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# Boundary layer and degenerate elliptic equation

Jean Rax

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Thèse présentée par **Jean Rax**

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**Couches limites fluides et  
équations elliptiques dégénérées**

Thèse dirigée par Anne-Laure DALIBARD

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**Keywords:** boundary layers, fluid dynamics, multiscale analysis, degenerate elliptic equations, magnetohydrodynamics

**Mots clés :** couches limites, mécanique des fluides, analyse multi-échelles, équations elliptiques dégénérées, magnétohydrodynamique



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*À mes parents, Jean-Marcel et Blandine.  
À mes grands parents Dominique, Simone, Baptiste et Andrée.*





**Couches limites fluides et équations elliptiques dégénérées****Abstract**

This thesis is devoted to the mathematical analysis of several boundary layer models in fluid dynamics. The starting point of our study is the Proudman problem, which describes the behavior of an incompressible highly rotating fluid between two spheres. In the low Rossby and low Ekman number limit, multiple nested boundary layers appear. They had been described, at a formal level, in previous works. The goal of the first part of this manuscript is to study rigorously several of them (in particular the equatorial Ekman layer) and to propose strategies of proof to justify the complete asymptotic expansion. In a second part we consider an MHD model, within which shear layers take place. They bear a strong resemblance to the boundary layers studied in the first part.

Eventually, the last part is dedicated to a stationary Burgers equation with transverse viscosity, for which we construct sign-changing solutions. Our long term goal is to construct solutions of the Prandtl system with a recirculation bubble, for which the present system play the role of a toy-model.

Remarkably, several of the systems studied in the present manuscript belong to the class of degenerate elliptic equations.

**Keywords:** boundary layers, fluid dynamics, multiscale analysis, degenerate elliptic equations, magnetohydrodynamics

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**Résumé**

Cette thèse est consacrée à l'analyse mathématique de plusieurs modèles de couches limites en mécanique des fluides.

Le point de départ de l'étude est le problème de Proudman qui modélise le comportement d'un fluide incompressible en rotation rapide entre deux sphères. Dans la limite où le nombre de Rossby et le nombre d'Ekman sont petits, de nombreuses couches limites imbriquées apparaissent, et avaient été décrites formellement dans des travaux antérieurs. Le but de la première partie de cette thèse est d'étudier rigoureusement plusieurs d'entre elles (en particulier la couche d'Ekman équatoriale) et de donner des pistes pour justifier le développement asymptotique.

Dans une seconde partie on s'intéresse à un modèle issu de la magnétohydrodynamique, au sein duquel apparaît une couche de cisaillement très similaire aux couches limites étudiées dans la première partie.

Enfin, en vue de construire des solutions stationnaires du système de Prandtl possédant des bulles de recirculation, on étudie dans la dernière partie une équation de type Burgers stationnaire avec viscosité transverse, pour laquelle on construit des solutions changeant de signe.

De façon remarquable plusieurs des systèmes étudiés entrent dans le cadre des équations elliptiques dégénérées.

**Mots clés :** couches limites, mécanique des fluides, analyse multi-échelles, équations elliptiques dégénérées, magnétohydrodynamique

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# Contents

<b>Abstract</b>	<b>ix</b>
<b>Contents</b>	<b>xi</b>
<b>1 Présentation générale (en français)</b>	<b>1</b>
1.1 Mécanique des fluides et petits paramètres . . . . .	1
1.2 Perturbations singulières et régulières . . . . .	2
1.3 Équations elliptiques . . . . .	7
1.4 Les couches limites: pourquoi et comment ? . . . . .	8
1.5 Organisation du manuscrit . . . . .	12
<b>2 Introduction</b>	<b>13</b>
2.1 Notations . . . . .	13
2.2 Context of the thesis . . . . .	17
2.3 About degenerate elliptic equations . . . . .	32
2.4 The one dimensional Proudman problem . . . . .	33
<b>3 Proudman problem</b>	<b>47</b>
3.1 Physical modelization . . . . .	47
3.2 Preliminary results on the Proudman problem . . . . .	54
3.3 Overall view of the construction . . . . .	59
3.4 The Ekman layers for two disks . . . . .	61
3.5 The shear layers . . . . .	69
3.6 Formal remarks on the equatorial problem . . . . .	101
3.7 Conclusion and perspectives . . . . .	105
<b>4 Mathematical study of the equatorial Ekman boundary layer</b>	<b>109</b>
4.1 Introduction . . . . .	109
4.2 Existence and properties of weak solutions . . . . .	111
4.3 Uniqueness and transparent boundary conditions . . . . .	119
4.4 Appendix . . . . .	125
Acknowledgements . . . . .	128
<b>5 MHD shear layer</b>	<b>129</b>
5.1 Introduction . . . . .	129
5.2 Preliminary results . . . . .	138
5.3 Hartmann layers . . . . .	140
5.4 Toy model for the shear layer . . . . .	148

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5.5	Conclusion and perspectives . . . . .	157
<b>6</b>	<b>Sign-changing solutions to the stationary Burgers equation with transverse viscosity</b>	<b>161</b>
6.1	Introduction . . . . .	161
6.2	Well-posedness results for the linear problem . . . . .	166
6.3	Fixed-point scheme for the nonlinear problem . . . . .	171
6.4	Local uniqueness of solutions to the nonlinear problem . . . . .	173
6.5	Strategy of proof of Conjecture 6.2.6 . . . . .	173
6.6	Appendix: Technical lemmas . . . . .	186
<b>A</b>	<b>Some basic methods and tools</b>	<b>187</b>
A.1	How to solve a linear ODE ? . . . . .	187
A.2	Duhamel formula . . . . .	188
A.3	Laplace transform . . . . .	189
A.4	Weighted Sobolev spaces . . . . .	189
A.5	Scalings, cut-offs and scaling . . . . .	190
	<b>Bibliography</b>	<b>195</b>
	<b>Contents</b>	<b>201</b>

# Chapitre 1

## Présentation générale (en français)

### Sommaire du présent chapitre

<b>1.1 Mécanique des fluides et petits paramètres</b>	<b>1</b>
<b>1.2 Perturbations singulières et régulières</b>	<b>2</b>
<b>1.3 Équations elliptiques</b>	<b>7</b>
<b>1.4 Les couches limites: pourquoi et comment ?</b>	<b>8</b>
<b>1.5 Organisation du manuscrit</b>	<b>12</b>

Cette thèse a pour sujet l'étude mathématique de plusieurs modèles de couches limites en mécanique des fluides : couche de Stewartson apparaissant entre deux sphères en rotation et couche de cisaillement en MHD. On construira également des solutions changeant de signe pour une équation de type Burgers avec viscosité transverse. Cette dernière équation peut être vue comme un toy-model pour l'équation de Prandtl stationnaire.

L'objectif de cette introduction et du chapitre qui suit est de présenter les différents modèles étudiés, les hypothèses physiques sous-jacentes ainsi que des outils mathématiques spécifiques pour les étudier.

### 1.1 Mécanique des fluides et petits paramètres

Lors de l'étude des systèmes fluides, il est fréquent d'adimensionner les équations et de faire apparaître des paramètres sans dimension, représentant les influences respectives des différents phénomènes physiques. Parmi les paramètres qui joueront un rôle important dans cette thèse, nous pouvons citer par exemple le nombre de Reynolds  $Re$  (mesurant l'influence des forces de viscosité par rapport au terme d'advection) ou le nombre de Rossby  $Ro$  (mesurant l'intensité de la rotation). D'autres paramètres fréquemment rencontrés mais qui n'interviendront pas ici sont par exemple le nombre de Froude  $Fr$  (mesurant l'intensité de la stratification) et le nombre de Mach  $Ma$  (mesurant le rapport entre la vitesse typique du fluide et la vitesse du son).

Lorsque ces paramètres sont petits, des phénomènes de pénalisation singulière peuvent apparaître. Les paragraphes suivants ont comme objectif d'expliquer ce phénomène de pénalisation singulière et les conséquences attendues.

## 1.2 Perturbations singulières et régulières

Mathématiquement le problème des petits paramètres adimensionnels en physique se traduit par une équation dépendant d'un paramètre que l'on notera dans ce chapitre  $\varepsilon$ . Si ce paramètre  $\varepsilon$  est très petit, peut-on raisonnablement le négliger face à 1, c'est-à-dire peut-on approximer le problème pour  $\varepsilon \ll 1$  par celui obtenu avec  $\varepsilon = 0$  ?

Autrement dit, en traduisant  $\varepsilon \ll 1$  par  $\varepsilon \rightarrow 0$  :

Comment relier la limite de la solution de l'équation avec la solution de l'équation limite ?

Avant d'aborder le cas des équations différentielles considérons pour commencer des équations polynomiales.

### 1.2.1 Premier exemple de développement asymptotique

#### Un trinôme du second degré

Un des exemples les plus simples est le suivant :

Quelles sont les racines du polynôme  $P_\varepsilon(X) = X^2 - \varepsilon X - 1$  lorsque  $\varepsilon$  est très petit devant 1 ?

Intuitivement les solutions sont proches de celles de  $P_0 = X^2 - 1$  c'est-à-dire  $\pm 1$ . Ici un calcul explicite est possible et les deux solutions exactes sont :

$$x_\varepsilon^\pm = \frac{\varepsilon \pm \sqrt{4 + \varepsilon^2}}{2} = \pm 1 + \frac{\varepsilon}{2} + o(\varepsilon)$$

À l'aide de la formule explicite précédente il est alors possible d'obtenir le développement limité à tout ordre de  $x_\varepsilon^\pm$ .

Mais comment faire sans cette formule explicite ?

Une idée pour établir un développement asymptotique est de raffiner progressivement les informations sur  $x_\varepsilon^\pm$ .

#### Bornes sur les solutions

Le point de départ de toute méthode est d'abord de montrer que pour  $\varepsilon$  assez petit on a bien deux solutions ainsi que d'obtenir une borne sur celles-ci.

Ici le théorème des valeurs intermédiaires nous suffit : pour  $1 > \varepsilon > 0$  on a  $P_\varepsilon(\pm 2) \geq 1 > 0$  et  $P_\varepsilon(\pm \frac{1}{2}) \leq -\frac{3}{4}$ . On en déduit qu'il existe une et unique solution dans  $[\frac{1}{2}, 2]$  et une et unique solution dans  $[-\frac{1}{2}, -2]$ . On note ces solutions  $x_\varepsilon^+$  et  $x_\varepsilon^-$  respectivement.

*Remarque 1.2.1.* Cette étape est cruciale et le point de départ de toute l'analyse. On remarquera que, pour le cas des EDP, ce sont les estimées d'énergie qui fournissent de telles informations.

#### Bootstraps

On effectue alors une méthode dite de *bootstrap* : on réinjecte successivement les approximations obtenues dans l'équation pour en améliorer la précision.

Pour simplifier on se concentrera sur  $x_\varepsilon^-$  que l'on notera simplement  $x_\varepsilon$  dans la suite, le traitement de  $x_\varepsilon^+$  étant identique.

Comme  $x_\varepsilon$  est bornée on en déduit que  $\varepsilon x_\varepsilon \rightarrow 0$  quand  $\varepsilon \rightarrow 0$ . De l'équation :

$$x_\varepsilon^2 - 1 = \varepsilon x^\varepsilon$$

on obtient donc  $x_\varepsilon^2 - 1 \rightarrow 0$  et comme  $x_\varepsilon \in [-2, -\frac{1}{2}]$  on conclut :

$$x_\varepsilon \rightarrow -1$$

*Remarque 1.2.2.* Un argument plus topologique, et plus générique, est de remarquer que comme  $x_\varepsilon$  est borné, on peut construire une sous-suite extraite convergeant vers un  $\tilde{x}_0$ . En passant à la limite l'équation on obtient  $\tilde{x}_0 = -1$  qui est indépendant de l'extraction d'où  $x_\varepsilon \rightarrow -1$ .

En adoptant un vocabulaire plus proche de l'analyse fonctionnelle on notera que, de par Bolzano-Weierstrass, la borne  $|x_\varepsilon| \leq 2$  permet de conclure directement la *convergence forte*. En pratique pour des EDPs il faudra se contenter, dans un premier temps, de *convergence faible*.

Pour obtenir l'ordre suivant on écrit ensuite  $x_\varepsilon = -1 + r_\varepsilon$  où  $r_\varepsilon = o(1)$  et on cherche à gagner des informations sur  $r_\varepsilon$ .

L'équation sur  $r_\varepsilon$  est :

$$(-1 + r_\varepsilon)^2 - \varepsilon(-1 + r_\varepsilon) - 1 = 0$$

c'est-à-dire :

$$-2r_\varepsilon + r_\varepsilon^2 + \varepsilon + \varepsilon r_\varepsilon = 0$$

sachant que  $r_\varepsilon = o(1)$  cela donne donc :

$$-2r_\varepsilon + o(r_\varepsilon) = -\varepsilon + o(\varepsilon)$$

et donc

$$r_\varepsilon = \frac{1}{2}\varepsilon + o(\varepsilon).$$

Pour obtenir l'ordre suivant la procédure est la même, avec  $r_\varepsilon = \frac{1}{2}\varepsilon + s_\varepsilon$  on obtient comme équation :

$$\left(-1 + \frac{1}{2}\varepsilon + s_\varepsilon\right)^2 - \varepsilon\left(-1 + \frac{1}{2}\varepsilon + s_\varepsilon\right) - 1$$

ce qui après simplification s'écrit :

$$-\frac{1}{4}\varepsilon^2 - 2s_\varepsilon + s_\varepsilon^2 = 0$$

et en utilisant le fait que  $s_\varepsilon = o(\varepsilon)$

$$s_\varepsilon = -\frac{1}{8}\varepsilon^2 + o(\varepsilon^2).$$

On peut ensuite continuer en posant  $s_\varepsilon = -\frac{1}{8}\varepsilon^2 + t_\varepsilon$ , puis en écrivant l'équation sur  $t_\varepsilon$  et en utilisant  $t_\varepsilon = o(\varepsilon^2)$ , etc.

### Utilisation d'ansatz

Plutôt que d'effectuer le développement comme précédemment une méthode utilisée est l'utilisation d'un *ansatz*. Il s'agit d'écrire la solution sous une forme faisant déjà ressortir les échelles caractéristiques.

Ainsi dans notre exemple si l'on écrit :

$$x_\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2)$$



on obtient comme équation :

$$a_0^2 + 2a_0a_1\varepsilon + (a_1^2 + 2a_0a_2)\varepsilon^2 - \varepsilon a_0 - \varepsilon^2 a_1 - 1 + o(\varepsilon^2) = 0.$$

On résout alors les équations pour chaque ordre en  $\varepsilon$  en commençant par l'ordre le plus bas :

- À l'ordre 0 l'équation est :

$$a_0^2 - 1 = 0$$

et donc on a bien  $a_0 = -1$ .

- À l'ordre 1 l'équation est :

$$2a_0a_1 - a_0 = 0$$

ce qui donne en utilisant  $a_0 = -1$  la valeur  $a_1 = \frac{1}{2}$ .

- À l'ordre 2 l'équation est :

$$a_1^2 + 2a_0a_2 - a_1 = 0$$

ce qui en utilisant les valeurs déjà trouvées permet d'obtenir  $a_2 = -\frac{1}{8}$ .

*Remarque 1.2.3.* Même si l'utilisation d'un ansatz semble bien plus rapide, il faut noter qu'il est nécessaire de trouver la bonne forme de l'ansatz, ce qui n'est pas forcément évident.

### Quelques autres exemples

Le principe utilisé ici s'applique dès qu'on a un polynôme en  $X$  avec des coefficients polynomiaux (ou analytiques) en  $\varepsilon$ ,

$$P_\varepsilon(X) = \sum_{k=0}^d a_k(\varepsilon)X^k$$

du moment que  $a_d(0) \neq 0$ .

Si cette dernière condition est vérifiée on parle de *perturbation régulière*. Le fait que les racines de  $P_\varepsilon$  tendent vers les racines de  $P_0$  est par exemple le fameux *théorème de continuité des racines par rapport aux coefficients*. De plus cette méthode est quantitative : elle permet d'obtenir de manière explicite le développement même si l'on ne connaît pas une forme fermée pour les racines (il s'agit d'une méthode de fonctions implicites).

Par exemple si l'on considère :

$$P_\varepsilon(X) = X^7 - \varepsilon X - 1$$

on sait que pour  $\varepsilon > 0$  assez petit il existe une unique solution réelle  $x_\varepsilon$ , mais il n'existe pas de formule explicite donnant  $x_\varepsilon$  en fonction de  $\varepsilon$ . Néanmoins, en procédant exactement comme précédemment, il est très facile d'obtenir par exemple :

$$x^\varepsilon = 1 + \frac{1}{7}\varepsilon + o(\varepsilon).$$

Enfin il faut noter que le développement n'est pas nécessairement en puissances entières de  $\varepsilon$  et qu'il faut donc être prudent au moment d'injecter des ansätze.

Par exemple considérons :

$$P_\varepsilon(X) = X^2 - 2X + (1 - \varepsilon)$$

alors si l'on essaye d'injecter un ansatz de la forme  $x_\varepsilon = a_0 + a_1\varepsilon + o(\varepsilon)$  on obtient :

- À l'ordre 0,  $a_0 = 1$ .
- À l'ordre 1,  $2a_0a_1 - 2a_1 - 1 = 0$  i.e  $(2 - 2)a_1 = 1$  ce qui est absurde.

La cause de cette contradiction est la non pertinence de l'ansatz. En effet les solutions sont  $x_\varepsilon^\pm = 1 \pm \sqrt{\varepsilon}$  ce qui ne correspond pas à l'ansatz précédent qui est uniquement en puissance entière de  $\sqrt{\varepsilon}$ . Néanmoins avec  $x_\varepsilon = 1 + r_\varepsilon$  on obtient bien  $r_\varepsilon^2 = \varepsilon$ . De plus l'ansatz (correct)  $x_\varepsilon = a_0 + a_1\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$  donne aussi  $a_0 = 1$ ,  $a_1 = \pm 1$ .

### 1.2.2 Une perturbation singulière

Considérons désormais un problème légèrement différent :

Quelles sont les racines du polynôme  $P_\varepsilon(X) = \varepsilon X^2 - X - 1$  lorsque  $\varepsilon$  est très petit devant 1 ?

On pourrait penser que la construction précédente s'adopte sans modification, mais ce n'est pas le cas.

Ici le problème est le suivant : pour tout  $\varepsilon > 0$  le polynôme  $P_\varepsilon$  possède deux racines. Néanmoins  $P_0$  n'en possédant qu'une il est impossible d'approximer naïvement celles de  $P_\varepsilon$  par celle de  $P_0$ .

On parle dans ce cas, où l'équation pour  $\varepsilon = 0$  n'a pas la même nature (ici degré) que l'équation pour  $\varepsilon > 0$ , de *perturbation singulière*.

Afin de voir ce qui se passe, notons que pour le trinôme du second degré proposé la résolution explicite est possible, et donne :

$$x_\varepsilon^\pm = \frac{1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}.$$

Un développement limité à l'ordre le plus élevé est :

$$x_\varepsilon^- = -1 + o(1)$$

$$x_\varepsilon^+ = \frac{1}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)$$

On a  $x_\varepsilon^-$  tend vers une racine de  $P_0$ , mais  $x_\varepsilon^+$  est non borné.

si l'on cherche à obtenir comme précédemment une méthode générale on observe qu'une fois le premier ordre du développement obtenu en posant  $x_\varepsilon^- = -1 + r_\varepsilon^-$  et  $x_\varepsilon^+ = \varepsilon^{-1} + r_\varepsilon^+$  où  $r_\varepsilon^- = o(1)$ ,  $r_\varepsilon^+ = o(\varepsilon^{-1})$  on obtient :

$$r_\varepsilon^- + 2\varepsilon r_\varepsilon^- - \varepsilon(r_\varepsilon^-)^2 - \varepsilon = 0$$

$$r_\varepsilon^+ + (\varepsilon r_\varepsilon^+)r_\varepsilon^+ - 1 = 0$$

d'où :

$$r_\varepsilon^- = \varepsilon + o(\varepsilon)$$

$$r_\varepsilon^+ = 1 + o(1).$$

On voit bien que le problème supplémentaire est en fait d'obtenir ce premier terme du développement.

Plus précisément, on peut montrer par le théorème des valeurs intermédiaires que  $x_\varepsilon^-$  est borné et en déduire  $x_\varepsilon^- = -1 + o(1)$  mais ce n'est pas possible pour  $x_\varepsilon^+$ . si l'on peut ici utiliser

des relations coefficients racines ou remarquer que  $\varepsilon x_\varepsilon^+ - 1 = 1/(x_\varepsilon^+)$  de manière générale il faut identifier l'échelle  $\frac{1}{\varepsilon}$  avant tout autre analyse.

Une fois cette échelle en  $\varepsilon^{-1}$  établie, le changement de variable  $x_\varepsilon = \varepsilon^{-1}\xi_\varepsilon$  permet d'obtenir :

$$\xi_\varepsilon^2 - \xi_\varepsilon - \varepsilon = 0$$

ce qui permet de montrer que  $\xi_\varepsilon$  est borné, i.e la première étape d'un développement asymptotique, puis de continuer à tous les ordres.

### Principe de moindre dégénérescence

De manière générale, par compacité, les racines bornées tendent vers des racines de  $P_0$ . La difficulté principale est donc d'obtenir le premier ordre du développement des racines non bornées.

Dans le cas de polynômes on sait par des arguments d'analyse complexe [2], ou bien par un calcul explicite, que si les coefficients sont des polynômes en  $\varepsilon$  alors à l'ordre le plus élevé les racines sont en puissance de  $\varepsilon$ .

On recherche donc un développement en  $x_\varepsilon \sim \xi_\varepsilon \varepsilon^{-\alpha}$  où  $\xi_\varepsilon$  est borné et loin de zéro, c'est-à-dire  $\xi_\varepsilon \rightarrow \xi_0 \in \mathbb{C} \setminus \{0\}$ . À cet effet, on pose  $x = \varepsilon^{-\alpha}\xi$  et on regarde ce qui se passe selon la valeur de  $\alpha$  si l'on suppose que  $\xi$  reste borné.

Pour le trinôme précédent l'équation sur  $\xi$  est :

$$\varepsilon^{1-2\alpha}\xi - \varepsilon^{-\alpha}\xi - 1 = 0$$

d'où lorsque  $\varepsilon \rightarrow 0$  :

- si  $\alpha < 0$  l'équation devient :

$$-1 = 0$$

ce qui est absurde.

- si  $\alpha = 0$  l'équation devient :

$$-\xi - 1 + o(1) = 0$$

ce qui est bien la racine en  $\varepsilon^0$  attendue.

- si  $0 < \alpha < 1$ , on a  $1 - 2\alpha > \alpha > 0$  et donc l'ordre principal de l'équation est :

$$\xi + o(1) = 0$$

ce qui ne permet pas d'obtenir davantage de précision.

- si  $\alpha = 1$  alors l'équation est :

$$\xi^2 - \xi + o(1) = 0$$

ce qui permet d'obtenir que  $\xi \rightarrow 1$ .

- enfin si  $\alpha > 1$  on obtient :

$$\xi^2 + o(1) = 0$$

ce qui encore une fois ne permet pas de conclure.

On voit donc que le principe pour gagner des informations sur l'ordre en  $\varepsilon$  est le suivant :

On trouve un changement de variables  $x = \varepsilon^{-\alpha}\xi$  tel que l'équation formelle sur  $\xi$  obtenue en  $\varepsilon = 0$  soit non triviale.

Où par triviale on entend une équation soit absurde ( $0 = 1$ ), soit ne permettant pas de pousser le développement à l'ordre suivant ( $\xi = 0 + o(1)$ ). D'où le nom de *principe de moindre dégénérescence* : il faut garder au moins deux termes dans l'équation.

Il faut noter que l'identification des échelles idoines en  $\varepsilon$  est essentielle car ce sont ces échelles qui permettent ensuite de commencer un bootstrap et qui dictent le type d'ansatz à utiliser.

### 1.2.3 Résumé

On considère une équation (polynomiale) dépendant d'un petit paramètre  $\varepsilon$  et on se demande quel est le comportement des solutions quand  $\varepsilon \rightarrow 0$ .

- Si l'équation pour  $\varepsilon = 0$  est de même nature que celle pour  $\varepsilon > 0$  (*perturbation régulière*), alors on peut attendre la convergence des solutions à  $\varepsilon > 0$  vers celles de  $\varepsilon = 0$ . De plus on peut obtenir un développement asymptotique en  $\varepsilon$  des solutions par bootstrap ou en utilisant un *ansatz* bien choisi.
- Si l'équation pour  $\varepsilon = 0$  est de nature différente de celles pour  $\varepsilon > 0$  (*perturbation singulière*) ce n'est plus le cas. Il est alors nécessaire d'identifier les échelles spécifiques au problème avant d'espérer établir des résultats de convergence suite à une *remise à l'échelle*.

## 1.3 Équations elliptiques

En pratique les équations de la mécanique des fluides (ou de la physique en général) ne sont pas des polynômes. De fait, une grande partie d'entre elles sont des équations aux dérivées partielles. La grande majorité des démonstrations de cette thèse se basent sur des méthodes issues de la théorie des équations différentielles *elliptiques*. Si l'étude de telles équations est extrêmement vaste, donnons néanmoins un exemple d'équation elliptique et quelques méthodes permettant de les étudier.

L'équation elliptique la plus simple est probablement la suivante : on se donne  $f$  une fonction régulière de  $(-1, 1)$  et on cherche  $u$  tel que

$$-\partial_x^2 u(x) + u(x) = f(x) \quad \forall x \in (-1, 1)$$

muni des conditions aux bords

$$u(-1) = u(1) = 0.$$

Ce problème est qualitativement différent d'un problème d'*équation différentielle ordinaire* (EDO), où le domaine serait une demi-droite, par exemple  $(0, \infty)$  et les conditions aux bords, appelées dans ce cas conditions initiales, seraient  $u(0) = 0$ ,  $u'(0) = 0$ . La théorie générale des EDOs ne s'applique donc pas directement.

Les deux remarques cruciales sont les suivantes

- Deux fonctions  $v$  et  $w$  de  $(-1, 1)$  sont égales ssi pour toute fonction  $\phi$  on a

$$\int_{-1}^1 v(x)\phi(x)dx = \int_{-1}^1 w(x)\phi(x).$$

- En effectuant une *intégration par parties* (IPP), pour tout  $\phi$  tel que  $\phi(-1) = \phi(1) = 0$  on obtient

$$-\int_{-1}^1 \partial_x^2 u(x)\phi(x)dx = \underbrace{[-\partial_x u\phi]_{-1}^1}_{=0} + \int_{-1}^1 \partial_x u(x)\partial_x \phi(x) = 0.$$

L'équation précédente peut alors se réécrire

$$\forall \phi \quad \underbrace{\int_{-1}^1 \partial_x u(x) \partial_x \phi(x) dx + \int_{-1}^1 u(x) \phi(x) dx}_{a(u, \phi)} = \underbrace{\int_{-1}^1 f(x) \phi(x) dx}_{L(\phi)}.$$

On s'est donc ramené à un simple problème d'algèbre linéaire  $\forall \phi, a(u, \phi) = L(\phi)$  où  $a(\cdot, \cdot)$  est une forme bilinéaire et  $L$  est une forme linéaire.

En pratique, comme il ne s'agit pas d'un problème de dimension finie, il faut être précautionneux sur les espaces auxquels appartiennent  $u$  et  $\phi$ , pour obtenir notamment la continuité et la coercivité des opérateurs.

En définissant l'espace fonctionnel

$$H_0^1 = \left\{ u \text{ t.q. } \int_{\Omega} (|u|^2 + |\nabla u|^2) < +\infty; u_{\partial\Omega} = 0 \right\}$$

Le théorème de Lax-Milgram nous assure que comme

- $a$  est bilinéaire continu sur  $H_0^1$
- $L$  est une forme linéaire continue sur  $H_0^1$ ,
- $a$  est coercive sur  $H_0^1$ , c'est-à-dire

$$a(u, u) \geq C \|u\|_{H_0^1}^2$$

pour une certaine constante  $C$ .

Il existe bien une et unique solution à ce problème.

Notons que cette méthode permet de traiter des problèmes en dimension et pour des géométries quelconques, des systèmes d'équations et des variantes avec des termes d'ordre inférieur. Le point clé étant la coercivité de la forme bilinéaire.

## 1.4 Les couches limites : pourquoi et comment ?

Même si les couches limites les plus connues, celles de Prandtl, ne sont pas issues d'un phénomène linéaire, le phénomène de couches limites peut être souvent compris d'un point de vue linéaire. Pour très brièvement reprendre l'explication de Gérard-Varet [31], les couches limites étant de petits correcteurs, les termes non-linéaires qu'ils créent sont encore plus petits. On va donc dans cette partie regarder ce qui se passe lorsque qu'une équation elliptique linéaire comporte un petit paramètre  $\varepsilon$ .

### 1.4.1 Perturbation régulière

Considérons l'exemple le plus simple d'équation elliptique perturbée de manière régulière. On cherche à résoudre sur un domaine régulier et borné  $\Omega$  l'équation

$$-\Delta u^\varepsilon + \varepsilon u^\varepsilon = f$$

avec  $f \in H^{-1}$ , et munie de conditions aux bords de Dirichlet

$$u|_{\partial\Omega} = 0. \tag{1.1}$$

L'estimée d'énergie naturelle associée à cette équation est

$$\int_{\Omega} |\nabla u|^2 + \varepsilon \int_{\Omega} |u|^2 \leq \int_{\Omega} f u.$$

On peut remarquer que  $u^\varepsilon$  est borné puis passer à la limite faible dans  $H^1$  et forte dans  $L^2$ . Mais ici considérons simplement  $u^0$  solution de

$$-\Delta u^0 = f$$

munie des conditions aux bords (1.1).

Comme  $u^\varepsilon - u^0$  est solution de

$$-\Delta(u^\varepsilon - u^0) + \varepsilon(u^\varepsilon - u^0) = -\varepsilon u^0$$

munie des conditions aux bords (1.1), et que  $\|u^0\|_{H^1} \leq C\|f\|_{H^{-1}}$ , on obtient directement

$$\|u^\varepsilon - u^0\|_{H^1} \leq C\varepsilon \rightarrow 0.$$

Il est tout aussi facile de continuer à tous les ordres, en écrivant

$$u^{app,K} = \sum_{k=0}^K \varepsilon^k u^k$$

avec  $u^k$  solution de

$$\Delta u^k = u^{k-1}$$

muni des conditions aux bords (1.1), on obtient

$$\left\| u^\varepsilon - \sum_{k=0}^K \varepsilon^k u^k \right\|_{H^1} \leq C\varepsilon^{K+1}.$$

*Remarque 1.4.1.* D'un point de vue formel, cela correspond à effectuer le développement asymptotique

$$(Id - \varepsilon\Delta^{-1})^{-1} = \sum_k (\varepsilon(\Delta)^{-1})^k$$

qui est valide d'un point de vue spectral pour  $|\varepsilon| < \lambda_0^{-1}$  avec  $\lambda_0 > 0$  la plus grande valeur propre de l'opérateur  $(-\Delta)^{-1}$ .

On voit donc que pour des exemples simples les perturbations régulières d'EDP ne posent pas de problèmes particuliers.

## 1.4.2 Un petit exemple de perturbation singulière

Le cas des perturbations singulières étant plus complexe que le précédent, pour une fonction  $f$  régulière, cherchons à résoudre sur le segment  $(-1, 1)$  l'équation

$$-\varepsilon \partial_x^2 u^\varepsilon + u^\varepsilon = f \tag{1.2}$$

munie des conditions aux bords

$$u^\varepsilon(-1) = u^\varepsilon(1) = 0.$$

On voit qu'en tant qu'équation elliptique (1.2) admet bien une unique solution vérifiant

$$\varepsilon \int_{-1}^1 |\partial_x u^\varepsilon|^2 dx + \int_{-1}^1 |u^\varepsilon|^2 dx \leq C \int_{-1}^1 |f|^2 dx.$$

Maintenant si  $\varepsilon \ll 1$  peut-on dire que  $u^\varepsilon \simeq u^0 = f$  ?

L'approche qui vient naturellement est de regarder l'équation vérifiée par  $u^\varepsilon - u^0$

$$-\varepsilon \partial_x^2 (u^\varepsilon - u^0) + (u^\varepsilon - u^0) = \varepsilon \partial_x^2 f.$$

Mais ici si l'on essaye de multiplier par  $u^\varepsilon - u^0$  et intégrer par parties on obtient

$$\varepsilon \int_{-1}^1 |\partial_x (u^\varepsilon - u^0)|^2 dx + [\varepsilon \partial_x (u^\varepsilon - u^0) u^0]_{-1}^1 + \int_{-1}^1 |(u^\varepsilon - u^0)|^2 dx = \int_{-1}^1 f (u^\varepsilon - u^0) dx$$

le terme de bords en rouge nous empêchant de conclure.

Une méthode efficace pour obtenir la convergence de  $u^\varepsilon$  vers  $u^0$  (et d'autres renseignements plus précis) est de remplacer dans les estimations que l'on vient d'essayer de faire  $u^0$  par

$$u^{app} = u^0 + u^{bl,g} \left( \frac{1+x}{\sqrt{\varepsilon}} \right) + u^{bl,d} \left( \frac{1-x}{\sqrt{\varepsilon}} \right)$$

où les termes dits de *couches limites*  $u^{bl,g/d}$  sont petits en norme, et donc ne se voient pas forcément dans la convergence ( $u^{app} \sim_{L^2} u^0$ ), mais permettent d'avoir  $u^{app}$  qui vérifie les conditions aux bords.

Notons que ces couches limites sont parfois plus qu'un simple artefact mathématique. En effet si l'on poursuit le développement asymptotique de  $u^\varepsilon$  que ce soit en puissance de  $\varepsilon$  ou en régularité, il est possible de voir de tels termes apparaître.

### 1.4.3 Bref historique

Prandtl proposa lors du 3ème congrès international des mathématiciens en 1904 le concept de couche, pour décrire la limite non visqueuse des équations de Navier-Stokes en présence d'un obstacle. Depuis une très large littérature mathématique sur les couches limites dites de *Prandtl* a vu le jour. Le but ici n'est pas de faire une présentation exhaustive de ces résultats. Nous citerons simplement quelques jalons parmi les développements récents

- dans le cas stationnaire, et tant que la vitesse tangentielle reste de signe constant, Oleinik [58] a montré le caractère bien posé des équations en s'appuyant sur une formulation parabolique du problème. Plus récemment l'Ansatz de Prandtl a été justifié dans ce cadre et pour deux situations différentes par Guo et Iyer [44], et par Gérard-Varet et Maekawa [34].
- Dans le cas dépendant du temps, l'équation est bien posée dans des espaces à forte régularité (voir les travaux pionniers de Sammartino et Caffisch [64], [65] dans un cadre analytique, récemment étendus à une régularité Gevrey par Gérard-Varet et Maekawa [36]). L'équation est également bien posée pour des données monotones voir les travaux d'Oleinik [58] récemment revisités par différents auteurs [57], [3], [50]. Néanmoins sans l'hypothèse de monotonie, des instabilités apparaissent dans les espaces de Sobolev [39], [33].

Concernant les couches limites en géophysique, leur observation physique date aussi du début du 20ème siècle, avec notamment l'explication par Ekman [25] dès 1905 de l'influence des couches

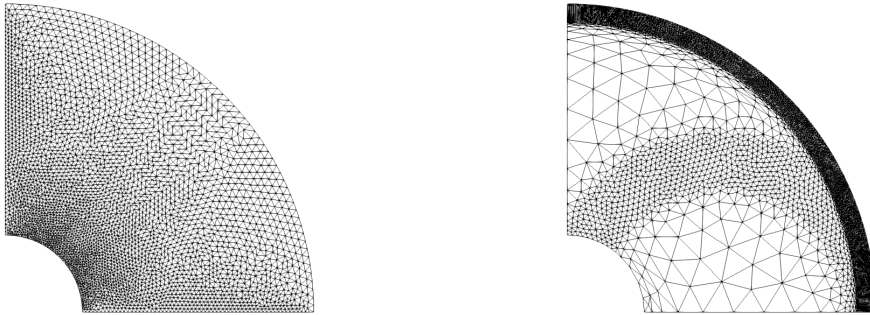


FIGURE 1.1 – Maille uniforme et maille adaptée.

limites qui porteront son nom sur les courants océaniques. Les premiers travaux sur l'étude mathématique de modèles océaniques remontent aux études de Lions, Temam et Wang [52] ainsi que Embid et Majda [26] (voir aussi les travaux de Babin, Mahalov et Nicolaenko, par exemple [4].) En ce qui concerne les couches d'Ekman on réfère à [12], [40], [56], [11], [21].

#### 1.4.4 Couches limites et simulations numériques

Enfin, commentons rapidement l'importance du phénomène de couches limites pour les simulations numériques. Comme des variations fortes ont lieu sur des petites distances, un schéma numérique doit prendre en compte un tel comportement afin d'être précis.

Considérons par exemple des équations issues de la MHD, modélisant un fluide conducteur en rotation se déplaçant dans un champ magnétique fixé. Les équations sur la vitesse  $v$  et sur la perturbation du champ magnétique  $b$  sont de la forme :

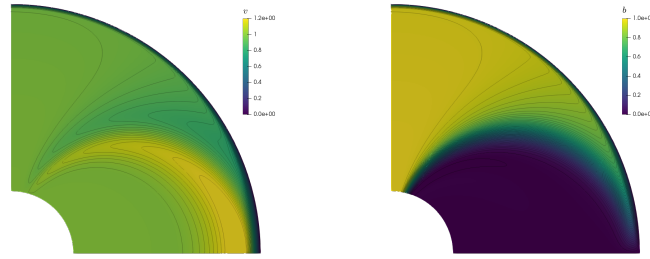
$$\begin{aligned} B \cdot \nabla v - M^{-1}(\Delta - 1)b &= 0 \\ B \cdot \nabla b - M^{-1}(\Delta - 1)v &= 0 \end{aligned}$$

où  $B$  est un champ magnétique fixé et  $M$  un nombre sans dimension appelé nombre d'Hartmann. Pour plus de détails sur la modélisation, notamment les conditions aux bords, et l'analyse mathématique, voir Chapitre 5.

Supposons que l'on cherche à résoudre numériquement l'équation pour un grand nombre d'Hartmann  $M$ . Alors il faut s'attendre à des variations de la solution sur des échelles très petites, nécessitant un pas de maille au moins aussi fin. En effet la Figure 1.2 obtenue avec un maillage uniforme très fin ( $3 \times 10^6$  degrés de libertés) montre le phénomène suivant : près des bords et le long d'une ligne de champ (appelée ligne de cisaillement) les fonctions  $v$  et  $b$  se mettent à varier très rapidement sur des échelles très courtes. Il s'agit bien entendu de couches limites : des couches d'Hartmann de taille  $M^{-1}$  près de la sphère extérieure et des couches de cisaillement de taille  $M^{-\frac{1}{2}}$  le long de la ligne de champ tangente à la sphère, on réfère de nouveau au chapitre 5 pour une description plus détaillée. La connaissance a priori de l'emplacement et de la taille de ces couches limites permet alors d'adapter le maillage en conséquence.

Pour illustrer ce dernier point, on a choisi comme les paramètre numérique  $M = 400$ , la résolution se faisant à l'aide d'éléments P1 et du logiciel `FreeFem++` [46].



FIGURE 1.2 – Comportement typique de  $v$  et  $b$ .

En calculant les erreurs effectuées, on observe, que ce soit en norme  $L^2$ ,  $H^1$  ou en terme d'énergie associée au système  $(\|v\|_{L^2}^2 + \|b\|_{L^2}^2) + \frac{1}{M} (\|v\|_{H^1}^2 + \|b\|_{H^1}^2)$ , que la connaissance a priori des couches limites permet de gagner un ordre de grandeur. Le tableau 1.1 montre bien ce résultat pour les deux mailles de la Figure 1.1.

	Degrés de liberté	Erreur $L^2$	Erreur $H^1$	Erreur en énergie
Maille uniforme	3707	$7 \times 10^{-3}$	$2 \times 10^{-1}$	$10^{-4}$
Maille adaptée	3256	$8 \times 10^{-2}$	$2 \times 10^0$	$10^{-2}$

TABLEAU 1.1 – Comparaison entre les mailles.

## 1.5 Organisation du manuscrit

Dans le chapitre suivant, on présente l'ensemble des modèles qui seront étudiés dans ce manuscrit, ainsi que quelques outils utilisés à cet effet.

Le Chapitre 3 est dédié à l'étude du problème de Proudman (fluide en rotation rapide entre deux sphères), ainsi qu'à l'analyse de plusieurs des couches limites qui apparaissent dans ce cas.

Le Chapitre 4 se concentre sur l'étude de la couche d'Ekman équatoriale, qui est précisément une couche limite du problème de Proudman. Ce chapitre correspond à un article, accepté dans *ZAMP* sous réserve de modifications mineures.

Le Chapitre 5 présente un problème de magnétohydrodynamique, dans lequel apparaissent des couches de cisaillement cousines des couches limites du problème de Proudman.

Enfin, dans le Chapitre 6 on construit des solutions changeant de signe pour une équation de type Burgers stationnaire avec viscosité transverse. Il s'agit d'un travail en collaboration avec Frédéric Marbach et Anne-Laure Dalibard.

# Chapter 2

## Introduction

### Outline of the current chapter

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<b>2.1 Notations</b>	<b>13</b>
<b>2.2 Context of the thesis</b>	<b>17</b>
<b>2.3 About degenerate elliptic equations</b>	<b>32</b>
<b>2.4 The one dimensional Proudman problem</b>	<b>33</b>

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### 2.1 Notations

Let us start by precisising some notations used in this thesis.

Perhaps the most currently used and nevertheless misleading one is the use of “ $C$ ” to denote any constant. As often, we will denote  $C$  any positive constant, regardless of its exact value, rather than tracking all the constants and their dependencies.

For example we can write, for a fixed polynomial  $P$ , something like

$$\int_0^\infty \left| P(\zeta) e^{-\zeta \varepsilon^{-1}} \right|^2 d\zeta \leq C \int_0^\infty e^{-2C^{-1} \zeta \varepsilon^{-1}} d\zeta \leq C\varepsilon.$$

In all generality, we will not specify the dependencies with respect to the parameters of the problem. However, if these dependencies are noteworthy, we will denote them by indices. For example we will write, for  $\varepsilon > 0$ :

$$\forall a, b \in \mathbb{R} \quad |ab| \leq C_\varepsilon a^2 + \varepsilon b^2.$$

#### 2.1.1 Functions

##### Functional spaces

Let  $\Omega$  be a sufficiently smooth subset of  $\mathbb{R}^d$ .

We will use the classical, see for example [8], functional spaces on  $\Omega$ . More precisely, for  $p \in [1, +\infty]$ ,  $s \in \mathbb{R}$ ,  $k \in \mathbb{N}$  we will denote  $L^p(\Omega)$  the usual Lebesgue space,  $H^s(\Omega)$  the usual Sobolev

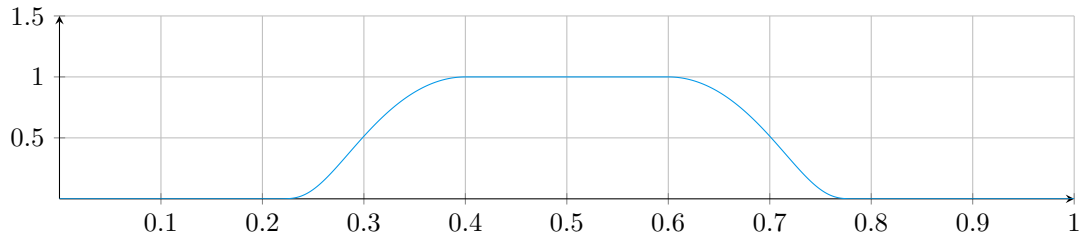


Figure 2.1 – A cut-off  $\chi$  such that  $\chi = 0$  on  $(0, 1/5) \cup (4/5, 1)$  and  $\chi = 1$  on  $(2/5, 3/5)$ .

space, and  $C^k(\Omega)$  the space of functions whose first through  $k$ th derivatives are continuous. Note that we will consider *inhomogeneous* Sobolev space, for example

$$\|u\|_{H^1(\Omega)}^2 = \int_{\Omega} (|\nabla u|^2 + |u|^2)$$

If there is no possible confusion about the domain we will simply denote these spaces  $L^p$ ,  $H^s$ ,  $C^k$ .

### Norms

Let  $(\mathcal{H}(\Omega), \|\cdot\|_{\mathcal{H}})$  be a usual functional space contained in  $\mathcal{D}'(\Omega)$ . We will extend  $\|\cdot\|_{\mathcal{H}}$  as a function from  $\mathcal{D}'(\Omega)$  to  $[0, +\infty]$  by setting  $\|f\|_{\mathcal{H}} = +\infty$  if  $f \notin \mathcal{H}$ .

This allow us to write for example

$$\mathcal{H}(\Omega) = \{f \text{ s.t. } \|f\|_{\mathcal{H}} < +\infty\}.$$

Note that, from a given norm, several spaces can be defined, notably when dealing with traces condition. For example, for  $\Omega$  a bounded domain and  $\|u\|_{H^1(\Omega)}^2 = \int_{\Omega} |u|^2 + |\nabla u|^2$ , we have

$$\begin{aligned} H^1(\Omega) &= \overline{C^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}} \\ H_0^1(\Omega) &= \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}. \end{aligned}$$

### Smooth functions

We will denote  $C^\infty(\Omega)$  the space of infinitely many times differentiable functions and  $C_c^\infty$  the subspace of  $C^\infty$  with compact support. If the domain is unbounded, we will denote by  $C_b^\infty = C^\infty \cap L^\infty$  and  $C_0^\infty$  the space of  $C^\infty$  functions such that  $f(x) \rightarrow 0$  when  $|x| \rightarrow \infty$ .

An important subspace of such functions will be *cut-off* functions. It is essentially a mollification of an indicator function, see figure 2.1. We will often denote by  $\chi$  such functions independently of the sets.

### Special functions

Some particular functions will be used repeatedly.

- For  $K$  a regular subset of  $\Omega$ ,  $\mathbf{1}_K$  will be the indicator function of  $K$ , i.e

$$\mathbf{1}_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K. \end{cases}$$

- We will sometimes use the notation  $f(x)$  instead of  $f$  to emphasize on the dependency on the variable  $x$ . A common example is that, for  $\eta \in \mathbb{C}$  we will denote  $e^{\eta\zeta}$  the function

$$e^{\eta\zeta} : (\sigma, \zeta) \mapsto e^{\eta\zeta}.$$

### Differential operators

We will denote by  $\partial$  the partial derivative. For example if  $f : (x, y, z) \mapsto f(x, y, z)$  we will denote

$$\frac{\partial f}{\partial y} = \partial_y f.$$

For multiple derivatives

$$\frac{\partial^2 f}{\partial y^2} = (\partial_y)^2 f = \partial_y^2 f.$$

In order to avoid confusion when the indices is a number, it will refer to the position of the variable. For example

$$\partial_2 f = \partial_y f.$$

Note that if no confusion are possible we will also use indices to denote derivatives, for example

$$\begin{aligned} \partial_y f &= f_y \\ \partial_y^2 f &= f_{yy}. \end{aligned}$$

We will also use indices on the Laplacian to specify the variables if needed.

$$\begin{aligned} \Delta f &= \partial_x^2 f + \partial_y^2 f + \partial_z^2 f \\ \Delta_{x,z} f &= \partial_x^2 f + \partial_z^2 f. \end{aligned}$$

For a function of only one real variable we will sometimes use  $'$  to denote the derivative. For example, for  $f : t \in \mathbb{R} \mapsto f(t)$ ,

$$\frac{df}{dt} = f' = \partial_t f.$$

### 2.1.2 Smallness

As our objective is to construct asymptotic expansions, the notations about smallness are crucial.

We will use the standard Landau notation for real-valued functions  $o$ ,  $O$  and  $\sim$ . For example

$$f(\varepsilon) = o_{\varepsilon \rightarrow 0}(g(\varepsilon))$$

means that there exists a real-valued function  $h$  such that  $\lim_{\varepsilon \rightarrow 0} h(\varepsilon) = 0$  and  $f = hg$ . If there is no possible confusion we will also denote it by

$$f(\varepsilon) = o_\varepsilon(g(\varepsilon))$$

or even

$$f(\varepsilon) = o(g(\varepsilon)).$$

We will also denote

$$f(\varepsilon) = o(\varepsilon^\infty)$$

which means that

$$\forall k \in \mathbb{N}, \quad f(\varepsilon) = o(\varepsilon^k).$$

For functions having value in a normed space of finite dimension, the choice of the norm does not modify the topology, so we will not precise it. But the choice of space is of utmost importance in the infinite dimensional setting. For  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  a normed space and  $\varepsilon \mapsto u^\varepsilon$  from a neighborhood of  $0^+$  to  $\mathcal{H}$  we will note

$$u^\varepsilon = o_{\mathcal{H}}(g(\varepsilon))$$

for a real-valued function  $g$  iff

$$\|u^\varepsilon\|_{\mathcal{H}} = o(g(\varepsilon)).$$

And we will note

$$u^\varepsilon = o_{H^\infty/C^\infty}(g(\varepsilon))$$

for

$$\forall k \in \mathbb{N}, \quad u^\varepsilon = o_{H^k/C^k}(g(\varepsilon)).$$

### 2.1.3 Some indications

We finish by some indications that are not mathematical notations, but rather guidelines that we have tried to follow.

- The variable  $\xi \in \mathbb{R}^d$  denotes the Fourier variable.
- The variable  $\zeta$  denotes the boundary layer variable, i.e the rescaled distance to the boundary.
- The variable  $\lambda$  will denote the Laplace variable, or exponential rate, like in  $e^{\lambda x}$ , or roots of a characteristic polynomial, since the three are related. Sometimes it will be denoted by  $\eta$  or  $\mu$ .
- The variable  $\delta$  will often be the size of the boundary layer.

When studying a function  $u$ , a certain number of “accents”, [bar/tilde/hat] and superscripts will be used.

- The solution of an equation depending on a parameter  $\varepsilon$  will be denoted  $u^\varepsilon$ .
- The *approximate solution* will be denoted  $u^{app}$ .
- The function  $\hat{u}$  will be the Fourier transform of  $u$ .
- The function  $\bar{u}$  will either be a “constant”  $u$  (either with respect of one variable or one iteration of a procedure) or a limit, for example  $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x) = \bar{u}(x)$ .
- The function  $\tilde{u}$  will sometimes be the Laplace transform, but mainly any modification of  $u$  for technical reasons. For example a cut-off of  $u$  will can be denoted  $\tilde{u} = u\chi$ .

- The function  $\mathbf{u}$  is a vector valued function. The coordinates will often be named after corresponding physical quantities. For example in the Proudman problem we will often denote  $\mathbf{u} = (u_v, u_\psi)$ .
- We will also sometimes use the capital form to represent traces. For example the trace of  $u$  can be named  $U$ .
- When making boundary layer expansions we will use two types of functions: interior (sometimes called outer by physicists) terms and boundary layer (sometimes called inner) terms. The interior terms will be typically denoted  $u^{int}$ . The boundary layer will be denoted  $u^{bl}$  when working in the global variable and  $u^{BL}$  when working in the boundary layer variables.

$$u^{bl}(x) = u^{BL}(\zeta) = u^{BL}\left(\frac{x}{\delta}\right)$$

- In order to distinguish between boundary layers, we will also add superscript corresponding to the geometrical position, for example  $l/r$  for left and right and  $t/b$  for top and bottom.
- When doing expansions we will denote the order of the term by a superscript  $k$ . Thus, an expansion can look like

$$\begin{aligned} u^{app} &= \sum_k \varepsilon^k u^{int,k}(x) + \varepsilon^k u^{bl,k,l}(x) + \varepsilon^k u^{bl,k,r}(x) \\ &= \sum_k \varepsilon^k u^{int,k}(x) + \varepsilon^k u^{BL,k,l}\left(\frac{x-x_l}{\delta_l}\right) + \varepsilon^k u^{BL,k,r}\left(\frac{x_r-x}{\delta_r}\right). \end{aligned}$$

The most important point to note is that we will often drop all these superscript and accents when focusing on specific term. In the previous expansion, if the focus of the paragraph is the study of  $x \mapsto u^{BL,k,l}\left(\frac{x-x_l}{\delta_l}\right)$ , then, for simplicity, we will instead denote it simply  $\zeta \mapsto u(\zeta)$ .

## 2.2 Context of the thesis

### 2.2.1 Quick historical context

In 1904, at the Third International Mathematics Congress, Prandtl proposed the notion of boundary layer to explain the the so-called d'Alembert paradox: incompressible and inviscid fluids does not produce any drag, contrary to what is experimentally observed, see Figure 2.2.

In modern terms, the problem was that the approximation for large Reynolds number  $Re \gg 1$  of

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{Re} \Delta \mathbf{u} + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}(t=0) &= u_0 \end{aligned} \tag{2.1}$$

by non-viscous fluids and Euler equation

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}(t=0) &= u_0 \end{aligned} \tag{2.2}$$

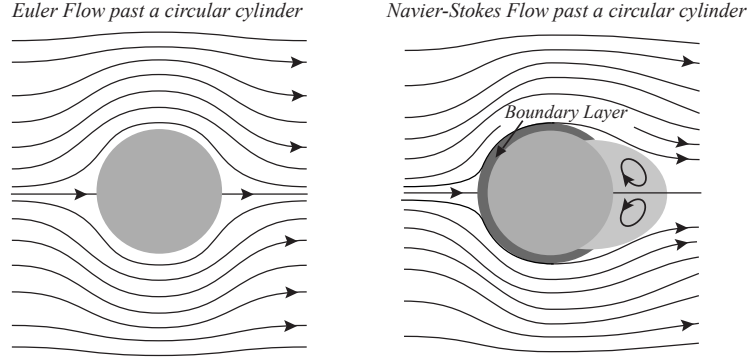


Figure 2.2 – Solution of Euler equation on the left, flow of a fluid with  $Re \gg 1$  on the right.

is valid for the upcoming flow, but becomes invalid near the the domain boundary. One of the main reason is that the boundary condition for the parabolic Navier-Stokes  $\mathbf{u}|_{\partial\Omega} = 0$  is of different nature than the one of the hyperbolic Euler equation  $(\mathbf{u} \cdot \mathbf{n})|_{\partial\Omega} = 0$ . The proposition of Prandtl was to keep the Euler equation (2.2) far from the boundary, and in a neighborhood of small size ( $\frac{1}{\sqrt{Re}}$ ) of the boundary to consider a *boundary layer*. For  $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$  this is to take

$$\mathbf{u}^{NS}(t, x, y) = \begin{cases} u^E(t, x, y) & \text{for } y \gg \sqrt{Re} \\ \left( u \left( t, x, \frac{y}{\sqrt{E}} \right), \sqrt{E}v \left( t, x, \frac{y}{\sqrt{E}} \right) \right) & \text{for } y \leq C\sqrt{Re}. \end{cases}$$

And the *boundary layer term*  $(u(x, \zeta), v(x, \zeta))$  is searched as a solution of the so-called Prandtl equation

$$\begin{aligned} \partial_t u + u \partial_x u + v \partial_\zeta u - \partial_\zeta^2 u &= -\partial_x p_e \\ \partial_x u + \partial_\zeta v &= 0 \end{aligned}$$

with ad-hoc boundary condition to ensure both the no-slip boundary condition and the reconnection with the main flow

$$(u, v)|_{\zeta=0} = 0, \quad \lim_{\zeta \rightarrow \infty} u = u|_{y=0}, \quad u|_{t=0} = u_0.$$

Since its first derivation, the Prandtl equation was widely studied, as was the reconnection between Prandtl flow and the main flow. We refer to the introduction of Chapter 6 for more details, but to summarize:

- for the stationary case, as long as  $u > 0$  Oleinik [58] proved the well-posedness of the equation, by using its parabolicity. And more recently, the Prandtl ansatz was justified for this case, by Guo and Iyer [44], and by Gérard-Varet and Maekawa [34].
- For the time-dependent case the equation is well-posed for high regularity spaces (see the seminal work of Sammartino and Caffisch [64], [65] in an analytic framework; these results were recently extended to Gevrey regularity by Gérard-Varet and Maekawa [36]). The equation is also well-posed for a monotonous initial data, see Oleinik's works [58] which was recently revisited by several authors [57], [3], [50]. However, without the assumption of monotonicity, instabilities can appear in a Sobolev setting [39], [33].

