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Kunkun Song

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Laboratoire d'Analyse et de Mathématiques Appliquées

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Présentée pour l'obtention du grade de Docteur

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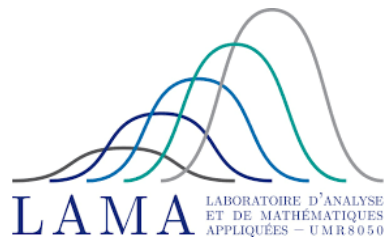
Kunkun SONG

**Quelques contributions à l'analyse multifractale en théorie
des fractions continues et à la transformée de Fourier des
mesures**

Spécialité : Mathématiques

Soutenue le 5 décembre 2020 devant le jury composé de :

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Liste de Publications

- (1) The growth rate of the partial quotients in Engel continued fractions, (avec L.-L Fang, Z.-H Li et Y.-L Yu), prépublication.
- (2) Multifractal analysis of the convergence exponent in continued fractions, (avec L.-L Fang, J.-H Ma et M. Wu), prépublication.
- (3) Some exceptional sets of Borel-Bernstein Theorem in continued fractions, (avec L.-L Fang et J.-H Ma), The Ramanujan Journal, à paraître.
- (4) Fourier decay rate of cion-tossing type measures, (avec X. Gao, J.-H Ma et Y.-F Zhang), J. Math. Anal. Appl. 484 (2020), no.1, 123706, 14pp.
- (5) Exceptional sets related to the run-length function of beta-expansions, (avec L.-L Fang et M. Wu), Fractals 26 (2018), no.4, 1850049,10pp.
- (6) Hausdorff dimension of level sets in Engel continued fraction, (avec L.-L Fang, Y.-Y Chang et J.-H Ma), Publicationes Mathematicae Debrecen 92 (2018), no.3-4, 317-330.
- (7) Level sets of partial maximal digits for Lüroth expansion, (avec L.-L Fang et J.-H Ma), International Journal of Number Theory Vol.13, No.10 (2017) 2777-2790.
- (8) Dimension of level sets in GCF expansion with parameters, (avec Y.-Y Chang), Journal of Number Theory 180 (2017) 743-755.

N.B.: Les articles (1), (5), (6), (7) et (8) ne sont pas joints dans la thèse.

摘要

本文研究与连分数部分商满足给定增长速度相关的集合的 Hausdorff 维数以及一类概率测度的 Fourier 变换的衰减速度。主要结果包含以下两个部分：

一，以连分数展式的部分商序列满足非减性的所有实数组成的集合为研究对象，我们通过集合的每一个元素的连分数展式的部分商序列定义的收敛指数来定义该元素的收敛指数。从重分形分析的观点出发，我们运用 Hausdorff 维数刻画这些实数的收敛指数定义的水平集的大小，结果表明，它的维数在水平 1 处是不可导的，这是维数中的不常见现象。与此同时，作为王保伟和吴军的结果的一个补充，通过对 Borel-Bernstein 定理例外集的研究，我们也得到了一类与部分商增长速度有关的水平集的 Hausdorff 维数。

二，给定区间 $[0, 1]$ 上具有卷积结构的概率测度

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{2^{-n}} \right),$$

其中 δ_x 是 Dirac 测度， ϕ 是定义在 \mathbb{N} 上并且取值为区间 $(0, 1)$ 的权重函数。对于任意给定的权重函数 $\phi(n)$ ，我们首先准确地刻画了测度 μ 的点态 Fourier 变换的衰减速度。该结论推广了 Hartman 和 Kershner 的一个经典结果。其次，基于 Cassels 和 Schmidt 的方法，我们利用 Davenport-Erdős-Leveque 判别法证明了几乎所有的实数关于测度 μ 都是绝对正规的。作为应用，我们给出了满足傅立叶变换的衰减速度很慢的一类奇异 Rajchman 测度，并且证明了几乎所有的实数关于这类测度依然是绝对正规的。该结论可以作为 Lyons 在研究非正规数集合上所得结果的一个补充。

关键词: Hausdorff 维数；连分数；收敛指数；Fourier 变换；正规数。

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Abstract

In this thesis, we are concerned with the Hausdorff dimension of sets of points with prescribed growth rate of partial quotients in continued fractions, and the decay rate of the Fourier transform of a class of probability measures.

In the first part, we consider the real numbers whose partial quotients of the continued fractions are non-decreasing. The convergence exponent of such a number is defined as the convergence exponent of its sequence of partial quotients. We do multifractal analysis of the convergence exponents of these real numbers. The multifractal spectrum, which is a function to each level associated the Hausdorff dimension of the level set of convergence exponents, has been determined. It turns out that the spectrum is not differentiable at the level of exponent 1. Thus an unusual phase transition of such spectrum appears. Meanwhile, as a complement of a result of Wang and Wu, we obtain the Hausdorff dimensions of some level sets related to the exceptional sets of Borel-Bernstein Theorem in continued fractions.

In the second part, we focus on the infinite convolution on $[0, 1]$ defined by

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{2^{-n}} \right),$$

where δ_x is Dirac measure and ϕ is a weight function defined on \mathbb{N} taking values in $(0, 1)$. For any given weight function $\phi(n)$, we obtain the explicit pointwise rate of decay of the Fourier transform of the measure μ . Our result generalizes a classical result of Hartman and Kershner. Using the methods of Cassels and Schmidt and applying the Davenport-Erdős-Leveque Theorem, we prove that almost all real numbers are absolutely normal with respect to μ . As an application, we provide some new examples of measures whose rates of decay of their Fourier transform could be very slow, yet almost all real numbers are absolutely normal, which complements a result of Lyons on the set of non-normal numbers.

Key Words: Hausdorff dimension; continued fractions; convergence exponent; Fourier transform; normal numbers.

Written by Kunkun SONG

Supervised by Prof. Lingmin LIAO

Supervised by Prof. Jihua MA

Résumé

Dans cette thèse on s'intéresse à la dimension de Hausdorff des points donc le taux de croissance des quotients partiels en fractions continues est prescrit, et au taux de décroissance pour la transformée de Fourier d'une classe de mesures de probabilité.

Dans la première partie, on s'intéresse aux nombres réels dont les quotients partiels des fractions continues sont croissants. L'exposant de convergence d'un tel nombre est défini par l'exposant de convergence de sa suite de quotients partiels. On cherche à réaliser l'analyse multifractale de cet exposant de convergence. Ce spectre multifractal, qui à un certain $h \in \mathbb{R}$ associe la dimension de Hausdorff des points d'exposant de convergence h , a été calculé. Il résulte de cette étude que le spectre multifractal n'est pas différentiable au point $h = 1$. En conséquence, on observe une transition de phase dans le spectre. D'autre part, en complément d'un résultat de Wang et Wu, on obtient les dimensions de Hausdorff de certains ensembles de niveaux associés aux ensembles exceptionnels issus du théorème de Borel-Bernstein sur les fractions continues.

La seconde partie traite la convolution infinie de mesures sur $[0, 1]$ définie par

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{2^{-n}} \right),$$

où δ_x désigne la mesure de Dirac au point x et ϕ est une fonction de poids définie sur \mathbb{N} et à valeurs dans $(0, 1)$. Pour toute fonction de poids ϕ , on obtient explicitement le taux de décroissance ponctuel de la transformée de Fourier de la mesure μ . Notre résultat généralise un résultat classique de Hartman et Kershner. En utilisant la méthode de Cassels et Schmidt et en appliquant le Théorème de Davenport-Erdős-Leveque, on prouve que μ -presque tout nombre réel est absolument normal. En application, on fournit de nouveaux exemples de mesures dont le taux de décroissance de la transformée de Fourier peut être très lent, et pour laquelle presque tout nombre est absolument normal, ce qui complète un résultat de R. Lyons concernant l'ensemble des nombres non normaux.

Mots clés: dimension de Hausdorff; fractions continues; exposant de convergence; transformée de Fourier; nombres normaux.

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1 Aperçu des travaux

La géométrie fractale, une branche relativement jeune des mathématiques, fournit de nouvelles méthodes d'analyse d'ensembles ou de fonctions qui ne sont pas suffisamment lisses ou réguliers. La notion de dimension de ces "fractals" est un concept clé en géométrie fractale. Deux des propriétés importantes des fractals sont l'auto-similarité ainsi que la notion de dimension non nécessairement entière. Récemment, l'intérêt pour l'étude d'ensembles irréguliers provenant de la théorie des nombres, de l'analyse harmonique, des systèmes dynamiques ainsi que de beaucoup d'autres champs mathématiques, a été croissant.

Cette thèse est une contribution à la recherche interdisciplinaire entre géométrie fractale ainsi que deux autres branches des mathématiques, la théorie des nombres, et l'analyse harmonique. Plus précisément, la première partie de cette thèse est dédiée à l'étude de la dimension de Hausdorff de certains ensembles de niveaux du taux de croissance des quotients partiels de la décomposition en fraction continue et la seconde partie de cette thèse étudie la transformée de Fourier des mesures de type coin-tossing et leurs applications. La première partie consiste en les chapitres 3, 5 et 6, dans lesquels on obtient le spectre multifractal de l'exposant de convergence définie à l'aide de la suite des quotients partiels du développement en fraction continue et la dimension de Hausdorff des ensembles exceptionnels associés au Théorème de Borel-Bernstein sur les fractions continues. La seconde partie consiste en les chapitres 4 et 7, où l'on étudie la transformée de fourrier de mesures, particulièrement celle d'une certaine classe de mesures de probabilité appelées mesures de type coin-tossing.

1.1 Analyse multifractale en théorie des fractions continues

Il est connu que tout nombre réel $x \in [0, 1)$ admet un *développement en fraction continue* de la forme

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}}$$

où $a_1(x), a_2(x), a_3(x), \dots$ sont des entiers positifs, appelés les *quotients partiels* de du développement en fraction continue de x . Un sujet important dans l'étude des fractions continues concerne l'étude des propriétés des quotients partiels de Lebesgue presque tout nombre. Borel [9] a démontré que pour Lebesgue presque tout $x \in [0, 1)$, $\limsup_{n \rightarrow \infty} a_n(x) = \infty$. Le Théorème de Borel-Bernstein démontre en fait plus (loi du "0-1"), voir [1, 9, 10], et joue un rôle clé en théorie métrique des fractions continues.

Theorem 1.1 (Théorème de Borel-Bernstein). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction. Alors*

$$\mathcal{L}\left(\left\{x \in [0, 1) : a_n(x) \geq \psi(n) \text{ , pour une infinité de } n\right\}\right) = \begin{cases} 0, & \sum_{n \geq 1} \frac{1}{\psi(n)} < \infty, \\ 1, & \sum_{n \geq 1} \frac{1}{\psi(n)} = \infty, \end{cases}$$

où \mathcal{L} désigne la mesure de Lebesgue.

Le Théorème de Borel-Bernstein implique que beaucoup d'ensembles de points pour lesquels les quotients partiels vérifient certaines restrictions sont de mesure de Lebesgue nulle. Cela conduit naturellement à considérer la taille de ces ensembles du point de vue de l'analyse multifractale. Les premiers travaux publiés sur le sujet sont dus à Jarník [53] qui démontra que l'ensemble des points dont les quotients partiels sont bornés est de dimension de Hausdorff 1. Plus tard, les ensembles de mesure de Lebesgue nulle du type

$$E_M := \left\{ x \in [0, 1) : 1 \leq a_n(x) \leq M \text{ pour tout } n \geq 1 \right\}, (M \geq 2)$$

particulièrement pour le cas $M = 2$, ont été étudié par Good [43], Bumby [14, 15], Hensley [46, 47], Mauldin et Urbański [76], Pollicott et Weiss [82], Jenkinson et Pollicott [55, 56], Jenkinson [54], Chousionis, Leykekhman et Urbański [19] et al.

Le problème ci-dessus concerne les fractions continues dont les quotients partiels restent petits. On peut aussi considérer les nombres dont les quotients partiels deviennent grands, en un certain sens. Des exemples typiques sont

$$(I) \left\{ x \in [0, 1) : a_n(x) \geq \psi(n), \text{ pour tout } n \geq 1 \right\}, (\psi(n) \rightarrow \infty \text{ quand } n \rightarrow \infty)$$

$$(II) \left\{ x \in [0, 1) : a_n(x) \geq \psi(n), \text{ pour une infinité de } n \right\}.$$

L'étude de ces ensembles remonte à Good [43] qui a démontré que l'ensemble

$$\left\{ x \in [0, 1) : a_n(x) \rightarrow \infty \text{ quand } n \rightarrow \infty \right\}$$

est de dimension de Hausdorff $\frac{1}{2}$. Suite à cela, beaucoup de chercheurs ont continué à étudier les ensembles de points pour lesquels le taux de croissance des quotients partiels vérifient des restrictions sous différents angles, voir par exemple, Hirst [49], Ramharther [85], Cusick [21], Łuczak [72], Feng, Wu, Liang et Tseng [40], Wang et Wu [93, 94], Wu et Xu [101], Xu [102], Fan, Liao, Wang et Wu [36, 38], Liao et Rams [69, 70].

Nous concentrerons notre attention sur certains ensembles associés aux ensembles typiques (I) et (II) concernant le taux de croissance des quotients partiels.

1.1.1 Ensembles de niveaux du taux de croissance associés aux ensembles typiques (I)

Soit $\tau(x)$ l'exposant de convergence associé à la suite de quotients partiels de x , i.e.,

$$\tau(x) := \inf \left\{ s \geq 0 : \sum_{n \geq 1} a_n^{-s}(x) < \infty \right\}.$$

Le Théorème ergodique de Birkhoff implique que $\tau(x)$ vaut $+\infty$ pour Lebesgue presque tout $x \in [0, 1)$. Il est naturel de considérer le spectre multifractal de l'exposant de convergence $\tau(x)$, i.e., les dimensions de Hausdorff des ensembles de niveaux

$$\Delta(\alpha) = \left\{ x \in [0, 1) : \tau(x) = \alpha \right\} \quad (0 \leq \alpha < \infty).$$

En fait, la définition de $\tau(x)$ montre que l'ensemble $\Delta(\alpha)$ dépend profondément de l'ensemble (I). En d'autres mots, l'ensemble $\Delta(\alpha)$ correspond à l'ensemble des points pour lesquels le taux de croissance tend vers $+\infty$ en un certain sens qui dépend de α (comparable à $n^{\frac{1}{\alpha}}$). Dans la suite on présente d'abord le spectre multifractal de l'exposant de convergence $\tau(x)$.

Theorem 1.2 (Chapitre 5, Théorème 5.9). *Pour tout $0 \leq \alpha < \infty$, on a*

$$\dim_{\text{H}} \Delta(\alpha) = \frac{1}{2}.$$

Le Théorème 1.2 montre que le spectre multifractal de l'exposant de convergence $\tau(x)$ est une fonction constante de $\alpha \in [0, \infty)$. Si la suite des quotients partiels $(a_n(x))_{n \geq 1}$ est croissante, alors

$$\tau(x) = \limsup_{n \rightarrow \infty} \frac{\log n}{\log a_n(x)},$$

voir Pólya and Szegő [84, p. 26]. On s'intéresse donc naturellement à l'étude du spectre multifractal de l'exposant de convergence $\tau(x)$ pour les points x dont la suite des quotients partiels est croissante. Plus explicitement, on cherche à calculer la dimension de Hausdorff des intersections entre des ensembles $\Delta(\alpha)$ et Λ , où

$$\Lambda = \left\{ x \in [0, 1) : a_n(x) \leq a_{n+1}(x), \forall n \geq 1 \right\}.$$

Il s'avère que le spectre multifractal de l'exposant de convergence $\tau(x)$ pour $x \in \Lambda$ est affine par morceaux.

Theorem 1.3 (Chapitre 5, Théorème 5.11). *Pour $\alpha \geq 0$, on a*

$$\dim_{\text{H}} (\Delta(\alpha) \cap \Lambda) = \begin{cases} \frac{1-\alpha}{2}, & 0 \leq \alpha \leq 1, \\ 0, & \alpha > 1. \end{cases}$$

Remarquons que pour tout $\alpha \geq 0$, on a

$$\Delta(\alpha) \cap \Lambda = \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = \frac{1}{\alpha} \right\} := E_{\text{inf}}(\Lambda, \frac{1}{\alpha}).$$

Cela signifie que le problème du calcul du spectre multifractal de l'exposant de convergence $\tau(x)$ pour $x \in \Lambda$ est précisément lié au problème du calcul de dimensions de Hausdorff d'ensembles de niveaux donnés par une certaine fonction de liminf. Ainsi le Théorème 1.3 est équivalent au Théorème suivant.

Theorem 1.4 (Chapitre 5, Théorème 5.12). *Pour $\alpha \geq 0$, on a*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

Plus généralement, on cherche à calculer la dimension de Hausdorff de l'ensemble

$$E_{\text{inf}}(\Lambda, \psi) := \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

où $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ est une fonction positive.

Theorem 1.5 (Chapitre 5, Théorème 5.13). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$.*

(1) *Si $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) quand $n \rightarrow \infty$, alors*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

(2) *Si $\psi(n)/\log n \rightarrow \infty$ quand $n \rightarrow \infty$ et $\psi(n)$ est croissante, alors*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \frac{1}{B+1},$$

où B est donné par

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Si on compare le Théorème 1.2 avec le Théorème 1.3, on comprend que la monotonie de la suite $(a_n(x))_{n \geq 1}$ joue un rôle crucial dans le Théorème 1.3. Motivé par cette remarque, en omettant l'hypothèse de monotonie de la suite $(a_n(x))_{n \geq 1}$ dans $E_{\text{inf}}(\Lambda, \psi)$ on définit

$$E_{\text{inf}}(\psi) := \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

On montre le résultat suivant concernant la dimension de Hausdorff de $E_{\text{inf}}(\psi)$.

Theorem 1.6 (Chapitre 5, Théorème 5.16). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$. Alors*

$$\dim_{\text{H}} E_{\text{inf}}(\psi) = \frac{1}{B+1},$$

où B est défini par

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Les démonstrations des Théorèmes 1.5 et 1.6, révèlent que la dimension de Hausdorff des ensembles de niveaux d'une fonction limite, i.e., des ensembles

$$E(\Lambda, \psi) := \left\{ x \in \Lambda : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\} \text{ et } E(\psi) := \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

est facile à obtenir.

Theorem 1.7 (Chapitre 5, Théorème 5.17). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$.*

(1) *Si $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) quand $n \rightarrow \infty$, alors*

$$\dim_{\text{H}} E(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

(2) *Si $\psi(n)/\log n \rightarrow \infty$ quand $n \rightarrow \infty$ et $\psi(n)$ est croissante, alors*

$$E(\Lambda, \psi) \neq \emptyset \quad \text{et} \quad \dim_{\text{H}} E(\Lambda, \psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1)+\dots+\psi(n)}}.$$

Theorem 1.8 (Chapitre 5, Théorème 5.18). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$. Alors*

$$\dim_{\text{H}} E(\psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1)+\dots+\psi(n)}}.$$

1.1.2 Ensembles exceptionnels associés aux ensembles typiques (II)

Le **Théorème de Borel-Bernstein**, a pour conséquence le fait suivant

$$\limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = 1$$

pour Lebesgue presque tout $x \in [0, 1)$. Ainsi il est naturel de considérer la dimension de Hausdorff de ces ensembles exceptionnels, i.e., des ensembles

$$E_{\text{sup}}(\psi) := \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

où $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ est une fonction positive.

Theorem 1.9 (Chapitre 6, Théorème 6.5). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$. On a les assertions suivantes.*

(1) *Si $\psi(n)/n \rightarrow 0$ quand $n \rightarrow \infty$, alors $\dim_{\text{H}} E_{\text{sup}}(\psi) = 1$.*

(2) *Si $\psi(n)/n \rightarrow \alpha$ ($0 < \alpha < \infty$) quand $n \rightarrow \infty$, alors $\dim_{\text{H}} E_{\text{sup}}(\psi) = S(\alpha)$, où*

$$S(\alpha) = \dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq (\exp(\alpha))^n, \text{ pour une infinité de } n \right\}.$$

(3) *Si $\psi(n)/n \rightarrow \infty$ quand $n \rightarrow \infty$, alors $\dim_{\text{H}} E_{\text{sup}}(\psi) = \frac{1}{A+1}$, où A est donné par*

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

De la même façon, on cherche à savoir si imposer une condition de monotonie de la suite des quotients partiels affecte la dimension de Hausdorff des ensembles de niveaux. À la façon du Théorème 1.9, on considère l'ensemble

$$E_{\text{sup}}(\Lambda, \psi) := \left\{ x \in \Lambda : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

où $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ est une fonction positive.

Theorem 1.10 (Chapitre 6, Théorème 6.10). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n)/\log n \rightarrow \infty$ quand $n \rightarrow \infty$. Alors*

$$\dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) = \frac{1}{A + 1},$$

où A est donné par

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Le Théorème 1.10 montre que l'hypothèse de monotonie des suites de quotients partiels $(a_n(x))_{n \geq 1}$ est redondante dans le cas où $\psi(n)$ croît plus vite que $\log n$. De plus, on souhaiterait étudier le cas où la limite dans la définition de l'ensemble $E(\psi)$ n'existe pas. Plus précisément, on s'intéresse à la dimension de Hausdorff de l'intersection de l'ensemble de niveaux correspondants à la \liminf et à la \limsup , i.e.,

$$H(\alpha, \beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \alpha, \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \beta \right\}.$$

Dans le cas $0 = \alpha < \beta < \infty$, on a

$$\dim_{\text{H}} H(\alpha, \beta) = \dim_{\text{H}} E_{\text{sup}}(\psi).$$

Dans le cas $0 < \alpha < \beta < \infty$, On obtient des résultats partiels concernant le calcul de $\dim_{\text{H}} H(\alpha, \beta)$.

Theorem 1.11 (Chapitre 6, Théorème 6.11). *Soit $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction satisfaisant $\psi(n) \rightarrow \infty$ quand $n \rightarrow \infty$. Si*

$$\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = 0 \text{ et } \lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = 1,$$

alors on a

$$\dim_{\text{H}} H(\alpha, \beta) = \frac{1}{2}.$$

Le calcul de ces dimensions soulève deux questions naturelles: l'une est de comparer ces résultats avec les dimensions des ensembles de niveaux correspondants aux \liminf , \lim and \limsup , et l'autre est de comprendre si l'hypothèse de monotonie des suites des quotients partiels affecte la dimension des ensembles ci-dessus.

1.2 Transformée de Fourier et applications

Soit μ une mesure borélienne finie sur \mathbb{R} . La transformée de Fourier de μ est définie par

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{2\pi ixt} d\mu(x), \quad t \in \mathbb{R}.$$

Une mesure μ est appelée une mesure de Rajchman si $\widehat{\mu}(t) \rightarrow 0$ quand $|t| \rightarrow \infty$. Les estimées sur la vitesse de décroissance de $\widehat{\mu}(t)$ quand $|t| \rightarrow \infty$ sont très utiles en analyse de Fourier, systèmes dynamiques et géométrie fractale. Par exemple, une majoration classique de la transformée de Fourier de mesure provenant de la théorie du potentiel (voir Mattila [75, p. 168]) montre que si la vitesse de décroissance de la transformée de Fourier de μ est polynomiale, c'est à dire $|\widehat{\mu}(t)| = O(|t|^{-\varepsilon})$ quand $|t| \rightarrow \infty$ pour une certaine constante positive ε , alors la dimension de Hausdorff de μ satisfait $\dim_{\text{H}} \mu \geq \min\{2\varepsilon, 1\}$. De plus, μ -presque tout point est absolument normal. En général, il n'est pas aisé d'obtenir des estimées de la vitesse de décroissance de $\widehat{\mu}(t)$ quand $|t| \rightarrow \infty$. Il peut arriver que $\widehat{\mu}(t)$ ne tende pas vers 0, alors que les moyennes asymptotiques de $\widehat{\mu}(t)$ tendent vers 0 quand $|t| \rightarrow \infty$. Cela a été prouvé en 1939 par Wiener et Wintner [98] pour la mesure uniforme sur le Cantor triadique, et généralisé par Strichartz [91] à des mesures auto-similaires plus générales. Un des exemples les plus importants de mesures auto-similaires pour lesquelles l'on a des estimations précises sont les convolutions de Bernoulli, i.e., la distribution de probabilité définie par $\sum_{n=0}^{\infty} \pm \lambda^n$, où les signes sont choisis indépendamment, avec probabilité $\frac{1}{2}$ et $0 < \lambda < 1$, voir par exemple, Kershner [61], Erdős [31] et Bufetov et Solomyak [13].

Notre attention se porte particulièrement sur les convolutions sur $[0, 1]$ définies par

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{2^{-n}} \right),$$

où δ_x est une mesure de Dirac et ϕ est une fonction de poids définie sur \mathbb{N} et à valeurs dans $(0, 1)$. La mesure μ est appelée une mesure de type coin-tossing. De telles mesures peuvent être interprétées de la façon suivante. Un nombre infini dénombrable de pièces sont lancées, avec probabilité $\frac{1}{2}(1 - \phi(n))$ la n -ième pièce donne face. Soit X_n la variable aléatoire qui vaut un si la n -ième pièce donne face et zero sinon. Alors μ est la distribution de probabilité de $\sum_{n=1}^{\infty} 2^{-n} X_n$. La loi classique des types purs de Jensen et Wiener [57] implique que μ est soit complètement singulière soit absolument continue. Salem [86] donna une caractérisation de la singularité de μ : μ est singulière si et seulement si la série $\sum \phi^2(n)$ diverge. De plus, il démontra aussi que $\phi(n) \rightarrow 0$ est une condition nécessaire pour que μ soit une mesure de Rajchman. Blum et Epstein [8] ont montré que $\phi(n) \rightarrow 0$ est aussi une condition suffisante pour que μ soit une mesure de Rajchman. Autant que l'on sache, aucun autre résultat quantitatif sur $\widehat{\mu}(t)$ n'était connu excepté celui de Hartman et Kershner [44], qui ont démontré que $|\widehat{\mu}(t)| = O(\log^{-\frac{1}{2}} |t|)$ quand $\phi(n) = n^{-\frac{1}{2}}$. En s'appuyant sur les travaux de Hartman et de Kershner [44], on obtient une caractérisation complète de la vitesse de décroissance ponctuelle de la transformée de Fourier pour les mesures de type coin-tossing pour des fonctions de poids générales.

Theorem 1.12 (Chapitre 7, Théorème 7.7). *Soit $\phi : \mathbb{N} \rightarrow (0, 1)$ une fonction décroissante satisfaisant $\phi(n) \rightarrow 0$ quand $n \rightarrow \infty$. Alors*

- (1) *si $\phi(n) < 2\phi(n+1)$ pour tout $n \geq 1$, alors $|\widehat{\mu}(t)| = O(\phi(\lceil \gamma \log_2 |t| \rceil))$, où $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ et $\delta \in (0, 1)$ est une constante dépendant seulement de ϕ ;*
- (2) *si $\phi(n) \geq 2\phi(n+1)$ pour tout $n \geq 1$, alors $|\widehat{\mu}(t)| = O(|t|^{-\gamma})$, où $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ et $\delta \in (0, 1)$ est une constante dépendant seulement de ϕ .*

Pour un nombre réel x , soit $b \geq 2$ un entier. Wall [92] a démontré que x est normal en base b si et seulement si la suite $(b^n x)_{n \geq 1}$ est uniformément distribuée modulo un. Invoquant le bien connu critère de Weyl concernant l'équidistribution modulo un, Davenport, Erdős et Leveque [25] ont établi un lien entre la vitesse de décroissance de la transformée de Fourier de mesures et l'équidistribution de certains nombres réels.

Theorem 1.13 (Davenport-Erdős-LeVeque). *Soit ν une mesure borélienne finie sur \mathbb{R} et $(s_n)_{n \geq 1}$ une suite de nombres réels. Si*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^N \widehat{\nu}(h(s_n - s_m)) < \infty$$

pour tout entiers $h \neq 0$, alors pour ν -presque tout nombre réel x , la suite $(s_n x)_{n \geq 1}$ est uniformément équidistribuée modulo un.

Ce théorème indique que quand la vitesse de décroissance de la transformée de Fourier est rapide (une décroissance en logarithme est suffisante), alors μ -presque tout nombre est normal. En se basant sur le Théorème de Davenport-Erdős-LeVeque, on utilise la méthode de Cassels [16] et Schmidt [87] pour démontrer que presque tout nombre réel est absolument normal pour les mesures de types coin-tossing dont la vitesse de décroissance de la transformée de Fourier obtenue dans le Théorème 1.12 peut être très lente.

Theorem 1.14 (Chapitre 7, Théorème 7.10). *Soit μ une mesure de type coin-tossing. Supposons que $\phi : \mathbb{N} \rightarrow (0, 1)$ est une fonction décroissante avec $\phi(n) \rightarrow 0$ quand $n \rightarrow \infty$. Alors μ -presque tout nombre réel est absolument normal.*

Grâce au théorème de Davenport-Erdős-LeVeque, on sait que la vitesse de décroissance de la transformée de Fourier d'une mesure donne juste une condition suffisante pour déterminer si presque tout nombre réel est absolument normal pour cette mesure. On se demande donc naturellement s'il existe d'une mesure singulière dont la vitesse de décroissance de la transformée de Fourier est lente mais pour laquelle presque tout nombre réel est absolument normal. On répond par la positive à cette question en construisant de nouveaux exemples non-triviaux de telles mesures dans le chapitre 7.

