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SORBONNE UNIVERSITÉ**

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Lions par

Nicolas CLOZEAU

Pour obtenir le grade de

DOCTEUR DE SORBONNE UNIVERSITÉ

**QUANTITATIVE ESTIMATES IN STOCHASTIC
HOMOGENIZATION OF ELLIPTIC EQUATIONS AND
SYSTEMS**

après avis des rapporteurs

Josselin GARNIER Professeur à l'École Polytechnique
Yu GU Associate Professor à Carnegie Mellon University

devant le jury composé de

Anne-Laure DALIBARD	Sorbonne Université-Examinatrice
Fabrice BETHUEL	Sorbonne Université-Examinateur
Isabelle GALLAGHER	Ecole Normale Supérieure-Examinatrice
Josselin GARNIER	Ecole Polytechnique-Rapporteur
Yu GU	Carnegie Mellon University-Rapporteur
Antoine GLORIA	Sorbonne Université-Directeur de thèse
Felix OTTO	Max Planck Institute for Mathematics in the Sciences-Examinateur

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Quand on débute une thèse, on ne sait pas trop par où commencer ou comment s’y prendre, l’exercice est tellement différent de ce qu’on a pu faire pendant le cycle licence-master. Mais quand on est bien entouré et guidé, on peut finalement apprendre et apprécier l’expérience de la recherche. Mes premiers mots vont naturellement à mes deux directeurs de thèse qui ont amplement rempli cette tâche et qui ont su me guider et me permettre de passer du statut d’étudiant que j’étais à celui de jeune chercheur en mathématiques appliqué que je suis aujourd’hui. Je remercie tout d’abord Antoine Gloria, avec qui j’ai pour la première fois découvert l’homogénéisation stochastique, pour m’avoir fait confiance pour cette thèse. Il a été d’une grande disponibilité et a su m’apporter l’aide et les conseils dont j’avais besoin pour mener à bien cette aventure. J’ai également appris beaucoup de notre collaboration, que ce soit à propos de ta méthode de travail ou ta manière d’aborder les problèmes. J’ai aussi eu la chance, grâce à toi, de passer une année au Max Planck Institut für Mathematik in den Naturwissenschaften à Leipzig sous la supervision de Felix Otto. Je le remercie pour son accueil si chaleureux. Le cadre de travail que propose l’institut est vraiment exceptionnel, favorisant les rencontres et les échanges. J’ai beaucoup appris au contact de Felix durant cette collaboration, d’une part pour sa manière d’appréhender les problèmes et d’autre part pour son recul et sa façon de comprendre les choses, qui sont une grande source d’inspiration pour moi. Vielen danke! J’ai vraiment hâte de collaborer davantage !

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Foreword

L'étude approfondie de la nature est la source la plus féconde des découvertes mathématiques.

Joseph Fourier - *Discours préliminaire à la théorie analytique de la chaleur*

This thesis was carried out starting on September 2018 during a three-year PhD program in Laboratoire Jacques-Louis Lions at Sorbonne Université in Paris and at the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig, under the co-supervision of Prof. Antoine Gloria and Prof. Felix Otto. The present manuscript contains different results in stochastic homogenization theory of elliptic equations and systems, aiming at a better understanding of the quantitative theory. Starting from quantitative estimates for linear elliptic systems with strongly correlated coefficient fields, we then develop a quantitative homogenization theory for nonlinear elliptic equations and systems with strongly monotone coefficients. In the last part, we analyse the so-called representative volume element method to compute the homogenized coefficient and we show optimal rate of convergence with respect to the size of the box. The Chapters 2 and 3 can be read independently of each others. The Chapter 5 make use of the results proved in Chapter 4.

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Chapter 1

Introduction

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1.1 Introduction to the homogenization theory

1.1.1 General introduction

The homogenization theory aims at deriving the effective properties of heterogeneous systems, when heterogeneities are small compared to the characteristic size of the system. Such systems are, at first sight, very complicated and the general question arising from the homogenization theory is : "Can we replace the complicated heterogeneous system by a simple equivalent homogeneous model (meaning that it is efficiently computable) keeping the relevant physical properties of the heterogeneous system ?".

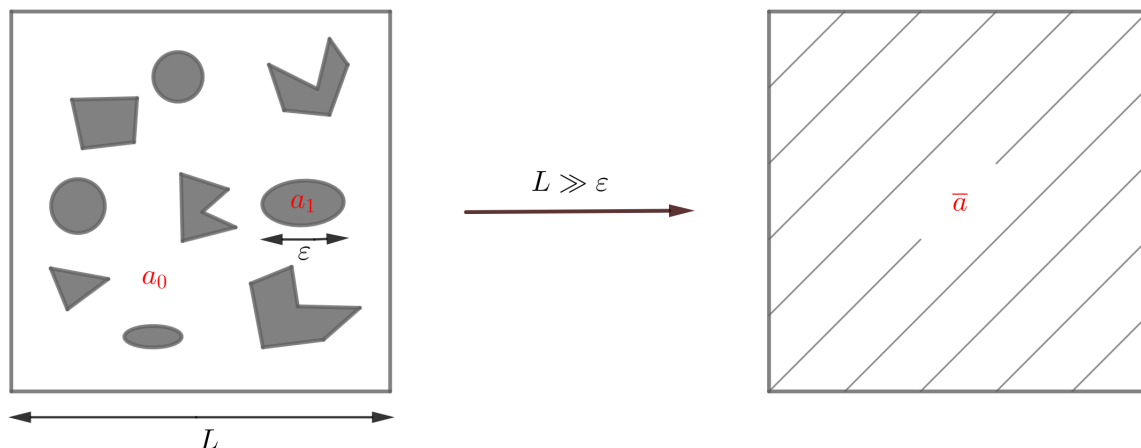


Figure 1.1: Left panel: A schematic of a two-phase material with properties a_0 and a_1 . L and ε represent the macroscopic and the microscopic length-scales, respectively. Right panel: When $L \gg \varepsilon$, the heterogeneous system can be replaced as a homogeneous one with effective property \bar{a} depending on a_0 , a_1 and the distribution of the two phases.

The challenge of the homogenization theory is to find a suitable formula for the effective property \bar{a} , as a function of a_0 , a_1 and of the distribution of the two phases. The understanding and the determination of effective properties of heterogeneous materials, such as transport, electromagnetic, heat conduction or mechanical, has a long history. We can cite, for instance, four pioneering works in that direction, back to the last two centuries. First, Clausius and Mossotti in [125, 126, 46] investigated the effective dielectric constant for isotropic spherical inclusions in an isotropic reference medium. Later, James Clerk Maxwell, in his Treatise on Electricity and Magnetism [121] popularized the works of Clausius and Mossotti and worked on the same equations. He investigated the effective thermal conductivity of composite materials, model as a continuous matrix of constant conductivity a_0 containing multiple spherical inclusions with identical radii of constant conductivity a_1 . He derived an expression for the effective thermal conductivity, via far-field perturbations to solutions of the steady heat equation, that is exact for dilute sphere concentrations. John William Strutt Rayleigh in [144] developed a formalism to compute the effective conductivity of regular arrays of spheres that is still used to this day. Finally, Einstein, in his PhD thesis [64], determined the effective viscosity of a dilute suspension of spheres, that is, assuming that the particles are scarce. It leads to the so-called Einstein's formula, which reads¹ $\mu = (1 + \frac{5}{2}\Phi)\mu_0$, where μ represents the

¹Einstein's original result contained an error, which was corrected with help of his student and collaborator Ludwig Hopf in [65].

effective viscosity of the composite system, μ_0 the viscosity of the pure liquid and Φ the volume concentration of the particles. This formula played an important role in the physic community at that time and, in particular, it served as a basis for Perrin's experiment to estimate the Avogadro number. More recently heterogeneous media have been extensively used in physics and engineering for their many interesting properties. Examples of manufactured heterogeneous materials include particulate composites, block copolymers or aligned and chopped fiber composites whereas heterogeneous materials in nature include polycrystals, Earth's crust or cell aggregates and tumors.

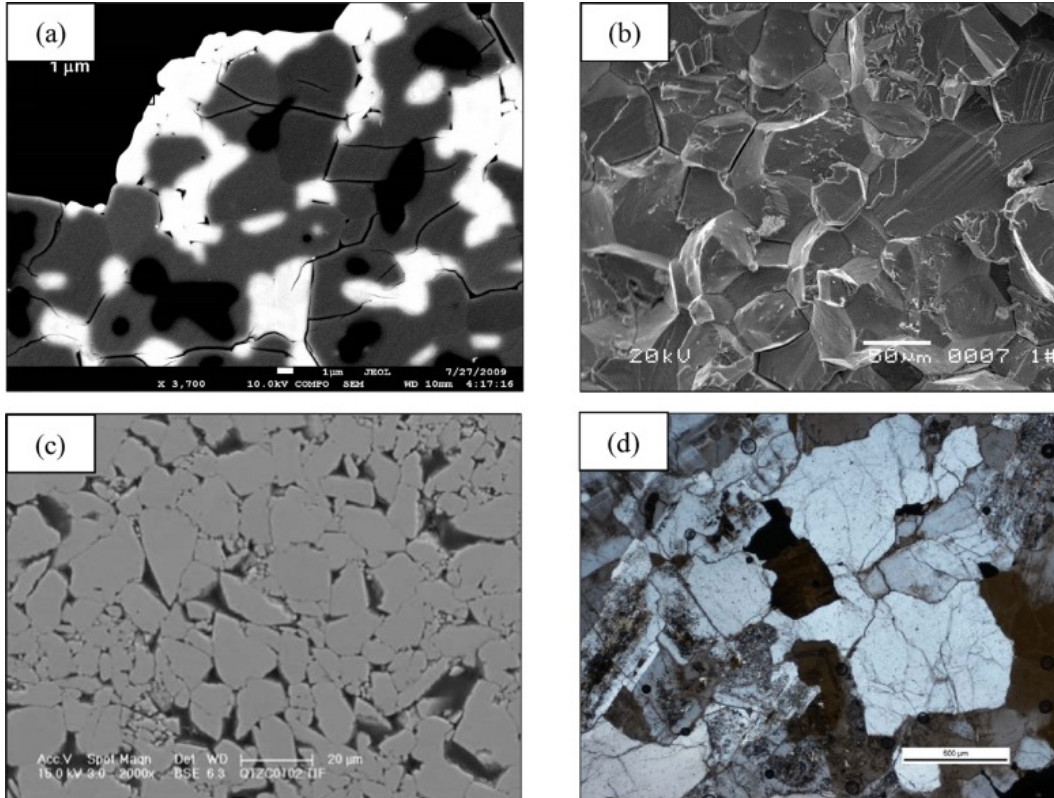


Figure 1.2: Example of polycrystals (see [146]): (a) aluminium titanate (b) mixed of titanium, molybdenum, vanadium, iron and aluminium (c) quartz (d) silesian granite.

The rich variety of examples encouraged the development of methods to characterise the effective medium and averaged fields, which took the form of the homogenization theory in the mathematical community. Important mechanical and physical properties include:

- The effective elasticity tensor \mathcal{C} , where pioneer works started from the celebrated papers [67] and [66] about a single ellipsoidal inhomogeneity. The quantity \mathcal{C} corresponds to the proportionality constant between the average stress and average strain, commonly referred to averaged Hooke's law for the composite material. More general nonlinear models describing the energy of deformation arise naturally in nonlinear elasticity, see for instance [44]. The elastic nonlinear energy model takes the form of an integral $\int_{\Omega} \overline{W}(\nabla u(x)) dx$ where ∇u denotes the deformation gradient and \overline{W} the effective stored elastic energy function. Computing \mathcal{C} and \overline{W} for heterogeneous material finds, for instance, its interest in biological material such as bone or tendon or also in various synthetic materials.
- The fluid permeability tensor κ , describing slow viscous flow through porous media. The

quantity κ corresponds to the proportionality tensor between the average fluid velocity \vec{V} and the applied pressure gradient in the porous medium ∇P , commonly referred to Darcy's law for the porous medium. In turbulent regimes, a nonlinear model of Darcy's law is often used and takes the form $\vec{V} = -\kappa(\nabla P) = -\kappa|\nabla P|^{p-2}\nabla P$, for some parameter $p > 1$ (see for instance [108]). Computing κ for porous media finds, for instance, its interest in extraction of oil or gas from porous rocks, chromatography or filtration.

- The effective thermal or conductivity tensor σ . The quantity σ corresponds to the proportionality tensor between the average of the local electric current (resp. heat flux \vec{j}_h) and the average of the local electric field (resp. temperature gradient ∇T) in the composite material, commonly referred to as averaged Ohm's law or Fourier's law in the electrical or thermal problems, respectively. For certain temperature range, we rather apply to nonlinear laws for heat transfers of the form $\vec{j}_h = -\sigma(T, \nabla T) = -\sigma T^q |\nabla T|^{p-2} \nabla T$, for some parameters $q > 0$ and $p \geq 2$ (see for instance [141, 23]). Computing σ for heterogeneous material finds, for instance, its interest in electrical components and oil drilling operations, where electric conductivity measurements of the brine-saturated rock are used to infer information about the permeability of the pore space. Thermal applications range from composites used for insulation, heat exchangers, and heat sinks for electronic cooling to geophysical problems (determination of the geothermal temperature gradient).

On large scales, microstructures are expected to average out due to a kind of a law of large numbers, in such a way that the heterogeneous physical properties can be replaced by homogeneous ones. This procedure is what we call the homogenization process. The homogenization theory can be applied to many situations and the most simple examples are obtained when the heterogeneities are periodically or quasi-periodically distributed through the material. However, in many instances, the microstructures can be characterized only statistically, and therefore are referred to as random heterogeneous materials, the main focus of this thesis. There is a vast family of random microstructures that are possible, both natural or synthetic, ranging from porous media, multilayer-structures (heterogeneous in one dimension) to complex connected multiphase media or perforated materials. Physical or mechanical phenomenas are usually described via partial differential equations or energy functionals, depending on the microstructure via a random variable (corresponding to the distribution of the heterogeneities) and the scale separation parameter $0 < \varepsilon \ll 1$. For a long time, homogenization problems for partial differential equations were being mostly considered by specialists in physics and mechanics. The general mathematical theory only emerged in the 1970's with the pioneer works of Murat and Tartar in [150, 151, 131] (in the periodic setting), Kozlov in [104], Papanicolaou and Varadhan in [139] (in the random setting) and later in the 1980's by Braides [35] and Müller in [130] (in the periodic setting) as well as Dal Maso in [119] (in the random setting) in the general framework of hyperelasticity, mixing tools from probability, calculus of variation and partial differential equation theory. Since then, the homogenization theory is a very active domain of research and a lot of references will be given in the course of this introduction.

In mathematical models of random heterogeneous media, constitutive properties are usually described by functions of the form $a(\frac{x}{\varepsilon})$, where x denotes the space variable and $\varepsilon > 0$ the characteristic size of the heterogeneities small compared to the characteristic size L of the material-see Figure 1.1.1. In this thesis, we are interested in heterogeneous random media, so that we always assume that $a = a(x)$ is a realization of a random distribution. The determination of the effective properties of the random heterogeneous material is a difficult task since the coefficients of the corresponding differential problem rapidly oscillate as $\varepsilon \downarrow 0$. However, by assuming stationarity and ergodicity of

the random distribution (in form of a law of large number, a precise statement will be given in section 1.1.2), meaning that the distribution of the heterogeneities is statistically invariant by translation and decorrelates on large scales, we expect an averaging procedure in the limit $\varepsilon \downarrow 0$. In general, the averaging that occurs is more subtle than the simple average $\langle a \rangle$ of the coefficients². We illustrate this by the following simple example. We consider the case of a one-dimensional heterogeneous material on the segment $[0, 1]$ with a series of rectangular inclusions of size $\varepsilon \ll 1$ (that we can see as resistors in series):



Figure 1.3: One-dimensional heterogeneous material model by resistors in series.

We are interested in the effective conductivity of the material as $\varepsilon \downarrow 0$, where for simplicity we impose that $\frac{1}{\varepsilon} \in \mathbb{N}$. The conductivity of the first phase (that is without the heterogeneities) is assumed to be constant equal to a_0 whereas for the second phase (corresponding to the heterogeneities) the conductivity is assumed to be constant equal to a_1 . Furthermore, we assume that the resistors are independently and identically distributed via a sequence of Bernoulli's random variables $(b_i)_{i \in \mathbb{N}} \subset \{0, 1\}^{\mathbb{N}}$, with parameter $q > 0$. More precisely, the conductivity a_ε of the heterogeneous material is given by

$$a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right) = a_0 + (a_1 - a_0) \sum_{i=1}^{+\infty} b_i \mathbf{1}_{[i-1, i]}\left(\frac{x}{\varepsilon}\right). \quad (1.1.1)$$

We impose a difference of potential (denoted by u) on this configuration of resistors, say $\int_0^1 \partial_x u = 1$, and we assume that there is no charge. Thus, the electrical flux is conserved, which reads:

$$\partial_x(a(\frac{x}{\varepsilon})\partial_x u) = 0.$$

By integrating once, we learn that the flux $F_\varepsilon = a(\frac{x}{\varepsilon})\partial_x u$ is constant. By averaging $\partial_x u = \frac{\bar{a}_\varepsilon}{a(\frac{x}{\varepsilon})}$ on $[0, 1]$, we obtain that the flux satisfy $F_\varepsilon = (\int_0^1 \frac{1}{a(\frac{x}{\varepsilon})} dx)^{-1}$, which turns into the following formula using (1.1.1),

$$F_\varepsilon = \left(\varepsilon \sum_{i=1}^{\frac{1}{\varepsilon}} \frac{1}{a_0 + (a_1 - a_0)b_i} \right)^{-1}. \quad (1.1.2)$$

Since the potential difference is 1, we have $F_\varepsilon = \bar{a}_\varepsilon \times 1$, where \bar{a}_ε represents the effective conductivity of the system. The formula (1.1.2) converges as $\varepsilon \downarrow 0$ by the law of large numbers, and the limit provides the formula for the effective conductivity $\bar{a} = \bar{a}(a_0, a_1)$ of the homogenized material:

$$\bar{a} := \lim_{\varepsilon \downarrow 0} \bar{a}_\varepsilon = \mathbb{E} \left[\frac{1}{a_0 + (a_1 - a_0)b_1} \right]^{-1} = \left(\frac{q}{a_1} + \frac{1-q}{a_0} \right)^{-1}. \quad (1.1.3)$$

In particular, $\bar{a} \neq \langle a \rangle$, which shows that the effective conductivity (1.1.3) is obtained by a more subtle averaging procedure than arithmetic averaging. Actually, the formula (1.1.3) is not surprising: the effective behaviour of a series of resistors is given by adding the resistivity of each resistor,

²The general notation $\langle \cdot \rangle$ is used to either mean the average on the periodic cell f_Q or the expectation $\mathbb{E}[\cdot]$ depending on the context.

expressed by $\frac{1}{a_1}$ and $\frac{1}{a_0}$, with a probability to contribute equal to q and $1 - q$ respectively (which is equivalent to (1.1.19) since the resistivity is the inverse of the conductivity).

The rest of the introduction is organized as follows. Section 1.1.2 is devoted to qualitative stochastic homogenization of elliptic systems in divergence form. In Section 1.1.3 we argue by a formal two-scale expansion on a simple nonlinear model, which allows us to introduce the classical objects of stochastic homogenization, such as the correctors and the homogenized operator. We then rigorously treat in Section 1.1.4 a nonlinear one-dimensional example and relate this result to the formal computations of Section 1.1.3. We also compute the homogenized operator as well as the two-scale expansion in a nonlinear two-dimensional example of a layered material. We then slowly turn to the contents of this thesis: we present in Section 1.1.5 quantitative assumptions and important tools that we use in this thesis. We show in particular how those tools can be applied on the simple one-dimensional example of Section 1.1.4 to get optimal estimates. Finally, in Section 1.2.1 and then in Section 1.2, we describe the questions and the contributions of this thesis.

1.1.2 Qualitative assumptions for a qualitative homogenization theory

We denote by $d \geq 1$ the dimension and we consider an open subset $\Omega \subset \mathbb{R}^d$. We fix two exponents $m \geq 1$ and $p \geq 2$ as well as $f \in L^p(\Omega, \mathbb{R}^{m \times d})$ and we consider the following system of equations with solution $u_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^m)$ in Ω :

$$-\nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) = \nabla \cdot f, \quad (1.1.4)$$

with either Neumann or Dirichlet boundary conditions and where $\varepsilon \ll 1$ denotes the microscopic scale³. Systems of type (1.1.4) model physical phenomena such as (nonlinear) conductivity (of thermal or electrical type) obtained by combining:

- A conservation law: $\nabla \cdot (q + f) = 0$, where q denotes either the heat flux or the local electric current and f some exterior forcing.
- A constitutive law (such as Fourier's or Ohm's law): $q = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$, where u_ε is either the temperature or the electric potential. The spatial dependence in the constitutive law in form of $\frac{x}{\varepsilon}$ comes from the composite structure of the material, made of different materials with different conductivities.⁴

The map $a : \mathbb{R}^d \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d}$ is assumed to be strongly monotone, that is: there exist two constants $\lambda \geq 1$ and $s \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\xi_1, \xi_2 \in \mathbb{R}^{m \times d}$, we have

$$(a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \frac{1}{\lambda} (s + |\xi_1|^{p-2} + |\xi_2|^{p-2}) |\xi_1 - \xi_2|^2, \quad (1.1.5)$$

and

$$|a(x, \xi_1) - a(x, \xi_2)| \leq \lambda (s + |\xi_1|^{p-2} + |\xi_2|^{p-2}) |\xi_1 - \xi_2|. \quad (1.1.6)$$

The well-posedness of (1.1.4), under (1.1.5) and (1.1.6), is ensured by the Hartman-Stampacchia's theorem (see for instance [98]). We are interested in random distributed microstructures and therefore the operators $a = a(x, \xi)$ are realizations of a probability distribution. We use the following

³The divergence form of the right-hand side allows us to consider also the case $\Omega = \mathbb{R}^d$. Note that if Ω is bounded, we can always rewrite the right-hand (provided it has zero mean) in divergence form. We precise that boundary layers are not discussed in this thesis and we fully focus on the homogenization error. In coordinates the system reads $-\sum_{j=1}^d \partial_j ((a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon))_{ij}) = \sum_{j=1}^d \partial_j f_{ij}$, for all $i \in \{1, \dots, m\}$.

⁴Note that we do not assume a dependence of q in u_ε and in particular our results do not apply to scalar conductivity models of the form $q = A\left(\frac{x}{\varepsilon}\right) u_\varepsilon^q (s + |\nabla u_\varepsilon|^{p-2}) \nabla u_\varepsilon$ for some $s \geq 0$, $q > 0$ and $p \geq 2$.

framework: we define the space of all admissible operators

$$\mathcal{M} := \{a : \mathbb{R}^d \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d} \text{ measurable} \mid (1.1.5) \text{ and } (1.1.6) \text{ are satisfied}\}. \quad (1.1.7)$$

We endow \mathcal{M} with the σ -algebra \mathcal{F} defined as the smallest σ -algebra that makes the evaluations⁵ $a \mapsto \int a(x, \xi) \psi(x) dx$ measurable for all $\xi \in \mathbb{R}^{m \times d}$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ (which is countably generated since $C_c^\infty(\mathbb{R}^d)$ is separable). We then equip the measurable space $(\mathcal{M}, \mathcal{F})$ with a probability measure \mathbb{P} . Two additional important conditions have to be added on the probability measure \mathbb{P} for the homogenization process of the equation (1.1.4) to hold, that we list hereafter.

- The action group $(\mathbb{R}^d, +)$ acts on \mathcal{M} and associates for all $z \in \mathbb{R}^d$ and $a \in \mathcal{M}$ the translated field $T_z a : (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^{m \times d} \mapsto a(x + z, \xi)$. We assume that \mathbb{P} is **stationary** with respect to \mathbb{R}^d -translations, that is:

$$\mathbb{P} \circ T_z = \mathbb{P} \text{ for every } z \in \mathbb{R}^d. \quad (1.1.8)$$

- \mathbb{P} is **ergodic** by the action of $(\mathbb{R}^d, +)$ in the sense that: for all $F \in L^1_{\text{loc}}(\mathbb{R}^d, L^1(d\mathbb{P}))$ ⁶ with the shift-invariant property for all $a \in \mathcal{M}$ and $z \in \mathbb{R}^d$, $F(\cdot + z, a) = F(\cdot, a(\cdot + z, \cdot))$, we have \mathbb{P} -almost surely⁷

$$\int_{B_r} F(x, a) dx \xrightarrow{r \uparrow +\infty} \mathbb{E}[F]. \quad (1.1.9)$$

The stationarity assumption (1.1.8) imposes that the statistics of $a(x)$ does not depend on x . The ergodic assumption (1.1.9) encodes decorrelation of a at large distances $r \uparrow +\infty$, such that statistical averages can be recovered by the spacial averages almost surely. This can be seen as a law of large numbers (extending the case of discrete i.i.d random variables) and we have already seen its importance in the homogenization process in the small example of resistors in series in Section 1.1.1.

The simplest way to construct examples satisfying the assumptions (1.1.5), (1.1.6), (1.1.8) and (1.1.9) is from a "random checkerboard" structure: we pave the space by unit-sized cubes and color each cube either white or black independently at random, where the origin $O \in [0, 1]^d$ is randomly distributed according to the uniform distribution on $[0, 1]^d$.⁸ Each color is then associated with a particular value of (for instance) a diffusivity matrix. More precisely, consider a sequence of independent and identically distributed Bernoulli's random variables $(b(z))_{z \in \mathbb{Z}^d} \subset \{0, 1\}^{\mathbb{Z}^d}$ with parameter $q \in [0, 1]$, namely for every $z \in \mathbb{Z}^d$,

$$\mathbb{P}(\{b(z) = 0\}) = 1 - \mathbb{P}(\{b(z) = 1\}) = q.$$

We consider also a random variable $O \in [0, 1]^d$ distributed according to the uniform distribution on $[0, 1]^d$. We fix two matrices a_0 and a_1 belonging to the set

$$\{\tilde{a} \in \mathbb{R}^{d \times d} \mid \text{for all } h \in \mathbb{R}^d, \frac{1}{\lambda} |h|^2 \leq \tilde{a} h \cdot h \leq \lambda |h|^2\}. \quad (1.1.10)$$

⁵We omit the domain of integration when we integrate on the whole space \mathbb{R}^d .

⁶We use the notation $L^1(d\mathbb{P}) := \{F \text{ measurable} \mid \mathbb{E}[|F|] < +\infty\}$.

⁷More generally, by density if $F \in L^q_{\text{loc}}(\mathbb{R}^d, L^1(d\mathbb{P}))$, $F(r \cdot, a) \xrightarrow{r \uparrow +\infty} \mathbb{E}[F]$ in $L^q_{\text{loc}}(\mathbb{R}^d)$, \mathbb{P} -almost surely.

⁸Without this choice, we only get the stationarity of the probability distribution with respect to \mathbb{Z}^d translations.

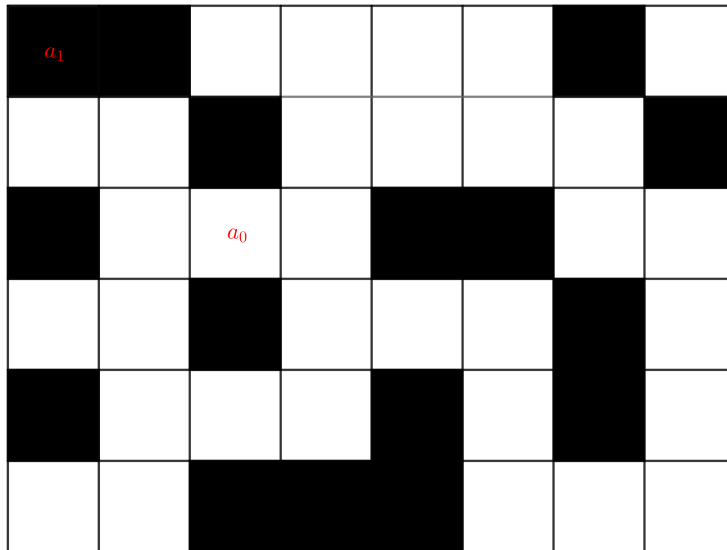


Figure 1.4: A piece of a sample of a random checkerboard. The conductivity matrix is equal to a_0 in the white region and a_1 in the black region.

We can define a random map $(x, \xi) \mapsto a(x, \xi)$ satisfying (1.1.5) and (1.1.6) with a law satisfying (1.1.8) and (1.1.9) by setting for all $z \in \mathbb{Z}^d$ and $x \in z + [0, 1)^d + O$

$$a(x, \xi) = a_{b(z)}(s + |\xi|^{p-2})\xi. \quad (1.1.11)$$

Another important class of examples are constructed using point processes. A simple case to illustrate this is given by Poisson point processes. We recall that a Poisson point process on a measurable space (X, \mathcal{A}) with (non-atomic, σ -finite) intensity measure μ is a random subset $\Pi \subset X$ such that the following properties hold:

- For every $A \in \mathcal{A}$, the number of points in $\Pi \cap A$, which is denoted by $N(A)$, follows a Poisson law of mean $\mu(A)$, that is: for all $k \in \mathbb{N}$

$$\mathbb{P}(\{N(A) = k\}) = \frac{(\mu(A))^k}{k!} e^{-\mu(A)}.$$

- For every pairwise disjoint measurable sets $(A_1, \dots, A_k) \in \mathcal{A}^k$, the random variables $N(A_1), \dots, N(A_k)$ are independent.

Let Π be a Poisson point process on \mathbb{R}^d with intensity measure $\mu = \gamma dx$, for some $\gamma > 0$. We fix a_0, a_1 in the set (1.1.10) and we define $(x, \xi) \mapsto a(x, \xi)$ by setting for all $x \in \mathbb{R}^d$

$$a(x, \xi) = (s + |\xi|^{p-2})\xi \times \begin{cases} a_0 & \text{if } \text{dist}(x, \Pi) \leq \frac{1}{2}, \\ a_1 & \text{otherwise.} \end{cases} \quad (1.1.12)$$

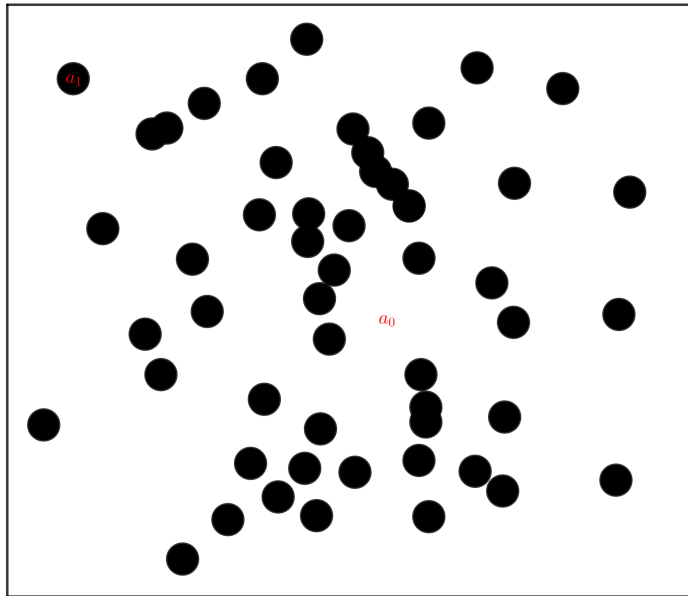


Figure 1.5: A sample of the coefficient field defined in (1.1.12) by the Poisson point cloud. The conductivity matrix is equal to a_1 in the black region and to a_0 in the white region.

A third important class of examples is obtained by defining the map $(x, \xi) \mapsto a(x, \xi)$ as a local function of a Gaussian field. We recall that a scalar white noise⁹ on \mathbb{R}^d of variance $\sigma^2 \in (0, +\infty)$ is a family of real random variables $(W(f), f \in L^2(\mathbb{R}^d))$ such that:

- W is a linear form almost surely.
- For all $f \in L^2(\mathbb{R}^d)$, $W(f) \sim$ ¹⁰ $\mathcal{N}(0, \sigma^2 \|f\|_{L^2(\mathbb{R}^d)}^2)$.

For a rigorous construction of a scalar white noise, we refer to [8, Proposition 5.14]. Now, for a given scalar white noise W of variance 1, we consider a kernel m satisfying for some $C > 0$, and $\beta > 0$, $|m(x)| \leq C(1 + |x|)^{-\frac{1}{2}(d+\beta)}$ as well as a smooth function A from \mathbb{R}^d into the set (1.1.10) and we define

$$a(x, \xi) = A(m \star W(x))(s + |\xi|^{p-2})\xi. \quad (1.1.13)$$

The convolution $g := m \star W$ corresponds to an infinite dimensional Gaussian field, with covariance function $c(x) = \mathbb{E}[g(x)g(0)] = m \star m(x)$. Thus, we call (1.1.13) a Gaussian model of randomness with correlation c . Note that the bound on m implies that¹¹ $|c(x)| \lesssim_{m,C} (1 + |x|)^{-\beta}$. This class includes for instance log-normal random coefficients of the form

$$A = b + \frac{e^{-\kappa(g-\gamma)}}{c + e^{-\kappa(g-\gamma)}},$$

where $b, c > 0$ and $\kappa, \gamma \in \mathbb{R}$.

The qualitative homogenization of the problem (1.1.4) aims at characterizing the asymptotic behaviour of $(u_\varepsilon, \nabla u_\varepsilon)$ as $\varepsilon \downarrow 0$. More precisely, we ask the following questions:

⁹Similar model can be consider as well using vector valued white noise, see [8, Definition 5.5].

¹⁰ $X \sim Y$ means that the random variables X and Y have the same distribution.

¹¹For $\beta = d$, a logarithmic correction $|m(x)| \leq c(1 + |x|)^{-\frac{1}{2}(d+\beta)} \log^{-\frac{1}{2}}(1 + |x|)$ is needed.

- Do we have the convergence of the sequence $(u_\varepsilon, \nabla u_\varepsilon)$? In which topology?
- If $(\bar{u}, \nabla \bar{u}) = \lim_{\varepsilon \downarrow 0} (u_\varepsilon, \nabla u_\varepsilon)$ exists, how can we characterize \bar{u} ?
- Does \bar{u} solve a problem of the type (1.1.4)? If yes, what is the structure of the operator?

At first sight, the convergence of $(u_\varepsilon, \nabla u_\varepsilon)$ (at least along a subsequence) is immediate: the monotonicity conditions (1.1.5) and (1.1.6) yield the energy estimate $\|\nabla u_\varepsilon\|_{L^p(\Omega)} \lesssim_{\lambda,s} \|f\|_{L^p(\Omega)}$, which in turn implies (together with Rellich–Kondrachov’s theorem) the existence of $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$ for which (up to an extraction) $u_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \bar{u}$ in $L^p(\Omega, \mathbb{R}^m)$ and $\nabla u_\varepsilon \rightharpoonup \nabla \bar{u}$ in $L^p(\Omega, \mathbb{R}^{m \times d})$. The characterization of \bar{u} , however, is more subtle. Rigorous mathematical studies started in the 1970’s, with contributions of the French, the Italian and the Russian schools, in the linear elliptic setting, that is for the map $a(x, \xi) = a(x)\xi$ with a periodic or realization of a random distribution. As references, we cite in particular the works of Murat and Tartar in [131, 151], Bakhvalov in [19, 20], De Giorgi and Spagnolo in [51], Tartar in [150], Kozlov in [104] and Papanicolaou and Varadhan in [139] as well as Allaire in [3]. In the general nonlinear setting (1.1.4), we cite in particular the works of Dal Maso and Modica in [119, 120] (in the variational framework $a = \nabla_\xi V$, with V convex), Braides in [35], Braides and Defranceschi in [36] as well as Müller in [130] (in the variational and general framework of integral functionals with p -growth and periodic integrands), Chiadò Piat, Dal Maso and Defranceschi in [142], Dal Maso and Defranceschi in [118]. For a nice summary of those results, we refer to the textbooks [148] of Shen and [100] of Jikov, Kozlov and Oleĭnik. In our context, the homogenization result can be summarized as follows:

Theorem 1 (Qualitative homogenization of (1.1.4)). *The sequence $(u_\varepsilon)_{\varepsilon>0}$ converges (strongly in $L^p(\Omega, \mathbb{R}^m)$ and weakly in $W^{1,p}(\Omega, \mathbb{R}^m)$) to some $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$ almost surely. In addition, there exists a deterministic map $\bar{a} : \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d}$ such that we have the weak convergence of the flux $a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon) \rightharpoonup \bar{a}(\nabla \bar{u})$ in $L^p(\Omega, \mathbb{R}^m)$ and \bar{u} solves*

$$-\nabla \cdot \bar{a}(\nabla \bar{u}) = \nabla \cdot f, \quad (1.1.14)$$

with the same boundary conditions than the one applied to (1.1.4). Furthermore, the map \bar{a} is monotone and satisfies, a priori, a weaker¹² version of (1.1.5), namely there exists a constant $C > 0$ depending on λ and s such that for all $\xi_1, \xi_2 \in \mathbb{R}^{m \times d}$

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \frac{1}{C}(s + |\xi_1 - \xi_2|^{p-2})|\xi_1 - \xi_2|^2, \quad (1.1.15)$$

and

$$|\bar{a}(\xi_1) - \bar{a}(\xi_2)| \leq C(s + |\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|. \quad (1.1.16)$$

The convergence of ∇u_ε in Theorem 1 is only weak in $L^p(\Omega, \mathbb{R}^{m \times d})$ and, in fact, cannot be expected to hold strongly in $L^p(\Omega, \mathbb{R}^{m \times d})$ due to the oscillations at scale ε of the variation of u_ε .

¹²Indeed, (1.1.5) implies (1.1.15).

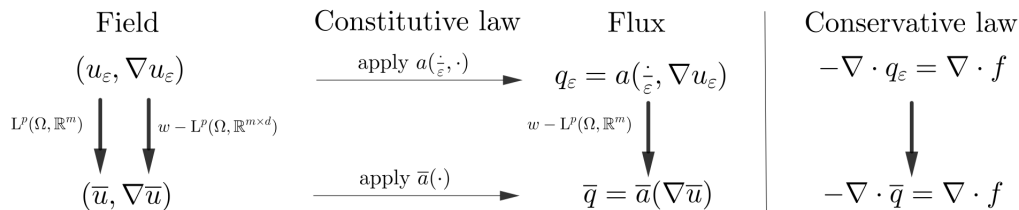
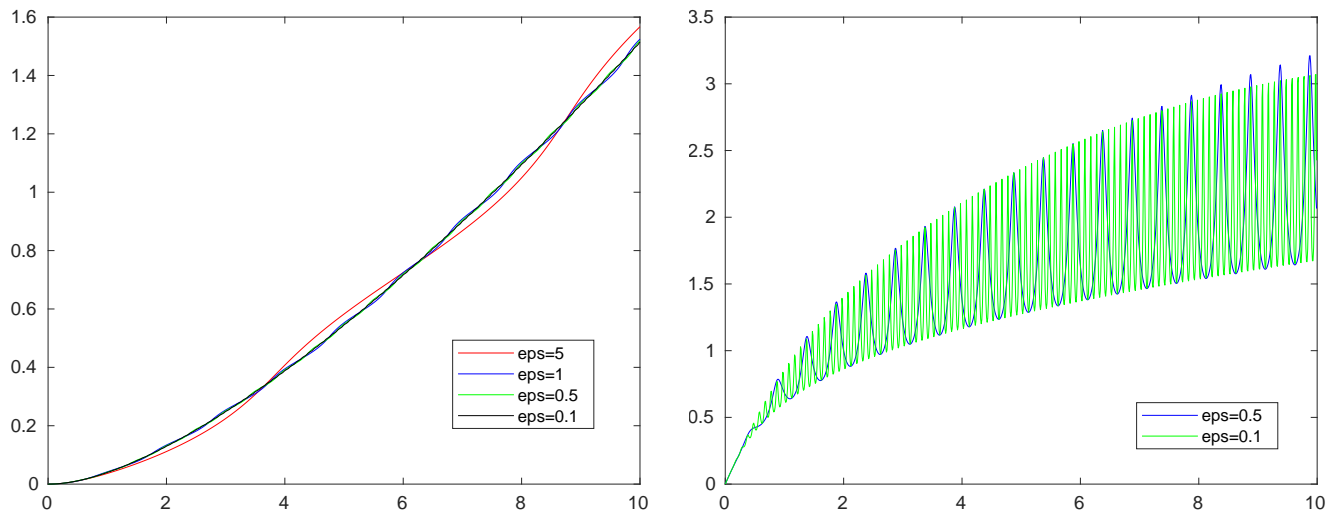


Figure 1.6: A diagram of the homogenization process of (1.1.4).

Figure 1.7: A one-dimensional illustration of the homogenization Theorem 1, for the equation $-u_\varepsilon'' - ((1 + 0.8 \sin(\frac{2\pi x}{\varepsilon}))(u'_\varepsilon)^3)' = -1$ with Neumann boundary conditions. The figure on the left is the plot of the solution u_ε and on the right the plot of the derivative u'_ε .

It tells us that a correction of ∇u_ε is needed to take into account the oscillations at the microscopic scale. We explain in the next section how we can formally reconstruct a posteriori the oscillations of ∇u_ε given $\nabla \bar{u}$ and some intrinsic quantities, in such a way that it is close to ∇u_ε in the strong topology of $L^p(\Omega, \mathbb{R}^{m \times d})$. A refinement of Theorem 1 will then be given.

1.1.3 The formal two-scale expansion

In this section, we develop formal arguments to reconstruct the oscillations of ∇u_ε as $\varepsilon \downarrow 0$, given $\nabla \bar{u}$. For the ease of the reading, we investigate the simple nonlinear model given by a p -Laplacian (for $m = 1$) regularized at 0:

$$a(x, \xi) := A(x)(1 + |\xi|^{p-2})\xi,$$

and we will use the short-hand notation $a_0(x, \xi) = A(x)|\xi|^{p-2}\xi$ for the p -Laplacian part. In addition, we use the general notation $\langle \cdot \rangle$ to denote either the average on the periodic cell (for periodic operator) or the expectation in the random setting.

The reconstruction of the behaviour of u_ε is based on a "two-scale expansion", that we formally establish hereafter. Based on Figure 1.1.2, we observe that u_ε varies at a macroscopic and a microscopic scale (at scale ε , revealed by the oscillations). We translate this observation by formally

assuming that u_ε depends on two variables: a slow variable x (macroscopic variable) and a fast-variable $y = \frac{x}{\varepsilon}$ (microscopic variable), that we treat as two independent variables. This dependence is made explicit through a formal expansion of u_ε into a power series in ε , where each term of the sum is a function depending on x and y . The expansion takes the form of the following Ansatz:

$$u_\varepsilon(x) \sim \sum_{n=0}^{+\infty} \varepsilon^n u_n(x, \frac{x}{\varepsilon}), \quad (1.1.17)$$

where we do not worry about summability at this stage. The functions $u_n = u_n(x, y)$ are the unknowns that we have to determine. For simplicity, we restrict ourself to identify the first order expansion in (1.1.17) which, in other words, means that we only identify u_0 and u_1 . For the formal derivation of higher expansion, we refer to [140].

First of all, we plug the Ansatz (1.1.17) into the equation (1.1.4). To do so, we observe that the chain rule, for all map $F = F(x, y)$, give us

$$\nabla[F(\cdot, \frac{\cdot}{\varepsilon})](x) = \nabla_x F(x, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_y F(x, \frac{x}{\varepsilon}). \quad (1.1.18)$$

By considering the second-order expansion in (1.1.17) and by computing the gradient of (1.1.17) using the rule (1.1.18), that in turn we plug into (1.1.32), we obtain

$$\begin{aligned} & \varepsilon^{-1} \nabla_y \cdot a(\frac{x}{\varepsilon}, \frac{1}{\varepsilon} \nabla_y u_0(x, \frac{x}{\varepsilon}) + \nabla_x u_0(x, \frac{x}{\varepsilon}) + \nabla_y u_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x u_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_y u_2(x, \frac{x}{\varepsilon}) + \varepsilon^2 \nabla_x u_2(x, \frac{x}{\varepsilon})) \\ & + \nabla_x \cdot a(\frac{x}{\varepsilon}, \frac{1}{\varepsilon} \nabla_y u_0(x, \frac{x}{\varepsilon}) + \nabla_x u_0(x, \frac{x}{\varepsilon}) + \nabla_y u_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x u_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_y u_2(x, \frac{x}{\varepsilon}) + \varepsilon^2 \nabla_x u_2(x, \frac{x}{\varepsilon})) \\ & = \nabla \cdot f. \end{aligned} \quad (1.1.19)$$

Now, the idea is to expand the left-hand side of (1.1.19) in powers of ε and then do a formal identification. By noticing that $\xi \mapsto a_0(\cdot, \xi)$ is smooth, our main tool is Taylor expansions. To begin with, using that a_0 , Da_0 and D^2a_0 are respectively $p-1$, $p-2$ and $p-3$ homogeneous¹³, one has for all $X_1, X_2 \in \mathbb{R}^{m \times d}$

$$\begin{aligned} a_0(y, \frac{1}{\varepsilon} X_1 + X_2) &= \varepsilon^{-p+1} a_0(y, X_1 + \varepsilon X_2) \\ &= \varepsilon^{-p+1} a_0(y, X_1) + \varepsilon^{-p+2} Da_0(y, X_1) X_2 + \mathcal{O}(\varepsilon^{-p+3}). \end{aligned} \quad (1.1.20)$$

By applying (1.1.20) with $X_1 = \nabla_y u_0(x, y)$ and $X_2 = \nabla_x u_0(x, y) + \nabla_y u_1(x, y) + \varepsilon \nabla_x u_1(x, y) + \varepsilon \nabla_y u_2(x, y) + \varepsilon^2 \nabla_x u_2(x, y)$, we observe that the term which appears with a factor ε^{-p} in (1.1.19) has to vanish, namely

$$-\nabla_y \cdot a_0(y, \nabla_y u_0) = 0,$$

so that the monotonicity of a_0 implies $\nabla_y u_0 = 0$ and therefore u_0 does not depend on the fast-variable y . Next, we perform a Taylor expansion in form of, for all $X_1, X_2, X_3 \in \mathbb{R}^{m \times d}$

$$a_0(y, X_1 + \varepsilon X_2 + \varepsilon^2 X_3) = a_0(y, X_1) + \varepsilon Da_0(y, X_1) X_2 + \varepsilon^2 (\frac{1}{2} D^2 a_0(y, X_1) X_2 \cdot X_2 + Da_0(y, X_1) X_3) + \mathcal{O}(\varepsilon^3),$$

that we use in (1.1.19) for $X_1 = \nabla_x u_0(x, \frac{x}{\varepsilon}) + \nabla_y u_1(x, \frac{x}{\varepsilon})$, $X_2 = \nabla_x u_1(x, \frac{x}{\varepsilon}) + \nabla_y u_2(x, \frac{x}{\varepsilon})$ and $X_3 = \nabla_x u_2(x, y)$, to deduce by a formal identification:

- At order ε^{-1} :

$$-\nabla_y \cdot a(y, \nabla_x u_0 + \nabla_y u_1) = 0, \quad (1.1.21)$$

¹³We denote by D the derivative with respect to ξ .

- At order ε^0 :

$$-\nabla_x \cdot a(y, \nabla_x u_0 + \nabla_y u_1) - \nabla_y \cdot Da(\nabla_x u_0 + \nabla_y u_1)(\nabla_x u_1 + \nabla_y u_2) = \nabla \cdot f. \quad (1.1.22)$$

In the equation (1.1.21), x is a parameter so that the equation (1.1.21) identifies u_1 and the latter can be seen as a function of $\nabla_x u_0$. Indeed, let us slightly reinterpret (1.1.21). We define a new function ϕ_ξ , depending on the direction $\xi \in \mathbb{R}^{m \times d}$, which solves the equation

$$-\nabla \cdot a(\cdot, \xi + \nabla \phi_\xi) = 0. \quad (1.1.23)$$

The function ϕ_ξ is called the corrector in the direction $\xi \in \mathbb{R}^{m \times d}$. From (1.1.21), one can see that u_1 can be generated with help of the corrector in the direction $\nabla_x u_0$, namely

$$u_1(x, y) = \phi_{\nabla_x u_0(x)}(y). \quad (1.1.24)$$

The complexity of the well-posedness of the equation (1.1.23) depends on the context. In the T -periodic setting, it is easy to give a sense to (1.1.23), since the equation is naturally posed on the torus $\mathbb{T}_T := \mathbb{R}^d / T\mathbb{Z}^d$. This way, using the monotonicity of a , the equation (1.1.23) has a unique periodic solution with vanishing mean. In the random setting, the story is different and (1.1.23) has to be set in \mathbb{R}^d . Therefore the well-posedness is more subtle and the stationarity assumption (1.1.8) on the random distribution plays an important role. Indeed, morally speaking, the stationarity assumption allows us to set the equation (1.1.23) on the probability space and construct, from the monotonicity of a , a solution such that $\nabla \phi_\xi$ is stationary¹⁴, of vanishing expectation, anchored by $\int_{\mathbb{B}_1} \phi_\xi = 0$ and satisfying

$$\mathbb{E}[|\nabla \phi_\xi|^2 + |\nabla \phi_\xi|^p] \lesssim_{\lambda, s} |\xi|^2 + |\xi|^p. \quad (1.1.25)$$

Hence, in the random setting, (1.1.23) has a distributional sense in \mathbb{R}^d (ϕ_ξ belongs to $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$) and a weak sense in probability. For more precision on the rigorous proof of the well-posedness of (1.1.23), we refer to [138, Chapter 15] and [80, Lemma 1]. Finally, the corrector ϕ_ξ is sublinear at infinity¹⁵, as a consequence of the fact that $\nabla \phi_\xi$ is stationary, $\mathbb{E}[\nabla \phi_\xi] = 0$ and (1.1.25) holds as well as the ergodicity assumption of the random distribution (1.1.9). For more details, we refer to [80, Proof of Corollary 1]. The sub-linear property of the correctors plays an important role in the homogenization theory, since it controls the error between u_ε and u_0 at first order. Indeed, the first order expansion in (1.1.17) together (1.1.24) gives $u_\varepsilon(x) - u_0(x) \approx \varepsilon \phi_{\nabla_x u_0(x)}(\frac{x}{\varepsilon})$ which vanishes as $\varepsilon \downarrow 0$ by sub-linearity.

It remains to identify u_0 and this is done via the equation (1.1.22). To do so, we average the equation (1.1.22) with respect to the fast-variable y . By noticing that $\langle \nabla_y \cdot Da_0(\nabla_x u_0 + \nabla_y u_1)(\nabla_x u_1 + \nabla_y u_2) \rangle = 0$ (which comes from an integration by parts in the periodic setting or the stationarity in the random setting¹⁶), averaging (1.1.22) has the effect of:

$$-\nabla_x \cdot \langle a(\cdot, \nabla_x u_0 + \nabla_y u_1) \rangle = \nabla \cdot f.$$

We slightly rewrite the latter equation, using (1.1.24), by making the homogenized operator $\bar{a}(\xi) = \langle a(\cdot, \xi + \nabla \phi_\xi) \rangle$ appear:

$$-\nabla \cdot \bar{a}(\nabla_x u_0) = \nabla \cdot f. \quad (1.1.26)$$

¹⁴We say that $F = F(a, x)$ is stationary if it has the shift-invariance property: for all $z \in \mathbb{R}^d$, $F(a, \cdot + z) = F(a(\cdot + z), \cdot)$.

¹⁵Meaning that $\varepsilon^2 \int_{\mathbb{B}_1(\frac{x}{\varepsilon})} |\phi_\xi|^2 \xrightarrow{\varepsilon \downarrow 0} 0$.

¹⁶Formally: if $F = F(a, y)$ is stationary, $\mathbb{E}[\partial_{y_i} F] = \mathbb{E}[\lim_{h \downarrow 0} \frac{F(a, h e_i) - F(a, 0)}{h}] = \lim_{h \downarrow 0} \mathbb{E}[\frac{F(a, h e_i) - F(a, 0)}{h}] = 0$, which is for instance rigorous if F is bounded.

The equation (1.1.26) (that we call the homogenized equation) combined with suitable boundary conditions uniquely defines u_0 , that we justify by showing that the homogenized operator \bar{a} is monotone (in the sense of (1.1.15) and (1.1.16)). We fix $\xi_1, \xi_2 \in \mathbb{R}^{m \times d}$. For the lower bound (1.1.15), we use the corrector equation (1.1.23), the strong monotonicity of a (1.1.5) and the vectorial inequality

$$|\xi_1 - \xi_2|^2(1 + |\xi_1|^{p-2} + |\xi_2|^{p-2}) \gtrsim_p |\xi_1 - \xi_2|^2(1 + |\xi_1 - \xi_2|^{p-2}), \quad (1.1.27)$$

in form of

$$\begin{aligned} (\bar{a}(\xi_1) - \bar{a}(\xi_2)) \cdot (\xi_1 - \xi_2) &= \langle (a(\cdot, \xi + \nabla\phi_{\xi_1}) - a(\cdot, \xi_2 + \nabla\phi_{\xi_2})) \cdot (\xi_1 - \xi_2) \rangle \\ &\stackrel{(1.1.23)}{=} \langle (a(\cdot, \xi + \nabla\phi_{\xi_1}) - a(\cdot, \xi_2 + \nabla\phi_{\xi_2})) \cdot (\xi_1 + \nabla\phi_{\xi_1} - (\nabla\phi_{\xi_2} + \xi_2)) \rangle \\ &\stackrel{(1.1.5), (1.1.27)}{\gtrsim_p} \frac{1}{\lambda} \langle (1 + |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})|^{p-2}) |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})|^2 \rangle, \end{aligned}$$

which turns into, by applying Jensen's inequality and using the fact that $\langle \nabla\phi_\xi \rangle = 0$ for all $\xi \in \mathbb{R}^{m \times d}$:

$$\begin{aligned} (\bar{a}(\xi_1) - \bar{a}(\xi_2)) \cdot (\xi_1 - \xi_2) &\gtrsim_p \frac{1}{\lambda} (1 + |\xi_1 + \langle \nabla\phi_{\xi_1} \rangle - (\xi_2 + \langle \nabla\phi_{\xi_2} \rangle)|^{p-2}) |\xi_1 + \langle \nabla\phi_{\xi_1} \rangle - (\xi_2 + \langle \nabla\phi_{\xi_2} \rangle)|^2 \\ &= \frac{1}{\lambda} (1 + |\xi_1 - \xi_2|^{p-2}) |\xi_1 - \xi_2|^2. \end{aligned}$$

For the upper bound (1.1.16), we use the following energy estimate

$$\langle |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})|^2 (1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2}) \rangle \lesssim_{p,\lambda,s} (s + |\xi_1|^{p-2} + |\xi_2|^{p-2}) |\xi_1 - \xi_2|^2, \quad (1.1.28)$$

which yields, together with the bound (1.1.6) of a and Cauchy-Schwarz' inequality as well as (1.1.25) (which holds in general for $\langle \cdot \rangle$)

$$\begin{aligned} |\bar{a}(\xi_1) - \bar{a}(\xi_2)| &\stackrel{(1.1.6)}{\lesssim} C \langle |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})| (1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2}) \rangle \\ &\leq \langle |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})| (1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2}) \rangle^{\frac{1}{2}} \\ &\quad \times \langle 1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2} \rangle^{\frac{1}{2}} \\ &\stackrel{(1.1.28), (1.1.25)}{\lesssim_{p,\lambda,s}} (1 + |\xi_1|^{p-2} + |\xi_2|^{p-2}) |\xi_1 - \xi_2|^2. \end{aligned}$$

We briefly argue in favour of (1.1.28). Using the corrector equation (1.1.23), we have $-\nabla \cdot (a(\cdot, \xi_1 + \nabla\phi_{\xi_1}) - a(\cdot, \xi_2 + \nabla\phi_{\xi_2})) = 0$, so that by testing the equation with $\phi_{\xi_1} - \phi_{\xi_2}$, we arrive at

$$\langle (a(\cdot, \xi + \nabla\phi_{\xi_1}) - a(\cdot, \xi_2 + \nabla\phi_{\xi_2})) \cdot (\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})) \rangle = \langle a(\cdot, \xi + \nabla\phi_{\xi_1}) - a(\cdot, \xi_2 + \nabla\phi_{\xi_2}) \cdot (\xi_1 - \xi_2) \rangle,$$

which, using (1.1.5) and (1.1.6), turns into

$$\begin{aligned} &\frac{1}{\lambda} \langle |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})|^2 (1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2}) \rangle \\ &\leq \lambda |\xi_1 - \xi_2| \langle |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})| (1 + |\xi_1 + \nabla\phi_{\xi_1}|^{p-2} + |\xi_2 + \nabla\phi_{\xi_2}|^{p-2}) \rangle, \end{aligned}$$

which gives (1.1.28) by Young's inequality in form of $|\xi_1 - \xi_2| |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})| \leq \frac{\lambda^2}{2} |\xi_1 - \xi_2|^2 + \frac{1}{2\lambda^2} |\xi_1 + \nabla\phi_{\xi_1} - (\xi_2 + \nabla\phi_{\xi_2})|^2$ and (1.1.25). Note that, a priori, except for the particular case of $p = 2$, the homogenized operator satisfies a weaker monotonicity condition than the original ones (1.1.5) and (1.1.6), which is enough for existence and uniqueness (under suitable boundary

conditions). It does also imply a regularity theory for the operator $-\nabla \cdot \bar{a}(\nabla u)$ in the case of scalar equations (that is for $m = 1$) for certain range of p (see for more precisions Section 3.1.1 of Chapter 3). The question of showing that the homogenized operator satisfies also the strong monotonicity conditions (1.1.5) and (1.1.6) is actually a challenging question, and we refer to Theorem 13 and Theorem 14 for a partial answer to this question and to Chapter 3 for a short discussion about this issue.

In conclusion, we have totally characterised the first order expansion in (1.1.17), called the (first-order) two-scale expansion of u_ε , which reads

$$u_\varepsilon(x) \underset{\varepsilon \downarrow 0}{\approx} u_0(x) + \varepsilon \phi_{\nabla u_0(x)}\left(\frac{x}{\varepsilon}\right). \quad (1.1.29)$$

Formula (1.1.29) give a drastic reduction of complexity since \bar{u} is the solution of a constant-coefficient equation and ϕ_ξ does not depend on f . We have the following refinement of Theorem 1.

Theorem 2 (Qualitative homogenization of (1.1.4)-Refinement). *For any $\xi \in \mathbb{R}^{m \times d}$, there exists a unique almost sure distributional solution $\phi_\xi \in W_{loc}^{1,p}(\mathbb{R}^d)$ of the corrector problem (1.1.23) posed in \mathbb{R}^d , anchored by $\int_{B_1} \phi_\xi = 0$ and its gradient $\nabla \phi_\xi$ is stationary, has vanishing expectation $\mathbb{E}[\nabla \phi_\xi] = 0$, and satisfies (1.1.25).*

The homogenized operator \bar{a} has an explicit formula which can be expressed with help of the correctors: for all $\xi \in \mathbb{R}^{m \times d}$

$$\bar{a}(\xi) = \mathbb{E}[a(0, \xi + \nabla \phi_\xi)]. \quad (1.1.30)$$

In addition, in the linear setting, we define the two-scale expansion¹⁷:

$$u_\varepsilon^{2sc} := \bar{u} + \varepsilon \phi_{\nabla \bar{u}}\left(\frac{\cdot}{\varepsilon}\right), \quad (1.1.31)$$

where \bar{u} is given in (1.1.14), and we have that $(\nabla u_\varepsilon^{2sc} - \nabla u_\varepsilon)_{\varepsilon > 0}$ converges, almost-surely, to 0 in $L^2(\Omega, \mathbb{R}^{m \times d})$.

We now illustrate the asymptotic (1.1.29) by a complete one-dimensional example ($d = m = 1$) and we compute the homogenized operator (1.1.30) as well as the two-scale expansion (1.1.31) in the two-dimensional example ($d = 2$ and $m = 1$) of a layered material.

1.1.4 Two examples of explicit homogenization

Example 1: A one-dimensional example

We illustrate the asymptotics (1.1.29) with the model $a(x, \xi) = \xi + A(x)|\xi|$ for $d = m = 1$. More precisely, we consider $A \in L^\infty([0, 1])$ such that $0 < c_1 \leq A \leq c_2$ (for some fixed constants $c_1, c_2 > 0$) seen as a realization of a random distribution \mathbb{P} satisfying (1.1.8) and (1.1.9). Let $f \in C^0([0, 1])$ such that $f \leq f(0)$ and we consider the sequence $(u_\varepsilon)_{\varepsilon > 0} \subset W^{1,3}([0, 1]) \subset C^0([0, 1])$ where for all $\varepsilon > 0$, u_ε is the weak solution of the nonlinear equation

$$\begin{cases} -u_\varepsilon'' - (A(\frac{\cdot}{\varepsilon})|u_\varepsilon'|u_\varepsilon')' = f' & \text{in } (0, 1], \\ u_\varepsilon(0) = u_\varepsilon'(0) = 0. \end{cases} \quad (1.1.32)$$

The equation (1.1.32) can be solved explicitly. Indeed, a first integration and $u_\varepsilon'(0) = 0$ give us $-u_\varepsilon' - A(\frac{\cdot}{\varepsilon})|u_\varepsilon'|u_\varepsilon' = f - f(0)$. Therefore, since $f \leq f(0)$, u_ε' is non-negative, which yields

¹⁷Because of measurability issues, the definition of the two-scale expansion (only in the nonlinear setting) has to be slightly modified using a piecewise constant approximation. We refer to Theorem 12 for more details.

$u'_\varepsilon + A(\frac{\cdot}{\varepsilon})(u'_\varepsilon)^2 = f(0) - f$. By solving this equation and by keeping only the relevant solution (that is the one compatible with $u'_\varepsilon \geq 0$) we obtain: for all $x \in [0, 1]$

$$u_\varepsilon(x) = -\frac{1}{2} \int_0^x \frac{1}{A(\frac{t}{\varepsilon})} dt + \int_0^x \sqrt{\frac{f(0) - f(t)}{A(\frac{t}{\varepsilon})} + \frac{1}{4A^2(\frac{t}{\varepsilon})}} dt. \quad (1.1.33)$$

From the explicit formula (1.1.33), the asymptotics of $(u_\varepsilon)_{\varepsilon>0}$ is clear: using the ergodicity assumption (1.1.9) on the distribution of A ¹⁸, one has

$$u_\varepsilon(x) \xrightarrow{\varepsilon \downarrow 0} \bar{u}(x),$$

with

$$\bar{u}(x) := -\frac{1}{2} \mathbb{E} \left[\frac{1}{A} \right] x + \int_0^x \mathbb{E} \left[\sqrt{\frac{f(0) - f(t)}{A} + \frac{1}{4A^2}} \right] dt. \quad (1.1.34)$$

We now identify the nonlinear equation solved by \bar{u} . Let us slightly rewrite \bar{u}' with help of the function $\Phi : \xi \in \mathbb{R}_+ \mapsto -\frac{1}{2} \mathbb{E}[\frac{1}{A}] + \mathbb{E} \left[\sqrt{\frac{\xi}{A} + \frac{1}{4A^2}} \right]$, namely $\bar{u}' = \Phi(f(0) - f)$. Now, observe that Φ is invertible (since $\Phi' > 0$), such a way that $\Phi^{-1}(\bar{u}') = f(0) - f$ and thus

$$\begin{cases} -(\bar{a}(\bar{u}'))' = f' & \text{in } (0, 1], \\ \bar{u}(0) = 0 \text{ and } \bar{u}'(0) = 0, \end{cases} \quad (1.1.35)$$

for $\bar{a} := \Phi^{-1}$. The structure of the equation (1.1.35) is the same as (1.1.32), namely $\bar{a} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strongly monotone in the sense of (1.1.5) and (1.1.6) (for $p = 3$), that we prove hereafter. Since $\bar{a} \in C^1(\mathbb{R}_+)$, we first reduce the proof to an estimate on \bar{a}' : the monotonicity will be inferred by showing that for all $\xi \in \mathbb{R}^+$,

$$\bar{a}'(\xi) = (\Phi^{-1})'(\xi) \sim^{19} 1 + 2\xi. \quad (1.1.36)$$

Indeed, knowing (1.1.36) implies the strong monotonicity via the fundamental theorem of calculus in form of: for all $\xi_1, \xi_2 \in \mathbb{R}_+$,

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2))(\xi_1 - \xi_2) = (\xi_1 - \xi_2) \int_{\xi_2}^{\xi_1} \bar{a}'(t) dt \stackrel{(1.1.36)}{\sim} (\xi_1 - \xi_2) \int_{\xi_2}^{\xi_1} 1 + 2t dt = (1 + \xi_1 + \xi_2)(\xi_1 - \xi_2)^2.$$

Also, since $(\Phi^{-1})' = \frac{1}{\Phi' \circ \Phi^{-1}}$, (1.1.36) will finally be justified by showing that $\Phi'(\Phi^{-1}(\xi)) \sim (1 + 2\xi)^{-1}$.

For $\alpha = (\mathbb{E}[\frac{1}{A}] \mathbb{E}[\frac{1}{\sqrt{A}}]^{-1})^2$, one has

$$\Phi(\xi^2 + \alpha) = -\frac{1}{2} \mathbb{E} \left[\frac{1}{A} \right] + \mathbb{E} \left[\sqrt{\frac{\xi^2 + \alpha}{A} + \frac{1}{4A^2}} \right] \geq \frac{1}{2} \xi \mathbb{E} \left[\frac{1}{\sqrt{A}} \right] + \frac{1}{2} \left(\sqrt{\alpha} \mathbb{E} \left[\frac{1}{\sqrt{A}} \right] - \mathbb{E} \left[\frac{1}{A} \right] \right) \gtrsim \xi,$$

and similarly, $\Phi(\xi^2 + \alpha) \lesssim \xi$. Therefore, since Φ^{-1} is non-decreasing (because Φ is), one has $\Phi^{-1}(\xi) \sim \xi^2 + \alpha$. Finally, since

$$\Phi'(\xi) = \frac{1}{2} \mathbb{E} \left[\left(A \sqrt{\frac{\xi}{A} + \frac{1}{4A^2}} \right)^{-1} \right] \sim (\xi + 1)^{-\frac{1}{2}},$$

¹⁸We can not directly apply the assumption (1.1.9). However, a standard piecewise constant approximation of f combined with the assumption (1.1.9) show the convergence. The details are left to the reader.

¹⁹The notation $A \sim B$ means that there exist a constant γ such that $\frac{1}{\gamma} B \leq A \leq \gamma B$.

we indeed have $\Phi'(\Phi^{-1}(\xi)) \sim (1 + 2\xi)^{-1}$, where the constant in \sim depends on α , c_1 and c_2 .

Thus, (1.1.36) holds and $\bar{u} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strongly monotone in the sense of (1.1.5) and (1.1.6) (for $p = 3$). Note that this proof is very specific to the one-dimensional case, and in general we cannot ensure such a property for $d \geq 2$ (see Chapter 3 for a discussion about this issue).

We now investigate the asymptotic behaviour of the derivative u'_ε . Differentiating Formula (1.1.33) yields: for all $x \in [0, 1]$

$$u'_\varepsilon(x) = -\frac{1}{2A(\frac{x}{\varepsilon})} + \sqrt{\frac{f(0) - f(t)}{A(\frac{x}{\varepsilon})} + \frac{1}{4A^2(\frac{x}{\varepsilon})}}, \quad (1.1.37)$$

and thus by the ergodic assumption (1.1.9),

$$u'_\varepsilon \rightharpoonup \bar{u}' \text{ in } L^3([0, 1]), \quad (1.1.38)$$

and the convergence is not strong in $L^3([0, 1])$. The structure of the oscillations of u'_ε can be made more precise with help of \bar{u}' and an explicit corrector. Indeed, observe that from (1.1.34), one has $f(0) - f = \Phi^{-1}(\bar{u}')$ so that we can rewrite \bar{u}'_ε given by (1.1.37) as

$$u'_\varepsilon(x) = -\frac{1}{2A(\frac{x}{\varepsilon})} + \sqrt{\frac{\Phi^{-1}(\bar{u}')}{A(\frac{x}{\varepsilon})} + \frac{1}{4A^2(\frac{x}{\varepsilon})}},$$

which can be itself rewritten as:

$$u'_\varepsilon(x) = \bar{u}'(x) + \phi'_{\bar{u}'}(\frac{x}{\varepsilon}), \quad (1.1.39)$$

where ϕ_ξ denotes the corrector²⁰, defined in the direction $\xi \in \mathbb{R}_+$ by

$$\phi_\xi(x) := -\frac{1}{2} \int_0^x \frac{1}{A(t)} dt + \int_0^x \sqrt{\frac{\Phi^{-1}(\xi)}{A(t)} + \frac{1}{4A^2(t)}} dt - \xi x. \quad (1.1.40)$$

Observe that ϕ_ξ is indeed the corrector (anchored at the origin via $\phi_\xi(0) = 0$) in the sense of (1.1.23), since a direct computation shows that for all $\xi \in \mathbb{R}_+$:

$$-\phi''_\xi - (A|\xi + \phi'_\xi|(\xi + \phi'_\xi))' = 0 \text{ in } \mathbb{R},$$

with, by definition of Φ , $\mathbb{E}[\phi'_\xi] = 0$. The equation (1.1.39) describes the oscillations of u'_ε , and this is a good hint for reconstructing the oscillations of u_ε via the two scale expansion:

$$u_\varepsilon^{2\text{sc}} := \bar{u} + \varepsilon \phi_{\bar{u}'}(\frac{\cdot}{\varepsilon}). \quad (1.1.41)$$

We now verify that (1.1.41) indeed reconstructs the oscillations of u_ε , in the sense that the sequence $(u_\varepsilon^{2\text{sc}} - u_\varepsilon)_{\varepsilon > 0}$ tends to 0 in $W^{1,3}([0, 1])$. First, the definition (1.1.40) of the corrector combined with the ergodic assumption (1.1.9) and the definition of Φ show that ϕ_ξ is sub-linear at infinity in the form of: for all $x \in [0, 1]$ and almost all A

$$\varepsilon \phi_\xi(\frac{x}{\varepsilon}) \xrightarrow{\varepsilon \downarrow 0} \int_0^x \Phi(\Phi^{-1}(\xi)) dt - \xi x = 0, \quad (1.1.42)$$

²⁰Note that since $\bar{u}' \geq 0$, we only need to construct the correctors in the directions $\xi \in \mathbb{R}^+$ in order to express the two-scale expansion (1.1.31), that we do for simplicity.

which in turn can be upgraded to $\varepsilon\phi_\xi(\frac{x}{\varepsilon}) \xrightarrow{\varepsilon\downarrow 0} 0$ in $L^3([0, 1])$ and almost surely by dominated convergence. Thus $u_\varepsilon^{2\text{sc}} - u_\varepsilon \xrightarrow{\varepsilon\downarrow 0} 0$ in $L^3([0, 1])$ and almost surely. Second, by differentiating (1.1.41) and using (1.1.39), we have

$$(u_\varepsilon^{2\text{sc}} - u_\varepsilon)' = \bar{u} + \phi'_{\bar{u}}(\frac{\cdot}{\varepsilon}) - u'_\varepsilon + \varepsilon \frac{\partial}{\partial \xi} \phi_\xi(\frac{\cdot}{\varepsilon})|_{\xi=\bar{u}'} \bar{u}'' \stackrel{(1.1.39)}{=} \varepsilon \frac{\partial}{\partial \xi} \phi_\xi(\frac{\cdot}{\varepsilon})|_{\xi=\bar{u}'} \bar{u}''.$$

The quantity $\frac{\partial}{\partial \xi} \phi_\xi$, that we call the linearized corrector, can be computed explicitly via (1.1.40): for all $x \in [0, 1]$

$$\begin{aligned} \frac{\partial}{\partial \xi} \phi_\xi(x) &= (\Phi^{-1})'(\xi) \int_0^x \frac{1}{2A(t) \sqrt{\frac{\Phi^{-1}(\xi)}{A(t)} + \frac{1}{4A^2(t)}}} dt - x \\ &= \frac{1}{\Phi'(\Phi^{-1}(\xi))} \int_0^x \frac{1}{2A(t) \sqrt{\frac{\Phi^{-1}(\xi)}{A(t)} + \frac{1}{4A^2(t)}}} dt - x \\ &= \mathbb{E} \left[\left(A \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}} \right)^{-1} \right]^{-1} \int_0^x \frac{1}{A(t) \sqrt{\frac{\Phi^{-1}(\xi)}{A(t)} + \frac{1}{4A^2(t)}}} dt - x, \end{aligned}$$

so that, from the ergodic assumption (1.1.9), the linearized corrector is sub-linear at infinity in form of: for all $x \in [0, 1]$ and almost all A

$$\varepsilon \frac{\partial}{\partial \xi} \phi_\xi(\frac{x}{\varepsilon}) \xrightarrow{\varepsilon\downarrow 0} \mathbb{E} \left[\left(A \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}} \right)^{-1} \right]^{-1} \int_0^x \mathbb{E} \left[\left(A \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}} \right)^{-1} \right] dx - x = 0, \quad (1.1.43)$$

and therefore $(u_\varepsilon^{2\text{sc}}(x) - u_\varepsilon(x))' \xrightarrow{\varepsilon\downarrow 0} 0$, which in turn can be upgraded to $(u_\varepsilon^{2\text{sc}} - u_\varepsilon)' \xrightarrow{\varepsilon\downarrow 0} 0$ in $L^3([0, 1])$ by dominated convergence. To conclude, we have shown that

$$\|u_\varepsilon^{2\text{sc}} - u_\varepsilon\|_{W^{1,3}([0,1])} \xrightarrow{\varepsilon\downarrow 0} 0 \text{ almost surely.} \quad (1.1.44)$$

Example 2: A two-dimensional example of a layered material

We study the effective conductivity of a material represented by the square $\Omega = [0, 1]^2$, for the nonlinear conductivity model

$$a((x_1, x_2), \xi) = (1 + |\xi|^2) \begin{pmatrix} A_1(x_1)\xi_1 \\ A_2(x_1)\xi_2 \end{pmatrix},$$

corresponding to the case of a layered material.



Figure 1.8: A layered material with conductivity matrix $\begin{pmatrix} a_0 & 0 \\ 0 & \tilde{a}_0 \end{pmatrix}$ in the white region and $\begin{pmatrix} a_1 & 0 \\ 0 & \tilde{a}_1 \end{pmatrix}$ in the black region, given conductivities $a_0, a_1, \tilde{a}_0, \tilde{a}_1 > 0$.

We start by computing the effective nonlinear conductivity model given by the formula (1.1.30). To do so, we solve the corrector equation (1.1.23) which, in our case, reads

$$\partial_{x_1}(A_1(x_1)(\xi_1 + \partial_{x_1}\phi_\xi)(1 + |\xi + \nabla\phi_\xi|^2)) + A_2(x_1)\partial_{x_2}((\xi_2 + \partial_{x_2}\phi_\xi)(1 + |\xi + \nabla\phi_\xi|^2)) = 0. \quad (1.1.45)$$

Since the material is heterogeneous in one direction, it is natural to seek a correction which depends only on x_1 , that is $\phi_\xi = \phi_\xi(x_1)$. Therefore, (1.1.45) turns into

$$\partial_{x_1}(A_1(x_1)(\xi_1 + \partial_{x_1}\phi_\xi)(1 + (\xi_1 + \partial_{x_1}\phi_\xi)^2 + \xi_2^2)) = 0,$$

which give us by integrating once,

$$A_1(x_1)(\xi_1 + \partial_{x_1}\phi_\xi)(1 + (\xi_1 + \partial_{x_1}\phi_\xi)^2 + \xi_2^2) = \bar{a}_1(\xi), \quad (1.1.46)$$

with $\bar{a}_1(\xi)$ that we have to determine. To do so, we define $\Psi_{\xi_2} : X \in \mathbb{R} \mapsto X(1 + X^2 + \xi_2^2)$. Observe that Ψ_{ξ_2} is invertible (since $\Psi'_{\xi_2} > 0$) and $\Psi_{\xi_2}^{-1}$ is given by Cardan's formula:

$$\Psi_{\xi_2}^{-1}(X) = \left(\frac{X}{2} + \frac{1}{2} \sqrt{X^2 + \frac{4(1 + \xi_2^2)^3}{27}} \right)^{\frac{1}{3}} + \left(\frac{X}{2} - \frac{1}{2} \sqrt{X^2 + \frac{4(1 + \xi_2^2)^3}{27}} \right)^{\frac{1}{3}}. \quad (1.1.47)$$

Therefore, by dividing the equation (1.1.46) by $A_1(x_1)$, applying $\Psi_{\xi_2}^{-1}$ and using that $\mathbb{E}[\partial_{x_1}\phi_\xi] = 0$, we deduce that $\bar{a}_1(\xi)$ is characterized by the nonlinear equation²¹

$$\xi_1 = \mathbb{E}[\Psi_{\xi_2}^{-1}(\frac{\bar{a}_1(\xi)}{A_1(0)})].$$

To conclude, using in addition that from (1.1.46) we have $\xi_1 + \partial_{x_1}\phi_\xi = \Psi_{\xi_2}^{-1}(\frac{\bar{a}_1(\xi)}{A_1(x_1)})$, the effective conductivity model is given by

$$\bar{a}(\xi) = \mathbb{E} \left[\begin{pmatrix} A_1(0)(\xi_1 + \partial_{x_1}\phi_\xi)(1 + (\xi_1 + \partial_{x_1}\phi_\xi)^2 + \xi_2^2) \\ A_2(0)\xi_2(1 + (\xi_1 + \partial_{x_1}\phi_\xi)^2 + \xi_2^2) \end{pmatrix} \right] = \begin{pmatrix} \bar{a}_1(\xi) \\ \bar{a}_2(\xi) \end{pmatrix}, \quad (1.1.48)$$

with $\bar{a}_2(\xi) = \mathbb{E}[A_2(0)\xi_2(1 + (\Psi_{\xi_2}^{-1}(\frac{\bar{a}_1(\xi)}{A_1(0)}))^2 + \xi_2^2)]$.

²¹The map $\Xi : X \mapsto \mathbb{E}[\Psi_{\xi_2}^{-1}(\frac{X}{A_1(0)})]$ is invertible since $\Xi' > 0$.

Finally, from (1.1.46), the corrector (anchored at the origin via $\phi_\xi(0) = 0$) can be expressed with help of $\bar{a}_1(\xi)$:

$$\phi_\xi(x_1) = \int_0^{x_1} \Psi_{\xi_2}^{-1}\left(\frac{\bar{a}_1(\xi)}{A_1(y)}\right)dy - \xi_1 x_1,$$

so that the two-scale expansion (1.1.31) is explicitly given by

$$u_\varepsilon^{2\text{sc}}(x_1, x_2) = \bar{u}(x_1, x_2) + \varepsilon \left(\int_0^{\frac{x_1}{\varepsilon}} \Psi_{\xi_2}^{-1}\left(\frac{\bar{a}_1(\nabla \bar{u}(x_1, x_2))}{A_1(y)}\right)dy - \xi_1 \frac{x_1}{\varepsilon} \right).$$

We see with this example that we have reduced the complexity of the problem, since now we only have to compute \bar{u} which solves a nonlinear elliptic equation in divergence form with a monotone operator given by (1.1.48).

In the next section, we turn to the quantitative theory of stochastic homogenization of (1.1.4) (the main focus of this work), and we present and explain the assumptions which lead to such a theory.

1.1.5 Quantitative ergodic assumptions for a quantitative homogenization theory

We present in this section the quantitative assumptions that allow to develop a quantitative theory of homogenization. First, we illustrate the quantitative theory for the model $a(x, \xi) = A(x)(1 + |\xi|^{p-2})\xi$ in the setting of the small contrast regime, in the sense that A is of the form $A = (1 + f(\delta \tilde{A}))\text{Id}$ with a smooth function f such that $f(0) = 0$ and for \tilde{A} stationary and ergodic (that we will specify to the Gaussian setting for the ease of the reading) and $\delta \ll 1$. Second, we present the general framework of quantitative homogenization.

The small contrast regime

We consider the particular model of the p -laplacian regularized at 0 (for $d \geq 2$)²²

$$a(x, \xi) = A(x)(1 + |\xi|^{p-2})\xi, \tag{1.1.49}$$

where A takes the specific form²³

$$A(x) = \left(1 + \frac{1}{\pi} \arctan(\pi \delta g(x))\right)\text{Id},$$

where g is a real valued Gaussian random variable with $\mathbb{E}[|g(0)|^2] = 1$ and $\delta \ll 1$. This perturbative regime allows us to linearise the problem with respect to δ in the corrector equation (1.1.23). It leads to a linear elliptic equation with constant coefficient where all the randomness is put in the right-hand side.

We have already seen that the ergodic assumption (1.1.9) is enough for the qualitative homogenization theory, as summarized in Theorem 1 and 2 and illustrated in Section 1.1.4. Naturally, for the quantitative homogenization, we need a quantification of ergodicity, which can be understood

²²For $d = 1$, one can make explicit computations (as in Subsection 1.1.4) and show that the convergence rate of the homogenization error is of order $\sqrt{\varepsilon}$.

²³The choice $f = \frac{1}{\pi} \arctan \pi \cdot$ is taken for convenience but any function such that $|f| < 1$, $f(0) = 0$ and $f'(0) = 1$ would work as well.

by imposing a convergence rate in (1.1.9). In our particular setting, we quantify the ergodicity, or in other words the decorrelation over large distances, by imposing an algebraic decay on the covariance function:

$$|c(x, y)| = |\mathbb{E}[g(x)g(y)]| \sim (1 + |x - y|)^{-d-\alpha}, \quad (1.1.50)$$

for some $\alpha > 0$.²⁴ Let us now show that, in the regime $\delta \downarrow 0$, this quantitative assumption implies a convergence rate on the homogenization error $u_\varepsilon^{2\text{sc}} - u_\varepsilon$, where we recall that the two-scale expansion $u_\varepsilon^{2\text{sc}}$ is defined in (1.1.31). In view of the analysis conducted in Section 1.1.4, the optimal convergence rate in (1.1.44) will follow from the optimal convergence rate in (1.1.42) and (1.1.43), namely from the quantification of the sub-linearity of ϕ_ξ and $\frac{\partial}{\partial \xi} \phi_\xi$. We start by investigating the growth of $x \mapsto \phi_\xi(x)$.

By linearizing around $\delta = 0$, we formally expand ϕ_ξ at first order in δ : $\phi_\xi = \delta \bar{\phi}_\xi + o(\delta)$,²⁵ where the equation for $\bar{\phi}_\xi$ is obtained by differentiating with respect to δ the equation (1.1.23) for the model (1.1.49), which reads

$$-\nabla \cdot \left(\text{Id} + (p-2) \frac{|\xi|^{p-4}}{(1+|\xi|)^{p-2}} \xi \otimes \xi \right) \nabla \bar{\phi}_\xi = \nabla \cdot (g\xi).$$

Since the above equation is a linear elliptic PDE with constant coefficients, we have an explicit formula for $\bar{\phi}_\xi$ which can be expressed with help of the Green's function associated to $a_\xi := \text{Id} + (p-2) \frac{|\xi|^{p-4}}{(1+|\xi|)^{p-2}} \xi \otimes \xi$: for all $x \in \mathbb{R}^d$ the Green's function reads

$$G_\xi(x) = \frac{1}{\sqrt{\det(a_\xi)}} \times \begin{cases} \frac{1}{4\pi} \log(x \cdot a_\xi^{-1} x) & \text{for } d = 2, \\ \frac{1}{d(2-d)|B_1|} (x \cdot a_\xi^{-1} x)^{-\frac{d-2}{2}} & \text{for } d \geq 3, \end{cases}$$

and we have

$$\bar{\phi}_\xi(x) = - \int \nabla G_\xi(x-y) \cdot g(y) \xi dy. \quad (1.1.51)$$

We first briefly justify that G_ξ (resp. its derivatives) can be dominated by the standard Green's function (resp. its derivatives) of the Laplace operator, uniformly in $|\xi|$. Indeed, since for all $x \in \mathbb{R}^d$, $x \cdot a_\xi x = |x|^2 + (p-2) \frac{|\xi|^{p-4}}{(1+|\xi|)^{p-2}} (\xi \cdot x)^2 \geq |x|^2$ and $\|a_\xi\| \leq 1 + (p-2) \frac{|\xi|^{p-4}}{(1+|\xi|)^{p-2}} \|\xi \otimes \xi\| \lesssim 1$, we thus have $x \cdot a_\xi^{-1} x \sim |x|^2$ uniformly in $|\xi|$. Also, it shows that the eigenvalues of a_ξ are contained in $[1, +\infty)$, and thus $\det(a_\xi) \geq 1$ (note that a_ξ is symmetric). We conclude that for all $x, \xi \in \mathbb{R}^d$

$$|\nabla G_\xi(x)| + |x| |\nabla^2 G_\xi| \lesssim |x|^{1-d}. \quad (1.1.52)$$

Because of the singularity of the Green's function at the origin, we have to treat differently the near-field contribution and the far-field contribution (denoted by $\bar{\phi}_n(x)$ and $\bar{\phi}_f(x)$ respectively), that we express with help of a smooth cut-off $\chi \in C_c^\infty(\mathbb{R}^d)$ for B_1 in B_2 :

$$\bar{\phi}_n(x) = - \int \chi(x-y) \nabla G_\xi(x-y) \cdot g(y) \xi dy \quad \text{and} \quad \bar{\phi}_f(x) = - \int (1 - \chi(x-y)) \nabla G_\xi(x-y) \cdot g(y) \xi dy.$$

For the near-field contribution, we use the triangle inequality combined with the fact that the singularity of ∇G_ξ is integrable (see (1.1.52)) and $\mathbb{E}[|g(0)|^2] = 1$, to obtain: for all $x \in \mathbb{R}^d$

$$\mathbb{E}[|\bar{\phi}_n(x)|^2]^{\frac{1}{2}} \leq |\xi| \int \chi |\nabla G_\xi| \lesssim |\xi|.$$

²⁴We also implicitly assume here that the law is isotropic since c is radially symmetric.

²⁵A rigorous analysis of the error term can be conducted by looking at the second order derivative in δ , but for the ease of the reading we only focus on the first term.

For the far-field contribution, one has to make use of stochastic cancellations, and this is here that our quantitative assumption (1.1.50) plays an important role. Using (1.1.50), we may express the second moment of $\bar{\phi}_f - \bar{\phi}_f(0)$ as:²⁶

$$\begin{aligned} & \mathbb{E}[|\bar{\phi}_f(x) - \bar{\phi}_f(0)|^2] \\ &= \int \int [(((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y)) \cdot \xi][(((1-\chi)\nabla G_\xi)(x-y') - ((1-\chi)\nabla G_\xi)(y')) \cdot \xi] \\ & \quad \times \mathbb{E}[g(y)g(y')] dy dy' \\ &\stackrel{(1.1.50)}{\lesssim} |\xi|^2 \int \int |((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))| |((1-\chi)\nabla G_\xi)(x-y') - ((1-\chi)\nabla G_\xi)(y'))| \\ & \quad \times (1 + |y-y'|)^{-d-\alpha} dy dy'. \end{aligned}$$

Now, using Cauchy-Schwarz' inequality (in the form of $\int \int f(y)f(y')h(y-y')dy dy' \leq \int \int |f(y)|^2|h(y-y')|dy dy'$) and the fact that the covariance function $c(x,0) : x \mapsto (1+|x|)^{-d-\alpha}$ is integrable, we arrive at

$$\mathbb{E}[|\bar{\phi}_f(x) - \bar{\phi}_f(0)|^2] \lesssim |\xi|^2 \int |((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))|^2 dy.$$

Then, the large-scale behaviour $|y-x| \geq 3|x|$ is treated by the Taylor expansion: $|((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))| \lesssim |x| \sup_{t \in [0,1]} |\nabla((1-\chi)\nabla G_\xi)(tx + (1-t)y)| \lesssim |x||y-x|^{-d}$ and gives the contribution:

$$\int_{\mathbb{R}^d \setminus B_{3|x|}(x)} |((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))|^2 dy \lesssim |x| \int_{\mathbb{R}^d \setminus B_{3|x|}} |y|^{-2d} dy \lesssim 1.$$

The near-scale behaviour $|y-x| < 3|x|$ (where here, without loss of generality, we may assume that $|x| \geq 1$) is treated by the triangle inequality: $|((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))| \lesssim (1-\chi)(x-y)|y-x|^{-d+1} + (1-\chi)(y)|y|^{-d+1}$ and gives the leading order contribution:

$$\int_{B_{3|x|}(x)} |((1-\chi)\nabla G_\xi)(x-y) - ((1-\chi)\nabla G_\xi)(y))|^2 dy \lesssim \int_{B_{4|x|} \setminus B_1} |y|^{-2(d-1)} dy \lesssim \begin{cases} \log(2+|x|) & \text{for } d=2, \\ 1 & \text{for } d=3. \end{cases}$$

To summarize, one has

$$\phi_\xi = \delta \bar{\phi}_\xi + o(\delta),$$

with

$$\mathbb{E}[|\bar{\phi}_\xi(x) - \bar{\phi}_\xi(0)|^2] \lesssim |\xi|^2 \begin{cases} \log(2+|x|) & \text{for } d=2, \\ 1 & \text{for } d>2. \end{cases} \quad (1.1.53)$$

For the linearized corrector $\frac{\partial}{\partial \xi} \phi_\xi$, similar computations can be done and yield the same conclusion. The growth (1.1.53) quantifies the sub-linearity of the corrector in the sense of $\varepsilon \phi_\xi(\frac{\cdot}{\varepsilon}) \xrightarrow{\varepsilon \downarrow 0} 0$ and $\varepsilon \frac{\partial}{\partial \xi} \phi_\xi(\frac{\cdot}{\varepsilon}) \xrightarrow{\varepsilon \downarrow 0} 0$.

Once we have understood the sub-linearity of the correctors, one has to find a suitable equation for the homogenization error $z_\varepsilon = u_\varepsilon^{2sc} - u_\varepsilon$. For the ease of the reading, we perform the computations for the linear setting (that is, by taking $p=2$ in (1.1.49)), which captures all the ideas. For

²⁶Since ϕ_ξ is only defined up to an additive constant, we choose to anchor it at the origin by subtracting $\phi_\xi(0)$.

the details in the nonlinear setting we refer to Section 3.6 in Chapter 3. By scaling, we may assume that $\varepsilon = 1$. We start by computing

$$\nabla z_1 = \nabla u_1 - (\nabla \bar{u} + \partial_{x_i} \bar{u} \nabla \phi_{e_i} + \phi_{e_i} \nabla \partial_{x_i} \bar{u}),$$

where we use Einstein's convention of implicit summation on repeated indices. Then, applying $-\nabla \cdot a$ together with equation (1.1.4) and the homogenized equation (1.1.14) has the effect of

$$\begin{aligned} -\nabla \cdot a \nabla z_1 &\stackrel{(1.1.4)}{=} \nabla \cdot f + \nabla \cdot (a(\nabla \phi_{e_i} + e_i) \partial_{x_i} \bar{u} + \phi_{e_i} a \nabla \partial_{x_i} \bar{u}) \\ &\stackrel{(1.1.14)}{=} \nabla \partial_{x_i} \bar{u} \cdot (a(\nabla \phi_{e_i} + e_i) - \bar{a} e_i) + \nabla \cdot (\phi_{e_i} a \nabla \partial_{x_i} \bar{u}). \end{aligned} \quad (1.1.54)$$

We now rewrite the first right-hand side term of (1.1.54) with help of a new corrector $\sigma = (\sigma_{ijk})_{i,j,k}$ (called the flux corrector), skew-symmetric in the last two-variables (that is $\sigma_{ijk} = -\sigma_{ikj}$) and that satisfies

$$\nabla \cdot \sigma_i := \partial_k \sigma_{ijk} = a(\nabla \phi_{e_i} + e_i) - \bar{a} e_i. \quad (1.1.55)$$

The flux corrector σ_i is a $d-1$ -form and (1.1.49) only defines σ_i up to gauge invariance. It is standard to fix the gauge by requiring σ_i to satisfy (which indeed yields the skew symmetry)

$$-\Delta \sigma_{ijk} = \partial_{x_j} (a(\nabla \phi_{e_i} + e_i) \cdot e_k) - \partial_{x_k} (a(\nabla \phi_{e_i} + e_i) \cdot e_j).$$

With help of the flux corrector and the chain rule, the first right-hand side of (1.1.54) can be rewritten as

$$\begin{aligned} \nabla \partial_{x_i} \bar{u} \cdot (a(\nabla \phi_{e_i} + e_i) - \bar{a} e_i) &= \partial_j \partial_i \bar{u} \partial_k \sigma_{ijk} = \partial_k (\partial_j \partial_i \bar{u} \sigma_{ijk}) - \partial_k \partial_j (\partial_i \bar{u}) \sigma_{ijk} \\ &= -\nabla \cdot (\sigma_i \nabla \partial_i \bar{u}), \end{aligned}$$

where the last line comes from the fact that from the skew-symmetry of $(\sigma_{ijk})_{j,k}$ and the symmetry of $(\partial_k \partial_j (\partial_i \bar{u}))_{j,k}$ one has $\partial_k \partial_j (\partial_i \bar{u}) \sigma_{ijk} = 0$. To conclude, one has the following equation for the homogenization error (after the rescaling $x \rightsquigarrow \frac{x}{\varepsilon}$)

$$-\nabla \cdot a(\frac{\cdot}{\varepsilon}) \nabla z_\varepsilon = \varepsilon \nabla \cdot ((a(\frac{\cdot}{\varepsilon}) \phi_{e_i}(\frac{x}{\varepsilon}) - \sigma_i(\frac{\cdot}{\varepsilon})) \nabla \partial_i \bar{u}).$$

In the regime of small ellipticity contrast, we get similarly the growth (1.1.53) for $\bar{\sigma}$, where the latter corresponds to the first order approximation of σ in $\delta \downarrow 0$.

By the energy estimate and (1.1.53), we deduce that the homogenization error satisfies (in the regime $\delta \downarrow 0$)

$$\begin{aligned} \mathbb{E} \left[\int |\nabla z_\varepsilon|^2 \right]^{\frac{1}{2}} &\lesssim \varepsilon \mathbb{E} \left[\int (|\phi(\frac{x}{\varepsilon})|^2 + |\sigma(\frac{x}{\varepsilon})|^2) |\nabla^2 \bar{u}(x)|^2 dx \right]^{\frac{1}{2}} \\ &\lesssim \|(1 + \log^{\frac{1}{2}}(\cdot) \mathbb{1}_{d=2}) \nabla^2 \bar{u}\|_{L^2(\mathbb{R}^d)} \times \begin{cases} \varepsilon \log^{\frac{1}{2}}(\varepsilon^{-1}) & \text{for } d = 2, \\ \varepsilon & \text{for } d > 2. \end{cases} \end{aligned}$$

The general framework of quantitative homogenization

We now present the tools which lead to a quantitative theory of homogenization. As mentioned in Subsection 1.1.5, the quantitative homogenization theory requires to quantify the ergodicity assumption (1.1.9). There are several approaches to encode quantitatively the decorrelation on large-scales.

To begin with, a fundamental example is given by discrete i.i.d coefficients, which means that the coordinate projections $a \in \mathcal{M} \mapsto a(x, \cdot) \in \mathcal{M}_0$ (corresponding to the set (1.1.7) with space independent operator), for $x \in \mathbb{Z}^d$, are independent and identically distributed. The associated probability distribution is given by $\mathbb{P} = \mathbb{P}_0^{\otimes \mathbb{Z}^d}$, the \mathbb{Z}^d -fold product of \mathbb{P}_0 , for \mathbb{P}_0 a probability measure on \mathcal{M}_0 . We can consider weaker assumptions as standard nonlinear mixing conditions that we define hereafter. For more details on linear mixing conditions, we refer to [52].

Definition 1 (Linear mixing conditions). *For all Borel subsets $\mathcal{U} \subset \mathbb{R}^d$, we define $\mathcal{F}_{\mathcal{U}}$ as the σ -algebra generated by the family $\{a \mapsto \int a(x, \xi)\psi(x)dx | \psi \in C_c^\infty(\mathcal{U}), \xi \in \mathbb{R}^d\}$. We say that the probability measure \mathbb{P} satisfies:*

- A α -mixing condition if there exist $C > 0$ and $\alpha > 0$ such that for all $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$ and $(A, B) \in \mathcal{F}_{\mathcal{U}} \times \mathcal{F}_{\mathcal{V}}$, we have²⁷

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq C(1 + \text{dist}(\mathcal{U}, \mathcal{V}))^{-\alpha}.$$

- A η -range of dependence (for some distance $\eta > 0$), if for all Borel subset $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$, one has

$$\text{dist}(\mathcal{U}, \mathcal{V}) \geq \eta \Rightarrow \mathcal{F}_{\mathcal{U}} \text{ and } \mathcal{F}_{\mathcal{V}} \text{ are independent.} \quad (1.1.56)$$

In other words, $a|_{\mathcal{U}}$ and $a|_{\mathcal{V}}$ are independent as soon as $\text{dist}(\mathcal{U}, \mathcal{V}) \geq \eta$.

The mixing conditions in Definition 1 are known to imply concentration properties and in particular the finite range of dependence (1.1.56) is satisfied by most examples of Section 1.1.2. More precisely, the discrete models (1.1.11) and (1.1.12) have by construction a unit range of dependence, that is they satisfy (1.1.56) for $\eta = 1$. The model (1.1.13) has a η -range of dependence provided we choose the kernel m be compactly supported in B_η . The quantitative homogenization theory of (1.1.4) within the above linear mixing conditions, either in the linear setting with symmetric coefficients or in the variational setting with $a(x, \xi) = \nabla_\xi W$ (with W strongly convex) and $p = 2$, has been extensively studied by Armstrong, Smart, Kuusi, Ferguson, Mourrat in [6, 5, 9, 10, 15, 12] and by Gloria and Otto in [81]. The strongest results are obtained in the case of finite range of dependence (1.1.56) where central limit theorem scalings are obtained. Linear mixing conditions require an involved analysis as they only allow to unravel local cancellations after iteration (see in particular the renormalization procedure in [12] and the notion of approximate locality in [81]). Importantly, such iterations lead to (nearly) optimal stochastic integrability. However, the quantitative homogenization of (1.1.4) under mixing conditions is so far restricted to the case $p = 2$ and the theory for the full range $p \geq 2$ is currently unknown. See the short discussion about this issue in Chapter 6. We refer to the textbook [8] for a summarize of the quantitative homogenization theory under finite range of dependence.

An other point of view, which is the one taken in this thesis, is based on functional inequalities in probability like spectral gap or logarithmic Sobolev inequalities which are powerful tools to prove nonlinear concentration properties and central limit theorem scalings, see for instance [107]. Such inequalities are well known to hold in the simple situation of i.i.d random variables, referred to as Efron-Stein' inequality (see [88, Lemma 7]) or also for the Boltzmann-Gibbs distribution $d\mu = Z_\mu^{-1}e^{-V}dx$ with V strongly convex (see [38]). In infinite dimension, standard spectral gap and logarithmic Sobolev inequalities depend on a suitable notion of derivative which applies to random variable $F = \mathcal{M} \rightarrow \mathbb{R}^k$ (depending on the structure of the randomness). Such standard inequalities take the following form.

²⁷We use the notation $\text{dist}(\mathcal{U}, \mathcal{V}) = \inf_{(x,y) \in \mathcal{U} \times \mathcal{V}} |x - y|$ for the distance between two sets.

Definition 2 (Standard functional inequalities). *We assume that there exists a suitable notion of derivative $x \mapsto \partial_{x,1}F$, which applies to random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$, depending on the structure of the randomness. Morally speaking, $\partial_{x,1}F$ "measures" how sensitive $F(a)$ reacts to changes of the value of $a(x, \xi)$ localized at x , for all $\xi \in \mathbb{R}^{m \times d}$.*

We say that the probability measure \mathbb{P} satisfies a logarithmic Sobolev inequality (LSI) if there exists $\rho > 0$ such that for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ (for some $k \geq 1$) with $\text{Ent}(F) := \mathbb{E}[F^2 \log(F^2)] - \mathbb{E}[F^2] \log(\mathbb{E}[F^2]) < +\infty$, the following inequality holds

$$\text{Ent}(F) \leq \frac{1}{\rho} \mathbb{E} \left[\int |\partial_{x,1}F|^2 dx \right]. \quad (1.1.57)$$

We say that the probability measure \mathbb{P} satisfies a spectral gap inequality (SG) if there exists $\rho > 0$ such that for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ with $\mathbb{E}[|F|^2] < +\infty$, the following inequality holds

$$\mathbb{E}[|F - \mathbb{E}[F]|^2]^{\frac{1}{2}} \leq \frac{1}{\rho} \mathbb{E} \left[\int |\partial_{x,1}F|^2 dx \right]^{\frac{1}{2}}. \quad (1.1.58)$$

For a discrete distribution $a = (a(x, \cdot))_{x \in \mathbb{Z}^d}$, we say that the probability distribution \mathbb{P} satisfies a spectral gap inequality if

$$\mathbb{E}[|F - \mathbb{E}[F]|^2]^{\frac{1}{2}} \leq \frac{1}{\rho} \mathbb{E} \left[\sum_{x \in \mathbb{Z}^d} |\partial_x F|^2 \right]^{\frac{1}{2}}. \quad (1.1.59)$$

By post-processing, if the probability measure \mathbb{P} satisfies LSI (1.1.57) or SG (1.1.58), we can actually control higher stochastic moments, see [59, Proposition 1.10]. More precisely:

Proposition 1. *Assume that the probability distribution \mathbb{P} satisfies SG (1.1.58). Then, there exists a constant $C > 0$ such that for all $q < +\infty$ and for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ with $\mathbb{E}[|F|^q] < +\infty$, the following inequality holds*

$$\mathbb{E}[|F - \mathbb{E}[F]|^q]^{\frac{1}{q}} \leq C q^2 \mathbb{E} \left[\left(\int |\partial_{x,1}F|^2 dx \right)^{\frac{q}{2}} \right]^{\frac{1}{q}},$$

where the multiplicative factor Cq^2 can be upgraded to Cp if LSI (1.1.57) is satisfied.

Functional inequalities of Definition 2 are satisfied by the all instances of Section 1.1.2. More precisely, the discrete model (1.1.11) satisfies (1.1.59) where the derivative ∂_x denotes the vertical derivative defined as:

$$\partial_x F = F - \mathbb{E}[F | \mathcal{F}_x],$$

with $\mathbb{E}[\cdot | \mathcal{F}_x]$ refers the conditional expectation where we condition on the σ -algebra \mathcal{F}_x generated by the family $\{a(z) := a_b(z) : z \neq x\}$. The derivative $\partial_x F$ is precisely the $L^2(\Omega)$ -orthogonal projection of F onto the subspace of random variables in $L^2(\mathcal{M})$ that do not depend on $a(x)$. In other words, $\mathbb{E}[\cdot | \mathcal{F}_x]$ denotes the conditional expectation, where we condition on the event that we know the value of $a(z)$ for all sites $z \neq x$.

The Poisson point process based model (1.1.12) satisfies (1.1.57) where the derivative $\partial_{x,1} := \partial_{x,1}^{\text{osc}}$

denotes the oscillation of $F = F(A)$ (taking values in \mathbb{R}^d) with respect to the variations of A on $B(x)$, that is formally²⁸,

$$\partial_{x,1}^{\text{osc}} F = \sup_{A': A'|_{\mathbb{R}^d \setminus B_1(x)} = A|_{\mathbb{R}^d \setminus B_1(x)}} F(A') - \inf_{A': A'|_{\mathbb{R}^d \setminus B_1(x)} = A|_{\mathbb{R}^d \setminus B_1(x)}} F(A'). \quad (1.1.60)$$

The Gaussian based model (1.1.13) satisfies (1.1.62) provided that A is Lipschitz and $\beta > d$, where the derivative $\partial_{x,1}^{\text{fct}} := \partial_{x,1}^{\text{fct}}$ denotes the functional derivative, defined as the Gâteaux derivative of $F(A)$ with respect to A , that is (we use the short-hand notation $A := A(m \star \xi)$)

$$\partial_{x,1}^{\text{fct}} F = \sup_{\delta A} \limsup_{h \downarrow 0} \frac{F(A + h\delta A) - F(A)}{h}, \quad (1.1.61)$$

where the supremum runs over coefficient fields δA that are compactly supported in $B_1(x)$ and bounded by 1 in $L^\infty(\mathbb{R}^d)$.

Functional inequalities imply a powerful calculus in probability, which is in particular convenient to unravel probabilistic cancellations. Optimal scaling can then be easily captured, but stochastic integrability often remains suboptimal, since functional analysis cannot be iterated nicely (compared to the first approach by mixing conditions). As pointed out in [59], standard functional inequalities of Definition 2 are very restrictive: the randomness essentially has to be either Gaussian with integrable covariance as in the model (1.1.13) (for $\beta > d$) or to display a product structure as for the models (1.1.11) and (1.1.12). This rules out most models of interest for heterogeneous materials considered in applied sciences in [152], such as random parking process (see [152, Chapter 3]), Voronoi and Delaunay tessellations of a Poisson point set (see [152, Chapter 8]) or Gaussian fields with long-range correlations. A way to relax Definition 2 is to introduce multiscale weighted generalizations of such standard functional inequalities, as introduced by Gloria and Duerinckx in [59, 60].

Definition 3 (Multiscale functional inequalities). *We say that the probability measure \mathbb{P} satisfies a multiscale logarithmic Sobolev inequality (MLSI) if there exist $\rho > 0$ and an integrable weight π such that for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ (for some $k \geq 1$) with $\text{Ent}(F) < +\infty$, the following inequality holds*

$$\text{Ent}(F) \leq \frac{1}{\rho} \mathbb{E} \left[\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell} F|^2 dx d\ell \right], \quad (1.1.62)$$

where now $x \mapsto \partial_{x,\ell} F$ depends on the multiscale parameter ℓ by looking also on how it depends on the ball of size ℓ in where the values of $x \mapsto a(x, \cdot)$ are localized.

We say that the probability measure \mathbb{P} satisfies a multiscale spectral gap inequality (MSG) if there exists an integrable weight π and a constant $\rho > 0$ for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ (for some $k \geq 1$) with $\mathbb{E}[|F|^2] < +\infty$, the following inequality holds

$$\mathbb{E}[|F - \mathbb{E}[F]|^2]^{\frac{1}{2}} \leq \frac{1}{\rho} \mathbb{E} \left[\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell} F|^2 dx d\ell \right]^{\frac{1}{2}}. \quad (1.1.63)$$

As for standard functional inequalities, if the probability measure \mathbb{P} satisfies MLSI (1.1.62) or MSG (1.1.63), we can control higher stochastic moments, see [59, Proposition 1.10]. More precisely:

²⁸We may rigorously define $\partial_{x,1}^{\text{osc}} F$ as the measurable envelope of (1.1.60).

Proposition 2. *Assume that the probability distribution \mathbb{P} satisfies MSG (1.1.63). Then, there exists a constant $C > 0$ such that for all $q < +\infty$ and for all random variables $F : \mathcal{M} \rightarrow \mathbb{R}^k$ with $\mathbb{E}[|F|^q] < +\infty$, the following inequality holds*

$$\mathbb{E}[|F - \mathbb{E}[F]|^q]^{\frac{1}{q}} \leq Cq^2 \mathbb{E} \left[\left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell} F|^2 dx d\ell \right)^{\frac{q}{2}} \right]^{\frac{1}{q}},$$

where the multiplicative factor Cq^2 can be upgraded to Cq if MLSI (1.1.62) is satisfied.

Multiscale functional inequalities allow us to consider long-range correlations, compared to standard functional inequalities of Definition 2. For instance, the Gaussian based model (1.1.13) with $\beta \leq d$ satisfies MLSI (1.1.62) with weight $\pi(\ell) = \ell^{-1-\beta}$ and where the functional derivative (1.1.61) has to be replaced by

$$\partial_{x,\ell}^{\text{fct}} F = \sup_{\delta A} \limsup_{h \downarrow 0} \frac{F(A + h\delta A) - F(A)}{h}, \quad (1.1.64)$$

where the supremum runs over coefficient fields δA that are compactly supported in $B_\ell(x)$ and bounded by 1 in $L^\infty(\mathbb{R}^d)$.²⁹

Voronoi tessellations of a Poisson point set satisfy MLSI (1.1.62) with weight $\pi(\ell) = Ce^{-\frac{1}{c}\ell^d}$ (for some $C > 0$) and with the oscillation derivative (1.1.60) replaced by

$$\partial_{x,\ell}^{\text{osc}} F = \sup_{A': A'|_{\mathbb{R}^d \setminus B_\ell(x)} = A|_{\mathbb{R}^d \setminus B_\ell(x)}} F(A') - \inf_{A': A'|_{\mathbb{R}^d \setminus B_\ell(x)} = A|_{\mathbb{R}^d \setminus B_\ell(x)}} F(A').$$

Random parking process with hardcore radius $R = 1$ satisfies MLSI (1.1.62) with weight $\pi(\ell) = Ce^{-\frac{1}{c}\ell}$ (for some $C > 0$) and with the oscillation derivative $\partial_{x,\ell} = \partial_{x,\ell}^{\text{osc}}$. For more precisions on those examples and for some generalizations, we refer to [60, Section 3]. In particular, all the examples of the textbook [152] satisfy a MLSI or a MSG. We also point out that since certain random coefficient fields (as Gaussians) satisfy only one of the assumptions in Definition 1 (mixing conditions) and in Definition 2 and 3 (functional inequalities), it is important to consider both separately.

We now focus on the application of MLSI/LSI/MSG/SG in stochastic homogenization of systems (1.1.4). We recall that the interest of MLSI/LSI/MSG/SG inequalities and what it makes them very powerful is that they entail fluctuations bounds for nonlinear functionals of a . To picture this, it is instructive to apply (1.1.62) to a simple example given by spatial average of a smooth nonlinear transformation of the scalar Gaussian model $A = A(m \star \xi) \in \mathbb{R}$, for $|m(x)| \leq (1 + |x|)^{-\frac{1}{2}(d+\beta)}$: $F_r := \int_{B_r} f(A(x)) dx$ with $r > 0$ and $f \in C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$. The functional derivative (1.1.64) is explicitly given by $\partial_{x,\ell}^{\text{fct}} F_r = |B_1|^{-1} \int_{B_\ell(x)} f'(A) r^{-d} \mathbb{1}_{B_r}$. We plug the derivatives in (1.1.62) and we use Jensen's inequality $\left(\int_{B_\ell(x)} f'(A) \mathbb{1}_{B_r} \right)^2 \lesssim \ell^d \int_{B_\ell(x)} |f'(A)|^2 \mathbb{1}_{B_r}$ in the regime $\ell \leq r$ and Minkowski's

²⁹More generally, if for all $x \in \mathbb{R}^d$, $|c(x)| \lesssim \gamma(|x|)$ holds for some non-increasing Lipschitz functions $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, then $A = A(m \star W)$ satisfies MLSI/MSG (1.1.62) and (1.1.63) with weight $\pi(\ell) \sim |\gamma'(\ell)|$. See [60, Section 3.1].

inequality $\int \left(\int_{B_\ell(x)} f'(A) \mathbb{1}_{B_r} \right)^2 \lesssim \ell^d \left(\int |f'(A)| \mathbb{1}_{B_r} \right)^2$ in the regime $\ell > r$, to obtain

$$\begin{aligned} \text{Ent}(F_r) &\leq \frac{1}{\rho} \mathbb{E} \left[\int_1^{+\infty} \ell^{-d} \pi(\ell) \int \left(|\mathbb{B}_1|^{-1} \int_{B_\ell(x)} f'(A) r^{-d} \mathbb{1}_{B_r} \right)^2 dx \right] \\ &\lesssim_d \sup |f'|^2 \left(r^{-d} \int_1^r \ell^{d-\beta-1} d\ell + \int_r^{+\infty} \ell^{-1-\beta} d\ell \right) \\ &\lesssim_{d,\beta} \sup |f'|^2 (r^{-\beta} \mathbb{1}_{\beta < d} + r^{-d} \log(r) \mathbb{1}_{\beta = d} + r^{-d} \mathbb{1}_{\beta > d}), \end{aligned} \quad (1.1.65)$$

which gives a convergence rate of (1.1.9) in this special case. For more general results on spatial averages, we refer to [59, Section 1.3]. Such estimates are expected to hold for more complex non-linear maps of the operator a , as the corrector defined in (1.1.23).

Inequalities of type MLSI/LSI/MSG/SG are very rich of applications in stochastic homogenization of (1.1.4) and in particular lead to optimal convergence rate of the homogenization error. To illustrate this, we prove how MSG (1.1.63) give the optimal convergence rate in the one-dimensional example of Section 1.1.4, in the particular scalar Gaussian case $A = A(m \star \xi)$, for $|m(x)| \leq (1 + |x|)^{-\frac{1}{2}(d+\beta)}$. In view of the analysis conducted in Section 1.1.4, the optimal convergence rate in (1.1.44) will follow from the optimal convergence rate in (1.1.42) and (1.1.43), namely the sub-linearity of ϕ_ξ and $\frac{\partial}{\partial \xi} \phi_\xi$. Let us show that (1.1.63) indeed quantify this two convergences. For simplicity, we only treat (1.1.42), (1.1.43) is treated the same way. In the following, we use the short-hand notation

$$\mathcal{F}_\varepsilon := \varepsilon \phi_\xi \left(\frac{x}{\varepsilon} \right) = -\frac{\varepsilon}{2} \int_0^{\frac{x}{\varepsilon}} \frac{1}{A(t)} dt + \varepsilon \int_0^{\frac{x}{\varepsilon}} \sqrt{\frac{\Phi^{-1}(\xi)}{A(t)} + \frac{1}{4A^2(t)}} dt - \xi x, \quad (1.1.66)$$

where we recall that $\Phi(\xi) = -\frac{1}{2} \mathbb{E}[\frac{1}{A}] + \mathbb{E} \left[\sqrt{\frac{\xi}{A} + \frac{1}{4A^2}} \right]$. We first compute the functional derivative of \mathcal{F}_ε in the sense of (1.1.61) and to do so, we fix $y \in \mathbb{R}^d$, $\ell \geq 1$ and a perturbation $\delta A \in L^\infty(\mathbb{R}^d)$ compactly supported in $[y - \ell, y + \ell]$ with $\|\delta A\|_{L^\infty(\mathbb{R}^d)} \leq 1$. A direct differentiation give

$$\left. \frac{d}{dt} \mathcal{F}_\varepsilon(A + t\delta A) \right|_{t=0} = \int \frac{\varepsilon}{2} \mathbb{1}_{[0, \frac{x}{\varepsilon}]} \left(\frac{1}{A^2} - \frac{2A\Phi^{-1}(\xi) + 1}{2A^3 \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}}} \right) \delta A,$$

which yields, by the definition of the functional derivative (1.1.64),

$$\partial_{y,\ell}^{\text{fct}} \mathcal{F}_\varepsilon = \int_{y-\ell}^{y+\ell} \frac{\varepsilon}{2} \mathbb{1}_{[0, \frac{x}{\varepsilon}]} \left(\frac{1}{A^2} - \frac{2A\Phi^{-1}(\xi) + 1}{2A^3 \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}}} \right).$$

Therefore, since $\mathbb{E}[\mathcal{F}_\varepsilon] = 0$, applying MSG (1.1.63) yields

$$\mathbb{E}[\mathcal{F}_\varepsilon^2]^{\frac{1}{2}} \leq \frac{1}{\rho} \mathbb{E} \left[\int_1^{+\infty} \ell^{-2-\beta} \int \left(\int_{y-\ell}^{y+\ell} \frac{\varepsilon}{2} \mathbb{1}_{[0, \frac{x}{\varepsilon}]} \left(\frac{1}{A^2} - \frac{2A\Phi^{-1}(\xi) + 1}{2A^3 \sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}}} \right) \right)^2 dy d\ell \right]^{\frac{1}{2}}. \quad (1.1.67)$$

We now bound the right-hand side of (1.1.67) as follows. We recall that we have shown in Section 1.1.4 that $\Phi^{-1}(\xi) \sim \xi^2 + \alpha$ (for $\alpha = (\mathbb{E}[\frac{1}{A}] \mathbb{E}[\frac{1}{\sqrt{A}}])^{-2}$), therefore by using in addition the bound

$0 < c_1 < A < c_2$, one has $\left| \frac{1}{A^2} - \frac{2A\Phi^{-1}(\xi)+1}{2A^3\sqrt{\frac{\Phi^{-1}(\xi)}{A} + \frac{1}{4A^2}}} \right| \lesssim_{c_1, c_2, \alpha} 1 + \xi$. We then split the integral as $\int_1^{+\infty} d\ell = \int_1^{\frac{1}{\varepsilon}} d\ell + \int_{\frac{1}{\varepsilon}}^{+\infty} d\ell$ and (1.1.67) turns into (using similarly Jensen's and Minkowski's inequality as in (1.1.65))

$$\begin{aligned} \mathbb{E}[\mathcal{F}_\varepsilon|^2]^{\frac{1}{2}} &\lesssim_\rho (1 + \xi)\varepsilon \left(\left(\int_1^{\frac{1}{\varepsilon}} + \int_{\frac{1}{\varepsilon}}^{+\infty} \right) \ell^{-2-\beta} \int \left(\int_{y-\ell}^{y+\ell} \mathbf{1}_{[0, \frac{x}{\varepsilon}]} \right)^2 dy d\ell \right)^{\frac{1}{2}} \\ &\lesssim (1 + \xi) \left(\sqrt{\varepsilon} \int_1^{\frac{1}{\varepsilon}} \ell^{-\beta} d\ell + \int_{\frac{1}{\varepsilon}}^{+\infty} \ell^{-1-\beta} d\ell \right)^{\frac{1}{2}} \\ &\lesssim (1 + \xi) \times \begin{cases} \varepsilon^{\frac{\beta}{2}} & \text{for } \beta \in (0, 1), \\ \sqrt{\varepsilon} \log(\varepsilon^{-1}) & \text{for } \beta = 1, \\ \sqrt{\varepsilon} & \text{for } \beta > 1. \end{cases} \end{aligned}$$

Similarly, $\mathbb{E}[|\varepsilon \frac{\partial}{\partial \xi} \phi_\xi(\frac{x}{\varepsilon})|^2]^{\frac{1}{2}}$ enjoys the same bound. To conclude, the convergence rate of the homogenization error is given by: for all $q < +\infty$

$$\mathbb{E}[\|u_\varepsilon - \bar{u} - \varepsilon \phi_{\bar{u}'}(\frac{\cdot}{\varepsilon})\|_{W^{1,3}([0,1])}^2]^{\frac{1}{2}} \lesssim_{\rho, c_1, c_2, \alpha} (1 + \|\bar{u}'\|_{L^\infty([0,1])}) \|u''\|_{L^\infty([0,1])} \times \begin{cases} \varepsilon^{\frac{\beta}{2}} & \text{for } \beta \in (0, 1), \\ \sqrt{\varepsilon} \log(\varepsilon^{-1}) & \text{for } \beta = 1, \\ \sqrt{\varepsilon} & \text{for } \beta > 1. \end{cases}$$

Note that in the case of weak correlations $\beta > 1$, we recover the central limit theorem scaling $\sqrt{\varepsilon}$. In higher dimension, the strategy is the same and the whole game is to quantify the sub-linearity of the correctors (for which we have no explicit formula in general).

The quantitative homogenization theory of (1.1.4) under functional inequalities has been intensively studied by Otto, Gloria, Neukamm, Fischer and their collaborators, inspired by an unpublished work by Naddaf and Spencer [132]. The linear model $a(x, \xi) = A(x)\xi$ is by now well understood, started from the works of Gloria and Otto in [82, 83] and Gloria, Neukamm and Otto in [87, 79] for discrete equation under the spectral gap inequality (1.1.59). Then, the continuum case has been studied by Gloria and Otto in [84, 80, 89] for more general coefficient fields which satisfy MLSI/MSG/LSI/SG. For the general (nonlinear) model (1.1.4), quantitative results were so far restricted to the case $p = 2$ and has been studied in the recent work of Fischer and Neukamm in [68] under LSI. The quantitative homogenization of (1.1.4) under LSI, for the full range $p \geq 2$, is studied in Chapter 3, however, the case of MLSI is currently out of reach. We summarize the results for the linear setting in the following theorem:

Theorem 3 (Quantitative homogenization of (1.1.4)-Linear case). *Assume that $a(x, \xi) = A(x)\xi$ and that the probability measure \mathbb{P} satisfies a MLSI (1.1.62) or a MSG (1.1.63) inequality with weight $\pi(\ell) = \ell^{-\beta-1}$, for some $\beta > 0$.*

The corrector $\phi = (\phi_{e_i})_{i \in [1, d]}$ admits the following growth. There exist a random variable $x \mapsto \mathcal{C}(x)$ and a constant $C > 0$ as well as an exponent $\alpha > 0$ such that $\sup_{x \in \mathbb{R}^d} \mathbb{E}[\exp(\frac{1}{C}\mathcal{C}^\alpha(x))] < +\infty$ and

$$\left(\int_{B(x)} |\phi|^2 + |\sigma|^2 \right)^{\frac{1}{2}} \leq \left| \int_B \phi \right| + \left| \int_B \sigma \right| + \mathcal{C}(x)\mu(|x|), \quad (1.1.68)$$

with

$$\mu(|x|) := \begin{cases} (|x| + 1)^{1-\frac{\beta}{2}} & \text{for } \beta < 2, \\ \log^{\frac{1}{2}}(|x| + 2) & \text{for } \beta = 2, d > 2 \text{ or } \beta > 2 \text{ and } d = 2, \\ \log(|x| + 2) & \text{for } \beta = d = 2, \\ 1 & \text{for } \beta > 2 \text{ and } d > 2. \end{cases}$$

The two-scale expansion $u_\varepsilon^{2sc} := \bar{u}_\varepsilon + \varepsilon \sum_{i=1}^d \phi_{e_i}(\frac{\cdot}{\varepsilon}) \partial_i \bar{u}_\varepsilon$ (where $\bar{u}_\varepsilon(x) := \int_{B_\varepsilon(x)} \bar{u}$) converges quantitatively with the following optimal rate: for all $\omega \subset \Omega$, there exist a random variable $\mathcal{C}_{\omega,\varepsilon,f}$, a constant $C > 0$ as well as an exponent $\alpha > 0$ such that $\mathbb{E}[\exp(\frac{1}{C} \mathcal{C}_{\omega,\varepsilon,f}^\alpha(x))] < +\infty$ (uniformly in ε and f) and

$$\|\nabla u_\varepsilon^{2sc} - \nabla u_\varepsilon\|_{L^2(\omega)} \leq \mathcal{C}_{\omega,\varepsilon,f} \varepsilon \mu(\varepsilon^{-1}) \left(\int \mu^2 |\nabla f|^2 \right)^{\frac{1}{2}}. \quad (1.1.69)$$

The need of local averages at scale ε of \bar{u} in Theorem 3 is due to the fact that we assume no regularity of the coefficients at small-scales which imposes that the growth of the correctors (1.1.68) has a no pointwise form. However, the standard De Giorgi-Nash-Moser theory (for $m = 1$) or the Schauder theory (in the case of systems with A is Hölder continuous), allow us to upgrade (1.1.68) into a pointwise estimate and therefore there is no need to take local average of \bar{u} in (1.1.69) in those cases. We prove similar results in Chapter 3 for the nonlinear setting (1.1.4) in the full range $p \geq 2$ (in dimension 3) under LSI (1.1.57). The quantitative homogenization of (1.1.4) under MLSI is currently unknown (see Chapter 6 for a short discussion about this issue). We now present the contributions of this thesis in the next section.

1.2 Contribution of the thesis

We present in this section the contributions of this thesis.

1.2.1 Questions arising in this thesis

Problem 1. *The first part aims for a better understanding of the quantitative homogenization theory of linear elliptic systems (namely, for the model $a(x, \xi) = A(x)\xi$ in (1.1.4)) under MLSI (1.1.62). We use a parabolic approach to quantitatively control the correctors defined in (1.1.23) and we deduce useful estimates in stochastic homogenization in that case (we provide also a different proof of the results in [89]). This approach give also convenient tools for the study of numerical methods. This problem is studied in Chapter 2.*

Problem 2. *The second part aims for a better understanding of the quantitative homogenization theory of the nonlinear system (1.1.4) in the all range $p \geq 2$ under LSI (1.1.10). We prove optimal stochastic estimates on the correctors and linearized correctors in the non-perturbative regime as well as an optimal convergence rate of the homogenization error. This work is the result of a collaboration with Antoine Gloria at the Laboratoire Jacques-Louis Lions. This problem is studied in Chapter 3.*

Problem 3. *The third part aims to study an efficient numerical method to compute the homogenized operator (1.1.30), in the particular linear model $a(x, \xi) = A(x)\xi$, which is characterized by the homogenized matrix, for all $e \in \mathbb{R}^d$, $\bar{a}e := \mathbb{E}[A(0)(\nabla\phi_e + e)]$. The method that we analyse is the so-called representative volume element method and we prove optimal convergence rates of the method with respect to the size of the box for two approaches of periodization, in the particular setting of Gaussian field. In particular, we extend the so called Price's formula for infinite dimensional Gaussian measures. In particular, the main message of this work is: rather periodize the probability*

law than the realizations. This work is the result of a collaboration with Marc Josien, Felix Otto and Qiang Xu at the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig. This problem is studied in Chapter 4 (for Price's formula) and Chapter 5 (for the representative volume element method).

1.2.2 List of Preprints

The contributions of this thesis are contained in the two following Preprints and in a work in preparation.

1. N. Clozeau. *Optimal decay of the parabolic semigroup in stochastic homogenization for correlated coefficient fields*, *Arxiv Preprint*, arXiv:2102.07452, 2021.
2. N. Clozeau and A. Gloria. *Quantitative nonlinear homogenization: control of oscillations*, *Arxiv Preprint*, arXiv:2104.04263v2, 2021.
3. N. Clozeau, M. Josien, F. Otto and Q. Xu. *Bias in the representative volume element method: periodize the ensemble instead of its realizations*, *In preparation*, 2021.

1.2.3 Optimal decay of the parabolic semigroup for linear elliptic systems with for correlated coefficient fields (Chapter 2)

In Chapter 2, we investigate the quantitative homogenization of linear elliptic systems with random coefficients, that is for $p = 2$ and when (1.1.4) takes the particular form of³⁰

$$-\nabla \cdot a(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla \cdot f \text{ in } \mathbb{R}^d. \quad (1.2.1)$$

We derive optimal estimates by following the ideas of the papers [92, 88] and [8, Sec.9] where the authors considered the semigroup associated with the corrector problem (1.1.23) (which, in the linear case, takes the form of $-\nabla \cdot a(e + \nabla \phi_e) = 0$ in \mathbb{R}^d , for a given $e \in \mathbb{R}^d$), namely the solution u_e of the parabolic system

$$\begin{cases} \partial_\tau u_e - \nabla \cdot a \nabla u_e = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u_e(0) = \nabla \cdot a(\cdot)e. \end{cases} \quad (1.2.2)$$

The relationship between the semigroup u_e and the corrector ϕ_e is the following formal integral formula

$$\phi_e = \int_0^{+\infty} u_e(t, \cdot) dt. \quad (1.2.3)$$

Indeed, provided we have $\lim_{t \rightarrow +\infty} u_e(t, \cdot) = 0$, integrating the equation (1.2.2) in time yields

$$-\nabla \cdot a e - \nabla \cdot a \nabla \int_0^{+\infty} u_e(t, \cdot) dt = 0,$$

and implies that $\int_0^{+\infty} u_e(t, \cdot) dt$ is a solution of the corrector equation (1.1.23) so that (1.2.3) follows by uniqueness. Formula (1.2.3) will be made rigorous by showing the optimal decay in time of stochastic moments of u_e . This theoretical relationship allow us to transfer optimal estimates on u_e into optimal estimates on the corrector ϕ_e . The semigroup u_e is also of numerical interest and can be used as a very convenient tool for the study of numerical method for approximating the correctors and the homogenized matrix \bar{a} . As examples, we present three situations where optimal estimates of u_e are used in the context of numerical methods.

³⁰This model corresponds to a stationary linear diffusion, corresponding to the linearization of (1.1.4). We also make the choice to work in the whole space to avoid boundary layer problems.

- (i) First, the study of the representative volume element method, where the method consists of replacing the corrector equation (1.1.23) by an equation posed in a large box $Q_L := [-\frac{L}{2}, \frac{L}{2}]^d$, for $L \gg 1$,

$$-\nabla \cdot a_L(\nabla \phi_e^L + e) = 0 \text{ in } Q_L,$$

with periodic boundary conditions, for a good choice of periodic realization a_L . We refer to [88] for an analysis via a semigroup approach.

- (ii) Second, the semigroup u_e has been used more recently in [1] for approximating ϕ_e via exponential regularization, that is we replace the corrector equation (1.1.23) by

$$\begin{cases} u_e(T) - \nabla \cdot a(\nabla \phi_{e,T,R} + e) = 0 & \text{in } Q_R, \\ \phi_{e,T,R} \equiv 0 & \text{on } \partial Q_R, \end{cases}$$

for $R \gg 1$ and $T \gg 1$. Optimal estimates on u_e are used to control the bias (or the systematic error).

- (iii) Third, we can derive optimal convergence rate in the massive term approximation as in [88, 92], and we propose a proof in Corollary 5 of Chapter 2.

The first result in the vein of this chapter is the optimal decay estimate in time of u_e and of its gradient in the case of discrete elliptic equation satisfying SG (1.1.59) proved by Gloria, Neukamm and Otto in [88]. In the case of finite range of dependence (1.1.56), similar results are obtained by Gloria and Otto in [92] and Armstrong, Kuusi and Mourrat in [8]. Their analysis strongly relies on the fast decay of correlations, and does not treat coefficients with fat tails. The aim of this Chapter is to provide an extension of those results to more correlated coefficient fields such as Gaussian coefficient fields with fat tails. Our quantitative analysis is based on functional inequalities of type MLSI (1.1.62).

The expected optimal time decay of u_e can be easily inferred in the small contrast regime, that is, when the coefficient field is given as a perturbation of the identity: $a^\delta = (1 + \frac{1}{\pi} \arctan(\delta a))\text{Id}$, for a small $\delta \ll 1$ and a scalar (as we did in Section 1.1.5). By linearizing around $\delta \downarrow 0$, the first order approximation of u_e is given by $u_e = \delta \bar{u} + o(\delta)$ where

$$\begin{cases} \partial_\tau \bar{u} - \Delta \bar{u} = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ \bar{u}(0) = \frac{1}{\pi} \nabla \cdot a(\cdot)e, \end{cases} \quad (1.2.4)$$

for which we have an explicit formula. Indeed, using the heat kernel $\Gamma : (T, x) \in \mathbb{R}^+ \times \mathbb{R}^d \mapsto \frac{1}{(4\pi T)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4T}}$, we have for all $(T, x) \in \mathbb{R}^+ \times \mathbb{R}^d$

$$\bar{u}(T, x) = \frac{1}{\pi} \int \nabla \Gamma(T, x - y) \cdot a(y) e dy. \quad (1.2.5)$$

Using a scalar Gaussian model (that is for g a scalar centered Gaussian) $a(x) = g(x)$ and imposing algebraic decay of the correlations in form of $|\mathbb{E}[g(x)g(y)]| \sim (1 + |x - y|)^{-\beta}$ (for some $\beta > 0$), we may express the variance as

$$\begin{aligned} \mathbb{E}[|\bar{u}(T, 0)|^2] &= \frac{1}{\pi^2} \int \int \nabla \Gamma(T, x) \cdot e \nabla \Gamma(T, y) \cdot e \mathbb{E}[g(x)g(y)] dx dy \\ &\lesssim \frac{1}{\pi^2} \int \int \nabla \Gamma(T, x) \cdot e \nabla \Gamma(T, y) \cdot e (1 + |x - y|)^{-\beta} dx dy. \end{aligned} \quad (1.2.6)$$

We then distinguish between the two regimes $\beta > d$ and $\beta \leq d$.

- In the regime $\beta > d$, the application $x \mapsto (1 + |x|)^{-\beta}$ is integrable, so that we may use the Cauchy-Schwarz inequality and bound the variance as:

$$\begin{aligned} \mathbb{E}[|\bar{u}(T, 0)|^2] &\lesssim \frac{1}{\pi^2} \int \int \nabla\Gamma(T, x) \cdot e(1 + |x - y|)^{-\frac{\beta}{2}} \nabla\Gamma(T, y) \cdot e(1 + |x - y|)^{-\frac{\beta}{2}} dx dy \\ &\leq \int \int |\nabla\Gamma(T, x)|^2 (1 + |x - y|)^{-\beta} dx dy \\ &\lesssim \int |\nabla\Gamma(T, x)|^2 dx \lesssim T^{-1-\frac{d}{2}}. \end{aligned}$$

This decay is the best that we can expect, since this is the one that we would obtain in the better situation of i.i.d coefficients (for which $\mathbb{E}[g(x)g(y)] = \delta(x - y)$).

- In the regime $\beta \leq d$, the decay becomes worse as soon as β gets smaller, and we show this by a more careful domination. We first estimate, for all $x \in \mathbb{R}^d$, $\int \nabla\Gamma(T, y) \cdot e(1 + |x - y|)^{-\beta} dy$, that we split into two contributions:

$$\begin{aligned} \int \nabla\Gamma(T, y) \cdot e(1 + |x - y|)^{-\beta} dy &= \int_{\mathbb{R}^d \setminus B_{\sqrt{T}}(x)} \nabla\Gamma(T, y) \cdot e(1 + |x - y|)^{-\beta} dy \\ &\quad + \int_{B_{\sqrt{T}}(x)} \nabla\Gamma(T, y) \cdot e(1 + |x - y|)^{-\beta} dy. \end{aligned}$$

The first contribution is directly of order $T^{-\frac{\beta}{2}} \int |\nabla\Gamma(T, x)| dx \lesssim T^{-\frac{1}{2}-\frac{\beta}{2}}$ whereas the second contribution, using $\sup |\nabla\Gamma(T, \cdot)| \lesssim T^{-1-\frac{d}{2}}$ and a polar change of coordinates, is of order of

$$T^{-\frac{1}{2}-\frac{d}{2}} \int_0^{\sqrt{T}} r^{-\beta} r^{d-1} dr \lesssim \begin{cases} T^{-\frac{1}{2}-\frac{\beta}{2}} & \text{if } \beta < d, \\ \log(T) T^{-\frac{1}{2}-\frac{d}{2}} & \text{if } \beta = d. \end{cases}$$

To conclude, using that $\int |\nabla\Gamma(T, x)| dx \lesssim T^{-\frac{1}{2}}$, we obtain from (1.2.6)

$$\mathbb{E}[|\bar{u}(T, 0)|^2] \lesssim \begin{cases} T^{-1-\frac{\beta}{2}} & \text{if } \beta < d, \\ \log(T) T^{-1-\frac{d}{2}} & \text{if } \beta = d. \end{cases}$$

Similar computations lead to the same decay for $\mathbb{E}[|\nabla\bar{u}(T, 0)|^2]$ with an additional T^{-1} factor.