On one hand, despite the recent advancement, notably for the stationary case with  $u > 0$ ,

it seems reasonable to state that the Prandtl expansion and the passage from Navier-Stokes to Euler remains an open problem in its full generality.

On the other hand, boundary layers are not restricted to Navier-Stokes with large Reynolds and Prandtl layer. Actually, this is a very generic phenomenon in singular perturbation of PDE. As explained by Gérard-Varet [31], on a domain  $\Omega$  ( $\partial\Omega \neq \emptyset$ ), for any problem of the form

$$\begin{aligned} \mathcal{D}u^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}u^\varepsilon + \mathcal{Q}(u^\varepsilon, u^\varepsilon) &= f^\varepsilon \text{ in } \Omega \\ + \text{ boundary conditions for } u^\varepsilon &\text{ on } \partial\Omega \end{aligned}$$

where  $\mathcal{D}, \mathcal{L}$  are differential operators, with  $\mathcal{D}$  of higher degree than  $\mathcal{L}$ , and  $\mathcal{Q}$  a non-linear term of lower order, boundary layers are to be expected. More precisely, even if the approximation  $u^\varepsilon \rightarrow \bar{u}$  with  $\bar{u}$  solution of  $\mathcal{L}\bar{u} = 0$  can be valid in a very weak sense, in the energy norms required to obtain traces and strong convergence such an approximation is insufficient. A symbolic analysis shows that  $u$  can formally be approximated by

$$\bar{u}(x) + \sum_k u_k^{BL} \left( \frac{d(x, \partial\Omega)}{\delta_k^\varepsilon} \right)$$

where  $u_k^{BL}(\zeta)$  being almost zero for  $\zeta \gg 1$  are called *boundary layer terms*, and  $\delta_k^\varepsilon$  going to 0 when  $\varepsilon \rightarrow 0$  are called the *boundary layer sizes*.

*Remark 2.2.1.* Prandtl equation does not fit into this framework. Indeed, the main balance is between  $\frac{1}{Re}\Delta u$  and the non-linear  $\partial_t u + (u \cdot \nabla)u + \nabla p$ . This is one of the difficulties of this equation.

Such singular perturbation problems appear naturally for geophysical fluids, as non-dimensional quantities can be very small (see Table 3.2), One such example is the Rossby number  $Ro$  quantifying the strength of inertia versus the earth rotation. For oceanic circulation  $Ro \ll 1$ , which added to the fact that  $Re \gg 1$  gives birth to the so called *Ekman layers* at the bottom and at the top of oceans. But if the Ekman layers emerged around the same time as Prandtl, their first description being in 1905 by Ekman [25], their mathematical study began much later. We can cite, among others, the work of Grenier and Masmoudi [40] in 1997, Desjardins and Grenier in 1999 [21], and Masmoudi in 2000 [56], for the Navier-Stokes-Coriolis equation

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \frac{1}{Ro}e_3 \times u - \frac{1}{Re_h}\Delta_h u - \frac{1}{Re_z}\partial_z^2 u + \nabla p &= f \\ \nabla \cdot u &= 0 \\ u|_{\partial\Omega} &= 0 \\ u(t=0) &= u_0. \end{aligned} \tag{2.3}$$

A simple version of the theorem they obtained can be stated as follows, in the *well prepared case*

**Theorem 2.2.1** (Grenier and Masmoudi [40]). *Let  $\Omega = \mathbb{T}^2 \times [0, 1]$ . Let us suppose  $Re_h$  is fixed,  $\frac{1}{Re_z}$  and  $Ro$  are bounded, that*

$$\frac{Ro}{Re_z} = E \rightarrow 0$$

*and that  $Re_z Ro \rightarrow \frac{1}{\beta}$ , where  $\beta \in [0, \infty]$  is called the Ekman pumping.*

*Let us suppose that the initial condition  $u_0 \in L^2(\Omega)$  and source term  $f \in L^2((0, \infty) \times \Omega)$  are*



well prepared i.e

$$\begin{aligned}\partial_z u_0 &= 0, \int u_0 = 0, u_0 \cdot e_3 = 0 \\ \partial_z f &= 0, f \cdot e_3 = 0.\end{aligned}$$

Then, there exists global weak solutions  $u^{Ro, Re_z}$  of (2.3) such that

- if  $\beta < +\infty$

$$\lim_{E \rightarrow 0} \|u^{Ro, Re_z} - (\bar{u}, 0)\|_{L^\infty((0, \infty), L^2(\Omega))} = 0$$

where  $\bar{u}$  is a two dimensional flow satisfying a damped Navier-Stokes equation

$$\begin{aligned}\partial_t \bar{u} + (\bar{u} \cdot \nabla_{x,y}) \bar{u} - \Delta_{x,y} \bar{u} + \sqrt{2\beta} \bar{u} + \nabla_{x,y} p &= f \\ \nabla \cdot_{x,y} \bar{u} &= 0 \\ \bar{u}|_{t=0} &= u_0.\end{aligned}$$

- if  $\beta = \infty$ , then,  $\forall t_0 > 0$

$$\lim_{E \rightarrow 0} \|u^{Ro, Re_z} - 0\|_{L^\infty((t_0, \infty), L^2(\Omega))} = 0.$$

*Remark 2.2.2.* For stationary Ekman layers, the main balance is between the two linear terms  $-\frac{1}{Re_z} \partial_z^2 u$  and  $\frac{1}{Ro} e_3 \times u + \nabla p$  (and the divergence condition). As a consequence, the boundary layer looks like a linear ODE in  $z$ , with an explicit solution called the *Ekman spiral*.

However, if we replace  $\mathbb{T}^2 \times [0, 1]$  by  $\Omega = \{x \in \mathbb{R}^3 \text{ s.t } r_0 < |x| < r_1\}$ , i.e the volume between two spheres, the question of the behavior of  $u$  in such limits remains open, due to difficulties near the equator. In fact, the size of Ekman layers depends on the latitude  $\theta$  and goes to  $+\infty$  when  $\theta \rightarrow \frac{\pi}{2}$ , causing the construction to break down.

More generally, Gérard-Varet and Paul [37] showed that, in the linear case, the main obstacles to the construction of the *boundary layer expansion* are the degeneracies:

- frequency degeneracy, when the size of the boundary layer depends strongly on the lateral frequency,
- geometric degeneracy, when the size of the boundary layer depends on the lateral coordinates.

Where in both cases *lateral* refers to the coordinates on  $\partial\Omega$ .

## 2.2.2 Objective and methods

In this thesis, we will consider three different physical problems, all originating from geophysics or fluid mechanics. In each case, a goal was to keep the mathematical justification as close as possible to similar work done by physicists.

For the Proudman problem and the equatorial Ekman layer it is the work of Dormy, Marcotte and Soward [54] and for the shear layer in Magnetohydrodynamics the one of Dormy and Soward [22]. Concerning the recirculation, the starting point was the existence of explicit recirculating solution of the stationary Prandtl equation, called Falkner-Skan solutions [27].

From a methodological and mathematical point of view, the main works on which this thesis is based are the following:

1. The article of Gérard-Varet [31] explaining the study program of these boundary layers of linear origin, completed by the one of Gérard-Varet and Paul [37] resolving the construction of these boundary layers outside the so-called degenerate cases (when the size of the boundary layer varies according to the tangential frequency or tangential coordinates).
2. The work of Fichera [28], showing how to adapt the classical tools of the theory of elliptic equations to degenerate equations. Here *degenerate* mean that the symbol of the operator is non-negative, but not necessary positive. For example

$$\partial_x u - \partial_y^2 u - 0 \cdot \partial_x^2 u = 0$$

can be seen as such an operator with specific boundary conditions. And using the theory developed notably by Grisvard [41] such boundary conditions can be various and in polygonal domains.

3. Finally, the works of Dalibard and Saint-Raymond [18], [17], highlighting among other things, two key points in the construction of boundary layers: the notion of an *approximate solution* in the sense of the energy balance and that of *boundary layer operator*.

Several methodological insights can be drawn from the experience.

The first is that formalism in terms of boundary layer/interior operators makes it possible to approach the construction of the different terms in an almost algorithmic way. This construction is very close to the widely used *matched asymptotic expansion* (see for example the monographs [13], [49]), as it decomposes the problem into boundary layer (inner) and interior (outer) regions. The main differences are the way we reconnect these regions, with the use of compatibility conditions, and the way we treat each region, as we solve a PDE with the exact number of degree of freedom of point 1, using methods from point 2.

As a consequence, this formalism is naturally adapted to the more general framework of point 3: the energy norm of the remainders at each step is clearly identified, and the knowledge of such norms is sufficient to pursue the expansion to the next order. Moreover, it is robust enough to adapt to degenerate boundary layers.

The second one is that the robustness and results of the classical elliptical framework can be used to carry out a thorough study of the so-called degenerate elliptic equations. This is of particular interest for boundary layer originating from elliptic PDE, as these are naturally of this type.

Let us now describe more precisely the organization of this manuscript. First, in Section 2.4, we illustrate how the boundary layer and interior operator can be used to obtain an asymptotic expansion. The example chosen is the one-dimensional variant of the Proudman problem, leading to a simple mathematical analysis but keeping most difficulties of the articulation between orders and regions. Then we will develop:

- Some results on the Proudman problem in Chapter 3, followed by a mathematical analysis of the equatorial Ekman layer in Chapter 4.
- A similar construction for an equation coming from Magnetohydrodynamics shear layers in Chapter 5.
- The study of a toy-model of recirculating Prandtl solution in Chapter 6.

### 2.2.3 Proudman problem and equatorial Ekman boundary layer

In Chapter 3, we look investigate the so-called Proudman problem. This problem represents a stationary and linearized version of the Navier-Stokes-Coriolis equations (2.3). With  $v$  the

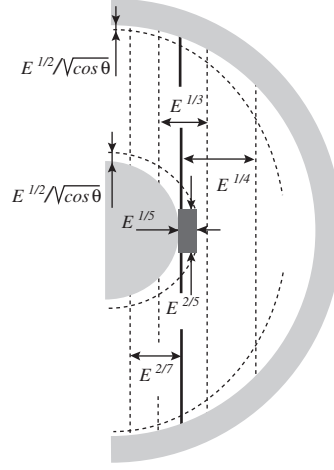


Figure 2.3 – Summary of the different boundary layers in the Proudman problem.

azimuthal velocity and  $\psi$  the stream function, the main equation reads as

$$\begin{aligned} \partial_z v - E \Delta_{x,z}^2 \psi &= 0 \\ \partial_z \psi + E \Delta_{x,z} v &= 0 \end{aligned} \quad (2.4)$$

with boundary conditions on  $v$ ,  $\partial_n \psi$  and  $\psi$

$$\begin{aligned} v|_{\partial\Omega} = v|_{\partial\Omega} \quad \partial_n \psi|_{\partial\Omega} &= 0 \\ \psi|_{\partial\Omega} &= 0. \end{aligned} \quad (2.5)$$

This equation may seem simple, but, in the case of a spherical geometry, a large number of boundary layers appear when  $E \rightarrow 0$ : the well-known Ekman layers of size  $E^{\frac{1}{2}}$  (far from the equator), several shear layers of sizes  $E^{\frac{1}{4}}$ ,  $E^{\frac{2}{7}}$  and  $E^{\frac{1}{3}}$  and an equatorial layer of size  $E^{\frac{2}{5}} \times E^{\frac{1}{5}}$ , see Figure 2.3.

### Fluid contained between two disks

Even though the Ekman layers are well-known, we start by re-deriving them in the case of a fluid contained between two disks, i. e. a domain

$$\Omega = \{(x, z) \in \mathbb{T} \times \mathbb{R}, \text{ s.t } \gamma_0(x) < z < \gamma_1(x)\}$$

with  $\gamma_0, \gamma_1$  smooths,  $\gamma_1 - \gamma_0 \geq c > 0$ . This allows us to use on a practical case the formalism of boundary layer and interior operators.

**Proposition 2.2.2** (Exemple of boundary layer operator). *Let  $(V(x), \Psi(x), \Upsilon(x))$  be smooth functions, satisfying the compatibility condition*

$$-\frac{(1 + \gamma_0'(x)^2)^{-\frac{1}{4}}}{\sqrt{E}} \Psi(x) + \frac{1}{\sqrt{2}} (V(x) - \Upsilon(x)) = 0. \quad (2.6)$$

*Then, let us define the boundary layer size  $\delta_0 = \sqrt{E}(1 + \gamma_0'(x)^2)^{\frac{1}{4}}$ . Let us define the boundary*

layer term  $\mathbf{u}^{bl} = (v, \delta_0 \psi) \left( \frac{d(x, \Gamma_0)}{\delta_0} \right)$ , where  $\mathbf{u} = (v, \psi)$  is solution boundary layer equation

$$\begin{aligned} \partial_\zeta - \partial_\zeta^4 \psi &= 0 \\ \partial_\zeta v + \partial_\zeta^2 v &= 0 \end{aligned} \tag{2.7}$$

with boundary conditions  $v|_{\zeta=0} = V$ ,  $\partial_n \psi|_{\zeta=0} = \Upsilon$ ,  $\psi|_{\zeta=0} = \delta_0 \Psi$ . Then  $u^{bl}$  solves

$$\begin{aligned} \partial_z v^{bl} - E \Delta_{x,z}^2 \psi^{bl} &= r_\psi \\ \partial_z v^{bl} + E \Delta_{x,z} v^{bl} &= r_v \end{aligned}$$

with boundaries conditions:

$$\begin{aligned} (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{\Gamma_0} &= (V, \Psi, \Upsilon) \\ (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{\Gamma_1} &= 0. \end{aligned}$$

And the remainders,  $r_v$  and  $r_\psi$  are acceptable remainders. In particular

$$\begin{aligned} \|r_v\|_{L^2} &\leq CE^{\frac{3}{4}} \\ \|r_\psi\|_{H^{-1}} &\leq CE^{\frac{1}{4}}. \end{aligned}$$

*Remark 2.2.3.* Note that, the compatibility condition (2.6) (which can be related to the *Ekman pumping*) and the corresponding boundary solution, depends strongly on the angle between the axis of rotation and the boundary and on  $E$ .

This means that we can lift-up imperfect boundary condition with boundary layers, as long as the conditions satisfies (2.6). All it remains is to solve an approximation of (2.4), but replacing the 6 boundary conditions of (2.5) by the 2 compatibility conditions (2.6) at the top and the bottom. This is simply achieved by considering an *interior operator*, similar to Proposition 2.2.2, but replacing the boundary layer scaling  $\zeta = d((x, z), \partial\Omega)E^{-\frac{1}{2}} \in [0, \infty)$  simply by the original scaling  $(x, z) \in \Omega$ , and with *interior equation*

$$\begin{aligned} \partial_z v &= 0 \\ \partial_z \psi &= 0. \end{aligned}$$

*Remark 2.2.4.* This construction can be further formalized to allow higher corrector, it is only a matter of taking the remainders of one order as a source term in the boundary layer/interior equation of the next order.

Assembling together these pieces allow us to conclude:

**Proposition 2.2.3.** *Let  $\gamma_0, \gamma_1 \in C^\infty(\mathbb{T})$ , such that  $C^{-1} < \gamma_1 - \gamma_0 < C$ . Then the solution of (2.4) with boundary conditions*

$$\begin{aligned} v|_{z=\gamma_i} &= v_i & \partial_n \psi|_{z=\gamma_i} &= 0 \\ \psi|_{z=\gamma_i} &= 0 \end{aligned}$$

*converges in  $L^2 \times H^1$  towards  $(\bar{v}, 0)$  where  $\bar{v}$  depends only on  $x$  (this is Taylor-Proudman theorem), and is a mean between the top and bottom boundary condition, weighted by the relative size of*

the Ekman boundary layers:

$$\bar{v}(x) = \frac{(1 + \gamma'_0(x)^2)^{-\frac{1}{4}} v_0(x) + (1 + \gamma'_1(x)^2)^{-\frac{1}{4}} v_1(x)}{(1 + \gamma'_0(x)^2)^{-\frac{1}{4}} + (1 + \gamma'_1(x)^2)^{-\frac{1}{4}}}.$$

### Case when the lower disk has a singular shape

Consequently, we then investigate the physical case of a geometric degeneracy, where the boundary admits a singularity with the shape  $\gamma_0(x) \sim (-x)^\alpha \mathbf{1}_{x < 0}$ , the sphere corresponding to  $\alpha = \frac{1}{2}$ . Indeed, for such  $\gamma_0$ , the corresponding  $\bar{v}$  is discontinuous so shear layers must appear, in order to ensure  $\bar{v} \in H^1$ . Using the first order Ekman boundary layer operator (2.6) as an effective boundary condition

$$\left( (-1)^{i+1} (1 + \gamma'_i(x)^2)^{\frac{1}{4}} \frac{1}{\sqrt{E}} \psi + \frac{1}{\sqrt{2}} (v - v_i) \right) \Big|_{z=\gamma_i} = 0,$$

we derive formally the equation satisfied by the first order of the flow far from the boundary

$$\left( (1 + \gamma_0'^2)^{-\frac{1}{4}} v_0 + (1 + \gamma_1'^2)^{-\frac{1}{4}} v_1 \right) - \left( (1 + \gamma_0'^2)^{-\frac{1}{4}} v_0 + (1 + \gamma_1'^2)^{-\frac{1}{4}} v_1 \right) \bar{v} - (\gamma_1 - \gamma_0) \sqrt{E} \partial_x^2 \bar{v} = 0. \quad (2.8)$$

Neglecting  $\sqrt{E} \partial_x^2$  leads to the weighted mean (2.8) of the previous proposition. But for a singular  $\gamma_0$ , such approximation is invalid. Further analysis show that two different boundary layers appears at  $x = 0$ :

- For  $x > 0$ , where  $\gamma_0$  is not singular, a  $E^{\frac{1}{4}}$  layer corresponding to the boundary layer equation

$$\partial_\zeta^2 u - u = f.$$

- For  $x < 0$  where  $\gamma_0$  is singular, a  $E^{\frac{1}{3+\alpha}}$  layer with boundary layer equation

$$\partial_\zeta^2 u - \zeta^{\frac{\alpha-1}{2}} u = f.$$

Even though these equations admit explicit solutions, we choose to treat them using the previous boundary layer operator framework and elliptic methods. As a consequence we are able to lift off the discontinuities, recovering for example the value of  $\bar{v}$  at  $x = 0$ ,  $\bar{v}(0) = v_0 + O(E^{\frac{1}{28}})$ .

But these layers are not enough to conclude the construction. Indeed,  $\psi$  still has a jump at  $x = 0$ . Another nested shear boundary layer is used to lift these discontinuities. It is the same boundary layer as the one for a vertical wall, called the Stewartson  $E^{\frac{1}{3}}$  layer, of size  $E^{\frac{1}{3}}$  and equation

$$\begin{aligned} \partial_z v - \partial_\zeta^4 \psi &= 0 \\ \partial_z \psi + \partial_\zeta^2 v &= 0. \end{aligned}$$

If this equation is close to what will be the equatorial Ekman equation, it avoids most of its problems. Thus we can solve it without difficulties, the main notable fact being that the layer size varies like  $(Ek)^{\frac{1}{3}}$ , where  $k$  is the vertical frequency (i.e with respect to  $z$ )

For each of these layers we display the corresponding boundary layer operator. If all the interconnections far from the equator are understood from a very formal point of view, we were not able to complete the construction, precisely because of the equatorial layer.

Chapter 4, based on an article accepted under minor modifications in *ZAMP*, then focuses

on this equatorial layer. The equation governing it is

$$\begin{aligned} \partial_z v + z \partial_y v - \frac{1}{2} \partial_y^4 \psi &= s_\psi \\ \partial_z \psi + z \partial_y \psi + \frac{1}{2} \partial_y^2 v &= s_v. \end{aligned} \tag{2.9}$$

For the domain  $y > 0, z > 0$ , it does not fit into strict frameworks already pre-established in PDE theory. However, a relevant angle of attack seems to be to use the theory of degenerate elliptic equations and a Lions-Lax-Milgram theorem. The Theorem 4.2.1 shows that there exists a weak solution in the energy space

**Theorem 2.2.4.** *For any  $V, \Upsilon \in H_0^{\frac{1}{2}}(\mathbb{R}_+)$ ,  $\Psi \in H_0^{\frac{3}{2}}(\mathbb{R}_+)$ , and for various conditions at  $z = 0$  ((I),(II),(III)) there exists a weak solution of (2.9) such that  $(v, \psi, \partial_y \psi)|_{y=0} = (V, \Psi, \Upsilon)$ . This solution is such that*

$$\|(v, \psi)\|_{E_0}^2 = \int_{\Omega} \left( |\partial_y v|^2 + \left| \frac{v}{1+y} \right|^2 \right) + \int_{\Omega} \left( |\partial_y^2 \psi|^2 + \left| \frac{\psi}{1+y^2} \right|^2 \right) < +\infty.$$

Moreover if  $v \in L^2, \psi \in L^2$  this solution is unique.

*Remark 2.2.5.* Thus, we are able to lift up to 3 boundary conditions at  $y = 0$ . For  $z = 0$  the right functional setting is to lift only one degree of freedom, for example  $v|_{z=0}, \psi|_{z=0}$  or  $\psi|_{z=0} - \Lambda v|_{z=0}$ .

*Remark 2.2.6.* The  $L^2$  integrability, which implies uniqueness, is recovered for multiple variants, for example with the addition zeroth order terms, or if  $y$  is bounded.

In addition, a transparent boundary condition is exhibited. Not only this is an important help for numerical simulations, but this also makes it possible to propose a credible candidate for the boundary layer operator's compatibility conditions. Unfortunately, technical difficulties of integrability do not allow the study to be concluded completely.

The Figure 2.4 summarizes the location of the various boundary layer in the manuscript.

## 2.2.4 About shear layers in Magnetohydrodynamics

In Chapter 5, we look at an equation coming from Magnetohydrodynamics theory. It describes a rotating conducting fluid between two shells, when a strong exterior magnetic field  $B$  is applied. Under the scalings described in Section 5.1, the equation is of the form

$$\begin{aligned} M(B \cdot \nabla + a)b + (\Delta - c)v &= 0 \\ M(B \cdot \nabla - a)v + (\Delta - c)b &= 0 \end{aligned}$$

where the unknowns are the azimuthal velocity  $v$  and the magnetic field  $b$ . The Hartmann number  $M$  being the origin of singular perturbation, as  $M \gg 1$ .

This problem is very closely related to the Proudman problem of the previous chapters. For example, in this setting, the equivalent of the Taylor-Proudman theorem is the Ferraro's law of iso-rotation, stating that when  $M \rightarrow \infty$ ,  $v$  and  $b$  depends only on the magnetic potential, i.e  $B \cdot \nabla v = 0, B \cdot \nabla b = 0$ .

Note that, up to taking the potential vector  $A$  and the scalar potential  $\phi$  as new coordinates, we can reduce the study to a uniform magnetic field  $B = B e_z$ .

Without surprise, the mathematical results are similar to those of the Proudman problem, the Ekman layers being replaced by the Hartman layers. Note that these Hartman layers are

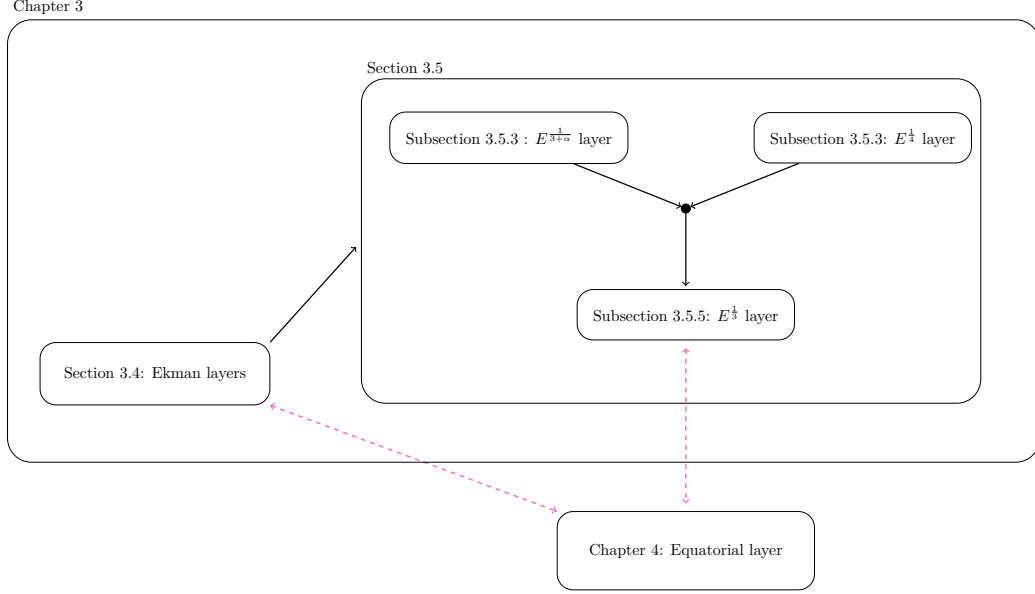


Figure 2.4 – The organization of the first part of this thesis.

simpler than the Ekman ones, as the compatibility condition we obtain does not depend on  $M$  or a normal derivative.

**Proposition 2.2.5.** *Let  $V, B \in H^2(\{r = r_0\})$ , and let  $\mathbf{f}$  be with exponential decay of rate  $\eta < 1$ , i.e*

$$\int_{\Gamma_0} \int_0^\infty \partial_\theta^2 \mathbf{f}(x, \zeta) e^{2\eta\zeta} d\zeta d\theta < +\infty$$

*Suppose that the following compatibility is satisfied*

$$V(\theta) + B(\theta) = \int_0^\infty (1 - e^{-\zeta})(f_v + f_b)(\theta, \zeta) d\zeta. \quad (2.10)$$

*And, in order to avoid degeneracy, let us suppose that  $V - B$  and  $\mathbf{f}$  cancels like  $\cos(\theta)^2$  when  $\theta \rightarrow \frac{\pi}{2}$ .*

*Then, there exist  $\mathbf{u}^{bl} = \mathbf{u}\left(\theta, \frac{r-r_0}{M \cos(\theta)}\right)$  solution of the approximate equation*

$$\begin{aligned} M(\partial_z + a)b + (\Delta - c)v &= M^2 f_v^{bl} + Mr_v^{bl} \\ M(\partial_z - a)v + (\Delta - c)b &= M^2 f_b^{bl} + Mr_b^{bl} \\ (v^{bl}, b^{bl})_{r=r_0} &= (V, B) \\ (v^{bl}, b^{bl})_{r=r_1} &= (0, 0) \end{aligned}$$

*with source term  $\mathbf{f}^{bl} = \mathbf{f}\left(\theta, \frac{r-r_0}{M \cos(\theta)}\right)$ . Moreover, the remainders are satisfying*

$$\|r_v^{bl}\|_{L^2} + \|r_b^{bl}\|_{L^2} \leq CM^{-\frac{1}{2}}.$$

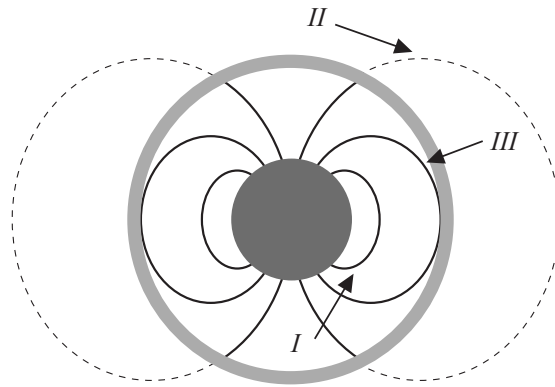


Figure 2.5 – The three different configuration of magnetic field line.

Boundary condition for $M < +\infty$	Effective boundary condition
Dirichlet on $b$ , Dirichlet on $v$	Dirichlet on $v + b$ or <sup>†</sup> on $v - b$
Dirichlet on $b$ , Neumann (or Robin) on $v$	Dirichlet for $b$
Neumann (or Robin) on $b$ , Dirichlet on $v$	Dirichlet for $v$

Table 2.1 – Effective boundary conditions.

The only additional step is to treat the boundary conditions for a conductive shell, as the usual Dirichlet condition is replaced by a Robin transparent condition

$$(\partial_n b + \Lambda b)|_{\Gamma} = 0.$$

But the formalism of boundary layer operators allows us to treat this without difficulty.

A consequence of Ferraro law of isototation is that we must distinguish three types of field line for the exterior magnetic field:

- (I) When the field line goes from one shell and then goes back inside it without intersecting the other shell.
- (II) When the field line goes from one shell and then intersect the other shell.
- (III) The tangent case, separating the two previous cases.

For the case (I),  $v$  and  $b$  are simply fixed by the compatibility condition of the Hartmann layer on the shell. For the case (II) it is a simple mean between the two conditions. But across the line of type (III) a discontinuity appears.

This discontinuity cannot be lifted by a simple use of the so-called Shercliff layers, equivalent to the  $E^{\frac{1}{3}}$  Stewartson layers. In fact, two degeneracies occur: a geometric degeneracy at the tangency point between the field line and the boundary, but also a boundary condition degeneracy. Indeed, from the analysis of the boundary layer operator, the effective boundary condition obtained for the interior term changes according to the main boundary condition, see table 2.1.

This creates a shear-layer of size  $E^{\frac{1}{2}}$  at the tangent magnetic field line (III), admitting a singular behavior at the tangency point between the field line and the outer shell, as can be seen in Figure 2.6.

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<sup>†</sup>Following the sign of  $n \cdot e_z$



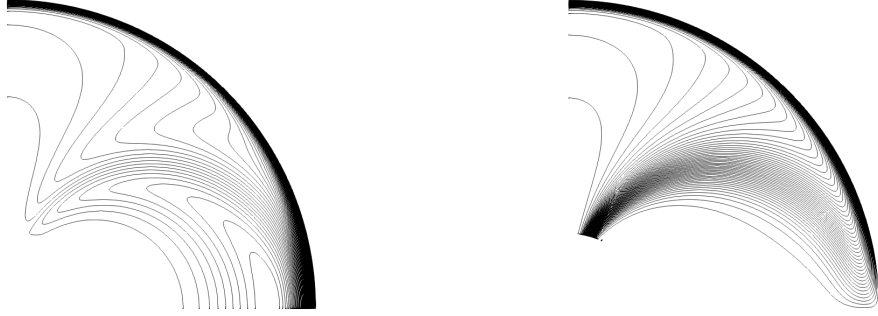


Figure 2.6 – Isovalues of  $v$  and  $b$  for a dipolar magnetic field  $B$  and  $M = 400$ .

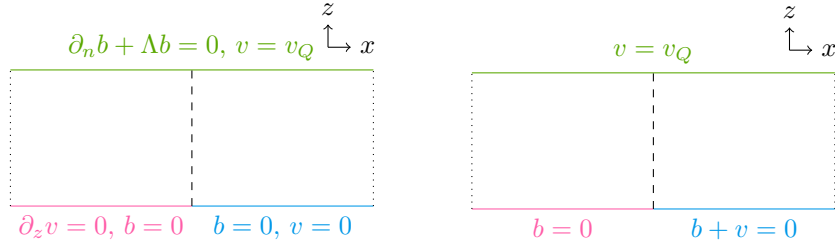


Figure 2.7 – Model for studying the boundary condition degeneracy, and effective boundary conditions on the right.

In order to focus on the singularity created by the boundary condition degeneracy, we consider a toy model, used by Dormy, Jault and Soward [22], where the geometric degeneracy is forgotten. The shear layer equation is now, in Alfvén variables  $v \pm b$ , the two coupled parabolic equations inside  $\mathbb{R} \times [0, 1]$

$$\begin{aligned} (\partial_z(v+b) - a(v-b)) - \partial_\zeta^2(v+b) &= 0 \\ (\partial_z(v-b) - a(v+b)) + \partial_\zeta^2(v-b) &= 0 \end{aligned} \quad (2.11)$$

with effective boundary conditions of Figure 2.7

$$v|_{\{z=1\}} = v_Q \quad b|_{\{z=0, x < 0\}} = 0 \quad (v+b)|_{\{z=0, x > 0\}} = 0. \quad (2.12)$$

Following the works of Grisvard [41], we know that the change of boundary condition type at  $(x, z) = (0, 0)$  does not hinder existence in low regularity spaces, but create singularities, preventing any higher order estimate.

For example, if we were to solve  $-\Delta u = f \in C_c^\infty$  on  $\mathbb{R} \times \mathbb{R}_+$  with boundary conditions  $u|_{y=0, x > 0} = 0$ ,  $\partial_n u|_{y=0, x < 0} = 0$ , we would obtain, for any  $R > 0$ ,  $u \in H^1(|r| < R)$  but  $u \notin$

$H^2(|r| < R)$ . In fact one can show using duality arguments that

$$u = \underbrace{u^{reg}}_{\in H^2(|r| < R)} + \underbrace{\gamma \sqrt{r} \sin\left(\frac{\theta}{2}\right)}_{\notin H^2(|r| < R)}$$

where  $\gamma \in \mathbb{R}$  and  $(r, \theta)$  are the polar coordinates.

For our problem we also have existence in anisotropic Sobolev spaces, and by using parabolic estimates we recover a similar behavior: the solution is the sum of a regular term and of a well identified singular term.

**Proposition 2.2.6.** *Let  $v_Q$  be smooth. There exist one and only one solution  $(v, b)$  of (2.11), (2.12) satisfying the estimate*

$$\int_{\Omega} (|v|^2 + |b|^2 + |\partial_{\zeta} v|^2 + |\partial_{\zeta} b|^2) \leq C.$$

*In fact  $v, b$  are in  $H^1(\Omega \setminus \{|(s, z)| \leq \epsilon\})$  for all  $\epsilon > 0$ . And, more precisely, there exist a constant  $\Xi \in \mathbb{R}$  and  $v^{reg}, b^{reg} \in H^2(\Omega)$  such that*

$$\begin{aligned} v &= \underbrace{v^{reg}}_{\in H^1(\Omega)} + \underbrace{\Xi u^{sing}}_{\notin H^1(\Omega)} \\ b &= \underbrace{b^{reg}}_{\in H^1(\Omega)} + \underbrace{\Xi u^{sing}}_{\notin H^1(\Omega)}. \end{aligned}$$

Where  $u^{sing}$  is a fixed function which look like  $\text{erf}\left(\frac{s}{\sqrt{z}}\right)$  near  $(0, 0)$ .

In the process, we also discuss some numerical aspects of this shear layer. Lastly, we turn our attention to the layer created by this singularity and show that it is in fact an equatorial layer.

## 2.2.5 The problem of recirculation for Prandtl layers

Let us start by briefly recalling the difficulties of the stationary Prandtl equation due to the change of sign of  $u$ .

$$u\partial_x + v\partial_y u - \nu\partial_y^2 u = -\partial_x p \quad (2.13)$$

$$\begin{aligned} u|_{x=0} &= u_0 & u|_{y=0} &= 0 & v|_{y=0} &= v_0 \\ u(x, y) &\rightarrow U(x) & \text{when } y &\rightarrow \infty. \end{aligned} \quad (2.14)$$

Indeed, since Oleinik's work, local existence of solutions holds under the assumption of positivity.

**Theorem 2.2.7** (Oleinik, Samokhin [58]). *Under the the following assumption*

1. (positivity of initial condition)  $u_0 > 0$ ,  $u_0(0) = 0$ ,  $u'_0(0) > 0$  and  $u_0(y) \rightarrow U(0)$  when  $y \rightarrow \infty$ .
2. (regularity assumption)  $\partial_x p$  and  $v_0$  are  $C^1$ ,  $u_0$ ,  $u'_0$  and  $u''_0$  are bounded and satisfy Hölder condition.
3. (compatibility condition) at the point  $(x, y) = (0, 0)$ ,  $v_0(0)\partial_y u_0 - \nu\partial_y^2 u_0 = -\partial_x p(0) + O(y^2)$ .

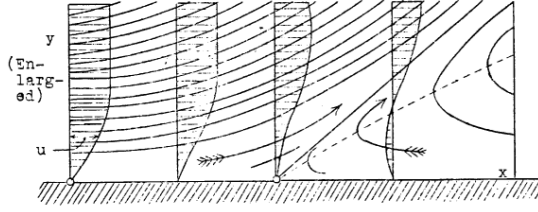


Figure 2.8 – The recirculation described by Prandtl in [59]

There exists a solution  $(u, v)$  of (2.13) with boundary conditions (2.14) on  $(0, X) \times (0, \infty)$  for some  $X > 0$ .

Moreover, a number of bounds on these solutions can be obtained, notably  $u > 0$  for  $y > 0$  and  $\partial_y u > c > 0$  for  $0 < y < C$ .

And, in the case  $\partial_x p \leq 0, v_0 \leq 0$  or  $\partial_x p < 0$  such result holds for any  $X > 0$ .

The constant sign of  $u$  being a crucial element of the proof, the question then arises of the physical adequacy of this hypothesis in the case of an *adverse pressure gradient*,  $\partial_x p > 0$ . Unfortunately, it has been known for a long time that a phenomenon called *boundary layer separation* might occur: for an adverse pressure gradient, in general there exist  $x^* > 0$  such that  $u_y(x^*, 0) = 0$ . Formal asymptotic expansion by Goldstein [38] followed by Stewartson [71] show that  $u$  may develop a singularity in the vicinity of  $x^*$ , namely

$$u_y(x, 0) \sim_{x \rightarrow x^*} C \sqrt{x^* - x}$$

for some positive  $C$ . The validity of this singularity was proved recently by Dalibard and Masmoudi [15] (see also the recent work [66]). However, to the best of our knowledge, there is no mathematical work addressing the behavior of the Prandtl equation beyond separation.

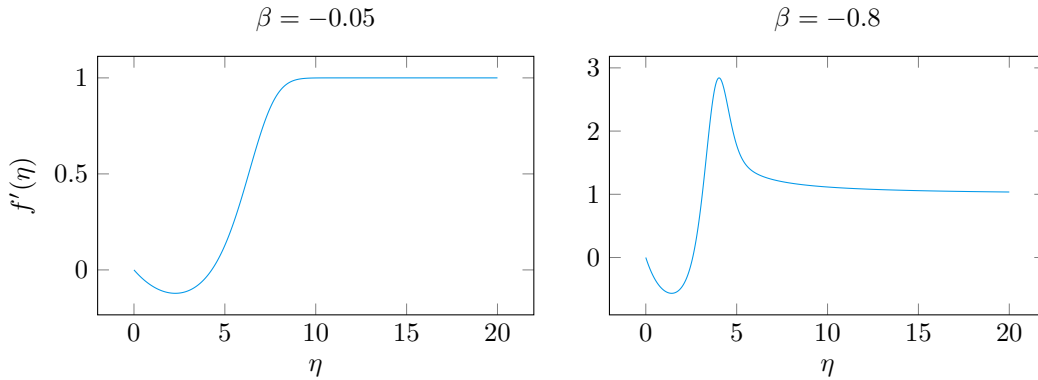
The rigorous construction of "recirculating" solutions, i.e. for which  $u$  changes sign, is therefore an important step in understanding the Prandtl equation.

Despite the theoretical difficulties that appear when such changes of sign occurs, there are explicit solutions of (2.13) with such recirculation. These self-similar solutions correspond to a Euler flow  $u_E = u_\infty x^m$  ( $m > -1$ ), i.e a pressure gradient  $p_E = m u_\infty^2 x^{2m+1}$  negative for  $m < 0$ . More precisely they are of the form  $u = u_\infty x^m f'(y x^{\frac{m-1}{2}})$  where  $f$  satisfies, in non-dimensional variables, the so-called Falkner-Skan ODE, see Figure 2.9,

$$f''' + f f'' + \underbrace{\frac{2m}{m+1}}_{\beta} (1 - (f')^2) = 0 \tag{2.15}$$

$$f(0) = 0, \quad f'(0) = 0, \quad \lim_{\eta \rightarrow \infty} f'(\eta) = 1$$

A method to build a whole family of recirculating solutions would then be to work in a perturbative way in the neighborhood of such solutions. However, this raises a number of difficulties. Some are linked to the non linearity of the Prandtl equation, and to the fact that the Prandtl equation around a recirculating flow can be considered as a degenerate elliptic system. In such a context, the correct linearization of the Prandtl system is not obvious. The other source of difficulty stems from the non-locality of the equation. In order to focus on the interplay between non-linearity and recirculation we decided to work on the following toy-model

Figure 2.9 – Numerical solutions of the Falkner-Skan equation for various  $\beta$ .

$$u\partial_x u - \partial_y^2 u = f. \quad (2.16)$$

If a linear version

$$\sigma\partial_x u - \partial_y^2 u = f. \quad (2.17)$$

where  $\sigma$  has no sign, called the forward-backward heat equation has already been considered, for example by Baouendi and Grisvard [6], to our knowledge (2.16) has never been studied precisely.

As long as  $u$  is far from zero, this can be seen as a quasi-linear version of the heat equation. But when  $u$  changes sign, the equation behaves like a forward parabolic equation in the zone  $u > 0$ , and like a backward parabolic equation in the zone  $u < 0$ .

The objective is then to construct, using an iterative method, a sign-changing solution  $u$ , in the neighborhood of an explicit solution  $\bar{u}$ . The key points are the following:

- A careful choice of boundary conditions. The idea is to perturb the boundary conditions  $\bar{u}$  by small functions  $\delta_i$  and to choose as boundary condition for  $u$ , at  $x = x_i$

$$u|_{\Sigma_i} = \bar{u} + \delta_i$$

where

$$\Sigma_i = \{x = x_i, (-1)^i (\bar{u} + \delta_i) > 0\}.$$

- The analysis of elliptic problems of the form (2.17). Since it is a degenerate elliptic problem, this is done through methods inspired by Fichera [28] in anisotropic spaces  $X^k = \{u \text{ s.t } \forall 0 \leq j \leq k, \partial_x^j u \in L^2, \partial_x^k \partial_y u \in L^2\}$ .
- Existence and estimate of weak solution in  $X^0$  of this elliptic problem.
- Under the hypothesis of existence of smooth solutions, existence and a priori estimates of strong solution in  $X^1$ . Then higher estimates (in  $X^2$ ). This is where compatibility conditions on  $\delta_i$  appears.
- The use of a well-chosen iterative scheme to conclude. Indeed, since the geometrical structure of  $u$  (the zone  $u > 0$  and  $u < 0$ ) plays an important role in the construction of a solution, a standard linearization does not work. We rather rely on the following iterative

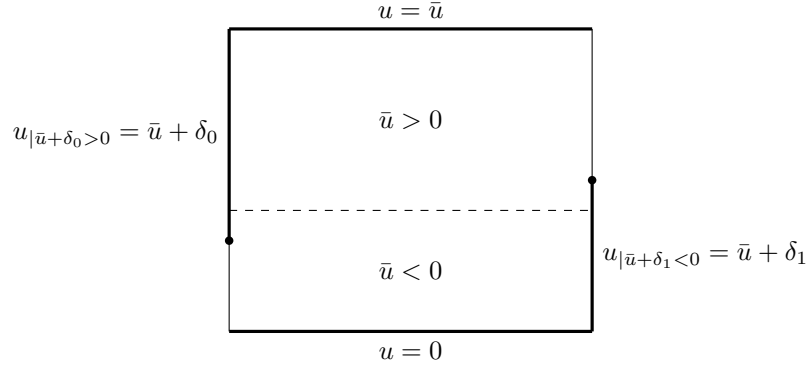


Figure 2.10 – Boundary conditions for (2.16).

scheme

$$\begin{aligned} (\bar{u} + u_n)\partial_x u_{n+1} - \partial_y^2 u_{n+1} &= 0 \\ u_{n+1}|_{\Sigma_i} &= \delta_i. \end{aligned}$$

This allows to obtain Theorem 6.1.1, which can be summarized as

**Sketch of Theorem 6.1.1.** *Assume that the existence in  $X^1$  for smooth data holds.*

*Let  $\delta_0, \delta_1 \in C^4$  satisfying some compatibility conditions.*

*Then, under the smallness condition on  $\delta_i$ , there exists a strong solution  $u$  in the class  $X^2$  to equation (2.16) endowed with the boundary conditions of Figure 2.10, which satisfies the stability estimate*

$$\|u - \bar{u}\|_{X^2} \leq C(\|\delta_0\| + \|\delta_1\|).$$

*Moreover, this solution is unique in a  $X^2$  neighborhood of  $\bar{u}$ .*

## 2.3 About degenerate elliptic equations

Several boundary layer problems studied in this manuscript fall into the scope of degenerate elliptic equations. We have mostly presented self-contained proofs in the rest of this manuscript. However, for the reader convenience let us recall the main ideas behind this theory. A typical degenerate elliptic equation can be written as

$$\sigma(y)\partial_x u - \partial_y^2 u = 0 \tag{2.18}$$

in a domain  $[x_0, x_1] \times [-1, 1]$  where  $\sigma(y)$  is allowed to change sign, and  $|\{\sigma = 0\}| = 0$ . The main ideas are the following:

- In order to prove existence we mainly need an a priori estimate.
- In order to obtain an energy inequality for degenerate elliptic equations, we need to prescribe adequate boundary conditions to obtain positivity of boundary terms obtained in the integration by parts.
- In order to prove uniqueness, classically we need to have existence of solutions of the dual problem which is similar to (2.18).

- When formulating the dual problem, *dual boundary conditions* must be identified.

We refer the interested reader to the work of Fichera [28], Oleinik [61] and Baouendi and Grisvard [6].

Assuming that  $\sigma$  is smooth, define the sets

$$\Sigma_i = \{(x = x_i; (-1)^i \sigma(x_i, y) > 0)\}.$$

Let  $D_i$  be two domains,  $D_i \subset \{x = x_i, y \in (-1, 1)\}$ , and  $u_i \in L^2(D_i)$ .

Then, it can be proved that the equation (2.18) has a solution a solution if the following boundary conditions are enforced

$$\begin{aligned} u|_{y=\pm 1} &= 0 \\ u|_{D_i} &= u_i \end{aligned}$$

with

$$D_i \subset \Sigma_i.$$

And uniqueness holds if

$$\Sigma_i \subset D_i.$$

This type of results will be used extensively in Chapters 4, 5 and 6. Notice that, in Chapters 4, 5, the degenerate elliptic equations we will consider are actually systems, so they do not fall exactly into the framework above. However, the strategy designed by Fichera [28] will allow us to identify the correct boundary conditions for these problems.

## 2.4 The one dimensional Proudman problem

In order to illustrate some basic techniques of boundary layers expansions and specificities of Ekman layers, let us focus first on a toy-model of the Proudman problem. We refer to the Section 3.1 for the physical derivation.

The equation is the following linear ODE with constant coefficients

$$\begin{aligned} \partial_z v - E \partial_z^4 \psi &= 0 \\ \partial_z \psi + E \partial_z^2 v &= 0 \end{aligned} \tag{2.19}$$

for  $z \in (0, 1)$  with boundary conditions

$$\begin{aligned} v(0) = v_0, v(1) = v_1, \partial_z \psi(0) = \partial_z \psi(1) &= 0 \\ \psi(0) = \psi(1) &= 0. \end{aligned} \tag{2.20}$$

Despite the fact that it is straightforward to obtain an explicit solution by solving a  $6 \times 6$  linear system (see Figure 2.11), and to obtain an asymptotic expansion from it, namely

$$\begin{aligned} v &= \frac{v_0 + v_1}{2} + \left( \frac{v_0 - v_1}{2} \right) e^{-\frac{z}{\sqrt{2E}}} \cos\left( \frac{z}{\sqrt{2E}} \right) + \left( \frac{v_1 - v_0}{2} \right) e^{-\frac{(1-z)}{\sqrt{2E}}} \cos\left( \frac{(1-z)}{\sqrt{2E}} \right) + O_{L^\infty}(\sqrt{E}) \\ \psi &= \sqrt{E} \left( \frac{v_1 - v_0}{2\sqrt{2}} \right) - \sqrt{E} \left( \frac{v_1 - v_0}{2} \right) e^{-\frac{z}{\sqrt{2E}}} \sin\left( \frac{z}{\sqrt{2E}} + \frac{\pi}{4} \right) \\ &\quad - \sqrt{E} \left( \frac{v_1 - v_0}{2} \right) e^{-\frac{(1-z)}{\sqrt{2E}}} \sin\left( \frac{(1-z)}{\sqrt{2E}} + \frac{\pi}{4} \right) + O_{L^\infty}(E) \end{aligned}$$

we will try, in this subsection, to use more advanced tools to obtain such an expansion.

*Remark 2.4.1.* Note that the explicit calculation is not uninteresting as it is the study of the symbol in  $z$  of the operator in the original Proudman problem.

More precisely, if we look for solutions as  $(v, \psi) = \sum_{\mu} (V_{\mu}, \Psi_{\mu}) e^{\mu z}$  ( $\mu = i\xi$  in a symbol analysis), then the equation (2.19), for each  $\mu$ , rewrites

$$\underbrace{\begin{pmatrix} E\mu^2 & \mu \\ \mu & -E\mu^4 \end{pmatrix}}_{A_{\mu}} \begin{pmatrix} V_{\mu} \\ \Psi_{\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So in order to obtain non-zero solutions we must have  $\det(A_{\mu}) = 0$  i.e

$$E^2\mu^6 + \mu^2 = 0.$$

This leads to 6 (counting the multiplicities) possible values of  $\mu$ , which are  $\frac{\pm 1 \pm i}{\sqrt{2E}}$  and 0. For each such  $\mu$  we then choose  $(V_{\mu}, \Psi_{\mu}) \in \ker(A_{\mu})$ . For  $\mu = \frac{\pm 1 \pm i}{\sqrt{2}}$  we obtain a kernel of dimension 1, when for  $\mu = 0$  we obtain a kernel of dimension 2. Thus, at the end, we have  $6 = 1 + 1 + 1 + 1 + 2$  linearly independent solutions, and there remains to write down the  $6 \times 6$  system associated with the 6 boundary conditions (2.20) for these solutions and check if it is invertible.

The step in the following analysis are quite generic:

- Establishing an energy estimate, and from it obtain a notion of approximate solution.
- Finding the boundary layer scalings, and the number of degree of freedom.
- Establishing the boundary layers operators, alongside their compatibility conditions, if needed.
- Establishing an interior operator, alongside its compatibility condition, if needed.
- Articulating the previous operator in order to obtain an approximate solution.

### 2.4.1 Energy estimate and stability

The existence and uniqueness in  $H^1 \times H^2$  of a solution of (2.19), (2.20) comes straight from Lax-Milgram theorem along with the estimate

$$E \int_0^1 (|\partial_z v|^2 + |\partial_z^2 \psi|^2) \leq C (|v_0|^2 + |v_1|^2).$$

In fact, we have a more precise estimate.

**Proposition 2.4.1** (Stability estimate). *Let  $(v, \psi)$  with homogeneous boundary conditions and satisfying (2.19) with source terms*

$$\begin{aligned} \partial_z v - E\partial_z^4 \psi &= f_{\psi} \\ \partial_z \psi + E\partial_z^2 v &= f_v. \end{aligned}$$

Then

$$E\|v\|_{H^1}^2 + E\|\psi\|_{H^2}^2 \leq C (\|f_v\|_{H^{-1}}^2 + \|f_{\psi}\|_{H^{-2}}^2)$$

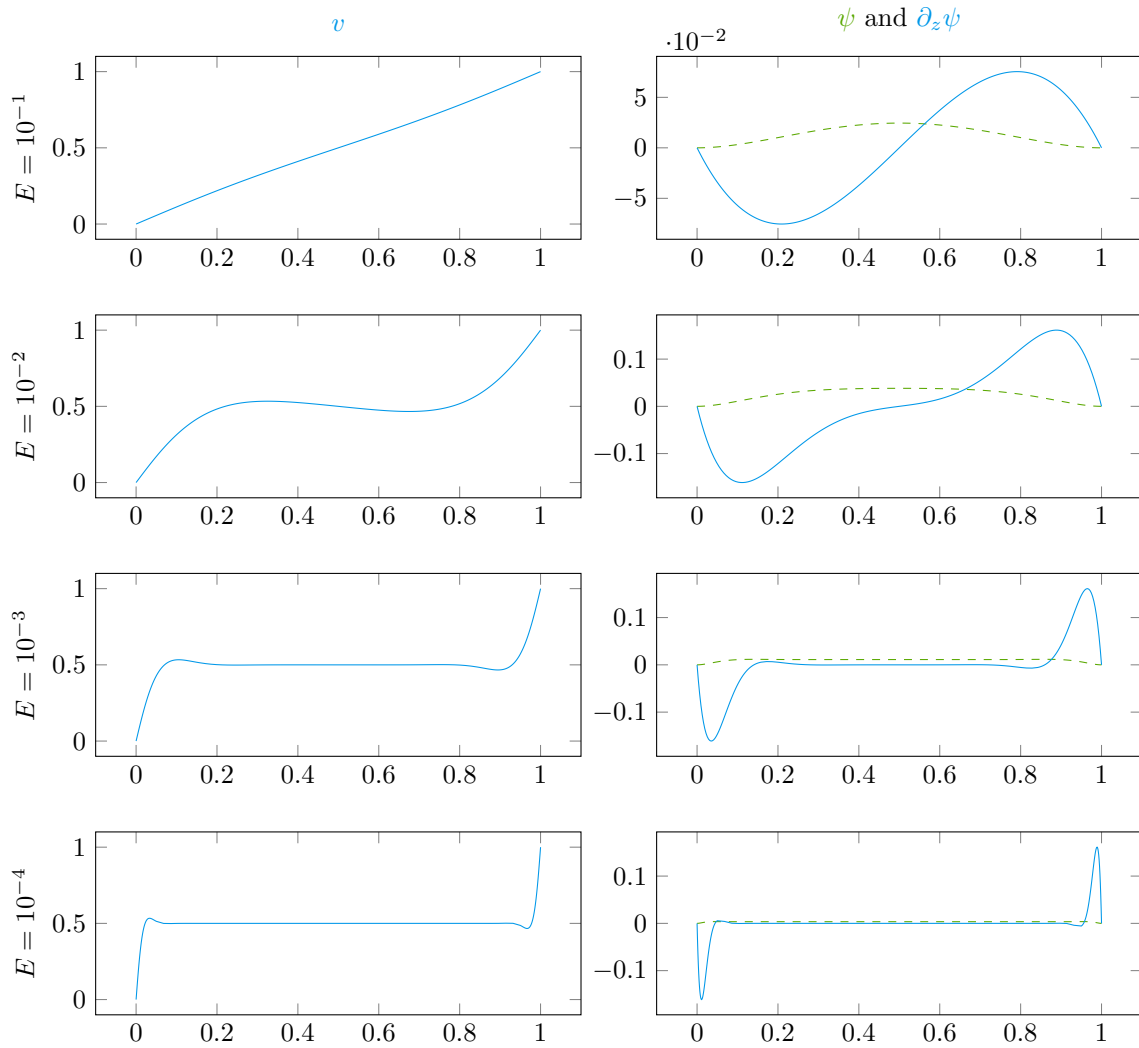


Figure 2.11 – Explicit solutions for  $v_0 = 0, v_1 = 1$  and various values of  $E$ .



and

$$\|v\|_{L^2}^2 + \|\psi\|_{H^1}^2 \leq C \left( \|f_v\|_{L^2}^2 + \|f_\psi\|_{H^{-1}}^2 + \frac{1}{E} \left( \int_0^1 f_v \right)^2 \right).$$

*Proof.* The first estimate is simply the one obtained by Lax-Milgram. The second one requires a more elaborate construction.

Let us consider the equation

$$w + iE\partial_z^2 w = \phi \tag{2.21}$$

with boundary conditions  $w(0) = w(1) = 0$ .

Then, as

$$\int_0^1 |w|^2 - iE \int_0^1 |\partial_z w|^2 = \int_0^1 \phi \bar{w}$$

it is sufficient to show that  $w = v + i\partial_z \psi$  solves (2.21) for a well chosen  $\phi$ .