2 Introduction

Fractal Geometry, a relatively young branch of mathematics, provides new ways of studying the structure of sets or functions that are not sufficiently smooth or regular. Dimension theory of these irregular objects called “fractals” is a key topic in the field of fractal geometry. Two of the most important properties of fractals are self-similarity and non-integer dimension. In recent years, there has been an increased interest in investigating various properties of irregular sets arising in number theory, harmonic analysis, dynamical systems and many other mathematical fields.

This thesis is a contribution to the interdisciplinary studies between fractal geometry and two other branches of mathematics, one is number theory, the other is harmonic analysis. To be specific, the first part of the thesis is devoted to investigating the Hausdorff dimension of some level sets on the growth rate of partial quotients in continued fractions and the second part of the thesis studies the Fourier transform of the coin-tossing type measures and its applications. The first part consists of Chapters 3, 5 and 6, where we focus on studying the multifractal spectrum of the convergence exponent defined by the sequence of partial quotients and calculating the Hausdorff dimension of the exceptional sets of Borel-Bernstein Theorem in continued fractions. The second part consists of Chapters 4 and 7, where we are concerned with the Fourier transform of measures, especially for a class of probability measures called coin-tossing type measures.

2.1 Multifractal analysis in continued fractions

It is well known that each irrational number $x \in [0, 1)$ admits a unique *continued fraction expansion* of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}}$$

where $a_1(x), a_2(x), a_3(x), \dots$ are positive integers, called the *partial quotients* of the continued fraction expansion of x . One of the major subjects in the study of continued fractions is concerned with the properties of the partial quotients for Lebesgue almost all real numbers. Borel [9] proved that for Lebesgue almost all $x \in [0, 1)$, $\limsup_{n \rightarrow \infty} a_n(x) = \infty$. A more precise result is the Borel-Bernstein Theorem (“0–1” law), see [1, 9, 10], which plays a key role in the metrical theory of continued fractions.

Theorem 2.1 (Borel-Bernstein Theorem). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Then*

$$\mathcal{L}\left(\{x \in [0, 1) : a_n(x) \geq \psi(n) \text{ for infinitely many } n\}\right) = \begin{cases} 0, & \sum_{n \geq 1} \frac{1}{\psi(n)} < \infty, \\ 1, & \sum_{n \geq 1} \frac{1}{\psi(n)} = \infty, \end{cases}$$

where \mathcal{L} denotes the Lebesgue measure.

Borel-Bernstein Theorem implies that many sets of continued fractions for which their partial quotients obeying some restrictions are of null Lebesgue measure. This naturally lead one to investigate the sizes of such null sets from the viewpoint of multifractal analysis. The first published work was due to Jarník [53] who proved that the set of continued fractions whose partial quotients are bounded has full Hausdorff dimension. Later on, Lebesgue null sets of type

$$E_M := \left\{ x \in [0, 1) : 1 \leq a_n(x) \leq M \text{ for any } n \geq 1 \right\}, (M \geq 2)$$

especially for the case when $M = 2$, were investigated by Good [43], Bumby [14, 15], Hensley [46, 47], Mauldin and Urbański [76], Pollicott and Weiss [82], Jenkinson and Pollicott [55, 56], Jenkinson [54], Chousionis, Leykekhman and Urbański [19] et al.

The problem discussed above is concerned with continued fractions whose partial quotients remain small. One can also consider the sets of continued fractions whose partial quotients become large in some sense. Typical sets are

- (I) $\left\{ x \in [0, 1) : a_n(x) \geq \psi(n) \text{ for any } n \geq 1 \right\}$, ($\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$)
- (II) $\left\{ x \in [0, 1) : a_n(x) \geq \psi(n) \text{ for infinitely many } n \right\}$.

The study of these sets goes back to Good [43] who proved that the set

$$\left\{ x \in [0, 1) : a_n(x) \rightarrow \infty \text{ as } n \rightarrow \infty \right\}$$

is of Hausdorff dimension one-half. After that, many authors continued to study the growth rate of partial quotients with some restrictions from various aspects, see for example, Hirst [49], Ramharther [85], Cusick [21], Łuczak [72], Feng, Wu, Liang and Tseng [40], Wang and Wu [93, 94], Wu and Xu [101], Xu [102], Fan, Liao, Wang and Wu [36, 38], Liao and Rams [69, 70].

We will pay our attention to a series of sets related to the typical sets (I) and (II) on the growth rate of partial quotients in continued fractions.

2.1.1 Level sets of the convergence exponent related to typical set (I)

Let $\tau(x)$ be the convergence exponent of the sequence of partial quotients of x , i.e.,

$$\tau(x) := \inf \left\{ s \geq 0 : \sum_{n \geq 1} a_n^{-s}(x) < \infty \right\}.$$

Birkhoff's Ergodic Theorem indicates that $\tau(x)$ equals to infinity for Lebesgue almost all $x \in [0, 1)$. It is natural to consider the multifractal spectrum of the convergence exponent $\tau(x)$, i.e., the Hausdorff dimension of the level sets

$$\Delta(\alpha) = \left\{ x \in [0, 1) : \tau(x) = \alpha \right\} \quad (0 \leq \alpha < \infty).$$

In fact, the definition of $\tau(x)$ shows that the set $\Delta(\alpha)$ deeply depends on the typical set (i). In other words, the set $\Delta(\alpha)$ reflects the growth rate of partial quotients tending to infinity in some sense. In the following we first present the multifractal spectrum of the convergence exponent $\tau(x)$.

Theorem 2.2 (Chapter 5, Theorem 5.9). *For any $0 \leq \alpha < \infty$, we have*

$$\dim_{\mathbb{H}} \Delta(\alpha) = \frac{1}{2}.$$

Theorem 2.2 indicates that the multifractal spectrum of the convergence exponent $\tau(x)$ is a constant function of $\alpha \in [0, \infty)$. If the partial quotients $(a_n(x))_{n \geq 1}$ is a non-decreasing sequence, then

$$\tau(x) = \limsup_{n \rightarrow \infty} \frac{\log n}{\log a_n(x)},$$

see Pólya and Szegő [84, p. 26]. Thus we are also interested in studying the multifractal spectrum of the convergence exponent $\tau(x)$ defined by a non-decreasing sequence of partial quotients. That is, the Hausdorff dimension of the intersection of the sets $\Delta(\alpha)$ and Λ , where

$$\Lambda = \left\{ x \in [0, 1) : a_n(x) \leq a_{n+1}(x), \forall n \geq 1 \right\}.$$

It turns out that the multifractal spectrum of the convergence exponent $\tau(x)$ defined by the non-decreasing sequence of partial quotients is piecewise affine.

Theorem 2.3 (Chapter 5, Theorem 5.11). *For any $\alpha \geq 0$, we have*

$$\dim_{\mathbb{H}} (\Delta(\alpha) \cap \Lambda) = \begin{cases} \frac{1-\alpha}{2}, & 0 \leq \alpha \leq 1, \\ 0, & \alpha > 1. \end{cases}$$

Note that for any $\alpha \geq 0$, we have

$$\Delta(\alpha) \cap \Lambda = \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = \frac{1}{\alpha} \right\} := E_{\text{inf}}(\Lambda, \frac{1}{\alpha}).$$

This means that the problem of the multifractal spectrum of the convergence exponent $\tau(x)$ defined by the non-decreasing sequence of partial quotients is exactly the problem of the Hausdorff dimension of level sets of some lim inf function. Thus Theorem 2.3 is equivalent to the following theorem.

Theorem 2.4 (Chapter 5, Theorem 5.12). *For any $\alpha \geq 0$, we have*

$$\dim_{\mathbb{H}} E_{\text{inf}}(\Lambda, \alpha) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

More generally, we consider the Hausdorff dimension of the set

$$E_{\text{inf}}(\Lambda, \psi) := \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ is a function.

Theorem 2.5 (Chapter 5, Theorem 5.13). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

(1) If $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, then

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

(2) If $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$ and $\psi(n)$ is non-decreasing, then

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \frac{1}{B+1},$$

where B is given by

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Comparing with Theorem 2.2 and Theorem 2.3, we know that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ plays an essential role in Theorem 2.3. Motivated by this point, we remove the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ in $E_{\text{inf}}(\Lambda, \psi)$ and then consider the Hausdorff dimension of the corresponding set, i.e.,

$$E_{\text{inf}}(\psi) := \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

Theorem 2.6 (Chapter 5, Theorem 5.16). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} E_{\text{inf}}(\psi) = \frac{1}{B+1},$$

where B is given by

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

In the proof of Theorems 2.5 and 2.6, we observe that the Hausdorff dimension of the corresponding level sets of limit are easily obtained, i.e., the sets

$$E(\Lambda, \psi) := \left\{ x \in \Lambda : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\} \text{ and } E(\psi) := \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

Theorem 2.7 (Chapter 5, Theorem 5.17). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

(1) If $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, then

$$\dim_{\text{H}} E(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

(2) If $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$ and $\psi(n)$ is non-decreasing, then we have

$$E(\Lambda, \psi) \neq \emptyset \text{ and } \dim_{\text{H}} E(\Lambda, \psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1)+\dots+\psi(n)}}.$$

Theorem 2.8 (Chapter 5, Theorem 5.18). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\mathbb{H}} E(\psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)}}.$$

2.1.2 The exceptional sets related to typical set (II)

As a consequence of **Borel-Bernstein Theorem**, we have

$$\limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = 1$$

for Lebesgue almost all $x \in [0, 1)$. Thus it is natural to consider the Hausdorff dimension of its exceptional sets, i.e.,

$$E_{\text{sup}}(\psi) := \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

Theorem 2.9 (Chapter 6, Theorem 6.5). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function with $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. We have*

(1) *if $\psi(n)/n \rightarrow 0$ as $n \rightarrow \infty$, then $\dim_{\mathbb{H}} E_{\text{sup}}(\psi) = 1$,*

(2) *if $\psi(n)/n \rightarrow \alpha$ ($0 < \alpha < \infty$) as $n \rightarrow \infty$, then $\dim_{\mathbb{H}} E_{\text{sup}}(\psi) = S(\alpha)$, where*

$$S(\alpha) = \dim_{\mathbb{H}} \left\{ x \in [0, 1) : a_n(x) \geq (\exp(\alpha))^n \text{ for infinitely many } n \right\},$$

(3) *if $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, then $\dim_{\mathbb{H}} E_{\text{sup}}(\psi) = \frac{1}{A+1}$, where A is given by*

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Similarly, we would like to investigate whether the monotonicity of the sequence of partial quotients affects the dimensional results. Comparing with Theorem 2.9, we consider the set

$$E_{\text{sup}}(\Lambda, \psi) := \left\{ x \in \Lambda : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

Theorem 2.10 (Chapter 6, Theorem 6.10). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function with $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\dim_{\mathbb{H}} E_{\text{sup}}(\Lambda, \psi) = \frac{1}{A+1},$$

where A is given by

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Theorem 2.10 shows that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ is redundant for the case when $\psi(n)$ grows faster than $\log n$. Besides, we also would like to study the case when the limit in set $E(\psi)$ does not exist. Precisely, we are interested in the Hausdorff dimension of the intersection of corresponding level sets of \liminf and \limsup , i.e.,

$$H(\alpha, \beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \alpha, \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \beta \right\}.$$

For the case $0 = \alpha < \beta < \infty$, we have

$$\dim_{\text{H}} H(\alpha, \beta) = \dim_{\text{H}} E_{\text{sup}}(\psi).$$

For the case $0 < \alpha < \beta < \infty$, We obtain some partial results about $\dim_{\text{H}} H(\alpha, \beta)$.

Theorem 2.11 (Chapter 6, Theorem 6.11). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. If*

$$\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = 1,$$

then we have

$$\dim_{\text{H}} H(\alpha, \beta) = \frac{1}{2}.$$

The purpose of this research mainly focus on two aspects: one is to compare with the Hausdorff dimension of the corresponding sets of \liminf , \lim and \limsup , and another one is to study whether the monotonicity of the sequence of partial quotients affects the dimensional results for the above sets.

2.2 Fourier transform of measures and its application

Let μ be a finite Borel measure on \mathbb{R} . The Fourier transform of μ is defined by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{2\pi ixt} d\mu(x), \quad t \in \mathbb{R}.$$

A measure μ is called a Rajchman measure if $\widehat{\mu}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Estimates for the rate of decay of $\widehat{\mu}(t)$ as $|t| \rightarrow \infty$ are very useful in Fourier analysis, dynamical systems and fractal geometry. For instance, a classical bound of Fourier transform of measures coming from potential theoretic methods (see Mattila [75, p. 168]) shows that if the rate of decay of Fourier transform of a measure μ is polynomial, that is $|\widehat{\mu}(t)| = O(|t|^{-\varepsilon})$ as $|t| \rightarrow \infty$ for some positive constant ε , then the Hausdorff dimension of μ satisfies $\dim_{\mathbb{H}} \mu \geq \min\{2\varepsilon, 1\}$. Moreover, μ -almost all points are absolutely normal. In general, it is not easy to obtain the rate of decay of $\widehat{\mu}(t)$ as $|t| \rightarrow \infty$. Sometimes it may also happen that $\widehat{\mu}(t)$ does not vanish at infinity, while the asymptotic behavior of averages of $\widehat{\mu}(t)$ tends to 0 as $|t| \rightarrow \infty$. This has been proved in 1939 by Wiener and Wintner [98] for the standard middle-third Cantor measure, and generalized by Strichartz [91] to more general self-similar measures. One of the most important examples of self-similar measures which can be explicitly estimated are Bernoulli convolutions, i.e., the probability distribution of the sum $\sum_{n=0}^{\infty} \pm \lambda^n$, where the signs are chosen independently with probability $\frac{1}{2}$ and $0 < \lambda < 1$, see for example, Kershner [61], Erdős [31] and Bufetov and Solomyak [13].

We mainly focus on another type of infinite convolution on $[0, 1]$ defined by

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{2^{-n}} \right),$$

where δ_x is Dirac measure and ϕ is a weight function defined on \mathbb{N} taking values in $(0, 1)$. The measure μ is called coin-tossing type measure. Such a measure may be interpreted in the following way. A countably infinite sets of coins is tossed, with $\frac{1}{2}(1 - \phi(n))$ the probability of the n -th coin being heads. Let X_n be the random variable which is one if the n -th coin is heads and zero otherwise. Then μ is the probability distribution of $\sum_{n=1}^{\infty} 2^{-n} X_n$. The classical Jessen-Wintner law of pure types [57] implies that μ is either absolutely continuous or singular. Salem [86] gave a characterization that μ is singular if and only if the series $\sum \phi^2(n)$ is divergent. Moreover, he showed that $\phi(n) \rightarrow 0$ is a necessary condition that μ is a Rajchman measure. Blum and Epstein [8] proved that $\phi(n) \rightarrow 0$ is also a sufficient condition of μ being a Rajchman measure. As far as we know, no quantitative results on $\widehat{\mu}(t)$ was known except the one due to Hartman and Kershner [44], who proved that $|\widehat{\mu}(t)| = O(\log^{-\frac{1}{2}} |t|)$ when $\phi(n) = n^{-\frac{1}{2}}$. Based on the work of Hartman and Kershner [44], we obtain a complete characterization of pointwise rate of decay of Fourier transform of coin-tossing type measures for general weight functions.

Theorem 2.12 (Chapter 7, Theorem 7.7). *Suppose $\phi : \mathbb{N} \rightarrow (0, 1)$ is a non-increasing function with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$, we have*

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- (1) if $\phi(n) < 2\phi(n+1)$ for any $n \geq 1$, then $|\widehat{\mu}(t)| = O(\phi(\lceil \gamma \log_2 |t| \rceil))$, where $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ and $\delta \in (0, 1)$ is a constant depending only on ϕ ;
- (2) if $\phi(n) \geq 2\phi(n+1)$ for any $n \geq 1$, then $|\widehat{\mu}(t)| = O(|t|^{-\gamma})$, where $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ and $\delta \in (0, 1)$ is a constant depending only on ϕ .

Let x be a real number and $b \geq 2$ be an integer. Wall [92] proved that x is normal to the base b if and only if the sequence $(b^n x)_{n \geq 1}$ is uniformly distributed modulo one. Invoking the famous Weyl's criterion for equidistribution with exponential sums, Davenport, Erdős and Leveque [25] established a relationship between the rate of decay of Fourier transform of measures and the equidistribution of sequence of real numbers.

Theorem 2.13 (Davenport-Erdős-LeVeque). *Let ν be a finite Borel measure on \mathbb{R} and $(s_n)_{n \geq 1}$ a sequence of real numbers. If*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^N \widehat{\nu}(h(s_n - s_m)) < \infty$$

for all integers $h \neq 0$, then for ν -almost all real numbers x , the sequence $(s_n x)_{n \geq 1}$ is uniformly distributed modulo one.

This theorem indicates that when the rate of decay of Fourier transform of a measure is fast (some appropriate logarithmic is enough), then almost all real numbers are normal with respect to the measure. Based on Davenport-Erdős-LeVeque Theorem, we use the method of Cassels [16] and Schmidt [87] to prove that almost all real numbers are absolutely normal with respect to coin-tossing type measures whose rates of decay of their Fourier transform obtained in Theorem 2.12 could be very slow.

Theorem 2.14 (Chapter 7, Theorem 7.10). *Let μ be a coin-tossing type measure. Suppose $\phi : \mathbb{N} \rightarrow (0, 1)$ is a non-increasing function with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$. Then μ -almost all real numbers are absolutely normal.*

By Davenport-Erdős-LeVeque Theorem, we know that the rate of decay of Fourier transform of a measure is just a sufficient condition to determine whether almost all real numbers are absolutely normal with respect to this measure. So one would ask whether there exists some singular measure whose rate of decay of its Fourier transform is slow, but almost all real numbers are absolutely normal with respect to that measure? We answer this question positively by presenting some nontrivial new examples of singular measures in Chapter 7.

3 Hausdorff dimension and continued fractions

In this chapter, we recall some fundamental knowledge of Hausdorff dimension and continued fractions. First we present the definition and basic properties of Hausdorff dimension. Then we collect some common methods for calculating Hausdorff dimension. Last we introduce the algorithm and elementary properties of continued fractions.

3.1 Hausdorff measure and Hausdorff dimension

Let U be any non-empty subset of n -dimensional Euclidean Space, \mathbb{R}^n . We denote the diameter of U by

$$\text{diam } U := \sup \left\{ |x - y| : x, y \in U \right\}.$$

For any subset E of \mathbb{R}^n and $\delta > 0$, $(U_i)_{i \geq 1}$ is called a δ -cover of E if $E \subseteq \bigcup_{i=1}^{\infty} U_i$ and $\text{diam } U_i \leq \delta$ for all i . Let s be a non-negative real number and let

$$\mathcal{H}^s(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : (U_i)_{i \geq 1} \text{ is a } \delta\text{-cover of } E \right\}. \quad (3.1)$$

Notice that the quantity

$$\mathcal{H}_{\delta}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : (U_i)_{i \geq 1} \text{ is a } \delta\text{-cover of } E \right\}$$

is increasing as δ decreases, thus the limit in (3.1) exists, although $\mathcal{H}^s(E)$ could be infinite. We call $\mathcal{H}^s(E)$ the s -dimensional Hausdorff outer measure of set E . It is called the s -dimensional Hausdorff measure when restricted on Borel sets. Indeed, Hausdorff measures generalize the ideas of Lebesgue measures. It could be shown that, the n -dimensional Hausdorff measure is, to within a constant multiple, the n -dimensional Lebesgue measure. In particular, the one-dimensional Hausdorff measure \mathcal{H}^1 coincides with the Lebesgue measure \mathcal{L} .

How does the Hausdorff outer measure $\mathcal{H}^s(E)$ behave as a function of s for a given set E ? We shall show that $\mathcal{H}^s(E)$ decreases as s increases. Let $(U_i)_{i \geq 1}$ be any δ -cover of E , then we easily obtain

$$\begin{cases} \sum_{i=1}^{\infty} (\text{diam } U_i)^s \leq \delta^{s-t} \sum_{i=1}^{\infty} (\text{diam } U_i)^t, & 0 < t < s; \\ \sum_{i=1}^{\infty} (\text{diam } U_i)^t \leq \delta^{t-s} \sum_{i=1}^{\infty} (\text{diam } U_i)^s, & t > s. \end{cases} \quad (3.2)$$

Taking infimum of sums and then letting $\delta \rightarrow 0$ in (3.2), if $0 < \mathcal{H}^s(E) < \infty$, we deduce from (3.1) that

$$\mathcal{H}^t(E) = \begin{cases} \infty, & 0 < t < s; \\ 0, & t > s. \end{cases} \quad (3.3)$$

The phenomenon happened in (3.3) indicates that there exists a critical value of s such that $\mathcal{H}^s(E)$ jumping from ∞ to 0. This value is called the Hausdorff dimension of E and written as $\dim_{\text{H}} E$, see (Figure 1).

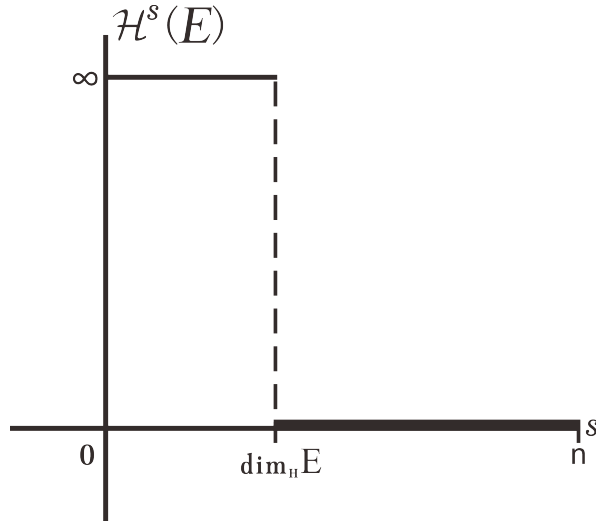


Figure 1 – Graph of $\mathcal{H}^s(E)$ with respect to s for a set E .

Definition 3.1. Let E be a subset of \mathbb{R}^n . The Hausdorff dimension of the set E is

$$\dim_{\mathbb{H}} E = \inf \left\{ s \geq 0 : \mathcal{H}^s(E) = 0 \right\} = \sup \left\{ s : \mathcal{H}^s(E) = \infty \right\}.$$

It is immediate from Definition 3.1 that

$$\mathcal{H}^s(E) = \begin{cases} \infty, & 0 \leq s < \dim_{\mathbb{H}} E, \\ [0, \infty], & s = \dim_{\mathbb{H}} E; \\ 0, & s > \dim_{\mathbb{H}} E. \end{cases} \quad (3.4)$$

Based on (3.4) and the monotonicity of outer measure, we obtain a series of fundamental properties of Hausdorff dimension.

Proposition 3.2 ([33]). *Hausdorff dimension satisfies the following properties.*

- (1) *Monotonicity:* if $E_1 \subseteq E_2$, then $\dim_{\mathbb{H}} E_1 \leq \dim_{\mathbb{H}} E_2$.
- (2) *Countable stability:* $\dim_{\mathbb{H}} (\bigcup_{i=1}^{\infty} E_i) = \sup_i \{\dim_{\mathbb{H}} E_i\}$.
- (3) *Countable set:* if E is countable, then $\dim_{\mathbb{H}} E = 0$.
- (4) *Hölder property:* $\dim_{\mathbb{H}} f(E) \leq \frac{1}{\alpha} \dim_{\mathbb{H}} E$, where f satisfies

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad (0 < c < \infty, \alpha > 0, x, y \in E).$$

- (5) *Bi-Lipschitz invariant:* $\dim_{\mathbb{H}} f(E) = \dim_{\mathbb{H}} E$, where f satisfies

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|, \quad (0 < c_1 \leq c_2 < \infty, x, y \in E).$$

For more details about Hausdorff measure and Hausdorff dimension, we refer the reader to the monographs of Falconer [33, 35], Edgar [28], Bishop and Peres [7].

3.2 Some methods for calculating Hausdorff dimension

Given a subset $E \subseteq \mathbb{R}^n$, we know that the Hausdorff dimension of the set E always exists. However, it is not easy to obtain its exact value. In the following we collect some fundamental and useful methods for calculating the Hausdorff dimension of sets.

Upper bound: In view of (3.1) and (3.4), to obtain an upper bound of $\dim_{\mathbb{H}} E$, it is enough to evaluate sums of the form $\sum_{i=1}^{\infty} (\text{diam } U_i)^s$ for a specific δ -cover of E . In other words, for $s \geq 0$ and a δ -cover of E denoted by $(U_i)_{i \geq 1}$, if $\sum_{i=1}^{\infty} (\text{diam } U_i)^s$ less than or equal to a non-negative real number independent of δ , then $\dim_{\mathbb{H}} E \leq s$. Usually, the upper bound of $\dim_{\mathbb{H}} E$ could be obtained by using the natural coverings of the set E .

Proposition 3.3 ([33, Proposition 4.1]). *Suppose E can be covered by $N_k(E)$ sets of diameter at most δ_k with $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. Then*

$$\dim_{\mathbb{H}} E \leq \liminf_{k \rightarrow \infty} \frac{\log N_k(E)}{-\log \delta_k}. \quad (3.5)$$

Lower bound: If $\sum_{i=1}^{\infty} (\text{diam } U_i)^s$ is greater than some positive constant for all δ -covers of E , then $\dim_{\mathbb{H}} E \geq s$. Clearly an enormous number of such δ -covers are available. To get around the difficulty, one could define a finite measure μ supported on E and compare measure $\mu(B(x, r))$ with r^s for each $B(x, r)$ with $x \in E$ and $r > 0$. The method is called ‘‘Mass Distribution Principle’’, see Bishop and Peres [7, p. 7].

Proposition 3.4 (Mass Distribution Principle). *Let μ be a finite measure supported on E . Suppose that there are numbers $s \geq 0$ and $c > 0$ such that $\mu(B(x, r)) \leq cr^s$ for each ball $B(x, r)$ with $x \in E$ and $r > 0$. We have $\mathcal{H}^s(E) \geq \mu(E)/c$ and thus $\dim_{\mathbb{H}} E \geq s$.*

Remark 3.5. *As a special case, a subset of \mathbb{R}^n with positive Lebesgue measure is of Hausdorff dimension n .*

Let μ be a finite measure on \mathbb{R}^n , we define the lower local dimension of μ at x by

$$\underline{\dim}_{\text{loc}} \mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

The following proposition is a refined version of the Mass Distribution Principle called Billingsley’s Lemma, see Falconer [35, p. 26].

Proposition 3.6 (Billingsley’s Lemma). *Let μ be a finite measure on \mathbb{R}^n and E be a Borel subset of \mathbb{R}^n .*

- (1) *If $\underline{\dim}_{\text{loc}} \mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_{\mathbb{H}} E \geq s$.*
- (2) *If $\underline{\dim}_{\text{loc}} \mu(x) \leq s$ for all $x \in E$, then $\dim_{\mathbb{H}} E \leq s$.*

Based on the monotonicity of Hausdorff dimension, one can also construct a suitable subset of E to obtain a lower bound of $\dim_{\mathbb{H}} E$. The following lemma provides a general method to construct Cantor type subsets.

Lemma 3.7 ([33, Example 4.6]). *Let $\mathbb{E} = \bigcap_{n \geq 0} \mathbb{E}_n$, where $[0, 1] = \mathbb{E}_0 \supseteq \mathbb{E}_1 \supseteq \dots$ is a decreasing sequence of subsets in $[0, 1]$ and \mathbb{E}_n is a union of a finite number of disjoint closed intervals such that each interval in \mathbb{E}_{n-1} contains at least m_n intervals in \mathbb{E}_n which are separated by gaps of lengths at least ε_n . If $m_n \geq 2$ and $\varepsilon_n > \varepsilon_{n+1} > 0$, then*

$$\dim_{\text{H}} \mathbb{E} \geq \liminf_{n \rightarrow \infty} \frac{\log(m_1 m_2 \cdots m_n)}{-\log(m_{n+1} \varepsilon_{n+1})}.$$

3.3 Continued fractions

The Gauss map $T_G : [0, 1) \rightarrow [0, 1)$ is defined by

$$T_G(0) := 0, \quad T_G(x) := 1/x - [1/x], \quad \text{for } x \in (0, 1), \quad (3.6)$$

where $[x]$ denotes the greatest integer not exceeding x . Let

$$a_1 = a_1(x) := [1/x] \quad \text{and} \quad a_n = a_n(x) := [1/T_G^{n-1}(x)].$$

Then each irrational number $x \in [0, 1)$ admits a unique *continued fraction expansion* of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}}, \quad (3.7)$$

where $a_1(x), a_2(x), a_3(x), \dots$ are positive integers, called the *partial quotients* of x . We denote (3.7) by $[a_1(x), a_2(x), a_3(x), \dots]$ for simplicity.

Definition 3.8. *For any $n \geq 1$ and $(a_1, \dots, a_n) \in \mathbb{N}^n$, we call*

$$I_n(a_1, \dots, a_n) := \left\{ x \in [0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n \right\}$$

a cylinder of order n of continued fractions.