Our goal is to extend the previous bound in the non-perturbative regime. Our approach is more in the vein of the series of work [80, 88, 89, 92] than in [12, 13, 8, 10, 15] (treating the case of finite range of dependence (1.1.56)). More precisely, as in [88], we use a sensitivity calculus and functional inequalities, albeit in the much weaker form of MLSI (1.1.62) introduced in [59, 60], as in [80, 89]. As in [89, 92] however, our starting point is to prove fluctuation bounds on the time dependent flux, for all $t \in \mathbb{R}^+$,

$$q_e(t, \cdot) := a(\cdot) \left(\int_0^t \nabla u_e(s, \cdot) ds + e \right), \quad (1.2.7)$$

after averaging at scales $r \leq \sqrt{t}$. Yet, since functional inequalities cannot be easily iterated, one cannot rely on the same approach as for coefficients with a finite range of dependence. To this aim, as in [89], we rely on large-scale regularity, this time in the parabolic setting.

The general idea of large-scale regularity is to make use of the nice regularity theory that enjoy the homogenized operator $\partial_\tau - \nabla \cdot \bar{a} \nabla$. Indeed, the proximity of the two resolvent of the operators $\partial_\tau - \nabla \cdot a \nabla$ and $\partial_\tau - \nabla \cdot \bar{a} \nabla$ provided by homogenization allows to infer an improvement of regularity for $\partial_\tau - \nabla a \nabla$ on large-scales (say, scale much larger than the correlation length or the period in the periodic setting). In other words, on large-scales, the heterogeneous linear parabolic operator $\partial_\tau - \nabla \cdot a \nabla$ "inherits" a suitable version of the regularity theory for the homogenized linear parabolic operator $\partial_\tau - \nabla \cdot \bar{a} \nabla$. In the seminal work [16], Avellaneda and Lin observed that such "transfer" of regularity is true for periodic coefficient field a , and proved intrinsic $C^{1,1-}$ -a priori estimates on a -harmonic functions, meaning that the estimates are formulated with help of the so-called harmonic coordinates $x \mapsto e \cdot x + \phi_\xi(x)$. Later, such a regularity theory has been extended to the case of linear elliptic operators $-\nabla \cdot a \nabla$ with stationary and ergodic coefficients in [80, 8], where the large-scales are characterized by a random minimal scale (called in the first reference the minimal radius), for which stretched exponential moments are established under MLSI/MSG (1.1.62) and (1.1.63) in [80] as well as under a finite range of dependence (1.1.56) in [8]. Finally, the theory has been extended to the parabolic case in [26, 11] and for completeness we recall the main results and extend them in Appendix 2.4.2.

We obtain (in the non-perturbative regime) optimal decay estimates in time of the semigroup u_e , in terms of scaling, both for mildly and strongly correlated coefficient fields, with good stochastic integrability (stretched exponential moments).

Theorem 4 (Main result of Chapter 2). *Let $T \geq 1$ and $e \in \mathbb{R}^d$ be a unit vector of \mathbb{R}^d . For an arbitrary map f and $r > 0$ we denote by $f \star g_r$ its convolution with respect to the Gaussian kernel $g_r := r^{-d} e^{-\frac{|\cdot|^2}{r^2}}$.*

- For all $1 \leq r \leq \sqrt{T}$, the average of the time dependent flux defined in (1.2.7) and the time dependent corrector $\phi_e(T) := \int_0^T u_e(s, \cdot) ds$ satisfy:

$$|((q_e)_r(T), \nabla(\phi_e)_r(T)) - \mathbb{E}[(q_e)_r(T), \nabla(\phi_e)_r(T)]| \leq \mathcal{C}_{\star, d, \lambda, \beta}(r) r^{-\frac{d}{2}} \mu_\beta(T) (1 + \log^2(\frac{\sqrt{T}}{r})),$$

with

$$\mu_\beta(T) := \begin{cases} T^{\frac{d}{4} - \frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) & \text{if } \beta = d, \\ 1 & \text{if } \beta > d. \end{cases}$$

- We define the exponential kernel $\eta_r = r^{-d} e^{-\frac{|\cdot|^2}{R}}$. The semigroup u_e defined in (1.2.4) satisfies the optimal decay estimates: there exists a constant $c < +\infty$ depending on λ and d such that for all $R \geq \sqrt{T}$

$$\left(\int \eta_R\left(\frac{y}{c}\right) |u_e(T, y)|^2 dy \right)^{\frac{1}{2}} + \sqrt{T} \left(\int \eta_R\left(\frac{y}{c}\right) |\nabla u_e(T, y)|^2 dy \right)^{\frac{1}{2}} \leq \mathcal{C}_{\star, d, \lambda, \beta}(T) \eta_\beta(T),$$

with for all $T \geq 1$

$$\eta_\beta(T) = \begin{cases} T^{-\frac{1}{2} - \frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) T^{-\frac{1}{2} - \frac{d}{4}} & \text{if } \beta = d, \\ T^{-\frac{1}{2} - \frac{d}{4}} & \text{if } \beta > d. \end{cases}$$

The random variable $X \in \mathbb{R}_*^+ \mapsto \mathcal{C}_{*,d,\lambda,\beta}(X)$ depends on d, λ, β and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + 2\frac{d+1}{\beta \wedge d}}$ there exists some constant $C < +\infty$ depending on d, λ, β and α such that

$$\sup_{X>0} \mathbb{E} \left[\exp\left(\frac{1}{C} \mathcal{C}_{*,d,\lambda,\beta}^\alpha(X)\right) \right] \leq 2.$$

As applications, the decay of the semigroup gives an alternative proof of the bounds on the correctors recently obtained in [89, Th.1] and yields other results of interest in stochastic homogenization, extending the results of [88, 92] for this setting.

1.2.4 Quantitative homogenization theory for nonlinear elliptic equations and systems (Chapter 3)

In Chapter 3, we investigate the quantitative homogenization of monotone elliptic systems with random coefficients of the general type (1.1.4). The aim is to make Theorem 2 quantitative, in the all range $p \geq 2$, which is so far restricted to the case $p = 2$.

Whereas for $p = 2$, the homogenized operator \bar{a} (see (1.1.30)) has a good regularity theory provided it is smooth (which is proved in [10, 5, 6]), this is not clear in general as soon as $p > 2$. Indeed, in general, the homogenized monotone map \bar{a} satisfies the weaker monotonicity condition (1.1.15) than the original one for the operator a in (1.1.5), and presumably the homogenized operator $\nabla \cdot \bar{a}(\nabla)$ might not possess $C^{1,\alpha}$ -regularity even if the monotone operator $\nabla \cdot a(\cdot, \nabla)$ does. Since elliptic regularity plays an important role in quantitative homogenization, this raises interesting questions and will impose restrictions on the operators we can consider. We refer to Section 3.1.1 for more precisions.

Quantitative estimates on the two-scale expansion (1.1.31) are derived by the optimal quantification of the sub-linearity of the corrector ϕ_ξ but also of the linearized corrector $\frac{\partial \phi_\xi}{\partial \xi}$ (as we have already seen in the one-dimensional example in Sections 1.1.4 and 1.1.5). It implies that we have to linearise at some point the nonlinear operator, essentially for the two following reasons:

- In order to define the linearized corrector $\frac{\partial \phi_\xi}{\partial \xi}$, which is formally given by differentiating the corrector equation (1.1.23):

$$-\nabla \cdot Da(\cdot, \xi + \nabla \phi_\xi) \left(\text{Id} + \frac{\partial \phi_\xi}{\partial \xi} \right) = 0, \quad (1.2.8)$$

where we recall that D denotes the derivative of $\xi \mapsto a(\cdot, \xi)$.

- In order to control the sub-linearities $\varepsilon \phi_\xi(\frac{\cdot}{\varepsilon}) \rightarrow 0$ and $\varepsilon \frac{\partial \phi_\xi}{\partial \xi}(\frac{\cdot}{\varepsilon}) \rightarrow 0$ via functional inequalities of type LSI (1.1.57), which essentially imposes to linearize the equation of the corrector with respect to the randomness.

When linearizing the nonlinear problem, one obtains the linear operator $-\nabla \cdot Da(\cdot, \xi + \nabla \phi_\xi) \nabla$, with coefficients that are heterogeneous and depend on the corrector $\xi + \nabla \phi_\xi$ itself. Can we prove perturbative regularity (typically in form of Meyers' estimates) for this operator? In the specific case $p = 2$ treated in [5, 6, 68], the coefficients of the linearized operator are bounded from above and below, so that Meyers' estimates are standard (which allowed Fischer and Neukamm to essentially follow the linear proof in [68]). This is not the case for $p > 2$, and this linear equation is hard to handle for two reasons:

- The coefficients may be degenerate (for the p -Laplacian e.g.), and despite recent strong contributions on degenerate models, this degeneracy is currently out of reach. Large-scale regularity has been established for the Laplacian on the percolation cluster by Armstrong and Dario in [4] (and optimal convergence rates by Dario in [50]) and for linear elliptic systems with degenerate and unbounded coefficients under moment bounds assumptions by Bella, Fehrman and Otto in [24]. The main new difficulty we face here is that, unlike in [4, 24], the degeneracy is not prescribed a priori: it is given by the critical set of harmonic coordinates (that is the set of $x \in \mathbb{R}^d$ such that $\nabla\phi_\xi(x) + \xi = 0$, cf. (3.1.7)). Precise information on this critical set is however currently unavailable for $d > 2$ (even for the p -Laplacian, the unique continuation principle is not known to hold for $d > 2$ and $p > 2$, e.g. [111]).
- The coefficients field $Da(\cdot, \xi + \nabla\phi_\xi)$ is unbounded. Our sole a priori control is given by the growth exponent $\frac{p}{p-2}$ (the larger p , the weaker the integrability), which can be seen directly from (1.1.5) and (1.1.6): there exists C depending on λ and p such that for all $h \in \mathbb{R}^d$,

$$\frac{1}{C}(s + |\xi + \nabla\phi_\xi|^{p-2})|h|^2 \leq h \cdot Da(\cdot, \xi + \nabla\phi_\xi)h \leq C(s + |\xi + \nabla\phi_\xi|^{p-2})|h|^2.$$

The only a priori bound on the coefficients is deduced from (1.1.25) and implies only

$$\mathbb{E}[|Da(0, \xi + \nabla\phi_\xi)|^{\frac{p}{p-2}}] \lesssim 1 + |\xi|^p, \quad (1.2.9)$$

which becomes weaker as p becomes larger.

In order to avoid the degeneracy of the the linearized operator, we impose that $s > 0$ in (1.1.5) and (1.1.6), and therefore $-\nabla \cdot Da(\cdot, \xi + \nabla\phi_\xi)\nabla$ becomes uniformly elliptic. It rules out the p -Laplacian, but not the p -Laplacian regularized at 0, and only yields the non-degeneracy of the linearized operator in a perturbative way (it disappears in the regime when the solution has a large gradient). Doing so, the remaining difficulty is on the unboundness of the coefficients. By following the standard strategy for Meyers' estimates, passing by Caccioppoli's inequality (in a refined version obtained by optimizing the cut-off function, following an original idea of Bella and Schäffner in [25]) and a reverse Hölder's inequality (usually obtained from the Poincaré-Sobolev inequality), we prove Meyers type estimates for the operator $-\nabla \cdot Da(\cdot, \xi + \nabla\phi_\xi)\nabla$ which holds for the all range $p \geq 2$ in dimension $d = 3$ and for $p < \frac{2(d-1)}{d-3}$ in higher dimension $d \geq 4$ (due the lack of integrability in (1.2.9) for large p). For more details on the strategy, we refer to Section 3.2. Combining the Meyer' estimates with functional calculus in form of LSI (1.1.57), we manage to optimally quantify the sub-linearities of the correctors and the linearized correctors which leads to the optimal estimate of the convergence of the homogenization error.

Theorem 5 (Main result of Chapter 3-1). *Let $\xi, e \in \mathbb{R}^d$ and $\varepsilon > 0$. There exists two random variables $\mathcal{C}_{x,\xi}$ and $\mathcal{C}_{\varepsilon,\bar{u}}$ such that*

- For all $x \in \mathbb{R}^d$,

$$|\frac{\partial}{\partial\xi}\sigma_\xi(x)| + |\sigma_\xi(x)| + |\frac{\partial}{\partial\xi}\phi_\xi(x)| + |\phi_\xi(x)| \lesssim \mathcal{C}_{x,\xi} \times \begin{cases} 1 + \sqrt{|x|} & \text{for } d = 1, \\ \log^{\frac{1}{2}}(2 + |x|) & \text{for } d = 2, \\ 1 & \text{for } d > 2. \end{cases}$$

- The two-scale expansion (1.1.31) converges to \bar{u} with the following convergence rate:

$$\|\nabla u_\varepsilon - \nabla u_\varepsilon^{2sc}\|_{L^2(\mathbb{R}^d)} \leq \mathcal{C}_{\varepsilon,\bar{u}} \times \begin{cases} \sqrt{\varepsilon} & \text{for } d = 1, \\ \varepsilon \log^{\frac{1}{2}}(\varepsilon^{-1}) & \text{for } d = 2, \\ \varepsilon & \text{for } d > 2. \end{cases}$$

The two random variables $\mathcal{C}_{x,\xi}$ and $\mathcal{C}_{\varepsilon,\bar{u}}$ satisfy for some $\alpha > 0$ and constants c_ξ (depending on ξ) and $c_{\bar{u}}$ (depending on \bar{u}), the stretched exponential moment bound

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}[c_\xi(\mathcal{C}_{x,\xi})^\alpha] + \sup_{\varepsilon > 0} \mathbb{E}[c_{\bar{u}}(\mathcal{C}_{\varepsilon,\bar{u}})^\alpha] \leq 4.$$

We also investigate in Chapter 3 whether one can infer more properties on \bar{a} if we make more assumptions on a . The answer is more subtle than one would expect and we provide two non trivial cases where it can be inferred, one in the periodic setting and another in the random setting.

Theorem 6 (Main result of Chapter 3-2). *Let A be a Q -periodic Lipschitz matrix field. For all $\xi \in \mathbb{R}^d$, denote by $\psi_\xi \in W_{\text{per}}^{1,p}(Q)$ the unique weak solution of*

$$-\nabla \cdot A(x) |\nabla \psi_\xi + \xi|^{p-2} (\nabla \psi_\xi + \xi) = 0.$$

Assume that for all $\xi \in \mathbb{R}^d$, there exists $r > 0$ such that the r -tubular neighborhood $\mathcal{T}_r(\xi) = \{x + B_r \mid x \in \mathcal{C}(\xi)\}$ of the critical set $\mathcal{C}(\xi) = \{x \in \mathbb{R}^d \mid \xi + \nabla \psi_\xi(x) = 0\}$ is such that $\mathbb{R}^d \setminus \mathcal{T}_r(\xi)$ is a connected set. Then there exists $c > 0$ such that \bar{a} satisfies the strong monotonicity conditions (1.1.5) and (1.1.6) with $\lambda = c$.

The assumptions of the previous Theorem are quite strong. They are satisfied in dimension $d = 2$ by [2] (which shows that $\mathcal{C}(\xi) \cap Q$ is indeed a finite union of points). For $d > 2$ this is a widely open problem (somewhat related to unique continuation). For linear equations, this follows from [41].

Theorem 7 (Main result of Chapter 3-3). *Assume that $a(x, \xi) = A(x)(s + |\xi|^{p-2})\xi$ with $A(x) = b(x)\text{Id}$ for some scalar-valued function b and that for all $R \in SO(d)$, $b(R \cdot)$ and b have the same (joint) distribution (in which case A is statistically isotropic). Then there exists $c > 0$ such that \bar{a} satisfies the strong monotonicity conditions (1.1.5) and (1.1.6) with $\lambda = c$.*

The theory that we develop in this chapter unfortunately does not apply to probability measure which satisfies MLSI (1.2.8) and we refer to Chapter 6 for a short discussion about this issue.

1.2.5 Price's formula for infinite dimensional Gaussian measures (Chapter 4)

In Chapter 4, we extend the so-called Price's formula for infinite dimensional Gaussian field. Originally introduced by Price in [143] (for finite dimensional Gaussian measures), it states, given a family of Gaussian measures $\langle \cdot \rangle_c$ with covariance c , a general formula for the derivative of $\langle F \rangle_c$ with respect to c for any smooth random variable F of the realizations.

In finite dimension, since the work of Price, the formula has been extended by McMahan in [122] and more recently by Voigtlaender in [154]. The formula, in its most simple version, is stated as follows. We consider a family of Gaussian measures $\langle \cdot \rangle_C$ in \mathbb{R}^N (with $N \in \mathbb{N}$) characterized by its covariance matrix (symmetric definite-positive) $C \in \mathbb{R}^{N \times N}$ and $F \in C^2(\mathbb{R}^N)$ with polynomial growth. There holds for all $i, j \in \llbracket 1, N \rrbracket$

$$\frac{\partial}{\partial C_{ij}} \langle F \rangle_C = \frac{1}{2} \langle \partial_{ij} F \rangle_C. \quad (1.2.10)$$

Formula (1.2.10) follows by a direct computation. Indeed, since F has polynomial growth, one has

$$\frac{\partial}{\partial C_{ij}} \langle F \rangle_C = \frac{1}{(2\pi)^d} \int F(x) \frac{\partial}{\partial C_{ij}} \left[\exp\left(-\frac{1}{2} x \cdot C^{-1} x\right) \sqrt{\det C^{-1}} \right] dx. \quad (1.2.11)$$

Then, for all $x \in \mathbb{R}^d$, we compute

$$\begin{aligned} \frac{\partial}{\partial C_{ij}} \left[\exp\left(-\frac{1}{2}x \cdot C^{-1}x\right) \sqrt{\det C^{-1}} \right] &= \frac{1}{2} \left[x \cdot C^{-1} \frac{\partial C}{\partial C_{ij}} C^{-1}x - \text{tr} C^{-1} \frac{\partial C}{\partial C_{ij}} \right] \exp\left(-\frac{1}{2}x \cdot C^{-1}x\right) \sqrt{\det C^{-1}} \\ &= \frac{1}{2} \text{tr} (C^{-1}x \otimes C^{-1}x - C^{-1}) \frac{\partial C}{\partial C_{ij}} \exp\left(-\frac{1}{2}x \cdot C^{-1}x\right) \sqrt{\det C^{-1}} \\ &= \frac{1}{2} \partial_{ij} \left[\exp\left(-\frac{1}{2}x \cdot C^{-1}x\right) \sqrt{\det C^{-1}} \right]. \end{aligned}$$

Therefore, by two integration by parts (which are justified since F is of polynomial growth), (1.2.11) turns into

$$\frac{\partial}{\partial C_{ij}} \langle F \rangle_C = \frac{1}{2} \int \partial_{ij} F(x) \frac{1}{(2\pi)^d} \exp\left(-\frac{1}{2}x \cdot C^{-1}x\right) \sqrt{\det C^{-1}} dx,$$

which is exactly (1.2.10).

In the infinite dimensional context, the family of measure $\langle \cdot \rangle_c$ are now Gaussian measures on a Banach space X . We show that a version of formula (1.2.10) holds if we assume an additional structure on X (which is classical in the theory, see [33]).

1.2.6 Bias in the Representative Volume Element method: periodize the ensemble instead of its realizations (Chapter 5)

In Chapter 5, we continue the investigation of the quantitative homogenization of linear elliptic systems (1.2.1). We analyse the so-called representative volume element method (RVE) that we use to approximate the homogenized matrix given by, for all $e \in \mathbb{R}^d$, $a_{\text{hom}} = \mathbb{E}[a(0)(\nabla\phi_e(0) + e)]$.

The RVE method. Suppose that the coefficient field a is L -periodic, meaning that $a(x + Lk) = a(x)$ for all x and $k \in \mathbb{Z}^d$. Given a Cartesian coordinate direction $i = 1, \dots, d$ and denoting by e_i the unit vector in the i -th direction, we consider the corrector (up to additive constants) $\phi_i^{(1)}$ as the L -periodic solution of

$$-\nabla \cdot a(\nabla\phi_i^{(1)} + e_i) = 0, \tag{1.2.12}$$

to which we associate the corresponding homogenized matrix:

$$\bar{a}e_i := \int_{[0,L]^d} a(\nabla\phi_i^{(1)} + e_i). \tag{1.2.13}$$

Observe that the notation \bar{a} without reference to the period L is legitimate since (1.2.13) is equivalent to $\bar{a}e_i = \lim_{R \uparrow \infty} \int_{[0,R]^d} a(\nabla\phi_i^{(1)} + e_i)$. In the periodic case, (1.2.13) in fact coincides with the homogenized coefficient a_{hom} . In the random case, we expect that (1.2.13) provides a good approximation of a_{hom} as $L \uparrow +\infty$. An important choice in the method is the strategy of periodization of the coefficients. Two strategies can be employed, whence we quantify the convergence for one of them.

Two strategies of periodizing. In order to apply the RVE method in form of (1.2.13), considered as an approximation for a_{hom} , one needs to produce samples a of L -periodic coefficient fields connected to the given ensemble $\langle \cdot \rangle$. The goal of this chapter is to prove that, to procure such L -periodic samples, the strategy of "periodizing the ensemble" leads to a smaller error than the

strategy of "periodizing the realizations". The first strategy of "periodizing the realizations" is a direct periodization procedure: Taking a coefficient field a in \mathbb{R}^d , we restrict it to the box $[0, L]^d$ and then periodically extend it. This defines a map $a \mapsto a_L$. We then take $\overline{a_L}$, cf. (1.2.13), as an approximation for a_{hom} . One unfavorable aspect of this strategy is obvious: The pushforward of $\langle \cdot \rangle$ under this map $a \mapsto a_L$ is no longer stationary – an imagined glance at a typical realization would reveal d families of parallel artificial hypersurfaces. Suboptimal convergence rates for this strategy has been establish in [34] and numerical experiments in [145] is in favour of a bias of order $O(L^{-1})$. To the best of our knowledge, no formal argument in favour of such a behaviour are established in the literature and such arguments will be added in the work in preparation [48].

The second strategy of "periodizing the ensemble" is more subtle: Given an ensemble $\langle \cdot \rangle$, one construct a "related" stationary ensemble $\langle \cdot \rangle_L$ of L -periodic fields samples a from $\langle \cdot \rangle_L$ and takes \bar{a} as an approximation. This second strategy was (probably not for the first time) laid out in [79, Remark 5]; we will give the construction of the map $\langle \cdot \rangle \rightsquigarrow \langle \cdot \rangle_L$ for a specific but relevant class of $\langle \cdot \rangle$ given in Assumption 8. This approach obviously capitalizes on the knowledge of the ensemble $\langle \cdot \rangle$ and not just of a single realization (a "snapshot"), in the sense of "known unknowns" as opposed to "unknown unknowns". This is in contrast with the numerical analysis on inferring a_{hom} in [127], or on constructing effective boundary conditions in [114] from a snapshot.

Fluctuations and bias. In this chapter, we are interested in the second strategy and in the study of its bias (also called systematic error): How much do the expected value $\langle \bar{a} \rangle_L$ deviate from a_{hom} , which by qualitative theory is their common limit for $L \uparrow \infty$. We shall prove that

$$\langle \bar{a} \rangle_L - a_{\text{hom}} = O(L^{-d}), \quad (1.2.14)$$

see Theorems 9. Here L should be thought of as the (non-dimensional) ratio between the actual period L and a suitably defined correlation length of $\langle \cdot \rangle$ set to unity. The quantification of the convergence in L is clearly of practical interest: After a discretization that resolves the correlation length, the number of unknowns of the linear algebra problem (1.2.12) scales with L^d for $L \gg 1$. Numerical experiments confirm the $O(L^{-d})$ scaling, see for instance [103].

We note that fluctuations (which are at the origin of the random part of the error), as for instance measured in terms of the square root of the variance, are in many situations proven to be of the order (see, *e.g.*, [88, Th. 2])

$$\langle |\bar{a} - \langle \bar{a} \rangle_L|^2 \rangle_L^{\frac{1}{2}} = O(L^{-\frac{d}{2}}), \quad (1.2.15)$$

and the same is expected to hold for the other strategy [63, (3.3)]

$$\langle |\overline{a_L} - \langle \overline{a_L} \rangle|^2 \rangle^{\frac{1}{2}} = O(L^{-\frac{d}{2}}).$$

Hence the variance scales like the inverse of the volume L^d of the periodic cell $[0, L]^d$, as if we were averaging over $[0, L]^d$ some field of unit range of dependence instead of the long-range correlated $a(e_i + \nabla \phi_i)$. In view of this identical fluctuation scaling for both strategies, the different bias scaling is significant in the most relevant dimension $d = 3$, which we mostly focus on in this paper: For the first strategy (by periodization of the realizations), the bias dominates, so that taking the empirical mean of $\overline{a_L}$ over many realizations a does not substantially reduce the total error. It does so in the second scenario, which suggests to use variance reduction methods [45, 69].

Theoretical results on the random error in RVE, at least for the second strategy like in (1.2.15), are by now abundant, starting from [82, Theorem 2.1] for a discrete medium with i. i. d. coefficient,

over [91, Theorem 1] for a class of continuum media based on the Poisson point process, to the leading-order identification of the variance in [55, Theorem 2]. The last result arises from the characterization of leading-order variances in stochastic homogenization in general, starting from [128, Theorem 2.1] for correctors, and is in the spirit of the general approach laid out in [94].

Theoretical results on the systematic error in RVE, again for the second strategy as in (1.2.14), seem to have been restricted to the case of a discrete medium with i. i. d. coefficients, see [88, Proposition 3], where the construction of $\langle \cdot \rangle_L$ is obvious. The argument for [88, Proposition 3] is based on a (necessarily non-stationary) coupling of $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$, and introduces a massive term into the corrector equation in order to screen the resulting boundary layer, which leads to a logarithmically worse estimate than (1.2.14). Our analysis avoids this coupling and suggests that such a logarithmic correction is artificial. (Incidentally, the phenomenon that the bias decays to an order that is twice the order of the fluctuation decay occurs also in the analysis of the homogenization error $(-\nabla \cdot a \nabla)^{-1} f - (-\nabla \cdot a_{\text{hom}} \nabla)^{-1} f$ itself: While the variance can be characterized to order $O(L^{-d})$, where $L \gg 1$ now is the ratio between the scale of f and the correlation length, see [56, Theorem 1], the expectation seems to be characterized to order $O(L^{-2d})$, see [61].)

From a practical point of view, the first strategy seems more appealing since it is less intrusive, because it only requires snapshots (which could be directly obtained from measures). However, it has been realized quite early in the numerical literature (see, e.g., [102]) that it is less accurate than the second strategy, because, as explained above, it suffers from a large bias. Based on experiments on periodic coefficient fields, oversampling [99] and filtering [32] strategies were first proposed in order to achieve a better accuracy. Unfortunately, their efficiency is limited for random media (they cannot perform better than $O(L^{-1})$). This motivated an alternative approach that screens the boundary effects by adding a massive zero-th order term to the equation (1.2.12) defining the correctors [86], which is nonetheless inspired from theoretical considerations [82]. This was combined with a sort of domain decomposition in [127] and proved to achieve (almost) the best possible accuracy for a given computational cost in the case where only snapshots are available [127, Prop. 1.1 & Th. 1.2], with an error $O(\delta)$ at the cost of $O(\delta^{-2})$ elementary operations (up to logarithms).

Assumptions and formulation of rigorous result We now introduce the class of stationary ensembles $\langle \cdot \rangle$ of λ -uniform coefficient fields a considered in this paper. Loosely speaking, it consists of regular pointwise transformation, see (1.2.17), of a stationary Gaussian field g with integrable covariance function. The ensemble is determined by the (translation invariant) covariance function c and the pointwise transformation A . Assumption 8 collects the two quantitative hypotheses on c and A needed, see (1.2.16) and (1.2.18), respectively.

Assumption 8. *Let $\langle \cdot \rangle$ be a stationary and centered Gaussian ensemble of scalar³¹ fields g on \mathbb{R}^d , as determined by the covariance function $c(x) := \langle g(x)g(0) \rangle$. We assume that there exists an $\alpha > 0$ such that*

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^{d+n+2\alpha} |\nabla^n c(x)| < \infty \quad \text{for } n \in \{0, \dots, d+1\}. \quad (1.2.16)$$

We identify $\langle \cdot \rangle$ with its push forward under the map

$$g \mapsto a := (x \mapsto A(g(x))), \quad (1.2.17)$$

³¹For notational simplicity, we consider scalar Gaussian field, however, the Gaussian field g may take values in any finite dimensional linear space.

where $A: \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is such that the coefficient field a is λ -uniformly elliptic. We assume that

$$\sup_{g \in \mathbb{R}} |A'(g)| + |A''(g)| < \infty. \quad (1.2.18)$$

The purely technical reason for going beyond first-order derivatives in characterizing a decay of order $d + \alpha$ in (1.2.16) is that it implies the following property on the level of the (automatically non-negative) Fourier transform

$$0 \leq \mathcal{F}c(k) \lesssim (1 + |k|)^{-d-2\alpha} \quad \text{for } k \in \mathbb{R}^d,$$

which allows us to appeal to [101]. However, the class of ensembles $\langle \cdot \rangle$ from Assumption 8 is mainly chosen because it allows for a straightforward periodization, namely through periodizing the covariance function. Note that since $\alpha > 0$, (1.2.16) in particular ensures that the sum in (1.2.19) converges absolutely.

We periodize the ensemble $\langle \cdot \rangle$ by periodizing the covariance function c (see (1.2.19) below). This periodization is natural because it formally amounts to considering realizations of the ensemble $\langle \cdot \rangle$ conditioned on $[0, L)^d$ -periodicity. We sketch a formal argument for it.³² We may indeed represent the Gaussian field g generated from $\langle \cdot \rangle$ by $g = m * \xi$, where ξ is the standard Gaussian white noise and where $m := \mathcal{F}^{-1}(\sqrt{\mathcal{F}c})$ is the model associated to c . Assuming that $\mathcal{F}(c) > 0$ (since c is a covariance function, we already have $\mathcal{F}(c) \geq 0$), we construct the $[0, L)^d$ -periodic realizations $g_L = m * \xi_L$, for the $[0, L)^d$ -periodic realization of the white noise $\xi_L = \sum_{k \in \mathbb{Z}^d} \xi(\cdot - kL) \mathbf{1}_{[0, L)^d}(\cdot - kL)$. Computing the covariance function for such realizations ξ yields (1.2.19).

Definition 1.2.1. For given $L < \infty$, let $\langle \cdot \rangle_L$ be the stationary and centered Gaussian ensemble of scalar fields g defined by the covariance function

$$c_L(x) := \sum_{k \in \mathbb{Z}^d} c(x + Lk). \quad (1.2.19)$$

Clearly, the covariance function c_L and thus the realizations g are L -periodic. As in Assumption 8 we identify $\langle \cdot \rangle_L$ with its push forward under (1.2.18).

Equipped with the ensemble $\langle \cdot \rangle_L$, we can state our main result.

Theorem 9. Let $d > 2$ and A be symmetric. Under the Assumption 8 on $\langle \cdot \rangle$, for all L , and with the Definition 1.2.1 of $\langle \cdot \rangle_L$ we have for the expectation $\langle \bar{a} \rangle_L$ of \bar{a} defined in (5.1.3)

$$\limsup_{L \uparrow \infty} L^d |\langle \bar{a} \rangle_L - a_{\text{hom}}| < \infty.$$

Note that the assumption that A is symmetric in Theorem 9 is only for convenience and allow us to simplify the use of second order correctors. For more precisions, we refer to Section. 5.2.2. We also give in Theorem 30 a refinement of Theorem 9 by characterizing the leading-order error term.

³²This formal argument can be made rigorous if the ensemble $\langle \cdot \rangle$ itself is discrete and periodic with a very large period with respect to L .

Chapter 2

Optimal decay of the parabolic semigroup for linear elliptic systems with correlated coefficient fields

We study the large scale behaviour of elliptic systems with stationary random coefficient that have only slowly decaying correlations. To this aim we analyse the so-called corrector equation, a degenerate elliptic equation posed in the probability space. In this chapter, we use a parabolic approach and optimally quantify the time decay of the semigroup. For the theoretical point of view, we prove an optimal decay estimate of the gradient and flux of the corrector when spatially averaged over a scale $R \geq 1$. For the numerical point of view, our results provide convenient tools for the analysis of various numerical methods. This chapter is based on the Preprint [47].

The chapter is organized as follows: In Section 2.1.1 we introduce notations and make the assumptions on the coefficient field precise. In Section 2.1.2 we state our main result and its consequences. Section 2.2 describes the strategy of the proof and list the auxiliary results needed in the proof of the main theorem. The results are proved in Section 2.3.

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2.1 Assumptions, notations and main results

We decide in this chapter to use scalar notations but the analysis remains true for systems. Also, we use the abbreviations "r.h.s" and "l.h.s" for right hand side and left hand side respectively.

2.1.1 Assumptions and notations

We define in this section the assumptions and notations that we use specifically in this chapter.

Assumptions on the coefficient field. We consider a coefficient field $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ of the form, for all $x \in \mathbb{R}^d$

$$a(x) := A(g(x)), \tag{2.1.1}$$

for a given Lipschitz map $A : \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, for some $k \geq 1$, which takes values in the set of uniformly elliptic and bounded matrices, and a given Gaussian field g . More precisely we assume that there exists $0 < \lambda \leq 1$ (fixed once for all) such that for all $s \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^d$

$$\lambda|\xi|^2 \leq \xi \cdot A(s)\xi \text{ and } |A(s)\xi| \leq |\xi|, \tag{2.1.2}$$

and that $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ is a stationary Gaussian field¹ on \mathbb{R}^d of zero mean on some probability space $(\Omega, \mathcal{A}, \langle \cdot \rangle)$, characterized by its covariance function $c : x \in \mathbb{R}^d \mapsto \langle g(0) \otimes g(x) \rangle$. The ensemble² $\langle \cdot \rangle$ satisfies the standard stationarity and ergodicity assumptions. We refer to Section 1.1.2 for precisions on the qualitative assumptions.

We assume that there exists a smooth non-increasing function γ such that for all $x \in \mathbb{R}^d$

$$|c(x)| \leq \gamma(|x|). \tag{2.1.3}$$

¹It can be generated via the model $g = m \star W$, for $m \in L^2(\mathbb{R}^d)$ and W a white noise. See Section 1.1.2.

²We use the symbol $\langle \cdot \rangle$ to address both the ensemble and to denote its expectation operator.

In this context, we recall that $\langle \cdot \rangle$ satisfies a multiscale logarithmic Sobolev inequality (see Section 1.1.5 for precisions), that we restate here. For all square integrable functional F of a (see [60, Th.3.1,(ii)]): there exists $\rho > 0$ such that

$$\text{Ent}(F(a)) := \langle F^2(a) \log(F^2(a)) \rangle - \langle F^2(a) \rangle \langle \log(F^2(a)) \rangle \leq \frac{1}{\rho} \left\langle \int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} F(a)|^2 dx d\ell \right\rangle, \quad (2.1.4)$$

where for all $x \in \mathbb{R}^d$ and $\ell \in [1, +\infty)$

$$\partial_{x,\ell}^{\text{fct}} F := \sup \left\{ \limsup_{h \rightarrow 0} \frac{F(a + h\delta a) - F(a)}{h}, \sup_{B_\ell(x)} |\delta a| \leq 1, \delta a = 0 \text{ outside } B_\ell(x) \right\}, \quad (2.1.5)$$

and where the weight π satisfies

$$\pi(\ell) = |\gamma'(\ell)|.$$

In this contribution, we specialize to algebraic decay and assume that there exists $\beta > 0$ such that for all $\ell \in [1, +\infty)$

$$\gamma(\ell) = (\ell + 1)^{-\beta}. \quad (2.1.6)$$

General notation. For $\mathcal{U} \subset \mathbb{R}^d$ open and $p \in [1, +\infty]$, we denote by $L^p(\mathcal{U})$ the Lebesgue space on \mathcal{U} with exponent p , that is, the set of measurable functions $f : \mathcal{U} \rightarrow \mathbb{R}^d$ satisfying

$$\|f\|_{L^p(\mathcal{U})} := \left(\int_{\mathcal{U}} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty,$$

and where for $p = +\infty$

$$\|f\|_{L^\infty(\mathcal{U})} := \inf\{C > 0 \mid |f(x)| \leq C \text{ for almost all } x \in \mathcal{U}\}.$$

The vector space of functions on \mathbb{R}^d which belongs to $L^p(\mathcal{U})$ whenever \mathcal{U} is bounded is denoted by $L_{\text{loc}}^p(\mathbb{R}^d)$. If $|\mathcal{U}| < +\infty$ and $f \in L^1(\mathcal{U})$, then we write

$$\int_{\mathcal{U}} f(x) dx := \frac{1}{|\mathcal{U}|} \int_{\mathcal{U}} f(x) dx.$$

For all \mathcal{U} , we denote by $H^1(\mathcal{U})$ the space of all measurable functions $f : \mathcal{U} \rightarrow \mathbb{R}^d$ in $L^2(\mathcal{U})$ such that ∇f is in $L^2(\mathcal{U})$. We also define $H_{\text{loc}}^1(\mathbb{R}^d)$ the space of functions which belongs to $H^1(\mathcal{U})$ whenever \mathcal{U} is bounded.

For all $p \in [1, +\infty)$, we denote by $L_{\langle \cdot \rangle}^p(\Omega)$ the space of random variables $X : \Omega \rightarrow \mathbb{R}^d$ satisfying

$$\langle X^p \rangle^{\frac{1}{p}} < +\infty.$$

If \mathcal{B} is a Banach space, then for all $p \in [1, +\infty)$, we denote by $L^p(\mathbb{R}^d, \mathcal{B})$ (resp. $L_{\text{loc}}^p(\mathbb{R}^d, \mathcal{B})$) the space of measurable functions $f : \mathbb{R}^d \rightarrow \mathcal{B}$ such that $\|f(\cdot)\|_{\mathcal{B}} \in L^p(\mathbb{R}^d)$ (resp. $\|f(\cdot)\|_{\mathcal{B}} \in L_{\text{loc}}^p(\mathbb{R}^d)$).

For all time interval $I := [t_1, t_2)$ and open subset $\mathcal{U} \subset \mathbb{R}^d$, we define the function space

$$H_{\text{par}}^1(I \times \mathcal{U}) := \{u \in L^2(I, H^1(\mathcal{U})) \mid \partial_\tau u \in L^2(I, H^{-1}(\mathcal{U}))\}.$$

and we say that $u \in H_{\text{par}}^1(I \times \mathcal{U})$ is a weak solution of

$$\begin{cases} \partial_\tau u - \nabla \cdot a \nabla u = \nabla \cdot f & \text{in } I \times \mathcal{U}, \\ u(t_1) = \nabla \cdot g, \end{cases}$$

for r.h.s $f \in L^2(I \times \mathcal{U})^d$ and initial data $g \in L^q(\mathcal{U})^d$ (for some $q \in [1, +\infty]$) if for all $\psi \in C_c^\infty(I \times \mathcal{U})$

$$\begin{aligned} & - \int_{\mathcal{U}} \int_{t_1}^{t_2} u(t, x) \partial_\tau \psi(t, x) dt dx + \int_{\mathcal{U}} \int_{t_1}^{t_2} \nabla u(t, x) \cdot a(x) \nabla \psi(t, x) dt dx + \int_{\mathcal{U}} g \cdot \nabla \psi(t_1, x) dt dx \\ & = - \int_{\mathcal{U}} \int_{t_1}^{t_2} f(t, x) \nabla \psi(t, x) dt dx. \end{aligned}$$

For all $R \geq 1$, we define the exponential kernel η_R by

$$\eta_R := \frac{1}{R^d} e^{-\frac{|\cdot|}{R}},$$

and the Gaussian kernel g_R by

$$g_R := \frac{1}{R^d} e^{-\frac{|\cdot|^2}{R^2}}.$$

For all measurable functions f and all $r > 0$, we denote by f_r the convolution with the Gaussian kernel g_r , namely

$$f_r := f \star g_r = \int f(y) g_r(\cdot - y) dy.$$

We say that a random field $X : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ is stationary if we have for all $x \in \mathbb{R}^d$

$$X(a, \cdot + x) = X(a(\cdot + x), \cdot) \text{ almost surely.} \quad (2.1.7)$$

For all $R > 0$ and $(s, x) \in \mathbb{R}^{d+1}$, we write $B_R(x) := \{y \in \mathbb{R}^d \mid |x - y| < R\}$ for the ball of radius R centered at x and $C_R(s, x) := (s - R^2, s) \times B_R(x)$ for the parabolic cylinder centered at (s, x) and of radius R (for $(s, x) = (0, 0)$, we do not write the dependance on (s, x)). We use the short-hand notation $\lesssim_{\alpha_1, \dots, \alpha_n}$ for $\leq C$ for a constant C which depends only on the parameters $(\alpha_i)_{i \in [1, n]}$.

We write for all $(a, b) \in \mathbb{R}^2$, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Homogenization theory. We denote by $\phi_e(\cdot, a) \in H_{\text{loc}}^1(\mathbb{R}^d)$ the corrector, in the direction of a unit vector e of \mathbb{R}^d , as the unique distributional solution of, for almost all realization of a

$$\begin{cases} -\nabla \cdot a(\nabla \phi_e + e) = 0 \text{ in } \mathbb{R}^d, \\ \text{with } \limsup_{R \rightarrow +\infty} \frac{1}{R} \left(\int_{B_R} |\phi_e(x)|^2 dx \right)^{\frac{1}{2}} = 0 \text{ and } \int_{B_1} \phi_e(x) dx = 0. \end{cases} \quad (2.1.8)$$

For the existence of correctors, we refer to [80, Lem. 1].

For all $T \geq 1$, we denote by $\phi_{e, T}$ the massive corrector, defined as the Lax-Milgram solution in $H_{\text{uloc}}^1(\mathbb{R}^d) := \{\psi \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \sup_{x \in \mathbb{R}^d} \int_{B_1(x)} |\psi|^2 + |\nabla \psi|^2 < +\infty\}$, to

$$\frac{1}{T} \phi_{e, T} - \nabla \cdot a(\nabla \phi_{e, T} + e) = 0 \text{ in } \mathbb{R}^d. \quad (2.1.9)$$

For the existence and uniqueness of the massive correctors, we refer to [84]. Likewise, we denote by ϕ_e^* and $\phi_{e,T}^*$ the solutions of (2.1.8) and (2.1.9) with a replaced by a^* , the transposed field of a . We denote by $u_e \in H_{\text{uloc}}^{1,\text{par}}$ the semigroup associated with the corrector problem (2.1.8) defined as the weak solution of

$$\begin{cases} \partial_T u_e - \nabla \cdot a \nabla u_e = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u_e(0) = \nabla \cdot a(\cdot)e, \end{cases} \quad (2.1.10)$$

with

$$H_{\text{uloc}}^{1,\text{par}} := \left\{ u \in C^0(\mathbb{R}_*^+, H_{\text{loc}}^1(\mathbb{R}^d)) \mid \sup_{T>0} \sup_{R \geq \sqrt{T}} \int_{B_R} |(T \nabla u(T, x), \sqrt{T} u(T, x))|^2 dx + \left| \int_0^T (\nabla u(s, x), \frac{1}{\sqrt{T}} u(s, x)) ds \right|^2 dx < +\infty \right\}.$$

For existence and uniqueness of u_e , we refer to [92, Lemma 1].

We also introduce the associated fluxes

$$q_e := a(\nabla \phi_e + e), \quad (2.1.11)$$

for all $T \geq 1$

$$q_{e,T} := a(\nabla \phi_{e,T} + e), \quad (2.1.12)$$

and for all $t \geq 0$

$$q_e(t, \cdot) := a(\cdot) \left(\int_0^t \nabla u_e(s, \cdot) ds + e \right), \quad (2.1.13)$$

as well as the associated time dependent corrector, for all $t \geq 0$

$$\phi_e(t, \cdot) := \int_0^t \nabla u_e(s, \cdot) ds. \quad (2.1.14)$$

We introduce the flux corrector $\sigma = (\sigma_{i,j,k})_{(i,j,k) \in \llbracket 1,d \rrbracket^3}$ as the unique distributional solution of, for almost all realization of a

$$\begin{cases} \nabla \cdot \sigma_i = q_{e_i}, \\ -\Delta \sigma_{i,j,k} = \partial_j (e_k \cdot q_{e_i}) - \partial_k (e_j \cdot q_{e_i}) \text{ in } \mathbb{R}^d, \\ \text{with } \limsup_{R \rightarrow +\infty} \frac{1}{R} \left(\int_{B_R} |\sigma_i(x)|^2 dx \right)^{\frac{1}{2}} = 0 \text{ and } \int_{B_1} \sigma_i(x) dx = 0, \end{cases} \quad (2.1.15)$$

where $(\nabla \cdot \sigma_i)_j = \sum_{k=0}^d \partial_k \sigma_{i,j,k}$ and ∂_i denotes the partial derivative with respect to the single coordinate x_i . For existence and uniqueness, we refer to [80, Lem. 1].

Finally, for all $T \geq 1$, we denote by $\sigma_T = (\sigma_{T,i,j,k})_{(i,j,k) \in \llbracket 1,d \rrbracket^3}$ the massive flux corrector, defined as the Lax-Milgram solution in $H_{\text{uloc}}^1(\mathbb{R}^d)$ to

$$\frac{1}{T} \sigma_{T,i,j,k} - \Delta \sigma_{T,i,j,k} = \partial_j (e_k \cdot q_{e_i,T}) - \partial_k (e_j \cdot q_{e_i,T}). \quad (2.1.16)$$

For the existence and uniqueness of the massive flux corrector, we refer to [84].

The quantities $u_e, \nabla \phi_e, \nabla \sigma, q_e, \phi_{e,T}$ and $q_{e,T}$ are stationary in the sense of (2.1.7), which implies that the distribution of their convolution with some smooth function f , under the stationary ensemble $\langle \cdot \rangle$, does not depend on the space variable. In the following, we do not distinguish between $F \star f(0)$ and $F \star f$ in our notation, for all stationary random fields F .

2.1.2 Quantitative results

Our first main result is split in two quantitative estimates on averages of the time dependent flux (2.1.13). First, we show that the fluctuations of $(q_e)_r(T)$ on scale $r \in [1, \sqrt{T}]$ decays as the central limit theorem scaling $r^{-\frac{d}{2}}$ times some growth in time which depends on the correlation (in particular, in the case $\beta > d$, we get exactly the central limit theorem scaling). Second, we show that the fluctuations of particular averages $q_e(r^2) \star f_r$, for all $r \geq 1$ and f_r which behaves like $\int_1^{r^2} \nabla g_{\sqrt{s}} ds$, has some growth in r depending on the correlation. The first result is a key estimate to obtain the optimal decay in time of the semigroup u_e , whereas the second is needed to get the optimal growth of the correctors state in Corollary 3. We prove those estimates for stretched exponential moments.

Theorem 10 (Fluctuations of averages of the time dependent flux). *Let $T \geq 1$ and $e \in \mathbb{R}^d$ be a unit vector of \mathbb{R}^d .*

- For all $1 \leq r \leq \sqrt{T}$, we have

$$|((q_e)_r(T), \nabla(\phi_e)_r(T)) - \langle ((q_e)_r(T), \nabla(\phi_e)_r(T)) \rangle| \leq C_{\star, d, \lambda, \beta}(r) r^{-\frac{d}{2}} \mu_\beta(T) (1 + \log^2(\frac{\sqrt{T}}{r})), \quad (2.1.17)$$

with

$$\mu_\beta(T) := \begin{cases} T^{\frac{d}{4} - \frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) & \text{if } \beta = d, \\ 1 & \text{if } \beta > d. \end{cases} \quad (2.1.18)$$

- For all $r \geq 1$ and function $f_r \in C_b^1(\mathbb{R}^d)$ which satisfies for all $x \in \mathbb{R}^d$

$$|f_r(x)| \lesssim |x| \int_1^{r^2} s^{-1} g_{\sqrt{s}}(x) ds \text{ and } |\nabla f_r(x)| \lesssim |x|^2 \int_1^{r^2} s^{-2} g_{\sqrt{s}}(x) ds, \quad (2.1.19)$$

we have

$$|(q_e(r^2), \nabla(\phi_e)(r^2)) \star f_r - \langle (q_e(r^2), \nabla(\phi_e)(r^2)) \star f_r \rangle| \leq C_{\star, d, \lambda, \beta}(r) \chi_{d, \beta}(r), \quad (2.1.20)$$

with

$$\chi_{d, \beta}(r) := \begin{cases} (r+1)^{1-\frac{\beta}{2}} & \text{for } \beta < 2 \text{ and } d > 2, \\ (r+1)^{1-\frac{\beta}{2}} \log(r+2) & \text{for } \beta \leq 2 \text{ and } d = 2, \\ \log^{\frac{1}{2}}(r+2) & \text{for } \beta = 2 \text{ and } d > 2 \text{ or } \beta > 2 \text{ and } d = 2, \\ 1 & \text{for } \beta > 2 \text{ and } d > 2. \end{cases} \quad (2.1.21)$$

The random variable $r \in \mathbb{R}^+ \mapsto C_{\star, d, \lambda, \beta}(r)$ depends on d, λ, β and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + 2\frac{d+1}{\beta \wedge d}}$ there exists some constant $C < +\infty$ depending on d, λ, β and α such that

$$\sup_{r \geq 0} \left\langle \exp\left(\frac{1}{C} C_{\star, d, \lambda}^\alpha(r)\right) \right\rangle \leq 2. \quad (2.1.22)$$

Theorem 10 implies the following optimal decay in time of the semigroup u_e (defined in (2.1.10)) and of its gradient. This result is in the spirit of [88, Th.1] established in the discrete setting and extends [92, Cor.4] and [8, Th.9.1] established in the case where the coefficients are randomly distributed according to a stationary ensemble of finite range of dependence to the Gaussian setting.

Corollary 1 (Decay of the semigroup). *There exists a constant $c < +\infty$ depending on λ and d such that for all $T \geq 1$, $R \geq \sqrt{T}$ and unit vector $e \in \mathbb{R}^d$*

$$\left(\int \eta_R\left(\frac{y}{c}\right) |u_e(T, y)|^2 dy \right)^{\frac{1}{2}} + \sqrt{T} \left(\int \eta_R\left(\frac{y}{c}\right) |\nabla u_e(T, y)|^2 dy \right)^{\frac{1}{2}} \leq \mathcal{C}_{*,d,\lambda,\beta}(T) \eta_\beta(T), \quad (2.1.23)$$

with for all $T \geq 1$

$$\eta_\beta(T) = \begin{cases} T^{-\frac{1}{2}-\frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) T^{-\frac{1}{2}-\frac{d}{4}} & \text{if } \beta = d, \\ T^{-\frac{1}{2}-\frac{d}{4}} & \text{if } \beta > d, \end{cases} \quad (2.1.24)$$

and for some random variable $T \in \mathbb{R}_*^+ \mapsto \mathcal{C}_{*,d,\lambda,\beta}(T)$ which depends on d, λ, β and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + 2\frac{d+1}{\beta \wedge d}}$ there exists some constant $C < +\infty$ depending on d, λ, β and α such that

$$\sup_{T>0} \left\langle \exp\left(\frac{1}{C} \mathcal{C}_{*,d,\lambda,\beta}^\alpha(T)\right) \right\rangle \leq 2.$$

In particular for all $x \in \mathbb{R}^d$

$$\langle |\nabla u_e(T, x)|^2 \rangle^{\frac{1}{2}} \lesssim_{d,\lambda} T^{-\frac{1}{2}} \eta_\beta(T). \quad (2.1.25)$$

Remark 1. Let us briefly comment on the results of Theorem 10 and Corollary 1.

1. Let us first comment on the scalings. The time decay $\eta_\beta(T)$ of Corollary 1 is optimal, in the sense that it is the optimal scaling in the case of the small contrast regime (see the computations done in Section 1.2.3). We now comment the scalings in Theorem 10. First, the decay $r^{-\frac{d}{2}} \mu_d(T)$ in (2.1.17) is also optimal in the sense that it is satisfied in the case of the small contrast regime (meaning that we consider a scalar model $a \in \mathbb{R}$ with $a^\delta = (1 + \frac{1}{\pi} \arctan(\pi \delta a)) \text{Id}$, for $\delta \ll 1$, using the notations of Section 1.2). Indeed, it may be verified by an explicit computation. We fix $1 \leq r \leq \sqrt{T}$. By linearizing around $\delta \downarrow 0$, the first order approximation of the time depend flux is given by $q_e \approx \delta \bar{q}$ where for all $T \in \mathbb{R}^+$

$$\bar{q}(T, \cdot) := \int_0^T \nabla \bar{u}(s, \cdot) ds + e, \quad (2.1.26)$$

and where we recall that \bar{u} is defined in (1.2.4). Using the fact that there exists a constant C depending on d such that $\Gamma(T, \cdot) = C g_{\sqrt{T}}$ and the semigroup property, for all $s \in [0, T]$, $g_{\sqrt{s}} \star g_r = g_{\sqrt{s+r^2}}$ as well as (1.2.5) and (2.1.26), we deduce

$$\begin{aligned} \bar{q}_r(T, \cdot) &\stackrel{(1.2.5), (2.1.26)}{=} C \int_0^T \nabla (\nabla g_{\sqrt{s}} \star a(\cdot) e \star g_r) ds + e \star g_r = C \int_0^T \nabla (\nabla g_{\sqrt{s+r^2}} \star a(\cdot) e) ds + e \star g_r \\ &\stackrel{(1.2.5)}{=} \int_0^T \nabla \bar{u}(s+r^2, \cdot) ds + e \star g_r. \end{aligned} \quad (2.1.27)$$

Consequently, using the moment bounds shown in Section 1.2.3 of $\nabla \bar{u}$ and the fact that $r \leq \sqrt{T}$, we get (noticing that $\langle \nabla \bar{u} \rangle = 0$)

$$\begin{aligned} \langle |\bar{q}_r(T) - \langle \bar{q}_r(T) \rangle|^2 \rangle^{\frac{1}{2}} &= \left\langle \left| \int_0^T \nabla \bar{u}(s+r^2, 0) ds \right|^2 \right\rangle^{\frac{1}{2}} \leq \int_0^T \langle |\nabla \bar{u}(s+r^2, 0)|^2 \rangle^{\frac{1}{2}} ds \\ &\lesssim_{d,\beta} \int_0^T (s+r^2)^{-\frac{1}{2}} \eta_\beta(s+r^2) ds \\ &\lesssim_{d,\beta} r^{-\frac{d}{2}} \mu_\beta(T), \end{aligned}$$

where $\mu_\beta(T)$ is defined in (2.1.18).

Second, the scaling $\chi_{d,\beta}(r)$ in (2.1.20) is optimal, except for $\beta < 2$ and $d = 2$. Indeed, it may be verified by an explicit computation as previously assuming for simplicity the more particular form $f_r = \int_1^{r^2} \nabla g_{\sqrt{\tau}} d\tau$ (which satisfies (2.1.19)). We may argue as in (2.1.27) to get $\bar{q}(r^2) \star f_r - \langle \bar{q}(r^2) \star f_r \rangle = \int_1^{r^2} \int_0^{r^2} \nabla^2 \bar{u}(s + \tau, \cdot) ds d\tau$. We then argue differently between the regimes of β and d .

In the regimes $\beta > 2, d > 2$ or $\beta < 2, d \geq 2$, we directly estimate

$$\begin{aligned} \langle |\bar{q}(r^2) \star f_r - \langle \bar{q}(r^2) \star f_r \rangle|^2 \rangle^{\frac{1}{2}} &= \left\langle \left| \int_1^{r^2} \int_0^{r^2} \nabla^2 \bar{u}(s + \tau, 0) ds d\tau \right|^2 \right\rangle^{\frac{1}{2}} \\ &\leq \int_1^{r^2} \int_0^{r^2} \langle |\nabla^2 \bar{u}(s + \tau, 0)|^2 \rangle^{\frac{1}{2}} ds d\tau, \end{aligned}$$

and we then conclude using that $\langle |\nabla^2 \bar{u}(s + \tau, 0)|^2 \rangle^{\frac{1}{2}} \lesssim_{d,\beta} \sqrt{p}(s + \tau)^{-1} \eta_\beta(s + \tau)$ (easily obtain by following the argument of Section 1.2.3). Note that, for $\beta < 2$ and $d = 2$, we obtain $r^{1-\frac{\beta}{2}}$ and thus the logarithmic contribution in (2.1.20) in this case is not optimal.

In the regimes $\beta > 2, d = 2, \beta = 2, d > 2$ and $\beta = d = 2$, this is more subtle and we have to expand the variance in order to capture the right behaviour in r . Assuming that $a = g, c(x - y) := \langle g(x)g(y) \rangle = (1 + |x - y|)^{-\beta}$ and applying ∇^2 to (1.2.5), we may express the variance as (we implicitly sum on repeated indices)

$$\begin{aligned} \langle |\bar{q}(r^2) \star f_r - \langle \bar{q}(r^2) \star f_r \rangle|^2 \rangle &= \int \int \int_1^{r^2} \int_0^{r^2} \partial_{ijk} \Gamma(s + \tau, x) e_k ds d\tau \\ &\quad \times \int_1^{r^2} \int_0^{r^2} \partial_{ijm} \Gamma(s' + \tau', y) e_m ds' d\tau' (1 + |x - y|)^{-\beta} dx dy. \end{aligned} \tag{2.1.28}$$

In what follows, we use the short-hand notation $\gamma_m := \int_1^{r^2} \int_0^{r^2} \partial_{ijm} \Gamma(s' + \tau', \cdot) e_m ds' d\tau'$ and note that a direct computation gives

$$\int |\gamma_m| \lesssim r \text{ and } |\gamma_m(y)| \lesssim \frac{r}{(|y| + 1)^d} \wedge \frac{1}{(|y| + 1)^{d-1}}. \tag{2.1.29}$$

On the one hand, in the case where $\beta > 2$ and $d = 2, x \mapsto (1 + |x|)^{-\beta}$ is integrable and thus by the Cauchy-Schwarz inequality the variance (2.1.28) is of order of

$$\begin{aligned} \int \int |\gamma_k(y)|^2 (1 + |x - y|)^{-\beta} dx dy &\stackrel{(2.1.29)}{\lesssim} \int_{B_r} \frac{1}{(|y| + 1)^2} dy + r^2 \int_{\mathbb{R}^d \setminus B_r} \frac{1}{(|y| + 1)^4} dy \\ &\sim \log(r + 2). \end{aligned}$$

On the other hand, in the case where $\beta = 2$ and $d \geq 2$, we split the variance into the far-field contribution $\int_{\mathbb{R}^d \setminus B_r} dx$ and the near-field contribution $\int_{B_r} dx$ where, for each contributions, we

bound $x \mapsto \int \gamma_m(y)(1 + |x - y|)^{-2} dy$. For the far-field contribution, we use

$$\begin{aligned} \int \gamma_m(y)(1 + |x - y|)^{-2} dy &= \int_{B_{\frac{|x|}{2}}(x)} \gamma_m(y)(1 + |x - y|)^{-2} dy + \int_{\mathbb{R}^d \setminus B_{\frac{|x|}{2}}(x)} \gamma_m(y)(1 + |x - y|)^{-2} dy \\ &\stackrel{(2.1.29)}{\lesssim} \frac{1}{(|x| + 1)^{d-1}} \int_{B_{\frac{|x|}{2}}} (|y| + 1)^{-2} dy + (|x| + 1)^{-2} \int |\gamma_m| \\ &\stackrel{(2.1.29)}{\lesssim} (|x| + 1)^{-1} (1 + \mathbf{1}_{d=2} \log(\frac{|x|}{2})) + r(|x| + 1)^{-2}, \end{aligned} \quad (2.1.30)$$

such a way that the far-field contribution is of order of, after a polar change of coordinates and using (2.1.29)

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_r} \gamma_k(x) \int \gamma_m(y)(1 + |x - y|)^{-2} dy dx &\lesssim r \int_r^{+\infty} (1 + \rho)^{-d-1} (1 + \mathbf{1}_{d=2} \log(\rho)) \rho^{d-1} d\rho \\ &\sim 1 + \mathbf{1}_{d=2} \log(r + 2). \end{aligned}$$

For the near-field contribution, we proceed the same way as in (2.1.30), expect for the contribution in $\int_{\mathbb{R}^d \setminus B_{\frac{|x|}{2}}(x)}$ that we estimate by a dyadic decomposition

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{\frac{|x|}{2}}(x)} \gamma_m(y)(1 + |x - y|)^{-2} dy &\leq \sum_{n=\lceil \log(\frac{|x|}{2}) \rceil}^{+\infty} 2^{-2n} \int_{B_{2^{n+1}}(x) \setminus B_{2^n}(x)} |\gamma_m| \\ &\leq \sum_{n=\lceil \log(\frac{|x|}{2}) \rceil}^{+\infty} 2^{-2n} \int_{B_{2^{n+1}+|x|}} |\gamma_m| \\ &\stackrel{(2.1.29)}{\lesssim} \sum_{n=\lceil \log(\frac{|x|}{2}) \rceil}^{+\infty} 2^{-2n} \int_{B_{2^{n+1}+|x|}} (|y| + 1)^{-d+1} dy \\ &\lesssim (|x| + 1)^{-1}, \end{aligned}$$

so that the near-field contribution is of order of, after a polar change of coordinates and using (2.1.29)

$$\begin{aligned} \int_{B_r} \gamma_k(x) \int \gamma_m(y)(1 + |x - y|)^{-2} dy dx &\lesssim \int_0^r (\rho + 1)^{-d} (1 + \mathbf{1}_{d=2} \log(\rho)) \rho^{d-1} d\rho \\ &\sim \log(r + 2) (1 + \mathbf{1}_{d=2} \log(r + 2)). \end{aligned}$$

2. Due to the computations done above, the logarithm correction in (2.1.17) is not optimal. In fact, this correction is here for technical reasons and mostly a consequence of the logarithm contribution in (2.2.39). However, in practice, it has no consequences in the proof of the optimal decay in time of ∇u_e and also in the proof to obtain the optimal growth of the correctors and their gradients (see Corollary 2 and 3) for which only the regime $T \sim r^2$ is needed.

Theorem 10 and Corollary 1 imply the following four results that are of interest in stochastic homogenization. The first one yields bounds on the gradient and flux of the extended corrector (ϕ_e, σ) , as well as the massive corrector $(\phi_{e,T}, \sigma_T)$, which gives an alternative proof of [89, Th.1]. Thanks

to the decay (2.1.23), the idea of the proof is clear: since $\lim_{T \rightarrow +\infty} u_e(T, \cdot) = 0$ and $\int_0^{+\infty} \nabla u_e(t, \cdot) dt$ is well defined in $L^2_{\text{loc}}(\mathbb{R}^d, L^2_{(\cdot)}(\Omega))$, we have by integrating the equation (2.1.10) in time

$$-\nabla \cdot a e - \nabla \cdot a \int_0^{+\infty} \nabla u_e(t, \cdot) dt = 0,$$

and we then recognize the corrector equation (2.1.8). By uniqueness, we then conclude that

$$\nabla \phi_e(\cdot) = \int_0^{+\infty} \nabla u_e(t, \cdot) dt. \quad (2.1.31)$$

Formula (2.1.31) combined with (2.1.23) then allow us to prove bounds on the gradient of correctors as well as on the flux.

Corollary 2 (Bounds on the flux and the gradient of correctors). *We have for all $r \geq 1$, $T \geq 1$ and unit vector $e \in \mathbb{R}^d$*

$$|(q_e)_r - \langle (q_e)_r \rangle| + |(q_{e,T})_r - \langle (q_{e,T})_r \rangle| + |\nabla(\phi_e)_r| + |\nabla \sigma_r| + |\nabla(\sigma_T)_r| + |\nabla(\phi_{e,T})_r| \leq C_{\star, d, \lambda, \beta}(r) \pi_{\star}^{-\frac{1}{2}}(r), \quad (2.1.32)$$

with some random variable $r \in [1, +\infty) \mapsto C_{\star, d, \lambda, \beta}(r)$ which depends on d, λ, β and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + \frac{\frac{1}{2}d+2}{\beta \wedge d}}$ there exists some constant $C < +\infty$ depending on d, λ, β and α such that

$$\sup_{r>0} \left\langle \exp\left(\frac{1}{C} C_{\star, d, \lambda, \beta}^{\alpha}(r)\right) \right\rangle \leq 2,$$

and for all $r \geq 1$

$$\pi_{\star}(r) = \begin{cases} r^{\beta} & \text{if } \beta < d, \\ r^d \log^{-1}(r) & \text{if } \beta = d, \\ r^d & \text{if } \beta > d. \end{cases}$$

In particular, for $\beta > d$, the quantities decay as the central limit theorem scaling $r^{-\frac{d}{2}}$.

Corollary 2 combined with Theorem 10 implies the following growth on the extended corrector (ϕ_e, σ) .

Corollary 3. *We have for all unit vector $e \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$*

$$|(\phi_e, \sigma) - (\phi_e, \sigma)_1(0)|_1^{\frac{1}{2}}(x) \leq C_{\star, d, \lambda, \beta}(x) \xi_{d, \beta}(|x|), \quad (2.1.33)$$

with for all $x \in \mathbb{R}^d$

$$\xi_{d, \beta}(|x|) := \begin{cases} (|x| + 1)^{1 - \frac{\beta}{2}} & \text{for } \beta < 2, \\ \log^{\frac{1}{2}}(|x| + 2) & \text{for } \beta = 2, d > 2 \text{ or } \beta > 2, d = 2, \\ \log(|x| + 2) & \text{for } \beta = d = 2, \\ 1 & \text{for } \beta > 2, d > 2, \end{cases} \quad (2.1.34)$$

and some random variable $x \in \mathbb{R}^d \mapsto C_{\star, d, \lambda, \beta}(x)$ which depends on d, λ, β and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + \frac{\frac{1}{2}d+2}{\beta \wedge d}}$ there exists some constant $C < +\infty$ depending on d, λ, β and α such that

$$\sup_{x \in \mathbb{R}^d} \left\langle \exp\left(\frac{1}{C} C_{\star, d, \lambda, \beta}^{\alpha}(x)\right) \right\rangle \leq 2.$$

Remark 2. The choice of the convolution with the Gaussian in (2.1.32) and (2.1.33) is not crucial. Indeed, for all $r \geq 1$ and $f_r := r^{-d} f(\frac{\cdot}{r})$ with $f \in W^{\alpha,1}(\mathbb{R}^d)$ (for some $\alpha > 0$), we may deduce from (2.1.32) and (2.1.33) that

$$\left| \int \psi(y) f_r(y) dy \right| \leq C_{1,\star,d,\lambda,\beta}(r) \pi_\star^{-\frac{1}{2}}(r),$$

where ψ can be replaced by one of the quantities which appears in (2.1.32), as well as for all $x \in \mathbb{R}^d$

$$(|(\phi_e, \sigma) - (\phi_e \star f, \sigma \star f)(0)|^2 \star f)^{\frac{1}{2}}(x) \leq C_{2,\star,d,\lambda,\beta}(x) \xi_{d,\beta}(|x|).$$

The two random variables $C_{1,\star,d,\lambda,\beta}(r)$ and $C_{2,\star,d,\lambda,\beta}(x)$ depend on f and have the same stochastic integrability as in Corollary 2 and Corollary 3. For more details, we refer to [8, Remark 4.28].

From Corollary 3, we obtain the following quantitative convergence of the two-scale expansion.

Corollary 4 (Convergence rate of the two-scale expansion). *Let $g \in H^1(\mathbb{R}^d)$ such that $\xi_{d,\beta}(|\cdot|) \nabla g \in L^2(\mathbb{R}^d)$, and for all $\varepsilon > 0$ let v_ε and v_{hom} be the Lax-Milgram solutions, in $\dot{H}^1(\mathbb{R}^d) := \{v \in H_{loc}^1(\mathbb{R}^d) | \nabla v \in L^2(\mathbb{R}^d)\} / \mathbb{R}$, of*

$$-\nabla \cdot a(\frac{\cdot}{\varepsilon}) \nabla v_\varepsilon = \nabla \cdot g \text{ and } -\nabla \cdot a_{hom} \nabla v = \nabla \cdot g,$$

with, for all $e \in \mathbb{R}^d$

$$a_{hom} = \langle a(\nabla \phi_e + e) \rangle. \quad (2.1.35)$$

Consider the two-scale expansion error

$$z_\varepsilon := v_\varepsilon - (v_{hom,\varepsilon} + \varepsilon \sum_{i=1}^d \phi_{e_i}(\frac{\cdot}{\varepsilon}) \partial_i v_{hom,\varepsilon}),$$

where $v_{hom,\varepsilon}$ is a simple moving average of v_{hom} at scale ε , that is $v_{hom,\varepsilon} = (v_{hom})_\varepsilon(0)$. Then

$$\left(\int |\nabla z_\varepsilon(x)|^2 dx \right)^{\frac{1}{2}} \leq C_{\star,d,\lambda,\beta,g}(\varepsilon) \varepsilon \xi_{d,\beta}(\varepsilon^{-1}) \left(\int \xi_{d,\beta}^2(|x|) |\nabla g(x)|^2 dx \right)^{\frac{1}{2}},$$

where $\xi_{d,\beta}$ is defined in (2.1.34) and for some random variable $\varepsilon \in \mathbb{R}_*^+ \mapsto C_{\star,d,\lambda,\beta,g}(\varepsilon)$ which depends on d, λ, β, g and satisfies: for all $\alpha < \frac{1}{\frac{1}{2} + \frac{d+2}{\beta \wedge d}}$, there exists some constant C depending on d, λ, β, g and α such that

$$\sup_{\varepsilon \in \mathbb{R}_*^+} \left\langle \exp\left(\frac{1}{C} C_{\star,d,\lambda,\beta,g}^\alpha(\varepsilon)\right) \right\rangle \leq 2.$$

Remark 3. The need for local averages at scale ε of v_{hom} is due to the fact that the corrector estimate (2.1.34) only holds for average of (ϕ_e, σ) under minimal regularity assumption on a . However, from De Giorgi-Nash-Moser theory in the case of equations and from classical Schauder theory in the case of systems with Hölder continuous coefficient field a , we may improve the estimate (2.1.34) into a pointwise estimate. Therefore, in both cases, there is no need to consider local averages of v_{hom} at scales ε .

For a proof of Corollary 4 based on the results of Corollary 3, we refer the reader to [80]. The second consequence of Corollary 1 is a new optimal control of the sub-systematic error, extending the bound obtained in [88, Lem.8] in the case of discrete elliptic equations and the one in [92, Th.3] for a finite range of dependence. This corollary is of numerical interest for approximating the homogenized matrix a_{hom} defined in (2.1.35).

Corollary 5 (Sub-systematic error). *Let $(\phi_{e_i, T})_{i \in \llbracket 1, d \rrbracket}$ be defined in (2.1.9). For all $(i, n) \in \llbracket 1, d \rrbracket \times \mathbb{N}$, we define the Richardson extrapolation of $\phi_{e_i, T}$ with respect to T by*

$$\begin{cases} \phi_{e_i, T}^{n+1} = \frac{1}{2^n - 1} (2^n \phi_{e_i, 2T}^n - \phi_{e_i, T}^n) & \text{for all } n \geq 1, \\ \phi_{e_i, T}^1 = \phi_{e_i, T}, \end{cases}$$

and likewise for $\phi_{e_i, T}^*$. We define the approximation $(\bar{a}_T^n)_{n \in \mathbb{N}}$ of the homogenized coefficients a_{hom} by: for all $(i, j, n) \in \llbracket 1, d \rrbracket^2 \times \mathbb{N}$

$$e_j \cdot \bar{a}_T^n e_i = \langle (\nabla \phi_{e_j, T}^{*, n} + e_j) \cdot a(\nabla \phi_{e_i, T}^n + e_i) \rangle. \quad (2.1.36)$$

We have the following estimates of the sub-systematic errors: for all $d \geq 2$ and $n > \frac{\beta \vee d}{4}$

$$\langle |\nabla \phi_{e_i, T}^n - \nabla \phi_{e_i}|^2 \rangle^{\frac{1}{2}} \lesssim T^{\frac{1}{2}} \eta_\beta(T), \quad (2.1.37)$$

and

$$|\bar{a}_T^n - a_{hom}| \lesssim T \eta_\beta^2(T), \quad (2.1.38)$$

where η_β is as in (2.1.24)

Finally, Corollary 5 implies the following bound on the bottom of the spectrum of $-\nabla \cdot a \nabla$ projected on $\nabla \cdot a(0)e$ and extends [92, Corollary 5], [88, Corollary 1] to correlated fields. Let us recall that stationarity allows us to define a differential calculus in probability through the correspondence for stationary fields: for all stationary fields $\psi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ we define for all $i \in \llbracket 1, d \rrbracket$:

$$D_i \psi(0) = \lim_{h \rightarrow 0} \frac{\psi(a(\cdot + h e_i), 0) - \psi(a, 0)}{h} = \lim_{h \rightarrow 0} \frac{\psi(a, h e_i) - \psi(a, 0)}{h} = \nabla_i \psi(a, 0),$$

and we set $D\psi := (D_i \psi(0))_{i \in \llbracket 1, d \rrbracket}$. We define the Hilbert space $\mathcal{H}^1 := \{\psi \in L^2_{(\cdot)}(\Omega) \mid \langle |D\psi|^2 \rangle < +\infty\}$. In the case when the coefficients a are symmetric, the operator $L := -D \cdot a(0)D$ defines a quadratic form on \mathcal{H}^1 . We denote by \mathcal{L} its Friedrichs extension on $L^2_{(\cdot)}(\Omega)$. Since \mathcal{L} is a self-adjoint non-negative operator, by the spectral theorem it admits a spectral resolution: for all $\Theta \in L^2_{(\cdot)}(\Omega)$, there exists a unique measure ν_Θ such that for all $g \in L^\infty(\mathbb{R}^+)$

$$\langle g(\mathcal{L})\Theta, \Theta \rangle = \int_0^{+\infty} g(\lambda) d\nu_\Theta(\lambda). \quad (2.1.39)$$

Corollary 6 (Spectral resolution). *Let assume that the map A defined in (2.1.1) takes values in the set of symmetric matrices and assume that $\Theta := D \cdot a(0)e$ is in $L^2_{(\cdot)}(\Omega)$ for some unit vector $e \in \mathbb{R}^d$. We denote by ν_Θ the spectral measure, defined in (2.1.39), of the operator $-D \cdot a(0) \cdot D$ associated to the vector Θ . We have for all $0 < \mu \leq 1$*

$$\left\langle \int_0^\mu d\nu_\Theta(\zeta) \right\rangle \lesssim \eta_\beta^2(\mu^{-1}),$$

where η_β is as in (2.1.24).

2.1.3 Extension to other model of coefficient field

The approach we develop here is not limited to the Gaussian setting. For coefficient field a for which the law satisfies multiscale functional inequalities with oscillation, similar result to the ones presented in this paper hold. More precisely, assume that there exists $\rho > 0$ such that for all square integrable functional F of a , we have

$$\text{Ent}(F(a)) \leq \frac{1}{\rho} \left\langle \int_1^{+\infty} \ell^d \pi(\ell) \int |\partial_{x,\ell}^{\text{osc}} F(a)|^2 dx d\ell \right\rangle, \quad (2.1.40)$$

with, for some $C > 0$ and $\beta > 0$

$$\pi(\ell) = e^{-\frac{1}{C}\ell^\beta}, \quad (2.1.41)$$

and for all $(x, \ell) \in \mathbb{R}^d \times [1, +\infty)$

$$\partial_{x,\ell}^{\text{osc}} F(a) := \sup\{F(a') - F(a'') \mid a' = a'' = a \text{ on } \mathbb{R}^d \setminus B_\ell(x)\}. \quad (2.1.42)$$

Then, with the notations $\mu_\beta(T) = 1$, $\eta_\beta(T) = T^{-\frac{1}{2}-\frac{d}{4}}$, $\pi_\star(r) = r^d$ and $\xi_{d,\beta}(|x|) = \log^{\frac{1}{2}}(|x| + 2)$ if $d = 2$ and $\xi_{d,\beta}(|x|) = 1$ if $d \geq 3$, the results of section 2.1.2 hold with a random variable \mathcal{C}_\star (possibly depending on $d, \lambda, \beta, g, x, r$ and T) with stretched exponential moments for some exponent α (depending on d and β) uniform in x, r and T when it depends on this parameters.

Multiscale logarithm Sobolev inequality of type (2.1.40) are satisfied, for instance, by random inclusions with random radii and random tessellations of Poisson points or the random parking measure. For more precise details, we refer to [59, 60] and Section 1.1.5. For completeness and to see the differences compared to the Gaussian setting, we provide in Appendix 2.4.4, a proof of Theorem 10 under the assumption (2.1.40), when u_e is real valued and a satisfy a regularity assumption. The proofs of the general case may be extended by following the arguments of Appendix 2.4.4 and Section 2.3.

2.2 Structure of the proof

Let us now describe the strategy of the proof of Theorem 10, together with a flow of auxiliary results. In the rest of the paper, for notational convenience, we do not write the dependance of all quantities on the unit vector e , fixed once for all.

2.2.1 Strategy of the proof

General strategy of the proof. The proof uses on two important quantities: for all $t \in \mathbb{R}_\star^+$

$$\forall r \geq \sqrt{t}, \mathcal{Q}_1^r := \sqrt{t} \left(\int \eta_{\sqrt{2r}}(y) |\nabla u(t, y)|^2 dy \right)^{\frac{1}{2}}, \quad (2.2.1)$$

and

$$\forall r \leq \sqrt{t}, \mathcal{Q}_2^r(t) : y \in \mathbb{R}^d \mapsto q_r(t, y) - \langle q_r(t, y) \rangle, \quad (2.2.2)$$

and their relationship. On the one hand, using the estimate from [92, Lemma 1], we have a deterministic relationship between (2.2.1) and averages in space and in r of (2.2.2), recalled in Lemma 3. On the other hand, using sensitivity estimates (see Lemma 7 and Proposition 3) and the multiscale logarithmic Sobolev inequality (2.1.4), we can control moments of (2.2.2) by moments of (2.2.1).

The main difficulty is that the estimates are coupled in an intricate way, which does not allow to buckle easily. We handle this difficulty by, first deriving nearly-optimal estimates in scaling in r, t of moments of (2.2.2) from a sub-optimal deterministic bound in t of (2.2.1), which is itself based on deterministic energy estimates (see Lemma 1). Second, from the nearly-optimal moment bounds of (2.2.2), we deduce a better decay in t of moments of (2.2.1) than the one provided by energy estimates, which allow us to deduce the optimal scaling in r, t of (2.2.2), which leads to Theorem 10. We then finally obtain the optimal decay in time of moments of (2.2.1), which leads to Corollary 1. Our main effort is to derive the sensitivity estimates and the control of moments of (2.2.2). We focus in the following on the main ideas of the proof of (2.1.17). For (2.1.20), the ideas are very closed and we only say few words on the differences at the end of this section

Sensitivity estimates. The proof of the sensitivity estimates combined two different types of arguments.

1. **Deterministic arguments.** There are two main ingredients. The first ingredient is the classical L^2 theory of parabolic systems in form of localized energy type estimates, see Lemmas 1 and 2. The second ingredient is the large-scale regularity theory for parabolic systems developed in [26] that we recall and extend in Appendix 2.4.2. It provides, in particular, a large-scale $C^{0,1}$ estimate: for all $x \in \mathbb{R}^d$, there exists a stationary random variable $r_*(x) \geq 1$ such that for all $t \in \mathbb{R}$ and weak solution v of, for $R \geq r_*(x)$, $\partial_\tau v - \nabla \cdot a \nabla v = \nabla \cdot g$ in $C_R(t, x)$, we have

$$\begin{aligned} \int_{C_{r_*(x)}(t,x)} |\nabla v(s, y)|^2 ds dy &\lesssim_{d,\lambda} \int_{C_R(t,x)} |\nabla v(s, y)|^2 ds dy \\ &+ \sup_{\rho \in [r_*(x), R]} \left(\frac{R}{\rho} \right)^{2\alpha} \int_{C_\rho} \left| g(s, y) - \int_{C_\rho(t,x)} g(s', z) ds' dz \right|^2 ds dy. \end{aligned} \quad (2.2.3)$$

This properties can be used provided r_* has good moment bounds, which have already been established in [89] in our context.

2. **Stochastic arguments.** Moment bounds on $q_r(T) - \langle q_r(T) \rangle$ will be obtained from the multiscale logarithmic Sobolev inequality (2.1.4), and more precisely in the version of Proposition 4 in order to control high moments: for all $p \in [1, +\infty)$, $T \geq 1$ and $r \in (0, \sqrt{T}]$,

$$\langle |q_r(T) - \langle q_r(T) \rangle|^p \rangle^{\frac{1}{p}} \lesssim \sqrt{p} \left\langle \left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}. \quad (2.2.4)$$

To use (2.2.4), we have to estimate the sensitivity of $q_r(T)$ with respect to the coefficient field a , namely the quantity $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell$. The method used here is inspired by the series of articles [70, 87, 80, 89] which treats the case of elliptic systems and proceeds by duality. The results are summarized in Lemma 7 for the computation of the functional derivative and in Proposition 3 for the sensitivity calculus and the control of moments.

The localized energy estimates of the deterministic part are classical and rely only on L^2 theory for parabolic systems. The contribution of this paper is more on the stochastic part. Let us now describe the main ideas of the proof of (2.1.17) in Theorem 10. For simplicity, we do this in a simpler case with two additional assumptions:

- (i) We assume that u is real-valued and a is symmetric. In that case, we get the uniform bound of ∇u : for all $t > 0$

$$\|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim_{d,\lambda} t^{-1}, \quad (2.2.5)$$

cf Remark 4 below for a proof. The bound (2.2.5) is our starting point to prove the sensitivity estimate.

- (ii) We assume that $\nabla \cdot ae \in L^\infty(\mathbb{R}^d)$ and

$$\|\nabla \cdot ae\|_{L^\infty(\mathbb{R}^d)} \leq 1. \quad (2.2.6)$$

This assumption allow us to avoid the singularity at $t = 0$ and to use the localized energy estimate in the form: for all $T \geq 1$, $R \geq \sqrt{T}$ and $z \in \mathbb{R}^d$

$$\int_0^T \int_{B_R(z)} |\nabla u(t, x)|^2 dx dt \lesssim_{d,\lambda} 1, \quad (2.2.7)$$

see for instance [92, Lemma 3] for a proof.

We proceed in three steps.

- I) The first step identifies the functional derivative of each components $k \in \llbracket 1, d \rrbracket$, $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \partial_{x,\ell}^{\text{fct}} q_r(T) \cdot e_k$, defined in (2.1.5) for $T \geq 1$ and $r \leq \sqrt{T}$. Formally, we have for all $(x, \ell) \in \mathbb{R}^d \times [1, +\infty)$,

$$\partial_{x,\ell}^{\text{fct}} q_r(T) \cdot e_k = \int_{B_\ell(x)} \left| \frac{\partial}{\partial a(y)} q_r(T) \cdot e_k \right| dy,$$

with, for all $y \in B_\ell(x)$, using that $\frac{\partial}{\partial a(y)} a = \delta_y$ and the chain rule

$$\frac{\partial}{\partial a(y)} q_r(T) \cdot e_k = g_r(y) e_k \cdot e + g_r(y) e_k \cdot \int_0^T \nabla u(t, y) dt + \int g_r(z) e_k \cdot a(z) \left(\int_0^T \nabla \frac{\partial}{\partial a(y)} u(t, z) dt \right) dz, \quad (2.2.8)$$

with from (2.1.10)

$$\begin{cases} \partial_\tau \frac{\partial}{\partial a(y)} u - \nabla \cdot a \nabla \frac{\partial}{\partial a(y)} u = \nabla \cdot \delta_y \nabla u & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ \frac{\partial}{\partial a(y)} u(0) = \nabla \cdot \delta_y e. \end{cases} \quad (2.2.9)$$

The first two r.h.s terms of (2.2.8) are directly controlled via (2.2.5) and (2.2.7), whereas the control of the last term is more technical. The idea is to rewrite this term by duality, in form of, for all $k \in \llbracket 1, d \rrbracket$

$$\int g_r(z) e_k \cdot a(z) \left(\int_0^T \nabla \frac{\partial}{\partial a(y)} u(t, z) dt \right) dz = \nabla v_k^T(0, y) \otimes e + \int_0^T \nabla u(t, y) \otimes \nabla v_k^T(t, y) dt, \quad (2.2.10)$$

where $v^T = (v_k^T)_{k \in \llbracket 1, d \rrbracket}$ solves the corresponding dual problem of (2.1.10) with final time T , that is the backward parabolic system

$$\begin{cases} \partial_\tau v_k^T + \nabla \cdot a^* \nabla v_k^T = \nabla \cdot a g_r e_k & \text{on } (-\infty, T) \times \mathbb{R}^d, \\ v_k^T(T) = 0. \end{cases} \quad (2.2.11)$$

Consequently, the crucial terms of $\partial_{x,\ell}^{\text{fct}} q_r(T) \cdot e_k$ are

$$\mathcal{M}_k(T, x, \ell) := \int_{B_\ell(x)} \left| \nabla v_k^T(0, y) \otimes e \right| dy + \int_{B_\ell(x)} \left| \int_0^T \nabla u(t, y) \otimes \nabla v_k^T(t, y) dt \right| dy. \quad (2.2.12)$$

This decomposition is summarized in Lemma 7.

- II) In the second step, we deduce the control of moments of $q_r(T) - \langle q_r(T) \rangle$ from the formula of $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \partial_{x,\ell}^{\text{fct}} q_r(T)$, for $T \geq 1$ and $1 \leq r \leq \sqrt{T}$, and the estimate (2.2.4). More precisely, at this stage, we are only able to reach a sub-optimal bound, with a $\log(T)$ correction in (2.1.17). This additional contribution is due to the, purely deterministic, sub-optimal uniform bound (2.2.5), as we will clearly see in the computations below. We provide the idea of the control of the main term (2.2.12), for all $\ell \geq 1$ and $k = 1$. In the following, we write \mathcal{M} and v^T for \mathcal{M}_1 and v_1^T respectively. As in Remark 1, we have to distinguish between the two regimes $\ell < \sqrt{T}$ and $\ell \geq \sqrt{T}$.

Regime $\ell < \sqrt{T}$. We start with the first r.h.s term of (2.2.12). We use the plain energy estimate: for all $t \leq T$

$$\int |\nabla v^T(t, y)|^2 dy \lesssim_{d,\lambda} \int g_r^2(y) dy \lesssim r^{-d}, \quad (2.2.13)$$

applied for $t = 0$ to get, using in addition the Jensen inequality and the estimate $\int_{B_\ell(x)} dx \lesssim_d \ell^d \int$,

$$\int \left(\int_{B_\ell(x)} |\nabla v_k^T(0, y) \otimes e| dy \right)^2 dx \lesssim_{d,\lambda} \ell^{2d} r^{-d}. \quad (2.2.14)$$

We now turn to the estimate of the second r.h.s term of (2.2.12). We start by splitting the time integral into the two contributions in $(0, 1)$ and $[1, T]$. For the contribution in $(0, 1)$, we make use of the Cauchy-Schwarz inequality, the energy estimate (2.2.7), the estimate $\int_{B_\ell(x)} dx \lesssim_d \ell^d \int$ and the plain energy estimate (2.2.13) to obtain

$$\begin{aligned} \int \left(\int_{B_\ell(x)} \left| \int_0^1 \nabla u(t, y) \otimes \nabla v^T(t, y) dt \right| dy \right)^2 dx &\leq \int \int_0^1 \int_{B_\ell(x)} |\nabla u(t, y)|^2 dy dt \quad (2.2.15) \\ &\quad \times \int_0^1 \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \\ &\stackrel{(2.2.7)}{\lesssim_{d,\lambda}} \ell^d \int \int_0^1 \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \\ &\stackrel{(2.2.13)}{\lesssim_{d,\lambda}} \ell^{2d} r^{-d}. \quad (2.2.16) \end{aligned}$$

For the contribution in $[1, T]$, we make use of the Jensen inequality, the estimate $\int_{B_\ell(x)} dx \lesssim_d \ell^d \int$, the deterministic uniform bound (2.2.5) and the Minkowski inequality in $L^2(\mathbb{R}^d)$ as well as the plain energy estimate (2.2.13) to obtain for all $\ell \geq 1$

$$\begin{aligned} \int \left(\int_{B_\ell(x)} \left| \int_1^T \nabla u(t, y) \otimes \nabla v^T(t, y) dt \right| dy \right)^2 dx &\stackrel{(2.2.5)}{\lesssim_{d,\lambda}} \ell^{2d} \int \left(\int_1^T t^{-1} |\nabla v^T(t, x)| dt \right)^2 dx \\ &\leq \left(\int_1^T t^{-1} \left(\int |\nabla v^T(t, x)|^2 dx \right)^{\frac{1}{2}} dt \right)^2 \\ &\lesssim_{d,\lambda} \log^2(T) r^{-d}. \quad (2.2.17) \end{aligned}$$

Therefore, the combination of (2.2.14), (2.2.16) and (2.2.17) yields

$$\int \mathcal{M}^2(T, x, \ell) dx \lesssim_{d, \lambda} \ell^{2d} r^{-d} (1 + \log^2(T)), \quad (2.2.18)$$

and gives the contribution in (2.2.4)

$$\begin{aligned} \int_1^{\sqrt{T}} \ell^{-d} \pi(\ell) \int \mathcal{M}^2(T, x, \ell) dx d\ell &\stackrel{(2.1.6), (2.2.18)}{\lesssim_{d, \lambda}} (1 + \log^2(T)) r^{-d} \int_1^{\sqrt{T}} \ell^{d-1-\beta} d\ell \\ &\lesssim_{\beta} (1 + \log^2(T)) r^{-d} \mu_{\beta}^2(T), \end{aligned}$$

where $\mu_{\beta}(T)$ is defined in (2.1.18).

Regime $\ell \geq \sqrt{T}$. Here, the bound (2.2.18) is of no use since $\ell \mapsto \ell^{-d} \pi(\ell) \int \mathcal{M}^2(T, x, \ell) dx$ needs to be integrable at infinity. This is why we treat this regime a different way. We start with the first r.h.s term of (2.2.12). This term is more subtle to control in this regime, even with the two additional assumptions (2.2.5) and (2.2.6). We present here the argument in the homogeneous case $a = \text{Id}$. In that case, we may express ∇v^T in terms of the Duhamel formula: for all $(t, x) \in (-\infty, T) \times \mathbb{R}^d$

$$\nabla v^T(t, x) = - \int_0^{T-t} \nabla^2 \Gamma(s, \cdot) \star g_r(x) ds,$$

where we recall that Γ denotes the heat kernel. Then, using the Minkowski inequality in $L^2(\mathbb{R}^d)$ and noticing that for all $s \in \mathbb{R}_*^+$, $\Gamma(s, \cdot) = C g_{\sqrt{s}}$ for some $C > 0$ and using the semigroup property $g_{\sqrt{s}} \star g_r = g_{\sqrt{s+r^2}}$

$$\begin{aligned} \int \left(\int_{B_{\ell}(x)} |\nabla v^T(0, y) \otimes e| dy \right)^2 dx &\lesssim_d \ell^d \left(\int |\nabla v^T(0, y)| dy \right)^2 \\ &= C \ell^d \left(\int \left| \int_0^T \nabla^2 g_{\sqrt{s+r^2}}(y) ds \right| dy \right)^2 \\ &\lesssim_d \ell^d \left(\int_0^T (s+r^2)^{-1} ds \right)^2 \lesssim_d \ell^d \log^2 \left(1 + \frac{T}{r^2} \right). \end{aligned}$$

In the heterogeneous case, we replace the use of the Green's function by appealing to large-scale regularity, in form of estimate (2.2.3), to get a pointwise bound of local average of ∇v^T , see Lemma 4, and we get, for all $t \in (-\infty, T]$

$$\int \left(\int_{B_{\ell}(x)} |\nabla v^T(t, y)| dy \right)^2 dx \leq C \ell^d \log^2 \left(1 + \frac{\ell}{r} \right), \quad (2.2.19)$$

with for all $p \geq 1$, $\langle C^p \rangle^{\frac{1}{p}} \lesssim_d p^{\frac{d+2}{\beta \wedge d}}$. We refer to the estimate of the first l.h.s term of (2.3.79) for more precisions. We now turn to the estimate of the second r.h.s term of (2.2.12). As before, we split the time integral into the two contributions $(0, 1)$ and $[1, T]$. For the contribution in $(0, 1)$, we make use of the Minkowski inequality in $L^2(\mathbb{R}^d)$ combined the identity $\int = \int \int_{B_r(z)} dz$, the Cauchy-Schwarz inequality, (2.2.7) applied with $R = r$ and $T = 1$ and [92, Lemma 3] applied to the equation (2.2.11) in form of

$$\int_0^1 \int_{B_r(z)} |\nabla v^T(t, y)|^2 dy dt \lesssim \int \eta_r(y-z) g_r^2(y) dy, \quad (2.2.20)$$

to get

$$\begin{aligned}
& \int \left(\int_{B_\ell(x)} \left| \int_0^1 \nabla u(t, y) \otimes \nabla v^T(t, y) dt \right| dy \right)^2 dx \\
& \lesssim_d \ell^d \left(\int \int_0^1 |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy \right)^2 \\
& \leq \ell^d \left(\int \left(\int_0^1 \int_{B_r(z)} |\nabla u(t, y)|^2 dy dt \right)^{\frac{1}{2}} \left(\int_0^1 \int_{B_r(z)} |\nabla v^T(t, y)|^2 dy dt \right)^{\frac{1}{2}} dz \right)^2 \\
& \stackrel{(2.2.7), (2.2.20)}{\lesssim} \ell^d \int \left(\int \eta_r(y-z) g_r^2(y) dy \right)^{\frac{1}{2}} dz \lesssim \ell^d. \tag{2.2.21}
\end{aligned}$$

For the contribution in $[1, T]$, we once again use the Minkowski inequality in $L^2(\mathbb{R}^d)$ combined with (2.2.5) and (2.2.19) to get for all $\ell \geq 1$

$$\begin{aligned}
\int \left(\int_{B_\ell(x)} \left| \int_1^T \nabla u(t, y) \otimes \nabla v^T(t, y) dt \right| dy \right)^2 dx & \lesssim \left(\int_1^T s^{-1} \left(\int \left(\int_{B_\ell(x)} |\nabla v^T(t, y)| dy \right)^2 dx \right)^{\frac{1}{2}} ds \right)^2 \\
& \stackrel{(2.2.19)}{\lesssim} \mathcal{C} \ell^d \log^2(T) \log^2\left(1 + \frac{\ell}{r}\right). \tag{2.2.22}
\end{aligned}$$

The combination of (2.2.19) applied with $t = 0$, (2.2.21) and (2.2.22) yields, using that $r \leq \sqrt{T}$ in the last line

$$\begin{aligned}
\int_{\sqrt{T}}^{+\infty} \ell^{-d} \pi(\ell) \int \mathcal{M}^2(T, x, \ell) dx d\ell & \lesssim \log^2(T) \int_{\sqrt{T}}^{+\infty} \ell^{-1-\beta} (1 + \mathcal{C} \log^2(\frac{\ell}{r})) d\ell \\
& \lesssim_\beta \log^2(T) T^{-\frac{\beta}{2}} (1 + \mathcal{C} \log^2(\frac{\sqrt{T}}{r})) \\
& \lesssim_\beta \log^2(T) r^{-d} \mu_\beta^2(T) (1 + \mathcal{C} \log^2(\frac{\sqrt{T}}{r})).
\end{aligned}$$

Let us now talk about the main difficulties and changes which occur in the general case, that is when we do not assume (2.2.5) and (2.2.6).

- (i) When the assumption (2.2.6) is not satisfied, u is now singular at $t = 0$ and thus the second r.h.s term of (2.2.12) is not well defined in the Lebesgue sense. In order to handle this singular part, we have to treat a different way the contribution in $(0, 1)$ of the time integral of the third r.h.s term of (2.2.8). This is done by using the localized energy estimates directly on the equation (2.2.9). As a consequence, we do not obtain an explicit formula for $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \partial_{x, \ell}^{\text{fct}} q_r(T)$ but rather a bound, see Lemmas 7 and 8. Note that the bound get much simpler in the case of fast decay of correlations, namely for $\beta > d$, where the case $\ell < \sqrt{T}$ has to be considered (and so the contribution $\mathcal{G}_{r, \ell}$ does not appear).
- (ii) When u is vector-valued, (2.2.5) fails and has to be replaced by: for all $R \geq \sqrt{t}$

$$\int_{B_R} |\nabla u(t, y)|^2 dy \lesssim t^{-2}. \tag{2.2.23}$$

This estimate is however not sufficient for our propose since we see in (2.2.12) that we need to bound the average of ∇u over all balls B_ℓ , for $\ell \in [1, +\infty)$. We have to appeal to

large-scale regularity theory in form of estimate (2.2.3) to obtain the improvement

$$\int_{B_\ell(x)} |\nabla u(t, y)|^2 dy \lesssim \left(\left(\frac{r_*(x)}{\ell} \vee 1 \right)^d \mathbb{1}_{\ell < \sqrt{t}} + \mathbb{1}_{\ell \geq \sqrt{t}} \right) t^{-2}. \quad (2.2.24)$$

Equipped with (2.2.24), we may control the second r.h.s term of (2.2.12) as in the scalar case. The only main change is that we cannot use the plain energy estimate (2.2.13) for the defining equation (2.2.11) as we did in (2.2.17). Instead, we prove a new lemma which states a pointwise bound (depending on the form of the r.h.s of (2.2.11)) on $\int_{B_{r_*(x)}} |\nabla v^T(t, y)|^2 dy$ for all $x \in \mathbb{R}^d$ and $\sqrt{T-t} \geq r_*(x)$, see Lemma 4. The sub-optimal estimate of moments of $q_r(T) - \langle q_r(T) \rangle$ is summarized in Proposition 3.

- III) In the final step, we remove the $\log(T)$ contribution which appears in the previous step. To this aim, we need a little more decay in time of the averages of ∇u than the one obtained in (2.2.24), since the $\log(T)$ contribution clearly comes from this sub-optimal bound. The idea is to use the L^2 - L^1 estimate of Lemma 3 which essentially says, by stationarity, that for all $R \geq \sqrt{t}$, $x \in \mathbb{R}^d$ and $p \in [1, +\infty)$

$$\left\langle \left(\int_{B_R(x)} |\nabla u(t, y)|^2 dy \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim_{d, \lambda} t^{-1-\frac{d}{4}} \int_{\frac{t}{4}}^{\frac{t}{2}} \int_0^{\sqrt{s}} r^{\frac{d}{2}} \left\langle |q_r(s) - \langle q_r(s) \rangle|^p \right\rangle^{\frac{1}{p}} dr ds.$$

It gives, using the sub-optimal moment bounds of $q_r(T) - \langle q_r(T) \rangle$ of the previous step that, for all $R \geq \sqrt{t}$

$$\int_{B_R(x)} |\nabla u(t, y)|^2 dy \lesssim \log^2(t) t^{-1} \eta_\beta^2(t) \mathcal{D}_*(x), \quad (2.2.25)$$

with η_β defined in (2.1.24) and where $\mathcal{D}_*(x)$ is a random variable with stretched exponential moment. By interpolating (2.2.24) and (2.2.25) we deduce for all $\varepsilon > 0$ and $\ell \in [1, +\infty)$

$$\int_{B_\ell(x)} |\nabla u(t, y)|^2 dy \lesssim \left(\left(\frac{r_*(x)}{\ell} \vee 1 \right)^d \mathbb{1}_{\ell < \sqrt{t}} + \mathbb{1}_{\ell \geq \sqrt{t}} \right) t^{-2(1-\varepsilon)} \eta_\beta^{2\varepsilon}(t) \mathcal{D}_*^{2\varepsilon}(x). \quad (2.2.26)$$

This improved decay allow us to prove the optimal estimates of Theorem 10. The price to pay in this step is a little loss of stochastic integrability due to the random variable $\mathcal{D}_*^{2\varepsilon}$. Note that, the exponent α that we get in (2.1.22), is neither optimal for $\beta > d$ (since [92] indicates that we expect nearly-Gaussian moments), nor for $0 < \beta \ll 1$ (since by [70] we can obtain nearly Gaussian moments).

- IV) The proof of (2.1.20) follows the same ideas and is even easier since we do not need to begin with a sub-optimal estimate as we need for (2.1.17). We use Lemma 8 for the estimate of $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \partial_{x, \ell}^{\text{fct}} q(r^2) \star f_r$, the pointwise bound (2.2.40) for local average of ∇v^{r^2} and the decay (2.2.26) of averages of ∇u .

We finally mention that in the case of fast decay correlation, that is for $\beta > d$, the proof is much simpler and only the regime $\ell < \sqrt{T}$ has to be considered.

Remark 4. When u is real-valued and a is symmetric, we can prove (2.2.5) by using the Nash-Aronson bounds on the parabolic Green function. Indeed, let $\psi : (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ be

the Green function of the operator $\partial_\tau - \nabla \cdot a \nabla$. Recall that the parabolic Green function ψ is a nonnegative function which solves in the distributional sense, for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, the equations

$$\begin{cases} \partial_\tau \psi(\cdot, \cdot, y) - \nabla \cdot a \nabla \psi(\cdot, \cdot, y) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ \psi(0, \cdot, y) = \delta_y, \end{cases} \quad (2.2.27)$$

and

$$\begin{cases} \partial_\tau \psi(\cdot, x, \cdot) - \nabla \cdot a \nabla \psi(\cdot, x, \cdot) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ \psi(0, x, \cdot) = \delta_x. \end{cases}$$

We have the following estimate (for a reference, see [8, Prop E1]): there exists a constant $C > 0$ which depends only on d and λ such that for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $t > 0$

$$\left(\int_{C_{\frac{3}{2}\sqrt{t}}(t,x)} |\nabla_z \psi(s, y, z)|^2 ds dz \right)^{\frac{1}{2}} \lesssim_{d,\lambda} T^{-\frac{1}{2}-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right). \quad (2.2.28)$$

By noticing that from (2.2.27), $\zeta : y \in \mathbb{R}^d \mapsto \nabla_y \psi(\cdot, \cdot, y)$ solves $\partial_\tau \zeta - \nabla \cdot a \nabla \zeta = 0$ in $(0, +\infty) \times \mathbb{R}^d$, we deduce from the Caccioppoli estimate (see Lemma 12) that for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left(\int_{C_{\sqrt{t}}(t,x)} |\nabla_z \nabla_y \psi(s, z, y)|^2 ds dz \right)^{\frac{1}{2}} \lesssim_{d,\lambda} t^{-\frac{1}{2}} \left(\int_{C_{\frac{3}{2}\sqrt{t}}(t,x)} |\nabla_y \psi(s, z, y)|^2 ds dz \right)^{\frac{1}{2}}. \quad (2.2.29)$$

Since a is symmetric, we have $\psi(s, z, y) = \psi(s, y, z)$ and so $\nabla_y \psi(s, z, y) = \nabla_z \psi(s, y, z)$. Consequently, the combination of (2.2.28), (2.2.29) and Lemma 2 yields for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left(\int_{B_{\sqrt{t}}(y)} |\nabla_x \nabla_z \psi(t, x, z)|^2 ds dz \right)^{\frac{1}{2}} \lesssim_{d,\lambda} t^{-1-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right). \quad (2.2.30)$$

This bound implies, using the explicit formula $\nabla u(t, x) = \int \nabla_x \nabla_y \psi(t, x, y) \cdot a(y) e dy$, that for all $(t, x) \in \mathbb{R}_*^+ \times \mathbb{R}^d$

$$|\nabla u(t, x)| \leq \int |\nabla_x \nabla_y \psi(t, x, y)| dy \leq \int \left(\int_{B_{\sqrt{t}}(y)} |\nabla_x \nabla_z \psi(t, x, z)|^2 dz \right)^{\frac{1}{2}} dy \stackrel{(2.2.30)}{\lesssim_{d,\lambda}} t^{-1},$$

and thus for all $t > 0$

$$\|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim_{d,\lambda} t^{-1}. \quad (2.2.31)$$

We now state the lemmas needed in the proof of Theorem 10. The first section lists the deterministic PDE ingredients, the second section the results derived from the large-scale regularity theory, and finally the third section the sub-optimal control of the fluctuations of the time dependent flux $q_r(\cdot, \cdot)$.

2.2.2 Deterministic results

This section displays the deterministic PDE ingredients needed in the proof of Theorem 10 and Corollary 1. We start with two classical results from standard L^2 regularity theory of parabolic system. The first one is the localized energy type estimates for parabolic systems.

Lemma 1 (Localized energy estimates). *Let v be the weak solution of the parabolic system*

$$\begin{cases} \partial_\tau v - \nabla \cdot a \nabla v = \nabla \cdot f & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ v(0) = \nabla \cdot q, \end{cases}$$

with $q \in L^2_{loc}(\mathbb{R}^d)$ and $f : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$T \in \mathbb{R}^+ \mapsto \int_0^T f(s, \cdot) ds \text{ is in } L^2_{loc}(\mathbb{R}^d)$$

and for all $(x, \ell) \in \mathbb{R}^d \times [1, +\infty)$,

$$\left(\int_{B_\ell(x)} |f(\cdot, y)|^2 dy \right)^{\frac{1}{2}} \text{ is in } L^1_{loc}(\mathbb{R}^+).$$

There exists a universal constant $c \in (0, +\infty)$ such that the three following estimates hold:

(i) Assume that $f \equiv 0$. We have for all $T > 0$, $R \geq \sqrt{T}$ and $x \in \mathbb{R}^d$

$$\begin{aligned} T \left(\int \eta_R \left(\frac{y-x}{c} \right) |\nabla v(T, y)|^2 dy \right)^{\frac{1}{2}} + \left(\int \eta_R \left(\frac{y-x}{c} \right) \left| \int_0^T \nabla v(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \\ \lesssim_{d,\lambda} \left(\int \eta_R \left(\frac{y-x}{c} \right) |q(y)|^2 dy \right)^{\frac{1}{2}}. \end{aligned} \quad (2.2.32)$$

(ii) Assume that $q \equiv 0$. We have for all $R \geq 1$ and $x \in \mathbb{R}^d$

$$\begin{aligned} \left(\int \eta_R \left(\frac{y-x}{c} \right) \left| \int_0^1 \nabla v(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \lesssim_{d,\lambda} \left(\int \eta_R \left(\frac{y-x}{c} \right) \left| \int_0^1 f(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \\ + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int \eta_R \left(\frac{y-x}{c} \right) |f(s, y)|^2 dy \right)^{\frac{1}{2}} ds dt. \end{aligned} \quad (2.2.33)$$

(iii) Assume that q and f are supported in $B_\ell(x)$ for some $x \in \mathbb{R}^d$ and $\ell \in [1, +\infty)$. Then we have

$$\begin{aligned} \left(\int_{\mathbb{R}^d \setminus B_\ell(x)} e^{\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} \left| \int_0^1 \nabla v(s, y) ds \right|^2 dy dz \right)^{\frac{1}{2}} \lesssim_{d,\lambda} \left(\int_{B_\ell(x)} |q(y)|^2 dy \right)^{\frac{1}{2}} \\ + \left(\int_{B_\ell(x)} \left| \int_0^1 f(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \\ + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell(x)} |f(s, y)|^2 dy \right)^{\frac{1}{2}} ds dt. \end{aligned} \quad (2.2.34)$$

We then state a technical lemma needed in order to obtain pointwise estimates in time.

Lemma 2. Fix $r > 0$, $(s, x) \in \mathbb{R}^{d+1}$ and $g \in L^2(\mathbb{R}^d)^d$. Assume that v is a weak solution of

$$\partial_\tau v - \nabla \cdot a \nabla v = \nabla \cdot g \text{ in } (s - 4r^2, s) \times B_{2r}(x),$$

then we have

$$\sup_{t \in (s-r^2, s)} \int_{B_r(x)} |\nabla v(t, y)|^2 dt dy \lesssim_{d, \lambda} \int_{s-4r^2}^s \int_{B_{2r}(x)} |\nabla v(s', y)|^2 ds' dy + \int_{B_{2r}(x)} |g(y)|^2 dy. \quad (2.2.35)$$

The same holds for the operator $\partial_\tau + \nabla \cdot a \nabla$ on $(s, s + 4r^2) \times B_{2r}(x)$ by a time reflexion $t \mapsto -t$.

We conclude this section by the relationship between spatial averages of $u(T, \cdot)$ and averages of $q_r - \langle q_r \rangle$ over scales $r \leq \sqrt{T}$. This lemma allow us to deduce Corollary 1 from Theorem 10. We refer the reader to [92, Lem. 6] for the original proof of this result.

Lemma 3 (L^2 - L^1 estimate). Let u defined in (2.1.10). There exists a universal constant $c > 0$ such that for all $T > 0$ and $R \geq \sqrt{T}$

$$\begin{aligned} \sqrt{T} \left(\int \eta_R \left(\frac{y}{c} \right) |\nabla u(T, y)|^2 dy \right)^{\frac{1}{2}} + \left(\int \eta_R \left(\frac{y}{c} \right) |u(T, y)|^2 dy \right)^{\frac{1}{2}} \\ \lesssim \frac{1}{\sqrt{T}} \int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^{\sqrt{t}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{d}{2}} \int \eta_{\sqrt{2}R} \left(\frac{y}{c} \right) |q_r(t, y) - \langle q_r(t, y) \rangle| dy dr dt, \end{aligned}$$

where $\eta_R := R^{-d} e^{-\frac{|\cdot|}{R}}$ and $q(\cdot, \cdot)$ is defined in (2.1.13).

2.2.3 Large-scale regularity results

We state in this section two estimates, needed in the proof of Theorem 10, which are obtained from the large-scale regularity theory recalled in the appendix 2.4.2. We start with a lemma which gives a pointwise bound on a local average of the solution of the dual problem (2.2.11), depending on the behavior of the r.h.s. It constitutes the parabolic version of Lemmas 2, 3 and 4 of [89] established for elliptic systems.

Lemma 4 (Pointwise estimates on the dual problem). Let $f_1 \in C_b^1(\mathbb{R}^d)$, \bar{e} a unit vector of \mathbb{R}^d and v_r satisfies, in the weak sense for some $r \geq 1$, the parabolic backward system

$$\begin{cases} \partial_\tau v_r + \nabla \cdot a \nabla v_r = \nabla \cdot a f_r \bar{e} & \text{in } (-\infty, 0) \times \mathbb{R}^d, \\ v_r(0) = 0, \end{cases} \quad (2.2.36)$$

with f_r which satisfies one of the two following assumptions:

- $f_r := r^{-d} f_1(\frac{\cdot}{r})$ such that for all $x \in \mathbb{R}^d$

$$|f_1(x)| \lesssim \frac{1}{(|x| + 1)^d} \text{ and } |\nabla f_1(x)| \lesssim \frac{1}{(|x| + 1)^{d+1}}. \quad (2.2.37)$$

- For all $x \in \mathbb{R}^d$

$$|f_r(x)| \lesssim \frac{r}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} \text{ and } |\nabla f_r(x)| \lesssim \frac{r}{(|x| + 1)^{d+1}} \wedge \frac{1}{(|x| + 1)^d}. \quad (2.2.38)$$

We have

1. If (2.2.37) holds, for all $x \in \mathbb{R}^d$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy dt \right)^{\frac{1}{2}} \lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d}. \quad (2.2.39)$$

2. If (2.2.38) holds, for all $x \in \mathbb{R}^d$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \right)^{\frac{1}{2}} \lesssim r_*^{\frac{d}{2}}(0) \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}}. \quad (2.2.40)$$

Let us briefly comment on Lemma 4.

1. The bound (2.2.39) is needed to replace the plain energy estimate for the solution v_r^T of (2.2.11), as we did in the heuristic argument (2.2.18). In the homogeneous case, i.e $a = \text{Id}$, and in the case where $f_1 = g_1$, the bound (2.2.39) takes the more natural form: for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$|\nabla v_r(t, x)| \lesssim (|x| + r)^{-d}. \quad (2.2.41)$$

Indeed, (2.2.41) is easy to see using the parabolic Green function Γ of the heat operator $\partial_t - \Delta$: we have for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$\nabla v_r(t, x) = - \int_t^0 \nabla^2 \Gamma(s - t, \cdot) \star g_r(x) ds.$$

Thus, using the fact that, for all $s \geq t$, $\Gamma(s - t, \cdot)$ is proportional to $g_{\sqrt{s-t}}$ and using the semigroup property of Gaussian kernel $g_{\sqrt{s-t}} \star g_r = g_{\sqrt{r^2+s-t}}$ as well as the estimate, for all $x \in \mathbb{R}^d$, $e^{-\frac{|x|^2}{r^2+s-t}} \lesssim (1 + \frac{|x|^2}{r^2+s-t})^{-\frac{d}{2}-2}$, we have for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$\begin{aligned} |\nabla v_r(t, x)| &\lesssim \int_t^0 |\nabla^2 g_{\sqrt{r^2+s-t}}(x)| ds \\ &\lesssim |x|^2 \int_t^0 (r^2 + s - t)^{-\frac{d}{2}-2} e^{-\frac{|x|^2}{r^2+s-t}} ds \\ &\lesssim |x|^2 \int_t^0 (|x|^2 + s - t + r^2)^{-\frac{d}{2}-2} ds \\ &\lesssim (|x| + r)^{-d}. \end{aligned}$$

Also, if we have the more precise structure $f_r = \int_1^{r^2} \nabla g_{\sqrt{s}}(\cdot) ds$ (which satisfies the assumption (2.2.38)), using the same ideas as before, the bound (2.2.40) takes the more natural form: for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$|\nabla v_r(t, x)| \lesssim \frac{r}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}}.$$

Therefore, since the bounds (2.2.39) and (2.2.40) are natural in the homogeneous case and we know from homogenization theory that on large-scales the heterogeneous parabolic operator $\partial_t - \nabla \cdot a \nabla$ inherits (in form on the $C^{0,1}$ estimate (2.4.5)) the regularity theory of the homogenized operator $\partial_t - \nabla \cdot a_{\text{hom}} \nabla$, it is natural to expect that the two estimates (2.2.39) and

(2.2.40) hold in the heterogeneous case once we fix the scale (characterized by the minimal radius r_*). Note that the logarithm contributions in (2.2.39) and (2.2.40) are due to the fact that we have less structure on the r.h.s of (2.2.36) than the two we took above. We also point out that the logarithm contribution in (2.2.40) may be removed (see for instance Lemmas 3 and 4 of [89] for elliptic systems). However, we prefer to keep it this way and provide simple arguments for (2.2.40) rather than going through additional technical difficulties.

2. We may deduce the result of Lemmas 3 and 4 of [89] from (2.2.39) by sending $t \downarrow -\infty$. Indeed, one may prove, from the localized energy of Lemma 1 that $v_r(t, \cdot) \xrightarrow{t \rightarrow -\infty} \tilde{v}_r$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ with $\nabla \cdot a \nabla \tilde{v}_r = \nabla \cdot a f_r \bar{e}$ and then pass to the limit in (2.2.39).

The next lemma allows us to control spatial averages of ∇u at scale $R < \sqrt{T}$. Combined with the energy estimate (2.2.32), it implies in particular the estimate (2.2.24) needed in the proof of Theorem 10.

Lemma 5 (Control of averages). *Let u be defined in (2.1.10). Assume that there exist two monotone functions f and g (increasing and decreasing respectively) such that for all $T \geq 1$ and for all $x \in \mathbb{R}^d$ there exists a constant $C(x, T) < +\infty$ for which*

$$\int_{B_{\sqrt{T}}(x)} |\nabla u(T, y)|^2 dy \leq C(x, T) f(T) g(T). \quad (2.2.42)$$

Then we have for all $T \geq 1$, $x \in \mathbb{R}^d$ and $R < \sqrt{T}$

$$\int_{B_R(x)} |\nabla u(T, y)|^2 dy \lesssim \left(\frac{r_*(x)}{R} \vee 1 \right)^d \tilde{C}(x, T) f(T) g\left(\frac{T}{2}\right),$$

with $\tilde{C}(x, T) := \max \left\{ C(x, T), \int_{\frac{T}{2}}^T C(x, s) ds \right\}$.

2.2.4 Suboptimal control of fluctuations of the time dependent flux

In this section, we state the suboptimal moment bounds of $q_r(T) - \langle q_r(T) \rangle$. We prove that it displays the central limit theorem scaling $r^{-\frac{d}{2}}$, a growth in T which depends on the parameter β defined in (2.1.6) and a $\log(T)$ correction (which makes it suboptimal and will be removed later). We first state the following bound on $q_r(\cdot, \cdot)$, for $r \leq 1$. It is only needed for technical reasons since, in view of the application of Lemma 3, r is allowed to be arbitrary close to 0.

Lemma 6. *Let $q(\cdot, \cdot)$ be defined in (2.1.13). For all $r \in (0, 1)$ and $x \in \mathbb{R}^d$, there exists a random variable $\mathcal{C}_*(r, x)$ such that for all $\sqrt{T} \geq \frac{1}{2}$ we have*

$$|q_r(T, x)| \leq \left(1 + r^{-\frac{d}{2}} \log\left(\frac{\sqrt{T}}{r}\right)\right) \mathcal{C}_*(r, x), \quad (2.2.43)$$

with $\sup_{(r, x) \in \mathbb{R}^+ \times \mathbb{R}^d} \left\langle \exp\left(\frac{1}{C} \mathcal{C}_*^{2 \wedge \frac{\beta \wedge d}{d}}(r, x)\right) \right\rangle \leq 2$ for some constant $C > 0$ depending on d and λ .

The next lemmas give an estimate of the functional derivative of averages of the flux $q_r(T)$ and $q(T) \star f_r$ for $T \geq 1$ and $r \leq \sqrt{T}$. This is the starting point for using our assumption (2.1.4).

Lemma 7 (Functional derivative). *Let $q(\cdot, \cdot)$ be defined in (2.1.13). There exists a universal constant $c \in (0, +\infty)$ such that for all $T \geq 1$, $r > 0$, $x \in \mathbb{R}^d$ and $\ell \in [1, +\infty)$, we have*

$$\begin{aligned} |\partial_{x,\ell}^{fct} q_r(T)| &\lesssim_{d,\lambda} \int_{B_\ell(x)} g_r(y) dy + \int_{B_\ell(x)} \left| \int_0^T \nabla u(s, y) ds \right| g_r(y) dy + \int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \\ &+ \mathcal{F}_{r,\ell}(x) \mathbb{1}_{\ell < \sqrt{T}} + \mathcal{G}_{r,\ell}(x) \mathbb{1}_{\ell \geq \sqrt{T}} + \int_{B_\ell(x)} \int_1^T |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy, \end{aligned} \quad (2.2.44)$$

where $v^T = (v_k^T)_{k \in [1,d]}$ is a weak solution of the backward parabolic system

$$\begin{cases} \partial_\tau v_k^T + \nabla \cdot a^* \nabla v_k^T = \nabla \cdot a g_r e_k & \text{on } (-\infty, T) \times \mathbb{R}^d, \\ v_k^T(T) = 0, \end{cases} \quad (2.2.45)$$

with

$$\mathcal{F}_{r,\ell}(x) = \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |g_r(y)|^2 dy dz \right)^{\frac{1}{2}} + \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |\nabla v^T(1, y)|^2 dy dz \right)^{\frac{1}{2}}, \quad (2.2.46)$$

and

$$\begin{aligned} \mathcal{G}_{r,\ell}(x) &= \mathcal{T}_{x,\ell}(\eta_r)(0) + \mathbb{1}_{\ell \geq r_*(0)} \left(\int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \right. \\ &+ \left. \int_{B_{7\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_{r_*(y)})(y) dy \right) \\ &+ \mathbb{1}_{\ell < r_*(0)} r_*^{\frac{d}{2}}(0) \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |\nabla v^T(1, y)|^2 dy dz \right)^{\frac{1}{2}}. \end{aligned} \quad (2.2.47)$$

as well as for all $y \in \mathbb{R}^d$ and $\rho > 0$

$$\begin{aligned} \mathcal{T}_{x,\ell}(\eta_\rho)(y) &= \left(\int_{B_\ell(x)} \eta_\rho(z-y) \left(1 + \left| \int_0^1 \nabla u(t, z) dt \right|^2 \right) dz \right)^{\frac{1}{2}} \\ &+ \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell(x)} \eta_\rho(z-y) |\nabla u(s, z)|^2 dz \right)^{\frac{1}{2}} ds dt. \end{aligned} \quad (2.2.48)$$

Lemma 8. *Let $q(\cdot, \cdot)$ be defined in (2.1.13) and for all $r > 0$ we consider $f_r \in C_b^1(\mathbb{R}^d)$ satisfying, for all $y \in \mathbb{R}^d$*

$$f_r(y) = \int_1^{r^2} \tilde{f}_r(s, y) ds \quad \text{with } |\tilde{f}_r(s, y)| \lesssim |y| s^{-1} g_{\sqrt{s}}(y). \quad (2.2.49)$$

There exists a universal constant $c \in (0, +\infty)$ such that for all $r \geq 2$, $x \in \mathbb{R}^d$ and $\ell \in [1, +\infty)$, we have

$$\begin{aligned} |\partial_{x,\ell}^{fct} q(r^2) \star f_r| &\lesssim \int_{B_\ell(x)} |f_r(y)| dy + \int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(s, y) ds \right| |f_r(y)| dy \\ &+ \int_{B_\ell(x)} |\nabla v^{r^2}(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \\ &+ \mathcal{K}_{r,\ell}(x) + \mathcal{G}_{r,\ell}(x) + \int_{B_\ell(x)} \int_1^{r^2} |\nabla u(t, y)| |\nabla v^{r^2}(t, y)| dt dy, \end{aligned} \quad (2.2.50)$$

where $v^{r^2} = (v_k^{r^2})_{k \in \llbracket 1, d \rrbracket}$ is a weak solution of the backward parabolic problem

$$\begin{cases} \partial_\tau v_k^{r^2} + \nabla \cdot a^* \nabla v_k^{r^2} = \nabla \cdot a f_r e_k & \text{on } (-\infty, r^2) \times \mathbb{R}^d, \\ v_k^{r^2}(r^2) = 0, \end{cases} \quad (2.2.51)$$

with

$$\begin{aligned} \mathcal{K}_{r, \ell}(x) &:= \mathbf{1}_{\ell \geq r} \int_1^{r^2} s^{-\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_{\sqrt{s}})(0) ds \\ &+ \mathbf{1}_{\ell < r} \left(\mathcal{T}_{x, \ell}(\eta_1)(0) + \sum_{n=0}^{\lceil \log_2(3\ell) \rceil} 2^n \mathcal{T}_{x, \ell}(\eta_{2^{n+1}})(0) + \int_{\mathbb{R}^d \setminus B_{2\ell}} \left(\int_{B_\ell(y)} |f_r(z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_\ell)(y) dy \right). \end{aligned} \quad (2.2.52)$$

and $\mathcal{G}_{r, \ell}$ as well as $\mathcal{T}_{x, \ell}$ are defined in (2.2.47) (for $T = r^2$) and (2.2.48) respectively.

We finally state the main result of this section, which is the sub-optimal moment bound on the time dependent flux $q_r(T, \cdot)$ for $\frac{1}{2} \leq r \leq \sqrt{T}$.

Proposition 3 (Sub-optimal fluctuation estimates). *Let $q(\cdot, \cdot)$ defined in (2.1.13). For all $T \geq 1$, $1 \leq r \leq \sqrt{T}$ and $p \in [1, +\infty)$*

$$\langle |q_r(T) - \langle q_r(T) \rangle|^p \rangle^{\frac{1}{p}} \lesssim_{d, \lambda, \beta} p^{\frac{1}{2} + \frac{d+2}{\beta \wedge d}} r^{-\frac{d}{2}} (1 + \log(T) + \log^2(\frac{\sqrt{T}}{r})) \mu_\beta(T), \quad (2.2.53)$$

with

$$\mu_\beta(T) := \begin{cases} T^{\frac{d}{4} - \frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) & \text{if } \beta = d, \\ 1 & \text{if } \beta > d. \end{cases}$$

2.3 Proofs

We give in the section the all proofs of the results stated in the sections 2.2.2, 2.2.3 and 2.2.4. For notational convenience, we shall assume that the results of Lemmas 1 and 3 hold for the universal constant $c = 1$. In the general case, it suffices to change the kernels g_r and η_r from line to line (by allowing a constant in the exponential). We also drop the dependance on d , λ and β in the inequalities.

2.3.1 Proof of the deterministic results

Proof of Lemma 1: Localized energy estimates

We only provide the arguments for (2.2.33) and (2.2.34), the proof of (2.2.32) can be found in [92, Lem.1]. Without loss of generality, we may assume that $x = 0$.

Step 1. Proof of (2.2.33). We set for all $t \in [0, 1]$, $w(t, \cdot) := \int_0^t v(s, \cdot) ds$ and we note that w is a weak solution of

$$\begin{cases} \partial_\tau w - \nabla \cdot a \nabla w = \nabla \cdot \int_0^\tau f(s, \cdot) ds & \text{on } (0, 1] \times \mathbb{R}^d, \\ w(0) = 0. \end{cases}$$

The idea of the proof is to use the estimate (2.2.32) by expressing w with help of the Duhamel formula. We denote by S the semigroup associated to the operator $-\nabla \cdot a \nabla$, namely $(S(t))_{t \in \mathbb{R}^+}$ is a family of operators such that for all Schwartz distributions ζ on \mathbb{R}^d , $z := S(\cdot)\zeta$ is the unique weak solution of

$$\begin{cases} \partial_\tau z - \nabla \cdot a \nabla z = 0 & \text{on } (0, 1] \times \mathbb{R}^d, \\ z(0) = \zeta. \end{cases}$$

We express $\nabla w(1, \cdot)$ with help of S in form of the Duhamel formula, that is

$$\nabla w(1, \cdot) = \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_0^t f(s, \cdot) ds \right) dt.$$

Thus, we write for all $R \geq 1$, by the triangle inequality and $\int_0^t f(s, \cdot) ds = \int_0^1 f(s, \cdot) ds - \int_t^1 f(s, \cdot) ds$ in the last line

$$\begin{aligned} \left(\int \eta_R(y) |\nabla w(1, y)|^2 dy \right)^{\frac{1}{2}} &= \left(\int \eta_R(y) \left| \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_0^t f(s, y) ds \right) dt \right|^2 dy \right)^{\frac{1}{2}} \\ &\leq \left(\int \eta_R(y) \left| \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_0^1 f(s, y) ds \right) dt \right|^2 dy \right)^{\frac{1}{2}} \\ &\quad + \left(\int \eta_R(y) \left| \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_t^1 f(s, y) ds \right) dt \right|^2 dy \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.1)$$

For the first r.h.s term of (2.3.1), we use (2.2.32) for $T = 1$ in form of

$$\left(\int \eta_R(y) \left| \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_0^1 f(s, y) ds \right) dt \right|^2 dy \right)^{\frac{1}{2}} \lesssim \left(\int \eta_R(y) \left| \int_0^1 f(s, y) ds \right|^2 dy \right)^{\frac{1}{2}},$$

which gives the first r.h.s term of (2.2.33). For the second r.h.s term of (2.3.1), we use (2.2.32) for $T = 1$, this time in the pointwise way, combined with the Minkowski inequality in $L^2(\mathbb{R}^d, \eta_R dx)$ to get

$$\begin{aligned} \left(\int \eta_R(y) \left| \int_0^1 \nabla \left(S(1-t) \nabla \cdot \int_t^1 f(s, y) ds \right) dt \right|^2 dy \right)^{\frac{1}{2}} &\leq \int_0^1 \left(\int \eta_R(y) \left| \nabla \left(S(1-t) \nabla \cdot \int_t^1 f(s, y) ds \right) \right|^2 dy \right)^{\frac{1}{2}} dt \\ &\stackrel{(2.2.32)}{\lesssim} \int_0^1 \frac{1}{1-t} \left(\int \eta_R(y) \left| \int_t^1 f(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} dt \\ &\leq \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int \eta_R(y) |f(s, y)|^2 dy \right)^{\frac{1}{2}} ds dt, \end{aligned}$$

which gives the second r.h.s term of (2.2.33).

Step 2. Poof of (2.2.34). Since, for all $(y, z) \in B_\ell \times \mathbb{R}^d \setminus B_\ell$, we have $e^{-\frac{|y-z|}{c\ell}} \leq e^{\frac{1}{c} - \frac{|z|}{c\ell}}$, we deduce from (2.2.32), (2.2.33) (applied with $R = \ell$) and the fact that f and q are compactly supported in B_ℓ :

$$\begin{aligned} e^{\frac{|z|}{2c\ell}} \int_{B_\ell(z)} \left| \int_0^1 \nabla v(s, y) ds \right|^2 dy &\lesssim e^{-\frac{|z|}{2c\ell}} \left(\int_{B_\ell} |q(y)|^2 dy + \int_{B_\ell} \left| \int_0^1 f(s, y) ds \right|^2 dy \right) \\ &\quad + e^{-\frac{|z|}{2c\ell}} \left(\int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell} |f(s, y)|^2 dy \right)^{\frac{1}{2}} ds dt \right)^2, \end{aligned}$$

which yields (2.2.34) by integrating over $\mathbb{R}^d \setminus B_\ell$.

Proof of Lemma 2

The arguments are the same as in [8, Lem 8.2] where a proof is given for $g \equiv 0$. For reader convenience, we repeat the proof with very small changes in order to consider non-zero r.h.s g . Without loss of generality, we may assume that $(s, x) = (0, 0)$.

Since g does not depend on time, $\partial_\tau v$ is also a weak solution of the equation and therefore the Caccioppoli inequality, see Lemma 12, yields for all $\rho \leq R$

$$\int_{C_\rho} |\nabla \partial_\tau v(s, y)|^2 dy ds \lesssim \frac{1}{(R - \rho)^2} \int_{C_R} |\partial_\tau v(s, y)|^2 dy ds. \quad (2.3.2)$$

To be entirely rigorous, we need to justify that $\partial_\tau u$ belongs to $H_{\text{par}}^1(C_R)$. One may justify it by considering difference quotients in time, obtaining a version of (2.3.2) by the Caccioppoli inequality for these difference quotients, which then allows to pass to weak limits in the difference quotient parameter to obtain that $\partial_\tau u$ does indeed belong to $H_{\text{par}}^1(C_R)$ and is a solution of the equation. Next, differentiating gives, for all $t \in (-r^2, 0)$

$$\partial_\tau \int_{B_r} |\nabla v(t, y)|^2 dy = 2 \int_{B_r} \nabla v(t, y) \cdot \nabla \partial_\tau v(t, y) dy,$$

and thus we have, after an integration in time and combined with a Young's inequality that

$$\begin{aligned} \sup_{t \in (-r^2, 0)} \int_{B_r} |\nabla v(t, y)|^2 dy &\leq \int_{C_r} |\nabla v(s, y)|^2 dy ds + 2r^2 \int_{C_r} |\nabla v(s, y) \cdot \nabla \partial_\tau v(s, y)| dy ds \\ &\leq 2 \int_{C_r} |\nabla v(s, y)|^2 dy ds + r^4 \int_{C_r} |\nabla \partial_\tau v(s, y)|^2 dy ds. \end{aligned}$$

Combining this with (2.3.2) applied with $\rho = r$ and $R = 2r$ yields

$$\sup_{t \in (-r^2, 0)} \int_{B_r} |\nabla v(t, y)|^2 dy \lesssim \int_{C_r} |\nabla v(s, y)|^2 dy ds + r^2 \int_{C_{2r}} |\partial_\tau v(s, y)|^2 dy ds. \quad (2.3.3)$$

It remains to estimate the second r.h.s term of (2.3.3). For this, fix $\rho < r$, $\eta \in C_c^\infty(B_{\frac{3}{2}\rho})$ such that $\eta = 1$ in B_ρ , $0 \leq \eta \leq 1$ and $|\nabla \eta| \lesssim \frac{1}{\rho}$. Testing the equation for v with the function $\eta \partial_\tau v$, we obtain

$$\begin{aligned} \int_{C_\rho} |\partial_\tau v(s, y)|^2 dy &\lesssim \frac{1}{\rho} \int_{C_{\frac{3}{2}\rho}} |\partial_\tau v(s, y)| |\nabla v(s, y)| dy ds + \int_{C_{\frac{3}{2}\rho}} |\nabla \partial_\tau v(s, y)| |\nabla v(s, y)| dy ds \\ &\quad + \frac{1}{\rho} \int_{C_{\frac{3}{2}\rho}} |g(y)| |\partial_\tau v(s, y)| dy ds + \int_{C_{\frac{3}{2}\rho}} |g(y)| |\nabla \partial_\tau v(s, y)| dy ds. \end{aligned} \quad (2.3.4)$$

Then, using the Cauchy-Schwarz inequality combined with (2.3.2), we have

$$\begin{aligned} &\int_{C_{\frac{3}{2}\rho}} |\nabla \partial_\tau v(s, y)| |\nabla v(s, y)| dy ds + \int_{C_{\frac{3}{2}\rho}} |g(y)| |\nabla \partial_\tau v(s, y)| dy ds \\ &\lesssim \frac{1}{\rho} \left(\int_{C_{2\rho}} |\partial_\tau v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \left(\left(\int_{B_{2\rho}} |g(y)|^2 dy \right)^{\frac{1}{2}} + \left(\int_{C_{2\rho}} |\nabla v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right). \end{aligned}$$

Therefore, (2.3.4) turns into

$$\int_{C_\rho} |\partial_\tau v(s, y)|^2 dy \lesssim \frac{1}{\rho} \left(\int_{C_{2\rho}} |\partial_\tau v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \left(\left(\int_{B_{2\rho}} |g(y)|^2 dy \right)^{\frac{1}{2}} + \left(\int_{C_{2\rho}} |\nabla v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right).$$

Choosing now $\frac{1}{2} \leq \sigma' < \sigma \leq 1$, we obtain by a covering argument (considering the decomposition $C_{\sigma'4r} \subset \bigcup_{i=1}^N (t_i - ((\sigma - \sigma')\frac{r}{2})^2, t_i) \times B_{(\sigma - \sigma')\frac{r}{2}}(x_i)$ for $(t_i, x_i)_{i \in \llbracket 1, N \rrbracket} \subset C_{\sigma'4r}$ and $N \sim (\sigma - \sigma')^{-d-2}$) that

$$\begin{aligned} & \int_{C_{\sigma'4r}} |\partial_\tau v(s, y)|^2 dy \\ & \lesssim \frac{1}{(\sigma - \sigma')^{d+3} r} \left(\int_{C_{\sigma 4r}} |\partial_\tau v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \left(\left(\int_{B_{4r}} |g(y)|^2 dy \right)^{\frac{1}{2}} + \left(\int_{C_{4r}} |\nabla v(s, y)|^2 dy ds \right)^{\frac{1}{2}} \right), \end{aligned}$$

which turns to, by a Young's inequality, for some constant C depending on d and λ

$$\begin{aligned} \int_{C_{\sigma'4r}} |\partial_\tau v(s, y)|^2 dy & \leq \frac{1}{2} \int_{C_{\sigma 4r}} |\partial_\tau v(s, y)|^2 dy ds \\ & \quad + \frac{C}{(\sigma - \sigma')^{2(d+3)} r^{-2}} \left(\int_{B_{4r}} |g(y)|^2 dy + \int_{C_{4r}} |\nabla v(s, y)|^2 dy ds \right). \end{aligned}$$

Lemma 11 applied with $\rho : \sigma \mapsto \int_{C_{\sigma 4r}} |\partial_\tau v(s, y)|^2 dy ds$ then implies that

$$\int_{C_{2r}} |\partial_\tau v(s, y)|^2 dy ds \lesssim \frac{1}{r^2} \left(\int_{C_{4r}} |\nabla v(s, y)|^2 dy ds + \int_{B_{4r}} |g(y)|^2 dy \right),$$

which gives, combining with (2.3.3)

$$\sup_{t \in (-r^2, 0)} \int_{B_r} |\nabla v(t, y)|^2 dy \lesssim \int_{C_{4r}} |\nabla v(s, y)|^2 dy ds + \int_{B_{4r}} |g(y)|^2 dy.$$

We conclude the proof by a covering argument by considering $\frac{r}{4}$ instead of r .