Let  $\phi = if_v + F_\psi$ , where  $\partial_z F_\psi = f_\psi$  such that  $\|F_\psi\|_{L^2} \leq C\|f_\psi\|_{H^{-1}}$ . We have

$$\begin{aligned} \Im(w + iE\partial_z^2 w) &= \partial_z \psi + E\partial_z^2 v = f_v \\ \partial_z \Re(w + iE\partial_z^2 w) &= \partial_z v - E\partial_z^4 \psi = f_v = \partial_z F_\psi. \end{aligned}$$

Meaning that  $w + iE\partial_z^2 w = \phi + K$ , with  $K \in \mathbb{R}$ . So all it remains is to bound  $|K|$ .

Let  $\omega$  be the solution on the dual equation

$$\omega - iE\partial_z^2 \omega = 1$$

with  $\omega(0) = \omega(1) = 0$ . Then,

$$\int (w + iE\partial_z^2 w) \bar{\omega} = \int w \overline{(\omega - iE\partial_z^2 \omega)} = \int (\phi + K) \bar{\omega}.$$

Taking the imaginary part, as  $\int_0^1 \omega = \int_0^1 v \in \mathbb{R}$ , we obtain

$$K = -\frac{\Im(\int \phi \bar{\omega})}{\Im(\int \bar{\omega})}.$$

But  $\omega$  is explicit,  $\omega = 1 - e^{\frac{-1+i}{\sqrt{2E}}z} - e^{\frac{-1+i}{\sqrt{2E}}(1-z)} + o_{C^\infty}(E^\infty)$ , so

$$|K| \leq C \left( \|\phi\|_{L^2} + \frac{1}{\sqrt{E}} \int f_v \right)$$

□

*Remark 2.4.2.* • The condition that  $\int f_v$  must be small is a natural one. Indeed we imposed  $\psi = 0$  on all boundaries, whereas the physical condition is that  $\psi$  is constant on each connected component of said boundaries.

- We already see that, in order to improve the results, the boundary layers, here of the dual equation, must be constructed.

In fact, the stability estimate can be rewritten as:

**Proposition 2.4.2** (Stability and approximate solutions). *For  $E > 0$  let  $\mathbf{u} = (v, \psi) \in H^1 \times H^2$  be the solution of (2.19), (2.20), and let  $\mathbf{u}^{app} = (v^{app}, \psi^{app}) \in H^1 \times H^2$  with (2.20) as boundary*

conditions and satisfying

$$\begin{aligned}\partial_z v - E\partial_z^4 \psi &= r_\psi \\ \partial_z \psi + E\partial_z^2 v &= r_v.\end{aligned}$$

Then

$$\begin{aligned}\|v - v^{app}\|_{L^2}^2 + \|\psi - \psi^{app}\|_{H^1}^2 &\leq C \left( \|r_v\|_{L^2}^2 + \|r_\psi\|_{H^{-1}}^2 + \frac{1}{E} \left( \int_0^1 \partial_z r_v \right)^2 \right) \\ \|v - v^{app}\|_{H^1}^2 + \|\psi - \psi^{app}\|_{H^2}^2 &\leq CE^{-1} \left( \|r_v\|_{H^{-1}}^2 + \|r_\psi\|_{H^{-2}}^2 \right)\end{aligned}\tag{2.22}$$

when  $E$  goes to 0.

Thus we see clearly the objective: to construct approximate solution  $\mathbf{u}^{app}$  such that the remainders  $\mathbf{r}$  are small, in order to conclude for the convergence of  $u$ .

### 2.4.2 Boundary layer ansatz, boundary layer operator

The construction of an approximate solution relies on a boundary ansatz. We look for an approximate solution as the sum of two terms:

- An interior term satisfying (up to an acceptable remainder) the equation in the whole domain but not all the boundary conditions. This will typically be the function  $\bar{\mathbf{u}}$ , with lower order terms if necessary. In our case, we see that  $\bar{\mathbf{u}}$  from the Taylor Proudman theorem fulfills the equation with an acceptable remainder, but we have only 2 degree of freedom for the boundary conditions when 6 of them must be satisfied.
- A boundary layer term (or multiple boundary layers), localized in a small region near the boundary, of size decreasing to 0 with  $E$ . These terms fulfill an approximate equation, and the associated boundary conditions are such that the sum of both the interior and the boundary layer terms satisfy the global boundary conditions. This boundary layer term, even if fundamental to the construction, is very small in low regularity norms (such as  $L^2$ ), and so may not appear directly in the convergence result.

This can be summarized (for only one boundary layer) as

$$\mathbf{u}^{app} = \mathbf{u}^{int}(z) + \underbrace{\mathbf{u}^{bl}(z)}_{= \mathbf{u}(d(z, \Gamma)/\delta_E)}\tag{2.23}$$

with  $\mathbf{u}^{int}$  satisfying the equation in the whole domain and  $\mathbf{u}^{bl}$  also solution of the equation but localized in a region of size going to 0 when  $E$  goes to 0,  $d(z, \Gamma) < \delta_E = o_{E \rightarrow 0}(1)$ .

The construction of  $\mathbf{u}^{bl}$  is such that  $\mathbf{u}^{bl} + \mathbf{u}^{int}$  satisfies the boundary conditions, this is why we will state that the boundary layer is *lifting the boundary conditions* of the interior term.

Following a formalism used by Dalibard and Saint-Raymond [18] we will see this lifting as a generic operator, constructing a boundary layer solution from boundary conditions. Here this very simple operator reads as

**Proposition 2.4.3** (Boundary layer operator). *For  $(V, \Psi, \Upsilon)$  such that*

$$-\sqrt{\frac{2}{E}}\Psi + V - \Upsilon = 0\tag{2.24}$$

there exist a solution  $\mathbf{u}^{bl} = \mathbf{u} \left( z/\sqrt{E} \right) \chi(z)$  of (2.19) satisfying

$$\begin{aligned} \|\mathbf{r}\|_{H^\infty} &= o(E^\infty)^\dagger \\ (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{z=0} &= (V, \Psi, \Upsilon) \\ (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{z=1} &= 0 \end{aligned}$$

and the following estimate

$$\|v^{bl}\|_{H^s} \leq CE^{\frac{1}{4} - \frac{s}{2}} \quad \|\psi^{bl}\|_{H^s} \leq CE^{\frac{3}{4} - \frac{s}{2}}.$$

*Remark 2.4.3.* Some basic remarks can be made about this operator:

- This operator is the bottom boundary layer operator. The exact same result holds for  $z = 1$  substituting  $z$  by  $1 - z$  and (2.24) by

$$\sqrt{\frac{2}{E}} \Psi + V - \Upsilon = 0 \quad (2.25)$$

- In this simple setting  $\mathbf{r}$  is essentially 0 in all norms, as the boundary layer equation is an exact rescaling of the main equation. For more complex models we will have to check the size of  $\mathbf{r}$  in the adequate norms and if necessary take  $\mathbf{r}$  as a source term for a higher order correctors.
- Quite generally, the boundary layer operator can depend on the tangential coordinate or frequency. In fact this formalism was introduced in [18], notably in order to consider the degeneracy (in frequency) created by resonant surface stress.
- The cut-off function  $\chi$  is a purely technical construction, and has close to no influence. This is why we will forget it when writing ansatz and scaling from now on (we will write  $\mathbf{u} \left( z/\sqrt{E} \right)$  instead of  $\mathbf{u} \left( z/\sqrt{E} \right) \chi(z)$ ).

*Proof.* The idea behind this corrector is a simple application of the generic method developed notably by Gérard-Varet in [45] and later precised by Gérard-Varet and Paul in [37].

The first step is to identify the scaling  $\sqrt{E}$  through the study of the characteristic "manifold" (here a finite set of points). With  $A(\xi)$  the symbol of the operator

$$A(\xi) = \begin{pmatrix} -E\xi^4 & i\xi \\ i\xi & -E\xi^2 \end{pmatrix}$$

this manifold is the set of  $\xi$  such that  $\det(A(\xi)) = E^2\xi^6 + \xi^2 = 0$ .

Thus, the scaling is  $\xi \sim \frac{C}{\sqrt{E}}$ , and so we have a boundary layer size of  $\sqrt{E}$ .

In such boundary layer scaling with  $\zeta = \frac{z}{\sqrt{E}}$  and

$$(v^{bl}(z), \psi^{bl}(z)) = (v(\zeta), \sqrt{E}\psi(\zeta)) \quad (2.26)$$

the equation is simply

$$\begin{pmatrix} -\partial_\zeta^4 & \partial_\zeta \\ \partial_\zeta & \partial_\zeta^2 \end{pmatrix} \begin{pmatrix} \psi \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.27)$$

---

<sup>†</sup>Meaning that  $\forall s, k \geq 0$  we have  $\|\mathbf{r}\|_{H^s} = o(E^k)$ , see the notations in Section 2.1.

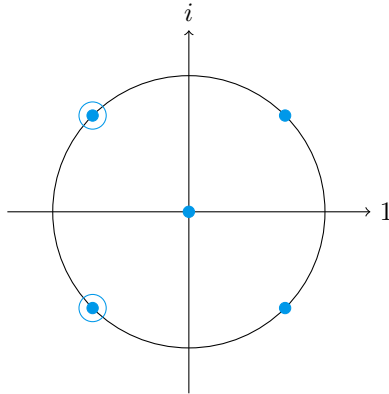


Figure 2.12 – The modes of the Ekman boundary layer operator.

with a rescaled characteristic polynomial

$$\lambda^6 + \lambda^2 = 0.$$

Here we can explicitly compute six roots:  $\frac{\pm 1 \pm i}{\sqrt{2}}$  and 0 with multiplicity 2, see Figure 2.12.

*Remark 2.4.4.* • A more usual approach would be to consider a Fourier transform and to compute the determinant of the symbol. We choose here to consider characteristic polynomial as it is an ODE, i.e we consider  $e^{\lambda\zeta}$  instead of  $e^{i\xi\zeta}$ . It not only allows to clearly see the decay at infinity in an ODE framework but, more importantly, it is closely related to the initial boundary condition problem and to the Laplace transform (see for example the use of such transform of Gérard-Varet and Paul [37]). Ultimately both approaches are equivalent with  $\lambda = i\xi$ .

- Note that the fact that  $\lambda = 0$  is a mode is sometimes referred to as degeneracy of Ekman layer. This mode creates an additional term in the interior flow, as constants (and polynomials) terms are relevant in both boundary layer and interior scalings. This mode is exactly the one creating the Ekman pumping as it allows the exchange of energy between both scalings. However, following the general framework described in [45], we will use specifically the terminology *degenerate* only for modes strongly dependent on tangent coordinates or non-normal frequencies.

As we need decay to 0 when  $\zeta \rightarrow +\infty$ , we only keep the two roots with negative real part:

$$\lambda_{\pm} = \frac{-1 \pm i}{\sqrt{2}}.$$

The next step is to compute the dimension of the kernels  $E_{\pm}$  for both modes. We obtain  $\dim(E_{\pm}) = 1$  so we can lift up to  $1 + 1 = 2$  boundary conditions, Thus, we need  $3 - 2 = 1$  compatibility conditions.

More explicitly, the solution is of the form

$$\begin{aligned} v(\zeta) &= \alpha_+ e^{\lambda_+ \zeta} + \alpha_- e^{\lambda_- \zeta} \\ \psi(\zeta) &= -\lambda_+ \alpha_+ e^{\lambda_+ \zeta} - \lambda_- \alpha_- e^{\lambda_- \zeta} \end{aligned}$$

where  $\alpha_{\pm} \in \mathbb{C}$ . As a consequence

$$\begin{pmatrix} v(0) \\ \psi(0) \\ \partial_{\zeta}\psi(0) \end{pmatrix} = \alpha_+ \begin{pmatrix} 1 \\ -\lambda_+ \\ -\lambda_+^2 \end{pmatrix} e^{\lambda_+ 0} + \alpha_- \begin{pmatrix} 1 \\ -\lambda_- \\ -\lambda_-^2 \end{pmatrix} e^{\lambda_- 0}.$$

Therefore, the compatibility condition, i.e the additional constraint we require to recover the right number of degrees of freedom, is to have boundary conditions in the 2 dimensional subspace of  $\mathbb{C}^3$  spanned by  $(1, -\lambda_{\pm}, -\lambda_{\pm}^2)$  i.e

$$-\sqrt{2}\psi(0) + v(0) - \partial_{\zeta}\psi(0) = 0 \quad (2.28)$$

which leads to (2.24) in rescaled variables.

Then, considering the scaling (2.26) with

$$\|v\|_{H^s} = O(1) \quad \|\psi\|_{H^s} = O(1)$$

we obtain the claimed estimates.

In fact to obtain traces vanishing exactly on  $\Gamma_1$  we must multiply the previous solution by a cut-off function  $\chi(z)$  such that  $\chi = 1$  on  $(0, 1/3)$  and  $\chi = 0$  on  $(2/3, 1)$

As  $\mathbf{u}^{bl}$  and all its derivatives are  $o_{L^\infty}(E^\infty)$  for  $z \in (1/3, 2/3)$ , the remainder term created by this cut-off is  $o(E^\infty)$  in all norms.  $\square$

### Construction of an approximate solution

Once both boundary layer operators have been identified, we can now look for an approximate solution of the form (2.23).

More precisely we consider the ansatz

$$\mathbf{u}^{app} = \sum_k E^{\frac{k}{2}} \left( \mathbf{u}^{int,k}(z) + \mathbf{u}^{BL,b,k} \left( \frac{z}{\sqrt{E}} \right) + \mathbf{u}^{BL,t,k} \left( \frac{1-z}{\sqrt{E}} \right) \right). \quad (2.29)$$

Identifying each order, the equations in the interior domain are

$$\begin{aligned} \partial_z v^{int,k} &= \partial_z^A \psi^{int,k-2} \\ \partial_z \psi^{int,k} &= -\partial_z^2 v^{int,k-2} \end{aligned} \quad (2.30)$$

and for the boundary layer terms we consider the equation (2.27).

The interior and boundary layer terms interact through the boundary conditions

$$\begin{aligned} v^{int,k}(0) + v^{BL,b,k}(0) = V_0^k &= \begin{cases} v_0 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases} & v^{int,k}(1) + v^{BL,t,k}(1) = V_1^k &= \begin{cases} v_1 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases} \\ \psi^{int,k}(0) + \psi^{BL,b,k-1}(0) = 0 & & \psi^{int,k}(1) + \psi^{BL,t,k-1}(0) = 0 & \\ \partial_z \psi^{int,k}(0) + \partial_z \psi^{BL,b,k}(0) = 0 & & \partial_z \psi^{int,k}(1) + \partial_z \psi^{BL,t,k}(0) = 0 & \end{aligned}$$

The previous study of the boundary layer operator allows us to decouple the interior and the boundary layer. Thus, we can rewrite the previous expansion in terms of the interior flow alone: the equation is (2.30) and the boundary conditions are given by

$$\begin{aligned} -\sqrt{2}\psi^{int,k+1}(0) + (v^{int,k}(0) - V_0^k) - \partial_z\psi^{int,k}(0) &= 0 \\ \sqrt{2}\psi^{int,k+1}(1) + (v^{int,k}(1) - V_1^k) - \partial_z\psi^{int,k}(1) &= 0. \end{aligned} \quad (2.31)$$

*Remark 2.4.5.* As expected, we obtain the right number of boundary conditions for the closed expansion of the interior flow. Here this is simply because out of the 6 modes we choose 2 for the top boundary layer, 2 for the bottom one and the two remaining are the ones associated with the 0 modes (invariant by scaling) and correspond in fact to the interior problem.

In particular, with  $k = 0$ , the equation leads to  $\psi^{int,0}$  and  $v^{int,0}$  being constant, and we obtain with  $k = -1$  that  $\psi^{int,0}(0) = \psi^{int,0}(1) = 0$  so  $\psi^{int,0} = 0$ .

With  $k = 1$  we obtain that  $\psi^{int,1}$  is also constant, and using again the boundary condition of  $k = 0$ , denoting the constant  $v^{int,0}$  and  $\psi^{int,1}$  by  $\mathfrak{V}^0$  and  $\mathfrak{F}^1$  respectively, (2.31) reads at the first order

$$\begin{aligned} -\sqrt{2}\mathfrak{F}^1 + (\mathfrak{V}^0 - v_0) &= 0 \\ \sqrt{2}\mathfrak{F}^1 + (\mathfrak{V}^0 - v_1) &= 0 \end{aligned}$$

leading to the announced values for the first order for  $\mathbf{u}$ .

As these interior terms and the associated boundary layer are an acceptable approximate solution (as defined in 3.2.2), in particular we can finally conclude that

$$\bar{\mathbf{u}} = (\mathfrak{V}^0, 0) = \left( \frac{v_0 + v_1}{2}, 0 \right).$$

The estimate on the remainder allowing us to derive the strong convergence in  $L^2$  (and even in the energy space).

Note that, in this simplified setting, we can pursue the asymptotic expansion up to any higher order without any additional difficulty.

### Generic framework and diagrams summary

This whole construction can be summarized as a construction with the *boundary layer operator* and *interior operator*.

This interior operator is the explicit integration of the equation (2.30), and can be expressed in a similar way to the boundary layer operator of Proposition 2.4.3.

**Proposition 2.4.4** (Interior operator for zero source term). *For  $V_0, \Psi_0, \Upsilon_0, V_1, \Psi_1, \Upsilon_1$  such that*

$$\begin{aligned} V_1 - V_0 &= 0 \\ \Psi_1 - \Psi_0 &= 0 \\ (-1)^i \Upsilon_i &= 0 \end{aligned} \quad (2.32)$$

*there exist  $\mathbf{u}^{int}$  solution of (3.9) satisfying*

$$\begin{aligned} \|\mathbf{r}\|_{H^\infty} &= 0 \\ (v^{int}, \psi^{int}, \partial_n \psi^{int})_{z=i} &= (V_i, \Psi_i, \Upsilon_i) \end{aligned}$$

*and the following estimate for all  $s$*

$$\|v^{int}\|_{H^s} \leq C \quad \|\psi^{int}\|_{H^s} = 0.$$

Indeed the interior equation is obtained by neglecting the  $E\partial_z^4\psi$  and the  $E\partial_z^2v$  and reads as

$$\begin{aligned}\partial_z v &= 0 \\ \partial_z \psi &= 0.\end{aligned}$$

Which is immediate to solve given the boundary conditions.

In fact it is as easy to solve for the equation with a source-term

**Proposition 2.4.5** (Interior operator for a non-zero source term). *Let  $\mathbf{f} = (f_v, f_\psi)$  be smooths function and  $V_0, \Psi_0, \Upsilon_0, V_1, \Psi_1, \Upsilon_1$  such that*

$$\begin{aligned}V_1 - V_0 &= \int_0^1 f_\psi(z) dz \\ \Psi_1 - \Psi_0 &= \int_0^1 f_v(z) dz \\ (-1)^i \Upsilon_i &= f_v(i).\end{aligned}\tag{2.33}$$

Then there exist  $\mathbf{u}^{int} = (v^{int}, \psi^{int})$  solution of

$$\begin{aligned}\partial_z v^{int} - E\partial_z^4 \psi^{int} &= f_\psi + Er_\psi^{int} \\ \partial_z \psi^{int} + E\partial_z^2 v^{int} &= f_v + Er_v^{int} \\ (v^{int}, \psi^{int}, \partial_n \psi^{int})|_{z=i} &= (V_i, \Psi_i, \Upsilon_i)\end{aligned}$$

and

$$\begin{aligned}\|r_v^{int}\|_{H^s} &\leq \|f_\psi\|_{H^{s+1}} \\ \|r_\psi^{int}\|_{H^s} &\leq \|f_v\|_{H^{s+3}}.\end{aligned}$$

The proof being once more the explicit integration of the *interior equation*

$$\begin{aligned}\partial_z v &= f_\psi \\ \partial_z \psi &= f_v.\end{aligned}$$

Note that the remainder  $\mathbf{r}^{int}$ , in this simple setting, is

$$\begin{aligned}r_v^{int} &= \partial_z f_\psi \\ r_\psi^{int} &= -\partial_z^3 f_v.\end{aligned}$$

*Remark 2.4.6.* The loss of regularity between the source and the remainder is a natural phenomenon in singular perturbation expansions, but is of no concern in this particular example. Indeed, in the following asymptotic expansion we have will  $\mathbf{f}^0 = 0 \in C^\infty$  so by induction we have in fact  $\mathbf{f}^k \in C^\infty$  for all  $k$ .

In order to articulate these operators at each order we can follow matching methods popularized by Van Dyke [72]. Here the boundary conditions for initializing the expansion are unknown, so we let  $v_{|z=0}^{int,0} = v_{|z=1}^{int,0} = \bar{v}$  be a free parameter.

- We construct an interior term satisfying the equation up to a lower order remainder.

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<sup>†</sup>Note that, in this case, as the resolution is explicit, there is no need for source term or remainder in the boundary layer. This is why these are in gray on the Figure 2.13.

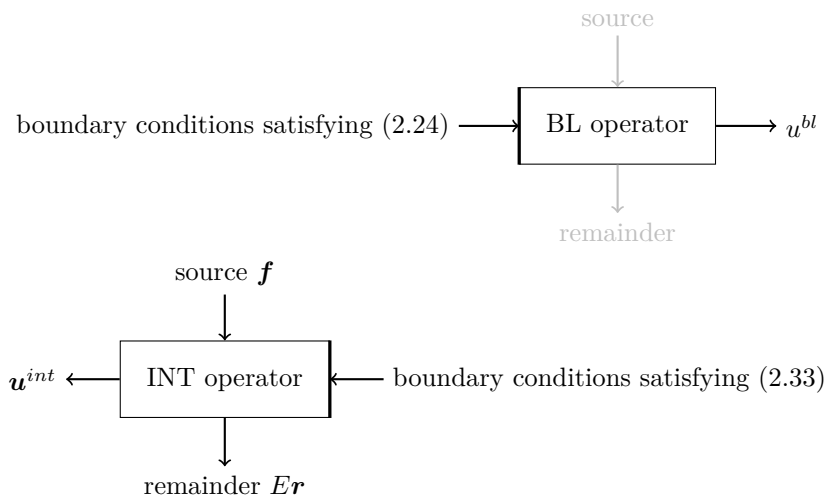


Figure 2.13 – Boundary layer and interior operator blocks<sup>†</sup>.

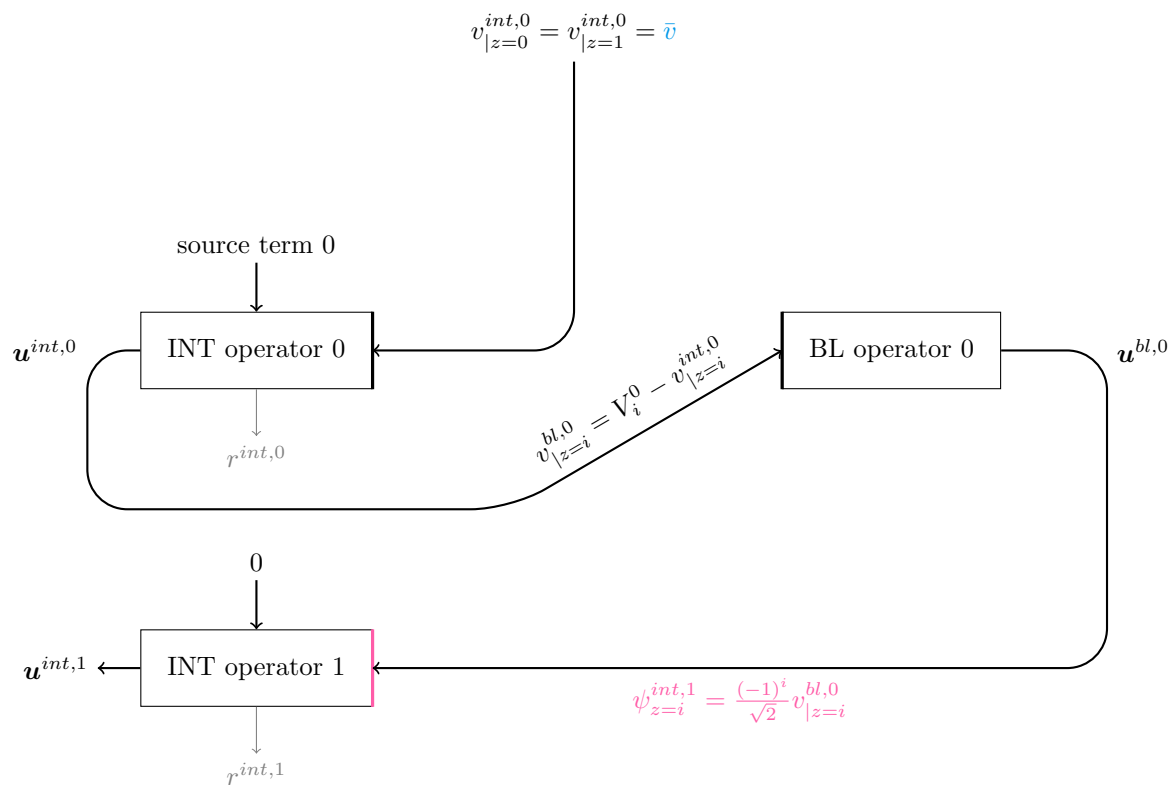


Figure 2.14 – The usual Van Dyke construction. The value of  $\bar{v}$  is chosen in order to have boundary conditions on  $\psi^{int,1}$  compatible with the interior operator.



- The constructed solution does not satisfy all boundary conditions, so we lift the mismatch in the boundary condition for  $v$  by a boundary layer, thus setting the values of  $v|_{z=i}^{bl,0}$  and  $\partial_z \psi|_{z=i}^{bl,0}$ .
- This creates a mismatch in the boundary conditions on  $\psi|_{z=0}$  and  $\psi|_{z=1}$ .
- This is lifted by another interior term  $\mathbf{u}^{int,1}$  of order  $\sqrt{E}$ . But in order to be compatible with  $\partial_z \psi^{int,1} = 0$  we need  $\psi|_{z=0} = \psi|_{z=1}$ .
- This condition leads to the choice of  $\bar{v}$ .

This construction, summarized in Figure 2.14 can then be iterated to higher orders.

*Remark 2.4.7.* Apart from the terminology, *interior* instead of *outer* and *boundary layer* instead of *inner*, there are two differences between this construction and the general framework of boundary layers as used by physicists.

- The first one is that rather than the matching principle, which can be summarized as,

$$\begin{aligned} u^{int}(z = \gamma_i) &= u^{bl}(\zeta = +\infty) \\ u^{bl}(\zeta = 0) &= v_i \end{aligned}$$

we use the following

$$\begin{aligned} u^{int}(z = \gamma_i) + u^{bl}(\zeta = 0) &= v_i \\ u^{bl}(\zeta = +\infty) &= 0. \end{aligned}$$

These formulations are very close, but the second one allows us to work with functions decaying at infinity rather than having a finite limit at infinity, leading to easier functional spaces.

- The second one is the matching orders, as we see the second order of the interior expansion exerts an influence on the first order of said expansion, through the choice of  $\bar{v}$ . This free parameter  $\bar{v}$  shows that the principle (for example put forward by Van Dyke in [72]) that higher order terms have no influence on lower order terms is not always satisfied.

But we see that the decomposition of the boundary conditions between the ones in the hyperplane (2.24) and its orthogonal allow us to close directly the interior expansion, without the need to navigate between multiple orders in order to set free parameters.

Thus, it is easier, at each step, to choose simultaneously the boundary condition for the interior and boundary layer term to satisfy all compatibility conditions.

As we have  $3 + 3 + 6 = 12$  traces to choose (3 for the bottom boundary layer operator, 3 for the top boundary layer operator and 6 for the interior operator), satisfying  $1 + 1 + 4 = 6$  compatibility conditions (1 for each boundary layer operator and 4 for the interior operator), a simple argument of dimension shows

**Proposition 2.4.6** (Figure 2.15). *Let  $(V_0, \Psi_0, \Upsilon_0, V_1, \Psi_1, \Upsilon_1) \in \mathbb{R}^6$ .*

*Then there exist one and only one  $(V_0^{bl}, \Psi_0^{bl}, \Upsilon_0^{bl}, V_1^{bl}, \Psi_1^{bl}, \Upsilon_1^{bl}, V_0^{int}, \Psi_0^{int}, \Upsilon_0^{int}, V_1^{int}, \Psi_1^{int}, \Upsilon_1^{int})$  such that*

- $V_0^{bl}, \Psi_0^{bl}, \Upsilon_0^{bl}$  satisfy (2.24);
- $V_1^{bl}, \Psi_1^{bl}, \Upsilon_1^{bl}$  satisfy (2.25);
- $V_0^{int}, \Psi_0^{int}, \Upsilon_0^{int}, V_1^{int}, \Psi_1^{int}, \Upsilon_1^{int}$  satisfy (2.32);

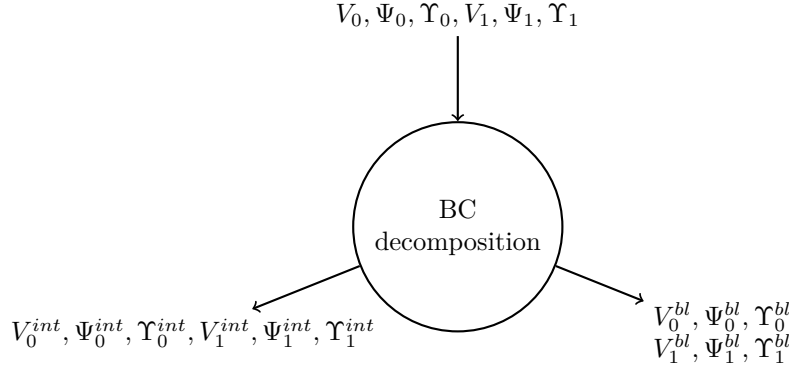


Figure 2.15 – The decomposition of boundary conditions.

- And

$$\begin{aligned} V_0^{int} + V_0^{bl} &= V_0, & V_1^{int} + V_1^{bl} &= V_1, \\ \Psi_0^{int} + \Psi_0^{bl} &= \Psi_0, & \Psi_1^{int} + \Psi_1^{bl} &= \Psi_1, \\ \Upsilon_0^{int} + \Upsilon_0^{bl} &= \Upsilon_0, & \Upsilon_1^{int} + \Upsilon_1^{bl} &= \Upsilon_1. \end{aligned}$$

*Remark 2.4.8.* It is important to note a crucial particularity of Ekman layers: this decomposition mixes scales, meaning that  $\Psi_{0/1}^{int}$  is one order higher (in  $\sqrt{E}$ ) than  $V_{0/1}^{int}$  and  $\Upsilon_{0/1}^{int}$ . This very phenomenon creates the dependency of the Ekman pumping with respect to the latitude. This mixing of scales originates from the order of the bi-laplacian: as it requires two boundary conditions,  $\psi = 0$  and  $\partial_n \psi = 0$ , but of different orders (as differential operators) it is impossible to satisfy both boundary conditions with only one scale.

This is a generic phenomenon: in order to lift a Dirichlet condition with a boundary layer of size  $\varepsilon$ , a term like  $u(x/\varepsilon)$  is to be used, when, in order to lift a Neumann condition, it would rather be  $\varepsilon u(x/\varepsilon)$ . The important consequence is the following: any Dirichlet corrector of order 1 creates a Neumann error of size  $\varepsilon^{-1}$  and any Neumann corrector of order 1 creates a Dirichlet error of order  $\varepsilon$ . The same effect appears notably in the Munk boundary layers as there is also a fourth order operator. Note that in the case of MHD, see Chapter 5, the equation is very similar but the  $\Delta^2 \psi$  is replaced by a  $\Delta b$ . As a consequence the interior flow does not depend on the latitude. However, such scale mixing still appears with Robin boundary conditions.

*Remark 2.4.9.* In this example we have done nothing more than choose a decomposition of boundary conditions into subspaces adapted to the eigenvectors of a one dimensional differential operator. Quite generally the compatibility conditions can be Dirichlet to Neumann like conditions, as both the interior and boundary layer equation will be PDE without explicit diagonalization.

With such decomposition, we can close both boundary layer and interior equation of each order as in equation (2.31). This is illustrated in Figure 2.16.

This allows us to prove the validity of this boundary layer expansion:

**Proposition 2.4.7.** For  $v_0, v_1 \in \mathbb{R}$  let  $u^E$  be the solution of (2.19). Let  $u^{app}$  be the result of the previous construction

$$\mathbf{u}^{app} = \mathbf{u}^{0,int} + \mathbf{u}^{bl,b,0} + \mathbf{u}^{bl,t,0} + \mathbf{u}^{int,1}.$$

Then when  $E \rightarrow 0$

$$\begin{aligned} \|\mathbf{u}^E - \mathbf{u}^{app}\|_{L^2} &= o(E^\infty) \\ \|\mathbf{u}^E - \mathbf{u}^{app}\|_{H^1} &= o(E^\infty) \end{aligned}$$

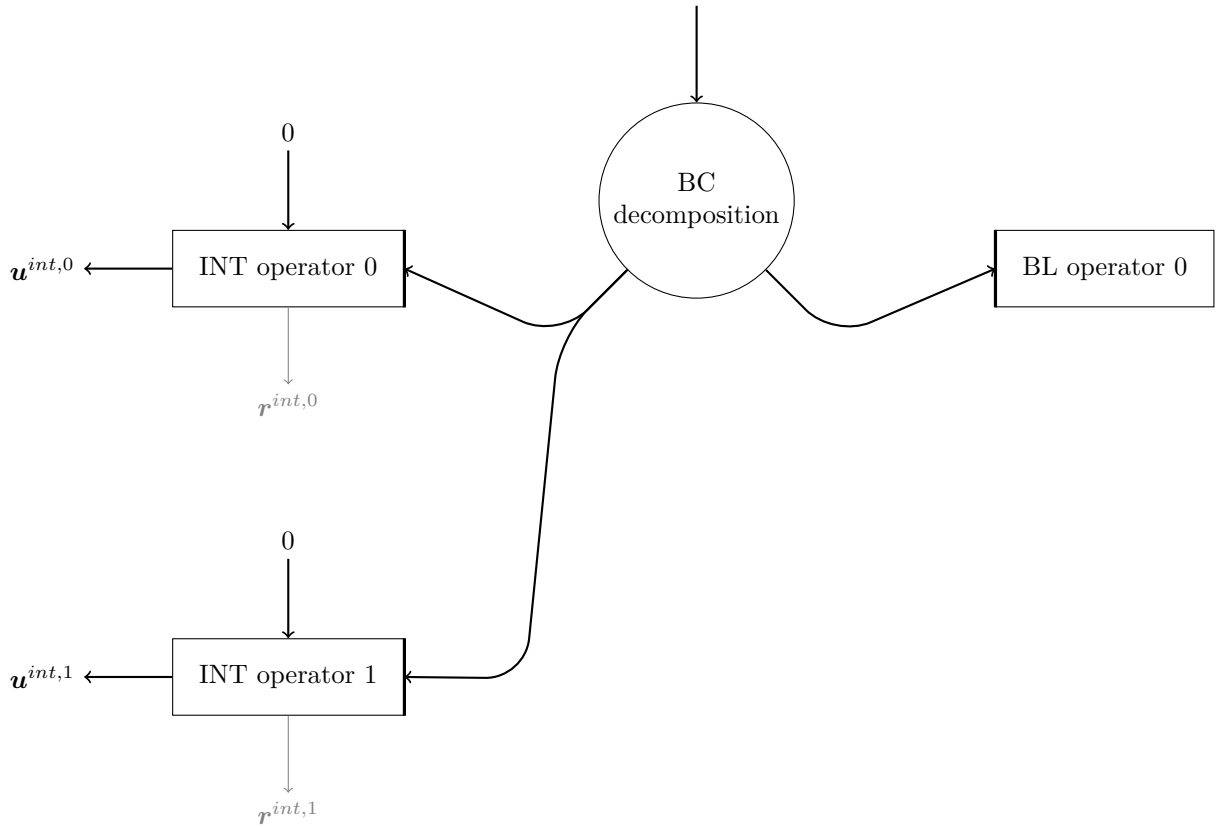


Figure 2.16 – The use of boundary condition decomposition.

*Proof.* This is only an application of the previous operators, in the very particular case  $\mathbf{f} = 0$ .  $\square$

*Remark 2.4.10.* In this very particular equation we can close the expansion after a finite number of application of the various operators. In all generality this is not the case and, as the total order of the expansion is limited, the order of convergence is never  $o(E^\infty)$ .

# Proudman problem

## Outline of the current chapter

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<b>3.1 Physical modelization</b>	<b>47</b>
<b>3.2 Preliminary results on the Proudman problem</b>	<b>54</b>
<b>3.3 Overall view of the construction</b>	<b>59</b>
<b>3.4 The Ekman layers for two disks</b>	<b>61</b>
<b>3.5 The shear layers</b>	<b>69</b>
<b>3.6 Formal remarks on the equatorial problem</b>	<b>101</b>
<b>3.7 Conclusion and perspectives</b>	<b>105</b>

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## 3.1 Physical modelization

### 3.1.1 Context

When describing geophysical fluids, the classical Navier-Stokes equation

$$\begin{aligned}\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= 0 \\ \nabla \cdot u &= 0\end{aligned}$$

taking into account both advective and diffusive terms for incompressible flow is not sufficient to capture the observed behavior of fluids. In fact, additional physical effects described by supplementary variables, such as temperature or salinity, come into play. Besides these thermodynamical effects, one major specificity of geophysical fluids flows is the occurrence of additional gravitational and inertial effects: the stratification and the Coriolis force arising from the rotation of Earth.

Among these various effects, the Coriolis force alone is sufficient to explain some crucial observations such as the intensification of currents near the west coasts of oceans (for example the Gulf Stream or the Kuroshio), the angle between the direction of the wind and the effective direction of surface currents (the Ekman transport) or some behaviors of dynamo effects associated with the magnetic field of Earth and other celestial bodies.

In order to assess the importance of rotation, we can set up the ordering of the various time scales associated with Earth's dynamic as in Table 3.1. This show that, contrary to everyday experiences with fluids, the effect Earth rotation cannot be neglected in a geophysical setting.

Physical effect	Earth rotation	Oceanic circulation
Typical time scale	1 day	months

Table 3.1 – The time scales of geophysical effects.

In order to have a more quantitative approach, we can write down the usual Navier Stokes equation in a rotating frame.

The derivation is standard, with the sum of three additional forces: Euler, centrifugal and Coriolis.

$$\underbrace{\frac{d\vec{\omega}}{dt} \times x}_{\text{Euler force}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times x)}_{\text{centrifugal force}} + \underbrace{2\vec{\omega} \times u}_{\text{Coriolis force}}$$

As we will be dealing with a stationary rotation, we will use a cartesian basis  $(e_1, e_2, e_3)$  such that  $\vec{\omega} = \frac{f}{2}e_3$ , so the Euler force is zero, and as all potential terms, the centrifugal force  $\vec{\omega} \times (\vec{\omega} \times x) = -\frac{1}{2}\nabla |\vec{\omega} \times x|^2$  can be absorbed into the pressure gradient term.

Thus, finally, within this framework only the Coriolis force matters and the resulting equation, called *Navier-Stokes-Coriolis*, is

$$\underbrace{\partial_t u + (u \cdot \nabla)u}_{\text{inertia}} - \underbrace{\nu \Delta u}_{\text{diffusion}} + \underbrace{f e_3 \times u}_{\text{Coriolis}} + \nabla p = 0 \quad (3.1)$$

$$\nabla \cdot u = 0$$

in a domain  $\Omega$  and completed with boundary conditions, for example no slip boundary conditions

$$u|_{\partial\Omega} = 0. \quad (3.2)$$

As in the remaining part of the chapter, we will treat the pressure force  $\nabla p$  as a Lagrange multiplier of the constraint  $\nabla \cdot u = 0$ . If needed this pressure force can be recovered, for smooth decaying solutions, through

$$-\Delta p = \nabla \cdot ((u \cdot \nabla)u + \omega e_3 \times u).$$

### 3.1.2 Non-dimensional parameters and orders of magnitudes

As usual in Physics the relative strength of each component of equation (3.1) is displayed through the identification of various non-dimensional parameters, Reynolds, Rossby, Ekman, etc.

Starting from now we will denote by  $\tilde{\cdot}$  the dimensional quantities and the non dimensional will be tilde free.

The dimensional equation (3.1) is on  $\tilde{\Omega}$

$$\begin{aligned} \partial_{\tilde{t}} \tilde{u} + (\tilde{u} \cdot \tilde{\nabla}) \tilde{u} - \tilde{\nu} \tilde{\Delta} \tilde{u} + \tilde{\omega} e_3 \times \tilde{u} + \tilde{\nabla} \tilde{p} &= 0 \\ \tilde{\nabla} \cdot \tilde{u} &= 0 \\ \tilde{u}|_{\partial \tilde{\Omega}} &= 0 \end{aligned}$$

with  $[\tilde{t}] = s^{-1}$ ,  $[\tilde{u}] = m s^{-1}$ ,  $[\tilde{\nabla}] = [\tilde{\nabla} \cdot] = m^{-1}$ ,  $[\tilde{\nu}] = m^2 s^{-1} [\tilde{\omega}] = s^{-1}$ ,  $[\tilde{p}] = m^2 s^{-2}$  and

the dimension of  $\tilde{\Omega}$  in m. So, given the fact that the fluid is incompressible this is a kinematic problem with only two independent physical dimensions at work.

The usual choice is to introduce  $L$ , the characteristic length of the problem (in m), and  $U$  the typical velocity of the flow (in  $\text{m s}^{-1}$ ). Thus, we define

- $\tilde{u} = Uu$
- $\tilde{x} = Lx$
- $\tilde{t} = \frac{L}{U}t$

and obtain

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u - \frac{1}{Re} \Delta u + \frac{1}{Ro} e \times u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{aligned} \quad (3.3)$$

where

- $Re = \frac{UL}{\nu}$  is called the Reynolds number
- $Ro = \frac{U}{Lf}$  is called the Rossby number.

Equation (3.3) is completed with non-dimensional boundary conditions on  $\partial\Omega$ , for example  $u|_{\partial\Omega} = 0$ . As we will see later, the balance between rotation and diffusion is the one driving the boundary layers. Therefore, we also introduce

$$E = \frac{Ro}{Re} = \frac{\nu}{fL^2}$$

called the Ekman number.

It is to be noted that oceanic circulation displays two length scales (depth and width) and a turbulent behavior, which can be modeled through anisotropic viscosity as written in the book [11] by Chemin, Desjardins, Gallagher and Grenier:

To take this effect into account, it is usual in meteorology and oceanography to replace the  $-\nu\Delta$  term by  $-\nu_h\Delta_h - \nu_v\partial_3$ .

Table 3.2 summarizes the definition and the typical values of the previous non dimensional parameters in various contexts.

	Ordinary fluid	Ocean	Geodynamo <sup>†</sup>	Solar dynamo <sup>†</sup>
$L$	1m	$\sim 10^3 - 10^5\text{m}$	$10^6\text{m}$	$10^9\text{m}$
$U$	$1\text{ms}^{-1}$	$\sim 10^{-3}\text{ms}^{-1}$	$10^{-4}\text{ms}^{-1}$	$10^1\text{ms}^{-1}$
$\nu$	$10^{-6}\text{m}^2\text{s}^{-1}$	$\sim 10^{-6}\text{m}^2\text{s}^{-1}$	$10^{-6}\text{m}^2\text{s}^{-1}$	$10^{-4}\text{m}^2\text{s}^{-1}$
$f$	$1.5 \times 10^{-4}\text{rads}^{-1}$	$1.5 \times 10^{-4}\text{rads}^{-1}$	$1.5 \times 10^{-4}\text{rads}^{-1}$	$5 \times 10^{-6}\text{rads}^{-1}$

Number	Expression	Balance	Ordinary fluid	Ocean	Geodynamo	Solar dynamo
Reynolds $Re$	$\frac{UL}{\nu}$	Inertia/Diffusion	$10^6$	$10^6$	$10^9$	$10^{13}$
Rossby $Ro$	$\frac{U}{Lf}$	Inertia/Rotation	$10^{15}$	$10^{-2}$	$10^{-6}$	$10^{-2}$
Ekman $E$	$\frac{\nu}{fL^2}$	Rotation/Diffusion	$10^7$	$10^{-8}$	$10^{-15}$	$10^{-15}$

Table 3.2 – The non-dimensional parameters for the Navier-Stokes-Coriolis system.

<sup>†</sup>The geodynamo is the movement of the inner liquid core of the earth, and the solar dynamo take place inside the sun. See for example the book of Dormy and Soward [67] for a physical description and analysis of all of these phenomenon.

### 3.1.3 Taylor-Proudman theorem and Ekman layer

These values clearly indicate the usual ordering  $Ro \ll 1 \ll Re$  for geophysical fluids, so one can expect that at lowest order (i.e  $Ro = 0$  and  $Re = +\infty$ ) the equation (3.3) reduces formally to

$$\begin{aligned} e_3 \times u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned} \quad (3.4)$$

with boundary conditions

$$(u \cdot n)_{\partial\Omega} = 0$$

i.e

$$\begin{aligned} \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_1 p \\ \partial_2 p \\ \partial_3 p \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 &= 0. \end{aligned}$$

This leads to  $\partial_3 u_1 = \partial_3 u_2 = 0$ . Moreover,  $\partial_1 u_1 + \partial_2 u_2 = 0$  and using the divergence free condition we obtain  $\partial_3 u_3 = 0$ .

As the remaining boundary condition is  $u \cdot n|_{\partial\Omega} = 0$ , in the case of a flat boundary the formal solution of (3.4) with  $Ro = 0$  is

$$u = \underbrace{(u_1, u_2, 0)}_{u_h} \quad \text{with } \partial_3 u_h = 0 \quad (3.5)$$

meaning that the flow is two-dimensional and independent of the height. This is the *Taylor-Proudman theorem* and is one of the key features of rotating fluids.

Another key feature comes from the fact that the Taylor-Proudman theorem is not compatible with most boundary conditions if  $\Omega$  is bounded in  $x_3$ . For example if the boundary conditions are  $u_h(x_3 = 0) = u_0, u_h(x_3 = 1) = u_1$  and if  $u_0 \neq u_1$  then  $\partial_3 u_h = 0$  cannot be fulfilled.

In fact near the horizontal boundaries the hypothesis  $Ro \ll 1$  i.e  $L \gg \frac{U}{\omega}$  is no longer valid as a new characteristic length is to be considered: the distance to the boundary. This problem can be resolved by introducing boundary layers lifting these trace incompatibilities. The size of these layers, called *Ekman layers*, scales typically as  $\sqrt{E}$ . The Ekman layers are responsible for both the *Ekman transport* (the angle between the wind forcing direction and the surface current) and a phenomenon called the *Ekman pumping*, a dampening effect as these layers “pump” energy from the main flow.

If Ekman pumping is not directly visible without writing down some equations it can be illustrated by a simple experience (proposed in [20]). When stirring a cup of coffee/tea, the beverage rapidly goes to rest. If the diffusion term  $-\frac{1}{Re} \Delta u$  is the major source of dissipation then the characteristic slowing down time is

$$\tau_{\text{diffusion}} = \frac{L^2}{\nu} \simeq \frac{(5\text{cm})^2}{10^{-6}\text{m}^2\text{s}^{-1}} \simeq 25\text{min}$$

this result clearly contradicts everyday experience. One can show that as the Ekman pumping is caused by the normal velocity of order  $\sqrt{E}$ , the associated characteristic time is

$$\tau_{\text{Ekman pumping}} = \left( \frac{\sqrt{E} R \omega}{\omega} \right)^{-1} = \frac{1}{\sqrt{E} \omega} = \frac{R}{\sqrt{\nu \omega}} \simeq \frac{5\text{cm}}{\sqrt{10^{-6}\text{m}^2\text{s}^{-1} \times 1\text{s}^{-1}}} \simeq 1\text{min}$$

a value easily observed experimentally.

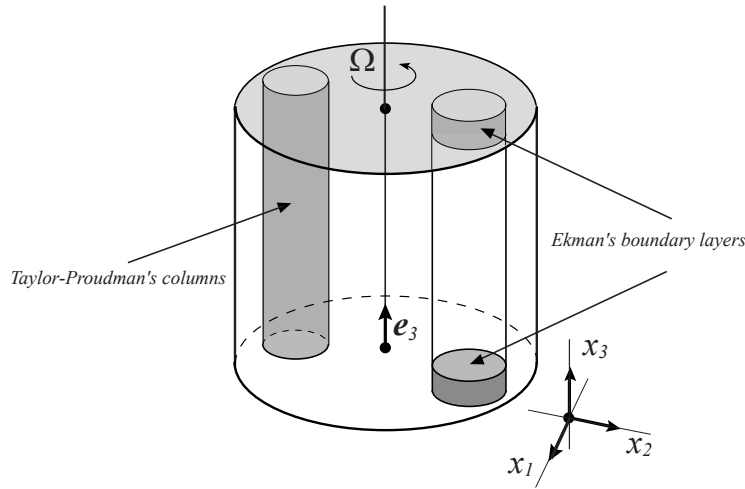


Figure 3.1 – Taylor-Proudman columns and Ekman layers.

### 3.1.4 Proudman problem

In [60] Proudman proposed a simplified model in order to understand the effect of the spherical geometry on rotating fluids. The idea is to consider a viscous fluid between two rotating spherical shells and to use the following set of hypotheses:

- We assume that non-linear terms can be neglected in front of the other terms as the driving mechanism is usually the linear balance between diffusion and rotation. The idea is to verify *a posteriori* that these non-linearities indeed do not modify the general behavior of the flow dynamics.
- Stationarity is also assumed system i.e  $\partial_t = 0$ . Note that this assumption changes the nature of the equation and disregards the possibility of waves created by the balance between inertia and Coriolis forces, the so-called *Rossby waves* (we refer to [11] for more details about these waves).
- Finally, we assume that the flows are axisymmetric. Indeed, if the problem is invariant through rotation around the axe  $e = e_3$ , so must be the function  $u$ .

Under these assumptions the problem writes

$$\begin{aligned} e_3 \times u - E\Delta u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned} \tag{3.6}$$

in the domain  $\Omega$ , the boundary conditions on  $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$  (where  $\Gamma_i, i \in \{0, 1\}$  are the two concentric spherical shell, see figure 3.2) are

$$\begin{aligned} (u \cdot n)|_{\Gamma} &= 0 \text{ (non penetration condition)} \\ (u_t)|_{\Gamma_i} &= v_i \text{ (no slip condition)} \end{aligned}$$

$v_i$  being the driving velocity on the boundary of the spheres. This problem, called the *Proudman problem* gives birth to various nested boundary layers of different scaling that we will discuss later.



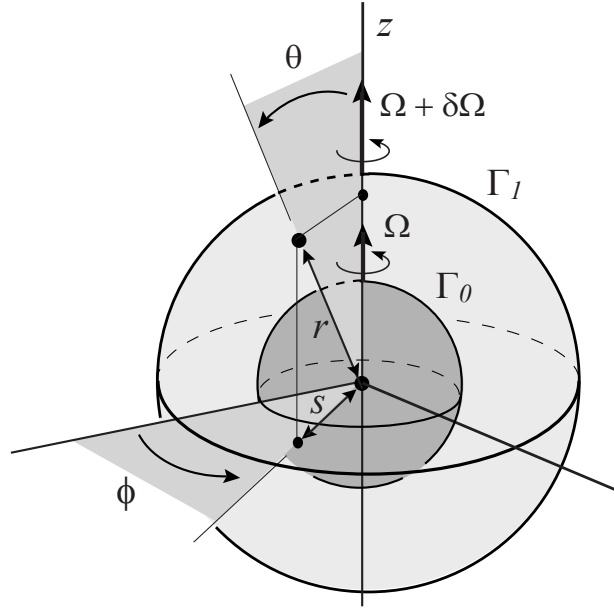


Figure 3.2 – The Proudman problem.

*Remark 3.1.1.* Note that these hypotheses are justified from mathematical point of view far from the equator, since in the fully non-linear and non-stationary problem the boundary layer are still constructed from this linear part, see for example [11].

These hypotheses and the resulting equation can be extended to other configuration where the flow is bounded by two rotating surfaces (for example two rotating wavy disks as in Section 3.4 or rotating shape looking like  $x^\alpha$  as in Section 3.5,  $\alpha = \frac{1}{2}$  being the case of the sphere).

Even in this simplified configuration, no explicit (or almost explicit) solution exists and, as the physical value of  $E$  is of order  $10^{-8}$ , direct numerical simulation is out of reach (presently such DNS can only consider cases up to  $E = 10^{-6} - 10^{-7}$ , see the discussion in the Phd thesis of Marcotte [53]).

A precise understanding and a careful analysis of boundary layers effects are clearly needed before any explicit or numerical resolution, because these layers can affect the main flow as the Ekman pumping does.

Such an analysis, coupling a detailed study of the scaling and behavior of boundary layers and a well-chosen numerical scheme for the most degenerate boundary layer was done recently by Marcotte, Dormy and Soward in [54]. The main idea is to identify the different boundary layers scales and equations and their coupling both between them and the main flow. The goal of this chapter and of the following is to provide a mathematical background to such analysis.

### 3.1.5 Reformulation of the Proudman problem

Let us first reformulate the Proudman problem in the variables used in [54]. In the cylindrical coordinates  $(s, \phi, z)$  of Figure 3.2, the axisymmetric hypothesis assumption leads to a reformulation of the problem. Since  $\partial_\phi u = 0$  the constraint  $\nabla \cdot u = 0$  reads in cylindrical coordinates

$$\frac{1}{s} \partial_s (s u_s) + \partial_z u_z = 0.$$

Thus there exist a scalar function  $\psi$ , the stream function, such that

$$u_s = -\partial_z \psi, u_z = \frac{1}{s} \partial_s (s\psi).$$

Denoting  $v$  the azimuthal velocity we then obtain on the cylindrical basis

$$u = \begin{pmatrix} -\partial_z \psi \\ v \\ \frac{1}{s} \partial_s (s\psi) \end{pmatrix}. \quad (3.7)$$

The Stokes-Coriolis equation (3.6) in the  $\phi$  component is

$$\partial_z \psi = -EDv$$

and taking the curl we obtain

$$\partial_z v = ED^2 \psi.$$

where we have denoted

$$D = \partial_s \left( \frac{1}{s} \partial_s (s \cdot) \right) + \partial_z^2.$$

An important point is the boundary conditions in these new variables. The no-slip boundary condition leads to  $v = 0, \partial_n \psi = 0$  on the boundary, when the non-penetration condition leads to  $\psi = 0$ .

So the problem is reduced to a 2-dimensional elliptic system in the two coordinates  $(s, z)$

$$\begin{aligned} \partial_z v - ED^2 \psi &= 0 \\ \partial_z \psi + EDv &= 0. \end{aligned} \quad (3.8)$$

The effect of the curvature, namely the  $\frac{1}{s} \partial_s - \frac{1}{s^2}$ , does not change the qualitative behavior near the equator  $s = r_0 > 0$ , and does not change the boundary layer as it is of lower order. Thus, we will focus, in this chapter, on the following equation

$$\begin{aligned} \partial_z v - E\Delta_{x,z}^2 \psi &= 0 \\ \partial_z \psi + E\Delta_{x,z} v &= 0 \end{aligned} \quad (3.9)$$

with boundary conditions

$$\begin{aligned} v|_{\Gamma_i} &= v_i, \quad \partial_n \psi|_{\Gamma_i} = 0 \\ \psi|_{\Gamma_i} &= 0. \end{aligned} \quad (3.10)$$

### 3.1.6 Boundary layers

With these variables, when  $E \ll 1$ , the Taylor Proudman theorem reads

$$\partial_z v = 0, \quad \psi = 0$$

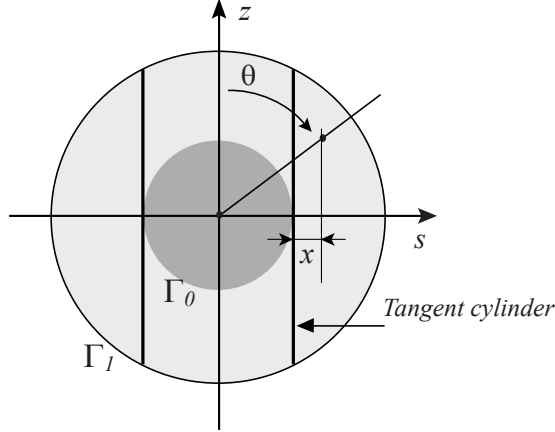


Figure 3.3 – The two-dimensional Proudman problem for spheres.

and, as previously, we recover the incompatibility between the fact that  $v$  is independent of  $z$  and the boundary conditions. Far from the equator this incompatibility is resolved by classical Ekman layers. But as a straightforward calculation (detailed in Section 3.4) shows that the size  $\delta$  of the boundary layer depends on the latitude  $\theta$ , more precisely

$$\delta = \sqrt{\frac{E}{\cos(\theta)}}.$$

Thus two consequences arise

- When approaching the equator, as  $\theta \rightarrow \pi/2$ , this size is no longer small with respect to 1, thus the core hypothesis of *boundary layer ansatz* is no longer valid. This is the so called *degeneracy* of the Ekman boundary layer near the equator. This ultimately leads to the equatorial Ekman boundary layer of size  $E^{\frac{1}{5}} \times E^{\frac{2}{5}}$  as we will see.
- Moreover, the strength of the Ekman pumping is in fact proportional to the size of the Ekman layer. Therefore, the equation on the interior displays a discontinuity (in fact a soft singularity) at the tangent cylinder  $s = R_0$  (see figure 3.3). This leads to additional nested boundary layers of sizes  $E^{\frac{2}{7}}, E^{\frac{1}{4}}$  and  $E^{\frac{1}{3}}$  named *Stewartson layers*. Note that the  $E^{\frac{1}{3}}$  layer is identical to the one near a vertical wall when the two other ones are shear layers driven by the Ekman pumping.

