmatrix}$  with  $\alpha : \Omega \rightarrow \mathbb{R}$  is the coupling function satisfying

$$\alpha \in L^\infty(\Omega), \quad \|\alpha\|_\infty < \frac{1}{C_0}, \quad (1.1.4)$$

where  $C_0$  denotes the Poincaré constant. More precisely,  $C_0$  is the smallest positive constant such that

$$\int_{\Omega} |\theta|^2 dx \leq C_0 \int_{\Omega} |\nabla \theta|^2 dx, \forall \theta \in H_0^1(\Omega).$$

System (1.1.1) models the vibration of bodies which have one part made of a Kelvin-Voigt type viscoelastic material and the other one is made of an elastic material. Naturally, system (1.1.1) is dissipative. Indeed, the energy of the system defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx + 2\text{Re} \int_{\Omega} \alpha u \bar{y} dx \quad (1.1.5)$$

is decreasing with respect to  $t \in (0, \infty)$  since

$$\frac{\partial E(t)}{\partial t} = E'(t) = - \int_{\Omega} (b(x) |\nabla u_t|^2 + c(x) |\nabla y_t|^2) dx \leq 0.$$

Besides, when no damping is applied to the system ( $b = c = 0$  on  $\Omega$ ), the energy is conserved, i.e.  $E(t) = E(0)$ , for all  $t > 0$ .

The stabilization of systems (simple or coupled) with Kelvin-Voigt damping has attracted the attention of many authors. In particular, in the one dimensional case, it was proved that the smoothness of the damping coefficient at the interface plays a critical role in the stability and regularity of the solution of the system (see Liu, 1997; Liu, Chen, and Liu, 1998; Liu and Liu, 1998; Liu and Liu, 2002; Liu and Rao, 2005; Zhang, 2010). However, there are only a few number of publications in the multi-dimensional setting. Let us start by recalling some previous studies in the case of simple wave equation system given by:

$$\begin{cases} u_{tt} - \text{div}(a \nabla u + b \nabla u_t) = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u(x, t) = 0, & \text{on } \Gamma \times \mathbb{R}_+^*, \\ u(x, 0) = u_0, u_t(x, 0) = u_1, & \text{in } \Omega. \end{cases} \quad (1.1.6)$$

In Huang, 1988, the author proved that when the Kelvin-Voigt damping  $\text{div}(b(x) \nabla u_t)$  is globally distributed i.e.  $b(x) \geq b_0 > 0$  for almost all  $x \in \Omega$ , the wave equation (1.1.6) generates an analytic semi-group. In Liu and Rao, 2006, the authors considered the wave equation with local visco-elastic damping distributed around the boundary of  $\Omega$ . They proved that the energy of the system

decays exponentially to zero as  $t$  goes to infinity for all usual initial data under the assumption that the damping coefficient satisfies:  $b \in C^{1,1}(\bar{\Omega})$ ,  $\Delta b \in L^\infty(\Omega)$  and  $|\nabla b(x)|^2 \leq M_0 b(x)$  for almost every  $x$  in  $\Omega$  where  $M_0$  is a positive constant. On the other hand, in Tebou, 2012, the author studied the stabilization of the wave equation with Kelvin-Voigt damping. He established a polynomial energy decay rate of type  $t^{-1}$  provided that the damping region is localized in a neighborhood of a part of the boundary and verifies the Piecewise Multiplier Geometric Condition (PMGC in short) from Liu, 1997.

Also in Nicaise and Pignotti, 2016, under the same assumption on  $b$ , the authors established the exponential stability of the wave equation with local Kelvin-Voigt damping localized around a part of the boundary and an extra boundary with time delay where they added an appropriate geometric condition. Later on, in Cavalcanti, Domingos Cavalcanti, and Tebou, 2017, the authors proved an exponential decay of the energy of a wave equation equation with two types of locally distributed mechanisms; a frictional damping and a Kelvin-Voigt damping where the location of each damping is such that none of them is able to exponentially stabilize the system. Under an appropriate geometric condition (PMGC) on a subset  $\omega$  of  $\Omega$  where the dissipation is effective, they proved that the energy of the system decays polynomially of type  $t^{-1}$  in the absence of regularity of the Kelvin-Voigt damping coefficient  $b$ . In Ammari, Hassine, and Robbiano, 2018, the wave equation with Kelvin-Voigt damping localized in a subdomain  $\omega$  far away from the boundary without any geometric conditions was considered. The authors established a logarithmic energy decay rate for smooth initial data.

Further, in Nasser, Noun, and Wehbe, 2019, the authors investigate the stabilization of the wave equation with Kelvin-Voigt damping localized via non smooth coefficient in a suitable sub-domain of the whole bounded domain. They proved a polynomial stability result in any space dimension, provided that the damping region satisfies some geometric conditions. Finally in Hassine, 2015 this system is considered in the interval  $(0, L)$ , with  $L > 0$ , when  $b = d\mathbb{1}_{(\alpha, \beta)}$ , with  $d > 0$ , and  $0 \leq \alpha < \beta < L$ , where an energy decay rate in  $t^{-4}$  is obtained. On the other hand, to the best of our knowledge, the stabilization of a system of weakly coupled wave equations with one locally Kelvin-Voigt damping was studied in Najdi, 2016; Oquendo and Pacheco, 2017; Hassine and Souayeh, 2019. In Najdi, 2016, the author proved a polynomial energy decay rate of type  $t^{-1/4}$  provided that the coupling coefficient is a constant real number, the damping coefficient  $b \in C^{1,1}(\bar{\Omega})$ , and the damping region is covering the whole boundary (that represent a quite restrictive assumption). In addition, it was proved that the polynomial energy decay rate of order  $t^{-1/4}$  is optimal in some sense. A decay rate of the energy in  $t^{-1/2}$  and an optimality result is proved in Oquendo and Pacheco, 2017 provided that the coupling function is a constant,  $c = 0$  and the visco-elastic coefficient  $b$  is a positive constant (that means that the damping acts on one equation but on the whole domain). A strongly coupled system (i.e. the term  $BZ$  is replaced by  $BZ_t$ ) in the interval  $(-1, 1)$  is considered in Hassine and Souayeh, 2019 when  $c = 0$  and  $b = d\mathbb{1}_{(0,1)}$ , with  $d > 0$ , where a decay rate of the energy in  $t^{-\frac{1}{6}}$  is obtained. Let us finally mention the paper Hassine, 2016, where a transmission wave/beam system with a local Kelvin-Voigt damping is studied. So, since the stabilization of the system (1.1.1) with more significant geometric situations and more flexible conditions on the damping coefficient is not covered yet and since the problem

of energy decay rate remains an open problem, our goal is to answer this open question.

In this chapter, we consider (1.1.1) with non smooth globally or locally distributed Kelvin-Voigt dampings in richer geometric situations than all previous results. First, we study the strong stability in several geometric situations. More precisely, we assumed that there exist three constants  $b_0, c_0$  and  $\alpha_0$  and three open sets  $\omega_b, \omega_\alpha$  and  $\omega_c$  contained in  $\Omega$  such that

$$b(x) \geq b_0 > 0, \quad \forall x \in \omega_b, \quad (1.1.7)$$

$$\alpha(x) \geq \alpha_0 > 0, \quad \forall x \in \omega_\alpha, \quad (1.1.8)$$

and

$$c(x) \geq c_0 > 0, \quad \forall x \in \omega_c. \quad (1.1.9)$$

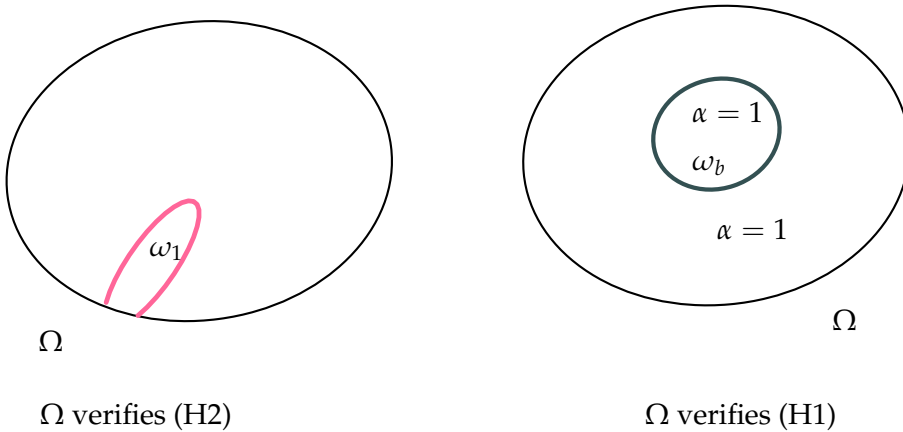
Then, we proved that the energy of the system (1.1.1) decays to zero as  $t$  tends to infinity if one of the following assumptions hold:

(H1) If  $\alpha = \alpha_0$  is a positive constant, then suppose that either  $\omega_b$  or  $\omega_c$  is a non-empty open subset of  $\Omega$  (see Figure 1.1).

(H2) If  $\alpha$  is a non-zero function, then suppose that  $\omega_b$  or  $\omega_c$  is a non-empty open subset of  $\Omega$ . Further, assume that either  $\omega_1 = \omega_b \cap \omega_\alpha$  satisfies  $\text{meas}(\overline{\omega_1} \cap \Gamma) > 0$  or  $\omega_2 = \omega_c \cap \omega_\alpha$  satisfies  $\text{meas}(\overline{\omega_2} \cap \Gamma) > 0$  (see Figure 1.1).

(H3)  $\omega_3 = \omega_b \cap \omega_c$  is a non-empty open subset of  $\Omega$ .

FIGURE 1.1: Examples of domains  $\Omega$



Next, we study the energy decay rate in distinguished several cases. In the first case, we proved that the corresponding semi-group is analytic when the Kelvin-Voigt dampings are globally distributed (i.e.  $\omega_b = \omega_c = \Omega$ ). Then, for the system (1.1.1) with one non-smooth local Kelvin-Voigt damping and under some conditions

specified below satisfied by  $\omega_b$  or  $\omega_c$  and  $\omega_\alpha$ , we showed that the energy of the system decays polynomially with different decay rates according to these conditions. In particular, we prove that the energy decay rate obtained in Najdi, 2016 is not optimal. We even improve this result by proving a polynomial energy decay rate of type  $t^{-1/2}$  when the system (1.1.1) is partially damped by one non-smooth locally Kelvin-Voigt damping ( $b \geq 0, c = 0$  or  $b = 0, c \geq 0$  on  $\Omega$ ) distributed in a flexible geometry that covers several situations (see Remark 5.6). Moreover, in the case where  $c = 0, b = 1$  and  $\alpha$  is constant, we prove that our energy decay rate of type  $t^{-1/2}$  is the optimal decay rate (better than the one obtained in Najdi, 2016).

Let us briefly outline the content of this chapter. First, in Section 1.2, we show that the system (1.1.1) is well-posed using semi-group theory. Then, using a unique continuation result based on a Carleman estimate and a general criteria of Arendt-Batty in Arendt and Batty, 1988, we proved the strong stability of the system in the absence of the compactness of the resolvent. In Section 1.3, we prove that the associated semi-group is analytic in the case of the global Kelvin-Voigt damping ( $\omega_b = \omega_c = \Omega$ ). In Section 1.4, using a spectral analysis, we prove the non uniform stability of the system in the case where  $c = 0, b = 1$  and  $\alpha$  is constant. In Section 1.5, when the system (1.1.1) is partially damped by one locally distributed Kelvin-Voigt damping ( $b \geq 0, c = 0$  or  $b = 0, c \geq 0$  on  $\Omega$ ), we established in different cases polynomial energy decay rates for smooth solutions using a frequency domain approach combined with a multiplier method. In Section 1.6, we prove a better polynomial decay rate in the case of two active damping. Finally, in Section 1.7, we prove the optimality of the polynomial decay rate  $t^{-1/2}$  in the case where  $c = 0, b = 1$  and  $\alpha$  is constant.

## 1.2 Well-posedness and strong stability

In this section, we will study the existence, uniqueness and regularity of the solution of system (1.1.1) using a semi-group approach. Later, we will show the strong stability of this system in the absence of the compactness of the resolvent.

Let us start by proving the wellposedness of system (1.1.1).

### 1.2.1 Well-posedness

First, we define the energy space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \quad (1.2.1)$$

equipped with the following inner product:

$$\begin{aligned} (U, V)_{\mathcal{H}} = & \int_{\Omega} (\nabla u \cdot \nabla \bar{\hat{u}} + \nabla y \cdot \nabla \bar{\hat{y}}) dx + \int_{\Omega} (v\bar{\hat{v}} + z\bar{\hat{z}}) dx \\ & + \operatorname{Re} \int_{\Omega} \alpha(x)(u\bar{\hat{y}} + y\bar{\hat{u}}) dx, \end{aligned} \quad (1.2.2)$$

for all  $U = (u, v, y, z)$ ,  $V = (\hat{u}, \hat{v}, \hat{y}, \hat{z}) \in \mathcal{H}$ . Note that (1.2.2) is an inner product due to condition (1.1.4).

Next, we define the unbounded linear operator  $\mathcal{A}$  by:

$$D(\mathcal{A}) = \left\{ (u, v, y, z) \in \mathcal{H} : \operatorname{div}(\nabla u + b\nabla v) \in L^2(\Omega), \operatorname{div}(\nabla y + c\nabla z) \in L^2(\Omega) \right. \\ \left. \text{and } v, z \in H_0^1(\Omega) \right\},$$

$$\mathcal{A}U = (v, \operatorname{div}(\nabla u + b(x)\nabla v) - \alpha y, z, \operatorname{div}(\nabla y + c(x)\nabla z) - \alpha u), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}).$$

By putting  $v = u_t, z = y_t$  and if  $U = (u, u_t, y, y_t)$  is a regular solution of system (1.1.1), we can rewrite this system as the following evolution equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (1.2.3)$$

where  $U_0 = (u_0, u_1, y_0, y_1)$ .

We recall that the energy of the system is given by

$$E(t) = \frac{1}{2} \left[ \int_{\Omega} (|u_t|^2 + |\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx + 2\operatorname{Re} \int_{\Omega} \alpha u \bar{y} dx \right], \quad (1.2.4)$$

and it is non-negative for any solution  $U$ . In fact,

$$E(t) \geq \frac{1}{2} (\|v\|_{L^2(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2) + \frac{1}{2} (1 - \|\alpha\|_{\infty} C_0) (\|u\|_{H_0^1(\Omega)}^2 + \|y\|_{H_0^1(\Omega)}^2).$$

Then, under assumption (1.1.4) we deduce that the energy  $E(t)$  is non-negative.

**Proposition 1.2.1.** *The unbounded linear operator  $\mathcal{A}$  generates a  $C_0$  semi-group of contractions on  $\mathcal{H}$ .*

*Proof.* Using Lumer-Phillips Theorem (see Pazy, 1983), it is sufficient to prove that  $\mathcal{A}$  is a maximal dissipative operator so that  $\mathcal{A}$  generates a  $C_0$  semi-group of contractions on  $\mathcal{H}$ . First, let  $U = (u, v, y, z) \in D(\mathcal{A})$ . Then, by Green's formula we have

$$\operatorname{Re}(AU, U)_{\mathcal{H}} = - \int_{\Omega} (b |\nabla v|^2 + c |\nabla z|^2) dx \leq 0. \quad (1.2.5)$$

This implies that  $\mathcal{A}$  is dissipative. Now, let us go on with maximality. Let  $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ , we look for  $U = (u, v, y, z) \in D(\mathcal{A})$  solution of

$$-AU = F. \quad (1.2.6)$$

Equivalently, we have the following system

$$-v = f_1, \quad (1.2.7)$$

$$-\operatorname{div}(\nabla u + b(x)\nabla v) + \alpha y = f_2, \quad (1.2.8)$$

$$-z = f_3, \quad (1.2.9)$$

$$-\operatorname{div}(\nabla y + c(x)\nabla z) + \alpha u = f_4. \quad (1.2.10)$$

Multiplying equation (1.2.8) by  $\bar{\varphi}_1 \in H_0^1(\Omega)$  and equation (1.2.10) by  $\bar{\varphi}_2 \in H_0^1(\Omega)$  respectively, integrating over  $\Omega$ , and using Green's formula we get

$$\int_{\Omega} \nabla u \cdot \nabla \bar{\varphi}_1 \, dx + \int_{\Omega} b \nabla v \cdot \nabla \bar{\varphi}_1 \, dx + \int_{\Omega} \alpha y \bar{\varphi}_1 \, dx = \int_{\Omega} f_2 \bar{\varphi}_1 \, dx, \quad (1.2.11)$$

and

$$\int_{\Omega} \nabla y \cdot \nabla \bar{\varphi}_2 \, dx + \int_{\Omega} c \nabla z \cdot \nabla \bar{\varphi}_2 \, dx + \int_{\Omega} \alpha u \bar{\varphi}_2 \, dx = \int_{\Omega} f_4 \bar{\varphi}_2 \, dx. \quad (1.2.12)$$

Replacing (1.2.7) in (1.2.11) and (1.2.9) in (1.2.12), then adding the resulting equations we obtain the following variational problem:

$$a((u, y)(\varphi_1, \varphi_2)) = L(\varphi_1, \varphi_2), \quad \forall (\varphi_1, \varphi_2) \in H_0^1(\Omega) \times H_0^1(\Omega), \quad (1.2.13)$$

where

$$a((u, y)(\varphi_1, \varphi_2)) = \int_{\Omega} (\nabla u \cdot \nabla \bar{\varphi}_1 + \nabla y \cdot \nabla \bar{\varphi}_2 + \alpha y \bar{\varphi}_1 + \alpha u \bar{\varphi}_2) \, dx$$

and

$$L(\varphi_1, \varphi_2) = \int_{\Omega} (f_2 \bar{\varphi}_1 + b \nabla f_1 \cdot \nabla \bar{\varphi}_1 + f_4 \bar{\varphi}_2 + c \nabla f_3 \cdot \nabla \bar{\varphi}_2) \, dx.$$

First, using (1.1.4),  $a$  is a continuous and coercive form on  $H_0^1(\Omega) \times H_0^1(\Omega)$  then, by Lax-Milgram theorem, the variational problem (1.2.13) admits a unique solution  $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$ . Next, taking  $\varphi_1 \in C_c^\infty(\Omega)$ ,  $\varphi_2 \equiv 0$  in (1.2.13) and applying Green's formula again, we deduce that

$$\int_{\Omega} (\operatorname{div}(\nabla u - b \nabla f_1) + \alpha y) \bar{\varphi}_1 \, dx = \int_{\Omega} f_2 \bar{\varphi}_1 \, dx, \quad \forall \varphi_1 \in C_c^\infty(\Omega). \quad (1.2.14)$$

This implies that  $\operatorname{div}(\nabla u - b \nabla f_1) = f_2 - \alpha y \in L^2(\Omega)$ . Similarly, by taking  $\varphi_1 = 0$  and  $\varphi_2 \in C_c^\infty(\Omega)$  in (1.2.13) we deduce that  $\operatorname{div}(\nabla y - c \nabla f_3) = f_4 - \alpha u \in L^2(\Omega)$ . Finally, by setting  $v = -f_1$  and  $z = -f_3$ , we conclude that  $U = (u, v, y, z) \in D(\mathcal{A})$  is solution of equation (1.2.6). To conclude, we need to show the uniqueness of such a solution. So, let  $U = (u, v, y, z) \in D(\mathcal{A})$  be a solution of equation (1.2.6) with  $f_1 = f_2 = f_3 = f_4 = 0$ , then we directly deduce that  $v = z = 0$  and therefore  $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$  satisfies (1.2.13) with  $L(\varphi_1, \varphi_2) = 0$ . As  $a$  is a sesquilinear, continuous coercive form on  $H_0^1(\Omega) \times H_0^1(\Omega)$ , we deduce that  $u = y = 0$ , in other words,  $\ker \mathcal{A} = \{0\}$  and 0 belongs to the resolvent set  $\rho(\mathcal{A})$  of  $\mathcal{A}$ . The proof is thus complete since  $\rho(\mathcal{A})$  is open.  $\square$

As  $\mathcal{A}$  generates a  $C_0$ -semigroup of contraction  $(e^{t\mathcal{A}})_{t \geq 0}$  (see Pazy, 1983), we have the following result:

**Theorem 1.2.2.** *(Existence and uniqueness of the solution)*

(1) If  $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ , then problem (1.2.3) admits a strong unique solution  $U = (u, v, y, z)$  satisfying:

$$U \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C^0(\mathbb{R}_+, D(\mathcal{A})).$$

(2) If  $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{H}$ , then problem (1.2.3) admits a unique weak solution  $U = (u, v, y, z)$  satisfying:

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$



Now, we are able to study the strong stability of system (1.1.1).

## 1.2.2 Strong Stability with non compact resolvent

In this section, we will prove the strong stability of the system using a unique continuation result based on a Carleman estimate and a general criteria of Arendt-Batty (see Arendt and Batty, 1988). But, before stating the main result of this section, our goal is to prove a local unique continuation result for a coupled system of wave equation.

We define the following elliptic operator  $P$  defined on a product space by

$$\begin{aligned} P : H^2(V) \times H^2(V) &\longrightarrow L^2(V) \times L^2(V) \\ (u, y) &\longrightarrow (\Delta u, \Delta y) \end{aligned} \quad (1.2.15)$$

and the following function  $g$  defined by

$$\begin{aligned} g : L^2(V) \times L^2(V) &\longrightarrow L^2(V) \times L^2(V) \\ (u, y) &\longrightarrow (-\lambda^2 u + \alpha y, -\lambda^2 y + \alpha u) \end{aligned} \quad (1.2.16)$$

In order to prove our result, we need a Carleman estimate represented in the following Lemma (see Hörmander, 1969, Hörmander, 2009 and Theorem 3.5 in Lebeau, 1996):

**Lemma 1.2.3.** *Let  $V$  be a bounded open set in  $\mathbb{R}^N$  and let  $\varphi = e^{\rho\psi}$  with  $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ ;  $|\nabla_x \psi| > 0$  and  $\rho > 0$  large enough. Then, there exist  $\tau_0 > 0$  large enough and  $C > 0$  such that*

$$\tau^3 \|e^{\tau\varphi} u\|_{L^2(V)}^2 + \tau \|e^{\tau\varphi} \nabla_x u\|_{L^2(V)}^2 \leq C \|e^{\tau\varphi} \Delta u\|_{L^2(V)}^2 \quad (1.2.17)$$

for all  $u \in H_0^2(V)$  and  $\tau > \tau_0$ .

Now, we are ready to state our first result in this section (see Section 4 in Lebeau, 1996).

**Proposition 1.2.4.** *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  and  $x_0$  be a point in  $\Omega$ . In a neighborhood  $V$  of  $x_0 \in \Omega$ , take a function  $f$  such that  $\nabla f \neq 0$  in  $\bar{V}$ . Moreover, let  $(u, y) \in H^2(V) \times H^2(V)$  be a solution of  $P(u, y) = g(u, y)$ . If  $u = y = 0$  in  $\{x \in V; f(x) \geq f(x_0)\}$  then,  $u = y = 0$  in a neighborhood of  $x_0$ .*

*Proof.* Set  $W := \{x \in V; f(x) \geq f(x_0)\}$ . Choose  $V'$  and  $V''$  neighborhoods of  $x_0$  such that  $V'' \subseteq V' \subseteq V$  (see Figure 1.2). Let  $\chi \in C_c^\infty(V')$  such that  $\chi = 1$  in  $V''$ . Set  $\tilde{u} = \chi u$  and  $\tilde{y} = \chi y$ . Then,  $(\tilde{u}, \tilde{y}) \in H_0^2(V) \times H_0^2(V)$ . Let  $\psi = f(x) - c|x - x_0|^2$  and set  $\varphi = e^{\rho\psi}$ . Then, apply the Carleman estimate of Lemma 1.2.3 to  $\tilde{u}$  and  $\tilde{y}$  respectively then add both inequalities to obtain

$$\begin{aligned} \tau^3 \int_{V'} e^{2\tau\varphi} (|\tilde{u}|^2 + |\tilde{y}|^2) dx + \tau \int_{V'} e^{2\tau\varphi} (|\nabla \tilde{u}|^2 + |\nabla \tilde{y}|^2) dx \\ \leq C \int_{V'} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \end{aligned} \quad (1.2.18)$$

As  $V'' \subseteq V'$  and  $\chi \in C_c^\infty(V')$  such that  $\chi = 1$  in  $V''$ , we get

$$\begin{aligned} \tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx + \tau \int_{V''} e^{2\tau\varphi} (|\nabla u|^2 + |\nabla y|^2) dx \leq C \int_{V''} e^{2\tau\varphi} (|\Delta u|^2 + |\Delta y|^2) dx \\ + C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \end{aligned}$$

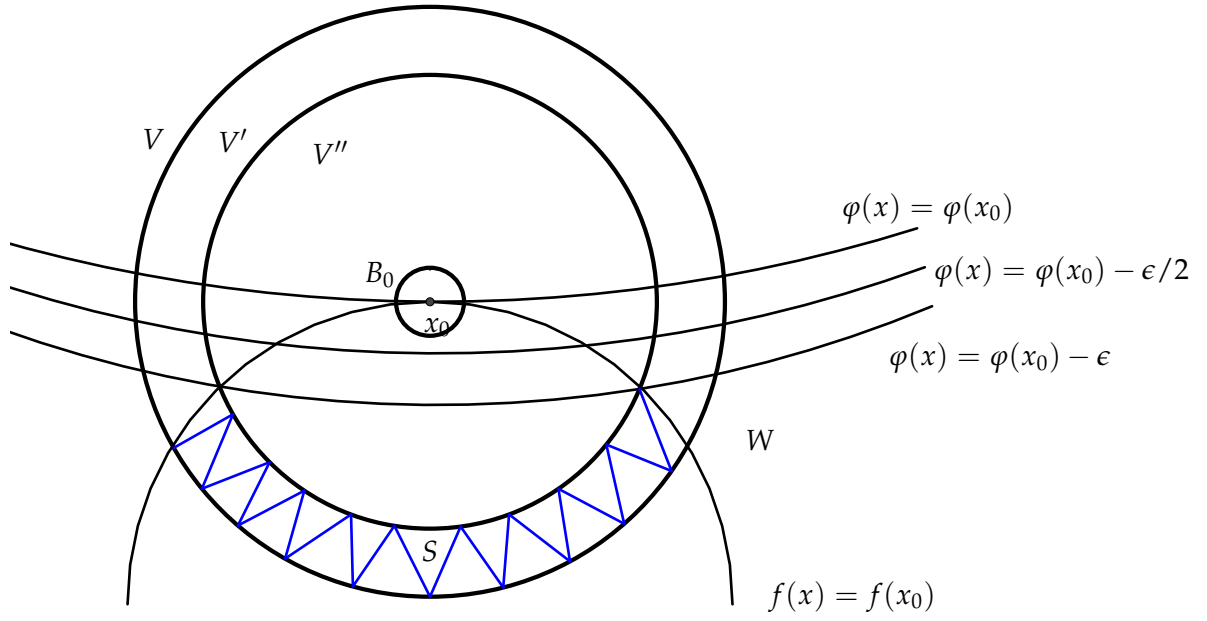


FIGURE 1.2:  $W$  is the region above the curve  $f(x) = f(x_0)$  and  $S$  is the region shaded with blue.

This implies that,

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V''} e^{2\tau\varphi} (|\Delta u|^2 + |\Delta y|^2) dx + C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx.$$

But  $\Delta u = -\lambda^2 u + \alpha y$  and  $\Delta y = -\lambda^2 y + \alpha u$ . Hence, there exists  $C_{\lambda\alpha} > 0$  such that

$$(\tau^3 - C_{\lambda\alpha}) \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx.$$

Then, there exist  $\tau > 0$  large enough and  $C > 0$  such that

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (1.2.19)$$

As  $u = y = 0$  in  $W$  we obtain,

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_S e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx, \quad (1.2.20)$$

where  $S = V' \setminus V'' \cup W$ .

Set  $J_\epsilon = \{x \in V; \varphi(x) \leq \varphi(x_0) - \epsilon\}$  and  $J'_\epsilon = \{x \in V; \varphi(x) \geq \varphi(x_0) - \frac{\epsilon}{2}\}$ . There exists  $\epsilon > 0$  such that  $S \subset J_\epsilon$ . Then choose a ball  $B_0$  with center  $x_0$  such that  $B_0 \subset V'' \cap J'_\epsilon$ . Then, using (1.2.20), we have

$$\int_{B_0} (|u|^2 + |y|^2) dx \leq \frac{C e^{-\epsilon\tau}}{\tau^3} \int_S (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (1.2.21)$$

Taking  $\tau$  tends to infinity, we obtain that  $u = y = 0$  in  $B_0$ . So the desired goal is achieved.  $\square$

**Theorem 1.2.5.** (*Calderón theorem*). Let  $\Omega$  be a connected open set in  $\mathbb{R}^N$  and let  $\omega \subset \Omega$ , with  $\omega \neq \emptyset$ . If  $(u, y)$  in  $H^2(\Omega) \times H^2(\Omega)$  satisfies  $P(u, y) = g(u, y)$  in  $\Omega$  and  $u = y = 0$  in  $\omega$ , then  $u$  and  $y$  vanishes in  $\Omega$ .

*Proof.* By setting  $F = \text{supp } u \cup \text{supp } y$  and using Proposition 1.2.4 instead of Proposition 4.1 in the proof of Theorem 4.2 in Rousseau and Lebeau, 2011 the result holds (see Figure 1.3).  $\square$

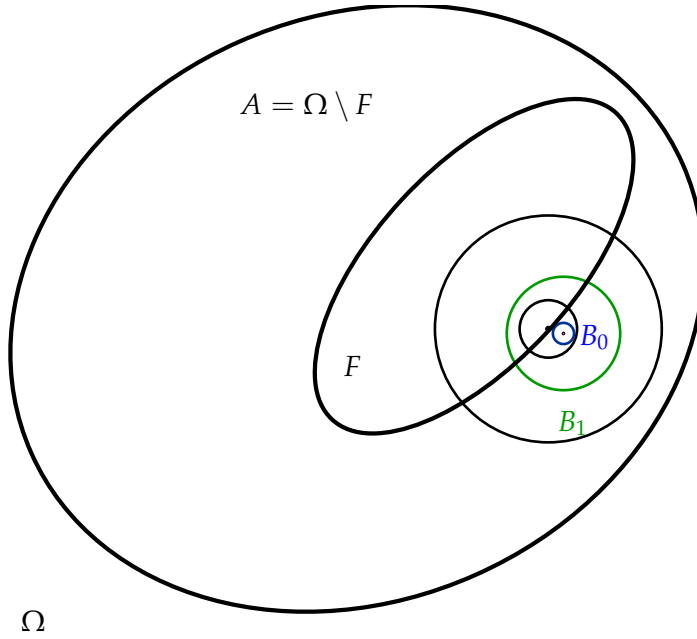


FIGURE 1.3: A dimonstrated figure

Now, we are in position to state our stability result.

**Theorem 1.2.6.** Let  $\Omega$  be a connected open set in  $\mathbb{R}^N$  and assume that either (H1), (H2) or (H3) holds. The  $C_0$ -semi group of contractions  $(e^{tA})_{t \geq 0}$  is strongly stable in the energy space  $\mathcal{H}$  in the sense that

$$\lim_{t \rightarrow \infty} \|e^{tA} U_0\| = 0, \quad \forall U_0 \in \mathcal{H}.$$

**Remark 1.2.7.** Theorem 1.2.6 shows that for the strong stability of the semi-group, we only need one of the damping coefficients  $b$  or  $c$  to be non-zero function.

The resolvent of  $\mathcal{A}$  is not compact. Then classical methods such as Lasalle's invariance principle in Slemrod, 1989 or the spectrum decomposition theory of Benchimol, 1978 are not applicable in this case. We prove the strong stability with a more general criteria of Arendt and Batty, 1988 that states that a  $C_0$ -semigroup of contractions in a reflexive Banach space is strongly stable, if there is no eigenvalues on the imaginary axis and the set  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  is countable. In order to prove Theorem 1.2.6 we need the following two Lemmas.

**Lemma 1.2.8.** Assume that either (H1), (H2) or (H3) holds. Then, we have

$$\ker(i\lambda - \mathcal{A}) = \{0\}, \forall \lambda \in \mathbb{R}.$$

*Proof.* From Proposition 1.2.1 we know that  $0 \in \rho(\mathcal{A})$ , then we may assume that  $\lambda \neq 0$  and let  $U = (u, v, y, z) \in D(\mathcal{A})$  such that

$$\mathcal{A}U = i\lambda U. \quad (1.2.22)$$

Using the identity (1.2.5) and the above assumption we get

$$0 = \operatorname{Re}(i\lambda \|U\|^2) = \operatorname{Re}(\mathcal{A}U, U) = - \int_{\Omega} (b(x)|\nabla v|^2 + c(x)|\nabla z|^2) dx. \quad (1.2.23)$$

This implies that,

$$\sqrt{b}\nabla v = 0 \text{ in } \Omega \text{ and } \sqrt{c}\nabla z = 0 \text{ in } \Omega. \quad (1.2.24)$$

Inserting (1.2.24) into (1.2.22), we get

$$\begin{cases} v = i\lambda u, & \text{in } \Omega, \\ \Delta u - \alpha y = i\lambda v, & \text{in } \Omega, \\ z = i\lambda y, & \text{in } \Omega, \\ \Delta y - \alpha u = i\lambda z, & \text{in } \Omega. \end{cases} \quad (1.2.25)$$

Here, we distinguish between the following three cases :

**Case 1.** If (H3) holds i.e.  $\omega_b \neq \emptyset$ ,  $\omega_c \neq \emptyset$  and  $\omega_3 = \omega_b \cap \omega_c$  satisfies  $\operatorname{meas}(\overline{\omega_3} \cap \Gamma) > 0$ . Then, by Poincaré inequality we have

$$v = z = 0 \text{ in } \omega_3.$$

So, the first and the third equation in (1.2.25) imply that  $u = 0$  and  $y = 0$  respectively in  $\omega_3$ . Consequently, we have the following system

$$\begin{cases} \lambda^2 u + \Delta u - \alpha y = 0, & \text{in } \Omega, \\ \lambda^2 y + \Delta y - \alpha u = 0, & \text{in } \Omega, \\ u = y = 0, & \text{in } \omega_3. \end{cases} \quad (1.2.26)$$

Then, by applying Theorem 1.2.5 we get  $u = y = 0$  in  $\Omega$ . Consequently, by the first and third equation of (1.2.25) we deduce that  $v = z = 0$  in  $\Omega$  and the desired goal  $U = 0$  holds.

**Case 2.** For simplicity, assume that  $\omega_b \neq \emptyset$  and  $\omega_c = \emptyset$  ( The same proof holds if  $\omega_c \neq \emptyset$  and  $\omega_b = \emptyset$ ). If (H2) holds then,  $\alpha$  is a non-zero function and  $\omega_1 = \omega_b \cap \omega_\alpha$  satisfies  $\operatorname{meas}(\overline{\omega_1} \cap \Gamma) > 0$ . Then, Poincaré inequality imply that

$$v = 0 \text{ in } \omega_1.$$

So, the first equation in (1.2.25) gives that  $u = 0$  in  $\omega_1$ . From the second and third equations in (1.2.25) and as  $\omega_1 \subseteq \omega_\alpha$  we deduce respectively that  $y \equiv 0$  and  $z \equiv 0$  in

$\omega_1$ . Then, we get the following system

$$\begin{cases} \lambda^2 u + \Delta u - \alpha y = 0, & \text{in } \Omega, \\ \lambda^2 y + \Delta y - \alpha u = 0, & \text{in } \Omega, \\ u = y = 0, & \text{in } \omega_1. \end{cases} \quad (1.2.27)$$

Similarly, by applying Theorem 1.2.5 we get  $u = y = 0$  in  $\Omega$ . Consequently, by the first and third equation of (1.2.25) we deduce that  $v = z = 0$  in  $\Omega$ .

**Case 3.** Otherwise, if (H1) holds, i.e.  $\alpha = \alpha_0$  is a positive constant and  $\omega_b$  is any open subset of  $\Omega$ . Then, by the first equation of (1.2.25) and using (1.2.24) we get

$$\nabla u = 0 \text{ in } \omega_b.$$

By differentiating the second equation of (1.2.25) we obtain

$$\partial_j(\alpha y) = 0 \text{ in } \omega_b, \forall j = 1, \dots, N,$$

but since  $\alpha$  is constant, we deduce that

$$\partial_j y = 0 \text{ in } \omega_b, \forall j = 1, \dots, N.$$

Then, for all  $j = 1, \dots, N$ , we have the following system

$$\begin{cases} \lambda^2 \partial_j u + \Delta \partial_j u - \alpha \partial_j y = 0, & \text{in } \Omega, \\ \lambda^2 \partial_j y + \Delta \partial_j y - \alpha \partial_j u = 0, & \text{in } \Omega, \\ \partial_j u = \partial_j y = 0, & \text{in } \omega_b. \end{cases} \quad (1.2.28)$$

Repeating the same argument of the first case and applying Theorem 1.2.5 we obtain

$$\partial_j u = \partial_j y = 0 \text{ in } \Omega, \forall j = 1, \dots, N.$$

Then,

$$u = y = c \text{ in } \Omega,$$

for some constant  $c$ . But as  $u|_\Gamma = y|_\Gamma = 0$ , we get  $u = y = 0$  in  $\Omega$ . Consequently,  $U = 0$  in  $\Omega$ .  $\square$

**Lemma 1.2.9.** *Under the assumptions of Lemma 1.2.8, we have*

$$\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}, \forall \lambda \in \mathbb{R}.$$

*Proof.* Let  $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ , we look for  $U \in D(\mathcal{A})$  such that

$$i\lambda U - \mathcal{A}U = F. \quad (1.2.29)$$

This is equivalent to  $v = i\lambda u - f_1, z = i\lambda y - f_3$  and to the following system

$$\begin{cases} \lambda^2 u + \operatorname{div}(\nabla u + i\lambda b(x)\nabla u) - \alpha y = f, & \text{in } \Omega, \\ \lambda^2 y + \operatorname{div}(\nabla y + i\lambda c(x)\nabla z) - \alpha u = g, & \text{in } \Omega, \end{cases} \quad (1.2.30)$$

where  $f = -f_2 - i\lambda f_1 + \operatorname{div}(b\nabla f_1) \in H^{-1}(\Omega)$  and  $g = -f_4 - i\lambda f_3 + \operatorname{div}(c\nabla f_3) \in H^{-1}(\Omega)$ . Now, define the linear operator  $\mathcal{L} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow H^{-1}(\Omega) \times$

$H^{-1}(\Omega)$  by

$$\mathcal{L} \begin{pmatrix} u \\ y \end{pmatrix} := \begin{pmatrix} -\operatorname{div}(\nabla u + i\lambda b \nabla u) + \alpha y \\ -\operatorname{div}(\nabla y + i\lambda c \nabla y) + \alpha u \end{pmatrix}. \quad (1.2.31)$$

Let  $U = (u, y)^T$  and  $F = (f, g)^T$ , then (1.2.30) is equivalent to

$$(\lambda^2 \mathcal{I} - \mathcal{L})U = F. \quad (1.2.32)$$

Since the operator  $\mathcal{L}$  is an isomorphism from  $H_0^1(\Omega) \times H_0^1(\Omega)$  onto  $H^{-1}(\Omega) \times H^{-1}(\Omega)$  and  $\mathcal{I}$  is a compact operator from  $H_0^1(\Omega) \times H_0^1(\Omega)$  onto  $H^{-1}(\Omega) \times H^{-1}(\Omega)$ . Then, using Fredholm's Alternative theorem, problem (1.2.32) admits a unique solution in  $H_0^1(\Omega) \times H_0^1(\Omega)$  if and only if  $\lambda^2 \mathcal{I} - \mathcal{L}$  is injective. For that purpose, let  $U = (u, y) \in \ker(\lambda^2 \mathcal{I} - \mathcal{L})$ . Then, if we set  $v = i\lambda u$  and  $z = i\lambda y$ , we deduce that  $U = (u, v, y, z) \in D(\mathcal{A})$  is a solution of

$$(i\lambda - \mathcal{A})U = 0.$$

Using Lemma 1.2.8, we deduce that  $u = v = y = z = 0$ . This implies that equation (1.2.32) admits a unique solution  $U = (u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$  and  $\operatorname{div}(\nabla y + i\lambda c \nabla y - c \nabla f_3)$ ,  $\operatorname{div}(\nabla u + i\lambda b \nabla u - b \nabla f_1) \in L^2(\Omega)$ . By setting  $v = i\lambda u - f_1$  and  $z = i\lambda y - f_3$ , we deduce that  $U = (u, v, y, z) \in D(\mathcal{A})$  is the unique solution of equation (1.2.29) and the proof is thus complete.  $\square$

**Proof of Theorem 1.2.6.** Using Lemma 1.2.8, the operator  $\mathcal{A}$  has no pure imaginary eigenvalues and by Lemma 1.2.9, we have  $\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}$ , for all  $\lambda$  in  $\mathbb{R}$ . Therefore, the closed graph theorem implies that  $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ . Following Arendt-Batty (see Arendt and Batty, 1988), the  $C_0$ -semi group of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable and the proof is complete.

### 1.3 Analytic Stability

In this section, under the condition that  $\omega_b = \omega_c = \Omega$ , we will prove that the associated semigroup of the system (1.1.1) is analytic. For this aim, we will use the following result (see Corollary 3.7.18 page 157 in Arendt et al., 2001).

**Theorem 1.3.1.** *Let  $(\mathcal{S}(t) = e^{t\mathcal{A}})_{t \geq 0}$  be a  $C_0$ -semi group of contractions in a Hilbert space. Assume that*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad (\text{A1}).$$

*Then,  $(e^{t\mathcal{A}})_{t \geq 0}$  is analytic if and only if*

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{-1}} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad (\text{A2}).$$

Now, we are ready to state our result.

**Theorem 1.3.2.** *(Analytic stability)*

*Assume that (1.1.7), (1.1.9) hold and  $\omega_b = \omega_c = \Omega$ . Then the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is analytic.*

*Proof.* We know that condition (A1) is verified. Our goal now is to prove that condition (A2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (A2) does not hold, then there exist a sequence  $(\lambda_n) \subset \mathbb{R}$  and a sequence

$(U_n) \subset D(\mathcal{A})$  such that

$$|\lambda_n| \longrightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1 \quad (1.3.1)$$

and

$$\lambda_n^{-1}(i\lambda_n - \mathcal{A})U_n = (f_1^n, g_1^n, f_2^n, g_2^n) \longrightarrow 0 \quad \text{in } \mathcal{H} \quad (1.3.2)$$

are satisfied.