2.3.2 Proof of the large-scale regularity results

We provide the proofs of Lemmas 4 and 5. Our main tool here in the large-scale regularity theory for parabolic system recalled in Appendix 2.4.2.

Proof of Lemma 4: Pointwise estimates on the dual problem

We prove Lemma 4 in two steps. The first step is devoted to prove (2.2.39) and we do it in two times. First, we treat the particular case where f_r is compactly supported in the ball B_r for some $r \geq 1$. Second, we treat the general case by decomposing \mathbb{R}^d into dyadic annuli $(\mathcal{B}_k)_{k \in \mathbb{N}}$, defined by $\mathcal{B}_k := B_{2^{k+1}r} \setminus B_{2^k r}$ for $k \geq 1$ and $\mathcal{B}_0 := B_r$, and writing $f_r = \sum_{k=0}^{+\infty} f_r \chi_k$, where $(\chi_k)_{k \in \mathbb{N}}$ is a partition of unity according to the decomposition $(\mathcal{B}_k)_{k \in \mathbb{N}}$. We then apply the result of the compact supported case for each $k \in \mathbb{N}$. The second step is devoted to prove (2.2.40) and this is done by using the results of the first step. This extends Lemmas 2, 3 and 4 of [89] from the elliptic to the parabolic setting.

Step 1. Proof of (2.2.39). We split the proof into two substeps.

Substep 1.1. We prove that under the assumptions

$$f_r \text{ is supported in } B_r \text{ and } r^d \sup_{x \in \mathbb{R}^d} |f_r(x)| + r^{d+1} \sup_{x \in \mathbb{R}^d} |\nabla f_r(x)| \lesssim 1, \quad (2.3.5)$$

we have for all $x \in \mathbb{R}^d$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_r(s, y)|^2 dy ds \right)^{\frac{1}{2}} \lesssim \frac{\left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}}}{(|x| + r)^d}. \quad (2.3.6)$$

The estimate (2.3.6) will come from the following four relations and estimates:

1. For all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$\nabla v_r(t, x) = \int_0^{-t} \nabla w_r(s, x) ds, \quad (2.3.7)$$

with w_r is the weak solution of

$$\begin{cases} \partial_t w_r - \nabla \cdot a \nabla w_r = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ w_r(0) = \nabla \cdot a f_r \bar{e}. \end{cases} \quad (2.3.8)$$

2. The plain energy estimate: for all $t \in \mathbb{R}_-$

$$\int_{\mathbb{R}^d} |\nabla v_r(t, x)|^2 dy \lesssim r^{-d}. \quad (2.3.9)$$

3. The large-scale regularity estimate: for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$ such that $|x| \geq 4(r_*(x) \vee r)$

$$\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \lesssim \frac{\left(\frac{r_*(0)}{r} \vee 1 \right)^d}{(|x| + r)^{2d}}. \quad (2.3.10)$$

4. The large-scale $C^{0,1}$ estimates: for all $x \in \mathbb{R}^d$, $\sqrt{-t} \geq 2r_*(x)$ and $r \geq r_*(x)$:

- For $\sqrt{-t} \leq r$,

$$\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \lesssim \int_t^0 \int_{B_{\sqrt{-t}}(x)} |\nabla v_r(s, y)|^2 dy ds + r^{-2d}. \quad (2.3.11)$$

- For $\sqrt{-t} \geq r$,

$$\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \lesssim \int_t^{t+r^2} \int_{B_r(x)} |\nabla v_r(s, y)|^2 dy ds + r^{-2d}. \quad (2.3.12)$$

Argument for (2.3.7). A direct computation shows that $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d \mapsto \int_0^{-t} w_r(s, x) ds$ is the weak solution of (2.2.36). Thus, by uniqueness, for all $t \in \mathbb{R}_-$, $v_r(t, \cdot) = \int_0^{-t} w_r(s, \cdot) ds$ and (2.3.7) follows.

Argument for (2.3.9). We have, by using the formula (2.3.7) combined with the localized energy estimate (2.2.32) applied to the equation (2.3.8) and the support condition (2.3.5) of f_r , for all $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$

$$\begin{aligned} \int \eta_{\sqrt{-t}}(y-x) |\nabla v_r(t, x)|^2 dy &\stackrel{(2.3.7)}{=} \int \eta_{\sqrt{-t}}(y-x) \left| \int_0^{-t} \nabla w_r(t, x) \right|^2 dy \\ &\stackrel{(2.2.32)}{\lesssim} \int \eta_{\sqrt{-t}}(y-x) |f_r(y)|^2 dy \\ &\stackrel{(2.3.5)}{\lesssim} r^{-2d} \int_{B_r} \eta_{\sqrt{-t}}(y-x) dy, \end{aligned} \quad (2.3.13)$$

which gives (2.3.9) by integrating the estimate over $x \in \mathbb{R}^d$.

Argument for (2.3.10). We first prove by a duality argument that for all $R \geq 2(r_*(0) \vee r)$ and $t \in \mathbb{R}_-$

$$\int_{\mathbb{R}^d \setminus B_R} |\nabla v_r(t, y)|^2 dy \lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^d R^{-d}. \quad (2.3.14)$$

Let $h \in C_c^\infty(\mathbb{R}^d)$ supported in $\mathbb{R}^d \setminus B_R$. Let $t \in \mathbb{R}_-$, $s \in [0, -t]$ and k^s be the weak solution of the backward parabolic system (corresponding to the dual system of (2.2.36) with final time s),

$$\begin{cases} \partial_\tau k^s + \nabla \cdot a^* \nabla k^s = 0 & \text{on } (-\infty, s) \times \mathbb{R}^d, \\ k^s(s) = \nabla \cdot h. \end{cases} \quad (2.3.15)$$

For all $\tau \in (-\infty, s)$, we have, by testing (2.3.15) with w_r

$$\int_{\mathbb{R}^d} w_r(\tau, y) \partial_\tau k^s(\tau, y) dy - \int_{\mathbb{R}^d} \nabla k^s(\tau, y) \cdot a(y) \nabla w_r(\tau, y) dy = 0,$$

and (2.3.8) with k^s

$$\int_{\mathbb{R}^d} k^s(\tau, y) \partial_\tau w_r(\tau, y) dy + \int_{\mathbb{R}^d} \nabla k^s(\tau, y) \cdot a(y) \nabla w_r(\tau, y) dy = 0.$$

By summing the two identities above, integrating in time over $\tau \in [0, s]$ and noticing that from the initial conditions of w_r and k^s we have

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^s w_r(\tau, y) \partial_\tau k^s(\tau, y) d\tau dy + \int_{\mathbb{R}^d} \int_0^s k^s(\tau, y) \partial_\tau w_r(\tau, y) d\tau dy \\ = \left[\int_{\mathbb{R}^d} w_r(\tau, y) k^s(\tau, y) dy \right]_0^s \\ = \int \nabla k^s(0, y) \cdot a(y) f_r(y) \bar{e} dy - \int \nabla w_r(s, y) \cdot h(y) dy, \end{aligned}$$

we get

$$\int \nabla w_r(t, y) \cdot h(y) dy = \int \nabla k^s(0, y) \cdot a(y) f_r(y) \bar{e} dy.$$

It follows by integrating over $s \in [0, -t]$, using the formula (2.3.7) combined with the Cauchy-Schwarz inequality and (2.3.5) that

$$\left| \int_{\mathbb{R}^d} h(y) \cdot \nabla v_r(t, y) dy \right| \leq \left(\int_{B_r} \left| \int_0^{-t} \nabla k^s(0, y) ds \right|^2 dy \right)^{\frac{1}{2}}. \quad (2.3.16)$$

The r.h.s of (2.3.16) is then dominated as follows. First, we set $\tilde{v}(t, \cdot) := \int_0^{-t} k^s(0, y) ds$ and by noticing that for all $s \in (0, -t)$, $k^s(0, \cdot) = k^0(-s, \cdot)$, we have

$$\begin{cases} \partial_t \tilde{v} + \nabla \cdot a^* \nabla \tilde{v} = \nabla \cdot h & \text{in } (-\infty, 0) \times \mathbb{R}^d, \\ \tilde{v}(0) = 0. \end{cases}$$

Second, we denote by \bar{v} the weak solution of

$$\begin{cases} \partial_t \bar{v} + \nabla \cdot a^* \nabla \bar{v} = \nabla \cdot h \mathbf{1}_{(-\infty, 0)} & \text{in } \mathbb{R}^{d+1}, \\ \bar{v}(0) = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d. \end{cases} \quad (2.3.17)$$

\bar{v} is an extension of \tilde{v} in the sense that $\nabla \tilde{v}(s, \cdot) = \nabla \bar{v}(s, \cdot)$ as long as $s \leq 0$. Now, since $h = 0$ in B_R , we have by using the estimate $\int_{B_r} \lesssim \left(\frac{r_*(0)}{r} \vee 1\right)^{\frac{d}{2}} \int_{B_{r_*(0) \vee r}}$, Lemma 2 and the large-scale $C^{0,1}$ estimate (2.4.6)

$$\begin{aligned} \left(\int_{B_r} \left| \int_0^{-t} \nabla k^s(0, y) ds \right|^2 dy \right)^{\frac{1}{2}} &= \left(\int_{B_r} |\nabla \bar{v}(t, y)|^2 dy \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} \left(\int_{B_{r_*(0) \vee r}} |\nabla \bar{v}(t, y)|^2 dy \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} \left(\int_t^{t+4(r_*(0) \vee r)^2} \int_{B_{2(r_*(0) \vee r)}} |\nabla \bar{v}_r(s, y)|^2 dy ds \right)^{\frac{1}{2}} \\ &\stackrel{(2.4.6)}{\lesssim} \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} \left(\int_t^{t+R^2} \int_{B_R} |\nabla \bar{v}(s, y)|^2 dy ds \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.18)$$

Now, since $\bar{v} \equiv 0$ in $\mathbb{R}^+ \times \mathbb{R}^d$, we have using the plain energy estimate (for which the proof is identical as (2.3.9)): for all $s \in \mathbb{R}_-$

$$\int |\nabla \bar{v}(s, y)|^2 dy \lesssim \int |h(y)|^2 dy,$$

(i) for $\sqrt{-t} \geq R$

$$\int_t^{t+R^2} \int_{B_R} |\nabla \bar{v}(s, y)|^2 dy ds = \int_t^{t+R^2} \int_{B_R} |\nabla \tilde{v}(s, y)|^2 dy ds \lesssim R^{-d} \int |h(y)|^2 dy. \quad (2.3.19)$$

(ii) for $R \geq \sqrt{-t}$

$$\begin{aligned} \int_t^{t+R^2} \int_{B_R} |\nabla \bar{v}(s, y)|^2 dy ds &= R^{-2} \int_t^0 \int_{B_R} |\nabla \tilde{v}(s, y)|^2 dy ds \lesssim \left(\frac{\sqrt{-t}}{R} \right)^2 R^{-d} \int |h(y)|^2 dy \\ &\lesssim R^{-d} \int |h(y)|^2 dy. \end{aligned} \quad (2.3.20)$$

The combination of (2.3.16), (2.3.18), (2.3.19) and (2.3.20) yields

$$\left| \int_{\mathbb{R}^d} h(y) \cdot \nabla v_r(t, y) dy \right| \lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} R^{-\frac{d}{2}} \left(\int |h(y)|^2 dy \right)^{\frac{1}{2}},$$

which gives (2.3.14) by the arbitrariness of h .

We now prove (2.3.10). Let $R := \frac{1}{2}|x|$ and assume that $R \geq 2(r_*(x) \vee r)$. Without loss of generality, we may assume that $R \geq 2r_*(0)$. Indeed, otherwise, we deduce from the $\frac{1}{8}$ -Lipschitz property of $r_*(x)$ in form of

$$r_*(0) \leq r_*(x) + \frac{|x|}{8} \rightarrow r_*(x) \geq \frac{|x|}{4},$$

and

$$r_*(0) \geq r_*(x) - \frac{|x|}{8} \rightarrow \frac{3}{2}r_*(0) \geq r_*(x),$$

as well as (2.3.9) that

$$\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \lesssim r_*^{-d}(x) \int |\nabla v_r(t, y)|^2 dy \lesssim \frac{r_*^d(0)}{(|x| + r)^d}.$$

We have $B_R(x) \subset \mathbb{R}^d \setminus B_R$. Indeed for all $y \in B_R(x)$, the triangle inequality yields

$$|y| \geq \left| |y - x| - |x| \right| \geq 2R - R = R,$$

so that $y \notin B_R$. We then argue once again by extension and we consider \bar{v}_r the weak solution of

$$\begin{cases} \partial_\tau \bar{v}_r + \nabla \cdot a \nabla \bar{v}_r = \nabla \cdot a f_r \bar{\mathbb{1}}_{(-\infty, 0)} & \text{in } \mathbb{R}^{d+1}, \\ \bar{v}_r = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \quad (2.3.21)$$

for which $v_r(s, \cdot) = \bar{v}_r(s, \cdot)$ as long as $s \leq 0$. It then follows from Lemma 2 applied to the equation (2.3.21) and the large-scale $C^{0,1}$ estimate (2.4.6) (since from the inclusion $B_R(x) \subset \mathbb{R}^d \setminus B_R$ and the assumption (2.3.5), $f_r \equiv 0$ on $B_R(x)$) that for all $t \in \mathbb{R}_-$

$$\begin{aligned} \int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy &= \int_{B_{r_*(x)}(x)} |\nabla \bar{v}_r(t, y)|^2 dy \\ &\lesssim \int_t^{t+4r_*^2(x)} \int_{B_{2r_*(x)}(x)} |\nabla \bar{v}_r(s, y)|^2 ds dy \\ &\stackrel{(2.4.6)}{\lesssim} \int_t^{t+R^2} \int_{B_R(x)} |\nabla \bar{v}_r(s, y)|^2 ds dy. \end{aligned} \quad (2.3.22)$$

Now, since $\bar{v}_r \equiv 0$ in $\mathbb{R}^+ \times \mathbb{R}^d$, we have from (2.3.14):

(i) For $\sqrt{-t} \geq R$,

$$\begin{aligned} \int_t^{t+R^2} \int_{B_R(x)} |\nabla \bar{v}_r(s, y)|^2 ds dy &= \int_t^{t+R^2} \int_{B_R(x)} |\nabla v_r(s, y)|^2 ds dy \\ &\leq R^{-d} \int_t^{t+R^2} \int_{\mathbb{R}^d \setminus B_R} |\nabla v_r(s, y)|^2 ds dy \\ &\stackrel{(2.3.14)}{\lesssim} \left(\frac{r_*(0)}{r} \vee 1 \right)^d R^{-2d}. \end{aligned} \quad (2.3.23)$$

(ii) For $R \geq \sqrt{-t}$,

$$\begin{aligned}
 \int_t^{t+R^2} \int_{B_R(x)} |\nabla \bar{v}_r(s, y)|^2 ds dy &= R^{-2} \int_t^0 \int_{B_R(x)} |\nabla v_r(s, y)|^2 ds dy \\
 &\leq R^{-d} \left(\frac{\sqrt{-t}}{R} \right)^2 \int_t^0 \int_{\mathbb{R}^d \setminus B_R} |\nabla v_r(s, y)|^2 ds dy \\
 &\stackrel{(2.3.14)}{\lesssim} \left(\frac{r_*(0)}{r} \vee 1 \right)^d R^{-2d}.
 \end{aligned} \tag{2.3.24}$$

The combination of (2.3.22), (2.3.23) and (2.3.24) concludes the argument for (2.3.10) since $|x| \geq 2r$ implies that $R = \frac{1}{2}|x| \geq \frac{1}{2}(\frac{1}{2}|x| + r)$.

Argument for (2.3.11) and (2.3.12). It follows directly from the combination of Lemma 2, the large-scale $C^{0,1}$ estimate (2.4.5), the Poincaré inequality in $B_\rho(x)$ and the assumption (2.3.5): for $r \geq r_*(x)$

(i) if $\sqrt{-t} \leq r$, we use (2.4.5) up to the scale $\sqrt{-t}$ in form of

$$\begin{aligned}
 \int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy &\lesssim \int_t^{t+4r_*^2(x)} \int_{B_{2r_*(x)}(x)} |\nabla v_r(s, y)|^2 ds dy + \int_{B_{2r_*(x)}(x)} |f_r(y)|^2 dy \\
 &\stackrel{(2.4.5), (2.3.5)}{\lesssim} \int_t^0 \int_{B_{\sqrt{-t}}(x)} |\nabla v_r(s, y)|^2 ds dy \\
 &\quad + \sup_{r_*(x) \leq \rho \leq \sqrt{-t}} \left(\frac{\sqrt{-t}}{\rho} \right)^2 \int_{B_\rho(x)} \left| f_r(y)\bar{e} - \int_{B_\rho(x)} f_r(z)\bar{e} dz \right|^2 dy + r^{-2d} \\
 &\stackrel{(2.3.5)}{\lesssim} \int_t^0 \int_{B_{\sqrt{-t}}(x)} |\nabla v_r(s, y)|^2 ds dy + r^{-2d} \left(\frac{\sqrt{-t}}{r} \right)^2 + r^{-2d} \\
 &\lesssim \int_t^0 \int_{B_{\sqrt{-t}}(x)} |\nabla v_r(s, y)|^2 ds dy + r^{-2d},
 \end{aligned}$$

which gives (2.3.11).

(ii) If $\sqrt{-t} \geq r$, we use (2.4.5) up to the scale r and we obtain the same way (2.3.12).

Argument for (2.3.6) from (2.3.9), (2.3.10), (2.3.11) and (2.3.12). The case $|x| \geq 4(r_*(x) \vee r)$ is done via (2.3.10). It remains to treat the case $|x| \leq 4(r_*(x) \vee r)$ and we distinguish two sub-cases:

(1) Assume that $r_*(x) \geq r$, which means that $|x| \leq 4r_*(x)$. We have by the $\frac{1}{8}$ -Lipschitz continuity property of r_*

$$r_*(x) \gtrsim r_*(0), \tag{2.3.25}$$

and

$$r_*(0) \geq r_*(x) - \frac{1}{8}|x| \geq \frac{r}{2} + \frac{r_*(x)}{2} - \frac{1}{8}|x| \gtrsim r + |x|. \tag{2.3.26}$$

Thus, from (2.3.9) we deduce that for all $t \in \mathbb{R}_-$

$$\begin{aligned} \int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy &\lesssim r_*^{-d}(x) \int_{\mathbb{R}^d} |\nabla v_r(t, y)|^2 dy \\ &\stackrel{(2.3.25)}{\lesssim} r_*^{-d}(0) \int_{\mathbb{R}^d} |\nabla v_r(t, y)|^2 dy \\ &\stackrel{(2.3.9), (2.3.26)}{\lesssim} \frac{r_*^d(0)}{(|x| + r)^{2d}}. \end{aligned}$$

(2) Assume that $r_*(x) \leq r$ which means that $|x| \leq 4r$. For all $\sqrt{-t} \geq 2r_*(x)$, we have

(i) If $\sqrt{-t} \leq r$, we use (2.3.11) and the plain energy estimate (2.3.13) to obtain

$$\begin{aligned} \int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy &\stackrel{(2.3.11)}{\lesssim} \int_t^0 \int_{B_{\sqrt{-t}}(x)} |v_r(s, y)|^2 dy ds + r^{-2d} \\ &\stackrel{(2.3.13)}{\lesssim} r^{-2d} \int_{B_r} \eta_{\sqrt{-t}}(y - x) dy + r^{-2d} \\ &\leq r^{-2d} \lesssim \frac{r_*^d(0)}{(|x| + r)^d}. \end{aligned}$$

(ii) If $\sqrt{-t} \geq r$, we use (2.3.12) and the plain energy estimate (2.3.9) to get

$$\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \stackrel{(2.3.12)}{\lesssim} \int_t^{t+r^2} \int_{B_r(x)} |\nabla v_r(s, y)|^2 dy ds + r^{-2d} \stackrel{(2.3.9)}{\lesssim} r^{-2d} \lesssim \frac{r_*^d(0)}{(|x| + r)^d}.$$

This concludes the proof of (2.3.6).

Substep 1.2. We prove (2.2.39) without the support condition (2.3.5) on f_r . We decompose the r.h.s of (2.2.36) according to a family of dyadic annuli $(\mathcal{B}_k)_{k \in \mathbb{N}}$, defined by $\mathcal{B}_k := B_{2^{k+1}r} \setminus B_{2^k r}$ for all $k \geq 1$ and $\mathcal{B}_0 := B_r$. Namely, we set for all $k \in \mathbb{N}$, $f_{r,k} := f_r \chi_k$, where $(\chi_k)_{k \in \mathbb{N}}$ is a partition of unity according to the decomposition $(\mathcal{B}_k)_{k \in \mathbb{N}}$, and we denote by $v_{r,k}$ the weak solution of (2.2.36) with r.h.s $\nabla \cdot a f_{r,k} \bar{e}$. By uniqueness, we have $\nabla v_r = \sum_{k=0}^{+\infty} \nabla v_{r,k}$. Hence, we get by the triangle inequality

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \right)^{\frac{1}{2}} \leq \sum_{k=0}^{+\infty} \left(\int_{B_{r_*(x)}(x)} |\nabla v_{r,k}(t, y)|^2 dy \right)^{\frac{1}{2}}.$$

Thanks to (2.2.37), $f_{r,k}$ satisfies (2.3.5) with radius $2^k r$, thus, by (2.3.6), we have for all $k \geq 0$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_{r,k}(t, y)|^2 dy \right)^{\frac{1}{2}} \lesssim \frac{\left(\frac{r_*(0)}{2^k r} \vee 1 \right)^{\frac{d}{2}}}{(|x| + 2^k r)^d}.$$

We deduce, setting $N_r := \lceil \log_2 \left(1 + \frac{|x|}{r} \right) \rceil$

$$\begin{aligned} \left(\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \right)^{\frac{1}{2}} &\lesssim \sum_{k=0}^{+\infty} \frac{\left(\frac{r_*(0)}{2^k r} \vee 1 \right)^{\frac{d}{2}}}{(|x| + 2^k r)^d} \\ &\lesssim \left(\frac{r_*(0)}{r} \vee 1 \right)^{\frac{d}{2}} \left((|x| + r)^{-d} N_r + r^{-d} \sum_{k=N_r+1}^{+\infty} 2^{-kd} \right), \end{aligned}$$

which gives (2.2.39).

Step 2. Proof of (2.2.40). We use the same type of decomposition as in Substep 1.2 : we have $\nabla v_r = \sum_{k=0}^{+\infty} \nabla v_{r,k}$ where this time, for all $k \geq 1$, $\mathcal{B}_k := B_{2^{k+1}} \setminus B_{2^k}$ and $\mathcal{B}_0 := B_1$. We then split the proof into two steps.

Substep 2.1. We argue in favor of the first alternative in (2.2.40), that is when the r.h.s is equal to $\frac{r^{\frac{d}{2}}(0)r \log(1+|x|)}{(|x|+1)^d}$. From the assumption (2.2.38) used in form of $|f_r(x)| + (|x| + 1)|\nabla f_r(x)| \lesssim \frac{r}{(|x|+1)^d}$, we note that $\frac{1}{r} f_{r,k}$ satisfies (2.3.5) with radius 2^k . Thus by (2.3.6), we have for all $k \geq 0$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_{r,k}(t, y)|^2 dy \right)^{\frac{1}{2}} \lesssim \frac{r \left(\frac{r_*(0)}{2^k} \vee 1 \right)^{\frac{d}{2}}}{(|x| + 2^k)^d}.$$

We then conclude exactly as in Substep 1.2.

Substep 2.2. We argue in favor of the second alternative in (2.2.40), that is when the r.h.s is equal to $\frac{r^{\frac{d}{2}}(0)}{(|x|+1)^{d-1}}$. From the assumption (2.2.38) used in form of $|f_r(x)| + (|x| + 1)|\nabla f_r(x)| \lesssim \frac{1}{(|x|+1)^{d-1}}$, we note that $\frac{1}{2^k} f_{r,k}$ satisfies (2.3.5) with radius 2^k . Thus by (2.3.6), we have for all $k \geq 0$ and $\sqrt{-t} \geq 2r_*(x)$

$$\left(\int_{B_{r_*(x)}(x)} |\nabla v_{r,k}(t, y)|^2 dy \right)^{\frac{1}{2}} \lesssim \frac{2^k \left(\frac{r_*(0)}{2^k} \vee 1 \right)^{\frac{d}{2}}}{(|x| + 2^k)^d}.$$

We then conclude by the same decomposition as in Substep 1.2 : setting $N := \lceil \log_2(1 + |x|) \rceil$

$$\begin{aligned} \left(\int_{B_{r_*(x)}(x)} |\nabla v_r(t, y)|^2 dy \right)^{\frac{1}{2}} &\lesssim \sum_{k=0}^{+\infty} \frac{2^k \left(\frac{r_*(0)}{2^k} \vee 1 \right)^{\frac{d}{2}}}{(|x| + 2^k)^d} \\ &\lesssim r_*^{\frac{d}{2}}(0) \left((|x| + 1)^{-d} \sum_{k=0}^N 2^k + \sum_{k=N+1}^{+\infty} 2^{(1-d)k} \right), \end{aligned}$$

which concludes the proof since

$$\sum_{k=0}^N 2^k \lesssim |x| + 1,$$

and

$$\sum_{k=N+1}^{+\infty} 2^{(1-d)k} \lesssim \frac{1}{(|x|+1)^{d-1}}.$$

Proof of Lemma 5: Control of averages

We consider the two regimes: the non-generic case $R \leq r_*(x)$ and the generic case $R \geq r_*(x)$.

1. Let us start with the non-generic case $R \leq r_*(x)$. We distinguish two sub-cases.

(i) In the case where $\sqrt{\frac{T}{2}} \leq 2r_*(x)$, we have, using (2.2.42) and $R < \sqrt{T}$

$$\int_{B_R(x)} |\nabla u(T, y)|^2 dy \lesssim \left(\frac{\sqrt{T}}{R}\right)^d \int_{B_{\sqrt{T}}(x)} |\nabla u(T, y)|^2 dy \stackrel{(2.2.42)}{\lesssim} \left(\frac{r_*(x)}{R}\right)^d C(x, T) f(T) g(T).$$

(ii) In the case where $\sqrt{\frac{T}{2}} \geq 2r_*(x)$, we have by noticing that $\partial_\tau u - \nabla \cdot a \nabla u = 0$ on $(\frac{T}{2}, T) \times B_{\sqrt{\frac{T}{2}}}(x)$ and from Lemma 2 combined with the large-scale $C^{0,1}$ estimate (2.4.6)

$$\begin{aligned} \int_{B_R(x)} |\nabla u(T, y)|^2 dy &\lesssim \left(\frac{r_*(x)}{R}\right)^d \int_{B_{r_*(x)}(x)} |\nabla u(T, y)|^2 dy \\ &\lesssim \left(\frac{r_*(x)}{R}\right)^d \int_{C_{2r_*(x)}(T, x)} |\nabla u(s, y)|^2 ds dy \\ &\stackrel{(2.4.6)}{\lesssim} \left(\frac{r_*(x)}{R}\right)^d \int_{\frac{T}{2}}^T \int_{B_{\sqrt{\frac{T}{2}}}(x)} |\nabla u(s, y)|^2 ds dy \\ &\lesssim \left(\frac{r_*(x)}{R}\right)^d \int_{\frac{T}{2}}^T \int_{B_{\sqrt{s}}(x)} |\nabla u(s, y)|^2 ds dy \\ &\stackrel{(2.2.42)}{\lesssim} \left(\frac{r_*(x)}{R}\right)^d f(T) g\left(\frac{T}{2}\right) \int_{\frac{T}{2}}^T C(x, s) ds. \end{aligned}$$

2. Let us now consider the generic case $R \geq r_*(x)$ and without loss of generality we may assume that $\sqrt{T} > 2\sqrt{2}R$ since otherwise $f_{B_R} \lesssim f_{B_{\sqrt{T}}}$ and the conclusion follows from (2.2.42). We have $\partial_\tau u - \nabla \cdot a \nabla u = 0$ on $(0, T) \times B_{\sqrt{T}}$ and $r_*(x) \leq 2R < \sqrt{\frac{T}{2}}$ so that from Lemma 2 and the large-scale $C^{0,1}$ estimate (2.4.6) we deduce that

$$\begin{aligned} \int_{B_R(x)} |\nabla u(T, y)|^2 dy &\lesssim \int_{T-4R^2}^T \int_{B_{2R}} |\nabla u(s, y)|^2 dy ds \lesssim \int_{\frac{T}{2}}^T \int_{B_{\sqrt{T}}(x)} |\nabla u(s, y)|^2 dy ds \\ &\lesssim C(x, T) f(T) g\left(\frac{T}{2}\right). \end{aligned}$$

2.3.3 Proof of the suboptimal control of fluctuations of the time dependent flux

We provide in this section the proofs of Lemmas 6, 7, 8 and Proposition 3 of Section 2.2.4.

Proof of Lemma 6

We prove the lemma in two steps. In the first step we prove a deterministic bound on $q_r(T, x)$, using the energy estimates of Lemma 1 and the control of averages of ∇u deduced from Lemma 5. The deterministic bound will depend on a random variable built from an average of r_* . In the second step, we prove that the random constant has stretched exponential moments, using the moment bound (2.4.3) of r_* .

Step 1. Proof that for all $r \leq \min\{1, \sqrt{T}\}$ and $x \in \mathbb{R}^d$

$$|q_r(T, x)| \lesssim 1 + r^{-\frac{d}{2}} \log\left(\frac{\sqrt{T}}{r}\right) \left(1 + r_*^{\frac{d}{2}}(0) + \int_{\mathbb{R}^d \setminus B_1(x)} \eta_1(y) r_*^{\frac{d}{2}}(ry + x) dy\right). \quad (2.3.27)$$

Without loss of generality, we may assume that $x = 0$. We have from the definition (2.1.13) of $q_r(T)$ followed by the triangle inequality $\int_0^T = \int_0^{r^2} + \int_{r^2}^T$, the continuous embedding $L^2(\mathbb{R}^d, \eta_r dx) \hookrightarrow L^1(\mathbb{R}^d, \eta_r dx)$ and the domination of the Gaussian kernel by the exponential kernel

$$\begin{aligned} |q_r(T)| &\stackrel{(2.1.13)}{\lesssim} \int g_r(y) dy + \int g_r(y) \left| \int_0^T \nabla u(s, y) ds \right| dy \\ &\lesssim 1 + \int \eta_r(y) \left| \int_0^{r^2} \nabla u(s, y) ds \right| dy + \int \eta_r(y) \left| \int_{r^2}^T \nabla u(s, y) ds \right| dy \\ &\lesssim 1 + \left(\int \eta_r(y) \left| \int_0^{r^2} \nabla u(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} + \int \eta_r(y) \left| \int_{r^2}^T \nabla u(s, y) ds \right| dy. \end{aligned} \quad (2.3.28)$$

For the first r.h.s term of (2.3.28), we use the energy estimate (2.2.32) in form of

$$\left(\int \eta_r(y) \left| \int_0^{r^2} \nabla u(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \lesssim 1.$$

For the second r.h.s term of (2.3.28), we write with $f = \int f_{B_r(x)}$ and the Cauchy-Schwarz inequality

$$\int \eta_r(y) \left| \int_{r^2}^T \nabla u(s, y) ds \right| dy \lesssim \int \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} \left(\int_{B_r(x)} \left| \int_{r^2}^T \nabla u(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} dx. \quad (2.3.29)$$

Using the energy estimate (2.2.32) applied to the equation (2.1.10) in form of, for all $T > 0$

$$\left(\int_{B_{\sqrt{T}}(x)} |\nabla u(T, y)|^2 dy \right)^{\frac{1}{2}} \lesssim T^{-1},$$

we deduce from the Minkowski inequality in $L^2(B_r(x))$ and Lemma 5 applied with $R = 1$, $f \equiv 1$ and $g : T \in \mathbb{R}^+ \mapsto T^{-1}$, that, since $r \leq 1$

$$\left(\int_{B_r(x)} \left| \int_{r^2}^T \nabla u(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \lesssim r^{-\frac{d}{2}} \int_{r^2}^T \left(\int_{B_1(x)} |\nabla u(s, y)|^2 dy \right)^{\frac{1}{2}} ds \lesssim r^{-\frac{d}{2}} r_*^{\frac{d}{2}}(x) \log\left(\frac{\sqrt{T}}{r}\right).$$

Consequently (2.3.29) turns into

$$\int \eta_r(y) \left| \int_{r^2}^T \nabla u(s, y) ds \right| dy \lesssim r^{-\frac{d}{2}} \log\left(\frac{\sqrt{T}}{r}\right) \int r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx.$$

It remains to show that

$$\int r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx \lesssim 1 + r_*^{\frac{d}{2}}(0) + \int_{\mathbb{R}^d \setminus B_1} \eta_1(x) r_*^{\frac{d}{2}}(rx) dx. \quad (2.3.30)$$

For this, we decompose \mathbb{R}^d into B_r and $\mathbb{R}^d \setminus B_r$ to the effect of

$$\int r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx = \int_{B_r} r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx + \int_{\mathbb{R}^d \setminus B_r} r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx. \quad (2.3.31)$$

The first r.h.s term of (2.3.31) is controlled by, using the $\frac{1}{8}$ -Lipschitz regularity of r_* in form of $\sup_{x \in B_r} r_*(x) \lesssim r_*(0) + r \lesssim r_*(0) + 1$ and the Cauchy-Schwarz inequality

$$\int_{B_r} r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx \lesssim (r_*^{\frac{d}{2}}(0) + 1) r^{\frac{d}{2}} \left(\int \eta_r^2(y) dy \right)^{\frac{1}{2}} \lesssim r_*^{\frac{d}{2}}(0) + 1.$$

For the second r.h.s term of (2.3.31), we note that for all $x \in \mathbb{R}^d \setminus B_r$ and $y \in B_r(x)$ we have $\eta_r(y) \lesssim \eta_r(x)$, so that

$$\int_{\mathbb{R}^d \setminus B_r} r_*^{\frac{d}{2}}(x) \left(\int_{B_r(x)} \eta_r^2(y) dy \right)^{\frac{1}{2}} dx \lesssim \int_{\mathbb{R}^d \setminus B_r} r_*^{\frac{d}{2}}(x) \eta_r(x) dx = \int_{\mathbb{R}^d \setminus B_1} r_*^{\frac{d}{2}}(rx) \eta_1(x) dx.$$

This concludes the proof of (2.3.30) and the argument for (2.3.27).

Step 2. We prove (2.2.43). It remains to show that $\mathcal{C}_*(r, x) := 1 + r_*^{\frac{d}{2}}(0) + \int_{\mathbb{R}^d \setminus B_1(x)} \eta_1(y) r_*^{\frac{d}{2}}(ry + x) dy$ has the desired stretched exponential moment. From the moment bound (2.4.3) of r_* and the equivalence between algebraic moments and exponential moments (see Lemma 9), we have for all $p \in [1, +\infty)$ and all $y \in \mathbb{R}^d$

$$\left\langle r_*^{dp}(y) \right\rangle^{\frac{1}{p}} \lesssim p^{\frac{d}{\beta \wedge d}}. \quad (2.3.32)$$

Therefore for all $p \in [1, +\infty)$ and $x \in \mathbb{R}^d$

$$\begin{aligned} \left\langle \left(1 + r_*^{\frac{d}{2}}(0) + \int_{\mathbb{R}^d \setminus B_1(x)} \eta_1(y) r_*^{\frac{d}{2}}(ry + x) dy \right)^p \right\rangle^{\frac{1}{p}} &\lesssim 1 + \left\langle r_*^{\frac{dp}{2}}(0) \right\rangle^{\frac{1}{p}} + \int_{\mathbb{R}^d \setminus B_1(x)} \eta_1(y) \left\langle r_*^{\frac{dp}{2}}(ry + x) \right\rangle^{\frac{1}{p}} dy \\ &\stackrel{(2.3.32)}{\lesssim} 1 + p^{\frac{1}{2} \frac{d}{\beta \wedge d}} \left(1 + \int \eta_1(y) dy \right), \end{aligned}$$

which gives (2.2.43) by applying once again Lemma 9.

Proof of Lemmas 7 and 8: Control of the functional derivatives

We prove Lemma 7 and Lemma 8 independently.

Proof of lemma 7. It is enough to prove (2.2.44) for the quantities $q_r(T) \cdot e_k$, for all $r > 0$, $T \geq 1$ and $k \in \llbracket 1, d \rrbracket$. In particular, we only treat the case $k = 1$, since the other contributions are controlled the same way. For notational convenience, we write v^T for v_1^T .

Let $x \in \mathbb{R}^d$, $h \in (0, 1)$, $T \geq 1$, $r > 0$, $\ell \in [1, +\infty)$ and δa be compactly supported in $B_\ell(x)$ such that $\sup_{y \in B_\ell(x)} |\delta a(y)| \leq 1$. We compute the finite difference

$$\begin{aligned} \delta^h q_r(T) \cdot e_1 &:= \frac{q_r(a + h\delta a, T) \cdot e_1 - q_r(a, T) \cdot e_1}{h} \\ &= \int g_r(y) e_1 \cdot \delta a(y) e \, dy + \int g_r(y) e_1 \cdot \delta a(y) \left(\int_0^T \nabla u(a + h\delta a, t, y) dt \right) dy \\ &\quad + \int_0^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt, \end{aligned} \quad (2.3.33)$$

where $\delta^h u(t, \cdot) := \frac{u(a+h\delta a, t, \cdot) - u(a, t, \cdot)}{h}$ is the weak solution of

$$\begin{cases} \partial_t \delta^h u - \nabla \cdot a \nabla \delta^h u = \nabla \cdot \delta a \nabla u(a + h\delta a, \cdot, \cdot) & \text{on } (0, +\infty) \times \mathbb{R}^d, \\ \delta^h u(0, \cdot) = \nabla \cdot \delta a(\cdot) e. \end{cases} \quad (2.3.34)$$

The first r.h.s term of (2.3.33) give directly the first r.h.s terms of (2.2.44). For the second r.h.s term of (2.3.33), we easily derive from the localized energy estimates (2.2.33) and (2.2.34) applied to the equation (2.3.34) combined with (2.2.32) with a replaced by $a + h\delta a$ (which control the norm of the r.h.s of (2.3.34)) that

$$\int g_r(y) e_1 \cdot \delta a(y) \left(\int_0^T \nabla u(a + h\delta a, t, y) dt \right) dy \xrightarrow{h \downarrow 0} \int g_r(y) e_1 \cdot \delta a(y) \left(\int_0^T \nabla u(t, y) dt \right) dy.$$

We thus obtain the second second r.h.s of (2.2.44). We now focus on the third r.h.s term of (2.3.33). We split the time integral into the singular part $t \leq 1$ and the regular part $t \geq 1$

$$\begin{aligned} \int_0^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt &= \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \\ &\quad + \int_1^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt. \end{aligned} \quad (2.3.35)$$

We now split the rest of the proof into two steps, and treat the two r.h.s terms of (2.3.35) separately.

Step 1. First r.h.s term of (2.3.35). We prove that

$$\begin{aligned} \limsup_{h \rightarrow 0} \left| \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| &\lesssim \mathbf{1}_{\ell < \sqrt{T}} \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |g_r(y)|^2 dy dz \right)^{\frac{1}{2}} \\ &\quad + \mathbf{1}_{\ell \geq \sqrt{T}} \mathcal{T}_{x, \ell}(\eta_r)(0), \end{aligned} \quad (2.3.36)$$

where $\mathcal{T}_{x, \ell}(\eta_r)$ is defined in (2.2.48).

Regime $\ell < \sqrt{T}$. Using that $\int = \int \int_{B_\ell(z)}$ and by splitting \mathbb{R}^d into $B_\ell(x)$ and $\mathbb{R}^d \setminus B_\ell(x)$ we have

$$\begin{aligned} \left| \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| &\lesssim \int \int_{B_\ell(z)} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| |g_r(y)| dy dz \\ &= \int_{B_\ell(x)} \int_{B_\ell(z)} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| |g_r(y)| dy dz \quad (=: I_1^\ell) \\ &\quad + \int_{\mathbb{R}^d \setminus B_\ell(x)} \int_{B_\ell(z)} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| |g_r(y)| dy dz. \quad (=: I_2^\ell) \end{aligned} \quad (2.3.37)$$

We will prove that

$$I_1^\ell + I_2^\ell \lesssim \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |g_r(y)|^2 dy dz \right)^{\frac{1}{2}}. \quad (2.3.38)$$

Since the arguments are similar we only give the details for I_2^ℓ . By the Cauchy-Schwarz inequality, we have

$$I_2^\ell \leq \left(\int_{\mathbb{R}^d \setminus B_\ell(x)} e^{\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} \left| \int_0^1 \nabla \delta^h u(s, y) ds \right|^2 dy dz \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d \setminus B_\ell(x)} e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |g_r(y)|^2 dy dz \right)^{\frac{1}{2}}. \quad (2.3.39)$$

It remains to estimate the first r.h.s factor of (2.3.39). First, by the localized energy estimate (2.2.34) applied to the equation (2.3.34) and $\left(\int_{B_\ell(x)} |\delta a(y) e|^2 dy \right)^{\frac{1}{2}} \lesssim \ell^{\frac{d}{2}}$, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^d \setminus B_\ell(x)} e^{\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} \left| \int_0^1 \nabla \delta^h u(s, y) ds \right|^2 dy dz \right)^{\frac{1}{2}} &\lesssim \ell^{\frac{d}{2}} + \left(\int_{B_\ell(x)} \left| \int_0^1 \delta a(y) \nabla u(a + h\delta a, s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \\ &\quad + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell(x)} |\delta a(y) \nabla u(a + h\delta a, s, y)|^2 dy \right)^{\frac{1}{2}} ds dt. \end{aligned} \quad (2.3.40)$$

Second, by the localized energy estimate (2.2.32) applied to the equation (2.1.10) with a replaced by $a + h\delta a$ and for $R = \ell \geq 1$, we obtain

$$\left(\int \eta_\ell(y-x) \left| \int_0^1 \nabla u(a + h\delta a, t, y) dt \right|^2 dy \right)^{\frac{1}{2}} \lesssim \ell^{\frac{d}{2}}, \quad (2.3.41)$$

and

$$\int_0^1 \frac{1}{1-t} \int_t^1 \left(\int \eta_\ell(y-x) |\nabla u(a + h\delta a, s, y)|^2 dy \right)^{\frac{1}{2}} ds dt \lesssim \ell^{\frac{d}{2}} \int_0^1 \frac{1}{1-t} \int_t^1 s^{-1} ds dt = \ell^{\frac{d}{2}} \int_0^1 \frac{-\log(t)}{1-t} dt \lesssim \ell^{\frac{d}{2}}. \quad (2.3.42)$$

Finally the combination of (2.3.39), (2.3.40), (2.3.41) and (2.3.42) yields (2.3.38). It then follows from (2.3.37) that

$$\left| \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| \lesssim \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |g_r(y)|^2 dy dz \right)^{\frac{1}{2}}. \quad (2.3.43)$$

Regime $\ell \geq \sqrt{T}$. Using the Cauchy-Schwarz inequality and by dominating the Gaussian kernel by the exponential kernel, we have

$$\left| \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| \leq \left(\int \eta_r(y) \left| \int_0^1 \nabla \delta^h u(t, y) dt \right|^2 dy \right)^{\frac{1}{2}}, \quad (2.3.44)$$

with, by applying the localized energy estimates (2.2.32) and (2.2.33) to (2.3.34) for $R = r$

$$\begin{aligned} \left(\int \eta_r(y) \left| \int_0^1 \nabla \delta^h u(t, y) dt \right|^2 dy \right)^{\frac{1}{2}} &\lesssim \left(\int \eta_r(y) |\delta a(y)|^2 dy \right)^{\frac{1}{2}} + \left(\int \eta_r(y) |\delta a(y)|^2 \left| \int_0^1 \nabla u(a + h\delta a, t, y) dt \right|^2 dy \right)^{\frac{1}{2}} \\ &\quad + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int \eta_r(y) |\delta a(y)|^2 |\nabla u(a + h\delta a, s, y)|^2 dy \right)^{\frac{1}{2}} ds dt. \end{aligned} \quad (2.3.45)$$

Now, since δa is supported in $B_\ell(x)$, the localized energy estimates (2.2.33) and (2.2.34) applied to the equation (2.3.34) with $R = \ell$ combined with (2.3.41) and (2.3.42) yield

$$\int_{B_\ell(x)} \left| \int_0^1 \nabla u(a+h\delta a, t, y) - \nabla u(a, t, y) dt \right|^2 dy + t^2 \left(\frac{\sqrt{t}}{\ell} \vee 1 \right)^{-d} \int_{B_\ell(x)} |\nabla u(a+h\delta a, t, y) - \nabla u(a, t, y)|^2 dy \lesssim h^2 \ell^d, \quad (2.3.46)$$

which allow us to pass to the limit when $h \downarrow 0$ in (2.3.44) and (2.3.45), and obtain

$$\limsup_{h \rightarrow 0} \left| \int g_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| \lesssim \mathcal{T}_{x, \ell}(\eta_r)(0).$$

It concludes the argument for (2.3.36).

Step 2. Second r.h.s term of (2.3.35). We prove that

$$\begin{aligned} & \limsup_{h \rightarrow 0} \left| \int_1^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt \right| \\ & \lesssim \int_{B_\ell(x)} \int_1^T |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy + \int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \\ & \quad + \mathcal{G}_{r, \ell}(x) \mathbf{1}_{\ell \geq \sqrt{T}} + \mathbf{1}_{\ell < \sqrt{T}} \ell^{\frac{d}{2}} \left(\int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} |\nabla v^T(1, y)|^2 dy dz \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.47)$$

Recall that v^T denotes the weak solution of the dual system associated with (2.3.34), which reads

$$\begin{cases} \partial_\tau v^T + \nabla \cdot a^* \nabla v^T = \nabla \cdot a g_r e_1 & \text{on } (-\infty, T) \times \mathbb{R}^d, \\ v^T(T) = 0. \end{cases} \quad (2.3.48)$$

We reformulate the l.h.s of (2.3.47) with help of the dual system (2.3.48). We have by testing the equation (2.3.48) with $\delta^h u$ and integrating in time

$$\int_1^T \int \delta^h u(t, y) \partial_t v^T(t, y) dy dt - \int_1^T \int \nabla \delta^h u(t, y) \cdot a^*(y) \nabla v^T(t, y) dy dt = - \int_1^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt, \quad (2.3.49)$$

and also, by testing (2.3.34) with v^T

$$\begin{aligned} & \int_1^T \int v^T(t, y) \partial_t \delta^h u(t, y) dy dt + \int_1^T \int \nabla v^T(t, y) \cdot a(y) \nabla \delta^h u(t, y) dy dt \\ & = - \int_1^T \int \nabla v^T(t, y) \cdot \delta a(y) \nabla u(a + h\delta a, t, y) dy dt. \end{aligned} \quad (2.3.50)$$

Consequently, by summing (2.3.49) and (2.3.50), using an integration by part in time and the fact that $v^T(T, \cdot) \equiv 0$, we get

$$\int_1^T \int g_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt = \int \delta^h u(1, y) v^T(1, y) dy - \int_1^T \int \nabla v^T(t, y) \cdot \delta a(y) \nabla u(a + h\delta a, t, y) dy dt. \quad (2.3.51)$$

Moreover, from (2.3.46) we can pass to the limit when $h \downarrow 0$ in the second r.h.s term of (2.3.51), namely

$$\lim_{h \rightarrow 0} \int_1^T \int \nabla v^T(t, y) \cdot \delta a(y) \nabla u(a + h\delta a, t, y) dy dt = \int_1^T \int \nabla v^T(t, y) \cdot \delta a(y) \nabla u(t, y) dy dt,$$

and obtain the first r.h.s term of (2.3.47). It remains to control the first r.h.s term of (2.3.51). To this aim, we integrate in time the equation (2.3.34) between 0 and 1:

$$\delta^h u(1, \cdot) - \nabla \cdot a \int_0^1 \nabla \delta^h u(t, \cdot) dt = \nabla \cdot \delta a \int_0^1 \nabla u(a + h\delta, t, \cdot) dt + \nabla \cdot \delta a e,$$

which provides by testing with $v^T(1, \cdot)$

$$\begin{aligned} \int \delta^h u(1, y) v^T(1, y) dy &= \int \nabla v^T(1, y) \cdot \delta a(y) \left(\int_0^1 \nabla u(a + h\delta a, t, y) dt \right) dy \\ &\quad - \int \nabla v^T(1, y) \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy - \int \nabla v^T(1, y) \cdot \delta a(y) e dy. \end{aligned} \quad (2.3.52)$$

The first r.h.s term of (2.3.52) combined with the third one and (2.3.46) give the second r.h.s term of (2.3.47). The second r.h.s term of (2.3.52) is then dominated in two ways.

Regime $\ell \geq \sqrt{T}$. For the generic case $\ell \geq r_*(0)$, we use the identity $f = \int f_{B_\ell(y)}$ and we split the integral into the two contributions $\int_{\mathbb{R}^d \setminus B_{4\ell}}$ and $\int_{B_{4\ell}}$ in form of

$$\begin{aligned} \left| \int \nabla v^T(1, y) \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| &\leq \int \int_{B_\ell(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy \\ &= \left(\int_{\mathbb{R}^d \setminus B_{4\ell}} + \int_{B_{4\ell}} \right) \int_{B_\ell(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy. \end{aligned}$$

For the far-field contribution $|x| \geq 4\ell$, we make use of the Cauchy-Schwarz inequality and the computations did in (2.3.45) as well as (2.3.46) to get

$$\limsup_{h \rightarrow 0} \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy \lesssim \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_\ell)(y) dy.$$

For the near-field contribution $|x| < 4\ell$, we first note that from the assumption $\ell > r_*(0)$ we have for all $y \in \mathbb{R}^d$

$$B_{r_*(y)}(y) \cap B_{5\ell} \neq \emptyset \Rightarrow y \in B_{7\ell}. \quad (2.3.53)$$

Indeed, from the $\frac{1}{8}$ -Lipschitz regularity property of r_* , if there exists $z \in B_{r_*(y)}(y) \cap B_{5\ell}$ then $|y| \leq |y - z| + |z| \leq r_*(y) + 5\ell \leq r_*(0) + \frac{|y|}{8} + 5\ell \leq 6\ell + \frac{|y|}{8}$ and thus $|y| \leq \frac{16}{3}\ell \leq 7\ell$. Therefore, using the property (2.4.20) combined with the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_{B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy &\leq \int_{B_{5\ell}} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz \\ &\stackrel{(2.4.20)}{\lesssim} \int \int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| \mathbb{1}_{B_{5\ell}}(z) dz dy \\ &\stackrel{(2.3.53)}{\leq} \int_{B_{7\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{B_{r_*(y)}(y)} \left| \int_0^1 \nabla \delta^h u(t, z) dt \right|^2 dz \right)^{\frac{1}{2}} dy, \end{aligned}$$

and we finally end up with, using (2.3.45) as well as (2.3.46)

$$\limsup_{h \rightarrow 0} \int_{B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1, z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy \lesssim \int_{B_{r\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_{r_*(y)})(y) dy.$$

For the non-generic regime $\ell < r_*(0)$, we use the estimate (2.3.43) which clearly holds by replacing g_r with $v^T(1, \cdot)$ and we bound $\ell^{\frac{d}{2}}$ by $r_*^{\frac{d}{2}}(0)$.

Regime $\ell < \sqrt{T}$. We use the estimate (2.3.43) which clearly holds by replacing g_r with $v^T(1, \cdot)$. \square

Proof of Lemma 8. We keep the notations of the previous proof and we give only the argument for $q(r^2) \star f_r \cdot e_1$. First, equality (2.3.33) holds and take the form

$$\begin{aligned} \delta^h q(r^2) \star f_r \cdot e_1 &= \int f_r(y) e_1 \cdot \delta a(y) dy + \int f_r(y) e_1 \cdot \delta a(y) \left(\int_0^{r^2} \nabla u(t, y) dt \right) dy \\ &\quad + \int_0^{r^2} \int f_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt. \end{aligned} \quad (2.3.54)$$

The two first r.h.s terms of (2.3.54) give directly the two first r.h.s terms of (2.2.50), respectively. As in (2.3.35), we make the decomposition

$$\begin{aligned} \int_0^{r^2} \int f_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt &= \int f_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \\ &\quad + \int_1^{r^2} \int f_r(y) e_1 \cdot a(y) \nabla \delta^h u(t, y) dy dt. \end{aligned} \quad (2.3.55)$$

Control of the second r.h.s term of (2.3.55). This term is controlled the same way as we did in Step 2 of the proof of Lemma 7 and provides the third, the fifth and the sixth r.h.s term of (2.2.50).

Control of the first r.h.s term of (2.3.55). It remains to argue that the first r.h.s term of (2.3.55) is dominated by the fourth r.h.s term of (2.2.50). We distinguish the two regimes $\ell \geq r$ and $\ell \leq r$.

Regime $\ell \geq r$. We use the assumption (2.2.49) in form of, for all $y \in \mathbb{R}^d$, $|f_r(y)| \lesssim \int_1^{r^2} s^{-\frac{1}{2}} g_{\sqrt{s}}(y) ds$ and (2.3.45) applied with $r = \sqrt{s}$ to get

$$\begin{aligned} \limsup_{h \rightarrow 0} \left| \int f_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| &\lesssim \limsup_{h \rightarrow 0} \int_1^{r^2} s^{-\frac{1}{2}} \int g_{\sqrt{s}}(y) \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| dy \\ &\lesssim \int_1^{r^2} s^{-\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_{\sqrt{s}})(0) ds. \end{aligned}$$

Regime $\ell < r$. We first note that from the assumption (2.2.49) we have, for all $y \in \mathbb{R}^d$

$$|f_r(y)| \lesssim |y| \int_1^{r^2} s^{-1-\frac{d}{2}} \left(1 + \frac{|y|^2}{s}\right)^{-\frac{d}{2}-\frac{3}{2}} \wedge \left(1 + \frac{|y|^2}{s}\right)^{-\frac{d}{2}-1} ds \lesssim \frac{r}{(|y|+1)^d} \wedge \frac{1}{(|y|+1)^{d-1}}, \quad (2.3.56)$$

We first make use of the identity $f = \int f_{B_\ell(y)}$ to get

$$\left| \int f_r(y) e_1 \cdot a(y) \left(\int_0^1 \nabla \delta^h u(t, y) dt \right) dy \right| \lesssim \int \int_{B_\ell(y)} |f_r(z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy.$$

We then split the integral into the far-field contribution $|x| \geq 2\ell$ and the near-field contribution $|x| < 2\ell$. For the near-field contribution, we make use of a dyadic decomposition and (2.3.56) in form of $|f_r(y)| \lesssim (|y| + 1)^{-d+1}$ to get

$$\begin{aligned}
\int_{B_{2\ell}} \int_{B_\ell(y)} |f_r(z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy &\lesssim \int_{B_{3\ell}} |f_r(y)| \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| dy \\
&\lesssim \int_{B_1} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| dy \\
&\quad + \sum_{n=0}^{\lceil \log_2(3\ell) \rceil} \int_{B_{2^{n+1}} \setminus B_{2^n}} |f_r(y)| \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| dy \\
&\stackrel{(2.3.56)}{\lesssim} \int_{B_1} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right| dy \\
&\quad + \sum_{n=0}^{\lceil \log_2(3\ell) \rceil} 2^n \left(\int_{B_{2^{n+1}}} \left| \int_0^1 \nabla \delta^h u(t, y) dt \right|^2 dy \right)^{\frac{1}{2}}
\end{aligned}$$

which gives the first term in the second r.h.s term of (2.2.52) by passing to the limsup and using (2.3.45) applied both with $R = 1$ and $R = 2^{n+1}$ as well as (2.3.46). For the far-field contribution, we make use of the Cauchy-Schwarz inequality in form of

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus B_{2\ell}} \int_{B_\ell(y)} |f_r(z)| \left| \int_0^1 \nabla \delta^h u(t, z) dt \right| dz dy &\lesssim \int_{\mathbb{R}^d \setminus B_{2\ell}} \left(\int_{B_\ell(y)} |f_r(z)|^2 dz \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{B_\ell(y)} \left| \int_0^1 \nabla \delta^h u(t, z) dt \right|^2 dz \right)^{\frac{1}{2}} dy,
\end{aligned}$$

and we get the second term in the second r.h.s term of (2.2.52) by passing to the limsup and using (2.3.45) applied with $R = \ell$ as well as (2.3.46). \square

Proof of Proposition 3: Suboptimal fluctuation estimates.

We split the proof into three steps. In the first two steps we control $\int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx$, using the bound of $\partial_{x,\ell}^{\text{fct}} q_r(T)$ proved in Lemma 7. More precisely, in the first step we treat the regime $\ell < \sqrt{T}$ and in the second step the regime $\ell \geq \sqrt{T}$. In the last step we deduce the desired moment bound (2.2.53) from the multiscale logarithmic Sobolev inequality, in form of (2.4.1), and the moment bound (2.4.3) on r_* . We start with preliminary estimates.

Step 0. Preliminary. First, we will use several times the following sub-optimal deterministic decay in time of averages of ∇u : for all $t \in (0, T]$

$$\int_{B_\ell(x)} |\nabla u(t, y)|^2 dy \lesssim \ell^d \left(\left(\frac{r_*(x)}{\ell} \vee 1 \right)^d \mathbf{1}_{\ell < \sqrt{t}} + \mathbf{1}_{\ell \geq \sqrt{t}} \right) t^{-2}. \quad (2.3.57)$$

This estimate is a direct consequence of the combination of the localized energy estimate (2.2.32) applied to (2.1.10) with $R = \sqrt{t}$ and Lemma 5 applied with $f \equiv 1$, $g : t \in \mathbb{R}^+ \mapsto t^{-2}$ and $R = \ell$.

Second, we will use several times the following large-scale regularity result: for all $t \in (-\infty, T]$ and $x \in \mathbb{R}^d$

$$\int_{B_{r_*(x)}(x)} |\nabla v^T(t, y)|^2 dy \lesssim r_*^d(0) \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} + r^{-d} \eta_{4r_*(x)}(x) + g_{2r}^2(x), \quad (2.3.58)$$

where we recall that v^T is defined in (2.2.45). This estimate is a consequence of the localized energy estimate (2.2.32) and the large-scale regularity estimate (2.2.39). Indeed, we first notice that $v^T = (\tilde{v}_k(\cdot - T, \cdot))_{k \in \llbracket 1, d \rrbracket}$ with \tilde{v}_k is the weak solution of (2.2.36) for $\bar{e} = e_k$ and $f_r = g_r$. In addition, from the identity (2.3.7) we get for all $(t, x) \in (-\infty, T] \times \mathbb{R}^d$

$$\nabla v^T(t, x) = \int_0^{T-t} \nabla w(s, x) ds, \quad (2.3.59)$$

with $w = (w_k)_{k \in \llbracket 1, d \rrbracket}$ and w_k solves (2.3.8) with $f_r = g_r$ and $\bar{e} = e_k$. Therefore, in the regime $\sqrt{T-t} \geq 2r_*(x)$, we use (2.2.39) (where we bound $(\frac{r_*(0)}{r} \vee 1)^d \leq 2r_*(0)$ since $r \geq 1$) in form of

$$\int_{B_{r_*(x)}(x)} |\nabla v^T(t, y)|^2 dy = \sum_{k=1}^d \int_{B_{r_*(x)}(x)} |\nabla \tilde{v}_k(t - T, y)|^2 dy \lesssim r_*^d(0) \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^d}.$$

In the regime $2r_*(x) \geq \sqrt{T-t}$, we use the formula (2.3.59) combined with the localized energy estimate (2.2.32) applied with $R = 2r_*(x)$ in form of

$$\int_{B_{r_*(x)}(x)} |\nabla v^T(t, y)|^2 dy \lesssim \int_{B_{2r_*(x)}(x)} \left| \int_0^{T-t} \nabla w(s, y) ds \right|^2 dy \lesssim \int \eta_{2r_*(x)}(x - y) g_r^2(y) dy.$$

Finally the r.h.s is dominated via

$$\int \eta_{2r_*(x)}(x - y) g_r^2(y) dy = \int_{B_{\frac{|x|}{2}}(x)} \eta_{2r_*(x)}(x - y) g_r^2(y) dy + \int_{\mathbb{R}^d \setminus B_{\frac{|x|}{2}}(x)} \eta_{2r_*(x)}(x - y) g_r^2(y) dy,$$

with

$$\int_{B_{\frac{|x|}{2}}(x)} \eta_{2r_*(x)}(x - y) g_r^2(y) dy \lesssim g_{2r}^2(x) \int \eta_{2r_*(x)}(y) dy \lesssim g_{2r}^2(x),$$

and

$$\int_{\mathbb{R}^d \setminus B_{\frac{|x|}{2}}(x)} \eta_{2r_*(x)}(x - y) g_r^2(y) dy \lesssim r^{-d} \eta_{4r_*(x)}(x) \int g_{\frac{1}{\sqrt{2}}r}(y) dy \lesssim r^{-d} \eta_{4r_*(x)}(x).$$

We now turn to the proof of (2.2.53).

Step 1. Regime $\ell < \sqrt{T}$. Proof that for all $\ell \in [1, \sqrt{T})$

$$\int |\partial_{x, \ell}^{\text{fct}} q_r(T)|^2 dx \lesssim \ell^{2d} r^{-d} (1 + \log^2(T) + \log^2(\frac{\sqrt{T}}{r})) \mathcal{C}_*(r, \ell), \quad (2.3.60)$$

with

$$\mathcal{C}_*(r, \ell) := \left(r_*^d(0) \int r_*^d(rx) \frac{\log^2(|x| + 1)}{(|x| + 1)^{2d}} dx + \int \frac{r_*^{2d}(x)}{(|x| + 1)^{2d}} dx + \int r_*^d(rx) g_{2r}^2(x) dx \right)^{\frac{1}{2}}. \quad (2.3.61)$$

We split this step into two parts. The first part is devoted to the control of the first fourth r.h.s terms of (2.2.44) and the second part is devoted to the control of the fifth term.

Substep 1.1. Proof that for all $\ell \in [1, \sqrt{T})$

$$\left(\int \left(\int_{\mathbb{B}_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right| g_r(y) dy \right)^2 dx \right)^{\frac{1}{2}} \lesssim \ell^d r^{-\frac{d}{2}} \left(1 + \log\left(\frac{\sqrt{T}}{r}\right) \left(\int r_*^d(rx) g_1^2(x) dx \right)^{\frac{1}{2}} \right), \quad (2.3.62)$$

and

$$\begin{aligned} \ell^{-d} \int \left(\int_{\mathbb{B}_\ell(x)} g_r(y) dy \right)^2 + \ell^{-d} \int \left(\int_{\mathbb{B}_\ell(x)} |\nabla v^T(1, y)| \left(1 + \int_0^1 \nabla u(s, y) ds \right) dy \right)^2 dx \\ + \int \int e^{-\frac{|x-z|}{2c\ell}} \int_{\mathbb{B}_\ell(z)} |g_r(y)|^2 + |\nabla v^T(1, y)|^2 dy dz dx \lesssim \ell^d r^{-d}. \end{aligned} \quad (2.3.63)$$

Argument for (2.3.62). Since $r \leq \sqrt{T}$, we split \int_0^T into $\int_0^{r^2} + \int_{r^2}^T$ and we apply the triangle inequality followed by the Jensen inequality, the identity $\int \int_{\mathbb{B}_\ell(x)} = \int$ as well as the Cauchy-Schwarz inequality to the effect of

$$\begin{aligned} \left(\int \left(\int_{\mathbb{B}_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right| g_r(y) dy \right)^2 dx \right)^{\frac{1}{2}} \lesssim \ell^d \left(\int \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 g_r^2(y) dy \right)^{\frac{1}{2}} \\ + \left(\int \int_{\mathbb{B}_\ell(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy \int_{\mathbb{B}_\ell(x)} g_r^2(y) dy dx \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.64)$$

For the first r.h.s term of (2.3.64), we dominate the Gaussian kernel by the exponential kernel and we use the localized energy estimate (2.2.32) applied to (2.1.10) in form of

$$\left(\int \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 g_r^2(y) dy \right)^{\frac{1}{2}} \lesssim r^{-\frac{d}{2}} \left(\int \eta_r(y) \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 dy \right)^{\frac{1}{2}} \stackrel{(2.2.32)}{\lesssim} r^{-\frac{d}{2}}. \quad (2.3.65)$$

For the second r.h.s term of (2.3.64), since $\ell < \sqrt{T}$, we have by the Minkowski inequality in $L^2(\mathbb{B}_\ell(x))$ and the decay estimate (2.3.57)

$$\int_{\mathbb{B}_\ell(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy \leq \left(\int_{r^2}^T \left(\int_{\mathbb{B}_\ell(x)} |\nabla u(t, y)|^2 dy \right)^{\frac{1}{2}} dt \right)^2 \stackrel{(2.3.57)}{\lesssim} (r_*(x) \vee \ell)^d \log^2\left(\frac{\sqrt{T}}{r}\right),$$

so that we finally get, using in the last line the Lipschitz property of r_* in form of $\sup_{y \in \mathbb{B}_\ell(x)} \left(\frac{r_*(y)}{\ell} \vee 1 \right)^d \lesssim \left(\frac{r_*(x)}{\ell} \vee 1 \right)^d$ combined with the identity $\int \int_{\mathbb{B}_\ell(x)} dx = \int$ and change of variables $x \mapsto \frac{x}{r}$

$$\begin{aligned} \left(\int \int_{\mathbb{B}_\ell(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy \int_{\mathbb{B}_\ell(x)} g_r^2(y) dy dx \right)^{\frac{1}{2}} \lesssim \log\left(\frac{\sqrt{T}}{r}\right) \left(\int (r_*(x) \vee \ell)^d \int_{\mathbb{B}_\ell(x)} g_r^2(y) dy dx \right)^{\frac{1}{2}} \\ \lesssim \ell^d r^{-\frac{d}{2}} \log\left(\frac{\sqrt{T}}{r}\right) \left(\int \left(\frac{r_*(rx)}{\ell} \vee 1 \right)^d g_1^2(x) dx \right)^{\frac{1}{2}}, \end{aligned}$$

Estimate (2.3.62) then follows in combination with the previous estimate, (2.3.64) and (2.3.65) as well as the estimate $\frac{r_*(rx)}{\ell} \vee 1 \lesssim r_*(rx)$.

Argument for (2.3.63). On the one hand, we have from the Jensen inequality, the identity $\int \mathbb{1}_{B_\ell(x)} = \int$ and the Cauchy-Schwarz inequality as well as $\int g_r^2(y)dy \lesssim r^{-d}$

$$\begin{aligned} & \int \left(\int_{B_\ell(x)} g_r(y)dy \right)^2 + \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \int_0^1 \nabla u(s, y)ds \right) dy \right)^2 dx \\ & \lesssim \ell^{2d} r^{-d} + \int \int_{B_\ell(x)} |\nabla v^T(1, y)|^2 dy \int_{B_\ell(x)} 1 + \left| \int_0^1 \nabla u(s, y)ds \right|^2 dx. \end{aligned} \quad (2.3.66)$$

Then, using the localized energy estimate (2.2.32) applied to (2.1.10) with $R = \ell$ and the plain energy estimate $\int |\nabla v^T(1, y)|^2 dy \lesssim r^{-d}$ (the proof is identical as for (2.3.9)) we get

$$\int \int_{B_\ell(x)} |\nabla v^T(1, y)|^2 dy \int_{B_\ell(x)} 1 + \left| \int_0^1 \nabla u(s, y)ds \right|^2 dx \stackrel{(2.2.32)}{\lesssim} \ell^d \int \int_{B_\ell(x)} |\nabla v^T(1, y)|^2 dy dx \lesssim \ell^{2d} r^{-d}. \quad (2.3.67)$$

On the other hand, by noticing that from the Fubini-Tonelli theorem, we have for all measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$

$$\int \int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} f(y)dy dz dx = \int f(y) \int \ell^{-d} \mathbb{1}_{B_\ell(y)}(z) \int e^{-\frac{|x-z|}{2c\ell}} dx dz dy \lesssim \ell^d \int f(y)dy,$$

we get

$$\int \int e^{-\frac{|x-z|}{2c\ell}} \int_{B_\ell(z)} g_r^2(y) + |\nabla v^T(1, y)|^2 dy dz dx \lesssim \ell^d \int g_r^2(y) + |\nabla v^T(1, y)|^2 dy \lesssim \ell^d r^{-d}. \quad (2.3.68)$$

The combination of (2.3.66), (2.3.67) and (2.3.68) give the desired (2.3.63).

Substep 1.2. Proof that for all $\ell \in [1, \sqrt{T}]$

$$\begin{aligned} & \left(\int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\ & \lesssim \ell^d r^{-\frac{d}{2}} \log(T) \left(r_*^d(0) \int r_*^d(rx) \frac{\log^2(|x|+1)}{(|x|+1)^{2d}} dx + \int \frac{r_*^{2d}(x)}{(|x|+1)^{2d}} dx + \int r_*^d(rx) g_2^2(x) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.69)$$

Let $N := \lceil \log_2(T) \rceil$. We start by decomposing the time interval $[1, T]$ into dyadic intervals $[2^j, 2^{j+1}]$ for $j \in \llbracket 1, N-1 \rrbracket$. By the triangle inequality, the Cauchy-Schwarz inequality and the decay estimate

(2.3.57)

$$\begin{aligned}
& \left(\int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\
& \leq \sum_{j=0}^{N-1} \left(\int \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(T, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\
& \leq \sum_{j=0}^{N-1} \left(\int \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)|^2 dy dt \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}} \tag{2.3.70}
\end{aligned}$$

$$\stackrel{(2.3.57)}{\lesssim} \sum_{j=0}^{N-1} 2^{-\frac{j}{2}} \left(\int (r_*(x) \vee \ell)^d \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt \right)^{\frac{1}{2}}. \tag{2.3.71}$$

In addition, by the $\frac{1}{8}$ -Lipschitz property of r_* in form of $r_*(x) \vee \ell \lesssim \inf_{y \in B_\ell(x)} r_*(y) \vee \ell$ and $\sup_{y \in B_{r_*(x)}(x)} r_*(y) \vee \ell \lesssim r_*(x) \vee \ell$ as well as the identity $\int \mathbf{f}_{B_\ell(x)} dx = \int$ and the property (2.4.20), we have for all $j \in \llbracket 0, N-1 \rrbracket$

$$\begin{aligned}
& \left(\int (r_*(x) \vee \ell)^d \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}} \\
& \lesssim \ell^{\frac{d}{2}} \left(\int \int_{2^j}^{2^{j+1}} (r_*(x) \vee \ell)^d |\nabla v^T(t, x)|^2 dt dx \right)^{\frac{1}{2}} \\
& \stackrel{(2.4.20)}{\lesssim} \ell^{\frac{d}{2}} \left(\int \int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} (r_*(y) \vee \ell)^d |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}} \\
& \lesssim \ell^{\frac{d}{2}} \left(\int (r_*(x) \vee \ell)^d \int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}}. \tag{2.3.72}
\end{aligned}$$

Then, using the large-scale estimate (2.3.58) we have

$$\int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} |\nabla v^T(s, y)|^2 dy ds \lesssim 2^j \left(r_*^d(0) \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} + r^{-d} \eta_{4r_*(x)}(x) + g_{2r}^2(x) \right). \tag{2.3.73}$$

Therefore, from (2.3.72), (2.3.73) and the change of variables $x \mapsto \frac{x}{r}$, we get

$$\begin{aligned}
& \left(\int (r_*(x) \vee \ell)^d \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}} \\
& \lesssim 2^{\frac{j}{2}} \ell^{\frac{d}{2}} r^{-\frac{d}{2}} \left(r_*^d(0) \int (r_*(rx) \vee \ell)^d \frac{\log^2(|x| + 1)}{(|x| + 1)^{2d}} dx + \int (r_*(x) \vee \ell)^d \eta_{4r_*(x)}(x) dx + \int (r_*(rx) \vee \ell)^d g_2^2(x) dx \right)^{\frac{1}{2}}, \tag{2.3.74}
\end{aligned}$$

which gives the estimate (2.3.69) by plugging (2.3.74) into (2.3.71) and using $\eta_{4r_*(x)}(x) \lesssim r_*^d(x) (|x| + 1)^{-2d}$ as well as $\frac{r_*(x)}{\ell} \vee 1 \lesssim r_*(x)$. This concludes the argument for (2.3.60).

Step 2. Regime $\ell \geq \sqrt{T}$. Proof that for all $\ell \in [\sqrt{T}, +\infty)$

$$\int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx \lesssim \ell^d \log^2(T) (r_*^{2d}(0) + \mathcal{D}_{*,1}(r, \ell) + \mathcal{D}_{*,2}(r) \log^2(\frac{\sqrt{T}}{r})), \tag{2.3.75}$$

with

$$\begin{aligned} \mathcal{D}_{\star,1}(r, \ell) = & r_*^d(0) \ell^d \int_{\mathbb{R}^d \setminus B_\ell} \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} dx + \ell^d \int_{\mathbb{R}^d \setminus B_\ell} r_*^d(x) |x|^{-2d} dx \\ & + r_*^d(0) \left(\int_{B_{7\ell}} \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d} dx \right)^2 + \left(\int_{B_{7\ell}} r_*^{\frac{d}{2}+1}(x) (|x| + 1)^{-d-1} dx \right)^2 + r_*^d(0) \int \frac{r_*^d(x)}{(|x| + 1)^{2d}} dx, \end{aligned} \quad (2.3.76)$$

and

$$\mathcal{D}_{\star,2}(r) = \int_{B_r} r_*^d(x) dx + \int_{\mathbb{R}^d \setminus B_1} r_*^d(rx) g_1(x) dx. \quad (2.3.77)$$

We split this step into two parts. The first one is devoted to the control of the first fourth r.h.s terms of (2.2.44) and the second one is devoted to the control of the fifth r.h.s term.

Substep 2.1. Proof that for all $\ell \in [\sqrt{T}, +\infty)$

$$\int \left(\int_{B_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right| g_r(y) dy \right)^2 dx \lesssim \ell^d \mathcal{D}_{\star,2}(r) \log^2\left(\frac{\sqrt{T}}{r}\right), \quad (2.3.78)$$

and

$$\int \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx + \int \mathcal{G}_{r,\ell}^2(x) dx \lesssim \ell^d \mathcal{D}_{\star,1}(r, \ell), \quad (2.3.79)$$

where we recall that $\mathcal{G}_{r,\ell}$ is defined in (2.2.47).

Argument for (2.3.78). Since for all $x \in \mathbb{R}^d$, $\int_{B_\ell(x)} g_r(y) dy \leq \int g_r(y) dy \lesssim 1$, we have by the Jensen inequality applied with the measure $g_r dy$ and the identity $\int f_{B_\ell(x)} = \int$

$$\begin{aligned} \int \left(\int_{B_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right| g_r(y) dy \right)^2 dx & \lesssim \int \int_{B_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right|^2 g_r(y) dy dx \\ & \lesssim \ell^d \int \left| \int_0^T \nabla u(t, y) dt \right|^2 g_r(y) dy. \end{aligned}$$

Since $r \leq \sqrt{T}$, we split the integral \int_0^T into $\int_0^{r^2} + \int_{r^2}^T$ and we use the localized energy estimate (2.2.32) applied to (2.1.10) in form of $\int \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 g_r(y) dy \lesssim 1$, to obtain

$$\begin{aligned} \int \left| \int_0^T \nabla u(t, y) dt \right|^2 g_r(y) dy & \lesssim \int \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 g_r(y) dy + \int \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 g_r(y) dy \\ & \stackrel{(2.2.32)}{\lesssim} 1 + \int \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 g_r(y) dy. \end{aligned} \quad (2.3.80)$$

Finally, using that $f = \int f_{B_r(x)} dx$ and for all $x \in \mathbb{R}^d \setminus B_r$, $\sup_{y \in B_r(x)} g_r(y) \lesssim g_r(x)$, in form of

$$\int \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 g_r(y) dy \lesssim \int_{B_r} \int_{B_r(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy dx + \int_{\mathbb{R}^d \setminus B_r} g_r(x) \int_{B_r(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy dx,$$

and the Minkowski inequality in $L^2(B_r(x))$ as well as the decay estimate (2.3.57) (where we use that, since $r \geq 1$, $\frac{r_*(x)}{r} \vee 1 \leq 2r_*(0)$), we have

$$\begin{aligned} \int \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 g_r(y) dy &\lesssim \int_{B_r} \left(\int_{r^2}^T \left(\int_{B_r(x)} |\nabla u(t, y)|^2 dy \right)^{\frac{1}{2}} dt \right)^2 dx \\ &\quad + \int_{\mathbb{R}^d \setminus B_r} g_r(x) \left(\int_{r^2}^T \left(\int_{B_r(x)} |\nabla u(t, y)|^2 dy \right)^{\frac{1}{2}} dt \right)^2 dx \\ &\stackrel{(2.3.57)}{\lesssim} \left(\int_{B_r} r_*^d(x) dx + \int_{\mathbb{R}^d \setminus B_1} r_*(rx) g_1(x) dx \right) \log^2\left(\frac{\sqrt{T}}{r}\right). \end{aligned} \quad (2.3.81)$$

Argument for (2.3.79). We start with the first l.h.s term. We distinguish between the generic case $\ell \geq r_*(0)$ and the non-generic case $\ell < r_*(0)$.

Regime $\ell \geq r_*(0)$. We split the integral into the far-field contribution $|x| \geq 4\ell$ and the near-field contribution $|x| < 4\ell$. For the far-field contribution, we write using the Cauchy-Schwarz inequality, the localized energy estimate (2.2.32) applied to (2.1.10) with $R = \ell$ and $T = 1$, the identity $\int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(x)} \lesssim \ell^d \int_{\mathbb{R}^d \setminus B_{3\ell}}$ as well as the property (2.4.20)

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left| 1 + \int_0^1 \nabla u(s, y) ds \right| dy \right)^2 dx &\leq \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(x)} |\nabla v^T(1, y)|^2 dy \\ &\quad \times \int_{B_\ell(x)} \left| 1 + \int_0^1 \nabla u(s, y) dy \right|^2 dx \\ &\stackrel{(2.2.32)}{\lesssim} \ell^d \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(x)} |\nabla v^T(1, y)|^2 dy \quad (2.3.82) \\ &\lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_{3\ell}} |\nabla v^T(1, y)|^2 dy \\ &\stackrel{(2.4.20)}{\lesssim} \ell^{2d} \int \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)|^2 \mathbf{1}_{\mathbb{R}^d \setminus B_{3\ell}}(y) dy dx. \end{aligned} \quad (2.3.83)$$

By the $\frac{1}{8}$ -Lipschitz continuity property of r_* and the assumption $r_*(0) \leq \ell$ one has $B_{r_*(x)}(x) \cap \mathbb{R}^d \setminus B_{3\ell} \neq \emptyset \rightarrow x \in \mathbb{R}^d \setminus B_\ell$. Indeed, by contradiction, if $x \in B_\ell$, then $r_*(x) \leq r_*(0) + \frac{|x|}{8} \leq \frac{15}{16}\ell$ so that $B_{r_*(x)}(x) \subset B_{\ell + \frac{15}{16}\ell} \subset B_{2\ell}$ and consequently $B_{r_*(x)}(x) \cap \mathbb{R}^d \setminus B_{3\ell} = \emptyset$. Hence,

$$\int \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)|^2 \mathbf{1}_{\mathbb{R}^d \setminus B_{3\ell}}(y) dy dx \leq \int_{\mathbb{R}^d \setminus B_\ell} \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)|^2 dy dx, \quad (2.3.84)$$

and the above inequality, with help of (2.3.58) reduces to

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left| 1 + \int_0^1 \nabla u(s, y) ds \right| dy \right)^2 dx \\ \lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_\ell} \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)|^2 dy dx \end{aligned} \quad (2.3.85)$$

$$\stackrel{(2.3.58)}{\lesssim} \ell^{2d} \left(r_*^d(0) \int_{\mathbb{R}^d \setminus B_\ell} \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} dx + \int_{\mathbb{R}^d \setminus B_\ell} \eta_{4r_*(x)}(x) dx + \int_{\mathbb{R}^d \setminus B_\ell} g_{2r}^2(x) dx \right), \quad (2.3.86)$$

which concludes since $\int_{\mathbb{R}^d \setminus B_\ell} g_{2r}^2(x) dx \lesssim \ell^{-d}$ and $\eta_{4r_*(x)}(x) \lesssim r_*^d(x) |x|^{-2d}$. For the near-field contribution $|x| < 4\ell$ we write, using the Minkowski inequality in $L^2(B_{4\ell})$, the Fubini-Tonnelli theorem, the property (2.4.20) and the assumption $\ell \geq r_*(0)$ in form of (2.3.53)

$$\begin{aligned}
 & \left(\int_{B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx \right)^{\frac{1}{2}} \\
 & \leq \int \left(\int_{B_{4\ell}} |\nabla v^T(1, y)|^2 \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right|^2 \right) \mathbb{1}_{B_\ell(x)}(y) dy \right)^{\frac{1}{2}} dx \\
 & \lesssim \ell^{\frac{d}{2}} \int_{B_{5\ell}} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \\
 & \stackrel{(2.4.20)}{\lesssim} \ell^{\frac{d}{2}} \int \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) \mathbb{1}_{B_{5\ell}}(y) dy dx \\
 & \stackrel{(2.3.53)}{\lesssim} \ell^{\frac{d}{2}} \int_{B_{7\ell}} \int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy dx. \tag{2.3.87}
 \end{aligned}$$

Then, from the localized energy estimate (2.2.32) applied to (2.1.10) with $R = r_*(x)$ and $T = 1$, we have

$$\left(\int_{B_{r_*(x)}(x)} 1 + \left| \int_0^1 \nabla u(s, y) ds \right|^2 dy \right)^{\frac{1}{2}} \lesssim 1,$$

consequently, by the Cauchy-Schwarz inequality and (2.3.58), (2.3.87) turns into

$$\begin{aligned}
 \left(\int_{B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^T(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx \right)^{\frac{1}{2}} & \lesssim \ell^{\frac{d}{2}} \int_{B_{7\ell}} \left(\int_{B_{r_*(x)}(x)} |\nabla v^T(1, y)|^2 dy \right)^{\frac{1}{2}} dx \\
 & \lesssim \ell^{\frac{d}{2}} \left(r_*^{\frac{d}{2}}(0) \int_{B_{7\ell}} \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d} dx \right. \\
 & \quad \left. + \int_{B_{7\ell}} \eta_{4r_*(x)}^{\frac{1}{2}}(x) dx + \int_{B_{7\ell}} g_r(y) dy \right), \tag{2.3.88}
 \end{aligned}$$

which concludes since $\int_{B_{7\ell}} g_r(y) dy \lesssim 1$ and $\eta_{4r_*(x)}^{\frac{1}{2}}(x) \lesssim r_*^{\frac{d}{2}+1}(x) (|x| + 1)^{-d-1}$.

Regime $\ell < r_*(0)$. We use the estimate (2.3.63) and we bound one ℓ^d by $r_*^d(0)$ and r^{-d} by 1.

We now turn to the second l.h.s term of (2.3.79). The first term in the definition (2.2.47) of $\mathcal{G}_{r,\ell}$ is bounded as follows. By definition of $\mathcal{T}_{x,\ell}(\eta_r)(0)$, that we recall here

$$\mathcal{T}_{x,\ell}(\eta_r)(0) = \left(\int_{B_\ell(x)} \eta_r(z) \left(1 + \left| \int_0^1 \nabla u(t, z) dt \right|^2 \right) dz \right)^{\frac{1}{2}} + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell(x)} \eta_r(z) |\nabla u(s, z)|^2 dz \right)^{\frac{1}{2}} ds dt,$$

and the localized energy estimates (2.2.32) as well as (2.2.33) applied to (2.1.10), the Minkowski inequality in $L^2(\mathbb{R}^d)$ and the estimate $\int \int_{B_\ell(x)} \lesssim \ell^d \int$, we have

$$\begin{aligned} \int \mathcal{T}_{x,\ell}^2(\eta_r)(0) dx &\lesssim \ell^d \left(1 + \int \eta_r(y) \left| \int_0^1 \nabla u(t,y) dt \right|^2 dy + \left(\int_0^1 \frac{1}{1-t} \int_t^1 \left(\int \eta_r(y) |\nabla u(s,y)|^2 dy \right)^{\frac{1}{2}} ds dt \right)^2 \right) \\ &\stackrel{(2.2.32), (2.2.33)}{\lesssim} \ell^d \left(1 + \left(\int_0^1 \frac{-\log(t)}{1-t} dt \right)^2 \right) \lesssim \ell^d. \end{aligned} \quad (2.3.89)$$

For the other term in (2.2.47), we can use previous estimates. To this aim, we need preliminary inequalities and we distinguish between the two regimes $\ell \geq r_*(0)$ and $\ell < r_*(0)$.

Regime $\ell \geq r_*(0)$. For the far-field contribution $|x| \geq 4\ell$, we make use of the Jensen inequality with the measure $\mathcal{T}_{x,\ell}(\eta_\ell)(y) dy$ in form of, for all $x \in \mathbb{R}^d$

$$\begin{aligned} \left(\int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(y)} |\nabla v^T(1,z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \right)^2 &\leq \int_{\mathbb{R}^d \setminus B_{4\ell}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \\ &\quad \times \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1,z)|^2 dz \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy. \end{aligned} \quad (2.3.90)$$

Next, we have

$$\int \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \lesssim \ell^d. \quad (2.3.91)$$

Indeed, we first split the integral into two contributions

$$\int \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy = \int_{B_{2\ell}(x)} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy + \int_{\mathbb{R}^d \setminus B_{2\ell}(x)} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy.$$

On the one hand, since from the localized energy estimate (2.2.32) applied to (2.1.10) with $R = \ell$ we have $\sup_{y \in \mathbb{R}^d} \mathcal{T}_{x,\ell}(\eta_\ell)(y) \lesssim 1 + \int_0^1 \frac{-\log(t)}{1-t} dt \lesssim 1$ (we bound the integral $\int_{B_\ell(x)}$ by \int in the definition (2.2.48) of $\mathcal{T}_{x,\ell}(\eta_\ell)(y)$), we get

$$\int_{B_{2\ell}(x)} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \lesssim \ell^d.$$

On the other hand, for all $y \in \mathbb{R}^d \setminus B_{2\ell}(x)$ and $z \in B_\ell(x)$ we have $|z - y| \geq |y - x| - \ell \geq \frac{|y-x|}{2}$ and thus $\eta_\ell(z - y) \lesssim \eta_{2\ell}(y - x)$. Therefore, the localized energy estimate (2.2.32) applied to (2.1.10) with $R = \ell$ yields

$$\begin{aligned} \mathcal{T}_{x,\ell}(\eta_\ell)(y) &\lesssim \eta_{2\ell}^{\frac{1}{2}}(y - x) \left(\left(\int_{B_\ell(x)} 1 + \left| \int_0^1 \nabla u(s,z) ds \right|^2 dz \right)^{\frac{1}{2}} + \int_0^1 \frac{1}{1-t} \int_t^1 \left(\int_{B_\ell(x)} |\nabla u(s,z)|^2 dz \right)^{\frac{1}{2}} dt \right) \\ &\lesssim \ell^{\frac{d}{2}} \eta_{2\ell}^{\frac{1}{2}}(y - x). \end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^d \setminus B_{2\ell}(x)} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \lesssim \ell^{\frac{d}{2}} \int_{\mathbb{R}^d \setminus B_{2\ell}(x)} \eta_{2\ell}^{\frac{1}{2}}(y - x) dy \lesssim \ell^d,$$

and it concludes the argument for (2.3.91). The combination of (2.3.90), (2.3.91) and the estimate $\sup_{y \in \mathbb{R}^d} \int \mathcal{T}_{x,\ell}(\eta_\ell)(y) dx \lesssim \ell^d$ (which can be proved with the same computation as (2.3.91) by exchanging the role of x and y) leads to

$$\begin{aligned} & \int \left(\int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \right)^2 dx \\ & \stackrel{(2.3.90), (2.3.91)}{\lesssim} \ell^d \int \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1, z)|^2 dz \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy dx \\ & \lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(y)} |\nabla v^T(1, z)|^2 dz dy dx, \end{aligned}$$

and we then proceed as we did from (2.3.82) to conclude. For the near-field contribution, we make use of the Minkowski inequality in $L^2(\mathbb{R}^d)$ and the estimate $\sup_{y \in \mathbb{R}^d} \int \mathcal{T}_{x,\ell}^2(\eta_{r_*(y)})(y) dx \lesssim \ell^d$ (obtained the same way as (2.3.89)) to obtain

$$\begin{aligned} & \int \left(\int_{B_{7\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_{r_*(y)})(y) dy \right)^2 dx \\ & \leq \left(\int_{B_{7\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} \left(\int \mathcal{T}_{x,\ell}^2(\eta_{r_*(y)})(y) dx \right)^{\frac{1}{2}} dy \right)^2 \\ & \lesssim \ell^d \left(\int_{B_{7\ell}} \left(\int_{B_{r_*(y)}(y)} |\nabla v^T(1, z)|^2 dz \right)^{\frac{1}{2}} dy \right)^2, \end{aligned} \tag{2.3.92}$$

and we then proceed as we did from (2.3.88).

Regime $\ell < r_*(0)$. In this regime, we use the estimate (2.3.68).

Substep 2.2. Proof that for $\ell \in [\sqrt{T}, +\infty)$

$$\begin{aligned} \int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy \right)^2 dx & \lesssim \ell^d \log^2(T) \left(\ell^d \int_{\mathbb{R}^d \setminus B_\ell} \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} dx + \ell^d \int_{\mathbb{R}^d \setminus B_\ell} r_*^d(x) |x|^{-2d} dx \right. \\ & \quad + r_*^d(0) \left(\int_{B_{7\ell}} \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d} dx \right)^2 \\ & \quad + \left(\int_{B_{7\ell}} r_*^{\frac{d}{2}+1}(x) (|x| + 1)^{-d-1} dx \right)^2 \\ & \quad \left. + r_*^{2d}(0) + r_*^d(0) \int \frac{r_*^d(x)}{(|x| + 1)^{2d}} dx \right). \end{aligned} \tag{2.3.93}$$

For the proof of (2.3.93), we argue as previously and we distinguish between the generic case $\ell \geq r_*(0)$ and the non-generic case $\ell < r_*(0)$. We mainly make use of previous ideas and estimates.

Regime $\ell \geq r_*(0)$. As in Substep 1.2, we decompose the time interval $[1, T]$ into dyadic intervals $[2^j, 2^{j+1}]$ for $j \in \llbracket 1, N-1 \rrbracket$ and $N := \lceil \log_2(T) \rceil$ and we write by the triangle inequality

$$\int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy \right)^2 dx \leq \left(\sum_{j=0}^{N-1} \left(\int \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \right)^2. \tag{2.3.94}$$

We split the integral in the r.h.s of (2.3.94) into the far-field contribution $|x| \geq 4\ell$ and the near-field contribution $|x| < 4\ell$. For the far-field contribution, we write for all $j \in \llbracket 1, N-1 \rrbracket$, using the Cauchy-Schwarz inequality, the decay estimate (2.3.57) applied for $\ell \geq \sqrt{T}$,

$$\int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \stackrel{(2.3.57)}{\leq} \ell^d 2^{-j} \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^d \setminus B_{4\ell}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dx dt.$$

We then argue as from (2.3.82) to (2.3.83), (2.3.84), (2.3.85) and (2.3.86) (noticing that the evaluation at time 1 plays no role in the estimates) and finally obtain

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx &\lesssim \ell^{2d} \left(r_*^d(0) \int_{\mathbb{R}^d \setminus B_\ell} \frac{\log^2(1 + \frac{|x|}{r})}{(|x| + r)^{2d}} dx \right. \\ &\quad \left. + \int_{\mathbb{R}^d \setminus B_\ell} r_*^d(x) |x|^{-2d} dx + \ell^{-d} \right), \end{aligned}$$

and we conclude by summing over $j \in \llbracket 1, N-1 \rrbracket$, which give the two first r.h.s terms of (2.3.93). For the near-field contribution $|x| \leq 4\ell$ we write, using the Minkowski inequality in $L^2(B_{4\ell})$, the Fubini-Tonnelli theorem, the property (2.4.20) and the assumption $\ell \geq r_*(0)$ in form of (2.3.53)

$$\begin{aligned} &\left(\int_{B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq \int_{2^j}^{2^{j+1}} \int \left(\int_{B_{4\ell}} |\nabla u(t, y)|^2 |\nabla v^T(t, y)|^2 \mathbb{1}_{B_\ell(x)}(y) dx \right)^{\frac{1}{2}} dy dt \\ &\lesssim \ell^{\frac{d}{2}} \int_{B_{5\ell}} \int_{2^j}^{2^{j+1}} |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy \\ &\stackrel{(2.4.20)}{\lesssim} \ell^{\frac{d}{2}} \int \int_{B_{r_*(x)}(x)} \int_{2^j}^{2^{j+1}} |\nabla u(t, y)| |\nabla v^T(t, y)| \mathbb{1}_{B_{5\ell}}(y) dt dy dx \\ &\stackrel{(2.3.53)}{\leq} \ell^{\frac{d}{2}} \int_{B_{7\ell}} \int_{B_{r_*(x)}(x)} \int_{2^j}^{2^{j+1}} |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy dx. \end{aligned} \tag{2.3.95}$$

Then, from the decay (2.3.57), we have

$$\left(\int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} |\nabla u(t, y)|^2 dy dt \right)^{\frac{1}{2}} \lesssim 2^{-\frac{j}{2}}, \tag{2.3.96}$$

consequently, by the Cauchy-Schwarz inequality and (2.3.73), (2.3.95) turns into

$$\begin{aligned} &\left(\int_{B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\ &\lesssim 2^{-\frac{j}{2}} \ell^{\frac{d}{2}} \int_{B_{7\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} |\nabla v^T(t, y)|^2 dy dt \right)^{\frac{1}{2}} dx \\ &\stackrel{(2.3.73)}{\lesssim} \ell^{\frac{d}{2}} \left(r_*^{\frac{d}{2}}(0) \int_{B_{7\ell}} \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d} dx + \int_{B_{7\ell}} \eta_{4r_*(x)}^{\frac{1}{2}}(x) dx + \int_{B_{7\ell}} g_{2r}(x) dx \right), \end{aligned}$$

and we conclude since $\int_{B_{7\ell}} g_{2r}(x) dx \lesssim 1$, $\eta_{4r_*}^{\frac{1}{2}}(x) \lesssim r_*^{\frac{d}{2}+1}(x)(|x|+1)^{-d-1}$ and summing over $j \in \llbracket 0, N-1 \rrbracket$.

Regime $\ell < r_*(0)$. We use the estimate (2.3.69) which holds for $\ell \geq \sqrt{T}$ by removing $r_*^d(rx)$ and $r_*^d(x)$ in each integral in the r.h.s (by using (2.3.57) for $\ell \geq \sqrt{T}$) and we estimate ℓ^d by $r_*^d(0)$ as well as $r^{-\frac{d}{2}}$ by 1.

Step 3. Proof of (2.2.53). We have from the multiscale logarithmic Sobolev inequality in form of (2.4.1): for all $p \in [1, +\infty)$

$$\langle |q_r(T) - \langle q_r(T) \rangle|^p \rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{2}} \left\langle \left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}. \quad (2.3.97)$$

We split the integral over ℓ into two parts.

(i) In the regime $\ell < \sqrt{T}$ we use (2.3.60):

$$\begin{aligned} & \left\langle \left(\int_1^{\sqrt{T}} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ & \stackrel{(2.3.60)}{\lesssim} r^{-\frac{d}{2}} (1 + \log(T) + \log(\frac{\sqrt{T}}{r})) \left\langle \left(\int_1^{\sqrt{T}} \ell^d \pi(\ell) \mathcal{C}_*(r, \ell) d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ & \stackrel{(2.1.6)}{\leq} r^{-\frac{d}{2}} (1 + \log(T) + \log(\frac{\sqrt{T}}{r})) \left(\int_1^{\sqrt{T}} \ell^{d-1-\beta} \langle \mathcal{C}_*^p(r, \ell) \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \\ & \lesssim r^{-\frac{d}{2}} (1 + \log(T) + \log(\frac{\sqrt{T}}{r})) \mu_\beta(T) \sup_{\ell \geq 1} \langle \mathcal{C}_*^p(r, \ell) \rangle^{\frac{1}{2p}}, \end{aligned} \quad (2.3.98)$$

by the definition (2.1.18) of μ_β .

(ii) In the regime $\ell \geq \sqrt{T}$ we use (2.3.75), the change of variables $\ell \mapsto \frac{\ell}{\sqrt{T}}$ and the fact that $r \leq \sqrt{T}$ in the last line:

$$\begin{aligned} & \left\langle \left(\int_{\sqrt{T}}^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ & \stackrel{(2.3.75), (2.1.6)}{\lesssim} \log(T) \left\langle \left(\int_{\sqrt{T}}^{+\infty} \ell^{-1-\beta} (r_*^d(0) + \mathcal{D}_{*,1}(r, \ell)) d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ & \quad + \log(\frac{\sqrt{T}}{r}) \left\langle \mathcal{D}_{*,2}^{\frac{p}{2}}(r) \right\rangle^{\frac{1}{p}} \left(\int_{\sqrt{T}}^{+\infty} \ell^{-1-\beta} d\ell \right)^{\frac{1}{2}} \\ & \lesssim T^{-\beta} \log(T) \left(\left(\int_1^{+\infty} \ell^{-1-\beta} \langle (r_*^d(0) + \mathcal{D}_{*,1}(r, \ell\sqrt{T}))^p \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \log(\frac{\sqrt{T}}{r}) \langle \mathcal{D}_{*,2}^{\frac{p}{2}}(r) \rangle^{\frac{1}{p}} \right) \\ & \lesssim r^{-\frac{d}{2}} \log(T) \mu_\beta(T) \left(\left(\int_1^{+\infty} \ell^{-1-\beta} \langle (r_*^d(0) + \mathcal{D}_{*,1}(r, \ell\sqrt{T}))^p \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \log(\frac{\sqrt{T}}{r}) \langle \mathcal{D}_{*,2}^{\frac{p}{2}}(r) \rangle^{\frac{1}{p}} \right). \end{aligned} \quad (2.3.99)$$

It remains to show that

$$\sup_{r \geq 1} \langle \mathcal{D}_{\star,2}^{\frac{p}{2}}(r) \rangle^{\frac{1}{p}} + \sup_{\ell \geq 1} \langle \mathcal{C}_{\star}^p(r, \ell) \rangle^{\frac{1}{2p}} \lesssim p^{\frac{d}{\beta \wedge d}}, \quad (2.3.100)$$

and

$$\left(\int_1^{+\infty} \ell^{-1-\beta} \langle \mathcal{D}_{\star,1}^p(r, \ell\sqrt{T}) \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \lesssim p^{\frac{d+2}{\beta \wedge d}} (1 + \log^2(\frac{\sqrt{T}}{r})). \quad (2.3.101)$$

We only provide the argument for (2.3.101), (2.3.100) is proved the same way. From the moment bound (2.4.3) on r_* and the equivalence between algebraic moments and exponential moments (see Lemma 9), we have for all $p \geq 1$

$$\langle r_*^{dp}(0) \rangle^{\frac{1}{p}} \lesssim p^{\frac{d}{\beta \wedge d}}. \quad (2.3.102)$$

We treat separately the bound of the two lines in the definition (2.3.76) of $\mathcal{D}_{\star,1}$. First, we have by the Minkowski inequality in $L_{\langle \cdot \rangle}^p(\Omega)$, the stationarity property of r_* and (2.3.102)

$$\begin{aligned} & \left(\int_1^{+\infty} \ell^{-1-\beta} \left\langle \left(r_*^d(0) (\ell\sqrt{T})^d \int_{\mathbb{R}^d \setminus B_{\ell\sqrt{T}}} \frac{\log^2(1 + \frac{|x|}{r})}{(|x|+r)^{2d}} dx + (\ell\sqrt{T})^d \int_{\mathbb{R}^d \setminus B_{\ell\sqrt{T}}} r_*^d(x) |x|^{-2d} dx \right)^p \right\rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \\ & \lesssim \langle r_*^{dp}(0) \rangle^{\frac{1}{2p}} T^{\frac{d}{4}} \left(\int_1^{+\infty} \ell^{d-1-\beta} \int_{\mathbb{R}^d \setminus B_{\ell\sqrt{T}}} \frac{\log^2(1 + \frac{|x|}{r})}{(|x|+r)^{2d}} dx + \int_{\mathbb{R}^d \setminus B_{\ell\sqrt{T}}} |x|^{-2d} dx d\ell \right)^{\frac{1}{2}} \\ & \stackrel{(2.3.102)}{\lesssim} p^{\frac{1}{2} \frac{d}{\beta \wedge d}} \left(\int_1^{+\infty} \ell^{-1-\beta} (1 + \log^2(\ell \frac{\sqrt{T}}{r})) d\ell \right)^{\frac{1}{2}} \lesssim p^{\frac{1}{2} \frac{d}{\beta \wedge d}} (1 + \log(\frac{\sqrt{T}}{r})). \end{aligned} \quad (2.3.103)$$

Second, we have by (2.3.102), a polar change of coordinates and the change of variables $\rho \mapsto \frac{\rho}{r}$

$$\begin{aligned} & \left(\int_1^{+\infty} \ell^{-1-\beta} \left\langle \left(r_*^d(0) \left(\int_{B_{7\ell\sqrt{T}}} \frac{\log(1 + \frac{|x|}{r})}{(|x|+r)^d} dx \right)^2 + \left(\int_{B_{7\ell\sqrt{T}}} r_*^{\frac{d}{2}+1}(x) (|x|+1)^{-d-1} dx \right)^2 \right. \right. \\ & \quad \left. \left. + r_*^d(0) \int \frac{r_*^d(x)}{(|x|+1)^{2d}} dx \right)^p \right\rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} \\ & \stackrel{(2.3.102)}{\lesssim} p^{\frac{d+2}{\beta \wedge d}} \left(\int_1^{+\infty} \ell^{-1-\beta} \left(\int_0^{7\ell\sqrt{T}} \frac{\log(1 + \frac{\rho}{r})}{(\rho+r)^d} \rho^{d-1} d\rho + 1 \right)^2 d\ell \right)^{\frac{1}{2}} \\ & \leq p^{\frac{d+2}{\beta \wedge d}} \left(\int_1^{+\infty} \ell^{-1-\beta} \log^2(1 + 7 \frac{\ell\sqrt{T}}{r}) \left(\int_0^{7 \frac{\ell\sqrt{T}}{r}} (\rho+1)^{-d} \rho^{d-1} d\rho + 1 \right)^2 d\ell \right)^{\frac{1}{2}}, \end{aligned} \quad (2.3.104)$$

and we estimate the integral in the integrand of the r.h.s of (2.3.104) by, for all $\ell \geq 1$

$$\begin{aligned} \int_0^{7 \frac{\ell\sqrt{T}}{r}} (\rho+1)^{-d} \rho^{d-1} d\rho & \lesssim \int_0^1 (1 + \log(1 + \frac{T}{r^2} \rho^{-2})) \rho^{d-1} d\rho + \int_1^{7 \frac{\ell\sqrt{T}}{r}} \rho^{-1} d\rho \\ & \lesssim 1 + \log(7 \frac{\ell\sqrt{T}}{r}). \end{aligned} \quad (2.3.105)$$

The combination of (2.3.103), (2.3.104) and (2.3.105) gives (2.3.101). To conclude, the estimates (2.3.97), (2.3.98), (2.3.99), (2.3.100) and (2.3.101) give the desired bound (2.2.53).

2.3.4 Proof of the main results

Proof of Theorem 10: Fluctuations of the time dependent flux

We only gives the argument for the flux $q(T)$, the computations for $\phi(T)$ are equivalent and are done by a straightforward adaptation of the argument of this proof and the ones to prove Lemmas 7 and 8.

Our first goal is to remove the $\log(T)$ correction in the r.h.s of (2.2.53), which will lead to (2.1.17). To this aim, we use the L^2 - L^1 type estimate of Lemma 3, which allow us to make the link between the $r^{-\frac{d}{2}}$ decay of the fluctuations of $q_r(T, \cdot)$, for $r \leq \sqrt{T}$, proved in Proposition 3 and the decay in T of moments of averages of $\nabla u(T, \cdot)$. It yields a better decay estimate in time T of spatial averages of $\nabla u(T, \cdot)$ than the one obtained in (2.3.57) (from which the $\log(T)$ contribution in (2.2.53) comes from). With this new decay in hand, we are able to obtain optimal estimate in scaling. The price to pay in this step is a little loss of stochastic integrability. Our second goal is to prove estimate (2.1.20). It does not require new ideas and this is done by dominating carefully the terms in the derivative (2.2.50) and by using some estimates already done in the proof of (2.1.17).

Proof of (2.1.17). We split the proof into three steps.

Step 1. Improvement of (2.3.57). Proof that for all $x \in \mathbb{R}^d$, $T \geq 1$ and $\varepsilon \in (0, 1)$

$$\int_{B_\ell(x)} |\nabla u(T, y)|^2 dy \leq \ell^d \left(r_*^d(x) \mathbf{1}_{\ell < \sqrt{T}} + \mathbf{1}_{\ell \geq \sqrt{T}} \right) \mathcal{D}_*^{2\varepsilon}(T, \ell, x) \eta_{\varepsilon, \beta}^2(T), \quad (2.3.106)$$

with

$$\eta_{\varepsilon, \beta}(T) = \begin{cases} (1 + \log^\varepsilon(T)) T^{-1 - \varepsilon \frac{\beta}{4}} & \text{if } \beta < d, \\ (1 + \log^{\frac{3}{2}\varepsilon}(T)) T^{-1 - \varepsilon \frac{d}{4}} & \text{if } \beta = d, \\ (1 + \log^\varepsilon(T)) T^{-1 - \varepsilon \frac{d}{4}} & \text{if } \beta > d, \end{cases} \quad (2.3.107)$$

and for some stationary random field $\mathcal{D}_*(T, \ell, \cdot)$ that satisfies for all $p \in [1, +\infty)$

$$\sup_{(T, \ell, x) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^d} \left\langle \mathcal{D}_*^p(T, \ell, x) \right\rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{2} + \frac{d+2}{\beta \wedge d}}. \quad (2.3.108)$$

We have from Lemma 3, the Minkowski inequality in $L_{\langle \cdot \rangle}^p(\Omega)$ and the stationarity of q_r : for all $p \in [1, +\infty)$, $T \geq 4$ and $R \geq \sqrt{T}$

$$\begin{aligned} \left\langle \left(\int \eta_{\sqrt{2R}}(y) |\nabla u(T, y)|^2 dy \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} &\lesssim \frac{1}{T} \left\langle \left(\int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^{\sqrt{t}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{d}{2}} \int \eta_{2R}(y) |q_r(t, y) - \langle q_r(t, y) \rangle| dy dr dt \right)^p \right\rangle^{\frac{1}{p}} \\ &\lesssim \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^{\sqrt{t}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt. \end{aligned} \quad (2.3.109)$$

Then, we split the integral over $[0, \sqrt{t}]$ into the two contributions $r \leq 1$ and $1 \leq r \leq \sqrt{t}$:

(i) For $r \leq 1$ we use (2.2.43) and the change of variable $r \mapsto \frac{\sqrt{t}}{r}$:

$$\begin{aligned} \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_0^1 \left(\frac{r}{\sqrt{t}} \right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt &\stackrel{(2.2.43)}{\lesssim} p^{\frac{1}{2} \frac{d}{\beta \wedge d}} T^{-1 - \frac{d}{4}} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_0^1 \left(r^{\frac{d}{2}} + \log\left(\frac{\sqrt{t}}{r}\right) \right) dr dt \\ &\lesssim p^{\frac{1}{2} \frac{d}{\beta \wedge d}} T^{-1 - \frac{d}{4}} \left(\int_{\sqrt{T}}^{+\infty} r^{-2} (1 + \log(r)) dr + \frac{1}{\sqrt{T}} \right) \\ &\lesssim p^{\frac{1}{2} \frac{d}{\beta \wedge d}} T^{-\frac{3}{2} - \frac{d}{4}} \log(T). \end{aligned} \quad (2.3.110)$$

(ii) For $1 \leq r \leq \sqrt{t}$ we use (2.2.53) and the change of variable $r \mapsto \frac{\sqrt{t}}{r}$:

$$\begin{aligned}
& \frac{1}{\bar{T}} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_{\frac{1}{4}}^{\sqrt{t}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt \\
& \stackrel{(2.2.53)}{\lesssim} p^{\frac{1}{2} + \frac{d+2}{\beta \wedge d}} T^{-1 - \frac{d}{4}} \mu_\beta(T) \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_1^{\sqrt{t}} \left(1 + \log(T) + \log^2\left(\frac{\sqrt{t}}{r}\right) \right) dr dt \\
& \leq p^{\frac{1}{2} + \frac{d+2}{\beta \wedge d}} \eta_\beta(T) \left(\log(T) + \int_1^{\sqrt{\frac{T}{2}}} r^{-2} (1 + \log^2(r)) dr \right) \\
& \lesssim p^{\frac{1}{2} + \frac{d+2}{\beta \wedge d}} (1 + \log(T)) \eta_\beta(T), \tag{2.3.111}
\end{aligned}$$

where η_β is defined in (2.1.24).

The combination of (2.3.109), (2.3.110) and (2.3.111) yields that for all $x \in \mathbb{R}^d$ and $R \geq \sqrt{T}$

$$\int_{B_{R(x)}} |\nabla u(T, y)|^2 dy \lesssim \tilde{\mathcal{D}}_\star^2(T, R, x) (1 + \log^2(T)) \eta_\beta^2(T), \tag{2.3.112}$$

where $\tilde{\mathcal{D}}_\star(T, R, \cdot) := \frac{\int_{B_{R(\cdot)}} |\nabla u(T, y)|^2 dy}{\log^2(T) \eta_\beta^2(T)}$ has the moment bound (2.3.108). It implies, from Lemma 5 applied with $f : t \in \mathbb{R}_*^+ \mapsto 1 + \log^2(t)$ and $g : t \in \mathbb{R}_*^+ \mapsto t^{-2 - \frac{d}{2}}$ that, for all $\ell < \sqrt{T}$

$$\int_{B_\ell(x)} |\nabla u(T, y)|^2 dy \lesssim (r_\star(x) \vee \ell)^d \left(\tilde{\mathcal{D}}_\star^2(T, \sqrt{T}, x) \vee \int_{\frac{T}{2}}^T \tilde{\mathcal{D}}_\star^2(s, \sqrt{s}, x) ds \right) (1 + \log^2(T)) \eta_\beta^2(T). \tag{2.3.113}$$

By interpolating between (2.3.57) and the combination of (2.3.112) and (2.3.113) as well as using that $\frac{r_\star(x)}{\ell} \vee 1 \leq 2r_\star(x)$ in the last line, we then obtain for all $\varepsilon \in (0, 1)$

$$\begin{aligned}
\int_{B_\ell(x)} |\nabla u(T, y)|^2 dy &= \left(\int_{B_\ell(x)} |\nabla u(T, y)|^2 dy \right)^{1-\varepsilon} \left(\int_{B_\ell(x)} |\nabla u(T, y)|^2 dy \right)^\varepsilon \\
&\stackrel{(2.3.57), (2.3.113)}{\lesssim} \ell^d \left(\left(\frac{r_\star(x)}{\ell} \vee 1 \right)^d \mathbf{1}_{\ell < \sqrt{T}} + \mathbf{1}_{\ell \geq \sqrt{T}} \right) T^{-2(1-\varepsilon)} \mathcal{D}_\star^{2\varepsilon}(T, \ell, x) (1 + \log^{2\varepsilon}(T)) \eta_\beta^{2\varepsilon}(T) \\
&\leq 2\ell^d \left(r_\star^d(x) \mathbf{1}_{\ell < \sqrt{T}} + \mathbf{1}_{\ell \geq \sqrt{T}} \right) \mathcal{D}_\star^{2\varepsilon}(T, \ell, x) \eta_{\varepsilon, \beta}^2(T),
\end{aligned}$$

with $\eta_{\varepsilon, \beta}$ defined in (2.3.107) and $\mathcal{D}_\star(T, \ell, x) := \max \left\{ \tilde{\mathcal{D}}_\star(T, \ell, x), \tilde{\mathcal{D}}_\star^2(T, \sqrt{T}, x) \vee \int_{\frac{T}{2}}^T \tilde{\mathcal{D}}_\star^2(s, \sqrt{s}, x) ds \right\}$ which satisfies the moment bound (2.3.108). Equipped with the new decay (2.3.106), we improve the estimates (2.3.60) and (2.3.75) for the control of $(x, \ell) \mapsto \int |\partial_{x, \ell}^{\text{fct}} q_r(T)|^2 dx$ (corresponding to the substeps 1 and 2 of the proof of Lemma 3).

Step 2. We split this step into two parts, one for the improvement of (2.3.60) and an other for (2.3.75), treating separately the two regimes $\ell < \sqrt{T}$ and $\ell \geq \sqrt{T}$.

Substep 2.1. Improvement of (2.3.60). Proof that for all $\varepsilon > 0$ and $\ell < \sqrt{T}$

$$\int |\partial_{x, \ell}^{\text{fct}} q_r(T)|^2 dx \leq \ell^{2d} r^{-d} \left(1 + \mathcal{E}_{\star, \varepsilon}(r, \ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{F}_{j, \star, \varepsilon}(r, \ell) \right)^2 \right), \tag{2.3.114}$$

where

$$\mathcal{E}_{*,\varepsilon}(r, \ell) := \int r_*^d(rx) \left(\int_{\frac{1}{16}}^{+\infty} \mathcal{D}_*^\varepsilon(t, \ell, rx) \eta_{\varepsilon, \beta}(t) dt \right)^2 \int_{B_{\frac{\ell}{r}}(x)} g_1^2(y) dy dx, \quad (2.3.115)$$

and

$$\begin{aligned} \mathcal{F}_{j,*,\varepsilon}(r, \ell) := & \left(\int r_*^d(x) \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_*^{2\varepsilon}(t, \ell, x) dt \right) \left(\int_{B_\ell(x)} r_*^d(0) r^d \frac{\log^2(1 + \frac{|y|}{r})}{(|y| + r)^{2d}} dy + \frac{r_*^d(y)}{(|y| + 1)^{2d}} + r^d g_{2r}^2(y) dy \right. \right. \\ & \left. \left. + r_*^{d+\varepsilon}(x) \ell^\varepsilon \int_{\mathbb{R}^d \setminus B_\ell(x)} |y - x|^{-d-\varepsilon} \left(r_*^d(0) r^d \frac{\log^2(1 + \frac{|y|}{r})}{(|y| + r)^{2d}} dy + \frac{r_*^d(y)}{(|y| + 1)^{2d}} + r^d g_{2r}^2(y) \right) dy \right) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.116)$$

The estimate (2.3.63) is unchanged and gives the first contribution in (2.3.114). We improve the estimates (2.3.62) and (2.3.69). On the one hand, noticing that from the Minkowski inequality in $L^2(\mathbb{R}^d)$ and (2.3.106) as well as $r \geq \frac{1}{2}$, we have for all $x \in \mathbb{R}^d$

$$\int_{B_\ell(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy \leq \left(\int_{r^2}^T \left(\int_{B_\ell(x)} |\nabla u(t, y)|^2 dy \right)^{\frac{1}{2}} dt \right)^2 \stackrel{(2.3.106)}{\lesssim} \ell^d r_*^d(x) \left(\int_{\frac{1}{16}}^{+\infty} \mathcal{D}_*^\varepsilon(t, \ell, x) \eta_{\varepsilon, \beta}(t) dt \right)^2,$$

thus we deduce from (2.3.64) and (2.3.65) that

$$\begin{aligned} & \left(\int \left(\int_{B_\ell(x)} \left| \int_0^T \nabla u(t, y) dt \right| g_r(y) dy \right)^2 dx \right)^{\frac{1}{2}} \\ & \stackrel{(2.3.64), (2.3.65)}{\leq} \ell^d r^{-\frac{d}{2}} + \left(\int \left(\int_{B_\ell(x)} \left| \int_{r^2}^T \nabla u(t, y) dt \right|^2 dy \int_{B_\ell(x)} g_r^2(y) dy \right) dx \right)^{\frac{1}{2}} \\ & \stackrel{(2.3.106)}{\lesssim} r^{-\frac{d}{2}} + \ell^{\frac{d}{2}} \left(\int r_*^d(x) \mathcal{J}_{*,\varepsilon}(x) \int_{B_\ell(x)} g_r^2(y) dy dx \right)^{\frac{1}{2}}, \end{aligned} \quad (2.3.117)$$

where

$$\mathcal{J}_{*,\varepsilon}(x) := \left(\int_{\frac{1}{16}}^{+\infty} \mathcal{D}_*^\varepsilon(t, \ell, x) \eta_{\varepsilon, \beta}(t) dt \right)^2.$$

Using the change of variables $x \mapsto \frac{x}{r}$ in the last r.h.s term of (2.3.117), it gives the term $\mathcal{E}_{*,\varepsilon}(r, \ell)$ defined in (2.3.115). On the other hand, noticing that, by monotonicity of $t \in \mathbb{R}_*^+ \mapsto \eta_{\varepsilon, \beta}(t)$, for all $j \in \mathbb{N}$, we have

$$\int_{2^j}^{2^{j+1}} \mathcal{D}_*^{2\varepsilon}(t, \ell, x) \eta_{\varepsilon, \beta}^2(t) dt \lesssim \eta_{\varepsilon, \beta}^2(2^j) \int_{2^j}^{2^{j+1}} \mathcal{D}_*^{2\varepsilon}(t, \ell, x) dt, \quad (2.3.118)$$

we deduce from (2.3.70) and (2.3.106)

$$\begin{aligned} & \left(\int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\ & \lesssim \ell^{\frac{d}{2}} \sum_{j=0}^{+\infty} \eta_{\varepsilon, \beta}(2^j) \left(\int r_*^d(x) \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_*^{2\varepsilon}(t, \ell, x) dt \right) \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx \right)^{\frac{1}{2}}. \end{aligned} \quad (2.3.119)$$

It remains to control the r.h.s integral of (2.3.119) by $2^j r^{-d} \mathcal{F}_{j,*,\varepsilon}(r, \ell)$, where $\mathcal{F}_{j,*,\varepsilon}(r, \ell)$ is defined in (2.3.116). To this aim, we note that, by the $\frac{1}{8}$ -Lipschitz property of r_* , we have for all $(x, y) \in \mathbb{R}^{2d}$

$$B_\ell(x) \cap B_{r_*(y)}(y) \neq \emptyset \rightarrow |y - x| \leq 3(r_*(x) \vee \ell). \quad (2.3.120)$$

Indeed if there exists $z \in B_\ell(x) \cap B_{r_*(y)}(y)$, we have by the triangle inequality

$$|y - x| \leq |y - z| + |z - x| \leq r_*(y) + \ell \leq r_*(x) + \frac{1}{8}|y - x| + \ell,$$

and thus (2.3.120) holds. Then, we use the property (2.4.20) and the decomposition, for all $x \in \mathbb{R}^d$, $\int = \int_{B_\ell(x)} + \int_{\mathbb{R}^d \setminus B_\ell(x)}$ to obtain

$$\begin{aligned} \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt &\stackrel{(2.4.20)}{\lesssim} \int_{2^j}^{2^{j+1}} \int \int_{B_{r_*(y)}(y)} \mathbb{1}_{B_\ell(x)}(z) |\nabla v^T(t, z)|^2 dz dy dt \\ &\leq \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} \int_{B_{r_*(y)}(y)} |\nabla v^T(t, z)|^2 dz dy dt \\ &\quad + \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^d \setminus B_\ell(x)} \int_{B_{r_*(y)}(y)} \mathbb{1}_{B_\ell(x)}(z) |\nabla v^T(t, z)|^2 dz dy dt. \end{aligned} \tag{2.3.121}$$

Next, we make use of (2.3.120) to bound the second r.h.s term of (2.3.121) with

$$\begin{aligned} &\int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^d \setminus B_\ell(x)} \int_{B_{r_*(y)}(y)} \mathbb{1}_{B_\ell(x)}(z) |\nabla v^T(t, z)|^2 dz dy dt \\ &\lesssim (r_*(x) \vee \ell)^{d+\varepsilon} \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^d \setminus B_\ell(x)} |y - x|^{-d-\varepsilon} \int_{B_{r_*(y)}(y)} |\nabla v^T(t, z)|^2 dz dy dt. \end{aligned} \tag{2.3.122}$$

The combination of (2.3.121) and (2.3.122) (where we bound $(r_*(x) \vee \ell)^{d+\varepsilon} \lesssim \ell^{d+\varepsilon} r_*^{d+\varepsilon}(x)$) as well as (2.3.73) (where we bound $\eta_{4r_*(x)}(x) \lesssim r_*^d(x) (|x| + 1)^{-2d}$) proves that the r.h.s integral of (2.3.119) is indeed bounded by $2^j r^{-d} \mathcal{F}_{j, \star, \varepsilon}(r, \ell)$ and concludes the argument for (2.3.114).

Substep 2.2. Improvement of (2.3.75). Proof that for all $\varepsilon > 0$ and $\ell \geq \sqrt{T}$

$$\int |\partial_{x, \ell}^{\text{fct}} q_r(T)|^2 dx \lesssim \ell^d \left(\mathcal{D}_{\star, 1}(r, \ell) + \mathcal{D}_{\star, 2}(r) \log^2 \left(\frac{\sqrt{T}}{r} \right) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{H}_{j, \star, \varepsilon}(r, \ell) \right)^2 \right), \tag{2.3.123}$$

with $\mathcal{D}_{\star, 1}(r, \ell)$ and $\mathcal{D}_{\star, 2}(r)$ defined respectively in (2.3.77) and (2.3.76),

$$\begin{aligned} \mathcal{H}_{j, \star, \varepsilon}(r, \ell) &= \ell^d \mathcal{H}_{j, \star, \varepsilon}(r, \ell, \mathbb{R}^d \setminus B_{4\ell}) + r_*^d(0) \mathcal{H}_{j, \star, \varepsilon}(r, \ell, \mathbb{R}^d) \\ &\quad + \int_{B_{7\ell}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}} \left(r_*^{\frac{d}{2}}(0) \frac{\log(1 + \frac{|x|}{r})}{(|x| + r)^d} + \frac{r_*^{\frac{d}{2}+1}(x)}{(|x| + 1)^{d+1}} + g_{2r}(x) \right) dx, \end{aligned} \tag{2.3.124}$$

as well as for all open subsets \mathcal{U} of \mathbb{R}^d

$$\begin{aligned} \mathcal{H}_{j, \star, \varepsilon}(r, \ell, \mathcal{U}) &= \left(\int_{\mathcal{U}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right) \left(\int_{B_\ell(x)} r_*^d(0) r^d \frac{\log^2(1 + \frac{|y|}{r})}{(|y| + r)^{2d}} dy + \frac{r_*^d(y)}{(|y| + 1)^{2d}} + r^d g_{2r}^2(y) dy \right. \right. \\ &\quad \left. \left. + r_*^{d+\varepsilon}(x) \ell^\varepsilon \int_{\mathbb{R}^d \setminus B_\ell(x)} |y - x|^{-d-\varepsilon} \left(r_*^d(0) r^d \frac{\log^2(1 + \frac{|y|}{r})}{(|y| + r)^{2d}} dy + \frac{r_*^d(y)}{(|y| + 1)^{2d}} + r^d g_{2r}^2(y) \right) dy \right) dx \right)^{\frac{1}{2}}. \end{aligned}$$

The estimates (2.3.78) and (2.3.79) are unchanged and give respectively the $\ell^d \mathcal{D}_{\star,2}(r) \log^2(\frac{\sqrt{T}}{r})$ and $\ell^d \mathcal{D}_{\star,1}(r, \ell)$ contributions in the r.h.s of (2.3.123). We improve the estimate (2.3.93). We argue differently with the generic case $\ell \geq r_*(0)$ and the non-generic case $\ell < r_*(0)$.

Regime $\ell \geq r_*(0)$. We have from (2.3.94)

$$\int \left(\int_1^T \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \leq \left(\sum_{j=0}^{+\infty} \left(\int \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \right)^2.$$

We then split the argument into the far-field regime $|x| \geq 4\ell$ and the near-field regime $|x| \leq 4\ell$. For the far-field regime, we use directly the Cauchy-Schwarz inequality combined with (2.3.106) (applied for $\ell \geq \sqrt{T}$) and (2.3.118) to the effect of

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \\ & \leq \int \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)|^2 dy dt \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt \right) dx \\ & \stackrel{(2.3.106), (2.3.118)}{\lesssim} \ell^d \eta_{\varepsilon, \beta}^2(2^j) \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right) \int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt dx. \end{aligned} \quad (2.3.125)$$

It then gives the first term of (2.3.124) by dominating $\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla v^T(t, y)|^2 dy dt$ using the arguments in (2.3.121) and (2.3.122). For the near-field regime, we use (2.3.106) and (2.3.118) in form of

$$\left(\int_{2^j}^{2^{j+1}} \int_{B_{r_*(x)}(x)} |\nabla u(t, y)|^2 dy dt \right)^{\frac{1}{2}} \stackrel{(2.3.106), (2.3.118)}{\lesssim} \eta_{\varepsilon, \beta}(2^j) \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}}, \quad (2.3.126)$$

which has the effect of, combined with (2.3.95) and the Cauchy-Schwarz inequality

$$\begin{aligned} & \left(\int_{B_{4\ell}} \left(\int_{2^j}^{2^{j+1}} \int_{B_\ell(x)} |\nabla u(t, y)| |\nabla v^T(t, y)| dy dt \right)^2 dx \right)^{\frac{1}{2}} \\ & \stackrel{(2.3.95)}{\lesssim} \ell^{\frac{d}{2}} \int_{B_{7\ell}} \int_{B_{r_*(x)}(x)} \int_{2^j}^{2^{j+1}} |\nabla u(t, y)| |\nabla v^T(t, y)| dt dy dx \\ & \stackrel{(2.3.126)}{\lesssim} \ell^{\frac{d}{2}} \eta_{\varepsilon, \beta}(2^j) \int_{B_{7\ell}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}} \left(\int_{B_{r_*(x)}(x)} \int_{2^j}^{2^{j+1}} |\nabla v^T(t, y)| dt dy \right)^{\frac{1}{2}} dx, \end{aligned} \quad (2.3.127)$$

and yields the third term of (2.3.124) by using (2.3.73) (where we use $\eta_{4r_*(x)}(x) \lesssim r_*^d(x) (|x|+1)^{-2d}$).

Regime $\ell < r_*(0)$. For the non-generic case $\ell < r_*(0)$, we use (2.3.119), (2.3.121), (2.3.122) and we bound one ℓ^d by $r_*^d(0)$ which gives the second term of (2.3.124).

Step 3. Proof of (2.1.22). We have from the logarithm Sobolev inequality in form of (2.4.1), for all $p \in [1, +\infty)$

$$\langle |q_r(T) - \langle q_r(T) \rangle|^p \rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{2}} \left\langle \left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \leq p^{\frac{1}{2}} (\mathcal{I}_{\sqrt{T}}^1 + \mathcal{I}_{\sqrt{T}}^2), \quad (2.3.128)$$

with

$$\mathcal{I}_{\sqrt{T}}^1 := \left\langle \left(\int_1^{\sqrt{T}} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}},$$

and

$$\mathcal{I}_{\sqrt{T}}^2 := \left\langle \left(\int_{\sqrt{T}}^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(T)|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}.$$

We then treat separately the two terms above.

(i) In the regime $\ell < \sqrt{T}$ we use (2.3.114) combined with the Minkowski inequality in $L_{(\cdot)}^p(\Omega)$:

$$\begin{aligned} \mathcal{I}_{\sqrt{T}}^1 &\stackrel{(2.3.114)}{\lesssim} r^{-\frac{d}{2}} \left\langle \left(\int_1^{\sqrt{T}} \ell^d \pi(\ell) \left(1 + \mathcal{E}_{\star,\varepsilon}(r,\ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \mathcal{F}_{j,\star,\varepsilon}(r,\ell) \right)^2 \right) d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\stackrel{(2.1.6)}{\leq} r^{-\frac{d}{2}} \left(\int_1^{\sqrt{T}} \ell^{d-1-\beta} \left(1 + \langle \mathcal{E}_{\star,\varepsilon}^p(r,\ell) \rangle^{\frac{1}{p}} + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{F}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \right)^2 \right) d\ell \right)^{\frac{1}{2}} \\ &\lesssim r^{-\frac{d}{2}} \mu_{\beta}(T) \left(1 + \sup_{\ell \geq 1} \langle \mathcal{E}_{\star,\varepsilon}^p(r,\ell) \rangle^{\frac{1}{2p}} + \sup_{\ell \geq 1} \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{F}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \right), \end{aligned} \quad (2.3.129)$$

by the definition (2.1.18) of μ_{β} .

(ii) In the regime $\ell \geq \sqrt{T}$ we use (2.3.123), the Minkowski inequality in $L_{(\cdot)}^p(\Omega)$ and the fact that $r \leq \sqrt{T}$ in the last line:

$$\begin{aligned} \mathcal{I}_{\sqrt{T}}^2 &\lesssim \left\langle \left(\int_{\sqrt{T}}^{+\infty} \pi(\ell) \left(\mathcal{D}_{\star,1}(r,\ell) + \mathcal{D}_{\star,2}(r) \log^2\left(\frac{\sqrt{T}}{r}\right) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \mathcal{H}_{j,\star,\varepsilon}(r,\ell) \right)^2 \right) d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\stackrel{(2.1.6)}{\lesssim} \left(\int_{\sqrt{T}}^{+\infty} \ell^{-1-\beta} \left(\langle \mathcal{D}_{\star,1}^p(r,\ell) \rangle^{\frac{1}{p}} + \langle \mathcal{D}_{\star,2}^p(r) \rangle^{\frac{1}{p}} \log^2\left(\frac{\sqrt{T}}{r}\right) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{H}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \right)^2 \right) d\ell \right)^{\frac{1}{2}} \\ &\lesssim r^{-\frac{d}{2}} \mu_{\beta}(T) \left(\left(\int_1^{+\infty} \ell^{-1-\beta} \langle \mathcal{D}_{\star,1}^p(r,\ell\sqrt{T}) \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} + \log\left(\frac{\sqrt{T}}{r}\right) \sup_{r \geq 1} \langle \mathcal{D}_{\star,2}^p(r) \rangle^{\frac{1}{2p}} \right. \\ &\quad \left. + \sup_{\ell \geq 1} \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{H}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \right). \end{aligned} \quad (2.3.130)$$

It remains to show that, for all $\varepsilon \in (0, 1)$

$$\sup_{\ell \geq 1} \langle \mathcal{E}_{\star,\varepsilon}^p(r,\ell) \rangle^{\frac{1}{2p}} + \sup_{\ell \geq 1} \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{F}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \lesssim p^{2\frac{d+1}{\beta\wedge d} + \varepsilon} (1 + \log\left(\frac{\sqrt{T}}{r}\right)), \quad (2.3.131)$$

and

$$\begin{aligned} &\left(\int_1^{+\infty} \ell^{-1-\beta} \langle \mathcal{D}_{\star,1}^p(r,\ell\sqrt{T}) \rangle^{\frac{1}{p}} d\ell \right)^{\frac{1}{2}} + \log\left(\frac{\sqrt{T}}{r}\right) \sup_{r \geq 1} \langle \mathcal{D}_{\star,2}^p(r) \rangle^{\frac{1}{2p}} + \sup_{\ell \geq 1} \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{H}_{j,\star,\varepsilon}^{2p}(r,\ell) \rangle^{\frac{1}{2p}} \\ &\lesssim p^{2\frac{d+1}{\beta\wedge d} + \varepsilon} (1 + \log^2\left(\frac{\sqrt{T}}{r}\right)). \end{aligned} \quad (2.3.132)$$

This is done the same way as for (2.3.100) and (2.3.101) using in addition (2.3.108) and $\sum_{j=0}^{+\infty} 2^j \eta_{\varepsilon,\beta}(2^j) \lesssim_{\varepsilon} 1$. \square

Proof of (2.1.20). We use the same notations as in the previous proof. We split the proof into two parts, treating separately the two regimes $\ell < r$ and $\ell \geq r$. We start with preliminary estimates.