A summary of these geometrical configuration of these boundary layers is depicted in Figure 3.4.

## 3.2 Preliminary results on the Proudman problem

### 3.2.1 Mathematical setting

In the following,  $\Omega$  is a smooth bounded set of  $\mathbb{R}^2$  with  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_i$ . For the spherical problem we will have  $\Omega = \{(x, z) \in \mathbb{R}^2 \text{ s.t. } r_0^2 < x^2 + z^2 < r^1\}$ ,  $\Gamma_i$  being the circle  $x^2 + z^2 = r_i$ . In fact, if our focus is the various boundary layers, we can, using symmetries reduce the problem to  $\Omega = \{(x, z) \in \mathbb{T} \times \mathbb{R} \text{ s.t. } \gamma_0(x) < z < \gamma_1(x)\}$ , see for example Figure 3.6.

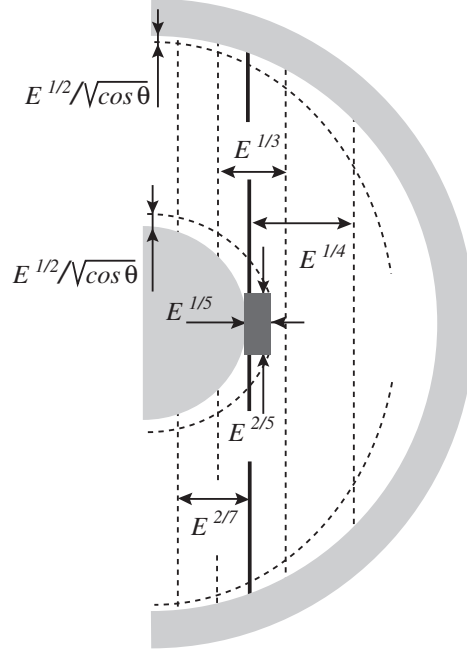


Figure 3.4 – Summary of the different boundary layers in the Proudman problem.

The mathematical setting is the following: for  $E > 0$  let us consider  $\mathbf{u}^E = (v, \psi)$  solution of (3.9):

$$\begin{aligned}\partial_z v - E \Delta_{x,z}^2 \psi &= 0 \\ \partial_z \psi + E \Delta_{x,z} v &= 0.\end{aligned}$$

We also recall the boundary conditions (3.10)

$$\begin{aligned}v|_{\Gamma_0} &= v_0 & v|_{\Gamma_1} &= v_1 \\ \partial_n \psi|_{\Gamma_0} &= 0 & \partial_n \psi|_{\Gamma_1} &= 0 \\ \psi|_{\Gamma_0} &= 0 & \psi|_{\Gamma_1} &= 0.\end{aligned}$$

We are interested in the behavior of  $\mathbf{u}^E$  when  $E \rightarrow 0$ , i.e we look for a convergence result or even an asymptotic expansion of  $\mathbf{u}^E$  in norms relevant to the physical problem.

Let us briefly go through standard estimates and methods.

### 3.2.2 Existence, energy estimate and weak convergence

The very first step is to notice that for all  $E > 0$  equation (3.9) is an elliptic equation, as a consequence

**Proposition 3.2.1.** *Let  $E > 0$ . There exists a unique solution  $\mathbf{u}^E = (v^E, \psi^E) \in H^1(\Omega) \times H^2(\Omega)$  of (3.9) with boundary conditions (3.10) fulfilling the energy inequality*

$$E \int_{\Omega} |\nabla v^E|^2 + E \int_{\Omega} |\nabla^2 \psi^E|^2 \leq C \left( \|v_0\|_{H^{\frac{1}{2}}(\Gamma_0)}^2 + \|v_1\|_{H^{\frac{1}{2}}(\Gamma_1)}^2 \right). \quad (3.11)$$

In fact, if  $v^i \in H^s(\Gamma_i)$  with  $s$  sufficiently large,  $\mathbf{u}^E$  satisfies the more precise estimate

$$\int_{\Omega} |v^E|^2 + \int_{\Omega} |\partial_z \psi^E|^2 + E \int_{\Omega} |\nabla v^E|^2 + E \int_{\Omega} |\nabla^2 \psi^E|^2 \leq C \left( \|v_0\|_{H^s(\Gamma_0)}^2 + \|v_1\|_{H^s(\Gamma_1)}^2 \right). \quad (3.12)$$

*Remark 3.2.1.* In terms of the velocity  $u$ , the estimate (3.12) is the analogous to

$$\|u\|_{L^2} + E \|\nabla u\|_{L^2} \leq C \|u|_{\Gamma}\|_{H^{\frac{1}{2}}}.$$

Where  $u$  is solution  $-E\Delta u + \nabla p = 0$  with the previous symmetries and boundary conditions.

Note that we will refer to the norm and the associated space of estimate (3.12) as the *energy norm* and the *energy space*.

*Proof.* The existence and uniqueness follows directly from the a priori estimate (3.11) and Lax-milgram theorem. Let us remark that the estimate (3.11) does not involve the Coriolis force as it is skew-symmetric

$$\int_{\Omega} (\partial_z v^E \cdot \psi^E + \partial_z \psi^E \cdot v^E) = 0$$

as

$$(v^E \psi^E)_{\partial\Omega} = 0.$$

The key point for the more precise estimate (3.12) in this estimate is the  $L^2$  control, and the key idea is to use the  $\partial_z$  terms to recover it.

The goal of this section is to be as simple as possible, but as this inequality is essential to the definition of approximate solution we will give a short sketch of proof, to highlight some general difficulties and ways to circumvent them. The first approach would be to multiply  $\partial_z \psi + E\Delta v = 0$  by  $\partial_z \psi$ . This leads to  $\|\partial_z \psi\|_{L^2} \leq C$ . And formally multiplying  $\partial_z v - E\Delta^2 \psi$  by  $e^{h(z)}v$  we recover  $-\frac{1}{2} \int |v^2| h'(z) e^{h(z)} \leq \int \Delta^2 \psi v e^{h(z)}$ . But due to the fact that  $\psi$  and  $v$  do not belong to the space functional space, not only  $v e^{h(z)}$  is not a valid test-function for this part of the equation, but more importantly there is a problem in boundary terms when performing integrations by parts. Therefore, we must use the structure of the equation more carefully.

Note that, along the way and due to the fact that the equation is homogeneous in  $\psi$ , we recover better powers of  $E$  than announced:

$$\int_{\Omega} |v^E|^2 + E^{-\frac{1}{2}} \int_{\Omega} |\partial_z \psi^E|^2 + E^{\frac{1}{2}} \int_{\Omega} |\nabla v^E|^2 + E^{\frac{1}{2}} \int_{\Omega} |\nabla^2 \psi^E|^2 \leq C \left( \|v_0\|_{H^s(\Gamma_0)}^2 + \|v_1\|_{H^s(\Gamma_1)}^2 \right). \quad (3.13)$$

But this is of no use in the rest of the chapter, since the cause of this improvement is the absence of source terms.

The first step is to multiply the  $\partial_z \psi^E + E\Delta v^E$  by  $\partial_z \psi^E$ . We then obtain, since  $\psi^E|_{\partial\Omega} = 0$ ,  $\partial_n \psi^E|_{\partial\Omega} = 0$

$$\|\partial_z \psi^E\|_{L^2}^2 \leq CE \|v^E\|_{H^1} \|\psi^E\|_{H^2}. \quad (3.14)$$

In particular equation (3.11) leads to  $\|\partial_z \psi^E\|_{L^2} \leq C \|v_{0/1}\|_{H^{\frac{1}{2}}}$ . As the domain is bounded in  $z$ , through Poincaré inequality we recover  $\|\psi^E\|_{L^2}$ .

Moreover, we can notice that the previous estimate can be refined with respect to the powers of  $E$ . We lift the non-homogeneous boundary conditions with a lifting term  $\mathbf{u}^l = (v^l, \psi^l) \in H^1 \times H^2$  (in this case  $\psi^l = 0$ ). The fact that  $\|v^l\|_{H^1} = O(1)$  led to (3.11). With  $\chi$  a cut-off function and  $\tilde{v}^l = v^l \chi \left( \frac{d(x, \partial\Omega)}{\sqrt{E}} \right)$ , then  $\tilde{v}^l$  is also a lifting of the boundary conditions but with  $\|\tilde{v}^l\|_{L^2} = O(E^{\frac{1}{4}})$ ,

$\|\tilde{v}^l\|_{H^1} = O(E^{-\frac{1}{4}})$ . We set  $(v^E, \psi^E) = (v, \psi) + (\tilde{v}^l, 0)$ , with  $(v, \psi)$  solution of the same equation but with homogeneous boundary condition and a non-zero right hand side.

As the exact estimate of (3.11) can be rewritten using (3.14)

$$\begin{aligned} E \int |\nabla v|^2 + E \int |\Delta \psi|^2 &\leq E \left| \int \nabla v \cdot \nabla \tilde{v}^l \right| + \left| \int \tilde{v}^l \partial_z \psi \right| \\ &\leq \left( \frac{1}{3} E \|v\|_{H^1}^2 + C E \|\tilde{v}^l\|_{H^1}^2 \right) + \left( \frac{1}{3} E (\|v + \tilde{v}^l\|_{H^1}^2 + \|\psi + 0\|_{H^2}^2) + C \|\tilde{v}^l\|_{L^2}^2 \right) \end{aligned}$$

with this lifting, we recover in fact

$$E^{\frac{1}{4}} \|v\|_{H^1} + E^{\frac{1}{4}} \|\psi\|_{H^2} \leq C \|v_{0/1}\|_{H^{\frac{1}{2}}}.$$

*Remark 3.2.2.* • This scaling  $\sqrt{E}$  is the one of the Ekman layer, but this construction does not comes from a boundary layer. In fact this scaling comes from the balance between  $\tilde{v}^l$  and  $E\Delta\tilde{v}^l$ , which is not the Ekman balance.

- We cannot hope to improve these estimates, in the sense that we cannot have better powers of  $E$  in front of each term of the sum. In fact, for the Ekman layers, all terms of (3.13) are of order  $O(1)$ .

We can then bootstrap higher regularity using elliptic regularity theory, provided that  $v_0$  and  $v_1$  are smooth enough. For example  $\|\Delta v^E\|_{L^2} = E^{-1} \|\partial_z \psi^E\|_{L^2} = O(E^{-\frac{3}{4}})$ , so  $\|v^E\|_{H^2} = O(E^{-\frac{3}{4}})$ , if  $v_i \in H^{\frac{3}{2}}(\Gamma_i)$ . Using this with  $\|\Delta(\Delta\psi^E)\|_{H^1} = E^{-1} \|\partial_z v\|_{H^1} = O(E^{-\frac{7}{4}})$  we obtain  $\|\psi^E\|_{H^5} = O(E^{-\frac{7}{4}})$ .

Using interpolation on  $k$ , we then obtain for all  $1 \leq k \leq 2$

$$\|v\|_{H^k} \leq C_k \|v_{0/1}\|_{H^s} E^{-\frac{2k-1}{4}} \quad \|\psi\|_{H^{k+1}} \leq C_k \|v_{0/1}\|_{H^s} E^{-\frac{2k-1}{4}}.$$

To obtain the inequality on  $\|v\|_{L^2}$ , now that we know that  $v$  has enough regularity, we consider  $\omega$  a weight (typically an exponential), and use  $(v^E - v^l)\omega$  as a test function on the equation  $\partial_z(v^E - v^l) - E\Delta^2\psi = -\partial_z v^l$ .

The only problematic term when integrating by parts  $\int \Delta^2(v^E - v^l)\omega$  is  $\int \nabla \Delta \psi \cdot \nabla(v^E - v^l)\omega$ , as we cannot proceed to further integrations by parts due to boundary conditions. But as  $|E \int \nabla \Delta \psi \cdot \nabla(v^E - v^l)\omega| \leq E \|\psi\|_{H^3} \|(v^E - v^l)\omega\|_{H^1} \leq C E^{1-\frac{3}{4}-\frac{1}{4}} \|v_{0/1}\|_{H^s}^2$  we finally obtain

$$\|v^E\|_{L^2} \leq C \|v_{0/1}\|_{H^s}.$$

□

So  $\mathbf{u}^E$  is bounded in  $L^2 \times H^1$  and there exist  $\mathbf{u} \in L^2 \times H^1$  so that, up to an extraction,  $\mathbf{u}^E \rightharpoonup \bar{\mathbf{u}}$  in this space.

With  $(w, \phi)$  a test function in  $H^2 \times H^3$  compactly supported in  $\Omega$ , integrating by parts and taking the limit  $E \rightarrow 0$  in the weak formulation allows us to obtain, in a weak sense,

$$\begin{aligned} \partial_z \bar{v} &= 0 \\ \bar{\psi} &= 0. \end{aligned}$$

This is indeed the Taylor-Proudman theorem of 3.5, the flow is independent of  $z$ , and the velocity of each column is in the plane orthogonal with the axe of rotation, as  $\psi = 0$ .

In order to obtain strong convergence and additional information on  $\bar{v}$  we must rely on a more elaborate construction.

### 3.2.3 Stability estimate, approximate solutions

The leading principle is to construct an approximate solution  $\mathbf{u}^{app}$  such that

$$\begin{aligned}\partial_z v^{app} - E\Delta^2 \psi^{app} &= r_\psi \\ \partial_z \psi^{app} + E\Delta v^{app} &= r_v\end{aligned}\tag{3.15}$$

satisfying the set of exact boundary conditions

$$\begin{aligned}v_{|\Gamma_0}^{app} &= v_0, v_{|\Gamma_1}^{app} = v_1, \quad \partial_z \psi_{|\Gamma_0}^{app} = \partial_n \psi_{|\Gamma_1}^{app} = 0 \\ \psi_{|\Gamma_0}^{app} &= \psi_{|\Gamma_1}^{app} = 0.\end{aligned}\tag{3.16}$$

As the problem is linear, the energy estimates (3.11),(3.12) lead to a stability result: with  $\mathbf{r} = (r_v, r_\psi) = \mathbf{r}^0 + \mathbf{r}^1$  we have

$$\begin{aligned}E \|v^E - v^{app}\|_{H^1}^2 + E \|\psi^E - \psi^{app}\|_{H^2}^2 &\leq C \left( \|r_v^0\|_{L^2}^2 + \|r_\psi^0\|_{L^2}^2 + E^{-1} \|r_v^1\|_{H^{-1}}^2 + E^{-1} \|r_\psi^1\|_{H^{-2}}^2 \right) \\ \|v^E - v^{app}\|_{L^2}^2 + \|\psi^E - \psi^{app}\|_{L^2}^2 &\leq C \left( \|r_v\|_{L^2}^2 + \|r_\psi\|_{L^2}^2 \right).\end{aligned}\tag{3.17}$$

Thus, following Dalibard and Saint-Raymond [17], we define an *approximate solution* from the energy balance:

**Definition 3.2.2.** *A function  $\mathbf{u}^{app} = (v^{app}, \psi^{app}) \in H^1 \times H^2$  is said to be an approximate solution of (3.9)-(3.10) if it is a solution of (3.15) with the exact boundary conditions (3.16), such that  $\mathbf{r} = (r_v, r_\psi)$  is an acceptable remainder i.e*

$$\|r_v\|_{L^2}^2 + \|r_\psi\|_{L^2}^2 = o \left( \|v^{app}\|_{L^2}^2 + \|\psi^{app}\|_{L^2}^2 + E \left( \|v^{app}\|_{H^1}^2 + \|\psi^{app}\|_{H^2}^2 \right) \right)$$

when  $E$  goes to 0.

Specifically, it is to be noted that if  $\mathbf{u}^{app}$  is an approximate solution and is convergent in  $L^2 \times H^1$  to  $\bar{\mathbf{u}}$  when  $E \rightarrow 0$ , then we also have  $\mathbf{u}^E \rightarrow \bar{\mathbf{u}}$ .

The goal of the following subsections will be to construct such approximate solution in different settings.

### 3.2.4 Some bibliographical remarks

This problem was first proposed by Proudman in [60], the formal analysis of the scales being completed by Stewartson [69, 70]. For both a physical and numerical detailed analysis we already cited the work of Marcotte, Dormy and Soward [54], and for a formal derivation the article of Gérard-Varet [31].

From a mathematical point of view, the Ekman layers (in the complete Navier-Stokes-Coriolis problem) are well understood since Grenier and Masmoudi [40], and the monograph of Chemin, Desjardins, Gallagher and Grenier [11].

Let us stress two main features of the complete Navier-Stokes-Coriolis problem, that does not appear in the simplified setting of the Proudman problem.

- First, contrary to Navier-Stokes-Coriolis, Proudman system is linear, removing the obvious additional difficulty of the non-linearity, and allowing a full use of linear methods. But as explained by Gérard-Varet [31], a quick a posteriori computation show that it does change the nature of the boundary layers:  $(u \cdot \nabla)u = u_h \cdot \nabla u + u_3 \partial_3 u$  so for a flat boundary all these terms are of low order, since  $u_3 \simeq \psi = O(\sqrt{E})$  compensate the  $\partial_3 = O(\frac{1}{\sqrt{E}})$ .
- Moreover, the removal of the time dependency has two effects. On the one hand, we lose an easy control of the  $L^2$  norm in the energy estimate at all scales. But on the other hand it avoids the difficulties of the dispersive nature of the limiting equation when time is present. These additional difficulties require attention to the domain (whole space, torus with or without resonance) and to the initial condition, distinguishing the part contained in the kernel of the singular operator (well-prepared case) from the part that is not (ill-prepared case). If we were to add  $\partial_t u$  to the equation, the time dependency would create waves. Indeed, as the Coriolis operator  $L : u \mapsto e \times u + \nabla p$  is skew-symmetric, the interaction between the two terms

$$\partial_t u + \frac{1}{Ro} Lu$$

create waves of frequency in  $\frac{1}{Ro}$ . Thus, all convergence results require the application of a filtering operator,  $e^{-\frac{t}{Ro}}$ .

Several refinements of the Navier-Stokes-Coriolis have also been studied from a mathematical point of view. Some of them, modifying and refining the boundary layer analysis, are the following:

- The  $\beta$ -plane model, taking into account the variation of angle between the axe of rotation and the normal of the surface, for example by Gallagher and Saint-Raymond [29], and by Dutrifoy, Schochet and Majda [23].
- The model of a rough bottom, taking into account the rapid variations on the ocean floor, see for example Gérard-Varet in [32].
- And the model taking into account the forcing of the wind at the top of the domain was studied by Masmoudi [56], possible resonance being treated by Dalibard and Saint-Raymond in [18].

### 3.3 Overall view of the construction

We first start with an analysis of the Ekman boundary layer for a non-flat, but non degenerate, boundary. The key point is that the analysis of the Ekman boundary layer equation

$$\begin{pmatrix} \partial_\zeta^2 & \partial_\zeta \\ \partial_\zeta & -\partial_\zeta^4 \end{pmatrix} \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

show that there is only two degree of freedom (corresponding to the two roots of  $\lambda^6 + \lambda^2$  such that  $\Re(\lambda) < 0$ ) for the solution. As we must lift three boundary conditions,  $v, \partial_n \psi$  and  $\psi$ , we obtain that we can lift such conditions if and only if we have one constraint satisfied, that is

$$\pm \frac{\sqrt{\cos(\theta)}}{\sqrt{E}} \psi + \frac{1}{\sqrt{2}} (v - \partial_n \psi) = 0 \tag{3.18}$$



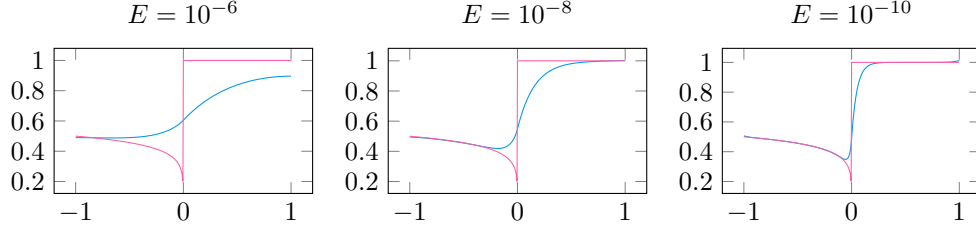


Figure 3.5 – Interior  $v$  for various values of  $E$ . The discontinuous  $\bar{v}$  is in red.

where  $\cos(\theta)$  is the angle between the axis of rotation  $e_z$  and the normal to the boundary.

Following the same type of construction as in Section 2.4, for a fluid between two shells  $\Gamma_0$ ,  $\Gamma_1$  rotating with respective speeds  $v_0$  and  $v_1$ , we obtain a convergence in  $L^2$ , when there is no geometric singularity (i.e  $\cos(\theta) \geq c > 0$ ) toward  $(\bar{v}, 0)$  with

$$\bar{v} = \frac{\frac{v_0}{\sqrt{\cos(\theta_0)}} + \frac{v_1}{\sqrt{\cos(\theta_1)}}}{\frac{1}{\sqrt{\cos(\theta_0)}} + \frac{1}{\sqrt{\cos(\theta_1)}}}. \quad (3.19)$$

We will consider this case in Section 3.4.

For a singular geometry, while the previous computation is formally justified, it cannot be a good candidate for an approximate solution as it is discontinuous.

In fact, using once more (3.18) we obtain an equation of the form

$$\left( \frac{v_0}{\sqrt{\cos(\theta_0)}} + \frac{v_1}{\sqrt{\cos(\theta_1)}} \right) - \left( \frac{1}{\sqrt{\cos(\theta_0)}} + \frac{1}{\sqrt{\cos(\theta_1)}} \right) v + h\sqrt{E}\partial_x^2 v = 0 \quad (3.20)$$

The Figure 3.5 is obtained, for  $\Gamma_0$  and  $\Gamma_1$  two circles, by solving equation (3.20) with finite elements. We can see that two boundary layers appear at the left and the right of  $x = 0$ . Computing the numerical solution for various values of  $E$  leads to the two sizes,  $E^{\frac{2}{7}}$  and  $E^{\frac{1}{4}}$  and a value  $v^E(0) \sim E^{\frac{1}{28}}$ .

In order to lift such discontinuity we need to introduce boundary layers at the right and left of  $x = 0$ , of respective sizes  $E^{\frac{1}{4}}$  and  $E^{\frac{2}{7}}$ .

At this stage, if the  $v$  obtained is a good choice as an approximate solution, the associated  $\psi$  is not. In order to lift it, we need to consider the equation

$$\begin{cases} \partial_z v - E\partial_x^4 \psi = 0 \\ \partial_z \psi + E\partial_x^2 v = 0 \end{cases} \quad (3.21)$$

and to construct another shear layer, of size  $E^{\frac{1}{3}}$ . This last shear layer will be the focus of Subsection 3.5.5

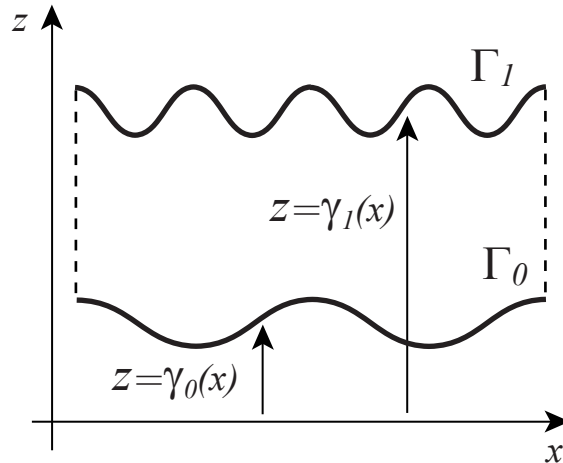


Figure 3.6 – Two non-flat disks.

### 3.4 The Ekman layers for two disks

Let us now consider two non-flat disks i.e a domain defined by

$$\Omega = \{(x, z), x \in \mathbb{T}, \gamma_0(x) < z < \gamma_1(x)\} \quad (3.22)$$

where  $\gamma_0, \gamma_1$  are smooth functions (see Figure 3.6), such that  $\gamma_1 - \gamma_0$  is far for  $0, 0 < C^{-1} \leq \gamma_1 - \gamma_0 \leq C$ .

The goal of this subsection is to show that *the Ekman pumping depends on the size of the Ekman layers*, size that increases drastically with the latitude.

Such results were already obtained by Chemin in a much more complete setting and can be found in lecture notes by Chemin, to be published in *Morningside Lectures in Mathematics* [10]. Thus, we will not provide too many details of the inner computations in order to focus on the expression of  $\bar{v}$ , as this expression will be the main cause of the shear layers.

We will denote by  $\theta_i(x)$  the angle between the normal vector and the vertical axis as showed in Figure 3.7 i.e

$$\cos(\theta_i(x)) = \frac{1}{\sqrt{1 + \gamma_i^2(x)}}.$$

We recall the objective: to construct an approximate solution of (3.15)-(3.16) as

$$\mathbf{u}^{app} = \mathbf{u}(x) + \mathbf{u}^{BL,t} \left( x, \frac{1-z}{\delta_1} \right) + \mathbf{u}^{BL,b} \left( x, \frac{z}{\delta_0} \right) + \dots$$

such that the remainder satisfies the Definition 3.2.2.

The approach is also the same as before:

- Identify the scaling and equation of the boundary layers and construct a boundary layer operator.
- Construct an interior operator.
- Articulate these operators in order to obtain an asymptotic expansion, up to a sufficiently small remainder.

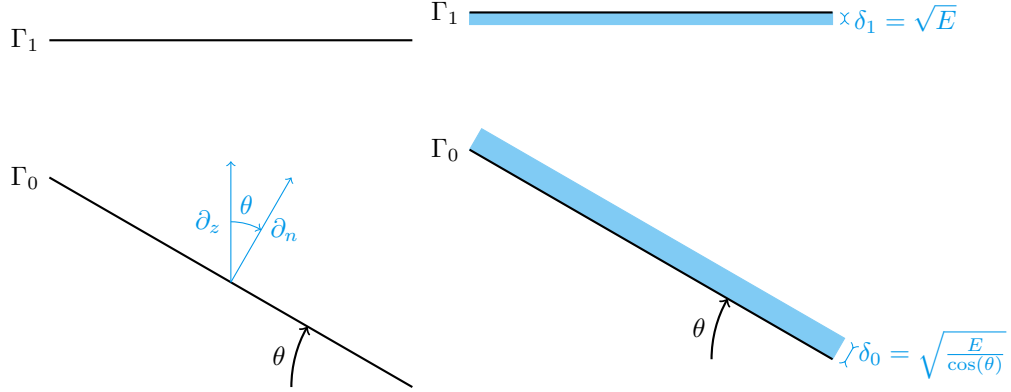


Figure 3.7 – Ekman layers for non-flat disks.

### Boundary layer operator

Because of the topography, the boundary layer variable  $\zeta$ , the rescaled distance to the boundary, is no longer a simple multiple of  $z$ . In fact we have in boundary layer variables (see Figure 3.7 and equation (3.24))

$$\Delta_{x,z} \sim \delta_i^{-2} \partial_\zeta^2 \quad \partial_z \sim \delta_i^{-1} \cos(\theta_i) \partial_\zeta$$

and the corresponding boundary layer symbol is

$$\begin{pmatrix} -E\delta_i^{-4}\xi^4 & \delta_i^{-1} \cos(\theta_i) i\xi \\ \delta_i^{-1} \cos(\theta_i) i\xi & -E\delta_i^{-2}\xi^2 \end{pmatrix}.$$

This leads to a boundary layer size of

$$\delta_i = \sqrt{\frac{E}{\cos(\theta_i)}}.$$

Note that the boundary layer size varies with the tangential coordinate, but since  $\gamma_{0,1}$  are smooth we always get  $\delta_i = O(E^{\frac{1}{2}})$ .

As soon as the scaling is identified we can define precisely the boundary layer operator

**Proposition 3.4.1.** *Let  $s_0 \geq 4$  and  $V, \Upsilon, \Psi \in H^{s_0}(\Gamma_0)$  such that*

$$-\frac{1}{\delta_0(x)} \Psi(x) + \frac{1}{\sqrt{2}} (V(x) - \Upsilon(x)) = 0. \quad (3.23)$$

*There exists  $\mathbf{u}^{bl} = \mathbf{u} \left( x, \frac{d((x,z), \Gamma_0)}{\delta_0} \right) \in H^1(\Omega) \times H^2(\Omega)$  solution of (3.15) satisfying*

$$\begin{aligned} (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{\Gamma_0} &= (V, \Psi, \Upsilon) \\ (v^{bl}, \psi^{bl}, \partial_z \psi^{bl})|_{\Gamma_1} &= 0 \end{aligned}$$

with remainder terms satisfying for all  $s \leq s_0 - 2$

$$\begin{aligned}\|r_v\|_{H^s} &\leq CE^{\frac{1}{4} + \frac{1}{2} - \frac{s}{2}} \\ \|r_\psi\|_{H^{s-2}} &\leq CE^{\frac{3}{4} - \frac{s}{2}}\end{aligned}$$

and the following estimate for all  $s \leq s_0$

$$\|v^{bl}\|_{H^s} \leq CE^{\frac{1}{4} - \frac{s}{2}} \quad \|\psi^{bl}\|_{H^s} \leq CE^{\frac{3}{4} - \frac{s}{2}}.$$

*Remark 3.4.1.* Note that the restriction on the regularity  $s$  for both the solutions and the remainders is only a restriction of regularity with respect to  $x$ , due to the regularity on  $V, \Upsilon, \Psi$ . If these functions are  $C^\infty$  then the same estimates hold for any  $s$ .

*Proof.* For this proof we will drop the index 0, writing for example  $\cos(\theta) = \frac{1}{\sqrt{1+\gamma'^2}}$  instead of  $\cos(\theta_0) = \frac{1}{\sqrt{1+\gamma_0'^2}}$ .

Since the boundary conditions are regular with respect to  $x$  and no degeneracy appears ( $\cos(\theta)$  far from 0) the idea is to construct the boundary layer for each  $x$  as if it were a plane with a constant slope.

More precisely, let  $\zeta$  be the rescaled distance to the boundary,  $\zeta = d((x, z), \Gamma) / \delta$  and  $\sigma$  the tangential coordinate.

By the local inversion theorem there exists  $\rho > 0$  such that, in the tubular neighborhood  $d((x, z), \Gamma) < \rho$ , the change of variables  $(x, z) \rightarrow (\sigma, \zeta)$  is well-defined, and the jacobian is

$$\begin{pmatrix} \frac{\partial x}{\partial \sigma} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial z}{\partial \sigma} & \frac{\partial z}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) + \partial_\sigma \theta \cos(\theta) \delta \zeta + \partial_\sigma \delta \sin(\theta) \zeta & \sin(\theta) \delta \\ -\sin(\theta) - \partial_\sigma \theta \sin(\theta) \delta \zeta + \partial_\sigma \delta \cos(\theta) \delta \zeta & \cos(\theta) \delta \end{pmatrix}. \quad (3.24)$$

Noticing that as  $\partial_\sigma \delta = \frac{1}{2} \partial_\sigma \theta \tan(\theta) \delta$ , as long as  $\theta$  is far from  $\frac{\pi}{2}$  (and  $\theta \in C^\infty$ ), we obtain

$$\begin{pmatrix} \frac{\partial \sigma}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \sigma}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} = \begin{pmatrix} (1 + o_{C^\infty}(1)) \cos(\theta) & (1 + o_{C^\infty}(1)) \delta^{-1} \sin(\theta) \\ -(1 + o_{C^\infty}(1)) \sin(\theta) & (1 + o_{C^\infty}(1)) \delta^{-1} \cos(\theta) \end{pmatrix}.$$

Keeping only the main terms, we get the boundary layer equations in boundary layer variables  $\mathbf{u}^{BL} = (v(\sigma, \zeta), \delta(\sigma)\psi(\sigma, \zeta))$

$$\begin{aligned}\cos(\theta) \delta^{-1} \partial_\zeta v - E \delta^{1-4} \partial_\zeta^4 \psi &= 0 \\ \cos(\theta) \delta^{1-1} \partial_\zeta \psi + E \delta^{-2} \partial_\zeta^2 v &= 0\end{aligned}$$

i.e considering the choice of  $\delta$ , equation (2.27):

$$\begin{pmatrix} -\partial_\zeta^4 & \partial_\zeta \\ \partial_\zeta & \partial_\zeta^2 \end{pmatrix} \begin{pmatrix} \psi \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The boundary conditions, in term of boundary layers variables, are

$$v|_{\zeta=0} = V, \quad \frac{1}{\delta} \psi|_{\zeta=0} = \Psi, \quad \partial_\zeta \psi|_{\zeta=0} = \Upsilon.$$

Hence, the analysis is the same as in the 1D Ekman boundary layer of the preceding subsection, since we consider that  $\sigma$  does not strongly vary. Thus, we have two modes with negative real part and, for each of them, the corresponding kernel is of dimension 1. Thus, we can lift up to 2

boundary conditions and, more precisely, any boundary condition in the hyperplane defined by equation (2.28), which is exactly the relation (3.23) up to the boundary layer scaling.

Once such construction is done, we need to evaluate the remainders. For example, for  $\mathbf{u} = (v(\sigma, \zeta), \delta(\sigma)\psi(\sigma, \zeta)) \in C^4$ , the explicit computation leads to

$$\begin{aligned} |r_v| &= |\partial_z(\delta\psi) + E\Delta_{x,z}v - \cos(\theta)(\partial_\zeta\psi + \partial_\zeta^2v)| \\ &\leq C_\theta \left( \sum_{k=0}^1 \sum_{j=0}^{2-k} E^{\frac{2-k}{2}} |\partial_\sigma^j \partial_\zeta^k v| + E^{\frac{1}{2}} |\partial_\zeta^2 v| + E^{\frac{1}{2}} \sum_{j=0}^1 |\partial_\sigma^j \psi| + E^{\frac{1}{2}} |\partial_\zeta \psi| \right) \end{aligned} \quad (3.25)$$

where  $C_\theta$  depends on the function  $\theta$  and is finite as long as  $\frac{1}{\cos(\theta)}$  and  $\partial_\sigma^j \theta, 1 \leq j \leq 4$  are finite.

*Remark 3.4.2.* • There are two main types of errors when approaching the whole equation by the boundary layer equation: the variations with respect to the tangential variable  $\sigma$  of the profile, i.e.  $\partial_\sigma v, \partial_\sigma \psi = O(1)$ , and the variation with respect to  $\sigma$  of the angle and boundary layer size i.e.  $\partial_\sigma \theta = O(1), \partial_\sigma \delta = O(\delta)$  (and all the lower order terms arising from the cross derivatives).

- Thus, all these estimates are no longer valid when  $\theta$  is too close to  $\frac{\pi}{2}$ , or if variations with respect to  $\sigma$  are no longer of order 1, i.e. if the boundary layer is degenerated.

There is a last technical step: we need to multiply the obtained function by a cut-off function  $\chi(d((x, z), \Gamma_0))$  where  $\chi(Y) = 0$  for  $Y > \rho$  and  $\chi(Y) = 1$  for  $Y < \rho/2$ . This ensures both the validity of the change of variables and the null boundary condition on  $\Gamma_1$ . As the functions display explicit exponential decay in  $\zeta$ , this only introduces  $o(E^\infty)$  error terms, and we obtain the claimed  $H^s$  bound from the scaling, the size of the remainder follows from (3.25) and a density argument. To see a more detailed justification of such scalings, we refer to the Appendix (A.5), keeping mind that the constructed profile has exponential decay.  $\square$

Note that if we want to pursue the asymptotic expansion, we need to consider the remainder terms as a source term for the next boundary layer corrector. The complete boundary layer operator is given in the following Proposition (written in boundary layer variables)

**Proposition 3.4.2.** *Let  $s_0 \geq 4$  and  $\mathbf{f} = (f_v, f_\psi)(\sigma, \zeta) \in H^{s_0}(\mathbb{T} \times \mathbb{R}^+)$  exponentially decreasing, i.e. such that*

$$\|\mathbf{f}e^{\eta\zeta}\|_{H^{s_0-4}(\mathbb{T} \times \mathbb{R}^+)} \leq C, \quad \forall \eta < \frac{1}{\sqrt{2}}. \quad (3.26)$$

*There exists a constant  $H_{\mathbf{f}}$ , depending linearly on  $\mathbf{f}$ , such that for any  $(V(x), \Psi(x), \Upsilon(x))$  in the affine hyperplane*

$$-\frac{1}{\delta(x)}\Psi(x) + \frac{1}{\sqrt{2}}(V(x) - \Upsilon(x)) = H_{\mathbf{f}}. \quad (3.27)$$

*there exists a solution  $\mathbf{u}^{bl} = (v(\sigma, \zeta), \delta(\sigma)\psi(\sigma, \zeta))$  of*

$$\begin{aligned} \partial_z v^{bl} - E\Delta_{x,z}^2 \psi^{bl} &= E^{-\frac{1}{2}} \cos(\theta)^{\frac{3}{2}} f_\psi(\sigma, \zeta) + r_\psi(\sigma, \zeta) \\ \partial_z \psi^{bl} + E\Delta_{x,z} v^{bl} &= \cos(\theta) f_v(\sigma, \zeta) + r_v(\sigma, \zeta) \end{aligned}$$

*with the same estimates as in Proposition 3.4.1. Moreover  $\mathbf{r}$  is also exponentially decreasing in the sense of (3.26).*

The proof is the same as in the previous proposition with  $f = 0$ , the only difference being

the source term in the equation solved by  $\mathbf{u}$ , namely

$$\begin{pmatrix} -\partial_\zeta^4 & \partial_\zeta \\ \partial_\zeta & \partial_\zeta^2 \end{pmatrix} \begin{pmatrix} \psi \\ v \end{pmatrix} = \begin{pmatrix} f_\psi(\sigma, \zeta) \\ f_v(\psi, \zeta) \end{pmatrix}$$

treated by the use of the explicit Duhamel formula or an explicit Laplace transform, both approaches leading to the condition (3.27).

For an example of explicit computation, see the example in Annex A or the construction of the  $E^{\frac{1}{4}}$  boundary layer operator in Proposition 3.5.3. Note that the condition to have exponential decay, present both condition (3.26) and Lemma A.5.1 is not too restrictive. In fact, this is the natural condition coming from the variational formulation of such ODE, see for example the use of adapted weighted Sobolev spaces in Subsection 3.5.3.

### Interior flow

Having obtained such conditions, we can consider the interior flow. The interior operator is still the same as the one in one dimension (i.e an integration with respect to  $z$ ) the conditions (2.33) being slightly modified

**Proposition 3.4.3** (Interior operator). *Let  $s_0 \geq 4$ ,  $\mathbf{f} = (f_v, f_\psi) \in H^{s_0}(\Omega) \times H^{s_0}(\Omega)$  and  $V_0, \Psi_0, \Upsilon_0 \in H^{s_0+4}(\Gamma_0), V_1, \Psi_1, \Upsilon_1 \in H^{s_0+4}(\Gamma_1)$ . Let us suppose that the following compatibility conditions are satisfied*

$$\begin{aligned} V_1(x) - V_0(x) &= \int_{\gamma_0(x)}^{\gamma_1(x)} f_\psi(x, z) dz \\ \Psi_1(x) - \Psi_0(x) &= \int_{\gamma_0(x)}^{\gamma_1(x)} f_v(x, z) dz \\ (-1)^i \Upsilon_i(x) &= \frac{1 - \gamma'_i(x)^2}{\sqrt{1 + \gamma'_i(x)^2}} f_v(\gamma_i) \end{aligned} \quad (3.28)$$

. Then there exist  $\mathbf{u}^{int} = (v^{int}, \psi^{int})$  solution of

$$\begin{aligned} \partial_z v^{int} - E\Delta^2 \psi^{int} &= f_\psi + Er_\psi^{int} \\ \partial_z \psi^{int} + E\Delta v^{int} &= f_v + Er_v^{int} \\ (v^{int}, \psi^{int}, \partial_n \psi^{int})|_{z=i} &= (V_i, \Psi_i, \Upsilon_i) \end{aligned}$$

and for  $s \leq s_0 - 4$

$$\begin{aligned} \|r_v^{int}\|_{H^{s+2}} &\leq CE \\ \|r_\psi^{int}\|_{H^s} &\leq CE. \end{aligned}$$

*Remark 3.4.3.* It is important to note that error terms are also coming from the fact that we consider  $\partial_x = O(1)$ . For example,  $\psi^{int}(x, z) = \Psi_0(x) + \int_{\gamma_0(x)}^z f_v^{int}(x, \tau) d\tau$ , so

$$r_\psi^{int} = \Delta_{x,z}^2 \left( \Psi_0(x) + \int_{\gamma_0(x)}^z f_v^{int}(x, \tau) d\tau \right).$$

So not only at each step do we have a loss in regularity with respect to  $z$  but also with respect to  $x$ .

More precisely we point out that the remainder  $r_v$  is equal to

$$r_v^{int} = -\partial_z^2 v^{int} - \partial_x^2 v^{int}$$

the blue term will be the source of the shear layers in the next subsection. Indeed, it will be no longer negligible when strong variations with respect to  $x$  appears, causing the various boundary layers of Section 3.5.

Nevertheless, we finally state the following result

**Theorem 3.4.4.** *Let  $\mathbf{u}^E$  be a solution of (3.9) in the domain  $\Omega$  defined by (3.22), where  $\gamma_0, \gamma_1 \in C^\infty(\mathbb{T})$ ,  $\gamma_1 - \gamma_0 \geq c \geq 0$ , with boundary conditions  $v_0, v_1 \in H^2(\mathbb{T})$ .*

*Then when  $E \rightarrow 0$  we have  $\mathbf{u}^E \rightarrow (v^{int,0}, 0)$  in the energy space (3.12) where*

$$v^{int,0}(x) = \frac{\delta_0(x)v_0(x) + \delta_1(x)v_1(x)}{\delta_0(x) + \delta_1(x)} = \frac{(1 + \gamma_0'(x)^2)^{-\frac{1}{4}}v_0(x) + (1 + \gamma_1'(x)^2)^{-\frac{1}{4}}v_1(x)}{(1 + \gamma_0'(x)^2)^{-\frac{1}{4}} + (1 + \gamma_1'(x)^2)^{-\frac{1}{4}}}. \quad (3.29)$$

*Proof.* The construction is exactly the same as the one in the 1D case, as we have sufficient regularity with respect to  $x$  for the first iteration. Indeed, it is sufficient to construct  $v^{int,0}, \psi^{int,0}, \psi^{int,1}, \mathbf{u}^{bl,b/t,0}$  in order to have the precision required to conclude. The influence is seen in the splitting of the boundary conditions. Here at first order the system is

$$\begin{aligned} \partial_n \psi|_{\Gamma_1}^{int,0} &= \partial_n \psi|_{\Gamma_0}^{int,0} = 0 \\ v|_{\Gamma_1}^{int,0} - v|_{\Gamma_0}^{int,0} &= 0 \\ \psi|_{\Gamma_1}^{int,0} - \psi|_{\Gamma_0}^{int,0} &= 0 \\ \psi|_{\Gamma_1}^{int,1} - \psi|_{\Gamma_0}^{int,1} &= 0 \\ -\frac{\sqrt{\cos(\theta_0)}}{\sqrt{E}} \psi|_{\Gamma_0}^{BL,b,0} + \frac{1}{\sqrt{2}}(v|_{\Gamma_0}^{BL,b,0} - \partial_n \psi|_{\Gamma_0}^{BL,b,0}) &= 0 \\ \frac{\sqrt{\cos(\theta_1)}}{\sqrt{E}} \psi|_{\Gamma_1}^{BL,b,1} + \frac{1}{\sqrt{2}}(v|_{\Gamma_1}^{BL,b,1} - \partial_n \psi|_{\Gamma_1}^{BL,b,1}) &= 0 \\ v|_{\Gamma_i}^{int,0} + v|_{\Gamma_i}^{BL,b/t,0} &= v_i \\ \psi|_{\Gamma_i}^{int,0} + 0 &= 0 \\ \partial_n \psi|_{\Gamma_i}^{int,0} + \partial_n \psi|_{\Gamma_i}^{BL,b/t,0} &= 0 \\ \psi|_{\Gamma_i}^{int,1} + \psi|_{\Gamma_i}^{BL,b/t,0} &= 0 \end{aligned}$$

so the closed equations on the interior term are

$$\begin{aligned} \partial_z \psi^{int,0} &= 0 \\ \psi|_{z=\gamma_0}^{int,0} &= \psi|_{z=\gamma_1}^{int,0} = 0 \\ \partial_z v^{int,0} &= 0 \\ \partial_z \psi^{int,1} &= 0 \\ (-1)^{i+1} \sqrt{\cos(\theta_i)} \psi|_{z=\gamma_i}^{int,1} + \frac{1}{\sqrt{2}}(v|_{z=\gamma_i}^{int,0} - v_i(x)) &= 0. \end{aligned}$$

From there we obtain

$$v^{int,0}(x) = \frac{\frac{v_0}{\sqrt{\cos(\theta_0)}} + \frac{v_1}{\sqrt{\cos(\theta_1)}}}{\frac{1}{\sqrt{\cos(\theta_0)}} + \frac{1}{\sqrt{\cos(\theta_1)}}}$$

$$\psi^{int,1}(x) = \frac{v_1 - v_0}{\sqrt{2 \cos(\theta_0)} + \sqrt{2 \cos(\theta_1)}}$$

□

Note that if we have enough regularity on  $v_{0,1}$  we can pursue the asymptotic expansion up to a higher order, see Figure 3.8.

In this case of a higher order expansion, we must keep track of the remainder in the boundary layer operator. In fact these remainders are the source terms of the next boundary layer operator. They modify the constant  $H_f$  of the next order boundary layer compatibility condition, and thus the interior flow. This is why the splitting of the null boundary conditions for higher orders is non-trivial.

### Ekman pumping for Navier-Stokes-Coriolis

Here, the Ekman pumping manifested through the fact that the limit flow satisfy, with  $\kappa_i = (1 + \gamma_i'^2)^{-\frac{1}{4}}$ ,

$$\kappa_0(x)v_0 + \kappa_1(x)v_1 - (\kappa_0(x) + \kappa_1(x))\bar{v} = 0.$$

In the time dependent case instead of we expect that the Ekman pumping manifest through a damping term in the evolution equation. And, formally, if we add the time dependency, by integrating with respect to  $z$  (i.e projecting over the kernel of Coriolis operator) the formal limit we obtain for  $w = \int_{\gamma_0}^{\gamma_1} v$  is

$$(\gamma_1 - \gamma_0)\partial_t w + (\kappa_0(x)(w - v_0) + \kappa_1(x)(w - v_1)) = 0.$$

As mentioned in the beginning of this section, the result (3.29) is only the stationary and linear case of general result proved by Chemin [10]: the convergence in the non-linear and time dependent case with an axisymmetric and well-prepared initial condition.

In our notation, with  $\Omega = \{(x_h, z) = (r \cos(\theta), r \sin(\theta), z) \in \mathbb{R}^3, \gamma_0(r) < z < 0 = \gamma_1\}$  the result of Chemin can be summarized as the following:

**Theorem 3.4.5** (Chemin). *For  $\varepsilon > 0$  let  $U^\varepsilon$  be a solution of*

$$\partial_t U^\varepsilon + \nabla \cdot U^\varepsilon \otimes U^\varepsilon - \nu \varepsilon \Delta U^\varepsilon + \frac{1}{\varepsilon} e_3 \times U^\varepsilon + \nabla P^\varepsilon = 0$$

$$\nabla \cdot U^\varepsilon = 0, \quad U^\varepsilon_{\partial\Omega} = 0, \quad U^\varepsilon(t=0) = u_0$$

*With  $U_0$  well-prepared, i.e  $U_0 = (0, \mathbf{v}_0(r), 0)$ . Then (up to a smallness condition due to the non-linearity)  $U^\varepsilon$  converges in  $L^2_{loc}(\mathbb{R}^+, \Omega)$  to  $(0, \bar{v}, 0)$  with  $\bar{v}$  solution of*

$$\partial_t \bar{v}(t, r) + \sqrt{2\nu} \frac{(1 + \gamma_0'^2(r))^{\frac{1}{4}} + 1}{0 - \gamma_0(r)} \bar{v} = 0$$

$$\bar{v}(t=0) = \mathbf{v}_0$$

Similar behavior was also derived by Rousset in [63] for the MHD system (and also with



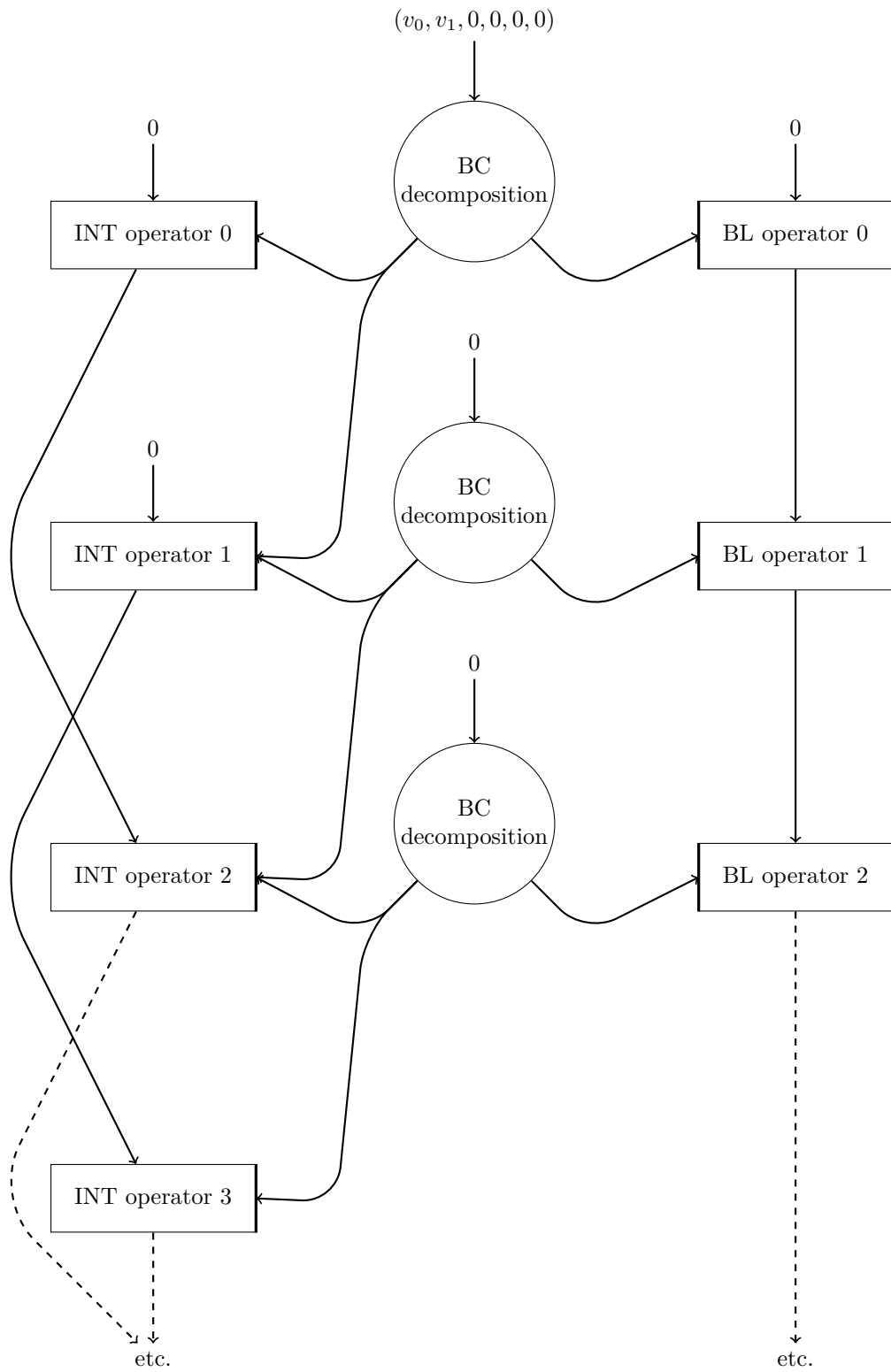


Figure 3.8 – Higher order expansion.

axisymmetric and well-prepared conditions) where the limit flow is solution of

$$\begin{aligned}\partial_t u + E(x)u &= 0 \\ u(t=0) &= u_0\end{aligned}$$

where  $E(x)$  is the Ekman pumping (depending in fact on both the Ekman number, and the Elsasser number  $\Lambda$  corresponding to the ration of magnetic and Coriolis forces), with strong variations with respect to the distance to the axis, even being singular close to the equator

$$E(x) = \frac{\sqrt{2}}{2|\cos(\theta)|^{\frac{3}{2}}} \left( \Lambda|\cos(\theta)| + \sqrt{1 + \Lambda^2|\cos(\theta)|^2} \right)^{\frac{1}{2}}.$$

It is to be noted that if the bottom topography is strongly varying,  $\gamma_0(x) = \varepsilon\tilde{\gamma}_0\left(\frac{x}{\varepsilon}\right)$  then the problem is very different. Indeed the boundary layer equation is a true PDE rather than an ODE, in a possibly unbounded domain. We refer to the work of Gérard-Varet [32], of Dalibard and Prange [16], and of Dalibard and Gérard-Varet [14]. Following Gérard-Varet and Masmoudi [35] these authors reduce the analysis to this boundary layer PDE in a bounded (in one direction) domain using a transparent boundary condition, relying on a Dirichlet to Neumann operator.

## 3.5 The shear layers

We have previously supposed that  $\gamma'_i$  is regular and bounded, leading to the validity of the boundary layer expansion.

If it not longer the case, for example in the sphere  $\gamma_0(x) \sim (-x)^{\frac{1}{2}}\mathbf{1}_{x<0}$ , the previous ordering and computation break down.

The goal of this section is to investigate the behavior of the main flow in such cases and, as the exponent  $\frac{1}{2}$  is not specific, we will be interested in  $\gamma_0$  admitting a singular behavior like  $x^\alpha$ ,  $\alpha \in (0, 1)$ .

### 3.5.1 About the validity of Ekman layers

One of the questions to be answered is to determine the critical latitude  $\theta_c$  up to which we can use the Ekman layer expansion. In this perspective let us recall the size of Ekman layers

$$\delta_i = \sqrt{\frac{E}{\cos(\theta_i)}} = \frac{\sqrt{E}}{(1 + \gamma'_i(x)^2)^{\frac{1}{4}}}.$$

Let us notice that one of the core boundary layer hypotheses, the localization near the boundary, namely  $\delta \ll 1$ , leads to the necessary (but not sufficient) condition  $\cos(\theta_c) \gg E$ . But long before this scaling, the predominant balance shifts from the Ekman balance to another one called the equatorial scaling.