In other words, $I_n(a_1, \dots, a_n)$ is the set of points beginning with (a_1, \dots, a_n) in their continued fraction expansions. For any $n \geq 1$, we denote the n -th convergent of the continued fraction expansion of x by

$$\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \dots, a_n(x)] = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \ddots + \frac{1}{a_n(x)}}}, \quad (3.8)$$

where $p_n(x)$ and $q_n(x)$ are positive integers and coprime. Notice that all points in $I_n(a_1, \dots, a_n)$ have the same $p_n(x)$ and $q_n(x)$. Thus, we usually write

$$p_n(a_1, \dots, a_n) = p_n = p_n(x) \quad \text{and} \quad q_n(a_1, \dots, a_n) = q_n = q_n(x).$$

It is well known (see [63, p. 4]) that p_n and q_n satisfy the following recursive formula:

$$\begin{cases} p_{-1} = 1, & p_0 = 0, & p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 1); \\ q_{-1} = 0, & q_0 = 1, & q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1). \end{cases} \quad (3.9)$$

As consequences, we have the following results.

Proposition 3.9 ([63]). *For any $(a_1, \dots, a_n) \in \mathbb{N}^n$, we have*

- (1) $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$,
- (2) $q_n \geq 2^{\frac{n-1}{2}}$, $\prod_{k=1}^n a_k \leq q_n \leq \prod_{k=1}^n (a_k + 1) \leq 2^n \prod_{k=1}^n a_k$.

Lemma 3.10 ([100]). *For any $n \geq 1$ and $1 \leq k \leq n$,*

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, \dots, a_n)}{q_{n-1}(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)} \leq a_k + 1.$$

For any real number $x \in I_n(a_1, \dots, a_n)$, we deduce from (3.7), (3.8) and (3.9) that

$$x = \frac{p_n(x) + T_G^n(x)p_{n-1}(x)}{q_n(x) + T_G^n(x)q_{n-1}(x)},$$

where $0 \leq T_G^n(x) < 1$. Then we can easily calculate the length of $I_n(a_1, \dots, a_n)$.

Proposition 3.11 ([51, p. 18]). *For any $(a_1, \dots, a_n) \in \mathbb{N}^n$, the cylinder $I_n(a_1, \dots, a_n)$ is an interval with the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. More precisely,*

$$I_n(a_1, \dots, a_n) = \begin{cases} [\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}}], & \text{if } n \text{ is even,} \\ (\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n}], & \text{if } n \text{ is odd.} \end{cases}$$

As a result, the length of $I_n(a_1, \dots, a_n)$ equals to

$$|I_n(a_1, \dots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}. \quad (3.10)$$

Note that the partial quotient a_{n+1} takes the integer value from 1 to ∞ in succession partitioning $I_n(a_1, \dots, a_n)$ into a countable set of cylinder of order $n+1$. The following result shows that the proportion of the length of cylinder of order $n+1$ occupying the length of cylinder of order n is determined by $1/a_{n+1}^2$.

Proposition 3.12 ([63, p. 60]). *For any $(a_1, \dots, a_n, a_{n+1}) \in \mathbb{N}^{n+1}$, we have*

$$\frac{1}{3a_{n+1}^2} \leq \frac{|I_{n+1}(a_1, \dots, a_n, a_{n+1})|}{|I_n(a_1, \dots, a_n)|} \leq \frac{2}{a_{n+1}^2}.$$

Let $B(x, \rho)$ denote the ball centered at x with radius ρ . For any $x \in I_n(a_1, \dots, a_n)$, the relationship between $B(x, |I_n(a_1, \dots, a_n)|)$ and $I_n(a_1, \dots, a_n)$ is called the regular property in [11], which is also found in [36].

Lemma 3.13 ([36, Lemma 2.5]). *Let $n \geq 2$ and $x = [a_1, \dots, a_n, \dots]$. If $|I_{n+1}(x)| \leq \rho < |I_n(x)|$, then*

$$I_{n+1}(x) \subseteq B(x, \rho) \subseteq I_{n-2}(x).$$

For more details about continued fractions, we refer the readers to the monographs of Dajani and Kraaikamp [23], Iosifescu and Kraaikamp [51], Khintchine [63].

4 Fourier transform and uniform distribution

In this chapter, we first present the definition of Fourier transform of measures and then recall some results for a class of self-similar measures, especially for Bernoulli convolutions, on the aspect of the absolute continuity, singularity and Fourier transform. The second part is to introduce the definition and fundamental properties of uniform distribution modulo one and normal numbers. It is Davenport-Erdős-LeVeque Theorem that reveals the connection between the rate of decay of Fourier transform of a measure and the sequence of real numbers being uniform distributed modulo one.

4.1 Fourier transform of measures

First we present the definition of Fourier transform of measures. Let $\mathcal{M}(\mathbb{R})$ be the set of all finite Borel measure on \mathbb{R} .

Definition 4.1. *Given a measure $\mu \in \mathcal{M}(\mathbb{R})$, we define the Fourier transform of μ by*

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{2\pi ixt} d\mu(x), \quad t \in \mathbb{R}. \quad (4.1)$$

Due to the Plancherel-Parseval formula, finite Borel measures are uniquely determined by their Fourier transforms. In other words, for any $\mu, \nu \in \mathcal{M}(\mathbb{R})$,

$$\widehat{\mu} = \widehat{\nu} \iff \mu = \nu. \quad (4.2)$$

A measure μ is called a Rajchman measure if $\widehat{\mu}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Combining Radon-Nikodym Theorem with Riemann-Lebesgue Lemma, we know that each measure μ which is absolutely continuous with respect to Lebesgue measure has its Fourier transform vanishing at infinity. Thus all absolutely continuous measures are Rajchman measures. On the contrary, not all Rajchman measures are absolutely continuous. A singular Rajchman measure was firstly constructed by Menshov [77].

The rate of decay of Fourier transform of a Rajchman measure provides lots of information about the measure. For instance, a classical bound of Fourier transform of measures coming from potential theoretic methods (see Mattila [75, p. 168]) shows that if the rate of decay of Fourier transform of a measure μ is polynomial, that is $|\widehat{\mu}(t)| = O(|t|^{-\varepsilon})$ as $|t| \rightarrow \infty$ for some positive constant ε , then the Hausdorff dimension of μ satisfies $\dim_{\text{H}} \mu \geq \min\{2\varepsilon, 1\}$. Moreover, μ -almost all points are absolutely normal. However, it is not easy to verify whether a measure is Rajchman, let alone for estimating the rate of decay of $\widehat{\mu}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. In fact, for a class of self-similar measures equipped with the structure of convolution, sometimes we could easily deal with their Fourier transform.

Definition 4.2. *For $\mu, \nu \in \mathcal{M}(\mathbb{R})$, the convolution of them on \mathbb{R} is defined by*

$$\int_{\mathbb{R}} \psi(z) d(\mu * \nu)(z) = \iint_{\mathbb{R}^2} \psi(x + y) d\mu(x) d\nu(y), \quad (4.3)$$

where ψ is a non-negative function with compact support on \mathbb{R} .

Equivalently, for any Borel subset $E \subseteq \mathbb{R}$,

$$(\mu * \nu)(E) = \int_{\mathbb{R}} \mu(E - x) d\nu(x) = \int_{\mathbb{R}} \nu(E - x) d\mu(x).$$

If μ and ν are probability measures, then the convolution $\mu * \nu$ has a simple probabilistic interpretation: it is the probability distribution of a random variable of the form $X + Y$, where X, Y are independent random variables taking values in \mathbb{R} with probability distribution μ, ν respectively. A simple application of the Fubini Theorem shows that

$$\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}. \quad (4.4)$$

Now let us focus on self-similar measures. Let $(\mathcal{S}_i)_{i=1}^m$ be an iterated function system of equicontractive similitudes on \mathbb{R} defined by

$$\mathcal{S}_i(x) = \lambda x + b_i, \quad 1 \leq i \leq m,$$

where $0 < \lambda < 1$ and $b_i \in \mathbb{R}$ for each i . It is well known (Hutchinson [50]) that there exists a unique non-empty compact set $E \subseteq \mathbb{R}$ such that $E = \bigcup_{i=1}^m \mathcal{S}_i(E)$. For any set of probability weights $(p_i)_{i=1}^m$ with $0 < p_i < 1$ and $\sum_{i=1}^m p_i = 1$, Hutchinson [50] also showed that there exists a unique Borel probability measure μ , known as a self-similar measure, which is supported on E and satisfies an identity of the form

$$\mu = \sum_{i=1}^m p_i \mu \circ \mathcal{S}_i^{-1}. \quad (4.5)$$

The measure μ can also be generated as follows. Let $(X_n)_{n \geq 0}$ be a sequence of independent identically distributed random variables each taking real values b_1, b_2, \dots, b_m with probability p_1, p_2, \dots, p_m respectively. For any $0 < \lambda < 1$, let $\mu_{\lambda, m}$ be the probability distribution of the random sum $\sum_{n=0}^{\infty} X_n \lambda^n$, then it can be verified that the measure $\mu_{\lambda, m}$ also satisfies (4.5). Thus by the uniqueness in Hutchinson's theorem, we have $\mu = \mu_{\lambda, m}$. Moreover, the probabilistic interpretation for the convolution of measures indicates that

$$\mu_{\lambda, m} = *_{n=0}^{\infty} (p_1 \delta_{b_1 \lambda^n} + p_2 \delta_{b_2 \lambda^n} + \dots + p_m \delta_{b_m \lambda^n}), \quad (4.6)$$

where δ_x is Dirac measure.

The simplest case for the self-similar measure $\mu_{\lambda, m}$ is when $m = 2$ and the random variable X_n takes ± 1 with the equal probability $1/2$. In this case, the measure

$$\mu_{\lambda, 2} = *_{n=0}^{\infty} \left(\frac{1}{2} \delta_{-\lambda^n} + \frac{1}{2} \delta_{\lambda^n} \right),$$

called Bernoulli convolutions. The classical Jessen-Wintner law of pure types [57] shows that $\mu_{\lambda, 2}$ is either singular or absolutely continuous with respect to \mathcal{L} .

- (i) For $0 < \lambda < 1/2$, Kershner and Wintner [60] observed that the support of $\mu_{\lambda, 2}$ is Cantor type set and thus singular.
- (ii) For $\lambda = 1/2$, Wintner [99] noticed that $\mu_{\lambda, 2} = \mathcal{L}|_{[-2, 2]}$ and thus absolutely continuous.

- (iii) For $1/2 < \lambda < 1$, the support of $\mu_{\lambda,2}$ is $[-1/(1-\lambda), 1/(1-\lambda)]$. It seems that $\mu_{\lambda,2}$ is absolutely continuous for all λ . However, to prove this conclusion is far away easy. In 1939, Erdős [30] proved that $\mu_{\lambda,2}$ is singular when λ is the reciprocal of a Pisot number. In 1995, Solomyak [89] proved that $\mu_{\lambda,2}$ is absolutely continuous for almost all $\lambda \in (1/2, 1)$. In 2014, Shmerkin [88] proved that the exceptional set $\{\lambda \in (1/2, 1) : \mu_{\lambda,2} \text{ is singular}\}$ is of Hausdorff dimension 0.

On the other hand, the Fourier transform of $\mu_{\lambda,2}$ is

$$\widehat{\mu_{\lambda,2}}(t) = \prod_{n=0}^{\infty} \cos 2\pi \lambda^n t.$$

It is Kershner [61] who first showed that the rate of the decay of $\widehat{\mu_{\lambda,2}}(t)$ has close relation with the arithmetic property of λ . More precisely, $\widehat{\mu_{\lambda,2}}(t)$ is logarithmic as $|t| \rightarrow \infty$ when λ is a rational number but not of the form $\frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{q}$, $q \in \mathbb{N}$. Later on, Erdős [31] proved that for almost every $\lambda \in (0, 1)$, the rate of decay of $\widehat{\mu_{\lambda,2}}(t)$ has some polynomial as $|t| \rightarrow \infty$. Dai, Feng and Wang [22] proved that the rate of decay of $\widehat{\mu_{\lambda,2}}(t)$ has some polynomial as $|t| \rightarrow \infty$ when λ is an algebraic integer whose conjugates are all outside the unit circle and the constant term for its minimal polynomial is ± 2 . Bufetov and Solomyak [13] proved that the rate of decay of $\widehat{\mu_{\lambda,2}}(t)$ is logarithmic as $|t| \rightarrow \infty$ when the reciprocal of λ is an algebraic integer but not a Pisot or Salem number. Gao and Ma [41] showed that the rate of decay of $\widehat{\mu_{\lambda,2}}(t)$ is logarithmic when the reciprocal of λ is a rational number but not of the form $\frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{q}$, $q \in \mathbb{N}$, or an algebraic integer but not a Pisot or Salem number. For more details about Rajchman measures and Bernoulli convolutions, see [74, 81].

The rate of decay of Fourier transform of other various measures was also investigated widely by many authors. For the images of smooth perturbation of self-similar measures, see Kaufman [59], Mosquera and Shmerkin [78], Chang and Gao [18]. For Gibbs measures associated to the Gauss map, see Jordan and Sahlsten [58]. For Patterson-Sullivan measures on the limit set of a co-compact subgroup of $SL(2, \mathbb{Z})$, see Bourgain and Dyatlov [12]. For Furstenberg measures on $SL(2, \mathbb{R})$, see Li [67]. For the non-homogeneous self-similar measures, see Li and Sahlsten [68], Solomyak [90].

In the end of this section, we present two examples of self-similar measures on $[0, 1]$.

Example 4.3. Let $(\mathcal{S}_i)_{i=1}^2$ be an iterated function system of equicontractive similitudes on $[0, 1]$ given by

$$\mathcal{S}_1(x) = \frac{1}{3}x, \quad \mathcal{S}_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

We know that the middle third Cantor set C is the unique compact set such that

$$C = \mathcal{S}_1(C) \cup \mathcal{S}_2(C).$$

Furthermore, for any set of probability weights p_1, p_2 with $0 < p_1, p_2 < 1$ and $p_1 + p_2 = 1$, there exists a unique self-similar measure μ supported on C satisfying

$$\mu = *_{n=0}^{\infty} (p_1 \delta_0 + p_2 \delta_{2/3^{n+1}}) = *_{n=1}^{\infty} (p_1 \delta_0 + p_2 \delta_{2/3^n}).$$

We claim that all the self-similar measures supported on C are singular. In fact,

$$|\widehat{\mu}(t)|^2 = \prod_{n=1}^{\infty} \left((2p_1 - 1)^2 \sin^2 \frac{2\pi t}{3^n} + \cos^2 \frac{2\pi t}{3^n} \right) \geq \prod_{n=1}^{\infty} \cos^2 \frac{2\pi t}{3^n}.$$

Choosing t along with the sequence $(3^k/2)_{k \geq 1}$, we have

$$|\widehat{\mu}(3^k/2)|^2 \geq \prod_{n=1}^{\infty} \cos^2 3^{-n} \pi \geq \frac{1}{4} \prod_{n=2}^{\infty} \cos^2 \frac{\pi}{2^n} = \frac{1}{\pi^2} > 0.$$

In particular, μ is called Cantor-Lebesgue measure when $p_1 = p_2 = 1/2$.

Example 4.4. Let $(\mathcal{S}_i)_{i=1}^2$ be an iterated function system of equicontractive similitudes on $[0, 1]$ given by

$$\mathcal{S}_1(x) = \frac{1}{2}x, \quad \mathcal{S}_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

For any set of probability weights p_1, p_2 with $0 < p_1, p_2 < 1$ and $p_1 + p_2 = 1$, there exists a unique self-similar measure μ supported on $[0, 1]$ satisfying

$$\mu = *_{n=0}^{\infty} (p_1 \delta_0 + p_2 \delta_{1/2^{n+1}}) = *_{n=1}^{\infty} (p_1 \delta_0 + p_2 \delta_{1/2^n}).$$

For the case $p_1 = p_2 = 1/2$, the corresponding measure μ is the Lebesgue measure on $[0, 1]$, i.e., $\mu = \mathcal{L}|_{[0,1]}$. Notice that

$$|\widehat{\mathcal{L}|_{[0,1]}}(t)| = \prod_{n=1}^{\infty} \left| \cos \frac{\pi t}{2^n} \right| = \left| \frac{\sin \pi t}{\pi t} \right| = O(|t|^{-1}) \quad \text{as } |t| \rightarrow \infty.$$

Then the measure $\mathcal{L}|_{[0,1]}$ is absolutely continuous.

Remark 4.5. Let $q \geq 2$ be an integer, then we can also write

$$\mathcal{L}|_{[0,1]} = *_{n=1}^{\infty} \left(\frac{1}{q} \delta_0 + \frac{1}{q} \delta_{\frac{1}{q^n}} + \cdots + \frac{1}{q} \delta_{\frac{q-1}{q^n}} \right).$$

4.2 Uniform distribution modulo one

The formal definition of uniform distribution modulo one was given by Weyl [96]. For a real number x , let $\{x\} = x - [x]$ be the fractional part of x . Let $\omega = (c_n)_{n \geq 1}$ be a sequence of real numbers. Given a positive integer N and a subset $E \subseteq [0, 1]$, the counting function $A(E; N; \omega)$ is defined by the number of terms c_n , $1 \leq n \leq N$, such that $\{c_n\} \in E$. That is,

$$A(E; N; \omega) := \#\left\{n \mid 1 \leq n \leq N, \{c_n\} \in E\right\}.$$

Definition 4.6. The sequence $\omega = (c_n)_{n \geq 1}$ is said to be uniformly distributed modulo one if for every pair a, d of real numbers with $0 \leq a < d \leq 1$ we have

$$\lim_{N \rightarrow \infty} \frac{A([a, d); N; \omega)}{N} = d - a. \quad (4.7)$$

In other words, given any interval $[a, d) \subseteq [0, 1]$, the sequence of real numbers $(c_n)_{n \geq 1}$ is uniformly distributed modulo one if the frequency with which the fractional parts of c_n lie in the interval $[a, d)$ is asymptotical equal to its length, $d - a$. Thus we can replace $[a, d)$ by $[a, d]$, $(a, d]$ or (a, d) without altering the definition. Notice that (4.7) can be written in the form

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \chi_{[a,d)}(\{c_n\}) = \int_0^1 \chi_{[a,d)}(x) dx.$$

Based on approximation theory, Weyl [96, 97] obtained a series of equivalent characterizations of the sequence of real numbers being uniformly distributed modulo one.

Theorem 4.7 ([96, 97]). *The following are equivalent.*

(1) *The sequence $(c_n)_{n \geq 1}$ is uniformly distributed modulo one.*

(2) *For every real-valued continuous function f on $[0, 1]$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{c_n\}) = \int_0^1 f(x) dx.$$

(3) *For every Riemann-integrable function f on $[0, 1]$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{c_n\}) = \int_0^1 f(x) dx.$$

(4) *For every complex-valued continuous function f on \mathbb{R} with period 1,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(c_n) = \int_0^1 f(x) dx.$$

(5) (**Weyl Criterion**) *For all integers $h \neq 0$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h c_n} = 0.$$

Remark 4.8. *By **Weyl Criterion**, to determine whether a sequence of real numbers is uniformly distributed modulo one, it suffices to test the functions f of the form $f(x) = e^{2\pi i h x}$, where $h \in \mathbb{Z} \setminus \{0\}$.*

Example 4.9. *Let θ be an irrational number, for all integers $h \neq 0$, we have*

$$\left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h n \theta} \right| = \frac{1}{N} \left| \frac{e^{2\pi i h N \theta} - 1}{e^{2\pi i h \theta} - 1} \right| \leq \frac{1}{N |\sin \pi h \theta|}.$$

Then the sequence $(n\theta)_{n \geq 1}$ is uniformly distributed modulo one.

Let $(c_n(x))_{n \geq 1}$ be a sequence of real numbers depending on a real parameter x lying in some given bounded or unbounded interval. The following metric theorem given by Weyl [97] describes the behavior of a parameterized sequence for Lebesgue almost all real numbers x .

Theorem 4.10 ([97]). *Let $(c_n)_{n \geq 1}$ be a sequence of distinct integers. Then the sequence $(c_n x)_{n \geq 1}$ is uniformly distributed modulo one for Lebesgue almost all real numbers x .*

Later on, Davenport, Erdős and LeVeque [25] generalized Weyl's result for a class of sequence $(c_n(x))_{n \geq 1}$, where each $c_n(x)$ is bounded and integrable for $c \leq x \leq d$.

Theorem 4.11 ([25]). *If the series*

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_a^d \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h c_n(x)} \right|^2 d\mathcal{L}(x) < \infty$$

for all integers $h \neq 0$, then for Lebesgue almost all x in $[a, d]$, the sequence $(c_n(x))_{n \geq 1}$ is uniformly distributed modulo one.

Remark 4.12. *Let $c_n(x) := c_n x$, then we have*

$$\begin{aligned} \sum_{N=1}^{\infty} \frac{1}{N} \int_a^d \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h c_n(x)} \right|^2 d\mathcal{L}(x) &= \sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^N \int_a^d e^{2\pi i h (c_n - c_m)x} d\mathcal{L}(x) \\ &= \sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^N \widehat{\mathcal{L}}(h(c_n - c_m)). \end{aligned}$$

Combining Theorem 4.11 with Remark 4.12, we have the following theorem.

Theorem 4.13 (Davenport-Erdős-LeVeque). *Let μ be a finite Borel measure on \mathbb{R} and $(c_n)_{n \geq 1}$ a sequence of real numbers. If*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n,m=1}^N \widehat{\mu}(h(c_n - c_m)) < \infty \quad (4.8)$$

for all integers $h \neq 0$, then for μ -almost all real numbers x , the sequence $(c_n x)_{n \geq 1}$ is uniformly distributed modulo one.

Proof. Let h be a fixed integer, write

$$S_h(N, x) = \frac{1}{N} \sum_{n=1}^N e^{2\pi i h c_n x} \quad \text{and} \quad I_h(N) = \frac{1}{N^2} \sum_{n,m=1}^N \widehat{\mu}(h(c_n - c_m)). \quad (4.9)$$

It follows from (4.8) and (4.9) that the series $\sum_{N=1}^{\infty} I_h(N)/N$ converges. Thus there exists an increasing sequence $(\lambda(N))_{N \geq 1}$ of real numbers with $\lambda(N) \rightarrow \infty$ such that

$$\sum_{N=1}^{\infty} \frac{I_h(N) \lambda(N)}{N} < \infty. \quad (4.10)$$

Let $M_1 < M_2 < \dots$ be positive integers such that

$$\lambda(M_1) > 1 \quad \text{and} \quad M_{r+1} = \lfloor \frac{\lambda(M_r)}{\lambda(M_r) - 1} M_r \rfloor + 1 \quad \text{for } r \geq 1.$$

Then it is easy to see that

$$\lim_{r \rightarrow \infty} \frac{M_{r+1}}{M_r} = 1 \quad \text{and} \quad \frac{M_{r+1}}{M_{r+1} - M_r} < \lambda(M_r) \quad \text{for } r \geq 1. \quad (4.11)$$

Let N_r be an integer in the range $M_r < N \leq M_{r+1}$ for which $I_h(N)$ obtains its least value. We deduce from (4.11) that

$$I_h(N_r) \leq \frac{1}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} I_h(N) \leq \frac{M_{r+1}}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} \frac{I_h(N)}{N} \leq \sum_{N=M_r+1}^{M_{r+1}} \frac{I_h(N)\lambda(N)}{N}.$$

Thus by (4.10), we have

$$\begin{aligned} \int_{\mathbb{R}} \sum_{r=1}^{\infty} |S_h(N_r, x)|^2 d\mu(x) &= \sum_{r=1}^{\infty} \int_{\mathbb{R}} |S_h(N_r, x)|^2 d\mu(x) \\ &= \sum_{r=1}^{\infty} I_h(N_r) \leq \sum_{N=M_1+1}^{\infty} \frac{I_h(N)\lambda(N)}{N} < \infty. \end{aligned}$$

Hence for μ -almost all real numbers x ,

$$\lim_{r \rightarrow \infty} S_h(N_r, x) = 0.$$

By the choice of the number N_r and (4.11), we have $N_{r+1}/N_r \rightarrow 1$ as $r \rightarrow \infty$. If $N_r < N \leq N_{r+1}$, then

$$\begin{aligned} |S_h(N, x)| &= \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h c_n x} \right| = \left| \frac{1}{N} \left(\sum_{n=1}^{N_r} e^{2\pi i h c_n x} + \sum_{n=N_r+1}^N e^{2\pi i h c_n x} \right) \right| \\ &\leq |S_h(N_r, x)| + \frac{N_{r+1} - N_r}{N_r}. \end{aligned}$$

Therefore for μ -almost all real numbers x ,

$$\lim_{N \rightarrow \infty} S_h(N, x) = 0. \quad (4.12)$$

Since the countable union of null sets is always null set, (4.12) holds for all integers $h \neq 0$. Hence by **Weyl Criterion**, for μ -almost all real numbers x , the sequence $(c_n x)_{n \geq 1}$ is uniformly distributed modulo one. \square

Let $b \geq 2$ be an integer. For the case when $c_n = b^{n-1}$ ($n \geq 1$) in Theorem 4.13, Wall [92] gave an equivalent characterization of uniform distribution modulo one for the sequence $(c_n x)_{n \geq 1}$.

Theorem 4.14 ([92]). *The sequence $(b^{n-1}x)_{n \geq 1}$ is uniformly distributed modulo one if and only if x is normal to the base b .*

The concept of normal numbers was introduced by Borel [9] in 1909. His original characterization of normality is that a real number α is normal to the base b if each of the numbers $\alpha, b\alpha, b^2\alpha, \dots$ is simple normal to all of the base b, b^2, b^3, \dots . Later on, Niven and Zuckerman [80] gave an equivalence of this characterization in 1951.

Let α be a real number and $b \geq 2$ be an integer. Then α has a unique b -adic expansion of the form

$$\alpha = [\alpha] + \sum_{n=1}^{\infty} \frac{a_n}{b^n} = [\alpha].a_1a_2 \cdots a_n \cdots, \quad (4.13)$$

where the digits a_n are integers with $0 \leq a_n \leq b-1$ for $n \geq 1$, and also $a_n < b-1$ for infinitely many n . Let a be an integer with $0 \leq a < b$, we denote $A_b(a; N; \alpha)$ the number of n , $1 \leq n \leq N$, such that $a_n = a$ in (4.13). That is,

$$A_b(a; N; \alpha) = \#\{n \mid 1 \leq n \leq N, a_n = a\}.$$

For any integer $k \geq 1$, let $D_k = d_1d_2 \cdots d_k$ be a given block of digits of length k and

$$A_b(D_k; N; \alpha) = \#\{n \mid 1 \leq n \leq N - k + 1, a_n a_{n+1} \cdots a_{n+k-1} = d_1 d_2 \cdots d_k\}.$$

Definition 4.15. *The number α is called simply normal to the base b if*

$$\lim_{N \rightarrow \infty} \frac{A_b(a; N; \alpha)}{N} = \frac{1}{b} \quad \text{for } a = 0, 1, 2, \dots, b-1.$$

The number α is called normal to the base b if

$$\lim_{N \rightarrow \infty} \frac{A_b(D_k; N; \alpha)}{N} = \frac{1}{b^k} \quad \text{for all } k \geq 1 \text{ and all blocks } D_k.$$

The number α is called absolutely normal if it is normal to all bases $b \geq 2$.

Let us recall a equivalent characterization of normal numbers which is due to Van der Corput [65, Theorem 7.1] and Pjateckiĭ-Šapiro [65, Lemma 8.1].

Theorem 4.16 ([65, Theorem 8.2]). *Let $k \geq 2$ be an integer. A real number α is normal to the base b if and only if α is normal to the base b^k .*

This theorem reveals a special relation of the normality with respect to different bases. In general, given two bases b_1 and b_2 , Schmidt [87] showed that if $\log b_1 / \log b_2$ is rational, then α normal to the base b_1 is also normal to the base b_2 , otherwise, the set of numbers α which are normal to the base b_1 but not even simply normal to the base b_2 has the power of the continuum. In fact, Pillai [83] also obtained a equivalent characterization of normal numbers.

Theorem 4.17 ([65, Theorem 8.3]). *The real number α is normal to the base b if and only if α is simply normal to all of the bases b, b^2, b^3, \dots .*

Theorem 4.18 (Borel's Theorem on normal numbers). *Lebesgue almost all real numbers are absolutely normal.*

Borel theorem indicates that the set of normal numbers is of full Lebesgue measure. However, it is not easy to construct normal numbers. There is no commonly used irrational number, such as π , e or even $\sqrt{2}$, that is known to be normal. It is Champernowne [17] who first showed that the number

$$0.1234567891011121314151617\dots$$

formed by concatenating all the natural numbers in order is normal to the base 10. For other works of this topic, see for example, Besicovitch [2], Copeland and Erdős [20], Davenport and Erdős [24]. On the other hand, the set of non-normal numbers is of null Lebesgue measure. Many authors further considered such null set by calculating its Hausdorff dimension, see for instance, Besicovitch [3] and Eggleston [29].

5 Multifractal analysis of the convergence exponent

In this chapter, we mainly investigate the Hausdorff dimension of sets of continued fractions with prescribed growth rate of partial quotients. To be specific, in this chapter, we first introduce the convergence exponent of the sequence of positive real numbers and then define the convergence exponent of a real number using its sequence of partial quotients in continued fractions. It turns out that the convergence exponent of real numbers defines a measurable function. We shall study the basic properties of this function, such as its range, denseness, continuity and etc. Besides, Birkhoff's Ergodic Theorem indicates that the function equals to infinity for Lebesgue almost all points, so we continue to study its multifractal properties. In the end, the Hausdorff dimensions of some sets related to the convergence exponent are computed.