**Step 1. (The dissipation).**

Multiply in  $\mathcal{H}$  equation (1.3.2) by the uniformly bounded sequence  $U^n = (u_n, v_n, y_n, z_n)$ , we get

$$\lambda_n^{-1} \int_{\Omega} (b|\nabla v_n|^2 + c|\nabla z_n|^2) dx = -\lambda_n^{-1} \text{Re}((i\lambda_n I - \mathcal{A})U^n, U^n)_{\mathcal{H}} = o(1).$$

Then, using (1.1.7), (1.1.9) and as  $\omega_b = \omega_c = \Omega$ , it follows that

$$\lambda_n^{-1/2} \|\nabla v_n\|_{L^2(\Omega)} = o(1), \quad (1.3.3)$$

and

$$\lambda_n^{-1/2} \|\nabla z_n\|_{L^2(\Omega)} = o(1). \quad (1.3.4)$$

Using Poincaré inequality, we get

$$\lambda_n^{-1/2} \|v_n\|_{L^2(\Omega)} = o(1), \quad \lambda_n^{-1/2} \|z_n\|_{L^2(\Omega)} = o(1). \quad (1.3.5)$$

In what follows, we drop the index  $n$  for simplicity.

**Step 2. (Information on  $\nabla u$  and  $\nabla y$ ).**

By detailing equation (1.3.2), we get the following system

$$\lambda^{-1}(i\lambda u - v) = f_1 \longrightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.3.6)$$

$$\lambda^{-1}(i\lambda v - \text{div}(\nabla u + b\nabla v) + \alpha y) = g_1 \longrightarrow 0 \quad \text{in } L^2(\Omega), \quad (1.3.7)$$

$$\lambda^{-1}(i\lambda y - z) = f_2 \longrightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.3.8)$$

$$\lambda^{-1}(i\lambda z - \text{div}(\nabla y + c\nabla z) + \alpha u) = g_2 \longrightarrow 0 \quad \text{in } L^2(\Omega). \quad (1.3.9)$$

From equations (1.3.3), (1.3.6), (1.3.8), (1.3.4) and (1.3.5), we get

$$\begin{cases} \|\nabla u\|_{L^2(\Omega)} = o(1) & \text{and} & \|\nabla y\|_{L^2(\Omega)} = o(1), \\ \|u\|_{L^2(\Omega)} = o(1) & \text{and} & \|y\|_{L^2(\Omega)} = o(1). \end{cases} \quad (1.3.10)$$

**Step 3. (Information on  $v$  and  $z$ ).**

Multiplying equation (1.3.7) by  $\bar{v}$ , then integrating over  $\Omega$  gives

$$\begin{aligned} i \int_{\Omega} |v|^2 dx + \lambda^{-1} \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx + \lambda^{-1} \int_{\Omega} b|\nabla v|^2 dx \\ + \lambda^{-1} \int_{\Omega} \alpha y \bar{v} dx = \int_{\Omega} g_1 \bar{v} dx. \end{aligned} \quad (1.3.11)$$

Then, taking the imaginary part of (1.3.11) and using (1.3.3), (1.3.10) and the fact that  $\|g_1\|_{H_0^1(\Omega)} = o(1)$ , we get

$$\|v\|_{L^2(\Omega)} = o(1). \quad (1.3.12)$$

Similarly, multiplying equation (1.3.9) by  $\bar{z}$ , then integrating over  $\Omega$  gives

$$\begin{aligned} i \int_{\Omega} |z|^2 dx + \lambda^{-1} \int_{\Omega} \nabla y \cdot \nabla \bar{z} dx + \lambda^{-1} \int_{\Omega} c |\nabla z|^2 dx \\ + \lambda^{-1} \int_{\Omega} \alpha u \bar{z} dx = \int_{\Omega} g_2 \bar{z} dx. \end{aligned} \quad (1.3.13)$$

Then, taking the imaginary part of (1.3.13) and using (1.3.4), (1.3.10) and the fact that  $\|g_2\|_{H_0^1(\Omega)} = o(1)$ , we get

$$\|z\|_{L^2(\Omega)} = o(1). \quad (1.3.14)$$

Consequently, by (1.3.10), (1.3.12) and (1.3.14) we deduce the desired contradiction.  $\square$

## 1.4 Non-Uniform stability of the system

In this section, we will prove the non-uniform stability of system (1.1.1). Our main result is represented in the following theorem.

**Theorem 1.4.1.** *Let  $c = 0, b = 1$  and  $\alpha = \alpha_0$  be a positive constant. Then the energy of the system (1.1.1) does not decay uniformly to zero as  $t$  tends to infinity.*

To prove Theorem 1.4.1 we study the asymptotic behavior of the eigenvalues of the operator  $\mathcal{A}$  near the imaginary axis. So, let  $\lambda$  be an eigenvalue of  $\mathcal{A}$  and  $\Phi = (u, v, y, z)$  be an associated eigenvector, i.e,

$$\mathcal{A}\Phi = \lambda\Phi, \quad (1.4.1)$$

equivalently,

$$\begin{cases} v = \lambda u, & \text{in } \Omega, \\ \Delta u + \lambda \Delta u - \alpha y = \lambda v, & \text{in } \Omega, \\ z = \lambda y, & \text{in } \Omega, \\ \Delta y - \alpha u = \lambda z, & \text{in } \Omega, \\ u = y = 0, & \text{on } \Gamma. \end{cases} \quad (1.4.2)$$

Eliminating  $v$  and  $z$  from (1.4.2), we get

$$\begin{cases} \lambda^2 u - \Delta u - \lambda \Delta u + \alpha y = 0, & \text{in } \Omega, \\ \lambda^2 y - \Delta y + \alpha u = 0, & \text{in } \Omega, \\ u = y = 0, & \text{on } \Gamma. \end{cases} \quad (1.4.3)$$

From the second equation of (1.4.3), we have

$$u = \frac{1}{\alpha} \left[ \Delta y - \lambda^2 y \right]. \quad (1.4.4)$$



Substituting (1.4.4) in the first equation of (1.4.3), we obtain

$$\begin{cases} \Delta^2 y - \lambda^2 \left( \frac{2 + \lambda}{1 + \lambda} \right) \Delta y + \frac{\lambda^4 - \alpha^4}{1 + \lambda} y = 0, & \text{in } \Omega, \\ y = \Delta y = 0, & \text{on } \Gamma. \end{cases} \quad (1.4.5)$$

Now, let  $\{\mu_k^2\}_{k=1}^\infty$  be the sequence of eigenvalues of the Laplace operator with Dirichlet boundary conditions in  $\Omega$  and eigenvectors  $\varphi_k$ , i.e,

$$\begin{cases} -\Delta \varphi_k = \mu_k^2 \varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \Gamma. \end{cases} \quad (1.4.6)$$

Then by taking  $y = \varphi_k$  in (1.4.5), we deduce that  $\lambda$  will be an eigenvalue of  $\mathcal{A}$  if  $\lambda$  is a root of

$$P(\lambda) = \lambda^4 + \mu_k^2 \lambda^3 + 2\mu_k^2 \lambda^2 + \mu_k^4 \lambda + \mu_k^4 - \alpha^2 = 0. \quad (1.4.7)$$

We have the following result,

**Proposition 1.4.2.** *Assume that  $c = 0$ ,  $b = 1$  and  $\alpha = \alpha_0$  is a positive constant. Then, there exists  $k_0 \in \mathbb{N}^*$  sufficiently large and two sequences  $(\lambda_k^+)$  and  $(\lambda_k^-)$  simple roots of  $P$  (that are also simple eigenvalues of  $\mathcal{A}$ ) satisfying the following asymptotic behavior*

$$\lambda_k^+ = i\mu_k - \frac{\alpha^2}{2\mu_k^4} + O\left(\frac{1}{\mu_k^9}\right), \quad \forall |k| \geq k_0 \quad (1.4.8)$$

and

$$\lambda_k^- = -i\mu_k - \frac{\alpha^2}{2\mu_k^4} + O\left(\frac{1}{\mu_k^9}\right), \quad \forall |k| \geq k_0. \quad (1.4.9)$$

*Proof.* First, set  $\xi = \frac{\lambda}{\mu_k}$  and  $\zeta_k = \frac{1}{\mu_k}$ . Then, multiply equation (1.4.7) by  $\frac{1}{\mu_k^5}$ , we get

$$h(\xi) = \xi^3 + \xi + \zeta_k + 2\xi^2 \zeta_k + \xi^4 \zeta_k - \alpha^2 \zeta_k^5 = 0. \quad (1.4.10)$$

Then, in order to find the roots of  $P$ , i.e. the eigenvalues of  $\mathcal{A}$  we need before to calculate the roots of  $h$ . We will continue the proof with three steps:

**Step 1.** Let

$$f(\xi) = \xi^3 + \xi$$

and

$$g(\xi) = \xi^4 \zeta_k + \zeta_k + 2\xi^2 \zeta_k - \alpha^2 \zeta_k^5.$$

We look for  $r_k$  sufficiently small such that

$$|f| > |h - f| = |g|, \quad \text{on } \partial D,$$

where,  $D = \{\xi \in \mathbb{C}; |\xi - i| \leq r_k\}$ .

Let  $\xi \in \partial D(i, r_k)$ , then  $\xi = i + r_k e^{i\theta}$  with  $0 \leq \theta \leq 2\pi$ . We have

$$f(\xi) = \xi^3 + \xi = \xi r_k (2ie^{i\theta} + r_k e^{2i\theta}).$$

But, if  $r_k \leq \frac{1}{2}$  then,

$$|\xi| \geq |1 - r_k| \geq \frac{1}{2}$$

and

$$|2ie^{i\theta} + r_k e^{2i\theta}| \geq |2ie^{i\theta}| - r_k \geq 2 - r_k \geq 3/2.$$

This gives that

$$|f| = |\zeta^3 + \zeta| \geq \frac{3r_k}{4}, \text{ if } r_k \leq 1/2.$$

On the other hand, since  $\zeta$  is bounded in  $D$  and  $\zeta_k \rightarrow 0$  we have,

$$|g(\zeta)| \leq c_1 \zeta_k, \text{ for some constant } c_1 > 0.$$

So, it is enough to choose  $r_k = \frac{4c_1}{3} \zeta_k$ .

Similarly, we can find  $r_k$  sufficiently small such that

$$|f| > |h - f| = |g|, \text{ on } \partial D' = \partial\{\zeta \in \mathbf{C}; |\zeta + i| \leq r_k\}.$$

**Step 2.** In this step, we prove that  $P$  admits an infinity of simple roots denoted by  $(\lambda_k^+)$  and  $(\lambda_k^-)$  for  $|k| \geq k_0$  with  $k_0$  large enough such that

$$\lambda_k^\pm = \pm \mu_k + \mu_k \epsilon_k^\pm, \text{ with } \lim_{|k| \rightarrow \infty} \epsilon_k^\pm = 0. \quad (1.4.11)$$

Using **Step 1** and thanks to Rouché's theorem, there exists  $k_0$  large enough such that for all  $|k| \geq k_0$  the roots of the polynomial  $h$  are close to the roots of the polynomial  $f(\zeta) = \zeta^3 + \zeta$ . Then,

$$\zeta_k^+ = i + \epsilon_k^+ \text{ and } \zeta_k^- = -i + \epsilon_k^-, \text{ with } \lim_{|k| \rightarrow \infty} \epsilon_k^\pm = 0. \quad (1.4.12)$$

Using the fact that  $\lambda_k^\pm = \mu_k \zeta_k^\pm$  we deduce (1.4.11).

**Step 3.** Asymptotic behavior of  $\epsilon_k^\pm$ . Inserting equation (1.4.12) into equation (1.4.10), we get

$$\epsilon_k^\pm \left[ -2 + \left( \pm 3i - \frac{4}{\mu_k} \right) \epsilon_k^\pm + \left( 1 \pm \frac{4i}{\mu_k} \right) (\epsilon_k^\pm)^2 + \frac{1}{\mu_k} (\epsilon_k^\pm)^3 \right] = \frac{\alpha^2}{\mu_k^5}. \quad (1.4.13)$$

From equation (1.4.13), we can predict that  $\epsilon_k^\pm$  is in the form

$$\epsilon_k^\pm = \frac{-\alpha^2}{2\mu_k^5} + \eta_k^\pm, \quad (1.4.14)$$

with  $\eta_k^\pm$  tends to zero as  $k$  tends to infinity.

Now, inserting expression (1.4.14) in (1.4.13), we find

$$a_{0,k}^\pm + a_{1,k}^\pm \eta_k^\pm + a_{2,k}^\pm (\eta_k^\pm)^2 + a_{3,k}^\pm (\eta_k^\pm)^3 + a_{4,k}^\pm (\eta_k^\pm)^4 = 0, \quad (1.4.15)$$

where,

$$\begin{cases} a_{0,k}^{\pm} = \frac{\alpha^8}{16\mu_k^{21}} - \frac{\alpha^6}{8\mu_k^{15}} - \frac{\alpha^4}{\mu_k^{11}} \pm \left(-\frac{\alpha^6}{2\mu_k^{16}} + \frac{3\alpha^4}{4\mu_k^{10}}\right)i, \\ a_{1,k}^{\pm} = \frac{-\alpha^6}{2\mu_k^{16}} + \frac{3\alpha^4}{4\mu_k^{10}} + \frac{4\alpha^2}{\mu_k^6} - 2 \pm \left(-\frac{3\alpha^2}{\mu_k^5} + \frac{3\alpha^4}{\mu_k^{11}}\right)i, \\ a_{2,k}^{\pm} = \frac{3\alpha^4}{2\mu_k^{11}} - \frac{3\alpha^2}{2\mu_k^5} - \frac{4}{\mu_k} \pm \left(3 - \frac{6\alpha^2}{\mu_k^6}\right)i, \\ a_{3,k}^{\pm} = \frac{-2\alpha^2}{\mu_k^6} + 1 \pm \frac{4i}{\mu_k}, \\ a_{4,k}^{\pm} = \frac{1}{\mu_k}. \end{cases} \quad (1.4.16)$$

We can notice that  $a_{1,k}^{\pm} \neq 0$  for  $k$  large enough,  $a_{i,k}^{\pm}$  is bounded uniformly for  $k$  large enough,  $i = 0, \dots, 4$  and

$$|a_{1,k}^{\pm}| \geq 1, \quad \forall k \geq k_0.$$

Therefore, using (1.4.15) there exists  $C > 0$  (independent of  $k$ ) such that

$$\left| \frac{a_{0,k}^{\pm}}{a_{1,k}^{\pm}} + \eta_k^{\pm} \right| \leq C |\eta_k^{\pm}|^2. \quad (1.4.17)$$

Hence, for  $k$  large enough, we deduce that

$$|\eta_k^{\pm}| \leq 2 \left| \frac{a_{0,k}^{\pm}}{a_{1,k}^{\pm}} \right|.$$

Simple calculations yields

$$\left| \frac{a_{0,k}^{\pm}}{a_{1,k}^{\pm}} \right| = O\left(\frac{1}{\mu_k^{10}}\right).$$

This implies that

$$\epsilon_k^{\pm} = \frac{-\alpha^2}{2\mu_k^5} + O\left(\frac{1}{\mu_k^{10}}\right).$$

Consequently

$$\tilde{\zeta}_k^{\pm} = \pm i + \frac{-\alpha^2}{2\mu_k^5} + O\left(\frac{1}{\mu_k^{10}}\right).$$

Finally, using the fact that  $\lambda_k^{\pm} = \mu_k \tilde{\zeta}_k^{\pm}$  we deduce that (1.4.8) and (1.4.9) hold.  $\square$

## 1.5 Polynomial Stability with one damping

From now on, we assume that  $\alpha(x) \geq 0$  for all  $x \in \Omega$ . In this section, we will study the polynomial decay of our system with only one damping term ( $b \geq 0, c = 0$  or  $b = 0, c \geq 0$  on  $\Omega$ ). One of the main ingredients is to use the exponential or polynomial decay of the wave equation with frictional damping in  $\omega_1 = \omega_b \cap \omega_\alpha$ , namely

$$\begin{cases} \varphi_{tt} - \Delta\varphi + \mathbb{1}_{\omega_1}\varphi_t = 0 & \text{in } \Omega \times (0, \infty), \\ \varphi = 0 & \text{on } \Gamma \times (0, \infty), \\ \varphi(t=0) = \varphi_0, \varphi_t(t=0) = \varphi_1. \end{cases} \quad (1.5.1)$$

Without loss of generality, assume that  $b \neq 0$  and  $c = 0$  (The same proof holds if  $c \neq 0$  and  $b = 0$ ).

**Theorem 1.5.1.** *(Polynomial energy decay rate)*

Let  $\omega_b$  and  $\omega_\alpha$  be non-empty open subsets of  $\Omega$  such that (1.1.7) and (1.1.8) hold. Also, suppose that  $\omega_1 = \omega_b \cap \omega_\alpha$  satisfies  $\text{meas}(\overline{\omega_1} \cap \Gamma) > 0$ . Further, assume that the energy of the system (1.5.1) is exponentially stable. Then, there exists a constant  $C > 0$  such that for every initial data  $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ , the energy of system (1.1.1) verifies the following estimate:

$$E(t) \leq C \frac{1}{\sqrt{t}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (1.5.2)$$

*Proof.* Following Borichev and Tomilov, 2009 (see also Liu and Rao, 2005, Batty and Duyckaerts, 2008), a  $C_0$  semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$  verifies (1.5.2) if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad (\text{S1})$$

and

$$\limsup_{|\lambda| \rightarrow \infty} \frac{1}{\lambda^4} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad (\text{S2}).$$

We know that condition (S1) is verified. Our goal now is to prove that condition (S2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S2) does not hold, then there exist a sequence  $(\lambda_n) \subset \mathbb{R}$  and a sequence  $(U_n) \subset D(\mathcal{A})$  such that

$$|\lambda_n| \longrightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1 \quad (1.5.3)$$

and

$$\lambda_n^4 (i\lambda_n - \mathcal{A})U_n = (f_1^n, g_1^n, f_2^n, g_2^n) \longrightarrow 0 \quad \text{in } \mathcal{H} \quad (1.5.4)$$

are satisfied.

In what follows, we drop the index  $n$  for simplicity.

**Lemma 1.5.2.** *Under all the above assumptions, we have*

$$\|\nabla v\|_{L^2(\omega_1)} = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \|v\|_{L^2(\omega_1)} = \frac{o(1)}{\lambda^2}. \quad (1.5.5)$$

*Proof.* Multiply in  $\mathcal{H}$  equation (1.5.4) by the uniformly bounded sequence  $U = (u, v, y, z)$ , we get

$$\int_{\Omega} b(x) |\nabla v|^2 dx = -\text{Re}((i\lambda I - \mathcal{A})U, U)_{\mathcal{H}} = \frac{o(1)}{\lambda^4}.$$

It follows that

$$\|\sqrt{b}\nabla v\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^2}. \quad (1.5.6)$$

Using (1.1.7), (1.5.6) and Poincaré inequality (since  $\text{meas}(\overline{\omega_1} \cap \Gamma) > 0$ ) we get (1.5.5).  $\square$

By detailing equation (1.5.4), we get the following system

$$\lambda^4(i\lambda u - v) = f_1 \longrightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.5.7)$$

$$\lambda^4(i\lambda v - \text{div}(\nabla u + b\nabla v) + \alpha y) = g_1 \longrightarrow 0 \quad \text{in } L^2(\Omega), \quad (1.5.8)$$

$$\lambda^4(i\lambda y - z) = f_2 \longrightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.5.9)$$

$$\lambda^4(i\lambda z - \Delta y + \alpha u) = g_2 \longrightarrow 0 \quad \text{in } L^2(\Omega). \quad (1.5.10)$$

From equations (1.5.3), (1.5.7) and (1.5.9) we get

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \quad (1.5.11)$$

and using (1.5.3), (1.5.5) and (1.5.7) we also have

$$\|\nabla u\|_{L^2(\omega_1)} = \frac{o(1)}{\lambda^3}, \quad \text{and} \quad \|u\|_{L^2(\omega_1)} = \frac{o(1)}{\lambda^3}. \quad (1.5.12)$$

**Lemma 1.5.3.** *Under all the above assumptions, we have*

$$\int_{\Omega} |\lambda u|^2 dx = o(1). \quad (1.5.13)$$

*Proof.* For all  $n \in \mathbb{N}$ , let  $\varphi_n, \psi_n \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of the following system

$$\lambda_n^2 \varphi_n + \Delta \varphi_n - i\lambda_n \mathbb{1}_{\omega_1} \varphi_n = u_n, \quad \text{in } \Omega, \quad (1.5.14)$$

$$\lambda_n^2 \psi_n + \Delta \psi_n - i\lambda_n \mathbb{1}_{\omega_1} \psi_n = y_n, \quad \text{in } \Omega, \quad (1.5.15)$$

$$\varphi_n = \psi_n = 0, \quad \text{in } \Gamma, \quad (1.5.16)$$

where  $(u_n, v_n, y_n, z_n)$  is the solution of (1.5.7)-(1.5.10). If (1.5.1) with the local viscous damping  $\mathbb{1}_{\omega_1} \varphi_t$  (or  $\mathbb{1}_{\omega_1} \psi_t$ ) is exponentially stable then, following Huang, 1985; Prüss, 1984, there exists  $M > 0$  independent of  $n$  such that

$$\begin{aligned} & \|\lambda \varphi_n\|_{L^2(\Omega)}^2 + \|\nabla \varphi_n\|_{L^2(\Omega)}^2 + \|\lambda \psi_n\|_{L^2(\Omega)}^2 + \|\nabla \psi_n\|_{L^2(\Omega)}^2 \\ & \leq M(\|u_n\|_{L^2(\Omega)}^2 + \|y_n\|_{L^2(\Omega)}^2). \end{aligned} \quad (1.5.17)$$

For simplicity, we drop the index  $n$ .

Now, multiplying (1.5.7) and (1.5.8) by  $i\lambda^3 \overline{\varphi}$  and  $\lambda^2 \overline{\varphi}$  respectively and applying Green's formula, we get

$$\begin{aligned} & - \int_{\Omega} \lambda^2 u (\lambda^2 \overline{\varphi} + \Delta \overline{\varphi}) dx + \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \overline{\varphi} dx + \int_{\Omega} \lambda^2 \alpha y \overline{\varphi} dx \\ & = \int_{\Omega} \left( \frac{g_1 \overline{\varphi}}{\lambda^2} + \frac{if_1 \overline{\varphi}}{\lambda} \right) dx. \end{aligned} \quad (1.5.18)$$

Using (1.5.14) we get

$$\begin{aligned} & - \int_{\Omega} |\lambda u|^2 dx + \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_1} u \overline{\varphi} dx + \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \overline{\varphi} dx + \int_{\Omega} \lambda^2 \alpha y \overline{\varphi} dx \\ & = \int_{\Omega} \left( \frac{g_1 \overline{\varphi}}{\lambda^2} + \frac{if_1 \overline{\varphi}}{\lambda} \right) dx. \end{aligned} \quad (1.5.19)$$

But, using Cauchy-Schwarz inequality, (1.5.12), (1.5.17) and (1.5.3) we have,

$$\left| \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_1} u \bar{\varphi} \, dx \right| \leq \|\lambda^3 u\|_{L^2(\omega_1)} \|\varphi\|_{L^2(\Omega)} = o(1) \quad (1.5.20)$$

and by (1.5.6), (1.5.17) and (1.5.3)

$$\left| \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} \, dx \right| \leq \|\lambda^2 b \nabla v\|_{L^2(\Omega)} \|\nabla \bar{\varphi}\|_{L^2(\Omega)} = o(1). \quad (1.5.21)$$

Again, using Cauchy-Schwarz inequality, (1.5.11), (1.5.17) and the fact  $\|u\|_{L^2(\Omega)} = o(1)$  we obtain,

$$\left| \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} \, dx \right| \leq \|\alpha\|_{\infty} \|\lambda y\|_{L^2(\Omega)} \|\lambda \varphi\|_{L^2(\Omega)} = o(1). \quad (1.5.22)$$

Consequently, by (1.5.19) and the fact that  $\varphi, f_1$  and  $g_1$  are bounded in  $L^2(\Omega)$  and using (1.5.20)-(1.5.22) we get

$$\int_{\Omega} |\lambda u|^2 \, dx = o(1).$$

□

**Lemma 1.5.4.** *Under all the above assumptions, we have*

$$\int_{\Omega} |\lambda y|^2 \, dx = o(1). \quad (1.5.23)$$

*Proof.* By inserting equation (1.5.7) in equation (1.5.8) we get

$$-\lambda^2 u - \operatorname{div}(\nabla u + b \nabla v) + \alpha y = \frac{g_1}{\lambda^4} + \frac{if_1}{\lambda^3}. \quad (1.5.24)$$

Similarly, inserting equation (1.5.9) in equation (1.5.10) we obtain

$$-\lambda^2 y - \Delta y + \alpha u = \frac{g_2}{\lambda^4} + \frac{if_2}{\lambda^3}. \quad (1.5.25)$$

Now multiplying (1.5.24) and (1.5.25) by  $\bar{y}$  and  $\bar{u}$  respectively then integrating over  $\Omega$  and applying Green's formula we obtain

$$\begin{aligned} - \int_{\Omega} \lambda^2 u \bar{y} \, dx + \int_{\Omega} \nabla u \cdot \nabla \bar{y} \, dx + \int_{\Omega} b \nabla v \cdot \nabla \bar{y} \, dx + \int_{\Omega} \alpha |y|^2 \, dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{y}}{\lambda^4} + \frac{if_1 \bar{y}}{\lambda^3} \right) \, dx, \end{aligned} \quad (1.5.26)$$

and

$$\begin{aligned} - \int_{\Omega} \lambda^2 y \bar{u} \, dx + \int_{\Omega} \nabla y \cdot \nabla \bar{u} \, dx + \int_{\Omega} \alpha |u|^2 \, dx \\ = \int_{\Omega} \left( \frac{g_2 \bar{u}}{\lambda^4} + \frac{if_2 \bar{u}}{\lambda^3} \right) \, dx. \end{aligned} \quad (1.5.27)$$

Taking the real part of (1.5.26) and (1.5.27) then subtracting both equations, using (1.5.11) and the fact that  $f_1, g_1, f_2$  and  $g_2$  are bounded in  $L^2(\Omega)$  we get

$$\int_{\Omega} \alpha |u|^2 dx + \operatorname{Re} \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx - \int_{\Omega} \alpha |y|^2 dx = \frac{o(1)}{\lambda^4}. \quad (1.5.28)$$

Using Cauchy-Schwarz inequality, (1.5.6) and the fact that  $\nabla y$  is bounded in  $L^2(\Omega)$  we have

$$\operatorname{Re} \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx \leq \left| \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx \right| \leq \|b \nabla v\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^2}. \quad (1.5.29)$$

This yields,

$$\int_{\Omega} \alpha |u|^2 dx - \int_{\Omega} \alpha |y|^2 dx = \frac{o(1)}{\lambda^2}. \quad (1.5.30)$$

Using (1.5.13) and the fact that  $\alpha \in L^\infty(\Omega)$  we obtain

$$\int_{\Omega} \alpha |\lambda y|^2 dx = o(1).$$

But, using condition (1.1.8) we have

$$\alpha_0 \int_{\omega_1} |\lambda y|^2 dx \leq \int_{\Omega} \alpha |\lambda y|^2 dx = o(1).$$

Consequently,

$$\int_{\omega_1} |\lambda y|^2 dx = o(1). \quad (1.5.31)$$

Now, multiplying (1.5.9) and (1.5.10) by  $i\lambda^3 \bar{\psi}$  and  $\lambda^2 \bar{\psi}$  respectively and applying Green's formula, we get

$$- \int_{\Omega} \lambda^2 y (\lambda^2 \bar{\psi} + \Delta \bar{\psi}) dx + \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx = \int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^2} + \frac{if_2 \bar{\psi}}{\lambda} \right) dx. \quad (1.5.32)$$

Using (1.5.15) we get

$$\begin{aligned} - \int_{\Omega} |\lambda y|^2 dx + \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_1} y \bar{\psi} dx + \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx \\ = \int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^2} + \frac{if_2 \bar{\psi}}{\lambda} \right) dx. \end{aligned} \quad (1.5.33)$$

But, using Cauchy-Schwarz inequality, (1.5.31), (1.5.17) and (1.5.3) we have

$$\left| \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_1} y \bar{\psi} dx \right| \leq \|\lambda^2 y\|_{L^2(\omega_1)} \|\lambda \psi\|_{L^2(\Omega)} = o(1) \quad (1.5.34)$$

and

$$\left| \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx \right| \leq \|\alpha\|_{\infty} \|\lambda u\|_{L^2(\Omega)} \|\lambda \psi\|_{L^2(\Omega)} = o(1). \quad (1.5.35)$$

Then, by (1.5.33), the fact that  $\psi$  is bounded in  $L^2(\Omega)$  by (1.5.11) and (1.5.17), the fact that  $f_2, g_2$  are bounded in  $L^2(\Omega)$  and using (1.5.34)-(1.5.35), we obtain

$$\int_{\Omega} |\lambda y|^2 dx = o(1).$$

□

**Lemma 1.5.5.** *Under all the above assumptions, we have*

$$\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1). \quad (1.5.36)$$

*Proof.* Multiply (1.5.24) and (1.5.25) by  $\bar{u}$  and  $\bar{y}$  respectively, then integrate over  $\Omega$  and apply Green's formula, we obtain

$$\begin{aligned} - \int_{\Omega} |\lambda u|^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} b \nabla v \cdot \nabla \bar{u} dx + \int_{\Omega} \alpha y \bar{u} dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{u}}{\lambda^4} + \frac{if_1 \bar{u}}{\lambda^3} \right) dx, \end{aligned} \quad (1.5.37)$$

and

$$- \int_{\Omega} |\lambda y|^2 dx + \int_{\Omega} |\nabla y|^2 dx + \int_{\Omega} \alpha u \bar{y} dx = \int_{\Omega} \left( \frac{g_2 \bar{y}}{\lambda^4} + \frac{if_2 \bar{y}}{\lambda^3} \right) dx. \quad (1.5.38)$$

But, using Cauchy-Schwarz inequality, (1.5.6) and (1.5.3) we have

$$\left| \int_{\Omega} b \nabla v \cdot \nabla \bar{u} dx \right| \leq \|b \nabla v\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} = o(1)$$

and

$$\left| \int_{\Omega} \alpha y \bar{u} dx \right| \leq \|\alpha\|_{\infty} \|u\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} = o(1).$$

Consequently, we get

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\lambda u|^2 dx + o(1)$$

and

$$\int_{\Omega} |\nabla y|^2 dx = \int_{\Omega} |\lambda y|^2 dx + o(1).$$

By (1.5.13) and (1.5.23) we deduce that

$$\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1).$$

□

In conclusion, by (1.5.13), (1.5.23) and (1.5.36) we deduce that  $\|U_n\|_{\mathcal{H}} = o(1)$  which leads to the desired contradiction. □

**Remark 1.5.6.**

1) Theorem 1.5.1 shows that we can have  $b \neq 0, c \geq 0$  or  $b \geq 0, c \neq 0$ . In other words, to prove the polynomial energy decay rate of type  $t^{-1/2}$  we only need one of the dampings to be effective.



2) From Zuazua, 1990, Theorem 1.1 and Remark 1.2 we deduce that if the boundary of  $\Omega$  is  $C^2$  and  $\omega_1$  is a neighbourhood of the boundary, then the solution of (1.5.1) is exponentially decaying. In fact, we know that given any point  $x^0 \in \mathbb{R}^N$ , (1.5.17) is satisfied when  $\omega_1$  is a neighborhood of  $\bar{\Gamma}(x^0)$  where

$$\Gamma(x^0) = \{x \in \Gamma : (x - x^0) \cdot \nu(x) > 0\},$$

where  $\nu(x)$  is the unit outward normal at  $x \in \Gamma$ .

3) Also, from Zuazua, 1990 and by Bardos, Lebeau, and Rauch, 1992 (see also Lebeau, 1993-1994) for domains with boundaries, we know that when  $\Omega$  is of class  $C^\infty$ , (1.5.17) is satisfied when  $\omega_1$  satisfies the Geometric Control Condition (GCC). Recall that the GCC can be formulated as follows: For a subset  $\omega$  of  $\Omega$ , we shall say that  $\omega$  satisfies the Geometric Control Condition if there exists  $T > 0$  such that every geodesic traveling at speed one issued from  $\Omega$  at time  $t = 0$  intersects  $\omega$  before time  $T$ .

4) In Remark 4.3 of Liu, 1997, further examples of pairs  $(\Omega, \omega_1)$  such that (1.5.17) is valid are given.

Now, under an additional condition on  $\omega_\alpha$ , we will show that if (1.5.1) decay exponentially ( $\beta = 0$ ) or polynomially as  $t^{-\frac{4}{\beta}}$  with  $\beta > 0$ , then the energy of the system (1.1.1) decays polynomially as  $t^{-\frac{2}{\ell}}$  with  $\ell = 2\beta + 4$ . In fact, under the additional assumption  $\bar{\omega}_\alpha = \text{supp } \alpha \subset \bar{\omega}_b$  we will find a better estimation of  $\|y_n\|_{L^2(\omega_\alpha)}$ .

**Theorem 1.5.7.** *(Polynomial energy decay rate)*

Let  $\omega_\alpha$  and  $\omega_b$  be non-empty subsets of  $\Omega$ . Assume that conditions (1.1.7) and (1.1.8) hold. Let  $\omega_\alpha$  satisfies  $\text{meas}(\bar{\omega}_\alpha \cap \Gamma) > 0$  and let  $\bar{\omega}_\alpha = \text{supp } \alpha \subset \bar{\omega}_b$ . Also, assume that the energy of the system (1.5.1) is exponentially stable ( $\beta = 0$ ) or polynomially stable as  $t^{-\frac{4}{\beta}}$ , with  $\beta > 0$ . Then, there exists a constant  $C > 0$  such that for every initial data  $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ , the energy of system (1.1.1) verifies the following estimate:

$$E(t) \leq \frac{C}{t^{\frac{2}{\ell}}} \|U_0\|_{D(\mathcal{A})}^2, \quad \ell = 2\beta + 4, \quad \beta \geq 0, \quad \forall t > 0. \quad (1.5.39)$$

*Proof.* Following Borichev and Tomilov, 2009, (see also Liu and Rao, 2005, Batty and Duyckaerts, 2008), a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$  verifies (1.5.39) if (S1) holds and

$$\limsup_{|\lambda| \rightarrow \infty} \frac{1}{\lambda^\ell} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad (\text{S3}).$$

We know that condition (S1) is verified. Our goal now is to prove that condition (S3) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S3) does not hold, then there exist a sequence  $(\lambda_n) \subset \mathbb{R}$  and a sequence  $(U_n) \subset D(\mathcal{A})$  such that (1.5.3) and

$$\lambda_n^\ell (i\lambda_n - \mathcal{A})U_n = (f_1^n, g_1^n, f_2^n, g_2^n) \longrightarrow 0 \quad \text{in } \mathcal{H} \quad (1.5.40)$$

are satisfied.

In what follows, we drop the index  $n$  for simplicity.

**Lemma 1.5.8.** *Under all the above assumptions, we have*

$$\|\nabla v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{\ell/2}} \text{ and } \|v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{\ell/2}}. \quad (1.5.41)$$

The proof is the same as the one of Lemma 1.5.2.

By detailing equation (1.5.40), we get the following system

$$\lambda^\ell(i\lambda u - v) = f_1 \longrightarrow 0 \text{ in } H_0^1(\Omega), \quad (1.5.42)$$

$$\lambda^\ell(i\lambda v - \operatorname{div}(\nabla u + b\nabla v) + \alpha y) = g_1 \longrightarrow 0 \text{ in } L^2(\Omega), \quad (1.5.43)$$

$$\lambda^\ell(i\lambda y - z) = f_2 \longrightarrow 0 \text{ in } H_0^1(\Omega), \quad (1.5.44)$$

$$\lambda^\ell(i\lambda z - \Delta y + \alpha u) = g_2 \longrightarrow 0 \text{ in } L^2(\Omega). \quad (1.5.45)$$

From equations (1.5.42) and (1.5.44) and using (1.5.3) we get (1.5.11), and using (1.5.41), (1.5.42), (1.5.3), the fact that  $\bar{\omega}_\alpha = \operatorname{supp} \alpha \subset \bar{\omega}_b$ , and Poincaré inequality (since  $\operatorname{meas}(\bar{\omega}_\alpha \cap \Gamma) > 0$ ), we also have

$$\|\nabla u\|_{L^2(\omega_\alpha)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}} \text{ and } \|u\|_{L^2(\omega_\alpha)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}. \quad (1.5.46)$$

**Lemma 1.5.9.** *Under all above assumptions, we have*

$$\int_{\omega_\alpha} |y|^2 dx = \frac{o(1)}{\lambda^{\ell/2}}. \quad (1.5.47)$$

*Proof.* By inserting equation (1.5.42) in equation (1.5.43) we get

$$-\lambda^2 u - \operatorname{div}(\nabla u + b\nabla v) + \alpha y = \frac{g_1}{\lambda^\ell} + \frac{if_1}{\lambda^{\ell-1}}. \quad (1.5.48)$$

Similarly, inserting equation (1.5.44) in equation (1.5.45) we obtain

$$-\lambda^2 y - \Delta y + \alpha u = \frac{g_2}{\lambda^\ell} + \frac{if_2}{\lambda^{\ell-1}}. \quad (1.5.49)$$

Now multiplying (1.5.48) and (1.5.49) by  $\bar{y}$  and  $\bar{u}$  respectively, then integrating over  $\Omega$  and applying Green's formula we obtain

$$\begin{aligned} - \int_{\Omega} \lambda^2 u \bar{y} dx + \int_{\Omega} \nabla u \cdot \nabla \bar{y} dx + \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx + \int_{\Omega} \alpha |y|^2 dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{y}}{\lambda^\ell} + \frac{if_1 \bar{y}}{\lambda^{\ell-1}} \right) dx, \end{aligned} \quad (1.5.50)$$

and

$$\begin{aligned} - \int_{\Omega} \lambda^2 y \bar{u} dx + \int_{\Omega} \nabla y \cdot \nabla \bar{u} dx + \int_{\Omega} \alpha |u|^2 dx \\ = \int_{\Omega} \left( \frac{g_2 \bar{u}}{\lambda^\ell} + \frac{if_2 \bar{u}}{\lambda^{\ell-1}} \right) dx. \end{aligned} \quad (1.5.51)$$

Taking the real part of (1.5.50) and (1.5.51) then subtracting both equations, using (1.5.11) and the fact that  $f_1, g_1, f_2$  and  $g_2$  converge to zero in  $L^2(\Omega)$  we get

$$\int_{\Omega} \alpha |u|^2 dx + \operatorname{Re} \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx - \int_{\Omega} \alpha |y|^2 dx = \frac{o(1)}{\lambda^\ell}. \quad (1.5.52)$$

Using Cauchy-Schwarz inequality, (1.5.41) and the fact that  $\nabla y$  is bounded in  $L^2(\Omega)$  we have

$$\operatorname{Re} \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx \leq \left| \int_{\Omega} b \nabla v \cdot \nabla \bar{y} dx \right| \leq \|b \nabla v\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\ell/2}}. \quad (1.5.53)$$

This yields,

$$\int_{\Omega} \alpha |u|^2 dx - \int_{\Omega} \alpha |y|^2 dx = \frac{o(1)}{\lambda^{\ell/2}}. \quad (1.5.54)$$

But, using (1.5.46) and the fact that  $\bar{\omega}_\alpha = \operatorname{supp} \alpha$ , we get

$$\int_{\Omega} \alpha |u|^2 dx \leq \|\alpha\|_\infty \int_{\omega_\alpha} |u|^2 dx = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}.$$

Then, (1.5.54) becomes

$$\int_{\Omega} \alpha |y|^2 dx = \frac{o(1)}{\lambda^{\ell/2}}.$$

But, using condition (1.1.8) and as  $\bar{\omega}_\alpha = \operatorname{supp} \alpha$ , we deduce that

$$\int_{\omega_\alpha} |y|^2 dx = \frac{o(1)}{\lambda^{\ell/2}}. \quad (1.5.55)$$

□

**Lemma 1.5.10.** *Under all the above assumptions, if  $\ell \geq \max\{\beta + 2, 2\beta\}$  we have*

$$\int_{\Omega} |\lambda u|^2 dx = o(1). \quad (1.5.56)$$

*Proof.* For all  $n \in \mathbb{N}$ , let  $\varphi_n, \psi_n \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of (1.5.14)-(1.5.16). If the system (1.5.1) with  $\omega_1 = \omega_\alpha$  decays exponentially ( $\beta = 0$ ) or polynomially as  $t^{-\frac{4}{\beta}}$ , with  $\beta > 0$ . Then, following Borichev and Tomilov, 2009, there exists  $M > 0$  independent of  $n$  such that

$$\begin{aligned} & \|\lambda \varphi_n\|_{L^2(\Omega)}^2 + \|\nabla \varphi_n\|_{L^2(\Omega)}^2 + \|\lambda \psi_n\|_{L^2(\Omega)}^2 + \|\nabla \psi_n\|_{L^2(\Omega)}^2 \\ & \leq M |\lambda|^\beta (\|u_n\|_{L^2(\Omega)}^2 + \|y_n\|_{L^2(\Omega)}^2). \end{aligned} \quad (1.5.57)$$

Hence, using (1.5.11) we have,

$$\begin{cases} \|\lambda \varphi\|_{L^2(\Omega)} = \frac{O(1)}{\lambda^{1-\frac{\beta}{2}}}, & \|\nabla \varphi\|_{L^2(\Omega)} = \frac{O(1)}{\lambda^{1-\frac{\beta}{2}}}, \\ \|\lambda \psi\|_{L^2(\Omega)} = \frac{O(1)}{\lambda^{1-\frac{\beta}{2}}}, & \|\nabla \psi\|_{L^2(\Omega)} = \frac{O(1)}{\lambda^{1-\frac{\beta}{2}}}. \end{cases} \quad (1.5.58)$$

Now, multiplying (1.5.42) and (1.5.43) by  $i\lambda^3\bar{\varphi}$  and  $\lambda^2\bar{\varphi}$  respectively and applying Green's formula, we get

$$\begin{aligned} - \int_{\Omega} \lambda^2 u (\lambda^2 \bar{\varphi} + \Delta \bar{\varphi}) dx + \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx + \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx. \end{aligned} \quad (1.5.59)$$