In fact the core assumption of the previous computation can be seen in the Jacobian (3.24), namely

$$\cos(\theta) \gg \frac{\partial_\sigma \theta \delta}{\cos(\theta)}. \quad (3.30)$$

But in curvilinear coordinates if  $\gamma'_0$  goes to  $-\infty$  as  $x^{\alpha-1}$  we have near  $x = 0$  (with  $\sigma = 0$ )

corresponding to  $x = 0$ )

$$\frac{\pi}{2} - \theta \sim \cos(\theta) \sim \frac{c}{\gamma_0} \sim cx^{1-\alpha}, \quad \sigma \sim cx^\alpha$$

so  $\partial_\sigma \theta \sim \cos(\theta)^{\frac{1-2\alpha}{1-\alpha}}$ , thus the previous assumption (3.30) is in fact

$$\cos(\theta)^{1+1+\frac{1}{2}-\frac{1-2\alpha}{1-\alpha}} = \cos(\theta)^{\frac{3-\alpha}{2(1-\alpha)}} \gg E^{\frac{1}{2}}$$

i.e

$$\cos(\theta) \gg E^{\frac{1-\alpha}{3-\alpha}} \Leftrightarrow x \gg E^{\frac{1}{3-\alpha}}. \quad (3.31)$$

Note that if  $\alpha = \frac{1}{2}$  we indeed recover the  $\frac{1}{5}$ -scaling of the sphere.

When  $\cos(\theta_i) \leq E^{\frac{1-\alpha}{3-\alpha}}$  another boundary layer appears, the so called *equatorial Ekman boundary layer*. This boundary layer is probably the main difficulty of understanding the Proudman problem for several reasons:

- Although the Ekman layers equation are simply linear ODEs with constant coefficients, the boundary layer equation at the equator is a PDE which does not fit well in already well understood classifications, see the discussion at the beginning of Chapter 4.
- The connection to the main flow is also less obvious. It would seem that a boundary layer of size  $(Ez)^{\frac{1}{3}}$  called by Marcotte, Dormy and Soward [54] a *similarity sublayer* is responsible for this connection.
- Finally, the transition between the classical Ekman layer and the equatorial Ekman layer is also non-trivial. The hope for this part, once the equatorial layer is sufficiently well understood, would be to apply techniques similar to the ones by Dalibard and Saint-Raymond [17] to connect two different kinds of boundary layers. Unfortunately we were unable to prove uniqueness and to obtain strong decay estimates for the equatorial Ekman layer, which prevented us from proving a convergence theorem in the spirit of [17].

A discussion about this layer will be the focus of the last section of this chapter and of Chapter 4.

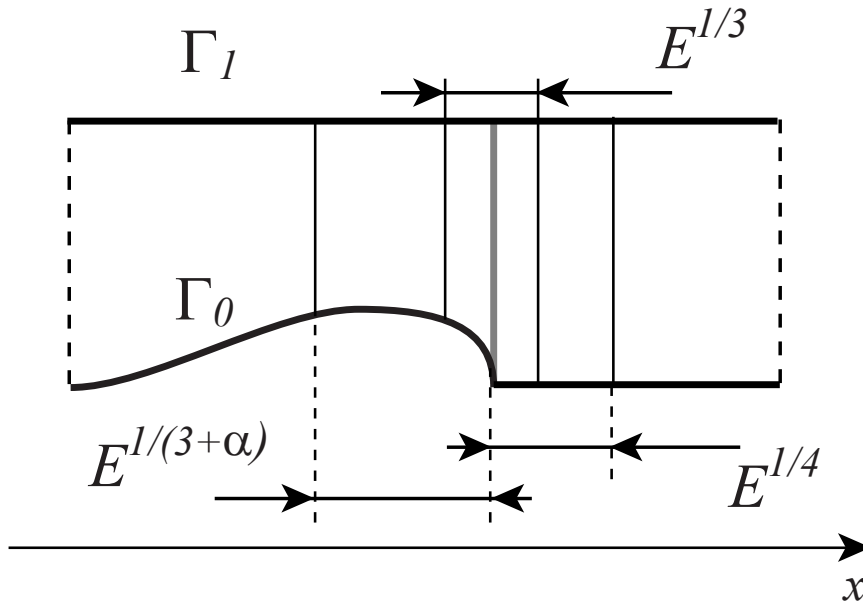
But as long as the distance and scalings to  $x = 0$  are far greater than  $E^{\frac{1}{3-\alpha}}$  it is reasonable to study the interior flow as solution of the equation obtained through the use of non degenerate Ekman layer. This study will, at least, lead to the description of 3 nested boundary layers, called the Stewartson  $\frac{1}{4}$ ,  $\frac{2}{7}$  and  $\frac{1}{3}$  layers in the setting of a sphere.

### 3.5.2 Objective

The goal of this section is to study the equation of the interior flow with Ekman type boundary condition, i.e assuming that the Ekman boundary condition is still valid.

For simplicity, we will work with  $x \in (-1, 1)$  identified as the torus  $\mathbb{T}$ , and consider only

- $x \mapsto \gamma_1(x)$  smooth,
- $\gamma_0(x) = x^\alpha \mathbf{1}_{x < 0} \chi(x)$ , where  $\chi$  is cut-off function equal to 1 on  $(-1/3, 0)$  and 0 outside  $(-2/3, 1/3)$ ,
- $C^{-1} < \gamma_1 - \gamma_0 < C$  for some  $C > 0$ .

Figure 3.9 – The shear layers when  $\gamma_0 \sim (-x)^\alpha$ .

For the horizontal variable we will take periodic boundary conditions in  $x$ . This setting is represented in Figure 3.9.

We will denote by  $\bar{v}$  the expression given by (3.29).

We recall that we are trying to construct an approximate solution as

$$\mathbf{u}^{app} = \mathbf{u}^{int}(x, z) + \mathbf{u}^{bl, Ekman} \left( x, \frac{d((x, z), \Gamma_i)}{\delta_i} \right).$$

If we keep all the terms in the main equation (3.9), and apply the boundary conditions corresponding to the Ekman boundary layer operator of Proposition 3.4.1, we obtain that  $\mathbf{u}^{int} = (v^{int}, \psi^{int})$  must satisfy, up to small enough remainders

$$\begin{aligned} \partial_z v^{int} - E \Delta_{x,z}^2 \psi^{int} &= 0 \\ \partial_z \psi^{int} + E \Delta_{x,z} v^{int} &= 0 \\ \left( (-1)^{i+1} \frac{1}{\delta_i} \psi^{int} + \frac{1}{\sqrt{2}} ((v^{int} - v_i) - \partial_n \psi^{int}) \right) \Big|_{z=\gamma_i} &= 0. \end{aligned} \tag{3.32}$$

*Remark 3.5.1.* • Let us stress that we were not able to rigorously derive such an equation for the interior flow, but that physical analysis and numerical simulations suggest that this is indeed the correct equation, see [54].

- The equation 3.32 seems undetermined, as there is too little boundary conditions for its order. This is indeed the case, but if we require  $\mathbf{u}^{int}$  to be bounded in  $H^1(\Omega) \times H^2(\Omega)$  uniformly in  $E$ , then the first order of the asymptotic behavior is unique. For example,

in the non-singular case, it is in fact the one described by the  $v^{int,0}, \psi^{int,0}, \psi^{int,1}$  of the previous subsection:  $\mathbf{u}^{int} = (v^{int,0}, 0) + O_{L^2}(\sqrt{E})$ .

The idea to construct an approximate solution of (3.32) is, as before, to identify the right asymptotic expansion and to compute the remaining terms.

But, for this case, since we have

$$\cos(\theta_0) \sim \begin{cases} \alpha^{-1}(-x)^{(1-\alpha)} & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

the  $\bar{v}$  given by (3.29) is discontinuous at  $x = 0$  if  $v_0(0^-) \neq \frac{\delta_0(0^+)v_0(0^+) + \delta_1(0^+)v_1(0^+)}{\delta_0(0^+) + \delta_1(0^+)}$ . The consequence is that we do not have  $\bar{v} \in H^1$  and cannot continue the asymptotic expansion, as the remainder from  $E\partial_x^2\bar{v}$  will not be in  $H^{-1}$  (it is in fact the derivative of a Dirac mass).

The assumption to neglect  $E\partial_x^2v$  in the interior operator is thus false, this term creates boundary layers near  $x = 0$ .

Note that we have a similar problem for the expression of  $\psi^1$  in the previous subsection, as it has a loss of regularity like  $x^{\frac{1}{4}}$  we can keep all the terms as remainders except  $\partial_x^4\psi^1$ .

Thus, in order to construct an approximate solution  $\mathbf{u}^{app}$  (in the sense of 3.2.2) of (3.32) satisfying the  $L^2$  convergence

$$\mathbf{u}^{app} \rightarrow (\bar{v}, 0)$$

we will in fact construct 3 boundary layers

$$\mathbf{u}^{app} = (\bar{v}, 0) + u^{BL,l} \left( \frac{x}{E^{\frac{1}{3+\alpha}}} \right) + u^{BL,r} \left( \frac{x}{E^{\frac{1}{4}}} \right) + u^{BL,c} \left( \frac{x}{E^{\frac{1}{3}}}, z \right) + o_{L^2} \left( E^{\frac{1-\alpha}{4(3+\alpha)}} \right).$$

- The  $u^{bl,l}$  and  $u^{bl,r}$  are boundary layers, respectively at the left and the right of  $x = 0$ , lifting the discontinuity of  $\bar{v}$ . These will be constructed in Subsection (3.5.3).
- The  $u^{bl,c}$  layer lifts the discontinuity of  $\psi$  and will be the subject of Subsection 3.5.5.

The rest of the section will be the study of the various layers. And Figure 3.10 illustrates the articulation of the various boundary layers. In this section we will use the previous construction of the Ekman layers, and focus on the [blue terms](#).

### 3.5.3 Stewartson 2/7 and 1/4 layers

The usual asymptotic expansion  $v^{int} = v + O(\sqrt{E})$ ,  $\psi^{int} = 0 + \sqrt{E}\psi + O(E)$  suggests that the rescaled equation we need to consider is in fact

$$\begin{aligned} \partial_z v - E^{\frac{3}{2}} \partial_x^4 \psi &= 0 \\ \partial_z \psi + \sqrt{E} \partial_x^2 v &= 0 \\ \left( (-1)^{i+1} \frac{\sqrt{E}}{\delta_i} \psi + \frac{1}{\sqrt{2}} (v - v_i) \right) \Big|_{z=\gamma_i} &= 0. \end{aligned} \tag{3.33}$$

Indeed, the lower order terms from  $\partial_n \psi|_{z=\gamma_i}$  and  $\partial_z^2 \partial_x^2$  can be treated as remainders, sufficiently small in the correct spaces.

If the variation of  $x$  occurs on a scale  $E^\beta$ , we expect  $E^{\frac{3}{2}} \partial_x^4 \simeq E^{\frac{3}{2}-4\beta}$  while  $\sqrt{E} \partial_x^2 \simeq E^{\frac{1}{2}-2\beta}$ . Thus, as long as  $\beta > \frac{1}{2}$  (the Ekman scaling),  $E^{\frac{3}{2}} \partial_x^4 \ll E^{\frac{1}{2}} \partial_x^2$

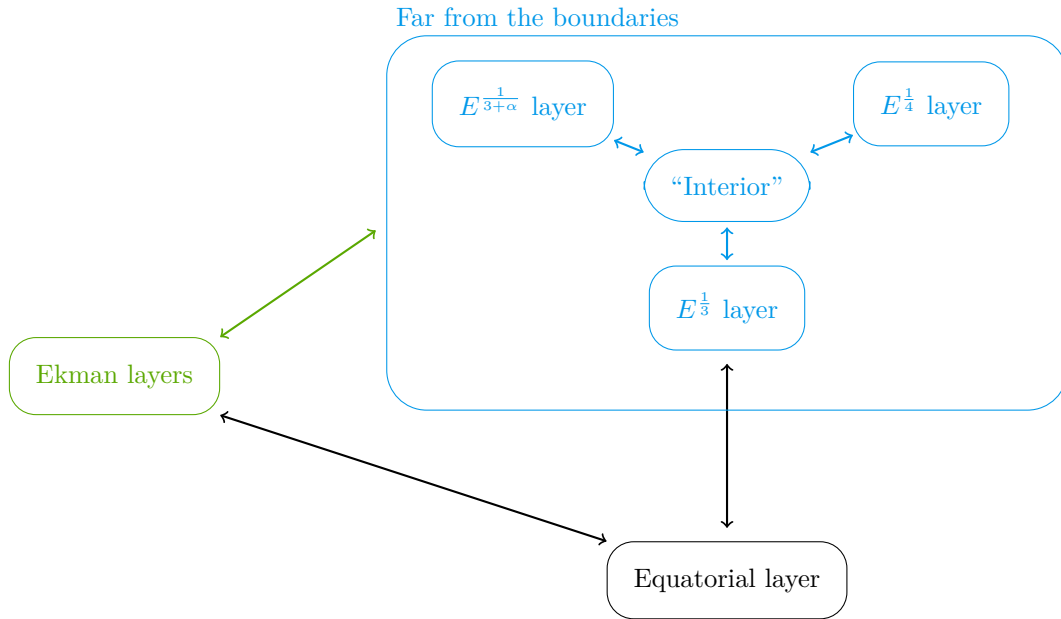


Figure 3.10 – Objective of Section 3.5.

But, even from a formal point of view, a necessary condition for this expansion to hold is that the scalings are of size  $E^\beta$  with  $\beta > \frac{1}{3-\alpha} > \frac{1}{2}$ . Therefore, this heuristic prompts us to consider first a constant (with respect to  $z$ )  $v$ .

This Taylor-Proudman theorem reads as  $\partial_z v^{int,0} = 0$ , and  $\partial_z \psi^{int,1} = C(x)$ . As a consequence, the integration of (3.33) with respect to  $z$  (i.e projection on the kernel of the singular operator) gives as equation

$$(\kappa_0(x)v_0(x) + \kappa_1(x)v_1(x)) - (\kappa_0(x) + \kappa_1(x))w(x) - (\gamma_1(x) - \gamma_0(x))\sqrt{E}\partial_x^2 w(x) = 0 \quad (3.34)$$

where, for simplicity, we denoted  $w = v^{int,0}$  the first order of the interior operator expansion corresponding to these heuristics, and with

$$\frac{1}{\sqrt{2}}\kappa_i(x) = \frac{\delta_i}{\sqrt{E}} = \frac{1}{\sqrt{\cos(\theta_i(x))}} = (1 + \gamma_i'(x)^2)^{-\frac{1}{4}}.$$

In the case where both  $\kappa_0$  and  $\kappa_1$  are smooth, we obtain the usual expansion. But here  $\kappa_0(x) \sim (-x)^{\frac{\alpha-1}{2}}$  for  $x < 0$  and 1 for  $x > 0$ .

*Remark 3.5.2.* Note that using  $\kappa_i$  instead of  $\gamma_i$  allows us to consider Neumann conditions for  $v$  on  $\Gamma_i$ . Since Neumann boundary conditions does not create boundary layers (at least at the first order), this indeed formally corresponds to  $\kappa_i = 0$ . This is the case for the Proudman problem on a sphere where the symmetry conditions leads in fact to  $\kappa_0(x) \simeq x^{-\frac{1}{4}}\mathbf{1}_{x<0}$ , as the symmetry leads to the condition  $\partial_z v = 0$  at  $z = 0$ .

The study of (3.34) in such a case will be the goal of this part. We will see that this ODE is the one creating the  $E^{\frac{1}{3+\alpha}}$  and  $E^{\frac{1}{4}}$  boundary layers.

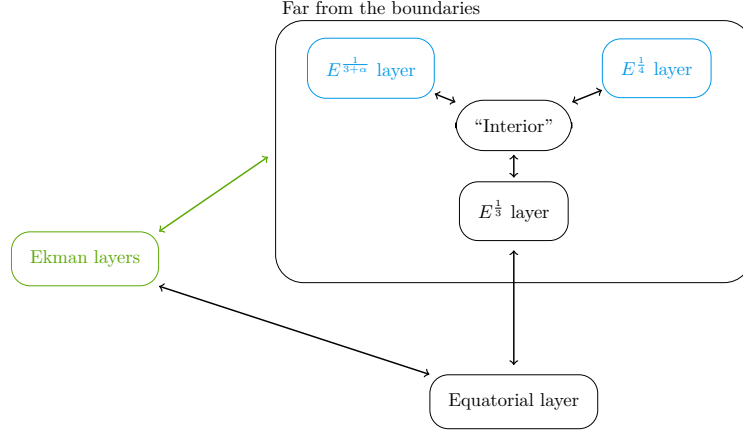


Figure 3.11 – Objective of Subsection 3.5.3 in blue.

**Proposition 3.5.1.** Let  $v_0, v_1 \in H^2$ . For  $E > 0$  let  $w$  be the solution of (3.34). Let

$$\bar{v}(x) = \frac{\kappa_0(x)v_0(x) + \kappa_1(x)v_1(x)}{\kappa_0(x) + \kappa_1(x)}.$$

Then, when  $E \rightarrow 0$

$$w(x) = \bar{v}(x) + E^{\frac{1-\alpha}{4(3+\alpha)}} w^{BL,l} \left( \frac{-x}{E^{\frac{1}{3+\alpha}}} \right) + w^{BL,r} \left( \frac{x}{E^{\frac{1}{4}}} \right) + O_{L^2}(E^{\frac{1}{6+2\alpha}}).$$

Where  $w^{BL,l}, w^{BL,r}$  are boundary layers terms defined on  $\mathbb{R}_+$ .

But, first, let us comment on the stability estimates of (3.34) with a remainder  $r$

$$(\kappa_0 + \kappa_1)w - (\gamma_1 - \gamma_0)\sqrt{E}\partial_x^2 w = r.$$

This equation is an elliptic equation and since  $(\kappa_0 + \kappa_1) \sim_{0^-} |x|^{\frac{\alpha-1}{2}}$ , by Hardy inequality

$$0 \leq \int |w|^2 \frac{(\kappa_0 + \kappa_1)}{\gamma_1 - \gamma_0} \leq C \|w\|_{H^1}^2$$

Thus, the problem is well-posed in  $H^1$  with the estimate

$$\int |w|^2 \frac{(\kappa_0 + \kappa_1)}{\gamma_1 - \gamma_0} dx + \sqrt{E} \int |\partial_x w|^2 \leq C \int wr.$$

If we had to study this equation on its own, the energy norm on the remainder  $r = r^0 + r^1$  would be

$$\|r^0(\kappa_0 + \kappa_1)^{-\frac{1}{2}}\|_{L^2} + E^{-\frac{1}{4}} \|r^1\|_{H^{-1}}.$$

Nevertheless, as our final goal is to study (3.32), we will not make use of the additional  $(\kappa_0 + \kappa_1)^{-\frac{1}{2}} \sim x^{\frac{1-\alpha}{4}}$ , even if difficulties come from  $x = 0$ .

### Scaling

Let us denote once more  $\zeta$  the boundary layer variable. With  $\zeta = E^{-\beta}x$ , we obtain for  $x > 0$  the equation

$$(\kappa_1 v_1 + \kappa_0 v_0)(E^\beta \zeta) - (\kappa_1 + \kappa_0)(E^\beta \zeta)v(\zeta) - (\gamma_1 - \gamma_0)(E^\beta \zeta)E^{\frac{1}{2}-2\beta}\partial_\zeta v(\zeta) = 0$$

and as all functions are smooth and the expected solution decays at an exponential rate we can write for example  $\kappa_0(E^\beta \zeta) = \kappa_0(0) + O(E^\beta)$ . Thus, the predominant balance leads to  $\beta = \frac{1}{4}$  i.e the boundary layer equation

$$(\kappa_0 v_0 + \kappa_1 v_1)(0^+) - (\kappa_0 + \kappa_1)(0^+)w(\zeta) - (\gamma_1 - \gamma_0)(0)\partial_\zeta^2 w(\zeta) = 0 \text{ for } \zeta > 0, \quad (3.35)$$

where  $\zeta = xE^{-\frac{1}{4}}$  is the boundary layer variable. And for  $x < 0$ , with  $\zeta = (-x)E^{-\beta}$ , as  $\kappa_0(x) \sim |x|^{\frac{\alpha-1}{2}}$  we obtain

$$\kappa_1(0)v_1(0) + E^{\beta \cdot \frac{\alpha-1}{2}} \zeta^{\frac{\alpha-1}{2}} v_0(0) - \left( \kappa_1(0) + E^{\beta \cdot \frac{\alpha-1}{2}} \zeta^{\frac{\alpha-1}{2}} \right) w(\zeta) - (\gamma_1 - \gamma_0)(E^\beta \zeta)E^{\frac{1}{2}-2\beta}\partial_\zeta^2 w(\zeta) = 0.$$

The predominant balance principle leads to

$$\beta \cdot \frac{\alpha-1}{2} = \frac{1}{2} - 2\beta \Leftrightarrow \beta = \frac{1}{3+\alpha}$$

i.e the boundary layer

$$\zeta = \frac{x}{E^{\frac{1}{3+\alpha}}} \quad (3.36)$$

$$\zeta^{\frac{\alpha-1}{2}} v_0(0) - \zeta^{\frac{\alpha-1}{2}} w(\zeta) - (\gamma_1 - \gamma_0)(0)\partial_\zeta^2 w(\zeta) = 0 \text{ for } \zeta > 0,$$

where  $\zeta = -xE^{-\frac{1}{3+\alpha}}$  is the boundary layer variable.

Note that in the case of most physical interest, namely the sphere  $\alpha = \frac{1}{2}$ , we recover the standard  $E^{\frac{2}{7}}$  boundary layer.

For simplicity, we will denote  $\kappa_0(0^+) + \kappa_1(0) = \kappa$  and  $\gamma_1(0) - \gamma_0(0) = h$ .

### Boundary layer operators

In order to quantify the decay via weighted spaces, let us also define for  $s \in \mathbb{N}$  and  $\lambda > 0$  the norm  $\mathcal{H}_\lambda^s(\mathbb{R}^+)$  by

$$\|f\|_{\mathcal{H}_\lambda^s(\mathbb{R}^+)}^2 = \int_0^\infty |\partial_\zeta^s f|^2 e^{2\lambda\zeta} d\zeta$$

and for  $p \in (0, 1]$

$$\|f\|_{\mathcal{H}_{\lambda,p}^s(\mathbb{R}^+)}^2 = \int_0^\infty |\partial_\zeta^s f|^2 e^{2\lambda\zeta^p} d\zeta.$$

The associated spaces are

$$\mathcal{H}_\lambda^s = \overline{C_c^\infty((0, \infty))}^{\mathcal{H}_\lambda^s}$$

$$\mathcal{H}_{\lambda,p}^s = \overline{C_c^\infty((0, \infty))}^{\mathcal{H}_{\lambda,p}^s}.$$

We will use  $p = 1$  for the  $\frac{1}{4}$  layer, and  $p < 1$  for the  $\frac{1}{3+\alpha}$  layer.



*Remark 3.5.3.* Note that  $\mathcal{H}_{\lambda,p}^s(\mathbb{R}^+) \hookrightarrow H^s((0,1))$  for all  $s \in \mathbb{N}$ . As a consequence, trace inequalities hold for all  $s \geq 1$ . In particular, if  $w \in \mathcal{H}_{\lambda,p}^s$  and  $0 \leq k \leq s-1$ , then  $w^{(k)}(0) = 0$ .

This allow us to easily derive a Hardy inequality with an explicit constant:

**Lemma 3.5.2.** *Let  $\lambda > 0$ ,  $p \in (0,1]$ . Then for all  $w \in \mathcal{H}_{\lambda,p}^1$  we have*

$$\int_0^\infty |\partial_\zeta w|^2 e^{2\lambda\zeta^p} d\zeta \geq p^2 \lambda^2 \int_0^\infty \frac{1}{\zeta^{2-2p}} |w|^2 e^{2\lambda\zeta^p} d\zeta - p(1-p)\lambda \int_0^\infty \frac{1}{\zeta^{2-p}} |w|^2 e^{2\lambda\zeta^p} d\zeta. \quad (3.37)$$

*Proof.* This is a straightforward computation, for  $w \in C_c^\infty((0,+\infty))$ ,

$$\begin{aligned} \int_0^\infty \left| \partial_\zeta (w e^{\lambda\zeta^p}) \right|^2 d\zeta &= \int_0^\infty |\partial_\zeta w|^2 e^{2\lambda\zeta^p} d\zeta - \int_0^\infty |w|^2 \partial_\zeta \left( e^{\lambda\zeta^p} \partial_\zeta \left( e^{\lambda\zeta^p} \right) \right) d\zeta + \int_0^\infty |w|^2 \left( \partial_\zeta e^{\lambda\zeta^p} \right)^2 d\zeta \\ &= \int_0^\infty |\partial_\zeta w|^2 e^{2\lambda\zeta^p} d\zeta - \int_0^\infty \left( p^2 \lambda^2 \zeta^{2(p-1)} + p(p-1)\zeta^{p-2}\lambda \right) |w|^2 e^{2\lambda\zeta^p} d\zeta \geq 0 \end{aligned}$$

*Remark 3.5.4.* The term  $p(1-p)\lambda \int_0^\infty \frac{1}{\zeta^{2-p}} |w|^2 e^{2\lambda\zeta^p} d\zeta$  is a lower order term far from 0, and can be absorbed by  $\int_0^\infty \frac{1}{\zeta^{2-2p}} |w|^2 e^{2\lambda\zeta^p} d\zeta$  for large  $\zeta$ . In fact, this term, due to the behavior of  $\zeta^p$  near  $\zeta = 0$ , disappear completely if we use as a norm

$$\|f\|_{\mathcal{H}_{\lambda,p}^s(\mathbb{R}^+)}^2 = \int_0^\infty |\partial_\zeta^s f|^2 e^{2\lambda(\zeta+\zeta_0)^p} d\zeta$$

with  $\zeta_0^p > \frac{1-p}{p\lambda}$ . As all of this is of no practical consequence, we will keep the more natural definition with a simple weight  $e^{\lambda\zeta^p}$ .

□

In the next two Propositions (3.5.3, 3.5.4), we will define the boundary layers operators. We start with the  $E^{\frac{1}{4}}$  boundary layer, which occurs on the right of the shear line  $x = 0$ .

**Proposition 3.5.3** (Stewartson  $E^{\frac{1}{4}}$  boundary layer operator). *Let  $0 < \lambda < \frac{\kappa}{h}$ . For all*

$$f \in \mathcal{H}_\eta^0(\mathbb{R}^+)$$

*there exists a constant  $H_f$ , depending linearly on  $f$ , given by*

$$H_f = - \int_0^\infty f(\tau) e^{-\sqrt{\frac{\kappa}{h}}\tau} d\tau$$

*such that for any  $W^0, W^1 \in \mathbb{R}$  satisfying*

$$\sqrt{\frac{\kappa}{h}} W_0 + E^{\frac{1}{4}} W_1 = H_f \quad (3.38)$$

*there exist  $w^{bl} = w\left(\frac{x}{E^{\frac{1}{4}}}\right)$  solution of*

$$(\gamma_1 - \gamma_0)(x) \sqrt{E} \partial_x^2 w^{bl} - (\kappa_0 + \kappa_1)(x) w^{bl} = f\left(\frac{x}{E^{\frac{1}{4}}}\right) + E^{\frac{1}{4}} r^{bl}(x)$$

with boundary conditions

$$\begin{aligned} w^{bl}(0^+) &= W_0 \\ \partial_x w^{bl}(0^+) &= W_1. \end{aligned}$$

The boundary layer profile  $w^{bl}(x)$  is identically zero outside  $(0, 1/2)$ . The remainder  $r^{bl} = r\left(\frac{x}{E^{\frac{1}{4}}}\right)$ , with  $r \in \cap_{\tilde{\lambda} < \lambda} \mathcal{H}_{\tilde{\lambda}}^0(\mathbb{R}^+)$  satisfies,

$$\|r^{bl}\|_{H^s((0, \frac{1}{2}))} \leq CE^{\frac{1}{8} - \frac{s}{4}} \quad \forall s \leq 0.$$

*Remark 3.5.5.* We can, in this proposition or in the following, replace  $\mathcal{H}_{\lambda}^0$  by  $\mathcal{H}_{\lambda}^{s_0}$  for  $s_0 \in \mathbb{N}$  and the estimate on the remainder is then true for all  $s \leq s_0$ . A way to see it is to derive  $s_0$  times the boundary layer equation with respect to  $\zeta$ . As this introduces further difficulties and the only estimations we need on the remainder are negative norms, we will not do so.

This can be derived from the explicit integration or Laplace transform of the boundary layer equation

$$\begin{aligned} h\partial_{\zeta}^2 w(\zeta) - \kappa w(\zeta) &= f(\zeta) \\ w(0) = W_0 \quad w'(0) &= E^{\frac{1}{4}} W_1. \end{aligned} \tag{3.39}$$

from where  $H_f = -\int_0^{\infty} f(\tau) e^{-\sqrt{\frac{\kappa}{h}}\tau} d\tau$ , which is indeed a continuous linear form on  $\mathcal{H}_{\lambda}^0$ . We refer to the Annex A.1 for this exact example.

However, in order to introduce a generic framework used for the next boundary layer operator, let us also briefly use methods based on the theory of elliptic PDEs.

*Proof.* Let

$$0 < \lambda < \sqrt{\frac{\kappa}{h}}. \tag{3.40}$$

We search  $w \in \mathcal{H}_{\lambda}^1$  solution of the elliptic equation (3.39). The solution will be a variational solution, coming from Lax-Milgram theorem. In this setting,  $W_0$  and  $W_1$  are respectively Dirichlet and Neumann traces.

First notice that up to a compactly supported lifting we can suppose that  $W_0 = 0$ , i.e consider an homogeneous Dirichlet problem.

Then, in order to use the Lax-Milgram theorem in our weighted setting, the only requirement is to check whether

$$\int_0^{\infty} \left(-\partial_{\zeta}^2 w + \frac{\kappa}{h} w\right) \cdot w e^{2\lambda\zeta} d\zeta \geq C \|w\|_{\mathcal{H}_{\lambda}^1}^2$$

holds for any  $w$  in a dense subset of  $\mathcal{H}_{\lambda}^1$ .

And integrating by parts, for  $w \in C_0^{\infty}(\Omega)$

$$\begin{aligned} \int_0^{\infty} \left(-\partial_{\zeta}^2 w + \frac{\kappa}{h} w\right) \cdot w e^{2\lambda\zeta} d\zeta &= \int_0^{\infty} \left(|\partial_{\zeta} w|^2 e^{2\lambda\zeta} + \left(-\frac{1}{2}\partial_{\zeta}^2 e^{2\lambda\zeta} + \frac{\kappa}{h} e^{2\lambda\zeta}\right) |w|^2\right) d\zeta \\ &= \int_0^{\infty} \left(|\partial_{\zeta} w|^2 + \left(-\frac{4}{2}\lambda^2 + \frac{\kappa}{h} e^{2\lambda\zeta}\right) |w|^2\right) e^{2\lambda\zeta} d\zeta \\ &= \|w\|_{\mathcal{H}_{\lambda}^1}^2 + \left(-2\lambda^2 + \frac{\kappa}{h}\right) \|w\|_{\mathcal{H}_{\lambda}^0}^2. \end{aligned}$$

Using the Hardy inequality (or simply Poincaré in this case) (3.37)

$$\|w\|_{\mathcal{H}_{\lambda}^1}^2 \geq \lambda^2 \|w\|_{\mathcal{H}_{\lambda}^0}^2.$$

We finally obtain the claimed coercivity as long as

$$-2\lambda^2 + \lambda^2 + \frac{\kappa}{h} > 0$$

i.e

$$\frac{\kappa}{h} > \lambda^2.$$

Once we have existence in  $\mathcal{H}_\lambda^1$ , using the equation  $\partial_\zeta^2 w = \frac{\kappa}{h}w + \frac{1}{h}f \in \mathcal{H}_\lambda^0$  we obtain the elliptic regularity

$$\|w\|_{\mathcal{H}_\lambda^2} \leq C \left( W_0 + \|f\|_{\mathcal{H}_\lambda^0} \right).$$

This allows us to define the Neumann trace  $\partial_\zeta w(0)$ . The function  $(W_0, f) \mapsto \partial_\zeta w(0)$  is obviously linear, leading to the condition like (3.38) in order to have  $W_1 = \partial_\zeta w(0)$ .

In order to obtain the explicit expression of this Dirichlet to Neumann operator, taking as a test function a solution of the dual equation, namely  $e^{-\sqrt{\frac{\kappa}{h}}\zeta}$  we have

$$\underbrace{\left[ -\partial_\zeta w e^{-\sqrt{\frac{\kappa}{h}}\zeta} + w \partial_\zeta e^{-\sqrt{\frac{\kappa}{h}}\zeta} \right]_0^\infty}_{W_1 + \sqrt{\frac{\kappa}{h}}W_0} + \underbrace{\int_0^\infty \left( -\partial_\zeta^2 + \frac{\kappa}{h} \right) e^{-\sqrt{\frac{\kappa}{h}}\zeta} \cdot w d\zeta}_0 = - \int_0^\infty f(\zeta) \cdot e^{-\sqrt{\frac{\kappa}{h}}\zeta} d\zeta.$$

After introducing the cut-off  $\chi$ , the remainder terms are

$$\begin{aligned} E^{\frac{1}{4}}r(\zeta) = & \underbrace{\left( \left( (\gamma_1 - \gamma_0)(E^{\frac{1}{4}}\zeta) - (\gamma_1 - \gamma_0)(0) \right) \partial_\zeta^2 w(\zeta) - \left( (\kappa_1 - \kappa_0)(E^{\frac{1}{4}}\zeta) - (\kappa_1 - \kappa_0)(0) \right) w(\zeta) \right) \chi(E^{\frac{1}{4}}\zeta)}_{\text{errors from the approximation of no variation w.r.t } x \text{ in the boundary layer}} \\ & + \underbrace{(\gamma_1 - \gamma_0)(E^{\frac{1}{4}}\zeta) \left( 2E^{\frac{1}{4}}\chi'(E^{\frac{1}{4}}\zeta)\partial_\zeta w + E^{\frac{1}{4}}\chi(E^{\frac{1}{4}}\zeta)w \right) + (1 - \chi(E^{\frac{1}{4}}\zeta))f(\zeta)}_{\text{errors from the cut-off } \chi} \end{aligned}$$

so, as  $\gamma_{0,1}, \kappa_{0,1}$  admit a Taylor expansion, we obtain  $r \in \mathcal{H}_\lambda^0$  for any  $\tilde{\lambda} < \lambda$ , with a control of the norm uniform in  $E$

$$\begin{aligned} \|r\|_{\mathcal{H}_\lambda^0} \leq & C_{\tilde{\lambda}, \lambda} (\|\gamma_1 - \gamma_0\|_{W^{1,\infty}} + \|\kappa_1 - \kappa_0\|_{W^{1,\infty}}) \|w\|_{\mathcal{H}_\lambda^2} \\ & + C_{\tilde{\lambda}, \lambda} \|\gamma_1 - \gamma_0\|_{L^\infty} \|w\|_{\mathcal{H}_\lambda^1} + C_{\tilde{\lambda}, \lambda} \|f\|_{\mathcal{H}_\lambda^0} \end{aligned}$$

As this is true for any  $\tilde{\lambda} < \lambda$  (but  $C_{\tilde{\lambda}, \lambda}$  blow up when  $\tilde{\lambda} \rightarrow \lambda$ ) we obtain the proposition, since the estimate on  $r^{bl}$  comes from the scaling as before (see Annex A.5).  $\square$

We now turn towards the so-called  $E^{\frac{2}{7}}$  boundary layer, of size  $E^{\frac{1}{3+\alpha}}$  for a generic boundary profile. This layer occurs for  $x < 0$ , and its particularity is that the rate of decay, previously  $\sqrt{\frac{\kappa}{h}}$  is now formally  $\sqrt{\frac{\kappa}{h}\zeta^{\frac{\alpha-1}{2}}}$  who depends on  $\zeta$ , and goes to 0 as  $\zeta \rightarrow \infty$  this is where the weighted spaces  $\mathcal{H}_{\lambda,p}^\alpha$  are used.

**Proposition 3.5.4** (Stewartson  $E^{\frac{1}{3+\alpha}}$  boundary layer operator). *Let*

$$0 < \lambda < \frac{4}{3+\alpha} h^{\frac{1}{2}}$$

and

$$f \in \mathcal{H}_{\lambda, \frac{3+\alpha}{4}}^0(\mathbb{R}^+).$$

There exists a constant  $\mathfrak{c}_\alpha$  and a constant  $H_f$ , depending linearly on  $f$  such that, for any  $W^0, W^1 \in \mathbb{R}$  satisfying

$$\mathfrak{c}_\alpha h^{-\frac{2}{\alpha+3}} W_0 + E^{\frac{1}{3+\alpha}} W_1 = H_f \quad (3.41)$$

there exists  $w^{bl} = w\left(-\frac{x}{E^{\frac{1}{3+\alpha}}}\right)$  solution of

$$(\gamma_1 - \gamma_0)(x) \sqrt{E} \partial_x^2 w^{bl} - (\kappa_0 + \kappa_1)(x) w^{bl} = E^{-\frac{1}{3+\alpha}} \left( f\left(-\frac{x}{E^{\frac{1}{3+\alpha}}}\right) + E^{\frac{1}{3+\alpha}} \right)$$

with boundary conditions

$$\begin{aligned} w^{bl}(0^-) &= W_0 \\ w^{bl'}(0^-) &= W_1 \end{aligned}$$

and  $w^{bl}(x)$  is identically zero outside  $(-1/2, 0)$ . The remainders  $r^{bl} = r\left(-\frac{x}{E^{\frac{1}{3+\alpha}}}\right)$ ,  $r \in \bigcap_{\tilde{\lambda} < \lambda} \mathcal{H}_{\tilde{\lambda}, \frac{3+\alpha}{4}}^0(\mathbb{R}^+)$  satisfies,  $\forall s \leq 0$

$$\|r^{bl}\|_{H^s((-\frac{1}{2}, 0))} \leq C E^{\frac{1}{6+2\alpha} - \frac{s}{3+\alpha}}$$

*Proof.* The boundary layer equation is

$$h \partial_\zeta^2 w(\zeta) - \zeta^{\frac{\alpha-1}{2}} w(\zeta) = f(\zeta).$$

Thanks to the change of variable  $\zeta \mapsto h^{\frac{2}{3+\alpha}} \zeta$  we can restrict ourselves to  $h = 1$ .

Here we can also compute an explicit solution of the homogeneous equation in terms of Bessel functions. With  $w(\zeta) = W_0 \sqrt{\zeta} G\left(\lambda \zeta^{\frac{3+\alpha}{4}}\right)$  the homogeneous equation becomes

$$(\lambda \zeta^{\frac{3+\alpha}{4}})^2 G''\left(\lambda \zeta^{\frac{3+\alpha}{4}}\right) + \lambda \zeta^{\frac{3+\alpha}{4}} G'\left(\lambda \zeta^{\frac{3+\alpha}{4}}\right) - \left( \left( \frac{4}{\alpha+3} \zeta^{\frac{3+\alpha}{4}} \right)^2 + \left( \frac{2}{\alpha+3} \right)^2 \right) G\left(\lambda \zeta^{\frac{3+\alpha}{4}}\right) = 0$$

with  $\lambda = \frac{4}{\alpha+3}$  we obtain the very definition of the modified Bessel function of the second kind ([1] p.374). Since we look for decaying solutions, we finally obtain the explicit solution for the homogeneous case

$$w(\zeta) = W_0 \underbrace{\sqrt{\zeta} K_{\frac{2}{\alpha+3}}\left(\frac{4}{\alpha+3} \zeta^{\frac{3+\alpha}{4}}\right)}_{\mathfrak{w}_\alpha}$$

note that we recover the result of [54] for  $\alpha = \frac{1}{2}$ .

We can then use Duhamel formula to obtain the generic solution and condition. The main interest of this explicit solution is less its existence than the explicit value of  $\mathfrak{c}_\alpha$ , namely

$$\mathfrak{c}_\alpha = \frac{\mathfrak{w}'_\alpha(0)}{\mathfrak{w}_\alpha(0)} = \left( \frac{\alpha+3}{2} \right)^{\frac{4}{\alpha+3}} \frac{\Gamma\left(\frac{2}{\alpha+3}\right)}{\Gamma\left(-\frac{2}{\alpha+3}\right)}$$

and thanks to the asymptotic behavior of Bessel  $K$  functions we see that the rate of decay is

$$|w(\zeta)| \leq e^{-\frac{4}{3+\alpha}\zeta^{\frac{3+\alpha}{4}}}.$$

which is not an exponential decay, but is still decaying faster than any rational function. In particular, we still have the size of  $o(E^\infty)$  for  $x \gg E^{\frac{1}{3+\alpha}}$ .

To use a weighted Sobolev space as we did previously, the only point of importance is the coercivity. Let us first notice that, to control the singularity of  $\zeta^{\frac{\alpha-1}{2}}$  near 0 is of no concern for such formulation as we have  $H^1$ -like regularity near 0, and  $\frac{\alpha-1}{2} > -2$ .

We will work in the space  $\mathcal{H}_{\lambda,p}^1$  for  $\lambda < \lambda_\alpha$ , where  $p$  will be chosen later and  $\lambda_\alpha$  will be fixed small enough.

The computation is very similar to the previous one. For  $w \in C_0^\infty((0, \infty))$ , we have

$$\begin{aligned} & \int_0^\infty \left( -\partial_\zeta^2 w + \zeta^{\frac{\alpha-1}{2}} w \right) \cdot w e^{2\lambda\zeta^p} d\zeta = \int_0^\infty \left( |\partial_\zeta w|^2 e^{2\lambda\zeta^p} + \left( -\frac{1}{2}\partial_\zeta^2 e^{2\lambda\zeta^p} + \zeta^{\frac{\alpha-1}{2}} e^{2\lambda\zeta^p} \right) |w|^2 \right) d\zeta \\ & = \int_0^\infty \left( |\partial_\zeta w|^2 + \left( -\frac{1}{2}(4\lambda^2 p^2 \zeta^{2p-2} - 2\lambda p(1-p)\zeta^{p-2}) + \zeta^{\frac{\alpha-1}{2}} \right) |w|^2 \right) e^{2\lambda\zeta^p} d\zeta \\ & = \|w\|_{\mathcal{H}_{\lambda,p}^1}^2 + \underbrace{p(1-p) \int_0^\infty \zeta^{p-2} |w|^2}_{\geq 0} + \int_0^\infty (-2\lambda^2 p^2 \zeta^{2p-2} + \zeta^{\frac{\alpha-1}{2}}) |w|^2 e^{2\lambda\zeta^p} d\zeta. \end{aligned}$$

We must ensure the positivity (or at least positivity when combined with the other ones) of the last term (in blue). Near 0 the two other terms can help, but at infinity the  $\mathcal{H}_{\lambda,p}^1$  norm contributes to  $\lambda^2 p^2 \zeta^{2p-2}$  at most<sup>†</sup>, and the other positive term is decaying too fast. More precisely, thanks to (3.37), we have

$$\begin{aligned} & (1-\epsilon) \left( \|w\|_{\mathcal{H}_{\lambda,p}^1}^2 + p(1-p) \int_0^\infty \zeta^{p-2} |w|^2 \right) + \int_0^\infty (-2\lambda^2 p^2 \zeta^{2p-2} + \zeta^{\frac{\alpha-1}{2}}) |w|^2 e^{2\lambda\zeta^p} d\zeta \\ & \geq \int_0^\infty (-(1+\epsilon)\lambda^2 p^2 \zeta^{2p-2} + \zeta^{\frac{\alpha-1}{2}}) |w|^2 e^{2\lambda\zeta^p} d\zeta. \end{aligned}$$

So we must at least have  $\zeta^{\frac{\alpha-1}{2}}$  stronger than  $\zeta^{2p-2}$  at infinity i.e

$$2p-2 \leq \frac{\alpha-1}{2} \Leftrightarrow p \leq \frac{3+\alpha}{4}.$$

Thus, we set  $p = p_\alpha = \frac{3+\alpha}{4}$ . For  $p < p_\alpha$  we then need to control the behavior near 0 by the  $\mathcal{H}_{\lambda,p}^1$  norm. But for  $p = p_\alpha$ , sufficient condition to have coercivity is then

$$-(1+\epsilon)\lambda^2 p_\alpha^2 + 1 \geq 0$$

for some  $\epsilon > 0$ . This leads to

$$\lambda < \lambda_\alpha = \frac{1}{p_\alpha} = \frac{4}{3+\alpha}.$$

The rest of the proof is then exactly the same as before.  $\square$

*Remark 3.5.6.* • Note that we obtain in fact a sharp estimate on the rate of decay, not only on the power of  $\zeta$  in the exponential but also on the  $\lambda$ .

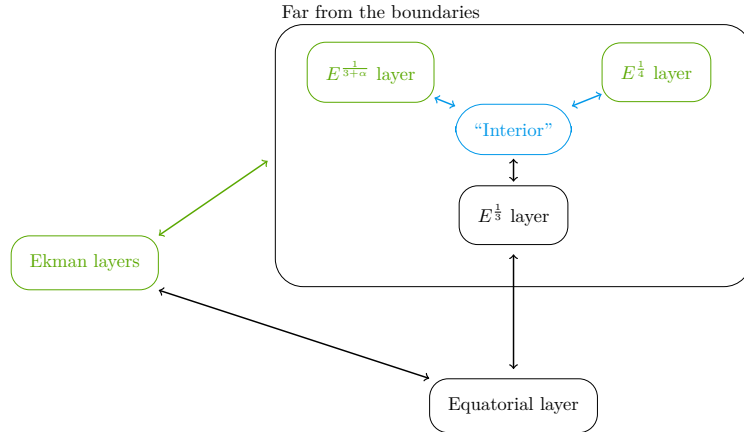


Figure 3.12 – Objective of Subsection 3.5.4 are in blue.

- This approach is ultimately the same as considering  $w = \mathbf{m}e^{\lambda\zeta^p}$  before working on  $\mathbf{m}$ .
- These proofs show that the angle of attack consisting in considering the boundary layer equation as elliptic-like equation is not only relevant but also leads to robust and sometimes sharp results.

### 3.5.4 Interior terms and connection between operators

Let us now focus on the interior terms. Since the limit problem is of order 0 as a differential operator there no need for boundary conditions.

But, as mentioned before, since  $\bar{v}$  is discontinuous, it is not in the energy space  $H^1$ . A first approach is to consider that  $\bar{v}$  is like a Heaviside function,

$$\bar{v} \sim_{x \rightarrow 0} \bar{v}(0^-)\mathbf{1}_{x < 0} + \bar{v}(0^+)\mathbf{1}_{x > 0} \quad (3.42)$$

and that we would be able to lift the discontinuity by two boundary layers and linear algebra:

$$w = \bar{v} + w^{bl,l} + w^{bl,r} \quad (3.43)$$

with  $w^{bl,l}$  and  $w^{bl,r}$  the results of the preceding boundary layer operators with respective traces  $W_0^l, W_1^l$  and  $W_0^r, W_1^r$  satisfying

$$\begin{aligned} \bar{v}(0^-) + W_0^l &= \bar{v}(0^+) + W_0^r \\ 0 + W_1^l &= 0 + W_1^r \\ c_l W_0^l + E^{\frac{1}{3+\alpha}} W_1^l &= 0 \\ c_r W_0^r + E^{\frac{1}{4}} W_1^r &= 0 \end{aligned}$$

from where we obtain a unique solution.

*Remark 3.5.7.* Note that the seconde condition  $W_1^l = W_1^r$  can be weakened. Indeed, if  $W_1^l \neq W_1^r$ , we would still have  $w$  of (3.43) in  $H^1$ , but it would introduce a remainder term

$$\sqrt{E} (W_1^l - W_1^r) \delta'_0$$

which must be either  $o(1)$  in  $L^2$  or  $o(E^{\frac{1}{2}})$  in  $H^{-1}$  in order to be an acceptable remainder. Thus, we can instead choose only  $W_1^l - W_1^r = o(1)$ .

The main point of this solution is that  $W_0^r = -\frac{[\bar{v}]_0}{1 - \frac{c_r}{c_l} E^{\frac{1}{3+\alpha} - \frac{1}{4}}}$ ,  $W_0^l = -\frac{[\bar{v}]_0 \frac{c_r}{c_l} E^{\frac{1}{3+\alpha} - \frac{1}{4}}}{1 - \frac{c_r}{c_l} E^{\frac{1}{3+\alpha} - \frac{1}{4}}}$  and that we obtain

$$w(0) = \frac{\bar{v}(0^-) + \epsilon \bar{v}(0^+)}{1 + \epsilon}$$

where

$$\epsilon = -\frac{c_r}{c_l} E^{\frac{1}{3+\alpha} - \frac{1}{4}} = -\frac{c_r}{c_l} E^{\frac{1-\alpha}{4(3+\alpha)}}.$$

*Remark 3.5.8.* In the relevant case for physics, we have  $\alpha = \frac{1}{2}$ , leading to  $\frac{1-\alpha}{4(3+\alpha)} = \frac{1}{28}$ . As a consequence even if  $E$  is small, typically  $E = 10^{-6}$ , the value  $E^{\frac{1}{28}}$  cannot be neglected for applications. However, from a mathematical point of view, it is still a small parameter.

Unfortunately, the previous expansion is not quite complete from a mathematical point of view. Indeed, if we expand  $\bar{v}$  near 0 we have

$$\begin{aligned} \bar{v} &= \frac{\kappa_0 v_0 + \kappa_1 v_1}{\kappa_0 + \kappa_1} \\ &= \underbrace{v_0(0^-) \mathbf{1}_{x < 0} + \frac{\kappa_0 v_0 + \kappa_1 v_1}{\kappa_0 + \kappa_1} (0^+) \mathbf{1}_{x > 0}}_{v^{disc} \text{ discontinuous}} + \underbrace{\kappa_1 (v_1 - v_0) (0^-) x^{\frac{1-\alpha}{2}} \mathbf{1}_{x < 0} + O(x^{1-\alpha})}_{v^{cont} \text{ continuous but } \notin H^1}. \end{aligned} \quad (3.44)$$

As a matter of fact, apart from the discontinuity, the lack of regularity is only near  $x = 0^-$ , and can be removed with a cut-off localized near  $0^-$ . The scaling of this cut-off is, fortunately, the same as the boundary layer, allowing us to lift the remainders by the left boundary layer operator. These difficulties are more linked to the singular  $x^{-\frac{1-\alpha}{2}}$  of the equation than to the discontinuity of  $\bar{v}$ .

To illustrate the difficulties let us look at the toy-model of the  $\frac{2}{7}$  layer, for  $\alpha = \frac{1}{2}$  on  $(-1, 0)$ ,

$$x^{-\frac{1}{4}} w - \sqrt{E} \partial_x^2 w = f.$$

Note that we do not prescribe any boundary conditions, as they will be lifted by the boundary layers. Then, when trying to construct without precautions a solution of the form  $w = \sum_k E^{\frac{k}{2}} w^k$ . We obtain, even at the first step

- If the source term has the same singularity than the coefficient,  $f = x^{-\frac{1}{4}}$  then  $w^0 = 1$  and the remainder is 0.
- If  $f \in C^1$  and  $f(0) = 0$  then  $w^0(x) = x^{\frac{1}{4}} f(x) \sim_{0^-} x^{\frac{5}{4}}$  and  $w^0 \in H^1$ .
- But if  $f = 1$  then  $w^0(x) = x^{\frac{1}{4}}$  i.e  $w_0 \notin H^1$ .

Our goal is to have as a source term  $\kappa_0 v_0 + \kappa_1 v_1$ . Following the decomposition (3.44), we first isolate the discontinuous part of  $\bar{v}$ , which we lift thanks to boundary layers of sizes  $E^{\frac{1}{4}}$  and  $E^{\frac{1}{3+\alpha}}$ , as described previously. There remains to treat the part  $v^{cont}$ . This correspond to a source term  $f^{cont} = (\kappa_0 + \kappa_1) v^{cont}$ . Note that not only  $v^{cont}$  is continuous, but also  $v^{cont}(0) = 0$ .

We cannot use  $v^{cont}$  it the approximate solution up to  $x = 0^-$ , since it will lead to  $v^{cont} \sim_{0^-}$

$(-x)^{\frac{1-\alpha}{2}} \notin H^1$ , so we isolate the singularity in this source term, by writing

$$f^{cont} = \chi\left(\frac{x}{E^\beta}\right) f^{cont} + \chi\left(\frac{x}{E^\beta}\right) f^{cont}.$$

But if we cut-off at a size  $E^\beta$ , the error created when differentiating two times the cut-off will be of  $L^\infty$  size  $E^{-2\beta}$ , compensated by the smallness of  $v^{cont}$ , of  $(E^\beta)^{\frac{1-\alpha}{2}}$ . The critical scaling is then, without surprise

$$\underbrace{\text{smallness of the singular perturbation}}_{E^{\frac{1}{2}}} \times \underbrace{E^{-2\beta}}_{\text{cut-off}} \times \underbrace{E^{\beta \cdot \frac{1-\alpha}{2}}}_{\text{smallness of } v^{cont}} = 1 \Leftrightarrow \beta = \frac{1}{3+\alpha}.$$

**Proposition 3.5.5** (Lifting of  $f^{cont}$ ). *There exists  $w^{int} \in H^2$  such that*

$$f^{cont} - (\kappa_0 + \kappa_1)w^{int} - (\gamma_1 - \gamma_0)\sqrt{E}\partial_x^2 w^{int} = r$$

with

$$w^{int} \in H^2, \quad \lim_{E \rightarrow 0} \left\| w^{int} - \frac{f^{cont}}{\kappa_0 + \kappa_1} \right\|_{L^2} = 0.$$

The remainder satisfy

$$\|r\|_{L^2} \leq CE^{\frac{1}{6+2\alpha}}$$

*Proof.* In order to prove this proposition, let  $\chi$  be a cut-off function such that  $\chi = 1$  on  $(0, 1)$  and  $\chi = 0$  on  $(2, +\infty)$ . Then with

$$w^{int} = \frac{f^{cont}}{\kappa_0 + \kappa_1} \left( 1 - \chi\left(\frac{-x}{E^{\frac{1}{3+\alpha}}}\right) \right)$$

we obtain

$$\begin{aligned} f^{cont} - (\kappa_0 + \kappa_1)w^{int} - (\gamma_1 - \gamma_0)\sqrt{E}\partial_x^2 w^{int} &= \chi\left(\frac{-x}{E^{\frac{1}{3+\alpha}}}\right) f^{cont}(x) \\ &\quad - E^{\frac{1}{2}}(\gamma_1 - \gamma_0) \left( 1 - \chi\left(\frac{-x}{E^{\frac{1}{3+\alpha}}}\right) \right) \partial_x^2 \left( \frac{f^{cont}}{\kappa_0 + \kappa_1} \right) \\ &\quad + 2(\gamma_1 - \gamma_0)E^{\frac{1}{2} - \frac{1}{3+\alpha}} \chi' \left( \frac{-x}{E^{\frac{1}{3+\alpha}}} \right) \partial_x \left( \frac{f^{cont}}{\kappa_1 + \kappa_0} \right) \\ &\quad + (\gamma_1 - \gamma_0)E^{\frac{1}{2} - \frac{2}{3+\alpha}} \chi'' \left( \frac{-x}{E^{\frac{1}{3+\alpha}}} \right) \left( \frac{f^{cont}}{\kappa_1 + \kappa_0} \right) \end{aligned}$$

*Remark 3.5.9.* If we were to iterate the construction, the second term (in blue) is  $r^{int}$  and all the others (in green) are  $r^{bl}$ .

All the terms are of same size in  $L^2$  et us compute for example the size of  $r^{int}$ . We have

$$\|r^{int}\|_{L^2}^2 \leq CE \int_{(-1,1)} \mathbf{1}_{(-E^{\frac{1}{3+\alpha}}, 0)} |\partial_x^2(v^{cont})|^2 dx$$



and as  $v^{cont}$  is uniformly smooth, except near  $0^-$  where it admits a singular expansion

$$v^{cont} = (-x)^{\frac{1-\alpha}{2}} + a_1(-x)^{1-\alpha} + a_2(-x)^{\frac{3(1-\alpha)}{2}} + \dots$$

Thus, we finally obtain

$$\|r^{int}\|_{L^2} \leq C \max\left(E^{\frac{1}{2} + \frac{-\alpha-2-2\cdot 0}{6+2\alpha}}, E^{\frac{1}{2}}\right) = CE^{\frac{1}{6+2\alpha}}.$$

□

We now have  $u^{app} = v^{disc} + v^{bl,l} + v^{bl,r} + w^{int}$ . Let us comment on the associated remainders. There is no difficulties for the ones coming from the shears layers used to lift the discontinuity. Concerning the remainder in  $(1-\chi)\sqrt{E}\partial_x^2 v^{cont}$  the same type of computation as above shows that we do not gain additional power of  $E$  iterating the procedure. The **green** remainders in the previous proof can be considered as source terms for the next order boundary layer. It is possible that this process could be iterated.