5.1 Convergence exponent

Let $(r_n)_{n \geq 1}$ be a sequence of positive real numbers. The convergence exponent of the sequence is defined by

$$\tau := \inf \left\{ s \geq 0 : \sum_{n \geq 1} r_n^{-s} < \infty \right\}.$$

When restricting some conditions on the sequence $(r_n)_{n \geq 1}$, we obtain a simpler formula for τ .

Proposition 5.1 ([84, p. 26]). *Let $(r_n)_{n \geq 1}$ be a non-decreasing sequence of positive real numbers with $\lim_{n \rightarrow \infty} r_n = \infty$. Then*

$$\tau = \limsup_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \sup \left\{ t \geq 0 : \sum_{n \geq 1} r_n^{-t} = \infty \right\}. \quad (5.1)$$

Proof. For convenience, write

$$v = \limsup_{n \rightarrow \infty} \frac{\log n}{\log r_n} \quad \text{and} \quad \lambda = \sup \left\{ t \geq 0 : \sum_{n \geq 1} r_n^{-t} = \infty \right\}.$$

So it is sufficient to prove $\tau = v$ and $\tau = \lambda$. We divide the proof into two parts.

(i) The proof of $\tau = \lambda$. By the definition of λ , for any $\varepsilon > 0$, we have

$$\sum_{n \geq 1} r_n^{-(\lambda + \varepsilon)} < \infty.$$

This implies $\tau \leq \lambda + \varepsilon$, and thus $\tau \leq \lambda$. On the other hand, we claim that for any $t \geq 0$ and $\sum_{n \geq 1} r_n^{-t} = \infty$, we always have $\tau \geq t$ and thus $\tau \geq \lambda$. If not, suppose $\tau < t$, by the definition of τ , there exists a number $s_0 \geq 0$ such that

$$s_0 < t \quad \text{and} \quad \sum_{n=1} r_n^{-s_0} < \infty.$$

Note that $\lim_{n \rightarrow \infty} r_n = \infty$. Then there exists a positive integer N such that $r_n > 1$ for any $n \geq N$. Thus

$$\sum_{n \geq N} r_n^{-t} \leq \sum_{n \geq N} r_n^{-s_0} < \sum_{n \geq 1} r_n^{-s_0} < \infty.$$

- (ii) The proof of $\tau = v$. For any $\varepsilon > 0$, choose $0 < \varepsilon_1 < \varepsilon$ and $N \geq 1$ such that for any $n \geq N$, we have $r_n^{v+\varepsilon_1} > n$, then

$$\sum_{n=N}^{\infty} r_n^{-(v+\varepsilon)} < \sum_{n=N}^{\infty} n^{-\frac{v+\varepsilon}{v+\varepsilon_1}} < \infty.$$

This implies $\tau \leq v + \varepsilon$, and thus $\tau \leq v$. On the other hand, there exists a lacunary sequence $(n_k)_{k \geq 1}$ such that $\log n_k / \log r_{n_k} > v - \varepsilon$ for any $k \geq 1$, i.e.,

$$n_{k+1} \geq \lambda n_k \quad (\lambda > 1) \quad \text{and} \quad n_k \cdot r_{n_k}^{-(v-\varepsilon)} \geq 1.$$

Then we have

$$\sum_{n=n_1}^{\infty} r_n^{-(v-\varepsilon)} = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} r_i^{-(v-\varepsilon)} \geq \left(1 - \frac{1}{\lambda}\right) \sum_{k=1}^{\infty} n_{k+1} r_{n_{k+1}}^{-(v-\varepsilon)} = \infty,$$

which means that $v \leq \tau$.

□

We give some remarks on the formula (5.1).

- (i) The formula (5.1) is false for some special cases when we remove the condition that $\lim_{n \rightarrow \infty} r_n = \infty$. For example, for some positive integer N_0 , choose $(r_n)_{n \geq 1}$ such that $r_n = 1/3$ for $1 \leq n < N_0$, and the rest is equal to $1/2$ for any $n \geq N_0$, then we can obtain $\tau = \lambda = \infty$, but $v = -\infty$.
- (ii) The formula (5.1) is also valid for the case when we replace $\lim_{n \rightarrow \infty} r_n = \infty$ by $r_n \geq 1$ for any $n \geq 1$. In fact, it is sufficient to check the formula under the condition that $(r_n)_{n \geq 1}$ is bounded. In this case, obviously $\tau = \lambda = v = \infty$.

The convergence exponent of various sequences appear quite often in the context of geometry of fractal sets and measures. It was used by Besicovitch and Taylor [4] to study the Hausdorff measure and Hausdorff dimension of cut-out sets; by Hawkes [45] to find the entropy dimension of some random sets determined by Lévy processes; by Dodson [26] to characterize the Hausdorff dimension of certain sets associated with Diophantine approximation; by Wang and Wu [93] to prove Hirst's conjecture on the Hausdorff dimension of certain sets of real numbers whose partial quotients lie in a fixed strictly increasing infinite sequence of integers; by Durand [27] and Li, Shieh and Xiao [66] to describe the Hausdorff dimension and hitting probabilities of random covering sets; by Fan, Liao, Ma and Wang [37] to determine the Hausdorff dimension of Besicovitch-Eggleston sets in countable symbolic space; by Fan, Li and Ma [39] to obtain the Billingsley dimension of the sets of generic points for shift-invariant measures in the countable symbolic space.

5.2 Convergence exponent of the sequence of partial quotients

Recall that each irrational number $x \in [0, 1)$ admits a unique *continued fraction expansion* of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}} := [a_1(x), a_2(x), a_3(x), \dots],$$

where $a_n(x) = \lfloor 1/T_G^{n-1}(x) \rfloor$ are *partial quotients* and T_G is the Gauss map. Let $\tau(x)$ be the convergence exponent of the sequence of partial quotients of x , i.e.,

$$\tau(x) := \inf \left\{ s \geq 0 : \sum_{n \geq 1} a_n^{-s}(x) < \infty \right\}. \quad (5.2)$$

Clearly, the function $x \rightarrow \tau(x)$ takes values in $[0, \infty]$ and that $\tau(x) = 0$ for any rational number x . In the following we give some basic properties of this function.

Theorem 5.2. *The following results hold.*

- (1) *The function $\tau : [0, 1) \rightarrow [0, \infty]$ is Borel measurable.*
- (2) *For any $0 \leq \alpha \leq \infty$, there exists an irrational number $x \in [0, 1)$ such that $\tau(x) = \alpha$.*
- (3) *For any $0 \leq \alpha \leq \infty$, $\{x \in [0, 1) : \tau(x) = \alpha\}$ is uncountable and dense in $[0, 1)$.*
- (4) *For any non trivial interval $I \subseteq [0, 1)$, we have $\tau(I) = [0, \infty]$.*
- (5) *The function $\tau : [0, 1) \rightarrow [0, \infty]$ is everywhere discontinuous.*

Proof. (1) It is sufficient to show that $\{x \in [0, 1) : \tau(x) < \alpha\}$ is a Borel set for any $\alpha \in (0, \infty)$. It is easy to verify that

$$\begin{aligned} \{x \in [0, 1) : \tau(x) < \alpha\} &= \bigcup_{k \geq k_0} \left\{ x \in [0, 1) : \sum_{j \geq 1} a_j^{-(\alpha-1/k)}(x) < \infty \right\} \\ &= \bigcup_{k \geq k_0} \bigcap_{\ell \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq 1} M_{k, \ell, m, n}^\alpha, \end{aligned} \quad (5.3)$$

where $k_0 := \lfloor 1/\alpha \rfloor + 1$ and the set $M_{k, \ell, m, n}^\alpha$ is given by

$$M_{k, \ell, m, n}^\alpha = \left\{ x \in [0, 1) : \sum_{j=m+1}^{m+n} a_j^{-(\alpha-1/k)}(x) \leq \frac{1}{\ell} \right\}.$$

For each $(k, \ell, m, n) \in \mathbb{N}^4$, we know that the set $M_{k, \ell, m, n}^\alpha$ is a countable union of cylinders of finite order, which are intervals and of the form $[a, b)$ or $(c, d]$ with rational numbers a, b, c, d in $[0, 1)$. Hence $M_{k, \ell, m, n}^\alpha$ is a Borel set. It follows from (5.3) that $\{x \in [0, 1) : \tau(x) < \alpha\}$ is a Borel set.

- (2) We divide $\alpha \in [0, \infty]$ into three cases. For the case $\alpha = 0$, let $a_n(x) = \lfloor \exp(n) \rfloor$ for any $n \geq 1$, and set $x = [a_1(x), \dots, a_n(x), \dots]$. Then $x \in [0, 1)$ is an irrational number and $\tau(x) = 0$. For the case $\alpha \in (0, \infty)$, let $a_n(x) = \lfloor n^{1/\alpha} \rfloor$ for any $n \geq 1$, and set $x = [a_1(x), \dots, a_n(x), \dots]$. Then x is also an irrational number in $[0, 1)$ and $\tau(x) = \alpha$. For the case $\alpha = \infty$, let $a_n(x) = 1$ for any $n \geq 1$, and set $x = [1, \dots, 1, \dots]$. Obviously x is an irrational number and $\tau(x) = \infty$.
- (3) By (2), there exists an irrational number $x_0 \in [0, 1)$ such that $\tau(x_0) = \alpha$. Let $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots) \in \{0, 1\}^{\mathbb{N}}$. For any $n \geq 1$, set

$$\tilde{x}_0 = [a_1(\tilde{x}_0), \dots, a_n(\tilde{x}_0), \dots] \text{ with } a_n(\tilde{x}_0) = a_n(x_0) + \varepsilon_n.$$

Then $\tilde{x}_0 \in [0, 1)$ and $\tau(\tilde{x}_0) = \alpha$. This implies that there are uncountably infinite many $x \in [0, 1)$ such that $\tau(x) = \alpha$. In the following, we will show that the set $\{x \in [0, 1) : \tau(x) = \alpha\}$ is dense in $[0, 1)$. Let

$$M(x_0) = \bigcup_{N \geq 1} \{x \in [0, 1) : a_n(x) = a_n(x_0), \forall n \geq N\}.$$

Then for any $x \in M(x_0)$ and $\sigma > 0$, the series $\sum_{n \geq 1} a_n^{-\sigma}(x)$ converges if and only if the series $\sum_{n \geq 1} a_n^{-\sigma}(x_0)$ converges. This yields that $\tau(x) = \tau(x_0) = \alpha$, i.e.,

$$M(x_0) \subseteq \{x \in [0, 1) : \tau(x) = \alpha\}.$$

Thus it is sufficient to prove that $M(x_0)$ is dense in $[0, 1)$. In fact, for any $y \in [0, 1)$, if y is a rational number, then there exist some integer $k \geq 1$ such that $y = [a_1(y), a_2(y), \dots, a_k(y)]$, so we construct a sequence of real numbers $(x_n)_{n \geq 1}$ such that $x_n = [a_1(y), a_2(y), \dots, a_k(y), n, a_{k+2}(x_0), a_{k+3}(x_0), \dots]$; if y is an irrational number, set $y = [a_1(y), a_2(y), \dots, a_n(y), \dots]$ and denote the corresponding sequence of real numbers $(x_n)_{n \geq 1}$ by

$$x_n = [a_1(y), a_2(y), \dots, a_n(y), a_{n+1}(x_0), a_{n+2}(x_0), \dots].$$

In both cases, we obtain the sequence $(x_n)_{n \geq 1} \in M(x_0)$ and $x_n \rightarrow y$ as $n \rightarrow \infty$. Hence the set $M(x_0)$ is dense in $[0, 1)$.

- (4) For any interval $I \subseteq [0, 1)$, it is trivial that $\tau(I) \subseteq [0, \infty]$. For any $\alpha \in [0, \infty]$, by (3), the set $\{x \in [0, 1) : \tau(x) = \alpha\}$ is dense in $[0, 1)$. Thus the intersection of the sets I and $\{x \in [0, 1) : \tau(x) = \alpha\}$ is not empty. Hence we can find an $x_0 \in I$ such that $\tau(x_0) = \alpha$, that is, $[0, \infty] \subseteq \tau(I)$.
- (5) Given a real number $y \in [0, 1)$ and set $\tau(y) = \alpha_0$. For any real number $\alpha \neq \alpha_0$, by (4), the level set $\{x \in [0, 1) : \tau(x) = \alpha\}$ is dense in $[0, 1)$. Then there exists a sequence of real number $(x_n)_{n \geq 1}$ such that $\tau(x_n) = \alpha$ for any $n \geq 1$ and $x_n \rightarrow y$ as $n \rightarrow \infty$. Note that $\tau(y) = \alpha_0 \neq \alpha = \tau(x_n)$, then the function τ is discontinuous at point y . Therefore, the function $\tau : [0, 1) \rightarrow [0, \infty]$ is everywhere discontinuous. \square

5.2.1 Multifractal spectrum of the convergence exponent

It is known that the continued fraction dynamical system $([0, 1], T_G, \mu)$ is ergodic (see [5, p. 45]), where T_G is the Gauss map and μ is the Gauss measure given by

$$\mu(C) = \frac{1}{\log 2} \int_C \frac{1}{1+x} dx$$

for any Borel measurable set $C \subseteq [0, 1]$. For any $t > 0$, by Birkhoff's Ergodic Theorem,

$$\lim_{n \rightarrow \infty} \frac{a_1^{-t}(x) + \cdots + a_n^{-t}(x)}{n} = \int_0^1 a_1^{-t}(x) d\mu(x) \quad (5.4)$$

for μ -almost all $x \in [0, 1]$. Note that

$$\frac{1}{2 \log 2} \sum_{k \geq 1} \frac{1}{k^{t+1}(k+1)} \leq \int_0^1 a_1^{-t}(x) d\mu(x) \leq \frac{1}{\log 2} \sum_{k \geq 1} \frac{1}{k^{t+1}(k+1)},$$

which, in combination with (5.2) and (5.4), implies that $\tau(x) = \infty$ for Lebesgue almost all $x \in [0, 1]$. For convenience, let

$$\Delta(\alpha) = \left\{ x \in [0, 1] : \tau(x) = \alpha \right\} \quad (0 \leq \alpha \leq \infty).$$

Then we have shown that the set $\Delta(\infty)$ is of full Lebesgue measure. In other words, $\Delta(\alpha)$ is of null Lebesgue measure for any $0 \leq \alpha < \infty$. A typical way of measuring the size of such null sets is then to calculate its Hausdorff dimension, i.e., the multifractal spectrum of the convergence exponent $\tau(x)$. In fact, the definition of $\tau(x)$ indicates that the set $\Delta(\alpha)$ is related to the sets of continued fractions for which their partial quotients become large in some sense and tend to infinity. The study of Hausdorff dimension of such sets of continued fractions with some restrictions on the growth rate of their partial quotients tending to infinity goes back to Good [43] in 1941.

Theorem 5.3 ([43, Theorem 1]). *Let $(a_n(x))_{n \geq 1}$ be the partial quotients of the continued fraction of $x \in [0, 1]$. Then*

$$\dim_{\text{H}} \left\{ x \in [0, 1] : a_n(x) \rightarrow \infty \text{ as } n \rightarrow \infty \right\} = \frac{1}{2}.$$

Indeed, Good also proved that

$$\dim_{\text{H}} \left\{ x \in [0, 1] : a_n(x) \geq f(n), \forall n \geq 1 \right\} = \frac{1}{2}, \quad (5.5)$$

where $f(n)$ is any function tending to infinity with n and satisfying $f(n) < (\log n)^t$ for some $t > 0$. In 1970, Hirst [48] extended the result in (5.5) when $f(n)$ increases much more rapidly. In particular, the set of continued fractions for which their partial quotients tend to infinity with polynomial growth rate is of Hausdorff dimension $1/2$.

Theorem 5.4 ([48]). *For any $b > 1$,*

$$\dim_{\text{H}} \left\{ x \in [0, 1] : a_n(x) \geq b^n, \forall n \geq 1 \right\} = \frac{1}{2}.$$

In 1985, Ramharter [85] proved that the set of continued fractions for which their partial quotients are non-decreasing is still of Hausdorff dimension $1/2$.

Theorem 5.5 ([85]). $\dim_{\text{H}} \left\{ x \in [0, 1) : a_{n+1}(x) \geq a_n(x), \forall n \geq 1 \right\} = \frac{1}{2}$.

In 1997, Łuczak [72] and Feng, Wu, Liang and Tseng [40] studied the set of continued fractions whose partial quotients tend to infinity with doubly exponential growth rate.

Theorem 5.6 ([40, 72]). *For any $a, b > 1$,*

$$\dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq a^{b^n}, \forall n \geq 1 \right\} = \frac{1}{b+1}.$$

In 2007, Wang [95] generalized the results of this type by Good [43], Hirst [48], Feng, Wu, Liang and Tseng [40] and Łuczak [72] in his PhD thesis.

Theorem 5.7 ([95]). *Let φ defined on \mathbb{N} be a positive function with $\varphi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq \varphi(n), \forall n \geq 1 \right\} = \frac{1}{b+1},$$

where b is given by

$$\log b := \limsup_{n \rightarrow \infty} \frac{\log \log \varphi(n)}{n}.$$

In 2009, Fan, Liao, Wang and Wu [36] investigated the set of continued fractions whose partial quotient $a_n(x)$ is bounded by t_n and Nt_n for any $n \geq 1$, where N is a constant and t_n tends to infinity as $n \rightarrow \infty$.

Lemma 5.8 ([36, Lemma 3.2]). *Let $(t_n)_{n \geq 1}$ be a sequence of positive integers tending to infinity with $t_n \geq 3$ for any $n \geq 1$. Then for any positive number $N \geq 2$,*

$$\dim_{\text{H}} \left\{ x \in [0, 1) : t_n \leq a_n(x) < Nt_n, \forall n \geq 1 \right\} = \liminf_{n \rightarrow \infty} \frac{\log(t_1 t_2 \cdots t_n)}{2 \log(t_1 t_2 \cdots t_n) + \log t_{n+1}}.$$

In the sequel, we shall concentrate on the multifractal spectrum of $\tau(x)$. Our first result is the following.

Theorem 5.9. *For any $0 \leq \alpha < \infty$, we have*

$$\dim_{\text{H}} \Delta(\alpha) = \frac{1}{2}.$$

Proof. Recall that

$$\Delta(\alpha) = \{x \in [0, 1) : \tau(x) = \alpha\}.$$

For any $x \in \Delta(\alpha)$, by the definition of $\tau(x)$, then there exists $\varepsilon > 0$ such that

$$\sum_{n \geq 1} a_n^{-(\alpha+\varepsilon)}(x) < \infty,$$

which implies $a_n(x) \rightarrow \infty$ as $n \rightarrow \infty$. Thus

$$\Delta(\alpha) \subseteq \left\{x \in [0, 1) : a_n(x) \rightarrow \infty \text{ as } n \rightarrow \infty\right\}.$$

It follows from Theorem 5.3 that

$$\dim_{\text{H}} \Delta(\alpha) \leq \frac{1}{2}.$$

For the lower bound of $\dim_{\text{H}} \Delta(\alpha)$, we construct the suitable subsets for $\Delta(\alpha)$.

(i) $\alpha = 0$, it is easy to verify that

$$\left\{x \in [0, 1) : \exp(n) \leq a_n(x) < 2 \exp(n), \forall n \geq 1\right\} \subseteq \Delta(\alpha).$$

(ii) $\alpha \in (0, \infty)$, it is also easy to check that

$$\left\{x \in [0, 1) : n^{1/\alpha} \leq a_n(x) < 2n^{1/\alpha}, \forall n \geq 1\right\} \subseteq \Delta(\alpha).$$

It follows from Lemma 5.8 that the above subsets of $\Delta(\alpha)$ are of Hausdorff dimension $1/2$. Therefore, for any $\alpha \in [0, \infty)$,

$$\dim_{\text{H}} \Delta(\alpha) \geq \frac{1}{2}.$$

□

Remark 5.10. *As far as we know, various exponents related to continued fractions have been investigated by many authors. To mention a few of them, we refer to Jarník [53] for the irrationality exponent, Pollicott and Weiss [82] for the Lyapunov exponent of the Gauss map, Kesseböhmer and Stratmann [62] for the Minkowski's question mark function, Fan, Liao, Wang and Wu [36] for the Khintchine exponent, Nicolay and Simons [79] for the Cantor bijection, Jaffard and Martin [52] for the Brjuno function. Theorem 5.9 shows that the multifractal spectrum of the convergence exponent $\tau(x)$ is*

a constant function of $\alpha \in [0, \infty)$. In other words, there is no multifractal phenomenon for the convergence exponent $\tau(x)$. Compare with the multifractal spectrums of others exponents in Continued fractions, such as Khintchine spectrum and Lyapunov spectrum given by

$$\dim_{\text{H}} \left\{ x \in [0, 1) : \frac{1}{n} \sum_{i=1}^n \log a_i(x) = \alpha \right\} \text{ and } \dim_{\text{H}} \left\{ x \in [0, 1) : \frac{1}{n} \sum_{i=1}^n \log |(T_G^n)'(x)| = \alpha \right\}$$

respectively, our result is different. To be specific, The Khintchine spectrum and Lyapunov spectrum are neither concave nor convex as functions of $\alpha \in [0, \infty)$, see Fan, Liao, Wang and Wu [36].

For the case when the sequence of partial quotients $(a_n(x))_{n \geq 1}$ is non-decreasing, we continue to study the multifractal spectrum of its convergence exponent, i.e., the Hausdorff dimension of the intersection of sets $\Delta(\alpha)$ and Λ , where

$$\Lambda = \left\{ x \in [0, 1) : a_n(x) \leq a_{n+1}(x), \forall n \geq 1 \right\}.$$

It turns out that the multifractal spectrum of the convergence exponent $\tau(x)$ defined by the non-decreasing sequence of partial quotients is piecewise linear.

Theorem 5.11. *For any $\alpha \geq 0$, we have*

$$\dim_{\text{H}} (\Delta(\alpha) \cap \Lambda) = \begin{cases} \frac{1-\alpha}{2}, & 0 \leq \alpha \leq 1; \\ 0, & \alpha > 1. \end{cases}$$

Before proceeding on, we give some remarks.

- (i) In the sense of Hausdorff dimension and Lebesgue measure,

$$\Lambda = \left\{ x \in [0, 1) : a_n(x) \leq a_{n+1}(x), \forall n \geq 1 \text{ and } a_n(x) \rightarrow \infty \text{ as } n \rightarrow \infty \right\}.$$

- (ii) For any $\alpha \in [0, \infty)$, with the same method in the proof of Theorem 5.9, we have

$$\dim_{\text{H}} \left\{ x \in [0, 1) : \tau(x) = \alpha, a_n(x) \rightarrow \infty \right\} = 1/2.$$

This means that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ plays an essential role in Theorem 5.11.

- (iii) Theorem 5.5 shows that $\dim_{\text{H}} \Lambda = 1/2$. Then the result in Theorem 5.11 provides an interesting example that the Hausdorff dimension of the intersection of two fractal sets with Hausdorff dimension $1/2$ can be any value between 0 and $1/2$.
- (iv) We note that the multifractal spectrum $\alpha \mapsto \dim_{\text{H}} (\Delta(\alpha) \cap \Lambda)$ is not differentiable at $\alpha = 1$. This unusual phenomenon can be considered as a phase transition of the multifractal spectrum.

By (5.1), for any $\alpha \geq 0$, we can rewrite the set $\Delta(\alpha) \cap \Lambda$ as follows:

$$\Delta(\alpha) \cap \Lambda = \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = \frac{1}{\alpha} \right\} := E_{\text{inf}}(\Lambda, \frac{1}{\alpha}). \quad (5.6)$$

This means the problem of the multifractal spectrum of the convergence exponent $\tau(x)$ defined by the non-decreasing sequence of partial quotients is exactly the problem of the Hausdorff dimension of level sets of \liminf . It follows (5.6) that the Theorem 5.11 is equivalent to the following theorem.

Theorem 5.12. *For any $\alpha \geq 0$, we have*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

More generally, we continue to consider the Hausdorff dimension of the set

$$E_{\text{inf}}(\Lambda, \psi) := \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where ψ is a positive function defined on \mathbb{N} .

Theorem 5.13. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

(1) *If $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, then*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

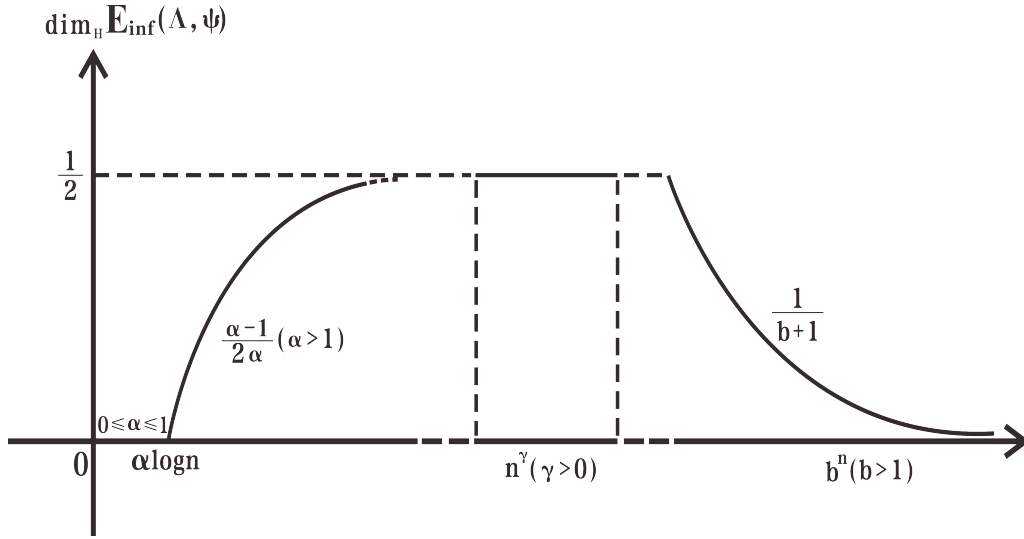
(2) *If $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$ and $\psi(n)$ is non-decreasing, then*

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) = \frac{1}{B+1},$$

where B is given by

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

As an application of Theorem 5.13, if ψ is a power function of the form $\psi(n) = n^\gamma$ with $\gamma > 0$, then $E_{\text{inf}}(\Lambda, \psi)$ is of Hausdorff dimension $1/2$; if ψ is an exponential function of the form $\psi(n) = b^n$ with $b > 1$, then $E_{\text{inf}}(\Lambda, \psi)$ is of Hausdorff dimension $1/(b+1)$. The following figure provides a full illustration of $\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi)$, which deeply reveals how it changes with different functions ψ .


 Figure 2 – $\dim_{\mathbb{H}} E_{\text{inf}}(\Lambda, \psi)$ for different functions ψ .

5.2.2 Proofs of Theorem 5.12 and Theorem 5.13

We first give a useful combinatorial formula on the cardinality of finite sets.

Lemma 5.14. *For any positive integer ℓ and $n \geq 1$, let*

$$D = \left\{ (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{N}^n : 1 \leq \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_n \leq \ell \right\}$$

and denote its cardinality by $\#D$. Then we have

$$\#D = \frac{(n + \ell - 1)!}{n!(\ell - 1)!}.$$

Proof. Let

$$\tilde{D} = \left\{ (r_1, r_2, \dots, r_n) \in \mathbb{N}^n : 1 \leq r_1 < r_2 < \dots < r_n \leq \ell + n - 1 \right\}$$

and consider a map $f : D \rightarrow \tilde{D}$ defined as follows: for any $(\sigma_1, \sigma_2, \dots, \sigma_n) \in D$,

$$f(\sigma_1, \sigma_2, \dots, \sigma_n) = (r_1, r_2, \dots, r_n) = (\sigma_1, \sigma_2 + 1, \dots, \sigma_n + n - 1).$$

It is easy to check that f is bijection. Thus the sets D and \tilde{D} have the same cardinality. The number of elements in \tilde{D} is equal to a combinatorial number, i.e., the number of n numbers can be chosen from among $\ell + n - 1$ numbers. That is,

$$\#\tilde{D} = \frac{(n + \ell - 1)!}{n!(\ell - 1)!}.$$

□

Notice that the set $\{x \in [0, 1] : t_n \leq a_n(x) < Nt_n, \forall n \geq 1\}$ in Lemma 5.8 cannot guarantee their partial quotients being non-decreasing for any sequence $(t_n)_{n \geq 1}$ and any positive number $N \geq 2$, so we cannot apply the result of Lemma 5.8 directly to establish the lower bound of $\dim_{\mathbb{H}} E_{\text{inf}}(\Lambda, \psi)$. This lead us to construct a set of points whose partial quotients could be a non-decreasing sequence. For general construction of set of this type in d -decaying Gauss like iterated function systems, see Liao and Rams [71]. Here we give a direct and different proof for the sake of completeness.