Using (1.5.14) we get

$$\begin{aligned} - \int_{\Omega} |\lambda u|^2 dx + \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_\alpha} u \bar{\varphi} dx + \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx + \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx. \end{aligned} \quad (1.5.60)$$

But, using Cauchy-Schwarz inequality, (1.5.46) and (1.5.58) we have,

$$\left| \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_\alpha} u \bar{\varphi} dx \right| \leq |\lambda|^2 \|u\|_{L^2(\omega_\alpha)} \|\lambda \varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-\beta}{2}}}. \quad (1.5.61)$$

This means that we need  $\frac{\ell-\beta}{2} \geq 0$ , i.e,  $\ell \geq \beta$  so that

$$\int_{\Omega} \lambda^3 \mathbb{1}_{\omega_\alpha} u \bar{\varphi} dx = o(1).$$

Also, by (1.5.41) and (1.5.58) we have

$$\left| \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx \right| \leq |\lambda|^2 \|b \nabla v\|_{L^2(\Omega)} \|\nabla \bar{\varphi}\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-\beta}{2}-1}}. \quad (1.5.62)$$

This means that we should have  $\frac{\ell-\beta}{2} - 1 \geq 0$ , i.e,  $\ell \geq \beta + 2$  so that

$$\int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx = o(1).$$

Again, using Cauchy-Schwarz inequality, (1.5.47), (1.5.58) and as  $\bar{\omega}_\alpha = \text{supp} \alpha$ , we obtain

$$\left| \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx \right| \leq \|\alpha\|_\infty |\lambda| \|y\|_{L^2(\omega_\alpha)} \|\lambda \varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-2\beta}{4}}}. \quad (1.5.63)$$

This means that we should have  $\frac{\ell-2\beta}{4} \geq 0$ , i.e,  $\ell \geq 2\beta$  so that

$$\int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx = o(1).$$

On the other hand, using Cauchy-Schwarz inequality, (1.5.58) and the fact that  $f_1$  and  $g_1$  converge to zero in  $L^2(\Omega)$  we have

$$\begin{aligned} \left| \int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx \right| \leq \left( \frac{1}{|\lambda|^{\ell-2}} \|g_1\|_{L^2(\Omega)} + \frac{1}{|\lambda|^{\ell-3}} \|f_1\|_{L^2(\Omega)} \right) \|\varphi\|_{L^2(\Omega)} \\ = \frac{o(1)}{\lambda^{\ell-\frac{\beta}{2}}} + \frac{o(1)}{\lambda^{\ell-1-\frac{\beta}{2}}}. \end{aligned} \quad (1.5.64)$$

This means that we need  $\ell - 1 - \frac{\beta}{2} \geq 0$ , i.e,  $\ell \geq \frac{\beta}{2} + 1$  so that

$$\int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx = o(1).$$

As a conclusion, if  $\ell \geq \max\{\beta + 2, 2\beta\}$  we get

$$\int_{\Omega} |\lambda u|^2 dx = o(1).$$

□

**Lemma 1.5.11.** *Under all above assumptions, if  $\ell \geq 2\beta + 4$  then, we have*

$$\int_{\Omega} |\lambda y|^2 dx = o(1). \quad (1.5.65)$$

*Proof.* Now, multiplying (1.5.44) and (1.5.45) by  $i\lambda^3 \bar{\psi}$  and  $\lambda^2 \bar{\psi}$  respectively and applying Green's formula, we get

$$- \int_{\Omega} \lambda^2 y (\lambda^2 \bar{\psi} + \Delta \bar{\psi}) dx + \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx = \int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^{\ell-2}} + \frac{if_2 \bar{\psi}}{\lambda^{\ell-3}} \right) dx. \quad (1.5.66)$$

Using (1.5.15) we get

$$\begin{aligned} - \int_{\Omega} |\lambda y|^2 dx + \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_{\alpha}} y \bar{\psi} dx + \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx \\ = \int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^{\ell-2}} + \frac{if_2 \bar{\psi}}{\lambda^{\ell-3}} \right) dx. \end{aligned} \quad (1.5.67)$$

But, using Cauchy-Schwarz inequality, (1.5.55) and (1.5.58) we have

$$\left| \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_{\alpha}} y \bar{\psi} dx \right| \leq |\lambda|^2 \|y\|_{L^2(\omega_{\alpha})} \|\lambda \psi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-2\beta}{4}-1}}. \quad (1.5.68)$$

This means that we need  $\ell \geq 2\beta + 4$  so that

$$\int_{\Omega} \lambda^3 \mathbb{1}_{\omega_{\alpha}} y \bar{\psi} dx = o(1).$$

Again using Cauchy-Schwarz inequality, (1.5.58), (1.5.46) and the fact that  $\bar{\omega}_{\alpha} = \text{supp } \alpha$ , we have

$$\left| \int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx \right| \leq \|\alpha\|_{\infty} \|\lambda u\|_{L^2(\omega_{\alpha})} \|\lambda \psi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{1+\frac{\ell-\beta}{2}}}. \quad (1.5.69)$$

This means that we need  $\ell \geq \beta - 2$  so that

$$\int_{\Omega} \lambda^2 \alpha u \bar{\psi} dx = o(1).$$

On the other hand, by Cauchy-Schwarz inequality, (1.5.58) and the fact that  $f_2, g_2$  converge to zero in  $L^2(\Omega)$  we have

$$\left| \int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^{\ell-2}} + \frac{if_2 \bar{\psi}}{\lambda^{\ell-3}} \right) dx \right| \leq \left( \frac{1}{|\lambda|^{\ell-1}} \|g_2\|_{L^2(\Omega)} + \frac{1}{|\lambda|^{\ell-2}} \|f_2\|_{L^2(\Omega)} \right) \|\lambda \psi\|_{L^2(\Omega)} \quad (1.5.70)$$

$$= \frac{o(1)}{\lambda^{\ell-\frac{\beta}{2}}} + \frac{o(1)}{\lambda^{\ell-1-\frac{\beta}{2}}}.$$

This means that we need  $\ell - 1 - \frac{\beta}{2} \geq 0$ , i.e,  $\ell \geq \frac{\beta}{2} + 1$  so that

$$\int_{\Omega} \left( \frac{g_2 \bar{\psi}}{\lambda^{\ell-2}} + \frac{if_2 \bar{\psi}}{\lambda^{\ell-3}} \right) dx = o(1).$$

As a conclusion, if  $\ell \geq 2\beta + 4$  we obtain

$$\int_{\Omega} |\lambda y|^2 dx = o(1).$$

□

Consequently, if  $\ell = 2\beta + 4$  we deduce that

$$\int_{\Omega} |\lambda u|^2 dx = o(1),$$

$$\int_{\Omega} |\lambda y|^2 dx = o(1),$$

and using Lemma 1.5.5, we obtain

$$\int_{\Omega} |\nabla u|^2 dx = o(1), \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1).$$

Thus, we conclude that  $\|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = o(1)$  which leads to the desired contradiction. □

**Remark 1.5.12.**

1) In Stahn, 2017, the damped wave equation (1.5.1) with Dirichlet boundary conditions was considered on the unit square  $(0,1)^2$ . In fact, it is proved that if  $\omega_1 = \{(x, y) \in (0,1)^2; x < \sigma\}$ , where  $\sigma$  is some fixed number from the interval  $(0,1)$ . Then, the energy of (1.5.1) decays polynomially as  $t^{-4/3}$ , which corresponds to  $\beta = 3$  in Theorem 1.5.7. The author noticed that the same decay remains true in higher dimensional spaces, namely in a hyper cube  $(0,1)^N$  and  $\omega_1 = \{(x_1, \dots, x_N) \in (0,1)^N; x_1 < \sigma\}$ .

2) In example 3 of Liu and Rao, 2005, the authors considered the wave equation with local viscous damping in the square  $(0, \pi)^2$ . Based on their results, if we impose that the damping region  $\omega_1$  contains a vertical strip of the square domain, i.e,

$$\omega_1 \supset \Omega_s = \{(x_1, x_2) / a < x_1 < b, 0 < x_2 < \pi\}.$$

Then, the energy of (1.5.1) decays polynomially as  $t^{-1}$ , i.e,  $\beta = 4$  in Theorem 1.5.7.

## 1.6 Polynomial stability with two dampings

In this section, we will improve the polynomial decay obtained in section 5, by using two damping terms. One of the main ingredients is to use the exponential or polynomial decay of the wave equation with frictional damping in  $\omega_3 = \omega_b \cap \omega_c$ , namely

$$\begin{cases} \varphi_{tt} - \Delta\varphi + \mathbb{1}_{\omega_3}\varphi_t = 0 & \text{in } \Omega \times (0, \infty), \\ \varphi = 0 & \text{on } \Gamma \times (0, \infty), \\ \varphi(t=0) = \varphi_0, \varphi_t(t=0) = \varphi_1. \end{cases} \quad (1.6.1)$$

**Theorem 1.6.1.** *(Polynomial energy decay rate)*

Let  $\omega_b, \omega_c$  and  $\omega_\alpha$  be non-empty open subsets of  $\Omega$ . Assume that conditions (1.1.7), (1.1.8) and (1.1.9) hold. Let  $\omega_3 = \omega_b \cap \omega_c$  satisfies  $\text{meas}(\overline{\omega_3} \cap \Gamma) > 0$  and let  $\text{supp } \alpha \subseteq \omega_3$ . Also, assume that the energy of the system (1.6.1) is exponentially stable ( $\beta = 0$ ) or polynomially stable as  $t^{-\frac{4}{\beta}}$ , with  $\beta > 0$ . Then, there exists a constant  $C > 0$  such that for every initial data  $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ , the energy of system (1.1.1) verifies the following estimate:

$$E(t) \leq \frac{C}{t^{\frac{2}{\beta+2}}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (1.6.2)$$

*Proof.* Following Borichev and Tomilov, 2009, (see also Liu and Rao, 2005, Batty and Duyckaerts, 2008), a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$  verifies (1.6.2) if (S1) holds and

$$\limsup_{|\lambda| \rightarrow \infty} \frac{1}{\lambda^\ell} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad \ell = \beta + 2 \quad (S4).$$

As (S1) already holds, our goal is to prove that condition (S4) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S4) does not hold, then there exist a sequence  $(\lambda_n) \subset \mathbb{R}$  and a sequence  $(U_n) \subset D(\mathcal{A})$  such that (1.5.3) and (1.5.40) are satisfied.

In what follows, we drop the index  $n$  for simplicity.

**Lemma 1.6.2.** *Under all the above assumptions, we have*

$$\begin{cases} \|\nabla v\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\ell/2}}, & \|v\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\ell/2}}, \\ \|\nabla z\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\ell/2}}, & \|z\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\ell/2}}. \end{cases} \quad (1.6.3)$$

*Proof.* Multiply in  $\mathcal{H}$  equation (1.5.40) by the uniformly bounded sequence  $U = (u, v, y, z)$ , we get

$$\int_{\Omega} (b(x)|\nabla v|^2 + c(x)|\nabla z|^2) dx = -\text{Re}((i\lambda I - \mathcal{A})U, U)_{\mathcal{H}} = \frac{o(1)}{\lambda^\ell}.$$

It follows that

$$\begin{cases} \|b^{1/2}\nabla v\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\ell/2}}, \\ \|c^{1/2}\nabla z\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\ell/2}}. \end{cases} \quad (1.6.4)$$

Using (1.1.7), (1.6.4) and Poincaré inequality (since  $\text{meas}(\overline{\omega_3} \cap \Gamma) > 0$ ) we get (1.6.3).  $\square$

By detailing equation (1.5.40), we get (1.5.42)-(1.5.44) and

$$\lambda^\ell (i\lambda z - \text{div}(\nabla y + c\nabla z) + \alpha u) = g_2 \longrightarrow 0 \quad \text{in } L^2(\Omega). \quad (1.6.5)$$

From equations (1.5.42) and (1.5.44) and using (1.5.3) we get (1.5.11). Using (1.5.3), (1.6.3), (1.5.42) and (1.5.44) we also have

$$\begin{cases} \|\nabla u\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}} & \text{and} & \|u\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}, \\ \|\nabla y\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}} & \text{and} & \|y\|_{L^2(\omega_3)} = \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}. \end{cases} \quad (1.6.6)$$

**Lemma 1.6.3.** *Under all the above assumptions, if  $\ell \geq \beta + 2$  then, we have*

$$\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 dx = o(1). \quad (1.6.7)$$

*Proof.* For all  $n \in \mathbb{N}$ , let  $\varphi_n, \psi_n \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of (1.5.14)-(1.5.16). If (1.6.1) decays exponentially ( $\beta = 0$ ) or polynomially as  $t^{-\frac{\beta}{2}}$ ,  $\beta > 0$ . Then, following Borichev and Tomilov, 2009 and using (1.5.11), we know that (1.5.57) and (1.5.58) hold.

As before we deduce that (1.5.59) holds, i.e.,

$$\begin{aligned} - \int_{\Omega} |\lambda u|^2 dx + \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_3} u \bar{\varphi} dx + \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx + \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx. \end{aligned} \quad (1.6.8)$$

But, using Cauchy-Schwarz inequality, (1.6.6) and (1.5.58) we have,

$$\left| \int_{\Omega} i\lambda^3 \mathbb{1}_{\omega_3} u \bar{\varphi} dx \right| \leq |\lambda|^2 \|u\|_{L^2(\omega_3)} \|\lambda \varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-\beta}{2}}}. \quad (1.6.9)$$

This means that we need  $\frac{\ell-\beta}{2} \geq 0$  i.e.  $\ell \geq \beta$  so that

$$\int_{\Omega} \lambda^3 \mathbb{1}_{\omega_3} u \bar{\varphi} dx = o(1).$$

Also, by (1.6.4) and (1.5.58) we have

$$\left| \int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx \right| \leq |\lambda|^2 \|b \nabla v\|_{L^2(\Omega)} \|\nabla \bar{\varphi}\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-\beta}{2}-1}}. \quad (1.6.10)$$

This means that we should have  $\frac{\ell-\beta}{2} - 1 \geq 0$  i.e.  $\ell \geq \beta + 2$  so that

$$\int_{\Omega} \lambda^2 b \nabla v \cdot \nabla \bar{\varphi} dx = o(1).$$



Again, using Cauchy-Schwarz inequality, (1.6.6) and (1.5.58) we obtain,

$$\left| \int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx \right| \leq \|\alpha\|_{\infty} |\lambda| \|y\|_{L^2(\omega_3)} \|\lambda \varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\frac{\ell-\beta}{2}+1}}. \quad (1.6.11)$$

This means that we should have  $\frac{\ell-\beta}{2} + 1 > 0$  i.e.  $\ell \geq \beta - 2$  so that

$$\int_{\Omega} \lambda^2 \alpha y \bar{\varphi} dx = o(1).$$

On the other hand, (1.5.64) holds, so we need  $\ell - 1 - \frac{\beta}{2} \geq 0$  i.e.  $\ell \geq \frac{\beta}{2} + 1$  so that

$$\int_{\Omega} \left( \frac{g_1 \bar{\varphi}}{\lambda^{\ell-2}} + \frac{if_1 \bar{\varphi}}{\lambda^{\ell-3}} \right) dx = o(1).$$

As a conclusion, if  $\ell \geq \beta + 2$  we get

$$\int_{\Omega} |\lambda u|^2 dx = o(1).$$

Similarly, multiplying (1.5.44) and (1.6.5) by  $i\lambda^3 \bar{\psi}$  and  $\lambda^2 \bar{\psi}$  respectively, applying Green's formula, and Cauchy-Schwarz inequality, we prove that

$$\int_{\Omega} |\lambda y|^2 dx = o(1),$$

if  $\ell \geq \beta + 2$ . □

**Lemma 1.6.4.** *Under all the above assumptions, if  $\ell \geq 0$  then,*

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\lambda u|^2 dx + o(1), \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = \int_{\Omega} |\lambda y|^2 dx + o(1).$$

*Proof.* By inserting equation (1.5.42) in equation (1.5.43) we get

$$-\lambda^2 u - \operatorname{div}(\nabla u + b \nabla v) + \alpha y = \frac{g_1}{\lambda^{\ell}} + \frac{if_1}{\lambda^{\ell-1}}. \quad (1.6.12)$$

Similarly, inserting equation (1.5.44) in equation (1.6.5) we obtain

$$-\lambda^2 y - \operatorname{div}(\nabla y + c \nabla z) + \alpha u = \frac{g_2}{\lambda^{\ell}} + \frac{if_2}{\lambda^{\ell-1}}. \quad (1.6.13)$$

Now, multiply (1.6.12) and (1.6.13) by  $\bar{u}$  and  $\bar{y}$  respectively then integrate over  $\Omega$  and apply Green's formula, we obtain

$$\begin{aligned} - \int_{\Omega} |\lambda u|^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} b \nabla v \cdot \nabla \bar{u} dx + \int_{\Omega} \alpha y \bar{u} dx \\ = \int_{\Omega} \left( \frac{g_1 \bar{u}}{\lambda^{\ell}} + \frac{if_1 \bar{u}}{\lambda^{\ell-1}} \right) dx \end{aligned} \quad (1.6.14)$$

and

$$\begin{aligned}
 - \int_{\Omega} |\lambda y|^2 dx + \int_{\Omega} |\nabla y|^2 dx + \int_{\Omega} c \nabla z \cdot \nabla \bar{y} dx + \int_{\Omega} \alpha u \bar{y} dx = \\
 \int_{\Omega} \left( \frac{g_2 \bar{y}}{\lambda^\ell} + \frac{if_1 \bar{y}}{\lambda^{\ell-1}} \right) dx.
 \end{aligned} \tag{1.6.15}$$

But, using Cauchy-Schwarz inequality, (1.6.4) and (1.5.3) we have

$$\left| \int_{\Omega} b \nabla v \cdot \nabla \bar{u} dx \right| \leq \|b \nabla v\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\ell/2}},$$

$$\left| \int_{\Omega} \alpha y \bar{u} dx \right| \leq \|\alpha\|_{\infty} \|u\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} = o(1),$$

and

$$\left| \int_{\Omega} c \nabla z \cdot \nabla \bar{y} dx \right| \leq \|c \nabla z\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{\ell/2}}.$$

Consequently, we get

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\lambda u|^2 dx + o(1),$$

and

$$\int_{\Omega} |\nabla y|^2 dx = \int_{\Omega} |\lambda y|^2 dx + o(1).$$

□

Consequently, if  $\ell = \beta + 2$  we deduce that

$$\int_{\Omega} |\lambda u|^2 dx = o(1),$$

$$\int_{\Omega} |\lambda y|^2 dx = o(1),$$

and using Lemma 1.6.4, we obtain

$$\int_{\Omega} |\nabla u|^2 dx = o(1), \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1).$$

Thus, we conclude that  $\|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = o(1)$  which leads to the desired contradiction.

□

**Remark 1.6.5.** In Theorem 1.6.1, if  $\text{supp } \alpha \not\subseteq \omega_3$  then, we can easily prove that the energy of system (1.1.1) decays polynomially as  $t^{\frac{-2}{\beta+2}}$ , with  $\beta < 2$ . Indeed, in order to prove that (compare with (1.6.11))

$$\int_{\Omega} \alpha \lambda^2 y \psi dx = o(1),$$

we will need  $\beta < 2$ .

## 1.7 Optimality of the polynomial decay rate

We study here the optimality of the polynomial decay rate obtained for the  $N$ -dimensional coupled wave system in Theorem 1.5.1.

**Theorem 1.7.1.** *Assume that  $c = 0$ ,  $b = 1$  and  $\alpha = \alpha_0 > 0$ . Then the energy decay rate (1.5.2) is optimal in the sense that for any  $\epsilon > 0$  we cannot expect the decay rate  $\frac{1}{t^{1/2+\epsilon}}$  for all initial data  $U_0 \in D(\mathcal{A})$ .*

*Proof.* Assume that there exist  $\epsilon > 0$  and  $C$  such that

$$\|e^{t\mathcal{A}}\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{4}+\epsilon}} \|U\|_{D(\mathcal{A})}, \quad \forall U \in D(\mathcal{A}). \quad (1.7.1)$$

Then, by Theorem 2.4 of Borichev and Tomilov, 2009, we deduce that there exists  $C' > 0$  such that

$$\|(is - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C'|s|^{\frac{4}{1+4\epsilon}}, \quad \forall s \in \mathbb{R}, \quad (1.7.2)$$

or equivalently,

$$\|U\|_{\mathcal{H}} \leq C'|s|^{\frac{4}{1+4\epsilon}} \|(is - \mathcal{A})U\|_{\mathcal{H}}, \quad \forall U \in D(\mathcal{A}). \quad (1.7.3)$$

Now, let  $\lambda_k^+$ , with  $k \geq k_0$ , be the sequence of eigenvalues of  $\mathcal{A}$  described in Proposition 1.4.2 and let  $U_n \in D(\mathcal{A})$  be the normalized eigenfunction. Moreover, set

$$\beta_k = \Im(\lambda_k^+), \quad \forall k \geq k_0. \quad (1.7.4)$$

Then, using (1.4.8) we have

$$\|(i\beta_k I - \mathcal{A})U_k\|_{\mathcal{H}} \sim \frac{1}{\mu_k^4}. \quad (1.7.5)$$

Consequently, by (1.7.3), we will have

$$1 = \|U_k\|_{\mathcal{H}} \lesssim \frac{C'|\beta_k|^{\frac{4}{1+4\epsilon}}}{\mu_k^4} \lesssim C'\mu_k^{\frac{4}{1+4\epsilon}-4} \lesssim C'\mu_k^{\frac{-16\epsilon}{1+4\epsilon}}.$$

This leads to a contradiction. □

## 1.8 Conclusion

The following table summarizes our main results.

Kelvin-Voigt Damping	Sufficient conditions	Obtained decay rate	Covered geometrical situations
One smooth non-smooth	(H2), (1.1.7) and (1.1.8), (1.5.1) exponentially decaying	pol. $t^{-1/2}$	1) $\Omega \in C^\infty$ , $\omega_1$ satisfies GCC. 2) $\Omega \in C^2$ , $\omega_1$ neighborhood of the boundary. 3) See Remark 1.5.6.
One smooth non-smooth	(1.1.7) and (1.1.8), $meas(\overline{\omega_\alpha} \cap \Gamma) > 0$ , $\overline{\omega_\alpha} = \text{supp } \alpha \subset \overline{\omega_b}$ , (1.5.1) exponentially stable or polynomially stable as $t^{-\frac{4}{\beta}}$ , with $\beta > 0$ .	pol. $t^{-\frac{1}{\beta+2}}$	1) $\Omega \in C^\infty$ , $\omega_\alpha$ satisfies GCC. 2) $\Omega \in C^2$ , $\omega_\alpha$ is a neighborhood of the boundary. 3) $\omega_\alpha$ contains a vertical strip of the square $(0, \pi)^2$ . 4) See Remark 1.5.6, 1.5.12.
Two smooth non-smooth	(1.1.7), (1.1.8) and (1.1.9), $\omega_3 = \omega_b \cap \omega_c$ satisfies $meas(\overline{\omega_3} \cap \Gamma) > 0$ , $\text{supp } \alpha \subseteq \omega_3$ , (1.6.1) is exponentially stable or polynomially stable as $t^{-\frac{4}{\beta}}$ , with $\beta > 0$ .	pol. $t^{-\frac{1}{\beta/2+1}}$	1) $\Omega \in C^\infty$ , $\omega_3$ satisfies GCC. 2) $\Omega \in C^2$ , $\omega_3$ is a neighborhood of the boundary. 3) $\omega_3$ contains a vertical strip of the square $(0, \pi)^2$ . 4) See Remark 1.5.6, 1.5.12.
Two smooth non-smooth	(1.1.7), (1.1.9), $\omega_b = \omega_c = \Omega$ .	Analytic	Lipschitz boundary



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## Part 3

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◇ Two coupled systems involving  
telegraph equation  
&  
thermoelastic/elastic transmission  
problem on network ◇



## Chapter 2

# Existence, uniqueness and stabilization of solutions of a generalized telegraph equation on star shaped Networks

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### Abstract

The existence, uniqueness, strong and exponential stability of generalized telegraph equation set on one dimensional star shaped networks are established. It is assumed that a dissipative boundary condition is applied at all the external vertices and an improved Kirchhoff law at the common internal vertex is considered. First, using a general criteria of Arendt-Batty (see Arendt and Batty, 1988), combined with a new uniqueness result, we prove that our system is strongly stable. Next, using a frequency domain approach, combined with a multiplier technique and the construction of a new multiplier satisfying some ordinary differential inequalities, we show that the energy of the system decays exponentially to zero.

### 2.1 Introduction

Over decades, telegraph equations have gained attention and interest among scientists due to their different applications in the transmission of electrical signals along transmission lines of all frequencies, in addition to many other physical, biological and engineering applications (see Bohme, 1987; Evans and Bulut, 2003; Jordan, Meyer, and Puri, 2000; Mohanty, 2009; Pascal, 1986). As a consequence, many mathematical models were set up, for instance in Imperiale and Joly, 2014, a general and realistic situation was considered and a mathematical model of electromagnetic wave propagation in heterogeneous lossy coaxial cables was derived. Recently, referring to Nicaise, 2015, S. Nicaise has considered the stabilization of the generalized telegraph equation set in a real interval (model on a cable from Imperiale and Joly, 2014):

$$\begin{cases} V_t + gV + aI_x + kW = 0, & \text{in } (0, L) \times (0, \infty), \\ I_t + rI + bV_x = 0, & \text{in } (0, L) \times (0, \infty), \\ W_t + cW = V, & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (2.1.1)$$



with the following boundary conditions

$$V(0, t) = V(L, t) = 0, \quad t \in \mathbb{R}_+^*,$$

and the following initial conditions

$$V(x, 0) = V_0(x), I(x, 0) = I_0(x), W(x, 0) = W_0(x), \quad x \in (0, L),$$

where,  $a, b, c, r, k$  and  $g$  are all non-negative functions in  $L^\infty(0, L)$  that verify some assumptions mentioned in Nicaise, 2015, see (2.1.2) below for the exact conditions. The generalized telegraph equation is a coupling between the usual telegraph equation where the electric unknowns are  $V$  and  $I$  representing the electric potential and the electric current respectively with a first order differential equation of parabolic type involving an auxiliary variable  $W$  representing the non-local effects. In Nicaise, 2015, the author was interested in studying the energy decay rate of system (2.1.1). In fact, under the additional condition that  $r + g > 0$  in  $(0, L)$ , an exponential energy decay rate was established. Otherwise, he proved a polynomial energy decay rate of type  $t^{-1}$  for smooth initial data. Moreover, the obtained polynomial decay rate is optimal in the case  $r = g = 0$ . On the other hand, first order hyperbolic systems set on graphs, also called networks, appear in recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks (see Bastin and Coron, 2016; Gugat, Dick, and Leugering, 2011; Leugering, Gugat, and Dick, 2010; Valein and Zuazua, 2009; Zhang and Xu, 2012; Kramar, Mugnolo, and Nicaise, 2020b; Kramar, Mugnolo, and Nicaise, 2020a; Kramar and Sikolya, 2004; Dorn et al., 2010; for heat, wave or beam equations on graphs, see Mehmeti, 1994; Below and De Coster, 2000; Gugat and Sigalotti, 2010; Mugnolo and Romanelli, 2007; Mugnolo, 2010; Zhang, Xu, and Mastorakis, 2009; Mugnolo, 2014) and the references there. In particular, referring to Nicaise, 2017, the usual telegraph equation was considered on a network with a general class of transmission conditions, including reasonable ones (like the Kirchhoff law) so that with the dissipative boundary conditions at the exterior vertices, an exponential stability was established. Note that some exact controllability results or stability properties have been obtained in Diagne, Bastin, and Coron, 2012; Gen Qi Xu and Liu, 2008; Lagnese, Leugering, and Schmidt, 1994; Maffucci and Miano, 2006; Perrollaz and Rosier, 2014; Zhou and Kriegsmann, 2009; Gugat and Sigalotti, 2010; Zhang, Xu, and Mastorakis, 2009 for instance. But to the best of our knowledge, the stability of the generalized telegraph equation on a network is an open problem. This motivates us to study this equation on star shaped graphs.

In the present work, we consider system (2.1.1) on a star shaped network  $\mathcal{S}$  made of  $N + 1$  cables of equal length  $L > 0$ , with a positive integer  $N$  (see Figure 2.1). For that purpose, we fix different real valued and non-negative functions  $\mathbf{a} = (a_\ell)_{\ell=0}^N$ ,  $\mathbf{b} = (b_\ell)_{\ell=0}^N$ ,  $\mathbf{c} = (c_\ell)_{\ell=0}^N$ ,  $\mathbf{k} = (k_\ell)_{\ell=0}^N$ ,  $\mathbf{r} = (r_\ell)_{\ell=0}^N$ , and  $\mathbf{g} = (g_\ell)_{\ell=0}^N$  in  $(L^\infty(0, L))^{N+1}$  satisfying the following assumption

$$a_\ell \geq \mu, b_\ell \geq \mu, c_\ell \geq \mu, k_\ell + g_\ell \geq \mu \quad a.e \text{ in } (0, L), \quad \forall \ell = 0, \dots, N, \quad (2.1.2)$$

where  $\mu > 0$  is a positive real number. These assumptions are in agreement with the physical setting from Imperiale and Joly, 2012; Imperiale and Joly, 2014. On each edge  $e_\ell$  with index  $\ell \in \{0, \dots, N\}$ , we consider the generalized telegraph

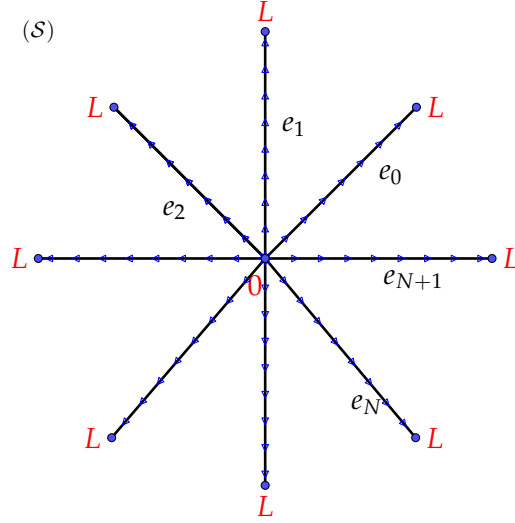


FIGURE 2.1: A star-shaped network

equation

$$\begin{cases} V_{\ell,t} + g_{\ell}V_{\ell} + a_{\ell}I_{\ell,x} + k_{\ell}W_{\ell} & = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ I_{\ell,t} + r_{\ell}I_{\ell} + b_{\ell}V_{\ell,x} & = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ W_{\ell,t} + c_{\ell}W_{\ell} & = V_{\ell}, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \end{cases} \quad (2.1.3)$$

where  $V_{\ell}$  represents the electric potential,  $I_{\ell}$  represents the electric current and  $W_{\ell}$  represents the non-local effects. Moreover, we denote by  $\mathbf{V} = (V_{\ell})_{\ell=0}^N$ ,  $\mathbf{I} = (I_{\ell})_{\ell=0}^N$ , and  $\mathbf{W} = (W_{\ell})_{\ell=0}^N$ .

Let us mention that the case when the edges have different lengths  $L_{\ell}$ ,  $\ell = 0, \dots, N$ , namely system

$$\begin{cases} V_{\ell,t} + g_{\ell}V_{\ell} + a_{\ell}I_{\ell,x} + k_{\ell}W_{\ell} & = 0, & (x, t) \in (0, L_{\ell}) \times \mathbb{R}_+^*, \\ I_{\ell,t} + r_{\ell}I_{\ell} + b_{\ell}V_{\ell,x} & = 0, & (x, t) \in (0, L_{\ell}) \times \mathbb{R}_+^*, \\ W_{\ell,t} + c_{\ell}W_{\ell} & = V_{\ell}, & (x, t) \in (0, L_{\ell}) \times \mathbb{R}_+^*, \end{cases}$$

can be treated similarly since it can be reduced to (2.1.3) as follows. Indeed by performing on edge  $\ell$ , the change of variable  $x_{\ell} = \frac{L_{\ell}}{L}x$ , with a fixed  $L > 0$ , and the change of unknowns

$$\hat{I}_{\ell}(x_{\ell}, t) = I_{\ell}\left(\frac{L_{\ell}}{L}x_{\ell}, t\right), \hat{V}_{\ell}(x_{\ell}, t) = V_{\ell}\left(\frac{L_{\ell}}{L}x_{\ell}, t\right), \hat{W}_{\ell}(x_{\ell}, t) = W_{\ell}\left(\frac{L_{\ell}}{L}x_{\ell}, t\right),$$

we transform this system into (2.1.3) where the coefficients  $a_{\ell}$  and  $b_{\ell}$  are changed into  $\frac{L}{L_{\ell}}a_{\ell}$  and  $\frac{L}{L_{\ell}}b_{\ell}$  respectively (and that continue to satisfy (2.1.2)).

System (2.1.3) is considered with the following dissipative boundary condition at the exterior vertices

$$V_{\ell}(L, t) - \alpha_{\ell}I_{\ell}(L, t) = 0, \quad \text{in } \mathbb{R}_+^*, \alpha_{\ell} > 0, \quad \forall \ell \in \{0, \dots, N\}, \quad (2.1.4)$$

and the transmission conditions from Beck, 2016, §8.2 or Joly and Semin, 2008 (called by these authors the improved Kirchhoff conditions) at the interior common vertex

$$\begin{cases} \sum_{k=1}^N Z_{\ell k} I_{k,t}(0, t) &= \frac{1}{\delta} (V_0(0, t) - V_\ell(0, t)), \quad \text{in } \mathbb{R}_+^*, \forall \ell \in \{1, \dots, N\}, \\ V_{0,t}(0, t) &= \frac{-1}{\delta Y} \sum_{\ell=0}^N I_\ell(0, t) \quad \text{in } \mathbb{R}_+^*, \end{cases} \quad (2.1.5)$$

where,  $Y$  and  $\delta$  are two positive constants and  $Z = (Z_{\ell k})_{N \times N}$  is a symmetric, positive definite matrix. These boundary conditions are derived in Beck, 2016, §8.2 or Joly and Semin, 2008 by using the method of matched asymptotics starting from thin coaxial cables of thickness  $\delta$ . The coefficient  $Y$  and the matrix  $Z$  are defined from the material properties of the medium and from 3D potentials defined in the reference domain.

**Remark 2.1.1.** If  $r_\ell = 0$ , then differentiating (2.1.4) with respect to  $t$  and using the second equation of (2.1.3), we exactly find the Neumann velocity feedback law

$$V_{\ell,t}(L, t) = -\alpha_\ell b_\ell V_{\ell,x}(L, t), \quad \text{in } \mathbb{R}_+^*, \forall \ell \in \{0, \dots, N\},$$

from Cox and Zuazua, 1995, (4) or Gugat, 2014, (4).

By setting  $v(t) = V_0(0, t)$  and  $\boldsymbol{\eta}(t) = (\eta_\ell(t))_{\ell=1}^N$  with  $\eta_\ell(t) = \sum_{k=1}^N Z_{\ell k} I_k(0, t)$ , System (2.1.3)-(2.1.5) is completed with the following initial conditions

$$\begin{cases} \mathbf{V}(x, 0) = \mathbf{V}^0(x), \mathbf{I}(x, 0) = \mathbf{I}^0(x), \mathbf{W}(x, 0) = \mathbf{W}^0(x), \quad x \in (0, L), \\ v(0) = v^0, \boldsymbol{\eta}(0) = \boldsymbol{\eta}^0. \end{cases} \quad (2.1.6)$$

As suggested before, our goal is then to find sufficient conditions on the functions  $a_\ell, b_\ell, c_\ell, k_\ell, r_\ell$ , and  $g_\ell$  that guarantee that the above system is strongly stable and then exponentially stable.

Let us briefly outline the content of this chapter. First, in Section 2.2, we show that System (2.1.3)-(2.1.6) admits a unique solution in an appropriate Hilbert space using semi-group theory. Next, using a general criteria of Arendt-Batty (see Arendt and Batty, 1988), we prove the strong stability of the system in the absence of the compactness of the resolvent. Then, in Section 2.3, based on a frequency domain approach combined with an ad-hoc multiplier technique, we establish an exponential energy decay rate. Finally, in Section 2.4, we discuss the stability of our system with some extensions, either by considering general dynamic boundary conditions or by considering general networks.

Let us finish this introduction with some notations used in the remainder of the paper: for all  $p \in [1, \infty]$ ,  $\mathbb{L}^p(\mathcal{S}) = L^p(0, L)^{N+1}$ ,  $1 \leq p \leq \infty$  and  $\mathbb{H}^1(\mathcal{S}) = H^1(0, L)^{N+1}$ . We denote by  $(\cdot, \cdot)_{\mathbb{C}^N}$  the inner product in  $\mathbb{C}^N$ .

## 2.2 Well posedness and strong stability

In this section, we will study the existence, uniqueness, regularity and asymptotic behavior of the solution of System (2.1.3)-(2.1.6).

### 2.2.1 Well posedness of the problem

First, we define the energy space

$$\mathcal{H} = (\mathbb{L}^2(\mathcal{S}))^3 \times \mathbb{C}^{N+1},$$

that is a Hilbert space equipped with the following inner product:

$$\begin{aligned} \langle (V, I, W, v, \eta)^\top, (V^*, I^*, W^*, v^*, \eta^*)^\top \rangle_{\mathcal{H}} &= \sum_{\ell=0}^N \int_0^L (\theta_\ell V_\ell \overline{V_\ell^*} + \beta_\ell I_\ell \overline{I_\ell^*} + \gamma_\ell W_\ell \overline{W_\ell^*}) dx \\ &\quad + \delta Y v \overline{v^*} + \delta (\eta, Z^{-1} \eta^*)_{\mathbb{C}^N} \end{aligned}$$

where  $\theta = (\theta_\ell)_{\ell=0}^N$ ,  $\beta = (\beta_\ell)_{\ell=0}^N$ ,  $\gamma = (\gamma_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$  will be fixed later but are such that

$$\theta_\ell \geq \mu_0, \quad \beta_\ell \geq \mu_0, \quad \gamma_\ell \geq \mu_0 \quad \text{a.e in } \Omega, \quad \ell = 0, \dots, N, \quad (2.2.1)$$

for some  $\mu_0 > 0$ .

Next, we define the unbounded linear operator  $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$  by

$$\begin{aligned} D(\mathcal{A}) &= \left\{ (V, I, W, v, \eta) \in \mathcal{H} : V \in \mathbb{H}^1(\mathcal{S}), I \in \mathbb{H}^1(\mathcal{S}), V_\ell(L) = \alpha_\ell I_\ell(L), \forall \ell \in \{0, \dots, N\}, \right. \\ &\quad \left. v = V_0(0), \text{ and } \eta = (\eta_\ell)_{\ell=1}^N \text{ with } \eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) \right\}, \end{aligned}$$

and

$$\mathcal{A} \begin{pmatrix} V \\ I \\ W \\ v \\ \eta \end{pmatrix} = - \left( \mathbf{g} \otimes V + \mathbf{a} \otimes I_x + \mathbf{k} \otimes W, \mathbf{r} \otimes I + \mathbf{b} \otimes V_x, \mathbf{c} \otimes W - V, \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0), \frac{1}{\delta} (V_\ell(0) - v)_{\ell=1}^N \right)^\top,$$

where for two vector functions  $\mathbf{P} = (p_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$ , and  $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{L}^2(\mathcal{S})$ , we set

$$\mathbf{P} \otimes \mathbf{Q} = (p_\ell q_\ell)_{\ell=0}^N,$$

while for  $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{H}^1(\mathcal{S})$ ,

$$\mathbf{Q}_x = (q_{\ell,x})_{\ell=0}^N.$$

Then, setting  $\mathbf{U}(t) = (V(\cdot, t), I(\cdot, t), W(\cdot, t), v(t), \eta(t))^\top$ , System (2.1.3)-(2.1.6) can be formally written as a linear evolution equation in the space  $\mathcal{H}$

$$\begin{cases} \mathbf{U}_t = \mathcal{A}\mathbf{U}, & t \in \mathbb{R}_+^*, \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \quad (2.2.2)$$

where  $\mathbf{U}_0 = (V_0, I_0, W_0, v_0, \eta_0)^\top$ .

For this system, we have the following existence result.

**Theorem 2.2.1.** *Assume that condition (2.1.2) holds. Then, there exist  $\theta, \beta, \gamma \in \mathbb{L}^\infty(\mathcal{S})$  satisfying (2.2.1) such that the operator  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on  $\mathcal{H}$ .*

*Proof.* Using Lumer-Phillips Theorem (see Pazy, 1983; Z. Liu, 1999), it is sufficient to prove that  $\mathcal{A}$  is a maximal dissipative operator so that  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ .

**Step 1.** Let us start by the dissipativity. Let  $\mathbf{U} = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  then, we have

$$\begin{aligned} \Re \langle \mathcal{A}\mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} &= \Re \left[ - \sum_{\ell=0}^N \int_0^L \theta_\ell (g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell) \overline{V}_\ell dx \right. \\ &\quad - \sum_{\ell=0}^N \int_0^L \beta_\ell (r_\ell I_\ell + b_\ell V_{\ell,x}) \overline{I}_\ell dx \\ &\quad \left. + \sum_{\ell=0}^N \int_0^L \gamma_\ell (V_\ell - c_\ell W_\ell) \overline{W}_\ell dx - V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) - \sum_{\ell=1}^N (V_\ell(0) - V_0(0)) \overline{I}_\ell(0) \right] \\ &= - \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + \beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \Re \sum_{\ell=0}^N \int_0^L \theta_\ell a_\ell I_{\ell,x} \overline{V}_\ell dx \\ &\quad - \Re \sum_{\ell=0}^N \int_0^L \beta_\ell b_\ell V_{\ell,x} \overline{I}_\ell dx + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell dx - \Re V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) \\ &\quad - \Re \sum_{\ell=1}^N (V_\ell(0) - V_0(0)) \overline{I}_\ell(0). \end{aligned}$$

Now, as in Nicaise, 2015, we chose  $\theta_\ell = a_\ell^{-1}$ ,  $\beta_\ell = b_\ell^{-1}$  for all  $\ell$ , and apply Green's formula on the third term of the right-hand side, we obtain

$$\begin{aligned} \Re \langle \mathcal{A}\mathbf{U}, \mathbf{U} \rangle &= - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \Re \sum_{\ell=0}^N \int_0^L I_{\ell,x} \overline{V}_\ell dx \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L \overline{I}_{\ell,x} V_\ell dx - \Re \sum_{\ell=0}^N V_\ell(L) \overline{I}_\ell(L) + \Re \sum_{\ell=0}^N V_\ell(0) \overline{I}_\ell(0) \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell dx - \Re V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) \\ &\quad + \Re V_0(0) \sum_{\ell=1}^N \overline{I}_\ell(0) - \Re \sum_{\ell=1}^N V_\ell(0) \overline{I}_\ell(0) \\ &= - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2. \end{aligned}$$

Finally, by estimating the second term of this right-hand side by Young's inequality we get,

$$\begin{aligned} \Re \langle \mathcal{A} \mathbf{U}, \mathbf{U} \rangle \leq & - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \\ & + \sum_{\ell=0}^N \int_0^L \frac{|\gamma_\ell - k_\ell \theta_\ell|}{2} (\epsilon_\ell |V_\ell|^2 + \frac{1}{\epsilon_\ell} |W_\ell|^2) dx, \end{aligned}$$

for all  $\epsilon(x) = (\epsilon_\ell(x))_{\ell=0}^N$  positive vector (i.e.  $\epsilon_\ell(x) > 0, \forall \ell \in \{0, \dots, N\}, x \in (0, L)$ ). Now, we need to find  $\epsilon(x) = (\epsilon_\ell(x))_{\ell=0}^N$  and  $\gamma = (\gamma_\ell)_{\ell=0}^N$  to satisfy for all  $\ell \in \{0, \dots, N\}$  :

$$\frac{|\gamma_\ell - k_\ell \theta_\ell| \epsilon_\ell}{2} - \theta_\ell g_\ell \leq 0$$

and

$$\frac{|\gamma_\ell - k_\ell \theta_\ell|}{2\epsilon_\ell} - \gamma_\ell c_\ell \leq 0.$$

But according to the proof of Theorem 2.1 of Nicaise, 2015 (see p. 3224-3225 in Nicaise, 2015) such a choice is always possible (we do not give the details and refer to that paper for the details). With such a choice, we find

$$\begin{aligned} \Re \langle \mathcal{A} \mathbf{U}, \mathbf{U} \rangle \leq & - \frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx \\ & - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \leq 0, \end{aligned} \tag{2.2.3}$$

and consequently,  $\mathcal{A}$  is dissipative.

**Step 2.** Now, let us go on with the maximality. Let  $\lambda > 0$  be fixed. Given  $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$ , we look for  $\mathbf{U} = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  solution of

$$(\lambda \mathcal{I} - \mathcal{A}) \mathbf{U} = F,$$

or equivalently,

$$\begin{cases} \lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = f_{\ell,1}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = f_{\ell,2}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda W_\ell + c_\ell W_\ell - V_\ell & = f_{\ell,3}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda v + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) & = f_4, \\ \lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - v) & = f_{\ell,5}, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \tag{2.2.4}$$

Assume for the moment that a solution  $\mathbf{U} \in D(\mathcal{A})$  of (2.2.4) exists. Then, the third equation is equivalent to

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{\lambda + c_\ell}, \quad \text{for all } \ell \in \{0, \dots, N\}. \tag{2.2.5}$$

Similarly, the last two equations of (2.2.4) allow to eliminate  $v$  and  $\eta_\ell$ , namely

$$v = \frac{f_4 - \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0)}{\lambda}, \quad (2.2.6)$$

and

$$\eta_\ell = \frac{f_{\ell,5} - \frac{1}{\delta}(V_\ell(0) - v)}{\lambda}, \text{ for all } \ell \in \{1, \dots, N\}. \quad (2.2.7)$$

Replacing (2.2.5) in the first equation of (2.2.4), we get

$$\begin{cases} V_\ell + a_{\ell,1}(\lambda)I_{\ell,x} = f_{\lambda,\ell}, \\ V_{\ell,x} + a_{\ell,2}(\lambda)I_\ell = g_{\lambda,\ell}, \end{cases} \quad (2.2.8)$$

where

$$\begin{cases} a_{\ell,1}(\lambda) = \frac{a_\ell}{\lambda + g_\ell + \frac{k_\ell}{\lambda + c_\ell}}, & a_{\ell,2}(\lambda) = \frac{\lambda + r_\ell}{b_\ell}, \\ f_{\lambda,\ell} = \frac{f_{\ell,1} - \frac{k_\ell f_{\ell,3}}{\lambda + c_\ell}}{\lambda + g_\ell + \frac{k_\ell}{\lambda + c_\ell}}, & g_{\lambda,\ell} = \frac{f_{\ell,2}}{b_\ell}, \end{cases} \quad (2.2.9)$$

for all  $\ell \in \{0, \dots, N\}$ .

Let  $\boldsymbol{\psi} = (\psi_\ell)_{\ell=0}^N \in \mathbb{H}^1(\mathcal{S})$ . Multiplying the second equation of (2.2.8) by  $\bar{\psi}_\ell$ , integrating over  $(0, L)$  and summing on  $\ell \in \{0, \dots, N\}$ , yields

$$\sum_{\ell=0}^N \int_0^L V_{\ell,x} \bar{\psi}_\ell \, dx + \sum_{\ell=0}^N \int_0^L a_{\ell,2}(\lambda) I_\ell \bar{\psi}_\ell \, dx = \sum_{\ell=0}^N \int_0^L g_{\lambda,\ell} \bar{\psi}_\ell \, dx. \quad (2.2.10)$$

Similarly, multiplying the first equation of (2.2.8) by  $\bar{\psi}_{\ell,x}$ , integrating over  $(0, L)$  and summing on  $\ell \in \{0, \dots, N\}$ , gives

$$\sum_{\ell=0}^N \int_0^L V_\ell \bar{\psi}_{\ell,x} \, dx + \sum_{\ell=0}^N \int_0^L a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} \, dx = \sum_{\ell=0}^N \int_0^L f_{\lambda,\ell} \bar{\psi}_{\ell,x} \, dx. \quad (2.2.11)$$

By integrating by parts the first term of the left hand side of (2.2.11), we obtain

$$\begin{aligned} & - \sum_{\ell=0}^N \int_0^L V_{\ell,x} \bar{\psi}_\ell \, dx + \sum_{\ell=0}^N V_\ell(L) \bar{\psi}_\ell(L) - \sum_{\ell=0}^N V_\ell(0) \bar{\psi}_\ell(0) \\ & + \sum_{\ell=0}^N \int_0^L a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} \, dx = \sum_{\ell=0}^N \int_0^L f_{\lambda,\ell} \bar{\psi}_{\ell,x} \, dx. \end{aligned} \quad (2.2.12)$$

Then, adding (2.2.10) and (2.2.12) leads to

$$\begin{aligned} \sum_{\ell=0}^N \int_0^L \left( a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(\lambda) I_{\ell} \bar{\psi}_{\ell} \right) dx + \sum_{\ell=0}^N V_{\ell}(L) \bar{\psi}_{\ell}(L) \\ - \sum_{\ell=0}^N V_{\ell}(0) \bar{\psi}_{\ell}(0) = \sum_{\ell=0}^N \int_0^L \left( f_{\lambda,\ell} \bar{\psi}_{\ell,x} + g_{\lambda,\ell} \bar{\psi}_{\ell} \right) dx. \end{aligned} \quad (2.2.13)$$

By taking into account the boundary conditions in  $D(\mathcal{A})$ , in addition to (2.2.6) and (2.2.7), we find

$$\begin{aligned} \sum_{\ell=0}^N V_{\ell}(0) \bar{\psi}_{\ell}(0) &= V_0(0) \bar{\psi}_0(0) + \sum_{\ell=1}^N V_{\ell}(0) \bar{\psi}_{\ell}(0) \\ &= V_0(0) \bar{\psi}_0(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0) \\ &\quad + V_0(0) \sum_{\ell=1}^N \bar{\psi}_{\ell}(0) - \delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0). \end{aligned}$$

Then,

$$\begin{aligned} \sum_{\ell=0}^N V_{\ell}(0) \bar{\psi}_{\ell}(0) &= V_0(0) \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0) \\ &\quad - \delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0) \\ &= \frac{f_4}{\lambda} \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) - \frac{1}{\delta Y \lambda} \left( \sum_{\ell=0}^N I_{\ell}(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) \\ &\quad + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0) - \delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0) \end{aligned}$$

and

$$\sum_{\ell=0}^N V_{\ell}(L) \bar{\psi}_{\ell}(L) = \sum_{\ell=0}^N \alpha_{\ell} I_{\ell}(L) \bar{\psi}_{\ell}(L).$$

By replacing these two identities in (2.2.13), we arrive at

$$\begin{aligned} \sum_{\ell=0}^N \int_0^L \left( a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(\lambda) I_{\ell} \bar{\psi}_{\ell} \right) dx + \sum_{\ell=0}^N \alpha_{\ell} I_{\ell}(L) \bar{\psi}_{\ell}(L) \\ + \frac{1}{\delta Y \lambda} \left( \sum_{\ell=0}^N I_{\ell}(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) + \delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0) \\ = \sum_{\ell=0}^N \int_0^L \left( f_{\lambda,\ell} \bar{\psi}_{\ell,x} + g_{\lambda,\ell} \bar{\psi}_{\ell} \right) dx + \frac{f_4}{\lambda} \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0), \end{aligned} \quad (2.2.14)$$

for all  $\psi \in \mathbb{H}^1(\mathcal{S})$ . Now, we notice that this problem has a unique solution  $I \in \mathbb{H}^1(\mathcal{S})$ , by the Lax-Milgram Lemma. First, this left hand side is a continuous and coercive sesquilinear form on  $\mathbb{H}^1(\mathcal{S})$ , since by our assumptions (in particular on



Z), there exists a positive constant  $C(\lambda)$  (that depends on  $\lambda$ ) such that

$$\sum_{\ell=0}^N \int_0^L (a_{\ell,1}(\lambda)|I_{\ell,x}|^2 + a_{\ell,2}(\lambda)|I_{\ell}|^2) dx + \sum_{\ell=0}^N \alpha_{\ell}|I_{\ell}(L)|^2 + \frac{1}{\delta Y \lambda} \left| \sum_{\ell=0}^N I_{\ell}(0) \right|^2 + \delta \lambda (Z \mathbf{I}_*(0), \mathbf{I}_*(0))_{\mathbb{C}^N} \geq C(\lambda) \|\mathbf{I}\|_{\mathbb{H}^1(\mathcal{S})}^2,$$

with  $\mathbf{I}_*(0) = (I_{\ell}(0))_{\ell=1}^N$ . Secondly, the right-hand side of (2.2.14) is a continuous form on  $\mathbb{H}^1(\mathcal{S})$ .

Now, according to the first equation of (2.2.8) we set

$$V_{\ell} = f_{\lambda,\ell} - a_{\ell,1}(\lambda)I_{\ell,x}, \quad (2.2.15)$$

that clearly belongs to  $L^2(0, L)$ , for all  $\ell$ . Then, taking in (2.2.14) a test function  $\psi$  such that  $\psi_{\ell} \in \mathcal{D}(0, L)$  for some  $\ell = 0, \dots, N$  and  $\psi_j = 0$ , for all  $j \neq \ell$ , we find

$$\begin{aligned} & \int_0^L a_{\ell,1}(\lambda)I_{\ell,x}\bar{\psi}_{\ell,x} dx + \int_0^L a_{\ell,2}(\lambda)I_{\ell}\bar{\psi}_{\ell} dx \\ &= \int_0^L f_{\lambda,\ell}\bar{\psi}_{\ell,x} dx + \int_0^L g_{\lambda,\ell}\bar{\psi}_{\ell} dx. \end{aligned} \quad (2.2.16)$$

Eliminating  $I_{\ell,x}$  from (2.2.15) and replacing it in (2.2.16), we get

$$- \int_0^L V_{\ell}\bar{\psi}_{\ell,x} dx + \int_0^L a_{\ell,2}(\lambda)I_{\ell}\bar{\psi}_{\ell} dx = \int_0^L g_{\lambda,\ell}\bar{\psi}_{\ell} dx. \quad (2.2.17)$$

Applying Green's formula on the first integral of the left-hand side of (2.2.17), we obtain

$$V_{\ell,x} + a_{\ell,2}(\lambda)I_{\ell} = g_{\lambda,\ell}, \quad \text{in } \mathcal{D}'(0, L). \quad (2.2.18)$$

As  $-a_{\ell,2}(\lambda)I_{\ell} + g_{\lambda,\ell} \in L^2(0, L)$ , we deduce that  $V_{\ell} \in H^1(0, L)$  and that (2.2.8) holds. Then, after defining  $\mathbf{W}$  by (2.2.5),  $\nu$  by (2.2.6) and  $\boldsymbol{\eta}$  by (2.2.7), it remains to check the boundary conditions appearing in  $D(\mathcal{A})$ . For this purpose, we introduce two Hilbert spaces

$$H_{\mathbb{R}}^1 = \{u \in H^1(0, L); u(0) = 0\},$$

and

$$H_{\mathbb{L}}^1 = \{u \in H^1(0, L); u(L) = 0\}.$$

Now, we first take different test functions  $\psi$  in (2.2.14). By Green's formula and taking into account (2.2.8), we have

$$\begin{aligned} & \sum_{\ell=0}^N V_{\ell}(L)\bar{\psi}_{\ell}(L) - \sum_{\ell=0}^N V_{\ell}(0)\bar{\psi}_{\ell}(0) - \sum_{\ell=0}^N \alpha_{\ell}I_{\ell}(L)\bar{\psi}_{\ell}(L) \\ & - \frac{1}{\delta Y \lambda} \left( \sum_{\ell=0}^N I_{\ell}(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) - \delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k}I_k(0) \right) \bar{\psi}_{\ell}(0) \\ & + \delta \sum_{\ell=1}^N f_{\ell,5}\bar{\psi}_{\ell}(0) + \frac{1}{\lambda} \sum_{\ell=0}^N f_4\bar{\psi}_{\ell}(0) = 0, \quad \forall \boldsymbol{\psi} \in \mathbb{H}^1(\mathcal{S}). \end{aligned} \quad (2.2.19)$$

First, we fix  $\ell$  and take  $\boldsymbol{\psi} \in \prod_{\ell=0}^N H_R^1$  such that  $\psi_\ell(L) = z$  arbitrary in  $\mathbb{C}$  and  $\psi_j(L) = 0, \forall j \neq \ell$ , we get

$$(V_\ell(L) - \alpha_\ell I_\ell(L)) \bar{z} = 0, \forall z \in \mathbb{C}.$$

Consequently,  $V_\ell(L) = \alpha_\ell I_\ell(L)$ . Second, we fix  $\ell$  and take  $\boldsymbol{\psi} \in \prod_{\ell=0}^N H_L^1$  such that  $\psi_\ell(0) = z$  arbitrary in  $\mathbb{C}$  and  $\psi_j(0) = 0, \forall j \neq \ell$ , we obtain

$$\left( -V_\ell(0) - \frac{1}{\delta Y \lambda} \left( \sum_{\ell=0}^N I_\ell(0) \right) - \delta \lambda \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) + \delta f_{\ell,5} + \frac{1}{\lambda} f_4 \right) \bar{z} = 0, \forall z \in \mathbb{C}.$$

Using (2.2.6) and (2.2.7) we deduce that  $\eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0)$ . Third, we fix  $\ell$  and take

$\boldsymbol{\psi} \in \prod_{\ell=0}^N H_L^1$  such that  $\psi_\ell = 0, \forall \ell \neq 0$ , and  $\psi_0(0) = z$ , arbitrary in  $\mathbb{C}$  we obtain

$$\frac{f_4}{\lambda} - \frac{1}{\delta Y \lambda} \sum_{\ell=0}^N I_\ell(0) = V_0(0).$$

Using (2.2.6), we deduce that  $v = V_0(0)$ . The proof of Theorem 2.2.1 is thus complete.  $\square$

As  $\mathcal{A}$  generates a  $C_0$ -semigroup on  $\mathcal{H}$ , problem (2.2.2) admits a unique solution given by

$$\mathbf{U}(t) = e^{t\mathcal{A}} \mathbf{U}_0, \quad t \geq 0.$$

Hence, the semi-group theory allows us to deduce the following existence and uniqueness results.

**Theorem 2.2.2.** *Assume that condition (2.1.2) holds. Then, for any initial data  $\mathbf{U}_0 \in \mathcal{H}$ , problem (2.2.2) admits a unique weak solution  $\mathbf{U} \in C^0([0, \infty); \mathcal{H})$ . Moreover if  $\mathbf{U}_0 \in D(\mathcal{A})$ , problem (2.2.2) admits a unique strong solution  $\mathbf{U} \in C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(\mathcal{A}))$ .*

As usual the energy associated with (2.2.2) is defined by

$$E(t) = \frac{1}{2} \|(\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})\|_{\mathcal{H}}^2$$

and for  $\mathbf{U} = (\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta}) \in C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(\mathcal{A}))$ , we have

$$E'(t) \leq -\frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_{\ell r_\ell} |I_\ell|^2 + \gamma_{\ell c_\ell} |W_\ell|^2) dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \leq 0.$$

For such a solution, we then have

$$E(t_2) \leq E(t_1), \forall 0 \leq t_1 \leq t_2,$$

in other words, our system is dissipative in the sense that its energy is non-increasing with respect to  $t$ . By the density of  $D(\mathcal{A})$  into  $\mathcal{H}$ , this last property remains valid for weak solution  $\mathbf{U} \in C^0([0, \infty); \mathcal{H})$ .

Now, we are able to study the strong stability of our system.

### 2.2.2 Strong stability

To show the strong stability of the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  we will rely on the following result due to Arendt and Batty, 1988.

**Theorem 2.2.3.** (Arendt and Batty, 1988). Let  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  generates a  $C_0$ -semigroup of contractions on the Hilbert space  $\mathcal{H}$ . If

- 1)  $\mathcal{A}$  has no pure imaginary eigenvalues,
- 2)  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  is countable,

where  $\sigma(\mathcal{A})$  denotes the spectrum of  $\mathcal{A}$ . Then, the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable, namely

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}}\mathbf{U}_0\|_{\mathcal{H}} = 0, \forall \mathbf{U}_0 \in \mathcal{H}.$$

Let us now state the main result of this subsection.

**Theorem 2.2.4.** Assume that assumption (2.1.2) holds. Then, the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable on the energy space  $\mathcal{H}$ .

Its proof is based on the use of Theorem 2.2.3. Since in our situation the resolvent of  $\mathcal{A}$  is not compact, we have to analyze its full spectrum on the imaginary axis. More precisely with the help of the following Lemmas, we will deduce that  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  is empty, hence the result.

**Lemma 2.2.5.** Under the same condition of Theorem 2.2.4, we have  $i\lambda - \mathcal{A}$  is injective for all  $\lambda \in \mathbb{R}$ , i.e.

$$\ker(i\lambda\mathcal{I} - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

*Proof.* Let  $\lambda \in \mathbb{R}$  and  $\mathbf{U} = (\mathbf{V}, \mathbf{I}, \mathbf{W}, \nu, \boldsymbol{\eta})^\top \in D(\mathcal{A})$  be such that

$$(i\lambda\mathcal{I} - \mathcal{A})\mathbf{U} = 0,$$

equivalently,

$$\begin{cases} i\lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda W_\ell + c_\ell W_\ell - V_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda \nu + \frac{1}{\delta \bar{Y}} \sum_{\ell=0}^N I_\ell(0) & = 0, \\ i\lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - \nu) & = 0, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \quad (2.2.20)$$

Hence, as  $\Re \langle \mathcal{A}\mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} = 0$ , using inequality (2.2.3) and the fact that  $c_\ell > 0$  and  $\alpha_\ell > 0$ , we get

$$g_\ell V_\ell = r_\ell I_\ell = W_\ell = I_\ell(L) = 0, \quad \forall \ell \in \{0, \dots, N\}. \quad (2.2.21)$$

Using the third identity of (2.2.20) we deduce that  $V_\ell = 0$ , for all  $\ell \in \{0, \dots, N\}$ . On the other hand, the first equation yields that  $I_{\ell,x} = 0$ , so  $I_\ell$  is constant on  $(0, L)$  for all  $\ell \in \{0, \dots, N\}$ . As  $I_\ell(L) = 0$ , we deduce that  $I_\ell = 0$ , for all  $\ell \in \{0, \dots, N\}$ . This implies that  $\nu = V_0(0) = 0$  and  $\eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) = 0$ , for all  $\ell \in \{1, \dots, N\}$ . Hence  $\mathbf{U} = 0$  and the desired goal holds.  $\square$

**Lemma 2.2.6.** *Under the same condition of Theorem 2.2.4, we have  $i\lambda - \mathcal{A}$  is surjective for all  $\lambda \in \mathbb{R}^*$ , i.e.*

$$\mathcal{R}(i\lambda - \mathcal{A}) = \mathcal{H}, \quad \forall \lambda \in \mathbb{R}^*.$$

*Proof.* For any  $\lambda \in \mathbb{R}^*$  and  $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$ , we look for a unique solution  $\mathbf{U} = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  of

$$(i\lambda\mathcal{I} - \mathcal{A})\mathbf{U} = F,$$

or equivalently,

$$\begin{cases} i\lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = f_{\ell,1}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = f_{\ell,2}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda W_\ell + c_\ell W_\ell - V_\ell & = f_{\ell,3}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda v + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) & = f_4, \\ i\lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - v) & = f_{\ell,5}, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \quad (2.2.22)$$

Suppose such a  $\mathbf{U} = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  exists, then

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{(i\lambda + c_\ell)}, \quad \forall \ell \in \{0, \dots, N\}, \quad (2.2.23)$$

$$v = \frac{f_4 - \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0)}{i\lambda}, \quad (2.2.24)$$

$$\eta_\ell = \frac{f_{\ell,5} - \frac{1}{\delta} (V_\ell(0) - v)}{i\lambda}, \quad \forall \ell \in \{1, \dots, N\}. \quad (2.2.25)$$

Replace  $W_\ell$  in the first and second identities of (2.2.22) to get

$$\begin{cases} V_\ell + \frac{a_\ell}{(i\lambda + g_\ell + \frac{k_\ell}{i\lambda + c_\ell})} I_{\ell,x} & = \frac{(f_{\ell,1} - \frac{k_\ell f_{\ell,3}}{i\lambda + c_\ell})}{(i\lambda + g_\ell + \frac{k_\ell}{i\lambda + c_\ell})}, \\ \frac{i\lambda + r_\ell}{b_\ell} I_\ell + V_{\ell,x} & = \frac{f_{\ell,2}}{b_\ell}, \quad \ell \in \{0, \dots, N\}. \end{cases}$$

Then, to find a solution of (2.2.22) it is enough to find a solution of

$$\begin{cases} V_\ell + a_{\ell,1}(i\lambda) I_{\ell,x} = f_{i\lambda,\ell}, \\ V_{\ell,x} + a_{\ell,2}(i\lambda) I_\ell = g_{i\lambda,\ell}, \end{cases} \quad (2.2.26)$$

where  $a_{\ell,1}(\lambda)$ ,  $a_{\ell,2}(\lambda)$ ,  $f_{\lambda,\ell}$  and  $g_{\lambda,\ell}$  are defined in (2.2.9).

Now following the arguments of the proof of Theorem 2.2.1, we deduce that  $I \in$

$\mathbb{H}^1(\mathcal{S})$  is a solution of

$$\begin{aligned} & \sum_{\ell=0}^N \int_0^L \left( a_{\ell,1}(i\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(i\lambda) I_{\ell} \bar{\psi}_{\ell} \right) dx + \sum_{\ell=0}^N \alpha_{\ell} I_{\ell}(L) \bar{\psi}_{\ell}(L) \\ & + \frac{1}{i\delta Y \lambda} \left( \sum_{\ell=0}^N I_{\ell}(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) + i\delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0) \quad (2.2.27) \\ & = \sum_{\ell=0}^N \int_0^L \left( f_{i\lambda,\ell} \bar{\psi}_{\ell,x} + g_{i\lambda,\ell} \bar{\psi}_{\ell} \right) dx + \frac{f_4}{i\lambda} \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0), \end{aligned}$$

for all  $\psi \in \mathbb{H}^1(\mathcal{S})$ . Here we note that Lax-Milgram Lemma cannot be applied because coercivity is not available. Therefore, we use a perturbation argument. For that purpose, let us introduce the sesquilinear form

$$\begin{aligned} b_{\lambda}(\mathbf{I}, \boldsymbol{\psi}) &= \sum_{\ell=0}^N \int_0^L a_{\ell,1}(i\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} dx + \sum_{\ell=0}^N \int_0^L \frac{1}{b_{\ell}} I_{\ell} \bar{\psi}_{\ell} dx + \sum_{\ell=0}^N \alpha_{\ell} I_{\ell}(L) \bar{\psi}_{\ell}(L) \\ & + \frac{1}{i\delta Y \lambda} \left( \sum_{\ell=0}^N I_{\ell}(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) + i\delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_{\ell}(0), \forall \mathbf{I}, \boldsymbol{\psi} \in \mathbb{H}^1(\mathcal{S}). \end{aligned}$$

This sesquilinear form  $b_{\lambda}$  is continuous and coercive on  $\mathbb{H}^1(\mathcal{S})$ , since for all  $\ell$

$$\Re(a_{\ell,1}(i\lambda)) = \frac{a_{\ell} (\lambda^2 g_{\ell} + c_{\ell} (c_{\ell} g_{\ell} + k_{\ell}))}{(c_{\ell} g_{\ell} + k_{\ell} - \lambda^2)^2 + \lambda^2 (c_{\ell} + g_{\ell})^2} \geq C(\lambda),$$

where  $C(\lambda)$  is a positive constant that depends only on  $\lambda$ . Then, by Lax-Milgram Lemma, the operator

$$B_{\lambda} : \mathbb{H}^1(\mathcal{S}) \rightarrow (\mathbb{H}^1(\mathcal{S}))^* : \mathbf{I} \rightarrow B_{\lambda} \mathbf{I},$$

with  $B_{\lambda} \mathbf{I}(\boldsymbol{\psi}) = b_{\lambda}(\mathbf{I}, \boldsymbol{\psi})$ , is an isomorphism.

Now, let us set

$$R_{\lambda} : \mathbb{H}^1(\mathcal{S}) \rightarrow (\mathbb{H}^1(\mathcal{S}))^* : \mathbf{I} \rightarrow R_{\lambda} \mathbf{I},$$

with

$$R_{\lambda} \mathbf{I}(\boldsymbol{\psi}) = \sum_{\ell=0}^N \int_0^L \frac{(i\lambda + r_{\ell} - 1)}{b_{\ell}} I_{\ell} \bar{\psi}_{\ell} dx.$$

As  $R_{\lambda}$  is a compact operator, we deduce that  $B_{\lambda} + R_{\lambda}$  is a Fredholm operator of index zero from  $\mathbb{H}^1(\mathcal{S})$  to  $(\mathbb{H}^1(\mathcal{S}))^*$ .

Now by setting

$$L_{\lambda}(\boldsymbol{\psi}) = \sum_{\ell=0}^N \int_0^L (g_{i\lambda,\ell} \bar{\psi}_{\ell} + f_{i\lambda,\ell} \bar{\psi}_{\ell,x}) dx + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_{\ell}(0) + \frac{1}{i\lambda} \sum_{\ell=0}^N f_4 \bar{\psi}_{\ell}(0),$$

we notice that (2.2.27) is equivalent to

$$(B_{\lambda} + R_{\lambda}) \mathbf{I} = L_{\lambda} \quad \text{in } (\mathbb{H}^1(\mathcal{S}))^*. \quad (2.2.28)$$

Hence problem (2.2.27) admits a unique solution  $\mathbf{I}$  if and only if  $B_{\lambda} + R_{\lambda}$  is invertible.  $B_{\lambda} + R_{\lambda}$  being a Fredholm operator it is enough to prove that  $B_{\lambda} + R_{\lambda}$  is injective,

i.e,

$$\ker(B_\lambda + R_\lambda) = \{0\}.$$

Let us now fix  $\mathbf{I} \in \ker(B_\lambda + R_\lambda)$ , then it satisfies

$$\begin{aligned} & \sum_{\ell=0}^N \int_0^L \left( a_{\ell,1}(i\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(i\lambda) I_\ell \bar{\psi}_\ell \right) dx + \sum_{\ell=0}^N \alpha_\ell I_\ell(L) \bar{\psi}_\ell(L) \\ & + \frac{1}{i\delta Y \lambda} \left( \sum_{\ell=0}^N I_\ell(0) \right) \left( \sum_{\ell=0}^N \bar{\psi}_\ell(0) \right) \\ & + i\delta \lambda \sum_{\ell=1}^N \left( \sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) = 0, \quad \forall \boldsymbol{\psi} \in \mathbb{H}^1(\mathcal{S}). \end{aligned}$$

Thus, if we set

$$V_\ell = -\frac{I_{\ell,x}}{a_{\ell,1}(i\lambda)}, W_\ell = \frac{V_\ell}{i\lambda + c_\ell}$$

and

$$v = \frac{-\sum_{\ell=0}^N I_\ell(0)}{i\lambda \delta Y} \quad \text{and} \quad \eta_\ell = \frac{-V_\ell(0) + v}{i\lambda \delta},$$

we conclude that  $(\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta}) \in D(\mathcal{A})$  is a solution of

$$(i\lambda - \mathcal{A})(\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})^\top = \mathbf{0}.$$

Using Lemma 2.2.5, we deduce that  $V_\ell = I_\ell = W_\ell = v = \eta_\ell = 0$  for all  $\ell \in \{0, \dots, N\}$ . This shows that  $B_\lambda + R_\lambda$  is invertible and therefore a unique solution  $(I_\ell)_\ell \in \mathbb{H}^1(\mathcal{S})$  of (2.2.28) exists. At this stage, by setting  $V_\ell = f_{i\lambda, \ell} - a_{\ell,1}(i\lambda) I_{\ell,x}$ , we conclude as in the proof of Theorem 2.2.1 that  $(\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta}) \in D(\mathcal{A})$  is a solution of (2.2.22) and the proof is thus complete.  $\square$

It remains to show the surjectivity of  $\mathcal{A}$ , but first let us introduce the following Hilbert space

$$\mathcal{V}_* = \left\{ (\mathbf{V}, \mathbf{I}) \in \mathbb{H}^1(\mathcal{S}) \times \mathbb{H}^1(\mathcal{S}) \text{ such that } V_\ell(L) = \alpha_\ell I_\ell(L), \forall \ell \in \{0, \dots, N\}, \right. \\ \left. \sum_{\ell=0}^N I_\ell(0) = 0, V_\ell(0) - V_0(0) = 0, \forall \ell \in \{1, \dots, N\} \right\}.$$

**Lemma 2.2.7.** *Let  $(\boldsymbol{\theta}, \boldsymbol{\Lambda}) \in \prod_{\ell=0}^N \mathcal{D}(0, L) \times \prod_{\ell=0}^N \mathcal{D}(0, L)$ , then there exists  $(\boldsymbol{\psi}, \boldsymbol{\chi}) \in \mathcal{V}_*$  such that*

$$\begin{cases} \psi_{\ell,x} = \theta_\ell, \\ \chi_{\ell,x} = \Lambda_\ell, \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \quad (2.2.29)$$

*Proof.* As (2.2.29) means that  $\psi_\ell$  (resp.  $\chi_\ell$ ) is a primitive of  $\theta_\ell$  (resp.  $\Lambda_\ell$ ), we have

$$\begin{aligned} \psi_\ell(x) &= \int_0^x \theta_\ell(y) dy + d_\ell, \\ \chi_\ell(x) &= \int_0^x \Lambda_\ell(y) dy + p_\ell, \end{aligned} \quad (2.2.30)$$

where  $d_\ell, p_\ell$  are two constants to be determined later on. Notice that  $(\psi, \chi) \in \mathcal{V}_*$  if and only if

$$\begin{cases} \sum_{\ell=0}^N \chi_\ell(0) &= 0, \\ \psi_\ell(0) &= \psi_0(0), \quad \forall \ell \in \{1, \dots, N\}, \\ \psi_\ell(L) &= \alpha_\ell \chi_\ell(L), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \quad (2.2.31)$$

Then, knowing that

$$\begin{aligned} \psi_\ell(0) &= d_\ell, \quad \psi_\ell(L) = \int_0^L \theta_\ell(y) dy + d_\ell, \\ \chi_\ell(0) &= p_\ell, \quad \chi_\ell(L) = \int_0^L \Lambda_\ell(y) dy + p_\ell, \end{aligned} \quad (2.2.32)$$

we see that (2.2.31) is equivalent to the following system with  $2N + 2$  equations

$$\begin{cases} \sum_{\ell=0}^N p_\ell &= 0, \\ d_\ell &= d_0, \quad \forall \ell \in \{1, \dots, N\}, \\ \int_0^L \theta_\ell(y) dy + d_\ell &= \alpha_\ell \left( \int_0^L \Lambda_\ell dy + p_\ell \right), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \quad (2.2.33)$$

By the second and third equations of (2.2.33) we find

$$p_\ell = \frac{1}{\alpha_\ell} \left( \int_0^L \theta_\ell(y) dy + d_0 \right) - \int_0^L \Lambda_\ell(y) dy, \quad \forall \ell \in \{0, \dots, N\}. \quad (2.2.34)$$

Then, replacing (2.2.34) in the first equation of (2.2.33) leads to

$$\sum_{\ell=0}^N \left( \frac{1}{\alpha_\ell} \int_0^L \theta_\ell(y) dy - \int_0^L \Lambda_\ell(y) dy \right) + \left( \sum_{\ell=0}^N \frac{1}{\alpha_\ell} \right) d_0 = 0. \quad (2.2.35)$$

As  $\sum_{\ell=0}^N \frac{1}{\alpha_\ell}$  is positive, the existence of  $d_0$  is guaranteed, which proves the Lemma.  $\square$

**Lemma 2.2.8.** *Under the same condition of Theorem 2.2.4,  $\mathcal{A}$  is surjective.*

*Proof.* Let  $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$ , we look for a unique solution  $U = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  of

$$- \mathcal{A}U = F,$$

or equivalently,

$$\begin{cases} a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell &= f_{\ell,1}, \\ b_\ell V_{\ell,x} + r_\ell I_\ell &= f_{\ell,2}, \\ c_\ell W_\ell - V_\ell &= f_{\ell,3}, \\ \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) &= f_4, \\ \frac{1}{\delta} (V_\ell(0) - v) &= f_{\ell,5}. \end{cases} \quad (2.2.36)$$

Suppose such a  $\mathbf{U} = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$  exists, then

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{c_\ell}, \quad \forall \ell \in \{0, \dots, N\}.$$

Replace  $W_\ell$  in the first and second identities of (2.2.36), we get

$$\begin{cases} V_\ell + a_{\ell,1}(0)I_{\ell,x} = f_{0,\ell}, \\ V_{\ell,x} + a_{\ell,2}(0)I_\ell = g_{0,\ell}, \end{cases} \quad (2.2.37)$$

where  $a_{\ell,1}(0), a_{\ell,2}(0), f_{0,\ell}$  and  $g_{0,\ell}$  are defined in (2.2.9).

We will proceed by dividing the proof into three steps.

**Step 1.** First, fix  $(\phi, \varphi) \in \mathbb{H}^1(\mathcal{S}) \times \mathbb{H}^1(\mathcal{S})$ , such that

$$\phi_\ell(0) = \delta f_{\ell,5} \quad \forall \ell \in \{1, \dots, N\}, \quad \phi_\ell = 0 \quad \text{on } [\epsilon_1, L] \text{ for some } \epsilon_1 \in (0, L), \quad \phi_0 = 0$$

and

$$\varphi_0(0) = \delta Y f_4, \quad \varphi_0 = 0 \text{ on } [\epsilon_2, L] \text{ for some } \epsilon_2 \in (0, L), \quad \varphi_\ell = 0 \quad \forall \ell \in \{1, \dots, N\}.$$

Then, setting  $\widehat{V}_\ell = V_\ell - \phi_\ell$  and  $\widehat{I}_\ell = I_\ell - \varphi_\ell$ , by (2.2.37) we deduce that they satisfy

$$\begin{cases} \widehat{V}_\ell + a_{\ell,1}(0)\widehat{I}_{\ell,x} = \widehat{f}_{0,\ell}, \\ \widehat{V}_{\ell,x} + a_{\ell,2}(0)\widehat{I}_\ell = \widehat{g}_{0,\ell}, \end{cases} \quad (2.2.38)$$

where,

$$\widehat{f}_{0,\ell} = f_{0,\ell} - (\phi_\ell + a_{\ell,1}(0)\varphi_{\ell,x}) \quad \text{and} \quad \widehat{g}_{0,\ell} = g_{0,\ell} - (\phi_{\ell,x} + a_{\ell,2}(0)\varphi_\ell),$$

for all  $\ell \in \{0, \dots, N\}$ . We further notice that

$$\begin{cases} \sum_{\ell=0}^N \widehat{I}_\ell(0) & = 0, \\ \widehat{V}_\ell(0) - \widehat{V}_0(0) & = 0, \quad \forall \ell \in \{1, \dots, N\}, \\ \widehat{V}_\ell(L) & = \alpha_\ell \widehat{I}_\ell(L), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \quad (2.2.39)$$

**Step 2.** Let  $(\psi, \chi) \in \mathcal{V}_*$ . Multiplying the first and the second equation of (2.2.38) by  $\bar{\chi}_{\ell,x}$  and  $\bar{\psi}_{\ell,x}$  respectively, then integrating in  $(0, L)$  gives

$$\begin{aligned} \sum_{\ell=0}^N \int_0^L (\widehat{V}_\ell \bar{\chi}_{\ell,x} + a_{\ell,1}(0)\widehat{I}_{\ell,x} \bar{\chi}_{\ell,x}) dx + \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(0)\widehat{I}_\ell \bar{\psi}_{\ell,x}) dx \\ = \sum_{\ell=0}^N \int_0^L (\widehat{f}_{0,\ell} \bar{\chi}_{\ell,x} + \widehat{g}_{0,\ell} \bar{\psi}_{\ell,x}) dx. \end{aligned} \quad (2.2.40)$$



Again the sesquilinear form from this left-hand side is not necessarily coercive, so again we introduce the sesquilinear form

$$\begin{aligned} q((\widehat{\mathbf{V}}, \widehat{\mathbf{I}}), (\boldsymbol{\psi}, \boldsymbol{\chi})) &= \sum_{\ell=0}^N \int_0^L a_{\ell,1}(0) \widehat{I}_{\ell,x} \bar{\chi}_{\ell,x} dx + \sum_{\ell=0}^N \int_0^L \widehat{V}_{\ell,x} \bar{\psi}_{\ell,x} dx \\ &+ \sum_{\ell=0}^N \int_0^L (I_{\ell} \bar{\chi}_{\ell} + V_{\ell} \bar{\psi}_{\ell}) dx, \forall (\widehat{\mathbf{V}}, \widehat{\mathbf{I}}), (\boldsymbol{\psi}, \boldsymbol{\chi}) \in \mathcal{V}_*. \end{aligned}$$

This sesquilinear form  $q$  is continuous and coercive on  $\mathcal{V}_*$  since

$$a_{\ell,1}(0) = \frac{a_{\ell} c_{\ell}}{(c_{\ell} g_{\ell} + k_{\ell})} > 0, \forall \ell \in \{0, \dots, N\}.$$

Then, by Lax-Milgram Lemma, the operator

$$Q : \mathcal{V}_* \rightarrow (\mathcal{V}_*)^* : (\widehat{\mathbf{V}}, \widehat{\mathbf{I}}) \rightarrow Q(\widehat{\mathbf{V}}, \widehat{\mathbf{I}}),$$

with

$$Q(\widehat{\mathbf{V}}, \widehat{\mathbf{I}})(\boldsymbol{\psi}, \boldsymbol{\chi}) = q((\widehat{\mathbf{V}}, \widehat{\mathbf{I}}), (\boldsymbol{\psi}, \boldsymbol{\chi})),$$

is an isomorphism. Now, the remainder

$$S : \mathcal{V}_* \rightarrow (\mathcal{V}_*)^* : (\widehat{\mathbf{V}}, \widehat{\mathbf{I}}) \rightarrow S(\widehat{\mathbf{V}}, \widehat{\mathbf{I}})(\boldsymbol{\psi}, \boldsymbol{\chi}),$$

with

$$S(\widehat{\mathbf{V}}, \widehat{\mathbf{I}})(\boldsymbol{\psi}, \boldsymbol{\chi}) = \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell} \bar{\chi}_{\ell,x} + a_{\ell,2}(0) \widehat{I}_{\ell} \bar{\psi}_{\ell,x}) dx - \sum_{\ell=0}^N \int_0^L (I_{\ell} \bar{\chi}_{\ell} + V_{\ell} \bar{\psi}_{\ell}) dx,$$

is a compact operator. Hence, we deduce that  $Q + S$  is a Fredholm operator of index zero from  $\mathcal{V}_*$  to  $(\mathcal{V}_*)^*$ .