Step 0. Preliminary. First, we will use several times the assumption (2.1.19) in form of (2.3.56), that we restate here: for all $y \in \mathbb{R}^d$

$$|f_r(y)| \lesssim \frac{r}{(|y|+1)^d} \wedge \frac{1}{(|y|+1)^{d-1}}. \quad (2.3.133)$$

Second, we note that from (2.3.133), we have

$$\int |f_r(y)|^2 dy \lesssim \int_{\mathbb{B}_r} \frac{1}{(|y|+1)^{2(d-1)}} dy + r^2 \int_{\mathbb{R}^d \setminus \mathbb{B}_r} \frac{1}{(|y|+1)^{2d}} dy \lesssim \begin{cases} 1 & \text{for } d \geq 3, \\ \log(r+1) & \text{for } d = 2, \end{cases} \quad (2.3.134)$$

and

$$\int_{\mathbb{B}_\ell} |f_r(y)| dy \lesssim \mathbf{1}_{\ell < r} \int_{\mathbb{B}_\ell} \frac{1}{(|y|+1)^{d-1}} dy + \mathbf{1}_{\ell \geq r} \left(\int_{\mathbb{B}_r} \frac{1}{(|y|+1)^{d-1}} dy + r \int_{\mathbb{B}_\ell \setminus \mathbb{B}_r} \frac{1}{(|y|+1)^d} dy \right) \quad (2.3.135)$$

$$\lesssim \ell \mathbf{1}_{\ell < r} + r(1 + \log(\frac{\ell}{r} + 1)) \mathbf{1}_{\ell \geq r}. \quad (2.3.136)$$

Therefore, by arguing the same way as for (2.3.58), using the system (2.2.51) and the estimate (2.2.40) instead of (2.2.39) (since from (2.3.133), (2.2.38) holds) as well as the estimate (2.3.134), we have the following large-scale regularity estimate: for all $t \in (-\infty, r^2)$ and $x \in \mathbb{R}^d$

$$\int_{\mathbb{B}_{r_*(x)}(x)} |\nabla v^{r^2}(t, x)|^2 dy \lesssim r_*^d(0) \frac{r^2 \log^2(1+|x|)}{(|x|+1)^{2d}} \wedge \frac{1}{(|x|+1)^{2(d-1)}} + (1 + \log(r+1) \mathbf{1}_{d=2}) \eta_{4r_*(x)}(x) + f_{2r}^2(x). \quad (2.3.137)$$

Step 1. Regime $\ell < r$. Proof that for all $\varepsilon > 0$ and $\ell < r$

$$\int |\partial_{x,\ell}^{\text{fct}} q(r^2) \star f_r|^2 dx \leq \ell^d \left(r_*^d(0) (1 + \log(r+1) \mathbf{1}_{d=2}) + \ell^2 (1 + \log(\frac{r}{\ell} + 1) \mathbf{1}_{d=2}) + \mathcal{M}_{\star,1}(r, \ell) + \mathcal{M}_{\star,2}(r, \ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \mathcal{K}_{j,\star,\varepsilon}(r, \ell) \right)^2 \right), \quad (2.3.138)$$

with

$$\begin{aligned} \mathcal{M}_{\star,1}(r, \ell) := & 1 + \mathcal{N}_{\star,\varepsilon}(r, \ell, 1) + \left(\sum_{n=0}^{\lceil \log_2(3\ell) \rceil} 2^n \mathcal{N}_{\star,\varepsilon}^{\frac{1}{2}}(r, \ell, 2^{n+1}) \right)^2 + \ell^d \sum_{n=\lceil \log_2(\ell) \rceil}^{\lceil \log_2(r) \rceil} 2^{-n(d-2)} \mathcal{N}_{\star,\varepsilon}(r, \ell, 2^{n+1}) \\ & + r^2 \ell^d \sum_{n=\lceil \log_2(r) \rceil}^{+\infty} 2^{-nd} \mathcal{N}_{\star,\varepsilon}(r, \ell, 2^{n+1}) \end{aligned} \quad (2.3.139)$$

where for all $\rho > 0$, $\mathcal{N}_{\star,\varepsilon}(r, \ell, \rho) = r_*^d(0) \left(\int_1^{r^2} \mathcal{D}_\star^\varepsilon(t, \rho, 0) \eta_{\varepsilon,\beta}(t) dt \right)^2$ (we recall that \mathcal{D}_\star is defined in

(2.3.108)) and

$$\begin{aligned}
\mathcal{M}_{\star,2}(r, \ell) &:= r_*^d(0) \ell^d \int_{\mathbb{R}^d \setminus B_\ell} \frac{r^2 \log^2(1 + |x|)}{(|x| + 1)^{2d}} \wedge \frac{1}{(|x| + 1)^{2(d-1)}} dx \\
&\quad + (1 + \log(r) \mathbf{1}_{d=2}) \ell^d \int_{\mathbb{R}^d \setminus B_\ell} \frac{r_*^d(x)}{(|x| + 1)^{2d}} dx + \ell^d \int_{\mathbb{R}^d \setminus B_\ell} f_{2r}^2(x) dx \\
&\quad + r_*^d(0) \left(\int_{B_{7\ell}} \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} dx \right)^2 \\
&\quad + (1 + \log(r) \mathbf{1}_{d=2}) \left(\int_{B_{7\ell}} \frac{r_*^{\frac{d}{2}+1}(x)}{(|x| + 1)^{d+1}} dx \right)^2 + \left(\int_{B_{7\ell}} f_{2r}(x) dx \right)^2. \tag{2.3.140}
\end{aligned}$$

as well as

$$\begin{aligned}
\mathcal{K}_{j,\star,\varepsilon}(r, \ell) &:= \mathcal{K}_{j,\star,\varepsilon}(r, \ell, \mathbb{R}^d \setminus B_{4\ell}) + r_*^{\frac{d}{2}}(0) \mathcal{K}_{j,\star,\varepsilon}(r, \ell, \mathbb{R}^d) \\
&\quad + \int_{B_{7\ell}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}} \left(r_*^{\frac{d}{2}}(0) \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} \right. \\
&\quad \left. + (1 + \log^{\frac{1}{2}}(r) \mathbf{1}_{d=2}) \frac{r_*^{\frac{d}{2}+1}(x)}{(|x| + 1)^d} + f_{2r}(x) \right) dx, \tag{2.3.141}
\end{aligned}$$

with for all open set \mathcal{U} of \mathbb{R}^d

$$\begin{aligned}
&\mathcal{K}_{j,\star,\varepsilon}(r, \ell, \mathcal{U}) \\
&:= \left(\int_{\mathcal{U}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}} \left(\int_{B_\ell(x)} r_*^d(0) \frac{r^2 \log^2(1 + |y|)}{(|y| + 1)^{2d}} \wedge \frac{1}{(|y| + 1)^{2(d-1)}} \right. \right. \\
&\quad \left. \left. + (1 + \log(r) \mathbf{1}_{d=2}) \frac{r_*^d(y)}{(|y| + 1)^{2d}} + f_{2r}^2(y) \right. \right. \\
&\quad \left. \left. + r_*^{d+\varepsilon}(x) \ell^\varepsilon \int_{\mathbb{R}^d \setminus B_\ell(x)} |y - x|^{-d-\varepsilon} \left(r_*^d(0) \frac{r^2 \log^2(1 + |y|)}{(|y| + 1)^{2d}} \wedge \frac{1}{(|y| + 1)^{2(d-1)}} \right. \right. \right. \\
&\quad \left. \left. \left. + (1 + \log(r) \mathbf{1}_{d=2}) \frac{r_*^d(y)}{(|y| + 1)^{2d}} + f_{2r}^2(y) \right) dy \right) dx \right)^{\frac{1}{2}}.
\end{aligned}$$

We split this step into two parts. The first part is devoted to the control of the first fifth r.h.s terms of (2.2.50).

Substep 1.1. Proof that for all $\ell \in [1, r)$

$$\int \left(\int_{B_\ell(x)} |f_r(y)| dy \right)^2 dx + \int \left(\int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(t, y) dt \right| |f_r(y)| dy \right)^2 dx \lesssim \ell^d \mathcal{M}_{\star,1}(r, \ell), \tag{2.3.142}$$

and

$$\begin{aligned}
&\int \mathcal{T}_{x,\ell}^2(\eta_1)(0) dx + \int \left(\sum_{n=0}^{\lceil \log_2(3\ell) \rceil} 2^n \mathcal{T}_{x,\ell}(\eta_{2^{n+1}})(0) \right)^2 dx \\
&\quad + \int \left(\int_{\mathbb{R}^d \setminus B_{2\ell}} \left(\int_{B_\ell(y)} |f_r(z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \right)^2 dx \lesssim \ell^d (1 + \ell^2 (1 + \log(\frac{r}{\ell} + 1) \mathbf{1}_{d=2})), \tag{2.3.143}
\end{aligned}$$

as well as

$$\int \left(\int_{B_\ell(x)} |\nabla v^{r^2}(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx + \int \mathcal{G}_{r, \ell}^2(x) dx \lesssim \ell^d (r_*^d(0)(1 + \log(r+1)) \mathbf{1}_{d=2}) + \mathcal{M}_{\star, 2}(r, \ell). \quad (2.3.144)$$

Proof of (2.3.142). We only give the argument for the second l.h.s term, the first one is dominated the same way. We split the argument into the far-field regime $|x| \geq 4\ell$ and the near-field regime $|x| < 4\ell$. For the near-field regime, we make use of a dyadic decomposition and the estimate (2.3.133) in form of $|f_r(y)| \lesssim (|y| + 1)^{-d+1}$ to get

$$\begin{aligned} \int_{B_{2\ell}} \left(\int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(t, y) dt \right| |f_r(y)| dy \right)^2 dx &\lesssim \ell^d \left(\int_{B_{3\ell}} \left| \int_0^{r^2} \nabla u(t, y) dt \right| |f_r(y)| dy \right)^2 \\ &\lesssim \ell^d \left(\left(\int_{B_1} \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy \right)^2 \right. \\ &\quad \left. + \left(\sum_{n=0}^{\lceil \log_2(3\ell) \rceil} \int_{B_{2^{n+1}} \setminus B_{2^n}} \left| \int_0^{r^2} \nabla u(t, y) dt \right| |f_r(y)| dy \right)^2 \right) \\ &\stackrel{(2.3.133)}{\lesssim} \ell^d \left(\left(\int_{B_1} \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy \right)^2 \right. \\ &\quad \left. + \left(\sum_{n=0}^{\lceil \log_2(3\ell) \rceil} 2^n \int_{B_{2^{n+1}}} \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy \right)^2 \right), \end{aligned}$$

which gives the second and third r.h.s terms of (2.3.139) by applying the localized energy estimate (2.2.32) to the equation (2.1.10) and the estimate (2.3.106) (applied for both $\ell = 1$ and $\ell = 2^{n+1}$) in form of

$$\begin{aligned} \left(\int_{B_1} \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy \right)^2 &\lesssim \left(\int_{B_1} \left| \int_0^1 \nabla u(t, y) dt \right| dy \right)^2 + \left(\int_{B_1} \left| \int_1^{r^2} \nabla u(t, y) dt \right| dy \right)^2 \\ &\lesssim 1 + r_*^d(0) \left(\int_1^{r^2} \mathcal{D}_\star^\varepsilon(t, 1, 0) \eta_{\varepsilon, \beta}(t) dt \right)^2, \end{aligned}$$

and for all $n \in \mathbb{N}$

$$\int_{B_{2^{n+1}}} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 dy \leq 1 + r^d(0) \left(\int_1^{r^2} \mathcal{D}_\star^\varepsilon(t, 2^{n+1}, 0) \eta_{\varepsilon, \beta}(t) dt \right)^2. \quad (2.3.145)$$

For the far-field contribution, we first make use of the Jensen inequality combined with the inequality $\int_{\mathbb{R}^d \setminus B_{2\ell}} \int_{B_\ell(x)} dx \lesssim \ell^d \int_{\mathbb{R}^d \setminus B_\ell}$ and the decomposition $\int_{\mathbb{R}^d \setminus B_\ell} = \int_{B_r \setminus B_\ell} + \int_{\mathbb{R}^d \setminus B_r}$ to write

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{2\ell}} \left(\int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(t, y) dt \right| |f_r(y)| dy \right)^2 dx &\lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_\ell} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy \\ &= \ell^{2d} \left(\int_{B_r \setminus B_\ell} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy \right. \\ &\quad \left. + \int_{\mathbb{R}^d \setminus B_r} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy \right). \end{aligned} \quad (2.3.146)$$

For the first r.h.s term of (2.3.146), we make use of a dyadic decomposition and the estimate (2.3.133) in form of $|f_r(y)| \lesssim (|y| + 1)^{-d+1}$ to obtain

$$\begin{aligned} \int_{B_r \setminus B_\ell} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy &= \sum_{\lfloor \log_2(\ell) \rfloor}^{\lfloor \log_2(r) \rfloor} \int_{B_{2^{n+1}} \setminus B_{2^n}} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy \\ &\stackrel{(2.3.133)}{\lesssim} \sum_{\lfloor \log_2(\ell) \rfloor}^{\lfloor \log_2(r) \rfloor} 2^{-n(d-2)} \int_{B_{2^{n+1}}} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 dy, \end{aligned}$$

which gives the fourth r.h.s term of (2.3.139) using (2.3.145). For the second r.h.s term of (2.3.146), we make use of a dyadic decomposition and (2.3.133) in form of $|f_r(y)| \lesssim r(|y| + 1)^{-d}$ to get

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_r} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy &= \sum_{n=\lfloor \log_2(r) \rfloor}^{+\infty} \int_{B_{2^{n+1}} \setminus B_{2^n}} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 |f_r(y)|^2 dy \\ &\stackrel{(2.3.133)}{\lesssim} r^2 \sum_{n=\lfloor \log_2(r) \rfloor}^{+\infty} 2^{-nd} \int_{B_{2^{n+1}}} \left| \int_0^{r^2} \nabla u(t, y) dt \right|^2 dy, \end{aligned}$$

which finally gives fifth r.h.s of (2.3.139) using once more (2.3.145).

Proof of (2.3.143). The estimate of the two first l.h.s terms is an immediate consequence of the Minkowski inequality in $L^2(\mathbb{R}^d)$ and the estimate (2.3.89) applied with $r = \sqrt{s}$, which provides the first r.h.s contribution in (2.3.143). For the third l.h.s term, we can use previous estimates. To this aim, we use the Jensen inequality with measure $\mathcal{T}_{x,\ell}(\eta_\ell)(y)dy$ and (2.3.91) as well as $\int \mathcal{T}_{x,\ell}(\eta_\ell)(y)dx \lesssim \ell^d$ (which may be obtained by changing the role of x and y in the proof of (2.3.91)) combined with the inequality $\int_{\mathbb{R}^d \setminus B_{2\ell}} \int_{B_\ell(y)} dy \lesssim \int_{\mathbb{R}^d \setminus B_\ell}$ to obtain

$$\begin{aligned} \int \left(\int_{\mathbb{R}^d \setminus B_{2\ell}} \left(\int_{B_\ell(y)} |f_r(z)|^2 dz \right)^{\frac{1}{2}} \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy \right)^2 dx &\stackrel{(2.3.91)}{\lesssim} \ell^d \int \int_{\mathbb{R}^d \setminus B_{2\ell}} \int_{B_\ell(y)} |f_r(z)|^2 dz \mathcal{T}_{x,\ell}(\eta_\ell)(y) dy dx \\ &\lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_{2\ell}} \int_{B_\ell(y)} |f_r(z)|^2 dz dy \\ &\lesssim \ell^{2d} \int_{\mathbb{R}^d \setminus B_\ell} |f_r(y)|^2 dy. \end{aligned}$$

We then get (2.3.143) using (2.3.133) in form of

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_\ell} |f_r(y)|^2 dy &\lesssim \int_{B_r \setminus B_\ell} (|y| + 1)^{-2(d-1)} dy + r^2 \int_{\mathbb{R}^d \setminus B_r} (|y| + 1)^{-2d} dy \\ &\lesssim \ell^{2-d} (1 + \log(\frac{r}{\ell} + 1) \mathbf{1}_{d=2}), \end{aligned} \tag{2.3.147}$$

Proof of (2.3.144). We start with the first l.h.s term. We distinguish between the generic case $\ell \geq r_*(0)$ and the non-generic case $\ell < r_*(0)$.

Regime $\ell \geq r_*(0)$. We split the integral into the far-field contribution $|x| \geq 4\ell$ and the near-field contribution $|x| < 4\ell$. For the far-field contribution, we make use of the estimates (2.3.85)

(applied with $T = r^2$) combined with (2.3.137) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^{r^2}(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx \\ & \lesssim \ell^{2d} \left(r_*^d(0) \int_{\mathbb{R}^d \setminus B_\ell} \frac{r^2 \log^2(1 + |x|)}{(|x| + 1)^{2d}} \wedge \frac{1}{(|x| + 1)^{2(d-1)}} dx \right. \\ & \quad \left. + (1 + \log(r + 1) \mathbf{1}_{d=2}) \int_{\mathbb{R}^d \setminus B_\ell} \eta_{4r_*(x)}(x) dx + \int_{\mathbb{R}^d \setminus B_\ell} f_{2r}^2(x) dx \right), \end{aligned}$$

which gives the first term in (2.3.140) using that $\eta_{4r_*(x)}(x) \lesssim r_*^d(x)(|x| + 1)^{-2d}$. For the near-field contribution, we make use of the estimate (2.3.88) combined with (2.3.137) to obtain

$$\begin{aligned} & \left(\int_{B_{4\ell}} \left(\int_{B_\ell(x)} |\nabla v^{r^2}(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx \right)^{\frac{1}{2}} \\ & \lesssim \ell^{\frac{d}{2}} \left(r_*^{\frac{d}{2}}(0) \int_{B_{7\ell}} \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} dx \right. \\ & \quad \left. + (1 + \log^{\frac{1}{2}}(r + 1) \mathbf{1}_{d=2}) \int_{B_{7\ell}} \eta_{4r_*(x)}^{\frac{1}{2}}(x) dx + \int_{B_{7\ell}} f_{2r}(x) dx \right), \end{aligned}$$

which gives the second term of (2.3.140) using that $\eta_{4r_*(x)}^{\frac{1}{2}}(x) \lesssim r_*^{\frac{d}{2}+1}(x)(|x| + 1)^{-d-1}$.

Regime $\ell < r_*(0)$. We use the Cauchy-Schwarz inequality, the identity $\int f_{B_\ell(x)} = f$ and the localized energy estimate (2.2.32) to get

$$\begin{aligned} & \int \left(\int_{B_\ell(x)} |\nabla v^{r^2}(1, y)| \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right| \right) dy \right)^2 dx \\ & \lesssim \int \int_{B_\ell(x)} |\nabla v^{r^2}(1, y)|^2 dy \int_{B_\ell(x)} \left(1 + \left| \int_0^1 \nabla u(s, y) ds \right|^2 \right) dy dx \\ & \lesssim \ell^{2d} \int |\nabla v^{r^2}(1, y)|^2 dy \lesssim \ell^d r_*^d(0) \int |\nabla v^{r^2}(1, y)|^2 dy, \end{aligned}$$

and we conclude with the plain energy estimate in the equation (2.2.51) (for which a proof is identical as the one for (2.3.9)) combined with (2.3.139)

$$\int |\nabla v^{r^2}(1, y)|^2 dy \lesssim \int |f_r(y)|^2 dy \lesssim 1 + \log(r + 1) \mathbf{1}_{d=2},$$

which gives the first contribution in (2.3.138). For the second l.h.s term of (2.3.144), we argue as in (2.3.90) and (2.3.92).

Substep 1.2. Proof that for all $\ell \in [1, r)$

$$\int \left(\int_{B_\ell(x)} \int_1^{r^2} |\nabla u(t, y)| |\nabla v^{r^2}(t, y)| dt dy \right)^2 dx \lesssim \ell^d \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{K}_{j, *, \varepsilon}(r, \ell) \right)^2. \quad (2.3.148)$$

We argue differently with the generic case $\ell \geq r_*(0)$ and the non-generic case $\ell < r_*(0)$ and we use several previous estimates.

Regime $\ell \geq r_*(0)$. We split the argument between the far-field regime $|x| \geq 4\ell$ and the near-field regime $|x| < 4\ell$. For the far-field contribution, we make use of (2.3.94), (2.3.125) and (2.3.121) as well as (2.3.122) combined with (2.3.137) to obtain

$$\int_{\mathbb{R}^d \setminus B_{4\ell}} \left(\int_{B_\ell(x)} \int_1^{r^2} |\nabla u(t, y)| |\nabla v^{r^2}(t, y)| dt dy \right)^2 dx \lesssim \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{K}_{j, \star, \varepsilon}(r, \ell, \mathbb{R}^d \setminus B_{4\ell}) \right)^2.$$

For the near-field contribution, we make use of (2.3.127) combined with (2.3.137) which leads to

$$\begin{aligned} & \int_{B_{4\ell}} \left(\int_{B_\ell(x)} \int_1^{r^2} |\nabla u(t, y)| |\nabla v^{r^2}(t, y)| dt dy \right)^2 dx \\ & \lesssim \left(\int_{B_{7\ell}} \left(\int_{2^j}^{2^{j+1}} \mathcal{D}_\star^{2\varepsilon}(t, \ell, x) dt \right)^{\frac{1}{2}} \left(r_\star^{\frac{d}{2}}(0) \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} \right. \right. \\ & \left. \left. + (1 + \log^{\frac{1}{2}}(r + 1) \mathbb{1}_{d=2}) \frac{r_\star^{\frac{d}{2}+1}(x)}{(|x| + 1)^d} + f_{2r}(x) \right) dx \right)^2, \end{aligned}$$

where we used that $\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) < +\infty$.

Regime $\ell < r_*(0)$. We make use of (2.3.119), (2.3.121) and (2.3.122) combined with (2.3.137) as well as we bound one ℓ^d by $r_\star^d(0)$ to obtain

$$\int \left(\int_{B_\ell(x)} \int_1^{r^2} |\nabla u(t, y)| |\nabla v^{r^2}(t, y)| dt dy \right)^2 dx \lesssim r_\star^d(0) \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{K}_{j, \star, \varepsilon}(r, \ell, \mathbb{R}^d) \right)^2.$$

Step 2. Regime $\ell \geq r$. Proof that for all $\varepsilon > 0$ and $\ell \geq r$

$$\int |\partial_{x, \ell}^{\text{fct}} q(r^2) \star f_r|^2 dx \leq \ell^d \left(r^2 + \mathcal{M}_{\star, 3}(r) + \mathcal{M}_{\star, 2}(r, \ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon, \beta}(2^j) \mathcal{K}_{j, \star, \varepsilon}(r, \ell) \right)^2 \right), \quad (2.3.149)$$

with

$$\mathcal{M}_{\star, 3}(r) = \left(\int_1^{r^2} s^{-\frac{1}{2}} (1 + \log(\frac{r}{\sqrt{s}})) \left(\int_{B_{\sqrt{s}}} r_\star^d(y) dy + \int_{\mathbb{R}^d \setminus B_1} r_\star^d(y) (\sqrt{s}y) g_{\sqrt{2}}(y) dy \right)^{\frac{1}{2}} ds \right)^2.$$

The estimates (2.3.144) and (2.3.148) are unchanged. We provide the arguments for the two first and the fourth r.h.s terms of (2.2.50) and prove that

$$\int \left(\int_1^{r^2} s^{-\frac{1}{2}} \mathcal{T}_{x, \ell}(\eta_{\sqrt{s}})(0) ds \right)^2 dx \lesssim \ell^d r^2. \quad (2.3.150)$$

and

$$\int \left(\int_{B_\ell(x)} |f_r(y)| dy \right)^2 dx + \int \left(\int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(s, y) ds \right| |f_r(y)| dy \right)^2 dx \lesssim \mathcal{M}_{\star, 3}(r). \quad (2.3.151)$$

First, (2.3.150) follows from the Minkowski inequality in $L^2(\mathbb{R}^d)$ and (2.3.89) applied with $r = \sqrt{s}$ (noticing that the evaluation at 0 plays no role in the estimate). Secondly, using the Minkowski inequality in $L^2(\mathbb{R}^d)$ and the assumption (2.1.19), we have

$$\begin{aligned} & \int \left(\int_{B_\ell(x)} |f_r(y)| dy \right)^2 dx + \int \left(\int_{B_\ell(x)} \left| \int_0^{r^2} \nabla u(s, y) ds \right| |f_r(y)| dy \right)^2 dx \\ & \lesssim \ell^d \left(\int_1^{r^2} s^{-1} \int |y| g_{\sqrt{s}}(y) \left(1 + \left| \int_0^{r^2} \nabla u(t, y) dt \right| \right) dy \right)^2, \end{aligned}$$

which gives (2.3.151) using that $|y| g_{\sqrt{s}}(y) \lesssim s^{\frac{1}{2}} g_{\sqrt{2s}}(y)$, the estimates (2.3.80) and (2.3.81) with (T, r) replaced by (r^2, s) to get

$$\begin{aligned} \int_1^{r^2} s^{-1} \int |y| g_{\sqrt{s}}(y) \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy & \lesssim \int_1^{r^2} s^{-\frac{1}{2}} \int g_{\sqrt{2s}}(y) \left| \int_0^{r^2} \nabla u(t, y) dt \right| dy \\ & \lesssim \int_1^{r^2} s^{-\frac{1}{2}} (1 + \log(\frac{r}{\sqrt{s}})) \left(\int_{B_{\sqrt{s}}} r_*^d(y) dy \right. \\ & \quad \left. + \int_{\mathbb{R}^d \setminus B_1} r_*^d(y) (\sqrt{s}y) g_{\sqrt{2}}(y) dy \right)^{\frac{1}{2}} ds. \end{aligned} \quad (2.3.152)$$

Step 3. Proof of (2.1.20). We have from the logarithm Sobolev inequality in form of (2.4.1), for all $p \in [1, +\infty)$

$$\langle |q(r^2) \star f_r - \langle q(r^2) \star f_r \rangle|^p \rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{2}} \left\langle \left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(r^2) \star f_r|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \leq p^{\frac{1}{2}} (\mathcal{I}_r^1 + \mathcal{I}_r^2), \quad (2.3.153)$$

with

$$\mathcal{I}_r^1 := \left\langle \left(\int_1^r \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(r^2) \star f_r|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}},$$

and

$$\mathcal{I}_r^2 := \left\langle \left(\int_r^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{\text{fct}} q_r(r^2) \star f_r|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}.$$

We then treat separately the two terms above.

- (i) In the regime $\ell < r$ we use (2.3.138) combined with the Minkowski inequality in $L^p_{(\cdot)}(\Omega)$ and the moment bound (2.3.102) of r_* as well as $\left(\int_1^r \ell^{1-\beta} (1 + \log(\frac{r}{\ell} + 1)) \mathbb{1}_{d=2} d\ell \right)^{\frac{1}{2}} \lesssim \chi_{d,\beta}(r)$ and

$1 + \log(r+1)\mathbf{1}_{d=2} \leq \chi_{d,\beta}(r)$ (where we recall that $\chi_{d,\beta}(r)$ is defined in (2.1.21)):

$$\begin{aligned} \mathcal{I}_r^1 &\leq \left\langle \left(\int_1^r \ell^{-1-\beta} \left(r_*^d(0)(1 + \log^2(r+1)\mathbf{1}_{d=2}) + \ell^2(1 + \log(\frac{r}{\ell} + 1)\mathbf{1}_{d=2}) \right. \right. \right. \\ &\quad \left. \left. \left. + \mathcal{M}_{\star,1}(r, \ell) + \mathcal{M}_{\star,2}(r, \ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \mathcal{K}_{j,\star,\varepsilon}(r, \ell) \right)^2 \right) \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\lesssim p^{\frac{d}{\beta \wedge d}} \left(1 + \chi_{d,\beta}(r) \right) + \left(\int_1^r \ell^{-1-\beta} \left(\langle \mathcal{M}_{\star,1}^p(r, \ell) \rangle^{\frac{1}{p}} + \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} \right. \right. \\ &\quad \left. \left. + \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} \right) d\ell \right)^{\frac{1}{2}}. \end{aligned}$$

(ii) In the regime $\ell \geq r$ we use (2.3.149) combined with the Minkowski inequality $L_{\langle \cdot \rangle}^p(\Omega)$:

$$\begin{aligned} \mathcal{I}_r^2 &\leq \left\langle \left(\int_r^{+\infty} \ell^{-1-\beta} \left(r^2 + \mathcal{M}_{\star,3}(r) + \mathcal{M}_{\star,2}(r, \ell) + \left(\sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \mathcal{K}_{j,\star,\varepsilon}(r, \ell) \right)^2 \right) d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\lesssim r^{1-\frac{\beta}{2}} + \left(\int_r^{+\infty} \ell^{-1-\beta} \left(\langle \mathcal{M}_{\star,3}^p(r, \ell) \rangle^{\frac{1}{p}} + \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} + \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} \right) d\ell \right)^{\frac{1}{2}}. \end{aligned}$$

It remains to prove that for all $\varepsilon \in (0, 1)$

$$\int_1^r \ell^{-1-\beta} \left(\langle \mathcal{M}_{\star,1}^p(r, \ell) \rangle^{\frac{1}{p}} + \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} + \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} \right) d\ell \lesssim p^{\frac{d+2}{\beta \wedge d} + \varepsilon} \chi_{d,\beta}^2(r), \quad (2.3.154)$$

and

$$\int_r^{+\infty} \ell^{-1-\beta} \left(\langle \mathcal{M}_{\star,3}^p(r, \ell) \rangle^{\frac{1}{p}} + \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} + \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} \right) d\ell \lesssim p^{\frac{d+2}{\beta \wedge d} + \varepsilon} \chi_{d,\beta}^2(r). \quad (2.3.155)$$

□

We start with (2.3.154). First, using the moment bound (2.3.108) of \mathcal{D}_\star and the definition of $\eta_{\varepsilon,\beta}$ in (2.3.107) as well as (2.3.102), we have for all $\rho > 0$, $\langle \mathcal{N}_{\star,\varepsilon}^p(r, \ell, \rho) \rangle^{\frac{1}{p}} \lesssim p^{\frac{d}{\beta \wedge d} + \varepsilon(1+2\frac{d+2}{\beta \wedge d})}$. Therefore, by making use of the triangle inequality, we get

$$\begin{aligned} \int_1^r \ell^{-1-\beta} \langle \mathcal{M}_{\star,1}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell &\lesssim 1 + p^{\frac{d}{\beta \wedge d} + \varepsilon(1+2\frac{d+2}{\beta \wedge d})} \int_1^r \ell^{-1-\beta} (1 + \ell^2(1 + \log(\frac{r}{\ell} + 1)\mathbf{1}_{d=2}) + \ell^d r^{2-d}) d\ell \\ &\lesssim p^{\frac{d}{\beta \wedge d} + \varepsilon(1+2\frac{d+2}{\beta \wedge d})} \chi_{d,\beta}^2(r). \end{aligned}$$

Secondly, from the triangle inequality, the moment bounds (2.3.102) on r_* and by splitting the first integral in the r.h.s of (2.3.140) in form of $\int_{\mathbb{R}^d \setminus B_\ell} = \int_{B_r \setminus B_\ell} + \int_{\mathbb{R}^d \setminus B_r}$ as well as (2.3.147) and (2.3.136),

we have for all $\ell < r$

$$\begin{aligned}
 \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} &\lesssim p^{\frac{d}{\beta \wedge d}} \left(\ell^d \int_{\mathbb{B}_r \setminus \mathbb{B}_\ell} \frac{1}{(|x|+1)^{2(d-1)}} dx + \ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_r} \frac{r^2 \log^2(1+|x|)}{(|x|+1)^{2d}} \wedge \frac{1}{(|x|+1)^{2(d-1)}} dx \right) \\
 &\quad + (1 + \log^2(r+1) \mathbf{1}_{d=2}) \ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} \frac{1}{(|x|+1)^{2d}} dx + \ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} |f_{2r}(y)|^2 dx + \left(\int_{\mathbb{B}_{7\ell}} \frac{1}{(|x|+1)^{d-1}} dx \right)^2 \\
 &\quad + (1 + \log^2(r+1) \mathbf{1}_{d=2}) p^{\frac{2}{\beta \wedge d}} \left(\int_{\mathbb{B}_{7\ell}} \frac{1}{(|x|+1)^{d+1}} dx \right)^2 + \left(\int_{\mathbb{B}_{7\ell}} f_{2r}(x) dx \right)^2 \\
 &\stackrel{(2.3.136), (2.3.147)}{\lesssim} p^{\frac{d}{\beta \wedge d}} \left(\ell^2 (1 + \log(\frac{r}{\ell} + 1) \mathbf{1}_{d=2}) + (1 + \ell^d r^{2-d} + p^{\frac{2}{\beta \wedge d}}) (1 + \log^2(r+1) \mathbf{1}_{d=2}) \right),
 \end{aligned}$$

which provides

$$\int_1^r \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell \lesssim p^{\frac{d+2}{\beta \wedge d}} \chi_{d,\beta}^2(r).$$

Finally, using the same decomposition as before and in addition the moment bound (2.3.108) of \mathcal{D}_\star we get

$$\int_1^r \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta} (2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell \lesssim p^{\frac{d+2}{\beta \wedge d} + \varepsilon(1+2\frac{d}{\beta \wedge d})} \chi_{d,\beta}^2(r),$$

which concludes the proof of (2.3.154) by replacing $\varepsilon(1+2\frac{d}{\beta \wedge d})$ by ε . We now turn to the proof of (2.3.155). First, using the moment bound (2.3.102) on r_\star we have

$$\int_r^{+\infty} \ell^{-1-\beta} \langle \mathcal{M}_{\star,3}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell \lesssim p^{\frac{d}{\beta \wedge d}} r^{2-\beta} \lesssim p^{\frac{d}{\beta \wedge d}} \chi_{d,\beta}^2(r).$$

Secondly, using the triangle inequality

$$\begin{aligned}
 &\langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} \\
 &\lesssim p^{\frac{d}{\beta \wedge d}} \left(\ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} \frac{r^2 \log^2(1+|x|)}{(|x|+1)^{2d}} \wedge \frac{1}{(|x|+1)^{2(d-1)}} dx + (1 + \log(r+1) \mathbf{1}_{d=2}) \ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} \frac{1}{(|x|+1)^{2d}} dx \right. \\
 &\quad \left. + \ell^d \int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} |f_{2r}(x)|^2 dx \right. \\
 &\quad \left. + \left(\int_{\mathbb{B}_{7\ell}} \frac{r \log(1+|x|)}{(|x|+1)^d} \wedge \frac{1}{(|x|+1)^{d-1}} dx \right)^2 + (1 + \log(r+1) \mathbf{1}_{d=2}) p^{\frac{2}{\beta \wedge d}} \left(\int_{\mathbb{B}_{7\ell}} \frac{1}{(|x|+1)^{d+1}} dx \right)^2 \right. \\
 &\quad \left. + \left(\int_{\mathbb{B}_{7\ell}} |f_{2r}(x)| dx \right)^2 \right) \\
 &\lesssim p^{\frac{d+2}{\beta \wedge d}} \left(\int_{\mathbb{R}^d \setminus \mathbb{B}_\ell} \frac{r^2 \log^2(1+|x|)}{(|x|+1)^{2d}} \wedge \frac{1}{(|x|+1)^{2(d-1)}} dx + \left(\int_{\mathbb{B}_{7\ell}} \frac{r \log(1+|x|)}{(|x|+1)^d} \wedge \frac{1}{(|x|+1)^{d-1}} dx \right)^2 \right. \\
 &\quad \left. + 1 + \log(r+1) \mathbf{1}_{d=2} + r^2 \right). \tag{2.3.156}
 \end{aligned}$$

We then argue differently, depending on the regime of β and d :

(i) For $\beta > 2$, we use

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus B_\ell} \frac{r^2 \log^2(1 + |x|)}{(|x| + 1)^{2d}} \wedge \frac{1}{(|x| + 1)^{2(d-1)}} dx + \left(\int_{B_{7\ell}} \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} dx \right)^2 \\ & \leq r^2 \int_{\mathbb{R}^d \setminus B_\ell} \frac{\log^2(1 + |x|)}{(|x| + 1)^{2d}} dx + r^2 \left(\int_{B_{7\ell}} \frac{\log(1 + |x|)}{(|x| + 1)^d} dx \right)^2 \\ & \lesssim r^2(1 + \log^2(\ell)), \end{aligned} \quad (2.3.157)$$

to deduce, combined with (2.3.156)

$$\begin{aligned} \int_r^{+\infty} \ell^{-1-\beta} \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell & \lesssim \int_r^{+\infty} \ell^{-1-\beta} (r^2 \log^2(\ell) + 1 + r^2 + \log(r+1) \mathbf{1}_{d=2}) d\ell \\ & \lesssim r^{-\beta} (r^2 + 1 + \log(r+1) \mathbf{1}_{d=2} + \log^2(r+1)) \\ & \lesssim 1, \end{aligned} \quad (2.3.158)$$

where we used in the last line that $\beta > 2$.

(ii) For $\beta \leq 2$ and $d > 2$, we use

$$\int_{\mathbb{R}^d \setminus B_\ell} \frac{r^2 \log^2(1 + |x|)}{(|x| + 1)^{2d}} \wedge \frac{1}{(|x| + 1)^{2(d-1)}} dx + \left(\int_{B_{7\ell}} \frac{r \log(1 + |x|)}{(|x| + 1)^d} \wedge \frac{1}{(|x| + 1)^{d-1}} dx \right)^2 \lesssim r^2 \log^2(\ell) \wedge \ell^2,$$

which yields combined with (2.3.156) and a dyadic decomposition

$$\begin{aligned} \int_r^{+\infty} \ell^{-1-\beta} \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell & \lesssim \sum_{n=0}^{\lceil \log_2(r) \rceil} \int_{2^n r}^{2^{n+1} r} \ell^{-1-\beta} (\ell^2 + 1 + r^2 + \log(r+1) \mathbf{1}_{d=2}) d\ell \\ & \quad + \sum_{n=\lceil \log_2(r) \rceil}^{+\infty} \int_{2^n r}^{2^{n+1} r} \ell^{-1-\beta} (r^2 \log^2(\ell) + 1 + r^2 + \log(r+1) \mathbf{1}_{d=2}) d\ell \\ & \lesssim \sum_{n=0}^{\lceil \log_2(r) \rceil} \left((2^n r)^{2-\beta} \mathbf{1}_{\beta < 2} + \mathbf{1}_{\beta=2} + r^{-\beta} 2^{-n\beta} (r^2 + \log(r+1) \mathbf{1}_{d=2}) \right) \\ & \quad + \sum_{n=\lceil \log_2(r) \rceil}^{+\infty} 2^{-n\beta} (r^{2-\beta} \log^2(2^{n+1} r) + 1 + r^{2-\beta} + \log(r+1) \mathbf{1}_{d=2}) \\ & \lesssim \chi_{\beta,d}(r). \end{aligned}$$

(iii) Finally, for $\beta \leq 2$ and $d = 2$, we combine (2.3.156), (2.3.157) and (2.3.158) to obtain

$$\int_r^{+\infty} \ell^{-1-\beta} \langle \mathcal{M}_{\star,2}^p(r, \ell) \rangle^{\frac{1}{p}} d\ell \lesssim r^{2-\beta} (1 + \log^2(r+1)) + 1 + \log(r+1).$$

To conclude, using the same decomposition as before and in addition the moment bound (2.3.108) of \mathcal{D}_\star we get

$$\int_r^{+\infty} \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \eta_{\varepsilon,\beta}(2^j) \langle \mathcal{K}_{j,\star,\varepsilon}^p(r, \ell) \rangle^{\frac{1}{p}} \lesssim p^{\frac{d+2}{\beta \wedge d} + \varepsilon(1+2\frac{d}{\beta \wedge d})} \chi_{d,\beta}^2(r).$$

Proof of Corollary 1: Decay of the semigroup.

We provide the argument for ∇u , the decay of u is proved the same way. We apply Lemma 3 and we make use of the Minkowski inequality in $L^p_{(\cdot)}(\Omega)$ and the stationarity of $q_r(T, \cdot)$ to the effect of: for all $p \in [1, +\infty)$, $T \geq 4$ and $R \geq \sqrt{T}$

$$\begin{aligned} \left\langle \left(\int \eta_{\sqrt{2}R}\left(\frac{y}{c}\right) |\nabla u(T, y)|^2 dy \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} &\lesssim \frac{1}{T} \left\langle \left(\int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^{\sqrt{t}} \left(\frac{r}{\sqrt{t}}\right)^{\frac{d}{2}} \int \eta_{2R}\left(\frac{y}{c}\right) |q_r(t, y) - \langle q_r(t, y) \rangle| dy dr dt \right)^p \right\rangle^{\frac{1}{p}} \\ &\lesssim \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^{\sqrt{t}} \left(\frac{r}{\sqrt{t}}\right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt. \end{aligned}$$

Then, we split the integral over $[0, \sqrt{t}]$ into the contributions $r \leq 1$ and $1 \leq r \leq \sqrt{t}$:

(i) For $r \leq 1$ we use (2.2.43):

$$\begin{aligned} \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_0^{\frac{1}{2}} \left(\frac{r}{\sqrt{t}}\right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt &\stackrel{(2.2.43)}{\lesssim} p^{\frac{1}{2}} \eta_{\beta}^{\frac{d}{2}} T^{-1-\frac{d}{4}} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_0^1 \left(r^{\frac{d}{2}} + \log\left(\frac{\sqrt{t}}{r}\right) \right) dr dt \\ &\lesssim p^{\frac{1}{2}} \eta_{\beta}^{\frac{d}{2}} T^{-\frac{3}{2}-\frac{d}{4}} \log(T). \end{aligned}$$

(ii) For $1 \leq r \leq \sqrt{t}$ we use Theorem 10 and the change of variable $r \mapsto \frac{\sqrt{t}}{r}$: for all $\alpha < \frac{1}{\frac{1}{2} + 2\frac{d+1}{\beta v d}}$

$$\begin{aligned} \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_1^{\sqrt{t}} \left(\frac{r}{\sqrt{t}}\right)^{\frac{d}{2}} \langle |q_r(t) - \langle q_r(t) \rangle|^p \rangle^{\frac{1}{p}} dr dt &\stackrel{(2.1.17)}{\lesssim} p^{\frac{1}{\alpha}} T^{-1-\frac{d}{4}} \mu_{\beta}(T) \int_{\frac{T}{4}}^{\frac{T}{2}} \frac{1}{\sqrt{t}} \int_1^{\sqrt{t}} \left(1 + \log^2\left(\frac{\sqrt{t}}{r}\right) \right) dr dt \\ &\leq p^{\frac{1}{\alpha}} \eta_{\beta}(T) \int_1^{\sqrt{\frac{T}{2}}} r^{-2} (1 + \log^2(r)) dr \\ &\lesssim p^{\frac{1}{\alpha}} \eta_{\beta}(T), \end{aligned}$$

where η_{β} is defined in (2.1.24). It concludes the argument for (2.1.23).

The estimate (2.1.25) is a direct consequence of (2.1.23) and the stationarity of ∇u : for all $x \in \mathbb{R}^d$ and $T \geq 1$

$$\langle |\nabla u(T, x)|^2 \rangle = \left\langle \int \eta_{\sqrt{2}R}\left(\frac{y}{c}\right) |\nabla u(T, y)|^2 dy \right\rangle \stackrel{(2.1.23)}{\lesssim} T^{-1} \eta_{\beta}^2(T) \langle \mathcal{C}_{*,d,\lambda,\beta}^2 \rangle \lesssim T^{-1} \eta_{\beta}^2(T).$$

Proof of Corollary 2 : Bounds on the flux and gradient of correctors.

We split the proof into two steps. The first one gives a rigorous proof of the formula (2.1.31). The second step prove (2.1.32).

Step 1. We prove the two following integral formulas

$$\nabla \phi(\cdot) = \int_0^{+\infty} \nabla u(t, \cdot) dt, \tag{2.3.159}$$

and for all $T \geq 1$

$$\nabla \phi_T(\cdot) = \int_0^{+\infty} e^{-\frac{t}{T}} \nabla u(t, \cdot) dt. \tag{2.3.160}$$

We first note that the r.h.s of (2.3.159) is well defined as a random variable in $L^2_{\langle \cdot \rangle}(\Omega)$, since from (2.1.25) and (2.2.32) we have for all $x \in \mathbb{R}^d$

$$\left\langle \left| \int_0^{+\infty} \nabla u(t, x) dt \right|^2 \right\rangle^{\frac{1}{2}} \leq \left\langle \left| \int_0^1 \nabla u(t, x) dt \right|^2 \right\rangle^{\frac{1}{2}} + \int_1^{+\infty} \left\langle |\nabla u(t, x)|^2 \right\rangle^{\frac{1}{2}} dt \lesssim 1.$$

We only provide the argument for (2.3.159), (2.3.160) follows the same way. To this aim, we prove that there exists a potential $\zeta \in L^2(\Omega \times \mathbb{R}^d)$, sub-linear at infinity, such that $\int_0^{+\infty} \nabla u(t, \cdot) dt = \nabla \zeta$ and solving $-\nabla \cdot a(\nabla \zeta + e) = 0$ in \mathbb{R}^d in the distributional sense. By uniqueness of $\nabla \phi$ defined by (2.1.8), it shall imply (2.3.159).

Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be supported in B_R for some $R > 0$ and let $\sqrt{T} > R$. We have by testing (2.1.10) with ψ and integrating in time from 0 to T

$$\int u(T, y) \psi(y) dy + \int \nabla \psi(y) \cdot a(y) e dy + \int \nabla \psi(y) \cdot a(y) \int_0^T \nabla u(s, y) ds dy = 0. \quad (2.3.161)$$

We now check that each term of (2.3.161) pass to the limit, almost surely, as $T \uparrow +\infty$. For the first l.h.s term of (2.3.161), we use the triangle inequality followed by the Poincaré inequality, (2.1.23) and (2.1.25):

$$\begin{aligned} \left\langle \int_{B_R} |u(T, y)|^2 dy \right\rangle^{\frac{1}{2}} &\leq \left\langle \int_{B_R} \left| u(T, y) - \int_{B_R} u(T, z) dz \right|^2 dy \right\rangle^{\frac{1}{2}} + \left\langle \left| \int_{B_{\sqrt{T}}} u(T, y) dy \right|^2 \right\rangle^{\frac{1}{2}} \\ &\quad + \left\langle \left| \int_{B_R} u(T, y) dy - \int_{B_{\sqrt{T}}} u(T, y) dy \right|^2 \right\rangle^{\frac{1}{2}} \\ &\lesssim R \left\langle \int_{B_R} |\nabla u(T, y)|^2 dy \right\rangle^{\frac{1}{2}} + \left\langle \int_{B_{\sqrt{T}}} |u(T, y)|^2 dy \right\rangle^{\frac{1}{2}} \\ &\quad + \left\langle \left| \int_{B_R} u(T, y) dy - \int_{B_{\sqrt{T}}} u(T, y) dy \right|^2 \right\rangle^{\frac{1}{2}} \\ &\stackrel{(2.1.23), (2.1.25)}{\lesssim} \left(\frac{R}{\sqrt{T}} + 1 \right) \eta_\beta(T) + \left\langle \left| \int_{B_R} u(T, y) dy - \int_{B_{\sqrt{T}}} u(T, y) dy \right|^2 \right\rangle^{\frac{1}{2}}. \end{aligned} \quad (2.3.162)$$

From the fundamental calculus theorem, the stationarity of ∇u and the application of (2.1.25), we also have

$$\begin{aligned} \left\langle \left| \int_{B_R} u(t, y) dy - \int_{B_{\sqrt{T}}} u(t, y) dy \right|^2 \right\rangle^{\frac{1}{2}} &= \left\langle \left| \int_R^{\sqrt{T}} \int_{B_1} \nabla u(t, \tau z) \cdot z dz d\tau \right|^2 \right\rangle^{\frac{1}{2}} \\ &\stackrel{(2.1.25)}{\lesssim} \left(1 - \frac{R}{\sqrt{T}} \right) \eta_\beta(T) \leq \eta_\beta(T). \end{aligned}$$

Hence, since $R < \sqrt{T}$, (2.3.162) turns into

$$\left\langle \int_{B_R} |u(T, y)|^2 dy \right\rangle^{\frac{1}{2}} \lesssim \eta_\beta(T),$$

and yields

$$\begin{aligned} \left\langle \left| \int u(T, y) \psi(y) dy \right|^2 \right\rangle^{\frac{1}{2}} &\leq R^{\frac{d}{2}} \|\psi\|_{L^\infty(\mathbb{R}^d)} \left\langle \int_{B_R} |u(T, y)|^2 dy \right\rangle^{\frac{1}{2}} \\ &\stackrel{(2.3.162)}{\lesssim} R^{\frac{d}{2}} \|\psi\|_{L^\infty(\mathbb{R}^d)} \eta_\beta(T). \end{aligned}$$

Consequently, we have in particular, almost surely

$$\lim_{T \rightarrow +\infty} \int u(T, y) \psi(y) dy = 0.$$

For the second l.h.s term of (2.3.161), we have directly using (2.1.25):

$$\begin{aligned} \left\langle \left| \int \nabla \psi(y) \cdot a(y) \int_T^{+\infty} \nabla u(s, y) ds dy \right|^2 \right\rangle^{\frac{1}{2}} &\leq \|\nabla \psi\|_{L^\infty(\mathbb{R}^d)} \int_{B_R} \int_T^{+\infty} \langle |\nabla u(s, y)|^2 \rangle^{\frac{1}{2}} ds dy \\ &\stackrel{(2.1.25)}{\lesssim} R^d \|\nabla \psi\|_{L^\infty(\mathbb{R}^d)} T^{\frac{1}{2}} \eta_\beta(T), \end{aligned}$$

with $\lim_{T \rightarrow +\infty} T^{\frac{1}{2}} \eta_\beta(T) = 0$, which yields almost surely

$$\lim_{T \rightarrow +\infty} \int \nabla \psi(y) \cdot a \int_0^T \nabla u(s, y) ds dy = \int \nabla \psi(y) \cdot a \int_0^{+\infty} \nabla u(s, y) ds dy.$$

To conclude, we can pass to the limit as $T \uparrow +\infty$ in (2.3.161) and obtain almost surely

$$\int \nabla \psi(y) \cdot a(y) e dy + \int \nabla \psi(y) \cdot a(y) \int_0^{+\infty} \nabla u(s, y) ds dy = 0. \quad (2.3.163)$$

Since $\int_0^{+\infty} \nabla u(t, \cdot) dt$ is curl free and belongs to $L^2(\Omega \times \mathbb{R}^d)$, there exists a potential $\zeta \in L^2(\Omega \times \mathbb{R}^d)$ such that $\int_0^{+\infty} \nabla u(t, \cdot) dt = \nabla \zeta$ and (2.3.163) takes the form

$$\int \nabla \psi(y) \cdot a(y) e dy + \int \nabla \psi(y) \cdot a(y) \nabla \zeta = 0,$$

which means that $-\nabla \cdot a(\nabla \zeta + e) = 0$ in the distributional sense in \mathbb{R}^d . Since $\int_0^{+\infty} \nabla u(t, \cdot) dt$ has finite second moment, it is well known that ζ own the sub-linear property. By the uniqueness of $\nabla \phi$ defined by (2.1.8), this concludes the argument for (2.3.159).

Step 2. We prove (2.1.32) and we split the proof into three steps. For the rest of the proof, we fix $\alpha < \frac{1}{\frac{1}{2} + 2\frac{d+1}{\beta v d}}$.

Substep 2.1. We start with the control of the flux and we only treat the control of $|q_r - \langle q_r \rangle|$, the control of $|(q_T)_r - \langle (q_T)_r \rangle|$ will be obtained the same way, using (2.3.160) instead of (2.3.159). We have directly by using the triangle inequality and Theorem 10 with $T = r^2$, for all $p \in [1, +\infty)$

$$\begin{aligned} \langle |q_r - \langle q_r \rangle|^p \rangle^{\frac{1}{p}} &\leq \langle |q_r - (q(r^2))_r|^p \rangle^{\frac{1}{p}} + \langle |(q(r^2))_r - \langle (q(r^2))_r \rangle|^p \rangle^{\frac{1}{p}} + \langle |q_r - (q(r^2))_r| \rangle \\ &\stackrel{(2.1.17)}{\lesssim} \langle |q_r - (q(r^2))_r|^p \rangle^{\frac{1}{p}} + \pi_*^{-\frac{1}{2}}(r) p^{\frac{1}{\alpha}}. \end{aligned}$$

It remains to control the first r.h.s term of the above inequality. To this aim, we write by dominating the Gaussian kernel by the exponential kernel and using the Minkowski inequality in $L^p_{(\cdot)}(\Omega)$ as well as the Jensen inequality

$$\begin{aligned} \langle |q_r - (q(r^2))_r|^p \rangle^{\frac{1}{p}} &\leq \left\langle \left(\int g_r(x) \left| \int_{r^2}^{+\infty} \nabla u(t, x) dt \right|^p \right)^{\frac{1}{p}} \right\rangle \\ &\leq \int_{r^2}^{+\infty} \left\langle \left(\int g_r(x) |\nabla u(t, y)| dy \right)^p \right\rangle^{\frac{1}{p}} dt \\ &\leq \int_{r^2}^{+\infty} \left\langle \left(\int \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} dt. \end{aligned} \quad (2.3.164)$$

It remains to control the space integral of the above inequality. We apply Lemma 5 using (2.1.23), $f \equiv 1$ and $g : t \in \mathbb{R}^+ \mapsto \eta_\beta(t)$ as well as the moment bound (2.4.3) of r_* to obtain, for all $r \geq 1$

$$\left\langle \left(\int_{B_r} |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{2} \frac{d}{\beta \wedge d} + \frac{1}{\alpha}} t^{-\frac{1}{2}} \eta_\beta(t). \quad (2.3.165)$$

Consequently, for all $t \geq r^2$

$$\begin{aligned} \left\langle \left(\int \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} &\leq \left\langle \left(\int_{B_r} |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} + \left\langle \left(\int_{\mathbb{R}^d \setminus B_r} \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\stackrel{(2.3.165)}{\lesssim} p^{\frac{1}{2} \frac{d}{\beta \wedge d} + \frac{1}{\alpha}} t^{-\frac{1}{2}} \eta_\beta(t) + \left\langle \left(\int_{\mathbb{R}^d \setminus B_r} \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}. \end{aligned} \quad (2.3.166)$$

For the second r.h.s term of the previous estimate, we decompose $\mathbb{R}^d \setminus B_r$ into the family of annuli $(B_{(n+1)r} \setminus B_{nr})_{n \geq 1}$ to obtain, with (2.3.165)

$$\begin{aligned} \left\langle \left(\int_{\mathbb{R}^d \setminus B_r} \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{+\infty} \left\langle \left(\int_{B_{(n+1)r} \setminus B_{nr}} \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{2}{p}} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{+\infty} e^{-n} n^d \left\langle \left(\int_{B_{(n+1)r}} |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{2}{p}} \right)^{\frac{1}{2}} \\ &\stackrel{(2.3.165)}{\lesssim} p^{\frac{1}{2} \frac{d}{\beta \wedge d} + \frac{1}{\alpha}} t^{-\frac{1}{2}} \eta_\beta(t). \end{aligned} \quad (2.3.167)$$

We conclude by plugging the two above inequalities into (2.3.164) and the fact $\int_{r^2}^{+\infty} t^{-\frac{1}{2}} \eta_\beta(t) dt \lesssim \pi_\star^{-\frac{1}{2}}(r)$. The bound on $(q_T)_r$ is obtained the same way since from (2.3.160) we have

$$(q_T)_r = \int_0^{+\infty} e^{-\frac{t}{T}} q_r(t, \cdot) dt.$$

Substep 2.2. We prove the control of $|\nabla \phi_r|$. We first notice that by integrating the equation (2.1.10) in time and using that, by stationarity, $\nabla \cdot \langle q \rangle = 0$, we have for all $t \geq 0$

$$u(t, \cdot) = \nabla \cdot (q(t, \cdot) - \langle q(t, \cdot) \rangle).$$

From the definition (2.1.14), we deduce that

$$(\nabla\phi(r^2))_r = \left(\nabla \left(\nabla \cdot \int_0^{r^2} (q(s, \cdot) - \langle q(s, \cdot) \rangle) ds \right) \right)_r. \quad (2.3.168)$$

By noticing that, from the semigroup property $g_r = g_{\frac{1}{\sqrt{2}}r} \star g_{\frac{1}{\sqrt{2}}r}$ we have for all $f \in H_{loc}^2(\mathbb{R}^d)$

$$|(\nabla^2 f)_r| = |(\nabla^2 f_{\frac{1}{\sqrt{2}}r})_{\frac{1}{\sqrt{2}}r}| \lesssim \frac{1}{r^2} |f_{\frac{1}{\sqrt{2}}r}|_{\frac{1}{\sqrt{2}}r}, \quad (2.3.169)$$

we deduce, from (2.3.168), the stationarity of q_r , (2.1.17) and (2.4.2) that, for all $p \geq 1$

$$\begin{aligned} \langle |(\nabla\phi(r^2))_r|^p \rangle^{\frac{1}{p}} &= \left\langle \left| \left(\nabla \left(\nabla \cdot \int_0^{r^2} (q(s, \cdot) - \langle q(s, \cdot) \rangle) ds \right) \right)_r \right|^p \right\rangle^{\frac{1}{p}} \\ &\stackrel{(2.3.169)}{\lesssim} \frac{1}{r^2} \int_0^{r^2} \langle |q_{\frac{1}{\sqrt{2}}r}(s) - \langle q_{\frac{1}{\sqrt{2}}r}(s) \rangle|^p \rangle^{\frac{1}{p}} ds \\ &\stackrel{(2.1.17), (2.4.2)}{\lesssim} p^{\frac{1}{\alpha}} r^{-\frac{d}{2}} \int_0^{r^2} \mu_\beta(s) (1 + \log^2(\frac{\sqrt{s}}{r})) ds \\ &\lesssim \pi_*^{-\frac{1}{2}}(r) p^{\frac{1}{\alpha}}. \end{aligned} \quad (2.3.170)$$

We finally deduce, from (2.3.159), (2.3.164), (2.3.166), (2.3.167) and (2.3.170), that for all $p \geq 1$

$$\begin{aligned} \langle |\nabla\phi_r|^p \rangle^{\frac{1}{p}} &\leq \langle |\nabla\phi_r - (\nabla\phi(r^2))_r|^p \rangle^{\frac{1}{p}} + \langle |(\nabla\phi(r^2))_r|^p \rangle^{\frac{1}{p}} \\ &\stackrel{(2.3.159), (2.3.170)}{\leq} \int_{r^2}^{+\infty} \left\langle \left(\int \eta_r(x) |\nabla u(t, x)|^2 dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} dt + \pi_*^{-\frac{1}{2}}(r) p^{\frac{1}{\alpha}} \\ &\stackrel{(2.3.164), (2.3.166), (2.3.167)}{\lesssim} \pi_*^{-\frac{1}{2}}(r) (p^{\frac{1}{2} \frac{d}{\beta \nabla d} + \frac{1}{\alpha}} + p^{\frac{1}{\alpha}}). \end{aligned}$$

Substep 2.3. We prove the control of $|\nabla(\sigma_T)_r|$ and $|\nabla\sigma_r|$. Let $i, j, k \in \llbracket 1, d \rrbracket$. Using the equation (2.1.16), we note that $(\sigma_{T, i, j, k})_r$ solves

$$\frac{1}{T} (\sigma_{T, i, j, k})_r - \Delta (\sigma_{T, i, j, k})_r = (e_k \cdot q_{e_i, T} - e_j \cdot q_{e_i, T}) \star (\partial_j g_r - \partial_k g_r).$$

Therefore, we may express $\nabla(\sigma_{T, i, j, k})_r$ with help of the Green function G_T of the massive Laplace operator $\frac{1}{T} - \Delta$ in \mathbb{R}^d

$$\nabla(\sigma_{T, i, j, k})_r = \nabla G_T \star ((e_k \cdot q_{e_i, T} - e_j \cdot q_{e_i, T}) \star (\partial_j g_r - \partial_k g_r)).$$

Then, using that there exists a constant C which depends on d such that

$$G_T = C \int_0^{+\infty} e^{-\frac{s}{T}} g_{\sqrt{s}} ds,$$

and by noticing that from the stationarity of $(e_k \cdot q_{e_i, T} - e_j \cdot q_{e_i, T})$ we have

$$\begin{aligned} &\left\langle \int_0^{+\infty} \int \int e^{-\frac{s}{T}} |\nabla g_{\sqrt{s}}(x)| |e_k \cdot q_{e_i, T}(x-y) - e_j \cdot q_{e_i, T}(x-y)| |\partial_j g_r(y) - \partial_k g_r(y)| dy dx ds \right\rangle \\ &\lesssim \langle |e_k \cdot q_{e_i, T} - e_j \cdot q_{e_i, T}| \rangle \|\partial_j g_r - \partial_k g_r\|_{L^1(\mathbb{R}^d)} \int_0^{+\infty} e^{-\frac{s}{T}} \|\nabla g_{\sqrt{s}}\|_{L^1(\mathbb{R}^d)} ds \\ &\lesssim \langle |e_k \cdot q_{e_i, T} - e_j \cdot q_{e_i, T}| \rangle \|\partial_j g_r - \partial_k g_r\|_{L^1(\mathbb{R}^d)} \int_0^{+\infty} s^{-\frac{1}{2}} e^{-\frac{s}{T}} ds < +\infty, \end{aligned}$$

we deduce from the Fubini theorem combined with the semigroup property $g_r \star g_{\sqrt{s}} = g_{\sqrt{s+r^2}}$ that, almost surely

$$\begin{aligned} \nabla(\sigma_{T,i,j,k})_r &= \int_0^{+\infty} e^{-\frac{s}{T}} (e_k \cdot q_{e_i,T} - e_j \cdot q_{e_i,T}) \star (\nabla(\partial_j g_{\sqrt{s+r^2}} - \partial_k g_{\sqrt{s+r^2}})) ds \\ &= \int_0^{+\infty} e^{-\frac{s}{T}} (\nabla(\partial_j(e_k \cdot q_{e_i,T}) - \partial_k(e_j \cdot q_{e_i,T})))_{\sqrt{s+r^2}} ds. \end{aligned}$$

Consequently, by making once again use of the stationarity of $q_{e_i,T}$ as well as (2.3.169) and (2.1.32) proved for $(q_{e_i,T})_r$ in substep 2.1, we obtain for all $p \geq 1$

$$\begin{aligned} \langle |\nabla(\sigma_{T,i,j,k})_r|^p \rangle^{\frac{1}{p}} &= C \left\langle \left| \int_0^{+\infty} e^{-\frac{s}{T}} (\nabla(\partial_j(e_k \cdot q_{e_i,T}) - \partial_k(e_j \cdot q_{e_i,T})))_{\sqrt{s+r^2}} ds \right|^p \right\rangle^{\frac{1}{p}} \\ &\stackrel{(2.3.169)}{\lesssim} \int_0^{+\infty} \frac{1}{s+r^2} \langle |(q_{e_i,T})_{\frac{1}{\sqrt{2}}\sqrt{s+r^2}} - \langle (q_{e_i,T})_{\frac{1}{\sqrt{2}}\sqrt{s+r^2}} \rangle|^p \rangle^{\frac{1}{p}} ds \\ &\lesssim p^{\frac{1}{2}+\frac{1}{\alpha}} \int_0^{+\infty} \frac{1}{s+r^2} \pi_*^{-\frac{1}{2}}(\sqrt{s+r^2}) ds \\ &\lesssim p^{\frac{1}{2}+\frac{1}{\alpha}} \pi_*^{-\frac{1}{2}}(r). \end{aligned} \tag{2.3.171}$$

which concludes (2.1.32) for $|\nabla(\sigma_T)_r|$ using Lemma 9. The bound on $|\nabla\sigma_r|$ then follows from the fact that $\nabla\sigma_T$ tends to $\nabla\sigma$ in $L^2_{(\cdot)}(\Omega)$ as $T \uparrow +\infty$ (see for instance [80, Proof of Proposition 2]), and thus also almost surely up to a subsequence, combined with (2.3.171) and Fatou's lemma.

Proof of Corollary 3 : Growth of the extended corrector (ϕ, σ)

We only give the arguments for ϕ . For the bound on σ , we may rewrite averages $\int \nabla\sigma_{i,j,k}(x) \cdot g(x) dx$ where g is assumed to be a gradient field; i.e., $g = \nabla\theta$ for some potential θ , using the second line of (2.1.15) to obtain

$$\int \nabla\sigma_{i,j,k}(x) \cdot g(x) dx = \int q(x) \cdot Sg(x) dx,$$

with $S =: e_j \otimes e_k - e_k \otimes e_j$ and $q = ((q_{e_i})_j)_{i,j}$. Since the averaging field Sg inherits the decay properties of g , we then conclude using Theorem 10 for q and the arguments for ϕ .

Let $\alpha < \frac{1}{\frac{1}{2} + \frac{\frac{5}{2}d+2}{\beta\sqrt{d}}}$ and $p \in [1, +\infty)$. On the one hand, by the triangle inequality combined with the Poincaré inequality in the space $L^2(\mathbb{R}^d, g_1 dx)$ and the stationarity property of $\nabla\phi$, we have for all $x \in \mathbb{R}^d$

$$\langle (|\phi - \phi_1(0)|^2)_1^{\frac{p}{2}}(x) \rangle^{\frac{1}{p}} \lesssim \langle |\phi_1(x) - \phi_1(0)|^p \rangle^{\frac{1}{p}} + \langle (|\nabla\phi|^2)_1^{\frac{p}{2}} \rangle^{\frac{1}{p}}. \tag{2.3.172}$$

Then, using the formula (2.3.159), the energy estimate (2.2.32) applied to the equation (2.1.10) in form of $\int \left| \int_0^1 \nabla u(t, x) dt \right|^2 g_1(x) dx \lesssim 1$, the Minkowski inequality in $L^p(\Omega, L^2(\mathbb{R}^d, g_1 dx))$ and the estimates (2.3.166) as well as (2.3.167) applied with $r = 1$ (after dominating the Gaussian kernel g_1

by the exponential kernel η_1), we have

$$\begin{aligned} \langle (|\nabla\phi|^2)_1^{\frac{p}{2}} \rangle^{\frac{1}{p}} &\stackrel{(2.3.159)}{=} \left\langle \left(\int_0^{+\infty} \left| \int_0^{\cdot} \nabla u(t, x) dt \right|^2 g_1(x) dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\leq \left\langle \left(\int_0^1 \left| \int_0^{\cdot} \nabla u(t, x) dt \right|^2 g_1(x) dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} + \int_1^{+\infty} \left\langle \left(\int_0^{\cdot} |\nabla u(t, x)|^2 g_1(x) dx \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} dt \\ &\stackrel{(2.3.166), (2.3.167)}{\lesssim} 1 + p^{\frac{1}{\alpha}}. \end{aligned}$$

Therefore, the estimate (2.3.172) turns into

$$\langle (|\phi - \phi_1(0)|^2)_1^{\frac{p}{2}}(x) \rangle^{\frac{1}{p}} \lesssim \langle |\phi_1(x) - \phi_1(0)|^p \rangle^{\frac{1}{p}} + p^{\frac{1}{\alpha}} + 1. \quad (2.3.173)$$

On the other hand, setting $R = |x| \geq 1$, we have by the triangle inequality

$$\langle |\phi_1(x) - \phi_1(0)|^p \rangle^{\frac{1}{p}} \leq \langle |\phi_R(x) - \phi_1(x)|^p \rangle^{\frac{1}{p}} + \langle |\phi_R(x) - \phi_R(0)|^p \rangle^{\frac{1}{p}} + \langle |\phi_R(0) - \phi_1(0)|^p \rangle^{\frac{1}{p}}. \quad (2.3.174)$$

The second r.h.s term of (2.3.174) is estimated via the fundamental calculus theorem combined with the Minkowski inequality in $L^p_{\langle \cdot \rangle}(\Omega)$, the stationary property of $\nabla\phi$ and (2.1.32)

$$\langle |\phi_R(x) - \phi_R(0)|^p \rangle^{\frac{1}{p}} = |x| \left\langle \left(\int_0^1 \frac{x}{|x|} \cdot \left(\int \nabla\phi(y + \tau x) g_R(y) dy \right) dx \right)^p \right\rangle^{\frac{1}{p}} \stackrel{(2.1.32)}{\lesssim} p^{\frac{1}{\alpha}} |x| \pi_{\star}^{-\frac{1}{2}}(R). \quad (2.3.175)$$

By stationarity, the first and the third r.h.s term of (2.3.174) are estimated the same way and we bound the third term in two different ways, depending on the regimes of β and d :

- (i) We consider the regimes $\beta < 2$, $\beta = d = 2$ and $\beta > 2$, $d > 2$. Our main tool here are the moment bounds on the gradients of correctors (2.1.32). We write by the fundamental calculus theorem

$$\phi_R(0) - \phi_1(0) = \int_1^R \frac{\partial}{\partial \tau} \phi_{\tau}(0) d\tau = \int_1^R \int \nabla\phi(x) \cdot g_{\tau}(x) \frac{x}{\tau} dx d\tau. \quad (2.3.176)$$

Then, by noticing that from the semigroup property of Gaussian field in form of $g_{\tau} = g_{\frac{\tau}{\sqrt{2}}} \star g_{\frac{\tau}{\sqrt{2}}}$, writing $\frac{x}{\tau} = \frac{y}{\tau} + \frac{x-y}{\tau}$ and applying the Fubini theorem, we have for all $\tau \in [1, R]$

$$\begin{aligned} \int \nabla\phi(x) \cdot g_{\tau}(x) \frac{x}{\tau} dx &= \int \int \nabla\phi(x) \cdot \frac{x}{\tau} g_{\frac{\tau}{\sqrt{2}}}(y) g_{\frac{\tau}{\sqrt{2}}}(x-y) dy dx \\ &= \int \int \nabla\phi(x) \cdot \frac{y}{\tau} g_{\frac{\tau}{\sqrt{2}}}(y) g_{\frac{\tau}{\sqrt{2}}}(x-y) dy dx \\ &\quad + \int \int \nabla\phi(x) \cdot \frac{x-y}{\tau} g_{\frac{\tau}{\sqrt{2}}}(y) g_{\frac{\tau}{\sqrt{2}}}(x-y) dy dx \\ &= 2 \int \frac{y}{\tau} g_{\frac{\tau}{\sqrt{2}}}(y) \cdot \int \nabla\phi(x) g_{\frac{\tau}{\sqrt{2}}}(x-y) dx dy \\ &= 2 \int \frac{y}{\tau} g_{\frac{\tau}{\sqrt{2}}}(y) \cdot \nabla\phi_{\frac{\tau}{\sqrt{2}}}(-y) dy, \end{aligned}$$

we deduce from the Minkowski inequality in $L^p_{\langle \cdot \rangle}(\Omega)$, the stationarity property of $\nabla\phi$, (2.3.176) and (2.1.32)

$$\langle |\phi_R(0) - \phi_1(0)|^p \rangle^{\frac{1}{p}} \lesssim \int_1^R \langle |\nabla\phi_{\frac{\tau}{\sqrt{2}}}|^p \rangle^{\frac{1}{p}} d\tau \stackrel{(2.1.32)}{\lesssim} p^{\frac{1}{\alpha}} \int_1^R \pi_{\star}^{-\frac{1}{2}}\left(\frac{\tau}{\sqrt{2}}\right) d\tau \lesssim p^{\frac{1}{\alpha}} \xi_{d,\beta}(R), \quad (2.3.177)$$

where we recall that $\xi_{d,\beta}$ is defined in (2.1.34). The combination of (2.3.173), (2.3.174), (2.3.175), (2.3.177) and Lemma 9 gives the desired estimate (2.1.33).

- (ii) We consider the regimes $\beta = 2, d > 2$ and $\beta > 2, d = 2$. Our main tools here are the fluctuation estimate (2.1.20) and the decay (2.1.23) of ∇u . We claim that

$$\phi_R(0) - \phi_1(0) = - \int \nabla H(x) \cdot \nabla \phi(x) dx, \quad (2.3.178)$$

with

$$H := \int_1^{R^2} g_{\sqrt{\tau}}(\cdot) d\tau.$$

Indeed, using that for all $\tau > 0$, $\partial_\tau g_{\sqrt{\tau}} = \Delta g_{\sqrt{\tau}}$, we have

$$\phi_R(0) - \phi_1(0) = \int (g_R(x) - g_1(x)) \phi(x) dx = \int \int_1^{R^2} \partial_\tau g_{\sqrt{\tau}}(x) d\tau \phi(x) dx = \int \int_1^{R^2} \Delta g_{\sqrt{\tau}}(x) d\tau \phi(x) dx,$$

and thus (2.3.178) follows from an integration by parts (which is justified by the sub-linearity property of the corrector ϕ). Now, using the formula (2.3.159), we get

$$\begin{aligned} \phi_R(0) - \phi_1(0) &= - \int \nabla H(x) \cdot \int_0^{+\infty} \nabla u(s, x) ds dx \\ &= - \int \nabla H(x) \cdot \nabla \phi(R^2, x) dx - \int \nabla H(x) \cdot \int_{R^2}^{+\infty} \nabla u(s, x) ds dx, \end{aligned} \quad (2.3.179)$$

where we recall that the time dependant corrector $\phi(\cdot, \cdot)$ is defined in (2.1.14). For the first r.h.s term of (2.3.179), we note that ∇H satisfies the assumption (2.1.19), therefore from Theorem 10 we have for all $p \in [1, +\infty)$

$$\left\langle \left| \int \nabla H(x) \cdot \nabla \phi(R^2, x) dx \right|^p \right\rangle^{\frac{1}{p}} \lesssim p^{\frac{1}{\alpha}} \log^{\frac{1}{2}}(R+2).$$

For the second r.h.s term of (2.3.179), we make use of the combination of (2.3.165), (2.3.166) and (2.3.167) as well as the following bound on ∇H : for all $x \in \mathbb{R}^d$

$$\begin{aligned} |\nabla H(x)| &\lesssim \int_1^{R^2} |x| \tau^{-\frac{d}{2}-1} e^{-\frac{|x|}{\tau}} d\tau \leq |x| e^{-\frac{|x|^2}{2R^2}} \int_1^{R^2} \tau^{-\frac{d}{2}-1} e^{-\frac{|x|^2}{2\tau}} d\tau \\ &\lesssim (|x|+1)^{1-d} e^{-\frac{|x|^2}{2R^2}} \lesssim R g_{2R}(x), \end{aligned}$$

to obtain for all $p \in [1, +\infty)$

$$\begin{aligned} \left\langle \left| \int \nabla H(x) \cdot \int_{R^2}^{+\infty} \nabla u(s, x) ds dx \right|^p \right\rangle^{\frac{1}{p}} &\leq R \int_{R^2}^{+\infty} \left\langle \left| \int g_{2R}(x) |\nabla u(s, y)|^2 \right|^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\lesssim p^{\frac{1}{\alpha}} R \int_{R^2}^{+\infty} s^{-\frac{1}{2}} \eta_\beta(s) ds \lesssim 1. \end{aligned}$$

Proof of Corollary 5: Sub-systematic error

We split the proof into two steps.

Step 1. Proof of (2.1.37). Using the two representation formulas (2.3.159) and (2.3.160), we have for all $n > \frac{\beta \wedge d}{4}$

$$\langle |\nabla \phi_{e_i, T}^n - \nabla \phi_{e_i}|^2 \rangle^{\frac{1}{2}} = \left\langle \left| \int_0^{+\infty} (1 - \exp_n(\tau, T)) \nabla u(\tau) d\tau \right|^2 \right\rangle^{\frac{1}{2}}, \quad (2.3.180)$$

where $(\exp_n(\cdot, T))_{n \in \mathbb{N}}$ is the Richardson extrapolation of $\exp_1(\cdot, T) := e^{-\frac{\cdot}{T}}$. Note that the extrapolation has the effect that for all $\tau \geq 0$

$$|1 - \exp_n(\tau, T)| \lesssim_n \left(\frac{\tau}{T}\right)^n \wedge 1 \text{ and } \left| \frac{\partial}{\partial \tau} \exp_n(\tau, T) \right| \lesssim_n \frac{1}{T} \left(\frac{\tau}{T}\right)^{n-1}. \quad (2.3.181)$$

We then split the integral (2.3.180) into three contributions. We start by the contribution on the interval $(0, 1)$. We write by an integration by parts

$$\int_0^1 (1 - \exp_n(\tau, T)) \nabla u(\tau) d\tau = \int_0^1 \frac{\partial}{\partial \tau} \exp_n(\tau, T) \int_0^\tau \nabla u(t) dt d\tau + (1 - \exp_n(1, T)) \int_0^1 \nabla u(\tau) d\tau.$$

Thus, by the Minkowski inequality in $L^2_{\langle \cdot \rangle}(\Omega)$ and the stationarity of ∇u , we get

$$\begin{aligned} \left\langle \left| \int_0^1 (1 - \exp_n(\tau, T)) \nabla u(\tau) d\tau \right|^2 \right\rangle^{\frac{1}{2}} &\lesssim \int_0^1 \left| \frac{\partial}{\partial \tau} \exp_n(\tau, T) \right|^2 d\tau \int_0^1 \left\langle \int \eta_{\sqrt{\tau}}(x) \left| \int_0^\tau \nabla u(t, x) dt \right|^2 dx \right\rangle^{\frac{1}{2}} d\tau \\ &\quad + |1 - \exp_n(1, T)|^2 \left\langle \int \eta_1(x) \left| \int_0^1 \nabla u(t, x) dt \right|^2 dx \right\rangle^{\frac{1}{2}}. \end{aligned}$$

Hence, using the localized energy estimate (2.2.32) combined with (2.3.181), we arrive at

$$\left\langle \left| \int_0^1 (1 - \exp_n(\tau, T)) \nabla u(\tau) d\tau \right|^2 \right\rangle^{\frac{1}{2}} \lesssim_n T^{-n},$$

which is of higher order than the r.h.s of (2.1.37). We turn now to the contributions on the intervals $(1, T)$ and $(T, +\infty)$, for which the estimate of the decay of the semigroup (2.1.25) and (2.3.181) yield

$$\begin{aligned} \left\langle \left| \int_1^{+\infty} (1 - \exp_n(\tau, T)) \nabla u(\tau) d\tau \right|^2 \right\rangle^{\frac{1}{2}} &\lesssim \int_1^T \left(\frac{\tau}{T}\right)^n \tau^{-1-\frac{d}{4}} d\tau + \int_T^{+\infty} \tau^{-1-\frac{d}{4}} d\tau \\ &\lesssim T^{\frac{1}{2}} \eta_\beta(T). \end{aligned}$$

It concludes the proof of (2.1.37).

Step 2. Proof of (2.1.38). This estimate is a direct consequence of (2.1.37). Indeed by the definition (2.1.36) of \bar{a}_T^n , we have

$$e_j \cdot (\bar{a}_T^n - a_{\text{hom}}) e_i = \langle (\nabla \phi_{e_j, T}^{*,n} - \nabla \phi_{e_j}^*) \cdot a(\nabla \phi_{e_i, T}^n + e_i) \rangle - \langle (\nabla \phi_{e_i}^* + e_j) \cdot a(\nabla \phi_{e_i} - \nabla \phi_{e_i, T}^n) \rangle.$$

Since we have

$$\langle (\nabla \phi_{e_j}^* + e_j) \cdot a(\nabla \phi_{e_i} - \nabla \phi_{e_i, T}^n) \rangle = \langle (\nabla \phi_{e_i} - \nabla \phi_{e_i, T}^n) \cdot a^*(\nabla \phi_{e_j}^* + e_j) \rangle,$$

the weak formulation of the corrector equation (2.1.8) for both $\phi_{e_i}^*$ and ϕ_{e_i} yields

$$\langle (\nabla \phi_{e_i} - \nabla \phi_{e_i,T}^n) \cdot a^*(\nabla \phi_{e_j}^* + e_j) \rangle = \langle (\nabla \phi_{e_j}^* - \nabla \phi_{e_j,T}^{*,n}) \cdot a(\nabla \phi_{e_i} + e_i) \rangle,$$

and we conclude that

$$|e_j \cdot (\bar{a}_T^n - a_{\text{hom}}) e_i| = |\langle (\nabla \phi_{e_j,T}^{*,n} - \nabla \phi_{e_j}^*) \cdot a(\nabla \phi_{e_i,T}^n - \nabla \phi_{e_i}) \rangle| \leq \langle |\nabla \phi_{e_j,T}^{*,n} - \nabla \phi_{e_j}^*|^2 \rangle^{\frac{1}{2}} \langle |\nabla \phi_{e_i,T}^n - \nabla \phi_{e_i}|^2 \rangle^{\frac{1}{2}},$$

so that the claim follows from (2.1.37), used for both a^* and a .

Proof of Corollary 6: Spectral resolution

Let $0 < \mu \leq 1$. The starting point is the use of the spectral theorem which allows one to rewrite the definition of $(\phi_{e,\mu^{-1}}^n)_{n \in \mathbb{N}}$, defined in Corollary 5, in the form, for all $n \in \mathbb{N}$

$$\phi_{e,T}^n = g_n(\mathcal{L}, \mu^{-1}) \Theta,$$

where $g_0 : \zeta \in (0, +\infty) \mapsto \frac{1}{\zeta}$, $g_1 : \zeta \in (0, +\infty) \mapsto (\zeta + \mu)^{-1}$, and $(g_n)_{n \in \mathbb{N}}$ is the Richardson extrapolation of g_1 with respect to μ^{-1} . Then, by the spectral theorem, we have for all $n \in \mathbb{N}$

$$\begin{aligned} \langle \nabla(\phi_{e,\mu^{-1}}^n - \phi_e) \cdot a \nabla(\phi_{e,\mu^{-1}}^n - \phi_e) \rangle &= \langle (\phi_{e,\mu^{-1}}^n - \phi_e) \mathcal{L}(\phi_{e,\mu^{-1}}^n - \phi_e) \rangle \\ &= \left\langle \int_0^{+\infty} \zeta (g_n(\zeta, \mu^{-1}) - g_0(\zeta))^2 d\nu_{\Theta}(\zeta) \right\rangle. \end{aligned} \quad (2.3.182)$$

On the one hand, for $n > \frac{\beta \wedge d}{4}$, Corollary 5 yields

$$\langle \nabla(\phi_{e,\mu^{-1}}^n - \phi_e) \cdot a \nabla(\phi_{e,\mu^{-1}}^n - \phi_e) \rangle \lesssim \mu^{-1} \eta_{\beta}^2(\mu^{-1}). \quad (2.3.183)$$

On the other hand, by induction on n (see for instance [90, Proof of Lemma 2.5]) we have for all $n \in \mathbb{N}$ and $\zeta \in (0, +\infty)$

$$|g_n(\zeta, \mu^{-1}) - g_0(\zeta)| \gtrsim \frac{\mu^n}{\zeta(\zeta + \mu)^n},$$

which we use in the form, for all $\zeta \leq \mu$

$$\zeta (g_n(\zeta, \mu^{-1}) - g_0(\zeta))^2 \gtrsim \frac{\mu^{2n}}{\zeta(\zeta + \mu)^{2n}}. \quad (2.3.184)$$

The combination of (2.3.182), (2.3.183) and (2.3.184) applied for some $n > \frac{\beta \wedge d}{4}$ gives

$$\begin{aligned} \left\langle \int_0^{\mu} d\nu_{\Theta}(\zeta) \right\rangle &\lesssim \mu \left\langle \int_0^{\mu} \frac{\mu^{2n}}{\zeta(\zeta + \mu)^{2n}} d\nu_{\Theta}(\zeta) \right\rangle \\ &\stackrel{(2.3.184)}{\lesssim} \mu \left\langle \int_0^{\mu} \zeta (g_n(\zeta, \mu^{-1}) - g_0(\zeta))^2 d\nu_{\Theta}(\zeta) \right\rangle \\ &\stackrel{(2.3.182), (2.3.183)}{\lesssim} \eta_{\beta}^2(\mu^{-1}). \end{aligned}$$

2.4 Appendix

2.4.1 Probabilistic tools

The following proposition shows that the multiscale logarithmic Sobolev inequality (2.1.4) gives a control of moments. For a reference, see [60, Prop 3.1].

Proposition 4. *Assume that the ensemble $\langle \cdot \rangle$ satisfies the multi-scale logarithm Sobolev inequality (2.1.4). For all $p \in [1, +\infty)$ and $F \in L^p_{(\cdot)}(\Omega)$*

$$\langle |F - \langle F \rangle|^p \rangle^{\frac{1}{p}} \lesssim_d \sqrt{p} \left\langle \left(\int_1^{+\infty} \ell^{-d} \pi(\ell) \int |\partial_{x,\ell}^{fct} F|^2 dx d\ell \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}}. \quad (2.4.1)$$

The following standard lemma gives the link between algebraic moment and exponential moment for non-negative random variables. The short proof is included for completeness.

Lemma 9. *Let $X : \Omega \rightarrow \mathbb{R}^+$ a non-negative random variable. We have the following equivalence:*

$$\exists C_1 > 0 \text{ such that } \left\langle \exp \left(\frac{1}{C_1} X \right) \right\rangle \leq 2 \Leftrightarrow \exists C_2 > 0 \text{ such that } \forall q \geq 1, \langle X^q \rangle^{\frac{1}{q}} \leq q C_2. \quad (2.4.2)$$

Proof. Let us suppose that there exists $C_2 > 0$ such that for all $q \geq 1$, $\langle X^q \rangle^{\frac{1}{q}} \leq q C_2$. We have, for all $C_1 > 0$

$$\left\langle \exp \left(\frac{1}{C_1} X \right) \right\rangle = \left\langle \sum_{n=0}^{+\infty} \frac{X^n}{n! C_1^n} \right\rangle \leq \sum_{n=0}^{+\infty} \frac{(\frac{C_2}{C_1} n)^n}{n!},$$

we then choose C_1 such that $\sum_{n=0}^{+\infty} \frac{(\frac{C_2}{C_1} n)^n}{n!} \leq 2$. Let us now suppose that there exists $C_1 > 0$ such that $\left\langle \exp \left(\frac{1}{C_1} X \right) \right\rangle \leq 2$. This implies that for all $q \geq 1$, $\langle X^q \rangle \leq C_1^q q!$. Since, from the Stirling formula, $q! \leq C q^q$ for some $C > 0$, we have for all $q \in \mathbb{N}$, $\langle X^q \rangle^{\frac{1}{q}} \leq C C_1 q$. \square

2.4.2 Large-scale regularity theory for parabolic system

In this section we recall the regularity theory for random parabolic operator of the form $\partial_\tau - \nabla \cdot a \nabla$ developed in the papers [26, 11] and draw some useful consequences. Here, we assume that a does not depend on time. However, the theory also holds with time dependent coefficients, using a time dependent minimal radius r_* different from the one defined in Theorem 11 but this is not needed in this paper. We start by recalling the excess decay property, which can be found in [26, Prop.4] and the moment bound on r_* which can be found in [80]. We then prove large-scale $C^{0,1}$ estimates, following the arguments of [80].

Theorem 11 (Excess decay). *There exists a $\frac{1}{8}$ -Lipschitz stationary random field $r_* : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ for which there exists a constant $C < +\infty$ such that for all $x \in \mathbb{R}^d$*

$$\left\langle \exp \left(\frac{1}{C} \pi_*(r_*(x)) \right) \right\rangle \leq 2, \quad (2.4.3)$$

with for all $r \geq 1$

$$\pi_*(r) = \begin{cases} r^\beta & \text{if } \beta < d, \\ r^d \log^{-1}(r) & \text{if } \beta = d, \\ r^d & \text{if } \beta > d. \end{cases}$$

In addition, for all distributional solution of

$$\partial_\tau u - \nabla \cdot a \nabla u = 0 \text{ in } C_R \text{ for } R \geq r_*,$$

we have for all $r \in [r_*, R]$ and $\alpha \in (0, 1)$

$$\text{Exc}(\nabla u, r) \lesssim_{d,\lambda,\alpha} \left(\frac{r}{R}\right)^{2\alpha} \text{Exc}(\nabla u, R), \quad (2.4.4)$$

with $\text{Exc}(\nabla u, r) := \inf_{\xi \in \mathbb{R}^d} \int_{C_r} |\nabla u(t, y) - \xi - \nabla \phi_\xi(y)|^2 dy$.

A direct consequence of the excess decay property of Theorem 11 is the following large-scale $C^{0,1}$ estimates, in the spirit of [80], state in the parabolic setting.

Corollary 7 (Large-scale $C^{0,1}$ estimates). *For the random field r_* defined in Theorem 11, we have for all $(s, x) \in \mathbb{R}^{d+1}$ and weak solution u of*

$$\partial_\tau u - \nabla \cdot (a(\nabla u + g)) = \nabla \cdot h \text{ in } C_R(s, x) \text{ for } R \geq r_*,$$

with $(g, h) \in L^2_{loc}(\mathbb{R}^d)$, for all $r \in [r_*(x), R]$ and $\alpha > 0$

$$\begin{aligned} \int_{C_r(s,x)} |\nabla u(t, y)|^2 dt dy &\lesssim_{d,\lambda,\alpha} \int_{C_R(s,x)} |\nabla u(t, y)|^2 dt dy \\ &+ \sup_{\rho \in [r_*, R]} \left(\frac{R}{\rho}\right)^{2\alpha} \int_{C_\rho(s,x)} \left(\left| h - \int_{C_\rho(s,x)} h \right|^2 + \left| g - \int_{C_\rho(s,x)} g \right|^2 \right). \end{aligned} \quad (2.4.5)$$

In particular, if $g \equiv h \equiv 0$, we have the following mean value property for a -caloric functions: for all $r \in [r_*(x), R]$

$$\int_{C_r(s,x)} |\nabla u(t, y)|^2 dt dy \lesssim_{d,\lambda} \int_{C_R(s,x)} |\nabla u(t, y)|^2 dt dy. \quad (2.4.6)$$

Proof. Without loss of generality, we may assume that $(s, x) = (0, 0)$. We split the proof into two steps.

Step 1. Proof of

$$\sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \lesssim_{d,\lambda,\alpha} \frac{1}{R^{2\alpha}} \text{Exc}(\nabla u + g, R) + \sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \int_{C_r} \left(|h - \int_{C_r} h|^2 + |g - \int_{C_r} g|^2 \right), \quad (2.4.7)$$

and if $R = +\infty$

$$\sup_{r \geq r_*} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \lesssim_{d,\lambda,\alpha} \sup_{r \geq r_*} \frac{1}{r^{2\alpha}} \int_{C_\rho} \left(|h - \int_{C_\rho} h|^2 + |g - \int_{C_\rho} g|^2 \right). \quad (2.4.8)$$

Let $\alpha' = \frac{1+\alpha}{2}$ and $r_* \leq r \leq \rho \leq R$. We prove that

$$\text{Exc}(\nabla u + g, r) \leq C_1 \left(\left(\frac{r}{\rho}\right)^{2\alpha'} \text{Exc}(\nabla u + g, \rho) + \left(\frac{\rho}{r}\right)^{d+2} \int_{C_\rho} \left(|h - \int_{C_\rho} h|^2 + |g - \int_{C_\rho} g|^2 \right) \right), \quad (2.4.9)$$

with some constant C_1 which depends on λ and d .

Set $\xi := \mathcal{F}_{C_\rho} g$ and let $w \in L^2((-\rho^2, \rho^2), H_0^1(B_\rho)) \cap H^1((-\rho^2, \rho^2), H^{-1}(B_\rho))$ be the weak solution of

$$\begin{cases} \partial_t w + \nabla \cdot a \nabla w = \nabla \cdot (a(g - \xi) + h) & \text{in } C_\rho, \\ w = 0 & \text{on } \partial_p C_\rho, \end{cases} \quad (2.4.10)$$

where $\partial_p C_\rho = (\partial B_\rho \times (-\rho^2, 0)) \cup B_\rho \times \{0\}$. Then, because $(t, x) \in \mathbb{R}^{d+1} \mapsto u(t, x) - w(t, x) + \xi \cdot x$ is a a -caloric function in C_ρ , we have by the Theorem 11 for the exponent α'

$$\text{Exc}(\nabla u - \nabla w + \xi, r) \lesssim \left(\frac{r}{\rho}\right)^{2\alpha'} \text{Exc}(\nabla u - \nabla w + \xi, \rho). \quad (2.4.11)$$

In addition, we have the following energy estimate

$$\int_{C_\rho} |\nabla w|^2 \lesssim \int_{C_\rho} \left| h - \mathcal{F}_{C_\rho} h \right|^2 + \int_{C_\rho} \left| g - \mathcal{F}_{C_\rho} g \right|^2. \quad (2.4.12)$$

Indeed, by testing (2.4.10) by w itself, we get

$$- \int_{C_\rho} w \partial_\tau w + \int_{C_\rho} \nabla w \cdot a \nabla w = \int_{C_\rho} \left(h - \mathcal{F}_{C_\rho} h + \nabla w \cdot a(g - \xi) \right).$$

Since

$$- \int_{C_\rho} w \partial_\tau w = - \int_{-\rho^2}^0 \frac{d}{dt} \|w(t, \cdot)\|_{L^2(B_\rho)}^2 dt = \|w(-\rho^2, \cdot)\|_{L^2(B_\rho)}^2 \geq 0,$$

this yields

$$\int_{C_\rho} \nabla w \cdot a \nabla w \leq \int_{C_\rho} \nabla w \cdot \left(h - \mathcal{F}_{C_\rho} h + a(g - \xi) \right).$$

By uniform ellipticity assumption (2.1.2) on a , (2.4.12) follows. The combination of (2.4.11), (2.4.12) and the triangle inequality yields (2.4.9). Now, we conclude by a Campanato iteration. Setting $0 < \theta = \frac{r}{\rho} \leq 1$, we rewrite (2.4.9) as

$$\text{Exc}(\nabla u + g, \theta \rho) \leq C_1 \left(\theta^{2\alpha'} \text{Exc}(\nabla u + g, \rho) + \theta^{-d-2} \mathcal{F}_{C_\rho} \left(|h - \mathcal{F}_{C_\rho} h|^2 + |g - \mathcal{F}_{C_\rho} g|^2 \right) \right).$$

We divide by $(\theta \rho)^{2\alpha}$ and take the supremum over $\rho \in [\frac{r_*}{\theta}, R]$:

$$\begin{aligned} \sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) &\leq C_1 \left(\theta^{2(\alpha' - \alpha)} \sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \right. \\ &\quad \left. + \theta^{-d-2-2\alpha} \sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \mathcal{F}_{C_r} \left(|h - \mathcal{F}_{C_r} h|^2 + |g - \mathcal{F}_{C_r} g|^2 \right) \right). \end{aligned} \quad (2.4.13)$$

We now choose $\theta = \theta(d, \lambda, \alpha) \leq 1$ so small that $C_1 \theta^{2(\alpha' - \alpha)} \leq \frac{1}{2}$. By using

$$\sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \leq \sup_{r \in [\theta R, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) + \sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r),$$

we may absorb the second r.h.s term of the previous inequality into the l.h.s of (2.4.13), which yields

$$\sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \lesssim \sup_{r \in [\theta R, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) + \sup_{r \in [r_*, R]} \frac{1}{r^{2\alpha}} \int_{C_r} \left(|h - \int_{C_r} h|^2 + |g - \int_{C_r} g|^2 \right).$$

Since

$$\sup_{r \in [\theta R, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g, r) \lesssim \frac{1}{R^{2\alpha}} \sup_{r \in [\theta R, R]} \frac{R^{d+2}}{r^{d+2}} \text{Exc}(\nabla u + g, R) \lesssim \frac{1}{R^{2\alpha}} \text{Exc}(\nabla u + g, R),$$

this yields (2.4.7) in the case $R < +\infty$. In the case $R = \infty$ we obtain (2.4.8) in the limit $R \rightarrow +\infty$ by the square integrability of $\nabla u + g$ on \mathbb{R}^{d+1} , in form of

$$\limsup_{R \rightarrow +\infty} \text{Exc}(\nabla u + g, R) \leq \limsup_{R \rightarrow +\infty} \int_{C_R} |\nabla u + g|^2 = 0.$$

Step 2. Proof of (2.4.5). We split this step into two parts.

Substep 2.1. Proof that for all $\rho > 0$ there exists a unique $\xi_\rho \in \mathbb{R}^d$ such that

$$\text{Exc}(\nabla u + g, \rho) = \int_{C_\rho} |\nabla u + g - (\xi_\rho + \nabla \phi_{\xi_\rho})|^2, \quad (2.4.14)$$

and for all $r_* \leq r \leq R$

$$|\xi_r - \xi_R|^2 \lesssim \sup_{\rho \in [r, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \text{Exc}(\nabla u + g, \rho) + \sup_{\rho \in [r_*, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \int_{C_\rho} \left(\left| h - \int_{C_\rho} h \right|^2 + \left| g - \int_{C_\rho} g \right|^2 \right). \quad (2.4.15)$$

We start by proving (2.4.14). Fix $\rho > 0$ and define

$$f : \xi \in \mathbb{R}^d \mapsto \int_{C_\rho} |\nabla u + g - (\xi + \nabla \phi_\xi)|^2.$$

f is a continuous function and the mean value property of ϕ , namely for all $R \geq r_*$:

$$\int_{B_R} |\nabla \phi_\xi + \xi|^2 \geq \frac{1}{2} |\xi|^2, \quad (2.4.16)$$

shows that f is coercive. Consequently, ξ_ρ in (2.4.14) exists. On the other hand, ξ_ρ is unique. Indeed, suppose that (2.4.14) is verified for two vectors ξ_1 and ξ_2 . We have

$$\text{Exc}(\nabla u + g, \rho) = \int_{C_\rho} |\nabla u + g - (\xi_1 + \nabla \phi_{\xi_1})|^2 = \int_{C_\rho} |\nabla u + g - (\xi_2 + \nabla \phi_{\xi_2})|^2,$$

and in particular

$$2\text{Exc}(\nabla u + g, \rho) = \int_{C_\rho} \left(|\nabla u + g - (\xi_1 + \nabla \phi_{\xi_1})|^2 + |\nabla u + g - (\xi_2 + \nabla \phi_{\xi_2})|^2 \right).$$

The parallelogram identity yields

$$2\text{Exc}(\nabla u + g, \rho) = \int_{C_\rho} \frac{1}{2} |\xi_1 - \xi_2 + \nabla \phi_{\xi_1 - \xi_2}|^2 + 2 |\nabla u + g - \left(\frac{\xi_1 + \xi_2}{2} + \nabla \phi_{\frac{\xi_1 + \xi_2}{2}} \right)|^2.$$

We infer that

$$\text{Exc}(\nabla u + g, \rho) \geq \int_{C_\rho} \frac{1}{4} |\xi_1 - \xi_2 + \nabla \phi_{\xi_1 - \xi_2}|^2 + \text{Exc}(\nabla u + g, \rho),$$

and so

$$\int_{C_\rho} |\xi_1 - \xi_2 + \nabla \phi_{\xi_1 - \xi_2}|^2 = 0,$$

which gives $\xi_1 = \xi_2$ using the estimate (2.4.16).

We turn to the proof of (2.4.15). It is enough to prove that

$$\forall r \leq R \leq 2r, |\xi_r - \xi_R|^2 \lesssim \text{Exc}(\nabla u + g, R). \quad (2.4.17)$$

Indeed, we argue by a dyadic argument. Let $N \in \mathbb{N}$ such that $2^{-(N+1)}R < r < 2^{-N}R$. By (2.4.17), we have for all $n \in \{0, \dots, N-1\}$

$$|\xi_r - \xi_{2^{-N}R}|^2 \lesssim \text{Exc}(\nabla u + g, 2^{-N}R) \text{ and } |\xi_{2^{-(n+1)}R} - \xi_{2^{-n}R}|^2 \lesssim \text{Exc}(\nabla u + g, 2^{-n}R).$$

Thus, by the triangle inequality followed by the excess decay (2.4.4) and the fact that $\sum_{n=0}^{+\infty} 2^{-n\alpha} < +\infty$, we have

$$\begin{aligned} |\xi_r - \xi_R|^2 &\lesssim \left(\sum_{n=0}^N \sqrt{\text{Exc}(\nabla u + g, 2^{-n}R)} \right)^2 \\ &\stackrel{(2.4.7)}{\lesssim} \left(\sum_{n=0}^N 2^{-n\alpha} \left(\sqrt{\text{Exc}(\nabla u + g, R)} + \left(\sup_{\rho \in [r^*, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \int_{C_\rho} \left| h - \int_{C_\rho} h \right|^2 + \left| g - \int_{C_\rho} g \right|^2 \right)^{\frac{1}{2}} \right) \right)^2 \\ &\lesssim \text{Exc}(\nabla u + g, R) + \sup_{\rho \in [r^*, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \int_{C_\rho} \left(\left| h - \int_{C_\rho} h \right|^2 + \left| g - \int_{C_\rho} g \right|^2 \right) \\ &\lesssim \sup_{\rho \in [r, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \text{Exc}(\nabla u + g, \rho) + \sup_{\rho \in [r^*, R]} \left(\frac{R}{\rho} \right)^{2\alpha} \int_{C_\rho} \left(\left| h - \int_{C_\rho} h \right|^2 + \left| g - \int_{C_\rho} g \right|^2 \right). \end{aligned}$$

We now turn to the argument for (2.4.17). By (2.4.16) we have

$$|\xi_r - \xi_R|^2 \lesssim \int_{C_r} |(\xi_\rho - \xi_R) + \nabla \phi_{\xi_r - \xi_R}|^2,$$

which, by linearity of $\xi \mapsto \phi_\xi$, we may rewrite as

$$|\xi_r - \xi_R|^2 \lesssim \int_{C_r} |(\xi_\rho + \nabla \phi_{\xi_r}) - (\xi_R + \nabla \phi_{\xi_R})|^2,$$

so that, by the triangle inequality in $L^2(C_\rho)$ and using that $r \leq R \leq 2r$, we obtain

$$|\xi_r - \xi_R|^2 \lesssim \int_{C_r} |\nabla u - (\xi_r + \nabla \phi_{\xi_r})|^2 + \int_{C_R} |\nabla u - (\xi_R + \nabla \phi_R)|^2.$$

By definition of Exc and using once more that $r \leq R \leq 2r$, this turns as desired into

$$|\xi_r - \xi_R|^2 \lesssim \text{Exc}(\nabla u + g, r) + \text{Exc}(\nabla u + g, R) \lesssim \text{Exc}(\nabla u + g, R).$$

Substep 2.2. We prove (2.4.5). The starting point is (2.4.7) in the more general form : for all $r \geq r_*$

$$\begin{aligned} \sup_{\rho \in [r, R]} \left(\frac{R}{\rho}\right)^{2\alpha} \text{Exc}(\nabla u + g, \rho) &\lesssim \text{Exc}(\nabla u + g, R) \\ &+ \sup_{\rho \in [r, R]} \left(\frac{R}{\rho}\right)^{2\alpha} \int_{C_\rho} \left(|g - \int_{C_\rho} g|^2 + |h - \int_{C_\rho} h|^2 \right). \end{aligned} \quad (2.4.18)$$

The estimates (2.4.15) and (2.4.18) combined with the triangle inequality yield

$$\begin{aligned} |\xi_r|^2 + \text{Exc}(\nabla u + g, r) &\lesssim |\xi_R|^2 + \text{Exc}(\nabla u + g, R) \\ &+ \sup_{\rho \in [r_*, R]} \left(\frac{R}{\rho}\right)^{2\alpha} \int_{C_\rho} \left(|g - \int_{C_\rho} g|^2 + |h - \int_{C_\rho} h|^2 \right). \end{aligned} \quad (2.4.19)$$

Using the triangle inequality in $L^2(C_r)$ and the definition of the excess in the form of

$$\int_{C_r} |\nabla u + g|^2 \lesssim |\xi_r|^2 + \text{Exc}(\nabla u + g, r),$$

and

$$|\xi_R|^2 + \text{Exc}(\nabla u + g, R) \lesssim \int_{C_R} |\nabla u + g|^2,$$

we may finally pass from (2.4.19) to (2.4.5). \square

We finally recall the following property of average of r_* . The proof can be found in [80, Esti. (139)].

Lemma 10. *For all measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$ there exists two constants c and C which depends only on the dimension d such that*

$$c \int \int_{B_{r_*(x)}(x)} f(y) dy dx \leq \int f(x) dx \leq C \int \int_{B_{r_*(x)}(x)} f(y) dy dx. \quad (2.4.20)$$

2.4.3 Technical lemmas

We state here two technical lemmas. The first one is needed in the proof of Lemma 2. For a proof, see [8, Lem. C.6].

Lemma 11. *Suppose that there exist $A, \xi \geq 0$ and $\rho : [\frac{1}{2}, 1) \rightarrow \mathbb{R}^+$ which satisfies*

$$\sup_{t \in [\frac{1}{2}, 1)} (1-t)^\xi \rho(t) < +\infty,$$

and for all $\frac{1}{2} \leq s < t < 1$,

$$\rho(s) \leq \frac{1}{2} \rho(t) + (t-s)^{-\xi} A.$$

Then, there exists a constant C depending on ξ such that $\rho(\frac{1}{2}) \leq CA$.

The second one is the Caccioppoli inequality for parabolic system. For a proof, see for instance [26, Lemma 5].

Lemma 12 (Caccioppoli estimate). *There exists a constant C depending on λ such that for every $\rho \leq R$ and weak solution u of*

$$\partial_t u - \nabla \cdot a \nabla u = 0 \text{ in } C_R,$$

we have

$$\int_{C_\rho} |\nabla u(x)|^2 dx \leq \frac{C}{(R - \rho)^2} \int_{C_R \setminus C_\rho} \left| u(x) - \fint_{C_R} u(y) dy \right|^2 dx.$$

2.4.4 Proof of Theorem 10 under a functional inequality with oscillation

We fix $T \geq 1$, $1 \leq r \leq \sqrt{T}$ and the unit vector $e \in \mathbb{R}^d$. We only give the argument for (2.1.17), (2.1.20) is obtained combining the ideas of this section and the proof of Section 2.3.4. We make for simplicity the two additional assumptions:

- (i) u_e is real-valued and a is symmetric. We recall that it implies

$$\|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-1}, \quad (2.4.21)$$

see Remark 4.

- (ii) The coefficient field a takes the form, for some $\chi \in C_c^\infty(\mathbb{R}^d)$ supported in B_1 ,

$$a := \chi \star \tilde{a},$$

with a field $\tilde{a} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ which takes value into the set of uniformly elliptic and bounded matrices and with a probability law which satisfies the logarithm Sobolev inequality with oscillation (2.1.40). In this setting, $\nabla \cdot a e \in L^\infty(\mathbb{R}^d)$ and

$$\|\nabla \cdot a e\|_{L^\infty(\mathbb{R}^d)} \lesssim_\chi 1.$$

We recall that it implies the following energy estimate: for all $R \geq 1$ and $z \in \mathbb{R}^d$

$$\int_0^1 \fint_{B_R(z)} |\nabla u(t, x)|^2 dx dt \lesssim_\chi 1, \quad (2.4.22)$$

where a proof can be found in [92, Lemma 3]. The first step is to estimate the derivative $(x, \ell) \in \mathbb{R}^d \times [1, +\infty) \mapsto \partial_{x, \ell}^{\text{osc}} q_r(T)$. We claim that for all $(x, \ell) \in \mathbb{R}^d \times [1, +\infty)$

$$\begin{aligned} |\partial_{x, \ell}^{\text{osc}} q_r(T)|^2 &\lesssim_\chi (\ell + 1)^{2d} \left((1 + \log^2(T)) \fint_{B_{\ell+1}(x)} g_r^2(y) dy + \fint_{B_{\ell+1}(x)} |\nabla v^T(0, y)|^2 dy \right. \\ &\quad \left. + \log(T) \int_1^T t^{-1} \fint_{B_{\ell+1}(x)} |\nabla v^T(t, y)|^2 dy dt \right), \end{aligned} \quad (2.4.23)$$

with v^T defined in (2.2.11).

We fix $(x, \ell) \in \mathbb{R}^d \times [1, +\infty)$ and we consider \tilde{a}' and \tilde{a}'' such that $\tilde{a}' = \tilde{a}'' = \tilde{a}$ on $\mathbb{R}^d \setminus B_\ell(x)$. We then set $a' := \chi \star \tilde{a}'$, $a'' := \chi \star \tilde{a}''$ and note that since χ is supported in B_1

$$a' = a'' = a \text{ on } \mathbb{R}^d \setminus B_{\ell+1}(x). \quad (2.4.24)$$

Using the notation $\delta u := u(a', \cdot) - u(a'', \cdot)$, we have

$$\begin{aligned}
q_r(a', T) - q_r(a'', T) &= \int g_r(y)(a'(y) - a''(y))e \, dy + \int g_r(y)a'(y) \int_0^T \nabla u(a', t, y) dt \, dy \\
&\quad - \int g_r(y)a''(y) \int_0^T \nabla u(a'', t, y) dt \, dy \\
&= \int g_r(y)(a'(y) - a''(y))e \, dy + \int g_r(y)(a'(y) - a(y)) \int_0^T \nabla u(a', t, y) dt \, dy \\
&\quad + \int g_r(y)(a''(y) - a(y)) \int_0^T \nabla u(a'', t, y) dt \, dy + \int g_r(y)a(y) \int_0^T \nabla \delta u(t, y) dt \, dy.
\end{aligned} \tag{2.4.25}$$

On the one hand, using (2.4.24) and the Jensen inequality, the first r.h.s term of (2.4.25) is dominated by

$$\left| \int g_r(y)(a'(y) - a''(y))e \, dy \right|^2 \stackrel{(2.4.25)}{=} \left| \int_{B_{\ell+1}(x)} g_r(y)(a'(y) - a''(y))e \, dy \right|^2 \lesssim_{\chi} (\ell+1)^{2d} \int_{B_{\ell+1}(x)} g_r^2(y) dy,$$

which contributes to the first r.h.s term of (2.4.23). On the other hand, the second and third r.h.s term of (2.4.25) are treated the same way (we estimate below the term with a') using (2.4.24), (2.4.22) (with a replaced by a'), (2.4.21), the Cauchy-Schwarz and Jensen's inequality

$$\begin{aligned}
\left| \int g_r(y)(a'(y) - a(y)) \int_0^T \nabla u(a', t, y) dt \, dy \right|^2 &\stackrel{(2.4.24)}{\lesssim_{\chi}} \left(\int_{B_{\ell+1}(x)} g_r(y) \left| \int_0^T \nabla u(a', t, y) dt \right| dy \right)^2 \\
&\lesssim \left(\int_{B_{\ell+1}(x)} g_r(y) \left| \int_0^1 \nabla u(a', t, y) dt \right| dy \right)^2 \\
&\quad + \left(\int_{B_{\ell+1}(x)} g_r(y) \left| \int_1^T \nabla u(a', t, y) dt \right| dy \right)^2 \\
&\stackrel{(2.4.21)}{\lesssim} (\ell+1)^{2d} \int_0^1 \int_{B_{\ell+1}(x)} |\nabla u(a', t, y)|^2 dy \, dt \int_{B_{\ell+1}(x)} g_r^2(y) dy \\
&\quad + \log^2(T) (\ell+1)^d \int_{B_{\ell+1}(x)} g_r^2(y) dy \\
&\stackrel{(2.4.22)}{\lesssim_{\chi}} (\ell+1)^{2d} (1 + \log^2(T)) \int_{B_{\ell+1}(x)} g_r^2(y) dy,
\end{aligned}$$

which contributes to the first r.h.s term of (2.4.23). It remains to control the fourth r.h.s term of (2.4.25). To do this, we first write the equation solved by δu , which we deduce from (2.1.10)

$$\begin{cases} \partial_{\tau} \delta u - \nabla \cdot a \nabla \delta u = \nabla \cdot (a - a'') \nabla u(a'', \cdot) - \nabla \cdot (a - a') \nabla u(a', \cdot) & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ \delta u(0) = \nabla \cdot (a' - a'')e. \end{cases} \tag{2.4.26}$$

Thus, by testing δu into (2.2.11) and v^T into (2.4.26), we deduce that

$$\begin{aligned} \int g_r(y)a(y) \int_0^T \nabla \delta u(t, y) dt dy &= \int \nabla v^T(0, y) \cdot (a'(y) - a''(y))e dy \\ &\quad + \int \int_0^T (a(y) - a''(y)) \nabla u(a'', t, y) \cdot \nabla v^T(t, y) dt dy \\ &\quad - \int \int_0^T (a(y) - a'(y)) \nabla u(a', t, y) \cdot \nabla v^T(t, y) dt dy. \end{aligned} \quad (2.4.27)$$

The first r.h.s term of (2.4.27) is dominated with (2.4.24) and gives the second r.h.s term of (2.4.23). The second and third r.h.s term of (2.4.27) are dominated the same way (we estimate below the term with a') using (2.4.21), (2.4.22), (2.4.24), (2.2.20) (applied with $r = \ell$ and $z = x$), the Cauchy-Schwarz and Jensen's inequality

$$\begin{aligned} &\left| \int \int_0^T (a(y) - a'(y)) \nabla u(a', t, y) \cdot \nabla v^T(t, y) dt dy \right|^2 \\ &\stackrel{(2.4.24)}{\lesssim_\chi} \left(\int_{B_{\ell+1}(x)} \int_0^T |\nabla u(a', t, y)| |\nabla v^T(t, y)| dt dy \right)^2 \\ &\stackrel{(2.4.21)}{\lesssim} (\ell + 1)^{2d} \int_0^1 \int_{B_{\ell+1}(x)} |\nabla u(a', t, y)|^2 dy dt \int_0^1 \int_{B_{\ell+1}(x)} |\nabla v^T(t, y)|^2 dy dt \\ &+ \left(\int_1^T t^{-1} \int_{B_{\ell+1}(x)} |\nabla v^T(t, y)| dy dt \right)^2 \\ &\stackrel{(2.4.22), (2.2.20)}{\lesssim_\chi} (\ell + 1)^{2d} \left(\int_{B_{\ell+1}(x)} g_r^2(y) dy + \log(T) \int_1^T t^{-1} \int_{B_{\ell+1}(x)} |\nabla v^T(t, y)|^2 dy dt \right), \end{aligned}$$

which contributes to the first and third r.h.s term of (2.4.23) and concludes the proof.

We now control the entropy of $q_r(T)$ by applying (2.1.40), using (2.4.23), the identity $\int \int_{B_{\ell+1}(x)} dx = \int, \int g_r^2(y) dy \lesssim r^{-d}$ and the plain energy estimate $\int |\nabla v^T(t, y)|^2 dy \lesssim \int g_r^2(y) dy$,

$$\begin{aligned} \text{Ent}(q_r(T)) &\lesssim_\chi \int_1^{+\infty} \ell^{-d} e^{-\frac{1}{C}\ell^\beta} (\ell + 1)^d (1 + \log^2(T)) \int \int_{B_{\ell+1}(x)} g_r^2(y) dy dx d\ell \\ &\quad + \int_1^{+\infty} \ell^{-d} e^{-\frac{1}{C}\ell^\beta} (\ell + 1)^d \int \int_{B_{\ell+1}(x)} |\nabla v^T(0, y)|^2 dy dx d\ell \\ &\quad + \log(T) \int_1^{+\infty} \ell^{-d} e^{-\frac{1}{C}\ell^\beta} (\ell + 1)^d \int_1^T t^{-1} \int \int_{B_{\ell+1}(x)} |\nabla v^T(t, y)|^2 dy dx dt d\ell \\ &\lesssim r^{-d} (1 + \log^2(T)) \int_1^{+\infty} \ell^d e^{-\frac{1}{C}\ell^\beta} d\ell \lesssim_{\beta, C} r^{-d} (1 + \log^2(T)). \end{aligned}$$

To conclude, the $\log^2(T)$ correction may be removed following the argument of subsection 2.3.4, and the control of the entropy yields control of higher moments and provide stretched exponential moments.

Chapter 3

Quantitative homogenization theory for nonlinear elliptic equations and systems

Quantitative stochastic homogenization of linear elliptic operators is by now well-understood. In this chapter we move forward to the nonlinear setting of monotone operators with p -growth. This first work is dedicated to a quantitative two-scale expansion result. Fluctuations will be addressed in companion articles. By treating the range of exponents $2 \leq p < \infty$ in dimensions $d \leq 3$, we are able to consider genuinely nonlinear elliptic equations and systems such as $-\nabla \cdot A(x)(1 + |\nabla u|^{p-2})\nabla u = f$ (with A random, non-necessarily symmetric) for the first time. When going from $p = 2$ to $p > 2$, the main difficulty is to analyze the associated linearized operator, whose coefficients are degenerate, unbounded, and depend on the random input A via the solution of a nonlinear equation. One of our main achievements is the control of this intricate nonlinear dependence, leading to annealed Meyers' estimates for the linearized operator, which are key to the quantitative two-scale expansion result. This Chapter is based on the Preprint [49] jointly write with Antoine Gloria.

The chapter is organized as follows: In Section 3.1 we present the main results. In Section 3.3, we describe and prove the perturbative regularity theory for the linearized operator. Sections 3.4, 3.5 and 3.6 are devoted to the proof of the quantitative estimates on the correctors and the quantitative two-scale expansion.

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3.1 Main results

In all the chapter, we decide to use scalar notations but the results also hold for sytem. For convenience, let us precisely recall the class of maps $\hat{a} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that we consider in this chapter. We assume that \hat{a} satisfies the following three properties: $\hat{a}(0) = 0$ and there exist $p \geq 2$ and $C > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$,

$$|\hat{a}(\xi_1) - \hat{a}(\xi_2)| \leq C(1 + |\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|, \quad (3.1.1)$$

$$(\hat{a}(\xi_1) - \hat{a}(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \frac{1}{C}|\xi_1 - \xi_2|^p. \quad (3.1.2)$$

Estimate (3.1.1) is a continuity or boundedness property, whereas estimate (3.1.2) is a monotonicity or coercivity property. With a map \hat{a} we associate the monotone differential operator $C^2(\mathbb{R}^d) \rightarrow C^0(\mathbb{R}^d), v \mapsto \nabla \cdot \hat{a}(\nabla v)$. Such operators do not necessarily possess (as far as anyone can prove) the structural conditions sufficient for $C^{1,\alpha}$ -regularity, which is why we also consider two strengthenings of (3.1.2), which read for all $\xi_1, \xi_2 \in \mathbb{R}^d$

$$(\hat{a}(\xi_1) - \hat{a}(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \frac{1}{C}(|\xi_1 - \xi_2|^2 + |\xi_1 - \xi_2|^p), \quad (3.1.3)$$

$$(\hat{a}(\xi_1) - \hat{a}(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \frac{1}{C}(|\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|^2. \quad (3.1.4)$$

We consider a map $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the strong monotonicity (3.1.1) and (3.1.4). For later references, we recall the equation for¹ $u_\varepsilon \in \dot{W}^{1,p}(\mathbb{R}^d)/\mathbb{R}$

$$-\nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) = \nabla \cdot f(x). \quad (3.1.5)$$

We also recall the homogenized equation with solution $\bar{u} \in \dot{W}^{1,p}(\mathbb{R}^d)/\mathbb{R}$,

$$-\nabla \cdot \bar{a}(\nabla \bar{u}(x)) = \nabla \cdot f(x) \quad (3.1.6)$$

as well as the formula for the homogenized operator \bar{a} characterized in direction $\xi \in \mathbb{R}^d$ by $\bar{a}(\xi) = \mathbb{E}[a(0, \nabla \phi_\xi(0) + \xi)]$, where ϕ_ξ is the corrector, defined as the unique almost sure distributional solution in $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$ of

$$-\nabla \cdot a(x, \nabla \phi_\xi(x) + \xi) = 0, \quad (3.1.7)$$

anchored at the origin via $\int_B \phi_\xi = 0$, and the gradient $\nabla \phi_\xi$ is stationary, has vanishing expectation $\mathbb{E}[\nabla \phi_\xi] = 0$, and satisfies

$$\mathbb{E}\left[|\nabla \phi_\xi|^2 + |\nabla \phi_\xi|^p\right] \lesssim |\xi|^2 + |\xi|^p. \quad (3.1.8)$$

For more precisions on the homogenization theory, we refer to Section 1.1.2 of the introduction of the thesis.

3.1.1 Growth conditions and classical regularity

In this section we recall what regularity theory one can expect for elliptic operators of the form $\nabla \cdot \hat{a}(\nabla)$ depending on properties of \hat{a} , which we then apply both to the random and the homogenized operator. There are essentially two classes of results:

- *Standard* growth conditions: If $\xi \mapsto \hat{a}(\xi)$ is smooth, and \hat{a} is strongly monotone (that is, it satisfies (3.1.1) and (3.1.4)), then $\nabla \cdot \hat{a}(\nabla)$ possesses $C^{k,\alpha}$ -regularity and nonlinear Calderón-Zygmund theory, cf. [105];
- *Non-standard* growth conditions: If $\xi \mapsto \hat{a}(\xi)$ is smooth, and \hat{a} is monotone and non-degenerate (that is, it satisfies (3.1.1) and (3.1.3)), then one expects that $\nabla \cdot \hat{a}(\nabla)$ possesses $C^{k,\alpha}$ -regularity and nonlinear Calderón-Zygmund theory provided $2 \leq p < \frac{2(d-1)}{d-3}$ (only active for $d \geq 4$), cf. [30]².

On the one hand, strongly monotone operators are nicer since they possess regularity without restriction on the exponent p , but strong monotonicity is not stable by homogenization (in the sense that the homogenized operator might not be strongly monotone). On the other hand, although non-degenerate operators do only possess regularity if p is close enough to 2 in high dimensions (there is no restriction for $d \leq 3$), this property is stable by homogenization.

As we shall see below, to establish quantitative homogenization results,

- The non-degeneracy condition (3.1.3) is needed for reasons that are independent of the regularity theory;

¹We recall that $\dot{W}^{1,p}(\mathbb{R}^d) := \{u \in L^p(\mathbb{R}^d) \mid \nabla u \in L^p(\mathbb{R}^d)\}$.

²The new feature in [30], which establishes $C^{k,\alpha}$ -regularity for local solutions of $\nabla \cdot a(\nabla u) = 0$, is the largest range of exponents $2 \leq p < \frac{2(d-1)}{d-3}$ compared to previous contributions – more general results with right-hand sides and nonlinear Calderón-Zygmund theory, which are expected to hold true as well, have not been established yet.

- We need local regularity theory for the random operator, which (provided $x \mapsto a(x, \xi)$ is smooth enough) is automatic for $2 \leq p < \frac{2(d-1)}{d-3}$ in view of the assumption (3.1.3), and follows from any $p \geq 2$ if we further assume (3.1.4);
- In the random setting, it turns out that the condition $2 \leq p < \frac{2(d-1)}{d-3}$ is also needed in the proof of (large-scale) Meyers' estimates for the linearized operator.

For random operators, the assumptions we need in this paper to establish Meyers estimates for the linearized operator also imply regularity theory for the homogenized operator, and we refer the reader to Section 3.2.2 for a further discussion of this observation. This is however not the case for periodic operators, and regularity theory for the homogenized operator might be an issue.

3.1.2 Quantitative assumptions and quantitative two-scale expansion

In view of the discussion above, to fix ideas and keep results and proofs readable, we consider the explicit class of p -Laplacians regularized at zero (see Section 3.2.3 for more general conditions).

Hypothesis 3.1.1. *Let $p \geq 2$, and consider the strongly monotone and non-degenerate operator*

$$a(x, \xi) := A(x)(1 + |\xi|^{p-2})\xi, \quad (3.1.9)$$

where A is a uniformly elliptic (non-necessarily symmetric) stationary ergodic matrix field. More precisely, we assume that A is smooth (uniformly wrt to the randomness) and satisfies the ellipticity conditions for some $0 < \lambda \leq 1$

$$\forall x, \xi \in \mathbb{R}^d : \quad \xi \cdot A(x)\xi \geq \lambda|\xi|^2 \text{ and } |A(x)\xi| \leq |\xi|.$$

Under Hypothesis 3.1.1, the monotone map a almost surely satisfies $a(\cdot, 0) \equiv 0$, (3.1.1), and (3.1.3) (and, incidently, also (3.1.4)) for some C depending only on p and λ , so that the qualitative homogenization result of Theorem 2 applies.

Let us now be more precise on the stochastic setting. It is convenient to define the probability space via $\Omega = \{A : \mathbb{R}^d \rightarrow \mathbb{M}_d(\lambda)\}$, endowed with some probability measure \mathbb{P} . In this setting, a random variable Y can be seen as a (measurable) function of $A \mapsto Y(A)$. We say that the measure \mathbb{P} is ergodic if we have the implication: $Y(A(\cdot + z)) = Y(A)$ for all $z \in \mathbb{R}^d \implies Y = \mathbb{E}[Y]$ almost surely. We say that a random field $X : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^k$ (for $k \in \mathbb{N}$) is stationary if for all $z \in \mathbb{R}^d$ and almost all $x \in \mathbb{R}^d$ we have $X(x + z, A) = X(x, A(\cdot + z))$, where $A(\cdot + z) : x \mapsto A(x + z)$. (Note that the expectation $\mathbb{E}[X(x)]$ of a stationary random field X does not depend on $x \in \mathbb{R}^d$ and we simply write $\mathbb{E}[X]$.) We use the notation $L^q(d\mathbb{P})$ for the space of q -integrable random variables.

In order to prove quantitative results, we need to quantify the ergodicity assumption, which we do by assuming Gaussianity of \mathbb{P} and the integrability of the covariance function in the following sense.

Hypothesis 3.1.2. *On top Hypothesis 3.1.1, assume that*

$$A(y) = \chi * B(G(y)), \quad (3.1.10)$$

where $B : \mathbb{R} \rightarrow \mathbb{M}_d$ is a Lipschitz map and G is a stationary centered random Gaussian field on \mathbb{R}^d (that is, $\mathbb{E}[G] = 0$) characterized by its covariance function $\mathcal{C} : \mathbb{R}^d \rightarrow \mathbb{R}, x \mapsto \mathcal{C}(x) := \mathbb{E}[G(x)G(0)]$, which we assume to be integrable on \mathbb{R}^d , and $\chi : \mathbb{R}^d \rightarrow [0, 1]$ is a smooth compactly supported

convolution kernel. In particular, A is smooth (uniformly wrt to the randomness). We further require $2 \leq p < \frac{2(d-1)}{d-3}$ in dimensions $d \geq 4$.³

Our main achievement is an optimal quantitative corrector result, which extends the results of [68] to the genuinely nonlinear setting of $p > 2$. Following Dal Maso and Defranceschi [118], we start with the suitable definition of the two-scale expansion. To this aim, we introduce a scale $\delta > 0$ (which we should think of as being ε in the upcoming result), set $\mathcal{K}_\delta := \delta\mathbb{Z}^d$ and for all $k \in \mathcal{K}_\delta$, we define the cube $Q_\delta(k) = k + [-\delta, \delta]^d$ centered at k and of sidelength 2δ . We also consider a partition $(\eta_k)_{k \in \mathcal{K}_\delta}$ of unity on \mathbb{R}^d with the following properties: $0 \leq \eta_k \leq 1$, $\eta_k \equiv 1$ on $Q_{\frac{\delta}{2d}}(k)$, $\eta_k \geq c$ on $Q_{(1-\frac{1}{3d})\delta}$, $\text{supp } \eta_k \subset Q_\delta(k)$, and $|\nabla \eta_k| \leq C\delta^{-1}$ (for some suitable $c, C > 0$ independent of δ). Given the solution \bar{u} of (3.1.6), we introduce local averages associated with the partition of unity in form for all $k \in \mathcal{K}_\delta$ of

$$(\nabla \bar{u})_{k,\delta} := \frac{\int_{\mathbb{R}^d} \eta_k \nabla \bar{u}}{\int_{\mathbb{R}^d} \eta_k},$$

and define the two-scale expansion $\bar{u}_{\varepsilon,\delta}^{2s}$ associated with \bar{u} via

$$\bar{u}_{\varepsilon,\delta}^{2s} := \bar{u} + \varepsilon \sum_{k \in \mathcal{K}_\delta} \eta_k \phi_{(\nabla \bar{u})_{k,\delta}}\left(\frac{\cdot}{\varepsilon}\right), \tag{3.1.11}$$

where ϕ_ξ denotes the corrector in direction $\xi \in \mathbb{R}^d$ (cf. (3.1.7)). This constitutes a convenient variant (introduced in [118] to deal with monotone operators) of the classical two-scale expansion $x \mapsto \bar{u}(x) + \varepsilon \phi_{\nabla \bar{u}(x)}(\frac{x}{\varepsilon})$, which may raise measurability issues. Based on this two-scale expansion, we have the following optimal convergence result.

Theorem 12. *Assume Hypothesis 3.1.2 and let $f \in L^p(\mathbb{R}^d)^d$. Let the weight $\mu_d : \mathbb{R}^k \rightarrow \mathbb{R}_+$ (for $k = 1$ and d) be given by*

$$\mu_d(z) = \begin{cases} d = 1 & : 1 + \sqrt{|z|}, \\ d = 2 & : \log(2 + |z|)^{\frac{1}{2}}, \\ d > 2 & : 1. \end{cases} \tag{3.1.12}$$

For all $\varepsilon > 0$ we denote by $u_\varepsilon \in \dot{W}^{1,p}(\mathbb{R}^d)/\mathbb{R}$ the unique weak solution of (3.1.5), by $\bar{u} \in \dot{W}^{1,p}(\mathbb{R}^d)/\mathbb{R}$ the unique weak solution of the homogenized equation (3.1.6), and by \bar{u}_ε^{2s} the two-scale expansion (3.1.11) for the choice $\delta = \varepsilon$. If the homogenized solution \bar{u} satisfies $\nabla \bar{u} \in L^\infty(\mathbb{R}^d)$ and $\mu_d \nabla^2 \bar{u} \in L^2(\mathbb{R}^d)$, then we have

$$\|\nabla u_\varepsilon - \nabla \bar{u}_\varepsilon^{2s}\|_{L^2(\mathbb{R}^d)} \leq C_{\varepsilon,\bar{u}} \varepsilon \mu_d\left(\frac{1}{\varepsilon}\right), \tag{3.1.13}$$

where $C_{\varepsilon,\bar{u}}$ denotes a random variable that satisfies

$$\mathbb{E} \left[\exp(cC_{\varepsilon,\bar{u}}^\alpha) \right] \leq 2, \tag{3.1.14}$$

for some exponent $\alpha > 0$ depending on d, p, λ , and $\|\nabla \bar{u}\|_{L^\infty(\mathbb{R}^d)}$, and some constant c further depending on $\|\mu_d \nabla^2 \bar{u}\|_{L^2(\mathbb{R}^d)}$, but not on ε .

Some comments are in order:

- This result also holds for nonlinear systems with Uhlenbeck structure under the same assumptions on p .

³This condition, which comes from an argument of [25] and is used in the main part of this paper dedicated to large-scale Meyers estimates, is also crucially used in [30].

- In the periodic setting, Theorem 12 holds without restrictions on $p \geq 2$ and with $\mu_d \equiv 1$ in any dimension. This result is sharper than [42], which contains the first quantitative two-scale expansion estimate for monotone periodic operators with $p > 2$ (there, one needs to know that \bar{a} satisfies (3.1.4) to construct a second-order two-scale expansion, which gives (3.1.13) after truncation and with a dependence of the constant on stronger norms of \bar{u}).
- The choice to work on the whole space with a right-hand side in divergence form allows one to avoid boundary layers (and therefore to truly focus on the homogenization error, in line with [89]) and to treat all dimensions at once. In particular one could state and prove a similar result on a bounded domain with Dirichlet boundary conditions, in which case the bound would be of the order of the square root of that in (3.1.13).
- This result takes the same form (with the same optimal rates) as for the linear case [89, 81, 9] and for the nonlinear case [68] with $p = 2$. As opposed to the latter, the stretched exponential exponent α in Theorem 12 depends on $\|\nabla\bar{u}\|_{L^\infty(\mathbb{R}^d)}$ itself. This intricate dependence could be made explicit (provided we make the exponent and constants explicit in Gehring's lemma) and is reminiscent of the way we treat the non-degeneracy of the linearized equation (that is, perturbatively).
- This result makes the a priori assumption that $\nabla\bar{u} \in L^\infty(\mathbb{R}^d)$ and $\mu_d\nabla^2\bar{u} \in L^2(\mathbb{R}^d)$:
 - In the scalar setting, under Hypothesis 3.1.2, since $\bar{a} \in C_{\text{loc}}^{1,1}$ (cf. Corollary 16 below), the conditions $\nabla\bar{u} \in L^\infty(\mathbb{R}^d)$ and $\mu_d\nabla^2\bar{u} \in L^2(\mathbb{R}^d)$ are not restrictive and hold under suitable assumptions on the right-hand side f , capitalizing on the results [30] by Bella and Schäffner.
 - Since the above result is local in nature, this estimate holds on domains of \mathbb{R}^d on which \bar{u} has the required regularity. In any case, if $\nabla\bar{u}$ develops some singularity somewhere, one does not expect the two-scale expansion to be accurate in that region. This applies in particular to systems and to the periodic setting if $p > \frac{2(d-1)}{d-3}$.
- The restriction $2 \leq p < \frac{2(d-1)}{d-3}$ on the exponent p in Hypothesis 3.1.2 (which is only active in high dimensions $d \geq 4$) is related to the perturbative regularity theory in the large that we develop for the linearized operator in Section 3.3. Indeed, the coefficient $a_\xi := Da(\cdot, \xi + \nabla\phi_\xi)$ of the operator a linearized at $\xi + \nabla\phi_\xi$ scales like $1 + |\xi + \nabla\phi_\xi|^{p-2}$ and therefore only satisfies $\mathbb{E}[|a_\xi|^{\frac{p}{p-2}}] < \infty$ a priori: as p increases, the stochastic integrability of the coefficients decreases. At some threshold (depending on dimension), this poor stochastic integrability cannot be compensated any longer by the Sobolev embedding — whence our restriction (even in dimension 3, the argument to get all the exponents $2 \leq p < \infty$ is not straightforward — see Sections 3.2 and 3.3).

3.1.3 Remarks on the strong monotonicity of \bar{a}

In this section we further investigate under which additional assumptions one could prove that \bar{a} satisfies (3.1.4) next to (3.1.2) and (3.1.3), which would ensure regularity theory for the homogenized operator (provided \bar{a} is smooth enough) independently of p and d . Let us first explain why this question is subtle, and assume for simplicity that A is symmetric. In what follows we denote by D_i the derivative with respect to the i -th entry of the vector $\xi \in \mathbb{R}^d$ (so that $D_i a(x, \xi) := \nabla_{\xi_i} a(x, \xi)$). Informal computations (that are made rigorous in this paper) suggest that $D\bar{a}(\xi) = \bar{a}_\xi$, where \bar{a}_ξ is

the homogenized matrix associated with the random coefficient field

$$a_\xi := Da(\cdot, \xi + \nabla\phi_\xi) = (1 + |\xi + \nabla\phi_\xi|^{p-2})A + (p-2)A \frac{(\xi + \nabla\phi_\xi) \otimes (\xi + \nabla\phi_\xi)}{|\xi + \nabla\phi_\xi|^2} |\xi + \nabla\phi_\xi|^{p-2}.$$

As a first attempt to control \bar{a}_ξ from below, we appeal to the Voigt-Reiss bounds (see [100, Section 1.6]), which yields after neglecting the second contribution to a_ξ

$$\bar{a}_\xi \geq \mathbb{E} \left[a_\xi^{-1} \right]^{-1} \geq A \mathbb{E} \left[(1 + |\xi + \nabla\phi_\xi|^{p-2})^{-1} \right]^{-1},$$

and amounts to controlling the harmonic average of $1 + |\xi + \nabla\phi_\xi|^{p-2}$ from below (and therefore to have information on the critical set of the harmonic coordinate $x \mapsto \xi \cdot x + \phi_\xi(x)$). A second attempt is to consider the specific direction $\xi \cdot \bar{a}_\xi \xi$. Starting point is the (informal) minimization problem

$$\begin{aligned} \xi \cdot \bar{a}_\xi \xi &= \inf_{\text{suitable } \nabla\psi} \mathbb{E} \left[(\xi + \nabla\psi) \cdot a_\xi(\xi + \nabla\psi) \right] \\ &\geq \inf_{\text{suitable } \nabla\psi} \mathbb{E} \left[(\xi + \nabla\psi) \cdot A(1 + |\xi + \nabla\phi_\xi|^{p-2})(\xi + \nabla\psi) \right]. \end{aligned}$$

Call $\nabla\psi^*$ a minimizer. By minimality of $\nabla\psi^*$, the definition (3.1.9) of the monotone map a , and the corrector equation (3.1.7), we then have

$$\begin{aligned} \xi \cdot \bar{a}_\xi \xi &\geq \mathbb{E} \left[(\xi + \nabla\psi^*) \cdot A(1 + |\xi + \nabla\phi_\xi|^{p-2})(\xi + \nabla\psi^*) \right] \\ &= \mathbb{E} \left[(\xi + \nabla\phi_\xi) \cdot A(1 + |\xi + \nabla\phi_\xi|^{p-2})(\xi + \nabla\psi^*) \right] \\ &= \mathbb{E} \left[(\xi + \nabla\psi^*) \cdot a(\xi + \nabla\phi_\xi) \right] \\ &\stackrel{(3.1.7)}{=} \mathbb{E} \left[(\xi + \nabla\phi_\xi) \cdot A(1 + |\xi + \nabla\phi_\xi|^{p-2})(\xi + \nabla\phi_\xi) \right] \gtrsim |\xi|^2 + |\xi|^p. \end{aligned}$$

In dimension $d = 1$, one directly has $D\bar{a}(\xi) \gtrsim 1 + |\xi|^{p-2}$, which yields (3.1.4). For $d > 1$ this is different since from the a priori estimate $\xi \cdot D\bar{a}(\xi)\xi \gtrsim |\xi|^2(1 + |\xi|^{p-2})$ we cannot deduce $e \cdot D\bar{a}(\xi)e \gtrsim |e|^2(1 + |\xi|^{p-2})$ for general $e \in \mathbb{R}^d$, unless combined with some isotropy arguments (which would ensure that controlling one direction is enough to control all).