Lemma 5.15. *Let $(s_n)_{n \geq 1}$ be a sequence of positive integers tending to infinity with $s_n \geq 2$ for any $n \geq 1$. Set*

$$\mathbb{F} = \left\{ x \in [0, 1] : ns_n \leq a_n(x) < (n+1)s_n, \forall n \geq 1 \right\}.$$

Then

$$\dim_{\mathbb{H}} \mathbb{F} = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}}.$$

Proof. Lower bound: To get the lower bound of $\dim_{\mathbb{H}} \mathbb{F}$, we shall make use of the symbolic space. Let

$$\mathcal{D}_n = \left\{ (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n : ks_k \leq \sigma_k < (k+1)s_k, \forall 1 \leq k \leq n \right\}.$$

For any $n \geq 1$ and $(\sigma_1, \dots, \sigma_n) \in \mathcal{D}_n$, we call

$$J_n(\sigma_1, \dots, \sigma_n) = \bigcup_{\sigma_{n+1}} \text{cl}(I_{n+1}(\sigma_1, \dots, \sigma_n, \sigma_{n+1}))$$

a basic interval of order n , where ‘cl’ stands for the closure of a set and the union is take over all σ_{n+1} such that $(\sigma_1, \dots, \sigma_n, \sigma_{n+1}) \in \mathcal{D}_{n+1}$. Put $\mathbb{E}_0 = [0, 1]$ and

$$\mathbb{E}_n = \bigcup_{(\sigma_1, \dots, \sigma_n) \in \mathcal{D}_n} J_n(\sigma_1, \dots, \sigma_n) \quad \text{for all } n \geq 1.$$

Then

$$\mathbb{F} = \bigcap_{n=0} \mathbb{E}_n = \bigcap_{n=0} \bigcup_{(\sigma_1, \dots, \sigma_n) \in \mathcal{D}_n} J_n(\sigma_1, \dots, \sigma_n).$$

The construction of \mathbb{F} indicates that each interval in \mathbb{E}_{n-1} contains at least m_n of the basic intervals of order n in \mathbb{E}_n , where

$$m_n = (n+1)s_n - ns_n = s_n \geq 2.$$

Let $J_n(\tau_1, \tau_2, \dots, \tau_n)$ and $J_n(\sigma_1, \sigma_2, \dots, \sigma_n)$ be the two distinct basic intervals in \mathbb{E}_n . We know that the intervals $J_n(\tau_1, \tau_2, \dots, \tau_n)$ and $J_n(\sigma_1, \sigma_2, \dots, \sigma_n)$ are separated by the $(n+1)$ -th order cylinder

$$I_{n+1}(\tau_1, \tau_2, \dots, \tau_n, 1) \quad \text{or} \quad I_{n+1}(\sigma_1, \sigma_2, \dots, \sigma_n, 1)$$

according to the location between $J_n(\tau_1, \tau_2, \dots, \tau_n)$ and $J_n(\sigma_1, \sigma_2, \dots, \sigma_n)$. In the following we estimate the length of the above $(n+1)$ -th order cylinder. By (3.10),

$$\begin{aligned} |I_{n+1}(\sigma_1, \sigma_2, \dots, \sigma_n, 1)| &\geq \frac{1}{2q_{n+1}^2} \geq \frac{1}{8} \left(\prod_{k=1}^n (\sigma_k + 1) \right)^{-2} \\ &\geq \frac{1}{8} \left(\prod_{k=1}^n ((k+1)s_k) \right)^{-2} := \varepsilon_n. \end{aligned}$$

Similarly, we also have $|I_{n+1}(\tau_1, \dots, \tau_n, 1)| \geq \varepsilon_n$. Clearly $0 < \varepsilon_{n+1} < \varepsilon_n$ for any $n \geq 1$. Then by Lemma 3.7, we have

$$\begin{aligned} \dim_{\mathbb{H}} \mathbb{F} &\geq \liminf_{n \rightarrow \infty} \frac{\log(m_1 m_2 \cdots m_n)}{-\log(m_{n+1} \varepsilon_{n+1})} \\ &= \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \log s_k}{-\log s_{n+1} + \log 8 + 2 \sum_{k=1}^{n+1} \log((k+1)s_k)} \\ &= \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}}. \end{aligned}$$

Upper bound: Let

$$J_n(a_1, a_2, \dots, a_n) = \bigcup_{a_{n+1} \geq (n+1)s_{n+1}} cl(I_{n+1}(a_1, a_2, \dots, a_n, a_{n+1})).$$

Then we have

$$\mathbb{F} \subseteq \bigcap_{n=1}^{\infty} \bigcup_{ks_k \leq a_k < (k+1)s_k, 1 \leq k \leq n} J_n(a_1, a_2, \dots, a_n).$$

It means that for any $n \geq 1$,

$$\bigcup_{ks_k \leq a_k < (k+1)s_k, 1 \leq k \leq n} J_n(a_1, a_2, \dots, a_n)$$

is a cover of \mathbb{F} . The number of such basic intervals is

$$\prod_{k=1}^n ((k+1)s_k - ks_k) = \prod_{k=1}^n s_k.$$

By Proposition 3.12, the length of such a basic interval is

$$\begin{aligned} |J_n(a_1, a_2, \dots, a_n)| &= \sum_{a_{n+1} \geq (n+1)s_{n+1}} |I_{n+1}(a_1, a_2, \dots, a_n, a_{n+1})| \\ &\leq |I_n(a_1, a_2, \dots, a_n)| \cdot \sum_{a_{n+1}(x) \geq (n+1)s_{n+1}} \frac{2}{a_{n+1}^2(x)} \\ &\leq \frac{1}{q_n^2} \cdot \frac{4}{(n+1)s_{n+1}} \leq \frac{4}{(n+1)s_{n+1} \cdot \left(\prod_{k=1}^n (ks_k) \right)^2}. \end{aligned}$$

Therefore, we deduce from Proposition 3.3 that

$$\begin{aligned}
 \dim_{\text{H}} \mathbb{F} &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \log s_k}{-\log |J_n(a_1, a_2, \dots, a_n)|} \\
 &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \log s_k}{-\log(4(n+1)) + \log s_{n+1} + 2 \sum_{k=1}^n \log s_k + 2 \log(n+1)!} \\
 &= \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}}.
 \end{aligned}$$

□

The proof of Theorem 5.12. Recall that

$$E_{\text{inf}}(\Lambda, \alpha) := \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = \alpha \right\} \quad (\alpha \geq 0).$$

The case $\alpha \geq 1$. Upper bound: By the definition of \liminf , if $x \in E_{\text{inf}}(\Lambda, \alpha)$, then for any $0 < \varepsilon < \alpha$, there exists $N \geq 1$ such that $a_j(x) \geq j^{\alpha-\varepsilon}$ for any $j \geq N$, and for any $n \geq 1$, there exists $k \geq n$ such that $a_k(x) \leq k^{\alpha+\varepsilon}$. Set

$$C_N(\alpha, \varepsilon) = \{x \in \Lambda : a_j(x) \geq j^{\alpha-\varepsilon}, \forall j \geq N\} \text{ and } D_n(\alpha, \varepsilon) = \bigcup_{k=n}^{\infty} \{x \in \Lambda : a_k(x) \leq k^{\alpha+\varepsilon}\}.$$

Then we have

$$\begin{aligned}
 E_{\text{inf}}(\Lambda, \alpha) &\subseteq \left(\bigcup_{N=1}^{\infty} C_N(\alpha, \varepsilon) \right) \cap \left(\bigcap_{n=1}^{\infty} D_n(\alpha, \varepsilon) \right) \subseteq \left(\bigcup_{N=1}^{\infty} C_N(\alpha, \varepsilon) \right) \cap \left(\bigcap_{n=N}^{\infty} D_n(\alpha, \varepsilon) \right) \\
 &= \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left(\{x \in \Lambda : a_j(x) \geq j^{\alpha-\varepsilon}, \forall j \geq N\} \cap \left(\bigcup_{k=n}^{\infty} \{x \in \Lambda : a_k(x) \leq k^{\alpha+\varepsilon}\} \right) \right) \\
 &\subseteq \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{k=n}^{\infty} \left\{ x \in \Lambda : a_j(x) \geq j^{\alpha-\varepsilon}, a_k(x) \leq k^{\alpha+\varepsilon}, \forall N \leq j \leq k \right\}.
 \end{aligned}$$

Let

$$B_N(\alpha, \varepsilon) = \bigcap_{n=N}^{\infty} \bigcup_{k=n}^{\infty} \{x \in \Lambda : a_k(x) \leq k^{\alpha+\varepsilon}, a_j(x) \geq j^{\alpha-\varepsilon}, \forall N \leq j \leq k\}. \quad (5.7)$$

Then we obtain

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) \leq \sup_{N \geq 1} \{ \dim_{\text{H}} B_N(\alpha, \varepsilon) \}. \quad (5.8)$$

In the following we only deal with the Hausdorff dimension of $B_1(\alpha, \varepsilon)$ since the proofs for other cases are similar. In view of (5.7), we rewrite the set $B_1(\alpha, \varepsilon)$ as

$$B_1(\alpha, \varepsilon) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\sigma_1, \dots, \sigma_k) \in \mathcal{C}_k} I_k(\sigma_1, \dots, \sigma_k), \quad (5.9)$$

where

$$\mathcal{C}_k = \left\{ (\sigma_1, \dots, \sigma_k) \in \mathbb{N}^k : 1 \leq \sigma_1 \leq \dots \leq \sigma_k \leq k^{\alpha+\varepsilon}, \sigma_j \geq j^{\alpha-\varepsilon}, \forall 1 \leq j \leq k \right\}.$$

Applying Lemma 5.14, we can estimate the cardinality of set \mathcal{C}_k as follows:

$$\begin{aligned} \#\mathcal{C}_k &\leq \frac{(k + \lfloor k^{\alpha+\varepsilon} \rfloor - 1)!}{k! (\lfloor k^{\alpha+\varepsilon} \rfloor - 1)!} \\ &= \frac{\lfloor k^{\alpha+\varepsilon} \rfloor \cdot (\lfloor k^{\alpha+\varepsilon} \rfloor + 1) \cdots (\lfloor k^{\alpha+\varepsilon} \rfloor + k - 1)}{k!} \\ &= \frac{k^{k(\alpha+\varepsilon)}}{k!} \cdot \left(1 + \frac{1}{k^{\alpha+\varepsilon}}\right) \cdots \left(1 + \frac{k-1}{k^{\alpha+\varepsilon}}\right) \leq \frac{2^k \cdot k^{k(\alpha+\varepsilon)}}{k!}. \end{aligned} \quad (5.10)$$

By the Stirling formula, for any $n \geq 1$,

$$\sqrt{2\pi n} n^{n+\frac{1}{2}} \exp(-n) \leq n! \leq \exp(1) n^{n+\frac{1}{2}} \exp(-n). \quad (5.11)$$

Combining (5.10) with (5.11), we deduce that

$$\#\mathcal{C}_k \leq 2^k \cdot (k!)^{\alpha+\varepsilon-1} \cdot \left(\frac{\exp(k)}{\sqrt{2\pi k}}\right)^{\alpha+\varepsilon}. \quad (5.12)$$

By (3.10),

$$|I_k(\sigma_1, \dots, \sigma_k)| \leq \frac{1}{q_k^2} \leq \prod_{i=1}^k \sigma_i^{-2} \leq (k!)^{-2(\alpha-\varepsilon)}. \quad (5.13)$$

Taking

$$s = \frac{\alpha + 2\varepsilon - 1}{2(\alpha - \varepsilon)},$$

by the definition of s -dimensional Hausdorff measure, we deduce from (5.9), (5.12) and (5.13) that

$$\begin{aligned} \mathcal{H}^s(B_1(\alpha, \varepsilon)) &\leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \sum_{(\sigma_1, \dots, \sigma_k) \in \mathcal{C}_k} |I_k(\sigma_1, \dots, \sigma_k)|^s \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{\#\mathcal{C}_k}{(k!)^{2s(\alpha-\varepsilon)}} \leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{2^k \exp(k(\alpha+\varepsilon))}{(k!)^\varepsilon} = 0. \end{aligned}$$

This shows

$$\dim_{\text{H}} B_1(\alpha, \varepsilon) \leq \frac{\alpha + 2\varepsilon - 1}{2(\alpha - \varepsilon)}.$$

It follows (5.8) that

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) \leq \frac{\alpha - 1}{2\alpha}.$$

Lower bound: According to the proof of its upper bound, it is obvious that

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) = 0 \text{ for } \alpha = 1.$$

In the following we always assume that $\alpha > 1$. Let M be the positive integer such that $s_n = M \lfloor n^{\alpha-1} \rfloor \geq 2$ for any $n \geq 1$. Set

$$\mathbb{F}(\alpha) = \left\{ x \in [0, 1) : n(M \lfloor n^{\alpha-1} \rfloor) \leq a_n(x) < (n+1)(M \lfloor n^{\alpha-1} \rfloor), \forall n \geq 1 \right\}.$$

Then we claim that

$$\mathbb{F}(\alpha) \subseteq E_{\text{inf}}(\Lambda, \alpha). \quad (5.14)$$

Indeed, on one hand, for any $x \in \mathbb{F}(\alpha)$ and $n \geq 1$, it is easy to see that

$$a_n(x) < (n+1)(M \lfloor n^{\alpha-1} \rfloor) \leq (n+1)(M \lfloor (n+1)^{\alpha-1} \rfloor) \leq a_{n+1}(x),$$

then we have $x \in \Lambda$. On the other hand, we also obtain

$$\begin{aligned} \alpha &= \liminf_{n \rightarrow \infty} \frac{\log(nM \lfloor n^{\alpha-1} \rfloor)}{\log n} \leq \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log((n+1)M \lfloor (n+1)^{\alpha-1} \rfloor)}{\log n} = \alpha. \end{aligned}$$

Notice that

$$\limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)} = \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log(M \lfloor (n+1)^{\alpha-1} \rfloor)}{\sum_{i=1}^n \log(M \lfloor n^{\alpha-1} \rfloor)} = \frac{2}{\alpha - 1}.$$

By Lemma 5.15, we have

$$\dim_{\mathbb{H}} \mathbb{F}(\alpha) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}} = \frac{\alpha - 1}{2\alpha}. \quad (5.15)$$

It follows (5.14) and (5.15) that

$$\dim_{\mathbb{H}} E_{\text{inf}}(\Lambda, \alpha) \geq \frac{\alpha - 1}{2\alpha}.$$

The case $0 \leq \alpha < 1$. By the definition of \liminf , for any $0 < \varepsilon < 1 - \alpha$, we have

$$E_{\text{inf}}(\Lambda, \alpha) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{x \in \Lambda : a_k(x) \leq k^{\alpha+\varepsilon}\}. \quad (5.16)$$

Notice that

$$\{x \in \Lambda : a_k(x) \leq k^{\alpha+\varepsilon}\} \subseteq \bigcup_{(\sigma_1, \dots, \sigma_k) \in \tilde{\mathcal{C}}_k} I_k(\sigma_1, \dots, \sigma_k),$$

where

$$\tilde{\mathcal{C}}_k = \left\{ (\sigma_1, \dots, \sigma_k) \in \mathbb{N}^k : 1 \leq \sigma_1 \leq \dots \leq \sigma_k \leq k^{\alpha+\varepsilon} \right\}.$$

Since $0 < \varepsilon < 1 - \alpha$, it follows from Lemma 5.14 that

$$\begin{aligned} \#\tilde{\mathcal{C}}_k &= \frac{(k + \lfloor k^{\alpha+\varepsilon} \rfloor - 1)!}{k!(\lfloor k^{\alpha+\varepsilon} \rfloor - 1)!} \\ &\leq (k+1) \cdot (k+2) \cdots (k + \lfloor k^{\alpha+\varepsilon} \rfloor - 1) \\ &\leq (k + k^{\alpha+\varepsilon})^{k^{\alpha+\varepsilon}} \leq \exp\left((1 + \log k)k^{\alpha+\varepsilon}\right). \end{aligned} \quad (5.17)$$

The second formula in (3.9) implies that $q_n \geq q_{n-1} + q_{n-2}$ for any $n \geq 1$. Thus

$$q_n \geq \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n \geq \frac{1}{2\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n. \quad (5.18)$$

By (3.10) and (5.18),

$$|I_k(\sigma_1, \dots, \sigma_k)| \leq \frac{1}{q_k^2} \leq 20 \left(\frac{1 + \sqrt{5}}{2}\right)^{-2k}. \quad (5.19)$$

By the definition of s -dimensional Hausdorff measure, taking $s = \varepsilon$ and then combining (5.16), (5.17) with (5.19), we deduce that

$$\begin{aligned} \mathcal{H}^\varepsilon(E_{\text{inf}}(\Lambda, \alpha)) &\leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \sum_{(\sigma_1, \dots, \sigma_k) \in \tilde{\mathcal{C}}_k} |I_k(\sigma_1, \sigma_2, \dots, \sigma_k)|^\varepsilon \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \#\tilde{\mathcal{C}}_k \cdot 20^\varepsilon \left(\frac{1 + \sqrt{5}}{2}\right)^{-2k\varepsilon} \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=n}^{\infty} \exp\left(k^{\alpha+\varepsilon}(1 + \log k)\right) \cdot 20^\varepsilon \left(\frac{1 + \sqrt{5}}{2}\right)^{-2k\varepsilon} = 0. \end{aligned}$$

This shows

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \alpha) = 0.$$

The proof of Theorem 5.13. Recall that

$$E_{\text{inf}}(\Lambda, \psi) := \left\{ x \in \Lambda : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

(1) For the case when $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, it is obvious that

$$E_{\text{inf}}(\Lambda, \psi) = E_{\text{inf}}(\Lambda, \alpha).$$

Thus the result follows from Theorem 5.12.

(2) **Upper bound:** For any $0 < \varepsilon < 1$, we have $E_{\text{inf}}(\Lambda, \psi) \subseteq \bigcup_{N=1}^{\infty} F_\psi(N)$, where

$$F_\psi(N) = \left\{ x \in [0, 1) : a_n(x) \geq \exp((1 - \varepsilon)\psi(n)), \forall n \geq N \right\}.$$

Then we obtain

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) \leq \sup_{N \geq 1} \{ \dim_{\text{H}} F_{\psi}(N) \}. \quad (5.20)$$

Similar to (5.8), we only consider the upper bound of Hausdorff dimension of $F_{\psi}(1)$. It follows from Theorem 5.7 that

$$\dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq \exp((1 - \varepsilon)\psi(n)), \forall n \geq 1 \right\} = \frac{1}{B + 1}.$$

Thus we deduce from (5.20) that

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) \leq \frac{1}{B + 1}.$$

Lower bound: It is trivial for the case when $B = \infty$. In the following we always assume that $1 \leq B < \infty$. Since

$$\limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n} = \log B,$$

for any $\varepsilon > 0$, we have $\psi(n) \leq (B + \varepsilon/2)^n$ for n large enough. This implies that

$$\text{for any } j \geq 1, \psi(n)(B + \varepsilon)^{j-n} \leq (B + \varepsilon/2)^n (B + \varepsilon)^{j-n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Let

$$T_j = \sup_{n \geq j} \{ \exp(\psi(n)(B + \varepsilon)^{j-n}) \}, \quad j = 1, 2, \dots.$$

Then we claim that

$$\liminf_{n \rightarrow \infty} \frac{\log T_n}{\psi(n)} = 1. \quad (5.21)$$

In fact, by the definition of T_j , we have $T_j \geq \exp \psi(j)$ for any $j \geq 1$. That is,

$$\liminf_{n \rightarrow \infty} \frac{\log T_n}{\psi(n)} \geq 1.$$

On the other hand, the supremum in the definition of T_j is achieved, we denote by $t_j \geq j$ the smallest number for which

$$T_j = \exp(\psi(t_j)(B + \varepsilon)^{j-t_j}).$$

Notice that

$$\begin{cases} T_j = \sup \{ \exp \psi(j), \dots, \exp(\psi(t_j)(B + \varepsilon)^{j-t_j}), \dots \}, \\ T_{j+1} = \sup \{ \exp \psi(j+1), \dots, \exp(\psi(t_j)(B + \varepsilon)^{j+1-t_j}), \dots \}, \\ \dots \dots \dots \\ T_{t_j} = \sup \{ \exp \psi(t_j), \exp(\psi(t_j+1)(B + \varepsilon)^{-1}), \dots \}. \end{cases}$$

Then for many consecutive j 's, the number t_j will be the same. More precisely,

$$t_j = t_{j+1} = \dots = t_{t_j}.$$

Let $(\ell_k)_{k \geq 1}$ be a strictly increasing sequence of $(t_j)_{j \geq 1}$. Then for these ℓ_k , we have

$$T_{\ell_k} = \exp \psi(\ell_k).$$

It is immediate that

$$\liminf_{n \rightarrow \infty} \frac{\log T_n}{\psi(n)} \leq \liminf_{k \rightarrow \infty} \frac{\log T_{\ell_k}}{\psi(\ell_k)} = 1.$$

Since ψ is a non-decreasing function, it is easy to check that

$$T_j \leq T_{j+1} \quad \text{and} \quad T_{j+1} \leq T_j^{B+\varepsilon}. \quad (5.22)$$

Now we use the sequence $(T_j)_{j \geq 1}$ to construct a suitable subset of $E_{\text{inf}}(\Lambda, \psi)$. Let M be the positive integer such that $s_n = M \lfloor T_n \rfloor \geq 2$ for any $n \geq 1$. Set

$$\mathbb{F}(T) = \left\{ x \in [0, 1) : n(M \lfloor T_n \rfloor) \leq a_n(x) < (n+1)(M \lfloor T_n \rfloor), \forall n \geq 1 \right\}.$$

Then by (5.21) and (5.22), we can verify that

$$\mathbb{F}(T) \subseteq E_{\text{inf}}(\Lambda, \psi). \quad (5.23)$$

Since $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$, combining (5.21) with (5.22), we have

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\log T_n} = 0 \quad \text{and} \quad \log T_{n+1} - \log T_1 \leq (B + \varepsilon - 1) \sum_{k=1}^n \log T_k. \quad (5.24)$$

We deduce from (5.24) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)} &= \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log(M \lfloor T_{n+1} \rfloor)}{\sum_{k=1}^n \log(M \lfloor T_k \rfloor)} \\ &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)!}{\sum_{k=1}^n \log T_k} + \limsup_{n \rightarrow \infty} \frac{\log T_{n+1}}{\sum_{k=1}^n \log T_k} \\ &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)}{\log T_n} + (B + \varepsilon - 1) = B + \varepsilon - 1. \end{aligned}$$

It follows from Lemma 5.15 and (5.23) that

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) \geq \dim_{\text{H}} \mathbb{F}(T) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}} \geq \frac{1}{B + 1 + \varepsilon}.$$

This shows

$$\dim_{\text{H}} E_{\text{inf}}(\Lambda, \psi) \geq \frac{1}{B + 1}.$$

5.3 Hausdorff dimension of the sets related to the convergence exponent

In this section, we mainly deal with the Hausdorff dimension of a series of sets related to the convergence exponent.

Compared with Theorem 5.9 and Theorem 5.11, we know that the dimensional result changes essentially under the monotonicity of sequence $(a_n(x))_{n \geq 1}$. Motivated by this point, we remove the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ in $E_{\text{inf}}(\Lambda, \psi)$ and then consider the Hausdorff dimension of its corresponding set, i.e.,

$$E_{\text{inf}}(\psi) := \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where ψ is a positive function defined on \mathbb{N} .

Theorem 5.16. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} E_{\text{inf}}(\psi) = \frac{1}{B + 1},$$

where B is given by

$$\log B := \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Proof. Using the same proof in the second part of Theorem 5.13, we can easily obtain the upper bound of $\dim_{\text{H}} E_{\text{inf}}(\psi)$. In fact,

$$E_{\text{inf}}(\psi) \subseteq \bigcup_{N=1}^{\infty} \left\{ x \in [0, 1) : a_n(x) \geq \exp((1 - \varepsilon)\psi(n)), \forall n \geq N \right\}.$$

Thus we obtain

$$\dim_{\text{H}} E_{\text{inf}}(\psi) \leq \frac{1}{B + 1}.$$

Lower bound: Let $(L_j)_{j \geq 1}$ be the sequence defined as follows:

$$L_j = \sup_{n \geq j} \left\{ \exp(\psi(n)(B + \varepsilon)^{j-n}) \right\}, \quad j = 1, 2, \dots \quad (5.25)$$

Using the same analysis in the proof of the second part in Theorem 5.13, we have

$$\liminf_{n \rightarrow \infty} \frac{\log L_n}{\psi(n)} = 1. \quad (5.26)$$

By (5.25), we have $L_{j+1} \leq L_j^{B+\varepsilon}$ and thus

$$\log L_{n+1} - \log L_1 \leq (B + \varepsilon - 1) \sum_{j=1}^n \log L_j. \quad (5.27)$$

Let M be the positive integer such that $t_n = M \lfloor L_n \rfloor \geq 3$ for any $n \geq 1$. Set

$$E_{\text{inf}}(\{L_n\}_{n \geq 1}) = \left\{ x \in [0, 1) : M \lfloor L_n \rfloor \leq a_n(x) < 2M \lfloor L_n \rfloor, \forall n \geq 1 \right\}.$$

Then by (5.26), we obtain

$$E_{\text{inf}}((L_n)_{n \geq 1}) \subseteq E_{\text{inf}}(\psi). \quad (5.28)$$

By Lemma 5.8, we deduce from (5.27) and (5.28) that

$$\begin{aligned} \dim_{\text{H}} E_{\text{inf}}(\psi) &\geq \dim_{\text{H}} E_{\text{inf}}((L_n)_{n \geq 1}) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log t_{n+1}}{\log(t_1 t_2 \cdots t_n)}} \\ &= \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log L_{n+1}}{\log L_1 + \cdots + \log L_n}} \geq \frac{1}{B + 1 + \varepsilon}. \end{aligned}$$

This shows

$$\dim_{\text{H}} E_{\text{inf}}(\psi) \geq \frac{1}{B + 1}.$$

□

The dimensional results in Theorem 5.13 and Theorem 5.16 indicate that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ is redundant in some sense when $\psi(n)$ grows faster than $\log n$.

On the other hand, motivated by the proofs of Theorem 5.12 and Theorem 5.13, we observe that the upper bounds of the Hausdorff dimension of the corresponding sets of limit are obvious. Consider the set

$$E(\Lambda, \alpha) := \left\{ x \in \Lambda : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = \alpha \right\} \quad (\alpha \geq 0),$$

we have

$$\mathbb{F}(\alpha) \subseteq E(\Lambda, \alpha) \subseteq E_{\text{inf}}(\Lambda, \alpha),$$

where $\mathbb{F}(\alpha)$ is the same as in (5.14). It follows Lemma 5.15 and Theorem 5.12 that

$$\dim_{\text{H}} E(\Lambda, \alpha) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases} \quad (5.29)$$

More generally, we further study the Hausdorff dimension of the set

$$E(\Lambda, \psi) := \left\{ x \in \Lambda : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where ψ is a positive function defined on \mathbb{N} .

Theorem 5.17. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

(1) *If $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, then*

$$\dim_{\text{H}} E(\Lambda, \psi) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \frac{\alpha-1}{2\alpha}, & \alpha \geq 1. \end{cases}$$

(2) If $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$ and $\psi(n)$ is non-decreasing, then we have

$$E(\Lambda, \psi) \neq \emptyset \quad \text{and} \quad \dim_{\mathbb{H}} E(\Lambda, \psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)}}.$$

Proof. (1) For the case when $\psi(n)/\log n \rightarrow \alpha$ ($0 \leq \alpha < \infty$) as $n \rightarrow \infty$, it is obvious that $E(\Lambda, \psi) = E(\Lambda, \alpha)$. Then the result follows from (5.29).

(2) If $\psi(n)$ is non-decreasing, define a point $x \in [0, 1]$ by taking

$$a_n(x) = \exp(\lfloor \psi(n) \rfloor), \quad n \geq 1.$$

Then we have

$$a_n(x) \leq a_{n+1}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1.$$

Thus $E(\Lambda, \psi)$ is non-empty. Next we calculate $\dim_{\mathbb{H}} E(\Lambda, \psi)$.