By setting

$$L(\boldsymbol{\psi}, \boldsymbol{\chi}) = \sum_{\ell=0}^N \int_0^L (\widehat{f}_{0,\ell} \bar{\chi}_{\ell,x} + \widehat{g}_{0,\ell} \bar{\psi}_{\ell,x}) dx,$$

we again see that (2.2.40) is equivalent to

$$(Q + S)(\widehat{\mathbf{V}}, \widehat{\mathbf{I}}) = L \quad \text{in } (\mathcal{V}_*)^*. \quad (2.2.41)$$

**Step 3.** Now, let us prove that the system (2.2.38)-(2.2.39) is equivalent to (2.2.41). First, let  $(\boldsymbol{\theta}, \boldsymbol{\Lambda})$  be an arbitrary test function, such that  $\theta_{\ell'} = 0 \quad \forall \ell' \in \{0, \dots, N\}$ , and  $\Lambda_{\ell} \in \mathcal{D}(0, L)$ , for a fixed  $\ell$  and  $\Lambda_j = 0, \quad \forall j \neq \ell$ . Then, by Lemma 2.2.7, we deduce that there exists  $(\boldsymbol{\psi}, \boldsymbol{\chi}) \in \mathcal{V}_*$  such that

$$\begin{cases} \psi_{\ell,x} = 0, \\ \chi_{\ell,x} = \Lambda_{\ell}, \quad \forall \ell \in \{0, \dots, N\}. \end{cases}$$

Replacing this  $(\boldsymbol{\psi}, \boldsymbol{\chi})$  in (2.2.41) we get,

$$\int_0^L (\widehat{V}_{\ell} \Lambda_{\ell} + a_{\ell,1}(0) \widehat{I}_{\ell,x} \Lambda_{\ell}) dx = \int_0^L \widehat{f}_{0,\ell} \Lambda_{\ell} dx.$$

Then,

$$\widehat{V}_\ell + a_{\ell,1}(0)\widehat{I}_{\ell,x} = \widehat{f}_{0,\ell} \text{ in } \mathcal{D}'(0, L),$$

which means that the first equation of (2.2.38) holds. Second, let  $(\theta, \Lambda)$  be an arbitrary test function, such that  $\Lambda_\ell = 0 \forall \ell \in \{0, \dots, N\}$ ,  $\theta_\ell \in \mathcal{D}(0, L)$ , for a fixed  $\ell$  and  $\theta_j = 0, \forall j \neq \ell$ . Using Lemma 2.2.7 and following the same arguments as above we deduce that the second equation of (2.2.38) holds. In conclusion, the system (2.2.38)-(2.2.39) admits a unique solution  $(\widehat{V}, \widehat{I})$  if and only if  $Q + S$  is invertible. As  $Q + S$  is a Fredholm operator, it is enough to prove that it is injective, i.e.,

$$\ker(Q + S) = \{0\}.$$

Let  $(\widehat{V}, \widehat{I}) \in \ker(Q + S)$  then

$$\begin{aligned} \sum_{\ell=0}^N \int_0^L (\widehat{V}_\ell \bar{\chi}_{\ell,x} + a_{\ell,1}(0)\widehat{I}_{\ell,x} \bar{\chi}_{\ell,x}) dx + \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(0)\widehat{I}_\ell \bar{\psi}_{\ell,x}) dx \\ = 0, \quad \forall (\psi, \chi) \in \mathcal{V}_*. \end{aligned} \quad (2.2.42)$$

Thus, if we set

$$\widehat{W}_\ell = \frac{\widehat{V}_\ell}{c_\ell}, \quad \widehat{v} = \widehat{V}_0(0) \quad \text{and} \quad \widehat{\eta}_\ell = \sum_{k=0}^N Z_{\ell,k} \widehat{I}_k(0).$$

We conclude that  $(\widehat{V}, \widehat{I}, \widehat{W}, \widehat{v}, \widehat{\eta}) \in D(\mathcal{A})$  is a solution of

$$-\mathcal{A}(\widehat{V}, \widehat{I}, \widehat{W}, \widehat{v}, \widehat{\eta})^\top = 0.$$

Using Lemma 2.2.5, we deduce that  $\widehat{V}_\ell = \widehat{I}_\ell = \widehat{W}_\ell = \widehat{v} = \widehat{\eta}_\ell = 0$  for all  $\ell \in \{0, \dots, N\}$ .

So,  $Q + S$  is invertible and a unique solution  $(\widehat{V}, \widehat{I}) \in \mathcal{V}_*$  of (2.2.41) exists. Then, by the previous arguments we deduce that (2.2.38) and (2.2.39) hold. As we set  $\widehat{V}_\ell = V_\ell - \phi_\ell$  and  $\widehat{I}_\ell = I_\ell - \varphi_\ell$ , we deduce that a unique solution  $(V, I) \in \mathbb{H}^1(\mathcal{S}) \times \mathbb{H}^1(\mathcal{S})$  of (2.2.37) exists, and satisfies

$$\begin{cases} \sum_{\ell=0}^N I_\ell(0) & = \delta Y f_4, \\ V_\ell(0) - V_0(0) & = \delta f_{\ell,5}, \quad \forall \ell \in \{1, \dots, N\}, \\ V_\ell(L) & = \alpha_\ell I_\ell(L), \quad \forall \ell \in \{0, \dots, N\}. \end{cases}$$

Finally, by defining

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{c_\ell} \quad \forall \ell \in \{0, \dots, N\}, \quad v = V_0(0), \quad \eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) \quad \forall \ell \in \{1, \dots, N\},$$

we deduce that a solution  $(V, I, W, v, \eta) \in D(\mathcal{A})$  of (2.2.36) exists. The proof is complete.  $\square$

## 2.3 Exponential energy decay rate

In this section, we will prove the exponential stability of System (2.1.3)-(2.1.6) based on a frequency domain approach. Here contrary to Nicaise, 2015, we do not require that  $r_\ell + g_\ell$  is uniformly bounded from below for each  $\ell$  (see the assumption (25) in

Nicaise, 2015) to get exponential decay of the energy but rather exploit the dissipative boundary conditions (2.1.4) at the exterior nodes. Hence our proof fully differs from the proof of Theorem 4.5.4. from Nicaise, 2015. We here need to combine the frequency domain approach with an ad-hoc multiplier method (see Lemma 2.3.5 above).

Our main result is the following one.

**Theorem 2.3.1.** *Assume that condition (2.1.2) holds. Also, assume that  $a_\ell, b_\ell \in W^{1,\infty}(0, L)$ , for all  $\ell \in \{0, \dots, N\}$ . Then, the semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is exponentially stable, i.e. there exist two positive constants  $M$  and  $\epsilon > 0$  such that*

$$\|e^{t\mathcal{A}}\mathbf{U}_0\|_{\mathcal{H}} \leq Me^{-\epsilon t} \|\mathbf{U}_0\|_{\mathcal{H}}, \quad \forall t \geq 0, \forall \mathbf{U}_0 \in \mathcal{H}.$$

*Proof.* Following Huang., 1985 and Prüss, 1984, our  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  in  $\mathcal{H}$  is uniformly (exponentially) stable if and only if

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{2.3.1}$$

and

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < \infty. \tag{2.3.2}$$

Since (2.3.1) was already proved, the proof of Theorem 2.3.1 is reduced to show that condition (2.3.2) holds. This is checked by using a contradiction argument. Indeed, suppose that (2.3.2) is false, then there exist a sequence of real number  $\lambda_n \in \mathbb{R}$  and a sequence of vectors  $\mathbf{U}_n = (\mathbf{V}_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)$  in  $D(\mathcal{A})$ , such that

$$|\lambda_n| \rightarrow \infty, \quad \|(\mathbf{V}_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)\|_{\mathcal{H}} = 1, \tag{2.3.3}$$

and

$$(i\lambda_n - \mathcal{A})\mathbf{U}_n \rightarrow 0 \text{ in } \mathcal{H} \quad \text{as } n \rightarrow \infty. \tag{2.3.4}$$

Our aim is to show that  $\|(\mathbf{V}_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)\|_{\mathcal{H}} \rightarrow 0$ . This condition permits to conclude a contradiction with (2.3.3). From now on, for simplicity we drop the index  $n$ . Now, writing (2.3.4) in a detailed form we get

$$(i\lambda + g_\ell)V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell = f_{\ell,1} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{2.3.5}$$

$$(i\lambda + r_\ell)I_\ell + b_\ell V_{\ell,x} = f_{\ell,2} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{2.3.6}$$

$$(i\lambda + c_\ell)W_\ell - V_\ell = f_{\ell,3} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{2.3.7}$$

$$i\lambda V_0(0) + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) = f_4 \rightarrow 0 \text{ in } \mathbb{C}, \tag{2.3.8}$$

$$i\lambda \eta_\ell + \frac{1}{\delta}(V_\ell(0) - V_0(0)) = f_{\ell,5} \rightarrow 0 \text{ in } \mathbb{C}, \quad \forall \ell \in \{1, \dots, N\}. \tag{2.3.9}$$

**Lemma 2.3.2.** *Under the above assumptions, we have the following estimations for all  $\ell \in \{0, \dots, N\}$  :*

$$\sqrt{g_\ell}V_\ell = o(1) \text{ in } L^2(0, L), \tag{2.3.10}$$

$$I_\ell(L) = o(1) \text{ in } \mathbb{C}, \tag{2.3.11}$$

$$\sqrt{r_\ell}I_\ell = o(1) \text{ in } L^2(0, L), \tag{2.3.12}$$

$$W_\ell = o(1) \text{ in } L^2(0, L). \tag{2.3.13}$$

*Proof.* Taking the inner product in  $\mathcal{H}$  of  $(i\lambda - \mathcal{A})\mathbf{U}$  with  $\mathbf{U}$  and using (2.2.3) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} \frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_{\ell r_\ell} |I_\ell|^2 + \gamma_{\ell c_\ell} |W_\ell|^2) dx + \alpha_\ell |I_\ell(L)|^2 &\leq \Re \langle (i\lambda - \mathcal{A})\mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} \\ &\leq \|(i\lambda - \mathcal{A})\mathbf{U}\|_{\mathcal{H}} \|\mathbf{U}\|_{\mathcal{H}}. \end{aligned}$$

As  $\|(i\lambda - \mathcal{A})\mathbf{U}\|_{\mathcal{H}} \rightarrow 0$  and  $\|\mathbf{U}\|_{\mathcal{H}} = 1$ , we obtain

$$\frac{1}{2} \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_{\ell r_\ell} |I_\ell|^2 + \gamma_{\ell c_\ell} |W_\ell|^2) dx + \alpha_\ell |I_\ell(L)|^2 = o(1), \quad \forall \ell \in \{0, \dots, N\},$$

and the desired results hold.  $\square$

From (2.3.3), (2.3.6) and (2.3.7) we have

$$\frac{\|V_{\ell,x}\|_{L^2}}{\lambda} = O(1), \quad \frac{\|I_{\ell,x}\|_{L^2}}{\lambda} = O(1). \quad (2.3.14)$$

**Lemma 2.3.3.** *Under the above assumptions, for all  $\ell \in \{0, \dots, N\}$ , we have*

$$|V_\ell(0)| = O(1), \quad (2.3.15)$$

and

$$|I_\ell(0)| = O(1). \quad (2.3.16)$$

*Proof.* As  $Z$  is an invertible matrix matrix,  $\mathbf{I}_*(0) = Z^{-1}\boldsymbol{\eta}$  and therefore, there exists  $\zeta > 0$  such that

$$\|\mathbf{I}_*(0)\|_{\mathbb{C}^N} \leq \zeta \|\boldsymbol{\eta}\|_{\mathbb{C}^N}. \quad (2.3.17)$$

Then, using (2.3.3), we directly deduce that (2.3.16) holds for  $\ell \in \{1, \dots, N\}$ .

Now, multiply (2.3.6) by  $2\bar{I}_\ell$ , and then integrate over  $(0, L)$ , to obtain

$$\begin{aligned} 2i \int_0^L \lambda V_\ell \bar{I}_\ell dx + 2 \int_0^L g_\ell V_\ell \bar{I}_\ell dx + 2 \int_0^L a_\ell I_{\ell,x} \bar{I}_\ell dx + 2 \int_0^L k_\ell W_\ell \bar{I}_\ell dx \\ = 2 \int_0^L f_{\ell,1} \bar{I}_\ell dx. \end{aligned} \quad (2.3.18)$$

The second and fourth integral on the left-hand side of (2.3.18) converge to zero by (2.3.3), (2.3.10) and (2.3.13), similarly the right-hand side converge to zero, using (2.3.3) and the fact that  $f_{\ell,1}$  converge to zero in  $L^2(0, L)$ . This yields

$$2i \int_0^L \lambda V_\ell \bar{I}_\ell dx + 2 \int_0^L a_\ell I_{\ell,x} \bar{I}_\ell dx = o(1). \quad (2.3.19)$$

Taking the real part of (2.3.19) and then applying Green's formula on the second integral of the left side leads to,

$$-2 \Im \int_0^L \lambda V_\ell \bar{I}_\ell dx - \int_0^L a_{\ell,x} |I_\ell|^2 dx + a_\ell(L) |I_\ell(L)|^2 - a_\ell(0) |I_\ell(0)|^2 = o(1), \quad (2.3.20)$$

Similarly, multiply (2.3.7) by  $2\bar{V}_\ell$ , and then integrate over  $(0, L)$ , we get

$$2i \int_0^L \lambda I_\ell \bar{V}_\ell dx + 2 \int_0^L r_\ell I_\ell \bar{V}_\ell dx + 2 \int_0^L b_\ell V_{\ell,x} \bar{V}_\ell dx = 2 \int_0^L f_{\ell,2} \bar{V}_\ell dx. \quad (2.3.21)$$

The right hand-side of (2.3.21) converge to zero by (2.3.6) and (2.3.3), similarly the second integral of the left hand-side converge to zero by (2.3.3) and (2.3.12). This yields

$$2i \int_0^L \lambda I_\ell \bar{V}_\ell dx + 2 \int_0^L b_\ell V_{\ell,x} \bar{V}_\ell dx = o(1). \quad (2.3.22)$$

Again, taking the real part of (2.3.22) and then, applying Green's formula in the second integral of the left hand side we get,

$$-2 \Im \int_0^L \lambda I_\ell \bar{V}_\ell dx - \int_0^L b_{\ell,x} |V_\ell|^2 dx + b_\ell(L) |V_\ell(L)|^2 - b_\ell(0) |V_\ell(0)|^2 = o(1). \quad (2.3.23)$$

Adding equations (2.3.20) and (2.3.23) leads to

$$\begin{aligned} & - \int_0^L a_{\ell,x} |I_\ell|^2 dx - \int_0^L b_{\ell,x} |V_\ell|^2 dx + a_\ell(L) |I_\ell(L)|^2 - a_\ell(0) |I_\ell(0)|^2 \\ & + b_\ell(L) |V_\ell(L)|^2 - b_\ell(0) |V_\ell(0)|^2 = o(1), \quad \forall \ell \in \{0, \dots, N\}. \end{aligned} \quad (2.3.24)$$

Under the assumption that  $a_\ell, b_\ell \in W^{1,\infty}(0, L)$ , using (2.1.2), (2.3.3), (2.3.11), (2.3.17) and the fact that  $V_\ell(L) = \alpha_\ell I_\ell(L)$ , we deduce from (2.3.24) that

$$|V_\ell(0)| = O(1), \quad \forall \ell \in \{1, \dots, N\}, \quad (2.3.25)$$

and as  $|V_0(0)| = O(1)$  using (2.3.3), we conclude that (2.3.15) holds. On the other hand, for  $\ell = 0$  and under the assumption that  $a_0, b_0 \in W^{1,\infty}(0, L)$  and using (2.1.2), (2.3.3), (2.3.11) and as  $V_0(L) = \alpha_0 I_0(L)$ , we deduce from (2.3.24) that  $|I_0(0)|^2 = O(1)$ . Consequently, with (2.3.17), we conclude that (2.3.16) holds.  $\square$

**Lemma 2.3.4.** *Under the above assumptions, we have*

$$v = V_0(0) = o(1), \quad (2.3.26)$$

and

$$\eta_\ell = o(1) \quad \forall \ell \in \{1, \dots, N\}. \quad (2.3.27)$$

*Proof.* Using (2.3.8), we have

$$V_0(0) = -\frac{1}{i\lambda\delta Y} \sum_{\ell=0}^N I_\ell(0) + \frac{f_4}{i\lambda},$$

by (2.3.16) and the fact that  $|f_4| = O(1)$ , we deduce that  $V_0(0) = o(1)$ . Similarly, using (2.3.9), we have

$$\eta_\ell = -\frac{1}{i\lambda\delta} V_\ell(0) + \frac{1}{i\lambda\delta} V_0(0) + \frac{1}{i\lambda} f_{\ell,5},$$

by (2.3.15) and the fact that  $|f_{\ell,5}| = O(1)$ , we deduce that (2.3.27) holds.  $\square$

**Lemma 2.3.5.** *Let  $\ell \in \{0, \dots, N\}$  be fixed and assume that the functions  $a_\ell, b_\ell \in W^{1,\infty}(0, L)$ . Then, there exists  $h_\ell \in W^{1,\infty}(0, L)$  and a positive real number  $\rho_\ell$  such that*

$$h_\ell(0) = 0, \quad (2.3.28a)$$

$$h_{\ell,x}(x) \geq \rho_\ell > 0, \text{ for a.a. } x \in (0, L), \quad (2.3.28b)$$

$$h_{\ell,x}(x) \left( \frac{a_\ell(x)}{b_\ell(x)} \right) + h_\ell(x) \left( \frac{a_\ell(x)}{b_\ell(x)} \right)_x \geq 0, \text{ for a.a. } x \in (0, L). \quad (2.3.28c)$$

*Proof.* We first notice that (2.3.28c) is equivalent to

$$\frac{h_{\ell,x}(x)}{h_\ell(x)} + \frac{\left( \frac{a_\ell(x)}{b_\ell(x)} \right)_x}{\left( \frac{a_\ell(x)}{b_\ell(x)} \right)} \geq 0, \text{ for a.a. } x \in (0, L), \quad (2.3.29)$$

or equivalently

$$\frac{h_{\ell,x}(x)}{h_\ell(x)} \geq \omega_\ell(x) := - \frac{\left( \frac{a_\ell(x)}{b_\ell(x)} \right)_x}{\left( \frac{a_\ell(x)}{b_\ell(x)} \right)}, \text{ for a.a. } x \in (0, L). \quad (2.3.30)$$

Since  $\omega_\ell(x)$  is bounded on  $(0, L)$ , then if  $h_\ell(x) = x$  near zero, there exists a positive real number  $\epsilon$  (sufficiently small) such that

$$\frac{1}{x} \geq \omega_\ell(x), \quad \forall x \in (0, \epsilon]. \quad (2.3.31)$$

So, we define

$$M_\ell = \max_{x \in (0, L)} \omega_\ell(x),$$

and we distinguish the following two cases:

i) If  $M_\ell \leq \frac{1}{L}$ , then (2.3.31) holds with  $\epsilon = L$ , and we can choose  $h_\ell(x) = x, \forall x \in (0, L)$ . In this case, (2.3.28a), (2.3.28b) and (2.3.28c) hold.

ii) If  $M_\ell > \frac{1}{L}$ , then we choose  $0 < \epsilon \leq \frac{1}{M_\ell}$ , and define  $h_\ell(x) = x, \forall x \leq \epsilon$ , so that (2.3.31) holds. Now, for  $x \geq \epsilon$ , we impose that

$$h_\ell(x) = \epsilon e^{\int_\epsilon^x m_\ell(y) dy}, \quad \forall x \geq \epsilon, \quad (2.3.32)$$

with

$$m_\ell(x) := \max\{\omega_\ell(x), \kappa\}, \quad (2.3.33)$$

for some  $\kappa > 0$ . Notice that  $\lim_{x \rightarrow \epsilon} h_\ell(x) = \epsilon$ , i.e.,  $h_\ell$  is continuous on  $(0, L)$ . Now from the expression (2.3.32), we have

$$h_{\ell,x}(x) = \epsilon m_\ell(x) e^{\int_\epsilon^x m_\ell(y) dy} = h_\ell(x) m_\ell(x), \quad \forall x > \epsilon. \quad (2.3.34)$$

As  $m_\ell(x) \geq \omega_\ell(x)$ , we deduce that (2.3.30) holds (i.e. (2.3.28c) holds). Finally, by (2.3.32), we have that

$$h_\ell(x) \geq \epsilon, \quad \forall x > \epsilon, \quad (2.3.35)$$

which by (2.3.34) and (2.3.33) implies that

$$h_{\ell,x}(x) \geq \epsilon\kappa > 0, \quad \forall x > \epsilon.$$

In other words, (2.3.28b) holds for almost all  $x \in (0, L)$ .  $\square$

**Lemma 2.3.6.** *Under the above assumptions, we have for all  $\ell \in \{0, \dots, N\}$ ,*

$$V_\ell = o(1) \text{ in } L^2(0, L). \quad (2.3.36)$$

*Proof.* Fix  $\ell \in \{0, \dots, N\}$  and let  $h_\ell \in W^{1,\infty}(0, L)$  be the function defined in Lemma 2.3.5.

First, by multiplying (2.3.5) by  $(2h_\ell \bar{V}_{\ell,x})$  and then integrating over  $(0, L)$ , we get

$$\begin{aligned} 2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L h_\ell g_\ell V_\ell \frac{\bar{V}_{\ell,x}}{\lambda} dx - 2i \int_0^L a_\ell h_\ell \bar{V}_{\ell,x} \frac{I_{\ell,x}}{\lambda} dx \\ - 2i \int_0^L h_\ell k_\ell W_\ell \frac{\bar{V}_{\ell,x}}{\lambda} dx = -2i \int_0^L h_\ell f_{\ell,1} \frac{\bar{V}_{\ell,x}}{\lambda} dx. \end{aligned} \quad (2.3.37)$$

The second and fourth integral on the left hand-side of (2.3.37) converge to zero by (2.3.10), (2.3.13) and (2.3.14). Similarly, the right hand-side converge to zero by (2.3.14) and the fact that  $f_{\ell,1}$  converge to zero in  $L^2(0, L)$ , for all  $\ell$ . This yields

$$2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L a_\ell h_\ell \bar{V}_{\ell,x} \frac{I_{\ell,x}}{\lambda} dx = o(1). \quad (2.3.38)$$

By eliminating  $V_{\ell,x}$  from (2.3.6) in the second term of (2.3.38), we obtain

$$\begin{aligned} 2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L \frac{a_\ell}{b_\ell} h_\ell \bar{f}_{\ell,2} \frac{I_{\ell,x}}{\lambda} dx + 2 \int_0^L \frac{a_\ell}{b_\ell} h_\ell \bar{I}_\ell I_{\ell,x} dx \\ + 2i \int_0^L \frac{a_\ell}{b_\ell} h_\ell r_\ell \bar{I}_\ell \frac{I_{\ell,x}}{\lambda} dx = o(1). \end{aligned} \quad (2.3.39)$$

The second and fourth integral converge to zero by (2.3.14), (2.3.12) and the fact that  $f_{\ell,2}$  converge to zero in  $L^2(0, L)$ ,  $\forall \ell$ . Now, by taking the real part of (2.3.39) and applying Green's formula on the remaining two integrals, we deduce

$$\begin{aligned} - \int_0^L h_{\ell,x} |V_\ell|^2 dx + h_\ell(L) |V_\ell(L)|^2 - h_\ell(0) |V_\ell(0)|^2 - \int_0^L \left( \frac{a_\ell}{b_\ell} \right)_x h_\ell |I_\ell|^2 dx \\ - \int_0^L \frac{a_\ell}{b_\ell} h_{\ell,x} |I_\ell|^2 dx + \frac{a_\ell(L)}{b_\ell(L)} h_\ell(L) |I_\ell(L)|^2 - \frac{a_\ell(0)}{b_\ell(0)} h_\ell(0) |I_\ell(0)|^2 = o(1). \end{aligned} \quad (2.3.40)$$

Then, due to the fact that  $V_\ell(L) = \alpha_\ell I_\ell(L)$  and using (2.3.11) and (2.3.28a), we obtain

$$\int_0^L h_{\ell,x} |V_\ell|^2 dx + \int_0^L \left( \left( \frac{a_\ell}{b_\ell} \right)_x h_\ell + \frac{a_\ell}{b_\ell} h_{\ell,x} \right) |I_\ell|^2 dx = o(1). \quad (2.3.41)$$

By using (2.3.28b) and (2.3.28c), the result holds.  $\square$

**Lemma 2.3.7.** *Under the above assumptions, we have for all  $\ell \in \{0, \dots, N\}$ ,*

$$I_\ell = o(1) \text{ in } L^2(0, L). \quad (2.3.42)$$

*Proof.* First, multiplying (2.3.6) by  $\frac{1}{b_\ell} \bar{I}_\ell$  and then integrating over  $(0, L)$ , we find

$$\int_0^L \frac{i\lambda}{b_\ell} |I_\ell|^2 dx + \int_0^L \frac{r_\ell}{b_\ell} |I_\ell|^2 dx + \int_0^L V_{\ell,x} \bar{I}_\ell dx = \int_0^L \frac{f_{\ell,2}}{b_\ell} \bar{I}_\ell dx. \quad (2.3.43)$$

The second integral of the left hand-side of (2.3.43) converge to zero by (2.3.3) and (2.3.13). Similarly, the integral of the right hand side converge to zero by (2.3.3) and the fact that  $f_{\ell,2}$  converge to zero in  $L^2(0, L)$ , for all  $\ell$ . This yields

$$\int_0^L \frac{i\lambda}{b_\ell} |I_\ell|^2 dx + \int_0^L V_{\ell,x} \bar{I}_\ell dx = o(1). \quad (2.3.44)$$

Next, by applying Green's formula on the second integral of the left-hand side of (2.3.44), eliminating  $I_{\ell,x}$  from (2.3.5) and then dividing by  $i\lambda$ , we obtain

$$\begin{aligned} \int_0^L \frac{|I_\ell|^2}{b_\ell} dx - \int_0^L \frac{|V_\ell|^2}{a_\ell} dx + \int_0^L \frac{g_\ell}{i\lambda a_\ell} |V_\ell|^2 dx + \int_0^L \frac{k_\ell}{i\lambda a_\ell} V_\ell \bar{W}_\ell dx \\ + \frac{V_\ell(L) \bar{I}_\ell(L)}{i\lambda} - \frac{V_\ell(0) \bar{I}_\ell(0)}{i\lambda} = \int_0^L V_\ell \frac{\bar{f}_{\ell,1}}{i\lambda a_\ell} dx. \end{aligned} \quad (2.3.45)$$

As  $V_\ell(L) = \alpha_\ell I_\ell(L)$  and using (2.1.2), (2.3.3), (2.3.11), (2.3.13) and Lemmas 2.3.3 and 2.3.6, the result holds.  $\square$

Consequently, using (2.3.13), Lemma 2.3.4, Lemma 2.3.6 and Lemma 2.3.7, we deduce the desired contradiction. This ends the proof.  $\square$

**Remark 2.3.8.** Contrary to Bastin and Coron, 2016, §5.6 (see also Gugat and Gerster, 2019 for second order hyperbolic systems), our boundary conditions are chosen in such a way that there is no limit of stabilization, namely under our previous assumptions, exponential decay occurs for all lengths  $L$  but the decay rate  $\epsilon$  in Theorem 2.3.1 may depend on  $L$ . If  $r_\ell = g_\ell = 0$ , we conjecture that this decay rate deteriorates as  $L$  goes to infinity.

## 2.4 Some extensions and open problems

### 2.4.1 Other dynamical boundary conditions

The standard Kirchhoff conditions at the interior common vertex are

$$\begin{cases} V_0(0, t) - V_\ell(0, t) = 0, & \text{in } \mathbb{R}_+^*, \forall \ell \in \{1, \dots, N\}, \\ \sum_{\ell=0}^N I_\ell(0, t) = 0 & \text{in } \mathbb{R}_+^*, \end{cases} \quad (2.4.1)$$

that corresponds to the limit case of (2.1.5) as  $\delta$  goes to zero. These boundary conditions are stationary ones but can be also used instead of (2.1.5). More generally in the spirit of Kramar, Mugnolo, and Nicaise, 2020a, we can mix up stationary and dynamical boundary conditions at the common vertex  $\mathbf{v}$  in the following way. First if  $\mathbf{u}(\cdot, t) = ((V_\ell(\cdot, t), I_\ell(\cdot, t), W_\ell(\cdot, t))_{\ell=0}^N)^\top$  represents the unknowns in (2.1.3), we denote by

$$\gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)) = ((V_\ell(0, t))_{\ell=0}^N, (I_\ell(0, t))_{\ell=0}^N)^\top$$

its trace at  $\mathbf{v}$  that is a vector in  $\mathbb{C}^{2(N+1)}$ . Obviously we only keep the trace of  $V_\ell$  and  $I_\ell$  since  $W_\ell$  is not regular enough in space (see the definition of  $D(\mathcal{A})$ ). Then we fix



a subspace  $Y_{\mathbf{v}}$  of  $\mathbb{C}^{2(N+1)}$ , another subspace  $Y_{\mathbf{v}}^{(d)}$  of  $Y_{\mathbf{v}}$  and finally a linear operator  $B_{\mathbf{v}}$  from  $Y_{\mathbf{v}}$  to  $Y_{\mathbf{v}}^{(d)}$  and set

$$x_{\mathbf{v}}(t) = P_{\mathbf{v}}\gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)),$$

where  $P_{\mathbf{v}}$  is a projection from  $\mathbb{C}^{2(N+1)}$  into  $Y_{\mathbf{v}}^{(d)}$  (for an appropriate inner product of  $\mathbb{C}^{2(N+1)}$ ).

Hence at  $\mathbf{v}$ , we can consider the boundary condition

$$\begin{cases} \gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)) \in Y_{\mathbf{v}}, & \text{in } \mathbb{R}_+^*, \\ x_{\mathbf{v},t}(t) = B_{\mathbf{v}}\gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)), & \text{in } \mathbb{R}_+^*. \end{cases} \quad (2.4.2)$$

The boundary conditions (2.1.5) and (2.4.1) enter in this setting. Namely in the first case, we need to take  $Y_{\mathbf{v}} = \mathbb{C}^{2(N+1)}$  and

$$Y_{\mathbf{v}}^{(d)} = \{((v_{\ell})_{\ell=0}^N, (i_{\ell})_{\ell=0}^N)^{\top} : i_0 = v_{\ell} = 0, \forall \ell = 1, \dots, N\},$$

$P_{\mathbf{v}}$  is the orthogonal projection into  $Y_{\mathbf{v}}^{(d)}$  with respect to the Euclidean inner product, and

$$B_{\mathbf{v}}((v_{\ell})_{\ell=0}^N, (i_{\ell})_{\ell=0}^N)^{\top} = \frac{1}{\delta} \left( \left( -\frac{1}{Y} \sum_{\ell=0}^N i_{\ell}, 0, \dots, 0 \right)^{\top}, \left( 0, Z^{-1}(v_0 \mathbf{1}_N - \mathbf{v}) \right)^{\top} \right)^{\top},$$

where  $\mathbf{1}_N$  is the vector of  $\mathbb{C}^N$  with all entries equal to 1, while  $\mathbf{v} = (v_{\ell})_{\ell=1}^N$ . On the contrary in the case (2.4.1), we simply need to take  $Y_{\mathbf{v}}^{(d)} = \{0\}$  and

$$Y_{\mathbf{v}} = \{((v_{\ell})_{\ell=0}^N, (i_{\ell})_{\ell=0}^N)^{\top} : \sum_{\ell=0}^N i_{\ell} = 0 \text{ and } v_0 = v_{\ell}, \forall \ell = 1, \dots, N\}.$$

First, under some assumptions on  $Y_{\mathbf{v}}$  and  $B_{\mathbf{v}}$ , one can show that (2.2.3) remains valid and therefore the associated operator  $\mathcal{A}$  is dissipative. For the maximality, in some particular situations, using the arguments of subsection 2.2.1 (eliminating the variables  $W_{\ell}$  and one of the other variables to obtain a variational formulation), we can prove that the associated operator is maximal dissipative under additional conditions on  $Y_{\mathbf{v}}$  and  $B_{\mathbf{v}}$  and therefore system (2.1.3) with the boundary conditions (2.1.4) and (2.4.2) is well-posed and is governed by a semigroup of contractions. In the general case, using a bounded perturbation argument as in Kramar, Mugnolo, and Nicaise, 2020a, Theorem 3.3 we can prove that a bounded perturbation of  $\mathcal{A}$  is maximal dissipative under additional conditions on  $Y_{\mathbf{v}}$  and  $B_{\mathbf{v}}$  and therefore system (2.1.3) with the boundary conditions (2.1.4) and (2.4.2) is well-posed and is governed by a  $C_0$  semigroup. But using the dissipativeness of  $\mathcal{A}$ , one can deduce that  $\mathcal{A}$  generates a semigroup of contractions.

Once the well-posedness of our system holds, one can analyze its stability. First, we notice that Lemma 2.2.5 remains valid because the proof presented before yields first  $V_{\ell} = I_{\ell} = W_{\ell} = 0$ , for all  $\ell$  and hence  $x_{\mathbf{v}} = P_{\mathbf{v}}\gamma_{\mathbf{v}}(\mathbf{u})$  will be zero as well. On the contrary, Lemma 2.2.6 is problematic because a compact perturbation argument cannot be invoked due to the third identity in (2.2.22). But if the surjectivity of  $i\lambda - \mathcal{A}$  holds for all  $\lambda \in \mathbb{R}$ , then the system will be exponentially stable. Indeed, in

this case the proof of Theorem 2.3.1 remains valid because the use of the multipliers  $h_\ell$  from Lemma 2.3.5 allows to skip the use of the boundary condition (2.4.2).

In summary the exponential stability results for system (2.1.3) with the boundary conditions (2.1.4) and (2.4.2) are reduced to an existence result and the surjectivity of the resolvent on the imaginary axis.

## 2.4.2 General networks

Obviously, we can consider system (2.1.3) on an arbitrary network  $\mathcal{G}$  with dissipative boundary conditions similar to (2.1.4) at exterior vertices (it is (2.1.4) if the edge having the exterior vertex  $\mathbf{v}$  as extremity is outgoing, otherwise it is  $V_\ell(\mathbf{v}, t) + \alpha_\ell I_\ell(\mathbf{v}, t) = 0$ ) and boundary conditions like (2.4.2) at the interior vertices. As in the previous subsection, existence results can then be obtained under appropriate conditions on  $Y_{\mathbf{v}}$  and  $B_{\mathbf{v}}$ . Obviously the surjectivity of  $i\lambda - \mathcal{A}$  for all  $\lambda \in \mathbb{R}$  will remain problematic, but here even its injectivity is not immediate.

The sole case for which injectivity of  $i\lambda - \mathcal{A}$  holds for all  $\lambda \in \mathbb{R}^*$  is the case of trees with the boundary conditions (2.1.5) or (2.4.1) at the interior vertices. Indeed, in this case the arguments of Lemma 2.2.5 allow to show that  $V_\ell = I_\ell = W_\ell = 0$ , for all edges  $\ell$  of the last generation of the tree. But then using (2.1.5) or (2.4.1), we will get that  $V_{\ell'}(\mathbf{v}') = I_{\ell'}(\mathbf{v}') = 0$  for all edges  $\ell'$  of the penultimate generation and the vertex  $\mathbf{v}'$  in common with the last generation. This means that we can use again the arguments of Lemma 2.2.5 to these edges  $\ell'$  and by iteration we will find the injectivity result. Finally, the exponential stability results for system (2.1.3) with the boundary conditions (2.1.4) at exterior vertices and boundary conditions (2.1.5) or (2.4.1) at interior vertices is not immediate even for trees.

Note that for general networks, as in Nicaise, 2015, once  $i\lambda - \mathcal{A}$  is bijective for all  $\lambda \in \mathbb{R}$ , exponential stability will be valid if we assume that  $r_\ell + g_\ell$  is uniformly bounded from below for each  $\ell$  (see the assumption (25) in Nicaise, 2015); without this condition, polynomial stability can be achieved. Nevertheless the type of the decay rate of such systems on general networks remains largely an open problem. Does it remain exponential or polynomial? How many boundary conditions can we remove from the exterior nodes to preserve stability? What will happen if we add some damping terms localized at the interior nodes?

Our proof of Theorem 2.3.1 is non constructive (since it uses a contradiction argument), hence it does not give the dependency of the decay rate with respect to the involved parameters, hence its degeneracy as one parameter tends to zero or to infinity (see Remark 2.3.8) cannot be detected. Therefore, it would be interesting to characterize such dependencies by performing some spectral analysis for instance. In power grids, failure of some parts can occur from time to time, this corresponds to switch off one of the boundary controllers ( $\alpha_\ell = 0$ ) from time to time. As in Gugat and Sigalotti, 2010 for the wave equation, is it possible to find sufficient conditions on the controllers to achieve exponential decay? All of these problems are challenging and merit to be investigated.



## Chapter 3

# A transmission problem of a thermo-elastic system on Networks

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### Abstract

**I**n this chapter, we investigate a network of elastic and thermo-elastic materials. On each thermo-elastic edge, we consider two coupled wave equations such that one of them is damped via a coupling with a heat equation. On each elastic edge (undamped), we consider two coupled conservative wave equations. Under some conditions, we prove that the thermal damping is enough to stabilize the whole system. If the two waves propagate with the same speed on each thermo-elastic edge, we show that the energy of the system decays exponentially. Otherwise, a polynomial energy decay is attained. Finally, we present some other boundary conditions and we show that under sufficient conditions on the lengths of some elastic edges, the energy of the system decays exponentially on some particular networks similar to the ones considered in Shel, 2012.

### 3.1 Introduction

Thermoelasticity is a principle concerned with predicting the thermo-mechanical behaviour of elastic solids. Understanding such a principle is needed by many engineers to design different materials. Thus, several scientists were motivated to study the thermoelastic system described by the coupling between the mechanical vibration and the heat (thermal) effect of materials. Mathematically, a linear one-dimensional thermo-elastic system satisfied by a thermoelastic bar  $(0, L)$  is represented by the following two equations:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = 0, & \text{in } (0, L) \times (0, \infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} = 0, & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (3.1.1)$$

with the initial conditions

$$u(x, 0) = u_0, u_t(x, 0) = u_1, \theta(x, 0) = \theta_0, \quad x \in (0, L), \quad (3.1.2)$$

where,  $u$  is the displacement,  $\theta$  is the temperature deviation from the reference temperature and the mechanical-thermal coupling  $\alpha$  is a positive constant. The existence and asymptotic behavior of the solution of the linear thermo-elastic system was firstly studied in Dafermos, 1968 but, no decay rate was given. In the one

dimensional case, the stabilization of the linear thermo-elastic system satisfied by thermo-elastic materials (damped by thermal effect) with various boundary conditions was investigated by several authors. We will recall some of these results. In Hansen, 1992, the author considered the stabilization of system (3.1.1)-(3.1.2) on a thermo-elastic rod with  $u$  and  $\theta$  satisfying the Dirichlet and Neumann condition respectively (or vice versa). He succeeded in proving the exponential stability of the system. More precisely, the author established the following energy estimate: There exist two positive constants  $M$  and  $\epsilon$  such that

$$E(t) \leq Me^{-\epsilon t}E(0), \quad \forall t > 0. \quad (3.1.3)$$

Similarly, when  $u$  and  $\theta$  satisfy both the Dirichlet condition, it was shown that the estimate (3.1.3) still holds in Liu and Zheng, 1993. Then, the method of Liu and Zheng, 1993 was extended in Burns, Liu, and Zheng, 1993 to prove (3.1.3) when  $u_x - \alpha\theta$  satisfies Dirichlet condition on both ends and  $\theta_x(0) = \theta_x(L) = 0$  or  $\theta_x(0) = 0, \theta(L) = 0$ . Later on, the importance of damping and controlling the vibrations of materials composed of both elastic (undamped) and thermo-elastic (damped by thermal effect) parts appears in several physical applications and consequently in several mathematical papers. The main questions that received the interest of the researchers is the kind of stability of the thermo-elastic system on such composite materials and how should the thermo-elastic damping be localized to get the best decay rate or what is the energy decay rate in different localizations of the thermal damping? Such questions were answered in several ways. For example, in Marzocchi, Rivera, and Naso, 2002, it was considered a one dimensional body which is configurated in  $[0, L_3] \subset \mathbb{R}$  and for a given  $L_1 < L_2$  in  $]0, L_3[$ , they assumed that the material is thermo-elastic over  $]0, L_1[ \cup ]L_2, L_3[$  and elastic over  $]L_1, L_2[$ . The authors proved that the whole system is exponentially stable, i.e., (3.1.3) holds. Then, in Fatori, Lueders, and Rivera, 2003, the authors considered the stabilization of a transmission problem for the thermo-elastic system with local thermal effect which is effective only over the interval  $[0, L_0], L_0 \in [0, L]$ , this corresponds to the following system:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = 0, & \text{in } (0, L_0) \times (0, \infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} = 0, & \text{in } (0, L_0) \times (0, \infty), \\ v_{tt} - v_{xx} = 0, & \text{in } (L_0, L) \times (0, \infty), \end{cases} \quad (3.1.4)$$

with the initial conditions

$$u(x, 0) = u_0, u_t(x, 0) = u_1, \theta(x, 0) = \theta_0, v(x, 0) = v_0, v_t(x, 0) = v_1, \quad x \in (0, L),$$

where  $u$  is the displacement in the thermo-elastic part,  $v$  is the displacement in the elastic part and  $\theta$  is the temperature difference from a reference value. The system is completed with the following boundary conditions

$$u(0, t) = v(L, t) = \theta(0, t) = 0, \quad t \in (0, \infty),$$

and the following transmission conditions

$$u_x(L_0, t) - \alpha\theta(L_0, t) = v_x(L_0, t) \quad \text{and} \quad \theta_x(L_0, t) = 0.$$

The authors proved that the localized dissipation due to the thermal effect is strong enough to prove the exponential decay to zero of the energy. We also refer to Lebeau

and Zuazua, 1999 and Oliveira and Charão, 2008 for the study of the stabilization of multi-dimensional linear thermo-elastic systems.

On the other hand, there are only few publications on the stabilization of networks of thermo-elastic materials. Let us recall some of these results. In Abdallah and Shel, 2012, an exponential stability was proved on a network of thermo-elastic materials under both Fourier's law and Cattaneo's law. In Shel, 2012, the author studied the stability problem of a thermo-elastic system on particular cases of networks of elastic and thermo-elastic materials. Under the continuity condition of the displacement, the Neumann condition for the temperature at the internal nodes, and the balance condition, an exponential stability was proved (see also, Shel, 2014 for the network of elastic and thermo-elastic beams). Later on, in Han and Zuazua, 2017 the authors discussed the asymptotic behaviour of a transmission problem of the thermo-elastic system on star shaped networks of elastic and thermo-elastic rods. The uniform exponential decay rate was proved by a frequency domain analysis when only one purely elastic edge was present. Otherwise, a polynomial decay rate was deduced under a suitable irrationality condition on the lengths of the rods when more than one purely elastic edge is involved. After the review of these results that investigated the stabilization of a thermo-elastic system composed of the coupling between one wave equation and a heat equation, a remarkable question can be asked. What happens if we consider a network of elastic and thermo-elastic materials such that:

- On the thermo-elastic edges, we have a system of two wave equations coupled by velocity, such that one wave equation is coupled to a heat equation with a thermal effect.
- On the purely undamped elastic edges, we have only a system of two conservative wave equations coupled by velocity.

Hence our main question is the following one: Will the dissipation due to the thermal effect be also strong enough to prove the exponential stability of the energy of the whole system? To the best of our knowledge, the answer to this question remains an open problem. Therefore, our aim is to solve this open question.