Remainder		$r^{blue}$	$r^{green}$	$r^{bl,r}$
$L^2$ norm	$E^{\frac{1}{6+2\alpha}}$	$E^{\frac{1}{6+2\alpha}}$	$E^{\frac{3}{8}}$	$E^{\frac{3}{6+2\alpha}}$
Can be used as a source for the next order operator	No	Probably	Yes	Yes

Table 3.3 – The different remainders.

We can remark that, except for  $r^{blue}$  all terms can be used as source term for next order operators. This is unfortunate because the current expansion is not precise enough to ensure the  $H^1$  convergence. If this difficulty was absent, then with an order sufficient to have all remainders  $o_{L^2}(E^{\frac{1}{2}})$ , the stability estimate will ensure the  $H^1$  convergence, and in particular that  $u^E(x=0) \sim C_\alpha E^{\frac{1}{28}}$ , a fact of physical importance.

At this point of the construction, we have thus constructed corrector for  $\bar{v}$  such that  $v^{int}$  is continuous. But, if we go back to equation (3.32), we see that the associated  $\psi^{int}$  can have jumps across  $x=0$ , as we will explain below. Lifting these jumps is the purpose of the next subsection.

### 3.5.5 Stewartson $\frac{1}{3}$ layer

Indeed, if the previous shear layers are enough in order to construct  $v$ , the corresponding  $\bar{\psi}$  is

$$\bar{\psi} = \frac{\kappa_0 \kappa_1}{\sqrt{2}(\kappa_0 + \kappa_1)}(v_1 - v_0).$$

Thus, far from  $x=0$  there is no problem but there is a discontinuity,  $\bar{\psi}(0^-) = \frac{\kappa_1(0)}{\sqrt{2}}(v_1 - v_0)$  when  $\bar{\psi}(0^+) = \frac{\kappa_1(0)}{\sqrt{2}\left(1 + \frac{\kappa_1(0)}{\kappa_0(0^+)}\right)}(v_1 - v_0)$ .

In order to lift this discontinuity, related to the fact that  $\partial_z v = E^{\frac{3}{2}}\partial_x^4 \psi$  was neglected, we need another boundary layer. If we write the whole symbol of (3.32) its determinant is

$$\partial_z^2 + E^2 \partial_x^6 = 0$$

prompting the scaling

$$\zeta = \frac{x}{E^{\frac{1}{3}}}.$$

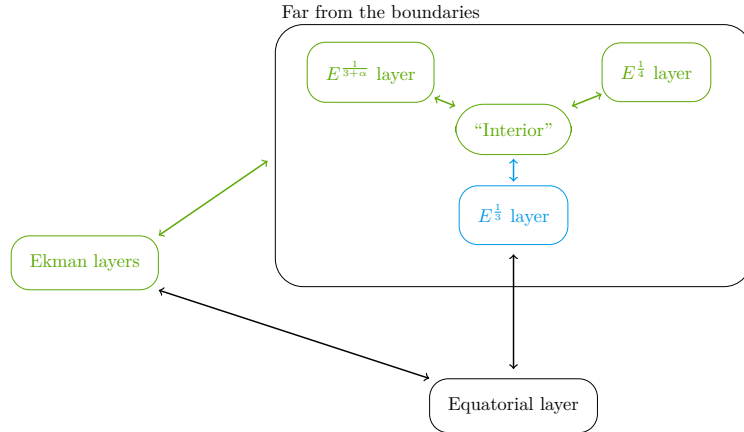


Figure 3.13 – Objective of Subsection 3.5.5 are in blue.

And the corresponding kernel leads to

$$\begin{aligned}\psi^{bl} &= \sqrt{E}\psi(\zeta, z) \\ v^{bl} &= E^{1/6}v(\zeta, z).\end{aligned}$$

*Remark 3.5.10.* Note that the  $\sqrt{E}$  is the exact scaling of the first order of the interior problem. This means that, if we consider the scaling and equation of equation (3.33) this factor disappears. The notable point is the  $E^{1/6}$  in front of the  $v$ . Indeed, this shows that any error created in the  $v$  variable will be small, leading to a possible correction with next order shear layers.

The boundary layer equation is thus

$$\begin{aligned}\partial_z v - \partial_\zeta^4 \psi &= 0 \\ \partial_z \psi + \partial_\zeta^2 v &= 0.\end{aligned}\tag{3.45}$$

This is in fact the same boundary layer than the one near a vertical wall, called the Stewartson layer. For the boundary condition we need the Ekman homogeneous condition. But as it write

$$(-1)^{i+1} \frac{1}{\kappa_i} \psi + \frac{1}{\sqrt{2}} \left( E^{1/6} v - \sqrt{E} \partial_z \psi \right) = 0\tag{3.46}$$

we will consider only the first order condition,  $\psi = 0$ . This approximation is only valid up to  $x^{\frac{1-\alpha}{2}} \sim E^{1/6}$  i.e  $x \sim E^{\frac{1}{3(1-\alpha)}} \ll E^{\frac{1}{3-\alpha}}$ , so such approximation is acceptable within our framework.

The goal of this subsection will be to show that the boundary layer equation

$$\begin{aligned}\partial_z v - \partial_\zeta^4 \psi &= 0 \\ \partial_z \psi + \partial_\zeta^2 v &= 0.\end{aligned}\tag{3.47}$$

with boundary condition  $\psi|_{z=\gamma_i} = 0$  can lift the discontinuity on  $\psi$  at  $x = 0$ .

Unfortunately such layer is deeply connected to the equatorial layer. Therefore, we will still try to lift discontinuity on  $\psi$ , but our proposition will only hold rigorously in the non degenerate setting,  $\kappa_0, \kappa_1$  smooth. For more explanations on the link between the vertical shear layer and the equatorial layer, see the discussion at the end of this chapter.

Let  $h = \gamma_1(0) - \gamma_0(0)$ . The main ingredient in the construction of said boundary layer is the flat case, i.e  $\gamma_0 = 0$ ,  $\gamma_1 = h$ , near a vertical wall:

**Proposition 3.5.6** (Flat Stewartson  $\frac{1}{3}$  layer near a wall). *Let  $\Omega_{flat} = (0, 1) \times (0, h)$ . Let  $0 < \eta < \frac{1}{2} \left(\frac{\pi}{h}\right)^{\frac{1}{3}}$  and  $s_0 \geq 1$ .*

*Let  $\Psi, V, \Upsilon \in H^{s_0}((0, h)) \times H^{s_0 - \frac{1}{3}}((0, h)) \times H^{s_0 - \frac{1}{3}}((0, h))$ . Suppose*

$$\begin{aligned} \Psi(0) &= \Psi(h) = 0 \\ \Upsilon(0) &= \Upsilon(h) = 0 \end{aligned} \tag{3.48}$$

and

$$\int_0^h V(z) dz = 0. \tag{3.49}$$

Let  $\mathbf{f} = (f_\psi, f_v)$  such that

$$(f_\psi)|_{z=0} = (f_\psi)|_{z=h} = 0, \quad \int_0^h f_v(\zeta, z) dz = 0 \tag{3.50}$$

with regularity

$$\mathbf{f} = (f_v, f_\psi) \in \mathcal{H}_\eta^{\max(0, s_0 - 2)} \times \mathcal{H}_\eta^{\max(0, s_0 - 4)}. \tag{3.51}$$

Then, with the boundary layer scaling  $\zeta = E^{-\frac{1}{3}}x$ , there exists

$$\mathbf{u}^{bl}(x, z) = \left( E^{\frac{1}{6}}v(\zeta, z), E^{\frac{1}{2}}\psi(\zeta, z) \right)$$

solution on  $\Omega_{flat}$  of

$$\begin{aligned} \partial_z v^{bl} - E\Delta^2 \psi^{bl} &= E^{\frac{1}{6}} \left( f_\psi \left( \frac{\cdot}{E^{\frac{1}{3}}}, \cdot \right) + r_\psi \left( \frac{\cdot}{E^{\frac{1}{3}}}, \cdot \right) \right) \\ \partial_z \psi^{bl} + E\Delta v^{bl} &= E^{\frac{1}{2}} \left( f_v \left( \frac{\cdot}{E^{\frac{1}{3}}}, \cdot \right) + r_v \left( \frac{\cdot}{E^{\frac{1}{3}}}, \cdot \right) \right) \end{aligned}$$

with boundary conditions

$$\begin{aligned} \psi|_{z=0} &= \psi|_{z=h} = 0 \\ (\psi^{bl}, v^{bl}, \partial_x \psi^{bl})|_{x=0} &= (E^{\frac{1}{2}}\Psi, E^{\frac{1}{6}}V, E^{\frac{1}{6}}\Upsilon) \\ \mathbf{u} &= 0 \text{ for } x > \frac{1}{2}. \end{aligned}$$

With the estimates

$$\begin{aligned} \|v^{bl}\|_{H^s(\Omega_h)} &\leq CE^{\frac{1}{6} + \frac{1}{6} - \frac{s}{3}}, \quad \forall s \leq s_0 + \frac{1}{6} \\ \|\psi^{bl}\|_{H^s(\Omega_h)} &\leq CE^{\frac{1}{2} + \frac{1}{6} - \frac{s}{3}}, \quad \forall s \leq s_0 - \frac{1}{6} \end{aligned}$$

and  $\mathbf{r}$  satisfying the estimates

$$\begin{aligned} \|r_v\|_{\mathcal{H}_\eta^s((0, +\infty) \times (0, h))} &\leq E^1, \quad \forall s \leq s_0 - \frac{13}{6} \\ \|r_\psi\|_{\mathcal{H}_\eta^s((0, +\infty) \times (0, h))} &\leq E^{\frac{1}{3}}, \quad \forall s \leq s_0 - \frac{23}{3}. \end{aligned}$$

*Remark 3.5.11.* The condition (3.48) on  $\psi$  is a compatibility condition between  $\psi|_{x=0}$  and  $\psi|_{z=0}$ . And the one on  $V$  is not restrictive. Indeed, we can lift constant  $V(x) = \bar{V}$  simply by

$$\begin{aligned}\psi(x, z) &= 0 \\ v(x, z) &= \bar{V}.\end{aligned}$$

Note that, in this case, this is not a boundary layer profile but rather an interior one. We will see in the proof that for a Fourier mode  $k$  in  $z$  the boundary layer size is actually  $(\frac{E}{k})^3$ , so this is no surprise that  $k = 0$  correspond to an interior term.

*Proof.* We will use a boundary layer term satisfying (3.47), and compute the remainders.

As, contrary to the boundary layer operator we have constructed until this subsection, this is a true PDE and not an ODE we will first focus on the homogeneous case  $\mathbf{f} = 0$ .

We can, as previously, rely either on an explicit computation or an abstract argument.

For the explicit computation, as  $\Psi(0) = \Psi(h) = 0$  and  $\int_0^h V dz = 0$ , let us use the decomposition

$$\begin{aligned}\psi(\zeta, z) &= \sum_{k=1}^{\infty} a_k(\zeta) \sin(\pi k h^{-1} z) \\ v(\zeta, z) &= \sum_{k=1}^{\infty} b_k(\zeta) \cos(\pi k h^{-1} z)\end{aligned}$$

Without loss of generality we can suppose that  $h = \pi$ .

Note that, for a non-flat  $\Omega$ , such functions does not verify exactly the boundary condition, as the boundary is not perfectly flat. Nevertheless, we will work as if  $\gamma_0$  and  $\gamma_1$  were constant, computing the trace remainder later.

The homogeneous equations on  $a_k, b_k$  are

$$\begin{aligned}-k b_k - \partial_{\zeta}^4 a_k &= 0 \\ k a_k + \partial_{\zeta}^2 b_k &= 0.\end{aligned}$$

This is a simple ODE, for each  $k \in \mathbb{N}$ , and the characteristic polynomial is

$$\lambda^6 - k^2$$

with three roots with negative real part, and three with positive real part. In particular there is no  $\lambda = 0$  modes thanks to the boundary conditions at  $z = \gamma_i$ .

Thus, with

$$\lambda_l = \begin{cases} -1 & \text{if } l = 0 \\ \frac{-1 \pm l\sqrt{3}i}{2} & \text{if } l = \pm 1 \end{cases}$$

we obtain that any functions

$$\begin{aligned}\psi &= \sum_{k=1}^{\infty} \sum_{l \in \{-1, 0, +1\}} a_{k,l} e^{\lambda_l |k|^{\frac{1}{3}} \zeta} \sin(kz) \\ v &= \sum_{k=1}^{\infty} \sum_{l \in \{-1, 0, +1\}} b_{k,l} e^{\lambda_l |k|^{\frac{1}{3}} \zeta} \cos(kz)\end{aligned}$$

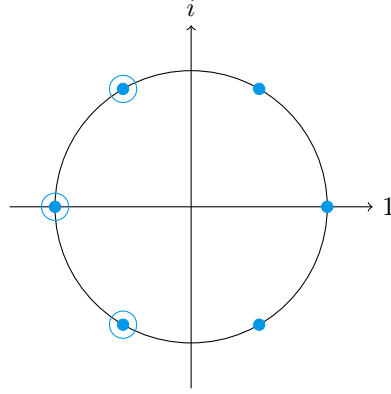


Figure 3.14 – The modes of the Stewartson boundary layer operator.

is solution, as long as

$$\forall k \in \mathbb{N}, \forall l \in \{-1, 0, 1\}, \quad b_{k,l} + \frac{k^{\frac{1}{3}}}{\lambda_l^2} a_{k,l} = 0. \quad (3.52)$$

*Remark 3.5.12.* It is important to observe that the size of the boundary layer is not  $E^{\frac{1}{3}}$  but  $\left(\frac{E}{k}\right)^{\frac{1}{3}}$ . Thus, at high frequency this layer is too small to satisfy the assumption presiding the derivation of the model. This was to be expected as the critical scaling is

$$\left(\frac{E}{k}\right)^{\frac{1}{3}} \geq E^{\frac{1}{3-\alpha}} \Leftrightarrow k \leq E^{\frac{\alpha}{3-\alpha}}$$

i.e variation at scales  $z \geq E^{\frac{\alpha}{3-\alpha}}$ . This is exactly the equatorial scaling (3.31), as  $z = x^\alpha$ . This is why this layer, called the similarity sublayer since its size can be seen also as  $(Ez)^{\frac{1}{3}}$ , is a candidate for the connection between the equatorial zone and the main flow.

More precisely, with  $V(z) = \sum_{k \geq 1} V_k \cos(kz)$ ,  $\Psi(z) = \sum_{k \geq 1} \Psi_k \sin(kz)$ ,  $\Upsilon = \sum_{k \geq 1} \Upsilon_k \cos(kz)$  we obtain for all  $k \geq 1$

$$\begin{aligned} \begin{pmatrix} a_{k,-1} \\ a_{k,0} \\ a_{k,1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ \frac{(1-\sqrt{3}i)k^{\frac{1}{3}}}{2} & -k^{\frac{1}{3}} & \frac{(1+\sqrt{3}i)k^{\frac{1}{3}}}{2} \\ \frac{(-1-\sqrt{3}i)k^{\frac{1}{3}}}{2} & -k^{\frac{1}{3}} & \frac{(-1+\sqrt{3}i)k^{\frac{1}{3}}}{2} \end{pmatrix}^{-1} \begin{pmatrix} \Psi_k \\ V_k \\ \Upsilon_k \end{pmatrix} \\ &= \Psi_k \begin{pmatrix} \frac{i}{\sqrt{3}} \\ 1 \\ -\frac{i}{\sqrt{3}} \end{pmatrix} + \frac{1}{k^{\frac{1}{3}}} V_k \begin{pmatrix} \frac{3-i\sqrt{3}}{6} \\ -1 \\ \frac{3+i\sqrt{3}}{6} \end{pmatrix} + \frac{1}{k^{\frac{1}{3}}} \Upsilon_k \begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \\ \frac{-1-\sqrt{3}i}{2} \end{pmatrix}. \end{aligned}$$

The computation of the size in Sobolev norms is then explicit, for  $\eta < \frac{1}{2}$ ,

$$\begin{aligned} \int_0^\infty \int_0^\pi e^{2\eta\zeta} \left| \partial_\zeta^{m_\zeta} \partial_z^{m_z} \psi \right|^2 dz d\zeta &\leq C \sum_{k \geq 1} \int_0^\infty \sum_l \left| a_{k,l} k^{\frac{(m_\zeta + 3m_z)}{3}} e^{(\lambda_l k^{\frac{1}{3}} - \eta)z} \right|^2 dz d\zeta \\ &\leq C \sum_{k \geq 1} \frac{k^{\frac{2(k_\zeta + 3k_z)}{3}}}{k^{\frac{1}{3}} - 2\eta} \left( |\Psi_k|^2 + \frac{1}{k^{\frac{2}{3}}} |V_k|^2 + \frac{1}{k^{\frac{2}{3}}} |\Upsilon_k|^2 \right). \end{aligned}$$

*Remark 3.5.13.* Notice that, once more, the limiting regularity is with respect to the tangential variable  $z$ . A more precise result could be obtained by considering anisotropic Sobolev spaces. But as the final norms for the whole problem are isotropic we will not do so.

So for  $s \in \mathbb{N}$

$$\|\psi e^{\eta\zeta}\|_{H^s((0,1) \times (0,+\infty))} \leq C_\eta \left( \|\Psi\|_{H^{s-\frac{1}{6}}((0,\pi))} + \|V\|_{H^{s-\frac{1}{2}}((0,\pi))} + \|\Upsilon\|_{H^{s-\frac{1}{2}}((0,\pi))} \right)$$

*Remark 3.5.14.* Note that, since the high frequencies in  $z$  are localized in a region of size  $\frac{1}{k^{\frac{1}{3}}}$ , we obtain in fact a slightly better ( $+\frac{1}{6}$ ) regularity than expected.

We obtain similar results for  $v$ , noticing that  $b_{k,l} = O(k^{\frac{1}{3}} a_{k,l})$ .

And the remainders are, in boundary layer variables

$$r_v = E \partial_z^2 v = -E \sum_{k \geq 1} \sum_l \lambda_l^{-2} k^2 k^{\frac{1}{3}} a_{k,l} e^{\lambda_l k^{\frac{1}{3}} \zeta} \cos(kz)$$

so, as  $2 \cdot (2 + \frac{1}{3}) - \frac{1}{3} = 13/3$ ,

$$\|r_v e^{\eta\zeta}\|_{H^s((0,1) \times (0,+\infty))}^2 \leq C_\eta E^2 \sum_{k \geq 1} k^{2s + \frac{13}{3}} \left( |\Psi_k|^2 + \frac{1}{k^{\frac{2}{3}}} |V_k|^2 + \frac{1}{k^{\frac{2}{3}}} |\Upsilon_k|^2 \right).$$

and similarly, as  $r_\psi$  is due to the fact that we neglect  $E \partial_z^4 + E^{1-\frac{2}{3}} \partial_z^2 \partial_\zeta^2$

$$\|r_\psi e^{\eta\zeta}\|_{H^s}^2 \leq C_\eta \sum_{k \geq 1} \left( E^2 k^{2s+8-\frac{1}{3}} + E^{\frac{2}{3}} k^{2s+4+\frac{4}{3}-\frac{1}{3}} \right) \left( |\Psi_k|^2 + \frac{1}{k^{\frac{2}{3}}} |V_k|^2 + \frac{1}{k^{\frac{2}{3}}} |\Upsilon_k|^2 \right).$$

After the cut-off and the scaling (see Lemmas A.5.1, A.5.2) we obtain the announced estimates. For an source term  $\mathbf{f} \neq 0$ , one can simply solve the ODE with non zero right hand side

$$\begin{aligned} -kb_k - \partial_\zeta^4 a_k &= f_{\psi,k}(\zeta) \\ ka_k + \partial_\zeta^2 b_k &= f_{v,k}(\zeta) \end{aligned}$$

where  $f_{\psi,k}, f_{v,k}$  are the decomposition in sines and cosines of  $f_v, f_\psi$  with respect to  $z$ .

But let us treat the case of arbitrary  $\mathbf{f}$  and homogeneous traces with a more abstract framework. Indeed, this will show us that the well-posedness of this system is not due to the explicit Fourier transform and ODE resolution, but rather to the boundedness in the tangential variable  $z$ .

Note that the estimates on regularity and decay obtained are not as good as the ones coming from the previous explicit resolution. But this is more robust as  $(0,1) \times (0,+\infty)$  can be replaced

by most domains bounded in one direction.

These methods are very close to the ones used for the equatorial Ekman layer, and are detailed in Chapter 4. Therefore, we will briefly use them and insist on the simplifications stemming from the boundedness in  $z$ .

First, multiplying the equation by  $(\psi, v)$  we obtain the a-priori estimate

$$\underbrace{\int_0^\infty \int_0^1 \partial_z(v\psi) dz d\zeta}_{=0} + \int_0^\infty \int_0^1 |\partial_\zeta v|^2 dz d\zeta + \int_0^\infty \int_0^1 |\partial_\zeta^2 \psi|^2 dz d\zeta \leq \int_0^\infty \int_0^1 \mathbf{f} \cdot \mathbf{u}.$$

If  $\mathbf{f}$  has sufficient decay this allow to us to conclude the existence in the space

$$\int |\partial_\zeta v|^2 + \int |\partial_\zeta^2 \psi|^2 < +\infty$$

and Hardy inequality allow us to obtain the decay

$$\int \left( |\partial_\zeta v|^2 + \left| \frac{v}{\zeta} \right|^2 \right) + \int \left( |\partial_\zeta^2 \psi|^2 + \left| \frac{\psi}{\zeta^2} \right|^2 \right) < C \int (|\zeta f_v|^2 + |\zeta^2 f_\psi|^2) < +\infty.$$

But this not enough to obtain a small scaling or the uniqueness. Indeed, need at least a  $L^2$  decay at  $\zeta \rightarrow \infty$ .

As this is an elliptic problem in  $\zeta$  we can derive Caccioppoli inequalities. For a cut-off function  $\chi \in C_c^\infty$  such that  $\chi(\zeta) = 0$  for  $\zeta < \frac{1}{2}$  and  $\zeta > M + 1$ , and  $\chi(\zeta) = 1$  for  $1 < \zeta < M$  we have

$$\int \chi(\zeta)^4 (|\partial_\zeta^4 \psi|^2 + |\partial_\zeta^3 v|^2) \leq C \left( \int (|\partial_\zeta v|^2 + |\partial_\zeta^2 \psi|^2) + \int (|\zeta(\partial_\zeta^2 f_v + f_v)|^2 + |\zeta^2(\partial_\zeta^2 f_\psi + f_\psi)|^2) \right).$$

And using the equation

$$\begin{aligned} \partial_z v &= \partial_\zeta^4 \psi + f_\psi \\ \partial_z \psi &= -\partial_\zeta^2 v + f_v \end{aligned}$$

we thus obtain, for any  $\eta > 0$

$$\int_1^M \int_0^1 (|\partial_z v|^2 + |\partial_z \psi|^2) dz d\zeta \leq \underbrace{C}_{\text{uniform in } M} \left( \int_\Omega (|\partial_\zeta v|^2 + |\partial_\zeta^2 \psi|^2) + \int e^{\eta\zeta} |\partial_\zeta^2 \mathbf{f}|^2 \right).$$

As  $\psi$  has Dirichlet conditions on  $z$ , we can conclude by Poincaré inequality with respect to the  $z$  variable

$$\int_1^\infty \int_0^1 |\psi|^2 d\zeta dz \leq C \int_1^\infty \int_0^1 |\partial_z \psi|^2 d\zeta dz < +\infty.$$

Using Poincaré inequality again, but with respect to  $\zeta \in (0, 1)$  we have

$$\int_0^1 \int_0^1 |\psi|^2 \leq C \int_0^1 \int_0^1 |\partial_\zeta \psi|^2 d\zeta dz < +\infty.$$

Combining the two estimates, we obtain a control of the  $L^2$  norm,  $\int_0^\infty \int_0^1 |\psi|^2 dz d\zeta < +\infty$ .

For  $v$ , we have

$$\partial_\zeta^2 \int_0^1 v(\zeta, z) dz = \int_0^1 \partial_\zeta^2 v dz = - \underbrace{\int_0^1 \partial_z \psi dz}_{\psi(\zeta, 1) - \psi(\zeta, 0) = 0} + \underbrace{\int_0^1 f_v dz}_{=0}$$

so, as  $\int_0^1 v(0, z) dz = 0$  we obtain  $\int_0^1 v(\zeta, z) dz = a\zeta + 0$ . But the growth is controlled using the fact that  $L^1((0, 1)) \subset L^2((0, 1))$  and a Hardy inequality

$$\left| \frac{1}{\zeta} \int_0^1 v(\zeta, z) dz \right| \leq C \sqrt{\int_0^1 \left| \frac{v(\zeta, z)}{\zeta} \right|^2 dz} \in L^2((0, +\infty)).$$

The only possibility is thus  $a = 0$ . For almost any  $\zeta$ ,  $\int_0^1 v(\zeta, z) dz = 0$ , therefore we can also use a Poincaré-Wirtinger inequality to conclude

$$\int_1^\infty \int_0^1 |v|^2 d\zeta dz \leq C \int_1^\infty \int_0^1 |\partial_z v|^2 \leq C.$$

To obtain finally  $\|v\|_{L^2((0, \infty) \times (0, 1))} \leq C$ .

This  $L^2$  control of the solution is very useful: it ensures decay of the solution and is used to rigorously prove uniqueness. In fact, if  $\mathbf{u}$  is a valid test function, then the energy estimate leads to uniqueness for 0 source term and homogeneous boundary conditions, the general uniqueness follows from linearity. But, as detailed in Section 4.3 of the next chapter, the key obstacle to such analysis is the integrability.

Note that we can in fact recover more than just  $L^2$  decay. Indeed, as the Caccioppoli inequalities are localized in  $\zeta$  and Poincaré inequalities are pointwise in  $\zeta$ , we have

$$\int_1^M \int_0^1 (|v^2| + |\psi^2|) \leq \underbrace{C}_{\text{uniform in } M} \int_0^{M+1} \int_0^1 (|\partial_\zeta v|^2 + |\partial_\zeta^2 \psi|^2 + e^{\eta\zeta} |\partial_\zeta^2 f|^2).$$

*Remark 3.5.15.* The only dependency of  $C$  is with respect to the size of the domain, through the use of Poincaré inequality. If the domain is  $(0, \infty) \times (0, h)$ , then

$$\int_1^M \int_0^h (|v^2| + |\psi^2|) \leq C_1(1 + h^2) \int_0^{M+1} \int_0^h (|\partial_\zeta v|^2 + |\partial_\zeta^2 \psi|^2 + e^{\eta\zeta} |\partial_\zeta^2 f|^2)$$

This shows that the  $L^2$  analysis is no longer possible if  $h = +\infty$ . Moreover we will see that the constant  $C$  leads the decay of the boundary layer, so the rate of decay will decrease when  $h$  increase.

For an arbitrary weight  $\omega$ , the energy inequality formally obtained by taking  $\mathbf{u}\omega$  as a test function is, if well defined,

$$\int (-v\psi\partial_z\omega) + \int \left( |\partial_\zeta^2 \psi|^2 \omega - 2|\partial_\zeta \psi|^2 \partial_\zeta^2 \omega + \frac{1}{2} |\psi|^2 \partial_\zeta^4 \omega \right) + \int \left( |\partial_\zeta v|^2 \omega - \frac{1}{2} |v|^2 \partial_\zeta^2 \omega \right) \leq \int (\mathbf{f} \cdot \mathbf{u}) \omega.$$

Thus, for  $\lambda > 0$  and  $\chi_M$  a cut-off function such that  $\chi_M(\zeta) = 1$  for  $\zeta < M$  and is zero for  $\zeta > M + 1$ , taking  $e^{\lambda\zeta} \chi_M$  as a weight,



$$\begin{aligned}
\int_0^M \int_0^1 (|\partial_\zeta^2 \psi|^2 + |\partial_\zeta v|^2) e^{\lambda\zeta} dz d\zeta &\leq C \int_M^{M+1} \int_0^1 (|\partial_\zeta \psi|^2 + |\psi|^2 + |v|^2) e^{\lambda\zeta} dz d\zeta \\
&+ C\lambda \int_0^{M+1} \int_0^1 (|\partial_\zeta \psi|^2 + |\psi|^2 + |v|^2) e^{\lambda\zeta} dz d\zeta \\
&+ \frac{C}{\lambda} \int_0^{M+1} \int_0^1 |f|^2 e^{\lambda\zeta} dz d\zeta
\end{aligned}$$

where  $C$  is uniformly bounded when  $M \rightarrow +\infty$  and  $\lambda \rightarrow 0$ . So with

$$\begin{aligned}
E_0(M) &= \int_0^M \int_0^1 (|\partial_\zeta \psi|^2 + |\psi|^2 + |v|^2) e^{\lambda\zeta} dz d\zeta \\
E_1(M) &= \int_0^M \int_0^1 (|\partial_\zeta^2 \psi|^2 + |\partial_\zeta v|^2) e^{\lambda\zeta} dz d\zeta \\
F(M) &= \int_0^M \int_0^1 |f|^2 e^{\lambda\zeta} dz d\zeta
\end{aligned}$$

this can be rewritten as

$$E_1(M) \leq C(E_0(M+1) - E_0(M)) + C\lambda E_1(M+1) + \frac{C}{\lambda} F(M+1). \quad (3.53)$$

The important point is that we are able to control  $E_0$  by  $E_1$ . More precisely, thanks to the localized Caccioppoli

$$\begin{aligned}
\int_m^{m+1} \int_0^1 (|\partial_\zeta \psi|^2 + |\psi|^2 + |v|^2) e^{\lambda\zeta} dz d\zeta &\leq e^{\lambda m} \int_m^{m+1} \int_0^1 (|\partial_\zeta \psi|^2 + |\psi|^2 + |v|^2) dz d\zeta \\
&\leq C e^{\lambda m} \int_{m-1}^{m+2} \int_0^1 (|\partial_\zeta^2 \psi|^2 + |\partial_\zeta v|^2) dz d\zeta \\
&\leq C e^\lambda \int_{m-1}^{m+2} \int_0^1 (|\partial_\zeta^2 \psi|^2 + |\partial_\zeta v|^2) e^{\lambda\zeta} dz d\zeta
\end{aligned}$$

meaning that

$$E_0(M+1) - E_0(M) \leq C(E_1(M+2) - E_1(M-1)).$$

So combining this inequality with (3.53) and the monotony of  $M \mapsto E_1(M)$  we obtain

$$E_1(M) \leq \mathcal{C}(E_1(M+3) - E_1(M)) + \mathfrak{C}\lambda E_1(M+3) + \frac{C}{\lambda} F(M+3).$$

*Remark 3.5.16.* Note that we do not explicitly compute the constants. The main point is that they are uniformly bounded near  $M = 0$  and  $\lambda = 0$ . The exact value of  $\mathcal{C}$  and  $\mathfrak{C}$  will prescribe the rate of decay we can obtain, but here we are only trying to obtain any rate  $\lambda > 0$ .

$$E_1(M) \leq \underbrace{\left( \frac{\mathcal{C}}{1+\mathcal{C}} + \mathfrak{C}\lambda \right)}_{q_\lambda} E_1(M+3) + \frac{C}{\lambda} F(M+3).$$

So, with  $M$  fixed, for all  $K \in \mathbb{N}$ , using the fact that  $E_1(N) \leq C_\lambda e^{2\lambda N}$

$$\begin{aligned} E_1(M) &\leq q_\lambda^K E_1(M + 3K) + \frac{C}{\lambda} \sum_{k=1}^K q_\lambda^{K-k} F(M + 3k) \\ &\leq C_\lambda (q_\lambda e^{6\lambda})^K e^{2\lambda M} + \frac{C}{\lambda} \sum_{k=1}^K q_\lambda^{K-k} F(\infty). \end{aligned}$$

As a consequence, if  $\lambda > 0$  is small enough, since  $q_\lambda e^{6\lambda} < 1$  and  $F(\infty) < +\infty$ , we can deduce that  $E_1(M)$  is bounded (uniformly in  $M$ ), i.e that

$$\int_0^\infty \int_0^1 (|\partial_\zeta^2 \psi|^2 + |\partial_\zeta v|^2) e^{\lambda \zeta} dz d\zeta = \|v\|_{\mathcal{H}_\lambda^1}^2 + \|\psi\|_{\mathcal{H}_\lambda^2}^2 < +\infty.$$

□

Let us comment on the trace remainders. Indeed, in the previous boundary layer operator, we considered  $\gamma_0$  and  $\gamma_1$  to be constant. This is not the case, so the trace condition are not exactly satisfied. As  $\gamma_1$  is smooth, given the decay of the profile we have, with  $\psi^k$  corresponding to the  $k$ -mode in the previous Fourier decomposition

$$\begin{aligned} |\psi_{z=\gamma_1}^{bl,k}(x)| &\leq C \left( \sum_l |a_{k,l}| \right) E^{\frac{1}{2}} e^{-\eta k^{\frac{1}{3}} x E^{-\frac{1}{3}}} k (\gamma_1(x) - \gamma_1(0)) \\ &\leq C \left( \sum_l |a_{k,l}| \right) k^{\frac{2}{3}} E^{\frac{1}{2} + \frac{1}{3}} \|\gamma_1'\|_{L^\infty}. \end{aligned}$$

And as  $\gamma_0(x) - \gamma_0(0) \sim x^\alpha \mathbf{1}_{x < 0}$  we have similarly

$$\begin{aligned} |\psi_{z=\gamma_0}^{bl,k}(x)| &\leq C \left( \sum_l |a_{k,l}| \right) E^{\frac{1}{2}} e^{-\eta k^{\frac{1}{3}} x E^{-\frac{1}{3}}} k |x|^\alpha \\ &\leq C \left( \sum_l |a_{k,l}| \right) k^{1 - \frac{\alpha}{3}} E^{\frac{1}{2} + \frac{\alpha}{3}}. \end{aligned}$$

Using the fact that  $\sum_l |a_{k,l}| \leq C(|\Psi_k| + k^{\frac{1}{3}}|V^k| + k^{-\frac{1}{3}}|\Upsilon^k|)$ , we can then estimate precisely the error at  $z = \gamma_i(x)$ . But, as the way to correct these traces errors is through a  $\frac{1}{3}$ -boundary layer, the quantity of interest is

$$\left\| \psi_{|z=\gamma_i(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^s(0,+\infty)}.$$

for  $\tilde{\eta} < \eta$ . Note the choice of  $s$  instead of  $s - \frac{1}{2}$  as the limiting regularity is the vertical one. Therefore, lifting a trace  $t_i \in H^s((0, \infty))$  by  $\chi_i(z)t_i(\zeta)$  with  $\chi_i$  a  $C^\infty$  function is more than enough in term of regularity. A crude estimate leads to

$$\begin{aligned} \left\| \psi_{|z=\gamma_i(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^s(0,+\infty)} &\leq C_{\tilde{\eta},\eta} E^{\frac{1}{3}} \sum_k \sum_l |a_{k,l}| k^{\frac{2+s}{3}} \\ &\leq C_{\tilde{\eta},\eta} E^{\frac{1}{3}} \sqrt{\sum_k k^{2s_0} (|\Psi_k|^2 + k^{\frac{2}{3}}|V^k|^2 + k^{-\frac{2}{3}}|\Upsilon^k|^2)} \sqrt{\sum_k k^{2(\frac{2+s}{3} - s_0)}}. \end{aligned}$$

So we obtain a acceptable estimate as long as  $\frac{2+s}{3} - s_0 < -\frac{1}{2}$  i.e  $s < 3s_0 - \frac{7}{2}$ .

*Remark 3.5.17.* If  $s_0$  is large enough we obtain more regularity than  $s_0$ , but this is not a surprise. It is simply linked with the strong anisotropy of the regularity, and with the fact that higher frequencies decay faster than the lower ones.

We can do a similar analysis for  $\gamma_0$ . The main difference is that near  $\zeta = 0$  the trace admits a singularity like  $\zeta^\alpha$ . So the estimates on  $(\zeta_0, +\infty)$ ,  $\zeta_0 > 0$  are of the same nature for any  $s > 0$ , but we can only derive  $\mathcal{H}_{\tilde{\eta}}^0$  estimate near 0

$$\begin{aligned} \left\| \psi \Big|_{z=\gamma_0(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^0(0,+\infty)} &\leq C_{\tilde{\eta},\eta} E^{\frac{\alpha}{3}} \sum_k \sum_l |a_{k,l}| k^{\frac{3-\alpha}{3}} \\ &\leq C_{\tilde{\eta},\eta} E^{\frac{\alpha}{3}} \sqrt{\sum_k k^{2s_0} \left( |\Psi_k|^2 + k^{\frac{2}{3}} |V_k|^2 + k^{-\frac{2}{3}} |\Upsilon_k|^2 \right)} \sqrt{\sum_k k^{2(1-\frac{\alpha}{3}-s_0)}}. \end{aligned}$$

We have similar estimates for the traces of  $v$

$$\left\| v \Big|_{z=\gamma_1(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^s(0,+\infty)} \leq C_{\tilde{\eta},\eta} \left( \|\Psi\|_{H^{s_0}} + \|V\|_{H^{s_0-\frac{1}{3}}} + \|\Upsilon\|_{H^{s_0-\frac{1}{3}}} \right) \sqrt{\sum_k k^{2\left(\frac{3+s}{3}-s_0\right)}}$$

and for  $s = 0$ , at  $z = \gamma_0$

$$\left\| v \Big|_{z=\gamma_0(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^0(0,+\infty)} \leq C_{\tilde{\eta},\eta} \left( \|\Psi\|_{H^{s_0}} + \|V\|_{H^{s_0-\frac{1}{3}}} + \|\Upsilon\|_{H^{s_0-\frac{1}{3}}} \right) \sqrt{\sum_k k^{2\left(\frac{4-\alpha}{3}-s_0\right)}}.$$

*Remark 3.5.18.* For  $v$  the derivative is not as singular as one can expect. Indeed the worse term is  $\cos(kE^{\frac{\alpha}{3}}\zeta^\alpha)$ . So instead of a  $\zeta^\alpha$  singularity we obtain a  $\zeta^{2\alpha}$ , which is in  $H^1(0,1)$  for  $\alpha > \frac{1}{4}$ . As a consequence, for  $\alpha > \frac{1}{4}$  we can obtain estimates in  $\mathcal{H}_{\tilde{\eta}}^1$  as long as  $s_0 \geq 2$ .

And for  $\partial_n \psi$  we also obtain

$$\begin{aligned} \left\| \partial_n \psi \Big|_{z=\gamma_1(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^s(0,+\infty)} &\leq C \quad \forall s < 3s_0 - \frac{13}{2} \\ \left\| \partial_n \psi \Big|_{z=\gamma_0(E^{\frac{1}{3}}\zeta)} \right\|_{\mathcal{H}_{\tilde{\eta}}^0(0,+\infty)} &\leq C \quad \forall s_0 > \frac{5}{2} - \frac{\alpha}{3}. \end{aligned}$$

So with  $s_0 \geq 4$  we obtain sufficient estimate on the traces on  $v$ ,  $\psi$ ,  $\partial_n \psi$  to bootstrap the construction to get as close as wanted to the exact boundary condition (3.46)

$$(-1)^{i+1} \frac{1}{\kappa_i} \psi + \frac{1}{\sqrt{2}} \left( E^{\frac{1}{6}} v - \sqrt{E} \partial_z \psi \right) = 0.$$

**Proposition 3.5.7.** *Let  $\gamma_0, \gamma_1$  be two smooths functions. Let  $K \in \mathbb{N}$ .*

*Then under the exact same hypotheses as Proposition 3.5.6, with the modifications*

- $\Omega_{flat} = (0,1) \times (0,h)$  is replaced by  $\Omega = \{(x,z) \text{ s.t } 0 \leq x \leq 1, \gamma_0(x) \leq z \leq \gamma_1(x)\}$ .
- $s_0 \geq 1$  is replaced by  $s_0 \geq 4$ .

*we obtain the same results as Proposition 3.5.6, except for the boundary condition (3.5.6),  $\psi|_{z=0} =$*

$\psi|_{z=h} = 0$ , which is replaced by,  $\forall \tilde{\eta} < \eta$

$$\left\| (-1)^{i+1} \frac{1}{\kappa_i} \psi + \frac{1}{\sqrt{2}} \left( E^{\frac{1}{6}} v - \sqrt{E} \partial_n \psi \right) \right\|_{\mathcal{H}_{\tilde{\eta}}^{s_0}} \leq C E^{\frac{\kappa+1}{6}}.$$

*Proof.* For the construction, let  $\varepsilon > 0$  small enough,  $\chi_0, \chi_1$  be  $C^\infty$  function such that

$$\begin{aligned} \chi_i|_{z=\gamma_i} &= 1 \\ \chi_i|_{z=\gamma_{1-i}} &= 0. \end{aligned}$$

The proof is a simple induction on  $K$ . For  $K = 0$  this is only the construction of Proposition 3.5.6 and the estimate of the trace remainder.

For  $K \geq 0$  let us suppose we can construct such approximate solution  $(v^K, \psi^K)$ , with  $T_i^K = (-1)^{i+1} \frac{1}{\kappa_i} \psi^K + \frac{1}{\sqrt{2}} \left( E^{\frac{1}{6}} v^K - \sqrt{E} \partial_n \psi^K \right) \in \mathcal{H}_{\tilde{\eta}+\varepsilon}^{s_0}(\Gamma_i)$  such that  $\|T^k\| \leq C E^{\frac{\kappa+1}{6}}$ .

Then let

$$\psi^l = \kappa_0 T_0^K \chi_0 - \kappa_1 T_1^K \chi_1.$$

We have  $(v^K, \psi^K) + (0, \psi^l)$  satisfying exactly (3.5.6). And the error  $\mathbf{f}^l$  generated in the equation by adding this lifting is such that

$$\|\mathbf{f}^l\|_{\mathcal{H}_{\tilde{\eta}+\varepsilon/2}^{s_0-2} \times \mathcal{H}_{\tilde{\eta}+\varepsilon/2}^{s_0-4}} \leq C \|\psi^l\|_{\mathcal{H}_{\tilde{\eta}+\varepsilon/2}^{s_0}} \leq C E^{\frac{\kappa+1}{6}}.$$

When the solve the boundary layer equation with  $V = \Psi = \Upsilon = 0$  at  $\zeta = 0$ ,  $\psi = 0$  at the top and the bottom and with  $\mathbf{f}^l$  as a source term. And finally with  $\mathbf{u}^l$  such solution we have

$$\mathbf{u}^{K+1} = \mathbf{u}^K + \underbrace{\mathbf{u}^l}_{\text{lift the trace}} + \underbrace{\mathbf{u}^{ll}}_{\text{lift the error created by } u^l}$$

satisfies the equation up to an acceptable remainder, and has traces

$$\|T^{K+1}\|_{\mathcal{H}_{\tilde{\eta}}^{s_0}} = \left\| (-1)^{i+1} \frac{1}{\kappa_i} \underbrace{\psi^{ll}}_0 + \frac{1}{\sqrt{2}} \left( E^{\frac{1}{6}} v^{ll} - \sqrt{E} \partial_n \psi^{ll} \right) \right\|_{\mathcal{H}_{\tilde{\eta}}^{s_0}} \leq C E^{\frac{1}{6}} \|u^{ll}\|_{\mathcal{H}_{\tilde{\eta}+\varepsilon/2}^{s_0}} \leq C E^{\frac{\kappa+1}{6}} E^{\frac{1}{6}}.$$

□

*Remark 3.5.19.* In fact, using the previous remark, for  $\alpha > \frac{1}{4}$ , the source term corresponding to this lifting,  $\mathbf{f} = (f_v, f_\psi)$  is in  $\mathcal{H}_{\tilde{\eta}}^1 \times \mathcal{H}_{\tilde{\eta}}^0$ . So the loss of regularity is completely compensated by the elliptic regularity, since  $\mathbf{f} \in \mathcal{H}_{\tilde{\eta}}^1 \times \mathcal{H}_{\tilde{\eta}}^0$  is enough to consider  $s_0 = 3$ .

But, our goal is not to construct a boundary layer near a vertical wall. It is to lift the discontinuity of  $\bar{\psi}$  without creating another discontinuity. The idea is to consider an approximate solution  $\mathbf{u} = \mathbf{u}_- \mathbf{1}_{x < 0} + \mathbf{u}_+ \mathbf{1}_{x > 0}$ , and it seems that the conditions in order to have  $(\bar{v}, \bar{\psi}) + (u, \psi)$  in the right functional space is

$$[\psi]_{x=0} = -[\bar{\psi}]_{x=0} \quad [v]_{x=0} = -[\bar{v}]_{x=0} \quad [\partial_x \psi]_{x=0} = -[\partial_x \bar{\psi}]_{x=0}$$

i.e, 3 conditions when we have  $3+3 = 6$  degrees of freedom. Using only this approach our problem is under-determined. Nevertheless, we want not only to be in the right functional space, but also to have a small remainder. And when computing  $(\partial_z(v + \bar{v}) - \Delta^2(\psi + \bar{\psi}), \partial_z(\psi + \bar{\psi}) + \Delta(v + \bar{v}))$ ,

due to the construction of  $\mathbf{u}$ , terms like

$$\begin{aligned} r_\psi &= [\partial_x^3(\psi + \bar{\psi})]_{x=0} \underbrace{\delta_{x=0}}_{\in H^{-2}} + [\partial_x^2(\psi + \bar{\psi})]_{x=0} \underbrace{\delta'_{x=0}}_{\in H^{-2}} + \underbrace{[\partial_x(\psi + \bar{\psi})]_{x=0}}_{=0} \underbrace{\delta''_{x=0}}_{\notin H^{-2}} + \underbrace{[(\psi + \bar{\psi})]_{x=0}}_{=0} \underbrace{\delta_{x=0}^{(3)}}_{\notin H^{-2}} \\ r_v &= [\partial_x(v + \bar{v})]_{x=0} \underbrace{\delta_{x=0}}_{\in H^{-1}} + \underbrace{[(v + \bar{v})]_{x=0}}_{=0} \underbrace{\delta'_{x=0}}_{\notin H^{-1}} \end{aligned}$$

appears. If the term in red must be exactly 0 to have a remainder in an acceptable space, the ones in blue are also problematic. In fact, the Definition 3.2.2 of an approximate solution allows the remainder to be in negative Sobolev spaces, but the trade-off is that their norms must be small, for example we need  $\|r_v\|_{L^2}$  or  $E^{-1}\|r_v\|_{H^{-1}}$  to be smaller than the energy norm of the approximate solution. But  $\delta_0 \notin L^2$ , so we need to take  $[\partial_x(v + \bar{v})]_{x=0} = 0$  (or at least  $o(E)$ , which is not the case for an arbitrary jump).

Note that, in all practical cases the jump created by the previous boundary layer are in  $C^\infty((0, 1))$  (and even constant), so we can choose an arbitrarily high regularity. This will be the setting of the following proposition, even if one can replace  $C^\infty$  with a high order Sobolev space.

**Proposition 3.5.8** (Flat Stewartson  $\frac{1}{3}$  layer for a jump). *Let  $\bar{\mathbf{u}} = (\bar{v}, \bar{\psi})$  such that  $(\bar{v}, \bar{\psi}) \in H^1(\Omega) \times H^2(\Omega)$ ,  $\bar{\mathbf{u}}$  is an approximate solution of (3.32) on  $\Omega \setminus \{x = 0\}$ , and*

$$\begin{aligned} |[\bar{v}]_{C^\infty}| &\leq CE^{\frac{1}{6}} & |[\partial_x \bar{v}]_{C^\infty}| &\leq CE^{-\frac{1}{6}} \\ |[\bar{\psi}]_{C^\infty}| &\leq CE^{\frac{1}{2}} & |[\partial_x \bar{\psi}]_{C^\infty}| &\leq CE^{\frac{1}{6}} \\ |[\partial_x^2 \bar{\psi}]_{C^\infty}| &\leq CE^{-\frac{1}{6}} & |[\partial_x^3 \bar{\psi}]_{C^\infty}| &\leq CE^{-\frac{1}{2}}. \end{aligned}$$

Then there exists  $\mathbf{u}^{bl}$ , such that

- $\bar{\mathbf{u}} + \mathbf{u}^{bl} \in H^1(\Omega) \times H^2(\Omega)$ ;
- $\bar{\mathbf{u}} + \mathbf{u}^{bl}$  is an approximate solution of (3.32) on the whole domain  $\Omega$ .

*Proof.* The first step is to consider a version of the Stewartson layer for a wall (Proposition 3.5.6) taking into account the higher order normal derivatives:

**Lemma 3.5.9.** *Keeping the notation and hypothesis of Proposition 3.5.6, there exist  $\mathbb{H}_f$  an hyperplane of  $E = H^{s_0 - \frac{1}{3}} \times H^{s_0 - \frac{2}{3}} \times H^{s_0} \times H^{s_0 - \frac{1}{3}} \times H^{s_0 - \frac{2}{3}} \times H^{s_0 - 1}$  with codimension 3, such that for any  $(V^0, V^1, \Psi^0, \Psi^1, \Psi^2, \Psi^3) \in E$  satisfying*

$$(V^0, V^1, \Psi^0, \Psi^1, \Psi^2, \Psi^3) \in \mathbb{H}_f \tag{3.54}$$

there exists an approximate solution satisfying

$$(v, \partial_x v, \psi, \partial_x \psi, \partial_x^2 \psi, \partial_x^3 \psi)|_{x=0} = (E^{\frac{1}{6}} V^0, E^{-\frac{1}{6}} V^1, E^{\frac{1}{2}} \Psi^0, E^{\frac{1}{6}} \Psi^1, E^{-\frac{1}{6}} \Psi^2, E^{-\frac{1}{2}} \Psi^3).$$

There is multiple ways to prove this lemma. For example, if  $f = 0$  it follows directly from

the explicit expression

$$\begin{aligned}\psi &= \sum_{k=1}^{\infty} \sum_{l \in \{-1,0,+1\}} a_{k,l} e^{\lambda_l |k|^{\frac{1}{3}} \zeta} \sin(kz) \\ v &= - \sum_{k=1}^{\infty} \sum_{l \in \{-1,0,+1\}} a_{k,l} \lambda_l^4 k^{\frac{1}{3}} e^{\lambda_l |k|^{\frac{1}{3}} \zeta} \cos(kz).\end{aligned}$$

and the computation, for all  $k$ , of the kernel of the modified Vandermonde matrix

$$\begin{pmatrix} -\lambda_{-1}^4 k^{\frac{1}{3}} & \lambda_{-1}^5 k^{\frac{2}{3}} & 1 & \lambda_{-1} k^{\frac{1}{3}} & \lambda_{-1}^2 k^{\frac{2}{3}} & k \lambda_{-1}^3 \\ \lambda_0^4 k^{\frac{1}{3}} & \lambda_0^5 k^{\frac{2}{3}} & 1 & \lambda_0 k^{\frac{1}{3}} & \lambda_0^2 k^{\frac{2}{3}} & k \lambda_0^3 \\ \lambda_{+1}^4 k^{\frac{1}{3}} & \lambda_{+1}^5 k^{\frac{2}{3}} & 1 & \lambda_{+1} k^{\frac{1}{3}} & \lambda_{+1}^2 k^{\frac{2}{3}} & k \lambda_{+1}^3 \end{pmatrix}.$$

Whatever the method, we obtain three conditions defining  $\mathbb{H}_f$ :

$$\forall \lambda \in \left\{ 1, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2} \right\}$$

$$\lambda^5 \Psi^0 + \lambda^4 k^{-\frac{1}{3}} \Psi^1 + \lambda^3 k^{-\frac{2}{3}} \Psi^2 + \lambda^2 k^{-1} \Psi^3 - \lambda k^{-\frac{1}{3}} V^0 - k^{-\frac{2}{3}} V^1 = \int_0^{\infty} e^{-\lambda k^{\frac{1}{3}} \zeta} \left( \lambda^2 k^{-1} f_{\psi,k} - k^{-\frac{2}{3}} f_{v,k} \right) (\zeta) d\zeta.$$

Note that the existence of such operator can, as in Subsection 3.5.3, be derived from traces operators once enough regularity on the solution is obtained, but it seems difficult to obtain the claimed regularity on the traces using this method.

With this Lemma obtained, all it remains is to choose the  $6 + 6 = 12$  degrees of freedom satisfying the  $6 + 3 + 3 = 12$  conditions

$$\begin{aligned}\Psi_+^0 - \Psi_-^0 &= -[\bar{\psi}] & V_+^0 - V_-^0 &= -[\bar{v}] & \Psi_+^1 - \Psi_-^1 &= -[\partial_x \bar{\psi}] \\ \Psi_+^2 - \Psi_-^2 &= -[\partial_x^2 \bar{\psi}] & V_+^1 - V_-^1 &= -[\partial_x \bar{v}] & \Psi_+^3 - \Psi_-^3 &= -[\partial_x^3 \bar{\psi}] \\ (V_+^0, V_+^1, \Psi_+^0, \Psi_+^1, \Psi_+^2, \Psi_+^3) &\in \mathbb{H}_{f_+} \\ (V_-^0, -V_-^1, \Psi_-^0, -\Psi_-^1, \Psi_-^2, -\Psi_-^3) &\in \mathbb{H}_{f_-}.\end{aligned}$$

Comparing to the previous computation, we need one more verification. Indeed, as  $\{V^i, \Psi^j\}_{i=1,2,j=1,\dots,4}$  are function of  $z$  we have to check that the solution obtained by computing each mode  $k$  is in the correct functional space. More precisely, we have to check that

$$([\bar{v}], [\partial_x \bar{v}], [\bar{\psi}], [\partial_x \bar{\psi}], [\partial_x^2 \bar{\psi}], [\partial_x^3 \bar{\psi}]) \in H^{s_0 - \frac{1}{3}} \times H^{s_0 - \frac{2}{3}} \times H^{s_0} \times H^{s_0 - \frac{1}{3}} \times H^{s_0 - \frac{2}{3}} \times H^{s_0 - 1}.$$

This is indeed the case, and in particular we can take  $s_0 \geq 4$ . Then, thanks to the previous analysis we can obtain traces errors as small as needed. We then lift these remaining errors by an arbitrary lifting, which is of size  $E^{\frac{\kappa+1}{6}}$  in all pertinent norms.  $\square$

### 3.5.6 How to put the pieces together ?

- For the reduced equation (3.32), which is justified at a formal level far from the  $x = 0$ , we have constructed every building block, as long as  $\delta_0$  remains small. It seems that the only remaining difficulty is the change of boundary conditions when  $\delta_0^{-1} \rightarrow 0$ . Indeed, for the  $E^{\frac{1}{3}}$  layer, if  $\delta_0$  is small, the first order of the effective boundary condition is  $\psi|_{z=\gamma_0} = 0$ , while it is  $v - \partial_n \psi = 0$  for a large  $\delta_0$ . And we will see in Section 5.4 that this change of

boundary condition create a singularity.

As a consequence, we could hope for an approximation result for (3.32) if  $\delta_0$  is modified in a neighborhood of 0 into a  $\tilde{\delta}_0$  such that  $\tilde{\delta}_0$  can be greater than  $\sqrt{E}$  but does not go to  $\infty$ .

- For the Proudman problem between two spheres we must avoid two obstacles: the degeneracy of the Ekman layer, creating among other the difficulties related to (3.32) we just raised, and the equatorial Ekman layer. However, there are at least two ways to make the previous computations sufficient to conclude in the whole Proudman problem.
  1. The first one is to drop the degeneracy, but to keep the shear layers. This correspond to either  $\alpha = 1$  (from a formal point of view) or a smooth  $\kappa_0$  but a  $v_0$  admitting a discontinuity at  $x = 0$ . We will study this setting in the Subsection 3.5.6.
  2. The second one is to choose carefully  $v_0$  and  $v_1$  such that the problematic boundary layers does not appear at the relevant scalings. This correspond to a high order of cancellation of  $v_1 - v_0$  at  $x = 0$ , or a  $v_1 - v_0$  supported outside a small equatorial zone. Note that we were no able to do so in a way that let the shear layers appears. Nevertheless, the computation of the following paragraphs suggest that these obstacles are indeed local obstacles.