Upper bound: For any $0 < \varepsilon < 1$, we have $E(\Lambda, \psi) \subseteq \bigcup_{N=1}^{\infty} E_{\psi}(N)$, where

$$E_{\psi}(N) = \left\{ x \in [0, 1] : \exp((1 - \varepsilon)\psi(n)) \leq a_n(x) \leq \exp((1 + \varepsilon)\psi(n)), \forall n \geq N \right\}.$$

Then we obtain

$$\dim_{\mathbb{H}} E(\Lambda, \psi) \leq \sup_{N \geq 1} \left\{ \dim_{\mathbb{H}} E_{\psi}(N) \right\}. \quad (5.30)$$

Similar to (5.8), we only consider the upper bound of $\dim_{\mathbb{H}} E_{\psi}(1)$. Let

$$D_n(\varepsilon) = \left\{ (a_1, \dots, a_n) \in \mathbb{N}^n : \exp((1 - \varepsilon)\psi(n)) \leq a_n(x) \leq \exp((1 + \varepsilon)\psi(n)) \right\}.$$

Then we deduce that

$$\#D_n(\varepsilon) \leq \prod_{k=1}^n \left(2\varepsilon\psi(k) \exp((1 + \varepsilon)\psi(k)) \right). \quad (5.31)$$

For any $n \geq 1$ and $(\sigma_1, \dots, \sigma_n) \in D_n(\varepsilon)$, put

$$J_n(\sigma_1, \dots, \sigma_n) = \bigcup_{\sigma_{n+1}} I_{n+1}(\sigma_1, \dots, \sigma_n, \sigma_{n+1}),$$

where the union is taken over all σ_{n+1} such that $(\sigma_1, \dots, \sigma_n, \sigma_{n+1}) \in D_{n+1}(\varepsilon)$. Notice that for any $n \geq 1$,

$$E_{\psi}(1) \subseteq \bigcup_{(\sigma_1, \dots, \sigma_n) \in D_n(\varepsilon)} J_n(\sigma_1, \dots, \sigma_n).$$

Thus we get a cover of $E_\psi(1)$, *i.e.*, $\{J_n(\sigma_1, \dots, \sigma_n) : (\sigma_1, \dots, \sigma_n) \in D_n(\varepsilon)\}$. It follows from Proposition 3.9 and Proposition 3.11 that

$$\begin{aligned}
 |J_n(\sigma_1, \dots, \sigma_n)| &\leq \sum_{\sigma_{n+1} \geq \exp((1-\varepsilon)\psi(n+1))} |I_{n+1}(\sigma_1, \dots, \sigma_n, \sigma_{n+1})| \\
 &\leq \sum_{\sigma_{n+1} \geq \exp((1-\varepsilon)\psi(n+1))} \frac{1}{\sigma_1^2 \cdots \sigma_n^2 \sigma_{n+1}^2} \\
 &\leq \left(\sum_{\sigma_{n+1} \geq \exp((1-\varepsilon)\psi(n+1))} \frac{1}{\sigma_{n+1}^2} \right) \cdot \exp \left(-2(1-\varepsilon) \sum_{k=1}^n \psi(k) \right) \\
 &\leq \exp \left(-(1-\varepsilon)\psi(n+1) - 2(1-\varepsilon) \cdot \sum_{k=1}^n \psi(k) \right). \tag{5.32}
 \end{aligned}$$

Combining (5.31) with (5.32), we deduce from Proposition 3.3 that

$$\dim_{\text{H}} E_\psi(1) \leq \liminf_{n \rightarrow \infty} \frac{\log \#D_n(\varepsilon)}{-\log |J_n(\sigma_1, \dots, \sigma_n)|} \leq \frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1)+\cdots+\psi(n)}}.$$

It follows (5.30) that

$$\dim_{\text{H}} E(\Lambda, \psi) \leq \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1)+\cdots+\psi(n)}}.$$

Lower bound: Let M be the positive integer such that $s_n = M \lfloor \exp \psi(n) \rfloor \geq 2$ for any $n \geq 1$. Set

$$\mathcal{E}(\Lambda, \psi) = \left\{ x \in [0, 1] : n(M \lfloor \exp \psi(n) \rfloor) \leq a_n(x) < (n+1)(M \lfloor \exp \psi(n) \rfloor), \forall n \geq 1 \right\}.$$

Under the condition that $\psi(n)$ is non-decreasing, we can easily obtain

$$\mathcal{E}(\Lambda, \psi) \subseteq E(\Lambda, \psi). \tag{5.33}$$

Notice that $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$, then we have

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)} &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)!}{\log(s_1 s_2 \cdots s_n)} + \limsup_{n \rightarrow \infty} \frac{\log s_{n+1}}{\log(s_1 s_2 \cdots s_n)} \\
 &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+2)}{\psi(n)} + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \cdots + \psi(n)} \\
 &= \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \cdots + \psi(n)}.
 \end{aligned}$$

We conclude from Lemma 5.15 and (5.33) that

$$\begin{aligned}
 \dim_{\text{H}} E(\Lambda, \psi) &\geq \dim_{\text{H}} \mathcal{E}(\Lambda, \psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_1 s_2 \cdots s_n)}} \\
 &\geq \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \cdots + \psi(n)}}.
 \end{aligned}$$

□

Similarly, we also would like to investigate the Hausdorff dimension of the corresponding set of $E(\Lambda, \psi)$ without the monotonicity of the sequence $(a_n(x))_{n \geq 1}$, i.e.,

$$E(\psi) := \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where ψ is a positive function defined on \mathbb{N} .

Theorem 5.18. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} E(\psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)}}.$$

Proof. Using the same proof for the upper bound of $\dim_{\text{H}} E(\Lambda, \psi)$, we also have

$$E(\psi) \subseteq \bigcup_{N=1}^{\infty} \left\{ x \in [0, 1) : \exp((1 - \varepsilon)\psi(n)) \leq a_n(x) \leq \exp((1 + \varepsilon)\psi(n)), \forall n \geq N \right\}.$$

Thus we obtain

$$\dim_{\text{H}} E(\psi) \leq \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)}}.$$

Lower bound: Let M be the positive integer such that $t_n = M \lfloor \exp \psi(n) \rfloor \geq 3$ for any $n \geq 1$. Set

$$\mathcal{E}(\psi) = \left\{ x \in [0, 1) : M \lfloor \exp \psi(n) \rfloor \leq a_n(x) < 2M \lfloor \exp \psi(n) \rfloor, \forall n \geq 1 \right\}.$$

Then we have

$$\mathcal{E}(\psi) \subseteq E(\psi). \tag{5.34}$$

It follows from Lemma 5.8 and (5.34) that

$$\begin{aligned} \dim_{\text{H}} E(\psi) &\geq \dim_{\text{H}} \mathcal{E}(\psi) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log t_{n+1}}{\log(t_1 t_2 \dots t_n)}} \\ &= \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)}}. \end{aligned}$$

□

The dimensional results in Theorem 5.17 and Theorem 5.18 also show that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ is redundant in some sense when $\psi(n)$ grows faster than $\log n$.

6 Exceptional sets of Borel-Bernstein Theorem

In this chapter, we mainly study the exceptional the Hausdorff dimension of sets of Borel-Bernstein Theorem in continued fractions. In fact, these exceptional sets are turned out some level sets of lim sup functions. The intersection of the level sets of lim sup and lim inf is also investigated.

6.1 Borel-Bernstein Theorem

Let $[a_1(x), a_2(x), \dots]$ be the continued fraction expansion of $x \in [0, 1)$. In 1909, Borel [9] proved that for Lebesgue almost all $x \in [0, 1)$,

$$\limsup_{n \rightarrow \infty} a_n(x) = \infty.$$

In 1912, Borel [10] and Bernstein [1] independently gave a more precise result called Borel-Bernstein Theorem or “0–1” law on describing the growth rate of partial quotients in the sense of Lebesgue measure.

Theorem 6.1 (Borel-Bernstein). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function and*

$$\mathbb{F}(\psi) = \left\{ x \in [0, 1) : a_n(x) \geq \psi(n) \text{ for infinitely many } n \right\}.$$

Then we have

$$\mathcal{L}(\mathbb{F}(\psi)) = \begin{cases} 0, & \sum_{n \geq 1} \frac{1}{\psi(n)} < \infty, \\ 1, & \sum_{n \geq 1} \frac{1}{\psi(n)} = \infty. \end{cases}$$

Borel-Bernstein Theorem shows that many sets of points restricted some conditions on their partial quotients are of null Lebesgue measure. A typical way of refining the size of such null sets is then to calculate its Hausdorff dimension. The first published work was due to Jarník [53]. For any $M \in \mathbb{N}$, let E_M be the set of points in $[0, 1)$ whose partial quotients in continued fraction expansion do not exceed M . That is,

$$E_M = \left\{ x \in [0, 1) : 1 \leq a_n(x) \leq M, \forall n \geq 1 \right\}.$$

Theorem 6.2 ([53]). *For any $M \geq 8$,*

$$1 - \frac{1}{M \log 2} \leq \dim_{\text{H}} E_M \leq 1 - \frac{1}{8M \log M}.$$

In particular, the set E whose partial quotients are bounded, i.e.,

$$E = \bigcup_{M \geq 2} E_M = \left\{ x \in [0, 1) : \sup_{n \geq 1} a_n(x) < +\infty \right\}$$

is of Hausdorff dimension one.

Indeed, the set E_M is concerned with continued fractions whose partial quotients remain small in some sense. It also can be viewed as the attractor of the iterated function systems given by

$$f_i(x) = \frac{1}{i+x}, \quad i \in \{1, 2, \dots, M\}.$$

Many authors investigated problems of this type, especially for estimating the Hausdorff dimension of E_2 . These include the work of Good [43], Bumby [14, 15], Hensley [46, 47], Pollicott and Weiss [82], Jenkinson and Pollicott [55, 56], Jenkinson [54], Chousionis, Leykekhman and Urbański [19], et.al.

The study of the Hausdorff dimension of the set $\mathbb{F}(\psi)$ goes back to Good. In 1941, Good [43] obtained some bound estimation for its Hausdorff dimension. In 1997, Łuczak [72] and Feng, Wu, Liang and Tseng [40] considered the case when $\psi(n) = a^{b^n}$ ($a, b > 1$) and obtained the following result.

Theorem 6.3 ([40, 72]). *For any $a, b > 1$,*

$$\dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq a^{b^n} \text{ for infinitely many } n \right\} = \frac{1}{b+1}.$$

In 2008, Wang and Wu [94] completely determined the Hausdorff dimension of the set $\mathbb{F}(\psi)$ for any positive function ψ defined on \mathbb{N} .

Theorem 6.4 ([94, Theorem 4.2]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Suppose*

$$\liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n} = \log b.$$

(1) *If $b = 1$, then $\dim_{\text{H}} \mathbb{F}(\psi) = 1$.*

(2) *If $1 < b < \infty$, then $\dim_{\text{H}} \mathbb{F}(\psi) = s_b$, where*

$$s_b := \dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq b^n \text{ for infinitely many } n \right\}.$$

Also, $s : (1, \infty) \rightarrow [\frac{1}{2}, 1]$ is a decreasing continuous function satisfying

$$\lim_{b \rightarrow 1} s_b = 1 \quad \text{and} \quad \lim_{b \rightarrow \infty} s_b = \frac{1}{2}.$$

(3) *If $b = \infty$, let*

$$\liminf_{n \rightarrow \infty} \frac{\log \log \psi(n)}{n} = \log d.$$

Then we have

$$\dim_{\text{H}} \mathbb{F}(\psi) = \begin{cases} \frac{1}{2}, & d = 1, \\ \frac{1}{d+1}, & 1 < d < \infty, \\ 0, & d = \infty. \end{cases}$$

6.2 Hausdorff dimension of the exceptional sets

It is immediate from the Borel-Bernstein Theorem that

$$\limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\log n} = 1 \quad (6.1)$$

for Lebesgue almost all $x \in [0, 1)$. The fractional dimensional theory provides an indication of the size and complexity in studying the exceptional sets of (6.1). More precisely, we would like to consider the Hausdorff dimension of the set

$$E_{\text{sup}}(\psi) = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\},$$

where ψ is a positive function defined on \mathbb{N} .

Theorem 6.5. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function with $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. We have*

(1) *if $\psi(n)/n \rightarrow 0$ as $n \rightarrow \infty$, then $\dim_{\text{H}} E_{\text{sup}}(\psi) = 1$,*

(2) *if $\psi(n)/n \rightarrow \alpha$ ($0 < \alpha < \infty$) as $n \rightarrow \infty$, then $\dim_{\text{H}} E_{\text{sup}}(\psi) = S(\alpha)$, where*

$$S(\alpha) = \dim_{\text{H}} \left\{ x \in [0, 1) : a_n(x) \geq (\exp(\alpha))^n \text{ for infinitely many } n \right\},$$

(3) *if $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, then $\dim_{\text{H}} E_{\text{sup}}(\psi) = \frac{1}{A+1}$, where A is given by*

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}. \quad (6.2)$$

Proof. The proof is divided into three parts.

(1) **The case** $\psi(n)/n \rightarrow 0$ as $n \rightarrow \infty$.

In this case, our strategy is to construct a suitable Cantor subset $E_M(\psi)$ of $E_{\text{sup}}(\psi)$ and then establish a connection between $E_M(\psi)$ and E_M by means of a $(1 + \varepsilon)$ Hölder function. Choose a sequence $(m_k)_{k \geq 1}$ with $m_k = 2^k$. For any $M \in \mathbb{N}$, define

$$E_M(\psi) = \left\{ x \in [0, 1) : a_{m_k}(x) = \lfloor \exp \psi(m_k) \rfloor \text{ and } 1 \leq a_i(x) \leq M \text{ (} i \neq m_k \text{), } \forall k \geq 1 \right\}.$$

From the definition of $E_M(\psi)$, we can easily obtain

$$E_M(\psi) \subseteq E_{\text{sup}}(\psi). \quad (6.3)$$

In the following we shall make use of a kind of symbolic space to estimate the Hausdorff dimension of $E_M(\psi)$. Let

$$C_n = \left\{ (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n : \sigma_{m_k} = \lfloor \exp \psi(m_k) \rfloor \text{ and } 1 \leq \sigma_i \leq M, 1 \leq i \neq m_k \leq n \right\}.$$

For any $n \geq 1$ and $(\sigma_1, \dots, \sigma_n) \in C_n$, we call

$$J_n(\sigma_1, \dots, \sigma_n) = \bigcup_{\sigma_{n+1}} I_{n+1}(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$$

a basic interval of order n , where $(\sigma_1, \dots, \sigma_n, \sigma_{n+1}) \in C_{n+1}$. Then

$$E_M(\psi) = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, \dots, \sigma_n) \in C_n} I_n(\sigma_1, \dots, \sigma_n) = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, \dots, \sigma_n) \in C_n} J_n(\sigma_1, \dots, \sigma_n).$$

Let $t(n) = \#\{k : 2^k \leq n\}$. Clearly we have

$$\lim_{n \rightarrow \infty} \frac{t(n)}{n} = 0. \quad (6.4)$$

For any $(\sigma_1, \dots, \sigma_n) \in C_n$, let $\overline{(\sigma_1, \dots, \sigma_n)}$ be the block obtained by eliminating the terms $\{\sigma_{m_k} : 1 \leq k \leq t(n)\}$ in $(\sigma_1, \dots, \sigma_n)$. Then we can write

$$\overline{(\sigma_1, \dots, \sigma_n)} \in \{1, 2, \dots, M\}^{n-t(n)}.$$

For convenience, write

$$\overline{I}_n(\sigma_1, \dots, \sigma_n) = I_{n-t(n)}(\overline{(\sigma_1, \dots, \sigma_n)}). \quad (6.5)$$

It is worth to note that $\lim_{n \rightarrow \infty} \psi(n)/n = 0$ and $m_k = 2^k$ ($k \geq 1$), then

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{t(n)} \psi(m_i) = \limsup_{n \rightarrow \infty} \frac{1}{n} (\psi(m_1) + \psi(m_2) + \dots + \psi(m_{t(n)})) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{2^{t(n)}} (\psi(2^1) + \psi(2^2) + \dots + \psi(2^{t(n)})) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{2^n} (\psi(2^1) + \psi(2^2) + \dots + \psi(2^n)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{\psi(2^{n+1})}{2^{n+1} - 2^n} = 2 \limsup_{n \rightarrow \infty} \frac{\psi(2^{n+1})}{2^{n+1}} = 0, \end{aligned}$$

where the last inequality holds for the general form of the Stolz-Cesàro Theorem. Thus we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{t(n)} \psi(m_i) = 0. \quad (6.6)$$

Lemma 6.6. *For any $\varepsilon > 0$, there exists N_1 such that for any $n \geq N_1$ and $(\sigma_1, \dots, \sigma_n) \in C_n$,*

$$|I_n(\sigma_1, \dots, \sigma_n)| \geq |\overline{I}_n(\sigma_1, \dots, \sigma_n)|^{1+\varepsilon}.$$

Proof. Let $\varepsilon > 0$, it follows from (6.4), (6.5), (6.6) and Lemma 3.10 that there exists N_1 such that for any $n \geq N_1$ and $(\sigma_1, \dots, \sigma_n) \in C_n$,

$$q_{n-t(n)}^{2\varepsilon} \overline{(\sigma_1, \dots, \sigma_n)} \geq 2^{(n-t(n)-1)\varepsilon} \geq 2 \prod_{i=1}^{t(n)} (\exp \psi(m_i) + 1)^2. \quad (6.7)$$

In view of (6.7), by Lemma 3.10 and Proposition 3.11, we have

$$\begin{aligned} |I_n(\sigma_1, \dots, \sigma_n)| &\geq \frac{1}{2q_n^2(\sigma_1, \dots, \sigma_n)} \geq \frac{1}{2 \left(q_{n-t(n)}(\sigma_1, \dots, \sigma_n) \prod_{i=1}^{t(n)} (\exp \psi(m_i) + 1) \right)^2} \\ &\geq \frac{1}{q_{n-t(n)}^{2+2\varepsilon}(\sigma_1, \dots, \sigma_n)} \geq |\overline{I}_n(\sigma_1, \dots, \sigma_n)|^{1+\varepsilon}. \end{aligned}$$

□

For any $x = [\eta_1, \eta_2, \dots] \in E_M(\psi)$, $y = [\tau_1, \tau_2, \dots] \in E_M(\psi)$ and $x \neq y$, then there exists the greatest integer n such that x, y are contained in the same cylinder of order n , i.e., $x, y \in I_n(\sigma_1, \dots, \sigma_n)$. For the case when $\eta_1 \neq \tau_1$, we write $x, y \in I_0 := [0, 1)$. Therefore there exist $l_{n+1} \neq r_{n+1}$ such that $(\sigma_1, \dots, \sigma_n, l_{n+1}) \in C_{n+1}$, $(\sigma_1, \dots, \sigma_n, r_{n+1}) \in C_{n+1}$ and $x \in I_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1})$, $y \in I_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1})$ respectively. We next compare $|x - y|$ with $|I_n(\sigma_1, \dots, \sigma_n)|$. Since

$$I_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1}) \cap E_M(\psi) = J_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1}) \cap E_M(\psi),$$

$$I_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1}) \cap E_M(\psi) = J_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1}) \cap E_M(\psi),$$

we have $x \in J_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1})$, $y \in J_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1})$. So $|x - y|$ is not less than the gap between $J_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1})$ and $J_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1})$.

Lemma 6.7.

$$|x - y| \geq \frac{1}{(M+2)^3} |I_n(\sigma_1, \dots, \sigma_n)|.$$

Proof. Without loss of generality, we assume that $x < y$ and n is even, for others cases, the proofs are similar. Notice that $n+1 \neq m_k$ for any $k \in \mathbb{N}$, otherwise $l_{n+1} = r_{n+1} = \lfloor \exp \psi(m_k) \rfloor$. So we divide the proof into two parts.

- (i) If $n+2 = m_k$ for some $k \geq 1$, then $y - x$ is greater than the distance between the left endpoint of $J_{n+1}(\sigma_1, \dots, \sigma_n, r_{n+1})$ and the right endpoint of $J_{n+1}(\sigma_1, \dots, \sigma_n, l_{n+1})$. Let $\lfloor \exp \psi(m_k) \rfloor = t$, then $t \geq 1$. We deduce from Proposition 3.11 that

$$\begin{aligned} |y - x| &> \left| \frac{(r_{n+1} + \frac{1}{t})p_n + p_{n-1}}{(r_{n+1} + \frac{1}{t})q_n + q_{n-1}} - \frac{(l_{n+1} + \frac{1}{t+1})p_n + p_{n-1}}{(l_{n+1} + \frac{1}{t+1})q_n + q_{n-1}} \right| \\ &\geq \frac{|l_{n+1} - r_{n+1} + \frac{1}{t+1} - \frac{1}{t}|}{(M+2)^2 q_n^2} \geq \frac{|l_{n+1} - r_{n+1}| - |\frac{1}{t+1} - \frac{1}{t}|}{(M+2)^2 q_n^2} \\ &\geq \frac{1}{2(M+2)^2 q_n^2} \geq \frac{1}{(M+2)^3} |I_n(\sigma_1, \dots, \sigma_n)|. \end{aligned}$$

(ii) If $n + 2 \neq m_k$ for any $k \geq 1$, similar to case (i), then we have

$$\begin{aligned} |y - x| &> \left| \frac{(r_{n+1} + 1)p_n + p_{n-1}}{(r_{n+1} + 1)q_n + q_{n-1}} - \frac{(l_{n+1} + \frac{1}{M+1})p_n + p_{n-1}}{(l_{n+1} + \frac{1}{M+1})q_n + q_{n-1}} \right| \\ &\geq \frac{|r_{n+1} + 1 - l_{n+1} - \frac{1}{M+1}|}{(M + 2)^2 q_n^2} \geq \frac{|r_{n+1} - l_{n+1}| - |1 - \frac{1}{M+1}|}{(M + 2)^2 q_n^2} \\ &\geq \frac{1}{(M + 2)^3 q_n^2} \geq \frac{1}{(M + 2)^3} |I_n(\sigma_1, \dots, \sigma_n)|. \end{aligned}$$

□

Consider a map $f : E_M(\psi) \rightarrow E_M$ defined as follows:

$$x = [\sigma_1, \dots, \sigma_n] \longrightarrow f(x) = \lim_{n \rightarrow \infty} \overline{[\sigma_1, \dots, \sigma_n]}.$$

For any $\varepsilon > 0$, by Lemma 6.6 and Lemma 6.7, when $x, y \in E_M(\psi)$ satisfying

$$|x - y| < \frac{1}{(M + 2)^3} \min_{(\sigma_1, \dots, \sigma_{N_1}) \in C_{N_1}} \{|I_{N_1}(\sigma_1, \dots, \sigma_{N_1})|\},$$

where N_1 is the same as in Lemma 6.6, we have

$$|f(x) - f(y)| \leq |\bar{I}_n(\sigma_1, \dots, \sigma_n)| \leq |I_n(\sigma_1, \dots, \sigma_n)|^{\frac{1}{1+\varepsilon}} \leq (M + 2)^{\frac{3}{1+\varepsilon}} |x - y|^{\frac{1}{1+\varepsilon}}.$$

It follows from Proposition 3.2 and (6.3) that

$$\dim_{\mathbb{H}} E_{\text{sup}}(\psi) \geq \dim_{\mathbb{H}} E_M(\psi) \geq \frac{1}{1 + \varepsilon} \dim_{\mathbb{H}} E_M.$$

Hence by Theorem 6.2 ($\varepsilon \rightarrow 0^+$ and then $M \rightarrow \infty$), we have

$$\dim_{\mathbb{H}} E_{\text{sup}}(\psi) = 1.$$

(2) **The case** $\psi(n)/n \rightarrow \alpha$ ($0 < \alpha < \infty$) as $n \rightarrow \infty$.

It is clear that $E_{\text{sup}}(\psi)$ equals to

$$\mathcal{F}(\alpha) := \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{n} = \alpha \right\}.$$

Upper bound: For any $0 < \varepsilon < \alpha$, the definition of set $\mathcal{F}(\alpha)$ implies that

$$\mathcal{F}(\alpha) \subseteq \left\{ x \in [0, 1) : a_n(x) \geq (\exp(\alpha - \varepsilon))^n \text{ for infinitely many } n \right\}.$$

We deduce from Theorem 6.4 (let $\varepsilon \rightarrow 0^+$) that

$$\dim_{\mathbb{H}} \mathcal{F}(\alpha) \leq S(\alpha).$$

Lower bound: Choose a sequence $(m_k)_{k \geq 1}$ satisfying

$$m_1 = 1, \quad m_1 + \cdots + m_k \leq \frac{1}{k+1} m_{k+1} \quad \text{for any } k \geq 2.$$

Fix $M \in \mathbb{N}$ and define

$$F_M(\exp(\alpha)) = \left\{ x \in [0, 1) : \lfloor \exp(\alpha m_k) \rfloor + 1 \leq a_{m_k}(x) \leq 2 \lfloor \exp(\alpha m_k) \rfloor \text{ for all } k \geq 1 \right. \\ \left. \text{and } 1 \leq a_i(x) \leq M \text{ for } i \neq m_k \text{ for any } k \geq 1 \right\}.$$

Using the same method in the proof of [94, Theorem 3.1], we can show that

$$F_M(\exp(\alpha)) \subseteq \mathcal{F}(\alpha).$$

It follows [94, Lemma 2.5] (let $M \rightarrow \infty$) that

$$\dim_{\mathbb{H}} \mathcal{F}(\alpha) \geq S(\alpha).$$

(3) **The case** $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$.

Upper bound: For any $0 < \varepsilon < 1$, we have $E_{\text{sup}}(\psi) \subseteq \hat{E}(\psi)$, where

$$\hat{E}(\psi) = \left\{ x \in [0, 1) : a_n(x) \geq \exp(\psi(n)(1 - \varepsilon)) \text{ for infinitely many } n \right\}.$$

Notice that $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, then it follows (6.2) and Theorem 6.4 that

$$\dim_{\mathbb{H}} E_{\text{sup}}(\psi) \leq \dim_{\mathbb{H}} \hat{E}(\psi) = \frac{1}{A+1}.$$

Lower bound: We construct a suitable Cantor type subset of $E_{\text{sup}}(\psi)$ in three steps. Firstly, we define function

$$\theta(n) = \min_{k \geq n} \psi(k) \quad \text{for any } n \geq 1. \quad (6.8)$$

So $\theta(n)$ is well defined for $\psi(n) \rightarrow \infty$. It follows (6.8) that

$$\theta(n) \leq \psi(n) \quad \text{and} \quad \theta(n) \leq \theta(n+1) \quad \text{for any } n \geq 1. \quad (6.9)$$

Furthermore, we claim that

$$\theta(n) = \psi(n) \quad \text{for infinitely many } n. \quad (6.10)$$

If not, there exist $N \in \mathbb{N}$ such that for any $n \geq N$, $\theta(n) < \psi(n)$. In view of (6.9),

$$\theta(n) < \min_{k \geq n} \psi(k) \quad \text{for any } n \geq N,$$

which contradicts to (6.8). Secondly, we define a sequence $(d_n)_{n \geq 1}$ as follows:

$$d_1 = \exp \theta(1) \quad \text{and} \quad d_n = \min \left\{ \exp \theta(n), \prod_{k=1}^{n-1} d_k^{A-1+\varepsilon} \right\} \quad (n \geq 2). \quad (6.11)$$

Then we easily obtain

$$\limsup_{n \rightarrow \infty} \frac{\log d_{n+1}}{\log d_1 + \cdots + \log d_n} \leq \limsup_{n \rightarrow \infty} \frac{\log \left(\prod_{k=1}^n d_k^{A-1+\varepsilon} \right)}{\log \prod_{k=1}^n d_k} = A - 1 + \varepsilon. \quad (6.12)$$

By (6.11), we also claim that

$$d_n = \exp \theta(n) \quad \text{for infinitely many } n. \quad (6.13)$$

If not, then there exist $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\exp \theta(n) > d_n \quad \text{and} \quad d_n = \prod_{k=1}^{n-1} d_k^{A-1+\varepsilon}, \quad (6.14)$$

From (6.14), it follows that $d_n = d_{n-1}^{A+\varepsilon}$ ($n \geq N$) and

$$\begin{aligned} \prod_{k=1}^n d_k &= \left(\prod_{k=1}^N d_k \right) \cdot d_{N+1} \cdot d_{N+2} \cdots d_n \\ &= \left(\prod_{k=1}^N d_k \right) \cdot \left(\prod_{k=1}^N d_k \right)^{A-1+\varepsilon} \cdot \left(\prod_{k=1}^N d_k \right)^{(A-1+\varepsilon)(A+\varepsilon)} \cdots \left(\prod_{k=1}^N d_k \right)^{(A-1+\varepsilon)(A+\varepsilon)^{n-N-1}} \\ &= \left(\prod_{k=1}^N d_k \right)^{1+(A-1+\varepsilon)+(A-1+\varepsilon)(A+\varepsilon)+\cdots+(A-1+\varepsilon)(A+\varepsilon)^{n-N-1}} = \left(\prod_{k=1}^N d_k \right)^{(A+\varepsilon)^{n-N}}. \end{aligned} \quad (6.15)$$

Combining (6.14) with (6.15), we have

$$\frac{\log \theta(n+1)}{n+1} > \frac{n+1-N}{n+1} \log(A+\varepsilon) + \frac{\log \left(\frac{A-1+\varepsilon}{A+\varepsilon} \sum_{k=1}^N \log d_k \right)}{n+1}, \quad (6.16)$$

then by (6.9) and (6.16),

$$\liminf_{n \rightarrow \infty} \frac{\log \psi(n+1)}{n+1} \geq \liminf_{n \rightarrow \infty} \frac{\log \theta(n+1)}{n+1} \geq \log(A+\varepsilon) > \log A,$$

which contradicts to (6.2). Thirdly, we claim that

$$\limsup_{n \rightarrow \infty} \frac{\log d_n}{\psi(n)} = 1 \quad \text{and} \quad d_n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (6.17)$$

In fact, on one hand, by (6.9) and (6.11), we have

$$\limsup_{n \rightarrow \infty} \frac{\log d_n}{\psi(n)} \leq 1 \quad \text{and} \quad d_n \leq d_{n+1} \quad (n \geq 2).$$

On the other hand, by (6.10) and (6.13), we obtain

$$\limsup_{n \rightarrow \infty} \frac{\log d_n}{\psi(n)} \geq 1.$$

Now we use the sequence $(d_n)_{n \geq 1}$ to construct a subset of $E_{\text{sup}}(\psi)$. Let M be the positive integer such that $t_n = M \lfloor d_n \rfloor \geq 3$ for any $n \geq 1$. Set

$$E((d_n)_{n \geq 1}) = \left\{ x \in [0, 1) : M \lfloor d_n \rfloor \leq a_n(x) < 2M \lfloor d_n \rfloor, \forall n \geq 1 \right\}.$$

By (6.17), we can check that

$$E((d_n)_{n \geq 1}) \subseteq E_{\text{sup}}(\psi). \quad (6.18)$$

It follows Lemma 5.8 that

$$\dim_{\text{H}} E((d_n)_{n \geq 1}) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log t_{n+1}}{\log t_1 + \dots + \log t_n}} = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log d_{n+1}}{\log d_1 + \dots + \log d_n}}. \quad (6.19)$$

We deduce from (6.12), (6.18) and (6.19) that

$$\dim_{\text{H}} E_{\text{sup}}(\psi) \geq \dim_{\text{H}} E((d_n)_{n \geq 1}) \geq \frac{1}{A + 1 + \varepsilon}.$$

This shows that

$$\dim_{\text{H}} E_{\text{sup}}(\psi) \geq \frac{1}{A + 1}.$$

□

Remark 6.8. Comparing Theorem 5.16, Theorem 5.18 with Theorem 6.5, we have

$$\dim_{\text{H}} E(\psi) \leq \dim_{\text{H}} E_{\text{inf}}(\psi) \leq \dim_{\text{H}} E_{\text{sup}}(\psi)$$

under the condition that $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. More precisely,

(i) $\psi(n)/n \rightarrow 0$ as $n \rightarrow \infty$,

$$\dim_{\text{H}} E(\psi) \leq \dim_{\text{H}} E_{\text{inf}}(\psi) = \frac{1}{2} < \dim_{\text{H}} E_{\text{sup}}(\psi) = 1.$$

(ii) $\psi(n)/n \rightarrow \alpha$ ($\alpha > 0$) as $n \rightarrow \infty$,

$$\dim_{\text{H}} E(\psi) = \dim_{\text{H}} E_{\text{inf}}(\psi) = \frac{1}{2} \leq \dim_{\text{H}} E_{\text{sup}}(\psi) \leq 1.$$

(iii) $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$,

$$\dim_{\text{H}} E(\psi) \leq \dim_{\text{H}} E_{\text{inf}}(\psi) \leq \dim_{\text{H}} E_{\text{sup}}(\psi) \leq \frac{1}{2}.$$

In the following we give an example to show that the dimensional result changes essentially when we replace \lim by \liminf or \limsup . The phenomenon also happens in Liao and Rams [69].