In this work, we investigate the stabilization of the above described transmission problem on networks of elastic and thermo-elastic materials. We prove the exponential stability of the whole system under the condition that the two waves propagate with the same speed on all the thermo-elastic edges of the network. On the other hand, if there exists an exterior thermo-elastic edge such that the two waves propagate with different speed on this edge, we show the polynomial stability of the whole system. Our main tool is a frequency domain approach, namely to prove the exponential stability we use a result due to Huang., 1985 and Prüss, 1984 and to show the polynomial stability we use a result due to Borichev and Tomilov, 2009.

Now, let us introduce some notations needed to formulate the problem under consideration, refer to Valein and Zuazua, 2009 and Abdallah and Shel, 2012 for more details. Let  $\mathcal{N}$  be a network embedded in the Euclidean space  $\mathbb{R}^m, m \in \mathbb{N}^*$ , with  $n$  vertices  $\mathcal{V} = \{a_0, a_2, \dots, a_{n-1}\}$  and  $N$  edges  $E = \{e_1, \dots, e_N\}$ , with  $\mathcal{I}(\mathcal{N}) = \{1, \dots, N\}$ , the set of indices of edges. Each edge  $e_j$  is a curve, parametrized by

$$\pi_j : [0, \ell_j] \rightarrow e_j : x_j \rightarrow \pi_j(x_j). \quad (3.1.5)$$

The degree of a vertex is the number of incident edges at the vertex. A vertex with degree 1 is called an exterior vertex. On the other hand, a vertex with degree greater than 1 is called an interior vertex.

We assume that the network is made of thermo-elastic edges and elastic ones, this means that  $\mathcal{I}(\mathcal{N})$  is split up into  $\mathcal{I}(\mathcal{N}) = \mathcal{I}_{te} \cup \mathcal{I}_e$ , with  $\mathcal{I}_e \cap \mathcal{I}_{te} = \emptyset$ , in other words,  $\mathcal{I}_{te}$  (resp.  $\mathcal{I}_e$ ) is the set of thermo-elastic (resp. elastic) edges.

We further denote by:

$V_{\text{ext}} :=$  set of exterior vertices of  $\mathcal{N}$ .

$V_{\text{int}} :=$  set of interior vertices of  $\mathcal{N}$ .

$\mathcal{I}(a_k) :=$  set of indices of edges incident to  $a_k$ .

$\mathcal{I}_{te}(a_k) :=$  set of indices of thermo-elastic edges adjacent to  $a_k$ .

$\mathcal{I}_e(a_k) :=$  set of indices of elastic edges incident to  $a_k$ .

$\mathcal{I}_{\text{ext}} :=$  set of indices of edges adjacent to an exterior vertex of  $\mathcal{N}$ .

The incidence matrix  $D = (d_{kj})_{n \times N}$  of  $\mathcal{N}$  is defined by

$$d_{kj} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = a_k, \\ -1 & \text{if } \pi_j(0) = a_k, \\ 0 & \text{otherwise,} \end{cases} \quad (3.1.6)$$

and for a function  $f : \mathcal{N} \rightarrow \mathbb{C}$ , we set  $f^j = f \circ \pi_j$  its restriction to the edge  $e_j$ . For simplicity, we will write  $f = (f^1, \dots, f^N)$  and we will denote  $f^j(x) = f^j(\pi_j(x))$  for any  $x$  in  $(0, \ell_j)$ . We consider a network of elastic and thermo-elastic materials that coincides with the graph  $\mathcal{N}$ . We assume that  $\mathcal{N}$  contains at least one thermo-elastic edge, that  $V_{\text{ext}} \neq \emptyset$ , that every maximal subgraph of elastic edges is a tree whose all of its exterior vertices except one are attached to thermo-elastic edges and that every subgraph of thermo-elastic edges is not a circuit.

Let  $u^j = u^j(x, t)$  and  $y^j = y^j(x, t)$  be the functions describing the displacement at time  $t$  of the edge  $e_j$ ,  $j \in \mathcal{I}(\mathcal{N})$  and  $\theta^j = \theta^j(x, t)$  be the temperature difference to a fixed reference temperature of  $e_j$ ,  $j \in \mathcal{I}_{te}$  at time  $t$ .

Our system is described as follows:

- On every thermo-elastic edge ( $j \in \mathcal{I}_{te}$ ) the following equations hold:

$$\begin{cases} u_{tt}^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j y_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ \theta_t^j - \kappa_j \theta_{xx}^j + \alpha_j u_{tx}^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (3.1.7)$$

where  $\alpha_j$ ,  $\rho_j$ ,  $\kappa_j$  and  $\beta_j$  are positive constants.

- On every elastic edge ( $j \in \mathcal{I}_e$ ) one has:

$$\begin{cases} u_{tt}^j - u_{xx}^j - \beta_j y_t^j = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (3.1.8)$$

where  $\beta_j$  and  $\rho_j$  are positive constants.

We assume that the initial data on the network  $\mathcal{N}$  are

$$\begin{cases} u^j(x, 0) = u_0^j(x), u_t^j(x, 0) = u_1^j(x), \forall j \in \mathcal{I}(\mathcal{N}), \\ y^j(x, 0) = y_0^j(x), y_t^j(x, 0) = y_1^j(x), \forall j \in \mathcal{I}(\mathcal{N}), \\ \theta^j(x, 0) = \theta_0^j(x), \forall j \in \mathcal{I}_{te}(\mathcal{N}). \end{cases} \quad (3.1.9)$$

We denote by  $V'_{\text{ext}}$  (resp.  $V'_{\text{int}}$ ) the set of exterior (resp. interior) nodes of maximal subgraph of thermo-elastic edges. Then, the boundary condition on  $\mathcal{N}$  are described as follows:

The displacement and temperature satisfies the Dirichlet boundary condition,

$$\begin{cases} u^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{\text{ext}}, \\ y^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), a_k \in V_{\text{ext}}, \\ \theta^j(a_k, t) = 0, & j \in \mathcal{I}_{te}(a_k), a_k \in V'_{\text{ext}}. \end{cases} \quad (3.1.10)$$

The displacement and temperature are continuous,

$$\begin{cases} u^j(a_k, t) = u^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{\text{int}}, \\ y^j(a_k, t) = y^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), a_k \in V_{\text{int}}, \\ \theta^j(a_k, t) = \theta^\ell(a_k, t), & j, \ell \in \mathcal{I}_{te}(a_k), a_k \in V'_{\text{int}}. \end{cases} \quad (3.1.11)$$

The system satisfies the balance condition on  $y$  at every interior node,

$$\sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k, t) = 0, \quad a_k \in V_{\text{int}}. \quad (3.1.12)$$

The system satisfies the following balance conditions on  $u$  and  $\theta$ ,

$$\begin{cases} \sum_{j \in \mathcal{I}_{te}(a_k)} d_{kj} \kappa_j \theta_x^j(a_k, t) = 0, & a_k \in V'_{\text{int}}, \\ \sum_{j \in \mathcal{I}_{te}(a_k)} d_{kj} (u_x^j(a_k, t) - \alpha_j \theta^j(a_k, t)) + \sum_{j \in \mathcal{I}_e(a_k)} d_{kj} u_x^j(a_k, t) = 0, & a_k \in V_{\text{int}}. \end{cases} \quad (3.1.13)$$

Remark that  $\alpha_j > 0$  and  $\kappa_j > 0$ , for all  $j \in \mathcal{I}_{te}$  while, on each elastic edge only two conservative wave equations hold, i.e, the two wave equations on each elastic edge are neither coupled to a heat equation nor affected by a thermal damping. Hence for  $j \in \mathcal{I}_e$ , we may set  $\alpha_j = \kappa_j = 0$ .

This chapter is organized as follows. In Section 3.2, we prove that system (3.1.7)-(3.1.13) admits a unique solution in an appropriate Hilbert space using semi-group theory. Next, in Section 3.3, using a general criteria of Arendt and Batty, 1988, we discuss the strong stability of the system. In Section 3.4, under the condition that the two waves propagate with the same speed on each thermo-elastic edge of the network, we prove the exponential stability of the system using a frequency domain approach combined with a multiplier technique. Otherwise, we establish



a polynomial decay. Finally, in Section 3.5, we present the Neumann boundary condition at the interior nodes of some particular networks, some of which being considered in Shel, 2012. We show that under some sufficient conditions, the same results as the ones from Section 3.4 hold.

### 3.2 Well-posedness

In this section, we will study the existence, uniqueness and regularity of the solution of system (3.1.7)-(3.1.13), using a semigroup approach.

First, denote by

$$\mathbb{L}^2 = \prod_{j=1}^N L^2(0, \ell_j), \quad \mathbb{H}^m = \prod_{j=1}^N H^m(0, \ell_j), \quad m = 1, 2,$$

and

$$\mathbb{V} = \prod_{j \in \mathcal{I}_{te}} L^2(0, \ell_j), \quad \mathbb{V}^m = \prod_{j \in \mathcal{I}_{te}} H^m(0, \ell_j), \quad m = 1, 2.$$

Set

$$\begin{aligned} \mathbb{H}_0^1 = \{ & u = (u^j)_{j \in \mathcal{I}(\mathcal{N})} \in \mathbb{H}^1 / u^j(a_k) = 0, \forall j \in \mathcal{I}(a_k), a_k \in V_{\text{ext}} \\ & \text{and } u^j(a_k) = u^\ell(a_k), \forall j, \ell \in \mathcal{I}(a_k), a_k \in V_{\text{int}} \}. \end{aligned} \quad (3.2.1)$$

We define the energy space  $\mathcal{H}$  associated with system (3.1.7)-(3.1.13), by

$$\mathcal{H} = \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{V} \quad (3.2.2)$$

equipped with the following inner product:

$$\begin{aligned} (U, \tilde{U})_{\mathcal{H}} = \sum_{j=1}^N \int_0^{\ell_j} ( & u_x^j \tilde{u}_x^j + v^j \tilde{v}^j + \rho_j y_x^j \tilde{y}_x^j + z^j \tilde{z}^j) dx \\ & + \sum_{j \in \mathcal{I}_{te}} \int_0^{\ell_j} \theta^j \tilde{\theta}^j dx, \end{aligned} \quad (3.2.3)$$

for all  $U = (u, v, y, z, \theta)$ ,  $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\theta}) \in \mathcal{H}$ . Next, we define the unbounded linear operator  $\mathcal{A}$  associated to system (3.1.7)-(3.1.13) by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ y \\ z \\ \theta \end{pmatrix} = \left( \begin{pmatrix} v^j \\ u_{xx}^j - \alpha_j \theta_x^j + \beta_j z^j \\ z^j \\ \rho_j y_{xx}^j - \beta_j v^j \\ \kappa_j \theta_{xx}^j - \alpha_j v_x^j \end{pmatrix} \right)_{j \in \mathcal{I}(\mathcal{N})} \quad (3.2.4)$$

whose domain  $D(\mathcal{A})$  is given by

$$D(\mathcal{A}) = \{ (u, v, y, z, \theta) \in \mathcal{H} \cap [\mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{V}^2] \text{ satisfying (3.2.5) below} \},$$

$$\left\{ \begin{array}{l} \theta^j(a_k) = 0, \quad j \in \mathcal{I}_{\text{te}}(a_k), \quad a_k \in V'_{\text{ext}}, \\ \theta^\ell(a_k) = \theta^\ell(a_k), \quad j, \ell \in \mathcal{I}_{\text{te}}(a_k), \quad a_k \in V'_{\text{int}}, \\ \sum_{j \in \mathcal{I}_{\text{te}}(a_k)} d_{kj} \kappa_j \theta_x^j(a_k) = 0, \quad a_k \in V'_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} (u_x^j(a_k) - \alpha_j \theta^j(a_k)) = 0, \quad a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, \quad a_k \in V_{\text{int}}. \end{array} \right. \quad (3.2.5)$$

If  $(u, y, \theta)$  is a regular solution of (3.1.7)-(3.1.9), then by setting  $U = (u, u_t, y, y_t, \theta)$ , we can rewrite this system as the following evolution equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (3.2.6)$$

where  $U_0 = (u_0, u_1, y_0, y_1, \theta_0)$ .

We recall that the energy associated with system (3.1.7)-(3.1.13) is given by

$$E(t) = \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (|u_x^j|^2 + |u_t^j|^2 + \rho_j |y_x^j|^2 + |y_t^j|^2) dx + \frac{1}{2} \sum_{j \in \mathcal{I}_{\text{te}}} \int_0^{\ell_j} |\theta_j|^2 dx \quad (3.2.7)$$

and we will see that

$$E'(t) = - \sum_{j \in \mathcal{I}_{\text{te}}} \int_0^{\ell_j} \kappa_j |\theta_x^j|^2 dx, \quad (3.2.8)$$

for regular solutions. Hence, the system is dissipative in the sense that its energy is non-increasing.

**Theorem 3.2.1.** *The unbounded linear operator  $\mathcal{A}$  associated with system (3.1.7)-(3.1.13) generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ .*

*Proof.* Using Lumer-Phillips Theorem (see Pazy, 1983), it is sufficient to prove that  $\mathcal{A}$  is a maximal dissipative operator so that  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ . First, let  $U = (u, v, y, z, \theta) \in D(\mathcal{A})$ . We have,

$$\begin{aligned} \text{Re}(\mathcal{A}U, U)_{\mathcal{H}} &= \text{Re} \left[ \sum_{j=1}^N \int_0^{\ell_j} v_x^j \overline{u_x^j} dx + \sum_{j=1}^N \int_0^{\ell_j} (u_{xx}^j - \alpha_j \theta_x^j + \beta_j z^j) \overline{v} dx \right. \\ &\quad \left. + \sum_{j=1}^N \int_0^{\ell_j} \rho_j z_x^j \overline{y_x^j} dx + \sum_{j=1}^N \int_0^{\ell_j} (\rho_j y_{xx}^j - \beta_j v^j) \overline{z} dx + \sum_{j \in \mathcal{I}_{\text{te}}} \int_0^{\ell_j} (\kappa_j \theta_{xx}^j - \alpha_j v_x^j) \overline{\theta^j} dx \right]. \end{aligned} \quad (3.2.9)$$

Using Green's formula, boundary and transmission conditions (3.1.10)-(3.1.13), we get

$$\text{Re}(\mathcal{A}U, U)_{\mathcal{H}} = - \sum_{j \in \mathcal{I}_{\text{te}}} \int_0^{\ell_j} \kappa_j |\theta_x^j|^2 dx \leq 0. \quad (3.2.10)$$

Thus, the operator  $\mathcal{A}$  is dissipative. Now, in order to prove that  $\mathcal{A}$  is maximal it is sufficient to show that  $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$ . So, for  $F = (f, \tilde{f}, g, \tilde{g}, h) \in \mathcal{H}$ , we look for  $U \in D(\mathcal{A})$  such that

$$(I - \mathcal{A})U = F. \quad (3.2.11)$$

Equivalently, for all  $j \in \mathcal{I}(\mathcal{N})$ ,

$$u^j - v^j = f^j, \quad (3.2.12)$$

$$v^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j z^j = \tilde{f}^j, \quad (3.2.13)$$

$$y^j - z^j = g^j, \quad (3.2.14)$$

$$z^j - \rho_j y_{xx}^j + \beta_j v^j = \tilde{g}^j, \quad (3.2.15)$$

$$\theta^j - \kappa_j \theta_{xx}^j + \alpha_j v_x^j = h^j. \quad (3.2.16)$$

Assume that  $U \in D(\mathcal{A})$  exists, then by using equation (3.2.12) and (3.2.14) we obtain for all  $j \in \mathcal{I}(\mathcal{N})$ ,

$$v^j = u^j - f^j, \quad (3.2.17)$$

$$z^j = y^j - g^j. \quad (3.2.18)$$

Inserting (3.2.17)-(3.2.18) in equations (3.2.13), (3.2.15) and (3.2.16), we get the following system for all  $j \in \mathcal{I}(\mathcal{N})$ ,

$$u^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j y^j = F_1^j, \quad (3.2.19)$$

$$y^j - \rho_j y_{xx}^j + \beta_j u^j = F_2^j, \quad (3.2.20)$$

$$\theta^j - \kappa_j \theta_{xx}^j + \alpha_j u_x^j = F_3^j \quad (3.2.21)$$

where,  $F_1^j = \tilde{f}^j + f^j - \beta_j g^j$ ,  $F_2^j = \tilde{g}^j + g^j + \beta_j f^j$ ,  $F_3^j = h^j + \alpha_j f_x^j$ .

Set

$$\begin{aligned} \mathcal{X} = \{ & (\varphi_1, \varphi_2, \varphi_3) \in \mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{V}^1 / \varphi_3^j(a_k) = 0, j \in \mathcal{I}_{\text{te}}(a_k), a_k \in V'_{\text{ext}} \\ & \text{and } \varphi_3^j(a_k, t) = \varphi_3^\ell(a_k, t), j, \ell \in \mathcal{I}_{\text{te}}(a_k), a_k \in V'_{\text{int}} \}. \end{aligned} \quad (3.2.22)$$

Let  $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}$ , multiply (3.2.19) by  $\overline{\varphi_1^j}$ , (3.2.20) by  $\overline{\varphi_2^j}$  and (3.2.21) by  $\overline{\varphi_3^j}$ , then integrate over  $(0, \ell_j)$  we get,

$$\begin{aligned} \int_0^{\ell_j} u^j \overline{\varphi_1^j} dx - \int_0^{\ell_j} u_{xx}^j \overline{\varphi_1^j} dx + \int_0^{\ell_j} \alpha_j \theta_x^j \overline{\varphi_1^j} dx - \int_0^{\ell_j} \beta_j y^j \overline{\varphi_1^j} dx \\ = \int_0^{\ell_j} F_1^j \overline{\varphi_1^j} dx, \end{aligned} \quad (3.2.23)$$

$$\int_0^{\ell_j} y^j \overline{\varphi_2^j} dx - \int_0^{\ell_j} \rho_j y_{xx}^j \overline{\varphi_2^j} dx + \int_0^{\ell_j} \beta_j u^j \overline{\varphi_2^j} dx = \int_0^{\ell_j} F_2^j \overline{\varphi_2^j} dx, \quad (3.2.24)$$

and

$$\int_0^{\ell_j} \theta^j \overline{\varphi_3^j} dx - \int_0^{\ell_j} \kappa_j \theta_{xx}^j \overline{\varphi_3^j} dx + \int_0^{\ell_j} \alpha_j u_x^j \overline{\varphi_3^j} dx = \int_0^{\ell_j} F_3^j \overline{\varphi_3^j} dx. \quad (3.2.25)$$

Applying Green's formula on the second and third term of (3.2.23) and taking the sum over  $\mathcal{I}(\mathcal{N})$ , we obtain using (3.1.13),

$$\begin{aligned} \sum_{j=1}^N \int_0^{\ell_j} u^j \overline{\varphi_1^j} dx + \sum_{j=1}^N \int_0^{\ell_j} u_x^j \overline{\varphi_{1,x}^j} dx - \sum_{j=1}^N \int_0^{\ell_j} \alpha_j \theta^j \overline{\varphi_{1,x}^j} dx \\ - \sum_{j=1}^N \int_0^{\ell_j} \beta_j y^j \overline{\varphi_1^j} dx = \sum_{j=1}^N \int_0^{\ell_j} F_1^j \overline{\varphi_1^j} dx. \end{aligned} \quad (3.2.26)$$

Again applying Green's formula on the second term of (3.2.24) and taking the sum over  $\mathcal{I}(\mathcal{N})$ , using (3.1.12), we get

$$\begin{aligned} \sum_{j=1}^N \int_0^{\ell_j} y^j \overline{\varphi_2^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \rho_j y_x^j \overline{\varphi_{2,x}^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \beta_j u^j \overline{\varphi_2^j} dx \\ = \sum_{j=1}^N \int_0^{\ell_j} F_2^j \overline{\varphi_2^j} dx. \end{aligned} \quad (3.2.27)$$

Similarly, applying Green's formula on the second term of (3.2.25) and taking the sum over  $\mathcal{I}(\mathcal{N})$ , condition (3.1.13) yields that,

$$\begin{aligned} \sum_{j=1}^N \int_0^{\ell_j} \theta^j \overline{\varphi_3^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \kappa_j \theta_x^j \overline{\varphi_{3,x}^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \alpha_j u_x^j \overline{\varphi_3^j} dx \\ = \sum_{j=1}^N \int_0^{\ell_j} F_3^j \overline{\varphi_3^j} dx. \end{aligned} \quad (3.2.28)$$

Adding equations (3.2.26), (3.2.27) and (3.2.28) we obtain

$$a((u, y, \theta), (\varphi_1, \varphi_2, \varphi_3)) = L(\varphi_1, \varphi_2, \varphi_3), \quad \forall (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}, \quad (3.2.29)$$

where,

$$\begin{aligned} a((u, y, \theta), (\varphi_1, \varphi_2, \varphi_3)) = \sum_{j=1}^N \int_0^{\ell_j} u^j \overline{\varphi_1^j} dx + \sum_{j=1}^N \int_0^{\ell_j} u_x^j \overline{\varphi_{1,x}^j} dx - \sum_{j=1}^N \int_0^{\ell_j} \alpha_j \theta^j \overline{\varphi_{1,x}^j} dx \\ - \sum_{j=1}^N \int_0^{\ell_j} \beta_j y^j \overline{\varphi_1^j} dx + \sum_{j=1}^N \int_0^{\ell_j} y^j \overline{\varphi_2^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \rho_j y_x^j \overline{\varphi_{2,x}^j} dx \\ + \sum_{j=1}^N \int_0^{\ell_j} \beta_j u^j \overline{\varphi_2^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \theta^j \overline{\varphi_3^j} dx + \sum_{j=1}^N \int_0^{\ell_j} \kappa_j \theta_x^j \overline{\varphi_{3,x}^j} dx \\ + \sum_{j=1}^N \int_0^{\ell_j} \alpha_j u_x^j \overline{\varphi_3^j} dx \end{aligned}$$

and

$$L(\varphi_1, \varphi_2, \varphi_3) = \sum_{j=1}^N \int_0^{\ell_j} F_1^j \overline{\varphi_1^j} dx + \sum_{j=1}^N \int_0^{\ell_j} F_2^j \overline{\varphi_2^j} dx + \sum_{j=1}^N \int_0^{\ell_j} F_3^j \overline{\varphi_3^j} dx.$$

As  $a$  is a continuous, coercive form on  $\mathcal{X} \times \mathcal{X}$  and  $L$  is a continuous form on  $\mathcal{X}$ , then using Lax-Milgram Theorem there exists a unique solution  $(u, y, \theta) \in \mathcal{X}$  of (3.2.29). Now, take in (3.2.29) the test function  $(\varphi_1, 0, 0)$  such that  $\varphi_1^j \in C_c^\infty(0, \ell_j)$ , for some

fixed  $j \in \mathcal{I}(\mathcal{N})$  and  $\varphi_1^k = 0$  for all  $k \neq j$ , we obtain

$$\begin{aligned} & \int_0^{\ell_j} u^j \overline{\varphi_1^j} dx + \int_0^{\ell_j} u_x^j \overline{\varphi_{1,x}^j} dx - \int_0^{\ell_j} \alpha_j \theta^j \overline{\varphi_{1,x}^j} dx \\ & - \int_0^{\ell_j} \beta_j y^j \overline{\varphi_1^j} dx = \int_0^{\ell_j} F_1^j \overline{\varphi_1^j} dx, \quad \forall \varphi_1^j \in C_c^\infty(0, \ell_j) \text{ for a fixed } j. \end{aligned} \quad (3.2.30)$$

Applying Green's formula on the second and third term of (3.2.31) we get

$$\begin{aligned} & \int_0^{\ell_j} u^j \overline{\varphi_1^j} dx - \int_0^{\ell_j} u_{xx}^j \overline{\varphi_1^j} dx + \int_0^{\ell_j} \alpha_j \theta_x^j \overline{\varphi_1^j} dx - \int_0^{\ell_j} \beta_j y^j \overline{\varphi_1^j} dx \\ & = \int_0^{\ell_j} F_1^j \overline{\varphi_1^j} dx, \quad \forall \varphi_1^j \in C_c^\infty(0, \ell_j) \text{ for a fixed } j. \end{aligned} \quad (3.2.31)$$

This implies that

$$u^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j y^j = F_1^j, \quad \text{in } \mathcal{D}'(0, \ell_j)$$

where,  $\mathcal{D}'(0, \ell_j)$  is the associated space of distributions.

As  $u^j + \alpha_j \theta_x^j - \beta_j y^j - F_1^j \in L^2(0, \ell_j)$ , we deduce that  $u^j \in H^2(0, \ell_j)$ . Similarly, we can prove that

$$\begin{aligned} y^j - \rho_j y_{xx}^j + \beta_j u^j &= F_2^j, \\ \theta^j - \kappa_j \theta_{xx}^j + \alpha_j u_x^j &= F_3^j \end{aligned} \quad (3.2.32)$$

and  $y^j, \theta^j \in H^2(0, \ell_j)$ . Now, it remains to prove the transmission conditions in (3.1.12)-(3.1.13). For that aim, fix  $a_k \in V_{\text{int}}$ . Let,

$$\varphi_1^j = \begin{cases} \frac{x}{\ell_j}, & \text{if } j \in \mathcal{I}(a_k) \text{ and } \pi_j(\ell_j) = a_k, \\ \frac{\ell_j - x}{\ell_j}, & \text{if } j \in \mathcal{I}(a_k) \text{ and } \pi_j(0) = a_k, \\ 0, & \text{if } j \notin \mathcal{I}(a_k). \end{cases} \quad (3.2.33)$$

Then, take in (3.2.29), a test function  $(\varphi_1, 0, 0) \in \mathcal{X}$ , apply Green's formula and take into account (3.2.19)-(3.2.21), to get

$$\sum_{j \in \mathcal{I}(a_k)} d_{kj} u_x^j(a_k) - \sum_{j \in \mathcal{I}(a_k)} d_{kj} \alpha_j \theta^j(a_k) = 0. \quad (3.2.34)$$

Similarly, by taking in (3.2.29) the test function  $(0, \varphi_1, 0) \in \mathcal{X}$  then, using Green's formula and taking into account (3.2.19)-(3.2.21) we obtain (3.1.12). Finally, we fix  $a_k \in V'_{\text{int}}$ , take  $(0, 0, \varphi_1)$  in (3.2.29), apply Green's formula and take into account (3.2.19)-(3.2.21), we get (3.1.13). By defining  $v^j$  by (3.2.17) and  $z^j$  by (3.2.18), for all  $j \in \mathcal{I}(\mathcal{N})$ , we deduce that  $(u, v, y, z, \theta) \in D(\mathcal{A})$  is a solution of (3.2.12)-(3.2.16) exists and the desired goal is attained.  $\square$

As  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  (see Pazy, 1983), we have the following result:

**Theorem 3.2.2.** (*Existence and uniqueness of the solution*)

(1) If  $U_0 = (u_0, u_1, y_0, y_1, \theta_0) \in D(\mathcal{A})$ , then problem (3.2.6) admits a strong unique solution  $U = (u, v, y, z, \theta)$  satisfying

$$U \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C^0(\mathbb{R}_+, D(\mathcal{A})).$$

(2) If  $U_0 = (u_0, u_1, y_0, y_1, \theta_0) \in \mathcal{H}$ , then problem (3.2.6) admits a unique weak solution  $U = (u, v, y, z, \theta)$  satisfying

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

### 3.3 Strong stability

In this section, we will give sufficient conditions that guarantee the strong stability of the system (3.1.7)-(3.1.13) in the sense that the energy  $E(t)$ , of the associated system decreases to zero as  $t$  tends to infinity. To show the strong stability of the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  we will rely on the following result obtained by Arendt and Batty, 1988.

Now, we are in position to state the main result of this section.

**Theorem 3.3.1.** *Consider the system (3.1.7)-(3.1.13) on  $\mathcal{N}$ . Assume additionally that one of the following conditions holds,*

1) *Each maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to  $V_{\text{ext}}$ .*

2) *There exists a maximal subgraph of thermo-elastic edges with no exterior vertices that belong to  $V_{\text{ext}}$  and  $\beta_j = \beta$ , for all  $j \in \mathcal{I}(\mathcal{N})$ .*

Then,

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (S1)$$

and therefore  $\lim_{t \rightarrow \infty} E(t) \rightarrow 0$ .

*Proof.* Using Sobolev embedding Theorem, we deduce that  $(I - \mathcal{A})^{-1}$  is a compact operator. Then, the spectrum  $\sigma(\mathcal{A})$  of  $\mathcal{A}$  is reduced to its discrete spectrum  $\sigma_p(\mathcal{A})$ . Hence, using Arendt and Batty, 1988, it is sufficient to prove that  $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ , since it implies that (S1) holds. Let  $\lambda \in \mathbb{R}$  and  $U = (u, v, y, z, \theta) \in D(\mathcal{A})$  be such that

$$\mathcal{A}U = i\lambda U,$$

equivalently, for all  $j \in \mathcal{I}(\mathcal{N})$  we have,

$$v^j = i\lambda u^j, \quad (3.3.1)$$

$$u_{xx}^j - \alpha_j \theta_x^j + \beta_j z^j = i\lambda v^j, \quad (3.3.2)$$

$$z^j = i\lambda y^j, \quad (3.3.3)$$

$$\rho_j y_{xx}^j - \beta_j v^j = i\lambda z^j, \quad (3.3.4)$$

$$\kappa_j \theta_{xx}^j - \alpha_j v_x^j = i\lambda \theta^j. \quad (3.3.5)$$

Eliminating  $v^j$  (resp.  $z^j$ ) using (3.3.1) (resp. (3.3.3)) and inserting them in (3.3.2), (3.3.4) and (3.3.5) we get the following system for all  $j \in \mathcal{I}(\mathcal{N})$ ,

$$\lambda^2 u^j + u_{xx}^j - \alpha_j \theta_x^j + i\lambda \beta_j y^j = 0, \quad (3.3.6)$$

$$\lambda^2 y^j + \rho_j y_{xx}^j - i\lambda \beta_j u^j = 0, \quad (3.3.7)$$

$$\kappa_j \theta_{xx}^j - i\lambda \alpha_j u_x^j - i\lambda \theta^j = 0. \quad (3.3.8)$$

Since we have

$$\sum_{j \in \mathcal{I}_{te}} \kappa_j \int_0^{\ell_j} |\theta_x^j|^2 dx = \operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = \operatorname{Re}(i\lambda U, U)_{\mathcal{H}} = 0,$$

we deduce that

$$\theta_x^j = 0, \quad \forall j \in \mathcal{I}_{te}. \quad (3.3.9)$$

Thus,  $\theta^j$  is constant for all  $j \in \mathcal{I}_{te}$ . But, using the fact that every maximal subgraph of thermo-elastic edges is not a circuit and using (3.1.10) and (3.1.11), we deduce that

$$\theta^j = 0, \quad \forall j \in \mathcal{I}_{te}. \quad (3.3.10)$$

Suppose that  $\lambda = 0$ . Then, (3.3.6), (3.3.7) and (3.3.9) implies that

$$\begin{cases} u_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ \rho_j y_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}). \end{cases} \quad (3.3.11)$$

Multiplying the first equation and second equation of (3.3.11) by  $\overline{u^j}$  and  $\overline{y^j}$ , respectively. Then, integrating over  $(0, \ell_j)$ , summing over  $j \in \mathcal{I}(\mathcal{N})$  and applying Green's formula, we get

$$\begin{cases} \sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 dx - \sum_{j=1}^N u_x^j \overline{u^j} \Big|_0^{\ell_j} = 0, \\ \sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 dx - \sum_{j=1}^N \rho_j y_x^j \overline{y^j} \Big|_0^{\ell_j} = 0. \end{cases} \quad (3.3.12)$$

But using (3.3.10) and the boundary conditions (3.1.10)-(3.1.13), the boundary terms are zero, hence (3.3.12) becomes

$$\begin{cases} \sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 dx = 0, \\ \sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 dx = 0. \end{cases} \quad (3.3.13)$$

By the fact that  $\rho_j > 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ , we obtain that  $u_x^j = y_x^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ . Again, by (3.1.10), (3.1.11) and using the fact that  $V_{\text{ext}} \neq \emptyset$ , we deduce that  $u^j = y^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ . Consequently, using (3.3.1) and (3.3.3), we conclude that  $v^j = z^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$  and therefore,  $U = 0$ .

Now, suppose that  $\lambda \neq 0$ . We will distinguish between two cases.

**Case 1.** Assume that each maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to  $V_{\text{ext}}$ . Using (3.3.10) and (3.3.8), we have

$$u_x^j = 0, \forall j \in \mathcal{I}_{\text{te}}. \quad (3.3.14)$$

This means that  $u^j$  is constant for all  $j \in \mathcal{I}_{\text{te}}$ . But, using (3.1.10), (3.1.11) and the fact that every maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to  $V_{\text{ext}}$ , we deduce that  $u^j = 0$ , for all  $j \in \mathcal{I}_{\text{te}}$ . Thus, by (3.3.1), we have  $v^j = 0$  and by (3.3.6), (3.3.14) and (3.3.9), we obtain that  $y^j = 0$ , for all  $j \in \mathcal{I}_{\text{te}}$ . Consequently, by (3.3.3), we get  $z^j = 0$ , for all  $j \in \mathcal{I}_{\text{te}}$ . Hence,  $u^j = v^j = y^j = z^j = \theta^j = 0$  on both ends of  $e_j$ , for all  $j \in \mathcal{I}_{\text{te}}$ .

Now, let  $e_j$  be an elastic edge attached only to thermo-elastic edge. As  $e_j$  is identified by  $[0, \ell_j]$ , assume that  $\ell_j$  is the extremity in common with the thermo-elastic edge. Then, using (3.1.11), (3.3.10), (3.1.12) and (3.1.13), we have the following system

$$\begin{cases} \lambda^2 u^j + u_{xx}^j + i\lambda\beta_j y^j &= 0, \\ \lambda^2 y^j + \rho_j y_{xx}^j - i\lambda\beta_j u^j &= 0, \\ u^j(\ell_j) = y^j(\ell_j) &= 0, \\ u_x^j(\ell_j) = y_x^j(\ell_j) &= 0. \end{cases} \quad (3.3.15)$$

Let

$$\tilde{u}^j = \begin{cases} u^j, & \text{on } (0, \ell_j), \\ 0, & \text{on } (\ell_j, \ell_j + 1), \end{cases}$$

and

$$\tilde{y}^j = \begin{cases} y^j, & \text{on } (0, \ell_j), \\ 0, & \text{on } (\ell_j, \ell_j + 1). \end{cases}$$

Then, using the boundary conditions of (3.3.15), we deduce that  $(\tilde{u}^j, \tilde{y}^j)$  belongs to  $H^2(0, \ell_j + 1) \times H^2(0, \ell_j + 1)$  and satisfies the first two equations of (3.3.15). Consequently, using Theorem 2.5 of Hayek et al., 2020, we deduce that  $\tilde{u}^j = \tilde{y}^j = 0$  on  $(0, \ell_j + 1)$  and hence,  $u^j = y^j = 0$  on  $(0, \ell_j)$ . Then,  $v^j = z^j = 0$  by equation (3.3.1) and (3.3.3) respectively. We repeat this technique to every elastic edge connected only to thermo-elastic edges and we proceed by iteration the same method on each maximal subgraph of purely elastic edges (from the leaves to the root), so that  $u^j = v^j = y^j = z^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ .

**Case 2.** Assume that there exists a maximal subgraph of thermo-elastic edges with no exterior vertices that belong to  $V_{\text{ext}}$  and  $\beta_j = \beta$ , for all  $j \in \mathcal{I}(\mathcal{N})$ . First, notice that (3.3.14) holds and thus,

$$u_{xx}^j = 0, \forall j \in \mathcal{I}_{\text{te}}. \quad (3.3.16)$$

Then, using (3.3.16), (3.3.9) and the fact that  $\lambda \neq 0, \beta_j = \beta$ , equation (3.3.6) becomes

$$\lambda u^j + i\beta y^j = 0, \forall j \in \mathcal{I}_{\text{te}}. \quad (3.3.17)$$

Differentiating (3.3.17) twice with respect to  $x$  and using (3.3.16), we deduce that

$$y_{xx}^j = 0, \forall j \in \mathcal{I}_{\text{te}}. \quad (3.3.18)$$



Then, using (3.3.18) and as  $\lambda \neq 0, \beta_j = \beta$ , (3.3.7) becomes

$$\lambda y^j - i\beta u^j = 0, \forall j \in \mathcal{I}_{te}. \quad (3.3.19)$$

Eliminating  $u^j$  from (3.3.17) and replacing it in (3.3.19) we obtain

$$(\lambda^2 - \beta^2)y^j = 0, \forall j \in \mathcal{I}_{te}. \quad (3.3.20)$$

Then, for  $\lambda \neq \pm\beta$  we deduce that  $y^j = 0$ , for all  $j \in \mathcal{I}_{te}$  and thus by equation (3.3.19) we get  $u^j = 0$ , for all  $j \in \mathcal{I}_{te}$ . Again, we proceed using unique continuation Theorem from Hayek et al., 2020 and iteration technique used in **Case 1** to conclude that  $u^j = v^j = y^j = z^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ .

On the other hand, if  $\lambda = \pm\beta$ . Without loss of generality, assume that  $\lambda = \beta$ . First, using (3.3.19), we have

$$y^\ell = iu^\ell, \forall \ell \in \mathcal{I}_{te} \quad (3.3.21)$$

and thus using (3.3.14), (3.3.21) implies that

$$y_x^\ell = iu_x^\ell = 0, \forall \ell \in \mathcal{I}_{te}. \quad (3.3.22)$$

Our aim is to prove that

$$\begin{cases} u_{xx}^j = 0, \forall j \in \mathcal{I}(\mathcal{N}), \\ y_{xx}^j = 0, \forall j \in \mathcal{I}(\mathcal{N}). \end{cases} \quad (3.3.23)$$

This would end the proof as in the case  $\lambda = 0$ . As (3.3.16) and (3.3.18) hold, it is enough to prove that (3.3.23) holds for each elastic edge. Let  $e_j$  be an elastic edge attached to a thermo-elastic edge at the vertex  $a_k$ . As  $\lambda = \beta$ , then (3.3.6) and (3.3.7) lead to

$$\beta^2 u^j + u_{xx}^j + i\beta^2 y^j = 0, \quad (3.3.24)$$

$$\beta^2 y^j + \rho_j y_{xx}^j - i\beta^2 u^j = 0. \quad (3.3.25)$$

By eliminating  $u^j$  from (3.3.24) and inserting it in (3.3.25), we obtain the following equation

$$y_{xxxx}^j + \beta^2 \frac{(\rho_j + 1)}{\rho_j} y_{xx}^j = 0. \quad (3.3.26)$$

Moreover, using (3.3.25), (3.1.11) and (3.3.21), we have

$$y_{xx}^j(a_k) = 0 \quad (3.3.27)$$

and using (3.3.25), (3.1.12), (3.1.13), (3.3.10) and (3.3.22), we get

$$y_{xxx}^j(a_k) = 0. \quad (3.3.28)$$

Consequently, by setting  $Z^j = y_{xx}^j$  and using (3.3.24)-(3.3.25), (3.3.27) and (3.3.28) we have the following system

$$\begin{cases} Z_{xx}^j + \beta^2 \frac{(\rho_j + 1)}{\rho_j} Z^j = 0, \\ Z^j(a_k) = 0, \\ Z_x^j(a_k) = 0. \end{cases} \quad (3.3.29)$$

Therefore,  $Z^j = 0$  and then  $y_{xx}^j = 0$ . This means that  $y_x^j$  is constant. But using (3.1.12) and (3.3.22), we deduce that  $y_x^j(a_k) = 0$ . Hence,  $y_x^j = 0$ . Therefore, using (3.3.25), we obtain that  $y^j = iu^j$  and then  $y_x^j = iu_x^j = 0$ . Again, by iteration on each maximal subgraph of purely elastic edges (from the leaves to the root), we repeat the same procedure to prove that (3.3.23) holds. Whenever (3.3.23) is attained, we can proceed as the case  $\lambda = 0$  which finishes the proof. The same procedure can be used in the case  $\lambda = -\beta$ .  $\square$

Let us finish this section by introducing some notations that will be used in the next section. let  $\mathcal{I}'_{\text{ext}}$  denotes the set of indices of edges adjacent to a vertex in  $V'_{\text{ext}}$  and  $\mathcal{G}'_{\text{int}}$  denotes the set of indices of edges adjacent to two vertices in  $V'_{\text{int}}$ .

### 3.4 Energy decay estimates

Take an arbitrary network  $\mathcal{N}$  for which the System (3.1.7)-(3.1.13) is stable. In this section, we will prove that under the condition that the two coupled wave equations propagate with the same speed on each thermo-elastic edge, i.e.,  $\rho_j = 1$ , for all  $j \in \mathcal{I}_{\text{te}}$ , and using a frequency domain approach combined with a multiplier method, the energy of the system decays exponentially to zero. Otherwise, if there exist  $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$  such that  $\rho_j \neq 1$ , we prove a polynomial decay rate of type  $t^{-1/3}$ , (see Borichev and Tomilov, 2009; Prüss, 1984). The main results are presented in Theorem 3.4.1 and Theorem 3.4.13.

**Theorem 3.4.1.** *Let  $\mathcal{N}$  be an arbitrary network for which the operator  $\mathcal{A}$  associated with System (3.1.7)-(3.1.13) satisfies (S1). If  $\rho_j = 1$ , for all  $j \in \mathcal{I}_{\text{te}}$ , then the energy of the system decays exponentially in  $\mathcal{H}$ . In other words, there exist two positive constants  $M$  and  $\epsilon$  such that*

$$\|e^{t\mathcal{A}}x_0\|_{\mathcal{H}} \leq Me^{-\epsilon t}\|x_0\|_{\mathcal{H}}, \quad \forall t > 0, \forall x_0 \in \mathcal{H}.$$

*Proof.* Following Huang, 1985 and Prüss, 1984, a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$  is exponentially stable if and only if (S1) and

$$\limsup_{|\lambda| \rightarrow \infty} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad (\text{S2})$$

hold. As we have assumed that (S1) is satisfied, it remains to prove that condition (S2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S2) does not hold, then there exist a sequence of real numbers  $\lambda_n \in \mathbb{R}$  and a sequence of vectors  $U_n = (u_n, v_n, y_n, z_n, \theta_n) \in D(\mathcal{A})$  such that

$$|\lambda_n| \longrightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n, \theta_n)\|_{\mathcal{H}} = 1, \quad (3.4.1)$$

and

$$(i\lambda_n - \mathcal{A})U_n = (f_n, \tilde{f}_n, g_n, \tilde{g}_n, h_n) \longrightarrow 0 \quad \text{in } \mathcal{H}, \quad (3.4.2)$$

are satisfied.