The upcoming results follow both paths. First we show that \bar{a} satisfies (3.1.4) provided we have a quantitative version of unique continuation, at least for periodic coefficients (as essentially noticed by Cherednichenko and Smyshlyaev in [42]).

Theorem 13. *Let A be a Q -periodic Lipschitz matrix field. For all $\xi \in \mathbb{R}^d$, denote by $\psi_\xi \in W_{\text{per}}^{1,p}(Q)$ the unique weak solution of*

$$-\nabla \cdot A(x) |\nabla\psi_\xi + \xi|^{p-2} (\nabla\psi_\xi + \xi) = 0.$$

Assume that for all $\xi \in \mathbb{R}^d$, there exists $r > 0$ such that the r -tubular neighborhood $\mathcal{T}_r(\xi) = \{x + B_r \mid x \in \mathcal{C}(\xi)\}$ of the critical set $\mathcal{C}(\xi) = \{x \in \mathbb{R}^d \mid \xi + \nabla\psi_\xi(x) = 0\}$ is such that $\mathbb{R}^d \setminus \mathcal{T}_r(\xi)$ is a connected set. Then there exists $C > 0$ such that \bar{a} satisfies (3.1.3) and (3.1.4).

Remark 3.1.1. *The assumptions of Theorem 13 are quite strong. They are satisfied in dimension $d = 2$ by [2] (which shows that $\mathcal{C}(\xi) \cap Q$ is indeed a finite union of points) – but, in this setting, regularity for the homogenized operator also holds because \bar{a} is non-degenerate, cf. [115]. For $d > 2$ this is a widely open problem. For linear equations, this follows from [41].*

In the random setting, we have a positive result assuming the statistical isotropy of A , which is new and holds in any dimension.

Theorem 14. *On top of Hypothesis 3.1.1, assume that $A(x) = b(x)\text{Id}$ for some scalar-valued function b and that for all $R \in SO(d)$, $b(R\cdot)$ and b have the same (joint) distribution (in which case A is statistically isotropic). Then there exists $C > 0$ such that \bar{a} satisfies (3.1.3) and (3.1.4).*

These results are proved in Appendices 3.7.5 and 3.7.6. We suspect that Theorems 13 and 14 hold under weaker assumptions but we are currently unable to establish this (even using the quantitative estimates proved in this paper).

3.2 Strategy of the proof, extensions, and limitations

3.2.1 General strategy and auxiliary results

In this section, we motivate our general strategy by comparison to the linear setting. To be more precise, we also consider the linear homogenization problem on \mathbb{R}^d

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla v_\varepsilon(x) = \nabla \cdot f(x), \quad (3.2.1)$$

with the same assumptions on A as in Hypothesis 3.1.2.

We start by defining the notion of (nonlinear) flux corrector.

Definition 3.2.1. *For all $\xi \in \mathbb{R}^d$, there exists a unique skew-symmetric random matrix field $(\sigma_{\xi,ij})_{1 \leq i,j \leq d}$, which solves almost surely in the distributional sense in \mathbb{R}^d the flux corrector equation*

$$-\Delta \sigma_{\xi,ij} = \partial_i(a(\cdot, \xi + \nabla \phi_\xi) \cdot e_j) - \partial_j(a(\cdot, \xi + \nabla \phi_\xi) \cdot e_i), \quad (3.2.2)$$

which is anchored at the origin via $\int_B \sigma_\xi = 0$, and whose gradient $\nabla \sigma_\xi$ is stationary, has vanishing expectation $\mathbb{E}[\nabla \sigma_\xi] = 0$, and is bounded in the sense

$$\mathbb{E}\left[|\nabla \sigma_\xi|^{\frac{p}{p-1}}\right] \lesssim |\xi|^2 + |\xi|^p.$$

In addition we have

$$\nabla \cdot \sigma_\xi = a(\cdot, \xi + \nabla \phi_\xi) - \bar{a}(\xi), \quad (3.2.3)$$

where the divergence of a matrix field σ is understood as $(\nabla \cdot \sigma)_i = \sum_{j=1}^d \partial_j \sigma_{\xi,ij}$.

The proof of existence and uniqueness of σ_ξ is essentially the same as in the linear setting in [80] when $p = 2$, provided the adaptations of [24, Lemma 1] for $p > 2$.

Note that in the linear case (3.2.1), we have the scaling relation $(\phi_{t\xi}, \sigma_{t\xi}) = t(\phi_\xi, \sigma_\xi)$ (in the nonlinear setting, the homogeneities of ϕ_ξ and of σ_ξ with respect to ξ are different, and not explicit since a has no homogeneity due to the regularization at 0, cf. (3.1.9)).

The interest of the flux corrector is that it allows to put the remainder in the equation satisfied by the two-scale expansion error in conservative form (which is convenient to use energy estimates). More precisely, in the linear setting, the two-scale expansion of v_ε takes the simpler form

$$\bar{v}_\varepsilon^{2s} := \bar{v} + \varepsilon \phi_i\left(\frac{\cdot}{\varepsilon}\right) \partial_i \bar{v}, \quad (3.2.4)$$

where \bar{v} solves the homogenized equation $-\nabla \cdot \bar{A} \nabla \bar{v} = \nabla \cdot f$, ϕ_i denotes the corrector in the canonical direction e_i , and where we use implicit summation on the repeated index i . Since $\phi_i\left(\frac{x}{\varepsilon}\right) \partial_i \bar{v}(x) =$

$\phi_{\nabla\bar{v}(x)}(\frac{x}{\varepsilon})$, (3.2.4) is indeed an elementary version of (3.1.11). In this case, one can prove (making a crucial use of the skew-symmetry of σ – see the proof of Theorem 12 in our nonlinear setting) that $v_\varepsilon - \bar{v}_\varepsilon^{2s}$ satisfies the equation

$$-\nabla \cdot A(\frac{\cdot}{\varepsilon})\nabla(v_\varepsilon - \bar{v}_\varepsilon^{2s}) = \varepsilon\nabla \cdot ((A\phi_i - \sigma_i)(\frac{\cdot}{\varepsilon})\nabla\partial_i\bar{v}) \quad (3.2.5)$$

(the factor ε comes by scaling since there are two gradients on \bar{v} on the right-hand side). This yields the bound (3.1.13) by an energy estimate provided we control the growth of (ϕ, σ) . In the nonlinear setting, we rather expect a term of the form $a_\xi\phi_\xi - \sigma_\xi$ in the right-hand side of (3.2.5) (see the proof of Theorem 12 for the precise statement). Note that $a_\xi\phi_\xi$ and σ_ξ both scale like $(|\xi|^2 + |\xi|^p)^{\frac{p-1}{p}}$ (which has to be compared to the difference of scalings of ϕ_ξ and σ_ξ themselves, cf. Definition 3.2.1).

The main result on the extended nonlinear corrector (ϕ_ξ, σ_ξ) is as follows. (Despite the above discussion, we do not make a difference in the scalings of ϕ_ξ and σ_ξ wrt ξ . This dependence is indeed not explicit as one could have expected, due to the nonlinear nature of the problem.)

Theorem 15. *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, the stationary extended corrector gradient $\nabla(\phi_\xi, \sigma_\xi)$ satisfies for some exponent $\alpha > 0$ depending on λ, p , and d , and some constant $c_\xi > 0$ depending further on $|\xi|$,*

$$\mathbb{E}\left[\exp(c_\xi|\nabla(\phi_\xi, \sigma_\xi)|^\alpha)\right] \leq 2. \quad (3.2.6)$$

For all $g \in L^2(\mathbb{R}^d)$, averages of $(\nabla\phi_\xi, \nabla\sigma_{\xi,ij})$ display the CLT scaling⁴ in the form

$$\left|\int g(\nabla\phi_\xi, \nabla\sigma_\xi)\right| \leq C_{\xi,g}\left(\int |g|^2\right)^{\frac{1}{2}}, \quad (3.2.7)$$

where $C_{\xi,g}$ is a random variable with finite stretched exponential moment

$$\mathbb{E}\left[\exp(c_\xi C_{\xi,g}^\alpha)\right] \leq 2,$$

for some exponent $\alpha > 0$ depending on p, λ , and d , and some constant $c_\xi > 0$ further depending on $|\xi|$ (but all independent of g). This directly implies the following bounds on the growth of (ϕ_ξ, σ_ξ) : For all $x \in \mathbb{R}^d$,

$$|(\phi_\xi, \sigma_\xi)(x)| \leq C_{x,\xi}\mu_d(x), \quad (3.2.8)$$

where μ_d is defined in (3.1.12) and $C_{x,\xi}$ is a random variable with the same stochastic integrability as $C_{\xi,g}$.

Remark 3.2.1. *Under Hypothesis 3.1.1, for Q -periodic matrix fields A , the nonlinear correctors are bounded in $C^{1,\alpha}(Q)$ (no restriction on $p \geq 2$).*

As pointed out in [5, 6] and used in [68] for $p = 2$, controlling the growth of correctors is not enough in the nonlinear setting. This should not come as a surprise when comparing (3.2.4) to (3.1.11). The additional gradient on $\nabla\bar{v}$ in the right-hand side of (3.2.5) (at the origin of the factor ε) indeed comes for local differences

$$\phi_{\nabla\bar{v}(x_1)} - \phi_{\nabla\bar{v}(x_2)} = |x_1 - x_2|\phi_{\frac{\nabla\bar{v}(x_1) - \nabla\bar{v}(x_2)}{|x_1 - x_2|}},$$

when reformulated by taking advantage of the linearity of the corrector. In the nonlinear setting, this identity does not hold any longer for ϕ_ξ (and even less for σ_ξ by homogeneity). It is however replaced by the following Lipschitz-continuity results.

⁴Indeed, for $g = |B_R|^{-1}\mathbb{1}_{B_R}$, the right-hand side of (3.2.7) scales like $R^{-\frac{d}{2}}$.

Corollary 16 (Control of nonlinear corrector differences). *Under Hypothesis 3.1.2, for all $K > 0$ and all $\xi_1, \xi_2 \in \mathbb{R}^d$ with $|\xi_1|, |\xi_2| \leq K$, we have for all $x \in \mathbb{R}^d$,*

$$|\nabla(\phi_{\xi_1} - \phi_{\xi_2}, \sigma_{\xi_1} - \sigma_{\xi_2})(x)| \leq C_{x,K}|\xi_1 - \xi_2|, \quad |(\phi_{\xi_1} - \phi_{\xi_2}, \sigma_{\xi_1} - \sigma_{\xi_2})(x)| \leq C_{x,K}|\xi_1 - \xi_2|\mu_d(x),$$

where $C_{x,K}$ is a random variable with finite stretched exponential moment depending only on d, p, λ , and K . In particular, $\xi \mapsto \bar{a}(\xi)$ is locally $C^{1,1}$.

Remark 3.2.2. *Under Hypothesis 3.1.1, for Q -periodic matrix fields A , corrector differences are controlled by $|\xi_1 - \xi_2|$ in $C^{1,\alpha}(Q)$ (no restriction on $p \geq 2$), and \bar{a} is $C^{1,1}$ as well.*

To prove such a result, we have to analyze the dependence of correctors with respect to the direction ξ , which leads us to the notion of linearized correctors. The control of (nonlinear) corrector differences is obtained as a corollary of bounds on these linearized correctors.

The following lemma (which is only used in an approximation argument) defines these linearized correctors. It is a consequence of [43, Section 4] and [24, Lemma 1] (which is devoted to the existence and uniqueness for linear corrector equations with prescribed in advance unbounded and degenerate coefficients). In the actual proofs, we shall only consider linearized correctors on bounded domains (see the discussion on periodization below), and the definition and control of the whole-space linearized corrector is given for completeness.

Lemma 13. *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$ set $a_\xi := Da(\cdot, \xi + \nabla\phi_\xi)$. For all $e \in \mathbb{R}^d$ there exists a unique random field $\tilde{\phi}_{\xi,e}$ that solves almost surely in the distributional sense in \mathbb{R}^d the linearized corrector equation*

$$-\nabla \cdot a_\xi(e + \nabla\tilde{\phi}_{\xi,e}) = 0, \tag{3.2.9}$$

anchored at the origin via $\int_B \tilde{\phi}_{\xi,e} = 0$, and whose gradient $\nabla\tilde{\phi}_{\xi,e}$ is stationary, has vanishing expectation $\mathbb{E}[\nabla\tilde{\phi}_{\xi,e}] = 0$, and is bounded in the sense of

$$\mathbb{E}\left[|\nabla\tilde{\phi}_{\xi,e}|^2(1 + |\xi + \nabla\phi_\xi|)^{p-2}\right] \lesssim (1 + |\xi|^{p-2})|e|^2.$$

In addition, there exists a skew-symmetric random matrix field $(\sigma_{\xi,e,ij})_{1 \leq i,j \leq d}$, which solves almost surely in the distributional sense in \mathbb{R}^d the linearized flux corrector equation

$$-\Delta\tilde{\sigma}_{\xi,e,ij} = \partial_i(a_\xi(e + \nabla\tilde{\phi}_{\xi,e}) \cdot e_j) - \partial_j(a_\xi(e + \nabla\tilde{\phi}_{\xi,e}) \cdot e_i), \tag{3.2.10}$$

which is anchored via $\int_B \tilde{\sigma}_{\xi,e} = 0$ almost surely, whose gradient $\nabla\tilde{\sigma}_{\xi,e}$ is stationary and is bounded in the sense

$$\mathbb{E}\left[|\nabla\tilde{\sigma}_{\xi,e}|^{\frac{p}{p-1}}\right] \lesssim (1 + |\xi|^{p-2})^{\frac{p}{p-1}}|e|^{\frac{p}{p-1}},$$

and which satisfies the property

$$\nabla \cdot \tilde{\sigma}_{\xi,e} = a_\xi(e + \nabla\tilde{\phi}_{\xi,e}) - \bar{a}_\xi e,$$

where $\bar{a}_\xi e = \mathbb{E}\left[a_\xi(e + \nabla\tilde{\phi}_{\xi,e})\right]$.

The upcoming theorem gives further information on the linearized correctors, in line with Theorem 15 for the nonlinear correctors.

Theorem 17. *Under Hypothesis 3.1.2, for all $\xi, e \in \mathbb{R}^d$ with $|e| = 1$, the stationary extended linearized corrector gradient $\nabla(\tilde{\phi}_{\xi,e}, \tilde{\sigma}_{\xi,e})$ satisfies for some exponent $\alpha_\xi > 0$ and some constant $c_\xi > 0$ depending on λ, p, d , and $|\xi|$,*

$$\mathbb{E} \left[\exp(c_\xi |\nabla(\tilde{\phi}_{\xi,e}, \tilde{\sigma}_{\xi,e})|^{\alpha_\xi}) \right] \leq 2. \quad (3.2.11)$$

For all $g \in L^2(\mathbb{R}^d)$, averages of $(\nabla\tilde{\phi}_{\xi,e}, \nabla\tilde{\sigma}_{\xi,e})$ display the CLT scaling in the form

$$\left| \int g(\nabla\tilde{\phi}_{\xi,e}, \nabla\tilde{\sigma}_{\xi,e}) \right| \leq C_{\xi,g} \left(\int |g|^2 \right)^{\frac{1}{2}}, \quad (3.2.12)$$

where $C_{\xi,g}$ is a random variable with stretched exponential moments $\mathbb{E} \left[\exp(c_\xi C_{\xi,g}^{\alpha_\xi}) \right] \leq 2$, for some exponent $\alpha_\xi > 0$ and some constant $c_\xi > 0$ depending on p, λ, d , and $|\xi|$. This directly implies that for all $x \in \mathbb{R}^d$, we have $|(\tilde{\phi}_{\xi,e}, \tilde{\sigma}_{\xi,e})(x)| \leq C_{x,\xi} \mu_d(x)$, where $C_{x,\xi}$ is a random variable with the same moment bounds as $C_{\xi,g}$.

Remark 3.2.3. *Under Hypothesis 3.1.1, for Q -periodic matrix fields A , the linearized correctors exist and are bounded in $C^{1,\alpha}(Q)$ (no restriction on $p \geq 2$).*

The general strategy we described above is essentially the same as in [68] for $p = 2$, itself very close to the strategy in the linear setting [89]. In line with [82, 83, 84, 79, 137, 101], we are after Meyers' type estimates for the linearized operator $-\nabla \cdot a_\xi \nabla$ (defined in Lemma 13). The main difference between this work and [5, 6, 68] is the way we obtain these estimates – that is, in the annealed version of Theorem 22. For $p = 2$, annealed Meyers estimates (even without loss of stochastic integrability) follow rather directly from the boundedness of a_ξ from above and below. In our genuinely nonlinear setting, these estimates are difficult to establish since a_ξ may be degenerate and unbounded (recall that the original proof of Meyers' estimates argues by perturbation and requires that $\frac{\inf a_\xi}{\sup a_\xi} > 0$, whereas we have $\frac{\inf a_\xi}{\sup a_\xi} = 0$ almost surely). There are two reasons why $\frac{\inf a_\xi}{\sup a_\xi} = 0$: the degeneracy of a_ξ and the unboundedness of a_ξ . The upcoming technical discussion points out the difficulties in the analysis, give hints on how to treat them, and explains how it leads to the above results.

First, in view of the difficulty to control the critical set of harmonic coordinates, we have imposed a non-degeneracy condition from the very beginning and assumed that $a(x, \cdot)$ satisfies (3.1.4) for $\kappa = 1$ (which rules out the p -Laplacian, but not the p -Laplacian regularized at 0). Let us emphasize that this only yields the *non-degeneracy* of the linearized operator in a *perturbative* way (it disappears in the regime when the solution has a large gradient). Doing so, the main remaining (and most important) difficulty is the unboundedness of the coefficients of the linearized operator.

Meyers estimates are the object of Section 3.3. We start with Subsections 3.3.1 & 3.3.2 and the quenched Meyers estimates in the large. The starting point is the energy estimate (3.1.8) in form of $\mathbb{E} \left[|a_\xi|^{\frac{p}{p-2}} \right] \lesssim 1 + |\xi|^p$, which yields two challenges: it is stochastically-averaged and gives a poor integrability for large p . We begin with the stochastically-averaged part. The idea is to relax the condition $\frac{\inf a_\xi}{\sup a_\xi} > 0$ into the milder requirement $\frac{\inf \int_{B_r(x)} a_\xi}{\sup \int_{B_r(x)} a_\xi} > 0$ for some $r > 0$, which, in turn, yields the weaker (yet sufficient) Meyers' estimate at scale $r > 0$ in form of: For u, g related via $-\nabla \cdot a_\xi \nabla u = \nabla \cdot g$, we have for some Meyers' exponent $m > 2$

$$\int_{\mathbb{R}^d} \left(\int_{B_r(x)} |\nabla u|^2 \right)^{\frac{m}{2}} dx \lesssim \left(\int_{\mathbb{R}^d} \left(\int_{B_r(x)} |\nabla u|^2 \right) dx \right)^{\frac{m}{2}} + \int_{\mathbb{R}^d} \left(\int_{B_r(x)} |g|^2 \right)^{\frac{m}{2}} dx. \quad (3.2.13)$$

Such a weakening of Meyers' estimates still does not hold in our setting because estimate (3.1.8) cannot be turned into an almost sure bound (with a uniform choice of r). A further weakening consists in letting the radius of the ball $B_r(x)$ in (3.2.13) be random and depend on the point $x \in \mathbb{R}^d$. We thus introduce in Definition 3.3.1 the Meyers minimal radius r_\star , a random field on \mathbb{R}^d which essentially ensures that, with the notation $B_\star(x) := B_{r_\star(x)}(x)$, $\frac{\inf_x \int_{B_\star(x)} a_\xi}{\sup_x \int_{B_\star(x)} a_\xi} > 0$ (with a deterministic positive lower bound). Such a form of the Meyers' estimates with a random scale r_\star was first used by Armstrong and Dario in [4] to deal with homogenization in percolation. To obtain an estimate in the spirit of (3.2.13) (with $B_r(x)$ replaced by $B_\star(x)$), we rely on the standard proof of Meyers' estimates going through a reverse Hölder inequality and Gehring's lemma. For uniformly elliptic equations, the reverse Hölder inequality is a consequence of Caccioppoli's inequality and of the Sobolev embedding. Caccioppoli's inequality for a_ξ -harmonic functions u (say, $-\nabla \cdot a_\xi \nabla u = 0$ in the ball B_{2R} centered at 0) typically takes the form

$$\int_{B_R} \nabla u \cdot a_\xi \nabla u \lesssim \frac{1}{R^2} \int_{B_{2R}} |u|^2 |a_\xi|.$$

Assuming that $\int_{B_{2R}} u = 0$, the next step is to appeal to the Poincaré-Sobolev inequality $\int_{B_{2R}} |u|^2 \lesssim R^2 \left(\int_{B_{2R}} |\nabla u|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}}$. The random unbounded coefficient $|a_\xi| \sim 1 + |\nabla \phi_\xi + \xi|^{p-2}$ is however in the way (and not of the Muckenhoupt class), and the estimate (3.1.8) only yields $\int_{B_{2R}} |a_\xi|^{\frac{p}{p-2}} \sim \int_{B_{2R}} 1 + |\nabla \phi_\xi + \xi|^p \lesssim 1 + |\xi|^p$ provided $2R \geq r_\star(0)$ (as a consequence of the definition of r_\star). We thus need to first appeal to Hölder's inequality with exponents $(\frac{p}{p-2}, \frac{p}{2})$ to upgrade the Caccioppoli inequality into

$$\int_{B_R} \nabla u \cdot a_\xi \nabla u \lesssim \frac{1}{R^2} \left(\int_{B_{2R}} |u|^p \right)^{\frac{2}{p}}.$$

Then, assuming that $p < \frac{2d}{d-2}$, the Poincaré-Sobolev inequality yields the desired reverse Hölder's inequality $\int_{B_R} \nabla u \cdot a_\xi \nabla u \lesssim \left(\int_{B_{2R}} |\nabla u|^{p_*} \right)^{\frac{2}{p_*}}$ with exponent $p_* = \frac{d+p}{dp} < 2$. In dimension $d = 3$, the condition on p reads $p < 6$. To reach the larger range of exponents of Theorem 12, the main observation is that one can improve Caccioppoli's inequality by choosing wisely the cut-off function – following an idea by Bella and Schäffner [25] (see Lemma 17). Doing so, we are able to use the Sobolev embedding in dimension $d - 1$ rather than d , and therefore treat all exponents $p \geq 2$ in the physically-relevant dimension $d = 3$. This yields the large scale Meyers' estimates of Theorem 18 (with a condition on p in dimensions $d > 3$).

In the form of Theorem 18, the Meyers estimates are only useful if we have a good control of the Meyers minimal radius r_\star (the larger r_\star , the weaker the estimate), which we obtain in Subsection 3.3.3. Since r_\star is a stationary random field, by control we mean moment bounds in probability. This is where our contribution further differs from other contributions of the literature on degenerate or unbounded coefficients: the statistics of a_ξ (which drives the moments of r_\star) are not given a priori (as opposed to the percolation cluster in [4], or to the moment bounds on a_ξ in [24]) but part of the problem – estimate (3.2.6), which is essentially equivalent to the control of r_\star in Theorem 21, is the very output of the analysis. Indeed, the coefficients a_ξ are a function of $\nabla \phi_\xi$, which depends itself on the random input A as the solution of the nonlinear corrector equation (3.1.7) (and therefore far from explicit).

Here comes the second ingredient to our approach: sensitivity calculus and concentration of measures (see Appendix 3.7.7). On the one hand, using the Meyers estimate in the large (in its

improved weighted form of Theorem 19 based on the hole-filling estimate) and sensitivity calculus, we control the stochastic moments of averages of $\nabla\phi_\xi$ by the CLT scaling and moments of r_\star – see Proposition 5. On the other hand, by Caccioppoli’s inequality for the nonlinear corrector equation, we control super level sets of r_\star by moments of averages of $\nabla\phi_\xi$. The desired control of the moments of r_\star of Theorem 21 follows by combining these two nonlinear estimates and taking advantage of the CLT scaling and the small room given by the hole-filling to buckle and single out r_\star .

Once we have good control of r_\star , the quenched large scale Meyers estimates of Theorem 18 can finally be upgraded to the annealed Meyers estimates of Theorem 22 (as introduced by Duerinckx and Otto in [56], using Shen’s lemma [147]), cf. Subsection 3.3.4.

Having these estimates at hand, Theorem 15 follows from another application of sensitivity calculus, cf. Section 3.4. The proof of Theorem 17 is similar, cf. Section 3.5. Although we have Theorem 22, we cannot use the elegant and efficient buckling argument of [137] for the linearized corrector either (due to unboundedness), and we have to pass again via the super level sets of another minimal radius. We then conclude with the routine proof of Theorem 12 in Section 3.6.

In order to establish these estimates on nonlinear and linearized correctors, we first use an approximation argument which allows us to discard the long-range correlations induced by the elliptic character of the equation, and actually define the associated approximations of the nonlinear flux corrector and of the linearized correctors by elementary deterministic arguments. In this contribution, we proceed by periodization in law, which has the advantage to keep differential relations neat in the approximation (in particular the identity (3.2.3)). For all $L > 0$, we introduce in Definition 3.7.2 (see Appendix 3.7.7) a probability measure \mathbb{P}_L taking values in $Q_L = [-\frac{L}{2}, \frac{L}{2}]^d$ -periodic functions. The associated maps $x \mapsto a(x, \xi)$ are therefore Q_L -periodic \mathbb{P}_L -almost surely, and the corrector equations are posed on the bounded domain Q_L . The coupling between \mathbb{P} and \mathbb{P}_L given in Lemma 29 then allows us to infer results on \mathbb{P} from corresponding results on \mathbb{P}_L , see in particular Proposition 10. The choice of periodization in law is convenient but not essential. In the linear setting one often adds a massive term to the equation (which yields an exponential cut-off for long-range interactions) [82, 83, 84, 80, 89] or disintegrates scales via a semi-group approach [89, 81, 47]. All our estimates are proved for fixed periodization and the above results follow by letting the periodization parameter go to infinity.

3.2.2 Towards large-scale regularity and a nonlinear theory of fluctuations

As discussed in Chapter 1 and in Subsection 6.2.1 in Chapter 6 (see the paragraph on fluctuations), there are three main types of results in (quantitative) stochastic homogenization of the linear elliptic equation (3.2.1):

- Control of oscillations of the solution via a quantitative two-scale expansion (here in form of Theorem 12 in the nonlinear setting);
- Large-scale regularity for the operator $-\nabla \cdot A(\frac{x}{\varepsilon})\nabla$;
- Control of the fluctuations of observables of the form $\int g \cdot \nabla u_\varepsilon$ and $\int g \cdot A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon$ (using the so-called homogenization commutator).

Let us start with large-scale regularity. The general principle [16, 17, 10, 80] is that the heterogeneous equation should possess the same regularity properties as the homogenized operator at the scale at which homogenization kicks in (characterized by the size of the corrector). In our range $2 \leq p < \frac{2(d-1)}{d-3}$ of exponents (for which we already control the growth of correctors) and if we restrict ourselves

to scalar equations, the results [30] by Bella and Schäffner (combined with the $C_{\text{loc}}^{1,1}$ regularity of $\xi \mapsto \bar{a}(\xi)$) should ensure that $-\nabla \cdot \bar{a}(\nabla)$ does possess nice regularity theory (both in terms $C^{1,\alpha}$ regularity and nonlinear Calderón-Zygmund theory), so that there is no obstruction to large-scale regularity in this setting. Such large-scale regularity would typically allow to upgrade the quantitative two-scale expansion of (3.1.13) stated in $L^2(\mathbb{R}^d)$ to any $L^q(\mathbb{R}^d)$ with $1 < q < \infty$ (with the same convergence rate in ε), and in particular cover the natural $L^p(\mathbb{R}^d)$ -norm.

Next to the large-scale regularity for the nonlinear operator $-\nabla \cdot a(\cdot, \nabla)$, one may wish to establish large-scale regularity for the linearized operator $-\nabla \cdot a_\xi \nabla$. In Section 3.3 below, large-scale Meyers estimates are proved (in their convenient annealed form of Theorem 22). Although this perturbative result is enough to prove the quantitative two-scale expansion of Theorem 12, Theorem 22 should hold for all exponents $1 < q < \infty$ by adapting the arguments of [80] to mildly unbounded coefficients (a_ξ indeed has finite stretched exponential moments by Lemma 21 and Theorem 21) and using the bounds of Theorem 17 on the linearized correctors (to control the scale at which homogenization kicks in).

We now turn fluctuations, the second main topic of stochastic homogenization. In the linear setting, the theory of fluctuations relies on the quantity $\Xi_\varepsilon := (A(\frac{\cdot}{\varepsilon}) - \bar{A})\nabla u_\varepsilon$, called the homogenization commutator in [55] (this object first appeared in a slightly different form in [10] in the context of large-scale regularity). The homogenization commutator is a natural object to consider since observables of Ξ_ε can be post-processed into observables of the field and flux (the two cornerstones of homogenization), that is, $\int g \cdot \nabla u_\varepsilon$ and $\int g \cdot A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon$ – see below for the argument in the nonlinear setting. The theory splits into two parts. On the one hand, fluctuations of the homogenization commutator Ξ_ε can be accurately described by the fluctuations of its two-scale expansion based on the standard commutator $\Xi := (A - \bar{A})(\nabla\phi + \text{Id})$, cf. [55, 62, 56]. On the other hand, the standard commutator Ξ behaves (in law) on large scales as a Gaussian random field, cf. [55, 56, 57]. The proofs of these results in the linear setting make heavy use of large-scale regularity. In order to extend this theory to the nonlinear setting, one mainly needs large-scale regularity for the linearized operator and a suitable notion of nonlinear homogenization commutator. A naïve guess would be to define the nonlinear commutator as $a_\varepsilon(\nabla u_\varepsilon) - \bar{a}(\nabla u_\varepsilon)$. This quantity however does not weakly converge to zero since \bar{a} is nonlinear. We have to devise a quantity that is compatible with weak convergence and still encapsulates the diagram of Figure 1.1.2. To this aim we reformulate the constitutive law by linearizing (assuming that \bar{a} is differentiable): Since $a_\varepsilon(0) = \bar{a}(0) = 0$, we then have

$$a(\frac{\cdot}{\varepsilon}, \xi) = a(\frac{\cdot}{\varepsilon}, \xi) - a(\frac{\cdot}{\varepsilon}, 0) = \left(\int_0^1 Da(\frac{\cdot}{\varepsilon}, t\xi) dt \right) \xi, \quad \bar{a}(\xi) = \bar{a}(\xi) - \bar{a}(0) = \left(\int_0^1 D\bar{a}(t\xi) dt \right) \xi,$$

so that the diagram of Figure 1.1.2 takes the equivalent form of Figure 3.1.

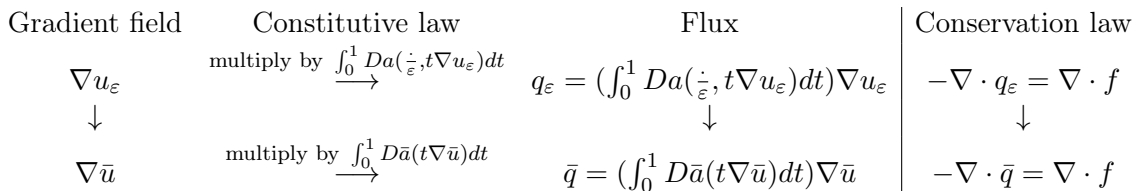


Figure 3.1: Reformulation of the commutative diagram

In this way, homogenization can be concisely reduced to the single condition $(\int_0^1 Da(\frac{\cdot}{\varepsilon}, t\nabla u_\varepsilon)dt)\nabla u_\varepsilon - (\int_0^1 D\bar{a}(t\nabla\bar{u})dt)\nabla u_\varepsilon \rightarrow 0$, and we define the nonlinear homogenization commutator of u_ε as

$$\Xi_\varepsilon(f) := \left(\int_0^1 Da(\frac{\cdot}{\varepsilon}, t\nabla u_\varepsilon)dt \right) \nabla u_\varepsilon - \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right) \nabla u_\varepsilon. \quad (3.2.14)$$

Let us argue that, as in the linear setting, the homogenization commutator contains both the fluctuations of the field ∇u_ε and of the flux $a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon)$. We start with ∇u_ε and consider fluctuations of the observable $\int g \cdot \nabla u_\varepsilon$. We introduce the auxiliary map \bar{v} solution of the linear equation

$$\nabla \cdot \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right)^* \nabla \bar{v} = \nabla \cdot g, \quad (3.2.15)$$

(which we assume to be well-posed in this discussion – the notation $*$ is used for the transposition). Then we have

$$\begin{aligned} \int g \cdot \nabla u_\varepsilon &\stackrel{(3.2.15)}{=} - \int \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right)^* \nabla \bar{v} \cdot \nabla u_\varepsilon \\ &\stackrel{(3.2.14)}{=} - \int \nabla \bar{v} \cdot (\Xi_\varepsilon(f) - a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon)) \\ &\stackrel{(3.1.5)}{=} - \int \nabla \bar{v} \cdot \Xi_\varepsilon(f) - \int \nabla \bar{v} \cdot f, \end{aligned}$$

so that the fluctuations of $\int g \cdot \nabla u_\varepsilon$ are given by those of $-\int \nabla \bar{v} \cdot \Xi_\varepsilon(f)$ (since the additional term $-\int \nabla \bar{v} \cdot f$ is deterministic). Likewise, for the flux we introduce the solution \bar{w} of

$$\nabla \cdot \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right)^* \nabla \bar{w} = \nabla \cdot \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right)^* g, \quad (3.2.16)$$

and obtain

$$\begin{aligned} \int g \cdot a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon) &\stackrel{(3.1.5)}{=} \int (g - \nabla \bar{w}) \cdot a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon) - \int \nabla \bar{w} \cdot f \\ &\stackrel{(3.2.14)}{=} \int (g - \nabla \bar{w}) \cdot \Xi_\varepsilon(f) + \int (g - \nabla \bar{w}) \cdot \left(\int_0^1 D\bar{a}(t\nabla\bar{u})dt \right) \nabla u_\varepsilon - \int \nabla \bar{w} \cdot f \\ &\stackrel{(3.2.16)}{=} \int (g - \nabla \bar{w}) \cdot \Xi_\varepsilon(f) - \int \nabla \bar{w} \cdot f, \end{aligned}$$

so that the fluctuations of $\int g \cdot a(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon)$ are given by those of $-\int (g - \nabla \bar{w}) \cdot \Xi_\varepsilon(f)$ (since the additional term $-\int \nabla \bar{w} \cdot f$ is deterministic).

Next to the homogenization commutator of the solution, we introduce the standard homogenization commutator, associated with the corrector. For all $\xi \in \mathbb{R}^d$, we define

$$\Xi_\xi := \left(\int_0^1 Da(\cdot, t(\xi + \nabla\phi_\xi))dt \right) (\xi + \nabla\phi_\xi) - \left(\int_0^1 D\bar{a}(t\xi)dt \right) (\xi + \nabla\phi_\xi). \quad (3.2.17)$$

The pathwise structure of fluctuations (in the form of a quantitative two-scale expansion of commutators in the spirit of (3.1.11) and accurate in the fluctuation scaling – see [55, 62, 56] in the linear setting) and the scaling limit of the standard commutator (see [55, 56, 57] in the linear setting) will be the object of companion articles.

3.2.3 Extensions and limitations

Hypothesis 3.1.2 makes several assumptions on the monotone operator and the randomness:

- The underlying probability law is Gaussian with integrable correlations;
- The monotone map $a(x, \xi)$ is a multiple of $(1 + |\xi|^{p-2})\xi$, the randomness is multiplicative (in form a random matrix field), and the admissible range of p depends on d ;
- The spatial dependence $x \mapsto a(x, \xi)$ is smooth on a deterministic level;
- If it admits a variational form, the operator is associated with a convex energy functional.

Several of these assumptions can be slightly relaxed, while others are crucial. They are discussed in the following paragraphs.

Probability laws

Consider our multiplicative model. Our approach is based on a sensitivity calculus which allows us to linearize quantities with respect to the randomness (say, wrt A) and on functional inequalities which allow us to control variances using this sensitivity calculus. In Hypothesis 3.1.2 we consider a Gaussian random field with integrable covariance function, and one might wonder to what extent Gaussianity and the integrability of the covariance functions are necessary. Our argument strongly relies on the CLT scaling $r^{-\frac{d}{2}}$ of spatial averages $\int_{B_r} \nabla \phi_\xi$ of the corrector gradient, which essentially follow from the same property for $a(x, \xi) - \mathbb{E}[a(x, \xi)]$. On the one hand, sensitivity calculus, functional inequalities, and CLT scaling are not limited to Gaussian fields: they can be developed as soon as the stationary field A is constructed via a “hidden” product structure. In particular, the random checkerboard and various Poisson-based processes also enjoy such tools, and we refer the reader to [60, 59] for a systematic study of sensitivity calculus and (multiscale) functional inequalities for random fields commonly used in the mechanics of composite materials [152]. Such models could be considered here as well. On the other hand, the CLT scaling indeed requires the integrability of the covariance function. (Since there is some little room in the argument, one could consider a covariance function such that $\int_{\mathbb{R}^d} |c(x)|(1 + |x|)^{-\beta} dx < \infty$ provided $0 < \beta \ll 1$, but this is detail.) In order to address Gaussian coefficients with heavier tail, one would first need to establish (nonlinear and linear) large-scale regularity for the random operator (and its linearized version), as in [80, 89, 47]. By [30] and (3.1.3), there is no obstruction to this approach in the range $2 \leq p < \frac{2(d-1)}{d-3}$ for scalar equations.

Another setting would ensure that spatial averages of $a(x, \xi) - \mathbb{E}[a(x, \xi)]$ decay at the CLT scaling: if A has finite range of dependence – as addressed in [10, 5, 6] for $p = 2$. The quenched Meyers estimates of Theorem 18 (and its weighted version of Theorem 19) proved below do hold for general stationary ergodic coefficients – and therefore in the setting of finite range of dependence. They are however of little use without a good control of the Meyers minimal radius (provided by Theorem 21 for Gaussian coefficients with integrable covariance). In [10, 5, 6] for $p = 2$, estimates of moments of the corrector gradient (which would control the Meyers minimal radius) are obtained by combining a rate of convergence (any would do) for the Dirichlet problem with a Campanato argument based on $C^{1,\alpha}$ -regularity for the homogenized operator. In particular, one would have to adapt the duality arguments of [10, 5, 6] to $p > 2$ to prove convergence rates. Since the natural object considered in [10, 5, 6] is close to the homogenization commutator for $p = 2$, the nonlinear commutators we introduced in (3.2.14) and (3.2.17) might be good objects to start with for $p > 2$.

Form of the monotone map

There are three different assumptions when considering a monotone map of the form $(x, \xi) \mapsto a(x, \xi) = A(x)(1 + |\xi|^{p-2})\xi$: coercivity conditions, regularity with respect to ξ , and multiplicative character of the randomness. To start with, we must assume that $\xi \mapsto a(x, \xi)$ is twice-differentiable (for all x) in order to apply sensitivity calculus to the linearized corrector.

Multiplicative models. The form of a is such that one can easily differentiate a with respect to the randomness. This is not strictly necessary but quite convenient. Any model having such a property would do, and we can consider coefficients of the form $a(x, \xi) = \rho(A(x), \xi)\xi$ provided $M \mapsto \rho(M, \cdot)$ satisfies $|D_M \rho(M, \xi)| \lesssim 1 + |\xi|^{p-1}$ and $|D_M \partial_\xi \rho(M, \xi)| \lesssim 1 + |\xi|^{p-2}$. This holds for instance for

$$a(x, \xi) = \chi(x)a_1(\xi) + (1 - \chi(x))a_2(\xi), \quad (3.2.18)$$

where $\chi : \mathbb{R}^d \rightarrow [0, 1]$ is a smooth random field (with a sensitivity calculus and a suitable functional inequality) and a_1 and a_2 are two given (suitable) monotone maps. This model is more in line with composite materials.

Coercivity conditions. What is crucial is the strong monotonicity and non-degeneracy (3.1.4) for some $\kappa > 0$ (which we take to be 1 without loss of generality). Whereas local regularity would hold with $\kappa = 0$, the choice $\kappa > 0$ is forced upon us to rule out the degeneracy of the linearized operator (cf. the critical set of $\nabla \phi_\xi$). In particular, this condition does not hold for the p -Laplacian, to which our results do not apply. To our opinion, relaxing this condition constitutes a very challenging problem.

Restriction on p . Our assumptions (3.1.1) and (3.1.4) make the the operator have p -growth from above and below. In dimensions $d > 3$, we further impose the condition $2 \leq p < \frac{2(d-1)}{d-3}$. Doing so allows us to prove a quenched Meyers estimate in the large with an almost surely finite radius r_\star . In order to relax this condition, a possible starting point could be to introduce (yet another) minimal radius, say ρ_\star , which would be defined such that $\int_{B_r} r_\star(x)^q dx \lesssim 1$ for some suitable q depending on d, p and all $r \geq \rho_\star$. The additional difficulty is that one would have to deal with the three quantities $r_\star, \rho_\star, \nabla \phi_\xi$ together rather than the two quantities $r_\star, \nabla \phi_\xi$. Since there is only little room in the exponents for the buckling argument (see proof of Theorem 21), it is not clear to us whether this might indeed work.

Local regularity

It is quite tempting to assert that quantitative homogenization is a matter of large scales (or say, low frequencies), and that local regularity assumptions might be convenient but are not necessary. This is indeed quite relevant provided small scales do not interact with large scales. A convincing counterexample of that is the quasiperiodic (and almost periodic) setting, where small and large scales indeed interact via a weak Poincaré inequality in a high-dimensional torus, cf. [7]. In our nonlinear setting, local regularity is not so much needed for the nonlinear correctors, but it seems unavoidable for the linearization part. This regularity requirement could be weakened in several directions:

- Only a local C^α -control of the spatial dependence is needed for some $\alpha > 0$, and the control of this local norm can be random itself provided the latter has good moment bounds. In particular, with the same notation as in Hypothesis 3.1.2, this is the case for coefficients of the form $A(y) = B(G(y))$ provided the (non-negative) Fourier transform \hat{c} of the covariance function satisfies $\hat{c}(k) \leq (1 + |k|)^{-d-2\alpha'}$ (for some $\alpha' > \alpha$). Then $x \mapsto \|A\|_{C^\alpha(B(x))}$ is stationary

and has finite Gaussian moments (as a slight quantification of [101, Appendix A.3] shows). All our arguments can be adapted to this setting.

- In the proofs we use local regularity to control pointwise values of the (nonlinear and linear) corrector gradient by its local averages, and therefore control a local supremum by a local C^α -norm. Such a control would also follow from a local broken C^α -norm, so that one could in principle be able to deal with some A (or χ in (3.2.18)) that would be piecewise smooth (and a fortiori piecewise constant with smooth boundaries, covering the case of smooth inclusions in a background material). This constitutes a question of classical regularity theory. For linear equations and systems, this is proved in [110, 109] and for monotone operators and $p = 2$ in [134]. The case $p > 2$ constitutes an interesting independent problem.
- The state of the art of local regularity is as follows. For scalar equations, the structure can be quite general, and only requires the Hölder continuity of the map $x \mapsto a(x, \cdot)$ in the sense (see [105, Theorem 13])

$$\sup_{r>0} \int_0^r \frac{(\omega(\rho))^{\frac{2}{p}}}{\rho^\alpha} \frac{d\rho}{\rho} < +\infty, \tag{3.2.19}$$

for

$$\omega : r \in (0, +\infty) \mapsto \left(\sup_{z \in \mathbb{R}^d, B_r(x) \subset \mathbb{R}^d} \int_{B_r(x)} \left(\frac{a(y, z) - (a)_{x,r}(z)}{(|z| + 1)^{p-1}} \right)^2 dy \right)^{\frac{1}{2}},$$

for some $\alpha > 0$ and $(a)_{x,r}(z) := \int_{B_r(x)} a(y, z) dy$. For systems however, we are restricted to quasi-diagonal structures of the form $a(x, \xi) = \rho(x, |\xi|)\xi$, for some $\rho : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ (the so-called Uhlenbeck structure, see [153]).

Non-convex energy functionals?

It would be natural to try to extend these results to the setting of nonlinear elasticity, for which a large part of the qualitative theory has been established (cf. [130, 35, 123], and [58] for the most general results in this context). Besides the much more delicate regularity theory (cf. [124]), non-convexity essentially prevents us from using the corrector equation efficiently (cf. the counterexamples to the cell formula in the periodic setting by Müller [130], see also [22]), and may cause loss of ellipticity upon linearization (see [75] at the nonlinear level, and [95, 39, 71, 85] at the linear level) – except in the vicinity of the identity (cf. [129, 78], and the further use of rigidity [72] to establish quantitative results in this regime [133]). Hence, quantitative results in homogenization of nonlinear nonconvex models of elasticity remain widely out of reach today.

Perspectives

We refer to Section 6 for some possible other extension and open questions about this work.

3.3 Perturbative regularity theory for the linearized operator

In this section we consider periodized random operators a_L distributed according to the law \mathbb{P}_L given in Definition 3.7.2. In particular, for all $L \geq 1$, a_L is almost surely Q_L -periodic in its space variable, and remains random and stationary (this owes to the fact that we use periodization in law rather than naive periodization, cf. Appendix 3.7.7). This implies that ϕ_ξ and σ_ξ are necessarily Q_L -periodic fields almost surely, so that the equations (3.1.7) and (3.2.2) can be posed on Q_L rather

than \mathbb{R}^d – and likewise for the linearized correctors. For all $L \geq 1$ we use the notation $H_{\text{per}}^1(Q_L)$ (resp. $W_{\text{per}}^{1,p}(Q_L)$) for Q_L -periodic fields of $H_{\text{loc}}^1(\mathbb{R}^d)$ (resp. $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$) with vanishing average. Our aim is to prove regularity statements and bounds that are uniform in the periodization parameter $L \geq 1$.

3.3.1 The Meyers minimal radius

In this paragraph we introduce the notion of Meyers minimal radius, a stationary random field which quantifies the scale at which Meyers' estimates hold for the linearized operator. We start with a definition.

Definition 3.3.1 (Meyers minimal radius). *Let $\xi \in \mathbb{R}^d$, $L \geq 1$ and $c > 0$. If it exists, the (Q_L -periodic) minimal radius $r_{\star,\xi,L}(\cdot, c)$ is defined for all $x \in \mathbb{R}^d$ via*

$$r_{\star,\xi,L}(\cdot, c) : x \in \mathbb{R}^d \mapsto \inf_{y \in \mathbb{R}^d} \left(r_{\star,\xi,L}(y, c) + \ell|x - y| \right), \quad (3.3.1)$$

where $\ell = \frac{1}{9C\sqrt{d}} \wedge \frac{1}{16}$ (with C defined in Lemma 34) and for all $y \in \mathbb{R}^d$

$$r_{\star,\xi,L}(y, c) := \inf_{r=2^N, N \in \mathbb{N}} \left\{ \forall R \geq r, \int_{B_R(y)} |\nabla \phi_\xi|^p \leq c(1 + |\xi|^p) \right\}. \quad (3.3.2)$$

We now argue that $r_{\star,\xi,L}(\cdot, c)$ is a well-defined bounded random field if c is chosen large enough.

Lemma 14 (Well posedness of $r_{\star,\xi,L}$). *Let $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ and $L \geq 1$. There exist two constants $c_1, c_2 > 0$ depending on p and d such that, \mathbb{P}_L -almost surely, $r_{\star,\xi,L}$ satisfies*

$$r_{\star,\xi,L}(x, c_2) \leq r_{\star,\xi,L}(x, c_1) \leq r_{\star,\xi,L}(x, c_1), \quad (3.3.3)$$

and

$$r_{\star,\xi,L}(x, c_1) \leq L. \quad (3.3.4)$$

Proof. Without loss of generality, we may assume that $x = 0$. We start with the proof of (3.3.4), and then turn to the proof of (3.3.3). We let c denote a constant depending only on d , λ , and p , that may change from line to line.

Step 1. Proof of (3.3.4).

From the defining equation (3.1.7) for ϕ_ξ , we have

$$-\nabla \cdot (a(\cdot, \xi + \nabla \phi_\xi) - a(\cdot, \xi)) = \nabla \cdot a(\cdot, \xi) \text{ in } Q_L,$$

so that by testing the equation with ϕ_ξ and using the monotonicity (3.1.4) and boundedness (3.1.1), we obtain for some constant c depending on λ and d

$$\int_{Q_L} |\nabla \phi_\xi(x)|^2 (1 + |\xi|^{p-2} + |\xi + \nabla \phi_\xi(x)|^{p-2}) dx \leq c \int_{Q_L} |\xi| (1 + |\xi|^{p-2}) |\nabla \phi_\xi|.$$

By absorbing part of the right-hand side into the left-hand side, this yields

$$\int_{Q_L} |\nabla \phi_\xi(x)|^2 (1 + |\xi|^{p-2} + |\xi + \nabla \phi_\xi(x)|^{p-2}) dx \leq c \int_{Q_L} |\xi|^2 (1 + |\xi|^{p-2}).$$

By the triangle inequality in form of $|\xi + \nabla\phi_\xi(x)|^{p-2} \gtrsim |\nabla\phi_\xi(x)|^{p-2} - |\xi|^{p-2}$, and using the above twice, we obtain

$$\int_{Q_L} |\nabla\phi_\xi(x)|^2 (1 + |\nabla\phi_\xi(x)|^{p-2}) dx \leq c \int_{Q_L} |\xi|^2 (1 + |\xi|^{p-2}) \leq c(1 + |\xi|^p).$$

Assume that L is dyadic. Given now $R \geq L$, we cover B_R by $N_{L,R} \leq c_d (\frac{R}{L})^d$ translations of Q_L (where c_d only depends on dimension), which we denote by Q_L^j for $1 \leq j \leq N_{L,R}$. This yields

$$\begin{aligned} \int_{B_R(y)} |\nabla\phi_\xi|^2 + |\nabla\phi_\xi|^p &\leq \frac{L^d}{|B_R|} \sum_{j=1}^{N_{L,R}} \int_{Q_L^j} |\nabla\phi_\xi|^2 + |\nabla\phi_\xi|^p \\ &\leq c_d \frac{R^d}{L^d} \frac{L^d}{|B_R|} c(1 + |\xi|^p) = c_1(1 + |\xi|^p) \end{aligned}$$

for the choice $c_1 := c_d |B|^{-1}$, which only depends on d and λ . This yields (3.3.4). If L is not dyadic, we cover B_R by cubes of sidelength 2^l with l such that $2^l \leq L < 2^{l+1}$, and obtain the result at the price of increasing c_1 .

Step 2. Proof of (3.3.3).

By definition (3.3.1) of $r_{\star,\xi,L}$, we have $r_{\star,\xi,L}(0) \leq \underline{r}_{\star,\xi,L}(0)$ by testing the infimum problem with $y = 0$. Let us now prove that there exists c_2 such that for all $R \geq 1$ we have the implication $r_{\star,\xi,L}(0, c_1) \leq R \implies \underline{r}_{\star,\xi,L}(0, c_2) \leq R$, from which we deduce (3.3.3). By definition (3.3.1) of $r_{\star,\xi,L}$, if $r_{\star,\xi,L}(0, c_1) \leq R$, there exists $y \in \mathbb{R}^d$ such that $|y| \leq \frac{R}{\ell}$ and $\underline{r}_{\star,\xi,L}(y, c_1) \leq R$. This implies that $B_R \subset B_{\bar{R}}(y)$ with $\bar{R} := (\frac{1}{\ell} + 1)R$ so that

$$\int_{B_R} |\nabla\phi_\xi|^p \leq \left(\frac{\bar{R}}{R}\right)^d \int_{B_{\bar{R}}} |\nabla\phi_\xi|^p \leq \left(\frac{1}{\ell} + 1\right)^d c_1.$$

Hence, with $c_2 := (\frac{1}{\ell} + 1)^d c_1$, this yields $\underline{r}_{\star,\xi,L}(y, c_2) \leq R$, and therefore (3.3.3). \square

In the rest of the paper, the notation $r_{\star,\xi,L}$ refers to the minimal scales $r_{\star,\xi,L}(\cdot, c_1)$ for which Lemma 14 holds. When no confusion occurs, we simply write r_\star for $r_{\star,\xi,L}$, and use the short-hand notation $B_\star(x)$ for $B_{r_{\star,\xi,L}(x)}(x)$.

We conclude this paragraph by showing that the Meyers minimal radius controls local averages of the nonlinear corrector.

Lemma 15 (Control of averages of the nonlinear correctors). *There exists a nonlinear hole-filling exponent $0 < \delta \leq d$ depending on d, p , and λ such that for all $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we have for all $r > 0$*

$$\int_{B_r(x)} |\xi + \nabla\phi_\xi|^2 + |\xi + \nabla\phi_\xi|^p \lesssim_{d,\lambda,p} (1 + |\xi|^p) \left(\frac{r_\star(x) \vee r}{r} \right)^{d-\delta}. \quad (3.3.5)$$

Proof. Without loss of generality, we may assume that $x = 0$. We use the short-hand notation $\rho := r_\star \vee r \geq r_\star$. By the hole-filling estimate (3.7.3) applied to the defining equation (3.1.7) for ϕ_ξ , there exists $\delta > 0$ depending on d and λ such that

$$\int_{B_r} |\xi + \nabla\phi_\xi|^2 + |\xi + \nabla\phi_\xi|^p \lesssim \left(\frac{\rho}{r}\right)^{d-\delta} \int_{B_\rho} |\xi + \nabla\phi_\xi|^2 + |\xi + \nabla\phi_\xi|^p \lesssim \left(\frac{\rho}{r}\right)^{d-\delta} \int_{B_\rho} |\xi|^2 + |\xi|^p + |\nabla\phi_\xi|^2 + |\nabla\phi_\xi|^p.$$

Using then (3.3.3) in form of $\rho \geq r_\star(0) \geq \underline{r}_{\star, \xi, L}(0, c_2)$, the definition (3.3.2), and Jensen's inequality, this yields the reformulation of (3.3.5)

$$\int_{B_r} |\xi + \nabla \phi_\xi|^2 + |\xi + \nabla \phi_\xi|^p \lesssim \left(\frac{\rho}{r}\right)^{d-\delta} (c_2 + 1)(1 + |\xi|^p).$$

□

3.3.2 Quenched perturbative regularity in the large

Quenched Meyers' estimate in the large

Recall that $a_\xi := Da(\cdot, \xi + \nabla \phi_\xi)$. The elliptic operator $-\nabla \cdot a_\xi \nabla$ has unbounded coefficients, whose growth depends on the nonlinear corrector $\nabla \phi_\xi$: There exists $(c, C) \in \mathbb{R}_+ \times \mathbb{R}_+$, depending on λ and p , such that for all $h \in \mathbb{R}^d$

$$c|h|^2 \mu_\xi \leq h \cdot a_\xi h \leq C|h|^2 \mu_\xi, \quad (3.3.6)$$

where

$$\mu_\xi := 1 + |\xi + \nabla \phi_\xi|^{p-2}. \quad (3.3.7)$$

In addition, by (3.3.5) in Lemma 15 we have for all $r \geq r_\star$

$$\|\mu_\xi\|_{L^{\frac{p}{p-2}}(B_r)}^{\frac{p}{p-2}} \lesssim_{d, \lambda, p} r^d (1 + |\xi|^p). \quad (3.3.8)$$

The main result of this section is the following quenched Meyers estimate in the large.

Theorem 18 (Quenched Meyers' estimate in the large). *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, there exists $\bar{m} > 2$ depending on d, p and $|\xi|$, such that for all exponents $2 \leq m \leq \bar{m}$, and all Q_L -periodic functions g and u related via*

$$-\nabla \cdot a_\xi \nabla u = \nabla \cdot (g \sqrt{\mu_\xi}), \quad (3.3.9)$$

and all $r > 0$ we have

$$\int_{B_r} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim_{|\xi|} \left(\int_{B_{2r}} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \right)^{\frac{m}{2}} + \int_{B_{2r}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx. \quad (3.3.10)$$

In particular,

$$\int_{Q_L} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim_{|\xi|} \int_{Q_L} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx. \quad (3.3.11)$$

The same result holds with a_ξ replaced by a_ξ^* (the pointwise transpose field).

We follow the standard strategy based on a reverse Hölder inequality and Gehring's lemma to prove this Meyers estimate. We start with the reverse Hölder inequality:

Lemma 16 (Reverse Hölder inequality). *Let Hypothesis 3.1.2 hold. Set $q = \frac{p}{p-2}$. For all $x \in \mathbb{R}^d$, $r \geq r_\star(x)$, and all g and u related via*

$$-\nabla \cdot a_\xi \nabla u = \nabla \cdot (g \sqrt{\mu_\xi}) \text{ in } B_{\frac{17}{12}r}(x), \quad (3.3.12)$$

we have

$$\left(\int_{B_{\frac{67}{48}r}(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{1}{2}} \lesssim (1 + |\xi|^p)^{\frac{p-2}{2p}} \left(\int_{B_{\frac{17}{12}r}(x)} |\nabla u|^{q^*} \right)^{\frac{1}{q^*}} + \left(\int_{B_{\frac{17}{12}r}(x)} |g|^2 \right)^{\frac{1}{2}}, \quad (3.3.13)$$

with $1 \leq q_* < 2$ given by

$$\frac{1}{q_*} = \begin{cases} 1 & \text{for } d = 2, \\ \frac{1}{2} - \frac{1}{2q} + \frac{1}{d-1} & \text{for } d \geq 3. \end{cases} \quad (3.3.14)$$

(The choice of $\frac{67}{48}$ and $\frac{17}{12}$ is convenient for the sequel, but obviously not essential.) The same result holds with a_ξ replaced by a_ξ^* (the pointwise transpose field).

Not surprisingly, this estimate follows from the Caccioppoli and the Poincaré-Sobolev inequalities. As opposed to the case of uniformly bounded coefficients, the weight μ_ξ is in the way (and cannot be treated as a Muckenhoupt weight, which it is not). In order to get the entire range of exponents $2 \leq p < \infty$ in dimension $d = 3$, we have to be careful in the Caccioppoli inequality. Inspired by [25, Lemma 1], we optimize with respect to the cut-off in Caccioppoli's inequality, which allows us to appeal to Poincaré-Sobolev in dimension $d - 1$ rather than d (and therefore improve the integrability).

Lemma 17. *Let $q \in [1, +\infty)$, assume that $q > \frac{d-1}{2}$ if $d \geq 3$, and let q_* be given by (3.3.14). For $0 < \rho < \sigma < +\infty$, $v \in W^{1,q_*(B_\sigma)}$ and $\mu \in L^q_{loc}(\mathbb{R}^d)$, the quantity*

$$\mathcal{J}(\rho, \sigma, \mu, v) := \inf \left\{ \int_{B_\sigma} \mu v^2 |\nabla \eta|^2 \mid \eta \in C_c^1(B_\sigma), 0 \leq \eta \leq 1, \eta \equiv 1 \text{ in } B_\rho \right\} \quad (3.3.15)$$

satisfies

$$\mathcal{J}(\rho, \sigma, \mu, v) \lesssim (\sigma - \rho)^{-\frac{2d}{d-1}} \|\mu\|_{L^q(B_\sigma \setminus B_\rho)} \left(\|\nabla v\|_{L^{q_*(B_\sigma \setminus B_\rho)}}^2 + \rho^{-2} \|v\|_{L^{q_*(B_\sigma \setminus B_\rho)}}^2 \right). \quad (3.3.16)$$

The proof of Lemma 17, which closely follows the proof of [25, Lemma 1], is postponed to Appendix 3.7.1. We now prove Lemma 16.

Proof of Lemma 16. Without loss of generality, we may assume $x = 0$ and $\int_{B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r}} u = 0$. We first apply the Caccioppoli inequality (3.7.6) with $\mu = \mu_\xi$ and $c_1 = \frac{67}{48} < \frac{17}{12} = c_2$, and obtain with the notation (3.3.15)

$$\int_{B_{\frac{67}{48}r}} |\nabla u|^2 \mu_\xi \lesssim \mathcal{J}\left(\frac{67}{48}r, \frac{17}{12}r, \mu_\xi, u\right) + \int_{B_{\frac{17}{12}r}} |g|^2. \quad (3.3.17)$$

We then apply Lemma 17 with exponent $q = \frac{p}{p-2}$ for $d \geq 3$ and $q = 1$ for $d = 2$, to the effect that

$$\mathcal{J}\left(\frac{67}{48}r, \frac{17}{12}r, \mu_\xi, u\right) \lesssim r^{-\frac{2d}{d-1}} \|\mu_\xi\|_{L^q(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})} \left(\|\nabla u\|_{L^{q_*(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})}}^2 + r^{-2} \|u\|_{L^{q_*(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})}}^2 \right). \quad (3.3.18)$$

Since $r \geq r_*(0)$, (3.3.8) yields $\|\mu_\xi\|_{L^q(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})} \lesssim (1 + |\xi|^p)^{\frac{p-2}{p}} r^{\frac{d}{q}}$, whereas Poincaré's inequality in $L^{q_*(B_{2r} \setminus B_r)}$ yields $r^{-2} \|u\|_{L^{q_*(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})}}^2 \lesssim \|\nabla u\|_{L^{q_*(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})}}^2$. Hence, (3.3.18) turns into

$$\mathcal{J}\left(\frac{67}{48}r, \frac{17}{12}r, \mu_\xi, u\right) \lesssim r^{-\frac{2d}{d-1} + \frac{d}{q}} (1 + |\xi|^p)^{\frac{p-2}{p}} \|\nabla u\|_{L^{q_*(B_{\frac{17}{12}r} \setminus B_{\frac{67}{48}r})}}^2.$$

Combined with (3.3.17), this entails

$$\int_{B_{\frac{67}{48}r}} |\nabla u|^2 \mu_\xi \lesssim r^{-\frac{2d}{d-1} + \frac{d}{q} - d + \frac{2d}{q_*}} (1 + |\xi|^p)^{\frac{p-2}{p}} \left(\int_{B_{\frac{17}{12}r}} |\nabla u|^{q_*} \right)^{\frac{2}{q_*}} + \int_{B_{\frac{17}{12}r}} |g|^2,$$

which concludes the proof since, by definition (3.3.14) of q_* , $-\frac{2d}{d-1} + \frac{d}{q} - d + \frac{2d}{q_*} = 0$. \square

Theorem 18 relies on the combination of Lemma 16 with Gehring's inequality in form of

Lemma 18 (Gehring's lemma). *Let $s > 1$, and let f and h be two non-negative measurable functions in $L^q_{\text{loc}}(\mathbb{R}^d)$ such that there exists $C > 0$ for which for all $r > 0$ and $x \in \mathbb{R}^d$*

$$\left(\int_{B_r(x)} f^s \right)^{\frac{1}{s}} \leq C \left(\int_{B_{2r}(x)} f + \left(\int_{B_{2r}(x)} h^s \right)^{\frac{1}{s}} \right).$$

Then, there exists $\bar{s} > s$ depending on q and C such that for all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\left(\int_{B_r(x)} f^{\bar{s}} \right)^{\frac{1}{\bar{s}}} \lesssim \int_{B_{2r}(x)} f + \left(\int_{B_{2r}(x)} h^{\bar{s}} \right)^{\frac{1}{\bar{s}}}.$$

We are now in the position to prove Theorem 18.

Proof of Theorem 18. Let $1 \leq q_* < 2$ be given by (3.3.14). We first prove that for all $r > 0$

$$\int_{B_r} \left(\int_{B_*(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \left(\int_{B_{2r}} \left(\int_{B_*(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q_*}{2}} dx \right)^{\frac{2}{q_*}} + \int_{B_{2r}} \left(\int_{B_*(x)} |g|^2 \right) dx. \quad (3.3.19)$$

If $r \leq 3r_*(0)$ this estimate follows from Lemma 33, and it remains to treat the case $r \geq 3r_*(0)$. We first use (3.7.62) with $f = |\nabla u|^2 \mu_\xi$ to the effect of

$$\int_{B_r} \left(\int_{B_*(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \int_{B_{\frac{67}{48}r}} |\nabla u|^2 \mu_\xi. \quad (3.3.20)$$

Then, by the reverse Hölder inequality (3.3.13) followed by (3.7.63), we obtain

$$\begin{aligned} \int_{B_{\frac{67}{48}r}} |\nabla u|^2 \mu_\xi &\stackrel{(3.3.13)}{\lesssim} (1 + |\xi|^p)^{\frac{p}{p-2}} \left(\int_{B_{\frac{17}{12}r}} |\nabla u|^{q_*} \right)^{\frac{2}{q_*}} + \int_{B_{\frac{17}{12}r}} |g|^2 \\ &\stackrel{(3.7.63)}{\lesssim} (1 + |\xi|^p)^{\frac{p}{p-2}} \left(\int_{B_{2r}} \left(\int_{B_*(x)} |\nabla u|^{q_*} \right) dx \right)^{\frac{2}{q_*}} + \int_{B_{2r}} \left(\int_{B_*(x)} |g|^2 \right) dx \end{aligned} \quad (3.3.21)$$

We then slightly reformulate the first right-hand side term using Jensen's inequality in the inner integral (since $q_* < 2$) and the lower bound $\mu_\xi \geq 1$, so that

$$\begin{aligned} \left(\int_{B_{2r}} \int_{B_*(x)} |\nabla u|^{q_*} dx \right)^{\frac{2}{q_*}} &\leq \left(\int_{B_{2r}} \left(\int_{B_*(x)} |\nabla u|^2 \right)^{\frac{q_*}{2}} dx \right)^{\frac{2}{q_*}} \\ &\leq \left(\int_{B_{2r}} \left(\int_{B_*(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q_*}{2}} dx \right)^{\frac{2}{q_*}}. \end{aligned} \quad (3.3.22)$$

The combination of (3.3.20), (3.3.21), (3.3.22) yields the claimed estimate (3.3.19). To conclude, we apply Lemma 18 with

$$f : x \mapsto \left(\int_{B_*(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q_*}{2}}, \quad h : x \mapsto \left(\int_{B_*(x)} |g|^2 \right)^{\frac{q_*}{2}}, \quad s = \frac{2}{q_*} > 1.$$

This yields (3.3.10), whereas (3.3.11) follows by applying (3.3.10) for B_r with $r = \frac{\sqrt{d}}{2}L$ and using the periodicity of the quantities involved together with the plain energy estimate $\int_{Q_L} |\nabla u|^2 \mu_\xi \lesssim \int_{Q_L} |g|^2$. \square

Quenched weighted Meyers' estimate in the large

The main result of this paragraph is the following upgrade of Theorem 18.

Theorem 19 (Quenched weighted Meyers estimates in the large). *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, there exists $\beta > 0$ depending only on $|\xi|$ and d such that for all $2 \leq m \leq \bar{m}$ (cf. Theorem 18), $0 \leq 2\varepsilon \leq \beta$ and all Q_L -periodic fields g and u related via (3.3.9), we have*

$$\int_{Q_L} \omega_{\varepsilon,r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim_{|\xi|} \int_{Q_L} \omega_{2\varepsilon,r}(x) \left(\fint_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx, \quad (3.3.23)$$

where for all $x \in Q_L$

$$\omega_{\varepsilon,r}(x) := \left(1 + \frac{|x| + r_\star(0)}{r} \right)^\varepsilon. \quad (3.3.24)$$

The same result holds with a_ξ replaced by a_ξ^* (the pointwise transpose field).

We proceed in two steps: From Theorem 18 we first prove a suitable hole-filling estimate which we use in turn to upgrade Theorem 18.

Corollary 20 (Linear hole-filling estimate in the large). *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$ there exist an exponent $\beta > 0$, depending only on d , p and $|\xi|$, and a constant $c_d \geq 1$ with the following properties. Let u be a Q_L -periodic function which is a_ξ -harmonic in $Q_R(x)$ for some $x \in \mathbb{R}^d$ and $L \geq R \geq c_d r_\star(x)$, that is*

$$-\nabla \cdot a_\xi \nabla u = 0 \text{ in } Q_R(x).$$

Then for all $r_\star(x) \leq r \leq R$,

$$\int_{Q_r(x)} |\nabla u|^2 \mu_\xi \lesssim_{|\xi|} \left(\frac{r}{R} \right)^\beta \int_{Q_R(x)} |\nabla u|^2 \mu_\xi. \quad (3.3.25)$$

The same result holds with a_ξ replaced by a_ξ^* (the pointwise transpose field).

Proof of Corollary 20. Without loss of generality, we may assume that $x = 0$, $r \geq r_\star(0)$, and that $2cr \leq \frac{R}{4}$ with $c = 3\sqrt{\frac{d}{2}}$. By (3.7.62), (3.7.63), the Hölder inequality with exponents $(\bar{m}, \frac{\bar{m}}{\bar{m}-1})$ (with \bar{m} as in Theorem 18) and the unweighted Meyers estimate (3.3.10), we have with $\beta := d(1 - \frac{1}{\bar{m}})$

$$\begin{aligned} \int_{Q_r} |\nabla u|^2 \mu_\xi &\leq \int_{B_{cr}} |\nabla u|^2 \mu_\xi \\ &\stackrel{(3.7.63)}{\lesssim} r^d \fint_{B_{2cr}} \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \\ &\leq r^d \left(\fint_{B_{2cr}} \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\bar{m}} dx \right)^{\frac{1}{\bar{m}}} \\ &\leq r^d \left(\frac{R}{r} \right)^{\frac{d}{\bar{m}}} \left(\fint_{B_{\frac{R}{4}}} \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\bar{m}} dx \right)^{\frac{1}{\bar{m}}} \\ &\stackrel{(3.3.10)}{\lesssim} r^d \left(\frac{R}{r} \right)^{\frac{d}{\bar{m}}} \fint_{B_{\frac{R}{4}}} \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \stackrel{(3.7.62)}{\lesssim} \left(\frac{r}{R} \right)^\beta \int_{B_{\frac{R}{2}}} |\nabla u|^2 \mu_\xi \leq \left(\frac{r}{R} \right)^\beta \int_{Q_R} |\nabla u|^2 \mu_\xi. \end{aligned}$$

□

We now prove Theorem 19.

Proof of Theorem 19. We split the proof into four steps. In the first step, we show that for compactly supported right-hand sides g the solution gradient decays algebraically away from the source in L^2 , based on hole-filling. We then upgrade this L^2 estimate into an L^m estimate for some $m > 2$ using Meyers' estimate (3.3.10). In the third step, we remove the assumption that g be compactly supported by using a dyadic decomposition of scales. In the last step we exploit the algebraic decay to add the desired weight. Since the proof relies on a dyadic decomposition of the torus, it is convenient to work with cubes rather than balls when taking averages (which makes constants slightly cumbersome).

Step 1. L^2 algebraic decay rate.

We prove that there exists $\delta > 0$, depending on d, p and $|\xi|$, such that for all $L \geq R \geq r \geq c_d r_*(0)$ and all g compactly supported in Q_r we have

$$\int_{Q_L \setminus Q_R} |\nabla u|^2 \mu_\xi \lesssim \left(\frac{r}{R}\right)^\beta \int_{Q_r} |g|^2. \quad (3.3.26)$$

We proceed by duality in form of

$$\int_{Q_L \setminus Q_R} |\nabla u|^2 \mu_\xi = \sup_h \int_{Q_L \setminus Q_R} h \cdot \nabla u \sqrt{\mu_\xi}, \quad (3.3.27)$$

where the supremum runs over functions $h \in L^2(Q_L \setminus Q_R)^d$ with $\|h\|_{L^2(Q_L \setminus Q_R)^d} = 1$. Consider such a test function h (implicitly extended by zero on Q_R) and denote by v the unique weak solution in $H_{\text{per}}^1(Q_L)$ of

$$-\nabla \cdot a_\xi^* \nabla v = \nabla \cdot (h \sqrt{\mu_\xi}), \quad (3.3.28)$$

which is well-posed since μ_ξ is bounded on Q_L almost surely by Lemma 25. By testing (3.3.28) with u and (3.3.9) with v , we obtain by Cauchy-Schwarz' inequality and the support condition on g

$$\left| \int_{Q_L} h \cdot \nabla u \sqrt{\mu_\xi} \right| = \left| \int_{Q_L} g \cdot \nabla v \sqrt{\mu_\xi} \right| \leq \left(\int_{Q_r} |g|^2 \right)^{\frac{1}{2}} \left(\int_{Q_r} |\nabla v|^2 \mu_\xi \right)^{\frac{1}{2}}. \quad (3.3.29)$$

Since h vanishes on Q_R , v is a_ξ^* -harmonic in Q_R , and the hole filling estimate (3.3.25) with exponent $\delta > 0$ yields in combination with the plain energy estimate $\int_{Q_L} |\nabla v|^2 \mu_\xi \lesssim \int_{Q_L} |h|^2$ and the assumption $\int_{Q_L} |h|^2 = 1$

$$\int_{Q_r} |\nabla v|^2 \mu_\xi \lesssim \left(\frac{r}{R}\right)^\beta \int_{Q_R} |\nabla v|^2 \mu_\xi \lesssim \left(\frac{r}{R}\right)^\beta \int_{Q_L} |h|^2 = \left(\frac{r}{R}\right)^\beta. \quad (3.3.30)$$

The claim (3.3.26) now follows from (3.3.27), (3.3.29) and (3.3.30).

Step 2. L^m algebraic decay rate for $2 \leq m \leq \bar{m}$.

In this step, we prove that, with $C_d = 4C \vee c_d \vee 16$ (with $C \geq 1$ as in (3.7.64)), for all $L > R \geq 2r$ with $r \geq C_d r_*(0)$, and all g compactly supported in Q_r , we may upgrade (3.3.26) to

$$\int_{Q_L \setminus Q_R} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim R^{d(1-\frac{m}{2})} \left(\frac{r}{R}\right)^{\delta \frac{m}{2}} \left(\int_{Q_r} |g|^2 \right)^{\frac{m}{2}}. \quad (3.3.31)$$

for all $2 \leq m \leq \bar{m}$, where \bar{m} is the Meyers exponent of Theorem 18.

Let $J \in \mathbb{N}$ be such that $2^J R < L \leq 2^{J+1} R$. By writing $Q_L \setminus Q_R = (Q_L \setminus Q_{2^J R}) \cup \cup_{j=1}^J (Q_{2^j R} \setminus Q_{2^{j-1} R})$ (with the convention that the second union is empty if $J = 0$), it is enough to prove that for all $1 \leq j \leq J + 1$, we have

$$\int_{Q_{(2^j R) \wedge L} \setminus Q_{2^{j-1} R}} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim (2^j R)^{d(1-\frac{m}{2})} \left(\frac{r}{2^j R} \right)^{\beta \frac{m}{2}} \left(\int_{Q_r} |g|^2 \right)^{\frac{m}{2}}. \quad (3.3.32)$$

Indeed, for all $m \geq 2$, $d(1 - \frac{m}{2}) - \delta \frac{m}{2} \leq -\delta$, so that the dyadic terms sum to (3.3.31).

To start with, reverting from balls to cubes, one may reformulate Theorem 18 with cubes instead of balls, and replace B_r and B_{2r} by Q_r and $Q_{C_1 r}$, respectively (for some C_1 depending only on dimension). Let $1 \leq j \leq J$ be fixed (the case $j = J + 1$ can be treated similarly). We partition $Q_{2^j R} \setminus Q_{2^{j-1} R}$ into the union of cubes $\{Q^k\}_{k=1, \dots, N}$ of side-length $\frac{1}{C_2} 2^j R$ for some C_2 to be fixed later (the number N of such cubes then depends on d and C_2 , but not on j or R), to the effect that for all $m > 2$ we have

$$\int_{Q_{2^j R} \setminus Q_{2^{j-1} R}} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx = \sum_{k=1}^N \int_{Q^k} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx. \quad (3.3.33)$$

By Theorem 18, for all $2 \leq m \leq \bar{m}$ and $1 \leq k \leq N$,

$$\int_{Q^k} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim \left(\int_{\bar{Q}^k} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \right)^{\frac{m}{2}} + \int_{\bar{Q}^k} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx, \quad (3.3.34)$$

where $\bar{Q}^k \supset Q^k$ denotes the cube of side-length $\frac{C_1}{C_2} 2^j R$ centered at the center $x_k \in Q_{2^j R} \setminus Q_{2^{j-1} R}$ of Q^k . We now control the two right-hand side terms of (3.3.34). On the one hand, by the ℓ -Lipschitz property of r_\star and the assumption $R \geq 2C_d r_\star(0)$, for all $x \in \bar{Q}^k$, we have $|x| \leq |x_k| + \frac{\sqrt{d}}{2} \frac{C_1}{C_2} 2^j R \leq \frac{\sqrt{d}}{2} (1 + \frac{C_1}{C_2}) 2^j R$, and therefore

$$r_\star(x) \leq r_\star(0) + \ell|x| \leq R \left(\frac{1}{2C_d} + \ell \frac{\sqrt{d}}{2} 2^j (1 + \frac{C_1}{C_2}) \right). \quad (3.3.35)$$

Recall the constant C in (3.7.64) and that C_1 only depends on dimension. We now choose $C_2 := 8C_1$. For our choice $C_d = 4C \vee c_d \vee 16$ and $R \geq 2C_d r_\star(0)$, and since $0 < \ell = \frac{1}{9C\sqrt{d}} \wedge \frac{1}{16}$, we have $\frac{C_1}{C_2} 2^j R = 2^{j-3} R$ and

$$C \left(\frac{1}{2C_d} + \frac{\ell\sqrt{d}}{2} 2^j (1 + \frac{C_1}{C_2}) \right) \leq C \left(\frac{1}{8C} + \frac{1}{18C} 2^j (1 + \frac{1}{8}) \right) \leq 2^{j-3},$$

which, by (3.3.35), entails $\frac{C_1}{C_2} 2^j R \geq C r_\star(x)$, condition under which (3.7.64) yields

$$\int_{\bar{Q}^k} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \int_{\bar{Q}^k} |\nabla u|^2 \mu_\xi,$$

where \bar{Q}^k denotes the cube of side-length $\frac{C_1}{C_2} 2^{j+1} R = 2^{j-2} R$ centered at the x_k , so that $\bar{Q}^k \bmod LZ^d \subset Q_{2^{j+1} R \wedge L} \setminus Q_{2^{j-2} R}$. Hence, by (3.3.26),

$$\int_{\bar{Q}^k} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \int_{Q_{2^{j+1} R \wedge L} \setminus Q_{2^{j-2} R}} |\nabla u|^2 \mu_\xi \lesssim \left(\frac{r}{2^j R} \right)^\beta \int_{Q_r} |g|^2. \quad (3.3.36)$$

On the other hand, the same argument implies

$$\int_{\bar{Q}^k} \left(\int_{B_\star(x)} |g|^2 \right) dx \lesssim \int_{Q_{2^{j+1}R \wedge L} \setminus Q_{2^{j-2}R}} |g|^2 = 0, \quad (3.3.37)$$

where we used that g is supported in Q_r and $r \leq \frac{R}{2}$. The claim (3.3.32) then follows from (3.3.33), (3.3.34), (3.3.36), and (3.3.37), and the identity $|Q^k| = (2^{j-3}R)^d$.

Step 3. Extension to general g .

In this step, we relax the support assumption on g in (3.3.31), and claim that for all $L \geq R \geq 2C_d r_\star(0)$ and all $2 \leq m \leq \bar{m}$,

$$\begin{aligned} \left(\int_{Q_L \setminus Q_R} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} &\lesssim \left(\int_{Q_L \setminus Q_{\frac{R}{4}}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\ &\quad + \left(\int_{Q_R} \left(\frac{|x| + r_\star(0)}{R} \right)^{\frac{\beta m}{4}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \end{aligned} \quad (3.3.38)$$

Let $N \in \mathbb{N}$ be such that $2^N C_d r_\star(0) \leq R < 2^{N+1} C_d r_\star(0)$ (note that $N \geq 0$ since $R \geq 2C_d r_\star(0)$). We decompose g as $g = \sum_{i=0}^N g_i$ with $g_0 := g \mathbb{1}_{Q_{C_d r_\star(0)}}$, $g_i := g \mathbb{1}_{Q_{2^i C_d r_\star(0)} \setminus Q_{2^{i-1} C_d r_\star(0)}}$ for all $1 \leq i \leq N-1$, and $g_N := g \mathbb{1}_{Q_L \setminus Q_{2^{N-1} C_d r_\star(0)}}$. By linearity (and uniqueness of the solution) of the equation, we have $u = \sum_{i=0}^N u_i$ where u_i denotes the (unique) weak solution in $H_{\text{per}}^1(Q_L)$ of

$$-\nabla \cdot a_\xi \nabla u_i = \nabla \cdot (g_i \sqrt{\mu_\xi}).$$

By the triangle inequality, we then have for $2 \leq m \leq \bar{m}$,

$$\left(\int_{Q_L \setminus Q_R} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \leq \sum_{i=0}^N \left(\int_{Q_L \setminus Q_R} \left(\int_{B_\star(x)} |\nabla u_i(y)|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \quad (3.3.39)$$

We start by estimating the term for $i = N$, for which we use the Meyers estimate (3.3.11) to the effect that

$$\left(\int_{Q_L} \left(\int_{B_\star(x)} |\nabla u_N|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \stackrel{(3.3.11)}{\lesssim} \left(\int_{Q_L} \left(\int_{B_\star(x)} |g_N|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}.$$

We then reformulate the right-hand side using the support condition on g_N . For $x \in Q_{2^{N-2} C_d r_\star(0)}$, since $\ell = \frac{1}{9C\sqrt{d}} \wedge \frac{1}{16}$, $C \geq 1$, $N \geq 0$, and $C_d \geq 16$, we have

$$r_\star(x) \leq r_\star(0) + \ell|x| \leq r_\star(0)(1 + \ell \frac{\sqrt{d}}{2} 2^{N-2} C_d) \leq 2^{N-2} C_d r_\star(0) \left(\frac{1}{4} + \frac{1}{9} \right) \leq 2^{N-3} C_d r_\star(0),$$

so that we have the implication

$$y \in B_\star(x) \implies y \in Q_{2^{N-2} C_d r_\star(0)}(x) \implies y \in Q_{2^{N-1} C_d r_\star(0)}(0) \implies g_N(y) = 0.$$

Since $R < 2^{N+1} C_d r_\star(0) = 4 \cdot 2^{N-1} C_d r_\star(0)$, $Q_{\frac{R}{4}} \subset Q_{2^{N-1} C_d r_\star(0)}$, and the above implies

$$\left(\int_{Q_L} \left(\int_{B_\star(x)} |\nabla u_N|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \lesssim \left(\int_{Q_L \setminus Q_{\frac{R}{4}}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \quad (3.3.40)$$

We then turn to the contributions for $0 \leq i \leq N - 1$, for which we appeal to (3.3.31) with $r = 2^i C_d r_\star(0) \geq C_d r_\star(0)$ and $R \geq 2^N C_d r_\star(0) \geq 2r$, and obtain

$$\begin{aligned} \int_{Q_L \setminus Q_R} \left(\fint_{B_\star(x)} |\nabla u_i|^2 \mu_\xi \right)^{\frac{m}{2}} dx &\stackrel{(3.3.31)}{\lesssim} R^{d(1-\frac{m}{2})} \left(\frac{r}{R}\right)^{\beta \frac{m}{2}} \left(\int_{Q_r \setminus Q_{r/2}} |g|^2 \right)^{\frac{m}{2}} \\ &\lesssim R^{d(1-\frac{m}{2})} \left(\frac{r}{R}\right)^{\beta \frac{m}{4}} \left(\int_{Q_r \setminus Q_{r/2}} \left(\frac{|y|+r_\star(0)}{R}\right)^{\frac{\beta}{2}} |g(y)|^2 dy \right)^{\frac{m}{2}}. \end{aligned}$$

We then appeal to (3.7.65) (which holds for r since $r = 2^i C_d r_\star(0) \geq 2^i C r_\star(0)$ by definition of C_d), to Jensen's inequality, and to the Lipschitz regularity of r_\star in form of $r_\star(x) \leq r_\star(0) + |x|$, and get

$$\begin{aligned} &\left(\int_{Q_L \setminus Q_R} \left(\fint_{B_\star(x)} |\nabla u_i|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\ &\stackrel{(3.7.65)}{\lesssim} R^{d\frac{2-m}{2m}} \left(\frac{r}{R}\right)^{\frac{\beta}{4}} \left(\int_{Q_{2r}} \fint_{B_\star(x)} \left(\frac{|y|+r_\star(0)}{R}\right)^{\frac{\beta}{2}} |g(y)|^2 dy dx \right)^{\frac{1}{2}} \\ &\leq R^{d\frac{2-m}{2m}} \left(\frac{r}{R}\right)^{\frac{\beta}{4}} (2r)^{d(\frac{1}{2}-\frac{1}{m})} \left(\int_{Q_{2r}} \left(\fint_{B_\star(x)} \left(\frac{|y|+r_\star(0)}{R}\right)^{\frac{\beta}{2}} |g(y)|^2 dy \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\ &\lesssim \left(\frac{r}{R}\right)^{\frac{\beta}{4}+d\frac{m-2}{2m}} \left(\int_{Q_R} \left(\frac{|x|+r_\star(0)}{R}\right)^{\frac{\beta m}{4}} \left(\fint_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\ &\leq (2^{\frac{\beta}{4}+d\frac{m-2}{2m}})^{i-N} \left(\int_{Q_R} \left(\frac{|x|+r_\star(0)}{R}\right)^{\frac{\beta m}{4}} \left(\fint_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \end{aligned} \tag{3.3.41}$$

The claimed estimate (3.3.38) then follows from (3.3.39), (3.3.40), and (3.3.41).

Step 4. Proof of (3.3.23).

If $L \leq 2C_d r_\star(0) \leq 2C_d L$, then the weight is essentially constant, for all $x \in Q_L$, $\omega_{r,\varepsilon}(x) \simeq (1 + \frac{L}{r})^\varepsilon$, and the conclusion (3.3.23) is obviously satisfied. In the rest of this step we thus assume that $L > 2C_d r_\star(0)$. Let $2C_d r_\star(0) < r \leq L$ (the case $0 < r \leq 2C_d r_\star(0)$ reduces to the case $r = 2C_d r_\star(0)$ by homogeneity). Let $N \in \mathbb{N}$ be such that $2^N C_d r_\star(0) \leq L < 2^{N+1} C_d r_\star(0)$ and let $N_0 \leq N$ be such that $2^{N_0} C_d r_\star(0) \leq r < 2^{N_0+1} C_d r_\star(0)$. We then have

$$\begin{aligned} \int_{Q_L} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx &= \int_{Q_{2^{N_0} C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \\ &\quad + \sum_{i=N_0}^{N-1} \int_{Q_{2^{i+1} C_d r_\star(0)} \setminus Q_{2^i C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \\ &\quad + \int_{Q_L \setminus Q_{2^N C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx. \end{aligned} \tag{3.3.42}$$

We then control each right-hand side term separately. For the first term, we have

$$\sup_{Q_{2^{N_0} C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2},r} \lesssim \omega_{\frac{\varepsilon}{2},r}(0) \lesssim \omega_{\frac{\varepsilon}{2},r}(x) \quad \forall x \in Q_L,$$

so that by Theorem 18

$$\begin{aligned} \int_{Q_{2^{N_0} C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx &\lesssim \omega_{\frac{\varepsilon}{2},r}(0) \int_{Q_L} \left(\fint_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \\ &\lesssim \int_{Q_L} \omega_{\frac{\varepsilon}{2},r}(x) \left(\fint_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx. \end{aligned} \tag{3.3.43}$$

For all $N_0 \leq i \leq N - 1$, we combine the bound $\omega_{\frac{\varepsilon}{2}, r}|_{Q_{2^{i+1}C_d r_\star(0)} \setminus Q_{2^i C_d r_\star(0)}} \simeq 2^{\frac{\varepsilon}{2}(i-N_0)}$ with (3.3.38) to the effect that (using that $2\varepsilon \leq \beta$)

$$\begin{aligned}
 & \left(\int_{Q_{2^{i+1}C_d r_\star(0)} \setminus Q_{2^i C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2}, r}(x) \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\
 & \lesssim 2^{\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_L \setminus Q_{2^i C_d r_\star(0)}} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\
 & \stackrel{(3.3.38), 2\varepsilon \leq \delta}{\lesssim} 2^{\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_L \setminus Q_{2^{i-2} C_d r_\star(0)}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \\
 & \quad + 2^{\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_{2^i C_d r_\star(0)}} \left(\frac{|x|+r_\star(0)}{2^i C_d r_\star(0)} \right)^{\frac{\varepsilon m}{2}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \tag{3.3.44}
 \end{aligned}$$

For the first right-hand side term, we use that for all $x \in Q_L \setminus Q_{2^{i-2} C_d r_\star(0)}$ we have $2^{\frac{\varepsilon}{2}(i-N_0)} \lesssim 2^{-\frac{\varepsilon}{2}(i-N_0)} \omega_{\varepsilon, r}(x)$, so that

$$2^{\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_L \setminus Q_{2^{i-2} C_d r_\star(0)}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \lesssim 2^{-\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_L} \omega_{\varepsilon, r} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \tag{3.3.45}$$

For the second term, we rather use that for all $x \in Q_{2^i C_d r_\star(0)}$ we have by definition of N_0 and since $m \geq 2$

$$\begin{aligned}
 2^{\frac{\varepsilon}{2}(i-N_0)} \left(\frac{|x|+r_\star(0)}{2^i C_d r_\star(0)} \right)^{\frac{\varepsilon m}{2}} & \lesssim 2^{\frac{\varepsilon}{2}(i-N_0)} \left(\frac{|x|+r_\star(0)}{2^i C_d r_\star(0)} \right)^\varepsilon \\
 & \lesssim 2^{\frac{\varepsilon}{2}(i-N_0)} 2^{-\varepsilon(i-N_0)} \left(\frac{|x|+r_\star(0)}{r} \right)^\varepsilon \lesssim 2^{-\frac{\varepsilon}{2}(i-N_0)} \omega_{\varepsilon, r}(x),
 \end{aligned}$$

so that

$$2^{\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_{2^i C_d r_\star(0)}} \left(\frac{|x|+r_\star(0)}{2^i C_d r_\star(0)} \right)^{\frac{\varepsilon m}{2}} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}} \lesssim 2^{-\frac{\varepsilon}{2m}(i-N_0)} \left(\int_{Q_L} \omega_{\varepsilon, r} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}. \tag{3.3.46}$$

Summing (3.3.44)–(3.3.46) over i from N_0 to $N - 1$ we then obtain

$$\sum_{i=N_0}^{N-1} \int_{Q_{2^{i+1}C_d r_\star(0)} \setminus Q_{2^i C_d r_\star(0)}} \omega_{\frac{\varepsilon}{2}, r}(x) \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \lesssim \int_{Q_L} \omega_{\varepsilon, r} \left(\int_{B_\star(x)} |g|^2 \right)^{\frac{m}{2}} dx. \tag{3.3.47}$$

Controlling the last right-hand side of (3.3.42) the same way, (3.3.23) follows from (3.3.43) and (3.3.47). \square

3.3.3 Control of the Meyers minimal radius: sensitivity estimate and buckling

The main result of this section is the following control of the Meyers minimal radius.

Theorem 21. *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, there exist an exponent $\gamma > 0$ depending on d , λ , and p , and a constant c_ξ depending additionally on $|\xi|$ (and all independent of $L \geq 1$) such that*

$$\mathbb{E}_L \left[\exp(c_\xi r_{\star, \xi, L}^\gamma) \right] \leq 2. \tag{3.3.48}$$

The proof of Theorem 21 relies on the combination of the following sensitivity estimate (based on the quenched weighted Meyers estimate of Theorem 19) with the Caccioppoli inequality via a buckling argument.

Proposition 5. *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, denote by $\bar{m} > 2$, $\delta > 0$, and $\beta > 0$ the Meyers and nonlinear and linear hole-filling exponents, respectively (cf. Lemma 15 and Theorem 19). Then, for all $r \geq 1$ and $0 < \tau < 1$, the random variable $\mathcal{F} := \int_{B_r} \nabla \phi_\xi$ satisfies*

$$\mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} \lesssim_{|\xi|} q r^{-d} \mathbb{E}_L \left[r_{\star, \xi, L}^{\frac{d-\delta}{1-\tau} q} \right]^{\frac{1-\tau}{q}} \quad (3.3.49)$$

for all $q \geq 1 + \frac{d+1}{\varepsilon}$, where

$$\varepsilon := \left(\frac{\beta}{2}\right) \wedge \left(\frac{(d+1)(\bar{m}-2)}{2}\right) \wedge \left(\frac{\tau(d-\delta)}{4(1-\tau)}\right). \quad (3.3.50)$$

We start with the proof of Theorem 21, and then turn to the proof of Proposition 5.

Proof of Theorem 21. We use the short-hand notation $r_\star := r_{\star, \xi, L}(0, c_1)$ (cf. (3.3.2) and Lemma 14). We split the proof into two steps. In the first step, we control the probability of the level set $\{r_\star = R\}$ for all dyadic $R \in [1, L]$ using averages of $\nabla \phi_\xi$, which we combine with Proposition 5 and the bound $r_\star \leq r_\star$ to buckle on moments of r_\star , and therefore on r_\star in the second step.

Step 1. We claim that there exist $\theta \in (0, 1)$ and $c > 0$, depending on p, d and λ such that for all dyadic $R \in [1, L]$ and all exponents $q \geq 1$,

$$\mathbb{P}_L[r_\star = R] \leq c^q (1 + |\xi|^p)^{-q} \mathbb{E}_L \left[\left| \int_{B_{\theta R}} \nabla \phi_\xi \right|^{pq} \right]. \quad (3.3.51)$$

Assume that $r_\star = R$. By the definition (3.3.2) of r_\star , we then have

$$c_2(1 + |\xi|^p) \geq \int_{B_{2R}} |\nabla \phi_\xi|^p, \quad (3.3.52)$$

$$\int_{B_{\frac{R}{2}}} |\nabla \phi_\xi|^p > c_2(1 + |\xi|^p). \quad (3.3.53)$$

By the Caccioppoli inequality (3.7.2), (3.3.53) turns into

$$\inf_{\eta \in \mathbb{R}} \int_{B_R} \frac{1}{R^2} |\phi_\xi(x) - \eta|^2 + \frac{1}{R^p} |\phi_\xi(x) - \eta|^p \gtrsim 1 + |\xi|^p, \quad (3.3.54)$$

which we shall use in the stronger form

$$\inf_{\eta \in \mathbb{R}} \int_{B_R} \frac{1}{R^p} |\phi_\xi(x) - \eta|^p \gtrsim 1 + |\xi|^p. \quad (3.3.55)$$

Indeed, by Jensen's inequality, with the short-hand notation $\alpha := \inf_{\eta \in \mathbb{R}} \int_{B_R} \frac{1}{R^p} |\phi_\xi(x) - \eta|^p$, (3.3.54) yields $\alpha^{\frac{2}{p}} + \alpha \gtrsim 1 + |\xi|^p$ so that $\alpha \gtrsim 1$, which implies $\alpha \gtrsim \alpha^{\frac{2}{p}}$, whence the reformulation (3.3.55).

Let $\theta \in (0, 1)$ (the value of which we shall choose below), and set $c_R := \int_{B_R} \int_{B_{\theta R}(x)} \phi_\xi(y) dy dx$. By the triangle inequality and Poincaré's inequality in $L^p(B_R)$, we obtain

$$\begin{aligned} \inf_{\eta \in \mathbb{R}} \int_{B_R} \frac{1}{R^p} |\phi_\xi - \eta|^p &\lesssim \int_{B_R} \frac{1}{R^p} \left| \phi_\xi(x) - \int_{B_{\theta R}(x)} \phi_\xi \right|^p dx + \int_{B_R} \frac{1}{R^p} \left| \int_{B_{\theta R}(x)} \phi_\xi - c_R \right|^p dx \\ &\lesssim \theta^p \int_{B_{2R}} |\nabla \phi_\xi(x)|^p dx + \int_{B_R} \left| \int_{B_{\theta R}(x)} \nabla \phi_\xi \right|^p dx. \end{aligned} \quad (3.3.56)$$

Combined with (3.3.56), (3.3.55) turns into

$$1 + |\xi|^p \lesssim \theta^p \int_{B_{2R}} |\nabla \phi_\xi|^p + \int_{B_R} \left| \int_{B_{\theta R}(x)} \nabla \phi_\xi \right|^p dx. \tag{3.3.57}$$

Using now (3.3.52), we may absorb the first right-hand side term into the left-hand side for θ small enough (independent of R), and therefore conclude that for some $c > 0$ (depending only on d, p, λ)

$$\{r_\star = R\} \subset \left\{ \int_{B_R} \left| \int_{B_{\theta R}(x)} \nabla \phi_\xi \right|^p dx \geq \frac{1}{c} (1 + |\xi|^p) \right\},$$

which yields (3.3.51) by Markov’s inequality and the stationarity of $\nabla \phi_\xi$.

Step 2. Buckling argument.

Fix $\tau := 1 - \frac{d-\delta}{d-\frac{\delta}{2}} = \frac{\delta}{2d-\delta} > 0$, to the effect that

$$\frac{d-\delta}{1-\tau} = d - \frac{\delta}{2}, \quad 1 - \frac{\tau}{2} = 1 - \frac{\delta}{2(2d-\delta)}, \quad \varepsilon := \left(\frac{(d+1)(\bar{m}-2)}{2} \right) \wedge \left(\frac{\delta}{8} \right).$$

For all dyadic $1 \leq R \leq L$, by (3.3.51) and by Proposition 5 with this choice of τ and $r = \theta R$, we obtain for all q with $q^{\frac{p}{2}} \geq 1 + \frac{d+1}{\varepsilon}$,

$$\mathbb{P}_L[r_\star = R] \leq c_\xi^q q^{\frac{p}{2}q} R^{-d\frac{p}{2}q} \mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right]^{1-\frac{\delta}{2(2d-\delta)}} \stackrel{(3.3.3)}{\leq} c_\xi^q q^{\frac{p}{2}q} R^{-d\frac{p}{2}q} \mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right]^{1-\frac{\delta}{2(2d-\delta)}}. \tag{3.3.58}$$

Therefore, using a dyadic decomposition (the sum is actually finite since $r_\star \leq L$), we deduce that (up to changing the value of c_ξ)

$$\begin{aligned} \mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right] &\leq 1 + \sum_{n=1}^{+\infty} (2^n)^{(d-\frac{\delta}{2})\frac{p}{2}q} \mathbb{P}[r_\star = 2^n] \\ &\stackrel{(3.3.58)}{\leq} 1 + c_\xi^q q^{\frac{p}{2}q} \mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right]^{1-\frac{\delta}{2(2d-\delta)}} \sum_{n=1}^{+\infty} 2^{(d-\frac{\delta}{2})q\frac{p}{2}n} 2^{-dq\frac{p}{2}n} \\ &\leq 1 + c_\xi^q q^{\frac{p}{2}q} \mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right]^{1-\frac{\delta}{2(2d-\delta)}}. \end{aligned}$$

Since both terms of this inequality are finite, this gives by Young’s inequality provided $q^{\frac{p}{2}} \geq 1 + \frac{d+1}{\varepsilon}$

$$\mathbb{E}_L \left[r_\star^{(d-\frac{\delta}{2})\frac{p}{2}q} \right]^{\frac{1}{q}} \lesssim c_\xi q^{p\frac{2d-\delta}{\delta}},$$

from which the stretched exponential moment bound (3.3.48) follows with $\gamma := \frac{\delta}{8}$ (cf. Lemma 31), which is not expected to be sharp. \square

We conclude this section with the proof of Proposition 5.

Proof of Proposition 5. We split the proof into three steps. In the first step, we compute the functional derivative of \mathcal{F} , in the sense of (3.7.43), and apply the logarithmic-Sobolev inequality in the second step to control moments of \mathcal{F} . In the third step, we then control these moments by suitable moments of r_\star using the quenched weighted Meyers estimate in the large of Theorem 19.

Step 1. Sensitivity calculus.

In this step, we take a slightly more general version of \mathcal{F} (this will be further used in the proof of Theorem 15), which we define, for some given $g \in L^2(Q_L)^d$ (extended by periodicity on \mathbb{R}^d), by

$$\mathcal{F} := \int_{Q_L} \nabla \phi_\xi \cdot g.$$

We then argue that for all $x \in Q_L$,

$$\partial_x \mathcal{F} = \int_{B(x)} |a_\circ(\xi + \nabla \phi_\xi) \otimes \nabla u|, \quad (3.3.59)$$

with the short-hand notation $a_\circ(\zeta) := (1 + |\zeta|^{p-2})\zeta$ and where u is the unique weak Q_L -periodic solution (with zero average) of

$$-\nabla \cdot a_\xi^* \nabla u = \nabla \cdot g, \quad (3.3.60)$$

we recall that a_ξ^* is bounded from above and below, since a is assumed to be smooth, and satisfies (3.3.6), cf. Lemma 25. Let denote by h a sequence that goes to zero and by δA a coefficient field supported in $B(x)$ (and extended by Q_L -periodicity) such that $\|\delta A\|_{L^\infty(\mathbb{R}^d)} \leq 1$. We let h be small enough so that $A + h\delta A$ is uniformly elliptic, and define

$$\begin{aligned} \delta^h \mathcal{F} &:= \frac{\mathcal{F}(A + h\delta A) - \mathcal{F}(A)}{h}, \\ \delta^h \phi_\xi &:= \frac{\phi_\xi(A + h\delta A) - \phi_\xi(A)}{h}, \end{aligned} \quad (3.3.61)$$

$$a_\xi^h := \int_0^1 Da(\cdot, \xi + t\nabla \phi_\xi(A + h\delta A) + (1-t)\nabla \phi_\xi(A)) dt. \quad (3.3.62)$$

By the definition of \mathcal{F} , we have $\delta^h \mathcal{F} = \int_{Q_L} \nabla \delta^h \phi_\xi \cdot g$, and we need to characterize $\delta^h \phi_\xi$. By the defining equation (3.1.7), we obtain

$$-\nabla \cdot \left(a(\cdot, \xi + \nabla \phi_\xi + h\nabla \delta^h \phi) - a(\cdot, \xi + \nabla \phi_\xi) \right) = h\nabla \cdot \delta A a_\circ(\xi + \nabla \phi_\xi(A + h\delta A)), \quad (3.3.63)$$

which we rewrite, by the fundamental theorem of calculus and the definition of a_ξ^h , as

$$-\nabla \cdot a_\xi^h \nabla \delta^h \phi_\xi = \nabla \cdot \delta A a_\circ(\xi + \nabla \phi_\xi(A + h\delta A)). \quad (3.3.64)$$

Assume that $\delta^h \phi_\xi$ converges weakly in $H_{\text{per}}^1(Q_L)$ to the solution $\delta \phi_\xi \in H_{\text{per}}^1(Q_L)$ of

$$-\nabla \cdot a_\xi \nabla \delta \phi_\xi = \nabla \cdot \delta A a_\circ(\xi + \nabla \phi_\xi). \quad (3.3.65)$$

Then $\lim_{h \downarrow 0} \delta^h \mathcal{F} = \delta \mathcal{F} = \int g \cdot \nabla \delta \phi_\xi$, which we now rewrite by duality. Testing (3.3.60) with $\delta \phi_\xi$ and then (3.3.65) with u , we obtain

$$\delta \mathcal{F} = \int \nabla u \cdot \delta A a_\circ(\xi + \nabla \phi_\xi),$$

and the claim (3.3.59) follows by taking the supremum over δA . It remains to argue in favor of the convergence of $\delta^h \phi_\xi$ to $\delta \phi_\xi$, which actually holds in $C^{1,\alpha}(Q_L)$. First, recall that $\{\phi_\xi(A + h\delta A)\}_h$ is a bounded set in $C^{1,\alpha}(Q_L)$ by Lemma 25. By testing (3.3.63) with $h\delta^h \phi_\xi$, we obtain by monotonicity

$$\int_{Q_L} |\nabla \phi_\xi(A + h\delta A) - \nabla \phi_\xi(A)|^2 + |\nabla \phi_\xi(A + h\delta A) - \nabla \phi_\xi(A)|^p \lesssim h^2,$$

so that $\nabla\phi_\xi(A + h\delta A) \rightarrow \nabla\phi_\xi(A)$ in $L^p(Q_L)$, and therefore $\phi_\xi(A + h\delta A) \rightarrow \phi_\xi(A)$ in $C^{1,\alpha}(Q_L)$ by Arzela-Ascoli's theorem as claimed.

Step 2. Application of the logarithmic-Sobolev inequality: For all $q \geq 1$,

$$\mathbb{E}_L \left[\mathcal{F}^{2q} \right]^{\frac{1}{q}} \lesssim q(1 + |\xi|^p) \mathbb{E}_L \left[\left(\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right) dx \right)^q \right]^{\frac{1}{q}}. \quad (3.3.66)$$

Since $\mathbb{E}_L[\nabla\phi_\xi] = 0$, by (3.7.44) and (3.3.59), we have for all $q \geq 1$

$$\mathbb{E}_L \left[\mathcal{F}^{2q} \right]^{\frac{1}{q}} \lesssim q \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |a_\circ(\xi + \nabla\phi_\xi)| |\nabla u|^2 \right) dx \right)^q \right]^{\frac{1}{q}}. \quad (3.3.67)$$

By Cauchy-Schwarz' inequality, the definition (3.3.7) of μ_ξ , and (3.3.5), we have for all $x \in Q_L$

$$\begin{aligned} & \left(\int_{B(x)} |a_\circ(\xi + \nabla\phi_\xi)| |\nabla u|^2 \right)^2 \\ & \lesssim \int_{B(x)} |\xi + \nabla\phi_\xi|^2 (1 + |\xi + \nabla\phi_\xi|^{p-2}) \int_{B(x)} |\nabla u|^2 \mu_\xi \\ & \stackrel{(3.3.5)}{\lesssim} (1 + |\xi|^p) r_\star(x)^{d-\delta} \int_{B(x)} |\nabla u|^2 \mu_\xi. \end{aligned}$$

The claim (3.3.66) then follows in combination with (3.3.67).

Step 3. Proof of (3.3.49).

For $0 < \tau < 1$ given, we define ε as in (3.3.50), and set $m := 2 + \frac{2\varepsilon}{d+1}$, to the effect that $m \leq \bar{m}$ and $\frac{2\varepsilon}{m-2} = d+1$. Since r_\star is $\frac{1}{16}$ -Lipschitz and $r_\star \geq 1$, we have

$$\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \int_{Q_L} \left(\int_{B(x)} r_\star^{d-\delta} |\nabla u|^2 \mu_\xi \right) dx,$$

so that by (3.7.65) combined with the estimate $r_\star \leq L$, with periodicity, and using again the Lipschitz-continuity of r_\star , we obtain

$$\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right) dx \lesssim \int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right) dx.$$

Inserting the weight $(1 + \frac{|x|}{r})^{\frac{2\varepsilon}{m}} (1 + \frac{|x|}{r})^{-\frac{2\varepsilon}{m}}$, and using Hölder's inequality in space with exponents $(\frac{m}{2}, \frac{m}{m-2})$ followed by Hölder's inequality in probability with exponents $(\frac{1}{1-\tau}, \frac{1}{\tau})$, (3.3.66) turns into

$$\begin{aligned} & \frac{1}{q(1 + |\xi|^p)} \mathbb{E}_L \left[\mathcal{F}^{2q} \right]^{\frac{1}{q}} \\ & \lesssim \mathbb{E}_L \left[\left(\int_{Q_L} (1 + \frac{|x|}{r})^{-d-1} r_\star(x)^{\frac{m-2}{m-2}(d-\delta)} dx \right)^q \right]^{\frac{m-2}{m}} \left(\int_{Q_L} (1 + \frac{|x|}{r})^\varepsilon \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{2}{m}} \right]^{\frac{1}{q}} \\ & \leq \mathbb{E}_L \left[\left(\int_{Q_L} (1 + \frac{|x|}{r})^{-d-1} r_\star(x)^{\frac{m-2}{m-2}(d-\delta)} dx \right)^q \right]^{\frac{1-\tau}{m(1-\tau)}} \mathbb{E}_L \left[\left(\int_{Q_L} (1 + \frac{|x|}{r})^\varepsilon \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^q \right]^{\frac{2}{m\tau}} \right]^{\frac{\tau}{q}}. \end{aligned} \quad (3.3.68)$$

By the change of variables $\frac{x}{r} \rightsquigarrow x$, Jensen's inequality in space provided $q \geq \frac{m-2}{m-2} = 1 + \frac{d+1}{\varepsilon}$, and the stationarity of r_\star , we control the first right-hand side term of (3.3.68) by

$$\mathbb{E}_L \left[\left(\int_{Q_L} (1 + \frac{|x|}{r})^{-d-1} r_\star(x)^{\frac{m-2}{m-2}(d-\delta)} dx \right)^q \right]^{\frac{1-\tau}{m(1-\tau)}} \lesssim r^{d\frac{m-2}{m}} \mathbb{E}_L \left[r_\star^q \right]^{\frac{d-\delta}{1-\tau}} \right]^{\frac{1-\tau}{q}}. \quad (3.3.69)$$

For the second right-hand side term of (3.3.68), we appeal to the quenched weighted Meyers estimate (3.3.23), which we may apply to equation (3.3.60) (rewriting the right-hand side as $\frac{1}{\sqrt{\mu_\xi}}g\sqrt{\mu_\xi}$) with weight $\omega_{\varepsilon,r}$ since $\varepsilon \leq \frac{\beta}{2}$. By stationarity of r_\star , this yields

$$\begin{aligned} \mathbb{E}_L \left[\left(\int_{Q_L} \left(1 + \frac{|x|}{r}\right)^\varepsilon \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{2}{m\tau}q} \right]^{\frac{\tau}{q}} &\lesssim_{|\xi|} \mathbb{E}_L \left[\left(\int_{Q_L} \omega_{2\varepsilon,r}(x) \left(\int_{B_\star(x)} |g|^2 \frac{1}{\mu_\xi} \right)^{\frac{m}{2}} dx \right)^{\frac{2}{m\tau}q} \right]^{\frac{\tau}{q}} \\ &\lesssim r^{-2d + \frac{2}{m}d} \mathbb{E}_L \left[r_\star^{\frac{4\varepsilon}{m\tau}q} \right]^{\frac{\tau}{q}}, \end{aligned} \quad (3.3.70)$$

where we used that $g = |B_r|^{-1} \mathbf{1}_{B_r}$, that $\mu_\xi \geq 1$, and (3.7.65). By (3.3.50) and our choice $m = 2 + \frac{2\varepsilon}{d+1}$, $\frac{4\varepsilon}{m\tau} \leq \frac{d-\delta}{1-\tau}$, to the effect that $\mathbb{E}_L \left[r_\star^{\frac{4\varepsilon}{m\tau}q} \right]^{\frac{\tau}{q}} \leq \mathbb{E}_L \left[r_\star^{q \frac{d-\delta}{1-\tau}} \right]^{\frac{4\varepsilon(1-\tau)}{qm(d-\delta)}}$ by Hölder's inequality. Using (3.3.50) again, this time in form of $\frac{4\varepsilon(1-\tau)}{m(d-\delta)} \leq \frac{\tau}{2}$, and the lower bound $r_\star \geq 1$, (3.3.70) turns into

$$\mathbb{E}_L \left[\left(\int_{Q_L} \left(1 + \frac{|x|}{r}\right)^\varepsilon \left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{m}{2}} dx \right)^{\frac{2}{m\tau}q} \right]^{\frac{\tau}{q}} \lesssim_{|\xi|} r^{-2d + \frac{2}{m}d} \mathbb{E}_L \left[r_\star^{q \frac{d-\delta}{1-\tau}} \right]^{\frac{\tau}{2q}}. \quad (3.3.71)$$

The claim (3.3.49) then follows from (3.3.68), (3.3.69) and (3.3.71). \square

3.3.4 Annealed Meyers' estimate

The annealed Meyers (or perturbative Calderón-Zygmund) estimates recently introduced by Duerinckx and Otto in [56] (see also [101]) constitute a very versatile upgrade of their quenched counterpart in stochastic homogenization. In the present setting the annealed Meyers estimates take the following form.

Theorem 22 (Annealed Meyers' estimate). *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, with $\kappa := \frac{(\bar{m}-2) \wedge 1}{8} > 0$ (where \bar{m} is the Meyers exponent of Theorem 18), for all Q_L -periodic random fields g and u related via (3.3.9), we have for all exponents $2 - \kappa \leq q, m \leq 2 + \kappa$ and $0 < \delta \leq \frac{1}{2}$,*

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \lesssim_{|\xi|} \delta^{-\frac{1}{4}} |\log \delta|^{\frac{1}{2}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q(1+\delta)}{2}} \right]^{\frac{m}{q(1+\delta)}} dx. \quad (3.3.72)$$

The same result holds with a_ξ replaced by a_ξ^* (the pointwise transpose field).

The proof is based on the quenched Meyers estimate in the large of Theorem 18, on the moment bounds of Theorem 21 on the Meyers minimal radius (which allows us to use duality at the price of a loss of stochastic integrability), real interpolation, and the following refined dual version of the Calderón-Zygmund lemma due to Shen [147, Theorem 3.2], based on ideas by Caffarelli and Peral [40].

Lemma 19 ([40, 147]). *Given $1 \leq q < m \leq \infty$, let $F, G \in L^q \cap L^m(Q_L)$ be nonnegative Q_L -periodic functions and let $C_0 > 0$. Assume that for all balls D (of radius $\lesssim L$) there exist measurable functions $F_{D,1}$ and $F_{D,2}$ such that $F \leq F_{D,1} + F_{D,2}$ and $F_{D,2} \leq F + F_{D,1}$ on D , and such that*

$$\left(\int_D F_{D,1}^q \right)^{\frac{1}{q}} \leq C_0 \left(\int_{C_0 D} G^q \right)^{\frac{1}{q}}, \quad \left(\int_{\frac{1}{C_0} D} F_{D,2}^m \right)^{\frac{1}{m}} \leq C_0 \left(\int_D F_{D,2}^q \right)^{\frac{1}{q}}.$$

Then, for all $q < s < m$,

$$\left(\int_{Q_L} F^s \right)^{\frac{1}{s}} \lesssim_{C_0, q, s, m} \left(\int_{Q_L} G^s \right)^{\frac{1}{s}}.$$

Before we prove Theorem 22, let us note that Theorem 21 allows one to pass from averages on $B_\star(x)$ to averages on $B(x)$ using [56, Lemma 6.7] in the (slightly more general) form of

Lemma 20. *Let r_\star be a stationary random field satisfying $\mathbb{E}_L \left[\exp(cr_\star^\alpha) \right] \leq 2$ for some $\alpha > 0$ and $c \simeq 1$. Set $B_\star(x) := B_{r_\star(x)}(x)$ for all $x \in Q_L$. For all $f \in C_{\text{per}}^\infty(Q_L; L^\infty(\Omega))$ and $1 \leq q_1 \leq q_2 < \infty$, we have*

(i) for all $r > q_1$,

$$\left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B_\star(x)} |f|^2 \right)^{\frac{q_1}{2}} \right]^{\frac{q_2}{q_1}} dx \right)^{\frac{1}{q_2}} \lesssim \left(\frac{1}{q_1} - \frac{1}{r} \right)^{-\left(\frac{1}{q_1} - \frac{1}{2}\right)_+} \zeta \left(\frac{1}{q_1} - \frac{1}{r} \right)^{\frac{1}{q_1} - \frac{1}{q_2}} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |f|^2 \right)^{\frac{r}{2}} \right]^{\frac{q_2}{r}} \right)^{\frac{1}{q_2}};$$

(ii) for all $r < q_1$,

$$\left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B_\star(x)} |f|^2 \right)^{\frac{q_1}{2}} \right]^{\frac{q_2}{q_1}} dx \right)^{\frac{1}{q_2}} \gtrsim \left(\frac{1}{r} - \frac{1}{q_1} \right)^{\left(\frac{1}{2} - \frac{1}{q_2}\right)_+} \zeta \left(\frac{1}{r} - \frac{1}{q_1} \right)^{-\left(\frac{1}{q_1} - \frac{1}{q_2}\right)} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |f|^2 \right)^{\frac{r}{2}} \right]^{\frac{q_2}{r}} \right)^{\frac{1}{q_2}};$$

where we have set $\zeta(t) := \log(2 + \frac{1}{t})$, and the multiplicative constants depend on q_1, q_2, α .

The proof of this result is identical to that of [56, Lemma 6.7], noting that the assumption $\mathbb{E}_L \left[\exp(\frac{1}{C} r_\star^d) \right] \leq 2$ can be weakened to $\mathbb{E}_L \left[\exp(\frac{1}{C} r_\star^\alpha) \right] \leq 2$ for any $\alpha > 0$ at the price of adding a dependence on α in the multiplicative factors in the estimates, and \mathbb{R}^d can be replaced by Q_L .

Proof of Theorem 22. We split the proof into three steps. In the first step, we upgrade Theorem 18 by adding expectations using Lemma 19 in a suitable way. At the price of a loss of stochastic integrability we then remove the local averages at scale r_\star in Step 2 by using Lemma 20. The formulation with local averages at unit scale allows us to conclude using a standard duality argument, and real interpolation.

Step 1. Proof that for all $2 \leq q < m < \bar{m}$, we have

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \lesssim_{|\xi|} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B_{5\star}(x)} |g|^2 \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \quad (3.3.73)$$

with the short hand notation $B_{5\star}(x) := B_{5r_\star(x)}(x)$.

Let $2 \leq q_1 \leq m_1 \leq \bar{m}$. Let D be a ball centered at $x \in Q_L$ and of radius $0 < r_D \lesssim L$, we define $D_\star := B_{r_D \vee (2r_\star(x))}(x)$, and let N be the smallest integer so that $D_\star \subset Q_{NL}$. We then decompose u as $u = u_{D,1} + u_{D,2}$, where $u_{D,1}$ is the Q_{NL} -periodic solution of $-\nabla \cdot a_\xi \nabla u_{D,1} = \nabla \cdot g \sqrt{\mu_\xi} \mathbf{1}_{D_\star}$. Note that $u_{D,2}$ is a_ξ -harmonic on D_\star . We start with the control of $u_{D,1}$ and claim that

$$\int_D \mathbb{E}_L \left[\left(\int_{B_\star(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy \lesssim \mathbb{E}_L \left[\int_{8D} \left(\int_{B_{5\star}(y)} |g|^2 \right)^{\frac{q_1}{2}} dy \right]. \quad (3.3.74)$$

Assume first that $r_D \geq 2r_\star(x)$, so that $D_\star = D$. By taking the expectation in Theorem 18, we have

$$\begin{aligned} \int_D \mathbb{E}_L \left[\left(\int_{B_\star(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy &\leq \mathbb{E}_L \left[\int_{Q_{NL}} \left(\int_{B_\star(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} dy \right] \\ &\stackrel{(3.3.11)}{\lesssim} \mathbb{E}_L \left[\int_{Q_{NL}} \left(\int_{B_\star(y)} |g|^2 \mathbf{1}_D \right)^{\frac{q_1}{2}} dy \right]. \end{aligned}$$

By the $\frac{1}{16}$ -Lipschitz property of r_* , we have the implication for all $z \in B_*(y)$

$$\begin{aligned} |y - x| \geq 2r_D \implies |z - x| &\geq |y - x| - r_*(y) \geq |y - x| - (r_*(x) + \frac{1}{16}|y - x|) \\ &\geq \frac{15}{16}|y - x| - r_*(x) \geq (\frac{15}{8} - \frac{1}{2})r_D \geq r_D \implies \mathbf{1}_D(z) = 0, \end{aligned}$$

so that (3.3.74) follows for $r_D \geq 2r_*(x)$ in the stronger form

$$\int_D \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy \lesssim \mathbb{E}_L \left[\int_{2D} \left(\int_{B_*(y)} |g|^2 \right)^{\frac{q_1}{2}} dy \right].$$

If $r_D \leq 2r_*(x)$, then $\sup_D r_* \lesssim \inf_D r_*$, $D_* = B_{2r_*(x)}(x) =: B_{2*}(x)$, and a plain energy estimate yields

$$\begin{aligned} \int_D \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy &\lesssim \mathbb{E}_L \left[|D|r_*(x)^{-d\frac{q_1}{2}} \left(\int_{Q_{NL}} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] \\ &\lesssim \mathbb{E}_L \left[|D|r_*(x)^{-d\frac{q_1}{2}} \left(\int_{D_*} |g|^2 \right)^{\frac{q_1}{2}} \right] \lesssim \mathbb{E}_L \left[|D| \left(\int_{B_{2*}(x)} |g|^2 \right)^{\frac{q_1}{2}} \right], \end{aligned}$$

and it remains to turn the right-hand side into an integral over D . For all $y \in D$, we have $r_*(y) \geq r_*(x) - \frac{1}{16}r_D \geq \frac{7}{8}r_*(x)$, and therefore for all $z \in B_{2*}(x)$, $|z - y| \leq |z - x| + |x - y| \leq 4r_*(x) \leq 5r_*(y)$, to the effect that $B_{2*}(x) \subset B_{5*}(y)$. Recalling that $\sup_D r_* \lesssim \inf_D r_*$, this implies the following stronger form of (3.3.74)

$$\int_D \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy \lesssim \mathbb{E}_L \left[\int_D \left(\int_{B_{5*}(y)} |g|^2 \right)^{\frac{q_1}{2}} dy \right].$$

We now turn to the control of $u_{D,2}$, and claim that

$$\left(\int_{\frac{1}{8}D} \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{m_1}{q_1}} dy \right)^{\frac{1}{m_1}} \lesssim \left(\int_D \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right] dy \right)^{\frac{1}{q_1}}. \quad (3.3.75)$$

The starting point is the Minkowski inequality: Since $\frac{m_1}{q_1} \geq 1$,

$$\left(\int_{\frac{1}{8}D} \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{m_1}{q_1}} dy \right)^{\frac{1}{m_1}} \leq \mathbb{E}_L \left[\left(\int_{\frac{1}{8}D} \left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{m_1}{2}} dy \right)^{\frac{q_1}{m_1}} \right]^{\frac{1}{q_1}}. \quad (3.3.76)$$

We then appeal to the local Meyers estimate (3.3.10) to bound the right-hand side

$$\int_{\frac{1}{8}D} \left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{m_1}{2}} dy \lesssim_{|\xi|} \left(\int_{\frac{1}{4}D} \left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right) dy \right)^{\frac{m_1}{2}} + \int_{\frac{1}{4}D} \left(\int_{B_*(y)} |g|^2 (1 - \mathbf{1}_{D_*}) \right)^{\frac{m_1}{2}} dy.$$

Since for all $y \in \frac{1}{4}D$, one has $r_*(y) \leq r_*(x) + \frac{1}{16}\frac{1}{4}r_D \leq \frac{3}{4}(r_D \vee (2r_*(x)))$, $B_*(y) \subset D_*$ and the second right hand side term vanishes identically. Combined with (3.3.76) and Jensen's inequality in space (using that $\frac{q_1}{2} \geq 1$), this entails

$$\begin{aligned} \left(\int_{\frac{1}{8}D} \mathbb{E}_L \left[\left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{m_1}{q_1}} dy \right)^{\frac{1}{m_1}} &\leq \mathbb{E}_L \left[\left(\int_{\frac{1}{4}D} \left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right) dy \right)^{\frac{q_1}{2}} \right]^{\frac{1}{q_1}} \\ &\leq \mathbb{E}_L \left[\int_{\frac{1}{4}D} \left(\int_{B_*(y)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} dy \right]^{\frac{1}{q_1}}, \end{aligned}$$

from which (3.3.75) follows.

We are in the position to conclude. Setting $F : x \mapsto \mathbb{E}_L \left[\left(\int_{B_\star(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{1}{q_1}}$, $G : x \mapsto \mathbb{E}_L \left[\left(\int_{B_{5\star}(x)} |g|^2 \right)^{\frac{q_1}{2}} \right]^{\frac{1}{q_1}}$, $F_{D,1} : x \mapsto \mathbb{E}_L \left[\left(\int_{B_\star(x)} |\nabla u_{D,1}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{1}{q_1}}$, and $F_{D,2} : x \mapsto \mathbb{E}_L \left[\left(\int_{B_\star(x)} |\nabla u_{D,2}|^2 \mu_\xi \right)^{\frac{q_1}{2}} \right]^{\frac{1}{q_1}}$, the assumptions of Lemma 19, and the claimed estimate (3.3.73) follows.

Step 2. Reformulation of (3.3.73).

Since both r_\star and $5r_\star$ satisfy stretched exponential moment bounds, Lemma 20 allows us to reformulate (3.3.73) as: For all $2 \leq q < m < \bar{m}$ and $0 < r \leq \frac{1}{2}$,

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \lesssim_{|\xi|, r} r^{-\frac{m-2}{2m}} |\log r|^{\frac{2(m-q)}{qm}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q+r}{2}} \right]^{\frac{m}{q+r}} dx. \quad (3.3.77)$$

Step 3. Proof of (3.3.72).

First, we show that for all $\bar{m}' < m < q \leq 2$ and $0 < r \ll 1$,

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \lesssim_{|\xi|} r^{-\frac{2-m}{2m}} |\log r|^{\frac{2(q-m)}{qm}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q-r}{2}} \right]^{\frac{m}{q-r}} dx. \quad (3.3.78)$$

Indeed, by duality we have

$$\left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \right)^{\frac{1}{m}} = \sup_h \left\{ \mathbb{E}_L \left[\int_{Q_L} \nabla u \cdot h \sqrt{\mu_\xi} \right] \right\},$$

where the supremum runs over maps $h \in C_{\text{per}}^\infty(Q_L, L^\infty(d\mathbb{P}_L))^d$ such that $\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |h|^2 \right)^{\frac{q'}{2}} \right]^{\frac{m'}{q'}} dx = 1$. For such h , denote by v_h the unique Q_L -periodic solution of $-\nabla \cdot a_\xi^* \nabla v_h = \nabla \cdot (h \sqrt{\mu_\xi})$. Testing this equation with u and the defining equation (3.3.9) for u by v_h , we obtain (using periodicity in the last equality)

$$\int_{Q_L} \nabla u \cdot h \sqrt{\mu_\xi} = \int_{Q_L} \nabla v_h \cdot g \sqrt{\mu_\xi} = \int_{Q_L} \left(\int_{B(x)} \nabla v_h \cdot g \sqrt{\mu_\xi} \right) dx.$$

By Cauchy-Schwarz' inequality on $B(x)$, followed by Hölder's inequality with exponents $(q-r, \frac{q-r}{q-r-1})$ on Q_L and with exponent (m, m') in probability, this yields

$$\left| \mathbb{E}_L \left[\int_{Q_L} \nabla v_h \cdot g \sqrt{\mu_\xi} \right] \right| \leq \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q-r}{2}} \right]^{\frac{m}{q-r}} dx \right)^{\frac{1}{m}} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla v_h|^2 \mu_\xi \right)^{\frac{(q-r)'}{2}} \right]^{\frac{m'}{(q-r)'}} dx \right)^{\frac{1}{m'}}.$$

Since $(q-r)' - q' = \frac{r}{(q-1)(q-1-r)}$, we may apply (3.3.77) to ∇v_h to the effect that

$$\begin{aligned} & \left| \mathbb{E}_L \left[\int_{Q_L} \nabla v_h \cdot g \sqrt{\mu_\xi} \right] \right| \\ & \lesssim r^{-\frac{m'-2}{2m'}} |\log r|^{\frac{2(m'-q')}{q'm'}} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q-r}{2}} \right]^{\frac{m}{q-r}} dx \right)^{\frac{1}{m}} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |h|^2 \right)^{\frac{q'}{2}} \right]^{\frac{m'}{q'}} dx \right)^{\frac{1}{m'}}, \end{aligned}$$

from which (3.3.78) follows by the arbitrariness of h and the identities $\frac{m'-2}{2m'} = \frac{2-m}{2m}$ and $\frac{2(m'-q')}{q'm'} = \frac{2(q-m)}{qm}$.

Replacing r by qr in (3.3.77) and (3.3.78), and using the bounds $\frac{m-2}{2m} \leq \frac{1}{4}$ and $\frac{2(m-q)}{qm} \leq \frac{1}{2}$ for $2 \leq q \leq m \leq 3$ and $\frac{2-m}{2m} \leq \frac{1}{4}$ and $\frac{2(q-m)}{qm} \leq \frac{1}{2}$ for $\frac{3}{2} \leq m \leq q \leq 2$, we have thus proved that (3.3.72) holds for all $2 \leq q < m < \bar{m} \wedge 3$ and for all $\bar{m}' \vee \frac{3}{2} < m < q \leq 2$. By choosing $\kappa = \frac{(\bar{m}-2) \wedge 1}{8}$, the validity of (3.3.72) in the full range of exponents $2 - \kappa \leq m, q \leq 2 + \kappa$ then follows by real interpolation. \square

We conclude this subsection by the annealed version of the maximal regularity for the Laplacian.

Theorem 23. *Let $L \geq 1$. For all Q_L -periodic random fields g and u related via*

$$-\Delta u = \nabla \cdot g,$$

we have for all exponents $1 < m, q < \infty$,

$$\left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla u|^2 \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \right)^{\frac{1}{m}} \lesssim_{m,q} \left(\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |g|^2 \right)^{\frac{q}{2}} \right]^{\frac{m}{q}} dx \right)^{\frac{1}{m}}. \quad (3.3.79)$$

A proof of this result can be found in [101, Section 7.1]. A simpler argument (based on CZ estimates for Hilbert-valued operators and interpolation) would show that the multiplicative constant in (3.3.79) is of the order $m + m' + q + q'$ (this finer result will not be used here).

3.4 Control of correctors: Proof of Theorem 15

The proof relies on the following upgrade of Proposition 5 based on Theorem 21 and on Theorem 22.

Corollary 24. *Under Hypothesis 3.1.2, there exists $\gamma > 0$ such that for all $\xi \in \mathbb{R}^d$, $L \geq 1$, and all $g \in L^2(\mathbb{R}^d)$ compactly supported in Q_L , the random field $\mathcal{F} := \int_{Q_L} g(\nabla \phi_\xi, \nabla \sigma_\xi)$ satisfies for all $q \geq 1$*

$$\mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} \lesssim_{|\xi|} q^\gamma \int_{Q_L} |g|^2. \quad (3.4.1)$$

For future reference, we state the following consequence of local regularity and of the hole-filling estimate.

Lemma 21. *Under Hypothesis 3.1.2, with $0 < \delta \leq d$ the nonlinear hole-filling exponent of Lemma 15, we have for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$*

$$\|\xi + \nabla \phi_\xi\|_{C^\alpha(B(x))} \lesssim (1 + |\xi|)(r_\star(x))^{\frac{d-\delta}{p}}. \quad (3.4.2)$$

Proof. By the deterministic regularity theory of Lemma 25 applied to the equation (3.1.7) combined with the estimate (3.3.5), we indeed have

$$\begin{aligned} \|\xi + \nabla \phi_\xi\|_{C^\alpha(B(x))} &\lesssim_{\|A\|_{C^{0,\alpha}(\mathbb{R}^d)}} \left(\int_{B_2(x)} |\xi + \nabla \phi_\xi|^p \right)^{\frac{1}{p}} \\ &\stackrel{(3.3.5)}{\leq} \|A\|_{C^{0,\alpha}(\mathbb{R}^d)} (1 + |\xi|)(r_\star(x))^{\frac{d-\delta}{p}}. \end{aligned}$$

\square

Before we turn to the proof of Corollary 24, let us quickly argue that it yields Theorem 15.

Proof of Theorem 15. By (3.4.2) and Theorem 21, assumption (3.7.49) in Proposition 10 is satisfied for $\nabla\phi_\xi$. Let us show that this also yields assumption (3.7.49) for $\nabla\sigma_\xi$. Indeed, by maximal regularity for the Laplacian applied to equation (3.2.2) we have for all $q > 1$, $\left(\int_{Q_L} |\nabla\sigma_\xi|^q\right)^{\frac{1}{q}} \lesssim q\left(\int_{Q_L} |\xi + \nabla\phi_\xi|^q\right)^{\frac{1}{q}}$, so that assumption (3.7.49) for $\nabla\sigma_\xi$ follows from taking the expectation of the q -th power of this estimate and using the stationarity of the extended corrector gradient together with the moment bound on $\nabla\phi_\xi$. By (3.7.50), we can then pass to the limit $L \uparrow +\infty$ in the moment bounds on the extended corrector gradient for the periodized ensemble, and obtain (3.2.6). Likewise, the claimed estimate (3.2.7) follows from Corollary 24 for g compactly supported by passing to the limit $L \uparrow \infty$ using (3.7.50). The result for general $g \in L^2(\mathbb{R}^d)$ is then obtained by approximation. The control (3.2.8) of the growth of the extended corrector is a direct consequence of (3.2.7) by “integration” (see for instance [54, Proof of Theorem 4.2, Step 3] – the argument is also displayed in the proof of Corollary 16). \square

It remains to prove Corollary 24.

Proof of Corollary 24. We split the proof into two steps, first treat averages of $\nabla\phi_\xi$ and then turn to averages of $\nabla\sigma_\xi$.

Step 1. Averages of $\nabla\phi_\xi$.

In this step we set $\mathcal{F} := \int_{Q_L} g \cdot \nabla\phi_\xi$ for some $g \in L^2(\mathbb{R}^d)^d$ compactly supported in Q_L . The starting point is the estimate (3.3.66) in the proof of Proposition 5, which takes the form for all $q \geq 1$ of

$$\mathbb{E}_L \left[\mathcal{F}^{2q} \right]^{\frac{1}{q}} \lesssim q(1 + |\xi|^p) \mathbb{E}_L \left[\left(\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right) dx \right)^q \right]^{\frac{1}{q}},$$

where u is the unique weak Q_L -periodic solution (with zero average) of (3.3.60), that is, $-\nabla \cdot a_\xi^* \nabla u = \nabla \cdot g$. By duality, we may reformulate the right-hand side as

$$\mathbb{E}_L \left[\left(\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u|^2 \mu_\xi \right) dx \right)^q \right]^{\frac{1}{q}} = \sup_{\mathbb{E}_L[|X|^{2q'}]=1} \mathbb{E}_L \left[\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla Xu|^2 \mu_\xi \right) dx \right],$$

where the supremum runs over random variables $X \in L^{2q'}(d\mathbb{P}_L)$ which are independent of the space variable. Let $0 < \eta < 1$ be some exponent (to be fixed later) small enough so that $\frac{q'}{1+\eta} > 1$. We then appeal to Hölder’s inequality with exponents $(\frac{q'}{q'-1-\eta}, \frac{q'}{1+\eta})$ and to the stationarity of r_\star to the effect that

$$\mathbb{E}_L \left[\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla Xu|^2 \mu_\xi \right) dx \right] \leq \mathbb{E} \left[r_\star^{\frac{q'}{q'-1-\eta}(d-\delta)} \right]^{\frac{q'-1-\eta}{q'}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla Xu|^2 \mu_\xi \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} dx.$$

Provided $2q' \leq 2 + \kappa$, we may appeal to Theorem 22 on the second right-hand side factor, which yields (recall that X does not depend on the space variable, that $\mathbb{E}_L[|X|^{2q'}] = 1$ and that $\mu_\xi \geq 1$)

$$\begin{aligned} & \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla Xu|^2 \mu_\xi \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} dx \\ & \lesssim \eta^{-\frac{1}{4}} |\log(\eta)|^{\frac{1}{2}} \int_{Q_L} \mathbb{E}_L \left[|X|^{2q'} \left(\int_{B(x)} |g|^2 \frac{1}{\mu_\xi} \right)^{q'} \right]^{\frac{1}{q'}} dx \lesssim \eta^{-\frac{1}{4}} |\log(\eta)|^{\frac{1}{2}} \int |g|^2. \end{aligned}$$

The choice $\eta = \frac{1}{2}(q' - 1) = \frac{1}{2(q-1)}$ is legitimate provided $q \gg 1$, in which case the above combined with the moment bound on r_* of Theorem 21 yields

$$\mathbb{E}_L \left[\mathcal{F}^{2q} \right]^{\frac{1}{q}} \lesssim q^\nu (1 + |\xi|^p) \int |g|^2.$$

for some exponent $\nu > 0$ independent of q . This entails (3.4.1) for $\nabla \phi_\xi$ for a suitable exponent $\gamma > 0$ (depending only on ν).