### Smooth $\kappa_i$ and discontinuous $v_i$

In this subsection we will consider that  $\kappa_0, \kappa_1$  are smooth but that  $v_0$  has a discontinuity at  $x = 0$ . Note that the following result can be adapted in the case where  $\kappa_0$  is discontinuous but not singular at  $x = 0$ . The case of physical interest is  $\kappa_0 = \mathbf{1}_{x < 0}$  corresponding to a rotating cylinder inside another rotating cylinder of larger radius,  $\kappa_0 = 0$  being the Neumann condition born from the symmetry of the problem.

This problem was described by Van der Vooren in [73] and already highlight the interaction between the  $\frac{1}{4}$  and  $\frac{1}{3}$  layers, and the similarity sublayer.

We will make use the  $\frac{1}{4}$ -boundary layer operator of Proposition 3.5.3 and  $\frac{1}{3}$ -boundary layer operator of Proposition 3.5.6.

In fact with these two operators identified the construction is almost immediate.

**Proposition 3.5.10.** *Let  $\gamma_0, \gamma_1 \in C^\infty(\mathbb{T})$  and  $v_0, v_1$  such that  $v_1 \in C^\infty(\mathbb{T})$  and*

$$v_0 = \underbrace{v_0^{reg}}_{\in C^\infty(\mathbb{T})} + \mathbf{1}_{x < 0}$$

. *Let  $\mathbf{u}^E$  be the solution of (3.9) with boundary conditions (3.10). Then*

$$\lim_{E \rightarrow 0} \|\mathbf{u}^E - (\bar{v}, 0)\|_{L^2} = 0$$

with

$$\bar{v} = \frac{\kappa_0 v_0 + \kappa_1 v_1}{\kappa_0 + \kappa_1}.$$

Moreover we have in fact the asymptotic expansion

$$\mathbf{u}^E = (\bar{v}, 0) + u^{bl, \frac{1}{4}} \left( \frac{x}{E^{\frac{1}{4}}} \right) + u^{bl, \frac{1}{3}} \left( \frac{x}{E^{\frac{1}{3}}} \right) + \text{lower order terms (in } L^2).$$

*Sketch of proof.* This proof is only an application of the boundary layer operators of previous section in the following order:

1. We first construct an approximate solution from the Ekman and standard interior operator of Section 3.4. The main problem is the fact that  $u^{app} \notin H^1 \times H^2$ .
2. We then construct two  $E^{\frac{1}{4}}$  operators, lifting the discontinuity on  $\bar{v}$ . However we have still a discontinuity on  $\bar{\psi}$ .
3. We construct the  $E^{\frac{1}{3}}$  operators in order to lift the discontinuity on  $\bar{\psi}$ .

Note that we can continue up to any order with this configuration.  $\square$

**The case**  $v_0 \simeq v_1$

As all the shear layers at  $x = 0$  are driven by  $(v_0 - v_1)$ , we can avoid them by requiring that  $v_0 - v_1$  has a high order of cancellation near  $x = 0$ . This is the goal of this subsection.

In order to quantify more precisely how close to the equatorial zone we can get let us develop the explicit calculation of Proposition 3.4.1 in the case of a sphere, i.e  $\gamma_0(x) = (1 - \sqrt{-x}) \mathbf{1}_{x < 0}$ .

In this case  $(\zeta, \sigma) = \left( \sqrt{\frac{\cos(\theta)}{E}} R, \theta \right)$  where  $R$  is the usual radius of the polar coordinates.

Denoting by  $\alpha_{\pm}(\theta)$  the two functions used to characterize the boundary layer functions of Proposition 3.4.1, we have

$$v(r, \theta) = \alpha_+(\theta) e^{-\frac{1+i}{\sqrt{2}} E^{-\frac{1}{2}} \sqrt{\cos(\theta)}(R-1)} + \alpha_-(\theta) e^{-\frac{1-i}{\sqrt{2}} E^{-\frac{1}{2}} \sqrt{\cos(\theta)}(R-1)}$$

$$\psi(r, \theta) = -\sqrt{\frac{E}{\cos(\theta)}} \left( \frac{1+i}{\sqrt{2}} \alpha_+(\theta) e^{-\frac{1+i}{\sqrt{2}} E^{-\frac{1}{2}} \sqrt{\cos(\theta)}(R-1)} + \frac{1-i}{\sqrt{2}} \alpha_-(\theta) e^{-\frac{1-i}{\sqrt{2}} E^{-\frac{1}{2}} \sqrt{\cos(\theta)}(R-1)} \right)$$

so, dropping the  $\pm$  for the sake of the computation,

$$\begin{aligned} \partial_z v &= \cos(\theta) \partial_r v - r^{-1} \sin(\theta) \partial_\theta v \\ &= \frac{1}{\sqrt{E}} \cos(\theta)^{\frac{3}{2}} \lambda v - \frac{1}{r} \left( \frac{\alpha'(\theta)}{\alpha(\theta)} \sin(\theta) - \frac{\sin(\theta)^2}{2 \cos(\theta)} \underbrace{\frac{\sqrt{\cos(\theta)}(r-1)}{\sqrt{E}}}_{\zeta} \right) v \end{aligned}$$

and

$$\begin{aligned} \Delta_{x,z} v &= r^{-1} \partial_r (r \partial_r v) + r^{-2} \partial_\theta^2 v \\ &= \frac{\lambda^2}{E} \cos(\theta) v + \frac{\sqrt{\cos(\theta)}}{r} \frac{\lambda}{\sqrt{E}} v + \frac{1}{r^2} \left( \frac{\alpha''(\theta)}{\alpha(\theta)} - \lambda \frac{\alpha'(\theta)}{\alpha(\theta)} \frac{\sin(\theta)}{\cos(\theta)} \zeta + \frac{\sin(\theta)^2}{4 \cos(\theta)^2} (\lambda^2 \zeta^2 - \lambda \zeta) - \frac{\lambda}{2} \zeta \right) v. \end{aligned}$$

Similarly

$$\partial_z \psi = -\lambda^2 \cos(\theta) v - \frac{1}{r} \left( -\frac{\sqrt{E} \lambda \sin(\theta)}{\sqrt{\cos(\theta)}} \frac{\alpha'(\theta)}{\alpha(\theta)} + \frac{\lambda \sqrt{E} \sin(\theta)^2}{2 \cos(\theta)^{\frac{3}{2}}} (\lambda \zeta - 1) \right) v$$

and a explicit expression for  $\Delta_{x,z}^2 \psi$ .

This allows us to compute explicitly the size of the boundary layer solutions. For example

$$\|\psi\|_{L^2}^2 = \iint \frac{E}{\cos(\theta)} \alpha^2(\theta) e^{-2\lambda \sqrt{\cos(\theta)} E^{-\frac{1}{2}} (r-1)} r dr d\theta = C \left( E^{\frac{3}{2}} \int \frac{\alpha^2(\theta)}{\cos(\theta)^{\frac{3}{2}}} d\theta + E^2 \int \frac{\alpha^2(\theta)}{\cos(\theta)^2} d\theta \right).$$



From there we can conclude the convergence but only in a setting that exclude any shear layer

**Proposition 3.5.11.** *Let  $v_0, v_1 \in C^4$  such that  $v_0 - v_1$  cancels itself to the order 3 at  $x = 0$ , i.e  $\alpha_{\pm}(0) = \alpha'_{\pm}(0) = \alpha''_{\pm}(0) = \alpha'''_{\pm}(0) = 0$ .*

*Then when  $E \rightarrow 0$  we have  $\mathbf{u}^E \rightarrow (\bar{v}, 0)$  in the energy space, with*

$$\bar{v} = \begin{cases} \frac{\kappa_0 v_0 + \kappa_1 v_1}{\kappa_0 + \kappa_1} & \text{for } x < 0 \\ v_1 & \text{for } x > 0. \end{cases}$$

*Remark 3.5.20.* We have

$$\frac{\kappa_0 v_0 + \kappa_1 v_1}{\kappa_0 + \kappa_1} = v_1 + \underbrace{(v_0 - v_1) \frac{1}{1 + \frac{\kappa_1}{\kappa_0}}}_{\sim \frac{v_0(x) - v_1(x)}{1 + x^{\frac{1}{4}}}}.$$

The singularity of the red term, source of all the various shear layers, being compensated by the cancellation of  $v_0 - v_1$ .

*Sketch of proof.* Constructing the interior and boundary layers up to the first order, as in the case of two disks, and computing explicitly the remainders, we obtain the result.

The three points to keep in mind are the following:

- From the Ekman condition (2.24) we obtain, as  $v^{int} = \bar{v}$

$$\alpha_{\pm} = \frac{1}{2}(v_0 - \bar{v}) = \frac{\kappa_1(v_0 - v_1)}{2(\kappa_1 + \kappa_0)}.$$

- The associated  $\psi^{int} = 0+$  does not create any additional difficulty.
- We cannot use a usual cut-off for the boundary layer created near the equator. As a consequence, when the boundary layer size on  $\Gamma_0$  is greater than 1 this create a trace on  $\Gamma_1$ . This trace is at least  $C^3$  and of size (in terms of  $|V| + E^{-\frac{1}{2}}|\Psi| + |\partial_n \Psi|$ , which is the adequate  $L^\infty$  norm on  $\Gamma_1$ )

$$\frac{1}{\sqrt{\cos(\theta)}} e^{-C\sqrt{\cos(\theta)}E^{-\frac{1}{2}}} |\alpha_{\pm}(\theta)| \leq C \left(\frac{\pi}{2} - \theta\right)^{4-\frac{1}{2}} e^{-C\sqrt{\frac{\pi-\theta}{E}}} \leq CE^{\frac{5}{2}}$$

which is sufficiently small to lift by any function without perturbing the energy estimates.

□

*Remark 3.5.21.* This result is far from optimal. For example Rousset in [63] managed to prove the nonlinear stability of Ekman-Hartmann boundary layers with a well-prepared initial condition  $u_0$  satisfying an assumption of cancellation near the equator, the hypothesis (**H**):

$$|u_0(x)| \leq C |\cos(\theta)|^{\frac{1}{2}}.$$

### The use of a localized cut-off

Maybe the best approach to justify the construction would be to stop the Ekman layers at an angle  $\theta$  such that  $(\frac{\pi}{2} - \theta) \leq E^\beta$ . More precisely, rather than considering  $\alpha_\pm$ , i.e  $v_0 - v_1$ , to be like  $(\frac{\pi}{2} - \theta)^k$  for  $k$  large enough, a more natural construction would be to choose  $\alpha_\pm$  with compact support. Indeed, our ultimate goal would be to use the methods proposed by Dalibard and Saint-Raymond in [17]. In this study, they displayed that the best way to connect such layers is not through a global boundary layer with varying size and amplitude, but the use of cut-offs in different subregions.

$$\chi_{Ekman} \mathbf{u}^{Ekman} + \chi_{equatorial} \mathbf{u}^{equatorial} + \dots$$

Where  $\chi_{Ekman}$  and  $\chi_{equatorial}$  are localized so that  $\chi_{Ekman}(\theta) = 0$  for  $\theta > \frac{\pi}{2} - CE^\beta$ , and  $\chi_{equatorial}(\theta) = 0$  for  $\theta < \frac{\pi}{2} - CE^\beta$

So we must evaluate the error with  $\alpha_\pm$  of the form  $\alpha_\pm = \chi_\pm \left( \frac{\cos(\theta)}{E^\beta} \right)$ , with  $\chi_\pm$  smooths and with support far from 0. This can be a perspective for future works.

## 3.6 Formal remarks on the equatorial problem

In the case relevant to geophysical physics the previous assumptions do not hold as the model often studied is typically  $v_0 = 0$ ,  $v_1 = \mathbf{1}_{X < 0}$ , and the sphere is  $\alpha = \frac{1}{2}$ .

In order to find the equatorial scaling let  $(X, Z)$  be the global variables. We must look for the new scaling and boundary layer near the equator  $X = 0, Z = 0$ . This layer has the particularity to be of small size both in the distance to the boundary and the angle  $\theta$ . From this observation we must expect two scalings and a PDE in a 2D domain instead of an ODE. In order to have a simple fixed domain let us choose as coordinates  $y = X + Z^{\frac{1}{\alpha}}$  (which behaves like the distance to the boundary) and  $z = Z$  (rather than the exact distance to the boundary and  $\theta$ ).

With this change of variables followed by the rescaling  $y \rightarrow E^{\beta_y} y$  and  $z \rightarrow E^{\beta_z} z$  we obtain

$$\begin{aligned} \partial_Z &= E^{-\beta_z} \partial_z + E^{-\beta_y + \frac{1-\alpha}{\alpha} \cdot \beta_z} \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \\ \Delta_{X,Z} &= E^{-2\beta_z} \partial_z^2 + E^{-2\beta_y} \partial_y^2 + \frac{1}{\alpha^2} E^{-2\beta_y + 2\frac{1-\alpha}{\alpha} \cdot \beta_z} z^{2\frac{1-\alpha}{\alpha}} \partial_y^2 + E^{-\beta_y + \frac{1-2\alpha}{\alpha} \cdot \beta_z} \left( 2 \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \partial_z + \frac{1-\alpha}{\alpha^2} z^{\frac{1}{\alpha}-2} \partial_y \right). \end{aligned}$$

As  $\beta_z > 0$  and  $1 > \alpha > 0$  we have  $-2\beta_y + 2\frac{1-\alpha}{\alpha} \cdot \beta_z > -2\beta_y$  and  $-\beta_y + \frac{1-2\alpha}{\alpha} \cdot \beta_z > -2\max(\beta_y, \beta_z)$ . As always, we then neglect (i.e these will be put in the remainder) the lower order differential operators if their power of  $E$  is greater than the one of higher order terms. This leads to the approximation

$$\begin{aligned} \partial_Z &= E^{-\beta_z} \partial_z + E^{-\beta_y + \frac{1-\alpha}{\alpha} \cdot \beta_z} \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \\ \Delta_{X,Z} &= E^{-2\beta_z} \partial_z^2 + E^{-2\beta_y} \partial_y^2 + E^{-\beta_y + \frac{1-2\alpha}{\alpha} \cdot \beta_z} 2 \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \partial_z + \text{l.o.t.} \end{aligned}$$

And the operator to be considered is

$$\left( \begin{array}{c} \left( E^{1-2\beta_z} \partial_z^2 + E^{1-2\beta_y} \partial_y^2 + E^{1-\beta_y + \frac{1-2\alpha}{\alpha} \cdot \beta_z} 2 \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \partial_z \right) \\ E^{-\beta_z} \partial_z + E^{-\beta_y + \frac{1-\alpha}{\alpha} \cdot \beta_z} \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \end{array} \begin{array}{c} E^{-\beta_z} \partial_z + E^{-\beta_y + \frac{1-\alpha}{\alpha} \cdot \beta_z} \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \\ \left( E^{1-2\beta_z} \partial_z^2 + E^{1-2\beta_y} \partial_y^2 + E^{1-\beta_y + \frac{1-2\alpha}{\alpha} \cdot \beta_z} 2 \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} \partial_y \partial_z \right)^2 \end{array} \right).$$

The predominant balance leads to  $\beta_y = \frac{1}{3-\alpha}$ ,  $\beta_z = \frac{\alpha}{3-\alpha}$ . The scaling is  $v = v$ ,  $\psi = E^{\frac{1}{3-\alpha}} \psi$ , and

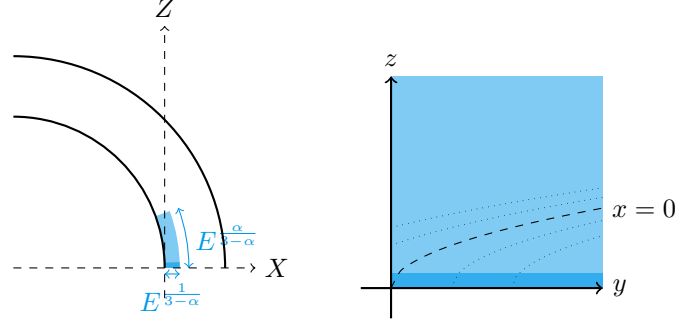


Figure 3.15 – The change of variables. The additional vertical Ekman layer is in darker blue.

the equation in this new scaling (keeping the same names for the rescaled variables)

$$\begin{aligned} \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) v - \partial_y^4 \psi - E^{4 \cdot \frac{1-\alpha}{3-\alpha}} \partial_z^4 \psi + \text{l.o.t} &= 0 \\ \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) \psi + \partial_y^2 v + E^{2 \cdot \frac{1-\alpha}{3-\alpha}} \partial_z^2 v + \text{l.o.t} &= 0. \end{aligned}$$

Note that, to neglect a priori the higher derivatives with respect to  $z$  (in blue) may leads to the formation of a new nested boundary layer in  $z$ . This is indeed the case, and this boundary layer is of size  $E^\beta$  where  $-2\beta = 6 \cdot \frac{1-\alpha}{3-\alpha} - 6\beta$  i.e  $\beta = \frac{3}{2} \cdot \frac{1-\alpha}{3-\alpha}$ . Therefore, in the interior variables the vertical boundary layer size is

$$E^{\frac{3}{2} \cdot \frac{1-\alpha}{3-\alpha}} \times E^{\frac{\alpha}{3-\alpha}} = E^{\frac{1}{2}}.$$

And the associated equation is

$$\begin{aligned} \partial_{\bar{z}} - \partial_{\bar{z}}^4 \psi + \text{l.o.t} &= 0 \\ \partial_{\bar{z}} \psi + \partial_{\bar{z}}^2 v + \text{l.o.t} &= 0. \end{aligned}$$

So this is no more than a standard Ekman layer, as the geometry is flat. This additional layer can be seen in Figure 3.15.

The final boundary layer equation we must investigate is thus

$$\boxed{\begin{aligned} \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) v - \partial_y^4 \psi &= 0 \\ \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) \psi + \partial_y^2 v &= 0 \end{aligned}} \quad (3.55)$$

on the domain  $y > 0, z > 0$ . This is the goal of the next chapter, but let us make some preliminary formal remarks.

First, some heuristics of what happens when one term of the equation becomes negligible:

- If we forget about the  $\partial_z$  term, i.e the degenerate nature of this layer, we can work as if  $z$

is constant, and end up with

$$\begin{aligned} z^{\frac{1-\alpha}{2\alpha}} \partial_y v - \partial_y^4 \psi &= 0 \\ z^{\frac{1-\alpha}{2\alpha}} \partial_y \psi + \partial_y^2 v &= 0. \end{aligned}$$

Therefore, in boundary layer variables, with the characteristic length  $\tilde{\delta}_z = z^{-\frac{1-\alpha}{2\alpha}}$ , and with  $y = \tilde{\delta}_z \zeta$  we recover equation (2.27), i.e the classical Ekman layers equation. As the error from making such assumption is  $\partial_z \mathbf{u}^{Ekman} \left( y z^{\frac{1-\alpha}{2\alpha}} \right)$ , i.e like

$$\frac{1}{z} y z^{\frac{1-\alpha}{2\alpha}} \exp \left( -\lambda y z^{\frac{1-\alpha}{2\alpha}} \right)$$

we see that we can hope to justify such approximation, for example via a series expansion in inverse powers of  $z$ , only for  $z \gg 1$ .

Moreover in  $(r, \theta)$  variables, as  $y = (r - r_0) E^{-\frac{1}{3-\alpha}}$  and  $z \simeq \left( \cos(\theta) \right)^{\frac{1}{1-\alpha}} E^{-\frac{\alpha}{3-\alpha}}$  we obtain

$$\begin{aligned} u \left( y z^{\frac{1-\alpha}{2\alpha}} \right) &= u \left( (r - r_0) E^{-\frac{1}{3-\alpha}} \cos(\theta)^{\frac{\alpha}{1-\alpha} \cdot \frac{1-\alpha}{2\alpha}} E^{-\frac{\alpha}{3-\alpha} \cdot \frac{1-\alpha}{2\alpha}} \right) \\ &= u \left( \frac{(r - r_0)}{\sqrt{\frac{E}{\cos(\theta)}}} \right). \end{aligned}$$

This show that we reconnect, at least formally with the Ekman layers for large  $z$ .

- On the contrary if we forget about the  $z \partial_y$  term, i.e the influence of the geometry, we obtain on the domain  $(0, \infty) \times (0, \infty)$

$$\begin{aligned} \partial_z v - \partial_y^4 \psi &= 0 \\ \partial_z \psi + \partial_y^2 v &= 0 \end{aligned} \tag{3.56}$$

which is the unbounded version of the  $\frac{1}{3}$ -layer. Using a Laplace transform with respect to  $z$  we obtain an ODE in  $y$  with characteristic polynomial

$$\lambda^6 + p^2 = 0$$

where  $p$  is the Laplace variable. Thus, as long as the traces at  $y = 0$  (and source terms) are in some space  $\mathcal{H}_\eta^s$  with  $s \geq 0$  and  $\eta > 0$ , we can solve explicitly the equation and lift up to three boundary conditions.

But, the striking feature of this approximation is less its explicit resolution than the asymptotic scaling. In fact, if we scale down to the equatorial layer but keep the orthogonal basis  $(x, z)$  we recover (3.56) (with  $x$  instead of  $y$ ) on the domain  $z \geq 0, x \geq -cz^{\frac{1}{\alpha}}$ . Note that this is the same as considering (3.55) along the characteristic of the transport  $\partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y$ . Anyway, in these variables the limit  $z \rightarrow \infty$  is the connection to the  $\frac{1}{3}$ -layer. And the *parabolic scaling* we can derive formally is

$$z^2 \sim x^6$$

i.e in the variables of the whole space

$$E^{\frac{2\alpha}{3-\alpha}} z^2 \sim E^{\frac{6}{3-\alpha}} x^6.$$

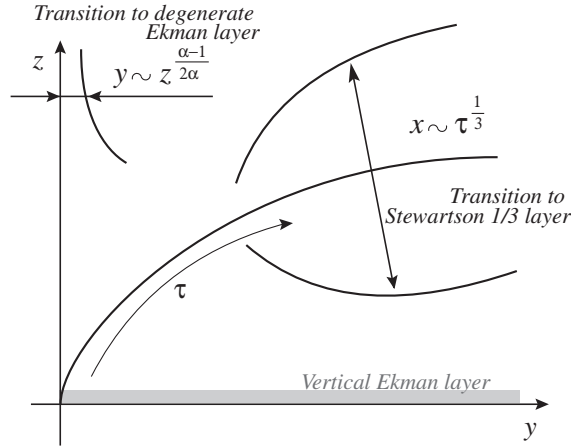


Figure 3.16 – The connections between the equatorial layer and the others boundary layers.

This is exactly the scaling  $x \sim (Ez)^{\frac{1}{3}}$  of the  $\frac{1}{3}$ -layer.

This connection between the asymptotic parabolic scaling of the equatorial layer and the similarity sublayer is also present in the MHD problem of Chapter 5, where we will give more precisions.

- Lastly, if we forget about the higher derivatives in  $y$ , we recover a transport equation whose characteristic are the iso- $x$ . This is just the Taylor-proudman columns, but has no effect in our study where we search only the boundary layer terms.

Figure 3.16 summarize the connections to the others layers in the variables  $(y, z)$ .

Let us now make remarks on the domain and boundary conditions of equation (3.55) in its entirety.

- If the domain was rather  $y \in \mathbb{R}, z > 0$  than  $y > 0, z > 0$  a simple Fourier transform in  $y$  would leads to two equations on  $w_{\pm} = v \pm (-\Delta)^{\frac{1}{2}} \psi$ ,

$$\partial_z w_{\pm} + z \partial_y w_{\pm} = \pm (-\Delta)^{\frac{3}{2}} w_{\pm}$$

so one behaves like a solution of a parabolic equation, the other as the solution of a retrograde parabolic equation. Therefore, we must cancel the retrograde solution, that is, in terms of degree of freedom, to constrain one with respect to  $z$  propagation. This is why we can only prescribe  $2 - 1 = 1$  condition at  $z = 0$ . It is also to be noted that this analysis is less obvious for the domain  $y > 0$  due to the fractional character of the differential operator and the importance of boundary conditions at  $y = 0$ . We nevertheless expect to have a differential condition when taking traces in  $z$ . This will be the idea behind the *transparent boundary condition*.

- If we consider from a very formal point of view the characteristic polynomial in  $y$  of this equation we obtain

$$\lambda^6 - (z\lambda + \partial_z)^2$$

Therefore, as in Fourier  $\partial_z = i\xi$  we see that the mode  $\lambda = 0$  appears only for low frequencies in  $z$ . But the degenerated character is linked with the high frequencies. So it would seem

reasonable to lift up to 3 boundary conditions on  $y = 0$  (and not only 2 as for the Ekman scaling).

*Remark 3.6.1.* The previous point shows that, contrary to what can be expected in a bounded domain, the main obstacle here will be integrability rather than regularity.

- To give an example of this integrability obstruction, we were unable to prove existence and uniqueness in the same space, for this equation and domain. However, we recover the well-posedness as soon as an additional zeroth order term ensure a control of the  $L^2$  norms. This additional control can probably be with time dependency, giving a  $L_t^\infty(L^2(\Omega))$  control and replacing  $H^s$  estimates by  $L_t^2(H^s(\Omega))$ . But this introduce many more problems outside this particular scaling, from internal waves to boundary layer stability.
- A less radical way would be to introduce damping terms, for example the equation

$$\begin{aligned} \epsilon_\psi \partial_y^2 \psi + \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) v - \partial_y^4 \psi &= 0 \\ -\epsilon_v v + \left( \partial_z + z^{\frac{1-\alpha}{\alpha}} \partial_y \right) \psi + \partial_y^2 v &= 0. \end{aligned}$$

This may seems innocuous in the interior scaling as this correspond to

$$\begin{aligned} \epsilon_\psi E^{\frac{1-\alpha}{3-\alpha}} \Delta \psi + \partial_Z v - E \Delta^2 \psi &= 0 \\ -\epsilon_v E^{\frac{1-\alpha}{3-\alpha}} v + \partial_Z \psi + E \Delta^2 v &= 0. \end{aligned}$$

However, this create modifications the boundary layer scalings, and has no physical ground.

### 3.7 Conclusion and perspectives

We were able to recover the several boundary layers scalings of a variant of the Proudman problem and to study for each one of them the boundary layer equation associated.

- The very well-known Ekman layer of size  $\sqrt{\frac{E}{\cos(\theta)}}$ .
- Two shear layers of respective sizes  $E^{\frac{1}{3+\alpha}}$  and  $E^{\frac{1}{4}}$  ( $\frac{1}{4}$  can be seen as the very particular case  $\alpha = 1$ ).
- A layer, corresponding to the  $\frac{1}{3}$ -Stewartson layer near a wall. This is to be noted that the actual size is rather  $(Ek)^{\frac{1}{3}}$ , explaining the connection to the equatorial problem.
- The  $E^{\frac{1}{3-\alpha}} \times E^{\frac{\alpha}{3-\alpha}}$  equatorial layer. This last one is of a different nature from the preceding, as it is a true PDE and cannot be reduced to a one dimensional problem.

From a methodological point of view, the two main ingredients are:

- The use of the *boundary layer operator* framework, introduced by Dalibard and Saint-Raymond in [18]. This not only allows for a much simpler approach when closing equations, but also is essential for the connection of the shear layers, especially when the size are different.
- The program proposed by Gérard-Varet [31] in order to study these linear boundary layers operators. And, even for simple ODE, the use of classical elliptic theory, rather than explicit resolution, for more robustness.

However, we were not able to conclude for the study of the whole problem, for three main reasons:

- (I) First, the interior operator associated with the  $E^{\frac{1}{3+\alpha}}$  boundary layer, even if precise enough to conclude the  $L^2$  study of (3.34), is not precise enough in high norms to pursue the expansion and to connect to the others layers at higher order.
- (II) Secondly, the Stewartson  $E^{\frac{1}{3}}$  boundary layer operator seems no longer valid in a singular geometry near  $z = 0$ , i.e when it connects to the equatorial zone.
- (III) Lastly, the connection between the Ekman layers and the equatorial zone is also missing.

Note that if the first difficulty seems to be a technical one, probably due to a wrong choice of cut-off, the two other ones are deeply connected to the equatorial Ekman boundary layer.

For this equatorial layer, we were able to obtain, up to a small variation of the problem, existence and uniqueness. We were also able to obtain elliptic-like interior regularity. But the main point is that we were able to identify a transparent boundary condition. This is highly probable that this transparent boundary condition ensures the connection with the Ekman layer. A clue to the connection with the Stewartson layer would have been estimates alongside transport characteristic, but we were unable to find any.

The Figure 3.17 summarize the zones of difficulty.

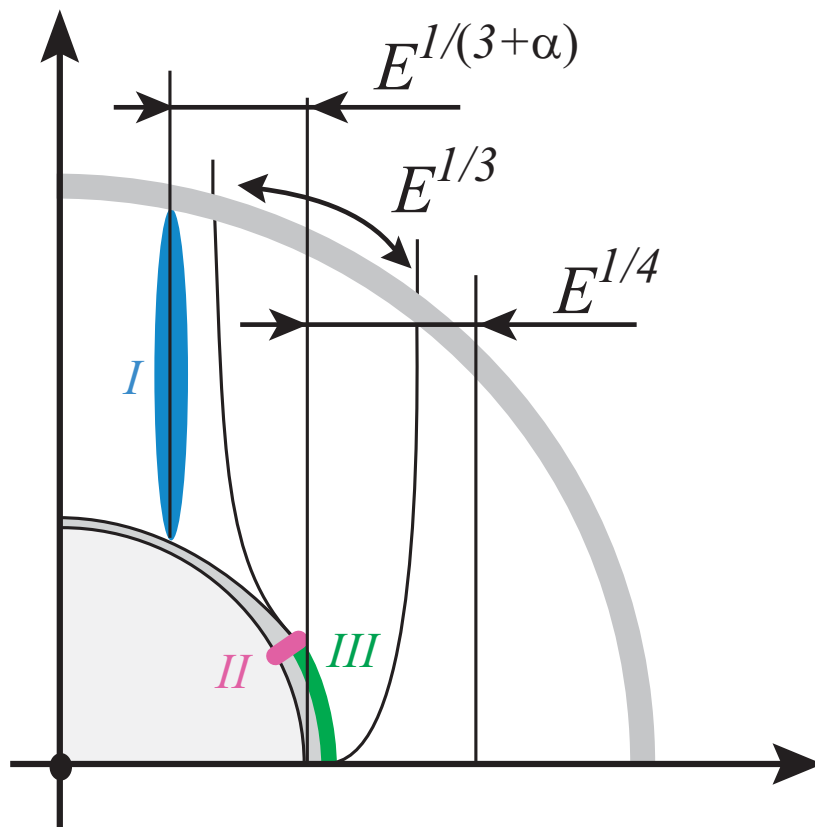


Figure 3.17 – Zones of difficulties for Proudman problem.





# Mathematical study of the equatorial Ekman boundary layer

## Outline of the current chapter

<b>4.1 Introduction</b>	<b>109</b>
<b>4.2 Existence and properties of weak solutions</b>	<b>111</b>
<b>4.3 Uniqueness and transparent boundary conditions</b>	<b>119</b>
<b>4.4 Appendix</b>	<b>125</b>
<b>Acknowledgements</b>	<b>128</b>

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In this paper we study the well-posedness of a simple model of boundary layer for rotating fluids between two concentric spheres near the equator. We show that this model can be seen as a degenerate elliptic equation, for which we prove an existence result thanks to a Lax-Milgram type lemma. We also prove uniqueness under an additional integrability assumption and present a transparent boundary condition for such layers.

## 4.1 Introduction

In this article we will study the linear Ekman boundary layer near the equator for a rotating fluid between two concentric spheres. With  $v$  the azimuthal flow velocity and  $\psi$  the stream function, the equation we will consider is

$$\begin{aligned} \partial_z v + z \partial_y v - \frac{1}{2} \partial_y^4 \psi &= s_\psi \\ \partial_z \psi + z \partial_y \psi + \frac{1}{2} \partial_y^2 v &= s_v. \end{aligned} \tag{4.1}$$

We will mainly consider three domains and boundary conditions:

- (I) The domain is  $y > 0, z > 0$  and the boundary condition at  $z = 0$  is  $\psi|_{z=0} = 0$ .
- (II) The domain is  $y > 0, H > z > 0$ , the boundary condition at  $z = 0$  is  $\psi|_{z=0} = 0$  and the one at  $z = H$  is  $\psi|_{z=H} = \Lambda v$ , where  $\Lambda : H_0^{\frac{1}{2}} \rightarrow H^{-\frac{1}{2}}$  is a non-positive operator.
- (III) The domain is  $y > 0, z > H$  and the boundary condition at  $z = H$  is  $v|_{z=H} = v_H \in H_0^{\frac{1}{2}}$ .

Other cases can be obtained by altering the  $z$  boundary conditions (for example  $\psi_{z=0} = -\Lambda v|_{z=0}$ ). The boundary condition at  $y = 0$  will be

$$v|_{y=0} = V, \partial_y \psi|_{y=0} = \Upsilon, \psi|_{y=0} = \Psi$$

and when not especially specified we will take  $V = \Upsilon = 0, \Psi = 0$ .

Since their description by Proudman [60] and their formal analysis by Stewartson [69, 70] the behavior of highly rotating fluids have been widely mathematically studied. We refer to the book of Chemin, Desjardins, Gallagher and Grenier [11] for more details. Although the case of a horizontal surface (and the resulting  $E^{\frac{1}{2}}$  boundary layer called Ekman layer) is now well understood, especially since Grenier and Masmoudi [40], several other geometries have been considered. For a vertical surface (i.e the axis of rotation is perpendicular to the normal) the resulting boundary layer of size  $E^{\frac{1}{3}}$  is well known and analysed (for a formal analysis see for example Van de Vooren [73], and for a detailed analysis with anisotropic viscosity see Bresch, Desjardins and Gérard-Varet [7]). In the spherical case the main difficulty is near the equator: as the latitude goes to 0 the Ekman boundary layer degenerates and the classical analysis becomes invalid, leading to the need for an additional assumption of smallness near the equatorial area (as in the article by Rousset [63]). It is to be noted that for small latitudes the  $\beta$ -plane model is used to take into account the variations of the angle between the axe of rotation and the normal of the surface as done by Dalibard and Saint-Raymond [19].

In this paper we will focus on a linear and time independent model taking into account the spherical geometry (or any other similar geometry) in the vicinity of the equator. The resulting boundary layer (of size  $E^{\frac{1}{3}} \times E^{\frac{2}{5}}$ ) was first derived by Stewartson [69] and is a typical example of so called degenerate boundary layer [45, 37]. The derivation of the equation and its numerical analysis have been done notably by Marcotte, Dormy and Soward [54], and is briefly recalled in the appendix, but up to our knowledge no formal proof of the well posedness of the problem exists.

For equation (4.1) we will prove the existence in the natural energy space. We will also prove the uniqueness assuming additional integrability. For case (II) and for variants these additional assumptions are redundant and we have one and only one solution, however for case (I) and (III) the resulting space is smaller leading to an incomplete result.

A simplified statement of the existence and uniqueness result for  $s_v = s_\psi = 0$  is:

**Theorem 4.1.1.** *For any  $V, \Upsilon \in H_0^{\frac{1}{2}}(\mathbb{R}_+)$ ,  $\Psi \in H_0^{\frac{3}{2}}(\mathbb{R}_+)$  there exists a weak solution of (4.1) in cases (I),(II),(III). This solution is such that*

$$\partial_y v \in L^2(\Omega), \partial_y^2 \psi \in L^2(\Omega).$$

*Moreover if  $v \in L^2, \psi \in L^2$  this solution is unique.*

When uniqueness holds, a transparent boundary layer operator (similar to the one in [54]) will be described. Such an operator is of great interest for numerical analysis or for connecting the boundary layer to the interior solution (or in this case to other boundary layers).

The main difficulty of the problem is that each approach to prove well-posedness stumbles on a different term. Let us observe the influence of each part of the equation:

- The  $\partial_z$  term is the obvious source of the degenerate character of the equation as a boundary layer equation as without it we recover a simple ODE with respect to  $y$ . More precisely at each  $z$  we recover the classical Ekman layer. Even if its size diverges as  $z \rightarrow 0$  one can make a formal expansion in powers of  $\sqrt{z}$  and  $y/\sqrt{z}$  which is the same as doing an expansion in  $\frac{1}{\sqrt{\cos(\theta)}}$  for the Ekman problem.
- The  $z\partial_y$  term associated to the fact that  $y > 0$  is also a major source of difficulties as it renders the spherical geometry. It prevents any simplification using symmetry arguments and as a counterpart of a simple domain it creates transport along characteristics  $z - y^2/2 = c$  which will create problems when trying to prove uniqueness. A possible approach would be to use well chosen weighted spaces that follow transport along those characteristics but we were unable to obtain satisfactory results with it. Without this transport term we recover the simple case of a vertical geometry.
- The fact that the equation is a system and without a maximum principle prevents us from directly using modulated energy methods or entropy estimates that are usually helpful in such situations, for example in cross-diffusion problems.
- Another main difference with standard cross-diffusion is the order of the operator in  $y$ : one term is a laplacian but the other is a bilaplacian. This asymmetry coupled with the boundary at  $y = 0$  is the main obstacle when trying to find better variables for the problem as the different regularity leads to mismatches in boundary conditions. The same problem arises when trying to use the decomposition between symmetric and skew-symmetric term for Carleman like estimate.

For these reasons, and the fact that the domain in  $y$  is unbounded, our approach will be to consider the problem as a degenerate elliptic one. As a drawback this overlooks the structure of the skew-symmetric term containing the  $\partial_z$  and transport terms and leads to sub-optimal results in terms of regularity with respect to  $z$ .

In the first part we will deal with existence with a proof similar to the ones used by Fichera [28]. The main point is the use of a well-chosen space of test functions for a duality approach. This space must ensure both a coercivity condition for the adjoint via a positivity of boundary terms and the recovery of the boundary conditions which are weakly formulated. These two constraints dictate the set of admissible horizontal boundary conditions.

In the second part we will show the uniqueness by standard energy methods. We will also propose variations of the main problem allowing a uniqueness result in the same space as existence. For such variants we will define a transparent boundary operator similar to a Dirichlet to Neumann operator and of great importance for numerical simulation.

## 4.2 Existence and properties of weak solutions

In this section we prove the existence of weak solutions of (4.1) using duality and energy methods for degenerate elliptic equations similar to the ones used by Fichera [28]. Thus, multiple boundary conditions can be weakly prescribed at  $z = 0$  and  $z = H$ , but the energy space is not regular enough to guarantee proper traces.

In the rest of the article we will denote  $\mathbf{u} = (v, \psi)$  and  $\mathbf{s} = (s_\psi, s_v)$ . The equation can then be formulated as  $L\mathbf{u} = (T - \frac{1}{2}D)\mathbf{u} = \mathbf{s}$  where we defined the positive symmetric operator  $D$  and

the skew-symmetric operator  $T$  as

$$T = \begin{pmatrix} 0 & \partial_z + z\partial_y \\ \partial_z + z\partial_y & 0 \end{pmatrix}, D = \begin{pmatrix} \partial_y^4 & 0 \\ 0 & -\partial_y^2 \end{pmatrix}.$$

The operator  $D$  leads to the choice of the energy space  $E_0$  and the operator  $T$  prescribes both the allowed horizontal boundary conditions and the choice of the test function space. It is to be noted that the  $z$  derivative and dependence is only involved in  $T$ , so it has no corresponding term in the energy space.

We will provide a detailed analysis for cases (I) and (II) and for homogeneous boundary conditions.

The other cases follow the exact same analysis, except for the choice of the space of test functions, which must be adapted to the horizontal boundary conditions. We will discuss non-homogeneous boundary conditions in the next subsection.

### 4.2.1 Statement of the result

We define the Banach space  $E_0$  by

$$\|\mathbf{u}\|_{E_0}^2 = \int_{\Omega} \left( |\partial_y v|^2 + \left| \frac{v}{1+y} \right|^2 \right) + \int_{\Omega} \left( |\partial_y^2 \psi|^2 + \left| \frac{\psi}{1+y^2} \right|^2 \right)$$

and to enforce homogeneous boundary conditions at  $y = 0$  we define

$$\|\mathbf{u}\|_{E_{0,0}}^2 = \int_{\Omega} \left( |\partial_y v|^2 + \left| \frac{v}{y} \right|^2 \right) + \int_{\Omega} \left( |\partial_y^2 \psi|^2 + \left| \frac{\psi}{y^2} \right|^2 \right).$$

Lastly for the weak formulation we denote the graph norm

$$\|\mathbf{u}\|_{E_1} = \|\mathbf{u}\|_{E_{0,0}} + \|T\mathbf{u}\|_{E'_{0,0}}.$$

Note that  $E_0$  lacks regularity with respect to  $z$  to have traces at  $z = 0$  (or  $z = H$ ). Moreover,  $\mathbf{u} \in E_1$  requires not only some weak (negative) regularity on  $\partial_z \mathbf{u}$  but also a better integrability than just  $E_0$ .

An important point is that these  $z$  boundary conditions are derived from the space of test functions. Let us consider

$$\mathcal{D} = \{\boldsymbol{\varpi}(y, z) = (w(y, z), \phi(y, z)); w \in C_c^\infty((0, +\infty) \times [0, +\infty)), \phi \in C_c^\infty((0, +\infty) \times (0, +\infty))\}$$

for case (I) and

$$\mathcal{D} = \{\boldsymbol{\varpi} = (w, \phi); w \in C_c^\infty((0, +\infty) \times [0, H]), \phi \in C_c^\infty((0, +\infty) \times (0, H)) \text{ s.t } \phi|_{z=H} = -\Lambda^* w|_{z=H}\}$$

for case (II). Note that in fact we can replace  $\mathcal{D}$  by its closure under the  $E_1$  norm.

Given this set of definitions, the following existence theorem holds, where as for the rest of the paper  $C$  denotes a numerical constant

**Theorem 4.2.1.** *Let  $\mathbf{s} = (s_v, s_\psi) \in E'_1$ .*

(i) (existence of weak solutions) There exists  $\mathbf{u} \in E_{0,0}$  such that  $\forall \varpi = (w, \phi) \in \mathcal{D}$

$$\int_{\Omega} -v\partial_z\phi - zv\partial_y\phi - \frac{1}{2}\partial_y^2\psi\partial_y^2\phi + \int_{\Omega} -\psi\partial_zw - z\psi\partial_yw - \frac{1}{2}\partial_yv\partial_yw = \int_{\Omega} s_{\psi}\phi + \int_{\Omega} s_vw \quad (4.2)$$

and

$$\|\mathbf{u}\|_{E_{0,0}} \leq C \|\mathbf{s}\|_{E'_1}$$

(ii) (boundary conditions) If  $\mathbf{u} \in E_0 \cap H_{loc}^2$  then  $\psi|_{z=0} = 0$  and in case (II),  $\psi|_{z=H} = \Lambda v|_{z=H}$ .

(iii) (interior regularity) If  $\partial_y^2\mathbf{s} \in E'_0$  we have a Caccioppoli type inequality: for all  $y_0 > 0, z_1 > 0$  there exist  $C_{y_0, z_1} > 0$  such that

$$\int_{(y_0, \infty) \times (0, z_1)} |\partial_y^4\psi|^2 + |\partial_y^3v|^2 \leq C_{y_0, z_1} \left( \|\mathbf{u}\|_{E_0}^2 + \|\partial_y^2\mathbf{u}\|_{E'_0} \right)$$

*Remark 4.2.1.* as we have

$$\begin{aligned} \partial_zv &= -z\partial_yv + \frac{1}{2}\partial_y^4\psi + s_{\psi} \\ \partial_z\partial_y\psi &= -z\partial_y^2\psi - \frac{1}{2}\partial_y^3v + \partial_y s_v \end{aligned}$$

from the interior regularity with respect to  $y$  we can obtain interior regularity with respect to  $z$ .

Points (i), (ii) and (iii) will be proved in subsection 4.2.3, 4.2.4 and 4.2.5 respectively.

## 4.2.2 Remarks on nonhomogeneous boundary conditions

The previous result only considers homogeneous boundary conditions. As usual we can recover nonhomogeneous boundary condition by lifting these boundary conditions. In this subsection we will briefly discuss this lifting.

Note that an important difference with Ekman boundary layers is that we are able to impose 3 boundary conditions at  $y = 0$  whereas in classical Ekman boundary layers only 2 boundary conditions can be imposed. This difference does not come from any particularity of our system as the same number can be prescribed if we replace the transport  $\partial_z + z\partial_y$  by  $\lambda u + c\partial_y$  with  $\lambda, c \neq 0$ . On the contrary, one can only prescribe 2 conditions for Ekman layers. this comes from a degeneracy of the Ekman system, which causes the Ekman pumping.

In order to consider nonhomogeneous boundary conditions we will need the following lemma:

**Lemma 4.2.2.** • Let  $\Psi \in H_0^{\frac{3}{2}}(\mathbb{R}_+)$  i.e  $\Psi \in H^{\frac{3}{2}}$  and  $\Psi(0) = 0$ . Moreover suppose that  $z\Psi \in H^{\frac{3}{2}}$ . Then there exist  $\mathbf{r} \in E_0$  such that

$$\begin{aligned} L\mathbf{r} &\in E'_1 \text{ and } \|L\mathbf{r}\|_{E'_1} \leq C \|(1+z)\Psi\|_{H_0^{\frac{3}{2}}} \\ \mathbf{r}|_{z=0} &= 0, r_v|_{y=0} = 0, r_{\psi}|_{y=0} = \Psi, \partial_y r_{\psi}|_{y=0} = 0 \end{aligned}$$

- Let  $\Upsilon \in H_0^{\frac{1}{2}}(\mathbb{R}_+)$  i.e such that  $\Upsilon \in H^{\frac{1}{2}}$  and

$$\int_0^1 \frac{|\Upsilon|^2(z)}{z} dz < +\infty. \quad (4.3)$$

Let suppose moreover  $z\Upsilon \in H^{\frac{1}{2}}(\mathbb{R}^+)$ . Then there exist  $\mathbf{r} = (r_v, r_\psi) \in E_0$  verifying

$$\begin{aligned} L\mathbf{r} \in E'_1 \text{ and } \|L\mathbf{r}\|_{E'_1} &\leq C \|(1+z)\Upsilon\|_{H_0^{\frac{1}{2}}} \\ \mathbf{r}|_{z=0} = 0, r_v|_{y=0} = 0, r_\psi|_{y=0} = 0, \partial_y r_\psi|_{y=0} &= \Upsilon \end{aligned}$$

- Let  $V, v_0 \in H^{\frac{1}{2}}(\mathbb{R}_+)$  such that

$$\int_0^1 \frac{|V(\zeta) - v_0(\zeta)|^2}{\zeta} d\zeta < +\infty \quad (4.4)$$

and  $zV \in H^{\frac{1}{2}}$ . Then there exist  $\mathbf{r} = (r_v, r_\psi) \in E_0$  verifying

$$\begin{aligned} L\mathbf{r} \in E'_1 \\ r_v|_{z=0} = v_0, r_\psi|_{z=0} = 0, r_v|_{y=0} = V, r_\psi|_{y=0} = 0, \partial_y r_\psi|_{y=0} = 0 \end{aligned}$$

and

$$\|L\mathbf{r}\|_{E'_1} \leq C \left( \|(1+z)V\|_{H^{\frac{1}{2}}} + \|v_0\|_{H^{\frac{1}{2}}} + \sqrt{\int_0^1 \frac{|V(\zeta) - v_0(\zeta)|^2}{\zeta} d\zeta} \right).$$

The proof is exactly the same as the one of theorem 1.5.2.4 in Grisvard's book [41]. Once the compatibility conditions (4.3),(4.4) are verified, one can find  $r_v \in H^1(\Omega)$  and  $r_\psi \in H^2(\Omega)$  verifying the boundary conditions. The only difference is that we first need to lift  $(1+z)(V, \Upsilon, \Psi)$  and then divide the lifting by  $(1+z)$  to obtain the correct integrability of  $z\partial_y r$ .

Note that there is no physical sense of non zero  $\psi|_{y=0}$  in our problem. In fact this corresponds to the non penetration condition, and a non zero  $\Psi$  will create a pumping similar to the Ekman pumping. However we included this case for the sake of mathematical completeness.

Moreover these hypotheses are far from optimal, in fact we recover more regularity with respect to  $z$  than needed for  $L\mathbf{r}$ .

Once this lemma is established, by linearity, considering the equation for  $\mathbf{u} - \mathbf{r}$  with source term  $L\mathbf{r}$  we can solve the equation with source terms satisfying the hypothesis of the lemma.

In the rest of the paper we will thus consider only homogeneous boundary conditions at  $y = 0$ .

For a nonhomogeneous horizontal boundary condition in case (III) condition (4.4) then becomes

$$v|_{z=H} = v_H \in H_0^{\frac{1}{2}}.$$

This condition will be used in the formulation of the transparent boundary condition.

### 4.2.3 Duality principle (proof of (i))

To prove the first part of theorem 4.2.1 we will consider the equation as an elliptic equation, albeit a degenerate one. It will allow us to use classical functional analysis and to carefully encode the boundary conditions in the functional spaces.

Equation (4.2) can be seen formally as the following problem: find  $\mathbf{u} \in E_{0,0}$  such that  $\forall \boldsymbol{\varpi} = (w, \phi) \in \mathcal{D} \subset E_1$ ,

$$\begin{aligned} \langle L\mathbf{u}, \boldsymbol{\varpi} \rangle_{E'_1, E_1} &= \int_{\Omega} -v\partial_z\phi - zv\partial_y\phi - \frac{1}{2}\partial_y^2\psi\partial_y^2\phi + \int_{\Omega} -\psi\partial_zw - z\psi\partial_yw - \frac{1}{2}\partial_yv\partial_yw \\ &= + \int_{\Omega} s_\psi\phi + \int_{\Omega} s_vw = \langle \mathbf{s}, \boldsymbol{\varpi} \rangle_{E'_1, E_1} \end{aligned}$$

where  $L : E_{0,0} \rightarrow E'_1$  is a continuous linear operator as  $\|\mathbf{u}\|_{E_0} \leq \|\mathbf{u}\|_{E_{0,0}}$  and

$$\left| \langle L\mathbf{u}, \boldsymbol{\varpi} \rangle_{E_1, E'_1} \right| = \left| \int_{\Omega} \mathbf{u} \cdot (T\boldsymbol{\varpi}) + \frac{1}{2} \int_{\Omega} (\partial_y^2v\partial_y^2w + \partial_y\psi\partial_y\phi) \right| \leq (\|\mathbf{u}\|_{E_{0,0}} \|T\boldsymbol{\varpi}\|_{E'_{0,0}} + \|\mathbf{u}\|_{E_0} \|\boldsymbol{\varpi}\|_{E_0}).$$

Through usual functional analysis methods (typically Lions-Lax-Milgram theorem, see lemma 4.4.1 for details) we have at least one solution of  $L\mathbf{u} = \mathbf{s}$  as long there exist a coercivity inequality for the adjoint operator  $L^* : E_1 \rightarrow E'_{0,0}$ , i.e a constant  $C$  such that

$$\begin{aligned} \forall \boldsymbol{\varpi} \in \mathcal{D} \subset E_1 : \\ \|L^*\boldsymbol{\varpi}\|_{E'_{0,0}} \geq C \|\boldsymbol{\varpi}\|_{E_1}. \end{aligned}$$

We have for  $\boldsymbol{\varpi} \in \mathcal{D}$

$$\begin{aligned} \langle L^*\boldsymbol{\varpi}, \boldsymbol{\varpi} \rangle_{E'_{0,0}, E_{0,0}} &= \int_{\Omega} -\partial_zw\phi - z\partial_yw\phi - \frac{1}{2}\partial_y^2\phi\partial_y^2\phi + \int_{\Omega} -\partial_z\phi w - z\partial_y\phi w - \frac{1}{2}\partial_yw\partial_yw \\ &= -\frac{1}{2} \int_{\Omega} |\partial_y^2\phi|^2 + |\partial_yw|^2 - \int_{\Omega} \partial_z(w\phi) + z\partial_y(w\phi). \end{aligned}$$

Using the fact that  $w$  and  $\phi$  are in  $C_c^\infty((0, +\infty) \times [0, +\infty))$  Hardy's inequality (see for example [55]) reads as

$$\frac{1}{2} \int_{\Omega} |\partial_y^2\phi|^2 + |\partial_yw|^2 \geq C \|\boldsymbol{\varpi}\|_{E_{0,0}}^2.$$

The first term arising from the skew-symmetric part  $T$  is

$$\int_{\Omega} z\partial_y(w\phi) = 0.$$

The last term arising from the skew-symmetric part  $T$  is

$$- \int_{\Omega} \partial_z(w\phi).$$

For case (I) this term is 0 thanks to the boundary condition i.e the fact that  $\boldsymbol{\varpi} \in \mathcal{D}$ .

For case (II) we have

$$- \int_{\Omega} \partial_z(w\phi) = \int_y w|_{z=H} \Lambda^* w|_{z=H} dy \leq 0.$$



So for all cases

$$\langle L^* \varpi, \varpi \rangle_{E'_{0,0}, E_{0,0}} \leq -C \|\varpi\|_{E_{0,0}}^2$$

leading to the inequality

$$\|\varpi\|_{E_{0,0}} \leq C \|L^* \varpi\|_{E'_{0,0}}.$$

As  $\varpi \in \mathcal{D}$  we have  $T\mathbf{u} = -L^*\mathbf{u} - \frac{1}{2}D\mathbf{u}$  and as  $\|D\varpi\|_{E'_0} \leq c\|\varpi\|_{E_{0,0}}$

$$\|\varpi\|_{E_1} = \|\varpi\|_{E_{0,0}} + \|T\varpi\|_{E'_{0,0}} \leq \|\varpi\|_{E_{0,0}} + \|L^*\varpi\|_{E'_{0,0}} + c\|\varpi\|_{E_{0,0}} \leq C \|L^*\varpi\|_{E'_{0,0}}.$$

We recognize the coercivity inequality needed to prove the point (i) of the theorem.

It can be checked that all the other cases can be analyzed along the very same lines, the main point and only part where  $z$  boundary conditions appear being the sign of  $-\int_{\Omega} \partial_z(w\phi)$ .

#### 4.2.4 Boundary conditions (proof of (ii))

As functions in the energy space  $E_0$  do not display sufficient regularity to have traces at  $z = 0$ , we used the duality formulation to prescribe such boundary conditions in a weak sense. For example  $\psi|_{z=0} = 0$  means that for all  $w \in C_c^\infty((0, +\infty) \times [0, \infty])$  we have

$$\int_{\Omega} \psi \partial_z w = 0$$

It remains to demonstrate that for a sufficiently regular solution this formulation is equivalent to the aforementioned boundary conditions.

To do so, let us consider  $\mathbf{u} \in E_0 \cap H_{loc}^2$  a solution of (4.2) (note that all weak solutions for a smooth source term have interior regularity by the point (iii)). Then all considered traces are well defined.

Let  $h$  be a regular function such that  $h(0) = 1$ ,  $\text{supp } h \subset [0, 1)$  and  $g \in C_c^\infty((0, +\infty))$ . With  $\varpi_\eta = (w_\eta, \phi_\eta) = \left(g(y)h\left(\frac{z}{\eta}\right), 0\right) = (g(y)h_\eta(z), 0) \in \mathcal{D}$  used as a test function we get

$$-\int_{\Omega} \psi \partial_z w_\eta + \int_{\Omega} \left(-z \partial_y w_\eta \psi + \frac{1}{2} \partial_y^2 w_\eta v\right) = \int_{\Omega} s_v w_\eta.$$

As  $\|\varpi_\eta\|_{E_0} \rightarrow 0$  when  $\eta \rightarrow 0$ ,

$$\lim_{\eta \rightarrow 0} \int_{\Omega} \psi \partial_z w_\eta = 0.$$

But  $\partial_z h_\eta$  is approximating the identity, so

$$\int_{\Omega} \partial_z \psi w_\eta \rightarrow \int_0^\infty g(y) \psi(y, 0) dy$$

thus for all smooth  $g$

$$\int_0^\infty g(y)\psi(y,0)dy = 0$$

i.e  $\psi(y,0) = 0$ .