In what follows, we drop the index  $n$  for simplicity.

Now by detailing (3.4.2), we get for all  $j \in \mathcal{I}(\mathcal{N})$

$$i\lambda u^j - v^j = f^j \rightarrow 0 \text{ in } H^1(0, \ell_j), \quad (3.4.3)$$

$$i\lambda v^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j z^j = \tilde{f}^j \rightarrow 0 \text{ in } L^2(0, \ell_j), \quad (3.4.4)$$

$$i\lambda y^j - z^j = g^j \rightarrow 0 \text{ in } H^1(0, \ell_j), \quad (3.4.5)$$

$$i\lambda z^j - \rho_j y_{xx}^j + \beta_j v^j = \tilde{g}^j \rightarrow 0 \text{ in } L^2(0, \ell_j), \quad (3.4.6)$$

$$i\lambda \theta^j - \kappa_j \theta_{xx}^j + \alpha_j v_x^j = h^j \rightarrow 0 \text{ in } L^2(0, \ell_j). \quad (3.4.7)$$

Then, by eliminating  $v^j$  and  $z^j$  from equations (3.4.3) and (3.4.5) respectively, (3.4.3)-(3.4.7) imply

$$\lambda^2 u^j + u_{xx}^j - \alpha_j \theta_x^j + i\lambda \beta_j y^j = \beta_j g^j - \tilde{f}^j - i\lambda f^j, \quad (3.4.8)$$

$$\lambda^2 y^j + \rho_j y_{xx}^j - i\lambda \beta_j u^j = -\beta_j f^j - \tilde{g}^j - i\lambda g^j, \quad (3.4.9)$$

$$i\lambda \theta^j - \kappa_j \theta_{xx}^j + i\lambda \alpha_j u_x^j = h^j + \alpha_j f_x^j, \quad (3.4.10)$$

where,  $\rho_j = 1$ , for all  $j \in \mathcal{I}_{te}$ . Now, we will proceed by dividing the proof into different Lemmas.

**Lemma 3.4.2.** *Under all above assumptions, we have*

$$\int_0^{\ell_j} |\theta_x^j|^2 dx = o(1), \quad \forall j \in \mathcal{I}_{te}. \quad (3.4.11)$$

*Proof.* Taking the inner product in  $\mathcal{H}$  of equation (3.4.2) with the uniformly bounded sequence  $U = (u, v, y, z, \theta)$ , we get

$$\sum_{j \in \mathcal{I}_{te}} \kappa_j \int_0^{\ell_j} |\theta_x^j|^2 dx = -\text{Re}((i\lambda I - \mathcal{A})U, U)_{\mathcal{H}} = o(1).$$

As  $\kappa_j > 0, \forall j \in \mathcal{I}_{te}$ , it follows that

$$\|\theta_x^j\|_{L^2(0, \ell_j)}^2 = o(1), \quad \forall j \in \mathcal{I}_{te}.$$

□

Using (3.4.3), (3.4.5) and (3.4.1), we have for all  $j \in \mathcal{I}(\mathcal{N})$

$$\|\lambda u^j\|_{L^2(0, \ell_j)} = O(1), \quad \|u_x^j\|_{L^2(0, \ell_j)} = O(1), \quad (3.4.12)$$

$$\|\lambda y^j\|_{L^2(0, \ell_j)} = O(1), \quad \|y_x^j\|_{L^2(0, \ell_j)} = O(1). \quad (3.4.13)$$

Also, using (3.4.8), (3.4.9) and (3.4.10), we have for all  $j \in \mathcal{I}(\mathcal{N})$

$$\left\| \frac{u_{xx}^j}{\lambda} \right\|_{L^2(0, \ell_j)} = O(1), \quad \left\| \frac{y_{xx}^j}{\lambda} \right\|_{L^2(0, \ell_j)} = O(1), \quad \left\| \frac{\theta_{xx}^j}{\lambda} \right\|_{L^2(0, \ell_j)} = O(1). \quad (3.4.14)$$

**Lemma 3.4.3.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te}$*

$$\lambda u^j(\ell_j) = O(1), \quad u_x^j(\ell_j) = O(1), \quad (3.4.15)$$

$$\lambda u^j(0) = O(1), \quad u_x^j(0) = O(1), \quad (3.4.16)$$

$$\lambda y^j(\ell_j) = O(1), \quad y_x^j(\ell_j) = O(1), \quad (3.4.17)$$

$$\lambda y^j(0) = O(1), \quad y_x^j(0) = O(1). \quad (3.4.18)$$

*Proof.* For all  $j \in \mathcal{I}_{te}$ , let  $\Phi^j$  be a function in  $W^{1,\infty}(0, \ell_j)$ , then multiply (3.4.8) by  $2\Phi^j \overline{u_x^j}$ , integrate over  $(0, \ell_j)$ , take the real part and apply Green's formula, we get

$$\begin{aligned} & - \int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2 dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2 dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\ & \quad - 2\operatorname{Re} \alpha_j \int_0^{\ell_j} \theta_x^j \Phi^j \overline{u_x^j} dx + \operatorname{Re} 2i \int_0^{\ell_j} \beta_j \Phi^j \lambda y^j \overline{u_x^j} dx \\ & = 2\operatorname{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \overline{u_x^j} dx - 2\operatorname{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \overline{u_x^j} dx - 2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \overline{u_x^j} dx. \end{aligned} \quad (3.4.19)$$

Using (3.4.12), (3.4.14) and (3.4.11) we obtain,

$$\Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} = O(1) - 2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \overline{u_x^j} dx. \quad (3.4.20)$$

But,

$$\begin{aligned} -2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \overline{u_x^j} dx & = 2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi_x^j \overline{u^j} dx + 2\operatorname{Re} \int_0^{\ell_j} i \lambda \Phi^j \overline{u^j} f_x^j dx \\ & \quad - 2\operatorname{Re} (i \lambda f^j \Phi^j \overline{u^j}) \Big|_{x=0}^{x=\ell_j}. \end{aligned} \quad (3.4.21)$$

Using (3.4.12) and the fact that  $f^j$  converges to zero in  $H^1(0, \ell_j)$ , (3.4.21) becomes

$$-2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \overline{u_x^j} dx = -2\operatorname{Re} (i \lambda f^j \Phi^j \overline{u^j}) \Big|_{x=0}^{x=\ell_j} + o(1). \quad (3.4.22)$$

Let  $\Phi^j = x$ , for all  $j \in \mathcal{I}_{te}$ . Using Young's inequality, we get

$$\left| 2\operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \overline{u_x^j} dx \right| \leq \ell_j^2 \epsilon_j |\lambda u^j(\ell_j)|^2 + \frac{|f^j(\ell_j)|^2}{\epsilon_j} + o(1). \quad (3.4.23)$$

Recalling the Gagliardo-Nirenberg inequality in Liu and Zheng, 1999: For all  $\ell > 0$ , there are two positive constants  $C_1$  and  $C_2$  depending on  $\ell$  such that for any  $\Psi$  in  $H^1(0, \ell) \subset C([0, \ell])$ ,

$$\|\Psi\|_{L^\infty(0, \ell)} \leq C_1 \|\Psi_x\|_{L^2(0, \ell)}^{1/2} \|\Psi\|_{L^2(0, \ell)}^{1/2} + C_2 \|\Psi\|_{L^2(0, \ell)}. \quad (3.4.24)$$

Applying (3.4.24) to  $\Psi = f^j$  and using the fact that  $f^j$  converge to zero in  $H^1(0, \ell_j)$ , we deduce that  $f^j(\ell_j) = o(1)$ . Thus, (3.4.23) yields that

$$\left| 2\operatorname{Re} \int_0^{\ell_j} i\lambda f^j \Phi^j \overline{u_x^j} dx \right| \leq \ell_j^2 \epsilon_j |\lambda u^j(\ell_j)|^2 + o(1). \quad (3.4.25)$$

By inserting (3.4.25) in (3.4.20) and as  $\Phi^j = x$ , we obtain

$$(\ell_j - \ell_j^2 \epsilon_j) |\lambda u^j(\ell_j)|^2 + \ell_j |u_x^j(\ell_j)|^2 = O(1).$$

By taking  $\epsilon_j = \frac{1}{2\ell_j}$ , we deduce that (3.4.15) holds. Similarly, by taking  $\Phi^j = (x - \ell_j)$ , for all  $j \in \mathcal{I}_{te}$ , we conclude that (3.4.16) holds. Also, multiplying (3.4.9) by  $2x\overline{y_x^j}$  and  $2(x - \ell_j)\overline{y_x^j}$  respectively, we deduce that (3.4.17) and (3.4.18) hold.  $\square$

**Lemma 3.4.4.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te}$ ,*

$$\int_0^{\ell_j} |u_x^j|^2 dx = o(1), \quad \int_0^{\ell_j} |\theta^j|^2 dx = o(1). \quad (3.4.26)$$

*Proof.* Let  $j \in \mathcal{I}_{te}$ . Multiply (3.4.10) by  $\frac{\overline{u_x^j}}{\lambda}$  and integrate over  $(0, \ell_j)$  we get

$$\begin{aligned} i \int_0^{\ell_j} \theta^j \overline{u_x^j} dx - \int_0^{\ell_j} \kappa_j \theta_{xx}^j \frac{\overline{u_x^j}}{\lambda} dx + i\alpha_j \int_0^{\ell_j} |u_x^j|^2 dx &= \int_0^{\ell_j} h^j \frac{\overline{u_x^j}}{\lambda} dx \\ &+ \alpha_j \int_0^{\ell_j} f_x^j \frac{\overline{u_x^j}}{\lambda} dx. \end{aligned} \quad (3.4.27)$$

Applying Green's formula on the first and second term of (3.4.27) and using (3.4.12), we obtain

$$\begin{aligned} -i \int_0^{\ell_j} \theta_x^j \overline{u^j} dx + i\theta^j \overline{u^j} \Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j} \kappa_j \theta_x^j \frac{\overline{u_{xx}^j}}{\lambda} dx - \kappa_j \theta_x^j \frac{\overline{u_x^j}}{\lambda} \Big|_{x=0}^{x=\ell_j} \\ + i\alpha_j \int_0^{\ell_j} |u_x^j|^2 dx = o(1). \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.4.1), (3.4.11) and (3.4.14), we have

$$i\alpha_j \int_0^{\ell_j} |u_x^j|^2 dx + i\theta^j \overline{u^j} \Big|_{x=0}^{x=\ell_j} - \kappa_j \theta_x^j \frac{\overline{u_x^j}}{\lambda} \Big|_{x=0}^{x=\ell_j} = o(1). \quad (3.4.28)$$

Then, by applying Gagliardo-Nirenberg inequality for  $\Psi = \frac{\theta^j}{\sqrt{\lambda}}$ ,  $\Psi = \frac{\theta_x^j}{\sqrt{\lambda}}$  and  $\frac{u_x^j}{\sqrt{\lambda}}$  and again using (3.4.11), (3.4.14), (3.4.15), (3.4.16), we deduce that (3.4.28) yields

$$i\alpha_j \int_0^{\ell_j} |u_x^j|^2 dx = o(1).$$

Taking the imaginary part and using the fact that  $\alpha_j$  is a positive constant for all  $j \in \mathcal{I}_{te}$ , we deduce that

$$\int_0^{\ell_j} |u_x^j|^2 dx = o(1). \quad (3.4.29)$$

Now, multiply (3.4.10) by  $\frac{\overline{\theta^j + \alpha_j u_x^j}}{\lambda}$  and integrate over  $(0, \ell_j)$ , we get

$$\begin{aligned} i \int_0^{\ell_j} |\theta^j + \alpha_j u_x^j|^2 dx - \int_0^{\ell_j} \frac{\kappa_j \theta_{xx}^j}{\lambda} \overline{(\theta^j + \alpha_j u_x^j)} dx &= \int_0^{\ell_j} \frac{h^j}{\lambda} \overline{(\theta^j + \alpha_j u_x^j)} dx \\ &+ \int_0^{\ell_j} \frac{\alpha_j f_x^j}{\lambda} \overline{(\theta^j + \alpha_j u_x^j)} dx. \end{aligned} \quad (3.4.30)$$

By applying Green's formula on the second term of (3.4.30), using Cauchy-Schwarz inequality on the integrals of the right hand side, (3.4.1) and the fact that  $h^j$  and  $f_x^j$  converge to zero in  $L^2(0, \ell_j)$ , we obtain

$$\begin{aligned} i \int_0^{\ell_j} |\theta^j + \alpha_j u_x^j|^2 dx + \int_0^{\ell_j} \kappa_j \frac{|\theta_x^j|^2}{\lambda} dx + \int_0^{\ell_j} \alpha_j \kappa_j \theta_x^j \frac{\overline{u_{xx}^j}}{\lambda} dx \\ - \frac{\kappa_j \theta_x^j}{\sqrt{\lambda}} \left( \frac{\theta^j}{\sqrt{\lambda}} + \frac{\alpha_j u_x^j}{\sqrt{\lambda}} \right) \Big|_{x=0}^{x=\ell_j} = o(1). \end{aligned} \quad (3.4.31)$$

Again, by using Gagliardo-Nirenberg inequality for  $\Psi = \frac{\theta^j}{\sqrt{\lambda}}$ ,  $\Psi = \frac{\theta_x^j}{\sqrt{\lambda}}$  and  $\Psi = \frac{u_x^j}{\sqrt{\lambda}}$ , we deduce that the boundary term in (3.4.31) converges to zero. Moreover, using (3.4.11) and (3.4.14), the second and third terms of (3.4.31) converge to zero. Consequently, using (3.4.29), we conclude (3.4.26).  $\square$

**Lemma 3.4.5.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te}$*

$$\int_0^{\ell_j} |\lambda u^j|^2 dx = o(1) \text{ and } \int_0^{\ell_j} |v^j|^2 dx = o(1). \quad (3.4.32)$$

*Proof.* Let  $j \in \mathcal{I}_{te}$ . Multiply (3.4.8) by  $\overline{u^j}$ , integrate over  $(0, \ell_j)$  and apply Green's formula, we get

$$\begin{aligned} \int_0^{\ell_j} |\lambda u^j|^2 dx - \int_0^{\ell_j} |u_x^j|^2 dx + u_x^j \overline{u^j} \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \alpha_j \theta_x^j \overline{u^j} dx \\ + \int_0^{\ell_j} i \beta_j \lambda y^j \overline{u^j} dx = \int_0^{\ell_j} \beta_j g^j \overline{u^j} dx - \int_0^{\ell_j} \tilde{f}^j \overline{u^j} dx - i \lambda \int_0^{\ell_j} f^j \overline{u^j} dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.4.26), (3.4.15), (3.4.16), (3.4.11), (3.4.12) and (3.4.13), we deduce that

$$\int_0^{\ell_j} |\lambda u^j|^2 dx = o(1).$$

Using (3.4.3) we conclude that (3.4.32) holds.  $\square$

As a conclusion, we have for every  $j \in \mathcal{I}_{te}$

$$\begin{aligned} v^j &\rightarrow 0, \text{ in } L^2(0, \ell_j), \\ u^j &\rightarrow 0, \text{ in } H^1(0, \ell_j), \\ \theta^j &\rightarrow 0, \text{ in } H^1(0, \ell_j). \end{aligned}$$

**Lemma 3.4.6.** *Assume that all above assumptions hold. Then, for every thermo-elastic edge, we have*

$$\lambda u^j(\ell_j) = o(1), u_x^j(\ell_j) = o(1), \operatorname{Re}(i\lambda f^j(\ell_j)u^j(\ell_j)) = o(1), \quad (3.4.33)$$

$$\lambda u^j(0) = o(1), u_x^j(0) = o(1), \operatorname{Re}(i\lambda f^j(0)u^j(0)) = o(1), \quad (3.4.34)$$

$$\theta^j(0) = o(1), \theta^j(\ell_j) = o(1). \quad (3.4.35)$$

*Proof.* By the proof of Lemma 3.4.3, for all  $j \in \mathcal{I}_{te}$ , and any  $\Phi^j$  in  $W^{1,\infty}(0, \ell_j)$ , (3.4.19) holds. Then, using (3.4.26), (3.4.32), (3.4.11) and (3.4.13) we obtain

$$\Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} = o(1) - 2\operatorname{Re} \int_0^{\ell_j} i\lambda f^j \Phi^j \overline{u_x^j} dx. \quad (3.4.36)$$

Then, by taking  $\Phi^j = x$ , for all  $j \in \mathcal{I}_{te}$ , and using (3.4.25), we deduce that (3.4.36) becomes,

$$(\ell_j - \ell_j^2 \epsilon_j) |\lambda u^j(\ell_j)|^2 + \ell_j |u_x^j(\ell_j)|^2 = o(1).$$

Taking  $\epsilon_j = \frac{1}{2\ell_j}$ , we deduce that

$$\lambda u^j(\ell_j) = o(1) \text{ and } u_x^j(\ell_j) = o(1).$$

Consequently, by (3.4.36) and (3.4.22), we conclude that (3.4.33) holds. Similarly, by taking  $\Phi^j = (x - \ell_j)$ , for all  $j \in \mathcal{I}_{te}$ , we conclude that (3.4.34) holds. On the other hand, applying Gagliardo-Nirenberg inequality for  $\Psi = \theta^j$ , using (3.4.26) and (3.4.11) we deduce that (3.4.35) holds.  $\square$

**Lemma 3.4.7.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te}$ ,*

$$\int_0^{\ell_j} |y_x^j|^2 dx = o(1). \quad (3.4.37)$$

*Proof.* Multiply (3.4.8) by  $\frac{\overline{y_{xx}^j}}{\lambda}$ , then integrate over  $(0, \ell_j)$  we get

$$\begin{aligned} \int_0^{\ell_j} \lambda u^j \overline{y_{xx}^j} dx + \int_0^{\ell_j} u_{xx}^j \frac{\overline{y_{xx}^j}}{\lambda} dx - \int_0^{\ell_j} \alpha_j \theta_x^j \frac{\overline{y_{xx}^j}}{\lambda} dx + i \int_0^{\ell_j} \beta_j y^j \overline{y_{xx}^j} dx \\ = \int_0^{\ell_j} \beta_j g^j \frac{\overline{y_{xx}^j}}{\lambda} dx - \int_0^{\ell_j} \tilde{f}^j \frac{\overline{y_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} f^j \overline{y_{xx}^j} dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.4.11), (3.4.14) and the fact that  $g^j$  and  $\tilde{f}^j$  converge to zero in  $L^2(0, \ell_j)$ , we obtain

$$\int_0^{\ell_j} \lambda u^j \overline{y_{xx}^j} dx + \int_0^{\ell_j} u_{xx}^j \frac{\overline{y_{xx}^j}}{\lambda} dx + i \int_0^{\ell_j} \beta_j y^j \overline{y_{xx}^j} dx = o(1) - i \int_0^{\ell_j} f^j \overline{y_{xx}^j} dx. \quad (3.4.38)$$

Applying Green's formula on the first and third term of the left hand side and on the integral of the right hand side, we get

$$\begin{aligned} & - \int_0^{\ell_j} \lambda u_x^j \overline{y_x^j} dx + \lambda u^j \overline{y_x^j} \Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j} u_{xx}^j \frac{\overline{y_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} \beta_j |y_x^j|^2 dx \\ & + i \beta_j y^j \overline{y_x^j} \Big|_{x=0}^{x=\ell_j} = i \int_0^{\ell_j} f_x^j \overline{y_x^j} dx - i f^j \overline{y_x^j} \Big|_{x=0}^{x=\ell_j} + o(1). \end{aligned}$$

But, using (3.4.33), (3.4.34), (3.4.17), (3.4.18), (3.4.13) and the fact that  $f^j$  converge to zero in  $H^1(0, \ell_j)$ , we deduce that

$$- \int_0^{\ell_j} \lambda u_x^j \overline{y_x^j} dx + \int_0^{\ell_j} u_{xx}^j \frac{\overline{y_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} \beta_j |y_x^j|^2 dx = o(1). \quad (3.4.39)$$

Similarly, multiplying (3.4.9) by  $\frac{\overline{u_{xx}^j}}{\lambda}$ , integrating over  $(0, \ell_j)$  and using the fact that  $\rho_j = 1$  for all  $j \in \mathcal{I}_{te}$ , we get

$$\begin{aligned} & \int_0^{\ell_j} \lambda y^j \overline{u_{xx}^j} dx + \int_0^{\ell_j} y_{xx}^j \frac{\overline{u_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} \beta_j u^j \overline{u_{xx}^j} dx \\ & = - \int_0^{\ell_j} \beta_j f^j \frac{\overline{u_{xx}^j}}{\lambda} dx - \int_0^{\ell_j} \overline{g^j} \frac{\overline{u_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} g^j \overline{u_{xx}^j} dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.4.14) and the fact that  $f^j$  and  $\overline{g^j}$  converge to zero in  $L^2(0, \ell_j)$ , we obtain

$$\int_0^{\ell_j} \lambda y^j \overline{u_{xx}^j} dx + \int_0^{\ell_j} y_{xx}^j \frac{\overline{u_{xx}^j}}{\lambda} dx - i \int_0^{\ell_j} \beta_j u^j \overline{u_{xx}^j} dx = o(1) - i \int_0^{\ell_j} g^j \overline{u_{xx}^j} dx. \quad (3.4.40)$$

Applying Green's formula on the first and third term of the left hand side and on the integral of the right hand side, we obtain

$$\begin{aligned} & - \int_0^{\ell_j} \lambda y_x^j \overline{u_x^j} dx + \lambda y^j \overline{u_x^j} \Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j} y_{xx}^j \frac{\overline{u_{xx}^j}}{\lambda} dx + i \int_0^{\ell_j} \beta_j |u_x^j|^2 dx \\ & - i \beta_j u^j \overline{u_x^j} \Big|_{x=0}^{x=\ell_j} = i \int_0^{\ell_j} g_x^j \overline{u_x^j} dx - i g^j \overline{u_x^j} \Big|_{x=0}^{x=\ell_j} + o(1). \end{aligned}$$

But, using (3.4.17), (3.4.18), (3.4.26), (3.4.33), (3.4.34), and the fact that  $g^j$  converges to zero in  $H^1(0, \ell_j)$ , we deduce that

$$- \int_0^{\ell_j} \lambda y_x^j \overline{u_x^j} dx + \int_0^{\ell_j} y_{xx}^j \frac{\overline{u_{xx}^j}}{\lambda} dx = o(1). \quad (3.4.41)$$

Taking the imaginary part of equations (3.4.39) and (3.4.41) then, adding the two resulting equations, we conclude that

$$\int_0^{\ell_j} |y_x^j|^2 dx = o(1).$$



and the result holds.  $\square$

**Lemma 3.4.8.** *Under all above assumptions, we have*

$$\int_0^{\ell_j} |\lambda y^j|^2 dx = o(1), \quad \forall j \in \mathcal{I}_{te}. \quad (3.4.42)$$

*Proof.* Multiply (3.4.9) by  $\bar{y}^j$  then, integrate over  $(0, \ell_j)$  and apply Green's formula, we get

$$\begin{aligned} \int_0^{\ell_j} |\lambda y^j|^2 dx - \int_0^{\ell_j} |y_x^j|^2 dx + y^j \bar{y}_x^j \Big|_{x=0}^{x=\ell_j} - i \int_0^{\ell_j} \beta_j \lambda u^j \bar{y}^j dx \\ = - \int_0^{\ell_j} \beta_j f^j \bar{y}^j dx - \int_0^{\ell_j} \tilde{g}^j \bar{y}^j dx - i \int_0^{\ell_j} g^j \lambda \bar{y}^j dx. \end{aligned}$$

But, using Cauchy-Schwarz inequality, (3.4.12), (3.4.13), (3.4.17), (3.4.18), (3.4.37) and the fact that  $f^j, g^j$  converge to zero in  $H^1(0, \ell_j)$  and  $\tilde{g}^j$  converges to zero in  $L^2(0, \ell_j)$ , we deduce that (3.4.42) holds.  $\square$

**Lemma 3.4.9.** *Assume that all above assumptions hold. Then, for every thermo-elastic edge, we have*

$$\lambda y^j(\ell_j) = o(1), \quad y_x^j(\ell_j) = o(1), \quad \operatorname{Re}(i \lambda g^j(\ell_j) y^j(\ell_j)) = o(1), \quad (3.4.43)$$

$$\lambda y^j(0) = o(1), \quad y_x^j(0) = o(1), \quad \operatorname{Re}(i \lambda g^j(0) y^j(0)) = o(1). \quad (3.4.44)$$

*Proof.* The proof is the same as the one of Lemma 3.4.3 or Lemma 3.4.6, using (3.4.37) and (3.4.42), the result holds.  $\square$

**Lemma 3.4.10.** *Under all above assumptions, for each elastic edge we have*

$$\int_0^{\ell_j} |\lambda u^j|^2 dx = o(1), \quad \int_0^{\ell_j} |u_x^j|^2 dx = o(1), \quad (3.4.45)$$

$$\int_0^{\ell_j} |\lambda y^j|^2 dx = o(1), \quad \int_0^{\ell_j} |y_x^j|^2 dx = o(1). \quad (3.4.46)$$

*Proof.* Let  $e_j$  be an elastic edge attached to a thermoelastic one at an interior vertex  $a_k$ , where  $a_k$  is a leaf of a maximal subgraph of elastic edge. Recall that  $\alpha_j = 0$ , and let  $\Phi^j \in W^{1,\infty}(0, \ell_j)$ . Multiply (3.4.8) by  $2\Phi^j \bar{u}_x^j$  then integrate over  $(0, \ell_j)$ , take the real part and apply Green's formula, we obtain

$$\begin{aligned} - \int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2 dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2 dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\ + \operatorname{Re} 2i \int_0^{\ell_j} \lambda \beta_j \Phi^j y^j \bar{u}_x^j dx = 2 \operatorname{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \bar{u}_x^j dx \\ - 2 \operatorname{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \bar{u}_x^j dx - 2 \operatorname{Re} \int_0^{\ell_j} i \lambda f^j \Phi^j \bar{u}_x^j dx. \end{aligned}$$

Again applying Green's formula on the fifth term of the left hand side and on the third term of the right hand side, we get

$$\begin{aligned}
 & - \int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2 dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2 dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\
 & - \operatorname{Re} 2i \int_0^{\ell_j} \lambda \bar{w}^j \beta_j \Phi^j y_x^j dx - \operatorname{Re} 2i \int_0^{\ell_j} \lambda \bar{w}^j \beta_j \Phi_x^j y^j dx + \operatorname{Re} (2i \beta_j \Phi^j \lambda y^j \bar{w}^j) \Big|_{x=0}^{x=\ell_j} \\
 & = 2\operatorname{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \bar{u}_x^j dx - 2\operatorname{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \bar{u}_x^j dx + \operatorname{Re} 2i \int_0^{\ell_j} \lambda \bar{w}^j f_x^j \Phi^j dx \\
 & \quad + \operatorname{Re} 2i \int_0^{\ell_j} f^j \Phi_x^j \lambda \bar{w}^j dx - 2\operatorname{Re}(i \lambda f^j \Phi^j \bar{w}^j) \Big|_{x=0}^{x=\ell_j}.
 \end{aligned} \tag{3.4.47}$$

But using Cauchy-Schwarz inequality, (3.4.12) and (3.4.13), we deduce that

$$\operatorname{Re} 2i \int_0^{\ell_j} \lambda \bar{w}^j \beta_j \Phi_x^j y^j dx = o(1). \tag{3.4.48}$$

Also, using Cauchy-Schwarz inequality, (3.4.12), (3.4.13) and the fact that  $f^j$  converge to zero in  $H^1(0, \ell_j)$  and  $\tilde{f}^j, g^j$  converge to zero in  $L^2(0, \ell_j)$ , we have

$$\begin{aligned}
 & 2\operatorname{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \bar{u}_x^j dx - 2\operatorname{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \bar{u}_x^j dx + \operatorname{Re} 2i \int_0^{\ell_j} \lambda \bar{w}^j f_x^j \Phi^j dx \\
 & \quad + \operatorname{Re} 2i \int_0^{\ell_j} f^j \Phi_x^j \lambda \bar{w}^j dx = o(1).
 \end{aligned} \tag{3.4.49}$$

Inserting (3.4.48) and (3.4.49) in the identity (3.4.47), we get

$$\begin{aligned}
 & - \int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2 dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2 dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\
 & + \operatorname{Re} (2i \beta_j \Phi^j \lambda y^j \bar{w}^j) \Big|_{x=0}^{x=\ell_j} + 2 \operatorname{Im} \int_0^{\ell_j} \lambda \bar{w}^j \beta_j \Phi^j y_x^j dx = -2\operatorname{Re}(i \lambda f^j \Phi^j \bar{w}^j) \Big|_{x=0}^{x=\ell_j}.
 \end{aligned} \tag{3.4.50}$$

Similarly, multiply (3.4.9) by  $2\Phi^j \bar{y}_x^j$ , integrate over  $(0, \ell_j)$ , take the real part and apply Green's formula, we obtain

$$\begin{aligned}
 & - \int_0^{\ell_j} \Phi_x^j |\lambda y^j|^2 dx + \Phi^j |\lambda y^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \rho_j \Phi_x^j |y_x^j|^2 dx + \rho_j \Phi^j |y_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\
 & - \operatorname{Re} 2i \int_0^{\ell_j} \lambda \beta_j \Phi^j u^j \bar{y}_x^j dx = -\operatorname{Re} 2 \int_0^{\ell_j} \beta_j f^j \Phi^j \bar{y}_x^j dx \\
 & \quad - \operatorname{Re} 2 \int_0^{\ell_j} g^j \Phi^j \bar{y}_x^j dx - \operatorname{Re} 2i \int_0^{\ell_j} \lambda g^j \Phi^j \bar{y}_x^j dx.
 \end{aligned}$$

Applying Green's formula on the last integral of the right hand side, we get

$$\begin{aligned}
 & - \int_0^{\ell_j} \Phi_x^j |\lambda y^j|^2 dx + \Phi^j |\lambda y^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \rho_j \Phi_x^j |y_x^j|^2 dx + \rho_j \Phi^j |y_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\
 & \quad - \operatorname{Re} 2i \int_0^{\ell_j} \lambda \beta_j \Phi^j u^j \overline{y_x^j} dx = - \operatorname{Re} 2 \int_0^{\ell_j} \beta_j f^j \Phi^j \overline{y_x^j} dx \\
 & \quad - \operatorname{Re} 2 \int_0^{\ell_j} \tilde{g}^j \Phi^j \overline{y_x^j} dx + \operatorname{Re} 2i \int_0^{\ell_j} \lambda g_x^j \Phi^j \overline{y^j} dx + \operatorname{Re} 2i \int_0^{\ell_j} \lambda g^j \Phi_x^j \overline{y^j} dx \\
 & \quad \quad \quad - 2 \operatorname{Re}(i \lambda g^j \Phi^j \overline{y^j}) \Big|_{x=0}^{x=\ell_j}.
 \end{aligned} \tag{3.4.51}$$

But, using (3.4.13) and the fact that  $g^j$  converge to zero in  $H^1(0, \ell_j)$  and  $f^j, \tilde{g}^j$  converge to zero in  $L^2(0, \ell_j)$ , we conclude that

$$\begin{aligned}
 & - \operatorname{Re} 2 \int_0^{\ell_j} \beta_j f^j \Phi^j \overline{y_x^j} dx - \operatorname{Re} 2 \int_0^{\ell_j} \tilde{g}^j \Phi^j \overline{y_x^j} dx + \operatorname{Re} 2i \int_0^{\ell_j} \lambda g_x^j \Phi^j \overline{y^j} dx \\
 & \quad + \operatorname{Re} 2i \int_0^{\ell_j} \lambda g^j \Phi_x^j \overline{y^j} dx = o(1).
 \end{aligned} \tag{3.4.52}$$

Then, inserting (3.4.52) in equation (3.4.51), we get

$$\begin{aligned}
 & - \int_0^{\ell_j} \Phi_x^j |\lambda y^j|^2 dx + \Phi^j |\lambda y^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \rho_j \Phi_x^j |y_x^j|^2 dx + \rho_j \Phi^j |y_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\
 & \quad + 2 \operatorname{Im} \int_0^{\ell_j} \lambda \beta_j \Phi^j u^j \overline{y_x^j} dx = o(1) - 2 \operatorname{Re}(i \lambda g^j \Phi^j \overline{y^j}) \Big|_{x=0}^{x=\ell_j}.
 \end{aligned} \tag{3.4.53}$$

Without loss of generality, assume that  $\pi_j(a_k) = 0$  and let  $\Phi^j = x - \ell_j$  (otherwise, let  $\Phi^j = x$ ). Then, adding the two equations (3.4.50) and (3.4.53), using Lemma 3.4.6, Lemma 3.4.9, the fact that  $u, y, f$  and  $g$  satisfy the continuity conditions in (3.1.11) and  $u, \theta$  and  $y$  satisfy the balance conditions (3.1.12)-(3.1.13), we deduce that

$$\begin{aligned}
 & \int_0^{\ell_j} |\lambda u^j|^2 dx + \int_0^{\ell_j} |u_x^j|^2 dx \\
 & \quad + \int_0^{\ell_j} |\lambda y^j|^2 dx + \int_0^{\ell_j} \rho_j |y_x^j|^2 dx = o(1).
 \end{aligned}$$

Consequently, as  $\rho_j > 0$  for all  $j \in \mathcal{I}_e$ , (3.4.45) and (3.4.46) hold. Repeating the same technique of Lemma 3.4.6 and Lemma 3.4.9, we conclude that

$$\begin{aligned}
 & \lambda u^j(0) = o(1), u_x^j(0) = o(1), \operatorname{Re}(i \lambda f^j(0) u^j(0)) = o(1), \\
 & \lambda y^j(0) = o(1), y_x^j(0) = o(1), \operatorname{Re}(i \lambda g^j(0) y^j(0)) = o(1), \\
 & \lambda u^j(\ell_j) = o(1), u_x^j(\ell_j) = o(1), \operatorname{Re}(i \lambda f^j(\ell_j) u^j(\ell_j)) = o(1), \\
 & \lambda y^j(\ell_j) = o(1), y_x^j(\ell_j) = o(1), \operatorname{Re}(i \lambda g^j(\ell_j) y^j(\ell_j)) = o(1).
 \end{aligned}$$

Then, by iteration on each maximal subgraph of purely elastic edges (from leaves to the root), we prove that

$$\int_0^{\ell_j} |\lambda u^j|^2 dx + \int_0^{\ell_j} |u_x^j|^2 dx + \int_0^{\ell_j} |\lambda y^j|^2 dx + \int_0^{\ell_j} |y_x^j|^2 dx = o(1), \quad (3.4.54)$$

for all  $j \in \mathcal{I}_e$ .  $\square$

In conclusion, using Lemmas 3.4.4, 3.4.5, 3.4.7, 3.4.8 and 3.4.10, we conclude that  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (3.4.1).  $\square$

**Remark 3.4.11.** Examples of networks for which (S1) holds are given by Theorem 3.3.1.

**Remark 3.4.12.** If there exists an elastic edge ( $j \in \mathcal{I}_e$ ) such that  $\rho_j \neq 1$  then, using Lemma 3.4.10, we show that Theorem 3.4.1 holds (i.e., the energy of the system decays exponentially to zero). But, if there exists a thermo-elastic edge ( $j \in \mathcal{I}_{te}$ ) such that  $\rho_j \neq 1$  then, it seems that the energy of the system does not decay exponentially, but polynomially (see Theorem 3.4.13 below).

**Theorem 3.4.13.** *Let  $\mathcal{N}$  be an arbitrary network for which the operator associated with System (3.1.7)-(3.1.13) satisfies (S1). Assume that there exists  $j \in \mathcal{I}_{te} \cap \mathcal{I}_{ext}^l$  such that  $\rho_j \neq 1$ . Then, the energy of the system satisfies*

$$E(t) \leq \frac{C}{t^{1/3}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall U_0 \in D(\mathcal{A}), \quad t > 0, \quad (3.4.55)$$

for some positive constant  $C > 0$ .

*Proof.* Following Borichev and Tomilov, 2009, a  $C_0$ - semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$  is polynomially stable if and only if (S1) and

$$\limsup_{|\lambda| \rightarrow \infty} \frac{1}{\lambda^6} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad (S2)$$

hold. As we have assumed that (S1) is satisfied, it remains to prove that condition (S2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S2) does not hold, then there exist a sequence of real numbers  $\lambda_n$  and a sequence of vectors  $U_n = (u_n, v_n, y_n, z_n, \theta_n) \in D(\mathcal{A})$  such that (3.4.1) and

$$\lambda^6 (i\lambda_n - \mathcal{A})U_n = (f_n, \tilde{f}_n, g_n, \tilde{g}_n, h_n) \longrightarrow 0 \quad \text{in } \mathcal{H} \quad (3.4.56)$$

are satisfied.

In what follows, we drop the index  $n$  for simplicity.

Now by detailing (3.4.56), we get for all  $j \in \mathcal{I}(\mathcal{N})$

$$\lambda^6 (i\lambda u^j - v^j) = f^j \rightarrow 0 \quad \text{in } H^1(0, \ell_j), \quad (3.4.57)$$

$$\lambda^6 (i\lambda v^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j z^j) = \tilde{f}^j \rightarrow 0 \quad \text{in } L^2(0, \ell_j), \quad (3.4.58)$$

$$\lambda^6 (i\lambda y^j - z^j) = g^j \rightarrow 0 \quad \text{in } H^1(0, \ell_j), \quad (3.4.59)$$

$$\lambda^6 (i\lambda z^j - \rho_j y_{xx}^j + \beta_j v^j) = \tilde{g}^j \rightarrow 0 \quad \text{in } L^2(0, \ell_j), \quad (3.4.60)$$

$$\lambda^6 (i\lambda \theta^j - \kappa_j \theta_{xx}^j + \alpha_j v_x^j) = h^j \rightarrow 0 \quad \text{in } L^2(0, \ell_j). \quad (3.4.61)$$

Then, by eliminating  $v^j$  and  $z^j$  using (3.4.57) and (3.4.59) respectively, (3.4.57)-(3.4.61) becomes

$$\lambda^2 u^j + u_{xx}^j - \alpha_j \theta_x^j + i\lambda \beta_j y^j = \frac{\beta_j g^j}{\lambda^6} - \frac{\tilde{f}^j}{\lambda^6} - i \frac{f^j}{\lambda^5}, \quad (3.4.62)$$

$$\lambda^2 y^j + \rho_j y_{xx}^j - i\lambda \beta_j u^j = -\frac{\beta_j f^j}{\lambda^6} - \frac{\tilde{g}^j}{\lambda^6} - i \frac{g^j}{\lambda^5}, \quad (3.4.63)$$

$$i\lambda \theta^j - \kappa_j \theta_{xx}^j + i\lambda \alpha_j u_x^j = \frac{h^j}{\lambda^6} + \frac{\alpha_j f_x^j}{\lambda^6}. \quad (3.4.64)$$

First, our aim is to prove that for each thermo-elastic edge, we have

$$\begin{cases} v^j \rightarrow 0, & \text{in } L^2(0, \ell_j), \\ u^j \rightarrow 0, & \text{in } H^1(0, \ell_j), \\ \theta^j \rightarrow 0, & \text{in } H^1(0, \ell_j), \\ z^j \rightarrow 0, & \text{in } L^2(0, \ell_j), \\ y^j \rightarrow 0, & \text{in } H^1(0, \ell_j). \end{cases} \quad (3.4.65)$$

Following the same proof of Lemmas 3.4.4, 3.4.5, 3.4.7, 3.4.8, we can prove that (3.4.65) holds for all  $j \in \mathcal{I}_{te}$  with  $\rho_j = 1$ . Hence, we only need to prove that (3.4.65) holds for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$  with  $\rho_j \neq 1$ . We will proceed by dividing the proof into different Lemmas.

**Lemma 3.4.14.** *Under all above assumptions, we have*

$$\int_0^{\ell_j} |\theta_x^j|^2 dx = \frac{o(1)}{\lambda^6}, \quad \forall j \in \mathcal{I}_{te}. \quad (3.4.66)$$

*Proof.* Same proof as the one of Lemma 3.4.2.  $\square$

Using (3.4.1), (3.4.57) and (3.4.59), we can easily deduce that (3.4.12)-(3.4.13) holds, for all  $j \in \mathcal{I}(\mathcal{N})$ . Also, using (3.4.62), (3.4.63) and (3.4.64) we conclude (3.4.14) for all  $j \in \mathcal{I}(\mathcal{N})$ .