Example 6.9. *Let $1 < a < b < \infty$ and*

$$\psi(n) = \begin{cases} a^n, & \text{if } n \text{ is even,} \\ b^n, & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n} = \log a \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log \psi(n)}{n} = \log b.$$

Also, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(1) + \dots + \psi(n)} &= \lim_{k \rightarrow \infty} \frac{\psi(2k+1)}{\psi(1) + \dots + \psi(2k)} \\ &= \lim_{k \rightarrow \infty} \frac{b^{2k+1}}{a^2 + \dots + a^{2k} + b^1 + \dots + b^{2k-1}} = b^2 - 1. \end{aligned}$$

It follows from Theorem 5.16, Theorem 5.18 and Theorem 6.5 that

$$\frac{1}{b^2 + 1} = \dim_{\mathbb{H}} E(\psi) < \frac{1}{b + 1} = \dim_{\mathbb{H}} E_{\inf}(\psi) < \frac{1}{a + 1} = \dim_{\mathbb{H}} E_{\sup}(\psi).$$

6.3 Hausdorff dimension of sets related to the exceptional sets

In this section, comparing with Theorem 6.5, we continue to study the Hausdorff dimension of the exceptional sets when adding the hypothesis of the monotonicity of the sequence of partial quotients $(a_n(x))_{n \geq 1}$. We study the set

$$E_{\text{sup}}(\Lambda, \psi) := \left\{ x \in \Lambda : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = 1 \right\}.$$

where ψ is a positive function defined on \mathbb{N} . We show that the monotonicity of the sequence $(a_n(x))_{n \geq 1}$ is also redundant for the case when $\psi(n)$ grows faster than $\log n$.

Theorem 6.10. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function with $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) = \frac{1}{A+1},$$

where A is given by

$$\log A = \liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n}.$$

Proof. Upper bound: For any $0 < \varepsilon < 1$, we have

$$E_{\text{sup}}(\Lambda, \psi) \subseteq \Lambda \cap E(\exp \psi(n)),$$

where

$$E(\exp \psi(n)) := \left\{ x \in [0, 1) : a_n(x) \geq \exp(\psi(n)(1 - \varepsilon)) \text{ for infinitely many } n \right\}.$$

Then

$$\dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) \leq \min \left\{ \frac{1}{2}, \dim_{\text{H}} E(\exp \psi(n)) \right\}. \quad (6.20)$$

By Theorem 6.4, we have

$$\dim_{\text{H}} E(\exp \psi(n)) = \begin{cases} [\frac{1}{2}, 1], & 0 \leq \liminf_{n \rightarrow \infty} \frac{\psi(n)}{n} < \infty, \\ \frac{1}{A+1}, & \liminf_{n \rightarrow \infty} \frac{\psi(n)}{n} = \infty. \end{cases} \quad (6.21)$$

We remark that $0 \leq \liminf_{n \rightarrow \infty} \frac{\psi(n)}{n} < \infty$ implies

$$\liminf_{n \rightarrow \infty} \frac{\log \psi(n)}{n} = \log 1. \quad (6.22)$$

It follows from (6.20), (6.21) and (6.22) that

$$\dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) \leq \frac{1}{A+1}.$$

Lower bound: Let $(\tilde{d}_n)_{n \geq 1}$ be a sequence of real numbers defined as follows:

$$\tilde{d}_1 = 3 \quad \text{and} \quad \tilde{d}_n = \min \left\{ \exp \theta(n), \prod_{k=1}^{n-1} \tilde{d}_k^{A-1+\varepsilon} \right\} \quad (n \geq 2), \quad (6.23)$$

where $\theta(n)$ is the same as in (6.8). Then we easily obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_{n+1}}{\log \widetilde{d}_2 + \cdots + \log \widetilde{d}_n} &= \limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_{n+1}}{\log \widetilde{d}_1 + \cdots + \log \widetilde{d}_n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \left(\prod_{k=1}^n \widetilde{d}_k^{A-1+\varepsilon} \right)}{\log \prod_{k=1}^n \widetilde{d}_k} = A - 1 + \varepsilon. \end{aligned} \quad (6.24)$$

Using the same analysis in the proof of the part (3) in Theorem 6.5, we have

$$\limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_n}{\psi(n)} = 1 \quad \text{and} \quad \widetilde{d}_n \leq \widetilde{d}_{n+1}, \quad \forall n \geq 2. \quad (6.25)$$

This implies that

$$\widetilde{d}_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

Together with $\psi(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$, we deduce from (6.8) and (6.23) that

$$\lim_{n \rightarrow \infty} \frac{\log \widetilde{d}_n}{\log n} = \infty. \quad (6.26)$$

Let M be the positive integer such that $s_n = M \lfloor \widetilde{d}_n \rfloor \geq 2$ for any $n \geq 2$. Set

$$F((s_n)_{n \geq 2}) = \left\{ x \in [0, 1) : 1 \leq a_1(x) < 3, \quad ns_n \leq a_n(x) < (n+1)s_n, \quad \forall n \geq 2 \right\}.$$

By (6.25), we have

$$F((s_n)_{n \geq 2}) \subseteq E_{\text{sup}}(\Lambda, \psi). \quad (6.27)$$

From (6.26), it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_2 \cdots s_n)} &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)!}{\log(\widetilde{d}_2 \cdots \widetilde{d}_n)} + \limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_{n+1}}{\log(\widetilde{d}_2 \cdots \widetilde{d}_n)} \\ &\leq \limsup_{n \rightarrow \infty} \frac{2 \log(n+2)}{\log \widetilde{d}_{n+1}} + \limsup_{n \rightarrow \infty} \frac{\log d_{n+1}}{\log \widetilde{d}_2 + \cdots + \log \widetilde{d}_n} \\ &= \limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_{n+1}}{\log \widetilde{d}_2 + \cdots + \log \widetilde{d}_n}. \end{aligned}$$

By Lemma 5.15, we deduce from (6.24) and (6.27) that

$$\begin{aligned} \dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) &\geq \dim_{\text{H}} F((s_n)_{n \geq 2}) = \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{2 \log(n+1)! + \log s_{n+1}}{\log(s_2 \cdots s_n)}} \\ &\geq \frac{1}{2 + \limsup_{n \rightarrow \infty} \frac{\log \widetilde{d}_{n+1}}{\log \widetilde{d}_2 + \cdots + \log \widetilde{d}_n}} \geq \frac{1}{A + 1 + \varepsilon}. \end{aligned}$$

This shows that

$$\dim_{\text{H}} E_{\text{sup}}(\Lambda, \psi) \geq \frac{1}{A + 1}.$$

□

Besides, for the case when the limit does not exist in $E(\psi)$ in Theorem (5.18), it is natural to study the Hausdorff dimension of the intersection of the corresponding set of \limsup and \liminf , i.e.,

$$H(\alpha, \beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \alpha, \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \beta \right\}.$$

For the case $0 = \alpha < \beta < \infty$, we easily obtain

$$\dim_{\mathbb{H}} H(\alpha, \beta) = \dim_{\mathbb{H}} E_{\text{sup}}(\psi).$$

For the case $0 < \alpha < \beta < \infty$, We obtain some partial results about $\dim_{\mathbb{H}} H(\alpha, \beta)$.

Theorem 6.11. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. If*

$$\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = 1, \quad (6.28)$$

then we have

$$\dim_{\mathbb{H}} H(\alpha, \beta) = \frac{1}{2}.$$

Proof. Upper bound: For any $0 < \alpha < \beta < \infty$, obviously we have

$$H(\alpha, \beta) \subseteq \left\{ x \in [0, 1) : a_n(x) \rightarrow \infty \text{ as } n \rightarrow \infty \right\}.$$

Hence we deduce from Theorem 5.3 that

$$\dim_{\mathbb{H}} H(\alpha, \beta) \leq \frac{1}{2}.$$

Lower bound: Consider a sequence $(m_k)_{k \geq 1}$ with $m_k = 2^k$ and define

$$\begin{aligned} \underline{H}(\alpha, \beta) = \left\{ x \in [0, 1) : a_{m_k}(x) = \lfloor 4 \exp(\beta \psi(m_k)) \rfloor \text{ for all } k \geq 1 \text{ and} \right. \\ \left. \lfloor 2 \exp(\alpha \psi(i)) \rfloor \leq a_i(x) \leq \lfloor 4 \exp(\alpha \psi(i)) \rfloor \text{ (} i \neq m_k \text{) for any } k \geq 1 \right\}. \end{aligned}$$

Then the definition of $\underline{H}(\alpha, \beta)$ immediately shows

$$\underline{H}(\alpha, \beta) \subseteq H(\alpha, \beta). \quad (6.29)$$

To estimate $\dim_{\mathbb{H}} \underline{H}(\alpha, \beta)$, we shall make use of a kind of symbolic space. Let

$$\begin{aligned} C_n = \left\{ (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n : \sigma_{m_k} = \lfloor 4 \exp(\beta \psi(m_k)) \rfloor \text{ for any } k \geq 1 \text{ satisfying } m_k \leq n \right. \\ \left. \text{and } \lfloor 2 \exp(\alpha \psi(i)) \rfloor \leq \sigma_i(x) \leq \lfloor 4 \exp(\alpha \psi(i)) \rfloor \text{ for } 1 \leq i \neq m_k \leq n \right\}. \end{aligned}$$

Then it follows that

$$\underline{H}(\alpha, \beta) = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, \dots, \sigma_n) \in C_n} I_n(\sigma_1, \dots, \sigma_n). \quad (6.30)$$

Now we define a mass distribution μ which is supported on $\underline{H}(\alpha, \beta)$ such that for any $n \geq 1$ and $(\sigma_1, \dots, \sigma_n) \in C_n$,

$$\mu(I_n(\sigma_1, \dots, \sigma_n)) = \frac{1}{\#C_n}. \quad (6.31)$$

By the construction of C_n , we have

$$\prod_{i=1}^{\star n} (2 \exp(\alpha \psi(i)) - 1) \leq \#C_n \leq \prod_{i=1}^{\star n} (2 \exp(\alpha \psi(i)) + 1), \quad (6.32)$$

where \star signifies the absence of m_k in $[1, n]$ in the product. Next we aim to prove that for any $x \in \underline{H}(\alpha, \beta)$,

$$\liminf_{\rho \rightarrow 0} \frac{\log \mu(B(x, \rho))}{\log \rho} \geq \frac{1}{2}. \quad (6.33)$$

Indeed, for any $\rho > 0$, there exists an integer n such that

$$|I_{n+1}(\sigma_1, \dots, \sigma_{n+1})| < \rho \leq |I_n(\sigma_1, \dots, \sigma_n)|. \quad (6.34)$$

By Lemma 3.13, we have $B(x, \rho) \subseteq I_{n-2}(x)$. In view of (6.34), then

$$\liminf_{\rho \rightarrow 0} \frac{\log \mu(B(x, \rho))}{\log \rho} \geq \liminf_{n \rightarrow \infty} \frac{\log \mu(I_{n-2}(\sigma_1, \dots, \sigma_{n-2}))}{\log |I_{n+1}(\sigma_1, \dots, \sigma_{n+1})|}. \quad (6.35)$$

Thus by (6.33) and (6.35), it suffice to prove that

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I_{n-2}(\sigma_1, \dots, \sigma_{n-2}))}{\log |I_{n+1}(\sigma_1, \dots, \sigma_{n+1})|} \geq \frac{1}{2}.$$

Together with Proposition 3.9 and (3.10), we have

$$2^{-(2n+3)} \cdot \prod_{i=1}^{n+1} \sigma_i^{-2} \leq |I_{n+1}(\sigma_1, \dots, \sigma_{n+1})| \leq \prod_{i=1}^{n+1} \sigma_i^{-2}. \quad (6.36)$$

It is worth to note that $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=1}^{n+1} \psi(i) = \infty. \quad (6.37)$$

Let $t(n) = \#\{k : 2^k \leq n\}$. With the same estimate in (6.6), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=1}^{t(n+1)} \psi(m_i) = 0. \quad (6.38)$$

By (6.37) and (6.38),

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{t(n+1)} \psi(m_i)}{\sum_{i=1}^{n+1} \psi(i)} = 0. \quad (6.39)$$

Then we obtain

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{*(n-2)} \log(2 \exp(\alpha \psi(i)) - 1)}{\alpha \sum_{i=1}^{*(n+1)} \psi(i)} = 1. \quad (6.40)$$

In fact, by (6.28) and (6.39), on one hand,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{*(n-2)} \log(2 \exp(\alpha \psi(i)) - 1)}{\alpha \sum_{i=1}^{*(n+1)} \psi(i)} \\ & \geq \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^{n-2} \log(2 \exp(\alpha \psi(i)) - 1) - \sum_{i=1}^{t(n+1)} \log(2 \exp(\alpha \psi(m_i)) - 1)}{\alpha \sum_{i=1}^{n+1} \psi(i)} = 1, \end{aligned}$$

on the other hand,

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{*(n-2)} \log(2 \exp(\alpha \psi(i)) - 1)}{\alpha \sum_{i=1}^{*(n+1)} \psi(i)} \leq \limsup_{n \rightarrow \infty} \frac{\alpha \sum_{i=1}^{n-2} \psi(i)}{\alpha \sum_{i=1}^{n+1} \psi(i) - \alpha \sum_{i=1}^{t(n+1)} \psi(m_i)} = 1.$$

It follows (6.31), (6.32), (6.36), (6.39) and (6.40) that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\log \mu(I_{n-2}(\sigma_1, \dots, \sigma_{n-2}))}{\log |I_{n+1}(\sigma_1, \dots, \sigma_{n+1})|} \\ & \geq \liminf_{n \rightarrow \infty} \frac{-\sum_{i=2}^{*(n-2)} \log(2 \exp(\alpha \psi(i)) - 1)}{-(2n+3) \log 2 - 2 \sum_{i=1}^{n+1} \log \sigma_i} \\ & \geq \liminf_{n \rightarrow \infty} \frac{-\sum_{i=1}^{*(n-2)} \log(2 \exp(\alpha \psi(i)) - 1)}{-(10n+11) \log 2 - 2 \left(\alpha \sum_{i=1}^{*(n+1)} \psi(i) + \beta \sum_{i=1}^{t(n+1)} \psi(m_i) \right)} = \frac{1}{2}. \end{aligned}$$

We conclude from (6.29) and Proposition 3.6 that

$$\dim_{\mathbb{H}} H(\alpha, \beta) \geq \dim_{\mathbb{H}} \underline{H}(\alpha, \beta) \geq \frac{1}{2}.$$

□

Remark 6.12. For any $0 < \alpha < \beta < \infty$, denote

$$E_\alpha = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \alpha \right\} \text{ and } E_\beta = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_n(x)}{\psi(n)} = \beta \right\}.$$

By (6.28), we deduce from Theorem 5.16 and Theorem 6.5 that

$$\dim_{\mathbb{H}} E_\alpha = \frac{1}{2} \text{ and } \dim_{\mathbb{H}} E_\beta = 1.$$

Thus Theorem 6.11 indicates that

$$\dim_{\mathbb{H}} H(\alpha, \beta) = \dim_{\mathbb{H}}(E_\alpha \cap E_\beta) = \min \{ \dim_{\mathbb{H}} E_\alpha, \dim_{\mathbb{H}} E_\beta \}.$$

In general, the Hausdorff dimension of the intersection of two sets is expected to behave well. In fact, on one hand, Falconer [32, 34] introduced classes of sets with large intersection. They are composed of subsets of \mathbb{R}^n with Hausdorff dimension at least s satisfying the property that countable intersection of the sets still have dimension at least s . In this situation, for $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^n$, we have

$$\dim_{\mathbb{H}}(E \cap F) = \min \{ \dim_{\mathbb{H}} E, \dim_{\mathbb{H}} F \}.$$

On the other hand, the Hausdorff dimension of the intersection of two subsets of \mathbb{R}^n is expected to satisfy the codimension formula (see Falconer [33, Chapter 8]), i.e.,

$$\dim_{\mathbb{H}}(E \cap F) = \dim_{\mathbb{H}} E + \dim_{\mathbb{H}} F - n.$$

7 Fourier transform of coin-tossing type measures

In this chapter, we focus on a class of probability measures called coin-tossing type measures. The explicit pointwise rate of decay of Fourier transform of coin-tossing type measures for general weight functions is determined. Based on Davenport-Erdős-LeVeque Theorem, we use the method of Cassels and Schmidt to prove that almost all real numbers are absolutely normal with respect to coin-tossing type measures.

7.1 Coin-tossing type measures

The coin-tossing type measure is an infinite convolution on $[0, 1]$ defined by

$$\mu = *_{n=1}^{\infty} \left(\frac{1}{2} (1 + \phi(n)) \delta_0 + \frac{1}{2} (1 - \phi(n)) \delta_{2^{-n}} \right), \quad (7.1)$$

where ϕ is a weight function defined on \mathbb{N} taking values in $(0, 1)$. Such a measure has a simple probabilistic interpretation: Consider the random sum

$$S = \sum_{n=1}^{\infty} X_n 2^{-n},$$

where $(X_n)_{n \geq 1}$ is the sequence of independent random variables satisfying

$$\mathbb{P}(X_n = 0) = \frac{1}{2} (1 + \phi(n)) \quad \text{and} \quad \mathbb{P}(X_n = 1) = \frac{1}{2} (1 - \phi(n)).$$

Then μ is the probability distribution of random sum S . The classical Jessen-Wintner law of pure types [81] implies that μ is either absolutely continuous or singular. In 1943, Salem [86] gave a characterization that μ is singular if and only if the series $\sum \phi^2(n)$ is divergent. In the meantime, he obtained a necessary condition for which μ is a Rajchman measure.

Theorem 7.1 ([86]). *Let μ be the measure given by (7.1). If μ is a Rajchman measure, then we have $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof.

$$|\widehat{\mu}(t)|^2 = \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n} t + \cos^2 \pi 2^{-n} t).$$

Choosing t along with the sequence $(2^m)_{m \geq 1}$, we have

$$\begin{aligned} |\widehat{\mu}(2^m)|^2 &= \phi^2(m+1) \prod_{n=m+2}^{\infty} (\phi^2(n) \sin^2 \pi 2^{m-n} + \cos^2 \pi 2^{m-n}) \\ &\geq \phi^2(m+1) \prod_{n=2}^{\infty} \cos^2 \pi 2^{-n} = \frac{4}{\pi^2} \phi^2(m+1). \end{aligned}$$

The above fact shows that if $\phi(n) \not\rightarrow 0$ as $n \rightarrow \infty$, the measure μ is not a Rajchman measure. Thus the proof is complete. \square

In 1971, Blum and Epstein [8] proved that $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$ is also a sufficient condition of μ being a Rajchman measure.

Theorem 7.2 ([8]). *Let μ be the measure given by (7.1). Then we have*

$$\limsup_{t \rightarrow \infty} |\widehat{\mu}(t)| \leq \limsup_{n \rightarrow \infty} \phi(n).$$

Proof. For convenience, set $\delta = \limsup_{n \rightarrow \infty} \phi(n)$, then we have $\delta \leq 1$. It is trivial for the case when $\delta = 1$. For $\delta < 1$, let α be any real number satisfying $\delta^2 < \alpha \leq 1$, then there exists a positive integer M such that $\phi^2(n) < \alpha$ whenever $n > M$. Thus

$$\begin{aligned} |\widehat{\mu}(t)|^2 &= \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n} t + \cos^2 \pi 2^{-n} t) \\ &\leq \prod_{n=M+1}^{\infty} (\alpha \sin^2 \pi 2^{-n} t + \cos^2 \pi 2^{-n} t) \\ &= \prod_{n=1}^{\infty} (\alpha \sin^2 \pi 2^{-n} s + \cos^2 \pi 2^{-n} s), \quad s = 2^{-M} t. \end{aligned} \quad (7.2)$$

Since each factor in the product in formula (7.2) is nonnegative, the value of this product, denoted by $f(s, \alpha)$, is unaffected by multiplying out the factors and assembling them according to powers of α . We have

$$\alpha^{-1} f(s, \alpha) = \alpha^{-1} \prod_{n=1}^{\infty} \cos^2 \frac{\pi s}{2^n} + f_1(s) + f_2(s)\alpha + \dots, \quad (7.3)$$

where the exact nature of the functions $f_1(s), f_2(s) \dots$ is of no concern. Notice that

$$\prod_{n=1}^{\infty} \cos^2 \frac{\pi s}{2^n} = \frac{\sin^2 \pi s}{(\pi s)^2},$$

then it follows (7.3) that

$$\alpha^{-1} \limsup_{s \rightarrow \infty} f(s, \alpha) = \limsup_{s \rightarrow \infty} (f_1(s) + f_2(s)\alpha + \dots). \quad (7.4)$$

Since each of the functions $f_1(s), f_2(s) \dots$ is nonnegative, the sum appearing on the right side of (7.4) is not decreased when α is replaced by 1. Notice that $f(s, 1) = 1$,

$$\alpha^{-1} \limsup_{s \rightarrow \infty} f(s, \alpha) = \limsup_{s \rightarrow \infty} \left(1 - \frac{\sin^2 \pi s}{(\pi s)^2} \right) \leq 1,$$

which, in combination with $\delta^2 < \alpha \leq 1$ and (7.2), implies that

$$\limsup_{t \rightarrow \infty} |\widehat{\mu}(t)|^2 \leq \limsup_{s \rightarrow \infty} f(s, \alpha) \leq \delta^2.$$

□

It is well known that if the sequence $(2^{-n})_{n \geq 1}$ is replaced by $(a^{-n})_{n \geq 1}$ ($a \geq 3$) in the construction of the measure μ in (7.1), then the corresponding measure is always singular. We give a proof of it for the sake of completeness.

Proposition 7.3. *Let $a \geq 3$ be an integer and*

$$\mu_a = *_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n))\delta_0 + \frac{1}{2}(1 - \phi(n))\delta_{a^{-n}} \right).$$

Then μ_a is not a Rajchman measure.

Proof. Notice that

$$\begin{aligned} |\widehat{\mu}_a(t)|^2 &= \prod_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi^2(n)) + \frac{1}{2}(1 - \phi^2(n)) \cos 2\pi a^{-n}t \right) \\ &= \prod_{n=1}^{\infty} (\phi^2(n) \sin^2(\pi a^{-n}t) + \cos^2(\pi a^{-n}t)) \\ &\geq \prod_{n=1}^{\infty} \cos^2(\pi a^{-n}t). \end{aligned} \tag{7.5}$$

Choosing t along with the sequence $(a^k)_{k \geq 1}$, we have

$$|\widehat{\mu}_a(a^k)|^2 \geq \prod_{n=1}^{\infty} \cos^2 \pi a^{-n+k} = \prod_{n=1}^{\infty} \cos^2 a^{-n} \pi \geq \cos^2 \frac{\pi}{a} \prod_{n=2}^{\infty} \cos^2 \frac{\pi}{2^n} = \frac{4}{\pi^2} \cos^2 \frac{\pi}{a} > 0.$$

□

The study of Fourier transform of coin-tossing type measures goes back to Hartman and Kershner [44]. For the case $\phi(n) = n^{-\frac{1}{2}}$, they proved that

$$|\widehat{\mu}(t)| = O(\log^{-\frac{1}{2}} |t|) \quad \text{as } |t| \rightarrow \infty. \tag{7.6}$$

Bisbas [6] investigated the asymptotic behavior of averages of the Fourier transform of the coin-tossing type measures. Theorem 7.1 shows that $\widehat{\mu}(t)$ does not vanish at infinity under the condition that $\phi(n) \not\rightarrow 0$ as $n \rightarrow \infty$. Motivated by this point, Gao and Wen [42] recently proved that the Fourier-Stieltjes coefficients of the coin-tossing measures could vanish at infinity along with some integer subsequences.

Observe that for the case $\phi(n) = 0$, μ degenerates into the Lebesgue measure on $[0, 1]$. As we show in Example 4.4,

$$|\widehat{\mu}(t)| = O(|t|^{-1}) \quad \text{as } |t| \rightarrow \infty. \tag{7.7}$$

It seems that the more rapidly ϕ decreases to 0, the more rapidly the rate of decay of $\widehat{\mu}(t)$ tends to 0. We shall show that this is indeed the case. Recall that

$$|\widehat{\mu}(t)|^2 = \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n}t + \cos^2 \pi 2^{-n}t). \tag{7.8}$$

The right hand side of (7.8) shows that the Fourier transform $\widehat{\mu}(t)$ is related to the behavior of the fractional part of the sequence $(2^{-n}t)_{n \geq 1}$. For convenience, we denote the distance from t to the nearest integer by $\|t\| = \min_{q \in \mathbb{Z}} |t - q|$. Let

$$A_\phi := \left\{ t : \|t\| < 2^{-K_\phi} \right\},$$

where K_ϕ is a positive number depending on ϕ .

Lemma 7.4. *If $t \notin A_\phi$, then there exists a positive number $\delta := \delta_\phi < 1$ such that*

$$|\phi^2(n) \sin^2 \pi t + \cos^2 \pi t| \leq \delta < 1 \quad \text{for all } n \geq 1.$$

Proof. Notice that $\phi(n)$ is non-increasing, it is obvious that

$$\begin{aligned} |\phi^2(n) \sin^2 \pi t + \cos^2 \pi t| &= |1 - (1 - \phi^2(n)) \sin^2 \pi t| \\ &\leq 1 - (1 - \phi^2(1)) \sin^2 \pi 2^{-K_\phi} := \delta < 1. \end{aligned}$$

□

Let

$$R_n = \left\{ t \in (0, \infty) : \left| t - \frac{k}{2} \right| < 2^{-K_\phi - n}, \text{ for some odd } k \right\},$$

where K_ϕ is the same as in set A_ϕ . In other words, all points in R_n are at a distance at least $\frac{1}{2} - 2^{-K_\phi - n}$ from the nearest integer. Then the value of $\cos^2 \pi t$ is small when the point t falls into the set R_n . More precisely, we have

$$|\cos \pi t| \leq |\sin(\pi 2^{-K_\phi - n})| \leq \pi 2^{-K_\phi - n}, \quad \text{for } t \in R_n. \quad (7.9)$$

Now let $t \geq 2$ be fixed, suppose that $m \geq 1$ is the unique integer satisfying

$$2^m \leq t < 2^{m+1}.$$

Consider the set

$$\mathcal{A} := \left\{ 1 \leq n \leq m : 2^{-n}t \in A_\phi \right\}, \quad k := \#\mathcal{A}. \quad (7.10)$$

That is, there exist exactly k ($k \leq m$) indices in \mathcal{A} for which $2^{-n}t \in A_\phi$ among the indices of $1, 2, \dots, m$. We split \mathcal{A} into a disjoint union $\bigcup_{i=1}^j B_i$, where $B_i := \{n_i - l_i, n_i - l_i - 1, \dots, n_i - 1\}$, being a block of l_i consecutive integers, and thus

$$\sum_{i=1}^j l_i = k. \quad (7.11)$$

Note that n_i (the i -th good index) is the first index which does not fall into A_ϕ among the integers between B_i and B_{i+1} . Obviously,

$$n_i \geq \sum_{s=1}^i l_s + 1. \quad (7.12)$$

We refer to Figure 3 for a graphical description of the above statements.