Step 2. Averages of $\nabla \sigma_\xi$.

Fix $1 \leq i, j \leq d$. We proceed as for $\nabla \phi_\xi$: We first derive a representation formula for the sensitivity of $\mathcal{F} := \int_{Q_L} g \cdot \nabla \sigma_{\xi,ij}$ with respect to changes of the coefficient A , and then use the annealed estimates of Theorems 22 and 23, and the moment bounds on r_* to conclude.

Substep 2.1. Sensitivity calculus.

Recall the defining equation for $\sigma_{\xi,ij}$

$$-\Delta \sigma_{\xi,ij} = \partial_i(a(\cdot, \xi + \nabla \phi_\xi) \cdot e_j) - \partial_j(a(\cdot, \xi + \nabla \phi_\xi) \cdot e_i).$$

As in Step 1 of the proof of Proposition 5, we proceed by duality. This time we introduce two auxiliary functions u_1 and u_2 as Q_L -periodic solutions of

$$-\Delta u_1 = \nabla \cdot g, \quad -\nabla \cdot a_\xi^* \nabla u_2 = \nabla \cdot a_\xi^* (\partial_i u_1 e_j - \partial_j u_1 e_i),$$

and claim that

$$\delta_x \mathcal{F} = \int_{B(x)} |a_\circ(\xi + \nabla \phi_\xi) \otimes (\nabla u_2 + \partial_i u_1 e_j - \partial_j u_1 e_i)|. \quad (3.4.3)$$

Let us quickly argue in favor of (3.4.3). With the notation of Step 1 of the proof of Proposition 5, and δA an increment of A localized in $B(x)$, we have by the defining equations for $\sigma_{\xi,ij}$ and u_1

$$\delta^h \mathcal{F} := \frac{\mathcal{F}(A + h\delta A) - \mathcal{F}(A)}{h} = \int (\partial_i u_1 e_j - \partial_j u_1 e_i) \cdot \delta^h (a(\xi + \nabla \phi_\xi)),$$

where

$$\begin{aligned} \delta^h (a(\xi + \nabla \phi_\xi)) &= \frac{(A + h\delta A)a_\circ(\xi + \nabla \phi_\xi(A + h\delta A)) - Aa_\circ(\xi + \nabla \phi_\xi)}{h} \\ &= \delta A a_\circ(\xi + \nabla \phi_\xi(A + h\delta A)) + a_\xi^h \nabla \delta^h \phi_\xi. \end{aligned}$$

Passing to the limit $h \downarrow 0$, and testing the equation for u_2 with $\delta \phi_\xi$ and equation (3.3.65) with u_2 , we obtain

$$\begin{aligned} \delta \mathcal{F} &= \lim_{h \downarrow 0} \delta^h \mathcal{F} = \int (\partial_i u_1 e_j - \partial_j u_1 e_i) \cdot (\delta A a_\circ(\xi + \nabla \phi_\xi) + a_\xi \nabla \delta \phi_\xi) \\ &= \int (\nabla u_2 + \partial_i u_1 e_j - \partial_j u_1 e_i) \cdot \delta A a_\circ(\xi + \nabla \phi_\xi), \end{aligned}$$

and the claim follows by taking the supremum over δA .

Substep 2.2. Proof of (3.4.1).

Combining (3.4.3) with the logarithmic-Sobolev inequality, we obtain for all $q \geq 1$

$$\mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} \lesssim q \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |a_\circ(\xi + \nabla \phi_\xi)| |\nabla u_2 + \partial_i u_1 e_j - \partial_j u_1 e_i|^2 dx \right)^q \right)^{\frac{1}{q}} \right].$$

We treat differently the terms involving u_1 and u_2 . For u_2 we proceed as in Step 3 of the proof of Proposition 5 (using the definition (3.3.7) of μ_ξ and (3.3.5)), whereas for u_1 we directly use (3.4.2). This yields

$$\begin{aligned} \mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} &\lesssim_{|\xi|} q \mathbb{E}_L \left[\left(\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla u_2|^2 \mu_\xi \right) dx \right)^q \right]^{\frac{1}{q}} \\ &\quad + q \mathbb{E}_L \left[\left(\int_{Q_L} r_\star(x)^{\frac{2(p-1)}{p}(d-\delta)} \left(\int_{B(x)} |\nabla u_1|^2 \right) dx \right)^q \right]^{\frac{1}{q}}. \end{aligned}$$

As in Step 1, this entails

$$\begin{aligned} \mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} &\lesssim_{|\xi|} q \sup_{\mathbb{E}_L[|X|^{2q'}]=1} \mathbb{E}_L \left[\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla X u_2|^2 \mu_\xi \right) dx \right] \\ &\quad + q \sup_{\mathbb{E}_L[|X|^{2q'}]=1} \mathbb{E}_L \left[\int_{Q_L} r_\star(x)^{\frac{2(p-1)}{p}(d-\delta)} \left(\int_{B(x)} |\nabla X u_1|^2 \right) dx \right]. \end{aligned}$$

For the second right-hand side term, we proceed as in Step 1 (using Theorem 23 in place of Theorem 22), and it remains to treat the first right-hand side term. We then use Hölder's inequality with exponents $(\frac{q'}{q'-(1+\eta)^2}, \frac{q'}{(1+\eta)^2})$ for some $0 < \eta < 1$ (so that $q' > (1+\eta)^2$) to be chosen below to the effect that

$$\begin{aligned} \mathbb{E}_L \left[\int_{Q_L} r_\star(x)^{d-\delta} \left(\int_{B(x)} |\nabla X u_2|^2 \mu_\xi \right) dx \right] \\ \leq \mathbb{E} \left[r_\star^{\frac{q'}{q'-(1+\eta)^2}(d-\delta)} \right]^{\frac{q'-(1+\eta)^2}{q'}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X u_2|^2 \mu_\xi \right)^{\frac{q'}{(1+\eta)^2}} \right]^{\frac{(1+\eta)^2}{q'}} dx. \end{aligned}$$

We then appeal to the annealed Meyers estimate of Theorem 22 under the condition that $2 \leq \frac{2q'}{(1+\eta)^2} \leq 2 + \kappa$, and obtain

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X u_2|^2 \mu_\xi \right)^{\frac{q'}{(1+\eta)^2}} \right]^{\frac{(1+\eta)^2}{q'}} dx \lesssim \eta^{\frac{1}{4}} |\log \eta|^{\frac{1}{2}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\mu_\xi \nabla X u_1|^2 \frac{1}{\mu_\xi} \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} dx$$

since under the assumption $0 < \eta < \frac{1}{2}$, we have $(1+\eta)^2 - 1 \lesssim \eta$. Bounding μ_ξ by $r_\star^{\frac{p-2}{p}(d-\delta)}$ (cf. Lemma 15) and using Hölder's inequality with exponents $(\frac{1+\eta}{\eta}, 1+\eta)$, the integral in the right-hand side is controlled by

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\mu_\xi \nabla X u_1|^2 \frac{1}{\mu_\xi} \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} \lesssim \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta} \frac{p-2}{p}(d-\delta)} \right]^{\frac{\eta}{q'}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X u_1|^2 \right)^{q'} \right]^{\frac{1}{q'}}.$$

We finally estimate the integral term by Theorem 23, which yields (since there is no loss in the stochastic exponent, g is deterministic, and $1 \leq q' \leq 2$)

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X u_1|^2 \right)^{q'} \right]^{\frac{1}{q'}} \lesssim \mathbb{E}_L \left[|X|^{2q'} \right] \int_{Q_L} |g|^2 = \int_{Q_L} |g|^2.$$

The conclusion follows by choosing $\eta = \frac{1}{4}(q' - 1)$ and $q \gg 1$, and using the moment bound on r_\star of Theorem 21. \square

3.5 Control of corrector differences: Proof of Theorem 17

3.5.1 Reduction argument

As for nonlinear correctors, by Proposition 10 it is enough to prove estimates for L -periodic ensembles that are uniform with respect to L . We split the version of Corollary 24 for the linearized corrector into two statements: Proposition 6 below shows that averages of the gradient of the extended linearized corrector decay at the CLT scaling provided we have good control of moments of $\nabla\tilde{\phi}_{\xi,e}$, whereas Proposition 7 provides the latter.

Proposition 6. *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$ and all $0 < \theta < 1$ there exists $\gamma > 0$ (depending on $|\xi|$ and θ) such that for all $L \geq 1$, all $g \in L^2(\mathbb{R}^d)$ compactly supported in Q_L , and all unit vectors $e \in \mathbb{R}^d$, the random field $\mathcal{F} := \int_{Q_L} g(\nabla\tilde{\phi}_{\xi,e}, \nabla\tilde{\sigma}_{\xi,e})$ satisfies for all $q \geq 1$ such that $2q' \leq 2 + \kappa$ (where $\kappa > 0$ is as in Theorem 22)*

$$\mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} \lesssim_{|\xi|, \theta} q^\gamma \mathbb{E}_L \left[\left(\sup_B |\nabla\tilde{\phi}_{\xi,e} + e|^2 \mu_\xi \right)^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \left(\int_{Q_L} |g|^2 \right) \quad (3.5.1)$$

The proof of Proposition 6 relies on a sensitivity estimate by duality combined with the annealed Meyers estimate of Theorem 22.

Proposition 7 (Control of moments). *Under Hypothesis 3.1.2, for all $\xi \in \mathbb{R}^d$, there exists $\gamma > 0$ (depending on $|\xi|$) such that for all $L \geq 1$ and all unit vectors $e \in \mathbb{R}^d$ we have*

$$\mathbb{E}_L \left[\left(\sup_B |\nabla\tilde{\phi}_{\xi,e} + e|^2 \mu_\xi \right)^q \right]^{\frac{1}{q}} \lesssim q^\gamma. \quad (3.5.2)$$

The proof of Proposition 7 is based on Proposition 6 and a buckling argument. Because the linearized corrector equation has unbounded coefficients, we cannot use the elegant approach of [137] (see also [54, Proposition 4.5]) to buckle on moments of $\nabla\tilde{\phi}_{\xi,e}$ themselves. Instead, as we did for $r_{*,\xi,L}$, we have to go through the super levelsets of some minimal radius controlling the growth of averages of $|\nabla\tilde{\phi}_{\xi,e}|^2 \mu_\xi$.

Before we turn to the proofs, let us show how bounds on linearized correctors allow us to derive bounds on nonlinear corrector differences in form of Corollary 16.

Proof of Corollary 16. For simplicity, we only treat ϕ_ξ .

Step 1. Statement for differences of corrector gradients.

By (3.7.50) in Proposition 10 in form of (note the difference of expectations)

$$\mathbb{E} \left[|\nabla(\phi_\xi - \phi_{\xi'})|^q \right]^{\frac{1}{q}} = \lim_{L \uparrow +\infty} \mathbb{E}_L \left[|\nabla(\phi_\xi - \phi_{\xi'})|^q \right]^{\frac{1}{q}},$$

it suffices to prove the statement for the periodized ensemble. By Lemma 27, \mathbb{P}_L -almost surely, $\xi \mapsto \nabla\phi_\xi$ is differentiable and we have by the fundamental theorem of calculus for all $e \in \mathbb{R}^d$

$$e \cdot (\nabla\phi_\xi - \nabla\phi_{\xi'}) = \int_0^1 \nabla\tilde{\phi}_{\xi+t(\xi'-\xi),e} \cdot (\xi' - \xi) dt, \quad (3.5.3)$$

so that by taking the q -th moment and using Proposition 7, one obtains

$$\mathbb{E}_L \left[|\nabla\phi_\xi - \nabla\phi_{\xi'}|^q \right]^{\frac{1}{q}} \leq |\xi - \xi'| \sum_i \int_0^1 \mathbb{E}_L \left[|\nabla\phi_{\xi+t(\xi'-\xi),e_i}|^q \right]^{\frac{1}{q}} \lesssim q^\gamma |\xi - \xi'|, \quad (3.5.4)$$

which yields the claim by taking the limit $L \uparrow \infty$.

Step 2. Statement for corrector differences.

By (3.7.51), since $\int_B \phi_\xi = 0$, for all $x \in \mathbb{R}^d$ we have for all $q \geq 1$

$$\mathbb{E} \left[\left(\int_{B(x)} |\phi_\xi - \phi_{\xi'}|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} = \lim_{L \uparrow \infty} \mathbb{E}_L \left[\left(\int_{B(x)} \left| \phi_\xi - \phi_{\xi'} - \int_B \phi_\xi - \phi_{\xi'} \right|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}}. \quad (3.5.5)$$

To control the right-hand side, we shall bound moments of periodic random fields ζ by moments of averages of their gradients $\nabla \zeta$. Indeed, by Poincaré's inequality on $B(x)$ for $x \in Q_L$, we have for $c = \int_B \zeta$

$$\mathbb{E}_L \left[\left(\int_{B(x)} (\zeta - c)^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \lesssim \mathbb{E}_L \left[|\nabla \zeta|^q \right]^{\frac{1}{q}} + \mathbb{E}_L \left[\left| \int_{B(x)} \zeta - c \right|^q \right]^{\frac{1}{q}}, \quad (3.5.6)$$

and it remains to estimate the second right-hand side term. For that purpose, we write

$$\int_{B(x)} \zeta - \int_B \zeta = \int_{Q_L} \nabla \zeta \cdot \nabla h_x,$$

where h_x denotes the unique weak periodic solution in Q_L of $-\Delta h_x = \frac{1}{|B|}(\mathbf{1}_{B(x)} - \mathbf{1}_B)$. We apply this to $\zeta = \phi_\xi - \phi_{\xi'}$ and rewrite the gradient as $e \cdot \nabla \zeta = \int_0^1 \nabla \phi_{\xi+t(\xi'-\xi),e} \cdot (\xi' - \xi) dt$, to the effect that (with implicit sum on the repeated index i)

$$\int_{B(x)} (\phi_\xi - \phi_{\xi'}) - \int_B (\phi_\xi - \phi_{\xi'}) = (\xi' - \xi) \cdot \int_0^1 \int_{Q_L} \nabla \phi_{\xi+t(\xi'-\xi),e_i} \nabla_i h_x.$$

Using Propositions 6 and 7, this yields

$$\mathbb{E}_L \left[\left| \int_{B(x)} (\phi_\xi - \phi_{\xi'}) - \int_B (\phi_\xi - \phi_{\xi'}) \right|^q \right]^{\frac{1}{q}} \leq_{|\xi|,|\xi'|} |\xi - \xi'| q^\gamma \left(\int_{Q_L} |\nabla h_x|^2 \right)^{\frac{1}{2}}.$$

A direct computation with Green's kernel gives $\|\nabla h_x\|_{L^2(Q_L)} \lesssim \mu_d(x)$, and thus

$$\mathbb{E}_L \left[\left| \int_{B(x)} (\phi_\xi - \phi_{\xi'}) - \int_B (\phi_\xi - \phi_{\xi'}) \right|^q \right]^{\frac{1}{q}} \leq_{|\xi|,|\xi'|} |\xi - \xi'| q^\gamma \mu_d(x).$$

Combined with (3.5.6), (3.5.5), and (3.5.4), this entails $\mathbb{E} \left[\left(\int_{B(x)} |\phi_\xi - \phi_{\xi'}|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \lesssim |\xi - \xi'| q^\gamma \mu_d(x)$, from which the claim follows using local regularity in form of Lemma 25 and (3.5.4) in the limit $L \uparrow +\infty$.

Step 3. Regularity of $\xi \mapsto \bar{a}(\xi)$.

The starting point is the definition $\bar{a}(\xi) := \mathbb{E} \left[a(\xi + \nabla \phi_\xi) \right] = \mathbb{E} \left[A(0) a_\circ(\xi + \nabla \phi_\xi(0)) \right]$ and of its approximation by periodization $\bar{a}_L(\xi) := \mathbb{E}_L \left[A(0) a_\circ(\xi + \nabla \phi_\xi(0)) \right]$ for all $L \geq 1$. Since $\bar{a}_L(\xi) \rightarrow \bar{a}(\xi)$ as $L \uparrow +\infty$, it is enough to prove that $D\bar{a}_L$ is Lipschitz-continuous uniformly with respect to L and given for all $\xi, e \in \mathbb{R}^d$ by

$$D\bar{a}_L(\xi)e := \bar{a}_{L,\xi}e = \mathbb{E}_L \left[A(0) D a_\circ(\xi + \nabla \phi_\xi(0))(e + \nabla \tilde{\phi}_{\xi,e}(0)) \right].$$

The differentiability of $\xi \mapsto \bar{a}_L(\xi)$ and the formula for $D\bar{a}_L(\xi)$ follow from (3.5.3), the continuity of $\xi \mapsto \nabla \tilde{\phi}_{\xi,e}$, and the moment bounds on $\nabla \tilde{\phi}_{\xi,e}$. It remains to argue that $\xi \mapsto D\bar{a}_L(\xi)$ is Lipschitz-continuous. Since $\xi \mapsto \nabla \phi_\xi$ is continuously differentiable with stretched exponential moment bounds,

it is enough to prove that $\xi \mapsto \nabla \tilde{\phi}_{\xi,e}$ is itself Lipschitz-continuous in $L^2(d\mathbb{P})$. This is a direct consequence of the defining equation (3.2.9) in the form for all $\xi, \xi' \in \mathbb{R}^d$ of

$$-\nabla \cdot Da(\cdot, \xi + \nabla \phi_\xi) \nabla (\tilde{\phi}_{\xi,e} - \tilde{\phi}_{\xi',e}) = \nabla \cdot (Da(\cdot, \xi + \nabla \phi_\xi) - Da(\cdot, \xi' + \nabla \phi_{\xi'}))(e + \nabla \tilde{\phi}_{\xi',e})$$

combined with the differentiability of $\xi \mapsto \nabla \phi_\xi$, uniform moment bounds on $\nabla \tilde{\phi}_{\xi',e}$ and $\nabla \phi_\xi$, and an energy estimate. \square

3.5.2 CLT-scaling: Proof of Proposition 6

In this paragraph, we fix e and ξ , and use the short-hand notation r_\star for $r_{\star,\xi,L}$, ϕ for ϕ_ξ , μ for μ_ξ , $\tilde{\phi}$ for $\tilde{\phi}_{\xi,e}$, $\tilde{\sigma}$ for $\tilde{\sigma}_{\xi,e}$. We split the proof into three steps. In the first two steps, we derive sensitivity estimates for averages of $\nabla \tilde{\phi}$ and of $\nabla \tilde{\sigma}$, respectively, and then conclude in the third step using Theorems 22 and 23.

Step 1. Sensivity formula for $\nabla \tilde{\phi}$: The random variable $\mathcal{F}_1 := \int_{Q_L} g \cdot \nabla \tilde{\phi}$ (where g abusively denotes ge' for some unit vector $e' \in \mathbb{R}^d$) satisfies for all $x \in Q_L$

$$\delta_x \mathcal{F}_1 = \int_{B(x)} |Da_o(\xi + \nabla \phi)(e + \nabla \tilde{\phi}) \otimes \nabla u_1 + a_o(\xi + \nabla \phi) \otimes \nabla u_2|, \quad (3.5.7)$$

where we recall that $a_o : \xi \in \mathbb{R}^d \mapsto (1 + |\xi|^{p-2})\xi$, and where $u_1, u_2 \in H_{\text{per}}^1(Q_L)$ are the unique weak solutions of

$$-\nabla \cdot a_\xi^* \nabla u_1 = \nabla \cdot g \quad (3.5.8)$$

and (with an implicit sum over the repeated index k)

$$-\nabla \cdot a_\xi^* \nabla u_2 = \partial_k (D^2 a(\xi + \nabla \phi)(e + \nabla \tilde{\phi}) e_k \cdot \nabla u_1). \quad (3.5.9)$$

(These equations are well-posed since the Q_L -periodic maps $\nabla \phi$ and a_ξ^* are bounded almost surely.)

Let us give the quick argument. Following Step 1 of the proof of Proposition 5, we let δA be an increment of A localized in $B(x)$ and consider for h small enough

$$\begin{aligned} \delta^h \mathcal{F}_1 &:= \frac{\mathcal{F}_1(A + h\delta A) - \mathcal{F}_1(A)}{h} = \int_{Q_L} g \cdot \nabla \delta^h \tilde{\phi}, & \delta^h \tilde{\phi} &:= \frac{\tilde{\phi}(A + h\delta A) - \tilde{\phi}(A)}{h}, \\ b_\xi^h &:= A \int_0^1 D^2 a_o(\xi + t\nabla \phi(A + h\delta A) + (1-t)\nabla \phi(A)) dt, \end{aligned}$$

and recall the notation (3.3.61) and (3.3.62). By the defining equation (3.2.9) for the linearized corrector, we obtain

$$\begin{aligned} -\nabla \cdot a_\xi \nabla \delta^h \tilde{\phi} &= \nabla \cdot \delta A D a_o(\xi + \nabla \phi(A + h\delta A))(e + \nabla \tilde{\phi}(A + h\delta A)) \\ &\quad + \frac{1}{h} \nabla \cdot A (D a_o(\xi + \nabla \phi(A + h\delta A)) - D a_o(\xi + \nabla \phi(A)))(e + \nabla \tilde{\phi}(A + h\delta A)), \end{aligned}$$

which we rewrite, by the fundamental theorem of calculus and the definition of b_ξ^h , as

$$-\nabla \cdot a_\xi \nabla \delta^h \tilde{\phi} = \nabla \cdot \delta A D a_o(\xi + \nabla \phi(A + h\delta A))(e + \nabla \tilde{\phi}(A + h\delta A)) + \nabla \cdot b_\xi^h \nabla \delta^h \phi(e + \nabla \tilde{\phi}(A + h\delta A)).$$

As in Step 1 of the proof of Proposition 5, we can pass to the limit as $h \downarrow 0$ and obtain that $\delta^h \tilde{\phi}$ converges in $C^{1,\alpha}(Q_L)$ to the solution $\delta \tilde{\phi} \in H_{\text{per}}^1(Q_L)$ of

$$-\nabla \cdot a_\xi \nabla \delta \tilde{\phi} = \nabla \cdot \delta A D a_o(\xi + \nabla \phi)(e + \nabla \tilde{\phi}) + \nabla \cdot b_\xi \nabla \delta \phi(e + \nabla \tilde{\phi}), \quad (3.5.10)$$

with $b_\xi := D^2a(\xi + \nabla\phi)$.

We now proceed by duality. First, we test (3.5.10) with u_1 and (3.5.8) with $\delta\tilde{\phi}$ to obtain

$$\delta\mathcal{F}_1 = \lim_{h\downarrow 0} \delta^h \mathcal{F}_1 = \int_{Q_L} \nabla u_1 \cdot \delta A D a_o(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) + \int_{Q_L} \nabla u_1 \cdot b_\xi \nabla \delta\phi(e + \nabla\tilde{\phi}). \quad (3.5.11)$$

Second, we test (3.3.65) with u_2 and (3.5.9) with $\delta\phi$ to get

$$\int_{Q_L} \nabla u_1 \cdot b_\xi \nabla \delta\phi(e + \nabla\tilde{\phi}) = \int_{Q_L} \nabla u_2 \cdot \delta A a_o(\xi + \nabla\phi). \quad (3.5.12)$$

The combination of (3.5.11) and (3.5.12) then entails the claim (3.5.7) by taking the supremum over δA .

Step 2. Sensitivity formula for $\nabla\tilde{\sigma}_{ij}$ (for i, j fixed): The random variable $\mathcal{F}_2 := \int_{Q_L} g \cdot \nabla\tilde{\sigma}_{ij}$ satisfies for all $x \in Q_L$

$$\delta_x \mathcal{F}_2 = \int_{B(x)} |D a_o(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) \otimes (\nabla w_1 + \partial_i v e_j - \partial_j v e_i) + a_o(\xi + \nabla\phi) \otimes \nabla w_2|, \quad (3.5.13)$$

where the functions $v, w_1, w_2 \in H^1_{\text{per}}(Q_L)$ solve (with an implicit sum over the repeated index k)

$$-\Delta v = \nabla \cdot g, \quad (3.5.14)$$

$$-\nabla \cdot a_\xi^* \nabla w_1 = \nabla \cdot a_\xi^* (\partial_i v e_j - \partial_j v e_i), \quad (3.5.15)$$

$$-\nabla \cdot a_\xi^* \nabla w_2 = \partial_k (D^2 a(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) e_k \cdot (\nabla w_1 + \partial_i v e_j - \partial_j v e_i)). \quad (3.5.16)$$

We only display the algebra of the argument (passing already to the limit $h \downarrow 0$, which entails that $\delta = \lim_{h\downarrow 0} \delta^h$ satisfies the Leibniz rule). Recall the defining equation for $\tilde{\sigma}_{ij}$ with the notation $a_\xi = D a(\xi + \nabla\phi)$

$$-\Delta \tilde{\sigma}_{ij} = \partial_i (a_\xi(e + \nabla\tilde{\phi}) \cdot e_j) - \partial_j (a_\xi(e + \nabla\tilde{\phi}) \cdot e_i).$$

First, by (3.5.14),

$$\delta \mathcal{F}_2 = \int (\partial_i v e_j - \partial_j v e_i) \cdot \delta (D a(\xi + \nabla\phi)(e + \nabla\tilde{\phi})).$$

Since δ satisfies the Leibniz rule, we have

$$\delta (D a(\xi + \nabla\phi)(e + \nabla\tilde{\phi})) = \delta A D a_o(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) + D^2 a(\xi + \nabla\phi) \nabla \delta\phi(e + \nabla\tilde{\phi}) + a_\xi \nabla \delta\tilde{\phi}.$$

The first right-hand term directly gives the right-hand side contribution of (3.5.13) involving ∇v . For the second term, we introduce the solutions $w_{2,1}$ and $w_{2,2}$ of $-\nabla \cdot a_\xi^* \nabla w_{2,1} = \partial_k (D^2 a(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) e_k \cdot (\partial_i v e_j - \partial_j v e_i))$ and $-\nabla \cdot a_\xi^* \nabla w_{2,2} = \partial_k (D^2 a(\xi + \nabla\phi)(e + \nabla\tilde{\phi}) e_k \cdot \nabla w_1)$ to the effect that $w_2 = w_{2,1} + w_{2,2}$. By using (3.3.65), we obtain

$$\int (\partial_i v e_j - \partial_j v e_i) \cdot D^2 a(\xi + \nabla\phi) \nabla \delta\phi(e + \nabla\tilde{\phi}) = \int \nabla w_{2,1} \cdot \delta A a_o(\xi + \nabla\phi).$$

This yields part of the right-hand side contribution of (3.5.13) involving ∇w_2 . We conclude with the third term. Using first (3.5.15) we obtain

$$\int (\partial_i v e_j - \partial_j v e_i) \cdot a_\xi \nabla \delta\tilde{\phi} = - \int \nabla w_1 \cdot a_\xi \nabla \delta\tilde{\phi},$$

and therefore using (3.5.10)

$$\int (\partial_i u e_j - \partial_j u e_i) \cdot a_\xi \nabla \delta \tilde{\phi} = \int \nabla w_1 \cdot \left(\delta A D a_\circ(\xi + \nabla \phi)(e + \nabla \tilde{\phi}) + D^2 a(\xi + \nabla \phi) \nabla \delta \phi(e + \nabla \tilde{\phi}) \right).$$

The first right-hand side term yields the right-hand side contribution of (3.5.13) involving ∇w_1 . For the second term, we use $w_{2,2}$, and conclude using (3.3.65) that

$$\int \nabla w_1 \cdot D^2 a(\xi + \nabla \phi) \nabla \delta \phi(e + \nabla \tilde{\phi}) = \int \nabla w_{2,2} \cdot \delta A a_\circ(\xi + \nabla \phi).$$

This gives the second part of the right-hand side contribution of (3.5.13) involving ∇w_2 , recalling that $\nabla w_2 = \nabla w_{2,1} + \nabla w_{2,2}$.

Step 3. Proof of (3.5.1).

From the logarithmic-Sobolev inequality, and Steps 1 and 2, we deduce by the triangle inequality that for all $q \geq 1$,

$$\begin{aligned} \mathbb{E}_L \left[|\mathcal{F}|^{2q} \right]^{\frac{1}{q}} &\lesssim \underbrace{q \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |D a_\circ(\xi + \nabla \phi)| |e + \nabla \tilde{\phi}| (|\nabla u_1| + |\nabla v| + |\nabla w_1|) \right)^2 dx \right)^q \right]^{\frac{1}{q}}}_{=: I_1} \\ &\quad + \underbrace{q \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |a_\circ(\xi + \nabla \phi)| (|\nabla u_2| + |\nabla w_2|) \right)^2 dx \right)^q \right]^{\frac{1}{q}}}_{=: I_2}. \end{aligned}$$

To control these terms we proceed as in the proof of Corollary 24: using duality in probability and Theorems 22 and 23. We treat the two right-hand sides separately. (In what follows, γ denotes finite positive exponents independent of q , the precise value of which we are not interested in.)

Substep 3.1. Proof of

$$I_1 \lesssim q^\gamma \mathbb{E}_L \left[\left(\int_{B(x)} |e + \nabla \tilde{\phi}|^2 \mu_\xi \right)^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \int_{Q_L} |g|^2. \quad (3.5.17)$$

The most technical term to treat is the one involving w_1 (which is defined by solving two equations successively, whereas u_1 and v are defined by solving one equation only). By Cauchy-Schwarz' inequality, and the definitions of a_\circ and μ_ξ ,

$$\begin{aligned} \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |D a_\circ(\xi + \nabla \phi)| |e + \nabla \tilde{\phi}| |\nabla w_1| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \\ \lesssim \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |e + \nabla \tilde{\phi}|^2 \mu_\xi \right) \left(\int_{B(x)} |\nabla w_1|^2 \mu_\xi \right) dx \right)^q \right]^{\frac{1}{q}}. \end{aligned}$$

By duality (in probability), this entails

$$\begin{aligned} \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |D a_\circ(\xi + \nabla \phi)| |e + \nabla \tilde{\phi}| |\nabla w_1| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \\ \lesssim \sup_X \mathbb{E}_L \left[\int_{Q_L} \left(\int_{B(x)} |e + \nabla \tilde{\phi}|^2 \mu_\xi \right) \left(\int_{B(x)} |\nabla X w_1|^2 \mu_\xi \right) dx \right], \end{aligned}$$

where the supremum runs over random variables X (independent of the space variable) such that $\mathbb{E}[|X|^{2q'}] = 1$. To obtain the claimed dependence on the moments of $\int_{B(x)} |e + \nabla\tilde{\phi}|^2 \mu_\xi$, we set $\eta_\circ := \frac{\theta}{(1+\theta)(q-1)}$, to the effect that $q' > 1 + \eta_\circ$ and $\frac{q'}{q'-(1+\eta_\circ)} = q(1+\theta)$, and use Hölder's inequality with exponents $(\frac{q'}{q'-(1+\eta_\circ)}, \frac{q'}{1+\eta_\circ})$, so that the above turns into

$$\begin{aligned} & \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |Da_\circ(\xi + \nabla\phi)| |e + \nabla\tilde{\phi}| |\nabla w_1| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \\ & \lesssim \mathbb{E}_L \left[\left(\int_{B(x)} |e + \nabla\tilde{\phi}|^2 \mu_\xi \right)^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \sup_X \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X w_1|^2 \mu_\xi \right)^{\frac{q'}{1+\eta_\circ}} \right]^{\frac{1+\eta_\circ}{q'}} dx. \end{aligned}$$

For convenience, we rewrite $1 + \eta_\circ$ as $(1 + \eta)^2$, and apply Theorem 22 to (3.5.15), which yields provided $2q' \leq 2 + \kappa$,

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X w_1|^2 \mu_\xi \right)^{\frac{q'}{(1+\eta)^2}} \right]^{\frac{(1+\eta)^2}{q'}} dx \lesssim \zeta(\eta_\circ) \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X v|^2 \mu_\xi \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} dx,$$

where $\zeta : t \mapsto t^{-\frac{1}{4}} |\log t|^{\frac{1}{2}}$ (since for $0 < \eta_\circ < \frac{1}{2}$, $\zeta(\eta) = \zeta(\sqrt{1+\eta_\circ} - 1) \lesssim \zeta(\eta_\circ)$). By the bound $\mu_\xi \lesssim r_\star^{(d-\delta)\frac{p-2}{p}}$ and Hölder's inequality with exponents $(\frac{1+\eta}{\eta}, 1+\eta)$, followed by Theorem 23 applied to (3.5.14) (with exponent $q' \lesssim 1$) we further have

$$\begin{aligned} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X v|^2 \mu_\xi \right)^{\frac{q'}{1+\eta}} \right]^{\frac{1+\eta}{q'}} dx & \lesssim \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta}{q'}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X v|^2 \right)^{q'} \right]^{\frac{1}{q'}} dx \\ & \lesssim \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta}{q'}} \mathbb{E}_L \left[|X|^{2q'} \right]^{\frac{1}{q'}} \int_{Q_L} |g|^2 \\ & = \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta}{q'}} \int_{Q_L} |g|^2, \end{aligned}$$

where we used that g is deterministic and $\mathbb{E}[|X|^{2q'}] = 1$. We have thus proved that

$$\begin{aligned} & \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |Da_\circ(\xi + \nabla\phi)| |e + \nabla\tilde{\phi}| |\nabla w_1| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \\ & \lesssim \mathbb{E}_L \left[\left(\int_{B(x)} |e + \nabla\tilde{\phi}|^2 \mu_\xi \right)^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \zeta(\eta_\circ) \mathbb{E}_L \left[r_\star^{\frac{q'}{\sqrt{1+\eta_\circ}-1} \frac{p-2}{p} (d-\delta)} \right]^{\frac{\sqrt{1+\eta_\circ}-1}{q'}} \int_{Q_L} |g|^2. \end{aligned}$$

Since $\eta_\circ = \frac{\theta}{(1+\theta)(q-1)}$, by definition of ζ and by the moment bound on r_\star of Theorem 21,

$$q\zeta(\eta_\circ) \mathbb{E}_L \left[r_\star^{\frac{q'}{\sqrt{1+\eta_\circ}-1} \frac{p-2}{p} (d-\delta)} \right]^{\frac{\sqrt{1+\eta_\circ}-1}{q'}} \lesssim q^\gamma$$

for some exponent $\gamma > 0$ independent of q . This entails the claimed estimate (3.5.17).

Substep 3.2. Proof of

$$I_2 \lesssim q^\gamma \mathbb{E}_L \left[\sup_B \{ |e + \nabla\tilde{\phi}|^2 \mu_\xi \}^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \int_{Q_L} |g|^2. \quad (3.5.18)$$

We only display the argument for the term involving ∇w_2 , which is defined by solving three equations successively (which will compel us to appeal to Theorem 22 twice in a row, and then to Theorem 23). By Cauchy-Schwarz' inequality, and the definition of a_\circ and μ_ξ ,

$$\mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |a_\circ(\xi + \nabla\phi)| |\nabla w_2| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \lesssim \mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} \mu_\xi \right) \left(\int_{B(x)} \mu_\xi |\nabla w_2|^2 \right) dx \right)^q \right]^{\frac{1}{q}}.$$

By duality and the bound $\mu_\xi \lesssim r_\star^{(d-\delta)\frac{p-2}{p}}$, we have

$$\mathbb{E}_L \left[\left(\int_{Q_L} \left(\int_{B(x)} |a_\circ(\xi + \nabla\phi)| |\nabla w_2| \right)^2 dx \right)^q \right]^{\frac{1}{q}} \lesssim \sup_X \mathbb{E}_L \left[\int_{Q_L} r_\star^{(d-\delta)\frac{p-2}{p}} \left(\int_{B(x)} \mu_\xi |\nabla X w_2|^2 \right) dx \right],$$

where the supremum runs over random variables X (thus independent of the space variable) such that $\mathbb{E} \left[|X|^{2q'} \right] = 1$. We now introduce exponents: $\eta_2 := \frac{1}{q-1} \frac{\theta}{8(1+\theta)}$ and $\eta_1 := \frac{1}{(q-1)(1+\eta_2)^2(1+\theta)}$ which are chosen so that $\frac{q'}{(1+\eta_2)^2\eta_1} = q(1+\theta)$ and $\frac{q'}{(1+\eta_2)^3(1+\eta_1)} > 1$. Let us quickly check the second property:

$$(1+\eta_2)^3(1+\eta_1) = (1+\eta_2)^3 + \frac{1+\eta_2}{(q-1)(1+\theta)} \leq 1 + \left(7 + \frac{1}{(q-1)(1+\theta)}\right)\eta_2 + \frac{1}{(q-1)(1+\theta)} < 1 + \frac{1}{q-1} = q'.$$

With these exponents at hands, we first use Hölder's inequality with exponents $\left(\frac{q'}{q'-(1+\eta_2)^3(1+\eta_1)}, \frac{q'}{(1+\eta_2)^3(1+\eta_1)}\right)$ together with the stationarity of r_\star , and obtain

$$\begin{aligned} & \mathbb{E}_L \left[\int_{Q_L} r_\star^{(d-\delta)\frac{p-2}{p}} \left(\int_{B(x)} \mu_\xi |\nabla X w_2|^2 \right) dx \right] \\ & \lesssim \mathbb{E}_L \left[r_\star^{\frac{q'}{q'-(1+\eta_2)^3(1+\eta_1)}(d-\delta)\frac{p-2}{p}} \right]^{\frac{q'-(1+\eta_2)^3(1+\eta_1)}{q'}} \int_{Q_L} \mathbb{E} \left[\left(\int_{B(x)} \mu_\xi |\nabla X w_2|^2 \right)^{\frac{q'}{(1+\eta_2)^3(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^3(1+\eta_1)}{q'}} dx. \end{aligned}$$

Provided $2q' \leq 2 + \kappa$, Theorem 22 applied to (3.5.16) yields

$$\begin{aligned} & \int_{Q_L} \mathbb{E} \left[\left(\int_{B(x)} \mu_\xi |\nabla X w_2|^2 \right)^{\frac{q'}{(1+\eta_2)^3(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^3(1+\eta_1)}{q'}} dx \\ & \lesssim \zeta(\eta_2) \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} \mu_\xi^{-1} |D^2 a(\xi + \nabla\phi)|^2 |e + \nabla\tilde{\phi}|^2 (|\nabla X w_1|^2 + |\nabla X v|^2) \right)^{\frac{q'}{(1+\eta_2)^2(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^2(1+\eta_1)}{q'}} dx. \end{aligned}$$

Since $\mu_\xi \geq 1$ and $|D^2 a(\xi + \nabla\phi)| \leq \mu_\xi$, this yields

$$\begin{aligned} & \int_{Q_L} \mathbb{E} \left[\left(\int_{B(x)} \mu_\xi |\nabla X w_2|^2 \right)^{\frac{q'}{(1+\eta_2)^3(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^3(1+\eta_1)}{q'}} dx \\ & \lesssim \zeta(\eta_2) \int_{Q_L} \mathbb{E}_L \left[\sup_{B(x)} \{ |e + \nabla\tilde{\phi}|^2 \mu_\xi \}^{\frac{q'}{(1+\eta_2)^2(1+\eta_1)}} \left(\int_{B(x)} (|\nabla X w_1|^2 + |\nabla X v|^2) \right)^{\frac{q'}{(1+\eta_2)^2(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^2(1+\eta_1)}{q'}} dx. \end{aligned}$$

We only treat the term involving w_1 , which is the most subtle of the two. We then apply Hölder's inequality with exponents $\left(\frac{1+\eta_1}{\eta_1}, 1 + \eta_1\right)$, and use the stationarity of $x \mapsto \sup_{B(x)} \{ |e + \nabla\tilde{\phi}|^2 \mu_\xi \}$ and

the definition of η_1 and η_2 to the effect that

$$\begin{aligned} & \int_{Q_L} \mathbb{E}_L \left[\sup_{B(x)} \{ |e + \nabla \tilde{\phi}|^2 \mu_\xi \}^{\frac{q'}{(1+\eta_2)^2(1+\eta_1)}} \left(\int_{B(x)} (|\nabla X w_1|^2 + |\nabla X v|^2) \right)^{\frac{q'}{(1+\eta_2)^2(1+\eta_1)}} \right]^{\frac{(1+\eta_2)^2(1+\eta_1)}{q'}} dx \\ & \leq \mathbb{E}_L \left[\sup_B \{ |e + \nabla \tilde{\phi}|^2 \mu_\xi \}^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X w_1|^2 \right)^{\frac{q'}{(1+\eta_2)^2}} \right]^{\frac{(1+\eta_2)^2}{q'}} dx. \end{aligned}$$

In view of equation (3.5.15), one may appeal to Theorem 22, and obtain

$$\int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X w_1|^2 \right)^{\frac{q'}{(1+\eta_2)^2}} \right]^{\frac{(1+\eta_2)^2}{q'}} dx \lesssim \zeta(\eta_2) \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} \mu_\xi |\nabla X v|^2 \right)^{\frac{q'}{1+\eta_2}} \right]^{\frac{1+\eta_2}{q'}} dx.$$

We finally bound μ_ξ using r_\star , use Hölder's inequality with exponents $(\frac{1+\eta_2}{\eta_2}, 1 + \eta_2)$ and we apply Theorem 23 to equation (3.5.14)

$$\begin{aligned} & \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} \mu_\xi |\nabla X v|^2 \right)^{\frac{q'}{1+\eta_2}} \right]^{\frac{1+\eta_2}{q'}} dx \\ & \leq \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta_2}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta_2}{q'}} \int_{Q_L} \mathbb{E}_L \left[\left(\int_{B(x)} |\nabla X v|^2 \right)^{q'} \right]^{\frac{1}{q'}} dx \\ & \lesssim \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta_2}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta_2}{q'}} \mathbb{E}_L \left[|X'|^{2q'} \right] \int_{Q_L} |g|^2 = \mathbb{E}_L \left[r_\star^{\frac{q'}{\eta_2}(d-\delta)\frac{p-2}{p}} \right]^{\frac{\eta_2}{q'}} \int_{Q_L} |g|^2. \end{aligned}$$

As in Substep 3.1, the above estimates combine to (3.5.18) using Theorem 21 and our choice of η_2 .

3.5.3 Control of level sets: Proof of Proposition 7

As mentioned above, we do not buckle on moments of $\nabla \tilde{\phi}_{\xi,e}$ but rather on a minimal scale that controls the growth of $R \mapsto \int_{B_R} |\nabla \tilde{\phi}_{\xi,e}|^2 \mu_\xi$ by the growth of $\int_{B_{2R}} \mu_\xi$.

Definition 3.5.1 (Linear minimal scale). *Let $\xi \in \mathbb{R}^d$, $L \geq 1$, $|e| = 1$ and $C > 0$. For all $x \in Q_L$, we define the linear minimal scale $\tilde{r}_{\star,\xi,e,L}(x, C)$ via*

$$\tilde{r}_{\star,\xi,e,L}(x, C) := \inf_{r=2^N, N \in \mathbb{N}} \left\{ \forall R \geq r : \int_{B_R} |\nabla \tilde{\phi}_{\xi,e}|^2 \mu_\xi \leq C \int_{B_{2R}} \mu_\xi \right\}. \quad (3.5.19)$$

As for the Meyers minimal radius, $\tilde{r}_{\star,\xi,e,L}(\cdot, C)$ is bounded by L as soon as C is large enough, due to periodicity and to the plain energy estimates for $\tilde{\phi}_{\xi,e}$ in form of $\int_{Q_L} |\nabla \tilde{\phi}_{\xi,e}|^2 \mu_\xi \lesssim \int_{Q_L} \mu_\xi$. In what follows we fix such a constant C , fix e and ξ , and use the short-hand notation \tilde{r}_\star for $\tilde{r}_{\star,\xi,e,L}(\cdot, C)$, r_\star for $r_{\star,\xi,L}$, ϕ for ϕ_ξ , and $\tilde{\phi}$ for $\tilde{\phi}_{\xi,e}$. The upcoming lemma uses local regularity and hole-filling to control $\sup_B |\nabla \tilde{\phi} + e|^2 \mu_\xi$ by \tilde{r}_\star and r_\star .

Lemma 22 (Quenched bounds on the linearized correctors). *For all $\xi \in \mathbb{R}^d$, there exist two exponents $0 < \beta \leq d$ (the linear hole-filling exponent of Lemma 15) and $\gamma > 0$ and a non-negative stationary random field χ (depending on r_\star , $\|A\|_{C^{0,\alpha}(\mathbb{R}^d)}$ and $|\xi|$) with the following properties: For all $x \in \mathbb{R}^d$*

$$\sup_{B(x)} |e + \nabla \tilde{\phi}|^2 \mu_\xi \leq \chi(x) (\tilde{r}_\star(x))^{d-\beta}, \quad (3.5.20)$$

and all $q \geq 1$

$$\mathbb{E}_L [\chi^q]^{\frac{1}{q}} \lesssim_{|\xi|} q^\gamma. \quad (3.5.21)$$

Proof. We split the proof into two steps. In the first step, we control the C^α -norm of a_ξ that we use in the second step to control the linearized corrector via classical Schauder theory for elliptic systems. W.l.o.g we may assume that $x = 0$.

Step 1. Proof that

$$\|a_\xi\|_{C^\alpha(B)} \leq Cr_\star^{(d-\delta)\frac{p-2}{p}}, \quad (3.5.22)$$

for some constant $C > 0$ depending on $d, p, \|A\|_{C^{0,\alpha}(\mathbb{R}^d)}$, and $|\xi|$, where $0 < \delta \leq d$ is the nonlinear hole-filling exponent of Lemma 15. (We recall that $\|X\|_{C^\alpha(B)} = \|X\|_{L^\infty(B)} + \|X\|_{C^{0,\alpha}(B)}$.)

On the one hand, by Lemma 25 applied to the equation (3.1.7) combined with the estimate (3.3.5), we have

$$\|\xi + \nabla\phi\|_{C^\alpha(B)} \lesssim \|A\|_{C^{0,\alpha}(\mathbb{R}^d)} \left(\int_{B_2} |\xi + \nabla\phi|^p \right)^{\frac{1}{p}} \stackrel{(3.3.5)}{\leq} \|A\|_{C^{0,\alpha}(\mathbb{R}^d)} (1 + |\xi|) r_\star^{\frac{d-\delta}{p}}. \quad (3.5.23)$$

On the other hand, recall that $a_\xi = ADa_o(\xi + \nabla\phi)$ with $a_o : \zeta \in \mathbb{R}^d \mapsto (1 + |\zeta|^{p-2})\zeta$, and thus for all $\zeta \in \mathbb{R}^d$

$$|Da_o(\zeta)| \lesssim 1 + |\zeta|^{p-2} \text{ and } |D^2a_o(\zeta)| \lesssim 1 + |\zeta|^{p-3}. \quad (3.5.24)$$

Therefore, by (3.5.23) and (3.5.24),

$$\begin{aligned} \|Da_o(\xi + \nabla\phi)\|_{C^\alpha(B)} &\leq \|Da_o(\xi + \nabla\phi)\|_{L^\infty(B)} + \|D^2a_o(\xi + \nabla\phi)\|_{L^\infty(B)} \|\xi + \nabla\phi\|_{C^{0,\alpha}(B)} \\ &\stackrel{(3.5.24)}{\lesssim} 1 + \|\xi + \nabla\phi\|_{C^\alpha(B)}^{p-2} \stackrel{(3.4.2)}{\lesssim} \|A\|_{C^{0,\alpha}(\mathbb{R}^d)} (1 + |\xi|)^{p-2} r_\star^{(d-\delta)\frac{p-2}{p}}, \end{aligned} \quad (3.5.25)$$

from which the claim (3.5.22) follows since $\|a_\xi\|_{C^\alpha(B)} \leq \|A\|_{C^\alpha(B)} \|Da_o(\xi + \nabla\phi)\|_{C^\alpha(B)}$.

Step 2. Proof of (3.5.20).

We first argue that

$$\int_B |e + \nabla\tilde{\phi}|^2 \mu_\xi \lesssim \tilde{r}_\star^{d-\beta} r_\star^{\frac{p-2}{p}(d-\delta)+\beta}. \quad (3.5.26)$$

If $\tilde{r}_\star < r_\star$, the claim follows from the defining property (3.5.19) in form of

$$\int_B |e + \nabla\tilde{\phi}|^2 \mu_\xi \lesssim 2\tilde{r}_\star^d \int_{B_{\tilde{r}_\star}} (1 + |\nabla\tilde{\phi}|^2) \mu_\xi \leq 2(C+1)$$

If $\tilde{r}_\star \geq r_\star$, we appeal to the hole filling estimate (3.3.25), to the defining property (3.5.19), and use (3.3.2) & (3.3.3), to the effect that

$$\int_B |e + \nabla\tilde{\phi}|^2 \mu_\xi \lesssim r_\star^d \int_{B_{r_\star}} |e + \nabla\tilde{\phi}|^2 \mu_\xi \stackrel{(3.3.25)}{\lesssim} \tilde{r}_\star^{d-\beta} r_\star^\beta \int_{B_{\tilde{r}_\star}} |e + \nabla\tilde{\phi}|^2 \mu \lesssim \tilde{r}_\star^{d-\beta} r_\star^\beta \int_{B_{\tilde{r}_\star}} \mu_\xi \lesssim \tilde{r}_\star^{d-\beta} r_\star^\beta.$$

We now argue that (3.5.26) entails (3.5.20). By the Schauder estimate [76, Theorem 5.19] applied to (3.2.9) (for which the constant depends algebraically on the ellipticity ratio and the $C^{0,\alpha}$ -seminorm of the coefficients, which we may encapsulate in the C^α -norm since $\mu_\xi \geq 1$), and the bound (3.5.22) on the coefficient and (3.5.26), there is some $\gamma > 0$ (depending on α and d) such that

$$\|e + \nabla\tilde{\phi}\|_{L^\infty(B)} \lesssim \|a_\xi\|_{C^\alpha(B)}^\gamma \left(\int_{B_2} |e + \nabla\tilde{\phi}|^2 \right)^{\frac{1}{2}} \stackrel{(3.5.22), (3.5.26)}{\lesssim} r_\star^{\gamma(d-\delta)\frac{p-2}{p}} \tilde{r}_\star^{\frac{1}{2}(d-\beta)} r_\star^{\frac{1}{2}((d-\delta)\frac{p-2}{p}+\beta)},$$

which yields (3.5.20) for $\chi := Cr_\star^{2(\gamma+1)(d-\delta)\frac{p-2}{p}+\beta}$ (for some constant $C > 0$ depending on $d, p, |\xi|$ and $\|A\|_{C^{0,\alpha}(\mathbb{R}^d)}$). The claimed moment bounds on χ follow from Theorem 21 (for a suitable $\gamma > 0$). \square

The main result of this section is the following control of \tilde{r}_* , which implies Proposition 7 in combination with Lemma 22.

Proposition 8. *There exists an exponent $\gamma > 0$ depending on $|\xi|$ such that for all $q \geq 1$, $\mathbb{E}_L[\tilde{r}_*^q]^{\frac{1}{q}} \lesssim q^\gamma$.*

Proof. We split the proof into three steps. In the first step, we control the level set $\{\tilde{r}_* = R\}$ for all dyadic R using averages of the corrector gradient. In Step 2, we use Proposition 6 to reformulate the right-hand side using moments of \tilde{r}_* itself, and then buckle in Step 3 by exploiting the gain of integrability provided by the hole-filling exponent $\beta > 0$.

Step 1. Control of level sets of \tilde{r}_* .

We claim that there exists a constant $c > 0$ (depending on ξ, d, p) such that for all dyadic $R \in [1, L]$, and all $0 < \kappa, \varepsilon < 1$ and $q \geq 1$

$$\mathbb{P}_L[\tilde{r}_* = R] \leq c^q R^{-(d-\beta+2(1-\kappa)-\varepsilon)q} \mathbb{E}_L \left[\mathcal{C}_{*,R}^q \tilde{r}_*^{(d-\beta)q} \right] + c^q R^{\varepsilon q} \mathbb{E}_L \left[\mathcal{C}_{*,R}^q \left(\int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla \tilde{\phi} \right|^2 dx \right)^q \right], \quad (3.5.27)$$

where $\mathcal{C}_{*,R} := R^{-\varepsilon} \|\mu_\xi\|_{L^\infty(B_{4R})}^2$. By the defining property (3.5.19) of \tilde{r}_* (with a constant C to be chosen below), we have

$$\int_{B_{2R}} |\nabla \tilde{\phi}|^2 \mu_\xi \leq C \int_{B_{4R}} \mu_\xi, \quad (3.5.28)$$

$$\int_{B_{R/2}} |\nabla \tilde{\phi}|^2 \mu_\xi \geq C \int_{B_R} \mu_\xi. \quad (3.5.29)$$

By the Caccioppoli inequality of Lemma 26, (3.5.29) yields

$$\inf_{c \in \mathbb{R}^d} \frac{1}{R^2} \int_{B_R} |\tilde{\phi} - c|^2 \mu_\xi + \int_{B_R} \mu_\xi \gtrsim C \int_{B_R} \mu_\xi,$$

so that, provided C is chosen large enough in (3.5.19), we have

$$\inf_{c \in \mathbb{R}^d} \frac{1}{R^2} \int_{B_R} |\tilde{\phi} - c|^2 \mu_\xi \gtrsim \int_{B_R} \mu_\xi \gtrsim 1.$$

Set $c_R := \int_{B_R} \int_{B_{R^\kappa}(x)} \tilde{\phi}(y) dy dx$. By the triangle inequality, Poincaré's inequality in $L^2(B_R)$, and the definition of $\mathcal{C}_{*,R}$, the above turns into

$$\begin{aligned} 1 &\lesssim \inf_{c \in \mathbb{R}^d} \frac{1}{R^2} \int_{B_R} |\tilde{\phi} - c|^2 \mu_\xi &\leq \sqrt{\mathcal{C}_{*,R}} R^{\frac{\varepsilon}{2}} \frac{1}{R^2} \int_{B_R} |\tilde{\phi} - c_R|^2 \\ &&\lesssim \sqrt{\mathcal{C}_{*,R}} R^{\frac{\varepsilon}{2}} \left(\frac{1}{R^2} \int_{B_R} |\tilde{\phi}(x) - \int_{B_{R^\kappa}(x)} \tilde{\phi}|^2 dx + \frac{1}{R^2} \int_{B_R} \left| \int_{B_{R^\kappa}(x)} \tilde{\phi} - c_R \right|^2 dx \right) \\ &&\lesssim \sqrt{\mathcal{C}_{*,R}} R^{\frac{\varepsilon}{2}} \left(R^{2(\kappa-1)} \int_{B_{2R}} |\nabla \tilde{\phi}|^2 + \int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla \tilde{\phi} \right|^2 dx \right) \\ &\stackrel{(3.5.28)}{\lesssim} \sqrt{\mathcal{C}_{*,R}} R^{\frac{\varepsilon}{2}} \left(R^{2(\kappa-1)} \sqrt{\mathcal{C}_{*,R}} R^{\frac{\varepsilon}{2}} + \int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla \tilde{\phi} \right|^2 dx \right) \\ &\stackrel{\tilde{r}_* = R}{=} \mathcal{C}_{*,R} R^\varepsilon \left(R^{2(\kappa-1)} R^{-d+\beta} \tilde{r}_*^{d-\beta} + \int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla \tilde{\phi} \right|^2 dx \right). \end{aligned}$$

The claim now follows from Markov' inequality.

Step 2. Control of the right-hand side of (3.5.27): For all $0 < \varepsilon, \kappa, \theta < 1$, and all dyadic R and exponents $q \geq 1$

$$\mathbb{P}_L[\tilde{r}_\star = R] \leq c^q q^\gamma (R^{-(d-\beta+2(1-\kappa)-\varepsilon)q} + R^{-(d\kappa-\varepsilon)q}) \mathbb{E}_L \left[\tilde{r}_\star^{(d-\beta)(1+\theta)^3 q} \right]^{\frac{1}{(1+\theta)^3}}, \quad (3.5.30)$$

for some constant $c > 0$ depending on $|\xi|, p, d, \varepsilon, \kappa, \theta$, but not on R and q .

Since $\tilde{r}_\star \leq L$, it suffices to establish the statement for dyadic $R \leq L$. By Lemma 32 and Theorem 21, there exists $\gamma > 0$ such that for all $q \geq \frac{d}{\varepsilon}$ and $R \geq 1$, we have

$$\mathbb{E}_L[\mathcal{C}_{\star,R}^q]^{\frac{1}{q}} \lesssim q^\gamma, \quad (3.5.31)$$

where the multiplicative constant does not depend on R and ε . By Hölder's inequality with exponents $(\frac{1+\theta}{\theta}, 1+\theta)$, we then get for the first right hand side term of (3.5.27)

$$\mathbb{E}_L \left[\mathcal{C}_{\star,R}^q \tilde{r}_\star^{(d-\beta)q} \right]^{\frac{1}{q}} \leq \mathbb{E}_L \left[\mathcal{C}_{\star,R}^{q\frac{1+\theta}{\theta}} \right]^{\frac{\theta}{1+\theta}} \mathbb{E}_L \left[\tilde{r}_\star^{(d-\beta)(1+\theta)q} \right]^{\frac{1}{q(1+\theta)}} \lesssim_\theta q^\gamma \mathbb{E}_L \left[\tilde{r}_\star^{(d-\beta)(1+\theta)q} \right]^{\frac{1}{q(1+\theta)}}. \quad (3.5.32)$$

We turn to the second right hand side term of (3.5.27). By Hölder's inequality with exponents $(\frac{1+\theta}{\theta}, 1+\theta)$, stationarity of $\nabla\tilde{\phi}$, and (3.5.31), we first have

$$\begin{aligned} \mathbb{E}_L \left[\mathcal{C}_{\star,R}^q \left(\int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla\tilde{\phi} \right|^2 dx \right)^q \right]^{\frac{1}{q}} &\leq \mathbb{E}_L \left[\mathcal{C}_{\star,R}^{q\frac{1+\theta}{\theta}} \right]^{\frac{\theta}{q(1+\theta)}} \mathbb{E}_L \left[\left(\int_{B_R} \left| \int_{B_{R^\kappa}(x)} \nabla\tilde{\phi} \right|^2 dx \right)^{q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} \\ &\lesssim_\theta q^\gamma \mathbb{E}_L \left[\left| \int_{B_{R^\kappa}} \nabla\tilde{\phi} \right|^{2q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}}. \end{aligned}$$

Then, by Proposition 6 applied to $g = |B_{R^\kappa}|^{-1} \mathbf{1}_{B_{R^\kappa}}$, followed by Lemma 22, by Hölder's inequality with exponent $(\frac{1+\theta}{\theta}, 1+\theta)$, and (3.5.21), we have

$$\begin{aligned} \mathbb{E}_L \left[\left| \int_{B_{R^\kappa}} \nabla\tilde{\phi} \right|^{2q(1+\theta)} \right]^{\frac{1}{q(1+\theta)}} &\lesssim_{|\xi|, \theta} q^\gamma \mathbb{E}_L \left[\left(\sup_B |\nabla\tilde{\phi}_{\xi, \varepsilon} + e|^2 \mu_\xi \right)^{q(1+\theta)^2} \right]^{\frac{1}{q(1+\theta)^2}} \left(\int_{Q_L} |g|^2 \right) \\ &\lesssim q^\gamma \mathbb{E}_L \left[\chi^{q(1+\theta)^2} \tilde{r}_\star^{(d-\beta)q(1+\theta)^2} \right]^{\frac{1}{q(1+\theta)^2}} R^{-d\kappa} \\ &\lesssim q^\gamma R^{-d\kappa} \mathbb{E}_L \left[\tilde{r}_\star^{(d-\beta)q(1+\theta)^3} \right]^{\frac{1}{q(1+\theta)^3}} \end{aligned}$$

(where we changed the value of γ from one line to the other). Combined with (3.5.32), this entails (3.5.30) by redefining γ once more.

Step 3. Buckling argument.

Recall that all the quantities we consider are finite since $\tilde{r}_\star \leq L$. We now express moments of \tilde{r}_\star using its level sets and obtain by (3.5.30) for some $K > 1$ to be fixed below and all $q \geq 1$

$$\begin{aligned} \mathbb{E}_L \left[\tilde{r}_\star^{q(d-\frac{\beta}{K})} \right] &\leq 1 + \sum_{n=1}^{\infty} 2^{nq(d-\frac{\beta}{K})} \mathbb{P}_L[\tilde{r}_\star = 2^n] \\ &\stackrel{(3.5.30)}{\leq} 1 + \sum_{n=1}^{\infty} 2^{nq(d-\frac{\beta}{K})} c^q q^\gamma (2^{-nq(d-\beta+2(1-\kappa)-\varepsilon)} + 2^{-nq(d\kappa-\varepsilon)}) \mathbb{E}_L \left[\tilde{r}_\star^{q(d-\beta)(1+\theta)^3} \right]^{\frac{1}{(1+\theta)^3}} \\ &\leq 1 + \mathbb{E}_L \left[\tilde{r}_\star^{q(d-\beta)(1+\theta)^3} \right]^{\frac{1}{(1+\theta)^3}} c^q q^\gamma \sum_{n=1}^{\infty} (2^{nq(-\frac{\beta}{K}+\beta-2(1-\kappa)+\varepsilon)} + 2^{nq(d(1-\kappa)+\varepsilon-\frac{\beta}{K})}). \end{aligned}$$

We now choose the exponents. We first fix $0 \leq \kappa < 1$ so that $d(1 - \kappa) = \frac{\beta}{2}$, and then set $\varepsilon := \frac{\beta}{5d}$ and $\frac{1}{K} := 1 - \frac{1}{5d}$, to the effect that

$$\frac{1}{2} \left(2^{nq(-\frac{\beta}{K} + \beta - 2(1-\kappa) + \varepsilon)} + 2^{nq(d(1-\kappa) + \varepsilon - \frac{\beta}{K})} \right) \leq 2^{-nq\frac{\beta}{5d}}.$$

With this choice, the series is summable and the above turns into

$$\mathbb{E} \left[\tilde{r}_*^{q(d - \frac{\beta}{K})} \right] \leq 1 + c^q q^\gamma \mathbb{E}_L \left[\tilde{r}_*^{q(d-\beta)(1+\theta)^3} \right]^{\frac{1}{(1+\theta)^3}}$$

for some redefined constant c . We may then absorb part of the right-hand side into the left-hand side by Young's inequality upon choosing $0 < \theta < 1$ so small that $(d - \beta)(1 + \theta)^3 < d - \frac{\beta}{K}$ (which is possible since $K > 1$), and the claimed moment bound follows for some suitable choice of $\gamma > 0$. \square

3.6 Quantitative two-scale expansion: Proof of Theorem 12

We assume $\delta \leq 1$, and split the proof into four steps. In the first step, we show that the two-scale expansion error satisfies a nonlinear PDE in conservative form (crucially using the flux corrector). In the second step we give a bound for the $H^{-1}(\mathbb{R}^d)$ -norm of the right-hand side, the moments of which we control in the third step. We then conclude in the fourth step by using the monotonicity of the heterogeneous operator a_ε . In the following, we use the short-hand notation $\xi_k := (\nabla \bar{u})_{k,\delta}$.

Step 1. Equation for the two-scale expansion error:

$$-\nabla \cdot (a(\frac{x}{\varepsilon}, \nabla \bar{u}_{\varepsilon,\delta}^{2s}(x)) - a(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x))) = \nabla \cdot R_{\varepsilon,\delta}(x), \quad (3.6.1)$$

where

$$\begin{aligned} R_{\varepsilon,\delta}(x) &= \left(\sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (\bar{a}(\xi_k) - \bar{a}(\nabla \bar{u}(x))) \right) - \left(\sum_{k \in \delta\mathbb{Z}^d} \varepsilon \sigma_{\xi_k}(\frac{x}{\varepsilon}) \nabla \eta_k(x) \right) \\ &+ \left(\sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (a(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \nabla \phi_{\xi_k}(\frac{x}{\varepsilon})) - a(\frac{x}{\varepsilon}, \xi_k + \nabla \phi_{\xi_k}(\frac{x}{\varepsilon}))) \right) \\ &+ \left(a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}(\frac{x}{\varepsilon}) \eta_k(x)\right) - \sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \nabla \phi_{\xi_k}(\frac{x}{\varepsilon})\right) \right) \\ &+ \left(a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}(\frac{x}{\varepsilon}) \eta_k(x) + \varepsilon \phi_{\xi_k}(\frac{x}{\varepsilon}) \nabla \eta_k(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}(\frac{x}{\varepsilon}) \eta_k(x)\right) \right). \end{aligned}$$

To start with, we expand $\nabla \bar{u}_{\varepsilon,\delta}^{2s}$ as

$$\nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla \bar{u}_{\varepsilon,\delta}^{2s}\right) = \nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \varepsilon \phi_{\xi_k}(\frac{x}{\varepsilon}) \nabla \eta_k(x) + \nabla \phi_{\xi_k}(\frac{x}{\varepsilon}) \eta_k(x)\right),$$

which we rewrite in the form of the telescopic sum (using that $\sum_{k \in \delta\mathbb{Z}^d} \eta_k \equiv 1$)

$$\begin{aligned}
 & \nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla \bar{u}_{\varepsilon, \delta}^{2s}(x)\right) - \nabla \cdot \bar{a}(\nabla \bar{u}(x)) \\
 &= \nabla \cdot \left(\sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (\bar{a}(\xi_k) - \bar{a}(\nabla \bar{u}(x))) \right) + \nabla \cdot \left(\sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (a\left(\frac{x}{\varepsilon}, \xi_k + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right) - \bar{a}(\xi_k)) \right) \\
 & \quad + \nabla \cdot \left(\sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right) - a\left(\frac{x}{\varepsilon}, \xi_k + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right)) \right) \\
 & \quad + \nabla \cdot \left(a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right) \eta_k(x)\right) - \sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right) \right) \\
 & \quad + \nabla \cdot \left(a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right) \eta_k(x) + \varepsilon \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right) \nabla \eta_k(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla \bar{u}(x) + \sum_{k \in \delta\mathbb{Z}^d} \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right) \eta_k(x)\right) \right).
 \end{aligned}$$

First, using (3.1.5) and (3.1.6) we may replace $-\nabla \cdot \bar{a}(\nabla \bar{u}(x))$ by $-\nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right)$ in the left-hand side. In the right-hand side, all the terms obviously converge strongly to zero in $H^{-1}(\mathbb{R}^d)$ (and are present in the definition of $R_{\varepsilon, \delta}$) except the second term, which we need to reformulate. More precisely, using the flux corrector σ (see Definition 3.2.1) in form of the property (3.2.3), we have for all $k \in \delta\mathbb{Z}^d$ (implicitly summing on the repeated indices ij)

$$\begin{aligned}
 \nabla \cdot \eta_k(x) (a\left(\frac{x}{\varepsilon}, \xi_k + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right) - \bar{a}(\xi_k)) &= \nabla \cdot (\eta_k(x) \nabla \cdot \sigma_{\xi_k}\left(\frac{x}{\varepsilon}\right)) = \partial_j (\eta_k(x) \partial_i \sigma_{\xi_k, ji}) \\
 &= \varepsilon \partial_i (\partial_j \eta_k(x) \sigma_{\xi_k, ji}\left(\frac{x}{\varepsilon}\right)) - \partial_i \partial_j (\eta_k(x) \sigma_{\xi_k, ji}\left(\frac{x}{\varepsilon}\right)), \quad (3.6.2)
 \end{aligned}$$

where the last term vanishes thanks to the skew-symmetry of $(\sigma_{\xi_k, ji})_{j,i}$ and the symmetry of $(\partial_i \partial_j \eta_k)_{j,i}$. By the skew-symmetry of σ_ξ , one has $\varepsilon \partial_i (\partial_j \eta_k(x) \sigma_{\xi_k, ji}\left(\frac{x}{\varepsilon}\right)) = -\varepsilon \nabla \cdot (\sigma_{\xi_k}\left(\frac{x}{\varepsilon}\right) \nabla \eta_k(x))$, and we thus deduce

$$\nabla \cdot \sum_{k \in \delta\mathbb{Z}^d} \eta_k(x) (a\left(\frac{x}{\varepsilon}, \xi_k + \nabla \phi_{\xi_k}\left(\frac{x}{\varepsilon}\right)\right) - \bar{a}(\xi_k)) = -\varepsilon \nabla \cdot \left(\sum_{k \in \delta\mathbb{Z}^d} \sigma_{\xi_k}\left(\frac{x}{\varepsilon}\right) \nabla \eta_k(x) \right).$$

This yields (3.6.1).

Step 2. Control by continuity of the operators: The remainder $R_{\varepsilon, \delta}$ satisfies

$$\begin{aligned}
 \int_{\mathbb{R}^d} |R_{\varepsilon, \delta}|^2 &\lesssim \sum_k \int_{\mathbb{R}^d} \eta_k |\xi_k - \nabla \bar{u}|^2 (1 + |\xi_k| + |\nabla \bar{u}| + |\nabla \phi_{\xi_k}\left(\frac{\cdot}{\varepsilon}\right)|)^{2(p-2)} \\
 & \quad + \int_{\mathbb{R}^d} \sum_k \eta_k \left| \sum_{k'} \varepsilon (\phi_{\xi_k'} - \phi_{\xi_k}, \sigma_{\xi_k'} - \sigma_{\xi_k})\left(\frac{\cdot}{\varepsilon}\right) \nabla \eta_{k'} \right|^2 \left(1 + |\nabla \bar{u}| + \left| \sum_{k''} \nabla \phi_{\xi_{k''}}\left(\frac{\cdot}{\varepsilon}\right) \eta_{k''} \right| \right)^{2(p-2)} \\
 & \quad + \int_{\mathbb{R}^d} \sum_k \eta_k \left| \sum_{k'} \varepsilon (\phi_{\xi_k'} - \phi_{\xi_k})\left(\frac{\cdot}{\varepsilon}\right) \nabla \eta_{k'} \right|^{2(p-1)} + \sum_k \int_{\mathbb{R}^d} \eta_k \left| \sum_{k'} \nabla (\phi_{\xi_k} - \phi_{\xi_{k'}})\left(\frac{\cdot}{\varepsilon}\right) \eta_{k'} \right|^2 (1 + |\nabla \bar{u}|)^{2(p-2)} \\
 & \quad + \sum_{k \in \delta\mathbb{Z}^d} \int_{\mathbb{R}^d} \eta_k \left| \sum_{k' \in \delta\mathbb{Z}^d} \nabla (\phi_{\xi_k} - \phi_{\xi_{k'}})\left(\frac{\cdot}{\varepsilon}\right) \eta_{k'} \right|^{2(p-1)}. \quad (3.6.3)
 \end{aligned}$$

This estimate directly follows from the definition of $R_{\varepsilon, \delta}$ together with the continuity of the operator in form of

$$|\tilde{a}(\xi_1) - \tilde{a}(\xi_2)| \lesssim C |\xi_1 - \xi_2| (1 + |\xi_1| + |\xi_1 - \xi_2|)^{p-2}$$

for $\tilde{a} = a_\varepsilon$ and $\tilde{a} = \bar{a}$, and with the observation that $\sum_{k'} \nabla \eta_{k'} = 0$ so that for all maps $(\zeta_k)_k$ one has

$$\sum_{k'} \zeta_{k'} \nabla \eta_{k'} = \sum_{k'} (\zeta_{k'} - \zeta_k) \nabla \eta_{k'},$$

which we applied to $\zeta_k = \varepsilon(\phi_{\xi_k}, \sigma_{\xi_k})(\frac{\cdot}{\varepsilon})$.

Step 3. Control of moments of $\int_{\mathbb{R}^d} |R_{\varepsilon, \delta}|^2$: For all $q \geq 1$,

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} |R_{\varepsilon, \delta}|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \leq C q^\gamma (\varepsilon + \delta) \mu_d(\frac{1}{\varepsilon}) \|\mu_d \nabla^2 \bar{u}\|_{L^2(\mathbb{R}^d)}, \quad (3.6.4)$$

for some constant C and an exponent $\gamma > 0$ depending on $\|\nabla \bar{u}\|_{L^\infty(\mathbb{R}^d)}$. We treat the second right-hand side term of (3.6.3) (that we denote by $\tilde{R}_{\varepsilon, \delta}$) – the other terms are easier and can be treated similarly. Since for all k' , $|\nabla \eta_{k'}| \lesssim \delta^{-1} \mathbf{1}_{Q_\delta(k')}$, we have

$$\sum_k \eta_k \left| \sum_{k'} \varepsilon(\phi_{\xi_{k'}'} - \phi_{\xi_k}, \sigma_{\xi_{k'}'} - \sigma_{\xi_k})(\frac{\cdot}{\varepsilon}) \nabla \eta_{k'} \right|^2 \lesssim \left(\frac{\varepsilon}{\delta}\right)^2 \sum_k \eta_k \sum_{k'} \mathbf{1}_{Q_\delta(k')} |(\phi_{\xi_{k'}'} - \phi_{\xi_k}, \sigma_{\xi_{k'}'} - \sigma_{\xi_k})(\frac{\cdot}{\varepsilon})|^2.$$

Inserting this estimate in $\tilde{R}_{\varepsilon, \delta}$, and using the assumption $\nabla \bar{u} \in L^\infty(\mathbb{R}^d)$, we obtain for all $q \geq 1$ by Cauchy-Schwarz' inequality followed by Minkowski's inequality in probability, the support condition $Q_\delta(k) \cap Q_\delta(k') \neq \emptyset \Rightarrow |k - k'| < 2\delta$, and the stationarity of $\nabla \phi_{k''}$,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{\mathbb{R}^d} |\tilde{R}_{\varepsilon, \delta}|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \\ & \lesssim \frac{\varepsilon}{\delta} \left(\sum_k \sum_{k' \in Q_{2\delta}(k)} \int_{Q_\delta(k)} \mathbb{E} [|(\phi_{\xi_{k'}'} - \phi_{\xi_k}, \sigma_{\xi_{k'}'} - \sigma_{\xi_k})(\frac{\cdot}{\varepsilon})|^{2q}]^{\frac{1}{q}} \left(1 + \|\nabla \bar{u}\|_{L^\infty(\mathbb{R}^d)}^{2(p-2)} + \sum_{k'' \in Q_{2\delta}(k)} \mathbb{E} [|\nabla \phi_{\xi_{k''}}|^{2q(p-2)}]^{\frac{1}{q}} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

By Theorem 15 and Corollary 16, and using that μ_d satisfies $\mu_d(t_1 t_2) \lesssim \mu_d(t_1) \mu_d(t_2)$ and $\sup_{Q_{4\delta}(k)} \mu_d \lesssim \inf_{Q_{4\delta}(k)} \mu_d$, this turns into

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} |\tilde{R}_{\varepsilon, \delta}|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \leq C q^\gamma \left(\frac{\varepsilon}{\delta}\right) \mu_d(\frac{1}{\varepsilon}) \left(\sum_k \left(\inf_{Q_{4\delta}(k)} \mu_d \right) \sum_{k' \in Q_{2\delta}(k)} |\xi_{k'} - \xi_k|^2 |Q_\delta| \right)^{\frac{1}{2}}, \quad (3.6.5)$$

for some constant C and an exponent $\gamma > 0$ depending on $\|\nabla \bar{u}\|_{L^\infty(\mathbb{R}^d)}$. It remains to reformulate the right-hand side sum. By Poincaré's inequality on $Q_{4\delta}(k)$, we have

$$\sum_{k' \in Q_{2\delta}(k)} |\xi_k - \xi_{k'}|^2 |Q_\delta| \lesssim \delta^2 \int_{Q_{4\delta}(k)} |\nabla^2 \bar{u}|^2,$$

so that (3.6.4) follows from (3.6.5).

Step 4. Conclusion by monotonicity.

We test (3.6.1) with $u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s}$, and deduce by monotonicity of a_ε that

$$\int_{\mathbb{R}^d} |\nabla(u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s})|^2 + |\nabla(u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s})|^p \lesssim \int_{\mathbb{R}^d} R_{\varepsilon, \delta} \cdot \nabla(u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s}).$$

By Young's inequality, we may absorb part of the right-hand side into the left-hand side, and obtain after taking the q -th moment of this inequality

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} |\nabla(u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s})|^2 + |\nabla(u_\varepsilon - \bar{u}_{\varepsilon, \delta}^{2s})|^p \right)^q \right]^{\frac{1}{q}} \lesssim \mathbb{E} \left[\left(\int_{\mathbb{R}^d} |R_{\varepsilon, \delta}|^2 \right)^q \right]^{\frac{1}{q}}.$$

This entails the claim in combination with (3.6.4) and the choice $\delta = \varepsilon$.

3.7 Appendix

3.7.1 Deterministic PDE estimates and consequences

In this appendix, we recall mostly standard inequalities for nonlinear operators $-\nabla \cdot a(\cdot, \nabla)$ and linear operators $-\nabla \cdot a \nabla$ (with unbounded coefficients) needed in the proofs of the paper. Based on these results we also prove the qualitative differentiability of correctors (with respect to ξ) when the equation is posed on a bounded domain, and we prove part of Theorem 14 for statistically isotropic operators.

3.7.2 Nonlinear systems: Caccioppoli, hole-filling, and Schauder

We start with Caccioppoli's inequality for nonlinear elliptic systems.

Lemma 23 (Caccioppoli's inequality). *Let $r > 0, c_2 > 0, x \in \mathbb{R}^d$ and $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d)$ be a weak solution of*

$$-\nabla \cdot a(\cdot, \nabla u) = 0 \text{ in } B_{c_2 r}(x). \quad (3.7.1)$$

Then for all $0 < c_1 < c_2$,

$$\int_{B_{c_1 r}(x)} |\nabla u|^2 (1 + |\nabla u|^{p-2}) \lesssim_{c_1, c_2} \inf_{c \in \mathbb{R}^d} \int_{B_{c_2 r}(x) \setminus B_{c_1 r}(x)} \left(\frac{|u - c|}{r} \right)^2 + \left(\frac{|u - c|}{r} \right)^p. \quad (3.7.2)$$

Proof. Without loss of generality, we may assume that $x = 0$. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a standard cut-off for $B_{c_1 r}$ in $B_{c_2 r}$ and set $\zeta^2 = \eta^p$ to the effect that $2\zeta \nabla \zeta = p\eta^{p-1} \nabla \eta$. By testing the equation (3.7.1) with $\zeta^2(u - c)$ and by making use of the monotonicity (3.1.4) of a and the property $a(\cdot, 0) = 0$ in form of $\int \zeta^2 \nabla u \cdot a(\cdot, \nabla u) \gtrsim \int \zeta^2 |\nabla u|^2 (1 + |\nabla u|^{p-2})$, we have

$$\int \zeta^2 |\nabla u|^2 (1 + |\nabla u|^{p-2}) \lesssim \int |\zeta| |\nabla \zeta| |u - c| |\nabla u| + \int |\zeta| |\nabla \zeta| |u - c| |\nabla u|^{p-1}.$$

This implies the desired estimate (3.7.2) by Young's inequality, with exponents $(2, 2)$ and $(p, \frac{p}{p-1})$ for the first and second right-hand side terms, respectively, together with the identity $2\zeta \nabla \zeta = p\eta^{p-1} \nabla \eta$, and absorbing part of the right-hand side into the left-hand side. \square

The Widman hole-filling estimate for nonlinear systems follows from Lemma 23 by simple iteration (see e.g. [76, Section 4.4]).

Lemma 24 (Hole-filling estimate). *There exists $0 < \delta \leq d$ such that if $u \in W_{\text{loc}}^{1,p}(B_R)$ is a weak solution of $-\nabla \cdot a(\cdot, \nabla u) = 0$ in the ball B_R for some $R > 0$, then for all $0 < r \leq R$ we have*

$$\int_{B_r} |\nabla u|^2 (1 + |\nabla u|^{p-2}) \lesssim \left(\frac{R}{r} \right)^{d-\delta} \int_{B_R} |\nabla u|^2 (1 + |\nabla u|^{p-2}). \quad (3.7.3)$$

We finally state regularity results for nonlinear equations, which are direct consequences of [153] and [106, Theorem 4] (for the uniform bound on the gradient).

Lemma 25. *Let a be a monotone operator which has the form (3.1.9) and assume that $A \in C^\alpha(B_{4R})$, for some $R > 0$ and $\alpha \in (0, 1)$. Let $u \in W^{1,p}(B_{4R})$ be a distributional solution of*

$$-\nabla \cdot a(\cdot, \nabla u) = 0.$$

Then, $u \in C^{1,\alpha}(B_R)$ and there exists a constant c depending on R and $\|A\|_{C^\alpha(B_{4R})}$ such that

$$\|\nabla u\|_{C^\alpha(B_R)} \leq c \left(\int_{B_{4R}} |\nabla u|^p \right)^{\frac{1}{p}},$$

where we recall that $\|X\|_{C^\alpha} = \|X\|_{L^\infty} + \|X\|_{C^{0,\alpha}}$.

3.7.3 Linear elliptic systems: Caccioppoli and Lemma 17

We state and prove Caccioppoli's inequality for linear elliptic systems with unbounded coefficients, from which we deduce Lemma 17 by optimizing the cut-off.

Lemma 26 (Caccioppoli's inequality for linear elliptic systems with unbounded coefficients). *Let $R > 0$, $a : B_R \mapsto \mathbb{R}^{d \times d}$, and $\mu \in L^1(B_R)$ be such that there exists a constant $\kappa > 0$ for which we have for all $x \in B_R$ and all $h \in \mathbb{R}^d$*

$$h \cdot a(x)h \leq |h|^2 \mu(x) \leq \kappa h \cdot a(x)h. \quad (3.7.4)$$

For all functions g and u related (in the weak sense) in B_R via

$$-\nabla \cdot a \nabla u = \nabla \cdot (g \sqrt{\mu}), \quad (3.7.5)$$

we have for all $0 < \rho < \sigma \leq R$,

$$\int_{B_\rho} |\nabla u|^2 \mu \lesssim_\kappa \mathcal{J}(\rho, \sigma, \mu, u) + \int_{B_\sigma} |g|^2, \quad (3.7.6)$$

where

$$\mathcal{J}(\rho, \sigma, \mu, u, g) := \inf \left\{ \int_{B_\sigma} \mu |u - \int_{B_\sigma} u|^2 |\nabla \eta|^2 \mid \eta \in C_c^1(B_\sigma), 0 \leq \eta \leq 1, \eta \equiv 1 \text{ in } B_\rho \right\}. \quad (3.7.7)$$

Proof. Without loss of generality, we may assume that $\int_{B_\sigma} u = 0$. Let $\eta \in C_c^1(B_\sigma)$ be such that $\eta \equiv 1$ in B_ρ and $0 \leq \eta \leq 1$. Testing the equation (3.7.5) with $\eta^2 u$ and using the condition (3.7.4) yield

$$\int \eta^2 |\nabla u|^2 \mu \lesssim_\kappa \int \eta^2 |\nabla u| \sqrt{\mu} |g| + \int |\eta| |\nabla \eta| |u| |\nabla u| \mu + \int |\eta| |\nabla \eta| |u| \sqrt{\mu} |g|. \quad (3.7.8)$$

By Young's inequality with exponents (2, 2), after absorbing part of the right-hand side into the left-hand side, and using in addition the support condition on η , (3.7.8) turns into

$$\int_{B_\rho} |\nabla u|^2 \mu \lesssim_\kappa \int_{B_\sigma} \mu |u|^2 |\nabla \eta|^2 + \int_{B_\sigma} |g|^2,$$

which yields (3.7.6) by optimizing over η . \square

We then turn to the proof of Lemma 17.

Proof of Lemma 17. We split the proof into two steps.

Step 1. Proof that for all $\gamma > 0$

$$\mathcal{J}(\rho, \sigma, \mu, v) \leq (\sigma - \rho)^{-1 - \frac{1}{\gamma}} \left(\int_\rho^\sigma \left(\int_{S_r} \mu |v|^2 \right)^\gamma dr \right)^{\frac{1}{\gamma}}, \quad (3.7.9)$$

where $S_r := \partial B_r$. By scaling we may assume without loss of generality that $\rho = 1$ and $\sigma = 2$. Estimate (3.7.9) essentially follows by minimizing among radially symmetric cut-off functions. Indeed, for all $\varepsilon > 0$ we have

$$\mathcal{J}(1, 2, \mu, v) \leq \inf \left\{ \int_1^2 \eta'(r)^2 \left(\int_{S_r} \mu |v|^2 + \varepsilon \right) dr \mid \eta \in C^1(1, 2), 0 \leq \eta \leq 1, \eta(1) = 1, \eta(2) = 0 \right\}.$$

This one-dimensional minimization problem can be solved explicitly. Set $f(r) := \int_{S_r} \mu|v|^2 + \varepsilon$. Using the competitor $\eta(r) := 1 - \frac{\int_1^r f^{-1}}{\int_1^2 f^{-1}}$ yields a control of this minimum by the harmonic average of f ,

$$\mathcal{J}(1, 2, \mu, v) \leq \left(\int_1^2 \left(\int_{S_r} \mu|v|^2 + \varepsilon \right)^{-1} dr \right)^{-1}.$$

By standard relations between quasi-arithmetic means, since $\gamma > -1$,

$$\left(\int_1^2 \left(\int_{S_r} \mu|v|^2 + \varepsilon \right)^{-1} dr \right)^{-1} \leq \left(\int_1^2 \left(\int_{S_r} \mu|v|^2 + \varepsilon \right)^\gamma dr \right)^{\frac{1}{\gamma}},$$

and the claim (3.7.9) follows by letting $\varepsilon \downarrow 0$.

Step 2. Proof of (3.3.16).

Let us first assume $d \geq 3$ and note that $q > \frac{d-1}{2}$ implies $q_* \in [1, 2)$. Recall that for all $s \in [1, d-1)$, $r > 0$ and $\phi \in W^{1,s}(S_r)$, Poincaré-Sobolev' inequality yields for $s_* = \frac{(d-1)s}{d-1-s}$

$$\left(\int_{S_r} |\phi|^{s_*} \right)^{\frac{1}{s_*}} \lesssim r \left(\int_{S_r} |\nabla \phi|^s \right)^{\frac{1}{s}} + \left(\int_{S_r} |\phi|^s \right)^{\frac{1}{s}}. \quad (3.7.10)$$

By Hölder's inequality with exponents $(q, \frac{q}{q-1})$, followed by (3.7.10) with $s = q_*$ and $s_* = \frac{2q}{q-1}$, (3.7.9) turns into

$$\begin{aligned} \mathcal{J}(\rho, \sigma, \mu, v) &\leq \frac{1}{(\sigma - \rho)^{1+\frac{1}{\gamma}}} \left(\int_\rho^\sigma \left(\int_{S_r} \mu^q \right)^{\frac{\gamma}{q}} \left(\int_{S_r} |v|^{\frac{2q}{q-1}} \right)^{\frac{(q-1)\gamma}{q}} dr \right)^{\frac{1}{\gamma}} \\ &\stackrel{(3.7.10)}{\lesssim_q} \frac{1}{(\sigma - \rho)^{1+\frac{1}{\gamma}}} \left(\int_\rho^\sigma \left(\int_{S_r} \mu^q \right)^{\frac{\gamma}{q}} \left(\left(\int_{S_r} |\nabla v|^{q_*} \right)^{\frac{2\gamma}{q_*}} + r^{-2\gamma} \left(\int_{S_r} |v|^{q_*} \right)^{\frac{2\gamma}{q_*}} \right) dr \right)^{\frac{1}{\gamma}}. \end{aligned}$$

We then choose $\gamma = \frac{d-1}{d+1}$ to the effect that $\frac{\gamma}{q} + \frac{2\gamma}{q_*} = 1$, so that by Hölder's inequality with exponents $(\frac{\gamma}{q}, \frac{2\gamma}{q_*})$ we obtain

$$\mathcal{J}(\rho, \sigma, \mu, v) \lesssim_q \frac{1}{(\sigma - \rho)^{\frac{2d}{d-1}}} \left(\int_{B_\sigma \setminus B_\rho} \mu^q \right)^{\frac{1}{q}} \left(\left(\int_{B_\sigma \setminus B_\rho} |\nabla v|^{q_*} \right)^{\frac{1}{q_*}} + \frac{1}{\rho^2} \left(\int_{B_\sigma \setminus B_\rho} |v|^{q_*} \right)^{\frac{2}{q_*}} \right),$$

which is the desired estimate (3.3.16). For $d = 2$, in which case $q_* = 1$, we use the one-dimensional Sobolev inequality $\|\phi\|_{L^\infty(S_r)} \lesssim r \int_{S_r} |\nabla \phi| + \int_{S_r} |\phi|$ instead of (3.7.10), and (3.3.16) follows from (3.7.9). The case $d = 1$ is similar. \square

3.7.4 Qualitative differentiability of correctors on bounded domains

In this section, we consider the approximation of correctors on bounded domains, both with Dirichlet and periodic boundary conditions. More precisely, let D be a smooth bounded domain of \mathbb{R}^d (resp. a cube Q_L , $L > 0$), let $A : D \rightarrow \mathbb{M}_d(\lambda)$ be of class C^α (resp. $C_{\text{per}}^\alpha(Q_L)$), and set $a : D \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $(x, \xi) \mapsto A(x)(1 + |\xi|^{p-2})\xi$ for some $p \geq 2$. We show that the periodic corrector gradients $\xi \mapsto (\nabla \phi_\xi, \nabla \sigma_\xi)$, where (ϕ_ξ, σ_ξ) are solutions of (3.1.7) and (3.2.2) on D with homogeneous boundary conditions (resp. Q_L -periodic with vanishing average), are Fréchet-differentiable and that their derivatives are given by the linearized corrector gradients. More precisely:

Lemma 27 (Differentiability of correctors). *For all D the corrector gradients $\xi \mapsto (\nabla\sigma_\xi, \nabla\phi_\xi)$ are Fréchet-differentiable in $C^0(D)$ and for all directions $e \in \mathbb{R}^d$ we have*

$$(\partial_\xi \nabla\phi_\xi \cdot e, \partial_\xi \nabla\sigma_\xi \cdot e) = (\nabla\tilde{\phi}_{\xi,e}, \nabla\tilde{\sigma}_{\xi,e}),$$

where $(\tilde{\phi}_{\xi,e}, \tilde{\sigma}_{\xi,e})$ solve (3.2.9) and (3.2.10) with homogeneous Dirichlet boundary conditions (resp. Q_L -periodic with vanishing average).

Proof. We only give the arguments for $\xi \mapsto \nabla\phi_\xi$. The differentiability of $\xi \mapsto \nabla\sigma_\xi$ can be proved similarly. Let $\xi, e \in \mathbb{R}^d$, $h \subset (0, 1)$ be a sequence that goes to 0, and set $\delta^h\phi_\xi := \frac{\phi_{\xi+he} - \phi_\xi}{h}$. We first show that we can extract a converging subsequence of $(\nabla\delta^h\phi_\xi)$ by local regularity and Arzela-Ascoli's theorem. Then, we show that this limit coincides with $\nabla\tilde{\phi}_{\xi,e}$. The starting point is the corrector equation (3.1.7) in the form

$$-\nabla \cdot (a(x, \xi + \nabla\phi_{\xi+he}) - a(x, \xi + \nabla\phi_\xi)) = \nabla \cdot (a(x, \xi + \nabla\phi_{\xi+he}) - a(x, \xi + he + \nabla\phi_{\xi+he})), \quad (3.7.11)$$

that we rewrite, using the smoothness of $\xi \mapsto a(\cdot, \xi)$, as

$$-\nabla \cdot a_h^{(1)} \nabla\delta^h\phi_\xi = \nabla \cdot a_h^{(2)} e, \quad (3.7.12)$$

where

$$a_h^{(1)} := \int_0^1 Da(\cdot, \xi + \nabla\phi_\xi + t(\nabla\phi_{\xi+he} - \nabla\phi_\xi)) dt \text{ and } a_h^{(2)} := \int_0^1 Da(\cdot, \xi + \nabla\phi_{\xi+he} + (1-t)he) dt.$$

Next, for all $\tilde{\xi} \in \mathbb{R}^d$, using local regularity (this time up to the boundary) in form of Lemma 25 (applied to the equation (3.1.7)) and an energy estimate on D , we have

$$\|\tilde{\xi} + \nabla\phi_{\tilde{\xi}}\|_{C^\alpha(D)} \lesssim_{D, \|A\|_{C^\alpha(D)}} \left(\int_D |\tilde{\xi} + \nabla\phi_{\tilde{\xi}}|^p \right)^{\frac{1}{p}} \lesssim_D 1 + |\tilde{\xi}|^p. \quad (3.7.13)$$

Therefore, $a_h^{(1)} \in C^\alpha(Q_L)$ and by using (3.7.13) for both $\tilde{\xi} = \xi$ and $\tilde{\xi} = \xi + he$ and arguing as in (3.5.25), there exists a constant c depending on $|\xi|$, D , $\|A\|_{C^\alpha(D)}$ such that

$$\|a_h^{(1)}\|_{C^\alpha(D)} + \|a_h^{(2)}\|_{C^\alpha(D)} \leq c_1. \quad (3.7.14)$$

On the one hand, by testing the equation (3.7.12) with $\delta^h\phi_\xi$ and using that $a_h^{(1)}$ is uniformly elliptic, we deduce

$$\int_D |\nabla\delta^h\phi_\xi|^2 \lesssim \int_D |a_h^{(2)}|^2 \stackrel{(3.7.14)}{\lesssim} c_1. \quad (3.7.15)$$

On the other hand, by the Schauder estimate [76, Theorem 5.19] applied to (3.7.12), and the bounds (3.7.14) and (3.7.15), there exists some $\gamma > 0$ (depending on α and d) such that

$$\|\nabla\delta^h\phi_\xi\|_{C^\alpha(D)} \lesssim \|a_h^{(1)}\|_{C^\alpha(D)}^\gamma \left(\left(\int_D |\nabla\delta^h\phi_\xi|^2 \right)^{\frac{1}{2}} + \|a_h^{(2)}\|_{C^\alpha(D)} \right) \stackrel{(3.7.14), (3.7.15)}{\leq} c_2, \quad (3.7.16)$$

for a constant c_2 depending on c_1 , d and D . By (3.7.16) and Arzela-Ascoli's theorem, there exists $\tilde{\psi} \in C_0^{1,\alpha}(D)$ (resp. $C_{\text{per}}^{1,\alpha}(Q_L)$) such that (up to a subsequence that we do not relabel)

$$\nabla\delta^h\phi_\xi \xrightarrow{h \downarrow 0} \nabla\tilde{\psi} \text{ in } C^0(D). \quad (3.7.17)$$

It remains to show that $\nabla\tilde{\psi} = \nabla\tilde{\phi}_{\xi,e}$, which directly follows from the weak formulation of (3.7.12). For all $w \in H_0^1(D)$ (resp. $H_{\text{per}}^1(Q_L)$)

$$\int_D \nabla w \cdot a_h^{(1)} \nabla \delta^h \phi_\xi = - \int_D \nabla w \cdot a_h^{(2)} e. \quad (3.7.18)$$

Using the convergence of the gradient (3.7.17), we can pass to the limit as $h \downarrow 0$ in (3.7.18), which implies that $\tilde{\psi}$ solves (3.2.9). By uniqueness, $\tilde{\psi} = \tilde{\phi}_{\xi,e}$, and (3.7.17) holds without extracting a subsequence. \square

3.7.5 Periodic setting: Proof of Theorem 13

In this paragraph, we show that if we have a good control of the critical set of the corrector of the leading order operator (an anisotropic p -Laplacian), then the homogenized operator \bar{a} belongs to $\mathcal{M}(p, 1, 2, c)$ for some $c > 0$. In what follows we set $b : (x, \xi) \mapsto A(x)|\xi|^{p-2}\xi$ and $c : (x, \xi) \mapsto A(x)\xi$.

Since $\bar{a} \in \mathcal{M}(p, 1, p, c)$ by construction, it suffices to prove the improved monotonicity property: For all $\xi_1, \xi_2 \in \mathbb{R}^d$,

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2), \xi_1 - \xi_2) \geq c|\xi_1 - \xi_2|^2(|\xi_1|^{p-2} + |\xi_2|^{p-2}). \quad (3.7.19)$$

We first introduce $\bar{b} : \mathbb{R}^d \rightarrow \mathbb{R}^d, \xi \mapsto \int_Q b(x, \xi + \nabla\psi_\xi(x))dx$, where $\psi_\xi \in W_{\text{per}}^{1,p}(Q)$ solves the corrector equation

$$-\nabla \cdot b(x, \xi + \nabla\psi_\xi(x)) = 0.$$

By homogeneity, for all $t > 0$ and all $\xi \in \mathbb{R}^d$ we have $\bar{b}(t\xi) = t^{p-1}\bar{b}(\xi)$.

Step 1. Proof of (3.7.19) for $|\xi_1|, |\xi_2| \gg 1$.

Substep 1.1. Reformulation.

By Lemma 27, correctors are differentiable and we thus have for all $\xi, e \in \mathbb{R}^d$

$$e \cdot D\bar{a}(\xi)e = \int_Q (e + \nabla\tilde{\phi}_{\xi,e}) \cdot a_\xi(e + \nabla\tilde{\phi}_{\xi,e}),$$

where $\tilde{\phi}_{\xi,e} \in H_{\text{per}}^1(Q)$ solves $-\nabla \cdot a_\xi(e + \nabla\tilde{\phi}_{\xi,e}) = 0$ and $a_\xi : x \mapsto Da(x, \xi + \nabla\phi_\xi(x))$. Hence, for all $\xi_1, \xi_2 \in \mathbb{R}^d$ we have

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2), \xi_1 - \xi_2) = \int_0^1 (\xi_1 - \xi_2) \cdot D\bar{a}(\xi_1 + t(\xi_2 - \xi_1)) \cdot (\xi_1 - \xi_2) dt. \quad (3.7.20)$$

Since $D\bar{a}$ is non-negative, the claim (3.7.19) follows for $|\xi_1|, |\xi_2| \gg 1$ provided we prove that $e \cdot D\bar{a}(\xi)e \geq c|\xi|^{p-2}$ for all $|\xi| \gg 1$ and all $e \in \mathbb{R}^d$ with $|e| = 1$.

Fix such a direction e . For all $s > 0$ let $\xi_s \in \mathbb{R}^d$ be such that $|\xi_s| = 1$ and $e \cdot D\bar{a}(s\xi_s)e = \inf_{|\xi|=1} e \cdot D\bar{a}(\xi)e$, which exists since $\xi \mapsto D\bar{a}(\xi)$ is continuous. In the following two substeps we prove the needed estimate in form of

$$\liminf_{s \uparrow \infty} e \cdot \frac{1}{s^{p-2}} D\bar{a}(s\xi_s)e > 0. \quad (3.7.21)$$

Substep 1.2. Proof of

$$\lim_{s \uparrow \infty} \left\| \frac{1}{s^{p-2}} a_{s\xi_s} - b_{\xi_s} \right\|_{C^\alpha(Q)} = 0, \quad (3.7.22)$$

where $b_\xi : x \mapsto Db(x, \xi + \nabla\psi_\xi(x))$.

On the one hand, by the corrector equations for ϕ_ξ and ψ_ξ we have

$$-\nabla \cdot (a(\xi + \nabla\phi_\xi) - a(\xi + \nabla\psi_\xi)) = \nabla \cdot c(x, \xi + \nabla\psi_\xi),$$

so that, testing with $\psi_\xi - \phi_\xi$ and using the monotonicity of a , we obtain

$$\begin{aligned} \int_Q |\nabla(\psi_\xi - \phi_\xi)|^2 (1 + |\xi + \nabla\psi_\xi|^{p-2} + |\xi + \nabla\phi_\xi|^{p-2}) &\lesssim \int_Q |\nabla(\psi_\xi - \phi_\xi)| |\xi + \nabla\psi_\xi| \\ &\lesssim \int_Q |\nabla(\psi_\xi - \phi_\xi)| (1 + |\xi + \nabla\psi_\xi|)^{\frac{p-2}{2}} (1 + |\xi + \nabla\psi_\xi|)^{2-\frac{p}{2}} \end{aligned}$$

and therefore

$$\int_Q |\nabla(\psi_\xi - \phi_\xi)|^2 (1 + |\xi + \nabla\psi_\xi|^{p-2} + |\xi + \nabla\phi_\xi|^{p-2}) \lesssim \int_Q (1 + |\xi + \nabla\psi_\xi|)^{4-p}.$$

Applied to $\xi = s\xi_s$ this yields using that $\psi_{s\xi_s} = s\psi_{\xi_s}$

$$\int_Q |\nabla(\frac{1}{s}\phi_{s\xi_s} - \psi_{\xi_s})|^p \lesssim s^{2(2-p)} \int_Q (1 + |\xi_s + \nabla\psi_{\xi_s}|^p) \stackrel{|\xi_s|=1}{\lesssim} s^{2(2-p)}. \quad (3.7.23)$$

Next we argue that $\{\frac{1}{s}\phi_{s\xi_s}\}_{s \geq 1}$ is a bounded sequence in $C^\alpha(Q)$, in which case (3.7.22) follows from (3.7.23) by Arzela-Ascoli's theorem. To this end we rewrite the corrector equation for $\chi_s := \frac{1}{s}\phi_{s\xi_s}$ as

$$-\nabla \cdot \tilde{a}_s(x, \xi_s + \nabla\chi_s) = 0$$

with $\tilde{a}_s(x, \xi) := b(x, \xi) + \frac{1}{s^{p-1}}c(x, s\xi)$. Since by assumption, \tilde{a}_s satisfies (3.2.19) for some ω independent of $s \geq 1$, $\{\chi_s\}_{s \geq 1}$ is indeed bounded in $C^{1,\alpha}(Q)$ by [105, Theorem 13]. The conclusion then follows using that $\frac{1}{s^{p-2}}a_{s\xi_s} = D\tilde{a}_s(x, \xi_s + \nabla\chi_s)$ and that $\xi \mapsto D\tilde{a}_s(x, \xi)$ is continuous (uniformly wrt s, x).

Substep 1.3. Proof of (3.7.21).

We assume without loss of generality that $\xi_s \rightarrow \xi_\infty$. Since $\xi \mapsto \nabla\psi_\xi$ is Lipschitz from \mathbb{R}^d to $C^\alpha(Q)$, (3.7.22) can be upgraded to (along the subsequence giving the liminf)

$$\lim_{s \uparrow +\infty} \|\frac{1}{s^{p-2}}a_{s\xi_s} - b_{\xi_\infty}\|_{C^\alpha(Q)} = 0. \quad (3.7.24)$$

By definition we have

$$e \cdot \frac{1}{s^{p-2}}D\bar{a}(s\xi_s)e = \int_Q (e + \nabla\tilde{\phi}_{s\xi_s,e}) \cdot \frac{1}{s^{p-2}}a_{s\xi_s}^{\text{sym}}(e + \nabla\tilde{\phi}_{s\xi_s,e}),$$

where $a_{s\xi_s}^{\text{sym}}$ is the symmetric part of $a_{s\xi_s}$. By assumption, there exists $r > 0$ such that $\mathbb{R}^d \setminus \mathcal{T}_r(\xi_\infty)$ is connected (where $\mathcal{T}_r(\xi_\infty) := \{x + B_r \mid x \in \mathbb{R}^d, |\xi_\infty + \nabla\psi_{\xi_\infty}(x)| = 0\}$). Since $[0, 1]^d \cap \mathbb{R}^d \setminus \mathcal{T}_r(\xi_\infty)$ is closed and $\nabla\psi_{\xi_\infty}$ is continuous, there exists $\kappa > 0$ such that the symmetric part $b_{\xi_\infty}^{\text{sym}}$ of b_{ξ_∞} satisfies $b_{\xi_\infty}^{\text{sym}}|_{Q \setminus \mathcal{T}_r(\xi_\infty)} \geq \kappa \text{Id}$. Hence, by (3.7.24), there exists $s_* < \infty$ such that, for all $s \geq s_*$, $a_{s\xi_s}^{\text{sym}}|_{Q \setminus \mathcal{T}_r(\xi_\infty)} \geq \frac{1}{2}\kappa \text{Id}$. For all $s \geq s_*$ we thus have

$$e \cdot \frac{1}{s^{p-2}}D\bar{a}(s\xi_s)e \geq \frac{\kappa}{2} \int_{Q \setminus \mathcal{T}_r(\xi_\infty)} |e + \nabla\tilde{\phi}_{s\xi_s,e}|^2 \geq \frac{\kappa}{2} \inf_{\tilde{\phi} \in H_{\text{per}}^1(Q)} \int_{Q \setminus \mathcal{T}_r(\xi_\infty)} |e + \nabla\tilde{\phi}|^2,$$

which is positive since $\mathbb{R}^d \setminus \mathcal{T}_r(\xi_\infty)$ is connected in \mathbb{R}^d . This proves (3.7.21).

Step 2. Proof of (3.7.19) in the remaining range: $|\xi_1|, |\xi_2| \lesssim 1$ and $|\xi_2| \gg 1, |\xi_1| \lesssim 1$.

On the one hand, since $c(x, \xi)\xi \geq \frac{1}{C}|\xi|^2$, we have for all $\xi_1, \xi_2 \in \mathbb{R}^d$

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2), \xi_1 - \xi_2) \geq c|\xi_1 - \xi_2|^2,$$

from which (3.7.19) follows for $|\xi_1|, |\xi_2| \lesssim 1$. On the other, since $\bar{a} \in \mathcal{M}(p, 1, p, C)$, for all $\xi_1, \xi_2 \in \mathbb{R}^d$,

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2), \xi_1 - \xi_2) \geq c|\xi_1 - \xi_2|^p,$$

from which (3.7.19) follows for $|\xi_2| \gg 1, |\xi_1| \lesssim 1$ using $|\xi_1 - \xi_2|^p \gtrsim |\xi_1 - \xi_2|^2(|\xi_2|^{p-2} - |\xi_1|^{p-2}) \gtrsim |\xi_1 - \xi_2|^2(1 + |\xi_2|^{p-2} + |\xi_1|^{p-2})$.

Remark 5. The strong assumption on the critical set of ψ_ξ is solely used to ensure that

$$\liminf_{s \uparrow +\infty} \int_Q (e + \nabla \tilde{\phi}_{s\xi_s, e}) \cdot \frac{1}{s^{p-2}} a_{s\xi_s}^{\text{sym}}(e + \nabla \tilde{\phi}_{s\xi_s, e}) \geq \inf_{\tilde{\phi} \in H_{\text{per}}^1(Q)} \int_Q (e + \nabla \tilde{\phi}) \cdot b_{\xi_\infty}^{\text{sym}}(e + \nabla \tilde{\phi}),$$

which might not hold true in general due to the Lavrentieff phenomenon – see e.g. [155] in a similar context.

3.7.6 Statistically isotropic random setting: Proof of Theorem 14

In this subsection, we exhibit a class of random monotone operators a whose homogenized operator \bar{a} belongs to $\mathcal{M}(p, 1, 2, c)$ for some $c > 0$. If a is a p -Laplacian then \bar{a} is homogeneous of degree $p - 1$. The upcoming result relies on a perturbation of this property. To define this class we make both structural assumptions on $(x, \xi) \mapsto a(x, \xi)$ and on the probability law \mathbb{P} . We emphasize that our arguments are purely qualitative and do not require Hypothesis 3.1.2. We start with the structural assumption on the operator (which quantifies what we mean by perturbation of a p -Laplacian)

Definition 3.7.1. We define a class \mathcal{A} of nonlinear maps $\hat{a} : [\lambda, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ with quasi-diagonal structure, that is, such that for all $(\alpha, \xi) \in [\lambda, 1] \times \mathbb{R}^d$

$$\hat{a}(\alpha, \xi) = \rho(\alpha, |\xi|)\xi, \tag{3.7.25}$$

where $\rho : [\lambda, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuously differentiable, and such that the map $\xi \mapsto \hat{a}(\alpha, \xi)$ is asymptotically of p -Laplacian type. More precisely, we assume that $\inf_\alpha \rho(\alpha, t) \geq \lambda(1 + t^{p-2})$ and that there exist two differentiable functions $\rho_1 : [\lambda, 1] \rightarrow \mathbb{R}_+$ and $\rho_2 : [\lambda, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that for all $(\alpha, t) \in [\lambda, 1] \times \mathbb{R}_+$

$$\rho(\alpha, t) = \rho_1(\alpha)t^{p-2} + \rho_2(\alpha, t),$$

and that there exist a constant C and an exponent $0 \leq \beta < p - 2$ such that

$$|\rho_2(\alpha, t)| + t|\partial_t \rho_2(\alpha, t)| \leq C(1 + t^\beta).$$

As a consequence, \hat{a} is variational in the sense that $\hat{a}(\alpha, \xi) = D\hat{W}(\alpha, \xi)$, where \hat{W} is given by $\hat{W}(\alpha, \xi) := \int_0^{|\xi|} s\rho(\alpha, s)ds \geq \lambda(\frac{1}{2}|\xi|^2 + \frac{1}{p}|\xi|^p)$.

As the following examples show, \mathcal{A} is not empty.

Example 3.7.1. The following nonlinear maps belong to \mathcal{A} :

1. The non-degenerate p -Laplacian operator $\hat{a} : [\lambda, 1] \times \mathbb{R}^d \ni (\alpha, \xi) \mapsto \alpha(1 + |\xi|^{p-2})\xi$, with $\rho_1(\alpha) = \alpha$ and $\rho_2(\alpha, t) = \alpha$ (and therefore $C = \frac{1}{2}$ and $\beta = 0$).
2. The operator $\hat{a} : [\lambda, 1] \times \mathbb{R}^d \ni (\alpha, \xi) \mapsto \left(\frac{1 + |\xi|^{p+q-2}}{1 + \alpha|\xi|^q} + 1 \right)\xi$, for any $q \geq 0$, and with $\rho_1(\alpha) := \frac{1}{\alpha}$, $\rho_2(\alpha, t) := \frac{\alpha - t^{p-2}}{\alpha(1 + \alpha t^q)} + 1$, $C =$ and $\beta = (p - 2 - q) \vee 0$.

We are in position to state the main result of this section.

Theorem 25. *Let $\hat{a} \in \mathcal{A}$ and let $A : \mathbb{R}^d \rightarrow [\lambda, 1]$ be a stationary and ergodic random field (which is locally C^α – this is convenient but not necessary) which is statistically isotropic in the sense that for all rotations $R \in SO(d)$, A and $A(R \cdot)$ have the same (joint) distribution. Consider the random monotone operator $a : \mathbb{R}^d \times \mathbb{R}^d \ni (x, \xi) \mapsto \hat{a}(A(x), \xi)$. Then, the associated homogenized map \bar{a} belongs to $\mathcal{M}(p, 1, 2, c)$ for some c depending on λ , p and \hat{a} . In particular, for all $\xi_1, \xi_2 \in \mathbb{R}^d$*

$$(\bar{a}(\xi_1) - \bar{a}(\xi_2), \xi_1 - \xi_2) \geq \frac{1}{c}(1 + |\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|^2, \quad (3.7.26)$$

and

$$|\bar{a}(\xi_1) - \bar{a}(\xi_2)| \leq c(1 + |\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|. \quad (3.7.27)$$

Note that the example of Theorem 14 satisfies the assumptions of Theorem 25. The proof of Theorem 25 relies on an approximation argument, an ODE argument, and the following property.

Lemma 28. *Let $a_\circ : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be such that for all $\xi \in \mathbb{R}^d$*

$$a_\circ(\xi) = \rho(|\xi|)\xi, \quad (3.7.28)$$

where $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function that satisfies for some constant $c > 0$ and for all $t > 0$

$$\frac{d}{dt}(t\rho(t)) \geq c(1 + t^p)^{\frac{p-2}{p}}. \quad (3.7.29)$$

Then, there exists a constant $\tilde{c} > 0$ depending on c and p such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$

$$(a(\xi_1) - a(\xi_2), \xi_1 - \xi_2) \geq \tilde{c}(1 + |\xi_1|^{p-2} + |\xi_2|^{p-2})|\xi_1 - \xi_2|^2.$$

Proof. W.l.o.g we may assume that $|\xi_1| > |\xi_2| > 0$. We fix $s > 0$ and we define $f : [s, +\infty) \rightarrow \mathbb{R}$ as

$$f : t \mapsto t\rho(t) - s\rho(s) - \tilde{c}(1 + t^p + s^p)^{\frac{p-2}{p}}(t - s),$$

where \tilde{c} will be fixed later. Differentiating f and using the assumption (3.7.29), we obtain

$$\begin{aligned} f'(t) &= \frac{d}{dt}(t\rho(t)) - \tilde{c}(1 + t^p + s^p)^{-\frac{2}{p}}t^{p-1}(t - s) + (1 + t^p + s^p)^{\frac{p-2}{p}} \\ &\geq c(1 + t^p)^{\frac{p-2}{p}} - \tilde{c}(1 + t^p + s^p)^{-\frac{2}{p}}t^{p-1}(t - s) + (1 + t^p + s^p)^{\frac{p-2}{p}}. \end{aligned}$$

Since $(t - s)t^{p-1} \leq t^p$ and $1 + t^p + s^p \leq 2(1 + t^p)$, this yields $f'(t) \geq (c - 2\tilde{c}(p - 1))(1 + t^p)^{\frac{p-2}{p}}$. With the choice $\tilde{c} = \frac{c}{2(p-1)}$, this entails $f'(t) \geq 0$, and thus $f(t) \geq f(s)$, which takes the form

$$t\rho(t) - s\rho(s) \geq \tilde{c}(1 + t^p + s^p)^{\frac{p-2}{p}}(t - s). \quad (3.7.30)$$

Used with $s = 0$, this implies the lower bound for all $t_1 > 0$

$$\rho(t_1) \geq \tilde{c}(1 + t_1^p)^{\frac{p-2}{p}}. \quad (3.7.31)$$

By the definition (3.7.28) of a_\circ , we have with the notation $t_1 = |\xi_1| > t_2 = |\xi_2|$,

$$\begin{aligned} (a_\circ(\xi_1) - a_\circ(\xi_2), \xi_1 - \xi_2) &= t_1^2 \rho(t_1) + t_2^2 \rho(t_2) - (\xi_1, \xi_2)(\rho(t_1) + \rho(t_2)) \\ &= (t_1 \rho(t_1) - t_2 \rho(t_2))(t_1 - t_2) + (t_1 t_2 - (\xi_1, \xi_2))(\rho(t_1) + \rho(t_2)) \\ &\stackrel{(3.7.30), (3.7.31)}{\geq} \tilde{c}(1 + t_1^p)^{\frac{p-2}{p}} (t_1 - t_2)^2 + \tilde{c}(t_1 t_2 - (\xi_1, \xi_2))(1 + t_1^p)^{\frac{p-2}{p}} \\ &\geq \frac{\tilde{c}}{2}(1 + t_1^p)^{\frac{p-2}{p}} |\xi_1 - \xi_2|^2, \end{aligned}$$

and the claim follows by redefining \tilde{c} . □

We now prove Theorem 25.

Proof of Theorem 25. Since (3.7.27) is a direct consequence of the monotonicity of a and the bound (3.1.8) on the corrector gradient, the subtle part is the proof of (3.7.26). We split the proof into two steps. In the first step we argue that it suffices to prove a version of (3.7.26) obtained by an approximation of the corrector on bounded domains, which we prove in the second step.

Step 1. Approximation.

By assumption there exists $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $a(x, \xi) = DW(x, \xi)$. Likewise, there exists $\bar{W} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\bar{a}(\xi) = D\bar{W}(\xi)$. For all $L \geq 1$, we denote by ϕ_ξ^L the unique weak solution in $W_0^{1,p}(B_L)$ of (3.1.7), and define

$$a^L(\xi) := \int_{B_L} a(x, \xi + \nabla \phi_\xi^L(x)) dx, \quad W^L(\xi) := \int_{B_L} W(x, \xi + \nabla \phi_\xi^L(x)) dx,$$

which satisfy $a^L(\xi) = DW^L(\xi)$. As a direct consequence of the homogenization result we have almost-surely

$$\lim_{L \uparrow +\infty} (a^L(\xi), W^L(\xi)) = (\bar{a}(\xi), \bar{W}(\xi)).$$

Set $\bar{a}^L(\xi) := \mathbb{E}[a^L(\xi)]$ and $\bar{W}^L(\xi) := \mathbb{E}[W^L(\xi)]$. Since we have the uniform almost sure bound $|a^L(\xi)| + |W^L(\xi)| \leq C(1 + |\xi|^p)$, we obtain on the one hand that $\bar{a}^L(\xi) = D\bar{W}^L(\xi)$ and on the other hand (by dominated convergence) that

$$\lim_{L \uparrow +\infty} (\bar{a}^L(\xi), \bar{W}^L(\xi)) = (\bar{a}(\xi), \bar{W}(\xi)).$$

Hence, (3.7.26) follows if we prove that there exists a constant $C > 0$ independent of L such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ we have

$$(\bar{a}^L(\xi_1) - \bar{a}^L(\xi_2), \xi_1 - \xi_2) \geq \frac{1}{C}(1 + |\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|^2. \quad (3.7.32)$$

The advantage of (3.7.32) over (3.7.26) is that it involves correctors on a bounded domain rather than on the whole space, for which differentiability with respect to ξ can be easily established (cf. Lemma 27). The advantage of \bar{W}^L over W^L (which motivates the choice of the ball B_L for the domain) is that, as \bar{W} , $\xi \mapsto \bar{W}^L$ is isotropic in the sense that there exists $\zeta^L : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$\bar{W}^L(\xi) = \zeta^L(|\xi|)$. In particular, we necessarily have $\bar{a}^L(\xi) = (\zeta^L)'(|\xi|)\frac{\xi}{|\xi|}$, so that, by Lemma 28, (3.7.32) will follow provided we show that $t \mapsto (\zeta^L)'(t)$ is differentiable and satisfies for some $c > 0$ independent of L for all $t > 0$

$$(\zeta^L)''(t) \geq c(1+t^p)^{\frac{p-2}{p}}. \quad (3.7.33)$$

Fix a unit vector $e \in \mathbb{R}^d$. By definition, we have

$$(\zeta^L)'(t) = \bar{a}^L(te) \cdot e = \mathbb{E} \left[\int_{B_L} a(x, te + \nabla \phi_{te}^L(x)) \cdot e dx \right].$$

By Lemma 27, correctors are differentiable, and differentiating the above yields using (3.2.9) (posed on B_L with Dirichlet boundary conditions)

$$\begin{aligned} (\zeta^L)''(t) &= \mathbb{E} \left[\int_{B_L} Da(x, te + \nabla \phi_{te}^L(x))(e + \nabla \tilde{\phi}_{te,e}^L(x)) \cdot e dx \right] \\ &= \mathbb{E} \left[\int_{B_L} Da(x, te + \nabla \phi_{te}^L(x))(e + \nabla \tilde{\phi}_{te,e}^L(x)) \cdot (e + \nabla \tilde{\phi}_{te,e}^L(x)) dx \right]. \end{aligned}$$

Since $a(x, \xi) = \hat{a}(A(x), \xi)$, $\xi \mapsto a(x, \xi)$ is differentiable and satisfies for some $c > 0$ and for all $\xi, h \in \mathbb{R}^d$

$$Da(x, \xi) : h \otimes h \geq c(1 + |\xi|^{p-2})|h|^2.$$

Hence (using that $\int_{B_L} \nabla \tilde{\phi}_{te,e}^L = 0$), $(\zeta^L)''(t) \geq c \mathbb{E} \left[\int_{B_L} |e + \nabla \tilde{\phi}_{te,e}^L|^2 \right] \geq c$, which yields (3.7.33) provided $t \lesssim 1$. It remains to treat the case $t \gg 1$.

Step 2. Proof of (3.7.33).

If \bar{a}^L were the p -Laplacian, ζ^L would satisfy $(\zeta^L)'(t) = \frac{p}{t} \zeta^L(t)$. The idea is to derive a similar ODE in our setting based on the identity $\bar{a}^L(\xi) = D\bar{W}^L(\xi)$, from which we shall prove (3.7.33) by differentiating and using our structural assumptions on a .

Substep 2.1. Formula for ζ^L via an ODE argument.

On the one hand, by the weak formulation of (3.1.7) tested with ϕ_{te}^L , we have

$$(\zeta^L)'(t) = \frac{1}{t} \mathbb{E} \left[\int_{B_L} a(x, te + \nabla \phi_{te}^L(x)) \cdot (te + \nabla \phi_{te}^L(x)) dx \right],$$

which, in combination with the form of a , yields

$$\begin{aligned} t(\zeta^L)'(t) &= \mathbb{E} \left[\int_{B_L} \rho(A(x), |te + \nabla \phi_{te}^L(x)|) |te + \nabla \phi_{te}^L(x)|^2 dx \right] \\ &= \mathbb{E} \left[\int_{B_L} \rho_1(A(x)) |te + \nabla \phi_{te}^L(x)|^p dx \right] + \mathbb{E} \left[\int_{B_L} \rho_2(A(x), |te + \nabla \phi_{te}^L(x)|) |te + \nabla \phi_{te}^L(x)|^2 dx \right]. \end{aligned} \quad (3.7.34)$$

On the other hand, using that $W(x, \xi) = \int_0^{|\xi|} s \rho(A(x), s) ds$ and the decomposition of ρ , we also have

$$\begin{aligned} \zeta^L(t) = \bar{W}^L(te) &= \mathbb{E} \left[\int_{B_L} \int_0^{|te + \nabla \phi_{te}^L(x)|} s \rho(A(x), s) ds dx \right] \\ &= \frac{1}{p} \mathbb{E} \left[\int_{B_L} \rho_1(A(x)) |te + \nabla \phi_{te}^L(x)|^p dx \right] + \mathbb{E} \left[\int_{B_L} \int_0^{|te + \nabla \phi_{te}^L(x)|} s \rho_2(A(x), s) ds dx \right]. \end{aligned} \quad (3.7.35)$$

From (3.7.34) and (3.7.35) we infer that ζ^L satisfies the differential relation

$$t(\zeta^L)'(t) - p\zeta^L(t) = \mathbb{E} \left[\int_{B_L} \rho_2(A(x), |te + \nabla \phi_{te}^L(x)|) |te + \nabla \phi_{te}^L(x)|^2 dx - p \int_{B_L} \int_0^{|te + \nabla \phi_{te}^L(x)|} s \rho_2(A(x), s) ds dx \right],$$

which we rewrite as $(\zeta^L)'(t) - \frac{p}{t}\zeta^L(t) = \frac{1}{t}h(t)$ with

$$h(t) := \mathbb{E} \left[\int_{B_L} \rho_2(A(x), |te + \nabla \phi_{te}^L(x)|) |te + \nabla \phi_{te}^L(x)|^2 dx - p \int_{B_L} \int_0^{|te + \nabla \phi_{te}^L(x)|} s \rho_2(A(x), s) ds dx \right].$$

Let $t_\star \geq 1$ to be fixed later. The solution of this ODE is explicitly given for all $t \geq t_\star$ by $\zeta^L(t) = \frac{\zeta^L(t_\star)}{t^p} t^p + t^p \int_{t_\star}^t s^{-1-p} h(s) ds$. Differentiating twice (which we can since $t \mapsto \nabla \phi_{te}^L(x)$ is differentiable for all $x \in B_L$), this yields

$$\begin{aligned} & (\zeta^L)''(t) \\ &= p(p-1) \underbrace{\left(\frac{\zeta^L(t_\star)}{t_\star^p} + \int_{t_\star}^t s^{-1-p} h(s) ds \right)}_{=: \gamma^L(t)} t^{p-2} + \underbrace{\frac{d^2}{dt^2} \left(t^p \int_{t_\star}^t s^{-1-p} h(s) ds \right) - p(p-1)t^{p-2} \int_{t_\star}^t s^{-1-p} h(s) ds}_{=: R^L(t)} \end{aligned} \tag{3.7.36}$$

In the following two substeps we provide a bound from below for $\gamma^L(t)$ and a bound from above for $R^L(t)$.

Substep 2.2. Choice of t_\star^0 and lower bound on $\gamma^L(t)$.

On the one hand, recall that $\zeta^L(t) = \bar{W}^L(te)$ and that $W(x, \xi) = \int_0^{|\xi|} s \rho(A(x), s) ds \geq \lambda(\frac{1}{2}|\xi|^2 + \frac{1}{p}|\xi|^p)$, so that

$$\zeta^L(t) = \bar{W}^L(te) = \mathbb{E}_L \left[\inf_{v \in W_0^{1,p}(B_L)} \int_{B_L} W(x, te + \nabla v) dx \right] \geq \frac{\lambda}{p} \mathbb{E}_L \left[\inf_{v \in W_0^{1,p}(B_L)} \int_{B_L} |te + \nabla v|^p \right] = \frac{\lambda}{p} t^p.$$

On the other hand, the assumption on ρ_2 implies that there exists a constant $c > 0$ such that for all $t \geq 1$ and $\alpha \in [\lambda, 1]$,

$$|\rho_2(\alpha, t)| + t|\partial_t \rho_2(\alpha, t)| \leq c(1 + t^\beta). \tag{3.7.37}$$

(The constant c will change from line to line, but remains independent of t and L .) Hence, by the (deterministic) energy estimate $\int_{B_L} |\nabla \phi_\xi^L|^p \leq c(1 + |\xi|^p)$, (3.7.37) yields for all $t \geq 1$

$$|h(t)| \leq ct^{\beta+2}, \tag{3.7.38}$$

so that

$$\left| \int_t^\infty s^{-1-p} h(s) ds \right| \leq ct^{\beta-(p-2)}.$$

Since $\beta < p - 2$, we deduce that for $t_\star^0 \sim 1$ large enough (and independent of L) we have for all $t \geq t_\star \geq t_\star^0$

$$\gamma^L(t) \geq \frac{\lambda}{2p}. \tag{3.7.39}$$

Substep 2.3. Choice of t_\star^1 and upper bound on $R^L(t)$.

Since $R^L(t) = (p-1)t^{-2}h(t) + t^{-1}h'(t)$, it remains to estimate $h'(t)$. By differentiating h , using (3.7.37), and rearranging the terms, Cauchy-Schwarz' inequality yields

$$\begin{aligned} |h'(t)| &\leq c\mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|^\beta) |te + \nabla\phi_{te}^L| |e + \nabla\tilde{\phi}_{te,e}^L| \right] \\ &\leq c\mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|)^{\beta+1-\frac{p-2}{2}} (1 + |te + \nabla\phi_{te}^L|^{p-2})^{\frac{1}{2}} |e + \nabla\tilde{\phi}_{te,e}^L| \right] \\ &\leq c\mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|^{p-2}) |e + \nabla\tilde{\phi}_{te,e}^L|^2 \right]^{\frac{1}{2}} \mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|)^{p-2(p-2-\beta)} \right]^{\frac{1}{2}}. \end{aligned}$$

For the second right-hand side factor, we use the (deterministic) energy estimate on $\nabla\phi_\xi^L$, which yields for $t \geq 1$

$$\mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|)^{p-2(p-2-\beta)} \right]^{\frac{1}{2}} \leq ct^{\frac{p}{2}-(p-2-\beta)},$$

whereas for the first right-hand side factor we use the (deterministic) energy estimate on $\nabla\phi_{\xi,e}^L$ (in favor of which we shall argue below), which yields for $t \geq 1$

$$\mathbb{E}_L \left[\int_{B_L} (1 + |te + \nabla\phi_{te}^L|^{p-2}) |e + \nabla\tilde{\phi}_{te,e}^L|^2 \right]^{\frac{1}{2}} \leq ct^{\frac{p-2}{2}}. \quad (3.7.40)$$

These last three estimates then combine to $|h'(t)| \leq ct^{p-1-(p-2-\beta)}$. With (3.7.38) and the formula for $R^L(t)$, this entails for all $t \geq 1$ the control $|R^L(t)| \leq ct^\beta$. In particular, since $\beta < p-2$, there exists $t_\star^1 \geq 1$ such that for all $t \geq t_\star^1$, we have $|R^L(t)| \leq \frac{\lambda(p-1)}{4}t^{p-2}$.

We conclude with the argument in favor of (3.7.40). By testing (3.2.9) with $\tilde{\phi}_{te,e}^L \in H_0^1(B_L)$ we obtain

$$\int_{B_L} \nabla\tilde{\phi}_{te,e}^L \cdot Da(x, te + \nabla\phi_{te}^L) \nabla\tilde{\phi}_{te,e}^L = \int_{B_L} \nabla\tilde{\phi}_{te,e}^L \cdot Da(x, te + \nabla\phi_{te}^L) e,$$

which, by our assumptions on a , entails

$$\int_{B_L} |\nabla\tilde{\phi}_{te,e}^L|^2 (1 + |te + \nabla\phi_{te}^L|^{p-2}) \lesssim \int_{B_L} |\nabla\tilde{\phi}_{te,e}^L| (1 + |te + \nabla\phi_{te}^L|^{p-2}),$$

and therefore (3.7.40) by Cauchy-Schwarz' inequality and the energy estimate on $\nabla\phi_{te}^L$.

Substep 2.4. Definition of t_\star and proof of (3.7.33).

Set $t_\star = t_\star^0 \vee t_\star^1$, which is independent of L . Step 1 yields (3.7.33) in the regime $t \leq t_\star$, whereas in the regime $t \geq t_\star$, (3.7.33) follows from (3.7.36) in combination with Substeps 2.2 and 2.3. \square

3.7.7 Periodization in law and functional inequalities

Periodization in law of a

We start with the definition of the periodized ensemble \mathbb{P}_L .

Definition 3.7.2. *Let $L \geq 1$. The probability \mathbb{P}_L is the stationary and centered Gaussian ensemble of scalar fields G defined by the covariance function $c_L : x \in \mathbb{R}^d \mapsto \sum_{k \in \mathbb{Z}^d} c(x + Lk)$. Clearly, the covariance function c_L and thus the realizations G are Q_L -periodic. We identify \mathbb{P}_L with its push forward under the map $G \mapsto A := (x \mapsto B(G(x)))$, where B is defined in Hypothesis 3.1.2.*

We now recall the qualitative convergence as $L \uparrow +\infty$ (see Lemma 42 of Chapter 5 for a proof).

Lemma 29. *There exists a family of coupling probabilities \mathbb{P}_{Ω_L} indexed by $L \geq 1$ between the probabilities \mathbb{P}_L and \mathbb{P} , that generate Gaussian fields G_L and G of covariance functions c_L and c , respectively, and such that $A := \chi \star B(G)$ and $A_L := \chi \star B(G_L)$ satisfy for all $R > 0$ and $q \geq 1$*

$$\mathbb{E}_{\Omega_L} \left[\sup_{Q_R} |A - A_L|^q \right] \xrightarrow{L \uparrow +\infty} 0. \quad (3.7.41)$$

Functional calculus

The ensemble \mathbb{P}_L satisfies the following logarithmic-Sobolev inequality (see for instance [59, 48]).

Proposition 9. *There exists $\rho > 0$ such that for all functional F of A with $\mathbb{E}_L[|F|^2] < +\infty$*

$$\mathbb{E}_L[F^2 \log(F)] - \mathbb{E}_L[F^2] \mathbb{E}_L[\log(F)] \leq \frac{1}{\rho} \mathbb{E}_L \left[\int_{Q_L} |\partial_x F|^2 dx \right], \quad (3.7.42)$$

where for all $x \in Q_L$

$$\partial_x F(A) := \sup_{\delta A} \limsup_{h \downarrow 0} \frac{F(A + h\delta A) - F(A)}{h}, \quad (3.7.43)$$

and where the supremum runs over coefficient fields δA that are supported in $B(x)$ and bounded by 1 in $C^\alpha(B(x))$.⁵

The logarithmic Sobolev inequality (3.7.42) yields control of moments (see e.g. [60]).

Lemma 30. *For all $q \geq 1$ and all random variables F we have*

$$\mathbb{E}_L \left[|F - \mathbb{E}[F]|^q \right]^{\frac{1}{q}} \lesssim \sqrt{q} \mathbb{E}_L \left[\left(\int_{Q_L} |\partial_x F|^2 dx \right)^{\frac{q}{2}} \right]^{\frac{1}{q}}. \quad (3.7.44)$$

The following standard result gives the link between algebraic moments and stretched exponential moments for non-negative random variables.

Lemma 31. *Let X be a non-negative random variable. We have the following equivalence:*

$$\exists C_1 > 0 \text{ such that } \mathbb{E}_L[\exp(\frac{1}{C_1} X)] \leq 2 \quad \Leftrightarrow \quad \exists C_2 > 0 \text{ such that } \forall q \geq 1, \mathbb{E}_L[X^q]^{\frac{1}{q}} \leq q C_2. \quad (3.7.45)$$

The last result of this subsection allows us to exchange supremum and expectation.

Lemma 32. *Let X be a stationary random field. If there exists an exponent $\gamma > 0$ such that for all $q \geq 1$*

$$\mathbb{E}_L \left[\|X\|_{L^\infty(B)}^q \right]^{\frac{1}{q}} \leq q^\gamma, \quad (3.7.46)$$

then we have for all $\varepsilon > 0$, $R \geq 1$, and $q \geq 1$

$$\mathbb{E}_L \left[(R^{-\varepsilon} \|X\|_{L^\infty(B_R)})^q \right]^{\frac{1}{q}} \lesssim_{d,\varepsilon} q^\gamma. \quad (3.7.47)$$

⁵The fact one can assume $\|\delta A\|_{C^\alpha(B(x))} \leq 1$ comes from the convolution in (3.1.10).

Proof. Let $R \geq 1$ and $\varepsilon > 0$. Consider $N(R, d) \lesssim R^d$ points $(x_i)_{i \in \llbracket 1, N \rrbracket} \subset B_R$ such that $B_R \subset \bigcup_{i=1}^N B(x_i)$. We then have

$$\|X\|_{L^\infty(B_R)} \leq \max_{i \in \llbracket 1, N \rrbracket} \|X\|_{L^\infty(B(x_i))}. \quad (3.7.48)$$

By the discrete $\ell^q - \ell^\infty$ estimate for all $q \geq 1$, (3.7.48) turns into

$$\|X\|_{L^\infty(B_R(x))} \leq \left(\sum_{i=1}^N \|X\|_{L^\infty(B(x_i))} \right)^{\frac{1}{q}}.$$

Therefore, by taking the the q -th moment, using the triangle inequality, the stationarity of X and the assumption (3.7.46), we get

$$\mathbb{E}_L \left[\|X\|_{L^\infty(B_R)}^q \right]^{\frac{1}{q}} \leq \max_{i \in \llbracket 1, N \rrbracket} \mathbb{E}_L \left[\|X\|_{L^\infty(B(x_i))}^q \right]^{\frac{1}{q}} N^{\frac{1}{q}} \stackrel{(3.7.46)}{\lesssim} q^\gamma R^{\frac{d}{q}},$$

which yields the desired estimate (3.7.47) provided $q \geq \frac{d}{\varepsilon}$ (and therefore in the whole range of exponents by Hölder's inequality). \square

Convergence of the periodization in law of the correctors

In this subsection (and here only), we denote by $(\nabla \phi_\xi, \nabla \sigma_\xi)$ the nonlinear corrector gradients associated with the ensemble \mathbb{P} , and by $(\nabla \phi_\xi^L, \nabla \sigma_\xi^L)$ the nonlinear corrector gradients associated with the periodized ensemble \mathbb{P}_L .