**Lemma 3.4.15.** *Under all above assumptions, (3.4.15)-(3.4.18) holds.*

*Proof.* Let  $\Phi^j$  be a function in  $W^{1,\infty}(0, \ell_j)$ , for all  $j \in \mathcal{I}_{te}$ . Multiplying (3.4.62) by  $2\Phi^j \overline{u_x^j}$ , integrating over  $(0, \ell_j)$  then taking the real part and applying Green's Formula, we obtain

$$\begin{aligned} & - \int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2 dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2 dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\ & \quad - 2\text{Re} \alpha_j \int_0^{\ell_j} \theta_x^j \Phi^j \overline{u_x^j} dx + \text{Re} 2i \int_0^{\ell_j} \beta_j \lambda y^j \Phi^j \overline{u_x^j} dx \\ & = 2\text{Re} \int_0^{\ell_j} \frac{\beta_j \Phi^j g^j \overline{u_x^j}}{\lambda^6} dx - 2\text{Re} \int_0^{\ell_j} \frac{\tilde{f}^j \Phi^j \overline{u_x^j}}{\lambda^6} dx - 2\text{Re} \int_0^{\ell_j} i \frac{f^j \Phi^j \overline{u_x^j}}{\lambda^5} dx. \end{aligned} \quad (3.4.67)$$

Using (3.4.12), (3.4.13) and (3.4.66) we get,

$$\Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} = O(1).$$

Let  $\Phi^j = x$ , for all  $j \in \mathcal{I}_{te}$ . We deduce that (3.4.15) holds. Similarly, by taking  $\Phi^j = (x - \ell_j)$ , we conclude (3.4.16). Also, multiplying (3.4.9) by  $2x\overline{y_x^j}$  and  $2(x - \ell_j)\overline{y_x^j}$  respectively, we deduce that (3.4.17) and (3.4.18) hold.  $\square$

**Lemma 3.4.16.** *Under all above assumptions, we have*

$$\int_0^{\ell_j} |u_x^j|^2 dx = \frac{o(1)}{\lambda^2}, \quad \forall j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}. \quad (3.4.68)$$

*Proof.* Let  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ . Multiply (3.4.64) by  $\frac{\overline{u_x^j}}{\lambda}$ , and integrate over  $(0, \ell_j)$ , we get

$$\begin{aligned} i \int_0^{\ell_j} \theta^j \overline{u_x^j} dx - \int_0^{\ell_j} \kappa_j \theta_{xx}^j \frac{\overline{u_x^j}}{\lambda} dx + i \int_0^{\ell_j} \alpha_j |u_x^j|^2 dx \\ = \int_0^{\ell_j} \frac{h^j \overline{u_x^j}}{\lambda^7} dx + \int_0^{\ell_j} \frac{\alpha_j f_x^j \overline{u_x^j}}{\lambda^7} dx. \end{aligned} \quad (3.4.69)$$

Applying Green's formula on the first and second term of (3.4.69), using Cauchy-Schwarz inequality, (3.4.12) and the fact that  $f^j$  converge to zero in  $H^1(0, \ell_j)$  and  $h^j$  converge to zero in  $L^2(0, \ell_j)$ , we obtain

$$\begin{aligned} -i \int_0^{\ell_j} \theta_x^j \overline{u^j} dx + i \theta^j \overline{u^j} \Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j} \kappa_j \theta_x^j \frac{\overline{u_{xx}^j}}{\lambda} dx - \kappa_j \frac{\theta_x^j}{\lambda} \overline{u_x^j} \Big|_{x=0}^{x=\ell_j} \\ + i \int_0^{\ell_j} \alpha_j |u_x^j|^2 dx = \frac{o(1)}{\lambda^7}. \end{aligned} \quad (3.4.70)$$

But, using Cauchy-Schwarz inequality, (3.4.66), (3.4.12) and (3.4.14), we deduce that

$$\int_0^{\ell_j} \theta_x^j \overline{u^j} dx = \frac{o(1)}{\lambda^4}, \quad (3.4.71)$$

and

$$\int_0^{\ell_j} \theta_x^j \frac{\overline{u_{xx}^j}}{\lambda} dx = \frac{o(1)}{\lambda^3}. \quad (3.4.72)$$

On the other hand, by applying Gagliardo-Nirenberg inequality to  $\Psi = \theta^j$  and  $\Psi = \frac{\theta_x^j}{\lambda}$ , using Poincaré inequality, (3.4.66) and (3.4.14) we deduce that

$$\theta^j(0) = \frac{o(1)}{\lambda^3}, \quad \theta^j(\ell_j) = \frac{o(1)}{\lambda^3}, \quad j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}, \quad (3.4.73)$$

and

$$\frac{\theta_x^j(0)}{\lambda} = \frac{o(1)}{\lambda^2}, \quad \frac{\theta_x^j(\ell_j)}{\lambda} = \frac{o(1)}{\lambda^2}, \quad j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}. \quad (3.4.74)$$

Using (3.4.71), (3.4.72), (3.4.73), (3.4.74), (3.4.15) and (3.4.16) we conclude that (3.4.68) holds.  $\square$

**Lemma 3.4.17.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |\lambda u^j|^2 dx = o(1) \text{ and } \int_0^{\ell_j} |v^j|^2 dx = o(1). \quad (3.4.75)$$

*Proof.* Let  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ . Multiply (3.4.62) by  $\overline{u^j}$ , integrate over  $(0, \ell_j)$  and apply Green's formula, we get

$$\begin{aligned} & \int_0^{\ell_j} |\lambda u^j|^2 dx - \int_0^{\ell_j} |u_x^j|^2 dx + u_x^j \overline{u^j} \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \alpha_j \theta_x^j \overline{u^j} dx \\ & + \int_0^{\ell_j} i \beta_j \lambda y^j \overline{u^j} dx = \int_0^{\ell_j} \frac{\beta_j g^j \overline{u^j}}{\lambda^6} dx - \int_0^{\ell_j} \frac{\tilde{f}^j \overline{u^j}}{\lambda^6} dx - i \int_0^{\ell_j} \frac{f^j \overline{u^j}}{\lambda^5} dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, (3.4.12), (3.4.13), (3.4.66), (3.4.68), (3.4.15) and (3.4.16) we deduce that

$$\int_0^{\ell_j} |\lambda u^j|^2 dx = o(1).$$

Using (3.4.57), we conclude that (3.4.75) holds.  $\square$

**Lemma 3.4.18.** *Assume that all above assumptions hold. Then, for  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ , we have*

$$\lambda u^j(\ell_j) = o(1), \quad u_x^j(\ell_j) = o(1), \quad (3.4.76)$$

$$\lambda u^j(0) = o(1), \quad u_x^j(0) = o(1), \quad (3.4.77)$$

$$\theta^j(0) = o(1), \theta^j(\ell_j) = o(1). \quad (3.4.78)$$

*Proof.* By the proof of Lemma 3.4.15, for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ , and any  $\Phi^j$  in  $W^{1,\infty}(0, \ell_j)$ , (3.4.67) holds. Then, using (3.4.68), (3.4.75), (3.4.66) and (3.4.13) we obtain

$$\Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} = o(1).$$

Then, by taking  $\Phi_j = x$ , for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ , we deduce that (3.4.76) holds. Similarly, by taking  $\Phi^j = (x - \ell_j)$ , for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ , we conclude that (3.4.77) holds. On the other hand, applying Gagliardo-Nirenberg inequality, using (3.4.66) and Poincaré inequality, we deduce that (3.4.78) holds.  $\square$

**Lemma 3.4.19.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |\lambda y^j|^2 dx = o(1) \quad \text{and} \quad \int_0^{\ell_j} |z^j|^2 dx = o(1). \quad (3.4.79)$$

*Proof.* Let  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ . Multiply equation (3.4.63) by  $\lambda \overline{u^j}$ , then, integrate over  $(0, \ell_j)$ , we get

$$\begin{aligned} & \int_0^{\ell_j} \lambda^3 y^j \overline{u^j} dx + \rho_j \int_0^{\ell_j} y_{xx}^j \lambda \overline{u^j} dx - i \int_0^{\ell_j} \beta_j |\lambda u^j|^2 dx \\ & = - \int_0^{\ell_j} \frac{\beta_j f^j \overline{u^j}}{\lambda^5} dx - \int_0^{\ell_j} \frac{g^j \overline{u^j}}{\lambda^5} dx - i \int_0^{\ell_j} \frac{g^j \overline{u^j}}{\lambda^4} dx. \end{aligned} \quad (3.4.80)$$

Applying Green's formula on the second term of (3.4.80), using Cauchy-Schwarz inequality and (3.4.12), we obtain

$$\int_0^{\ell_j} \lambda^3 y^j \overline{u^j} dx - \rho_j \int_0^{\ell_j} y_x^j \lambda u_x^j \overline{u^j} dx + \rho_j y_x^j \lambda \overline{u^j} \Big|_{x=0}^{x=\ell_j} - i \int_0^{\ell_j} \beta_j |\lambda u^j|^2 dx = o(1).$$

But, using (3.4.76), (3.4.77), (3.4.68) (3.4.75), (3.4.17), (3.4.18) and (3.4.13), we deduce that

$$\int_0^{\ell_j} \lambda^3 u^j \bar{u}^j dx = o(1). \quad (3.4.81)$$

Similarly, multiply (3.4.62) by  $\lambda \bar{y}^j$  then integrate over  $(0, \ell_j)$  and apply Green's formula, we obtain

$$\begin{aligned} & \int_0^{\ell_j} \lambda^3 u^j \bar{y}^j dx - \int_0^{\ell_j} u_x^j \lambda \bar{y}_x^j dx + u_x^j \lambda \bar{y}^j \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \alpha_j \theta_x^j \lambda \bar{y}^j dx \\ & + i\beta_j \int_0^{\ell_j} |\lambda y^j|^2 dx = \int_0^{\ell_j} \frac{\beta_j g^j \bar{y}^j}{\lambda^5} dx - \int_0^{\ell_j} \frac{\check{f}^j \bar{y}^j}{\lambda^5} dx - i \int_0^{\ell_j} \frac{f^j \bar{y}^j}{\lambda^4} dx. \end{aligned}$$

Consequently, using (3.4.68), (3.4.76), (3.4.77), (3.4.17), (3.4.18) and (3.4.66) we get

$$\int_0^{\ell_j} \lambda^3 u^j \bar{y}^j dx + i\beta_j \int_0^{\ell_j} |\lambda y^j|^2 dx = o(1).$$

Then, taking the imaginary part of the above equality, using (3.4.81) and the fact that  $\beta_j$  is a positive constant for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$ , we deduce that

$$\int_0^{\ell_j} |\lambda y^j|^2 dx = o(1).$$

Finally, by (3.4.60), we conclude that (3.4.79) holds.  $\square$

**Lemma 3.4.20.** *Under all above assumptions, we have for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |y_x^j|^2 dx = o(1). \quad (3.4.82)$$

*Proof.* Multiply (3.4.63) by  $\bar{y}^j$ , integrate over  $(0, \ell_j)$  and apply Green's formula, we get

$$\begin{aligned} & \int_0^{\ell_j} |\lambda y^j|^2 dx - \rho_j \int_0^{\ell_j} |y_x^j|^2 dx + \rho_j y_x^j \bar{y}^j \Big|_{x=0}^{x=\ell_j} - i \int_0^{\ell_j} \beta_j u^j \bar{y}^j dx \\ & = - \int_0^{\ell_j} \frac{\beta_j f^j \bar{y}^j}{\lambda^6} dx - \int_0^{\ell_j} \frac{\check{g}^j \bar{y}^j}{\lambda^6} dx - i \int_0^{\ell_j} \frac{g^j \bar{y}^j}{\lambda^5} dx. \end{aligned}$$

Then, using (3.4.79), (3.4.17), (3.4.18) and (3.4.13), we deduce that for all  $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$  we have

$$\begin{cases} \lambda y^j(\ell_j) = o(1), y_x^j(\ell_j) = o(1), \\ \lambda y^j(0) = o(1), y_x^j(0) = o(1). \end{cases} \quad (3.4.83)$$

$\square$

Hence, using (??) and Lemma 3.4.18 and by iteration on each maximal subgraph of purely elastic edges (from leaves to root), the results of Lemma 3.4.10 holds for every elastic edge. Finally, using Lemmas 3.4.16, 3.4.17, 3.4.19, 3.4.20 and 3.4.10, we conclude that  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (3.4.1).  $\square$

**Remark 3.4.21.** 1) If every maximal subgraph of thermo-elastic edges is composed of maximum three thermo-elastic edges and there exist  $j \in \mathcal{I}_{te} \cap \mathcal{G}'_{int}$  such that



$\rho_j \neq 1$  then, the same result of Theorem 3.4.13 holds. The key step to prove that result is to show that (3.4.73) still holds. Due to (3.1.11) and as (3.4.73) holds for all  $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$ , we deduce that (3.4.73) is achieved for all  $j \in \mathcal{I}_{\text{te}}$  (in particular for  $j \in \mathcal{I}_{\text{te}} \cap \mathcal{G}'_{\text{int}}$ ). Hence, the desired result is attained.

2) If we replace condition (3.1.13) by the following condition

$$\begin{cases} \sum_{j \in \mathcal{I}(a_k)} d_{kj} u_x^j(a_k) = 0, & a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}_{\text{te}}(a_k)} d_{kj} (\alpha_j u_t^j(a_k) - \kappa_j \theta_x^j(a_k)) = 0, & a_k \in V'_{\text{int}}. \end{cases} \quad (3.4.84)$$

Then, we can also prove that the same results of Theorem 3.4.1 and Theorem 3.4.13 hold for system  $\{(3.1.7)-(3.1.12), (3.4.84)\}$ . Notice that due to condition (3.4.84), a slight gain appears when proving the system  $\{(3.1.7)-(3.1.12), (3.4.84)\}$  is strongly stable. In other words, the energy associated with system  $\{(3.1.7)-(3.1.12), (3.4.84)\}$  converges to zero if one of the following conditions holds,

- i) Each maximal subgraph of thermo-elastic edges contains at least one interior vertex or contains an exterior vertex that belongs to  $V_{\text{ext}}$ .
- ii) There exists a maximal subgraph of thermo-elastic edges that contains no interior vertices and contains no exterior vertex that belongs to  $V_{\text{ext}}$  and  $\beta_j = \beta$ , for all  $j \in \mathcal{I}(\mathcal{N})$ .

**Idea of the proof.** In comparison with Theorem 3.3.1, it is enough to prove that if each maximal subgraph of thermo-elastic edges contains at least one interior vertex, then the energy associated with system  $\{(3.1.7)-(3.1.12), (3.4.84)\}$  converges to zero. Due to the fact that  $(I - \mathcal{A}_*)^{-1}$  is compact where,  $\mathcal{A}_* = \mathcal{A}_{|D(\mathcal{A}_*)}$  is the operator associated with system  $\{(3.1.7)-(3.1.12), (3.4.84)\}$ , we have  $\sigma(\mathcal{A}_*) = \sigma_p(\mathcal{A}_*)$ . Following the same proof as the one of Theorem 3.3.1, let  $\lambda \in \mathbb{R}$  and  $U = (u, v, y, z, \theta) \in D(\mathcal{A}_*)$  be such that

$$\mathcal{A}_* U = i\lambda U.$$

As  $\mathcal{A}_* = \mathcal{A}_{|D(\mathcal{A}_*)}$ , we obtain, (3.3.1)-(3.3.10). If  $\lambda = 0$ , we proceed exactly as in the proof of Theorem 3.3.1. Otherwise, if  $\lambda \neq 0$  then, using (3.3.10) and (3.3.8) we have (3.3.14), which means that  $u^j$  is constant for every  $j \in \mathcal{I}_{\text{te}}$  but, due to the fact that each maximal subgraph of thermo-elastic edges contains at least one interior vertex (i.e.,  $V'_{\text{int}} \neq \emptyset$ ) and using (3.3.9), (3.3.1) and (3.1.11), the balance condition (3.4.84) asserts that  $u^j(a_k) = 0$  for some  $j \in \mathcal{I}_{\text{te}}(a_k)$ ,  $a_k \in V'_{\text{int}}$ . Again, using (3.1.11) we deduce that  $u^j = 0$ , for all  $j \in \mathcal{I}_{\text{te}}$ . Finally, we proceed exactly as **Case 1** of the proof of Theorem 3.3.1 to reach the desired aim.

### 3.5 Stabilization of Thermo-elastic System with Neumann Boundary condition at the interior nodes of some particular networks

In this section, we investigate the stabilization of a thermo-elastic system with Neumann boundary condition at the interior nodes of some particular networks (composed of elastic and thermoelastic materials) similar to the particular networks

considered in Shel, 2012. In the first case, we consider trees  $(G_1)$  for which all exterior edges (except one) are thermo-elastic. In the second case, we consider the path  $(\mathcal{P})$  composed of two exterior elastic edges and an interior thermoelastic edge. In the third case, we consider  $(G_2)$ , trees of elastic materials, whose leaves (exterior nodes of the last generation) are connected to thermoelastic materials as follows: the thermoelastic body connects two leaves issued from the same vertex, with the condition that each leaf is connected to only one thermoelastic body.

In fact, the considered networks  $(G_1)$ ,  $(\mathcal{P})$  and  $(G_2)$  are particular graphs of the general networks covered in Section 3.1-Section 3.4. Notice that, if we apply the boundary conditions of Section 3.1 on these particular networks, we can deduce that the stabilization of the thermo-elastic system on  $(G_1)$ ,  $(\mathcal{P})$  and  $(G_2)$  is achieved when  $\theta$  satisfies Dirichlet condition on each end of every thermo-elastic edge (see Section 3.1-Section 3.4). In this section, we discuss the stabilization of the thermo-elastic system on these particular networks such that  $\theta$  satisfies Neumann boundary condition at each interior node connected to a thermo-elastic edge (see Figure 3.1).

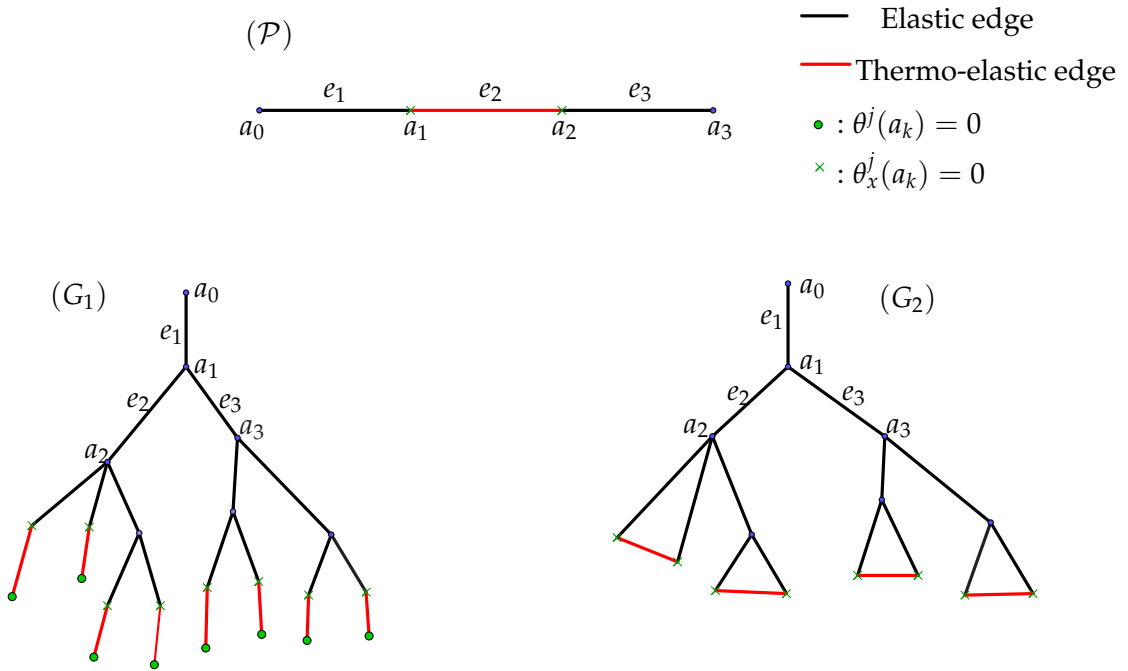


FIGURE 3.1: Some particular networks

The system is described as follows:

- On every thermo-elastic edge  $(j \in \mathcal{I}_{te})$  the following equations hold:

$$\begin{cases} u_{tt}^j - u_{xx}^j + \alpha_j \theta_x^j - \beta_j y_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ \theta_t^j - \kappa_j \theta_{xx}^j + \alpha_j u_{tx}^j & = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (3.5.1)$$

where  $\alpha_j$ ,  $\rho_j$ ,  $\kappa_j$  and  $\beta_j$  are positive constants.

- On every elastic edge ( $j \in \mathcal{I}_e$ ) one has:

$$\begin{cases} u_{tt}^j - u_{xx}^j - \beta_j y_t^j = 0 & \text{in } (0, \ell_j) \times (0, \infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j = 0 & \text{in } (0, \ell_j) \times (0, \infty), \end{cases} \quad (3.5.2)$$

where  $\beta_j$  and  $\rho_j$  are positive constants.

We assume that the initial data on the network  $\mathcal{N}$  are

$$\begin{cases} u^j(x, 0) = u_0^j(x), \quad u_t^j(x, 0) = u_1^j(x), \quad \forall j \in \mathcal{I}(\mathcal{N}) \\ y^j(x, 0) = y_0^j(x), \quad y_t^j(x, 0) = y_1^j(x), \quad \forall j \in \mathcal{I}(\mathcal{N}), \\ \theta^j(x, 0) = \theta_0^j(x), \quad \forall j \in \mathcal{I}_{te}(\mathcal{N}). \end{cases} \quad (3.5.3)$$

The boundary conditions of system (3.5.1)-(3.5.3) on the considered networks will be as follows.

The system satisfies the Dirichlet boundary condition for the displacement and temperature at the exterior nodes,

$$\begin{cases} u^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), \quad a_k \in V_{\text{ext}}, \\ y^j(a_k, t) = 0, & j \in \mathcal{I}(a_k), \quad a_k \in V_{\text{ext}}, \\ \theta^j(a_k, t) = 0, & j \in \mathcal{I}_{te}(a_k), \quad a_k \in V_{\text{ext}}. \end{cases} \quad (3.5.4)$$

The displacement is continuous at every interior node,

$$\begin{cases} u^j(a_k, t) = u^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), \quad a_k \in V_{\text{int}}, \\ y^j(a_k, t) = y^\ell(a_k, t), & j, \ell \in \mathcal{I}(a_k), \quad a_k \in V_{\text{int}}. \end{cases} \quad (3.5.5)$$

The temperature satisfies the Neumann condition at the interior nodes,

$$\theta_x^j(a_k, t) = 0, \quad j \in \mathcal{I}_{te}(a_k), \quad a_k \in V_{\text{int}}. \quad (3.5.6)$$

The system satisfies the balance condition at every interior node,

$$\begin{cases} \sum_{j \in \mathcal{I}_{te}(a_k)} d_{kj} (u_x^j(a_k, t) - \alpha_j \theta^j(a_k, t)) + \sum_{j \in \mathcal{I}_e(a_k)} d_{kj} u_x^j(a_k, t) = 0, & a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, & a_k \in V_{\text{int}}. \end{cases} \quad (3.5.7)$$

Mainly, we find sufficient conditions on the lengths of the purely elastic edges attached to the thermo-elastic ones so that the system is strongly stable and then exponentially stable on the above described networks.

Here the energy space  $\mathcal{H}_1$  is given by

$$\mathcal{H}_1 = \{(u, u_t, y, y_t, \theta) \in \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{V} \text{ satisfying (3.5.9) below}\}, \quad (3.5.8)$$

$$\int_0^{\ell_j} (\alpha_j u_x^j + \theta^j) dx = 0, \quad \forall j \in \mathcal{G}_{\text{int}} \cap \mathcal{I}_{te} \quad (3.5.9)$$

where,  $\mathcal{G}_{\text{int}}$  is the set of indices of edges adjacent to two interior vertices.

The Hilbert space  $\mathcal{H}_1$  is equipped with the inner product given in (3.2.3).

Next, we define the unbounded linear operator  $\mathcal{A}_1$  by:

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \\ y \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} v^j \\ u_{xx}^j - \alpha_j \theta_x^j + \beta_j z^j \\ z^j \\ \rho_j y_{xx}^j - \beta_j v^j \\ \kappa_j \theta_{xx}^j - \alpha_j v_x^j \end{pmatrix}_{j \in \mathcal{I}(\mathcal{N})} \quad (3.5.10)$$

with domain

$$D(\mathcal{A}_1) = \{(u, v, y, z, \theta) \in \mathcal{H}_1 \cap [\mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{V}^2], \text{ satisfying (3.5.11)}\}$$

$$\begin{cases} \theta^j(a_k) = 0, & j \in \mathcal{I}_{\text{te}}(a_k), & a_k \in V_{\text{ext}}, \\ \theta_x^j(a_k) = 0, & j \in \mathcal{I}_{\text{te}}(a_k), & a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} (u_x^j(a_k) - \alpha_j \theta^j(a_k)) = 0, & a_k \in V_{\text{int}}, \\ \sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, & a_k \in V_{\text{int}}. \end{cases} \quad (3.5.11)$$

Let  $\Omega$  denotes the set of indices of purely elastic edges attached to thermo-elastic edges in the network  $\mathcal{N}$ . The main results of this section are stated in the following Theorems:

**Theorem 3.5.1.** *The unbounded linear operator  $\mathcal{A}_1$  generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}_1$ .*

*Proof.* The same proof as the one of Theorem 3.2.1 implies that  $\mathcal{A}_1$  is a maximal dissipative operator. Then, using Lumer-Phillips Theorem (see Pazy, 1983),  $\mathcal{A}_1$  generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_1})_{t \geq 0}$  on  $\mathcal{H}_1$ .  $\square$

**Theorem 3.5.2.** *Consider the system (3.5.1)-(3.5.7) on  $\mathcal{N}$ . Assume that one of the following conditions holds:*

1)  $\mathcal{N}$  is the graph  $G_1$ ,

2)  $\mathcal{N}$  is the graph  $\mathcal{P}$ ,  $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(\mathcal{P})$ , and there exists  $j \in \{1, 3\}$  such that

$$\ell_j \neq \frac{m\pi}{\sqrt{2}\beta_j}, \quad \forall m \in \mathbb{N}^*, \quad (3.5.12)$$

3)  $\mathcal{N}$  is the graph  $G_2$ ,  $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(G_2)$ , and in every circuit  $C$ , for the unique  $j, k \in \Omega$  such that  $e_j$  and  $e_k$  are edges of  $C$ , we have

$$\sin(\sqrt{2}\beta_j \ell_j) + \sin(\sqrt{2}\beta_k \ell_k) \neq 0. \quad (3.5.13)$$

Then,  $i\mathbb{R} \subset \rho(\mathcal{A}_1)$  and therefore the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_1})_{t \geq 0}$  is strongly stable.

*Proof.* As before  $(I - \mathcal{A}_1)^{-1}$  being compact, then  $\sigma(\mathcal{A}_1) = \sigma_p(\mathcal{A}_1)$ . Thus, it is sufficient to prove that  $\sigma_p(\mathcal{A}_1) \cap i\mathbb{R} = \emptyset$ . Let  $\lambda \in \mathbb{R}$  and  $U = (u, v, y, z, \theta) \in D(\mathcal{A}_1)$  be such that

$$\mathcal{A}_1 U = i\lambda U,$$

equivalently, for all  $j \in \mathcal{I}(\mathcal{N})$  we have (3.3.1)-(3.3.5). Using (3.3.1), (3.3.3) to eliminate  $v^j$  and  $z^j$ , we obtain (3.3.6)-(3.3.8). Further, we easily check that

$$\sum_{j \in \mathcal{I}_{te}} \kappa_j \int_0^{\ell_j} |\theta_x^j|^2 dx = \operatorname{Re}(\mathcal{A}_1 U, U)_{\mathcal{H}_1} = \operatorname{Re}(i\lambda U, U)_{\mathcal{H}_1} = 0.$$

This implies that (3.3.9) holds. Thus,  $\theta^j$  is constant for all  $j \in \mathcal{I}_{te}$ .

If  $\lambda = 0$  then, using (3.3.6), (3.3.8) and (3.3.9), we have

$$\begin{cases} u_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ u_{xx}^j - \alpha_j \theta_x^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ \rho_j y_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \end{cases} \quad (3.5.14)$$

where  $\alpha_j = 0, \forall j \in \mathcal{I}_e$ .

Multiplying the second equation and third equation of (3.5.14) by  $\overline{u^j}$  and  $\overline{y^j}$ , respectively. Then, integrating over  $(0, \ell_j)$ , summing over  $j \in \mathcal{I}(\mathcal{N})$  and applying Green's formula, we get

$$\begin{cases} -\sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 dx + \sum_{j=1}^N u_x^j \overline{u^j} \Big|_0^{\ell_j} + \sum_{j=1}^N \int_0^{\ell_j} \alpha_j \theta^j \overline{u_x^j} dx - \sum_{j=1}^N \alpha_j \theta^j \overline{u^j} \Big|_0^{\ell_j} = 0, \\ \sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 dx - \sum_{j=1}^N \rho_j y_x^j \overline{y^j} \Big|_0^{\ell_j} = 0. \end{cases}$$

But, using (3.5.4), (3.5.5), (3.5.7) and (3.5.9), we obtain for all  $j \in \mathcal{I}(\mathcal{N})$ ,

$$\begin{cases} -\sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 dx - \sum_{j \in \mathcal{I}_{te}} \int_0^{\ell_j} |\theta^j|^2 dx = 0, \\ \sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 dx = 0. \end{cases}$$

This implies that  $u_x^j = y_x^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$  and  $\theta^j = 0$ , for all  $j \in \mathcal{I}_{te}$ . Again, by (3.5.4)-(3.5.7), we deduce that  $u^j = y^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ . Consequently, using (3.3.1) and (3.3.3), we conclude that  $U = 0$ .

Now, assume that  $\lambda \neq 0$ . We will proceed by distinguishing different cases:

**Case i.** Assume that  $\mathcal{N}$  is the graph  $G_1$ . Then, the proof in this case is exactly the same as the proof of Case 1 in Theorem 3.3.1. In fact, under the boundary conditions of Section 3.5, on each thermo-elastic edge  $e_j$  of  $(G_1)$ ,  $\theta^j$  satisfies the Dirichlet boundary condition on one end and the Neumann boundary condition on the other end. While, under the boundary conditions of Section 3.1,  $\theta^j$  satisfies the Dirichlet boundary condition on the both ends of each thermo-elastic edge  $e_j$  of  $(G_1)$ . This shows that on networks like  $(G_1)$ , if  $\theta^j$  satisfies Dirichlet boundary condition on only one end of each thermo-elastic edge  $e_j$ , then it is enough to prove that the system is strongly stable.

Now, before proceeding the other cases (**Case ii** and **Case iii** below), remark

that for an arbitrary network  $\mathcal{N}$  with  $\beta_j = \beta, \rho_j = 1$ , for all  $j \in \mathcal{I}(\mathcal{N})$ , and using (3.3.8) and (3.3.9), (3.3.16)-(3.3.20) hold for all  $j \in \mathcal{I}_{te}$ . Then, for  $\lambda \neq \pm\beta$  we have  $y^j = 0$ , for all  $j \in \mathcal{I}_{te}$  and thus by equation (3.3.19) we get  $u^j = 0$ , for all  $j \in \mathcal{I}_{te}$ . Then, by (3.3.8), we obtain that  $\theta^j = 0$ , for all  $j \in \mathcal{I}_{te}$ . Again, by proceeding using unique continuation Theorem in Hayek et al., 2020 and iteration technique used in Case 1 of the proof of Theorem 3.3.1, we conclude that  $u^j = v^j = y^j = z^j = 0$ , for all  $j \in \mathcal{I}(\mathcal{N})$ . So, it is enough to treat the remaining cases (**Case ii** and **Case iii** below) with  $\lambda = \pm\beta$ . Without loss of generality, assume that  $\lambda = \beta$ .

**Case ii.** Assume that  $\mathcal{N}$  is the path  $\mathcal{P}$ , composed of two exterior elastic edges and an interior thermo-elastic edge,  $\beta_j = \beta, \rho_j = 1$ , for all  $j \in \mathcal{I}(\mathcal{P})$  and there exists  $j \in \{1, 3\}$  such that (3.5.12) holds. Without loss of generality, assume that  $\pi_1(0) = a_0, \pi_2(0) = a_1$  and  $\pi_3(0) = a_2$ . For the thermoelastic edge  $e_2$ , we have  $u_{xx}^2 = y_{xx}^2 = 0$  but, using (3.3.17) and dividing by  $\lambda = \beta$ , we also have  $y^2 = iu^2$ . Thus,  $u^2$  and  $y^2$  can be written in the following form

$$y^2 = ax + b, \quad u^2 = -iax - ib, \quad \text{for some } a, b \in \mathbb{C}. \quad (3.5.15)$$

Moreover, using (3.3.9), (3.5.15) and (3.5.9), we can write

$$\theta^2 = i\alpha_2 a. \quad (3.5.16)$$

For the elastic edges  $\{e_1, e_3\}$ , (3.3.6)-(3.3.8) becomes,

$$\begin{cases} \beta^2 u^j + u_{xx}^j + i\beta^2 y^j = 0, & \text{on } (0, \ell_j), j = 1, 3, \\ \beta^2 y^j + y_{xx}^j - i\beta^2 u^j = 0, & \text{on } (0, \ell_j), j = 1, 3. \end{cases} \quad (3.5.17)$$

Using (3.5.4)-(3.5.7), (3.5.15) and (3.5.16), we have the following boundary conditions:

$$\begin{cases} u^1(0) = y^1(0) = 0, \\ u^1(\ell_1) = u^2(0) = -ib, \quad y^1(\ell_1) = y^2(0) = b, \\ u_x^1(\ell_1) = u_x^2(0) - \alpha_2 \theta^2(0) = -i(1 + \alpha_2^2)a, \quad y_x^1(\ell_1) = y_x^2(0) = a \end{cases} \quad (3.5.18)$$

and

$$\begin{cases} u^3(\ell_3) = y^3(\ell_3) = 0, \\ u^3(0) = u^2(\ell_2) = -ial_2 - ib, \quad y^3(0) = y^2(\ell_2) = al_2 + b, \\ u_x^3(0) = u_x^2(\ell_2) - \alpha_2 \theta^2(\ell_2) = -i(1 + \alpha_2^2)a, \quad y_x^3(0) = y_x^2(\ell_2) = a. \end{cases} \quad (3.5.19)$$

Consequently, (3.5.17), (3.5.18) and (3.5.19) leads to the following system

$$\Phi_x^j = A\Phi^j, \quad j = 1, 3 \quad (3.5.20)$$

where,

$$\Phi^j = \begin{pmatrix} u^j \\ u_x^j \\ y^j \\ y_x^j \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\beta^2 & 0 & -i\beta^2 & 0 \\ 0 & 0 & 0 & 1 \\ i\beta^2 & 0 & -\beta^2 & 0 \end{pmatrix}.$$

The solution of (3.5.20) is given by

$$\Phi^j = e^{Ax} \Phi^j(0), \quad j = 1, 3. \quad (3.5.21)$$

But using (3.5.18) and (3.5.19), we have

$$\Phi^1(0) = \begin{pmatrix} 0 \\ u_x^1(0) \\ 0 \\ y_x^1(0) \end{pmatrix}; \quad \Phi^1(\ell_1) = \begin{pmatrix} -ib \\ -i(1 + \alpha_2^2)a \\ b \\ a \end{pmatrix}, \quad (3.5.22)$$

and

$$\Phi^3(\ell_3) = \begin{pmatrix} 0 \\ u_x^3(\ell_3) \\ 0 \\ y_x^3(\ell_3) \end{pmatrix}; \quad \Phi^3(0) = \begin{pmatrix} -ial_2 - ib \\ -i(1 + \alpha_2^2)a \\ al_2 + b \\ a \end{pmatrix}. \quad (3.5.23)$$

Since we have

$$\Phi^1(0) = e^{-A\ell_1} \Phi^1(\ell_1) \text{ and } \Phi^3(\ell_3) = e^{A\ell_3} \Phi^3(0), \quad (3.5.24)$$

using (3.5.22), (3.5.23) and technical computations, we obtain

$$\begin{cases} \frac{i\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_1) + ial_1 - ib + \frac{i\alpha_2^2 \ell_1 a}{2} = 0, \\ \frac{\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_1) - al_1 + b - \frac{\alpha_2^2 \ell_1 a}{2} = 0, \end{cases} \quad (3.5.25)$$

and

$$\begin{cases} \frac{-i\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_3) - ia(\ell_2 + \ell_3) - ib - \frac{i\alpha_2^2 \ell_3 a}{2} = 0, \\ \frac{-\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_3) + a(\ell_2 + \ell_3) + b + \frac{\alpha_2^2 \ell_3 a}{2} = 0. \end{cases} \quad (3.5.26)$$

Multiplying the second equation of (3.5.25) (and (3.5.26) respectively) by  $i$ , then adding the resulting equation to the first equation of (3.5.25) (and (3.5.26) respectively) we obtain,

$$\frac{-i\alpha_2^2 a}{\beta\sqrt{2}} \sin(\sqrt{2}\beta\ell_j) = 0, \quad \forall j \in \{1, 3\}. \quad (3.5.27)$$

So, if there exists  $j \in \{1, 3\}$  such that  $\ell_j \neq \frac{m\pi}{\sqrt{2}\beta_j}$ ,  $\forall m \in \mathbb{N}^*$ , we deduce that  $a = 0$ .

Consequently, by (3.5.25) or (3.5.26) we deduce that  $b = 0$  and hence  $u^2 = y^2 = 0$  and  $\theta^2 = 0$ . By proceeding using unique continuation Theorem in Hayek et al., 2020 and iteration technique used in Case 1 of the proof of Theorem 3.3.1, we deduce that  $u^j = y^j = v^j = z^j = 0$ ,  $j = 1, 2, 3$ . The same procedure can be used in the case  $\lambda = -\beta$  so that the desired goal holds.

**Case iii.** Assume that  $\mathcal{N}$  is the graph  $G_2$ ,  $\beta_j = \beta, \rho_j = 1$ , for all  $j \in \mathcal{I}(G_2)$  and that in every circuit  $C$ , for the unique  $j, k \in \Omega$  such that  $e_j$  and  $e_k$  are edges of  $C$ , (3.5.13) holds. Notice that for all  $j \in \mathcal{I}_{te}$ , we have (3.3.21) holds. As in the proof of **Case 2** of Theorem 3.3.1, our aim is to prove (3.5.14). This would end the proof as in the case  $\lambda = 0$ . But since, (3.3.9), (3.3.16) and (3.3.18) hold, then it is enough to prove that (3.5.14) holds for every elastic edge. First, for a fixed circuit  $C$  of  $G_2$  without loss of generality, we may use the parametrizations from Figure 3.2.

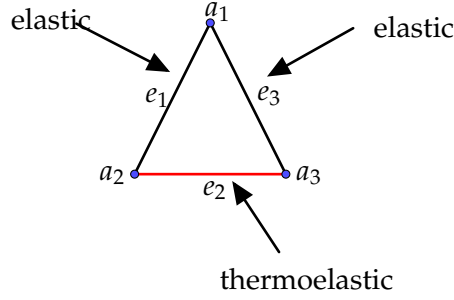


FIGURE 3.2: A circuit and its parametrizations:  $\pi_1(0) = a_1$ ,  $\pi_2(0) = a_2$ , and  $\pi_3(0) = a_3$ .

First, notice that for the thermoelastic edge  $e_2$ , (3.5.15) and (3.5.16) hold. Also, for the elastic edges  $\{e_1, e_3\}$ , (3.5.20) holds with

$$\Phi^1(\ell_1) = \begin{pmatrix} -ib \\ -i(1 + \alpha_2^2)a \\ b \\ a \end{pmatrix} \text{ and } \Phi^3(0) = \begin{pmatrix} -ial_2 - ib \\ -i(1 + \alpha_2^2)a \\ al_2 + b \\ a \end{pmatrix}. \quad (3.5.28)$$

Our aim is to find  $\{u^1(0), y^1(0), u^3(\ell_3), y^3(\ell_3)\}$ . For that purpose, we use (3.5.24) to find  $\Phi^1(0)$  and  $\Phi^3(\ell_3)$ .

Then, using (3.5.28) and technical computations, we have

$$\begin{cases} u^1(0) = \frac{i\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_1) + ial_1 - ib + \frac{i\alpha_2^2 \ell_1 a}{2}, \\ y^1(0) = \frac{\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_1) - al_1 + b - \frac{\alpha_2^2 \ell_1 a}{2}. \end{cases}$$

This implies that

$$y_{xx}^1(0) = \beta^2(y^1(0) - iu^1(0)) = \frac{\alpha_2^2 a \beta}{\sqrt{2}} \sin(\sqrt{2}\beta\ell_1). \quad (3.5.29)$$



Similarly, using (3.5.28) and technical computations, we have

$$\begin{cases} u^3(\ell_3) = \frac{-i\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_3) - ia(\ell_2 + \ell_3) - ib - \frac{i\alpha_2^2\ell_3 a}{2}, \\ y^3(\ell_3) = \frac{-\sqrt{2}\alpha_2^2 a}{4\beta} \sin(\sqrt{2}\beta\ell_3) + a(\ell_2 + \ell_3) + b + \frac{\alpha_2^2\ell_3 a}{2}. \end{cases}$$

This implies that

$$y_{xx}^3(0) = \beta^2(y^3(0) - iu^3(0)) = \frac{-\alpha_2^2 a \beta}{\sqrt{2}} \sin(\sqrt{2}\beta\ell_3). \quad (3.5.30)$$

On the other hand, using (3.5.4)-(3.5.7), we have  $u^1(0) = u^3(\ell_3)$  and  $y^1(0) = y^1(\ell_3)$ . Then,

$$y_{xx}^1(0) = y_{xx}^3(\ell_3). \quad (3.5.31)$$

This means that

$$\frac{\alpha_2^2 a \beta}{\sqrt{2}} \left( \sin(\sqrt{2}\beta\ell_1) + \sin(\sqrt{2}\beta\ell_3) \right) = 0.$$

Notice that if  $\sin(\sqrt{2}\beta\ell_1) + \sin(\sqrt{2}\beta\ell_3) \neq 0$ , then,  $a = 0$ , i.e.,  $\theta^2 = 0$ . Using (3.3.8), we obtain that  $u_x^2 = 0$ . Again, repeating the same strategy in every circuit of  $\mathcal{N}$  and using the fact that (3.5.13) holds, we deduce that

$$\theta^j = 0, \forall j \in \mathcal{I}_{te}. \quad (3.5.32)$$

Hence,  $u_x^j = 0$ , for all  $j \in \mathcal{I}_{te}$  and (3.3.22) holds. Consequently, using (3.5.4)-(3.5.7), (3.5.32), (3.3.21)-(3.3.22) and using iteration method from the leaves to the root, we prove that every elastic edge satisfies (3.3.29). This implies that  $y_{xx}^j = 0$ , for all  $j \in \mathcal{I}_e$ . Finally, we can proceed as the case  $\lambda = 0$ . This finishes the proof. The same procedure is used in the case  $\lambda = -\beta$  so that the desired goal holds.  $\square$

**Theorem 3.5.3.** *Let  $\mathcal{N}$  be an arbitrary network for which the operator  $(\mathbb{A}_1)$  associated with System (3.5.1)-(3.5.7) satisfies  $i\mathbb{R} \subset \rho(\mathcal{A}_1)$ . If  $\rho_j = 1$ , for all  $j \in \mathcal{I}(\mathcal{N})$ , then the energy of the system decays exponentially in  $\mathcal{H}_1$ . In other words, there exist two positive constants  $M$  and  $\epsilon$  such that*

$$\|e^{t\mathcal{A}_1} x_0\|_{\mathcal{H}_1} \leq M e^{-\epsilon t} \|x_0\|_{\mathcal{H}_1}, \quad \forall t > 0, \forall x_0 \in \mathcal{H}_1.$$

*Proof.* Same proof as the one of Theorem 3.4.1 holds.  $\square$

**Remark 3.5.4.** Examples of networks for which  $i\mathbb{R} \subset \rho(\mathcal{A}_1)$  are given by Theorem 3.5.2.

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