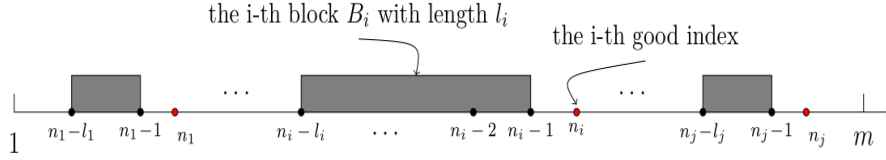


Figure 3 – Behaviour of fractional part of $(2^{-n}t)_{n \geq 1}$.

Lemma 7.5. *Let n_i, l_i be as above. Then we have $2^{-n_i}t \in R_{l_i}$.*

Proof. By the definition of B_i and n_i ,

$$2^{-n_i}t \notin A_\phi, 2^{-(n_i-1)}t \in A_\phi, 2^{-(n_i-2)}t \in A_\phi, \dots, 2^{-(n_i-l_i)}t \in A_\phi.$$

Note that

$$\begin{cases} |2^{-n_i}t - k_0| \geq 2^{-K_\phi} \\ |2^{-(n_i-1)}t - k_1| < 2^{-K_\phi} \\ |2^{-(n_i-2)}t - k_2| < 2^{-K_\phi} \\ \dots \\ |2^{-(n_i-l_i)}t - k_{l_i}| < 2^{-K_\phi} \end{cases} \quad (7.13)$$

where k_j is the integral part of $2^{-(n_i-j)}t$ ($0 \leq j \leq l_i$) respectively, we claim that

$$k_{l_i} = 2^{l_i-1}k_1. \quad (7.14)$$

Indeed, consider the two equations

$$2^{-(n_i-1)}t = k_1 + \varepsilon_1 \quad \text{and} \quad 2^{-(n_i-2)}t = k_2 + \varepsilon_2,$$

where $|\varepsilon_1| < 2^{-K_\phi}, |\varepsilon_2| < 2^{-K_\phi}$, we have

$$|2k_1 - k_2| = |2\varepsilon_1 - 2\varepsilon_2| < 3 \cdot 2^{-K_\phi} \leq 1. \quad (7.15)$$

Since the left-hand side of (7.15) is an integer, it follows that $2k_1 = k_2$. Similarly, (7.14) follows that there are l_i consecutive integers which fall into A_ϕ , i.e., $|\varepsilon_j| < 2^{-K_\phi}$ for all $1 \leq j \leq l_i$. Substituting (7.14) into the last inequality of (7.13), we have

$$|2^{-n_i}t - k_1/2| < 2^{-K_\phi-l_i}. \quad (7.16)$$

Combining the first inequality of (7.13) with (7.16), we know that k_1 is odd and thus

$$2^{-n_i}t \in R_{l_i}.$$

□

In view of Lemma 7.4 and Lemma 7.5, we could obtain an upper bound estimate of the rate of decay of $\widehat{\mu}(t)$.

Lemma 7.6. *Let μ be the measure in (7.1), then we have*

$$|\widehat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}).$$

Proof. We deduce from (7.8) that

$$\begin{aligned} |\widehat{\mu}(t)|^2 &= \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi 2^{-n} t + \cos^2 \pi 2^{-n} t) \\ &\leq \prod_{n=1}^m (\phi^2(n) \sin^2 \pi 2^{-n} t + \cos^2 \pi 2^{-n} t) \\ &\leq \delta^{m-k-j} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}), \end{aligned} \tag{7.17}$$

where the last inequality holds for the reasons listed as follows:

- (i) It follows (7.10) that there exist exactly $m - k$ values of n ($n = 1, 2, \dots, m$) such that $2^{-n} t \notin A_\phi$. Among these indices, there are j good indices n_i . For these good indices, we deduce from (7.9) and Lemma 7.5 that

$$\phi^2(n_i) \sin^2 \pi 2^{-n_i} t + \cos^2 \pi 2^{-n_i} t \leq \phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}.$$

The remaining $m - k - j$ factors are less than δ from Lemma 7.4.

- (ii) The rest k factors in the products in (7.17) are replaced by 1.

□

In the following we present explicit pointwise rate of decay of Fourier transform of coin-tossing type measures for general weight functions.

Theorem 7.7. *Let μ be the measure given by (7.1). Suppose $\phi : \mathbb{N} \rightarrow (0, 1)$ is a non-increasing function with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$, we have*

- (1) if $\phi(n) < 2\phi(n+1)$ for any $n \geq 1$, then $|\widehat{\mu}(t)| = O(\phi(\lceil \gamma \log_2 |t| \rceil))$, where $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ and $\delta \in (0, 1)$ is a constant depending only on ϕ ;
- (2) if $\phi(n) \geq 2\phi(n+1)$ for any $n \geq 1$, then $|\widehat{\mu}(t)| = O(|t|^{-\gamma})$, where $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ and $\delta \in (0, 1)$ is a constant depending only on ϕ .

Proof. We divide the proof into two parts.

(1) By Lemma 7.6, we have

$$|\widehat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}). \quad (7.18)$$

Under the condition that $\phi(n) < 2\phi(n+1)$, it is clear that

$$\phi(1+l_1) < 2^{l_2} \phi(1+l_1+l_2). \quad (7.19)$$

Notice that $\phi(n)$ is non-increasing, together with (7.12), we have

$$\phi(n_i) \leq \phi(1+l_1+l_2+\cdots+l_i). \quad (7.20)$$

Let K_ϕ be large enough such that

$$\phi^2(2) + \pi^2 4^{-1-K_\phi} + \pi^2 4^{-K_\phi} \leq 1 \text{ and } \pi 2^{-K_\phi} \leq 1.$$

Here, we may choose

$$K_\phi = K_{\phi,1} := \frac{1}{2} \log_2 \frac{5\pi^2}{4(1-\phi^2(2))}.$$

In view of (7.19), when $j=2$, we have

$$\begin{aligned} & \prod_{i=1}^2 (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}) \\ &= (\phi^2(n_1) + \pi^2 2^{-2(l_1+K_\phi)}) (\phi^2(n_2) + \pi^2 2^{-2(l_2+K_\phi)}) \\ &\leq (\phi^2(2) + \pi^2 4^{-1-K_\phi} + \pi^2 4^{-K_\phi}) \phi^2(1+l_1+l_2) + \pi^4 2^{-2(l_1+l_2+2K_\phi)} \\ &\leq \phi^2(1+l_1+l_2) + \pi^2 2^{-2(l_1+l_2+K_\phi)}. \end{aligned}$$

Then we estimate the right-hand side of (7.18) by induction on j .

$$\begin{aligned} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}) &\leq \phi^2(1+l_1+l_2+\cdots+l_j) + \pi^2 2^{-2(l_1+l_2+\cdots+l_j+K_\phi)} \\ &= \phi^2(1+k) + \pi^2 2^{-2(k+K_\phi)} \\ &\leq \left(1 + \frac{\pi^2}{4^{K_\phi} \phi^2(1)}\right) \phi^2(k). \end{aligned}$$

It follows (7.18) that

$$|\widehat{\mu}(t)|^2 \leq \delta^{m-k-j} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i+K_\phi)}) \leq C \delta^{m-k-j} \phi^2(k), \quad (7.21)$$

where $C = 1 + \pi^2 4^{-K_\phi} \phi^{-2}(1)$ is a constant depending on ϕ . Notice that

$$m-j-k \geq 0 \text{ and } j \leq k \leq m. \quad (7.22)$$

The second inequality in (7.22) comes from (7.11). In the following we estimate the right-hand side of inequality (7.21). Let

$$0 < \gamma_1 \leq \min\left\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\right\} := \gamma.$$

(i) If $k \geq \gamma_1 m$, note that ϕ is non-increasing, it follows (7.22) that

$$\delta^{m-k-j} \phi^2(k) \leq \delta^0 \phi^2(\lceil \gamma_1 m \rceil) = \phi^2(\lceil \gamma_1 m \rceil).$$

(ii) If $k < \gamma_1 m$, then

$$\delta^{m-k-j} \phi^2(k) \leq \delta^{m-2k} \phi^2(k) \leq \delta^{m-k/\gamma_1} \phi^2(k) = \phi^2(\lceil \gamma_1 m \rceil) \frac{\phi^2(k) \delta^m}{\phi^2(\lceil \gamma_1 m \rceil) \delta^{\frac{k}{\gamma_1}}}.$$

It follows $0 < \phi(n) < 2\phi(n+1)$ that

$$\begin{aligned} \frac{\phi^2(k) \delta^m}{\phi^2(\lceil \gamma_1 m \rceil) \delta^{\frac{k}{\gamma_1}}} &\leq \frac{\phi^2(k) \delta^{\frac{\lceil \gamma_1 m \rceil - 1}{\gamma_1}}}{\phi^2(\lceil \gamma_1 m \rceil) \delta^{\frac{k}{\gamma_1}}} = \delta^{\frac{\lceil \gamma_1 m \rceil - k}{\gamma_1}} \frac{\phi^2(k)}{\phi^2(\lceil \gamma_1 m \rceil)} \delta^{-\frac{1}{\gamma_1}} \\ &\leq \delta^{\frac{\lceil \gamma_1 m \rceil - k}{\gamma_1}} \cdot 4^{\lceil \gamma_1 m \rceil - k} \cdot \delta^{-\frac{1}{\gamma_1}} = (4\delta^{\frac{1}{\gamma_1}})^{\lceil \gamma_1 m \rceil - k} \cdot \delta^{-\frac{1}{\gamma_1}} \leq 4. \end{aligned}$$

Together with (i) and (ii), we have

$$|\widehat{\mu}(t)|^2 \leq 4C\phi^2(\lceil \gamma_1 m \rceil).$$

Recall that $2^m \leq t < 2^{m+1}$, therefore we obtain

$$|\widehat{\mu}(t)| = O(\phi(\lceil \gamma \log_2 |t| \rceil)). \quad (7.23)$$

(2) We can choose $K_\phi = K_{\phi,2} := 3$ such that

$$\pi^4 2^{-4K_\phi} + \pi^2 2^{-2K_\phi} \phi^2(1) + \pi^2 2^{-2(1+K_\phi)} \phi^2(1) \leq 1.$$

Under the condition that $\phi(n)$ is non-increasing and $\phi(n) \geq 2\phi(n+1)$,

$$\begin{aligned} &\prod_{i=1}^2 (\phi^2(n_i) + \pi^2 2^{-2(l_i + K_\phi)}) \\ &= (\pi^2 2^{-2(l_1 + K_\phi)} + \phi^2(n_1)) (\pi^2 2^{-2(l_2 + K_\phi)} + \phi^2(n_2)) \\ &\leq (\pi^4 2^{-4K_\phi} + \pi^2 2^{-2K_\phi} \phi^2(1) + \pi^2 2^{-2(1+K_\phi)} \phi^2(1)) 2^{-2(l_1 + l_2)} + \phi^2(1 + l_1 + l_2) \\ &\leq 2^{-2(l_1 + l_2)} + \phi^2(1 + l_1 + l_2). \end{aligned}$$

By induction on j ,

$$\begin{aligned} \prod_{i=1}^j (\phi^2(n_i) + \pi^2 2^{-2(l_i + K_\phi)}) &\leq 2^{-2(l_1 + l_2 + \dots + l_j)} + \phi^2(1 + l_1 + l_2 + \dots + l_j) \\ &= 2^{-2k} + \phi^2(1 + k) \leq (1 + \phi^2(1)) 2^{-2k}. \end{aligned} \quad (7.24)$$

Combining Lemma 7.6 with (7.24), we have

$$|\widehat{\mu}(t)|^2 \leq C_1 \delta^{m-k-j} 2^{-2k}, \quad (7.25)$$

where $C_1 = (1 + \phi^2(1))$ is a constant depending on ϕ . With the same method, let

$$0 < \gamma_2 \leq \min \left\{ \frac{1}{2}, \frac{-\log_2 \delta}{2} \right\} := \gamma.$$

(i) If $k \geq \gamma_2 m$, then

$$\delta^{m-k-j} 2^{-2k} \leq \delta^0 2^{-2\gamma_2 m} = 2^{-2\gamma_2 m}.$$

(ii) If $k < \gamma_2 m$, then

$$\begin{aligned} \delta^{m-k-j} 2^{-2k} &\leq \delta^{m-2k} 2^{-2k} \leq \delta^{m-k/\gamma_2} 2^{-2k} \\ &= 2^{-2\gamma_2 m} \frac{\delta^m 2^{2\gamma_2 m}}{\delta^{k/\gamma_2} 2^{2k}} \leq (4\delta^{1/\gamma_2})^{\gamma_2 m - k} \cdot 2^{-2\gamma_2 m} \leq 2^{-2\gamma_2 m}. \end{aligned}$$

Combining (i) with (ii), we get

$$|\widehat{\mu}(t)|^2 \leq C_1 2^{-2\gamma_2 m}.$$

Therefore we obtain

$$|\widehat{\mu}(t)| = O(|t|^{-\gamma}). \quad (7.26)$$

□

Remark 7.8. *Typical examples of weight functions ϕ in case (1) are $(\log n)^{-\tau_1}, n^{-\tau_2}$, where τ_1, τ_2 are positive constants, while for case (2) are κ^{-n} , with κ being a constant no smaller than 2. Note that from the theorem of Ivasev-Musatov [64], one can construct a singular measure with a given rate of decay satisfying some regular conditions. However, our result is different, we show that for any given weight function $\phi(n)$, the rate of decay of Fourier transform of corresponding measure in (7.1) is determined by the weight function $\phi(n)$.*

7.2 Normal numbers

The classical Borel's normal numbers theorem shows that almost all real numbers in $[0, 1)$ are absolutely normal with respect to Lebesgue measure. We know that Lebesgue measure is Rajchman measure. Motivated by the uniqueness problem in trigonometric series, Kahane and Salem in 1964 asked whether the classical Borel's normal numbers theorem is also true for any Rajchman measure. In 1986, Lyons [73] constructed a Rajchman measure ν (a variant of coin-tossing type measures) for which ν almost all real numbers are not normal to the base 2, and thus answered the Kahane-Salem problem in the negative.

Davenport-Erdős-LeVeque Theorem indicates that when the rate of decay of Fourier transform of a measure is fast (some appropriate logarithmic is enough), then almost all real numbers are normal with respect to the measure. Using the method of Cassels [16] and Schmidt [87], we shall show that the Kahane-Salem problem is positive for the coin-tossing type measure μ given by (7.1) whose rate of decay of its Fourier transform obtained in Theorem 7.7 could be very slow.

Recall that

$$\widehat{\mu}(t) = \prod_{n=1}^{\infty} \left(\frac{1}{2}(1 + \phi(n)) + \frac{1}{2}(1 - \phi(n))e^{2\pi i 2^{-n}t} \right). \quad (7.27)$$

The following lemma is useful for estimating the factors in the product in (7.27).

Lemma 7.9. *Let a_1, a_2 be two positive real numbers. Then*

$$|a_1 + a_2 e^{2\pi i t}| \leq a_1 + a_2 - 4 \min\{a_1, a_2\} \|t\|^2.$$

Proof. Without loss of generality, we may assume that $a_1 \geq a_2$, otherwise, replace t by $-t$. Thus it suffices to prove

$$a_1 + a_2 - |a_1 + a_2 e^{2\pi i t}| \geq 4a_2 \|t\|^2.$$

Using the basic inequality

$$\sin^2 t \geq \frac{4}{\pi^2} t^2, \text{ for all } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2},$$

we obtain

$$\sin^2 \pi t = \sin^2 \pi \|t\| \geq 4 \|t\|^2. \quad (7.28)$$

Notice that

$$(a_1 + a_2 + |a_1 + a_2 e^{2\pi i t}|)(a_1 + a_2 - |a_1 + a_2 e^{2\pi i t}|) = 4a_1 a_2 \sin^2 \pi t. \quad (7.29)$$

We deduce from (7.28) and (7.29) that

$$(a_1 + a_2 + |a_1 + a_2 e^{2\pi i t}|)(a_1 + a_2 - |a_1 + a_2 e^{2\pi i t}|) \geq 16a_1 a_2 \|t\|^2 = 4a_1 4a_2 \|t\|^2. \quad (7.30)$$

Since a_1, a_2 are positive real numbers, it is clear that

$$a_1 + a_2 + |a_1 + a_2 e^{2\pi i t}| \leq 4a_1. \quad (7.31)$$

Combing (7.30) with (7.31), we complete the proof. \square

Theorem 7.10. *Let μ be the measure given by (7.1). Suppose $\phi : \mathbb{N} \rightarrow (0, 1)$ is a non-increasing function with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$, then μ -almost all real numbers are absolutely normal.*

Proof. We divide the proof into two cases.

- (1) $\frac{\log b}{\log 2} \in \mathbb{Q}$. In this case, a result of Schmidt in [87] indicates that α is normal to the base b if and only if α is normal to the base 2. Then it suffices for us to show that μ almost all real numbers are normal to the base 2. Due to Theorem 4.17, it is equivalent to prove that μ -almost all real numbers are simply normal to all of the bases 2^k ($k \in \mathbb{N}$). Let α be a real number, then it admits a unique 2^k -adic expansion of the form

$$\alpha = [\alpha] + \sum_{n=1}^{\infty} \frac{\varepsilon_n^{(k)}(\alpha)}{(2^k)^n}, \quad \varepsilon_n^{(k)}(\alpha) \in \{0, 1, \dots, 2^k - 1\}, \quad (7.32)$$

where the digits $\varepsilon_n^{(k)}(\alpha) < 2^k - 1$ for infinitely many n . Given $k \in \mathbb{N}$ and $m \in \{0, 1, \dots, 2^k - 1\}$, we define

$$E_{m,k,n}(\alpha) = \{\alpha \in \mathbb{R} : \varepsilon_n^{(k)}(\alpha) = m\}.$$

For simplicity, we also write

$$\mathbb{P}(X_n = 0) = \frac{1}{2}(1 + \phi(n)) := p_0^{(n)} \quad \text{and} \quad \mathbb{P}(X_n = 1) = \frac{1}{2}(1 - \phi(n)) := p_1^{(n)}. \quad (7.33)$$

Then the sequence of characteristic functions $(\chi_{E_{m,k,n}}(\alpha))_{n \geq 1}$ is independent but not identically distributed random variables in the sense of the measure μ satisfying

$$\mu(E_{m,k,n}(\alpha)) = \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(n-1)k)}, \quad \text{where } m = \sum_{i=1}^k c_i 2^{i-1}, \quad c_i \in \{0, 1\}.$$

By Kolmogorov's strong law of large numbers, μ -almost all real numbers α ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\chi_{E_{m,k,n}}(\alpha) - \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(n-1)k)} \right) = 0. \quad (7.34)$$

By (7.33),

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \prod_{i=0}^{k-1} p_{c_{k-i}}^{(i+1+(n-1)k)} = \frac{1}{2^k}. \quad (7.35)$$

Together with (7.34) and (7.35), we obtain that μ -almost all real numbers α ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \chi_{E_{m,k,n}}(\alpha) = \frac{1}{2^k}.$$

This shows that μ -almost all real numbers α are simply normal to the base 2^k .

(2) $\frac{\log b}{\log 2} \notin \mathbb{Q}$. By Theorem 4.13, it suffices to prove

$$\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\widehat{\mu}(h(b^n - b^m))| = O(N^{2-\eta})$$

for all non-zero integer h , where $\eta > 0$ is a constant. It is easy to see that

$$\begin{aligned} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\widehat{\mu}(h(b^n - b^m))| &\leq 2 \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} |\widehat{\mu}(h(b^k - 1)b^n)| \\ &\leq 2N + 2 \sum_{k=1}^{N-1} \sum_{n=0}^{N-1} |\widehat{\mu}(h(b^k - 1)b^n)|. \end{aligned}$$

Thus, it is enough to prove

$$\sum_{n=0}^{N-1} |\widehat{\mu}(hb^n)| = O(N^{1-\eta}) \quad \text{for all } h \in \mathbb{Z} \setminus \{0\}. \quad (7.36)$$

Given a non-negative integer τ_0 , we follows (7.8) that

$$\begin{aligned} |\widehat{\mu}(h2^{\tau_0})|^2 &= \prod_{n=1}^{\infty} (\phi^2(n) \sin^2 \pi h 2^{\tau_0-n} + \cos^2 \pi h 2^{\tau_0-n}) \\ &= \prod_{k=1}^{\infty} (\phi^2(\tau_0 + k) \sin^2 \pi h 2^{-k} + \cos^2 \pi h 2^{-k}). \end{aligned}$$

Suppose that

$$b = b_1 2^{\tau_1} \quad \text{and} \quad h = h_2 2^{\tau_2},$$

where $2 \nmid b_1, 2 \nmid h_2$, and τ_1, τ_2 are two non-negative integers, then we also have

$$\begin{aligned} |\widehat{\mu}(hb^n)|^2 &= |\widehat{\mu}(h_2 2^{\tau_2} b_1^n 2^{n\tau_1})|^2 \\ &= \prod_{k=1}^{\infty} (\phi^2(k + \tau_2 + n\tau_1) \sin^2 \pi h_2 b_1^n 2^{-k} + \cos^2 \pi h_2 b_1^n 2^{-k}) \\ &\leq \prod_{k=1}^{\infty} (\phi^2(k) \sin^2 \pi h_2 b_1^n 2^{-k} + \cos^2 \pi h_2 b_1^n 2^{-k}) \leq |\widehat{\mu}(h_2 b_1^n)|^2. \end{aligned}$$

Thus by (7.27) and (7.36), it reduces to prove

$$\sum_{n=0}^{N-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^n 2^{-k}} \right) \right| = O(N^{1-\eta}) \quad (7.37)$$

under the condition that $2 \nmid b, 2 \nmid h$. First, we claim that for any $r \geq 1$,

$$\sum_{n=0}^{2^r-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^n 2^{-k}} \right) \right| = O(2^{r(1-\eta_0)}), \quad (7.38)$$

where η_0 is a positive real number. Let l be an integer defined by

$$b \equiv 1 \pmod{2^l} \quad \text{and} \quad b \not\equiv 1 \pmod{2^{l+1}}. \quad (7.39)$$

For the case when $l = 1$, there exists a positive integer $m \geq 2$ satisfying $b^2 \equiv 1 \pmod{2^m}$. Moreover, Theorem 4.16 indicates that α is normal to the base b if and only if α is normal to the base b^2 . Thus we only consider the case $l \geq 2$. Under the condition that $2 \nmid b, 2 \nmid h$, we deduce from (7.39) that

- (i) $hb^{n_1} \equiv hb^{n_2} \pmod{2^l}$ for all $0 \leq n_1 < n_2 \leq 2^r - 1$.
- (ii) $b^{2^r} \equiv 1 \pmod{2^{l+r}}$ and $b^t \not\equiv 1 \pmod{2^{l+r}}$ for all $1 \leq t \leq 2^r - 1$.
- (iii) $hb^{n_1} \not\equiv hb^{n_2} \pmod{2^{l+r}}$ for all $0 \leq n_1 < n_2 \leq 2^r - 1$.

For $0 \leq n \leq 2^r - 1$, if we have the expansion

$$hb^n = \sum_{k \geq 0} d_k(n) 2^k = d_0(n) + \cdots + d_{l-1}(n) 2^{l-1} + d_l(n) 2^l + \cdots + d_{l+r-1}(n) 2^{l+r-1} + \cdots,$$

where $d_k(n) \in \{0, 1\}$, then (i) and (iii) ensure that r -strings

$$d_l(n)d_{l+1}(n)\dots d_{l+r-1}(n) \quad (7.40)$$

take exactly one (and only one) of the 2^r possible sets of values as n runs from 0 to $2^r - 1$. For convenience, we call $(0, 1), (1, 0)$ regular pairs and denote the set

$$\mathcal{P} := \{(0, 1), (1, 0)\}.$$

In the following we divide the indices $n(0 \leq n \leq 2^r - 1)$ into two cases, denoted by I and II . The index n belongs to case I if in the string (7.40) there are at least $\lceil \varepsilon r \rceil (0 < \varepsilon < \frac{1}{4})$ pairs of $(d_k(n), d_{k+1}(n))$ belong to \mathcal{P} , where $l \leq k \leq l + r - 2$; otherwise n belongs to case II . For case I , we consider the range of the fractional part of $hb^n 2^{-k-2}$, clearly

$$\{hb^n 2^{-k-2}\} = \frac{d_0(n) + \dots + d_k(n)2^k + d_{k+1}(n)2^{k+1}}{2^{k+2}} \in \left[\frac{1}{4}, \frac{3}{4}\right).$$

In other words,

$$\|hb^n 2^{-k-2}\| \geq \frac{1}{4}. \quad (7.41)$$

Notice that ϕ is non-increasing and case I contains at most 2^r elements. Taking $\delta_2 := \frac{7}{8} + \frac{1}{8}\phi(1)$, by Lemma 7.9 and (7.41), we have

$$\begin{aligned} & \sum_{n \in I} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i hb^n 2^{-k}} \right) \right| \\ & \leq \sum_{n \in I} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k+2)) + \frac{1}{2}(1 - \phi(k+2))e^{2\pi i hb^n 2^{-k-2}} \right) \right| \\ & \leq \sum_{n \in I} \prod_{k=1}^{\infty} (1 - 2(1 - \phi(k+2))\|hb^n 2^{-k-2}\|^2) \\ & \leq \sum_{n \in I} \prod_{\lceil \varepsilon r \rceil \leq \#k \leq l+r-2} \delta_2 \leq 2^r \delta_2^{\varepsilon r} = 2^{r(1+\varepsilon \log_2 \delta_2)} = 2^{r(1-\eta_1)}, \end{aligned} \quad (7.42)$$

where $\eta_1 := -\varepsilon \log_2 \delta_2 \in (0, 1)$. For case II , we know that in the string (7.40) the number of all combinations of the string such that there are exactly $s (s < \lceil \varepsilon r \rceil)$ indices belong to \mathcal{P} is $2(r-s)$. Thus the number of n for which there are less than $\lceil \varepsilon r \rceil$ regular pairs in the string (7.40) can not exceed

$$\sum_{s=0}^{\lceil \varepsilon r \rceil} 2(r-s) \leq (2\varepsilon - \varepsilon^2)r^2 + \varepsilon r \leq 2^{r(1-\eta_2)}, \quad (0 < \eta_2 < 1).$$

Hence we have

$$\sum_{n \in II} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i hb^n 2^{-k}} \right) \right| \leq \sum_{n \in II} 1 \leq 2^{r(1-\eta_2)}. \quad (7.43)$$

Taking $\eta_0 = \min\{\eta_1, \eta_2\}$, we deduce (7.38) from (7.42) and (7.43). Let N be an any positive integer with expansion

$$N = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}, \quad 0 \leq r_1 < r_2 < \cdots < r_t. \quad (7.44)$$

The range of summation $0 \leq n \leq N - 1$ in (7.36) could be subdivided into t intervals of the type $N_0 \leq n \leq N_0 + 2^r - 1$, i.e.,

$$[0, 2^{r_1} - 1], [2^{r_1}, 2^{r_1} + 2^{r_2} - 1], \cdots, [2^{r_1} + 2^{r_2} + \cdots + 2^{r_{t-1}}, N - 1].$$

Notice that

$$\begin{aligned} & \sum_{n=N_0}^{N_0+2^r-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^n 2^{-k}} \right) \right| \\ &= \sum_{n=0}^{2^r-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^{N_0} b^n 2^{-k}} \right) \right| \\ &= O(2^{r(1-\eta_0)}) = O(N^{1-\eta_0}). \end{aligned} \quad (7.45)$$

The reason why the formula (7.45) holds is that the above is just a sum of type in (7.38) with hb^{N_0} instead of h . It follows (7.27), (7.44) and (7.45) that

$$\begin{aligned} \sum_{n=0}^{N-1} |\widehat{\mu}(hb^n)| &= \sum_{n=0}^{N-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^n 2^{-k}} \right) \right| \\ &\leq t \cdot \sum_{n=N_0}^{N_0+2^r-1} \prod_{k=1}^{\infty} \left| \left(\frac{1}{2}(1 + \phi(k)) + \frac{1}{2}(1 - \phi(k))e^{2\pi i h b^n 2^{-k}} \right) \right| \\ &\leq r_t \cdot O(N^{1-\eta_0}) \leq (\log_2 N) O(N^{1-\eta_0}) = O(N^{1-\eta}). \end{aligned}$$

□

Salem's characterization states that the divergence of the series $\sum \phi^2(n)$ is a necessary and sufficient condition for the measure μ being a singular measure. Combining Theorem 7.7, part (1) with Theorem 7.10, we obtain the following corollary.

Corollary 7.11. *Let $\phi : \mathbb{N} \rightarrow (0, 1)$ be a non-increasing function satisfying*

$$\sum_{n \geq 1} \phi^2(n) = +\infty, \quad \lim_{n \rightarrow \infty} \phi(n) = 0 \text{ and } \phi(n) < 2\phi(n+1) \text{ for any } n \geq 1.$$

Then μ is a singular Rajchman measure for which $|\widehat{\mu}(t)| = O(\phi(\lceil \gamma \log_2 |t| \rceil))$, where $\gamma = \min\{\frac{1}{2}, \frac{-\log_2 \delta}{2}\}$ and $\delta \in (0, 1)$ depending only on ϕ . Moreover, μ -almost all real numbers are absolutely normal.

Example 7.12. *Let $\phi(n) = 1/\log \log n$, then we obtain a singular Rajchman measure with slow rate of decay of its Fourier transform, i.e., $|\widehat{\mu}(t)| = O(1/\log \log(\lceil \log_2 |t| \rceil))$.*

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