Proposition 10. *For all $L \geq 1$, we consider the coupling ensemble $\mathbb{E}_{\Omega_L}[\cdot]$ defined in Lemma 29. Let $\xi \in \mathbb{R}^d$. If $(\nabla \phi_\xi^L, \nabla \sigma_\xi^L)$ satisfies for all $q \geq 1$*

$$\mathbb{E}_L \left[|(\nabla \phi_\xi^L, \nabla \sigma_\xi^L)|^q \right]^{\frac{1}{q}} \lesssim_q 1 \quad (3.7.49)$$

(where the multiplicative constant does not depend on L), then for all $R \geq 1$ and $q \geq 1$, we have

$$\sup_{Q_R} \mathbb{E}_{\Omega_L} \left[|(\nabla \phi_\xi, \nabla \sigma_\xi) - (\nabla \phi_\xi^L, \nabla \sigma_\xi^L)|^q \right]^{\frac{1}{q}} \xrightarrow{L \uparrow +\infty} 0. \quad (3.7.50)$$

As a consequence, for all $x \in \mathbb{R}^d$, we have

$$\mathbb{E}_{\Omega_L} \left[\left(\int_{B(x)} \left| (\phi_\xi, \sigma_\xi) - (\phi_\xi^L, \sigma_\xi^L) + \int_B (\phi_\xi^L, \sigma_\xi^L) \right|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \xrightarrow{L \uparrow +\infty} 0. \quad (3.7.51)$$

In addition, for all unit vectors $e \in \mathbb{R}^d$, the linearized correctors $(\nabla \tilde{\phi}_{\xi, e}, \nabla \tilde{\sigma}_{\xi, e})$ are well-defined, and if for all $q \geq 1$

$$\mathbb{E}_L \left[|(\nabla \phi_\xi^L, \nabla \sigma_\xi^L)|^q \right]^{\frac{1}{q}} \lesssim_q 1 \quad (3.7.52)$$

then we have for all $R \geq 1$ and $q \geq 1$,

$$\sup_{Q_R} \mathbb{E}_{\Omega_L} \left[|(\nabla \tilde{\phi}_{\xi, e}, \nabla \tilde{\sigma}_{\xi, e}) - (\nabla \tilde{\phi}_{\xi, e}^L, \nabla \tilde{\sigma}_{\xi, e}^L)|^q \right]^{\frac{1}{q}} \xrightarrow{L \uparrow +\infty} 0. \quad (3.7.53)$$

Proof. We first iteratively use Lemma 29 to construct the coupling ensemble $\mathbb{E}_\Omega[\cdot]$ between the ensembles $\mathbb{E}[\cdot]$ and $\mathbb{E}_{L_n}[\cdot]$, for $n \in \mathbb{N}$ (and a sequence $L_n \uparrow +\infty$) in such a way that it generates Gaussian fields G and G_{L_n} , and that the joint law of (G, G_{L_n}) is the same under $\mathbb{E}_\Omega[\cdot]$ and $\mathbb{E}_{L_n}[\cdot]$. In particular, we first generate G (with a law determined by $\mathbb{E}[\cdot]$), and then we successively build the Gaussian fields G_{L_n} in such a way that the conditional probabilities $\mathbb{P}_\Omega(G_{L_n}|G)$ are independent of one another. Hereafter, for simplicity, we abusely use L instead of L_n , and replace the limit $n \uparrow +\infty$ by $L \uparrow +\infty$. We split the proof into three steps.

Step 1. Proof of (3.7.50).

We start with the argument for $\nabla\phi_\xi^L$, and first show that we can extract a converging subsequence as $L \uparrow +\infty$. By Lemma 25 applied to (3.1.7) (with A replaced by A_L) combined with (3.7.49), we get for all $R \geq 1$ and $q \geq 1$

$$\sup_{L \geq 1} \mathbb{E}_\Omega \left[\|\xi + \nabla\phi_\xi^L\|_{C^\alpha(\overline{Q_R})}^q \right]^{\frac{1}{q}} \lesssim_R \sup_{L \geq 1} \mathbb{E}_\Omega \left[\left(\int_{Q_{2R}} |\xi + \nabla\phi_\xi^L|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \lesssim_{R, |\xi|, q} 1. \quad (3.7.54)$$

By (3.7.54) and Arzela-Ascoli's theorem, there exists $\psi \in C^{1, \alpha}(\overline{Q_R}, L^q(d\mathbb{P}_\Omega))$ such that (up to a subsequence that we do not relabel)

$$\sup_{\overline{Q_R}} \mathbb{E}_\Omega \left[|\nabla\phi_\xi^L - \nabla\psi|^q \right]^{\frac{1}{q}} \xrightarrow{L \uparrow +\infty} 0. \quad (3.7.55)$$

It remains to argue that $\nabla\psi = \nabla\phi_\xi$. Since, by construction, $\nabla\psi$ is stationary and has finite second moment, it suffices to show that ψ solves (3.1.7). Let $X \in L^{\frac{p}{p-1}}(d\mathbb{P}_\Omega)$, $\chi \in C_c^\infty(\mathbb{R}^d)$ and define $a_L(x, \zeta) = A_L(x)(1 + |\zeta|^{p-2})\zeta$. By equation (3.1.7) (with the ensemble \mathbb{P}_L) for ϕ_ξ^L , we have

$$\mathbb{E}_\Omega \left[X \int_{\mathbb{R}^d} \nabla\chi(x) \cdot a_L(x, \xi + \nabla\phi_\xi^L(x)) dx \right] = 0.$$

Thus, using Lemma 29 and the convergence (3.7.55), we can pass to the limit when $L \uparrow +\infty$ to obtain

$$\mathbb{E}_\Omega \left[X \int_{\mathbb{R}^d} \nabla\chi(x) \cdot a(x, \xi + \nabla\phi_\xi(x)) dx \right] = 0,$$

which shows that ψ solves (3.1.7), and therefore that $\nabla\psi = \nabla\phi_\xi$ and that (3.7.55) holds without taking a subsequence. We proceed the same way to prove the convergence (3.7.50) for $(\nabla\sigma_\xi^L)$, combining the equations (3.2.2) and (3.2.3) with the strong convergence

$$\sup_{\overline{Q_R}} \mathbb{E}_{\Omega_L} \left[|a(\cdot, \xi + \nabla\phi_\xi^L) - a(\cdot, \xi + \nabla\phi_\xi)|^q \right] \xrightarrow{L \uparrow +\infty} 0,$$

which follows from (3.7.50) for $(\nabla\phi_\xi^L)$.

Step 2. Proof of (3.7.51).

First we claim that for all $R \geq 1$ and all functions ζ on Q_R we have the Poincaré inequality

$$\int_{Q_R} \left| \zeta - \int_B \zeta \right|^2 \lesssim R^2 \left\{ \begin{array}{l} d = 1 : \sqrt{R} \\ d = 2 : \log(R + 1) \\ d > 2 : 1 \end{array} \right\} \int_{Q_R} |\nabla\zeta|^2. \quad (3.7.56)$$

This simply follows by summation over dyadic scales of the following standard estimates for all $0 \leq i \leq \log R$

$$\int_{Q_{2^{i+1}}} \left| \zeta - \int_{Q_{2^i}} \zeta \right|^2 \lesssim (2^i)^2 \int_{Q_{2^{i+1}}} |\nabla\zeta|^2, \quad \left| \int_{Q_{2^i}} \zeta - \int_{Q_{2^{i-1}}} \zeta \right|^2 \lesssim (2^i)^2 \int_{Q_{2^i}} |\nabla\zeta|^2.$$

Applying (3.7.56) to $\zeta = (\phi_\xi, \sigma_\xi) - (\phi_\xi^L, \sigma_\xi^L)$ and using the anchoring condition $f_B(\phi_\xi, \sigma_\xi) = 0$, for all $x \in \mathbb{R}^d$ we have with $R = 2(|x| + 1)$

$$\int_{B(x)} \left| (\phi_\xi, \sigma_\xi) - (\phi_\xi^L, \sigma_\xi^L) + \int_B (\phi_\xi^L, \sigma_\xi^L) \right|^2 \lesssim R^2 \begin{cases} d = 1 & : \sqrt{R} \\ d = 2 & : \log(R + 1) \\ d > 2 & : 1 \end{cases} \int_{Q_R} |\nabla(\phi_\xi, \sigma_\xi) - \nabla(\phi_\xi^L, \sigma_\xi^L)|^2,$$

so that (3.7.51) follows from (3.7.50).

Step 3. Proof of (3.7.53).

The argument is essentially the same as for (3.7.50), and we only argue for $\nabla \tilde{\phi}_{\xi,e}$. First, by (3.7.50) and Lemma 13, the linearized extended corrector $(\tilde{\phi}_{\xi,e}, \tilde{\sigma}_{\xi,e})$ is well-defined. By the Schauder theory in form of [76, Theorem 5.19] applied to the equation (3.2.9) (the Hölder norm of the coefficients is controlled by Lemma 25 and (3.7.54)) and assumption (3.7.52), $\sup_{L \geq 1} \mathbb{E}[\|e + \nabla \tilde{\phi}_{\xi,e}^L\|_{C^\alpha(\overline{Q}_R)}^q] \lesssim_{R,|\xi|} 1$ for all $q \geq 1$. We then extract a converging subsequence as before and identify the limit using the weak formulation of (3.2.9) together with the convergence $Da_L(\cdot, \xi + \nabla \phi_\xi^L) \xrightarrow{L \uparrow +\infty} Da(\cdot, \xi + \nabla \phi_\xi)$ in $C^0(\overline{Q}_R, L^q(d\mathbb{P}_\Omega))$ (which follows from (3.7.50)). \square

3.7.8 Large-scale averages

We prove in this section estimates used to control large-scale averages, which are variations around [101]. We fix $L \geq 1$ and $\xi \in \mathbb{R}^d$, and use the short-hand notation r_\star for $r_{\star,\xi,L}$.

Lemma 33. *Let $m \in (0, 1)$, $r \leq 3r_\star(0)$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ a measurable function. We have*

$$\int_{B_r} \left(\int_{B_\star(x)} f \right) dx \lesssim \left(\int_{B_{\frac{3}{2}r}} \left(\int_{B_\star(x)} f \right)^m dx \right)^{\frac{1}{m}}. \tag{3.7.57}$$

Proof. The estimate (3.7.57) follows from

$$\sup_{x_0 \in B_r} \int_{B_\star(x_0)} f \lesssim \left(\int_{B_{\frac{3}{2}r}} \left(\int_{B_\star(x)} f \right)^m dx \right)^{\frac{1}{m}}. \tag{3.7.58}$$

Let $x_0 \in B_r$ be fixed. Since $r_{\star,\xi,L}$ is $\frac{1}{16}$ -Lipschitz, we have for all $x \in B_{\frac{r}{8}}$

$$B_\star(x_0) \subset B_{r_\star(x_0) + \frac{9}{8}r}(x). \tag{3.7.59}$$

Indeed, if $y \in B_\star(x_0)$ and $x \in B_{\frac{r}{8}}$, we have

$$|y - x| \leq |y - x_0| + |x_0 - x| \leq r_{\star,\xi,L}(x_0) + r + \frac{r}{8} = r_\star(x_0) + \frac{9}{8}r.$$

By the $\frac{1}{16}$ -Lipschitz property of r_\star , we have $r_\star(0) - \frac{1}{16}r \leq r_\star(x_0) \leq r_\star(0) + \frac{1}{16}r$. Together with the assumption $r \leq 2r_\star(0)$, this entails $\frac{13}{16}r_\star(0) \leq r_\star(x_0)$ and $r_\star(x_0) + \frac{9}{8}r \leq \frac{35}{8}r_\star(0)$ so that for all $x \in B_{\frac{r}{8}}$,

$$|B_\star(x_0)| \sim |B_{r_\star(x_0) + \frac{9}{8}r}(x)|.$$

Combined with (3.7.59) this yields

$$\left(\int_{B_\star(x_0)} f \right)^m \lesssim \int_{B_{\frac{r}{8}}} \left(\int_{B_{r_\star(x_0) + \frac{9}{8}r}(x)} f \right)^m dx.$$

Now let $N \in \mathbb{N}$ depending only on d and $(x_i)_{i \in \llbracket 1, N \rrbracket} \subset B_{\frac{11}{8}r}$ be such that

$$B_{\frac{11}{8}r} \subset \bigcup_{i=1}^N B_{\frac{1}{16}r}(x_i). \quad (3.7.60)$$

We claim that

$$B_{r_*(x_0) + \frac{9}{8}r} \subset \bigcup_{i=1}^N B_{r_*(x_0) - \frac{3}{16}r}(x_i). \quad (3.7.61)$$

Indeed, let $y \in B_{r_*(x_0) + \frac{9}{8}r}$ and set $z = \frac{\frac{11}{8}r}{r_*(x_0) + \frac{9}{8}r}y$. Since $|z| \leq \frac{11}{8}r$, there exists $i \in \llbracket 1, N \rrbracket$ such that $|z - x_i| \leq \frac{1}{16}r$. Thus by the triangle inequality

$$|y - x_i| \leq |y - z| + |z - x_i| \leq |y| \left| 1 - \frac{\frac{11}{8}r}{r_*(x_0) + \frac{9}{8}r} \right| + \frac{1}{16}r.$$

Since $y \in B_{r_*(x_0) + \frac{9}{8}r}$ and since the $\frac{1}{16}$ -Lipschitz property of r_* entails $r_*(x_0) \geq r_*(0) - \frac{1}{16}|x_0| \geq \frac{r}{3} - \frac{r}{16} = \frac{13}{48}r \geq \frac{1}{4}r$, we have

$$|y| \left| 1 - \frac{\frac{11}{8}r}{r_{*,\xi,L}(x_0) + \frac{9}{8}r} \right| \leq r_*(x_0) - \frac{1}{4}r,$$

and consequently

$$|y - x_i| \leq \frac{1}{16}r + r_*(x_0) - \frac{1}{4}r = r_*(x_0) - \frac{3}{16}r,$$

which concludes the proof of (3.7.61). We deduce from the sub-additive property of $x \in \mathbb{R}^+ \mapsto x^m$ and the fact that for all $i \in \llbracket 1, N \rrbracket$, $x_i + B_{\frac{1}{8}r} \subset B_{\frac{3}{2}r}$

$$\int_{B_{\frac{1}{8}r}} \left(\int_{B_{r_*(x_0) + \frac{9}{8}r}(x)} f \right)^m dx \lesssim \sum_{i=1}^N \int_{B_{\frac{1}{8}r}} \left(\int_{B_{r_*(x_0) - \frac{3}{16}r}(x+x_i)} f \right)^m dx \lesssim \int_{B_{\frac{3}{2}r}} \left(\int_{B_{r_*(x_0) - \frac{3}{16}r}(x)} f \right)^m dx,$$

which concludes the proof of (3.7.58) since the $\frac{1}{16}$ -Lipschitz property of r_* entails for all $x_0 \in B_r$ and $x \in B_{\frac{3}{2}r}$

$$r_*(x_0) - \frac{3}{16}r \leq r_*(x) + \frac{1}{16}|x - x_0| - \frac{3}{16}r \leq r_*(x) + \frac{1}{16}(\frac{3}{2}r + r) - \frac{3}{16}r \leq r_*(x).$$

□

Lemma 34. *Let $x_0 \in \mathbb{R}^d$ and $r \geq 3r_*(x_0)$, $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ a measurable function. We have*

$$\int_{B_r(x_0)} \left(\int_{B_*(x)} f \right) dx \lesssim \int_{B_{\frac{67}{48}r}(x_0)} f, \quad (3.7.62)$$

and

$$\int_{B_{\frac{17}{12}r}(x_0)} f \lesssim \int_{B_{2r}(x_0)} \left(\int_{B_*(x)} f \right) dx. \quad (3.7.63)$$

By reverting from balls to cubes, this implies that there exists $C \geq 1$ depending only on d such that for all $r \geq Cr_*(x_0)$

$$\int_{Q_r(x_0)} \left(\fint_{B_*(x)} f \right) dx \lesssim \int_{Q_{2r}(x_0)} f, \quad (3.7.64)$$

and

$$\int_{Q_r(x_0)} f \lesssim \int_{Q_{2r}(x_0)} \left(\fint_{B_*(x)} f \right) dx. \quad (3.7.65)$$

This constant C (which only depends on d and our upper bound $\frac{1}{16}$ on the Lipschitz constant of r_*) will be used to define the best Lipschitz constant for r_* (cf. Definition 3.3.1).

Proof. Without loss of generality, we may assume that $x_0 = 0$. Since r_* is $\frac{1}{16}$ -Lipschitz, we have for all integrable non-negative functions g

$$\int_{\mathbb{R}^d} \left(\fint_{B_*(x)} g \right) dx \sim \int_{\mathbb{R}^d} g, \quad (3.7.66)$$

cf. [80, (140)] (which relies on the construction of a Calderón-Zygmund partition of \mathbb{R}^d based on r_*). We start with the proof of (3.7.62). Since r_* is $\frac{1}{16}$ -Lipschitz and $3r_*(0) \leq r$, for all $x \in B_r$, $r_*(x) \leq r_*(0) + \frac{r}{16} \leq \frac{19}{48}r$ so that $B_*(x) \subset B_{\frac{67}{48}r}$. This yields (3.7.62) in form of

$$\int_{B_r} \left(\fint_{B_*(x)} f \right) dx = \int_{B_r} \left(\fint_{B_*(x)} f \mathbf{1}_{B_{\frac{67}{48}r}} \right) dx \leq \int_{\mathbb{R}^d} \left(\fint_{B_*(x)} f \mathbf{1}_{B_{\frac{67}{48}r}} \right) dx \stackrel{(3.7.66)}{\sim} \int_{\mathbb{R}^d} f \mathbf{1}_{B_{\frac{67}{48}r}}.$$

We now turn to (3.7.63). By (3.7.66),

$$\int_{B_{\frac{17}{12}r}} f \sim \int_{\mathbb{R}^d} \left(\fint_{B_*(x)} f \mathbf{1}_{B_{\frac{17}{12}r}} \right) dx.$$

Since r_* is $\frac{1}{16}$ -Lipschitz and $3r_*(0) \leq r$, if $|x| \geq 2r$, $r_*(x) \leq r_*(0) + \frac{2}{16}r \leq \frac{11}{24}r \leq \frac{11}{48}|x|$, so that $B_*(x) \subset \mathbb{R}^d \setminus B_{\frac{85}{48}r} \subset \mathbb{R}^d \setminus B_{\frac{17}{12}r}$. Hence, exploiting the indicator function $\mathbf{1}_{B_{\frac{17}{12}r}}$, the above turns into

$$\int_{B_{\frac{17}{12}r}} f \sim \int_{B_{2r}} \left(\fint_{B_*(x)} f \mathbf{1}_{B_{\frac{17}{12}r}} \right) dx \leq \int_{B_{2r}} \left(\fint_{B_*(x)} f \right) dx,$$

that is, (3.7.63). □

Chapter 4

Price's formula for infinite dimensional Gaussian measure

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We extend the so-called Price's formula (well known in finite dimension) to the case infinite dimensional Gaussian field. It gives a general formula for the derivative of a functional with respect to the covariance function, which is one of the main tool of Chapter 5. This Chapter is based on joint work in preparation with Felix Otto.

The Chapter is organized as follows: In Section 4.1 we recall the framework of infinite dimensional Gaussian field. In Section 4.2 we state and prove the extension of the Price's formula to the infinite dimensional Gaussian setting of Section 4.1. In Section 4.3, we discuss about the paring between two bilinear forms and we prove that under additional assumptions, the paring can be rewriting in a natural way that will be used in the next chapter.

4.1 Reminder on infinite dimensional Gaussian measures

We recall in this section the framework of infinite dimensional Gaussian field that we will use in this chapter and in Chapter 5. For more details, we refer to the general book of Gaussian measures of Bogachev [33].

We consider a centred Gaussian measure¹ $\langle \cdot \rangle$ on a Banach space X , which is defined by imposing that the push-forward under any bounded linear form $\ell \in X^*$ is a centred one-dimensional Gaussian. By the Cauchy-Schwarz inequality in $L^2_{\langle \cdot \rangle}$, such a measure $\langle \cdot \rangle$ defines a symmetric positive semidefinite bilinear form c on X^* via

$$c(\ell, \ell') := \langle \ell \ell' \rangle \text{ for } \ell, \ell' \in X^*. \tag{4.1.1}$$

¹We use the symbol $\langle \cdot \rangle$ to address both the ensemble and to denote its expectation operator.

This bilinear form c determines the centred Gaussian measure $\langle \cdot \rangle$, see [116, Proposition 3.9]. We will assume that there exists an Hilbert space H densely embedded in X and that c is continuous as a bilinear form in H^* . More precisely, since by the inclusion $H \subset X$ we also mean that the topology of H is finer than that of X , an element $\ell \in X^*$ can be identified with a bounded linear form on H and thus, by Riesz's representation theorem, with an element of H itself. Hence we have

$$X^* \subset H^* \cong H \subset X$$

with also the first embedding being dense. What we mean by the domination of c is

$$c(\ell, \ell) \leq |\ell|_H^2 \quad \text{for all } \ell \in X^*, \quad (4.1.2)$$

where the norm on $H \cong H^*$ is denoted by $|\cdot|_H$ and the inner product by $(\cdot, \cdot)_H$.

It is well-known that such a Gaussian measure only exists provided the embedding $H \subset X$ is compact in a quantitative way. A simple sufficient criterion, see [33, Example 3.9.7.] serves as a convenient assumption for our application:

$$\|h\|_X \leq |Ah|_H, \quad (4.1.3)$$

for a symmetric Hilbert-Schmidt operator A on H ,² where $\|\cdot\|_X$ denotes the norm of X . In view of [33, Proposition A.2.6.(iii)], the assumption of symmetry is just for convenience. A quick way to see the connection between (4.1.3) and the existence of a Gaussian measure $\langle \cdot \rangle$ goes as follows: By compactness and symmetric property of the Hilbert-Schmidt operator A , there exists an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of H formed by eigenvectors of A with eigenvalues $(\mu_n)_{n \in \mathbb{N}}$. Hence we obtain from (4.1.3) that $\|h\|_X^2 \leq \sum_n \mu_n^2 (h_n, h)_H^2$ and thus from (4.1.1) and (4.1.2) that $\langle \|h\|_X^2 \rangle \leq \sum_n \mu_n^2$, which is finite by the characterization of the Hilbert-Schmidt property. Note that by Fernique's theorem [33, Theorem 2.8.5] we have

$$\langle \|g\|_X^p \rangle \leq C_p < \infty \quad \text{for all } p < \infty. \quad (4.1.4)$$

We now discuss about the pairing of two bilinear forms in this context, necessary to formulate the extension of Price's formula of the next section. We recall that under the assumption (4.1.3), the trace provides a pairing between a bounded symmetric bilinear form \dot{c} on $H \supset X^*$ and a bounded symmetric bilinear form b on $X \supset H$. More precisely, the trace provides a pairing between the Riesz representations \dot{C} and B as bounded symmetric linear operators on H . Even more precisely, the pairing is given by the trace of the product $\dot{C}B$, which requires $\dot{C}B$ to be of trace class³, see [33, Proposition A.2.9.]. In view of [33, Proposition A.2.10. (ii)], it suffices to argue that B is of trace class, which we will do now. By boundedness of b on X , B is bounded from X to H . Since as a consequence of (4.1.3), $H \subset X$ is compact, B is compact (with respect to H). Hence there exists an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of H formed by eigenvectors of B . In order to show that B is trace class, we need to show summability of the modulus of eigenvalues

$$\sum_{n=0}^{+\infty} |(h_n, Bh_n)_H| < \infty,$$

²We recall that it means that A is compact and for any orthonormal basis $(h_n)_{n \in \mathbb{N}}$, $\sum_n |Ah_n|_H^2 < +\infty$. Equivalently, if A is symmetric, $\sum_n \mu_n^2 < +\infty$ where $(\mu_n)_{n \in \mathbb{N}}$ denotes the eigenvalues of A .

³Recall that $\text{tr}(\dot{C}B) := \sum_{n=0}^{+\infty} (h_n, \dot{C}Bh_n)_H$, for an arbitrary orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of H . We say that $\dot{C}B$ is of trace class if $\text{tr}(\dot{C}B) < +\infty$ (and then, it is independent of the choice of the orthonormal basis).

see [33, Proposition A.2.9.(iii)]. The summability of $|h_n \cdot B h_n| = |b(h_n, h_n)|$ in turn follows via the boundedness of b (with respect to X) and the assumption (4.1.3) in form of

$$|b(h_n, h_n)| \leq \|b\| \|h_n\|_X \|h_n\|_X \stackrel{(4.1.3)}{\leq} \|b\| |Ah_n|_H |Ah_n|_H,$$

(where $\|b\|$ denotes the norm of the bilinear form b with respect to X) combined with the characterizing property $|A|_{HS}^2 := \sum_n |Ah_n|^2 < \infty$ of a Hilbert-Schmidt operator⁴ A .

It follows from the definition of the trace of $\dot{C}B$ that the pairing can be recovered by

$$b \cdot \dot{c} = \sum_{n=0}^{+\infty} \dot{c}(h_n, B h_n) \quad (4.1.5)$$

for an arbitrary orthonormal basis $(h_n)_{n \in \mathbb{N}}$ of H . By the symmetry of \dot{C} and B , which implies that the traces of $\dot{C}B$ and $B\dot{C}$ are equal, we also have

$$b \cdot \dot{c} = \sum_{n=0}^{+\infty} b(h_n, \dot{C} h_n). \quad (4.1.6)$$

We now recall the boundedness property of this pairing. For later use, we note that by (4.1.3), the summands in (4.1.6) are estimated as

$$|b(h_n, \dot{C} h_n)| \leq \|b\| |Ah_n|_H |A\dot{C} h_n|_H. \quad (4.1.7)$$

Using the fact that, since \dot{C} is a continuous operator in H , $A\dot{C}$ is an Hilbert-Schmidt operator in H with $|A\dot{C}|_{HS} \leq |A|_{HS} |\dot{C}|$ (where $|\dot{C}|$ denotes the operator norm on H), see [33, Proposition A.2.10.], we learn by Cauchy-Schwarz' inequality that the dominating series in (4.1.7) is summable

$$\sum_{n=0}^{+\infty} |Ah_n| |A\dot{C} h_n| \leq |A|_{HS}^2 |\dot{c}|, \quad (4.1.8)$$

where $|\dot{c}|$ denotes the norm of the bilinear form \dot{c} with respect to H . Inserting (4.1.7) into (4.1.6) we obtain from (4.1.8) the boundedness of the pairing:

$$|b \cdot \dot{c}| \leq |A|_{HS}^2 \|b\| |\dot{c}|. \quad (4.1.9)$$

The paring defined in (4.1.5) is canonical in the finite dimensional case, as a paring between a form b on X and a form \dot{c} on X^* (where in that case, we can obviously take $X = H$ and $A = \text{Id}$). Suppose that X is a space of dimension $N < +\infty$ and set $(e_i)_{i \in \llbracket 1, N \rrbracket}$ its canonical basis. Then, b admits the representation

$$b = \sum_{n=1}^N \sum_{m=1}^N \mu_{nm} e_m^* \otimes e_n^*, \quad (4.1.10)$$

where $(e_n^*)_{n \in \llbracket 1, N \rrbracket}$ denotes the canonical dual basis of X^* and $\mu_{nm} = b(e_n, e_m)$. We claim that in this case, the paring (4.1.5) specifies to

$$b \cdot \dot{c} = \sum_{n=1}^N \sum_{m=1}^N \mu_{nm} \dot{c}(e_m^*, e_n^*). \quad (4.1.11)$$

⁴This definition of the Hilbert-Schmidt norm is independent of the choice of orthonormal basis of H , see [33, Proposition A.2.6.].

Indeed, we fix an orthonormal basis $(h_k)_{k \in \llbracket 1, N \rrbracket}$ of X , and we observe that by definition of B (that is the Riesz representation of b in X), we have $Bh_k = \sum_{n=1}^N \sum_{m=1}^N \mu_{nm} e_n^*(h_k) e_m^*$. Consequently, since the sums commutes with the bilinear form \dot{c} , we obtain

$$b \cdot \dot{c} = \sum_{k=1}^N \dot{c}(h_k, Bh_k) = \sum_{k=1}^N \sum_{n=1}^N \sum_{m=1}^N \mu_{nm} e_n^*(h_k) \dot{c}(h_k, e_m^*) = \sum_{n=1}^N \sum_{m=1}^N \mu_{mn} \dot{c}\left(\sum_{k=1}^N e_n^*(h_k) h_k, e_m^*\right),$$

which gives (4.1.11). We will show in Section 4.3 that Formula (4.1.11) also holds in the infinite dimensional setting if we assume that b is compact and admits a decomposition of the form (4.1.10) (that we will specify when X has an Hilbert structure for simplicity).

4.2 Statement and proof of the extension of Price's formula

Equipped with the pairing (4.1.5), we are now ready to formulate a functional, i. e. coordinate-free, version of Price's formula in our abstract infinite dimensional context of Section 4.1.

Theorem 26. *Let the norm $\|\cdot\|_X$ of the Banach space X and the norm $|\cdot|_H$ of a Hilbert space $H \subset X$ be related by (4.1.3). Given a family of centered Gaussian measures $\langle \cdot \rangle_L$ on X depending on a parameter $L \in (0, \infty)$, we suppose that the covariance c_L satisfies (4.1.2) and is continuously differentiable in L with values in the space of bounded bilinear forms on H . Let F be a twice continuously Fréchet differentiable (but possibly unbounded) function on X with for all $g, h \in X$*

$$|d^2 F(g) \cdot (h, h)| \leq (\|g\|_X + 1)^p \|h\|_X^2, \quad (4.2.1)$$

for some exponent $p < \infty$. Then $L \mapsto \langle F \rangle_L$ is continuously differentiable with

$$\frac{d}{dL} \langle F \rangle_L = \frac{1}{2} \langle d^2 F \cdot \frac{dc_L}{dL} \rangle_L. \quad (4.2.2)$$

Proof. First of all, since (4.2.1) implies

$$|F(g)| \leq C(\|g\|_X + 1)^{p+2}, \quad (4.2.3)$$

we learn from (4.1.4) that $\langle F \rangle_L$ is well-defined. We then split the proof into two steps. We first prove by an approximation argument that it is enough to establish (4.2.2) when $d^2 F$ has a finite rank, that we prove in the second step.

Step 1. Finite rank approximation. We will establish (4.2.2) by making use of an finite rank approximation, which reduces to prove the classical Price formula in finite dimension. The finite rank approximation is established with help of the inclusion $X^* \subset H$ and projection. Indeed, since X^* is dense in H , there exists an orthonormal basis $(h_n)_{n \in \mathbb{N}} \subset X^*$ of H . We use it to define for every $N \in \mathbb{N}$ the linear (projection) map, for all $g \in X$

$$P_N g = \sum_{n=0}^N (h_n, g)_H h_n, \quad (4.2.4)$$

and recall that when restricted to H , by definition of $(h_n)_{n \in \mathbb{N}}$, we have for all $h \in H$

$$\lim_{N \uparrow \infty} |P_N h - h|_H^2 = 0. \quad (4.2.5)$$

We claim that this survives on X in form of

$$\lim_{N \uparrow \infty} \langle \|P_N g - g\|_X^2 \rangle_L = 0 \quad (4.2.6)$$

uniformly in L , where we made explicit the variable $g \in X$ that $\langle \cdot \rangle_L$ integrates over. Indeed, we obtain from (4.1.3) that

$$\|P_N g - g\|_X^2 \leq \sum_{n>N} \sum_{m>N} (h_n, g)_H (h_m, g)_H (Ah_n, Ah_m)_H,$$

and thus by definition⁵ (4.1.1)

$$\langle \|P_N g - g\|_X^2 \rangle_L \leq \sum_{n>N} \sum_{m>N} c_L(h_n, h_m) (Ah_n \cdot Ah_m)_H. \quad (4.2.7)$$

We now argue as for (4.1.9). Let C_L be the Riesz representation of c_L in H and observe that for all $n, m \in \mathbb{N}$ $c_L(h_n, h_m) = (h_m, C_L h_n)_H$. Thus, we deduce from the definition (4.2.4) of P_N and the fact that P_N is symmetric that, for all $n > N$ (writing A^* the adjoint operator of A)

$$\begin{aligned} \sum_{m>N} c_L(h_n, h_m) (Ah_n, Ah_m)_H &= \left(\sum_{m>N} (Ah_n, Ah_m)_H h_m, C_L h_n \right)_H \\ &= ((1 - P_N)A^* Ah_n, C_L h_n)_H \\ &= (Ah_n, A(1 - P_N)C_L h_n)_H. \end{aligned}$$

Hence, (4.2.6) follows by inserting the above identity in (4.2.7) and by using the fact that $A(1 - P_N)C_L$ is Hilbert-Schmidt (as a product of continuous maps and an Hilbert-Schmidt operator) with $|A(1 - P_N)C_L|_{HS} \leq |A|_{HS}$ (using (4.1.2)):

$$\begin{aligned} \langle \|P_N g - g\|_X^2 \rangle_L &\leq \sum_{n>N} (Ah_n, A(1 - P_N)C_L h_n)_H \leq \left(\sum_{n>N} |Ah_n|_H^2 \right)^{\frac{1}{2}} \left(\sum_{n>N} |A(1 - P_N)C_L h_n|_H^2 \right)^{\frac{1}{2}} \\ &\leq |A|_{HS} \left(\sum_{n>N} |Ah_n|_H^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which vanishes as $N \uparrow +\infty$ since A is Hilbert-Schmidt. In addition, by the same argument as for (4.2.6) it follows that $\langle \|P_N g\|_X^2 \rangle_L$ is uniformly bounded in N and L so that by Fernique's theorem [33, Theorem 2.8.5], for all $q < +\infty$ and uniformly in N and L

$$\langle \|P_N g\|_X^q \rangle_L \lesssim 1. \quad (4.2.8)$$

We now define our approximation F_N via pull-back under P_N :

$$F_N(g) = F(P_N g). \quad (4.2.9)$$

Since F is continuous on X and of algebraic growth, see (4.2.3), it follows from (4.2.8) and (4.2.6) that

$$\lim_{N \uparrow \infty} \langle F_N \rangle_L = \langle F \rangle_L, \quad (4.2.10)$$

⁵We identify here the Riesz representation of a linear form and the linear form itself via $H^* \cong H$.

uniformly in L . We now claim that also the right-hand side term of (4.2.2) converges, that is

$$\lim_{N \uparrow \infty} \langle d^2 F_N(g) \cdot \frac{dc_L}{dL} \rangle_L = \langle d^2 F(g) \cdot \frac{dc_L}{dL} \rangle_L, \quad (4.2.11)$$

uniformly in L . Indeed, on the level of the second Fréchet derivative, the linear transformation (4.2.9) translates into

$$d^2 F_N(g)(h, h') = d^2 F(P_N g)(P_N h, P_N h').$$

From this we obtain as for (4.1.6) (where \dot{C}_L denotes the Riesz representation of the bilinear form $\frac{dc_L}{dL}$ in H)

$$\begin{aligned} d^2 F_N(g) \cdot \frac{dc_L}{dL} &= \sum_{n=0}^{+\infty} d^2 F(P_N g)(P_N h_n, P_N \dot{C}_L h_n) \\ &= \sum_{n=0}^{+\infty} d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n), \end{aligned} \quad (4.2.12)$$

$$d^2 F(g) \cdot \frac{dc_L}{dL} = \sum_{n=0}^{+\infty} d^2 F(g)(h_n, \dot{C}_L h_n), \quad (4.2.13)$$

Since by assumption (4.2.1), $d^2 F$ is a continuous mapping of algebraic growth from X into the bounded bilinear forms on X and a fortiori on H , it follows from (4.2.5) in form of $\lim_{N \uparrow \infty} |P_N \dot{C}_L P_N h_n - \dot{C}_L h_n|_H = 0$ combined with (4.2.6) and (4.2.8) that for fixed $n \in \mathbb{N}$

$$\lim_{N \uparrow \infty} \langle d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n) \rangle_L = \langle d^2 F(g)(h_n, \dot{C}_L h_n) \rangle_L, \quad (4.2.14)$$

uniformly in L . In addition, using (4.2.1) and (4.1.3), one has

$$|d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n)| \leq (1 + \|P_N g\|_X)^p |Ah_n|_H |AP_N \dot{C}_L P_N h_n|_H,$$

which gives with (4.2.8)

$$|\langle d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n) \rangle_L| \leq C |Ah_n| |AP_N \dot{C}_L P_N h_n|_H$$

for some generic constant C independent of L and N . Hence by Cauchy-Schwarz and the fact that $AP_N \dot{C}_L P_N$ is Hilbert-Schmidt,

$$\begin{aligned} &\sum_{n>M} |\langle d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n) \rangle_L| \\ &\leq C \left(\sum_{n>M} |Ah_n|^2 \right)^{\frac{1}{2}} |AP_N \dot{C}_L P_N|_{HS}. \end{aligned}$$

Then, as for (4.1.9), we use $|AP_N \dot{C}_L P_N|_{HS} \leq |A|_{HS} |P_N \dot{C}_L P_N| \leq |A|_{HS} |\dot{C}_L| \leq C$ (for some generic constant C independent of L), so that

$$\sum_{n>M} |\langle d^2 F(P_N g)(h_n, P_N \dot{C}_L P_N h_n) \rangle_L| \leq C \left(\sum_{n>M} |Ah_n|^2 \right)^{\frac{1}{2}},$$

vanishes as $M \uparrow \infty$ uniformly in N and L . By the identities (4.2.12) and (4.2.13) together with (4.2.14) this establishes (4.2.11).

Step 2. Proof of (4.2.2) for finite rank functional. From (4.2.10) and (4.2.11) we learn that it is enough to establish (4.2.2) for F that are of the form (4.2.9), that is, of the form

$$F(g) = f(\ell_1(g), \dots, \ell_N(g)) \quad (4.2.15)$$

for $(\ell_n)_{n \in [1, N]} \subset X^*$ and f is twice continuously differentiable of at most algebraic growth in its variables x_n, \dots, x_N , which we assemble in $x \in \mathbb{R}^N$. We first rewrite the paring $d^2 F \cdot \frac{dc_L}{dL}$ in this context. By the chain rule we obtain from (4.2.15)

$$d^2 F(g) = \sum_{n=1}^N \sum_{m=1}^N (\partial_{nm} f)(x) \ell_n \otimes \ell_m. \quad (4.2.16)$$

We consider the centered Gaussian measure $\langle \cdot \rangle_{LN}$ on \mathbb{R}^N defined by the push-forward of $\langle \cdot \rangle_L$ under (4.2.4), that is,

$$\langle F(g) \rangle_L =: \langle f(x) \rangle_{LN}, \quad (4.2.17)$$

where we made the variable of integration explicit. With the standard identification $(\mathbb{R}^N)^* \cong \mathbb{R}^N$, the covariance of $\langle \cdot \rangle_{LN}$ is given by

$$c_{L, nm} := c_L(\ell_n, \ell_m), \quad (4.2.18)$$

where a difficulty arises from the fact that the so defined symmetric $N \times N$ -matrix may not be definite but only semi-definite. Hence we obtain from (4.2.16) and (4.2.18) as well as via the paring in finite dimension (4.1.11) that

$$d^2 F(g) \cdot \frac{dc}{dL} = \sum_{n=1}^N \sum_{m=1}^N (\partial_{nm} f)(x) \frac{dc_{L, nm}}{dL}. \quad (4.2.19)$$

We now check that Price's formula (4.2.2) holds with (4.2.19) and we essentially follow the computations done to prove (1.2.10). The crucial ingredient is the following identity for a symmetric positive definite $N \times N$ -matrix $L \mapsto C_L$ differentiable

$$\begin{aligned} & \frac{d}{dL} \exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right) \sqrt{\det C_L^{-1}} \\ &= \frac{1}{2} \left(x \cdot C_L^{-1} \frac{dC_L}{dL} C_L^{-1} x - \operatorname{tr} C_L^{-1} \frac{dC_L}{dL} \right) \exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right) \sqrt{\det C_L^{-1}} \\ &= \frac{1}{2} \operatorname{tr} \left(C_L^{-1} x \otimes C_L^{-1} x - C_L^{-1} \right) \frac{dC_L}{dL} \exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right) \sqrt{\det C_L^{-1}} \\ &= \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^N \partial_{nm} [\exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right)] \sqrt{\det C_L^{-1}} \frac{dc_{L, nm}}{dL}, \end{aligned}$$

Testing this identity with f we obtain, by two integration by parts (which are justified since f has polynomial growth)

$$\begin{aligned} & \frac{d}{dL} \int_{\mathbb{R}^N} dx f(x) \exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right) \sqrt{\det C_L^{-1}} \\ &= \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^N \int_{\mathbb{R}^N} dx (\partial_{nm} f)(x) \frac{dc_{L, nm}}{dL} \exp\left(-\frac{1}{2} x \cdot C_L^{-1} x\right) \sqrt{\det C_L^{-1}}. \end{aligned}$$

Approximating our symmetric semi-positive matrix $(c_{L,nm})_{n,m}$ by a definite one via $C_{L,\epsilon,nm} := c_{L,nm} + \epsilon\delta_{nm}$ and letting ϵ go to zero, we obtain

$$\frac{d}{dL}\langle f \rangle_{LN} = \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^N \langle (\partial_{nm} f) \frac{dc_{L,nm}}{dL} \rangle_{LN}.$$

Appealing (4.2.16) and (4.2.19) we learn from (4.2.17) that this assumes the form of the desired (4.2.2). \square

4.3 About the paring of bilinear forms and application

We discuss in this section about the paring (4.1.6). We first show that under the additional assumption that b in compact⁶, we can rewrite (4.1.6) in a more canonical way as we did for the finite dimensional case in (4.1.11). Then, in view of an application to the next chapter, we will specify the spaces X in H as L^2 -based spaces and show that if b is in addition Hilbert-Schmidt in X , the paring can be rewritten with help of a kernel for b .

4.3.1 Simplification of the paring (4.1.6)

For simplicity, we assume that X has an Hilbert structure and we strengthen the assumption (4.1.3) by assuming that

$$(\cdot, \cdot)_X = (A\cdot, A\cdot)_H, \quad (4.3.1)$$

where $(\cdot, \cdot)_X$ denotes the inner product in X . We consider a bounded symmetric bilinear form b on X (and we denote by B its Riesz representation in X) and a bounded symmetric bilinear form \dot{c} on $H \supset X^*$ (where we denote by \dot{C} its Riesz representation in H). We have the following lemma.

Lemma 35. *Assume that b is compact⁷ on X . There exist a bounded sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and an orthonormal basis $(f_n)_{n \in \mathbb{N}}$ of X such that*

$$b.\dot{c} = \sum_{n=0}^{+\infty} \lambda_n \dot{c}(\ell_n, \ell_n), \quad (4.3.2)$$

with for all $n \in \mathbb{N}$,

$$\ell_n : g \in X \mapsto (g, f_n)_X. \quad (4.3.3)$$

Proof. Since B is compact and symmetric on X (because b is), there exist a bounded sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and an orthonormal basis $(f_n)_{n \in \mathbb{N}}$ of X such that

$$B = \sum_{n=0}^{+\infty} \lambda_n \ell_n.$$

Therefore, we have the following decomposition of b

$$b = \sum_{n=0}^{+\infty} \lambda_n \ell_n \otimes \ell_n. \quad (4.3.4)$$

⁶Meaning that b is continuous with respect to the weak topology in X .

⁷Since X has an Hilbert structure, it is equivalent to say that B is a compact operator on X .

The goal is now to show that the computations done in the finite dimensional case to go from (4.1.6) to (4.1.11) hold here. To do so, we fix an orthonormal basis $(h_k)_{k \in \mathbb{N}}$ of H . By definition (4.1.6),

$$b.\dot{c} = \sum_{k=0}^{+\infty} b(h_k, \dot{C}h_k),$$

which turns into, using (4.3.4),

$$b.\dot{c} = \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} \lambda_n \ell_n(h_k) \ell_n(\dot{C}h_k). \quad (4.3.5)$$

We now show that we can exchange the two summations in (4.3.5). To do so, we check the summability of the summands: using that $(\lambda_n)_{n \in \mathbb{N}}$ is bounded, the Cauchy-Schwarz' inequality and the fact that $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis in X as well as (4.3.1), one obtains

$$\begin{aligned} \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} |\lambda_n \ell_n(h_k) \ell_n(\dot{C}h_k)| &\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sum_{k=0}^{+\infty} \left(\sum_{n=0}^{+\infty} |\ell_n(h_k)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{+\infty} |\ell_n(\dot{C}h_k)|^2 \right)^{\frac{1}{2}} \\ &= \sup_{n \in \mathbb{N}} |\lambda_n| \sum_{k=0}^{+\infty} \|h_k\|_X \|\dot{C}h_k\|_X \\ &\stackrel{(4.3.1)}{=} \sup_{n \in \mathbb{N}} |\lambda_n| \sum_{k=0}^{+\infty} |Ah_k|_H |A\dot{C}h_k|_H, \end{aligned}$$

which is finite since both A and $A\dot{C}$ are Hilbert-Schmidt in H . Therefore, we can exchange the summations in (4.3.5) to obtain (using also the definition of ℓ_n in (4.3.3))

$$\begin{aligned} b.\dot{c} &= \sum_{n=0}^{+\infty} \lambda_n \sum_{k=0}^{+\infty} \ell_n(h_k) \ell_n(\dot{C}h_k) \\ &\stackrel{(4.3.3)}{=} \sum_{n=0}^{+\infty} \lambda_n \sum_{k=0}^{+\infty} (h_k, f_n)_X (\dot{C}h_k, f_n)_X \\ &\stackrel{(4.3.1)}{=} \sum_{n=0}^{+\infty} \lambda_n \sum_{k=0}^{+\infty} (h_k, A^* A f_n)_H (\dot{C}h_k, A^* A f_n)_H \\ &= \sum_{n=0}^{+\infty} \lambda_n \left(\dot{C} \sum_{k=0}^{+\infty} (h_k, A^* A f_n)_H h_k, A^* A f_n \right)_H \\ &= \sum_{n=0}^{+\infty} \lambda_n \dot{c}(A^* A f_n, A^* A f_n), \end{aligned}$$

which gives (4.3.2) by noticing that $A^* A f_n$ is the Riesz representation of ℓ_n in H by the assumption (4.3.1). \square

4.3.2 Application for the Chapter 5

We now specify the spaces X and H that we will use for our application in the next chapter and we rewrite (4.3.2) in a very specific form that we will use in the next chapter.

In Chapter 5, the family of covariance function $L \mapsto c_L$ will be constructed by a periodization procedure from c . The purpose of our application of Formula (4.2.2) is to vary the period L . We identify a random variable g on the torus $[0, L]^d$ with an L -periodic random variable g on the whole space \mathbb{R}^d . Thus, we build X and H as weighted $L^2(\mathbb{R}^d)$ -based spaces. We consider an exponent $k > \frac{d}{4}$ and a weight $\mu : x \in \mathbb{R}^d \mapsto (1 + |x|^2)^{-4k}$ as well as the corresponding measure $d\mu = \mu(x)dx$. We define the spaces

$$X := \left\{ g : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable} \left| \int |g(x)|^2 d\mu(x) < \infty \right. \right\}, \quad (4.3.6)$$

and

$$H := \left\{ g : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable} \left| \int |(1 - \Delta)^k g(x)|^2 \sqrt{\mu(x)} dx < +\infty \right. \right\}, \quad (4.3.7)$$

that we endow with the two inner products, for all $g, g' \in X$ and $\tilde{g}, \tilde{g}' \in H$

$$(g, g')_X = \int g(x)g'(x)d\mu(x),$$

as well as

$$(\tilde{g}, \tilde{g}')_H = \int (1 - \Delta)^k \tilde{g}(x)(1 - \Delta)^k \tilde{g}'(x)\sqrt{\mu(x)}dx. \quad (4.3.8)$$

Obviously, (4.3.1) holds with

$$A : g \in H \mapsto (1 - \Delta)^{-k}[(1 + |\cdot|^2)^{-k}g].$$

We claim that A is an Hilbert-Schmidt operator in H . In view of the definition (4.3.8), it is equivalent to prove that the conjugate operator \tilde{A} of A given by: for all $g \in H$

$$\begin{aligned} \tilde{A}g &= (x \mapsto (1 + |x|^2)^{-k}(1 - \Delta)^k A(1 - \Delta)^{-k}(1 + |x|^2)^k g(x)) \\ &= (x \mapsto (1 + |x|^2)^{-2k}(1 - \Delta)^{-k}(1 + |x|^2)^k g(x)), \end{aligned}$$

is Hilbert-Schmidt in $L^2(\mathbb{R}^d)$. The operator \tilde{A} is the product of the multiplication operator $g \in L^2(\mathbb{R}^d) \mapsto (1 + |\cdot|^2)^{-2k}g$ and $g \in L^2(\mathbb{R}^d) \mapsto (1 - \Delta)^{-k}[(1 + |\cdot|^2)^k g]$, which can be seen as a convolution operator with a kernel $k : (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto K(x - y)(1 + |y|^2)^k$ where K is the kernel of $(1 - \Delta)^{-k}$ defined via its Fourier transform $\mathcal{F} : \xi \mapsto (1 + |\xi|^2)^{-2k}$. Therefore, \tilde{A} is an integral operator where its kernel is given by

$$\tilde{k} : (x, y) \mapsto (1 + |x|^2)^{-2k}K(x - y)(1 + |y|^2)^k.$$

In order to show that \tilde{A} is Hilbert-Schmidt in $L^2(\mathbb{R}^d)$, we thus have to check that \tilde{k} is square integrable in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Using the inequality $(1 + |y|^2)^k \lesssim (1 + |x - y|^2)^k + (1 + |x|^2)^k$, we have

$$\begin{aligned} \int \int |\tilde{k}(x, y)|^2 dx dy &= \int \int (1 + |x|^2)^{-4k} |K(x - y)|^2 (1 + |y|^2)^{2k} dx dy \\ &\lesssim \int (1 + |x|^2)^{-4k} dx \int |K(z)|^2 (1 + |z|^2)^{2k} dz + \int (1 + |x|^2)^{-2k} dx \int |K(z)|^2 dz, \end{aligned}$$

which is finite by the smoothness of K and the decay of its Fourier transform.

The assumptions of Subsection 4.3.1 are thus satisfied. We now rewrite the Formula (4.3.2) for the specific X and H defined above.

Lemma 36. *We keep the notations of Subsection 4.3.1 and Lemma 35. We consider the spaces X and H as in (4.3.6) and (4.3.7). Assume that B is an Hilbert-Schmidt operator in X . Then, b admits a (unique) kernel $k \in L^2_{d\mu \otimes d\mu}(\mathbb{R}^d \times \mathbb{R}^d)$, that is for all $g, g' \in X$*

$$b(g, g') = \int \int k(x, y)g(x)g(y)d\mu(x)d\mu(y). \tag{4.3.9}$$

Moreover, for all $x \in \mathbb{R}^d$, $\delta_x : g \in H \mapsto g(x)$ is a continuous linear form on H and Formula (4.3.2) specifies to

$$b.\dot{c} = \int \int k(x, y)\dot{c}(\delta_x, \delta_y)d\mu(x)d\mu(y). \tag{4.3.10}$$

Proof. We use the notations of the proof of Lemma 35. Recall that from (4.3.4), we have the decomposition

$$b = \sum_{n=0}^{+\infty} \lambda_n \ell_n \otimes \ell_n, \tag{4.3.11}$$

where ℓ_n is defined in (4.3.8) and the sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ is now squared summable. In addition, using the $L^2(\mathbb{R}^d)$ -based structure of X , we have for all $g \in X$

$$\ell_n(g) = \int h_n(x)g(x)d\mu(x). \tag{4.3.12}$$

We now prove that b admits a kernel in the sense of (4.3.9). For all $g, g' \in X$, we have from (4.3.11) and (4.3.12)

$$b(g, g') = \sum_{n=0}^{+\infty} \lambda_n \int \int h_n(x)h_n(y)g(x)g(y)d\mu(x)d\mu(y). \tag{4.3.13}$$

In addition, the function

$$k : (x, y) \mapsto \sum_{n=0}^{+\infty} \lambda_n h_n(x)h_n(y) \tag{4.3.14}$$

is well defined in $L^2_{d\mu \otimes d\mu}(\mathbb{R}^d \times \mathbb{R}^d)$. Indeed, using the fact that $(h_n)_{n \in \mathbb{N}}$ is orthonormal in X , we have for all $n \geq 0$ and $p \geq 0$

$$\int \int \left| \sum_{k=n}^{n+p} \lambda_k h_k(x)h_k(y) \right|^2 d\mu(x)d\mu(y) = \sum_{k=n}^{n+p} \lambda_k^2 \leq \sum_{k=n}^{+\infty} \lambda_k^2,$$

which vanishes as $n \uparrow +\infty$ since $(\lambda_n)_{n \in \mathbb{N}}$ is squared integrable. Therefore $k \in L^2_{d\mu \otimes d\mu}(\mathbb{R}^d \times \mathbb{R}^d)$ and (4.3.13) turns into

$$b(g, g') = \int \int k(x, y)g(x)g(y)d\mu(x)d\mu(y),$$

and thus k is the⁸ kernel of b in the sense of (4.3.9).

We now prove (4.3.10). We first reinterpret, for all $n \in \mathbb{N}$, the linear form ℓ_n given by (4.3.12) as the Bochner integral

$$\ell_n = \int \delta_x h_n(x)d\mu(x), \tag{4.3.15}$$

⁸The uniqueness is deduced by density of the family $\{(x, y) \mapsto \sum_{n=1}^N g_n(x)g_n(y) | (g_n)_n \subset L^2(\mathbb{R}^d), N \geq 1\}$ in $L^2(\mathbb{R}^d)$.

where $\delta_x : g \in H \mapsto g(x)$. This integral makes sense since, by definition⁹ of $|\cdot|_H$, $|\delta_x| \lesssim (1 + |x|^2)^k$ so that since $h_n \in X$, $\int (1 + |x|^2)^k h_n(x) d\mu(x) < +\infty$. Consequently, using Formula (4.3.2) from Lemma 35 combined with (4.3.9), we obtain

$$\begin{aligned} b.\dot{c} &\stackrel{(4.3.2)}{=} \sum_{n=0}^{+\infty} \lambda_n \dot{c}(\ell_n, \ell_n) \\ &\stackrel{(4.3.15)}{=} \sum_{n=0}^{+\infty} \lambda_n \int \int h_n(x) h_n(y) \dot{c}(\delta_x, \delta_y) d\mu(x) d\mu(y). \end{aligned} \quad (4.3.16)$$

Finally, by continuity of \dot{c} in H (see (4.1.2)) one has $|\dot{c}(\delta_x, \delta_y)| \leq |\delta_x|_H |\delta_y|_H \lesssim (1 + |x|^2)^k (1 + |y|^2)^k$ so that $(x, y) \mapsto \dot{c}(\delta_x, \delta_y)$ belongs to $L^2_{d\mu \otimes d\mu}(\mathbb{R}^d \times \mathbb{R}^d)$. To conclude, (4.3.16) simplifies to (4.3.10) by exchanging the sum and the integrals as well as the definition of the kernel k in (4.3.14). \square

We end this chapter by the following corollary, which is a direct consequence of Price's formula of Theorem 26 and the paring (4.3.10). It states the version of Price's formula that we use in Chapter 5.

Corollary 27. *We assume that the assumptions of Theorem 26 are satisfied and consider X and H as in (4.3.6) and (4.3.7). We assume in addition that d^2F is Hilbert-Schmidt in X and we denote by $\partial^2 F \in L^2_{d\mu \otimes d\mu}(\mathbb{R}^d \times \mathbb{R}^d)$ its kernel. We have the following version of Price's formula*

$$\frac{d}{dL} \langle F \rangle_L = \frac{1}{2} \int \int \langle \partial^2 F(x, y) \rangle_L \frac{dc_L}{dL}(\delta_x, \delta_y) d\mu(x) d\mu(y). \quad (4.3.17)$$

⁹We can see it via the inequality (which follows by the chain rule) $|(1 + |\cdot|^2)^k (1 - \Delta)^k g|_{L^2} \gtrsim |(1 + |\cdot|^2)^k (g, \nabla g, \dots, \nabla^{2k} g)|_{L^2}$ and the Sobolev embedding $|g(x)| \lesssim |(g, \nabla g, \dots, \nabla^{2k} g)|_{L^2(B_1(x))}$.

Chapter 5

Bias in the Representative Volume Element method: periodize the ensemble instead of its realizations

We study the Representative Volume Element (RVE) method, which is a method to approximately infer the effective behavior a_{hom} of a stationary random medium. The latter is described by a coefficient field $a(x)$ generated from a given ensemble $\langle \cdot \rangle$ and the corresponding linear elliptic operator $-\nabla \cdot a \nabla$. In line with the theory of homogenization, the method proceeds by computing $d = 3$ correctors (d denoting the space dimension). To be numerically tractable, this computation has to be done on a finite domain: the so-called “representative” volume element, i. e. a large box with, say, periodic boundary conditions. The main message of this chapter is: Periodize the ensemble instead of its realizations.

By this we mean that it is better to sample from a suitably periodized ensemble than to periodically extend the restriction of a realization $a(x)$ from the whole-space ensemble $\langle \cdot \rangle$. We make this point by investigating the bias (or systematic error), i. e. the difference between a_{hom} and the expected value of the RVE method, in terms of its scaling w. r. t. the lateral size L of the box. In case of a suitable periodization of $\langle \cdot \rangle$, we rigorously show that it is $O(L^{-d})$. In fact, we give a characterization of the leading-order error term for the strategy of periodizing $\langle \cdot \rangle$. For the strategy of periodizing the realizations, the behaviour is expected to be $O(L^{-1})$ (see the numerical experiments [103]) and formal argument will be added in the work in preparation in [48])

We carry out the rigorous analysis in the convenient setting of ensembles $\langle \cdot \rangle$ of Gaussian type, which allow for a straightforward periodization, passing via the (integrable) covariance function. This setting has also the advantage of making the Price’s theorem (see Chapter 4) and the Malliavin calculus available for optimal stochastic estimates of correctors. We actually need control of second-order correctors to capture the leading-order error term. This is due to inversion symmetry when applying the two-scale expansion to the Green’s function. As a bonus, we present a stream-lined strategy to estimate the error in a higher-order two-scale expansion of the Green’s function.

This Chapter is based on joint work in preparation with Felix Otto, Marc Josien and Qiang Xu.

The Chapter is organized as follows. In Section 5.1, we recall the assumptions of the chapter as well as our main result. In Section 5.2, we present a refinement of our main theorem and the main ideas. In Section 5.3 we present the structure of the proof as well as the auxiliary results.

Finally, in Section 5.4, we provide the proofs of the auxiliary results.

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5.1 Assumption and statement of rigorous result

We restate in this section the assumption of the chapter as well as our main result. For an introduction to this result, we refer to Section 1.2.6. We start by recalling the important quantities. Let $L \geq 1$ and $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be λ -uniformly elliptic (for some $\lambda > 0$): for all $x, \xi \in \mathbb{R}^d$

$$\lambda|\xi|^2 \leq \xi \cdot a(x)\xi \leq |\xi|^2 \tag{5.1.1}$$

Suppose that the coefficient field a is L -periodic, meaning that $a(x + Lk) = a(x)$ for all x and $k \in \mathbb{Z}^d$. Given a Cartesian coordinate direction $i = 1, \dots, d$ and denoting by e_i the unit vector in the i -th direction, we consider the corrector (up to additive constants) $\phi_i^{(1)}$ as the L -periodic solution of

$$-\nabla \cdot a(\nabla \phi_i^{(1)} + e_i) = 0, \tag{5.1.2}$$

to which we associate the corresponding homogenized matrix:

$$\bar{a}e_i := \int_{[0,L]^d} a(\nabla \phi_i^{(1)} + e_i). \tag{5.1.3}$$

We then recall the general assumptions.

Assumption 28. Let $\langle \cdot \rangle$ be a stationary and centered Gaussian ensemble of scalar¹ fields g on \mathbb{R}^d , as determined by the covariance function $c(x) := \langle g(x)g(0) \rangle$. We assume that there exists an $\alpha > 0$ such that

$$\sup_{x \in \mathbb{R}} (1 + |x|)^{d+n+2\alpha} |\nabla^n c(x)| < \infty \quad \text{for } n \in \{0, \dots, d+1\}. \quad (5.1.4)$$

We identify $\langle \cdot \rangle$ with its push forward under the map

$$g \mapsto a := (x \mapsto A(g(x))), \quad (5.1.5)$$

where $A: \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is such that the coefficient field a is λ -uniformly elliptic, see (5.1.1). We assume that

$$\sup_{g \in \mathbb{R}} |A'(g)| + |A''(g)| < \infty. \quad (5.1.6)$$

We then recall the definition of the periodize ensemble $\langle \cdot \rangle_L$.

Definition 5.1.1. For given $L < \infty$, let $\langle \cdot \rangle_L$ be the stationary and centered Gaussian ensemble of scalar fields g defined by the covariance function

$$c_L(x) := \sum_{k \in \mathbb{Z}^d} c(x + Lk). \quad (5.1.7)$$

Clearly, the covariance function c_L and thus the realizations g are L -periodic. As in Assumption 28 we identify $\langle \cdot \rangle_L$ with its push forward under (5.1.6).

The purpose of the chapter is to prove the following.

Theorem 29. Let $d > 2$ and A be symmetric. Under the Assumption 28 on $\langle \cdot \rangle$, for all L , and with the Definition 5.1.1 of $\langle \cdot \rangle_L$ we have for the expectation $\langle \bar{a} \rangle_L$ of \bar{a} defined in (5.1.3)

$$\limsup_{L \uparrow \infty} L^d |\langle \bar{a} \rangle_L - a_{\text{hom}}| < \infty.$$

More precisely, we prove the refinement version of Theorem 30.

5.2 Theorem 29: refinement and main ideas

The two ingredients for Theorem 29 are a suitable representation formula for $\langle \bar{a} \rangle_L$, see Subsection 5.2.1, and its asymptotics through stochastic homogenization, here on the level of the mixed derivatives of the Green's function, see Subsection 5.2.2. We need the second-order version of stochastic homogenization because of an inversion symmetry. We refine Theorem 29 in Subsection 5.2.3 by identifying the leading-order error term, see Theorem 30. In Subsection 5.2.4 we will argue that the leading-order error typically does not vanish, by exploring the regime of small ellipticity contrast.

¹For notational simplicity, we consider scalar Gaussian field, however, the Gaussian field g may take values in any finite dimensional linear space.

5.2.1 Representation formula

We start with an informal, but detailed, derivation of the representation formula, see (5.2.9), which might be the most conceptual piece of our work.

Let us fix two vectors ξ and ξ^* and focus on the component $\xi^* \cdot \bar{a}\xi$; we denote by ϕ the solution of (5.1.2) with e_i replaced by ξ .² By stationarity³ of $\nabla\phi$ and of $\langle \cdot \rangle_L$, we have

$$\langle \xi^* \cdot \bar{a}\xi \rangle_L = \langle F \rangle_L \quad \text{where} \quad F := (\xi^* \cdot a(\nabla\phi + \xi))(0). \quad (5.2.1)$$

Instead of directly estimating $\langle \xi^* \cdot \bar{a}\xi \rangle_L - \xi^* \cdot a_{\text{hom}}\xi$, we will estimate its derivative with respect to L , that is $\frac{d}{dL} \langle \xi^* \cdot \bar{a}\xi \rangle_L$. The reason is that by general Gaussian calculus (in form of Price's formula of Chapter 4 applied to the ensemble $\langle \cdot \rangle_L$ of (periodic) fields g that depends on a parameter L , we have for any $F = F(g)$

$$\frac{d}{dL} \langle F \rangle_L = \frac{1}{2} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \left\langle \frac{\partial^2 F}{\partial g(-x) \partial g(-y)} \right\rangle_L \frac{\partial c_L}{\partial L}(x-y), \quad (5.2.2)$$

where the two minus signs in the denominator are for later convenience.⁴ Here $\frac{\partial^2 F}{\partial g(x) \partial g(y)}$ denotes the kernel representing the second Fréchet derivative of F , seen as a bilinear form on the space of functions on \mathbb{R}^d . As a derivative w. r. t. the noise g , it can be seen as a Malliavin derivative.

We define F by (5.2.1). By the change of variables $z \rightsquigarrow x-y$, which capitalizes on the translation invariance of the covariance, and (more directly) by the stationarity of $\langle \cdot \rangle_L$ in conjunction with the stationarity of $\nabla\phi$ that leads to $\left\langle \frac{\partial^2 F}{\partial g(-x) \partial g(z-x)} \right\rangle_L = \left\langle \frac{\partial^2 [\xi^* \cdot (a(\nabla\phi^{(1)} + \xi))(x)]}{\partial g(0) \partial g(z)} \right\rangle_L$, we obtain

$$\frac{d}{dL} \langle \xi^* \cdot \bar{a}\xi \rangle_L = \frac{1}{2} \int_{\mathbb{R}^d} dz \left\langle \int_{\mathbb{R}^d} \xi^* \cdot \frac{\partial^2 [a(\nabla\phi^{(1)} + \xi)]}{\partial g(0) \partial g(z)} \right\rangle_L \frac{\partial c_L}{\partial L}(z). \quad (5.2.3)$$

With help of the corrector for the (pointwise) dual⁵ coefficient field a^* in direction ξ^* , i. e. the periodic solution $\phi^{*(1)}$ of

$$\nabla \cdot a^*(\nabla\phi^{*(1)} + \xi^*) = 0, \quad (5.2.4)$$

the inner integral can be rewritten more symmetrically as

$$\begin{aligned} & \int_{\mathbb{R}^d} \xi^* \cdot \frac{\partial^2 [a(\nabla\phi^{(1)} + \xi)]}{\partial g(0) \partial g(z)} \\ &= \int_{\mathbb{R}^d} (\nabla\phi^{*(1)} + \xi^*) \cdot \frac{\partial^2 [a(\nabla\phi^{(1)} + \xi)]}{\partial g(0) \partial g(z)} \\ &= \int_{\mathbb{R}^d} (\nabla\phi^{*(1)} + \xi^*) \cdot \left[\frac{\partial^2}{\partial g(0) \partial g(z)}, a \right] (\nabla\phi^{(1)} + \xi); \end{aligned}$$

²By uniqueness (up to additive constants), we have $\phi = \sum_i \xi_i \phi_i$.

³Stationarity is understood here in the sense of shift covariance $\nabla\phi(a(\cdot + h), x) = \nabla\phi(a, x + h)$, for any $h \in \mathbb{R}^d$.

⁴We use here formula (4.3.17) formally, where the kernel is formally given by the formal derivatives $\frac{\partial^2 F(x, y)}{\partial g(-x) \partial g(-y)}$.

⁵While we work with the assumption $A^* = A$ and thus have $a^* = a$, keeping primal and dual medium apart reveals more of the structure.

indeed, the first identity (formally) follows from applying $\frac{\partial^2}{\partial g(0)\partial g(z)}$ to (5.1.2) and then testing with $\phi^{*(1)}$, whereas the second identity follows from testing (5.2.4) with $\frac{\partial^2 \phi^{(1)}}{\partial g(0)\partial g(z)}$. Resolving the commutator $[\frac{\partial^2}{\partial g(0)\partial g(z)}, a]$ by Leibniz' rule we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \xi^* \cdot \frac{\partial^2 [a(\nabla\phi^{(1)} + \xi)]}{\partial g(0)\partial g(z)} \\ &= \int_{\mathbb{R}^d} (\nabla\phi^{*(1)} + \xi^*) \cdot \frac{\partial^2 a}{\partial g(0)\partial g(z)} (\nabla\phi^{(1)} + \xi) \\ &+ \int_{\mathbb{R}^d} (\nabla\phi^{*(1)} + \xi^*) \cdot \frac{\partial a}{\partial g(0)} \nabla \frac{\partial \phi^{(1)}}{\partial g(z)} + \text{term with } z, 0 \text{ exchanged.} \end{aligned}$$

Denoting $a' := A'(g)$ and $a'' := A''(g)$, we remark that by (5.1.5) we have $\frac{\partial a(x)}{\partial g(z)} = a'(z)\delta(x - z)$. Applying operator $\frac{\partial}{\partial g(z)}$ on (5.1.2), we thus obtain the representation

$$\frac{\partial \nabla \phi^{(1)}(x)}{\partial g(z)} = -\nabla \nabla G(x, z) a'(z) (\nabla \phi + e)(z) \quad (5.2.5)$$

in terms of the mixed derivatives of the *non-periodic* Green function⁶ $G = G(a, x, y)$ associated with the operator $-\nabla \cdot a \nabla$. Hence the above turns into

$$\begin{aligned} & \int_{\mathbb{R}^d} \xi^* \cdot \frac{\partial^2 a(\nabla\phi^{(1)} + \xi)}{\partial g(0)\partial g(z)} \\ &= \delta(z) \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a''(\nabla\phi^{(1)} + \xi) \rangle(0) \\ &- \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a'(0) \cdot \nabla \nabla G(0, z) (a'(\nabla\phi^{(1)} + \xi))(z) \rangle \\ &- \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a'(z) \cdot \nabla \nabla G(z, 0) (a'(\nabla\phi^{(1)} + \xi))(0) \rangle. \end{aligned}$$

Applying $\langle \cdot \rangle_L$ we obtain by stationarity

$$\begin{aligned} & \left\langle \int_{\mathbb{R}^d} \xi^* \cdot \frac{\partial^2 a(\nabla\phi^{(1)} + \xi)}{\partial g(0)\partial g(z)} \right\rangle_L \\ &= \delta(z) \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a''(\nabla\phi^{(1)} + \xi) \rangle_L \\ &- \langle \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a'(0) \cdot \nabla \nabla G(0, z) (a'(\nabla\phi^{(1)} + \xi))(z) \rangle_L \rangle \\ &- \langle \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a'(0) \cdot \nabla \nabla G(0, -z) (a'(\nabla\phi^{(1)} + \xi))(-z) \rangle_L \rangle. \end{aligned} \quad (5.2.6)$$

Inserting this into (5.2.3), and noting that since $\frac{\partial c_L}{\partial L}$, as derivative of a covariance function is even, the two last terms have the same contribution, we obtain

$$\begin{aligned} \frac{d}{dL} \langle \xi^* \cdot \bar{a} \xi \rangle_L &= - \int_{\mathbb{R}^d} dz \langle \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a'(0) \rangle_L \\ &\quad \cdot \nabla \nabla G(0, z) (a'(\nabla\phi^{(1)} + \xi))(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\ &\quad + \frac{1}{2} \langle \langle (\nabla\phi^{*(1)} + \xi^*) \cdot a''(\nabla\phi^{(1)} + \xi) \rangle_L \frac{\partial c_L}{\partial L}(0) \rangle. \end{aligned} \quad (5.2.7)$$

⁶Since we are only interested in the mixed gradient of the Green function, the dimension $d = 2$ poses no problems here.

We now insert (5.1.7) in form of

$$\frac{\partial c_L}{\partial L}(z) \stackrel{(5.1.7)}{=} \sum_{k \in \mathbb{Z}^d} k \cdot \nabla c(z + Lk). \quad (5.2.8)$$

This relation highlights that the z -integral in (5.2.7) is not absolutely convergent for $|z| \uparrow \infty$, not even borderline: While $\nabla \nabla G(0, z)$ decays as $|z|^{-d}$, a glance at (5.2.8) reveals that $\frac{\partial c_L}{\partial L}(z)$ grows as $|z|$. Part of the rigorous work is devoted to emulate this formal derivation of (5.2.7) by replacing the operator $-\nabla \cdot a \nabla$ by $\frac{1}{T} - \nabla \cdot a \nabla$, see Proposition 11.

In order to access the cancellations, we will perform a re-summation. Assuming for simplicity for this exposition that $\langle \cdot \rangle$ has unit range of dependence, so that c is supported in the unit ball, we have that $c_L(z = 0)$ does not depend on $L \geq 2$. Hence the second r. h. s. term in (5.2.7) does not contribute. By L -periodicity of the correctors, (5.2.7) can be re-summed to

$$\begin{aligned} \frac{d}{dL} \langle \xi^* \cdot \bar{a} \xi \rangle_L &= \int_{\mathbb{R}^d} dz \langle ((\nabla \phi^{*(1)} + \xi^*) \cdot a') (0) \\ &\cdot \left(\sum_{k \in \mathbb{Z}^d} k_n \nabla \nabla G(0, z + Lk) \right) (a'(\nabla \phi^{(1)} + \xi))(z) \rangle_L \partial_n c(z), \end{aligned} \quad (5.2.9)$$

where from now on we use Einstein's convention of summation over repeated spatial indices, here $n \in \{1, \dots, d\}$. Formula (5.2.9) is our final representation. Clearly, the sum over k is still not absolutely convergent. However, as we shall see in the next subsection, it converges after homogenization.

5.2.2 Approximation by second-order homogenization

In this subsection, we turn to the asymptotics of the representation (5.2.9) for $L \uparrow \infty$. In particular, we shall argue why first-order homogenization is not sufficient and give an efficient introduction into second-order correctors.

As there is no contribution from $k = 0$, and since by our finite range assumption (for the sake of this discussion), z is constrained to the unit ball, the argument $z + Lk$ of the Green's function satisfies $|z + Lk| \gtrsim L$. Hence we may appeal to homogenization to replace $G(x, y)$ by $\bar{G}(x - y)$, where \bar{G} denotes the fundamental solution of $-\nabla \cdot \bar{a} \nabla$. This appears like periodic homogenization as long as L is fixed, but in fact amounts to stochastic homogenization since we are interested in $L \uparrow \infty$. Since we are interested in its gradient, we need to replace G by the two-scale expansion of \bar{G} . (See below for more details on the two-scale expansion.) Since we are interested in the mixed gradient, the two-scale expansion acts on both variables. Hence in a first Ansatz, we approximate

$$\nabla \nabla G(0, x) \approx -\partial_{ij} \bar{G}(x) (e_i + \nabla \phi_i^{*(1)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(x), \quad (5.2.10)$$

where $\phi_j^{*(1)}$ denotes the solution of (5.2.4) with ξ^* replaced by e_j . To leading order, this yields by the periodicity of correctors

$$\begin{aligned} \nabla \nabla G(0, z + Lk) \\ \approx -\partial_{ij} \bar{G}(Lk) (e_i + \nabla \phi_i^{*(1)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(z). \end{aligned} \quad (5.2.11)$$

Applying $\sum_{k \in \mathbb{Z}^d} k_l$ to the r. h. s., we see that it vanishes by parity w. r. t. inversion $k \rightsquigarrow -k$. This is an indication that the *first order* two-scale expansion (5.2.10) is not sufficient and that we have to go to a second-order expansion, which we shall describe now.

We need to replace the first-order version of the two-scale expansion of \bar{G} by its second-order version. We recall the two-scale expansion in its first-order version: Given an \bar{a} -harmonic function \bar{u} , one considers $u = (1 + \phi_i^{(1)}\partial_i)\bar{u}$ as a good approximation to an a -harmonic function. Indeed, it follows from (5.1.2) that when \bar{u} is a first-order polynomial, u is exactly a -harmonic. In fact, this is a characterization of the first-order correctors $\phi_i^{(1)}$. Second-order correctors $\phi_{ij}^{(2)}$ can be characterized in a similar way: For every \bar{a} -harmonic second-order polynomial \bar{u} , we postulate that $u = (1 + \phi_i^{(1)}\partial_i + \phi_{ij}^{(2)}\partial_{ij})\bar{u}$ is a -harmonic⁷. It is clear from this characterization that $\phi_{ij}^{(2)}$ depends on the choice of the additive constant in $\phi_i^{(1)}$, which we now fix through

$$\int_{[0,L]^d} \phi_i^{(1)} = 0. \quad (5.2.12)$$

Since for our second-order polynomial \bar{u} we have

$$\nabla u = \partial_i \bar{u}(e_i + \nabla \phi_i^{(1)}) + \partial_{ij} \bar{u}(\phi_i^{(1)} e_j + \nabla \phi_{ij}^{(2)}), \quad (5.2.13)$$

so that $\nabla \cdot a \nabla u = 0$ turns into $\nabla \partial_i \bar{u} \cdot a(e_i + \nabla \phi_i^{(1)}) + \partial_{ij} \bar{u} \nabla \cdot a(\phi_i^{(1)} e_j + \nabla \phi_{ij}^{(2)}) = 0$, and using that $\nabla \cdot \bar{a} \nabla \bar{u} = 0$, we obtain the following standard PDE characterization of $\phi_{ij}^{(2)}$:

$$-\nabla \cdot a(\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j) = e_j \cdot (a(\nabla \phi_i^{(1)} + e_i) - \bar{a} e_i). \quad (5.2.14)$$

Note that (5.2.14) is uniquely solvable (up to additive constants) for a periodic $\phi_{ij}^{(2)}$ because the r. h. s. of (5.2.14) has vanishing average in view of (5.1.3). The definition of $\phi_{ij}^{*(2)}$ for the dual medium a^* is analogous.

In view of (5.2.13), we thus replace (5.2.10) by

$$\begin{aligned} \nabla \nabla G(0, x) &\approx -\partial_{ij} \bar{G}(x)(e_i + \nabla \phi_i^{(1)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(x) \\ &\quad -\partial_{ijm} \bar{G}(x)(\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(x) \\ &\quad +\partial_{ijm} \bar{G}(x)(e_i + \nabla \phi_i^{(1)})(0) \otimes (\phi_j^{*(1)} e_m + \nabla \phi_{jm}^{*(2)})(x). \end{aligned} \quad (5.2.15)$$

It is here that the assumption of symmetry of A is convenient: Otherwise, the instance of \bar{G} in the first r. h. s. term of (5.2.15) would have to be replaced by $\bar{G} + \bar{G}^{(2)}$ where $\bar{G}^{(2)}$ is the $(1-d)$ -homogeneous solution of $\nabla \cdot (\bar{a} \nabla \bar{G}^{(2)} + \bar{a}_m^{(2)} \nabla \partial_m \bar{G}) = 0$, where $\bar{a}^{(2)}$ is the second-order homogenized coefficient, see (5.3.20). Since $\bar{G}^{(2)}$, as a dipole, is odd w. r. t. point inversion, its contribution does not vanish as for \bar{G} , c. f. (5.2.11). For the analogue of (5.2.11) we now turn to the first-order Taylor expansion (recall $k \neq 0$)

$$\begin{aligned} \nabla \nabla G(0, z + Lk) &\approx -(\partial_{ij} \bar{G}(Lk) + z_m \partial_{ijm} \bar{G}(Lk))(e_i + \nabla \phi_i^{(1)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(z) \\ &\quad -\partial_{ijm} \bar{G}(Lk)(\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(z) \\ &\quad +\partial_{ijm} \bar{G}(Lk)(e_i + \nabla \phi_i^{(1)})(0) \otimes (\phi_j^{*(1)} e_m + \nabla \phi_{jm}^{*(2)})(z). \end{aligned}$$

⁷This does not characterize all components $\phi_{ij}^{(2)}$ separately but only the trace-free and symmetric part of this tensor, where the trace is defined w. r. t. \bar{a} . Since we apply the two scale expansion only to \bar{a} -harmonic functions like \bar{G} , this is not an issue.

By the inversion symmetry of \bar{G} and its homogeneity, this implies

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} k_n \nabla \nabla G(0, z + Lk) &\approx L^{-d-1} \sum_{k \in \mathbb{Z}^d} k_n \partial_{ijm} \bar{G}(k) \\ &\times \left(-z_m (e_i + \nabla \phi_i^{(1)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(z) \right. \\ &\quad - (\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)})(0) \otimes (e_j + \nabla \phi_j^{*(1)})(z) \\ &\quad \left. + (e_i + \nabla \phi_i^{(1)})(0) \otimes (\phi_j^{*(1)} e_m + \nabla \phi_{jm}^{*(2)})(z) \right). \end{aligned} \quad (5.2.16)$$

In view of $\bar{a} \approx a_{\text{hom}}$ we finally replace \bar{G} , which is still random, by the deterministic G_{hom} that may be pulled out of $\langle \cdot \rangle_L$ when inserting (5.2.16) into (5.2.9). Hence we obtain the approximation

$$\frac{d}{dL} \langle \xi^* \cdot \bar{a} \xi \rangle_L \approx L^{-d-1} \Gamma_{\text{hom}, ijmn} \int_{\mathbb{R}^d} dz \xi^* \cdot \mathcal{Q}_{Lijm}(z) \xi \partial_n c(z), \quad (5.2.17)$$

where the five-tensor field \mathcal{Q}_L is defined through a combination of three covariances of quadratic expressions in correctors, see Definition 5.2.1, and where the four-tensor Γ_{hom} is formally given by the (borderline) divergent lattice sum $\sum_{k \in \mathbb{Z}^d} k_n \partial_{ijm} G_{T, \text{hom}}(k)$, which in line with the remark at the end of Subsection 5.2.1 we replace by

$$\Gamma_{\text{hom}} = \lim_{T \uparrow \infty} \Gamma_{\text{hom}, T} \quad \text{where} \quad \Gamma_{\text{hom}, T ijmn} := \sum_{k \in \mathbb{Z}^d} k_n \partial_{ijm} G_{T, \text{hom}}(k), \quad (5.2.18)$$

with $G_{T, \text{hom}}$ denoting the fundamental solution of $\frac{1}{T} - \nabla \cdot a_{\text{hom}} \nabla$.

5.2.3 Refinement of rigorous result

We start with the full definition of the tensor field \mathcal{Q}_L appearing in (5.2.17).

Definition 5.2.1. Recall the definitions (5.1.2) & (5.2.14) of first and second-order correctors $\phi_i^{(1)}$ and $\phi_{ij}^{(2)}$, and their versions $\phi_i^{*(1)}$ and $\phi_{ij}^{*(2)}$ with a replaced by a^* . For given vectors ξ and ξ^* we continue to write $\phi^{(1)} = \xi_i \phi_i$ and $\phi^{*(1)} = \xi_i \phi_i^{*(1)}$. Consider the random tensor fields

$$\begin{aligned} \xi^* \cdot \mathcal{Q}_{ij}^{(1)}(z) \xi &:= ((\xi^* + \nabla \phi^{*(1)}) \cdot a'(e_i + \nabla \phi_i^{(1)}))(0) \\ &\quad \times ((e_j + \nabla \phi_j^{*(1)}) \cdot a'(\xi + \nabla \phi^{(1)}))(z), \end{aligned} \quad (5.2.19)$$

$$\begin{aligned} \xi^* \cdot \mathcal{Q}_{ijm}^{(2)}(z) \xi &:= -((\xi^* + \nabla \phi^{*(1)}) \cdot a'(\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)}))(0) \\ &\quad \times ((e_j + \nabla \phi_j^{*(1)}) \cdot a'(\xi + \nabla \phi^{(1)}))(z) \\ &\quad + ((\xi^* + \nabla \phi^{*(1)}) \cdot a'(e_i + \nabla \phi_i^{(1)}))(0) \\ &\quad \times ((\phi_j^{*(1)} e_m + \nabla \phi_{jm}^{*(2)}) \cdot a'(\xi + \nabla \phi^{(1)}))(z). \end{aligned} \quad (5.2.20)$$

For any L we consider the ensemble $\langle \cdot \rangle_L$ from Definition 5.1.1 and define

$$\mathcal{Q}_{Lijm}(z) := -z_m \langle \mathcal{Q}_{ij}^{(1)}(z) \rangle_L + \langle \mathcal{Q}_{ijm}^{(2)}(z) \rangle_L. \quad (5.2.21)$$

Here comes the more precise version of Theorem 29, which consists in making (5.2.17) rigorous:

Theorem 30. *Let $d > 2$ and A be symmetric. Suppose $\langle \cdot \rangle$ satisfies Assumption 28 and let a_{hom} denote the homogenized coefficient. For all L , let $\langle \cdot \rangle_L$ be as in Definition 5.1.1, \bar{a} be defined by (5.1.3), $\Gamma_{\text{hom},T}$ defined by (5.2.18), and \mathcal{Q}_L be as in Definition 5.2.1. Then the following limits exist:*

$$\Gamma_{\text{hom},ijmn} := \lim_{T \uparrow \infty} \Gamma_{\text{hom},Tijmn}, \tag{5.2.22}$$

$$\mathcal{Q}_{ijm}(z) := \lim_{L \uparrow \infty} \mathcal{Q}_{Lijm}(z) \quad \text{pointwise, uniformly bounded in } z, \tag{5.2.23}$$

and the latter only depends on $\langle \cdot \rangle$ (and not the lattice). Moreover, we have

$$\lim_{L \uparrow \infty} L^{d+1} \frac{d\langle \bar{a} \rangle_L}{dL} = \Gamma_{\text{hom},ijmn} \int_{\mathbb{R}^d} dz \mathcal{Q}_{ijm}(z) \partial_n c(z). \tag{5.2.24}$$

With the tools of this paper, the asymptotics of $\frac{d\langle \bar{a} \rangle_L}{dL}$ could be characterized up to order $O(L^{-d-\frac{d}{2}})$. Let us comment on the representation of the leading error term arising from (5.2.24), namely

$$d \lim_{L \uparrow 0} L^d (a_{\text{hom}} - \langle \bar{a} \rangle_L) = \Gamma_{\text{hom},ijmn} \int_{\mathbb{R}^d} dz \mathcal{Q}_{ijm}(z) \partial_n c(z). \tag{5.2.25}$$

This representation separates a first factor Γ_{hom} , which only depends on the type of the periodic lattice (here cubic) and the homogenized coefficient a_{hom} , from a second factor that only depends on the whole-space ensemble $\langle \cdot \rangle$, via its covariance function c and covariances involving its first- and second-order correctors.

5.2.4 Small contrast regime and non-degeneracy

In this subsection, we (formally) identify the leading order (5.2.26) of the r. h. s. of (5.2.25) in the small-contrast regime. We then argue that this leading-order error term typically does not vanish, even in the high-symmetry case of an isotropic ensemble.

We start with the derivation of (5.2.26): To leading order in a small ellipticity contrast $1 - \lambda$, the quantity $\nabla \phi_i^{(1)}$ may be neglected w. r. t. e_i ; likewise $\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)}$ may be neglected w. r. t. e_i . Hence to leading order, (5.2.21) reduces to

$$\xi^* \cdot \mathcal{Q}_{ijm}(z) \xi \approx -z_m \langle \xi^* \cdot a'(0) e_i e_j \cdot a'(z) \xi \rangle.$$

Restricting to the case of scalar A for convenience, the expression further simplifies to

$$\mathcal{Q}_{ijm}(z) \approx -z_m \langle a'(0) a'(z) \rangle e_i \otimes e_j.$$

Restricting ourselves w. l. o. g. to ensembles $\langle \cdot \rangle$ with $c(0) = \langle g^2(0) \rangle = \langle g^2(z) \rangle = 1$, we see that $\langle a'(0) a'(z) \rangle$ depends on the Gaussian ensemble $\langle \cdot \rangle$ only through $c(z)$. We thus write $\langle a'(0) a'(z) \rangle = \mathcal{A}'(c(z))$ for some function \mathcal{A} , so that by the chain rule

$$\mathcal{Q}_{ijm}(z) \partial_n c(z) \approx -z_m \partial_n \mathcal{A}(c(z)) e_i \otimes e_j.$$

Normalizing \mathcal{A} such that $\mathcal{A}(0) = 0$, we obtain by integration by parts

$$\int_{\mathbb{R}^d} dz \mathcal{Q}_{ijm}(z) \partial_n c(z) \approx \delta_{mn} \int_{\mathbb{R}^d} dz \mathcal{A}(c(z)) e_i \otimes e_j.$$

Hence the r. h. s. of (5.2.25) is given by

$$\left(\lim_{T \uparrow \infty} \sum_{k \in \mathbb{Z}^d} k_m \partial_m \nabla^2 G_{T, \text{hom}}(k) \right) \int_{\mathbb{R}^d} dz \mathcal{A}(c(z)) \quad (5.2.26)$$

to leading order in the contrast.

It remains to argue that the two factors in (5.2.26) typically do not vanish. The second factor in (5.2.26) does not vanish in the typical case of $A' > 0$ and $c \geq 0$. Indeed, by definition of \mathcal{A} , we then have $\mathcal{A}' > 0$ and thus $\mathcal{A}(c) > 0$ for $c > 0$, so that $\int_{\mathbb{R}^d} dz \mathcal{A}(c(z)) > 0$ because of $c(0) = 1$.

For the first factor in (5.2.26), we restrict ourselves to an isotropic ensemble, namely the case where c is radially symmetric, in addition to A being scalar. In line with this, we show that the trace of the first factor in (5.2.26) does not vanish:

$$\lim_{T \uparrow \infty} \sum_{k \in \mathbb{Z}^d} k_m \partial_m \Delta G_{T, \text{hom}}(k) \neq 0. \quad (5.2.27)$$

For our isotropic ensemble, the contravariant two-form a is invariant in law under orthogonal transformations, and so is a_{hom} , which thus is a multiple of the identity, so that Δ is a multiple of $\nabla \cdot a_{\text{hom}} \nabla$. Hence by definition of $G_{T, \text{hom}}$, (5.2.27) follows from

$$\lim_{T \uparrow \infty} \frac{1}{T} \sum_{k \in \mathbb{Z}^d} k_m \partial_m G_{T, \text{hom}}(k) \neq 0. \quad (5.2.28)$$

By scaling, we have $G_{T, \text{hom}}(k) = \frac{1}{\sqrt{T}^{d-2}} G_{1, \text{hom}}(\frac{k}{\sqrt{T}})$. Hence we see that the sum in (5.2.28) can be interpreted as a Riemann sum that in the limit $T \uparrow \infty$ converges to the integral

$$\int_{\mathbb{R}^d} dk k_m \partial_m G_{1, \text{hom}}(k) = -d \int_{\mathbb{R}^d} dk G_{1, \text{hom}}(k),$$

which indeed does not vanish.

5.3 Structure of the proof of Theorem 30

In this section, we formulate the main intermediate results that lead to Theorem 30: In Subsection 5.3.1, we introduce the massive approximation in order to rigorously derive the analogue of the representation formula (5.2.28) from Subsection 5.2.1, see Proposition 11. In Subsection 5.3.2 we argue, following Subsection 5.2.1, that a re-summation allows for removing the massive approximation in the representation formula, see Proposition 12. It relies on second-order homogenization, as introduced in Subsection 5.2.2. In Subsection 5.3.3 we sketch how to pass from the representation given by Proposition 12 to the asymptotics stated in Theorem 30. This essentially relies on corrector estimates and the estimate of the homogenization error, see Subsections 5.3.4 and 5.3.5. In Subsection 5.3.4, we formulate the uniform stochastic estimates on first and second-order correctors needed to capture the asymptotics $L \uparrow \infty$, see Proposition 13. In Subsection 5.3.5, we formulate the stochastic second-order estimate of the homogenization error, applied to the Green's function, see Proposition 14.

5.3.1 Massive approximation

As became apparent in Subsection 5.2.1, there is divergence in the sum over the periodic cells, see (5.2.7). We avoid it by replacing the operator $-\nabla \cdot a \nabla$ by $\frac{1}{T} - \nabla \cdot a \nabla$ where $T < \infty$ will eventually tend to infinity. This has the desired effect that the corresponding Green's function $G_T(a, x, y)$ and its derivatives now decay exponentially in $\frac{|y-x|}{\sqrt{T}}$, more precisely:

$$|G_T(a, x, y)| + |x - y| |G_T(a, x, y)| + |x - y|^2 |\nabla \nabla G_T(a, x, y)| \leq C(a) |x - y|^{-d+2} e^{-\frac{|x-y|}{\sqrt{T}}}, \quad (5.3.1)$$

where the constant C depends at most polynomially on the Hölder constant $[a]_{\alpha', B_1(x)}$ (next to depending on d and λ). In view of Assumption 28 (and Definition 5.1.1), which for $\alpha' < \alpha$ ensures Hölder continuous realizations g and thus a , any stochastic moment of $[a]_{\alpha', B_1(x)}$ is finite, that is for all $p < +\infty$

$$\langle [a]_{\alpha', B_1(x)}^p \rangle_L \lesssim 1, \quad (5.3.2)$$

and thus C in (5.3.1) is well controlled. We refer to Appendix 5.5.1 for a proof. The language of “massive” approximation arises from field theory where such a zero-order term is often introduced to suppress an infrared divergence, like here. Assimilating m^2 to the inverse of a time scale T however makes the connection to stochastic processes, since $\frac{1}{T} - \nabla \cdot a \nabla$ is the generator of a diffusion-desorption process where T is the time scale of desorption, and ultimately to parabolic intuition. As a collateral of the massive approximation, we have to replace the definitions (5.1.2) and (5.1.3) by

$$\frac{1}{T} \phi_{T_i}^{(1)} - \nabla \cdot a (\nabla \phi_{T_i}^{(1)} + e_i) = 0, \quad \bar{a}_T e_i := \int_{[0, L]^d} a (\nabla \phi_{T_i}^{(1)} + e_i); \quad (5.3.3)$$

with analogous definitions for the transposed medium a^* . From Schauder's theory and (5.3.2), $(\phi_T^{(1)}, \nabla \phi_T^{(1)})$ is Hölder continuous, more precisely: for any $p < +\infty$ and $\alpha' < \alpha$

$$\langle \|\phi_T^{(1)}, \nabla \phi_T^{(1)}\|_{C^{0, \alpha'}([0, L]^d)}^p \rangle_L \lesssim_{p, L} 1, \quad (5.3.4)$$

where $\|\cdot\|_{C^{0, \alpha'}([0, L]^d)} = \|\cdot\|_{L^\infty([0, L]^d)} + [\cdot]_{\alpha', [0, L]^d}$. We now can rigorously establish the massive version of formula (5.2.7):

Proposition 11. *It holds*

$$\begin{aligned} \frac{d}{dL} \langle \xi^* \cdot \bar{a}_T \xi \rangle_L &= \int_{\mathbb{R}^d} dz \langle (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot a'(0) \\ &\quad \times \nabla \nabla G_T(0, z) a'(z) (\nabla \phi_T^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\ &\quad + \frac{1}{2} \langle (\nabla \phi_T^{*(1)} + \xi^*) \cdot a''(\nabla \phi_T^{(1)} + \xi) \rangle_L \frac{\partial c_L}{\partial L}(0), \end{aligned} \quad (5.3.5)$$

where we recall that $\phi_T^{(1)} = \sum_i \xi_i \phi_{T_i}^{(1)}$.

The z -integral on the r. h. s. of (5.3.5) converges absolutely for $|z| \uparrow \infty$ since the exponential decay of $\nabla \nabla G_T(0, z)$ dominates the linear growth of $\frac{\partial c_L}{\partial L}(z)$, cf. (5.2.8). The singularity at $z = 0$ is to be interpreted by duality, using that the other factors are locally smooth in z .

5.3.2 Re-summation

Following Subsection 5.2.2, we now appeal to second-order homogenization, which allows for a re-summation. As a by-product of the re-summation, we may pass to the limit $T \uparrow \infty$ in (5.3.5). The difficulty with passing to the limit $T \uparrow \infty$ lies in the $\{|z| \geq L\}$ -part of the integral in (5.3.5). We thus fix a smooth cut-off function η for $B_{\frac{1}{2}}$ in B_1 , rescale according to

$$\eta_L(z) = \eta\left(\frac{z}{L}\right),$$

and to split the z -integral into the benign near-field part $\int_{\mathbb{R}^d} dz \eta_L(z)$ and the delicate far-field part $\int_{\mathbb{R}^d} dz (1 - \eta_L)(z)$. On the far-field part, we appeal to the two-scale expansion (5.2.15). Hence we have to monitor the homogenization error

$$\begin{aligned} \mathcal{E}(x, y) &:= \nabla \nabla G(x, y) + \partial_{ij} \bar{G}(x - y) (e_i + \nabla \phi_i^{(1)})(x) \otimes (e_j + \nabla \phi_j^{*(1)})(y) \\ &+ \partial_{ijm} \bar{G}(x - y) (\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)})(x) \otimes (e_j + \nabla \phi_j^{*(1)})(y) \\ &- \partial_{ijm} \bar{G}(x - y) (e_i + \nabla \phi_i^{(1)})(x) \otimes (\phi_j^{*(1)} e_m + \nabla \phi_{jm}^{*(2)})(y), \end{aligned} \quad (5.3.6)$$

where as before \bar{G} denotes the fundamental solution for the constant-coefficient operator $-\nabla \cdot \bar{a} \nabla$.

The translation invariance of \bar{G} together with the periodicity of $\phi^{(1)}$ and $\phi^{(2)}$ allows for a re-summation. As in Subsection 5.2.2, we feed in a zeroth- and first-order Taylor expansion of \bar{G} . This gives rise to the analogue of (5.2.18), namely

$$\bar{\Gamma}_{ijmn} = \lim_{T \uparrow \infty} \bar{\Gamma}_{Tijmn} \quad \text{where} \quad \bar{\Gamma}_{Tijmn} := \sum_{k \in \mathbb{Z}^d} k_n \partial_{ijm} \bar{G}_T(k), \quad (5.3.7)$$

where \bar{G}_T denotes the fundamental solution of $\frac{1}{T} - \nabla \cdot \bar{a} \nabla$. The existence of this limit follows by the same arguments given for (5.2.27). The Taylor expansion generates the additional error terms

$$\epsilon_{Lijn}^{(1)}(z) := \sum_{k \in \mathbb{Z}^d} k_n \left(((1 - \eta_L) \partial_{ij} \bar{G})(z + Lk) - \partial_{ij} \bar{G}(Lk) - z_m \partial_{ijm} \bar{G}(Lk) \right), \quad (5.3.8)$$

$$\epsilon_{Lijmn}^{(2)}(z) := \sum_{k \in \mathbb{Z}^d} k_n \left(((1 - \eta_L) \partial_{ijm} \bar{G})(z + Lk) - \partial_{ijm} \bar{G}(Lk) \right). \quad (5.3.9)$$

Thanks to this re-summation, the subtlety of the $T \uparrow \infty$ is limited to the not absolutely convergent sum in (5.3.7). The sums in (5.3.8) and (5.3.9) are absolutely convergent since both summands decay as $|k|^{-(d+1)}$ for $|k| \gg \frac{|z|}{L}$, see (5.3.13) and (5.3.14) for a more quantitative discussion. Equipped with these definitions, we are now able to express the limit $T \uparrow \infty$ of (5.3.5):

Proposition 12. *Let $\bar{\Gamma}$ be as in (5.3.7), $\epsilon^{(1)}$ and $\epsilon^{(2)}$ as in (5.3.8) & (5.3.9), and \mathcal{E} as in (5.3.6).*

Let $Q^{(1)}$ and $Q^{(2)}$ be defined as in (5.2.19) and (5.2.20). Then we have

$$\begin{aligned}
& \frac{d}{dL} \langle \xi^* \cdot \bar{a} \xi \rangle_L \\
&= L^{-(d+1)} \int_{\mathbb{R}^d} dz \langle \bar{\Gamma}_{ijmn}(\xi^* \cdot Q_{ijm}^{(2)}(z)\xi - z_m \xi^* \cdot Q_{ij}^{(1)}(z)\xi) \rangle_L \partial_n c(z) \\
&+ \int_{\mathbb{R}^d} dz \langle \epsilon_{Lijmn}^{(2)}(z)\xi^* \cdot Q_{ijm}^{(2)}(z)\xi + \epsilon_{Lijn}^{(1)}(z)\xi^* \cdot Q_{ij}^{(1)}(z)\xi \rangle_L \partial_n c(z) \\
&+ \int_{\mathbb{R}^d} dz (1 - \eta_L)(z) \langle (\nabla \phi^{*(1)} + \xi^*)(0) \cdot a'(0) \\
&\quad \times \mathcal{E}(0, z) a'(z) (\nabla \phi^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\
&+ \int_{\mathbb{R}^d} dz \eta_L(z) \langle (\nabla \phi^{*(1)} + \xi^*)(0) \cdot a'(0) \\
&\quad \times \nabla \nabla G(0, z) a'(z) (\nabla \phi^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\
&+ \frac{1}{2} \langle (\nabla \phi^{*(1)} + \xi^*) \cdot a''(\nabla \phi^{(1)} + \xi) \rangle_L \frac{\partial c_L}{\partial L}(0). \tag{5.3.10}
\end{aligned}$$

Periodic homogenization theory suffices to establish Proposition 12 and in particular to ensure that all five expressions on the r. h. s. of (5.3.10) are well-defined, including the third one. Indeed, it helps to momentarily think of having rescaled length by the fixed L . This puts us into the context of a 1-periodic coefficient field a , which in addition is Hölder continuous. By and rescaling back, we obtain

$$|\mathcal{E}(x, y)| \leq C(a) L^2 |y - x|^{-d-2},$$

where the constant C depends at most polynomially on the non-dimensionalized Hölder constant $L^\alpha [a]_{\alpha'}$ (next to depending on d and λ). In particular we have, from (5.3.2)

$$\sup_{x, y} |y - x|^{d+2} \langle |\mathcal{E}(x, y)| \rangle_L < \infty. \tag{5.3.11}$$

This estimate yields the absolute convergence of the third term on the r. h. s. of (5.3.10), since the decay (5.3.11) over-compensates the linear growth of $\frac{\partial c_L}{\partial L}$. Finally, periodic homogenization and (5.3.2) ensure the following convergences of the massive quantities: for any $p < +\infty$ and any x, y , the following convergences holds in $L^p_{(\cdot)}$

$$\begin{aligned}
& (\phi_T^{(1)}, \nabla \phi_T^{(1)}) \xrightarrow{T \uparrow +\infty} (\phi^{(1)}, \nabla \phi^{(1)}) \text{ and } \nabla \nabla G_T(x, y) \xrightarrow{T \uparrow +\infty} \nabla \nabla G(x, y) \\
& \text{as well as } (\nabla^3 \bar{G}_T(x), \nabla^2 \bar{G}_T(x)) \xrightarrow{T \uparrow +\infty} (\nabla^3 \bar{G}(x), \nabla^2 \bar{G}(x)). \tag{5.3.12}
\end{aligned}$$

We refer to Appendix 5.5.2 for a proof of (5.3.12).

5.3.3 From representation to asymptotics

In order to pass from the representation in Proposition 12 to the asymptotics in Theorem 30, we have to show that the first r. h. s. term of (5.3.10), up to the factor L^{d+1} , converges to the r. h. s. term of (5.2.24), and that the remaining terms are $o(L^{-(d+1)})$. Note that from (5.1.4), the fifth term of (5.3.10) is immediately of order $L^{-d-1-2\alpha}$. Below, we discuss the first fourth terms.

We start with the second term and estimate $\epsilon^{(1)}$, see (5.3.8): In the range $|k| \geq \frac{|z|}{L}$, we obtain from Taylor applied to $(1 - \eta_L)\partial_{ij}\bar{G}$ that the summand is estimated by $|z|^2(L|k|)^{-(d+2)} \leq |z|(L|k|)^{-(d+1)}$. Hence the contribution to the sum from this range is dominated by $\min\{|z|^2L^{-(d+2)}, |z|L^{-(d+1)}\}$. In the other range $|k| \leq \frac{|z|}{L}$, the contribution from the middle term vanishes by parity, the contribution from the last term is estimated by $|z|L^{-(d+1)}$ (by the same argument that shows that the limit (5.3.7) exists), and the first term in the summand is estimated by $|k|L^{-d}$ so that its contribution to the sum is also dominated by $|z|L^{-(d+1)}$. Since this second range is only present for $|z| \geq L$, we obtain in conclusion

$$|\epsilon_{Lijn}^{(1)}(z)| \lesssim \min\{|z|^2L^{-(d+2)}, |z|L^{-(d+1)}\}. \quad (5.3.13)$$

For the estimate of $\epsilon^{(2)}$, see (5.3.9), we proceed in a similar way and obtain the stronger estimate

$$|\epsilon_{Lijmn}^{(2)}(z)| \lesssim |z|L^{-(d+2)}. \quad (5.3.14)$$

We combine (5.3.13) and (5.3.14) with the corrector estimates of Proposition 13 i), which by definitions (5.3.22) and (5.2.20) yield

$$\langle |Q_{ij}^{(1)}(z)| \rangle_L + \langle |Q_{ijm}^{(2)}(z)| \rangle_L \lesssim 1. \quad (5.3.15)$$

We now see that Assumption 28 is just what we need: By (5.1.4) with $n = 1$ (and w. l. o. g. $\alpha \leq \frac{1}{2}$) we obtain for the second term in (5.3.10)

$$\left| \int_{\mathbb{R}^d} dz \langle \epsilon_{Lijmn}^{(2)}(z) Q_{ijm}^{(2)}(z) + \epsilon_{Lijn}^{(1)}(z) Q_{ij}^{(1)}(z) \rangle_L \partial_n c(z) \right| \lesssim L^{-(d+1+2\alpha)},$$

which as desired is $o(L^{-(d+1)})$. Here and in the sequel \lesssim means \leq up to a multiplicative constant that only depends on d , λ , and the constants implicit in (5.1.4) and (5.1.6) of Assumption 28.

We now turn to the third term on the r. h. s. of (5.3.10). It follows from Proposition 13 i) and Proposition 14, together with (5.1.6) in Assumption 28, that

$$\begin{aligned} & \left| \langle (\nabla\phi^{*(1)} + \xi^*)(0) \right. \\ & \left. \cdot a'(0)\mathcal{E}(0, z)a'(z)(\nabla\phi^{(1)} + \xi)(z) \rangle_L \right| \lesssim |z|^{-(d+\frac{3}{2})}. \end{aligned}$$

Inserting (5.2.8) we obtain the following estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} dz (1 - \eta_L(z)) \langle (\nabla\phi^{*(1)} + \xi^*)(0) \right. \\ & \left. \cdot a'(0)\mathcal{E}(0, z)a'(z)(\nabla\phi^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \right| \\ & \lesssim \sum_k |k| \int_{\mathbb{R}^d} dz (1 - \eta_L(z)) |z|^{-(d+\frac{3}{2})} |\nabla c(z + Lk)|. \end{aligned}$$

Since by integrability of $|\nabla c|$, which is ensured by (5.1.4), the z -integral is estimated by $(L|k|)^{-(d+\frac{3}{2})}$, the sum converges and is estimated by $L^{-(d+\frac{3}{2})}$, which as desired is $o(L^{-(d+1)})$.

We finally turn to the first term in (5.3.10). On the one hand, from the qualitative result (which can be seen as a consequence of Corollary 43)

$$\lim_{L \uparrow \infty} \langle |\bar{a} - a_{\text{hom}}| \rangle_L = 0$$

we obtain, by the explicit dependance of \bar{G}_T and thus $\bar{\Gamma}$ on \bar{a} ,

$$\lim_{L \uparrow \infty} \langle |\bar{\Gamma} - \Gamma_{\text{hom}}| \rangle_L = 0.$$

Since on the other hand, $\bar{\Gamma}$ is uniformly bounded (recall that \bar{a} is confined to the set (5.1.1)), and by (5.3.15), it is enough to show that the limits

$$\lim_{L \uparrow \infty} \langle Q_{ij}^{(1)}(z) \rangle_L \quad \text{and} \quad \lim_{L \uparrow \infty} \langle Q_{ijm}^{(2)}(z) \rangle_L$$

not only exist but can be characterized in terms of $\langle \cdot \rangle$. This is the content of Corollary 31 iii) in the following subsection.

We finally turn to the fourth term on the r. h. s of (5.3.10). We split the term into the near-zero contribution $\int_{[0,1]^d} dz$ and the far-zero contribution $\int_{\mathbb{R}^d \setminus [0,1]^d} dz \eta_L$. On the near-zero contribution, we decompose as

$$\begin{aligned} & \int_{[0,1]^d} dz \langle (\nabla \phi^{*(1)} + \xi^*)(0) \cdot a'(0) \nabla \nabla G(0, z) a'(0) (\nabla \phi^{(1)} + \xi)(0) \rangle_L \frac{\partial c_L}{\partial L}(0) \\ = & \int_{[0,1]^d} dz \langle (\nabla \phi^{*(1)} + \xi^*)(0) \cdot a'(0) \nabla \nabla G(0, z) a'(0) (\nabla \phi^{(1)} + \xi)(0) \rangle_L \frac{\partial c_L}{\partial L}(0) \\ & + \int_{[0,1]^d} dz \langle (\nabla \phi^{*(1)} + \xi^*)(0) \cdot a'(0) \nabla \nabla G(0, z) a'(z) (\nabla \phi^{(1)} + \xi)(z) \frac{\partial c_L}{\partial L}(z) - a'(0) (\nabla \phi^{(1)} + \xi)(0) \frac{\partial c_L}{\partial L}(0) \rangle_L. \end{aligned} \tag{5.3.16}$$

On the one hand, the first r. h. s term of (5.3.16), thanks to our assumption (5.1.4), is of order $L^{-d-1-2\alpha}$. On the other hand, by Hölder continuity of a , $\phi^{(1)}$ and $\frac{\partial c_L}{\partial L}$ as well as the decay of $\nabla \nabla G(0, z)$, the integrand of the second r. h. s term of (5.3.16) is dominated by $|z|^{-d+\gamma} L^{-d-1-2\alpha}$ (for some $\gamma > 0$ depending on α) and thus this term is of order $L^{-d-1-2\alpha}$. On the far-zero contribution, from the decay of $\nabla \nabla G(0, z)$ and (5.1.4), the integrand is dominated by $|z|^{-d} L^{-d-1-2\alpha}$ and thus this term is also of order $L^{-d-1-2\alpha}$.

5.3.4 Stochastic corrector estimates up to second order

As just discussed, the proof of Theorem 30 will rely on estimates of not only the first-order corrector $\phi_i^{(1)}$, but also its second-order version $\phi_{ij}^{(2)}$, see part i) of Proposition 13. Since the period L of the ensemble $\langle \cdot \rangle_L$ tends to infinity, these have to be of stochastic nature. This is the reason for the restriction to $d > 2$ (which is just a more telling way of saying $d \geq 3$ since it is rather $d = 2$ that is borderline): For $d = 2$, the first-order corrector in the whole-space ensemble $\langle \cdot \rangle$ is not stationary, so that one loses (pointwise) control even of a centered second-order corrector. Only for $d > 2$ one has the middle item in (5.3.23) (see for instance Theorem 15 of Chapter 3). For the (limiting) whole-space ensemble $\langle \cdot \rangle$, such higher-order corrector estimates have first been established in [94] (however sub-optimal in odd dimensions) and [27, Theorem 3.1] (see [56, Proposition 2.2] for a treatment of any order). These works, like ours, rely on Malliavin calculus and a suitable spectral gap estimate, as is available under Assumption 28 (see Proposition 9 of Chapter 3). Unfortunately, we cannot simply quote [27] since we need the estimate for the periodized ensembles $\langle \cdot \rangle_L$ (uniform for $L \uparrow \infty$, of course).

For Proposition 14, we need to also estimate the flux correctors, both first and second-order, which we shall recall now. (We also refer to [56, Section 2] for a compact introduction into all

higher-order correctors). It follows from (5.1.2) and (5.1.3) that $a(\nabla\phi_i^{(1)} + e_i) - \bar{a}e_i$ is divergence-free, periodic, and of zero average. Hence it allows for, in the language of $d = 2$, a periodic stream function, or in the language of $d = 3$, a periodic vector potential. For general d , it can be represented in terms of a periodic tensor field σ_i with

$$a(\nabla\phi_i^{(1)} + e_i) = \bar{a}e_i + \nabla \cdot \sigma_i^{(1)} \quad \text{and} \quad \sigma_{imn}^{(1)} = -\sigma_{inm}^{(1)}, \quad (5.3.17)$$

where for a (skew symmetric) tensor field σ , we write $(\nabla \cdot \sigma)_m := \partial_n \sigma_{mn}$, as an instance of an exterior derivative. Observe that (5.3.17) does not determine $\sigma_i^{(1)}$. Indeed, $\sigma_i^{(1)}$, which can be interpreted as an alternating $(d-2)$ -form, is only determined up to a $(d-3)$ -form. For estimates like in Proposition 13, we choose a suitable (and simple) gauge, that is

$$-\Delta\sigma_{imn}^{(1)} = \partial_j(e_n \cdot a(e_i + \nabla\phi_i^{(1)})) - \partial_n(e_m \cdot a(e_i + \nabla\phi_i^{(1)})). \quad (5.3.18)$$

Note also that (5.2.14) can be reformulated in divergence form

$$\nabla \cdot a(\nabla\phi_{ij}^{(2)} + \phi_i^{(1)}e_j) = \nabla \cdot \sigma_i^{(1)}e_j. \quad (5.3.19)$$

This shows that there is a second-order analogue of (5.3.17): For every coordinate direction i , let the matrix $\bar{a}^{(2)}$ be defined through

$$\bar{a}_i^{(2)}e_j := \int_{[0,L]^d} a(\nabla\phi_{ij}^{(2)} + \phi_i^{(1)}e_j) \quad (5.3.20)$$

for any $j = 1, \dots, d$, and the periodic tensor field $\sigma_{ij}^{(2)}$ through

$$\begin{aligned} a(\nabla\phi_{ij}^{(2)} + \phi_i^{(1)}e_j) &= \bar{a}_i^{(2)}e_j + \sigma_i^{(1)}e_j + \nabla \cdot \sigma_{ij}^{(2)} \\ \text{and} \quad \sigma_{ijmn}^{(2)} &= -\sigma_{ijnm}^{(2)}. \end{aligned} \quad (5.3.21)$$

The merits of the flux correctors $\sigma_i^{(1)}$ and $\sigma_{ij}^{(2)}$ will become clear in Subsection 5.3.5. In fact, in that context it will be convenient to have yet one more object, namely the periodic solution ω_i of

$$-\Delta\omega_i = \phi_i^{(1)}. \quad (5.3.22)$$

Proposition 13. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. Let $p < \infty$ be arbitrary.*

i) We have

$$\langle |\nabla\phi_i^{(1)}|^p \rangle_L^{\frac{1}{p}} + \langle |\phi_i^{(1)}|^p \rangle_L^{\frac{1}{p}} + \langle |\nabla\phi_{ij}^{(2)}|^p \rangle_L^{\frac{1}{p}} \lesssim_p 1. \quad (5.3.23)$$

ii) The random tensor fields $\sigma_i^{(1)}$ and $\sigma_{ij}^{(2)}$ can be constructed such that

$$\langle |\sigma_i^{(1)}|^p \rangle_L^{\frac{1}{p}} + \langle |\nabla\sigma_{ij}^{(2)}|^p \rangle_L^{\frac{1}{p}} \lesssim_p 1.$$

iii) We have for any periodic deterministic vector field g

$$\langle \left| \int_{[0,L]^d} g \cdot \nabla\phi_i^{(1)} \right|^p \rangle_L^{\frac{1}{p}} \lesssim_p \left(\int_{[0,L]^d} |g|^2 \right)^{\frac{1}{2}}. \quad (5.3.24)$$

iv) We have for all z

$$\langle |\phi_{ij}^{(2)}(z) - \phi_{ij}^{(2)}(0)|^p \rangle_L^{\frac{1}{p}} + \langle |\sigma_{ij}^{(2)}(z) - \sigma_{ij}^{(2)}(0)|^p \rangle_L^{\frac{1}{p}} \lesssim_p \mu_d^{(2)}(|z|), \quad (5.3.25)$$

where

$$\mu_d^{(2)}(r) := \left\{ \begin{array}{ll} r & \text{for } 2 > r \\ r^{\frac{1}{2}} & \text{for } 2 \leq r \text{ and } d = 3 \\ \ln^{\frac{1}{2}} r & \text{for } 2 \leq r \text{ and } d = 4 \\ 1 & \text{for } 2 \leq r \text{ and } d > 4 \end{array} \right\}. \quad (5.3.26)$$

v) We have for all z

$$\langle |\nabla \omega_i(z) - \nabla \omega_i(0)|^p \rangle_L^{\frac{1}{p}} \lesssim_p \mu_d^{(2)}(|z|). \quad (5.3.27)$$

Here \lesssim means \leq up to a multiplicative constant that only depends on d , λ , and the constants implicit in (5.1.4) and (5.1.6) of Assumption 28. The subscript p indicates an additional dependence.

While part i) of Proposition 13 is explicitly used in Section 5.3.3, the usage of the other parts more indirect: Part ii) is used in Corollary 31, part iii) is used to estimate the second-order homogenization error in Lemma 39; and part iv) is used to apply this to the Green's function, see Proposition 14.

The proof of Proposition 13 essentially follows the strategy of [101, Section 4] and extends it from first-order to second-order correctors; the passage from $\langle \cdot \rangle$ to $\langle \cdot \rangle_L$ is only a minor change. In this paper, we will only establish the most important ingredient for Proposition 13, namely the characterization of stochastic cancellations of the gradient of the correctors in Lemma 37. While (5.3.28) reproduces [101, Proposition 4.1], the new element is its second-order counterpart (5.3.29). The first item of (5.3.25) is a consequence of (5.3.29), adapting [101, Proposition 4.1, Part 1, Step 5]. The second item of (5.3.25) follows from the analogue of (5.3.29) on the level of the second-order flux (5.3.21), adapting [101, Proposition 4.1, Part 2].

Lemma 37. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. For any deterministic periodic vector field f and any $p < \infty$ we have*

$$\langle \left| \int_{[0,L]^d} f \cdot \nabla \phi_i^{(1)} \right|^p \rangle_L^{\frac{1}{p}} \lesssim_p \left(\int_{[0,L]^d} |f|^2 \right)^{\frac{1}{2}}, \quad (5.3.28)$$

$$\langle \left| \int_{[0,L]^d} f \cdot \nabla \phi_{ij}^{(2)} \right|^p \rangle_L^{\frac{1}{p}} \lesssim_p \left(\int_{[0,L]^d} |x_L^2| |f|^2 \right)^{\frac{1}{2}}, \quad (5.3.29)$$

where we denote by $|\cdot|_L$ the periodic distance on \mathbb{T}_L . We note that (5.3.29) also holds with the weighted L^2 -norm $(\int_{[0,L]^d} |x_L^2| |f|^2)^{\frac{1}{2}}$ replaced by the L^q -norm of the same scaling, namely $(\int_{[0,L]^d} |f|^q)^{\frac{1}{q}}$ with $q = \frac{2d}{d+2}$. However when passing from (5.3.29) to (5.3.25), we essentially choose $f = \nabla \bar{G}(\cdot - z) - \nabla \bar{G}$, and in the critical dimension $d = 4$, we thus would have $|f|^q = O(|x|^{-4})$ for $1 \ll |x| \ll |z|$ and thus would obtain a power $\frac{3}{4}$ on the logarithm $\ln |z|$ instead of the optimal power $\frac{1}{2}$.

In establishing (5.3.29), we use the same approach as [101, Proposition 4.1] for (5.3.28), namely we identify and estimate the Malliavin derivative of the l. h. s. and then appeal to the spectral gap estimate. However, while for the first-order result (5.3.28), a buckling is required, it is not necessary for its second-order counterpart (5.3.29). One can avoid it by appealing to the quenched Calderon-Zygmund estimate, see [56] and [101, Proposition 7.1 ii)], albeit in the weighted form of Lemma 38:

Lemma 38. *Let $d > 2$ and let $\langle \cdot \rangle_L$ be an ensemble of λ -uniformly elliptic coefficient fields that are L -periodic. Let the random periodic fields f and u be related by*

$$\nabla \cdot (a \nabla u + f) = 0.$$

For all $p \geq 1$ and $\gamma \in (-d, d(p-1))$, we have for $\mu : x \in \mathbb{R}^d \mapsto |x|_L^\gamma$

$$\left(\int_{[0,L]^d} \mu \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}} \right)^{\frac{1}{p}} \lesssim_p \left(\int_{[0,L]^d} \mu \langle |f|^{2p} \rangle^{\frac{1}{p}} \right)^{\frac{1}{p}}, \quad (5.3.30)$$

where we recall that $|x|_L := \inf_{k \in \mathbb{Z}^d} |x + kL|$.

An inspection of the proof of [101, Proposition 7.1 ii)] shows that the argument extends to the case with a weight in the corresponding Muckenhoupt class. Indeed, the only essential new ingredient is that this weighted annealed estimate holds for the constant coefficient operator, i. e. the analogue of [101, Lemma 7.4]. This in turn follows from [113, Theorem 7.1]. Alternatively, one can derive the weighted estimate from the unweighted one and the dualized Lipschitz estimate Lemma 41, following the strategy of [80, Corollary 5]. Precisions will be added in the future version [48].

The limit $L \uparrow \infty$ for the first r. h. s. term in (5.3.10) relies on the following purely qualitative consequence of Proposition 13.

Corollary 31. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1.*

i) For $i = 1, \dots, d$ there exists a unique stationary random field $\phi_i^{(1)}$ with $\langle (\phi_i^{(1)})^2 + |\nabla \phi_i^{(1)}|^2 \rangle < \infty$ and $\langle \phi_i^{(1)} \rangle = 0$ such that

$$\nabla \cdot a(\nabla \phi_i^{(1)} + e_i) = 0 \quad a. s..$$

ii) For $i, j = 1, \dots, d$ there exists a unique stationary random field $\nabla \phi_{ij}^{(2)}$ with $\langle |\nabla \phi_{ij}^{(2)}|^2 \rangle < \infty$ and $\langle \nabla \phi_{ij}^{(2)} \rangle = 0$ such that

$$-\nabla \cdot a(\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j) = e_j \cdot (a(\nabla \phi_i^{(1)} + e_i) - \bar{a} e_i) \quad a. s.. \quad (5.3.31)$$

iii) We have $\langle |\phi_i^{(1)}|^p + |\nabla \phi_i^{(1)}|^p + |\nabla \phi_{ij}^{(2)}|^p \rangle < \infty$ for all $p < \infty$ and for all z

$$\lim_{L \uparrow \infty} \langle Q_{ij}^{(1)}(z) \rangle_L = \langle Q_{ij}^{(1)}(z) \rangle \quad \text{and} \quad \lim_{L \uparrow \infty} \langle Q_{ijm}^{(2)}(z) \rangle_L = \langle Q_{ijm}^{(2)}(z) \rangle, \quad (5.3.32)$$

where also the r. h. s. integrands are defined by the formulas (5.3.22) and (5.2.20).

The important element of part i) of Corollary 31 is the stationarity of $\phi_i^{(1)}$ itself, not just of $\nabla \phi_i^{(1)}$. Such a result was first established in [82, Proposition 2.1] in the case of a discrete medium, see [91, Proposition 1] for the first result for a continuum medium. Part ii) is an easy consequence of the divergence-form reformulation (5.3.19) of (5.3.31) and the fact that $\sigma_i^{(1)}$, see (5.3.17), has the same properties as $\phi_i^{(1)}$, see part ii) of Proposition 14. Part iii) is new and relies on a soft argument based on a suitable coupling of $\langle \cdot \rangle_L$ with $\langle \cdot \rangle$, next to the uniform bounds of Proposition 13.

5.3.5 Estimate of homogenization error to second order, application to the Green function

A second main role of the corrector estimates of Proposition 13, in particular the estimate of the flux correctors, is to provide an estimate of the homogenization error. On our second-order level, this connection relies on identity (5.3.35) involving the two-scale expansion (5.3.34), which we recall now. Suppose that u and \bar{u} are related via

$$\nabla \cdot a \nabla u = \nabla \cdot \bar{a} \nabla \bar{u}$$

and that $\bar{u}^{(2)}$ is related to \bar{u} via

$$\nabla \cdot (\bar{a} \nabla \bar{u}^{(2)} + a_i^{(2)} \nabla \partial_i \bar{u}) = 0. \quad (5.3.33)$$

Consider the error in the second-order two-scale expansion

$$w := u - (1 + \phi_i^{(1)} \partial_i + \phi_{ij}^{(2)} \partial_{ij})(\bar{u} + \bar{u}^{(2)}). \quad (5.3.34)$$

Then $\sigma_{ij}^{(2)}$ allows to write the residuum in divergence form:

$$-\nabla \cdot a \nabla w = \nabla \cdot ((\phi_{ij}^{(2)} a - \sigma_{ij}^{(2)}) \nabla \partial_{ij} \bar{u} + \bar{a}_i^{(2)} \nabla \partial_i \bar{u}^{(2)}). \quad (5.3.35)$$

Now the advantage of A and thus a being symmetric becomes apparent: It implies that the symmetric part of the three-tensor with entries $\bar{a}_{imn}^{(2)}$ vanishes (see e. g. [56, Lemma 2.4]). Since (5.3.33) may be rewritten as $-\nabla \cdot \bar{a} \nabla \bar{u}^{(2)} = a_{imn}^{(2)} \partial_{imn} \bar{u}$, we may assume $\bar{u}^{(2)} = 0$ under our symmetry assumption. Hence (5.3.34) simplifies to

$$w := u - (1 + \phi_i^{(1)} \partial_i + (\phi_{ij}^{(2)} - \phi_{ij}^{(2)}(0)) \partial_{ij}) \bar{u} \quad (5.3.36)$$

and (5.3.35) may be rewritten as

$$-\nabla \cdot a \nabla w = \nabla \cdot ((\phi_{ij}^{(2)} - \phi_{ij}^{(2)}(0)) a - (\sigma_{ij}^{(2)} - \sigma_{ij}^{(2)}(0))) \nabla \partial_{ij} \bar{u}. \quad (5.3.37)$$

We are allowed to pass to the centered versions of the second order (flux) corrector, by which we mean that $(\phi_{ij}^{(2)}, \sigma_{ij}^{(2)})$ is replaced by $(\phi_{ij}^{(2)} - \phi_{ij}^{(2)}(0), \sigma_{ij}^{(2)} - \sigma_{ij}^{(2)}(0))$, which we do with (5.3.25) in mind, since a change by an additive constant does not affect anything stated so far, and in particular not formula (5.3.21), on which (5.3.35) solely relies.

Lemma 39. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28 with symmetric A ; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. Given a deterministic and smooth vector field g supported in $B_R(y)$ with $y \in \mathbb{R}^d$ and some $R < \infty$, let u and \bar{u} be the decaying solutions of*

$$\nabla \cdot a \nabla u = \nabla \cdot g = \nabla \cdot \bar{a} \nabla \bar{u}. \quad (5.3.38)$$

Then w defined in (5.3.36) satisfies for all $p < \infty$

$$\langle |\nabla w(0)|^p \rangle_L^{\frac{1}{p}} \lesssim_p \mu_3^{(2)}(R) \sup |\nabla^2 g|. \quad (5.3.39)$$

Here \lesssim_p has the same meaning as in Proposition 13.

This pointwise estimate (5.3.39) relies on a decomposition of the r. h. s. of (5.3.37) into pieces supported on dyadic annuli. For each piece, we first apply the energy estimate, into which we feed (5.3.25) and the energy estimate for $\nabla^2 \bar{u}$, see (5.3.38), and then apply Lemma 41. For simplicity, we have taken the worst case in (5.3.26), namely $d = 3$; the finer estimates would come with an additional logarithm.

The main goal of this subsection is to estimate the homogenization error on the level of the Green's function, see Proposition 14. This type of homogenization result with singular r. h. s. has been worked out on the level of the first-order approximation in [28, Corollary 3] and extended to second-order in [29, Theorem 1], where these estimates are derived from estimates on $(\phi_i^{(1)}, \sigma_i^{(1)})$ and $(\phi_{ij}^{(2)}, \sigma_{ij}^{(2)})$ of the type of Proposition 13, however in a pathwise way, see [29, Proposition 1]. While equipped with Proposition 13, we could post-process [29, Theorem 1] to obtain Proposition 14, we take a different, and shorter, route in this paper. Note that [29, Theorem 1] is not formulated in terms of the Green's function G , but in terms of decaying a -harmonic functions in exterior domains. Recovering a statement on the Green's function would require [28, Lemma 4], which we restate as Lemma 40 below for the convenience of the reader.

In this paper, we use Lemma 40, or rather Corollary 32, in a more substantial way than it is used in [28, Corollary 3]. Here comes an outline of the argument for Proposition 14: We apply Lemma 39 with the origin replaced by a general point x_0 . Writing $u(x) = \int_{\mathbb{R}^d} dy g(y) \cdot \nabla_y G(x, y)$ and $\bar{u}(x) = \int_{\mathbb{R}^d} dy g(y) \cdot \nabla_y \bar{G}(x, y)$, this provides control of

$$\int_{\mathbb{R}^d} dy (\nabla \cdot g)(y) (\nabla_x G(x_0, y) - \partial_i \bar{G}(x_0 - y)(e_i + \nabla \phi_i^{(1)}(x_0)) - \partial_{ij} \bar{G}(x_0 - y)(\phi_i^{(1)}(x_0)e_j + \nabla \phi_{ij}^{(2)}(x_0))),$$

in terms of $\mu_3^{(2)}(R) \sup |\nabla^2 g|$ with $2R$ the diameter of $\text{supp} g$; here we used the centering of $\phi_{ij}^{(2)}$ in x_0 . We now fix a point y_0 with $|y_0 - x_0| \geq 4$ and replace both instances of $\bar{G}(x_0 - y)$ by what we obtain from applying the two-scale expansion operator in the y -variable

$$1 + \phi_m^{*(1)}(y) \frac{\partial}{\partial y_m} + (\phi_{mn}^{*(2)}(y) - \phi_{mn}^{*(2)}(y_0)) \frac{\partial^2}{\partial y_m \partial y_n}.$$

Provided g is supported in $B_R(y_0)$ with $R := \frac{1}{2}|y_0 - x_0| \geq 2$, this preserves the estimate: While for three out of the four extra terms, this follows directly from parts i) through iii) of Proposition 13, we need part iv) and an integration by parts in y for the contribution coming from $\phi_i^{*(1)}(y) \partial_{im} \bar{G}(x_0 - y)(e_i + \nabla \phi_i^{(1)}(x_0))$. Keeping only first and second-order terms and recalling the definition (5.3.6), this yields

$$\langle |\int_{\mathbb{R}^d} dy g(y) \cdot \mathcal{E}_m(x_0, y)|^p \rangle_L^{\frac{1}{p}} \lesssim \mu_3^{(2)}(R) \sup |\nabla^2 g| \tag{5.3.40}$$

for any g supported in $B_R(y_0)$. By construction, up to third-order terms, $\mathbb{R}^d - \{x_0\} \ni y \mapsto \mathcal{E}_m(x_0, y)$ is a linear combination of a gradient of an a^* -harmonic function, namely $\frac{\partial G}{\partial x_m}(x_0, y)$, and gradients of two-scale expansions of \bar{a}^* -harmonic functions, namely of $\bar{u}(y) = \partial_i \bar{G}(x_0 - y)(\delta_{im} + \partial_m \phi_i^{(1)}(x_0))$ and of $\bar{u}(y) = \partial_{ij} \bar{G}(x_0 - y)(\phi_i^{(1)}(x_0)\delta_{jm} + \partial_m \phi_{ij}^{(2)}(x_0))$. Hence we may appeal once more to (5.3.37), this time in the y -variable and thus for the dual medium, and with the origin replaced by y_0 . We decompose the r. h. s. of (5.3.37) into a far-field supported on $\mathbb{R}^d - B_R(y_0)$ and a near-field

supported on dyadic annuli centered at y_0 of radii $R, \frac{R}{2}, \frac{R}{4}, \dots$. For the near-field contributions, we appeal to the energy estimate followed by Lemma 41. For the far-field contribution, we use (5.3.40) (in conjunction with the estimate of the near-field part) by appealing to Corollary 32, both with the origin replaced by y_0 . It is thus Corollary 32 that converts the weak control (5.3.40) into pointwise control (5.3.41).

Proposition 14. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28 with symmetric A ; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. Then we have for \mathcal{E} defined in (5.3.6)*

$$|y - x|^{d+2} \langle |\mathcal{E}(x, y)|^p \rangle_L^{\frac{1}{p}} \lesssim \mu_3^{(2)} (|y - x|) \tag{5.3.41}$$

provided $|y - x| < \infty$ and for all $p < \infty$. Here \lesssim_p has the same meaning as in Proposition 13.

Here comes the crucial Lemma that converts weak into strong control.

Lemma 40. *Let $\langle \cdot \rangle$ be an ensemble of λ -uniformly elliptic coefficient fields⁸. Let the random function u be a -harmonic in the ball B_R of radius R . Then we have for all $p < \infty$*

$$\left\langle \left(\int_{B_{\frac{R}{2}}} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim_{d,\lambda} \sup_{g \in C_0^\infty(B_R)} \frac{\langle | \int_{B_R} g \cdot \nabla u|^p \rangle^{\frac{1}{p}}}{R^2 \sup |\nabla^2 g|}. \tag{5.3.42}$$

Lemma 40 amounts to an inner regularity estimate for a -harmonic functions u , in terms of the norms $L_{\langle \cdot \rangle}^p L_x^2$ and $W_x^{-2,1} L_{\langle \cdot \rangle}^p$ on the level of the gradient ∇u . As [28, Lemma 4], estimate (5.3.42) is a consequence of an inner regularity estimate, uniform in a , with respect to norms L_x^2 and H_x^{-n} (the case $W_x^{-2,1}$ of (5.3.42) is obtained for $n > \frac{d}{2} + 2$). However, it strengthens [28, Lemma 4] by restricting the r. h. s. functional to smooth functions g with compact support, *i.e.*, functions that vanish to appropriate order at the boundary.

Nevertheless, it requires only a minor modification of the proof. It is obtained as a combination of two ingredients. First, by the Caccioppoli estimate and by an L_x^2 interpolation estimate, we may estimate the l. h. s. of (5.3.42) by the L_x^2 norm of w for $\Delta^{2n} w = u$. Second, appealing to the fact that the Dirichlet operator Δ^{2n} has finite trace for $2n > d$, we may obtain (5.3.42). This second step differs from [28, Lemma 4], where the Fourier decomposition was explicitly used to solve $\Delta^{2n} w = u$ (thus, losing the property of compact support). This argument also shows that the second derivative on g , that we need here for our second-order homogenization, could be replaced by any order (properly non-dimensionalized).

We use Lemma 40 only in combination with a second inner regularity estimate, Lemma 41, which amounts to a Lipschitz estimate. Lipschitz estimates are central in the large-scale regularity theory in homogenization as initiated by Avellaneda and Lin in the periodic context, and as introduced by Armstrong and Smart [10] to the random context.

Lemma 41. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. Let the random function u be a -harmonic in the ball B_R of radius R . Then we have for all $p', p < \infty$*

$$\langle |\nabla u(0)|^{p'} \rangle_L^{\frac{1}{p'}} \lesssim_{p',p} \left\langle \left(\int_{B_R} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle_L^{\frac{1}{p}} \quad \text{provided } p' < p. \tag{5.3.43}$$

Here $\lesssim_{p,p'}$ has the same meaning as in Proposition 13.

⁸We will apply it to $\langle \cdot \rangle_L$

Lemma 41 is an easy consequence of the pathwise Lipschitz estimate [80, Theorem 1]. More precisely, we refer to [80, (16)], which takes the form of

$$\left(\int_{B_1} |\nabla u|^2\right)^{\frac{1}{2}} \lesssim_{d,\lambda} r_*^{\frac{d}{2}} \left(\int_{B_R} |\nabla u|^2\right)^{\frac{1}{2}},$$

with the random radius r_* defined in [80, (12)]. It easily follows from the estimates on $(\phi_i^{(1)}, \sigma_i^{(1)})$ in Proposition 13 that $\langle r_*^p \rangle_L^{\frac{1}{p}} \lesssim_p 1$ for all $p < \infty$. On the other hand, by standard Schauder theory in $C^{\alpha'}$ we have

$$|\nabla u(0)| \leq C(a) \left(\int_{B_1} |\nabla u|^2\right)^{\frac{1}{2}},$$

where C depends at most polynomially on the local Hölder norm $[a]_{\alpha', B_1}$. Since $\alpha' < \alpha$, we obtain from Assumption 28 that $\langle [a]_{\alpha', B_1}^p \rangle_L^{\frac{1}{p}} < \infty$ for all $p < \infty$. Note (5.3.43) follows from combining both estimates; note that the loss in stochastic integrability is unavoidable, since it compensates the fact that both r_* and $[a]_{\alpha'}$ are not uniformly bounded.

As mentioned, we use Lemma 40 only in its form combined with Lemma 41

Corollary 32. *Let $d > 2$ and $\langle \cdot \rangle$ satisfy Assumptions 28; let $\langle \cdot \rangle_L$ be defined as in Definition 5.1.1. Let the random function u be a -harmonic in the ball B_R of radius R . Then we have for all $p', p < \infty$*

$$\langle |\nabla u(0)|^{p'} \rangle_L^{\frac{1}{p'}} \lesssim_{p', p} \sup_{g \in C_0^\infty(B_R)} \frac{\langle |\int_{B_R} g \cdot \nabla u|^p \rangle_L^{\frac{1}{p}}}{R^2 \sup |\nabla^2 g|} \quad \text{provided } p' < p. \quad (5.3.44)$$

Here $\lesssim_{p', p}$ has the same meaning as in Proposition 13.

Corollary 32 amounts to an inner regularity estimate for a -harmonic functions u , in terms of the norms $L_x^\infty L_{\langle \cdot \rangle}^p$ and $W_x^{-2,1} L_{\langle \cdot \rangle}^p$ on the level of the gradient ∇u . We call this estimate an annealed⁹ estimate, since now on both sides of (5.3.44), the probabilistic norm is inside.

5.4 Proofs

5.4.1 Proof of Proposition 11: Derivative in L

We prove (5.3.5) by an approximation argument on the flux $a(\nabla \phi_T + e)$ to which we apply the version of Price's formula (4.3.17) of Corollary 27 of Chapter 4. We finally let the approximation parameter goes to $+\infty$.

Fix $z \in \mathbb{R}^d$ and define

$$\hat{F}(\cdot, z) : a \in \hat{X} \mapsto \xi^* \cdot (a(\nabla \phi_T^{(1)} + \xi))(z), \quad (5.4.1)$$

where we recall that the massive corrector $\phi_T^{(1)}$ is defined in (5.3.3) and where

$$\hat{X} := \left\{ a \left| [a]_{\gamma, \alpha} := \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{-\gamma} \left(\sup_{y: |y-x| \leq 1} \frac{|a(y) - a(x)|}{|y-x|^\alpha} + |a(x)| \right) < +\infty \right. \right\},$$

⁹The language of quenched and annealed arises from metallurgy estimate and made its to model with disorder in statistical mechanics

for some fixed $\gamma > 0$. The derivative (5.3.5) is reinterpreted by $\frac{d}{dL} \langle \hat{F}(0) \rangle_L$ (where the dependence on a is made implicit). Let $(\eta_n)_{n \in \mathbb{N}}$ be a mollifying sequence and define the approximation of the coefficient field a by, for all $n \in \mathbb{N}$

$$a_n : g \in X \mapsto \eta_n \star A(\eta_n \star g). \quad (5.4.2)$$

Now define the approximation $F_n(\cdot, z)$ of $\hat{F}(\cdot, z)$ via

$$F_n(\cdot, z) := \hat{F}(a_n(\cdot), z). \quad (5.4.3)$$

We now show that we can apply the version of Price's formula of Corollary 27 to F_n , for all $n \in \mathbb{N}$. To do so, we check that F_n is twice Fréchet differentiable in the space¹⁰ X defined in (4.3.6), that $g \in X \mapsto d^2 F_n(g)$ is bounded (as a function taking value into the set of continuous bilinear form on X) and that $d^2 F(g)$ is Hilbert-Schmidt for all $g \in X$. We argue in the two following substeps.

Twice Fréchet differentiability and boundedness of $g \mapsto d^2 F(g)$. We prove that F_n is twice Fréchet differentiable in X and that $g \in X \mapsto d^2 F_n(g)$ is bounded. From (5.4.1) and (5.4.3), the second variation of F_n is given by (we use the notations $a'_n := \eta_n \star A'(\eta_n \star g)$ and $a''_n = \eta_n \star A''(\eta_n \star g)$) and skip the dependence in n for the massive corrector $\phi_T^{(1)}$ depending on a_n): for all $g, \delta g, \delta g' \in X$

$$\begin{aligned} d^2 F_n(g, z)(\delta g, \delta g') &= \xi^* \cdot (a''_n(\nabla \phi_T^{(1)} + \xi))(z)(\eta_n \star \delta g(z))(\eta_n \star \delta g'(z)) \\ &\quad + \xi^* \cdot (a'_n \nabla \delta \phi_T^{(1)}(\delta g', \cdot))(z)(\eta_n \star \delta g(z)) + \xi^* \cdot (a'_n \nabla \delta \phi_T^{(1)}(\delta g, \cdot))(z)(\eta_n \star \delta g'(z)) \\ &\quad + \xi^* \cdot (a_n \nabla \delta^2 \phi_T^{(1)}(\delta g, \delta g', \cdot))(z), \end{aligned} \quad (5.4.4)$$

where

$$\frac{1}{T} \delta \phi_T^{(1)}(\delta g) - \nabla \cdot a_n \nabla \delta \phi_T^{(1)}(\delta g) = \nabla \cdot a'_n(\nabla \phi_T^{(1)} + \xi)(\eta_n \star \delta g), \quad (5.4.5)$$

and

$$\begin{aligned} \frac{1}{T} \delta^2 \phi_T(\delta g, \delta g') - \nabla \cdot a_n \nabla \delta^2 \phi_T^{(1)}(\delta g, \delta g') &= \nabla \cdot a'_n \nabla \delta \phi_T^{(1)}(\delta g)(\eta_n \star \delta g') + \nabla \cdot a'_n(\nabla \delta \phi_T^{(1)}(\delta g'))(\eta_n \star \delta g) \\ &\quad + \nabla \cdot a''_n(\nabla \phi_T^{(1)} + \xi)(\eta_n \star \delta g)(\eta_n \star \delta g'). \end{aligned} \quad (5.4.6)$$

We now verify that for all $g \in X$, $d^2 F_n(g)$ is a continuous bilinear form on X . Fix $g \in X$. First of all, thanks to the outer convolution in (5.4.2), one has

$$[a_n]_{0, \alpha} \lesssim_{\eta_n} 1.$$

In addition, using local Schauder's theory to the equation (5.3.3) and the exponentially localized energy estimates (see Proposition 15) one has

$$[\nabla \phi_T^{(1)}]_{0, \alpha} \lesssim_{\alpha, [a]_{0, \alpha}, T} 1. \quad (5.4.7)$$

Then, using the exponentially localized energy estimate (see Proposition 15) followed by the local Schauder's theory to the equation (5.4.5), we obtain

$$|\nabla \delta \phi_T^{(1)}(z, \delta g)|_{\gamma, z} \leq [\nabla \delta \phi_T^{(1)}(\cdot, \delta g)]_{\gamma, \alpha} \lesssim_{T, \alpha, \gamma, [a_n]_{0, \alpha}} [\eta_n \star \delta g]_{\gamma, \alpha} \lesssim_{T, \alpha, \gamma, n} \|\delta g\|_X, \quad (5.4.8)$$

¹⁰We recall that $X = \left\{ g \left| \int |g|^2 d\mu < +\infty \right. \right\}$, where $\mu : x \in \mathbb{R}^d \mapsto (1 + |x|^2)^{-4k}$.

By the same ingredients, it follows from (5.4.6) that

$$\begin{aligned}
 |\nabla \delta^2 \phi_T^{(1)}(\delta g, \delta g', z)| &\lesssim_{T, \alpha, \gamma, [a_n]_{0, \alpha}} [\nabla \delta \phi_T^{(1)}(\delta g)]_{\gamma, \alpha} [\eta_n \star \delta g']_{\gamma, \alpha} + [\nabla \delta \phi_T^{(1)}(\delta g')]_{\gamma, \alpha} [\eta_n \star \delta g]_{\gamma, \alpha} \\
 &\quad + [\eta_n \star \delta g]_{\gamma, \alpha} [\eta_n \star \delta g']_{\gamma, \alpha} \\
 &\stackrel{(5.4.8)}{\lesssim_{T, \alpha, \gamma, n}} \|\delta g\|_X \|\delta g'\|_X.
 \end{aligned} \tag{5.4.9}$$

From the estimates¹¹ (5.4.7), (5.4.8), (5.4.9) and (5.4.5), we deduce that F_n is twice Fréchet differentiable in X with for all $g, \delta g, \delta g' \in X$

$$|d^2 F_n(g, z)(\delta g, \delta g')| \lesssim_{z, T, \alpha, \gamma, n} \|\delta g\|_X \|\delta g'\|_X.$$

Hilbert-Schmidt property. We now prove that for all $g \in X$, $d^2 F_n(g, z)$ in Hilbert-Schmidt. Fix $g \in X$. We prove that $d^2 F_n(g, z)$ is an integral operator in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ where its kernel $\partial^2 F_n(z, \cdot, \cdot)$ belongs to¹² $L^2_{\mu^{-1} dx \otimes \mu^{-1} dy}(\mathbb{R}^d \times \mathbb{R}^d)$ and is given by, for all $x, y \in \mathbb{R}^d \times \mathbb{R}^d$

$$\begin{aligned}
 \partial^2 F_n(z, x, y) &= \xi^* \cdot (a_n''(\nabla \phi_T^{(1)} + \xi))(z) \eta_n(z - x) \eta_n(z - y) \\
 &\quad + \xi^* \cdot (a_n' \nabla \delta \phi_T^{(1)}(\delta_y, \cdot))(z) \eta_n(z - x) + \xi^* \cdot (a_n' \nabla \delta \phi_T^{(1)}(\delta_x, \cdot))(z) \eta_n(z - y) \\
 &\quad + \xi^* \cdot (a_n \nabla \delta^2 \phi_T^{(1)}(\delta_x, \delta_y, \cdot))(z),
 \end{aligned} \tag{5.4.10}$$

Note that (5.4.10) is formally obtained by evaluating (5.4.4) for $\delta g = \delta_x$ and $\delta g' = \delta_y$ (where we recall that $\delta_x : \eta \in C_c^\infty(\mathbb{R}^d) \mapsto \eta(x)$). For notational convenience, we evaluate $\delta \phi_T^{(1)}$ and $\delta^2 \phi_T^{(1)}$ on Dirac distributions, which makes sense in (5.4.5) and (5.4.6) thanks to the convolution with η_n . The fact that $\partial^2 F_n(z, \cdot, \cdot)$ is the kernel of $d^2 F_n(g)$ in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ can be easily check by computing, for all $\delta g, \delta g' \in L^2(\mathbb{R}^d)$, $\int \int dx dy \partial^2 F_n(z, x, y) \delta g(x) \delta g'(y)$ and noticing that $\int dx \nabla \delta \phi_T^{(1)}(\delta_x, z) \delta g(x)$ as well as $\int dx dy \nabla \delta^2 \phi_T^{(1)}(\delta_x, \delta_y, z) \delta g(x) \delta g'(y)$ are the solution of (5.4.5) and (5.4.6) respectively. We now check that $\partial^2 F_n(z, \cdot, \cdot) \in L^2_{\mu^{-1} dx \otimes \mu^{-1} dy}(\mathbb{R}^d \times \mathbb{R}^d)$. We only treat the last right-hand side term of (5.4.10), the others are treated the same way. From the same ingredients as for (5.4.8) and (5.4.9), one has $|\nabla \delta^2 \phi_T^{(1)}(\delta_x, \delta_y, z)| \lesssim_{z, T, \alpha, \gamma, n} [\eta_n(\cdot - x)]_{\gamma, \alpha} [\eta_n(\cdot - y)]_{\gamma, \alpha}$ which implies that

$$\int \int dx dy \mu^{-1}(x) \mu^{-1}(y) |\nabla \delta^2 \phi_T^{(1)}(\delta_x, \delta_y, z)|^2 < +\infty, \tag{5.4.11}$$

since η_n is compactly supported.

We are now in position to apply the version of Price's formula of Corollary 27:

$$\frac{d}{dL} \langle F_n(z) \rangle_L = \frac{1}{2} \int \int dx dy \langle \partial^2 F_n(z, x, y) \rangle_L \frac{dc_L}{dL}(x, y), \tag{5.4.12}$$

where by definition $c_L(\delta_x, \delta_y) = c_L(x, y)$ (see Assumption 28). We now slightly rewrite (5.4.12) by using stationarity. Clearly, \hat{F} has the shift invariance property $\hat{F}(a, z + h) = \hat{F}(a(\cdot + h), z)$ (from the uniqueness of the solution of (5.3.3)). Combined with the trivial shift invariance of a_n , it translates to the shift invariance of F_n which in turn translates to $d^2 F_n$ in form of

$$d^2 F_n(g, z + h)(\delta g, \delta g') = d^2 F_n(g(\cdot + z), z)(\delta g(\cdot + h), \delta g'(\cdot + h)),$$

¹¹And a purely qualitative bit more which amounts to estimating the third derivatives.

¹²Note that if k is a kernel in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ with integrability in $L^2_{\mu^{-1} dx \otimes \mu^{-1} dy}(\mathbb{R}^d \times \mathbb{R}^d)$ then $(x, y) \mapsto \mu(x) \mu(y) k(x, y)$ is the kernel in $L^2_{\mu dx \otimes \mu dy}(\mathbb{R}^d \times \mathbb{R}^d)$ in the sense of (4.3.9).

for all $h \in \mathbb{R}^d$ and $g, \delta g, \delta g' \in X$. At the level of the kernel, it reads (where we make explicit the dependance in $g \in X$)

$$\partial^2 F_n(g, z + h, x + h, y + h) = \partial^2 F_n(g(\cdot + h), z, x, y).$$

Consequently, using the stationarity of $\langle \cdot \rangle_L$, one deduces that

$$\langle F(z + h) \rangle_L = \langle F(z) \rangle_L \text{ and } \langle \partial^2 F_n(z + h, x + h, y + h) \rangle_L = \langle \partial^2 F_n(z, x, y) \rangle_L. \quad (5.4.13)$$

With the identity (5.4.13) together with the fact that by stationarity of $\langle \cdot \rangle$ one has $c_L(x, y) = c_L(x - y)$, we arrive at (using the change $x - y \rightsquigarrow z$ and the fact that the argument for (5.4.11) also show that $\partial^2 F_n(z, \cdot, \cdot)$ is integrable.)

$$\frac{d}{dL} \langle F_n(0) \rangle_L = \frac{1}{2} \int dz \frac{dc_L}{dL}(z) \left\langle \int dx \partial^2 F_n(x, z, 0) \right\rangle_L. \quad (5.4.14)$$

We now compute $I := \langle \int dx \partial^2 F_n(x, z, 0) \rangle_L$ for a fixed $z \in \mathbb{R}^d$. We denote by $\partial^2 \tilde{F}_n$ the function such that $\partial^2 F_n = \xi^* \cdot \partial^2 \tilde{F}_n$ (see (5.4.10)). We proceed as in Subsection (5.2.1). We note that

$$I = \left\langle \int dx (\xi^* + \nabla \phi_T^{*(1)}) \cdot (\partial^2 \tilde{F}_n(z, x, 0) - a_n \nabla \delta^2 \phi_T^{(1)}(\delta_x, \delta_0, z)) \right\rangle. \quad (5.4.15)$$

Ideed, from (5.4.5) and (5.4.6) we have for $u : x \mapsto \partial^2 \tilde{F}_n(x, z, 0)$

$$\frac{1}{T} \delta^2 \phi_T(\delta_x, \delta_0) - \nabla \cdot u = 0.$$

Thus, by testing the above equation with $\phi_T^{*(1)}$ and by testing $\delta^2 \phi(\delta_x, \delta_0, \cdot)$ in (5.3.3) (with a_n replaced by a_n^*)¹³, we obtain

$$\frac{1}{T} \int \phi_T^{*(1)} \delta^2 \phi_T(\delta_x, \delta_0, \cdot) + \int \nabla \phi_T^{*(1)} \cdot \partial^2 \tilde{F}_n(x, \cdot, 0) = 0,$$

and

$$\frac{1}{T} \int \phi_T^{*(1)} \delta^2 \phi_T(\delta_x, \delta_0, \cdot) + \int (\xi^* + \nabla \phi_T^{*(1)}) \cdot a_n \nabla \delta^2 \phi_T(\delta_x, \delta_0, \cdot).$$

Taking the difference of the two identities above yields (5.4.15). Finally, using the definition of $\partial^2 \tilde{F}_n$ and (5.4.10), (5.4.15) turns into

$$\begin{aligned} I &= \left\langle \int dx (\xi^* + \nabla \phi_T^{*(1)}) \cdot (a'_n \nabla \delta \phi_T^{(1)}(\delta_0, \cdot))(x) \eta_m(x - z) + (\xi^* + \nabla \phi_T^{*(1)}) \cdot (a'_n \nabla \delta \phi_T^{(1)}(\delta_z, \cdot))(x) \eta_m(x) \right\rangle_L \\ &\quad + \left\langle \int dx (\xi^* + \nabla \phi_T^{*(1)}) \cdot (a''_n (\nabla \phi_T^{(1)} + \xi))(x) \eta_m(x - z) \eta_m(x) \right\rangle. \end{aligned} \quad (5.4.16)$$

By plugging (5.4.16) into (5.4.14), using the stationarity of $\langle \cdot \rangle_L$ (together with the shift invariance of $\nabla \phi_T^{*(1)}$) as well as the fact that c_L is even (which implies that the two contributions in the first right-hand side of (5.4.16) are identical), we arrive at

$$\begin{aligned} \frac{d}{dL} \langle F_n(0) \rangle_L &= \int \frac{dc_L}{dL} \star \eta_m \langle (\xi^* + \nabla \phi_T^{*(1)}) \cdot (a'_n \nabla \delta \phi_T^{(1)}(\delta_0, \cdot)) \rangle \\ &\quad + \frac{1}{2} \int \langle (\nabla \phi_T^{*(1)} + \xi^*) \cdot \left(\frac{dc_L}{dL} \star \eta_m \right) a''_n \eta_m (\nabla \phi_T^{(1)} + \xi) \rangle_L. \end{aligned}$$

¹³Which are both admissible since they decay exponentially by Schauder theory and the exponentially localized energy estimates, see Proposition 15

We conclude by rewriting the first term of the above equation by duality, using the solution of

$$\frac{1}{T}u_n - \nabla \cdot a_n \nabla u_n = \nabla \cdot a'_n \left(\eta_n \star \frac{dc_L}{dL} \right) (\nabla \phi_T^{*(1)} + \xi^*),$$

and equation (5.4.5), such a way that finally

$$\begin{aligned} \frac{d}{dL} \langle F_n(0) \rangle_L &= \int (\nabla \phi_T + \xi) \cdot (a'_n \eta) \nabla u_n \\ &\quad + \frac{1}{2} \int \langle (\nabla \phi_T^{*(1)} + \xi^*) \cdot \left(\frac{dc_L}{dL} \star \eta_n \right) a''_n \eta_n (\nabla \phi_T^{(1)} + \xi) \rangle_L. \end{aligned}$$

We then obtain (5.3.5) by letting $n \uparrow +\infty$, which easily follow from Schauder's theory (which gives compactness) and continuity of all the functions.

5.4.2 Proof of Proposition 12: Limit $T \uparrow \infty$

The strategy of proof is simple. First, we reorder the terms of the derivative of $\frac{d}{dL} \langle \xi^* \cdot \bar{a}_T \xi \rangle_L$ in order to make appear the "massive" analogue (that is, involving the massive operator $\frac{1}{T} - \nabla \cdot a \nabla$) of the r. h. s. of (5.3.10). For this first step, we essentially make precise the computations done in Section 5.2.2. Second, we systematically make use of the dominated convergence theorem to obtain the convergence of each term to its massless counterpart.

Reordering the derivative of $\langle \xi^* \cdot \bar{a}_T \xi \rangle_L$. We shall first establish that, under Assumptions 28, formula (5.3.10) holds on the level of the massive equation, that is

$$\begin{aligned} &\frac{d}{dL} \langle \xi^* \cdot \bar{a}_T \xi \rangle_L \\ &= L^{-(d+1)} \int dz \langle \bar{\Gamma}_{T/L^2 ijmn} (\xi^* \cdot Q_{Tijm}^{(2)}(z) \xi - z_m \xi^* \cdot Q_{Tij}^{(1)}(z) \xi) \rangle_L \partial_n c(z) \\ &\quad + \int dz \langle \epsilon_{TLijmn}^{(2)}(z) \xi^* \cdot Q_{Tijm}^{(2)}(z) \xi + \epsilon_{TLijn}^{(1)}(z) \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \rangle_L \partial_n c(z) \\ &\quad + \int dz (1 - \eta_L)(z) \langle (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot a'(0) \\ &\quad \quad \times \mathcal{E}_T(0, z) a'(z) (\nabla \phi_T^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\ &\quad + \int dz \eta_L(z) \langle (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot a'(0) \\ &\quad \quad \times \nabla \nabla G_T(0, z) a'(z) (\nabla \phi_T^{(1)} + \xi)(z) \rangle_L \frac{\partial c_L}{\partial L}(z) \\ &\quad + \frac{1}{2} \langle (\nabla \phi_T^{*(1)} + \xi^*) \cdot a'' (\nabla \phi_T^{(1)} + \xi) \rangle_L \frac{\partial c_L}{\partial L}(0), \end{aligned} \tag{5.4.17}$$

where the objects \mathcal{E}_T , $\epsilon_{TLijn}^{(1)}$, $\epsilon_{TLijlm}^{(2)}$, $Q_{Tij}^{(1)}$ and $Q_{Tijm}^{(2)}$ are defined by the "massive" analogue of (5.3.6), (5.3.8), (5.3.9), (5.2.19) and (5.2.20) respectively (where (G, \bar{G}) shall be replaced by (G_T, \bar{G}_T)). Moreover, $\bar{\Gamma}_{Tijmn}$ is defined by (5.3.7).

The proof of (5.4.17) follows from straightforward computations on (5.3.5), which are all legitimate on account of the exponential decay of the massive Green's function, see (5.3.1). Recall

Formula (5.3.5) established in Proposition 11, which directly provides the fifth term of (5.4.17). Thus, we only have to study the r. h. s. integral in (5.3.5). As explained in Section 5.3.2, we split the integral into the near-field contribution $\int dz \eta_L(z)$ (which gives the fourth r. h. s term of (5.4.17)) and the far-field contribution $\int dz(1 - \eta_L)(z)$ (that we denote by $\Pi_{L,T}$). On the far-field part, we appeal to the two-scale expansion of $\nabla \nabla G_T$ and, by the definition of the homogenization error \mathcal{E}_T , the tensors $Q_T^{(1)}$ and $Q_T^{(2)}$ naturally appear in

$$\begin{aligned} (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot a'(0) \nabla \nabla G_T(0, z) a'(z) (\nabla \phi_T^{(1)} + \xi)(z) &= \partial_{ij} \bar{G}_T(z) \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \\ &\quad + \partial_{ijm} \bar{G}_T(z) \xi^* \cdot Q_{Tijm}^{(2)}(z) \xi \\ &\quad + (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot a'(0) \mathcal{E}_T(0, z) a'(z) (\nabla \phi_T^{(1)} + \xi)(z), \end{aligned} \quad (5.4.18)$$

which we may insert in $\Pi_{L,T}$ to the effect of

$$\begin{aligned} \Pi_{L,T} &= - \int dz (1 - \eta_L(z)) \langle \partial_{ij} \bar{G}_T(z) \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \rangle_L \frac{\partial c_L}{\partial L}(z) && (=: \Pi_{L,T}^{(1)}) \\ &\quad + \int dz (1 - \eta_L(z)) \langle \partial_{ijm} \bar{G}_T(z) \xi^* \cdot Q_{Tijm}^{(2)}(z) \xi \rangle_L \frac{\partial c_L}{\partial L}(z) && (=: \Pi_{L,T}^{(2)}) \\ &\quad + \int (1 - \eta_L(z)) \left\langle (\nabla \phi_T^{*(1)} + \xi^*)(0) \cdot \mathcal{E}_T(0, z) (\nabla \phi_T^{(1)} + \xi)(0) \right\rangle_L \frac{\partial c_L}{\partial L}(z) dz, \end{aligned} \quad (5.4.19)$$

where we identify the third above r. h. s. term as the third r. h. s. term of (5.4.17). Then, we shall treat separately $\Pi_{L,T}^{(1)}$ and $\Pi_{L,T}^{(2)}$.

On the one hand, using (5.2.8), we may expand $\frac{\partial c_L}{\partial L}$ as in (5.4.19), in such a way that the sum on $k \neq 0$ may be transferred to the Green function. Namely, we obtain

$$\begin{aligned} \Pi_{L,T}^{(1)} &= \sum_{k \neq 0} \int dz (1 - \eta_L(z)) \langle \partial_{ij} \bar{G}_T(z) \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \rangle_L k_n \partial_n c(z - kL) \\ &= \int dz \left\langle \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \left(\sum_{k \neq 0} ((1 - \eta_L) \partial_{ij} \bar{G}_T)(z + kL) k_n \right) \right\rangle_L \partial_n c(z), \end{aligned} \quad (5.4.20)$$

where we used the change of variables $z \rightsquigarrow z + kL$ and the fact that $Q_T^{(1)}$ is L -periodic. Since $\partial_{ij} \bar{G}_T$ is an even function, the following identity holds:

$$\sum_{k \neq 0} \partial_{ij} \bar{G}_T(kL) k_n = 0.$$

Therefore, we deduce from (5.4.20) that

$$\Pi_{L,T}^{(1)} = \int dz \left\langle \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \left(\sum_{k \neq 0} (((1 - \eta_L) \partial_{ij} \bar{G}_T)(z + kL) - \partial_{ij} \bar{G}_T(kL)) k_n \right) \right\rangle_L \partial_n c(z). \quad (5.4.21)$$

Next, we reorganize the above r. h. s. term into brackets by means of a Taylor-like expansion

$$\sum_{k \neq 0} (((1 - \eta_L) \partial_{ij} \bar{G}_T)(z + kL) - \partial_{ij} \bar{G}_T(kL)) k_n = z_m \sum_{k \neq 0} \partial_{ijm} \bar{G}_T(kL) k_n + \epsilon_{TLijmn}^{(1)}(z),$$

(which defines $\epsilon_{TLijmn}^{(1)}$) so that (5.4.21) reads

$$\Pi_{L,T}^{(1)} = \int dz \left\langle \xi^* \cdot Q_{Tij}^{(1)}(z) \xi \left(z_m \sum_{k \neq 0} \partial_{ijm} \bar{G}_T(kL) k_n + \epsilon_{TLijmn}^{(1)}(z) \right) \right\rangle_L \partial_n c(z). \quad (5.4.22)$$

On the other hand, by (5.2.8) and by a change of variables, we similarly rewrite $\Pi_{L,T}^{(2)}$ as

$$\begin{aligned} \Pi_{L,T}^{(2)} &= - \int dz \left\langle \xi \cdot Q_{Tijm}^{(2)}(z) \xi \sum_{k \neq 0} ((1 - \eta_L) \partial_{ijm} \bar{G}_T)(z + kL) k_n \right\rangle_L \partial_n c(z) \\ &= - \int dz \left\langle \xi^* \cdot Q_{Tijm}^{(2)}(z) \xi \left(\sum_{k \neq 0} \partial_{ijm} \bar{G}_T(kL) k_n + \epsilon_{TLijmn}^{(2)} \right) \right\rangle_L \partial_n c(z). \end{aligned} \quad (5.4.23)$$

We finally remark that, in (5.4.22) and (5.4.23), we may use a scaling argument and replace

$$\partial_{ijm} \bar{G}_T(kL) = L^{-d-1} \partial_{ijm} \bar{G}_{T/L^2}(k). \quad (5.4.24)$$

As a conclusion, combining (5.3.5), (5.4.19), (5.4.22), (5.4.23) and (5.4.24), yields the far-field contribution and therefore (5.4.17).

Asymptotic analysis as $T \uparrow \infty$. We now show that each term in (5.4.17) pass to the limit as $T \uparrow \infty$ and converge to their massless counterpart. Thus, we need to establish that this limit makes sense for each of the five r. h. s. terms of (5.4.17); in this task, the dominated convergence theorem is our main tool. Note that the estimates (5.3.11), (5.3.13) and (5.3.14) hold, uniformly in T , at the level of the massive quantities \mathcal{E}_T , $\epsilon_{TLijmn}^{(2)}$ and $\epsilon_{TLijn}^{(1)}$ (thanks to (5.3.1) for the two latters). Therefore, combined with the convergences of the massive quantities (5.3.12), the second, third, fourth and fifth r. h. s term of (5.4.17) converge to their massive less counterpart as $T \uparrow +\infty$. Therefore, the subtle part is in the first r. h. s term of (5.4.17), that we treat in details.

In the sequel, $L \geq 1$ is fixed. We prove that

$$\begin{aligned} &\lim_{T \uparrow \infty} \int dz \left\langle \bar{\Gamma}_{T/L^2ijmn} (\xi^* \cdot Q_{Tijm}^{(2)}(z) \xi - z_m \xi^* \cdot Q_{Tij}^{(1)}(z) \xi) \right\rangle_L \partial_n c(z) \\ &= \int dz \left\langle \bar{\Gamma}_{ijmn} (\xi^* \cdot Q_{ijm}^{(2)}(z) \xi - z_m \xi^* \cdot Q_{ij}^{(1)}(z) \xi) \right\rangle_L \partial_n c(z). \end{aligned} \quad (5.4.25)$$

We claim that the only additional needed ingradient is the well-posedeness of $\bar{\Gamma}_{ijmn}$ given by definition (5.3.7) and $\langle \cdot \rangle_L$ -almost-surely)

$$|\bar{\Gamma}_{ijmn}| \leq \sup_{T \geq 1} |\bar{\Gamma}_{Tijmn}| \lesssim 1. \quad (5.4.26)$$

Indeed, thanks to the assumption (5.1.4) on c , (5.3.4) and (5.4.26), the integrand is bounded (uniformly in T) by $(1 + |z|)^{-d-2\alpha}$. We conclude using the convergences (5.3.12) and the Lebesgue convergence theorem.

Here comes the argument for (5.4.26). We fix a smooth compactly supported η with $\eta = 1$ on the

unit cube $(-\frac{1}{2}, \frac{1}{2})^d$; we use it to split the lattice, which we interpret as a Riemann sum, as follows:

$$\begin{aligned} & \sum_{k \neq 0} k_n \partial_{ijm} \bar{G}_T(k) \\ &= \int_{\mathbb{R}^d \setminus (-\frac{1}{2}, \frac{1}{2})^d} dx \eta(x) x_n \partial_{ijm} \bar{G}_T(x) + \int dx (1 - \eta)(x) x_n \partial_{ijm} \bar{G}_T(x) \\ &+ \sum_{k \neq 0} (k_n \partial_{ijm} \bar{G}_T(k) - \int_{k + (-\frac{1}{2}, \frac{1}{2})^d} dx x_n \partial_{ijm} \bar{G}_T(x)). \end{aligned} \tag{5.4.27}$$

The first r. h. s. integral effectively extends over a compact subset of $\mathbb{R}^d \setminus \{0\}$ and thus obviously converges for $T \uparrow \infty$, thanks to (5.3.12). On the second r. h. s. integral in (5.4.27) we perform two integrations by parts:

$$\begin{aligned} & \int dx (1 - \eta)(x) x_n \partial_{ijm} \bar{G}_T(x) \\ &= \int dx (-\partial_j \eta(x) \delta_{mn} \partial_i \bar{G}_T(x) + \partial_m \eta(x) x_n \partial_{ij} \bar{G}_T(x)). \end{aligned}$$

Again, the r. h. s. integral effectively extends over a compact subset of $\mathbb{R}^d \setminus \{0\}$ and converges for $T \uparrow \infty$, thanks to (5.3.12). We finally turn to the last contribution in (5.4.27); clearly each summand has a limit $T \uparrow \infty$. This extends to the sum because of dominated convergence: Each summand is dominated, in absolute value, by the Lipschitz norm of $x \mapsto x_n \partial_{ijm} \bar{G}_T(x)$ on the translated cube $k + (-\frac{1}{2}, \frac{1}{2})^d$, which by the uniform-in- T decay of \bar{G}_T is $\lesssim |k|^{-d-1}$, an expression that is summable in $k \in \mathbb{Z}^d \setminus \{0\}$.

5.4.3 Proof of Lemma 37: Fluctuation estimates

As announced above, we show only (5.3.29) by closely following [101]. The only difference is that we appeal not only to the annealed Calderón-Zygmund estimates as in [101], but also to the annealed weighted estimates contained in Lemma 38.

For a deterministic and periodic vector field f , we consider the random variable of zero average

$$F := \int_{[0,L]^d} f \cdot \nabla \phi_{ij}^{(2)}.$$

We employ on it the spectral gap inequality (cf. [101, Lem. 3.1]), which, combined with the Bochner estimate (assuming that $p \geq 2$), reads

$$\langle |F|^p \rangle^{\frac{1}{p}} \lesssim \left(\int_{[0,L]^d} \langle |\frac{\partial F}{\partial g}|^p \rangle^{\frac{2}{p}} \right)^{\frac{1}{2}}. \tag{5.4.28}$$

We split the proof into three steps. First, we establish that the Fréchet derivative of F is given by

$$\frac{\partial F}{\partial g} = \nabla v \cdot a'(\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j) - (\nabla w_j + v e_j) \cdot a'(\nabla \phi_i^{(1)} + e_i), \tag{5.4.29}$$

where v and w_j are defined through (5.4.34) and (5.4.36) below. Next, we show that the annealed estimates of Lemma 38 imply

$$\left(\int_{[0,L]^d} \langle |\nabla v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |f|^2 \right)^{\frac{1}{2}}, \tag{5.4.30}$$

$$\left(\int_{[0,L]^d} \langle |\nabla w_j + v e_j|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 |f|^2 \right)^{\frac{1}{2}}, \tag{5.4.31}$$

where we recall that $|x|_L = \inf_{k \in \mathbb{Z}^d} |x + kL|$. Last, we insert (5.4.29) into (5.4.28) and we appeal to the Cauchy-Schwarz inequality (we also employ (5.1.6)), to the effect of

$$\begin{aligned} \langle |F|^p \rangle^{\frac{1}{p}} &\lesssim \left(\int_{[0,L]^d} (|\nabla v|^{2p})^{\frac{1}{p}} \langle |\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j|^{2p} \rangle^{\frac{1}{p}} \right. \\ &\quad \left. + \langle |\nabla w_j + v e_j|^{2p} \rangle^{\frac{1}{p}} \langle |\nabla \phi_i^{(1)} + e_i|^{2p} \rangle^{\frac{1}{p}} \right)^{\frac{1}{2}}. \end{aligned}$$

Invoking (5.3.23) and recalling (5.4.30) and (5.4.31) finally yields the desired estimate (5.3.29).

Argument for (5.4.29) We give ourselves infinitesimal (periodic) perturbation δg of g . In view of (5.1.2) and (5.2.12), it generates a perturbation $\delta \phi_i^{(1)}$ characterized by

$$\nabla \cdot (a \nabla \delta \phi_i^{(1)} + \delta g a' (\nabla \phi_i^{(1)} + e_i)) = 0 \quad \text{and} \quad \int_{[0,L]^d} \delta \phi_i^{(1)} = 0. \quad (5.4.32)$$

In view of (5.2.14), this in turn generates the perturbation $\delta \phi_{ij}^{(2)}$ characterized by

$$\begin{aligned} -\nabla \cdot (a (\nabla \delta \phi_{ij}^{(2)} + \delta \phi_i^{(1)} e_j) + \delta g a' (\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j)) \\ = P e_j \cdot (a \nabla \delta \phi_i^{(1)} + \delta g a' (\nabla \phi_i^{(1)} + e_i)), \end{aligned} \quad (5.4.33)$$

where P denotes the (L^2 -orthogonal) projection onto functions of vanishing spatial mean, i. e. $Pf = f - \int_{[0,L]^d} f$. The form of (5.4.33) motivates the introduction of the periodic function v defined through

$$\nabla \cdot (a^* \nabla v + f) = 0 \quad \text{and} \quad \int_{[0,L]^d} v = 0, \quad (5.4.34)$$

so that, by testing (5.4.34) with $\delta \phi_{ij}^{(2)}$ and (5.4.33) with v , we obtain the representation (for $\delta F := \int_{[0,L]^d} f \cdot \nabla \delta \phi_{ij}^{(2)}$):

$$\begin{aligned} \delta F = \int_{[0,L]^d} \left(\nabla v \cdot (a \delta \phi_i^{(1)} e_j + \delta g a' (\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j)) \right. \\ \left. - v e_j \cdot (a \nabla \delta \phi_i^{(1)} + \delta g a' (\nabla \phi_i^{(1)} + e_i)) \right). \end{aligned} \quad (5.4.35)$$

This in turn prompts the introduction of a second auxiliary periodic function w_j of zero mean

$$-\nabla \cdot a^* (\nabla w_j + v e_j) = P e_j \cdot a^* \nabla v, \quad (5.4.36)$$

so that by testing (5.4.32) with w_j and (5.4.36) with $\delta \phi_i^{(1)}$, we may eliminate $\delta \phi_i^{(1)}$ in (5.4.35):

$$\delta F = \int_{[0,L]^d} \delta g \left(\nabla v \cdot a' (\nabla \phi_{ij}^{(2)} + \phi_i^{(1)} e_j) - (v e_j + \nabla w_j) \cdot a' (\nabla \phi_i^{(1)} + e_i) \right).$$

By definition of the Fréchet derivative, this amounts to (5.4.29).

Argument for (5.4.30) and (5.4.31) Notice first that (5.4.30) is a direct consequence of Lemma 38 applied to v , which satisfies (5.4.34), with weight 1. Therefore, it remains to establish (5.4.31). In this perspective, we introduce the (gradient) field h_j such that the r. h. s. of (5.4.36) reads $Pe_j \cdot a^* \nabla v = \nabla \cdot h_j$. As a consequence of annealed unweighted estimates on $\nabla(-\nabla \cdot a^* \nabla)^{-1} \nabla \cdot$ (namely, Lemma 38 with weight $\mu = 1$), we get

$$\left(\int_{[0,L]^d} \langle |\nabla w_j|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} \langle |h_j|^{2p} + |v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}}. \quad (5.4.37)$$

We now claim the following annealed Hardy inequality:

$$\left(\int_{[0,L]^d} \langle |v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 \langle |\nabla v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}}. \quad (5.4.38)$$

As a consequence of annealed weighted estimates on $\nabla(-\nabla \cdot a^* \nabla)^{-1} \nabla \cdot$ (namely, Lemma 38 with weight $\mu = |\cdot|_L^2$) applied to (5.4.34), we have

$$\left(\int_{[0,L]^d} |x|_L^2 \langle |\nabla v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 |f|^2 \right)^{\frac{1}{2}}, \quad (5.4.39)$$

and therefore, by (5.4.38), there holds

$$\left(\int_{[0,L]^d} \langle |v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 |f|^2 \right)^{\frac{1}{2}}. \quad (5.4.40)$$

Moreover, by the annealed weighted estimates on $\nabla^2(-\Delta)^{-1}$ [113, Theorem 7.1], we obtain

$$\left(\int_{[0,L]^d} |x|_L^2 \langle |\nabla h_j|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 \langle |\nabla v|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}}.$$

Combining it with the Hardy inequality (5.4.38) for h_j and with (5.4.39) yields

$$\left(\int_{[0,L]^d} \langle |h_j|^{2p} \rangle_L^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{[0,L]^d} |x|_L^2 |f|^2 \right)^{\frac{1}{2}}$$

Inserting this and (5.4.40) into (5.4.37), and employing once more (5.4.40) in the triangle inequality gives (5.4.31).

Argument for (5.4.38) W. l. o. g. we may assume that $L = 1$, and we consider random periodic functions of vanishing average u . The annealed Hardy inequality (5.4.38) relies on three ingredients. First, if $\langle |v|^{2p} \rangle^{\frac{1}{p}}$ is compactly supported, we have

$$\int \langle |v|^{2p} \rangle^{\frac{1}{p}} \lesssim \int |x|^2 \langle |\nabla v|^{2p} \rangle^{\frac{1}{p}}. \quad (5.4.41)$$

Next, for $\Omega := [-\frac{1}{2}, \frac{1}{2}]^d \setminus [-\frac{1}{4}, \frac{1}{4}]^d$, the following annealed Poincaré estimate holds:

$$\left(\int_{\Omega} \langle |u|^{2p} \rangle^{\frac{1}{p}} \right)^{\frac{1}{2}} \lesssim \left(\int_{\Omega} \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}} \right)^{\frac{1}{2}} + \int_{\Omega} \langle |u|^{2p} \rangle^{\frac{1}{2p}}. \quad (5.4.42)$$

Last, we make use of an annealed Poincaré-Wirtinger estimate

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |u|^{2p} \rangle^{\frac{1}{2p}} \lesssim \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |\nabla u|^{2p} \rangle^{\frac{1}{2p}}. \quad (5.4.43)$$

Using (5.4.41) for $v := \eta u$ where η is a cut-off function of $[-\frac{1}{2}, \frac{1}{2}]^d$ into $[-\frac{3}{4}, \frac{3}{4}]^d$, we have by periodicity of u

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |u|^{2p} \rangle^{\frac{1}{p}} \lesssim \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |x|_1^2 \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}} + \int_{\Omega} \langle |u|^{2p} \rangle^{\frac{1}{p}},$$

Inserting (5.4.42) and then (5.4.43) into the above estimate yields

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |u|^{2p} \rangle^{\frac{1}{p}} \lesssim \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |x|_1^2 \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}} + \left(\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |\nabla u|^{2p} \rangle^{\frac{1}{2p}} \right)^2. \quad (5.4.44)$$

Since $d > 2$, we may employ the Cauchy-Schwarz inequality to get

$$\begin{aligned} \left(\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |\nabla u|^{2p} \rangle^{\frac{1}{2p}} \right)^2 &\leq \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |x|_1^2 \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |x|_1^{-2} \\ &\lesssim \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |x|_1^2 \langle |\nabla u|^{2p} \rangle^{\frac{1}{p}}. \end{aligned}$$

Inserting this into (5.4.44) yields the desired (5.4.38) (noting that by periodicity, we can replace $[-\frac{1}{2}, \frac{1}{2}]$ by $[0, 1]^d$).

We now establish successively (5.4.41), (5.4.42), and (5.4.43). First, (5.4.41) comes by applying the following Hardy inequality for compactly supported functions v :

$$\int_{\mathbb{R}^d} |v|^2 \lesssim \int_{\mathbb{R}^d} |x|^2 |\nabla v|^2,$$

to $v \rightsquigarrow \langle |u|^{2p} \rangle^{\frac{1}{2p}}$, and noticing that by the Hölder inequality with exponents $(\frac{2p}{2p-1}, 2p)$

$$|\nabla \langle |v|^{2p} \rangle^{\frac{1}{2p}}| = |\langle |v|^{2p} \rangle^{\frac{1}{2p}-1} \langle |v|^{2p-1} \nabla |v| \rangle| \leq \langle |\nabla v|^{2p} \rangle^{\frac{1}{2p}}.$$

Similarly, we get (5.4.42) from the usual Poincaré inequality applied to the function $\langle |u|^{2p} \rangle^{\frac{1}{2p}}$. Last, we get (5.4.43) by recalling that u is periodic of vanishing average in $[-\frac{1}{2}, \frac{1}{2}]^d$, so that

$$\begin{aligned} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |u|^{2p} \rangle^{\frac{1}{2p}} &= \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \left\langle |u - \int_{[-\frac{1}{2}, \frac{1}{2}]^d} u|^{2p} \right\rangle^{\frac{1}{2p}} \\ &\leq \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dx \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dy \langle |u(x) - u(x+y)|^{2p} \rangle^{\frac{1}{2p}} \\ &\leq \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dx \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dy \langle (|y| \int_0^1 ds |\nabla u(x+sy)|)^{2p} \rangle^{\frac{1}{2p}} \\ &\lesssim \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dx \int_{[-\frac{1}{2}, \frac{1}{2}]^d} dz \langle |\nabla u(z)|^{2p} \rangle^{\frac{1}{2p}} = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \langle |\nabla u|^{2p} \rangle^{\frac{1}{2p}}. \end{aligned}$$

5.4.4 Proof of Corollary 31: Limit $L \uparrow \infty$

The proof of Corollary 31 rest on a coupling procedure, which jointly builds Gaussian fields of covariance functions c and c_L that become closer when $L \uparrow \infty$. More precisely:

Lemma 42. *Under the assumption 28, there exists a family of coupling ensembles $\langle \cdot \rangle_{\Omega_L}$ indexed by $L \in [1, +\infty)$ between the ensembles $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$, that generate Gaussian fields g_L and g of covariance functions c_L and c , respectively, and such that the following convergence holds, for any $R \geq 1$, $p < +\infty$*

$$\left\langle \sup_{B_R} |g - g_L|^p \right\rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0. \quad (5.4.45)$$

Proof. Let $R \geq 1$ and $p < +\infty$. We also set $L \gg 1$ and a discretization parameter $N \in \mathbb{N}^*$, the value of which will be fixed at the end of the proof. By (5.3.2) (which also holds for g), there holds for a fixed α'

$$\left\langle \|g\|_{C^{0,\alpha'}(B_R)}^p \right\rangle^{\frac{1}{p}} + \left\langle \|g\|_{C^{0,\alpha'}(B_R)}^p \right\rangle_L^{\frac{1}{p}} \lesssim_{R,p} 1, \quad (5.4.46)$$

where we recall that $\|\cdot\|_{C^{0,\alpha'}} = \sup|\cdot| + [\cdot]_{\alpha'}$. We introduce the discretization points $x_i := i/\sqrt{N}$ for $i \in \mathbb{Z}^d$, the Gaussian vectors G^N , and the Gaussian field g^N built from interpolating linearly w. r. t. the points x_i with¹⁴

$$G_i^N := g(x_i) \quad \text{for } i \in \{-N, \dots, N\}^d \quad \text{and} \quad g^N(x_i) := \begin{cases} G_i^N & \text{for } i \in \{-N, \dots, N\}^d, \\ 0 & \text{for } i \in \mathbb{Z}^d \setminus \{-N, \dots, N\}^d. \end{cases} \quad (5.4.47)$$

It is sufficient to build the coupling on the level of the Gaussian vectors G^N for N sufficiently large, since (5.4.46) implies that, if $\sqrt{N} \geq R$

$$\left\langle \sup_{B_R} |g - g^N|^p \right\rangle^{\frac{1}{p}} + \left\langle \sup_{B_R} |g - g^N|^p \right\rangle_L^{\frac{1}{p}} \lesssim_R N^{-\frac{\alpha'}{2}}. \quad (5.4.48)$$

We denote by C and C_L the covariance matrices associated with G^N under $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$ defined by: $C_{ij} := c(x_i - x_j)$ and $(C_L)_{ij} := c_L(x_i - x_j)$ for $i, j \in \{-N, \dots, N\}^d$. Now, by [53, Th.], there exists an ensemble $\langle \cdot \rangle_{\tilde{\Omega}_L}$ generating two Gaussian vectors G^N and G_L^N of covariance matrices C and C_L , respectively, such that the euclidean distance $\|G^N - G_L^N\|_2$ is minimal in the following sense:

$$\left\langle \|G^N - G_L^N\|_2^2 \right\rangle_{\tilde{\Omega}_L} = \text{trace}((\sqrt{C} - \sqrt{C_L})^2). \quad (5.4.49)$$

Using the operator norm $\|\cdot\|$ and the fact that the function $\sqrt{\cdot}$ is 1/2-Hölder continuous, we easily get the following string of inequalities

$$\begin{aligned} \text{trace}((\sqrt{C} - \sqrt{C_L})^2) &\leq (2N + 1)^d \left\| (\sqrt{C} - \sqrt{C_L})^2 \right\| \lesssim N^d \left\| \sqrt{C} - \sqrt{C_L} \right\|^2 \\ &\lesssim N^d \|C - C_L\| \\ &\lesssim N^{2d} \sup_{i,j \in \{-N, \dots, N\}^d} |C_{ij} - (C_L)_{ij}|. \end{aligned} \quad (5.4.50)$$

¹⁴Notice that the set of discretization points $\{x_i, i \in \{-N, \dots, N\}^d\}$ becomes denser and spreader in \mathbb{R}^d when $N \uparrow \infty$.

Therefore, inserting (5.4.50) into (5.4.49) and using the injection $\ell^2 \subset \ell^\infty$, we get

$$\left\langle \sup_{i \in \{-N, \dots, N\}^d} |G_i^N - (G_L^N)_i|^2 \right\rangle_{\tilde{\Omega}_L} \lesssim_d N^{2d} \sup_{[0, \sqrt{N}]^d} |c - c_L|. \quad (5.4.51)$$

Next, assuming that $L \geq 2\sqrt{N}$, from (5.1.7) and (5.1.4), we have for all $x \in [0, \sqrt{N}]^d$

$$|c(x) - c_L(x)| \leq \sum_{|k| \geq 1} \frac{1}{(1 + |x + kL|)^{d+\alpha}} \lesssim L^{-d-\alpha}, \quad (5.4.52)$$

which we may insert into (5.4.51), to the effect of

$$\left\langle \sup_{i \in \{-N, \dots, N\}^d} |G_i^N - (G_L^N)_i|^2 \right\rangle_{\tilde{\Omega}_L} \lesssim_\gamma N^{2d} L^{-d-\alpha}. \quad (5.4.53)$$

Next, we enrich the ensemble $\langle \cdot \rangle_{\tilde{\Omega}_L}$ into $\langle \cdot \rangle_{\Omega_L}$ in such a way that the latter generates Gaussian fields g and g_L of respective covariance c and c_L that are coherent with G^N and G_L^N , respectively, in the sense of the first equality of (5.4.47) (and in such a way that g conditioned by (G^N, G_L^N) is independent of g_L conditioned by (G^N, G_L^N)). By a triangle inequality, this coupling $\langle \cdot \rangle_{\Omega_L}$ between $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$ satisfies

$$\begin{aligned} \left\langle \sup_{B_R} |g - g_L|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} &\leq \left\langle \sup_{B_R} |g - g^N|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} + \left\langle \sup_{B_R} |g_L - g^N|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} + \left\langle \sup_{B_R} |g^N - g_L^N|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} \\ &\leq \left\langle \sup_{B_R} |g - g^N|^2 \right\rangle^{\frac{1}{2}} + \left\langle \sup_{B_R} |g - g^N|^2 \right\rangle_L^{\frac{1}{2}} + \left\langle \sup_{B_R} |G^N - G_L^N|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}}. \end{aligned}$$

Hence, we may now use (5.4.48) to cope with the first two r. h. s. term, whereas we appeal to (5.4.53) to estimate the third r. h. s. term, obtaining

$$\left\langle \sup_{B_R} |g - g_L|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} \lesssim_{\gamma, R} N^{-\frac{\alpha'}{2}} + N^d L^{-\frac{d}{2} - \frac{\alpha}{2}}$$

(where we assumed that $\sqrt{N} \geq R$ and $L \geq 2\sqrt{N}$). We now optimize the latter inequality by setting $N := \lceil L^{\frac{d+\alpha}{2d+\alpha'}} \rceil$ for $L \gg_R 1$, which gives

$$\left\langle \sup_{B_R} |g - g_L|^2 \right\rangle_{\Omega_L}^{\frac{1}{2}} \lesssim_{\gamma, R} L^{-\frac{\alpha'}{2} \frac{d+\alpha}{2d+\alpha'}} \xrightarrow{L \uparrow \infty} 0.$$

Interpolating this result with (5.4.46) concludes the proof. \square

Capitalizing on Lemma 42, we show that, under the coupling $\langle \cdot \rangle_{\Omega_L}$, the correctors $\phi^{(1)}$ and $\phi^{(2)}$ are as well close to their counterparts $\phi_L^{(1)}$ and $\phi_L^{(2)}$. More precisely:

Lemma 43. *For any $R \geq 1$ and $p < +\infty$ we have the following convergence:*

$$\left\langle |(a, \nabla \phi^{(1)}, \phi^{(1)}, \nabla \sigma^{(1)}, \sigma^{(1)}, \nabla \phi^{(2)}) - (a_L, \nabla \phi_L^{(1)}, \phi_L^{(1)}, \nabla \sigma_L^{(1)}, \sigma_L^{(1)}, \nabla \phi_L^{(2)})|^p \right\rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0, \quad (5.4.54)$$

where the functions with subscript L and with no subscript refer to the functions previously associated with $\langle \cdot \rangle_L$ and $\langle \cdot \rangle$, respectively.

The proof of (5.4.54) relies on a more general and abstract result (that we then apply successively to all quantities in (5.4.54)). We now prove the following general result:

Lemma 44. *Under Assumptions 28, suppose that, for all the ensembles $\langle \cdot \rangle_L$ with $L \in [1, +\infty)$, we are given two square-integrable stationary random fields on \mathbb{T}_L , namely f and ∇u , that are related through the equation*

$$-\nabla \cdot a \nabla u = \nabla \cdot f, \tag{5.4.55}$$

and that satisfy the following bounds

$$\sup_{L \geq 1} \langle |(\nabla u, f)|^2 \rangle_L \leq C, \quad \sup_{L \geq 1} \sup_{R \geq 1} \langle R^{-d-1} \|u\|_{L^2(B_R)}^2 \rangle_L \leq C, \tag{5.4.56}$$

for a fixed constant C . In addition, suppose that we have two stationary fields f and ∇u on \mathbb{R}^d associated with $\langle \cdot \rangle$ that are related through the equation (5.4.55) and satisfy

$$\langle |(\nabla u, f)|^2 \rangle \leq C, \tag{5.4.57}$$

along the sublinear growth property

$$\limsup_{R \uparrow \infty} \frac{1}{R} \left(\int_{B_R} |u|^2 \right)^{\frac{1}{2}} = 0, \quad \langle \cdot \rangle \text{-almost-surely.} \tag{5.4.58}$$

Let $\langle \cdot \rangle_{\Omega_L}$ denote the coupling ensembles between $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$ that are built in Lemma 42. (We denote with a subscript L and without a subscript the coupled random variables previously associated with $\langle \cdot \rangle_L$ and $\langle \cdot \rangle$, respectively.) If, given a sequence $L_n \uparrow \infty$, we have the convergence

$$\langle \|f_{L_n} - f\|_{L^2(B_R)}^2 \rangle_{\Omega_{L_n}} \xrightarrow{n \uparrow \infty} 0 \quad \text{for all } R \geq 0, \tag{5.4.59}$$

then there holds

$$\langle \|\nabla u_{L_n} - \nabla u\|_{L^2(B_R)}^2 \rangle_{\Omega_{L_n}} \xrightarrow{n \uparrow \infty} 0 \quad \text{for all } R \geq 0. \tag{5.4.60}$$

Proof of Lemma 44. The proof relies on classical mechanisms of weak convergence and elliptic theory. It falls in three steps. First, we build a global coupling ensemble $\langle \cdot \rangle_{\Omega}$ between all the ensembles $\langle \cdot \rangle$ and $\langle \cdot \rangle_{L_n}$, and we denote by Ω an associated probability space. Second, we extract a weak limit u_{∞} of u_{L_n} in $L^2(\Omega, H_{\text{loc}}^1(\mathbb{R}^d))$, and show it solves (5.4.55) along with (5.4.58); thus, by a Liouville theorem $u_{\infty} - u$ is constant. Last, we establish the strong local L^2 convergence of ∇u_{L_n} to ∇u , namely (5.4.60).

Step 1: We may iteratively use the construction of Lemma 42 to build the coupling ensemble $\langle \cdot \rangle_{\Omega}$ between the ensembles $\langle \cdot \rangle$ and $\langle \cdot \rangle_{L_n}$ for $n \in \mathbb{N}$ in such a way that it generates Gaussian fields g and g_{L_n} , and where the joint law of (g, g_{L_n}) is the same under $\langle \cdot \rangle_{\Omega}$ as under $\langle \cdot \rangle_{\Omega_{L_n}}$ (defined in Lemma 42). That is, we first generate g of law determined by $\langle \cdot \rangle$ and then we successively build the Gaussian fields g_{L_n} in such a way that the conditional probabilities $\mathbb{P}_{\Omega}(g_{L_n} | g)$ are all independent of each others.

Note that, since A is Lipschitz regular (see (5.1.6)), then (5.4.45) can be turned into

$$\langle \|a - a_{L_n}\|_{L^{\infty}(B_R)}^p \rangle_{\Omega} \xrightarrow{n \uparrow \infty} 0 \quad \text{for all } R \geq 1 \text{ and } p \in [1, +\infty). \tag{5.4.61}$$

Step 2: On account of (5.4.56), there exists a subsequence of $(u_{L_n})_n$ (that we do not relabel) and a function $u_\infty \in L^2(\Omega; H_{\text{loc}}^1(\mathbb{R}^d))$ such that, for any $R \geq 1$, the following weak convergences hold

$$\begin{aligned} u_{L_n} &\rightharpoonup_{n \uparrow \infty} u_\infty && \text{weakly in } L^2(\Omega \times B_R), \\ \nabla u_{L_n} &\rightharpoonup_{n \uparrow \infty} \nabla u_\infty && \text{weakly in } L^2(\Omega \times B_R). \end{aligned} \tag{5.4.62}$$

We show that $\nabla u_\infty = \nabla u$, almost-surely. Hereafter, for simplicity, we replace L_n by the lighter notation L and the limit $n \uparrow \infty$ by $L \uparrow \infty$.

We verify that u_∞ solves the equation (5.4.55) in the sense of distributions. To do so, let $X \in L^q(\Omega)$ for $q > 2$ be a random variable and $\zeta \in C_c^\infty(\mathbb{R}^d)$ be a smooth function. By (5.4.55), there holds

$$\left\langle X \int \nabla \zeta \cdot (a_L \nabla u_L + f_L) \right\rangle_\Omega = 0. \tag{5.4.63}$$

Thus, it follows from (5.4.61), (5.4.62) and (5.4.59) that, in the limit $L \uparrow \infty$, we have

$$\left\langle X \int \nabla \zeta \cdot (a \nabla u_\infty + f) \right\rangle_\Omega = 0,$$

which shows that u_∞ satisfies (5.4.55) in the sense of distributions, $\langle \cdot \rangle_\Omega$ -almost-surely.

Also, by means of the weak lower semi-continuity of the L^2 norm, we obtain

$$\left\langle \int_{B_R} |u_\infty|^2 \right\rangle_\Omega^{\frac{1}{2}} \leq \liminf_{L \uparrow \infty} \left\langle \int_{B_R} |u_L|^2 \right\rangle_\Omega^{\frac{1}{2}} \lesssim R^{\frac{1}{2}} \quad \text{for all } R \geq 1.$$

This together with the Markov inequality gives

$$\left\langle \frac{1}{2^m} \left(\int_{B_{2^m}} |u_\infty|^2 \right)^{\frac{1}{2}} > \nu \right\rangle_\Omega \leq \frac{\left\langle \left(\int_{B_{2^m}} |u_\infty|^2 \right)^{\frac{1}{2}} \right\rangle_\Omega}{2^{m\nu}} \lesssim \frac{1}{2^{\frac{m}{2}\nu}},$$

which is summable in $m \in \mathbb{N}$. Hence, it follows from the Borel-Cantelli lemma that

$$\left\langle \limsup_{m \uparrow \infty} \frac{1}{2^m} \left(\int_{B_{2^m}} |u_\infty|^2 \right)^{\frac{1}{2}} > \nu \right\rangle_\Omega = 0,$$

for any $\nu > 0$, so that u_∞ satisfies (5.4.58).

Recall that the solution to (5.4.55) with the condition (5.4.59) is unique up to the addition of a (random) constant (see the Liouville theorem [80, Cor. 1]). Whence $\nabla u_\infty = \nabla u$.

Step 3: Since we already know that $\nabla u_L \rightharpoonup \nabla u$ in $L^2(\Omega \times B_R)$, then by ellipticity of a , it suffices to show the conservation of the norm

$$\limsup_{L \uparrow \infty} \int_{B_R} \langle \nabla u_L \cdot a \nabla u_L \rangle_\Omega \leq \int_{B_R} \langle \nabla u \cdot a \nabla u \rangle_\Omega \tag{5.4.64}$$

to get the strong convergence $\nabla u_L \rightarrow \nabla u$ in $L^2(\Omega \times B_R)$. By the strong convergence (5.4.61) of a_L and by (5.4.56), we first obtain that

$$\limsup_{L \uparrow \infty} \int_{B_R} \langle \nabla u_L \cdot a \nabla u_L \rangle_\Omega \leq \limsup_{L \uparrow \infty} \int_{B_R} \langle \nabla u_L \cdot a_L \nabla u_L \rangle_\Omega.$$

Then, making use of the coupling properties $\langle \cdot \rangle_\Omega$ and (5.4.55) yields

$$\langle \nabla u_L \cdot a_L \nabla u_L \rangle_\Omega = \langle \nabla u \cdot a \nabla u \rangle_L \stackrel{(5.4.55)}{=} -\langle \nabla u \cdot f \rangle_L = -\langle \nabla u_L \cdot f_L \rangle_\Omega.$$

(The third above equality is due to the stationarity of f and u under $\langle \cdot \rangle_L$.) Whence

$$\limsup_{L \uparrow \infty} \int_{B_R} \langle \nabla u_L \cdot a \nabla u_L \rangle_\Omega \leq \limsup_{L \uparrow \infty} - \int_{B_R} \langle \nabla u_L \cdot f_L \rangle_\Omega = - \int_{B_R} \langle \nabla u \cdot f \rangle_\Omega,$$

where we appealed to the strong convergence of f_L given by (5.4.59) and the weak convergence of ∇u_L . Using once more the coupling properties of $\langle \cdot \rangle_\Omega$ and (5.4.55) yields

$$-\langle \nabla u \cdot f \rangle_\Omega = -\langle \nabla u \cdot f \rangle = \langle \nabla u \cdot a \nabla u \rangle = \langle \nabla u \cdot a \nabla u \rangle_\Omega.$$

As a consequence, we obtain (5.4.64), which establishes the strong convergence (5.4.60). \square

Equipped with Lemma 44, we proceed with the:

Proof of Lemma 43. Let $\langle \cdot \rangle_{\Omega_L}$ be the coupling ensembles between $\langle \cdot \rangle$ and $\langle \cdot \rangle_L$ that are built in Lemma 42. Notice that we have already established in (5.4.61) that a and a_L satisfy (5.4.54). The proof falls in four steps. First, we show that is sufficient to establish a version of (5.4.54) where $p = 2$ and where the spatial L^∞ norm is replaced by the L^2 norm. Then in Step 2, Step 3 and Step 4, we successively show that $(\phi^{(1)}, \nabla \phi^{(1)})$, $(\sigma^{(1)}, \nabla \sigma^{(1)})$ and $\nabla \phi^{(2)}$ satisfy the convergence (5.4.54) by appealing to Lemma 44.

Step 1. From Proposition 13, (5.3.2) and Schauder theory applied to (5.1.2), (5.3.18) and (5.3.19), one has for any $p < +\infty$ and $\alpha' < \alpha$

$$\langle \|\langle \nabla \phi_L^{(1)}, \phi_L^{(1)}, \nabla \sigma_L^{(1)}, \sigma_L^{(1)}, \nabla \phi_L^{(2)} \rangle\|_{C^{0,\alpha'}(B_R)}^p \rangle_{\Omega_L} \lesssim_{p,\alpha',R} 1, \quad (5.4.65)$$

Hence, by Arzela-Ascolie's theorem, it is sufficient to establish

$$\langle \|\langle \nabla \phi^{(1)}, \phi^{(1)}, \nabla \sigma^{(1)}, \sigma^{(1)}, \nabla \phi^{(2)} \rangle - \langle \nabla \phi_L^{(1)}, \phi_L^{(1)}, \nabla \sigma_L^{(1)}, \sigma_L^{(1)}, \nabla \phi_L^{(2)} \rangle\|_{L^2(B_R)}^2 \rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0. \quad (5.4.66)$$

Step 2. We first consider $f := a$ and $u := \phi^{(1)}$. Indeed, by Proposition 13, u and f satisfy (5.4.56) ((5.4.57) and (5.4.58) being consequences of the homogenization theory). Moreover, by Lemma 42 and 5.1.6, the condition (5.4.59) is also satisfied. As a consequence of (5.4.60) there holds

$$\langle \|\nabla \phi^{(1)} - \nabla \phi_L^{(1)}\|_{L^2(B_R)}^2 \rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0, \quad (5.4.67)$$

for any $R \geq 0$. Interpolating as described in Step 1, this implies that $\nabla \phi^{(1)} - \nabla \phi_L^{(1)}$ satisfies (5.4.54).

Next, by (5.3.24) in Proposition 13 (where a similar estimate holds for $\langle \cdot \rangle$) applied with $g = \nabla v$ and $-\Delta v = R^{-d} \mathbf{1}_{B_R}$, there holds

$$\left\langle \left| \int_{B_R} \phi^{(1)} \right|^2 \right\rangle_L^{\frac{1}{2}} + \left\langle \left| \int_{B_R} \phi^{(1)} \right|^2 \right\rangle^{\frac{1}{2}} \lesssim \left(\int |\nabla v|^2 \right)^{\frac{1}{2}} \lesssim \left(\int (R^{-d} \mathbf{1}_{B_R})^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}} \lesssim R^{1-\frac{d}{2}}, \quad (5.4.68)$$

for $L \geq R$. Since, by a Poincaré inequality combined with a triangle inequality, there holds

$$\left\langle \|\phi^{(1)} - \phi_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_L \lesssim_d R^2 \left\langle \|\nabla \phi^{(1)} - \nabla \phi_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_{\Omega_L} + \left\langle \left| \int_{B_R} \phi^{(1)} \right|^2 \right\rangle_L + \left\langle \left| \int_{B_R} \phi^{(1)} \right|^2 \right\rangle,$$

we deduce from the fact that $\nabla\phi^{(1)} - \nabla\phi_L^{(1)}$ satisfies (5.4.54) and from (5.4.68) that

$$\limsup_{L \uparrow \infty} \left\langle \|\phi^{(1)} - \phi_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_L \lesssim R^{2-d}, \quad (5.4.69)$$

and get the desired convergence

$$\limsup_{L \uparrow \infty} \left\langle \|\phi^{(1)} - \phi_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_L \lesssim \limsup_{R' \uparrow \infty} \limsup_{L \uparrow \infty} \left\langle \|\phi^{(1)} - \phi_L^{(1)}\|_{L^2(B_{R'})}^2 \right\rangle_L = 0. \quad (5.4.70)$$

Step 3. We justify that we may apply Lemma 44 (with the slight modification that a is replaced by the identity matrix in (5.4.55)) for $f := \partial_j(e_n \cdot a(e_i + \nabla\phi_i^{(1)})) - \partial_n(e_m \cdot a(e_i + \nabla\phi_i^{(1)}))$ and $u := \sigma_{imn}^{(1)}$. By Step 2 above, we know that f satisfies (5.4.59). Therefore, we can apply Lemma 44 and, by a similar reasoning than in Step 2, we obtain successively the two convergences

$$\left\langle \|\nabla\sigma^{(1)} - \nabla\sigma_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0 \text{ and } \left\langle \|\sigma^{(1)} - \sigma_L^{(1)}\|_{L^2(B_R)}^2 \right\rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0. \quad (5.4.71)$$

Step 4. Finally, we set $f := a\phi_i^{(1)}e_j - \sigma_i^{(1)}e_j$ and $u := \phi_{ij}^{(2)} - \phi_{ij}^{(2)}(0)$. We check that (5.4.55), (5.4.56), (5.4.57) and are satisfied by appealing to Proposition 13. Also, we deduce (5.4.59) from (5.4.70) and (5.4.71). Therefore, as in Step 2, we may apply Lemma 44, to the effect of

$$\left\langle \|\nabla\phi^{(2)} - \nabla\phi_L^{(2)}\|_{L^2(B_R)}^2 \right\rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0. \quad (5.4.72)$$

We conclude by gathering the estimates (5.4.67), (5.4.70), (5.4.71), and (5.4.72), which yields (5.4.66) and, thanks to Step 1, finally establish (5.4.54). \square

Equipped with Lemma 43, we finally turn to the proof of:

Proof of Corollary 31. Since the proof is the same for $Q_{ij}^{(1)}$ and $Q_{ijm}^{(2)}$, we only show that for all z and ξ

$$\lim_{L \uparrow \infty} \langle \xi^* \cdot Q_{ij}^{(1)}(z)\xi \rangle_L = \langle \xi^* \cdot Q_{ij}^{(1)}(z)\xi \rangle. \quad (5.4.73)$$

In this perspective, we introduce the coupling ensembles $\langle \cdot \rangle_{\Omega_L}$ of Lemma 42 as well as the notations $\psi := \xi + \nabla\phi^{(1)}$, $\zeta^{(1)i} := e_i + \nabla\phi_i^{(1)}$ and we compute

$$\begin{aligned} & \left| \langle \xi^* \cdot (Q_{ij}^{L,(1)}(x) - Q_{ij}^{(1)}(x))\xi \rangle_{\Omega_L} \right| \\ & \leq \langle |\psi_L^*(0) \cdot a' \zeta_L^{(1)i}(0) (\zeta^{(1)i}(z))^*_L \cdot a' \psi_L(z) - \psi^*(0) \cdot a' \zeta^{(1)i}(0) (\zeta^{(1)i}(z))^* \cdot a' \psi(z)| \rangle_{\Omega_L}. \end{aligned}$$

Thanks to Lemma 43, we have the following convergences: for all z and $p < +\infty$

$$\langle |\psi_L(z) - \psi(z)|^p \rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0 \text{ and } \langle |\zeta_L^{(1)i}(z) - \zeta^{(1)i}(z)|^p \rangle_{\Omega_L} \xrightarrow{L \uparrow \infty} 0.$$

Moreover, by (5.4.65), all the components of $Q_{ij}^{(1)}$ have finite p -moments; namely for all z

$$\langle |\psi(z)|^p \rangle_L + \langle |\psi(z)|^p \rangle + \langle |\zeta^{(1)i}(z)|^p \rangle_L + \langle |\zeta^{(1)i}(z)|^p \rangle \lesssim_p 1.$$

Therefore, the Hölder inequality entails

$$\lim_{L \uparrow \infty} \langle |\xi^* \cdot (Q_{ij}^{L,(1)}(z) - Q_{ij}^{(1)}(z))\xi| \rangle_{\Omega_L} = 0,$$

which implies in turn (5.4.73). This concludes the proof of Corollary 31. \square

5.4.5 Proof of Lemma 40: Improved Caccioppoli inequality

By scaling, it is enough to consider $R = 1$;

we fix a smooth cut-off function η for B_1 in B_2 . Starting point is the following localized version of a standard L^2 -based interpolation estimate, with $n \in \mathbb{N}$ to be fixed later, which we take from the proof of [29, Lemma 4]:

$$\int (\eta^{4n} \Delta^{2n} w)^2 \lesssim \left(\int |\eta^{4n+1} \nabla \Delta^{2n} w|^2 \right)^{\frac{4n}{4n+1}} \left(\int w^2 \right)^{\frac{1}{4n+1}} + \int w^2. \quad (5.4.74)$$

We apply it to $w \in H_0^{2n}(B_2)$ (as usual, $H_0^{2n}(B_2)$ denotes the closure of $C_0^\infty(B_2)$ w. r. t. to the $H^{2n}(B_2)$ -norm) that (weakly) solves

$$\Delta^{2n} w = u \quad \text{in } B_2. \quad (5.4.75)$$

We construct w with help of the Riesz representation theorem, so that we automatically have

$$\int uw = \int (\Delta^n w)^2 \sim \int |\nabla^{2n} w|^2 \gtrsim \int w^2, \quad (5.4.76)$$

where we used higher-order L^2 -regularity and a higher-order Poincaré estimate. We obtain from inserting (5.4.75) into (5.4.74)

$$\int (\eta^{4n} u)^2 \lesssim \left(\int |\eta^{4n+1} \nabla u|^2 \right)^{\frac{4n}{4n+1}} \left(\int w^2 \right)^{\frac{1}{4n+1}} + \int w^2.$$

Combining this with Caccioppoli's estimate $\int |\eta^{4n+1} \nabla u|^2 \lesssim \int (\eta^{4n} u)^2$ and Young's inequality, we obtain by the choice of η

$$\int_{B_1} |\nabla u|^2 \leq \int |\eta^{4n+1} \nabla u|^2 \lesssim \int w^2. \quad (5.4.77)$$

It remains to post-process this inner regularity estimate for an a -harmonic function u .

In route to an annealed estimate, we express the r. h. s. of (5.4.77) in terms of u , which is conveniently done in terms of the complete orthonormal system of eigenfunctions $\{w_k\}_{k \in \mathbb{N}} \subset H_0^{2n}(B_2)$ and eigenvalues $\{\lambda_k\}_k \subset (0, \infty)$ of the Dirichlet- Δ^{2n} , which is a positive operator with compact inverse:

$$\int w \sum_k \frac{1}{\lambda_k^2} \left(\int uw_k \right)^2 = \sum_k \frac{1}{\lambda_k} \frac{\left(\int uw_k \right)^2}{\int (\Delta^n w_k)^2} \stackrel{(5.4.76)}{\lesssim} \sum_k \frac{1}{\lambda_k} \frac{\left(\int uw_k \right)^2}{\int |\nabla^{2n} w_k|^2}.$$

We insert this into (5.4.77)

$$\int_{B_1} |\nabla u|^2 \lesssim \sum_k \frac{1}{\lambda_k} \frac{\left(\int uw_k \right)^2}{\int |\nabla^{2n} w_k|^2}$$

and apply $\langle (\cdot)^{\frac{p}{2}} \rangle$. By Hölder's inequality in k we obtain

$$\left\langle \left(\int_{B_1} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle \lesssim \left(\sum_{k'} \frac{1}{\lambda_{k'}} \right)^{\frac{p}{2}-1} \sum_k \frac{1}{\lambda_k} \frac{\langle |\int uw_k|^p \rangle}{\left(\int |\nabla^{2n} w_k|^2 \right)^{\frac{p}{2}}}.$$

In order to proceed, we need $\sum_k \frac{1}{\lambda_k} < \infty$, which means that the inverse of the Dirichlet- Δ^{2n} has finite trace, which in turn follows from the finiteness of the corresponding Green's function along the diagonal, which means that Dirac distributions are in $H^{-2n}(B_2)$, which amounts to the Sobolev embedding $H_0^{2n}(B_2) \subset C_0^0(B_2)$, and thus holds provided $2n > d$, which we henceforth assume. Hence by the density of $C_0^\infty(B_2)$ in $H_0^{2m}(B_2) \ni w_k$ we obtain the annealed inner regularity estimate

$$\left\langle \left(\int_{B_1} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle \lesssim \sup_{w \in C_0^\infty(B_2)} \frac{\langle | \int u w |^p \rangle}{\left(\int |\nabla^{2n} w|^2 \right)^{\frac{p}{2}}}. \quad (5.4.78)$$

It remains to post-process (5.4.78). Provided $2n > \frac{d}{2} + 2$ we may appeal to Sobolev's embedding applied to $\nabla^2 w$ in order to upgrade (5.4.78) to

$$\left\langle \left(\int_{B_1} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim \sup_{w \in C_0^\infty(B_2)} \frac{\langle | \int u w |^p \rangle^{\frac{1}{p}}}{\sup |\nabla^2 w|^2}. \quad (5.4.79)$$

Since we may w. l. o. g. assume $\int_{B_2} u = 0$, we may restrict to w with $\int w = 0$. A standard argument in the theory of distributions yields the existence of a vector field $g \in C_0^\infty((-2, 2)^d) \subset C_0^\infty(B_{2\sqrt{d}})$ such that

$$\nabla \cdot g = w \quad \text{and} \quad \sup |\nabla^2 g| \lesssim \sup |\nabla^2 w|.$$

Hence (5.4.79) may be upgraded to the desired

$$\left\langle \left(\int_{B_1} |\nabla u|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim \sup_{g \in C_0^\infty(B_{2\sqrt{d}})} \frac{\langle | \int g \cdot \nabla u |^p \rangle^{\frac{1}{p}}}{\sup |\nabla^2 g|}. \quad (5.4.80)$$

□

5.4.6 Proof of Lemma 39

We decompose the r. h. s. of (5.3.37) according to a family of dyadic annuli. Namely, we set

$$g_k := \mathbf{1}_{B_{2^k} \setminus B_{2^{k-1}}} \left((\phi_{ij}^{(2)} - \phi_{ij}^{(2)}(0))a - (\sigma_{ij}^{(2)} - \sigma_{ij}^{(2)}(0)) \right) \nabla \partial_{ij} \bar{u},$$

so that we can also decompose $\nabla w = \sum_{k=-\infty}^{\infty} \nabla w_k$, where ∇w_k is the square-integrable solution to

$$-\nabla \cdot a \nabla w_k = \nabla \cdot g_k.$$

Noting that for any k there holds $-\nabla \cdot a \nabla w_k = 0$ in $B_{2^{k-1}}$, we mainly applied stochastic corrector estimates (see Proposition 13) together with annealed estimates (see Lemma 41, and annealed

Calderón-Zygmund estimates [101, Proposition 7.1]) to the following computations.

$$\begin{aligned}
 \langle |\nabla w_k(0)|^{p'} \rangle_L^{\frac{1}{p'}} &\stackrel{(5.3.43)}{\lesssim} \left\langle \left(\int_{B_{2^{k-1}}(0)} |\nabla w_k|^2 \right)^{\frac{p''}{2}} \right\rangle_L^{\frac{1}{p''}} \leq \left(\int_{B_{2^{k-1}}(0)} \langle |\nabla w_k|^{p''} \rangle_L^{\frac{2}{p''}} \right)^{\frac{1}{2}} \\
 &\lesssim \left(\int_{B_{2^k}(0)} \langle |g_k|^{p'''} \rangle_L^{\frac{2}{p'''}} \right)^{\frac{1}{2}} \\
 &\stackrel{(5.3.25)}{\lesssim} \mu_3^{(2)}(2^k) \left(\int_{B_{2^k}(0)} \langle |\nabla^3 \bar{u}|^{p'''} \rangle_L^{\frac{2}{p'''}} \right)^{\frac{1}{2}} \leq \mu_3^{(2)}(2^k) \left(\int_{B_{2^k}(0)} \langle |\nabla^3 \bar{u}|^{p'''} \rangle_L^{\frac{q}{p'''}} \right)^{\frac{1}{q}} \\
 &\lesssim \mu_3^{(2)}(2^k) 2^{-\frac{dk}{q}} \left(\int \langle |\nabla^2 g|^{p'''} \rangle_L^{\frac{q}{p'''}} \right)^{\frac{1}{q}} \\
 &\lesssim \mu_3^{(2)}(2^k) 2^{-\frac{dk}{q}} R^{\frac{d}{q}} \sup |\nabla^2 g|,
 \end{aligned} \tag{5.4.81}$$

where $q \geq 2$ and $p' < p'' < p''' < p'''' < \infty$. We mention that the second line is due to [101, Proposition 7.1], while the fourth line follows from a (random) coordinate transformation¹⁵ coupled with [101, Lemma 7.4]. We distinguish the later proof into large-scale cases $R \geq 2$ and small-scale cases $R < 2$. Firstly, we handle the large scale case, and set k_0 to be an integer such that $2^{k_0-1} < R \leq 2^{k_0}$,

$$\begin{aligned}
 \langle |\nabla w(0)|^{p'} \rangle_L^{\frac{1}{p'}} &\lesssim \sum_{k=-\infty}^{\infty} \langle |\nabla w_k(0)|^{p'} \rangle_L^{\frac{1}{p'}} \\
 &\stackrel{(5.4.81)}{\lesssim} \left(R^{\frac{d}{q_1}} \sum_{k=-\infty}^{k_0-1} 2^{k(\frac{1}{2}-\frac{d}{q_1})} + R^{\frac{d}{q_2}} \sum_{k=k_0}^{\infty} 2^{k(\frac{1}{2}-\frac{d}{q_2})} \right) \sup |\nabla^2 g| \lesssim \mu_3^{(2)}(R) \sup |\nabla^2 g|,
 \end{aligned} \tag{5.4.82}$$

provided $q_1 > 2d$ and $2 \leq q_2 < 2d$. We can similarly address the small-scale case,

$$\langle |\nabla w(0)|^{p'} \rangle_L^{\frac{1}{p'}} \stackrel{(5.4.81)}{\lesssim} \left(R^{\frac{d}{q_3}} \sum_{k=-\infty}^{k_0-1} 2^{k(1-\frac{d}{q_3})} + R^{\frac{d}{q_4}} \sum_{k=k_0}^{\infty} 2^{k(1-\frac{d}{q_4})} \right) \sup |\nabla^2 g| \lesssim \mu_3^{(2)}(R) \sup |\nabla^2 g|,$$

whenever $q_3 > d$ and $2 \leq q_4 < d$. This completes the proof. □

5.4.7 Proof of Proposition 14: Homogenization error

We fix any $x_0, y_0 \in \mathbb{R}^d$, such that $2R := |x_0 - y_0|$ and $R \geq 1$. Let $g \in C_0^\infty(B_R(y_0))$, and $x, y \in \mathbb{R}^d$ with $x \neq y$. Let the error $V_{x_0, y_0}(x, y)$ be (fully) the two-scale expansion of the Green function, defined by

$$\begin{aligned}
 V_{x_0, y_0}(x, y) &:= G(x, y) - (1 + \phi_i^{(1)}(x) \partial_i + (\phi_{im}^{(2)}(x) - \phi_{im}^{(2)}(x_0)) \partial_{im}) \\
 &\quad \times (1 + \phi_j^{*(1)}(y) \partial_j + (\phi_{jn}^{*(2)}(y) - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{G}(x - y),
 \end{aligned} \tag{5.4.83}$$

¹⁵The ideas is to transform the coefficient \bar{a} to Δ and then appeal to annealed Calderón-Zygmund estimates on $-\Delta$. Consequently, the loss of stochastic integrability caused by the random transformations of the coordinate. Concretely, it relies on the following computations. Let $y = Qx$ such that $Q^{-1}\bar{a}(Q^*)^{-1} = \text{Id}$, and then there holds $\nabla_y = (Q^{-1})^* \nabla_x$. Let $\tilde{u}(x) = u(Qx)$ with $\tilde{g}(x) = Q^{-1}g(Qx)$. Thus, the equation $-\nabla \cdot \bar{a} \nabla \bar{u} = 0$ can be formulated by

$$\Delta \tilde{u} = \nabla \cdot \tilde{g}.$$

Then, one may apply [101, Lemma 7.4] to the above equation to have the desired estimate.

where we follow the similar rule of the notations in (5.2.15), and this leads to

$$\begin{aligned} \nabla \nabla V_{x_0, y_0}(x, y) &= \nabla \nabla G(x, y) \\ &- [(e_i + \nabla \phi_i^{(1)}(x)) \partial_i + (\nabla \phi_{im}^{(2)}(x) + \phi_i^{(1)} e_m) \partial_{im} + (\phi_{im}^{(2)}(x) - \phi_{im}^{(2)}(x_0)) e_k \partial_{imk}] \\ &\times [(e_j + \nabla \phi_j^{*(1)}(y)) \partial_j + (\nabla \phi_{jn}^{*(2)}(y) + \phi_j^{*(1)} e_n) \partial_{jn} + (\phi_{jn}^{*(2)}(y) - \phi_{jn}^{*(2)}(y_0)) e_l \partial_{jnl}] \bar{G}(x - y). \end{aligned} \quad (5.4.84)$$

Thus, recalling the definition of $\mathcal{E}(x, y)$ in (5.3.6), we have

$$\begin{aligned} \nabla \nabla V_{x_0, y_0}(x, y) - \mathcal{E}(x, y) &= \partial_{ijnl} \bar{G}(x - y) (\nabla \phi_i^{(1)} + e_i)(x) \otimes ((\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) e_l)(y) \\ &+ \partial_{imkj} \bar{G}(x - y) ((\phi_{im}^{(2)} - \phi_{im}^{(2)}(x_0)) e_k)(x) \otimes (\nabla \phi_j^{*(1)} + e_j)(y) \\ &- \partial_{imjn} \bar{G}(x - y) (\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)}(x)) \otimes (\phi_j^{*(1)} e_n + \nabla \phi_{jn}^{*(2)}(y)) \\ &+ \partial_{imjnl} \bar{G}(x - y) (\phi_i^{(1)} e_m + \nabla \phi_{im}^{(2)}(x)) \otimes ((\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) e_l)(y) \\ &- \partial_{imkjn} \bar{G}(x - y) ((\phi_{im}^{(2)} - \phi_{im}^{(2)}(x_0)) e_k)(x) \otimes (\phi_j^{*(1)} e_n + \nabla \phi_{jn}^{*(2)}(y)) \\ &+ \partial_{imkjnl} \bar{G}(x - y) ((\phi_{im}^{(2)} - \phi_{im}^{(2)}(x_0)) e_k)(x) \otimes ((\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) e_l)(y). \end{aligned}$$

We additionally mention that since the symmetry assumption on a , it is not very necessary to use the superscript “*” to indicate the adjoint objects. However, it is convenient to distinguish the variables in later computations. Then, taking $x = x_0$ and $y = y_0$, it follows from the above equality and Proposition 13 that

$$\langle |\nabla \nabla V_{x_0, y_0}(x_0, y_0) - \mathcal{E}(x_0, y_0)|^p \rangle_L^{\frac{1}{p}} \lesssim \mu_d^{(2)}(R) R^{-d-2} \quad \text{for all } p < \infty. \quad (5.4.85)$$

Obviously, to obtain the desire estimate (5.3.41) it suffices to show

$$\langle |\nabla \nabla V_{x_0, y_0}(x_0, y_0)|^p \rangle_L^{\frac{1}{p}} \lesssim \mu_3^{(2)}(R) R^{-d-2}. \quad (5.4.86)$$

Consequently, the stated estimate (5.3.41) simply follows from replacing x_0, y_0 with x, y , respectively.

Arguments for (5.4.86). We first consider the gradient of $V_{x_0, y_0}(x, y)$ in (5.4.83) with respect to the variable x , and then fix $x = x_0$. Hence, $\nabla V_{x_0, y_0}(x_0, \cdot)$ is expanded as follows:

$$\nabla V_{x_0, y_0}(x_0, \cdot) = \nabla G(x_0, \cdot) - (1 + \phi_j^{*(1)}(\cdot) \partial_j + (\phi_{jn}^{*(2)}(\cdot) - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, \cdot), \quad (5.4.87)$$

where $\bar{u}_{x_0}(\cdot, \cdot)$ is defined by

$$\bar{u}_{x_0}(x, \cdot) := ((e_i + \nabla \phi_i^{(1)}(x)) \partial_i + (\phi_i^{(1)}(x) e_m + \nabla \phi_{im}^{(2)}(x)) \partial_{im} + (\phi_{im}^{(2)}(x) - \phi_{im}^{(2)}(x_0)) e_k \partial_{imk}) \bar{G}(x - \cdot). \quad (5.4.88)$$

Moreover, we observe that

$$-\nabla \cdot a^* \nabla \nabla G(x_0, \cdot) = 0 = \nabla \cdot \bar{a}^* \nabla \bar{u}_{x_0}(x_0, \cdot) \quad \text{in } \mathbb{R}^d \setminus \{x_0\}.$$

By the two-scale expansion arguments introduced in Subsection 5.3.5, we arrive at

$$-\nabla \cdot a^* \nabla \nabla V_{x_0, y_0}(x_0, \cdot) = \nabla \cdot h \quad \text{in } \mathbb{R}^d \setminus \{x_0\}, \quad (5.4.89)$$

in which, the matrices h is given by

$$h := ((\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0))a^* - (\sigma_{jn}^{*(2)} - \sigma_{jn}^{*(2)}(y_0)))\nabla\partial_{jn}\bar{u}_{x_0}(x_0, \cdot). \tag{5.4.90}$$

Let k_0 be such that $2^{k_0-1} < R \leq 2^{k_0}$, and it suffices to study the equation (5.4.89) in a ball with the radius 2^{k_0} centered at y_0 (since we are then interesting in evaluating at $y = y_0$). To estimate it, we decompose its right-hand side according to a family of dyadic annuli, which are

$$-\nabla \cdot a^*\nabla Z_k = \nabla \cdot h_k \quad \text{in } \mathbb{R}^d \quad \text{for } h_k := \mathbb{1}_{B_{2^k}(y_0)\setminus B_{2^{k-1}}(y_0)}h \quad \text{with } k \leq k_0. \tag{5.4.91}$$

Then we set $W = \nabla V_{x_0, y_0}(x_0, \cdot) - \sum_{k=-\infty}^{k_0} Z_k$, and it satisfies a^* -harmonic condition on $B_{2^{k_0}}(y_0)$, i.e.,

$$-\nabla \cdot a^*\nabla W = 0 \quad \text{in } B_{2^{k_0}}(y_0). \tag{5.4.92}$$

We will handle the equations (5.4.91) and (5.4.92), separately.

As a preparation, recalling the definition of (5.4.88), for all positive integer j and $p < \infty$, we derive from Proposition 13 and decay estimates of \bar{G} that

$$\sup_{y \in B_R(y_0)} \langle |\nabla^j \bar{u}_{x_0}(x_0, y)|^p \rangle_L^{\frac{1}{p}} \lesssim R^{-j-d+1} \left(1 + \left\langle \left| \left(\nabla \phi^{(1)}(x_0), \frac{\phi^{(1)}(x_0)}{R}, \frac{\nabla \phi^{(2)}(x_0)}{R} \right) \right|^p \right\rangle_L^{\frac{1}{p}} \right) \lesssim R^{-j-d+1}, \tag{5.4.93}$$

where we require $R \geq 1$ in the last inequality.

Let $p''' > p'' > p' > p$. Appealing to Lemma 41, the energy estimate yields

$$\begin{aligned} \sum_{k=-\infty}^{k_0} \langle |\nabla Z_k(y_0)|^{p'} \rangle_L^{\frac{1}{p'}} &\stackrel{(5.3.43)}{\lesssim} \sum_{k=-\infty}^{k_0} \left\langle \left(\int_{B_{2^{k-1}}(y_0)} |\nabla Z_k|^2 \right)^{\frac{p''}{2}} \right\rangle_L^{\frac{1}{p''}} \\ &\lesssim \sum_{k=-\infty}^{k_0} \left\langle \left(\int_{B_{2^k}(y_0)} |h|^2 \right)^{\frac{p''}{2}} \right\rangle_L^{\frac{1}{p''}} \stackrel{(5.4.95)}{\lesssim} \mu_3^{(2)}(R)R^{-d-2}. \end{aligned} \tag{5.4.94}$$

For the last inequality above, in order to use Minkowski's inequality and Jensen's inequality, we will obtain the second line of the following computations (5.4.95), and then on account of Proposition 13, a similar calculation as given for (5.4.82) leads to the last line of (5.4.95).

$$\begin{aligned} \sum_{k=-\infty}^{k_0} \left\langle \left(\int_{B_{2^k}(y_0)} |h|^2 \right)^{\frac{p''}{2}} \right\rangle_L^{\frac{1}{p''}} &\leq \sum_{k=-\infty}^{k_0} \left(\int_{B_{2^k}(y_0)} \langle |h|^{p''} \rangle_L^{\frac{2}{p''}} \right)^{\frac{1}{2}} \\ &\leq \sum_{k=-\infty}^{k_0} \left(\int_{B_{2^k}(y_0)} \left\langle \left| \left(\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0), \sigma_{jn}^{*(2)} - \sigma_{jn}^{*(2)}(y_0) \right) \right|^{2p''} \right\rangle_L^{\frac{1}{p''}} \right)^{\frac{1}{2}} \left(\int_{B_{2^k}(y_0)} \langle |\nabla \partial_{jn} \bar{u}_{x_0}(x_0, \cdot)|^{2p''} \rangle_L^{\frac{1}{p''}} \right)^{\frac{1}{2}} \\ &\stackrel{(5.3.25), (5.3.26), (5.4.93)}{\lesssim} \sum_{k=-\infty}^{k_0} \mu_3^{(2)}(2^k)R^{-2-d} \lesssim \mu_3^{(2)}(R)R^{-2-d}, \end{aligned} \tag{5.4.95}$$

in which we also ask for the condition $2 \leq p'' < \infty$ in first inequality, and recall the relationship $2^{k_0-1} < R \leq 2^{k_0}$ for the last one. Beyond the pointwise estimate (5.4.94), we still require some weak norm estimate later on, and now establish it as follows. In fact, the argument is quite similar to that given (5.4.95), and the main difference is to employ annealed Calderón-Zygmund estimates

[101, Proposition 7.1] for the second line below.

$$\begin{aligned}
 \sum_{k=-\infty}^{k_0} \left\langle \left(\int_{B_R} |\nabla Z_k|^{p'} \right)^{\frac{1}{p'}} \right\rangle_L &\leq \sum_{k=-\infty}^{k_0} \int_{B_R} \langle |\nabla Z_k|^{p'} \rangle_L^{\frac{1}{p'}} \leq \sum_{k=-\infty}^{k_0} \left(\int_{B_R} \langle |\nabla Z_k|^{p'} \rangle_L^{\frac{q}{p'}} \right)^{\frac{1}{q}} \\
 &\lesssim R^{-\frac{d}{q}} \sum_{k=-\infty}^{k_0} \left(\int \langle |h_k|^{p''} \rangle_L^{\frac{q}{p''}} \right)^{\frac{1}{q}} \\
 &\lesssim R^{-\frac{d}{q}} \sum_{k=-\infty}^{k_0} (2^k)^{\frac{1}{2}-\frac{d}{q}} \sup_{y \in B_R(y_0)} \langle |\nabla^3 \bar{u}_{x_0}(y)|^{p'''} \rangle_L^{\frac{1}{p'''}} \stackrel{(5.4.93)}{\lesssim} \mu_3^{(2)}(R) R^{-2-d},
 \end{aligned}$$

provided $q \geq 2d$, and the third line is due to the same ideas given for (5.4.82). In the sequel, the above estimate implies the weak norm estimate, i.e., for any $g \in C_0^\infty(B_R(y_0))$,

$$\left\langle \left| \sum_{k=-\infty}^{k_0} \int g \cdot \nabla Z_k |^{p'} \right|^{\frac{1}{p'}} \right\rangle_L \leq \sup_{y \in B_R(y_0)} |g(y)| \sum_{k=-\infty}^{k_0} \left\langle \left(\int_{B_R(y_0)} |\nabla Z_k|^{p'} \right)^{\frac{1}{p'}} \right\rangle_L \lesssim \mu_3^{(2)}(R) \sup_{y \in B_R(y_0)} |\nabla^2 g(y)|, \quad (5.4.96)$$

where we also employ Poincaré's inequality in the last step.

We now turn to address the estimate on ∇W in the equation (5.4.92). In view of Lemmas 40 and 41, there holds

$$\begin{aligned}
 \left\langle |\nabla W(y_0)|^p \right\rangle_L^{\frac{1}{p}} &\stackrel{(5.3.43)}{\lesssim} \left\langle \left(\int_{B_{\frac{R}{2}}(y_0)} |\nabla W|^2 \right)^{p'} \right\rangle_L^{\frac{1}{2p'}} \stackrel{(5.3.42)}{\lesssim} \sup_{g \in C_0^\infty(B_R(y_0))} \frac{\langle \int_{B_R(y_0)} g \cdot \nabla W |^{p'} \rangle_L^{\frac{1}{p'}}}{R^2 \sup |\nabla^2 g|} \\
 &\leq \sup_{g \in C_0^\infty(B_R(y_0))} \frac{\langle \int g \cdot \nabla \nabla V_{x_0, y_0}(x_0, \cdot) |^{p'} \rangle_L^{\frac{1}{p'}}}{R^{d+2} \sup |\nabla^2 g|} + \sup_{g \in C_0^\infty(B_R(y_0))} \frac{\langle \sum_{k=-\infty}^{k_0} \int g \cdot \nabla Z_k |^{p'} \rangle_L^{\frac{1}{p'}}}{R^{d+2} \sup |\nabla^2 g|} \\
 &\stackrel{(5.4.98), (5.4.96)}{\lesssim} \mu_3^{(2)}(R) R^{-d-2},
 \end{aligned} \quad (5.4.97)$$

where the last step relies on the following estimate

$$\left\langle \left| \int dy \nabla \nabla V_{x_0, y_0}(x_0, y) \cdot g(y) \right|^p \right\rangle_L^{\frac{1}{p}} \lesssim_{\gamma, p} \mu_3^{(2)}(R) \sup |\nabla^2 g| \quad (5.4.98)$$

for any $p < \infty$. As a consequence, the estimates (5.4.94) and (5.4.97) lead to the desired estimate (5.4.86) by the triangle inequality.

Arguments for (5.4.98). Let u with \bar{u} and $g \in C_0^\infty(B_R(y_0))$ be associated by the equation (5.3.38). Recalling the expansion (5.3.36), we may define w_{x_0} as follows.

$$\begin{aligned}
 w_{x_0}(x) &:= u(x) - \left(1 + \phi_i^{(1)}(x) \partial_i + (\phi_{im}^{(2)}(x) - \phi_{im}^{(2)}(x_0)) \partial_{im} \right) \bar{u}(x) \\
 &\stackrel{(5.3.38)}{=} \int dy \left(G(x, y) - (1 + \phi_i^{(1)}(x) \partial_i + (\phi_{im}^{(2)}(x) - \phi_{im}^{(2)}(x_0)) \partial_{im}) \bar{G}(x - y) \right) (\nabla \cdot g)(y),
 \end{aligned}$$

and then by differentiating in both sides w. r. t. the x -variable combined with an integration by parts, we arrive at

$$\nabla w_{x_0}(x) \stackrel{(5.4.88)}{=} - \int dy (\nabla \nabla G(x, y) - \nabla_x \bar{u}_{x_0}(x, y)) g(y). \quad (5.4.99)$$

By the formula (5.4.88), we rewrite (5.4.84) as

$$\begin{aligned} \nabla \nabla V_{x_0, y_0}(x, y) &= \nabla \nabla G(x, y) \\ &- [(e_j + \nabla \phi_j^{*(1)}(y)) \partial_j + (\nabla \phi_{jn}^{*(2)}(y) + \phi_j^{*(1)} e_n) \partial_{jn} + (\phi_{jn}^{*(2)}(y) - \phi_{jn}^{*(2)}(y_0)) e_l \partial_{jnl}] \bar{u}_{x_0}(x, y). \end{aligned} \quad (5.4.100)$$

Thus, we split the integral in the l. h. s. of (5.4.98) as follows (where in the last line we use an integration by parts):

$$\begin{aligned} &\int dy \nabla \nabla V_{x_0, y_0}(x_0, y) \cdot g(y) \\ &\stackrel{(5.4.100)}{=} \int dy (\nabla \nabla G(x_0, y) - \nabla \bar{u}_{x_0}(x_0, y)) \cdot g(y) \\ &\quad - \int dy \nabla (\phi_j^{*(1)}(y) \partial_j + (\phi_{jn}^{*(2)}(y) - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, y) g(y) \\ &\stackrel{(5.4.99)}{=} -\nabla w_{x_0}(x_0) - \int dy (\phi_j^{*(1)}(y) \partial_j + (\phi_{jn}^{*(2)}(y) - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, y) (\nabla \cdot g)(y). \end{aligned} \quad (5.4.101)$$

It follows from Lemma 39 that¹⁶,

$$\langle |\nabla w_{x_0}(x_0)|^p \rangle^{\frac{1}{p}} \lesssim \mu_3^{(2)}(R) \sup |\nabla^2 g|, \quad (5.4.102)$$

We rewrite the second term of the right-hand side of (5.4.101) by means of ω_j^* defined by (5.3.22), so that an integration by parts yields

$$\begin{aligned} &\int dy (\phi_j^{*(1)} \partial_j + (\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, \cdot) (\nabla \cdot g) \\ &\stackrel{(5.3.22)}{=} \int dy (-\Delta \omega_j^* \partial_j + (\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, \cdot) (\nabla \cdot g) \\ &= \int dy (\partial_n \omega_j^* - \partial_n \omega_j^*(y_0) + \phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) \partial_{jn} \bar{u}_{x_0}(x_0, \cdot) (\nabla \cdot g) \\ &+ \int dy ((\nabla \omega_j^* - \nabla \omega_j^*(y_0)) \partial_j \bar{u}_{x_0}(x_0, \cdot)) \cdot \nabla (\nabla \cdot g). \end{aligned}$$

Therefore, there holds

$$\begin{aligned} &\left\langle \left| \int dy (\phi_j^{*(1)} \partial_j + (\phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0)) \partial_{jn}) \bar{u}_{x_0}(x_0, \cdot) (\nabla \cdot g) \right|^p \right\rangle_L^{\frac{1}{p}} \\ &\leq \int dy \langle |(\partial_n \omega_j^* - \partial_n \omega_j^*(y_0), \phi_{jn}^{*(2)} - \phi_{jn}^{*(2)}(y_0))|^{2p} \rangle_L^{\frac{1}{p}} \langle |\partial_{jn} \bar{u}_{x_0}(x_0, \cdot)|^{2p} \rangle_L^{\frac{1}{p}} |\nabla g| \\ &\quad + \int dy \langle |\nabla \omega_j^* - \nabla \omega_j^*(y_0)|^{2p} \rangle_L^{\frac{1}{2p}} \langle |\partial_j \bar{u}_{x_0}(x_0, \cdot)|^{2p} \rangle_L^{\frac{1}{2p}} |\nabla^2 g| \\ &\stackrel{(5.3.27), (5.4.93)}{\lesssim} \mu_d^{(2)}(R) R^{-1} \sup |\nabla g| + \mu_d^{(2)}(R) \sup |\nabla^2 g| \lesssim \mu_d^{(2)}(R) \sup |\nabla^2 g|. \end{aligned} \quad (5.4.103)$$

Inserting the estimates (5.4.102) and (5.4.103) into (5.4.101) entails (5.4.98). \square

¹⁶Observe that if we anchor the second correctors as in (5.3.36), we have to do the same procedure in Lemma 39 to obtain a similar estimate as (5.3.39) (where note that the location of the support of g is not important).

5.5 Appendix

5.5.1 Regularity of the coefficients

We give in this section the argument for (5.3.2). Without loss of generality, we may assume that $x = 0$.

By the stationarity of the ensemble $\langle \cdot \rangle$ and the inequality (5.1.4), we have

$$\langle |g(x) - g(0)|^2 \rangle_L = 2(c_L(0) - c_L(x)) \leq 2|\nabla c_L(0)| + \|\nabla^2 c_L\|_{L^\infty(\mathbb{R}^d)}|x|^2 \lesssim |x|^2, \quad (5.5.1)$$

since $\nabla c_L(0) = 0$ (as c_L is an even function). High moments of Gaussian variables are estimated by their second moments. Therefore, we have $\langle |g|^p \rangle_L \lesssim c_L^{\frac{p}{2}}(0) \lesssim 1$ and we deduce from (5.5.1) that, by stationarity of g , there holds

$$\left\langle \int_{B_2} \int_{B_2} \left(\frac{|g(x+h) - g(x)|}{|h|} \right)^p dh dx + \int_{B_2} |g|^p \right\rangle_L \lesssim 1 \quad \text{for all } p \in [1, +\infty).$$

By [18, Th. 2.36 p. 74], this implies that g is bounded in the Besov spaces $B_{p,p}^s$ for any $s \in (0, 1)$ in the sense of

$$\langle \|\eta g\|_{B_{p,p}^s(\mathbb{R}^d)}^p \rangle_L^{\frac{1}{p}} \lesssim 1, \quad (5.5.2)$$

where η is a smooth cut-off with support inside B_2 . By a Besov embedding [18, Prop. 2.71 p. 99], we can replace the Besov space $B_{p,p}^s(\mathbb{R}^d)$ in (5.5.2) by the Besov space $B_{\infty,\infty}^\beta$ for all $\beta \in (0, 1)$ which coincides with the Hölder space $C^{0,\beta}(\mathbb{R}^d)$ (see [18, p. 99, Examples]). Hence, g satisfies (5.3.2). Last, the relation $a = A(g)$ along with (5.1.6) implies that a also satisfies (5.3.2).

5.5.2 Convergence of the massive quantities

We give in this section the argument for (5.3.12).

We only prove the convergence of ∇G_T , the other convergences will follow by the same type of arguments. Let $\beta \in (0, 1)$, $x \neq y \in \mathbb{R}^d$ and $r := |x - y|/2$. In the following proof, the symbol \lesssim implicitly depends on γ, β, r and $\|a\|_{C^{0,\beta}(\mathbb{T}_L)}$ but not on T . Thanks to the Arzelà-Ascoli theorem and the dominated convergence theorem, it is enough to prove the two following statements:

$$\forall x \in \mathbb{R}^d, \nabla_x \nabla_y G_T(x, \cdot) \xrightarrow{T \uparrow \infty} \nabla_x \nabla_y G_T(x, \cdot) \text{ in the distributional sense in } B_r(y), \quad (5.5.3)$$

and

$$\limsup_{T \uparrow \infty} \|\nabla_x \nabla_y G_T(x, \cdot)\|_{C^{0,\beta}(B_r(y))} \lesssim 1. \quad (5.5.4)$$

We first give arguments for (5.5.3). Let $f \in C_c^\infty(B_r(y))$, and the square-integrable functions u_T and ∇u satisfying

$$\begin{cases} \frac{1}{T} u_T - \nabla \cdot a \nabla u_T = \nabla \cdot f, \\ -\nabla \cdot a \nabla u = \nabla \cdot f. \end{cases} \quad (5.5.5)$$

Using the energy estimates, there holds

$$\left\| \left(\frac{u_T}{\sqrt{T}}, \nabla u_T \right) \right\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla u\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \quad (5.5.6)$$

The difference $u_T - u$ solves

$$-\nabla \cdot a \nabla (u_T - u) = -\frac{1}{T} u_T. \quad (5.5.7)$$

Hence, by the interior Schauder estimate (see, *e.g.*, [77, Cor. 8.36 p. 212]) applied to the equation (5.5.7) we have

$$\begin{aligned} \|\nabla u_T - \nabla u\|_{C^{0,\beta}(B_r(x))} &\lesssim \|\nabla u_T - \nabla u\|_{L^2(B_{2r}(x))} + \frac{1}{T} \|u_T\|_{L^2(B_{2r}(x))} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (5.5.8)$$

Moreover, Since ∇u_T is uniformly bounded in $L^2(\mathbb{R}^d)$, we may extract a weak limit $\nabla u_\infty \in L^2(\mathbb{R}^d)$ when $T \uparrow \infty$. By (5.5.5) and using the above estimates, we deduce that $-\nabla \cdot a \nabla u_\infty = \nabla \cdot f$. By the Liouville principle associated with the operator $-\nabla \cdot a \nabla$ (recall that a is \mathbb{T}_L -periodic), we finally obtain that $\nabla u_\infty = \nabla u$. Therefore, we deduce that

$$\nabla u_T \xrightarrow{T \uparrow \infty} \nabla u \quad \text{in } L^2(\mathbb{R}^d).$$

Combining this with (5.5.8), we get that

$$\|\nabla(u_T - u)\|_{C^{0,\frac{\beta}{2}}(B_r(x))} \xrightarrow{T \uparrow \infty} 0.$$

From the pointwise convergence in x and the formula

$$\nabla u_T(x) - \nabla u(x) = \int (\nabla_x \nabla_y G_T(x, y') - \nabla_x \nabla_y G(x, y')) f(y') dy',$$

we obtain the convergence (5.5.3).

We now give the arguments for (5.5.4). We have from the definition of the Green function G_T (since a is symmetric)

$$-\nabla_y \cdot a(y) \nabla_y \nabla_x G_T(x, \cdot) = -\frac{1}{T} \nabla_x G_T(x, \cdot) \quad \text{in } \mathbb{R}^d \setminus B_{r/5}(x).$$

Therefore, using interior Schauder estimate combined with the Caccioppoli inequality we obtain

$$|\nabla_x \nabla_y G_T(x, y)| \lesssim \|G_T\|_{L^\infty(B_{r/4}(x) \times B_{r/4}(y))}. \quad (5.5.9)$$

Moreover, we already know from [93] (with the very slight difference that there is here an extra massive term, which is of no harm for the result) that, for all $x' \neq y'$ and for all $T \geq 1$, there holds $|G_T(x', y')| \lesssim |x' - y'|^{2-d}$. Therefore, (5.5.4) follows from inserting the above estimate into (5.5.9).

5.5.3 Localized energy estimates

We recall the exponential localization for the massive operator $\frac{1}{T} - \nabla \cdot a \nabla$. For a proof, we refer to [68, Lemma 36].

Proposition 15. *Let $T > 0$ and $L \geq \sqrt{T}$. Let $u \in H_{loc}^1(\mathbb{R}^d)$ and $f \in H_{loc}^1(\mathbb{R}^d)$ be related to*

$$\frac{1}{T} u - \nabla \cdot a \nabla u = \nabla \cdot f,$$

in the distributional sense in \mathbb{R}^d . Suppose that there exists a $k > 0$ such that

$$\limsup_{R \rightarrow +\infty} R^{-k} \left(\int_{B_R} (|u| + |\nabla u| + |f|)^2 \right)^{\frac{1}{2}} = 0.$$

Then, there exists a $\gamma > 0$ depending on and λ such that

$$\int dx (|\nabla u|^2 + \frac{1}{T}|u|^2) e^{-\frac{\gamma|x|}{L}} \lesssim_{d,\lambda} \int dx |f|^2 e^{-\frac{\gamma|x|}{L}}.$$

Chapter 6

Perspectives and remaining open questions

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6.1 On linear elliptic systems

6.1.1 Second order parabolic semigroup

In Chapter 2, we prove optimal time decay estimates for the first order semigroup defined in (1.2.2) related to the first order correctors. A natural extension would be to go one order further and to consider the second order semigroup $u^{(2)} = (u_{ij}^{(2)})_{ij}$ defined by:

$$\begin{cases} \partial_\tau u_{ij}^{(2)} - \nabla \cdot a \nabla u_{ij}^{(2)} = 0 & \text{in } \mathbb{R}^d \times (0, +\infty), \\ u^{(2)}(0) = \nabla \cdot ((a\phi_{e_i} - \sigma_i)e_j). \end{cases}$$

The second order semigroup is related to the second order correctors $(\phi_{ij}^{(2)})_{ij} \in H_{loc}^1(\mathbb{R}^d)$, which are the distributional and sub-quadratic¹ (up to an additive constant) solutions in \mathbb{R}^d of

$$-\nabla \cdot a \nabla \phi_{ij}^{(2)} = \nabla \cdot ((a\phi_{e_i} - \sigma_i)e_j). \tag{6.1.1}$$

The second order correctors provide an improvement of the convergence of the homogenization error whence the two-scale expansion is truncated at second order, see [27]. The second order approximation finds also its interest in the study of numerical methods, as in [114] where the authors mentioned

¹In the sense that $\frac{1}{R^2} \int_{B_R} |\phi_{ij}^{(2)}|^2 \xrightarrow{R \uparrow +\infty} 0$.

that the use of the second order correctors would allow to deal with artificial boundary conditions for approximating the homogenized matrix in dimension 3. In the latter application, we have to appeal to massive term approximation, defined by adding a massive term $\frac{1}{T}\phi_{ij}^{(2)}$ in (6.1.1). Optimal estimates on the massive term correctors are then necessary and they can be easily deduced from the optimal time decay estimates on $u^{(2)}$ (as we did in Chapter 2, see Corollary 5).

At first sight, in dimension 3 and under the assumption MSG (1.1.63), the gradient of second order correctors can be constructed with help of $u^{(2)}$ only for $\beta > 2$, which corresponds to the regime where ϕ can be constructed stationary itself (see Corollary 3). Indeed, this is easily inferred in the small contrast regime, that we perform hereafter. We assume that $a^\delta = (1 + \delta A)\text{Id}$, for a small $\delta \ll 1$ and for the scalar Gaussian model $A = A(m \star \xi) \in \mathbb{R}$ for $|m(x)| \leq (1 + |x|)^{-\frac{1}{2}(d+\beta)}$. By linearizing around $\delta \downarrow 0$, the first order approximation of $u^{(2)}$ is given by $u_{ij}^{(2)} = \delta \bar{u}^{(2)} + o(\delta)$ where

$$\begin{cases} \partial_\tau \bar{u}^{(2)} - \nabla \cdot a \nabla \bar{u}^{(2)} = 0 & \text{in } \mathbb{R}^d \times (0, +\infty), \\ \bar{u}^{(2)}(0) = \nabla \cdot (\bar{\phi}_{e_i} - \bar{\sigma}_i) e_j, \end{cases}$$

with

$$-\Delta \bar{\phi}_{e_i} = \nabla \cdot A e_i \text{ and } -\Delta \bar{\sigma}_{ijk} = \partial_j (e_k \cdot \nabla \bar{\phi}_{e_i}) - \partial_k (e_j \cdot \nabla \bar{\phi}_{e_i}).$$

In the following, we only consider the part of $\bar{u}^{(2)}$ corresponding to the initial value $\nabla \cdot (\bar{\phi}_{e_i} e_j)$ (the flux part is treated the same way) and for the ease of the reading we keep the notation $\bar{u}^{(2)}$. Recall that since $d = 3$ and $\beta > 2$, $\bar{\phi}_{e_i}$ can be expressed with help of the first order semigroup \bar{u}_{e_i} (see (1.2.4) with $e = e_i$):

$$\bar{\phi}_{e_i} = \int_0^{+\infty} \bar{u}_{e_i}(s, \cdot) ds = - \int_0^{+\infty} \int \nabla \Gamma(s, \cdot - y) \cdot A(y) e_i dy ds,$$

where $\Gamma(s, x) = (4\pi s)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4s}}$. Therefore, $\bar{u}^{(2)}$ is explicitly given by: for all $(T, x) \in (0, +\infty) \times \mathbb{R}^d$

$$\begin{aligned} \bar{u}^{(2)}(T, x) &= - \int \nabla \Gamma(T, x - y) \cdot \bar{\phi}_{e_i} e_j dy \\ &= - \int_0^{+\infty} \int \int \partial_j \Gamma(T, x - y) \partial_i \Gamma(s, y - y') A(z) dy' dy ds. \end{aligned}$$

By differentiating with respect to the randomness (in the sense of (1.1.61)) and using the semigroup property of Gaussian kernels in form of $\Gamma(T, \cdot) \star \Gamma(s, \cdot) = \frac{1}{\pi^{\frac{d}{2}}} \Gamma(T + s, \cdot)$, we have for all $(z, \ell) \in \mathbb{R}^d \times (1, +\infty)$

$$\begin{aligned} \partial_{z, \ell}^{\text{fct}} \bar{u}^{(2)}(T, x) &= \int_{\mathbb{B}_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T, \cdot) \star \Gamma(s, \cdot)(x - y) ds \right| dy \\ &= \frac{1}{\pi^{\frac{d}{2}}} \int_{\mathbb{B}_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T + s, x - y) ds \right| dy. \end{aligned}$$

We now apply MSG (1.1.63) to the effect of: for all $q < +\infty$

$$\mathbb{E}[(\bar{u}^{(2)})^q]^{\frac{1}{q}} \lesssim \left(\int_1^{+\infty} \ell^{-d-1-\beta} \int \left(\int_{\mathbb{B}_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T + s, x - y) ds \right| dy \right)^2 dz d\ell \right)^{\frac{1}{2}}. \quad (6.1.2)$$

We now consider two regimes in ℓ .

- **Regime $\ell \leq \sqrt{T}$:** Here, we use L^2 -estimates in form of Jensen's inequality $(\int_{B_\ell(z)} f(y)dy)^2 \lesssim \ell^d \int_{B_\ell(z)} |f(y)|^2 dy$, to the effect of

$$\begin{aligned} \int \left(\int_{B_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, x-y) ds \right| dy \right)^2 dz &\lesssim \ell^{2d} \int \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, x-y) ds \right|^2 dy \\ &\leq \left(\int_0^{+\infty} \left(\int (\partial_i \partial_j \Gamma(T+s, y))^2 dy \right)^{\frac{1}{2}} ds \right)^2 \\ &\lesssim \int_0^{+\infty} (T+s)^{-1-\frac{d}{2}} ds \lesssim T^{-\frac{d}{2}}. \end{aligned}$$

Therefore, the contribution of the right-hand side of (6.1.2) in the regime $\ell \leq \sqrt{T}$ is given by

$$\begin{aligned} \left(\int_1^{\sqrt{T}} \ell^{-d-1-\beta} \int \left(\int_{B_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, x-y) ds \right| dy \right)^2 dz d\ell \right)^{\frac{1}{2}} &\lesssim T^{-\frac{d}{2}} \left(\int_1^{\sqrt{T}} \ell^{d-1-\beta} d\ell \right)^{\frac{1}{2}} \\ &\lesssim \begin{cases} T^{-\frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) T^{-\frac{d}{4}} & \text{if } \beta = d, \\ T^{-\frac{d}{4}} & \text{if } \beta > d. \end{cases} \end{aligned} \quad (6.1.3)$$

- **Regime $\ell > \sqrt{T}$:** Here, we use L^1 -estimates in form of Minkowski's inequality $\int (\int_{B_\ell(z)} f)^2 \lesssim \ell^d \int |f|$:

$$\int \left(\int_{B_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, x-y) ds \right| dy \right)^2 dz \lesssim \ell^d \int \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, y) ds \right| dy \lesssim 2\ell^d.$$

Therefore, the contribution of the right-hand side of (6.1.2) in the regime $\ell > \sqrt{T}$ is given by:

$$\begin{aligned} \left(\int_{\sqrt{T}}^{+\infty} \ell^{-d-1-\beta} \int \left(\int_{B_\ell(z)} \left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, x-y) ds \right| dy \right)^2 dz d\ell \right)^{\frac{1}{2}} &\lesssim \left(\int_{\sqrt{T}}^{+\infty} \ell^{-1-\beta} d\ell \right)^{\frac{1}{2}} \\ &\lesssim T^{-\frac{\beta}{4}}. \end{aligned} \quad (6.1.4)$$

The combination of (6.1.3) and (6.1.4) finally gives

$$\mathbb{E}[(\bar{u}^{(2)})^q]^{\frac{1}{q}} \lesssim \begin{cases} T^{-\frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) T^{-\frac{d}{4}} & \text{if } \beta = d, \\ T^{-\frac{d}{4}} & \text{if } \beta > d. \end{cases}$$

Similarly, we can obtain

$$\mathbb{E}[|\nabla \bar{u}^{(2)}|^q]^{\frac{1}{q}} \lesssim \begin{cases} T^{-\frac{1}{2}-\frac{\beta}{4}} & \text{if } \beta < d, \\ \log^{\frac{1}{2}}(T) T^{-\frac{1}{2}-\frac{d}{4}} & \text{if } \beta = d, \\ T^{-\frac{1}{2}-\frac{d}{4}} & \text{if } \beta > d. \end{cases}$$

We see that for $\beta < d$, $\nabla \bar{u}^{(2)}$ is integrable only if $\beta > 2$. The extension in the non-perturbative regime as well as its application to artificial boundary conditions is left for future investigation.

²Where we use the bounds $\left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, y) ds \right| \lesssim T^{-\frac{d}{2}}$ for $|y| \leq \sqrt{T}$ and $\left| \int_0^{+\infty} \partial_i \partial_j \Gamma(T+s, y) ds \right| \lesssim |x|^{-d} e^{-\frac{|x|^2}{T}}$ for $|y| \geq \sqrt{T}$.

6.1.2 Representative volume element method: beyond the Gaussian setting with short-range correlations and linear equations

In Chapter 5, we prove optimal estimates for the representative volume element method for approximating the homogenized coefficient. The present work is however limited to Gaussian distributions with integrable covariance, essentially for two reasons:

- Because of the way we compute the derivative $\frac{d}{dL}\langle\bar{a}\rangle_L$, using Price's formula of Chapter 4, which is restricted to Gaussian fields.
- Because of the way we periodize the law: we strongly use the integrability of the covariance function allowing for a straightforward periodization (see (5.1.7)).

A natural extension consists of considering other type of random distributions, based for instance on point processes or Gaussian distributions with slowly decaying correlations. A first step would be to understand the periodization in law. Such a periodization in law is immediate, for instance, in the case of Poisson point processes or i.i.d coefficients (see [88]), whereas the case of slowly decaying correlations does not allow a straightforward periodization. Once we have define the periodization in law $\langle\cdot\rangle_L$, a second step would be to understand the derivative with respect to L . An alternative of the Price's formula of Chapter 4 has to be found in the non Gaussian setting.

An other possible work is to extend the results of Chapter 5 in the nonlinear setting (with the same Gaussianity of the coefficient). Such work in this direction has been done recently in [68] for quadratic nonlinearity (that is for the model (1.1.4) with $p = 2$). In [68], however, the proof is more in the vein of the work of Gloria, Neukamm and Otto in [88] where the authors introduce a massive term into the corrector equation in order to screen the resulting boundary layer, which leads to a logarithmically worse estimate. A possible way to avoid this loss is to adapt the strategy of Chapter 5. This would require several results, where one of them are already established in Chapter 3. More precisely:

- One needs stochastic estimates on the correctors for the periodized ensemble $\langle\cdot\rangle_L$. The stochastic estimates for the first order correctors has been already established in Chapter 3, but we will also need stochastic estimates on the second order correctors (also for the linearized ones). Such estimates may be established with help of the perturbative large-scale regularity developed in Chapter 3.
- One needs to define the Green's function of the linearized operator $-\nabla \cdot Da(\cdot, \xi + \nabla\phi)\nabla$ and to prove its second order two-scale expansion. To do so, the perturbative large-scale regularity theory of Chapter 3 will not be enough. However, since we prove good control of the correctors in Chapter 3, we have access to the non-perturbative regularity theory for the linearized operator, using the result in [24], but in the regime $p < \frac{2d}{d-2}$ in dimension 3. This may allow to prove the existence of the Green's function as well as (by combining the regularity theory for the linearized operator and the stochastic control of the first and second order correctors) its second order two-scale expansion.

6.2 On nonlinear elliptic systems

6.2.1 Nonlinear systems with monotone coefficients

In Chapter 3, we developed a quantitative homogenization theory for the nonlinear system (1.1.4) in the full range $p \geq 2$ for $d = 3$ and under LSI (1.1.57). Some questions and extensions, that we

list hereafter, are still open.

- Does such a quantitative homogenization theory hold with other mixing conditions as finite range of dependence (1.1.56) (as addressed in [10, 5, 6] for $p = 2$) or MLSI (1.1.62)? The quenched Meyers' estimates of Theorem 18 (and its weighted version of Theorem 19) do hold for general stationary ergodic coefficients – and therefore in the setting of finite range of dependence or MLSI. They are however useless without a good control of the Meyers minimal radius (provided by Theorem 21 for Gaussian coefficients with integrable correlation). As a direct consequence of the control of the Meyers minimal radius, we obtain a control of moments of the corrector gradient (cf. Lemma 15). For finite range of dependence [6, 10], such estimates are obtained by combining a rate of convergence (any would do) for the Dirichlet problem with a Campanato argument based on $C^{1,\alpha}$ -regularity for the homogenized operator. On the one hand, it is not completely clear how to adapt the duality arguments (the Fenchel transform of a map with p -growth has $\frac{p}{p-1}$ -growth – which we can also see to some extent at the level of the stochastic integrability of $\nabla\sigma_\xi$ in Definition 3.2.1) to prove convergence rates. On the other hand, without the a priori knowledge that \bar{a} satisfies the strong monotonicity (1.1.5), one cannot easily turn this convergence rate into a moment bound on the corrector gradient. For MLSI, the difficulty is that it does not provide the CLT scaling (see for instance (1.1.65) for $\beta < d$), which was crucial to buckle in the proof of Theorem 21. A way would be to use the powerful approach developed in the linear setting based on large-scale regularity, this time for the nonlinear equation (1.1.4). A possible first investigation may be to consider the example of Theorem 14. A possible second investigation is to consider the case of scalar equations with $2 \leq p < \frac{2(d-1)}{d-3}$ for which we now have a $C^{1,\alpha}$ regularity theory for the homogenized operator (see the discussion in Section 3.1.1).
- In Chapter 3, we investigate the oscillations of ∇u_ε that we reconstruct with help of $\nabla \bar{u}$ in form of the two-scale expansion (1.1.31). An other natural question is to understand the random fluctuations of the macroscopic observables $\int g \cdot \nabla u_\varepsilon$ that qualitatively converges to $\int g \cdot \nabla \bar{u}$. In the linear setting, this question has been successfully studied by Gloria, Duerinckx, Otto, Mourrat and Gu in [55, 56, 94, 57] where they proved that under LSI or finite range of dependence (more general results for the Gaussian setting with non-integrable covariance has been established in [57]) the centred rescaled observables $\varepsilon^{-\frac{d}{2}} \int g \cdot (\nabla u_\varepsilon - \mathbb{E}[\nabla u_\varepsilon])$ converge in law to a Gaussian. We can even look for a finer descriptions of this convergence by means of a two-scale expansion. It was observed in [94] that the naive two-scale expansion which consists to replace $\varepsilon^{-\frac{d}{2}} \int g \cdot \nabla u_\varepsilon$ by $\varepsilon^{-\frac{d}{2}} \int g \cdot (1 + \varepsilon \phi_{e_i} \partial_{x_i}) \bar{u}$ does not work. The two-scale expansion has to be done in the level of the so-called homogenization commutator. The homogenization commutator, defined by

$$\Xi_\varepsilon = (a(\frac{\cdot}{\varepsilon}) - \bar{a}) \nabla u_\varepsilon, \quad (6.2.1)$$

is a key quantity to understand the fluctuations, since it encodes the convergence of ∇u_ε and $a(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon$ in the sense that

$$\Xi_\varepsilon \rightharpoonup 0 \text{ if and only if } \nabla u_\varepsilon \rightharpoonup \nabla \bar{u} \text{ and } a(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon \rightharpoonup \bar{a} \nabla \bar{u}.$$

Now, by defining the standard homogenization commutator $\Xi = (\Xi_i)_{i \in [1,d]}$ by

$$\Xi_i := a(\nabla \phi_{e_i} + e_i) - \bar{a}(\nabla \phi_{e_i} + e_i),$$

the two-scale expansion of the homogenization commutator is accurate in the sense that

$$\mathbb{E} \left[\left| \varepsilon^{-\frac{d}{2}} \int g \cdot (\Xi_\varepsilon - \mathbb{E}[\Xi_\varepsilon]) - \varepsilon^{-\frac{d}{2}} \int g \cdot \Xi_i(\frac{\cdot}{\varepsilon}) \partial_{x_i} \bar{u} \right|^2 \right]^{\frac{1}{2}} \lesssim \begin{cases} \varepsilon^{\frac{1}{2}} & \text{for } d = 1, \\ \varepsilon \log^{\frac{1}{2}}(\varepsilon^{-1}) & \text{for } d = 2, \\ \varepsilon & \text{for } d > 2, \end{cases}$$

Those results are so far restricted to the linear setting. The investigation of the fluctuations in the nonlinear setting is left for future investigation (for ideas and details, we refer to Section 3.2.2 of Chapter 3).

- The convergence of the two-scale expansion that we prove in Chapter 3 (see Theorem 12) is stated in the whole space and therefore we truly focused on the homogenization error and we avoid the question of boundary layers. One could state and prove a similar result on a bounded domain with Dirichlet boundary conditions in which case the bound would be of the order of the square root of what we have in Theorem 12. A way to refine this result is to construct a correction up to the boundary and introduce a new "boundary" corrector $u_{\varepsilon, \text{bl}}$:

$$\begin{cases} -\nabla \cdot a(\frac{\cdot}{\varepsilon}, \nabla u_{\varepsilon, \text{bl}}) = 0 & \text{in } \Omega, \\ u_{\varepsilon, \text{bl}}(x) = -\phi_{\nabla \bar{u}}(\frac{x}{\varepsilon}) & \text{on } \partial\Omega. \end{cases} \quad (6.2.2)$$

One expect that the "modified" two-scale expansion:

$$u_\varepsilon^{2\text{sc}} = \bar{u} + \varepsilon(\phi_{\nabla \bar{u}}(\frac{\cdot}{\varepsilon}) + u_{\varepsilon, \text{bl}}),$$

reconstruct correctly the oscillations on the boundary of Ω , where the advantage is now that $\phi_{\nabla \bar{u}}(\frac{\cdot}{\varepsilon}) + u_{\varepsilon, \text{bl}}$ satisfies a Dirichlet boundary condition. In addition, since the boundary layer $u_{\varepsilon, \text{bl}}$ is not explicit with respect to ε , one has to capture the asymptotic behaviour of $u_{\varepsilon, \text{bl}}$ by homogenizing the system (6.2.2). The homogenization of (6.2.2) started in the linear and periodic case with the works of Bakhvalov, Panasenko, Bensoussan, Lions, Papanicolaou, Oleinik, Shamaev and Yosifian in [21, 31, 112, 135] for very specific domains, and was then extended for instance by Allaire, Gérard-Varet, Masmoudi, Shen, Zhuge, Armstrong, Kuusi, Mourrat and Prange [73, 74, 149, 14] for more general domains. In the random setting, the homogenization error up to the boundary has been addressed in [8, Section 6.4] however similar results in the nonlinear setting or the homogenization of (6.2.2) has been solved. An interesting work could be to investigate this question and it would provide a good complement to the quantitative theory developed in Chapter 3.

6.2.2 Non-convex variational models: beyond nonlinear systems with monotone coefficients

In the variational context, that is when $a = \nabla_\xi W$ and $m = d$ in (1.1.4), as it appears in the context of nonlinear elasticity, we essentially dealt with uniformly convex W in Chapter 3 and the system (1.1.4) can be seen as the Euler-Lagrange equation of the minimization of the functional (we consider Dirichlet boundary conditions for the discussion)

$$\mathcal{F}_\varepsilon : u \in W_0^{1,p}(\Omega) = \int_\Omega W(\frac{x}{\varepsilon}, \nabla u) dx. \quad (6.2.3)$$

A natural question is to investigate the case of non-convex integrand in (6.2.3). Such model of nonlinear elasticity are physically relevant if in addition the energy integrand W satisfy the two basic facts of hyperelasticity, namely:

- The necessity of an infinite amount of energy to compress a finite volume into zero volume:

$$\lim_{\det \xi \downarrow 0} W(x, \xi) = +\infty. \quad (6.2.4)$$

- The noninterpenetration of the matter:

$$W(x, \xi) = \infty \text{ if } \det \xi < 0. \quad (6.2.5)$$

The standard example of such integrand potential is given by

$$W(x, \xi) = V(x, \xi) + A(x)h(\det \xi), \quad (6.2.6)$$

with $\xi \mapsto V(\cdot, \xi)$ a convex stationary ergodic integrand, $A \in L^\infty(\Omega)$ an ergodic and stationary random field and $h : \mathbb{R} \rightarrow [0, +\infty]$ is a continuous convex function with $h(t) = +\infty$ for $t \leq 0$ and $h(t) < +\infty$ for $t > 0$.

In this model, we are interested by minimizing the value of the energy and we understand the homogenization process by finding a suitable limit to $(\inf \mathcal{F}_\varepsilon)_\varepsilon$. The natural notion of convergence in this variational context is the De Giorgi Γ -convergence (see for instance [117]). More precisely, the aim is to find a functional \mathcal{F}_{hom} such that $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ -converge to \mathcal{F}_{hom} (for the $L^p(\Omega)$ -topology), meaning that:

- For all $(u_\varepsilon)_\varepsilon \subset W_0^{1,p}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ such that $u_\varepsilon \xrightarrow{\varepsilon \downarrow 0} u$ in $L^p(\Omega)$ and

$$\mathcal{F}_{\text{hom}}(u) \leq \liminf_{\varepsilon \downarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

- For all $(u_\varepsilon)_\varepsilon \subset W_0^{1,p}(\Omega)$ there exists $u \in W_0^{1,p}(\Omega)$ such that $u_\varepsilon \xrightarrow{\varepsilon \downarrow 0} u$ in $L^p(\Omega)$ and

$$\mathcal{F}_{\text{hom}}(u) \geq \limsup_{\varepsilon \downarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

The qualitative homogenization of $(\mathcal{F}_\varepsilon)_\varepsilon$ has been intensively studied since the 1980', starting from the works of Braides and Müller in [35, 130] for integrand W with p -growth (that is for all $x \in \Omega$, $|W(x, \xi)| \sim 1 + |\xi|^p$), and then extended in a lot of works by, for instance, Braides, Defranceschi, Garroni, Messaoudi, Michaille, Jikov, Kozlov, Oleinik, Anza Hafsa, Mandallena, Gloria and Duerinckx in [36, 37, 123, 100, 97, 58]. However, those results do not take into account the two conditions (6.2.4) and (6.2.5), and it constitutes a very challenging and open question to prove an homogenization result which is compatible. Some works in this direction has been proved in [136, 96] in the periodic case, for model of type (6.2.6) for V with p -growth, the periodic functions $x \mapsto V(x, \cdot)$ and $x \mapsto A(x)$ are continuous almost-everywhere and there exists $T > 0$ such that $h : \mathbb{R} \rightarrow [0, +\infty]$ is continuous and convex on $(-T, +\infty)$, $h(t) = +\infty$ for $t \in [-T, 0]$ and $h(t) = T$ for $t < -T$. It would be interesting to extend this result to the stochastic setting. However, this result will not be physically relevant since the assumptions on h are only compatible with (6.2.4) and not with (6.2.5) (but only in the weaker form of $W(x, \xi) = +\infty$, if $\det \xi \in [-T, 0]$ for some T possibly very large).

Concerning the quantitative homogenization theory of $(\mathcal{F}_\varepsilon)_\varepsilon$, it is currently unknown in the all situations above and constitute a difficult open question. The main reason is that the non-convexity

of W does not imply the existence of a single corrector, which constitutes a strong obstruction to proving quantitative homogenization based on two-scale expansion (as we did in Chapter 3). Even some relaxation of convexity (as, for instance, polyconvexity) does not allow to prove the existence of a corrector (see for instance the counterexample of Müller in [130], see also [22]). However, some works in this direction has been done in [133, 134] for small deformations, in the periodic setting. The authors have observed that for small deformations, one can construct a single corrector (bound on the extensive use of null Lagrangians) and reconstruct the oscillations of a minimizer of \mathcal{F}_ε via a two-scale expansion. An interesting extension would be to understand this work in the stochastic setting of Chapter 3 and get quantitative estimates in the case of small deformations.

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