Similarly for case (II) with

$$\begin{aligned} w &= g(y)h_\eta(z) \\ \phi &= -\Lambda^*g(y)h_\eta(z) \end{aligned}$$

as  $\eta$  goes to 0 we obtain

$$\int_\Omega (\Lambda^*g(y)v - \psi g(y)) \partial_z h_\eta(z) = 0$$

which leads to

$$\int_0^\infty (\Lambda v_{y,H} - \psi_{y,H})(y)g(y)dy = 0$$

for all  $g$ . This is the expected result.

As for the previous point the other cases (notably  $v|_{z=H} = 0$ ) can be described along the same lines.

#### 4.2.5 Caccioppoli type inequality and interior regularity

In order to obtain interior regularity we use the elliptic character with respect to  $y$  (associated with the  $D$  part of the linear operator) to obtain Caccioppoli type inequalities with respect to  $y$ .

Let  $\theta$  a smooth function on  $\mathbb{R}$  such that

$$\theta(\zeta) \begin{cases} = 0 & \text{if } \zeta \in (-\infty, 0] \\ \in [0, 1] & \text{if } \zeta \in (0, 1) \\ = 1 & \text{if } \zeta \in [1, +\infty) \end{cases}$$

Let  $z - 1 > 0$ ,  $L > y_0 > 0$  and define

$$\chi(y, z) = \theta\left(\frac{2z_1 - z}{z_1}\right) \theta\left(\frac{2y - y_0}{y_0}\right) \theta\left(\frac{2L - y}{L}\right)$$

then  $\chi$  is a smooth cut-off function such that  $\chi = 0$  outside  $(y_0/2, 2L) \times [0, 2z_1]$  and  $\chi = 1$  inside  $[y_0, L] \times [0, z_1]$ .

Let  $\rho_\varepsilon$  an approximation of the identity with support inside  $\mathbb{R}^-$  and  $\mathbf{u}_\varepsilon = \rho_\varepsilon *_y \mathbf{u}$ . We have  $\mathbf{u}_\varepsilon$  smooth with respect to  $y$  and solution of equation (4.1) with a source term  $\mathbf{s}_\varepsilon = \rho_\varepsilon *_y \mathbf{s}$ . Using the equation we can deduce that  $\partial_z \mathbf{u}_\varepsilon$  is also smooth with respect to  $y$  so  $\partial_y^2 \mathbf{u}_\varepsilon \chi^4$  and its derivatives with respect to  $y$  are in  $E'_1$ .

With  $\partial_y^2(\partial_y^2 \mathbf{u}_\varepsilon \chi^4)$  as a test function, integrating by parts we obtain after cancellation of most skew-symmetric terms

$$\begin{aligned}
& - \int \partial_y^2 v_\varepsilon \partial_y^2 \psi_\varepsilon \partial_z \chi^4 - \int z \partial_y^2 v_\varepsilon \partial_y^2 \psi_\varepsilon \partial_y \chi^4 - \int \partial_y^6 \psi_\varepsilon \partial_y^2 \psi_\varepsilon \chi^4 + \int \partial_y^4 v_\varepsilon \partial_y^2 v_\varepsilon \chi^4 = \\
& \quad 2 \int \partial_y^2 S_{\varepsilon\psi} \partial_y^2 \psi_\varepsilon \chi^4 + \partial_y^2 S_{\varepsilon v} \partial_y^2 v_\varepsilon \chi^4.
\end{aligned}$$

Integrating by parts again, we get

$$\begin{aligned}
\int \partial_y^4 v_\varepsilon \partial_y^2 v_\varepsilon \chi^4 &= - \int |\partial_y^3 v_\varepsilon|^2 \chi^4 + \frac{1}{2} \int \partial_y^2 v_\varepsilon \partial_y^2 v_\varepsilon \partial_y^2 \chi^4 \\
&= - \int |\partial_y^3 v_\varepsilon|^2 \chi^4 - \frac{1}{2} \int \partial_y^3 v_\varepsilon \partial_y v_\varepsilon \partial_y^2 \chi^4 + \frac{1}{4} \int |\partial_y v_\varepsilon|^2 \partial_y^4 \chi^4 \\
\int \partial_y^6 \psi_\varepsilon \partial_y^2 \psi_\varepsilon \chi^4 &= \int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + \int \partial_y^4 \psi_\varepsilon \partial_y^2 \psi_\varepsilon \partial_y^2 \chi^4 - \int \partial_y^3 \psi_\varepsilon \partial_y^3 \psi_\varepsilon \partial_y^2 \chi^4 \\
&= \int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + 2 \int \partial_y^4 \psi_\varepsilon \partial_y^2 \psi_\varepsilon \partial_y^2 \chi^4 - \frac{1}{2} \int |\partial_y^2 \psi_\varepsilon|^2 \partial_y^4 \chi^4.
\end{aligned}$$

Moreover, defining  $\eta_1^{-1} = 16 \|\theta'\|_\infty^2 \left( \frac{1}{z_1} + z_1 \left( \frac{2}{y_0} + \frac{1}{L} \right) \right) \geq 16 \sup |(\partial_z + z \partial_y) \chi|^2$  and using Cauchy-Schwarz inequality

$$\begin{aligned}
\left| \int \partial_y^2 v_\varepsilon \partial_y^2 \psi_\varepsilon (\partial_z + z \partial_y) \chi^4 \right| &\leq \frac{1}{4\eta_1} \int (\partial_y^2 \psi_\varepsilon)^2 \chi^{4-2} + \eta_1 \int |\partial_y^2 v_\varepsilon|^2 4^2 |(\partial_z + z \partial_y) \chi|^2 \chi^4 \\
&\leq \frac{1}{4\eta_1} \int (\partial_y^2 \psi_\varepsilon)^2 \chi^{4-2} + \frac{\eta_1}{2} \int (|\partial_y v_\varepsilon|^2 + |\partial_y^3 v_\varepsilon|^2) 4^2 |(\partial_z + z \partial_y) \chi|^2 \chi^4 \\
&\leq \frac{1}{2} \int |\partial_y^3 v_\varepsilon|^2 \chi^4 + \int \left( \frac{1}{2} |\partial_y v_\varepsilon|^2 + \frac{1}{4\eta_1} |\partial_y^2 \psi_\varepsilon|^2 \chi^2 \right)
\end{aligned}$$

and similarly with  $\frac{1}{16} \eta_2^{-1} = \left( \|\theta''\|_\infty^2 \frac{4}{y_0^2} + \frac{1}{L^2} \right) + \left( \|\theta'\|_\infty^2 \frac{2}{y_0} + \frac{1}{L} \right)$

$$\begin{aligned}
\left| \int \partial_y^3 v_\varepsilon \partial_y v_\varepsilon \partial_y^2 \chi^4 \right| &\leq \frac{1}{4} \int |\partial_y^3 v_\varepsilon|^2 \chi^4 + \int |\partial_y v_\varepsilon|^2 (4^2 |\partial_y^2 \chi|^2 \chi^{4-2} + (4(4-1))^2 |\partial_y \chi|^4 \chi^{4-4}) \\
&\leq \frac{1}{4} \int |\partial_y^3 v_\varepsilon|^2 \chi^4 + \frac{1}{\eta_2} \int |\partial_y v_\varepsilon|^2
\end{aligned}$$

$$\begin{aligned}
\left| \int \partial_y^4 \psi_\varepsilon \partial_y^2 \psi_\varepsilon \partial_y^2 \chi^4 \right| &\leq \frac{1}{4} \int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + \int |\partial_y^2 \psi_\varepsilon|^2 (4^2 |\partial_y^2 \chi|^2 \chi^{4-2} + (4(4-1))^2 |\partial_y \chi|^4 \chi^{4-4}) \\
&\leq \frac{1}{4} \int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + \frac{1}{\eta_2} \int |\partial_y^2 \psi_\varepsilon|^2.
\end{aligned}$$

Therefore combining all these previous inequalities we end up with

$$\int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + \int |\partial_y^3 v_\varepsilon|^2 \chi^4 \leq c \int (\partial_y^2 S_{\varepsilon\psi} \partial_y^2 \psi_\varepsilon \chi^4 + \partial_y^2 S_{\varepsilon v} \partial_y^2 v_\varepsilon \chi^4) + C \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right) \int (|\partial_y v_\varepsilon|^2 + |\partial_y^2 \psi_\varepsilon|^2)$$

where  $c, C$  are numerical constants.

Using the fact that  $\chi \geq 0$ ,  $\chi = 1$  on  $(y_0, L) \times (0, z_1)$  and taking  $L \rightarrow +\infty$ , we finally obtain

$$\int |\partial_y^4 \psi_\varepsilon|^2 \chi^4 + \int |\partial_y^3 v_\varepsilon|^2 \chi^4 \leq C_{y_0, z_1} \left( \|\mathbf{u}_\varepsilon\|_{E_0}^2 + \|\partial_y^2 \mathbf{s}_\varepsilon\|_{E'_0}^2 \right).$$

The claimed estimate follows from  $\varepsilon \rightarrow 0$ .  
This concludes the proof of theorem 4.2.1.

### 4.3 Uniqueness and transparent boundary conditions

In order to prove that (4.1) admits a unique solution, we try to rely on an energy estimate. However the drawback of the weak formulation is that such an estimate makes no sense in the energy space as integrability with respect to  $z$  is missing. In other words, we cannot take  $\mathbf{u}$  as a test function. In this section we will show the uniqueness of the solution in a smaller space  $\widetilde{E}_0$ . It is to be noted that as the difficulties appear when  $z \rightarrow \infty$ , in case (II) we can recover uniqueness.

Once uniqueness is obtained we can reduce the study on the whole space to the study on a bounded (in  $z$ ) domain thanks to so called transparent boundary conditions. We will exhibit such boundary conditions and in the last part briefly see their explicit formulation in a simple setting.

#### 4.3.1 Uniqueness

The main obstacle to uniqueness is once more the lack of information with respect to  $z$  in the energy space. More precisely if, instead of a degenerate elliptic equation, we consider (4.1) as a transport equation, the transport term being  $\partial_z + z\partial_y$  with a cross-diffusion term  $\frac{1}{2} \begin{pmatrix} 0 & \partial_y^4 \\ -\partial_y^2 & 0 \end{pmatrix}$ , the main risk is the loss of mass along the characteristics  $y - \frac{z^2}{2} = c$ . Unfortunately, in the unbounded case we were not able to show that such a problem does not occur as such characteristics go to infinity. However up to a hypothesis of integrability we can show uniqueness of weak solutions.

Let  $\widetilde{E}_0$  be the set of all functions  $\mathbf{u} = (v, \psi) \in E_0$  such that  $v \in L^2$  and  $\psi \in L^2$  i.e

$$\|\mathbf{u}\|_{\widetilde{E}_0}^2 = \int_{\Omega} \left( |\partial_y v|^2 + \left| \frac{v}{1+y} \right|^2 + |v|^2 \right) + \int_{\Omega} \left( |\partial_y^2 \psi|^2 + \left| \frac{\psi}{1+y^2} \right|^2 + |\psi|^2 \right) \quad (4.5)$$

**Theorem 4.3.1.** *There exists at most one solution of (4.2) in  $\widetilde{E}_0$ .*

*Proof.* As before we will focus on the case (I), the other cases following similar analysis.

By linearity it is sufficient to show that if  $\mathbf{u} \in \widetilde{E}_0$  is a solution with homogeneous boundary conditions and  $\mathbf{s} = 0$  then  $\mathbf{u} = 0$ . Let  $\mathbf{u}$  be such a function.

The formal argument is the following. Define

$$\mathcal{E}(Z) = \int_0^\infty v(y, Z) \psi(y, Z) dy.$$

We obtain differentiating with respect to  $Z$

$$\frac{d\mathcal{E}}{dZ} = \frac{1}{2} \int_0^\infty |\partial_y^2 \psi|^2 (y, Z) + |\partial_y v|^2 (y, Z) dy \geq 0.$$

So  $\mathcal{E}$  is 0 at  $Z = 0$ ,  $L^1$  and non-decreasing. The only option is then  $\mathcal{E} = 0$  almost everywhere. This leads to

$$\int_{\Omega} |\partial_y^2 \psi|^2 + |\partial_y v|^2 = 0$$

i.e  $\psi = 0$  and  $v = 0$  considering the boundary conditions.

However we cannot apply directly this formal argument as it requires to use  $\mathbf{u}$  as a test function, which is not possible due to insufficient  $z$  regularity, i.e  $E_0 \not\subset E_1$ .

So let  $\mathbf{u}_\varepsilon$  be the convolution with respect to  $z$  of an approximation of the identity  $\rho_\varepsilon$  (with support in  $\mathbb{R}^-$ ) with  $\mathbf{u}$ .

Then  $\mathbf{u}_\varepsilon \in E_1 \cap \widetilde{E}_0$ , and the function

$$\mathcal{E}_\varepsilon(Z) = \int_0^\infty v_\varepsilon(y, Z) \psi_\varepsilon(y, Z) dy$$

is well defined in  $L^1$ . Moreover it is differentiable as  $v_\varepsilon, \psi_\varepsilon \in C_z^\infty(L_y^2)$  and using the fact that

$$\begin{aligned} \partial_z v_\varepsilon + z \partial_y v_\varepsilon - \frac{1}{2} \partial_y^4 \psi_\varepsilon &= r_\varepsilon^\psi \\ \partial_z \psi_\varepsilon + z \partial_y \psi_\varepsilon + \frac{1}{2} \partial_y^2 v_\varepsilon &= r_\varepsilon^v \end{aligned}$$

where  $\mathbf{r}_\varepsilon = z \partial_y \mathbf{u}_\varepsilon - \rho_\varepsilon *_z (z \partial_y \mathbf{u}) = (z \rho_\varepsilon(z)) *_z \partial_y \mathbf{u}$  (which goes to 0 in  $L^2$  when  $\varepsilon \rightarrow 0$ ), we obtain

$$\frac{d\mathcal{E}_\varepsilon}{dZ}(Z) = \frac{1}{2} \int_0^\infty (|\partial_y^2 \psi_\varepsilon|^2 + |\partial_y v_\varepsilon|^2) dy + \int_0^\infty (r_\varepsilon^\psi \psi_\varepsilon + r_\varepsilon^v v_\varepsilon) dy.$$

So

$$\mathcal{E}_\varepsilon \rightarrow \mathcal{E} \text{ in } L^1.$$

and

$$\frac{d\mathcal{E}_\varepsilon}{dZ} \rightarrow \frac{1}{2} \int_0^\infty (|\partial_y^2 \psi|^2 + |\partial_y v|^2) dy \text{ in } L^1.$$

From there  $\frac{d\mathcal{E}}{dZ} = \frac{1}{2} \int_0^\infty (|\partial_y^2 \psi|^2 + |\partial_y v|^2) dy$  as a distribution so

$$\mathcal{E} \in W^{1,1}$$

To conclude it remains to show that the now well-defined  $\mathcal{E}(0)$  is indeed 0.

By the Caccioppoli inequality of the theorem 4.2.1, for all  $a > 0$  the trace  $v|_{z=0, y>a}$  is well defined and  $\int_a^\infty v(y, 0) \psi(y, 0) dy = 0$ , so

$$\mathcal{E}(0) = 0$$

The previously formal argument can now be used to obtain uniqueness. □

It should be noted that we can obtain the uniqueness in  $E_0$  in the following variants:

- If the domain is bounded in  $z$  (case (II)) then using the interior  $z$  regularity, the boundaries at  $z = 0$  and  $z = H$  and Poincaré inequalities in the  $z$  variable we can recover a control of the  $L^2$  norm of  $u$ , the boundaries condition at  $z = H$  leading to  $\mathcal{E}(H) \leq 0$ .
- If the domain is  $y \in \mathbb{R}, z > 0$  then Fourier analysis leads easily to existence and uniqueness (see last subsection).

- If the equation includes additional zero order terms then the natural energy space (dictated by  $D$ ) is  $\widetilde{E}_0$  instead of  $E_0$  and thus include an  $L^2$  control so the existence and uniqueness is assured (see next subsection).
- If there is no transport term then the equation is the same as the one for the  $E^{\frac{1}{2}}$  Stewartson layer and uniqueness can once more be recovered with explicit Fourier analysis.

It is reasonable to hope that uniqueness indeed holds for the case (I) of (4.1) but we need to have a better control along characteristics to show it.

### 4.3.2 Transparent boundary conditions

Similarly to the Dirichlet to Neumann operator for elliptic problems (used for example by Gerard-Varet and Masmoudi [35] for Navier-Stokes equations), in this section we show that solving the equation on the whole space is equivalent to solving the same equation on the two subdomains ( $0 < z < H$  and  $z > H$ ) with adequate boundary conditions on both subdomains.

Such a decomposition can be used to focus the study in a bounded (with respect to  $z$ ) subdomain, which is especially useful for numerical analysis (as done in [54]) and for deriving boundary layer operators as in [18].

Unfortunately to make such a study a proper uniqueness result is needed. For this reason we will study variants of the initial problem, namely the one with additional zero order terms. It ensures that the energy norm controls the  $L^2$  norm. The modified equation reads as

$$\begin{aligned} \partial_z v + z \partial_y v - \frac{1}{2} \partial_y^4 \psi - \psi &= s_\psi \\ \partial_z \psi + z \partial_y \psi + \frac{1}{2} \partial_y^2 v - v &= s_v. \end{aligned} \tag{4.6}$$

As before, the boundary conditions at  $y = 0$  will always be  $v|_{y=0} = \partial_y \psi|_{y=0} = 0, \psi|_{y=0} = 0$ . The horizontal condition will be either (I), (II) or (III).

The previous analysis leads to both existence and uniqueness for (4.6). With  $\|\varpi\|_{\widetilde{E}_1} = \|\varpi\|_{\widetilde{E}_0} + \|\varpi\|_{\widetilde{E}_0'}$ , where  $\widetilde{E}_0$  is defined by (4.5), we have

**Lemma 4.3.2.** *There exist a weak solution of (4.6) in case (I), (II) and (III).*

*This solution is unique and*

$$\|\mathbf{u}\|_{\widetilde{E}_0} \leq \|\mathbf{u}\|_{\widetilde{E}_1'}.$$

*Moreover in case (I), if  $\partial_z \mathbf{s} \in \widetilde{E}_1'$  and  $\mathbf{s} \in \widetilde{E}_{0,0}$  we have  $\partial_z \mathbf{u} \in \widetilde{E}_0$  and*

$$\|\partial_z \mathbf{u}\|_{\widetilde{E}_0} \leq C \left( \|\partial_z \mathbf{s}\|_{\widetilde{E}_1'} + \|\mathbf{s}\|_{\widetilde{E}_0} \right).$$

*Proof.* The proof of this lemma is exactly the same as before, the only new point being the control on  $\partial_z \mathbf{u}$ . This comes from the fact that in this particular case we can deduce boundary conditions on  $\partial_z \mathbf{u}$ .

More precisely we have  $\partial_z \mathbf{u}$  verifying inside the domain

$$\begin{aligned} \partial_z(\partial_z v) + z \partial_y(\partial_z v) - \frac{1}{2} \partial_y^4(\partial_z \psi) - \partial_z \psi &= \partial_z s_\psi - \partial_y v \\ \partial_z(\partial_z \psi) + z \partial_y(\partial_z \psi) + \frac{1}{2} \partial_y^2(\partial_z v) - \partial_z v &= \partial_z s_v - \partial_y \psi \end{aligned} \tag{4.7}$$

and the boundary conditions at  $y = 0$  are  $\partial_z v = \partial_y \partial_z \psi = 0, \partial_z \psi = 0$ . Moreover, contrary to the original problem (4.1), we have  $(\partial_y \psi, \partial_y v) \in L^2$  and  $\widetilde{E}_1 \subset L^2$  so

$$\|(\partial_z s_v - \partial_y \psi, \partial_z s_\psi - \partial_y v)\|_{\widetilde{E}_1'} \leq C \left( \|\partial_z \mathbf{s}\|_{\widetilde{E}_1'} + \|\mathbf{s}\|_{\widetilde{E}_1'} \right).$$

All that remains is the boundary condition at  $z = 0$ . In case (I), the equation (4.1) leads to

$$\partial_z v|_{z=0} = s_\psi|_{z=0}$$

which is an admissible boundary condition.  $\square$

Once we have obtained this result we can now consider transparent boundary conditions.

**Theorem 4.3.3.** (i) (*v-to- $\psi$  operator*) For all  $H > 0$  there exists a non-positive operator  $\Lambda_H : H_0^{\frac{1}{2}} \rightarrow H^{-\frac{1}{2}}$  such that the only solution of (4.6) in the domain  $y > 0, z > H$  with boundary condition  $v|_{z=H} = V$  verifies  $\psi|_{z=H} = \Lambda_H V$ .

(ii) (*transparent BC*) Let  $H_0 > 0$  and let  $\mathbf{s}$  verifying the hypothesis of lemma 4.3.2 be a source term with support inside  $0 < z < H_0$ . For any  $H > H_0$  let  $\mathbf{u}^b$  be the solution of (4.6) on the domain  $y > 0, H > z > 0$  with boundary conditions of type (II)

$$\psi|_{z=0}^b = 0, \psi|_{z=H}^b = \Lambda_H v|_{z=H}^b$$

and let  $\mathbf{u}_t$  be the solution of (4.6) on  $y > 0, z > H$  with type (III) boundary condition

$$v|_{z=H}^t = v|_{z=H}^b$$

and zero source term. Then  $\mathbf{u}^b 1_{0 < z < H} + \mathbf{u}^t 1_{z \geq H}$  is the solution of (4.6) on the whole domain  $y > 0, z > 0$  with boundary condition  $\psi|_{z=0} = 0$ .

*Proof.* We start by the point (i), i.e the definition of the operator  $\Lambda_H$ .

For  $V \in H_0^{\frac{1}{2}}$  let  $\mathbf{u}^V$  the solution of (4.6) in case (III) with nonhomogeneous boundary condition  $v|_{z=H} = V$ . Let us recall that such a solution is obtained by considering homogeneous boundary condition but with a source term  $\mathbf{s}^V = L\mathbf{r}^V$  where  $\mathbf{r}^V$  is an appropriate lifting.

Similarly for any  $W \in H_0^{\frac{1}{2}}$  let  $\mathbf{u}^W$  be the solution of (4.6) with  $w|_{z=H} = W$ .

Using the same argument as in the proof of theorem 4.3.1

$$\mathcal{Q}(Z) = \int_0^\infty (v^W \psi^V)(y, Z) dy$$

is well defined and in  $W^{1,1}$  (note that  $\mathcal{E}$  is the quadratic form associated with the bilinear form  $\mathcal{Q}$ ) and

$$\frac{d\mathcal{Q}}{dZ} = \int_0^\infty (\partial_y^2 \psi^V \partial_y^2 \psi^W + \partial_y v^V \partial_y v^W + v^V v^W + \psi^V \psi^W)(y, Z) dy.$$

So as

$$\begin{aligned} \|\mathcal{Q}\|_{L^\infty} &\leq C\|\mathcal{Q}\|_{W^{1,1}} \leq C\left(\int |v^W\psi^V| + \int (\partial_y^2\psi^V\partial_y^2\psi^W + \partial_y v^V\partial_y v^W + v^V v^W + \psi^V\psi^W)\right) \\ &\leq C\|u^V\|_{\widetilde{E}_0}\|u^W\|_{\widetilde{E}_0} \leq C\|\mathbf{s}^V\|_{\widetilde{E}_1'}\|\mathbf{s}^W\|_{\widetilde{E}_1'} \\ &\leq C\|V\|_{H_0^{\frac{1}{2}}}\|W\|_{H_0^{\frac{1}{2}}} \end{aligned}$$

we obtain

$$\forall W \in H_0^{\frac{1}{2}}, \left| \int_0^\infty W\psi|_{z=H}^V dy \right| \leq C\|V\|_{H_0^{\frac{1}{2}}}\|W\|_{H_0^{\frac{1}{2}}}.$$

This means that  $\psi|_{z=H}^V \in H^{-\frac{1}{2}}$  and moreover the application  $\Lambda_H : V \mapsto \psi|_{z=H}^V$  is continuous from  $H_0^{\frac{1}{2}}$  to its dual space.

At last since  $\mathcal{Q}(Z) \rightarrow 0$  when  $Z \rightarrow \infty$  we have

$$\int_0^\infty V\Lambda_H V dy = \int_0^\infty V\psi|_{z=H}^V dy = - \int_H^\infty \int_0^\infty (|\partial_y^2\psi^V|^2 + |\partial_y v^V|^2 + |v^V|^2 + |\psi^V|^2) dy dz \leq 0.$$

and therefore  $\Lambda_H$  is a non-positive operator.

It remains to prove (ii) i.e that this condition is indeed a transparent boundary condition.

First of all let  $\mathbf{u}$  be the solution of (4.6) in case (I) and with source term  $\mathbf{s}$ .

Then by lemma 4.3.2  $v$  has proper trace in  $H_0^{\frac{1}{2}}$  and  $v|_{z=H}, \psi|_{z=H}$  are well defined. So  $\psi|_{z=H} = \Lambda_H v|_{z=H}$  and  $\mathbf{u}$  is a solution of (4.6) so by uniqueness in the case (II) we have  $\mathbf{u}|_{0 \leq z \leq H} = \mathbf{u}^b$ .

We deduce that  $\mathbf{u}|_{z=H}^b$  is well defined and is an admissible trace so  $\mathbf{u}^t$  is well defined and once more by uniqueness in case (III)  $\mathbf{u}|_{z \geq H} = \mathbf{u}^t$ .

We can prove this result without solving the problem on the whole space: by constructing  $\mathbf{u}$  from  $\mathbf{u}^b$  and  $\mathbf{u}^t$  in order to show that such  $v$ -to- $\psi$  operator is necessary to ensure the continuity of both  $v$  and  $\psi$ .

With  $\mathbf{u} = \mathbf{u}^b 1_{0 \leq z \leq H} + \mathbf{u}^t 1_{z \geq H}$  it is straightforward to see that the weak formulation on the whole space is verified for any test function with support inside  $0 < z < H$  or  $z > H$ .

Let  $\varpi = (w, \phi) \in \mathcal{D}$ . Let  $\chi$  be a smooth function such that  $\chi(s) = 0$  for  $|s| > 2$  and  $\chi(s) = 1$  for  $|s| < 1$ .

Then with  $\chi_\varepsilon(z) = \chi\left(\frac{z-H}{\varepsilon}\right)$  using the fact that  $\varpi = \varpi\chi_\varepsilon + \varpi(1 - \chi_\varepsilon)$  we obtain

$$\langle L\mathbf{u}, \varpi \rangle = \langle L\mathbf{u}, \varpi(1 - \chi_\varepsilon) \rangle + \langle L\mathbf{u}, \varpi\chi_\varepsilon \rangle = \langle \mathbf{s} + 0, \varpi(1 - \chi_\varepsilon) \rangle + \langle L\mathbf{u}, \varpi\chi_\varepsilon \rangle$$

as  $\varpi(1 - \chi_\varepsilon)$  is the sum of a function with support inside  $0 < z < H$  and a function with support inside  $z > H$ .

The last term is

$$\begin{aligned} &\int_\Omega -v(\partial_z\phi\chi_\varepsilon + \phi\partial_z\chi_\varepsilon) - zv\partial_y\phi\chi_\varepsilon - \frac{1}{2}\partial_y^2\psi\partial_y^2\phi\chi_\varepsilon + \psi\phi\chi_\varepsilon \\ &+ \int_\Omega -\psi(\partial_z w\chi_\varepsilon + w\partial_z\chi_\varepsilon) - z\psi\partial_y w\chi_\varepsilon - \frac{1}{2}\partial_y v\partial_y w\chi_\varepsilon + vw\chi_\varepsilon \end{aligned}$$

and we will show that it goes to 0 when  $\varepsilon \rightarrow 0$ .



Indeed as  $\mathbf{u} \in \widetilde{E}_0$  and  $\boldsymbol{\varpi} \in \widetilde{E}_1$  we have when  $\varepsilon \rightarrow 0$

$$\int_{\Omega} -v\partial_z\phi\chi_\varepsilon - zv\partial_y\phi\chi_\varepsilon - \frac{1}{2}\partial_y^2\psi\partial_y^2\phi\chi_\varepsilon + \psi\phi\chi_\varepsilon + \int_{\Omega} -\psi\partial_z w\chi_\varepsilon - z\psi\partial_y w\chi_\varepsilon - \frac{1}{2}\partial_y v\partial_y w\chi_\varepsilon + vw\chi_\varepsilon \rightarrow 0.$$

As  $\boldsymbol{\varpi}$  is identically 0 near  $y = 0$  and  $\mathbf{s} = 0$  near  $z = H$  using once more the same arguments as before

$$\int_{\Omega} v\phi\partial_z\chi_\varepsilon + \psi w\partial_z\chi_\varepsilon \rightarrow \int_{y=0}^{\infty} ((v^b - v^t)\phi + (\psi^b - \psi^t)w)(y, H)dy$$

which is zero, as the boundary conditions can be rewritten as  $v^b - v^t = 0$  and  $\psi^b - \psi^t = \Lambda_H v^b - \psi^t = \Lambda_H v^b - \Lambda_H v^t = 0$ .  $\square$

### 4.3.3 The case of the half plane

In the case where the domain is the half-plane  $z > 0$  existence and uniqueness are a lot more easier. In fact we can use Fourier transform. Denoting by  $\widehat{f}(\xi, z)$  the Fourier transform of  $f(y, z)$  with respect to  $y$  one can see that the problem can be rewritten as an ODE for each  $\xi$

$$\begin{pmatrix} \partial_z + zi\xi & -\xi^4 \\ -\xi^2 & \partial_z + zi\xi \end{pmatrix} \begin{pmatrix} \widehat{v} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} \widehat{s}_\psi \\ \widehat{s}_v \end{pmatrix}.$$

Hence with  $\widehat{w}_\pm = \widehat{v} \pm |\xi|\widehat{\psi}$  the problem is diagonalized

$$\partial_z \widehat{w}_\pm + (zi\xi \mp |\xi|^3)\widehat{w}_\pm = \widehat{s}_\pm$$

and the explicit solution is

$$\widehat{w}_\pm(\xi, z) = \widehat{w}_\pm(0)e^{-\frac{z^2}{2}i\xi}e^{\pm|\xi|^3z} + \int_0^z e^{-\frac{z^2-s^2}{2}i\xi}e^{\pm|\xi|^3(z-s)}\widehat{s}_\pm(s)ds$$

Note that two exponential modes appear: one in  $e^{-|\xi|^3z}$  and one in  $e^{|\xi|^3z}$ . To ensure that  $\widehat{w}_+$  and  $\widehat{w}_-$  are both in  $L^2$ , a necessary and sufficient condition is that the coefficient of  $\exp(|\xi|^3z)$  is 0. This offers another explanation of why only one condition at  $z = 0$  can be fixed.

For the transparent boundary condition, if there is no source term this condition simply becomes  $\widehat{w}_+(\xi, H) = 0$  i.e

$$\forall \xi, \widehat{v} + |\xi|\widehat{\psi} = 0$$

which in real space translates as  $\Lambda_H = -(-\Delta)^{\frac{-1}{2}}$ . This is exactly the condition used in [54] for the numerical approximation.

It is to be noted that in this case the operator  $\Lambda_H$  goes in fact from  $\dot{H}^{\frac{1}{2}}$  to  $\dot{H}^{\frac{3}{2}}$  which is the expected regularity as  $\partial_y\psi$  is of the same regularity as  $v$ .

But in our case because of the transport term we cannot use symmetries to extend (4.1) to the whole half space.

## 4.4 Appendix

### 4.4.1 Physical derivation

We recall here the main steps of the derivation of (4.1) and refer to [54] for details.

We consider the Stokes-Coriolis problem between two surfaces of revolution  $\Gamma_{\pm}$  (our main focus will be spheres of radius  $R_{\pm}$ ) and denote by  $(X, \Phi, Z)$  the cylindrical coordinates. The Stokes equation of an incompressible fluid rotating around the axis  $e_Z$  where we neglect the transport, in non-dimensional variables and with  $E$  the Ekman number, can be written as

$$\begin{aligned}\nabla p + e_Z \times U - E\Delta U &= 0 \\ \nabla \cdot U &= 0.\end{aligned}$$

We consider non-penetration boundary conditions on  $\Gamma_{\pm}$

$$\begin{aligned}U \cdot n &= 0 \\ U &= V_{\pm}e_{\Phi} + \Upsilon_{\pm}e_{\Phi} \times n.\end{aligned}$$

If we consider an axisymmetric flow,  $U = (U_X(X, Z), V(X, Z), U_Z(X, Z))$  then the incompressibility condition becomes  $\partial_X U_X + \partial_Z U_Z = 0$  so there exist a stream function  $\Psi$  such that

$$U = \begin{pmatrix} \partial_Z \Psi \\ V \\ -\partial_X \Psi \end{pmatrix}$$

The corresponding equations are

$$\begin{aligned}\partial_z V - E\Delta^2 \Psi &= 0 \\ \partial_z \Psi + E\Delta V &= 0\end{aligned}$$

and the boundary conditions

$$\begin{aligned}V|_{\Gamma^+} &= V_+, V|_{\Gamma^-} = V_- \\ \partial_n \Psi|_{\Gamma^+} &= \Upsilon_+, \partial_n \Psi|_{\Gamma^-} = \Upsilon_- \\ \Psi|_{\Gamma^{\pm}=0} &= 0\end{aligned}$$

When  $E \rightarrow 0$  we obtain the formal equations  $\partial_Z V = 0$ ,  $\partial_Z \Psi = 0$ . So at the main order in  $E$ , inside the domain

$$\begin{aligned}V(X, Z) &= V^0(X) + o(1) \\ \Psi(X, Z) &= 0 + o(1)\end{aligned}$$

In order to find  $V^0$  and pursue further the asymptotic expansion we must consider the boundary layer ensuring that the boundary conditions are satisfied.

Near a horizontal boundary (i.e constant  $Z$ ) we recover the classical Ekman scaling

$$\begin{aligned}V &= v \left( X, \frac{Z}{E^{\frac{1}{2}}} \right) \\ \Psi &= E^{\frac{1}{2}} \psi \left( X, \frac{Z}{E^{\frac{1}{2}}} \right)\end{aligned}$$

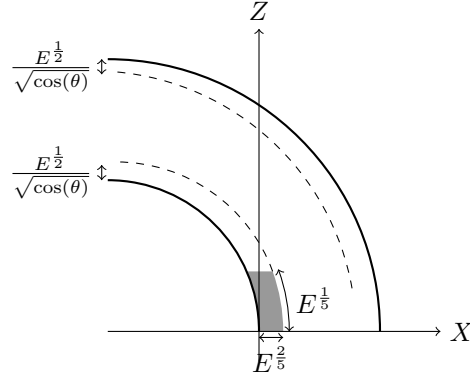


Figure 4.1 – The different scalings and boundary layers

and with  $(x, z)$  the rescaled variables the boundary equation is

$$\begin{aligned}\partial_z v - \partial_z^4 \psi &= 0 \\ \partial_z \psi + \partial_z^2 v &= 0.\end{aligned}$$

Note that the same equation holds for any boundary as long as  $\cos(\theta) = e_Z \cdot n$  does not approach 0 where  $\theta$  is the angle between the normal of the surface and the axis of rotation. In this case the scaling is

$$Z = \frac{z}{E^{1/2} \cos(\theta)^{-1/2}}$$

For a vertical boundary (i.e constant  $X$ ) the scaling is

$$\begin{aligned}V &= v \left( \frac{X}{E^{1/3}}, Z \right) \\ \Psi &= E^{1/3} \psi \left( \frac{X}{E^{1/3}}, Z \right)\end{aligned}$$

and the equation

$$\begin{aligned}\partial_z v - \partial_y^4 \psi &= 0 \\ \partial_z \psi + \partial_y^2 v &= 0.\end{aligned}$$

In the case of  $\cos(\theta)$  approaching 0 the previous scaling and equation are no longer correct.

If the boundary is  $Z = (-X)^\alpha 1_{X < 0}$ , denoting by  $Y = X + Z^{1/\alpha}$  the equation becomes

$$\begin{aligned}\left( \partial_Z + \alpha^{-1} Z^{1-\alpha} \partial_Y \right) V - E \left( \partial_Z^2 + \partial_Y^2 + 2\alpha^{-1} Z^{1-\alpha} \partial_Y \partial_Z \right)^2 \Psi &= 0 \\ \left( \partial_Z + \alpha^{-1} Z^{1-\alpha} \partial_Y \right) \Psi + E \left( \partial_Z^2 + \partial_Y^2 + 2\alpha^{-1} Z^{1-\alpha} \partial_Y \partial_Z \right) V &= 0\end{aligned}$$

The scaling is then

$$V = v \left( \frac{y}{E^{\frac{1}{3-\alpha}}}, \frac{z}{E^{\frac{\alpha}{3-\alpha}}} \right)$$

$$\Psi = E^{\frac{1}{3-\alpha}} \psi \left( \frac{y}{E^{\frac{1}{3-\alpha}}}, \frac{z}{E^{\frac{\alpha}{3-\alpha}}} \right)$$

and the associated equation becomes

$$\partial_z v + \alpha^{-1} z^{\frac{1-\alpha}{\alpha}} \partial_y v - \partial_y^4 \psi - E^{\frac{4(1-\alpha)}{(3-\alpha)}} \partial_z^4 \psi = 0$$

$$\partial_z \psi + \alpha^{-1} z^{\frac{1-\alpha}{\alpha}} \partial_y \psi + \partial_y^2 v + E^{\frac{2(1-\alpha)}{(3-\alpha)}} \partial_z^2 v = 0$$

with domain  $y > 0, z > 0$ .

The higher terms in  $\partial_z$  lead to another boundary layer of size  $E^{\frac{3(1-\alpha)}{2(3-\alpha)}}$  in  $z$  i.e a standard Ekman layer of size  $E^{\frac{3(1-\alpha)}{2(3-\alpha)} + \frac{\alpha}{1-\alpha}} = E^{\frac{1}{2}}$ . Note that this Ekman layer can be expressed in term of a boundary condition connecting  $v$  and  $\partial_z \psi$  but that in the physical case it is simply a symmetry condition,  $\psi = 0$ .

Considering only the higher order in  $E$  we obtain the announced equation for the spherical case  $\alpha = \frac{1}{2}$ .

Note that there are other boundary layers in the vicinity of the equator or of the cylinder  $X = R_-$ , but since we do not describe them in this paper we did not include them in figure 4.1. We refer to [69, 54] for a complete physical description.

#### 4.4.2 Duality argument

To prove existence of a solution we used a simpler version of Lions-Lax-Milgram [51] which can be rewritten as:

**Lemma 4.4.1.** *Let  $E$  and  $F$  two reflexive Banach spaces and*

$$L : E \rightarrow F'$$

*a continuous operator.*

*Let  $D \subset F$  a dense subspace of  $F$  and  $L^*$  the adjoint of  $L$  from  $F$  to  $E'$ .*

*If there exists a constant  $\gamma > 0$  such that*

$$\forall v \in D, \|L^*v\|_{E'} \geq \gamma \|v\|_F \quad (4.8)$$

*then for all  $f \in F'$  there exist a solution  $u$  of*

$$Lu = f$$

*with*

$$\|u\|_E \leq \frac{1}{\gamma} \|f\|_{F'}$$

The proof is elementary but as we did not find this exact formulation in the literature we detail the proof for the reader's convenience.

Let us first notice that the relation (4.8), also called observability inequality, ensures that  $L^*$

is injective. Thus the linear form

$$\begin{aligned}\phi : L^*D &\rightarrow \mathbb{R} \\ L^*v &\mapsto \langle f, v \rangle_{F',F}\end{aligned}$$

is well defined. Moreover it is continuous

$$|\phi(L^*v)| = |\langle f, v \rangle_{F',F}| \leq \|f\|_{F'} \|v\|_F \leq \frac{1}{\gamma} \|f\|_{F'} \|L^*v\|_{E'}.$$

As  $D$  is dense we can define  $\phi$  as a continuous form on  $L^*F \subset E'$ .

Using Hahn-Banach theorem, we then extend  $\phi$  as a linear continuous form on the whole  $E'$ . As  $E$  is a reflexive Banach space there exists  $u \in E$  such that

$$\forall g \in E', \langle u, g \rangle_{E,E'} = \phi(g)$$

and in particular

$$\forall L^*v \in L^*F, \langle u, L^*v \rangle_{E,E'} = \phi(L^*v) = \langle f, v \rangle_{F',F}$$

i.e

$$\forall v \in F, \langle Lu, v \rangle_{F',F} = \langle f, v \rangle_{F',F}.$$

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# MHD shear layer

## Outline of the current chapter

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<b>5.1 Introduction</b>	<b>129</b>
<b>5.2 Preliminary results</b>	<b>138</b>
<b>5.3 Hartmann layers</b>	<b>140</b>
<b>5.4 Toy model for the shear layer</b>	<b>148</b>
<b>5.5 Conclusion and perspectives</b>	<b>157</b>

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In this chapter we propose the study of a system closely related to the Proudman problem. From the point of view a physics we consider a different context (magnetohydrodynamics rather than geophysical fluids), although the governing equation share some similarities with the previous Chapter. This problem was suggested by Emmanuel Dormy during a discussion.

## 5.1 Introduction

### 5.1.1 Physical derivation and orders of magnitude

We consider a viscous incompressible and electrically conducting fluid between two rotating spheres and with an imposed exterior magnetic field, as proposed for example by Hollerbach [47]. This model can be seen as the simplest extension of the Proudman problem to MHD.

Let us briefly derive the equations of the problem. In addition to the fluid equation and Coriolis force, we must take into account:

- the magnetic force in Navier-Stokes equation,
- the Ohm law in a moving fluid,
- both Maxwell-Ampère and Maxwell-Faraday equations.

We end up with the equations of evolution of both the velocity  $u$  and magnetic fields  $B$  within the context of dissipative MHD

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u + \Omega e_3 \times u - \nu \Delta u - \frac{(\nabla \times B) \times B}{\rho \mu_0} + \frac{\nabla p}{\rho} &= 0 \\ \partial_t B + \nabla \times \left( \frac{\nabla \times B}{\sigma \mu_0} - u \times B \right) &= 0\end{aligned}$$

where  $\rho$  is the mass density and  $\mu_0$  is the magnetic permittivity of vacuum,  $\sigma$  the electrical conductivity, and  $\nu$  the kinematic viscosity ensuring ohmic and viscous dissipation.

The terms involving the rotational operators in both equations can be rearranged to obtain :

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u + \Omega e_3 \times u - \nu \Delta u - \frac{(B \cdot \nabla) B - \nabla B^2/2}{\rho \mu_0} + \frac{\nabla p}{\rho} &= 0 \\ \partial_t B - \nabla \times (u \times B) - \frac{\Delta B}{\sigma \mu_0} &= 0.\end{aligned}$$

Introducing the typical length  $L$  and velocity  $U$ , we can define four independent adimensional quantities describing the balances between the various dynamical processes described by these MHD equations, summarized in Table 5.1. Their typical values can be found in the book of Dormy and Soward [67].

Name	Symbol	Definition	Balance
Reynolds	$Re$	$UL/\nu$	Inertia/Viscosity
Rossby	$Ro$	$U/\Omega L$	Inertia/Rotation
Magnetic Reynold	$Rm$	$UL\mu_0\sigma$	Magnetic convection/Magnetic diffusion
Alfvenic Mach	$Ma$	$U\sqrt{\mu_0\rho}/B$	Mass velocity/Alfven velocity

Table 5.1 – Primary adimensional parameters.

The rescaled MHD equations in a rotating fluid are thus given by

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u + \frac{1}{Ro} e_3 \times u - \frac{1}{Re} \Delta u - \frac{1}{Ma^2} (B \cdot \nabla) B + \nabla p &= 0 \\ \partial_t B - \nabla \times (u \times B) - \frac{1}{Rm} \Delta B &= 0.\end{aligned}$$

Usually a set of three secondary adimensional parameters is introduced according to the definitions:

Name	Symbol	Definition	Relation
Ekman	$E$	$\nu/L^2\Omega$	$E = Ro/Re = \Lambda/M^2$
Elsasser	$\Lambda$	$\sigma B^2/\rho\Omega$	$\Lambda = RoRm/Ma^2 = M^2 E$
Hartmann	$M$	$L^2 B^2 \sigma / \rho \nu$	$M^2 = RmRe/Ma^2 = \Lambda/E$

Table 5.2 – Secondary adimensional parameters.

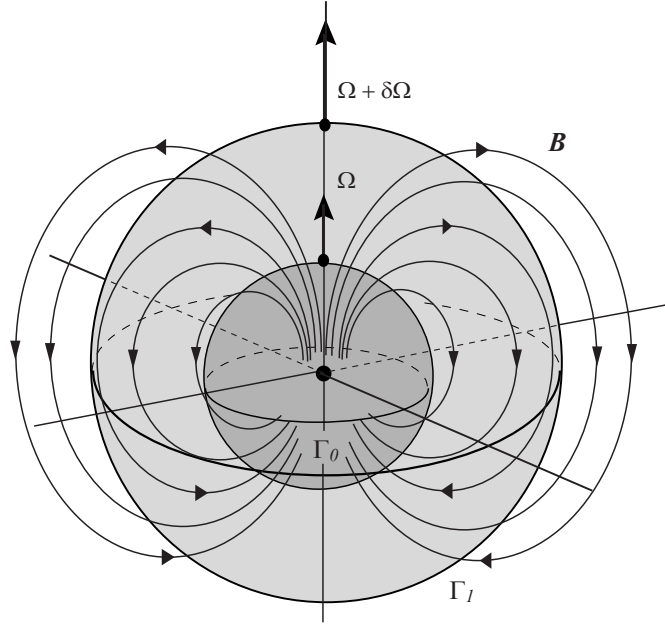


Figure 5.1 – A conductive fluid between two rotating spheres, and with a dipolar magnetic field.

We assume a static equilibrium between two rotating spheres

$$-\frac{\Lambda}{RoRm} (B \cdot \nabla) B + \nabla p = 0$$

$$\Delta B = 0$$

and consider a small differential in rotation between the two spheres,  $|\frac{\delta\Omega}{\Omega}| \ll 1$ . Then the linearization in an axisymmetric setting and in cylindrical coordinates  $(s, \phi, z)$  with a fluid velocity and induced magnetic field along  $e_\phi$ .

$$B = B + \frac{Rm}{M} b e_\phi$$

$$u = v e_\phi.$$

with  $|v| \ll 1$ ,  $|b| \ll 1$  leads to the coupled equations

$$-\Delta v e_\phi - M [(B \cdot \nabla) b e_\phi + (b e_\phi \cdot \nabla) B] = 0$$

$$-\Delta b e_\phi + M [(v e_\phi \cdot \nabla) B - (B \cdot \nabla) v e_\phi] = 0.$$

This can be rewritten as

$$\Delta v - \frac{v}{s^2} + M \left( \frac{B}{s} \cdot \nabla \right) s b = 0$$

$$\Delta b - \frac{b}{s^2} + M (s B \cdot \nabla) \frac{v}{s} = 0.$$

or

*Remark 5.1.1.* Following Kleorin & al. [48] and Starchenko [68], we can also consider a more



general linearization as it was done in the Proudman problem (3.7), with

$$u = ve_\phi + \begin{pmatrix} -\partial_z \\ 0 \\ s^{-1}\partial_s(s\cdot) \end{pmatrix} \phi$$

$$B = B + be_\phi + \begin{pmatrix} -\partial_z \\ 0 \\ s^{-1}\partial_s(s\cdot) \end{pmatrix} \alpha.$$

In this case the equations we obtain are

$$\begin{aligned} 2\partial_z\psi + E(\Delta - s^{-2})v + \Lambda\frac{1}{s}(B \cdot \nabla)(sb) &= 0 \\ 2\partial_zv - E(\Delta - s^{-2})^2\psi + s\Lambda(B \cdot \nabla)(s^{-1}(B \cdot \nabla)\psi) &= 0 \\ Ms(B \cdot \nabla)\left(\frac{1}{s}v\right) + (\Delta - s^{-2})b &= 0 \\ \frac{M}{s}(B \cdot \nabla)(s\psi) + (\Delta - s^{-2})\alpha &= 0 \end{aligned} \tag{5.1}$$

We remark that for  $\Lambda = 0$  we recover the Proudman equation (3.9), while our current setting is the formal  $|\psi| \ll 1$ .

*Remark 5.1.2.* Note the non zero commutators  $s^{\pm 1}(B \cdot \nabla)(s^{\mp 1}\bullet)$ . We will see that these terms create a coupling in the equations written in the so called Alfvén variables.

The boundary condition for the azimuthal velocity is simply the no-slip condition on a moving shell, i.e

$$v|_{\Gamma_0} = v_0, v|_{\Gamma_1} = v_1.$$

For the field  $b$  the main physical condition is to have the continuity of both  $b$  and its normal derivative. The associate boundary condition varies following the electrical properties of the shells. More precisely, according [68] and [22], we must distinguish the insulating shells from the conductive shells.

- For an insulating shell the boundary condition is

$$b|_\Gamma = 0$$

- For a conducting sphere we must have across the boundary

$$\begin{aligned} [b]_\Gamma &= 0 \\ \left[\frac{1}{\sigma}\partial_n b\right]_\Gamma &= 0 \end{aligned}$$

where  $\sigma$  is the conductivity. The magnetic field  $b$  inside the conductive sphere  $b$  satisfying the equation

$$(\Delta - a)b = 0 \tag{5.2}$$

where  $a(s, z) = \frac{1}{s^2}$  i.e the main equation with  $v = 0$ .

From a mathematical point of view, the continuity of  $b$  and  $\partial_n b$  can be expressed through the

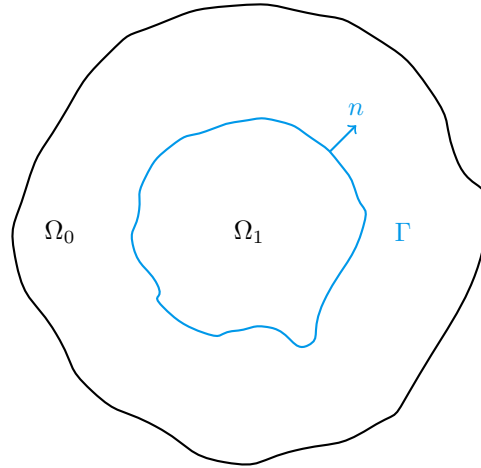


Figure 5.2 – Illustration of Lemma 5.1.1.

Dirichlet to Neumann operator of the equation (5.2)

$$\partial_n b|_{\Gamma} = -\Lambda b|_{\Gamma}$$

where  $-\Lambda$  is the Dirichlet to Neumann operator associated with the equation (5.2). As this result is very useful, let us recall the simple demonstration of this fact.

**Lemma 5.1.1.** *Let  $\Omega_1 \subset \Omega_0$  be two smooth connected domains and let  $\Gamma = \partial\Omega_1$  and  $n$  the normal vector from  $\Omega_1$  to  $\Omega_0$  (see Figure 5.2). Let  $a \in L^\infty(\Omega_1)$ . There exists a continuous and negative operator  $\Lambda : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$  such that*

- Any  $b \in H^1(\Omega_0)$  such that  $b$  is solution of (5.2) inside  $\Omega_1$  and  $[b]_{\Gamma} = 0$  and  $[\partial_n b]_{\Gamma} = 0$  satisfies

$$\partial_n b|_{\Gamma} + \Lambda b|_{\Gamma} = 0.$$

- For any  $\tilde{b} \in \Omega_0 \setminus \overline{\Omega_1}$  such that

$$\partial_n \tilde{b}|_{\Gamma} + \Lambda \tilde{b}|_{\Gamma} = 0.$$

there exists a unique  $b$  in  $H^1(\Omega_0)$  satisfying

- $b|_{\Omega_0 \setminus \overline{\Omega_1}} = \tilde{b}$ ,
- $b$  solution of (5.2) inside  $\Omega_1$ ,
- $[b]_{\Gamma} = 0$  and  $[\partial_n b]_{\Gamma} = 0$ .

*Remark 5.1.3.* In most practical cases  $\Lambda$  can be explicitly computed. For example, if  $\Omega_1$  is the unit disk and  $a = 1$  we obtain, in terms of Bessel functions,

$$-\Lambda e^{in\theta} = \frac{I'_{|n|}(1)}{I_{|n|}(1)} e^{in\theta}.$$

Note that, thanks to the recurrence relations of Bessel function, we have  $\frac{I'_{|n|}(1)}{I_{|n|}(1)} \sim n$ .

*Proof.* The main point is to establish the properties of the operator  $\Lambda$ .

Let  $T \in H^{\frac{1}{2}}(\Gamma)$ . Then, there exists  $b_T \in H^1(\Omega_1)$  solution of

$$\begin{aligned}\Delta b_T - ab_T &= 0 \text{ in } \Omega_1 \\ b_T|_{\Gamma} &= T \text{ on } \partial\Omega_1.\end{aligned}$$

Moreover,  $\|b_T\|_{H^1(\Omega_1)} \leq C\|T\|_{H^{\frac{1}{2}}(\Gamma)}$ .

We then define  $\Lambda T$  by

$$-\Lambda(T) = \partial_n b_T|_{\Gamma} \in H^{-\frac{1}{2}}(\Gamma).$$

The operator  $T$  is obviously linear. And, by definition, for any  $\beta \in H^1(\Omega_1)$  we have

$$-\int_{\Gamma} \Lambda(T)\beta|_{\Gamma} = \int_{\Omega_1} \nabla b_T \cdot \nabla \beta + \int_{\Omega_1} ab\beta.$$

For any  $\Theta \in H^{\frac{1}{2}}(\Gamma)$ , there exist  $\beta \in H^1(\Omega_1)$  such that  $\beta|_{\Gamma} = \Theta$  and  $\|\beta\|_{H^1(\Omega_1)} \leq C\|\Theta\|_{H^{\frac{1}{2}}(\Gamma)}$ , with  $C$  a universal constant depending only on  $\Omega_1$ . Thus

$$\begin{aligned}|\langle \Lambda T, \Theta \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} &\leq \int_{\Omega_1} \nabla b_T \cdot \nabla \beta + \int_{\Omega_1} a(s)b\beta \\ &\leq \|\beta\|_{H^1(\Omega_1)} \|b\|_{H^1(\Omega_1)} \\ &\leq C\|\Theta\|_{H^{\frac{1}{2}}(\Gamma)} \|b\|_{H^1(\Omega_1)} \leq C\|\Theta\|_{H^{\frac{1}{2}}(\Gamma)} \|T\|_{H^{\frac{1}{2}}(\Gamma)}.\end{aligned}$$

This is the case  $\forall \Theta \in H^{\frac{1}{2}}(\Gamma)$ , so  $\|\Lambda T\|_{H^{-\frac{1}{2}}} \leq C\|T\|_{H^{\frac{1}{2}}}$ , and  $\Lambda$  is continuous. And with  $\beta = b$  we recover the positivity of  $-\Lambda$ .

The first point of the Lemma is verified by definition, note that, thanks to the condition  $[\partial_n b] = 0$ , there is no difficulty when defining  $\partial_n b|_{\Gamma}$ .

For the second point, let  $\tilde{b} \in H^1(\Omega_0 \setminus \overline{\Omega_1})$  such that  $(\partial_n \tilde{b} + \Lambda \tilde{b})|_{\Gamma} = 0$ .

We define  $b$  by  $b|_{\Omega_0 \setminus \overline{\Omega_1}} = \tilde{b}$ , and  $b|_{\Omega_1}$  solution of

$$\begin{aligned}\Delta b|_{\Omega_1} - a(s)b|_{\Omega_1} &= 0 \text{ in } \Omega_1 \\ b|_{\Omega_1} &= \tilde{b}|_{\Gamma} \text{ on } \partial\Omega_1.\end{aligned}\tag{5.3}$$

Then  $[b]_{\Gamma} = 0$  by definition. By the jump-formula, as  $b$  is  $H^1$  on both subdomains and has no Dirichlet jump  $b \in H^1(\Omega_0)$ . And as  $\partial_n b|_{\Omega_1} = -\Lambda b|_{\Omega_1}$ ,  $\partial_n b|_{\Omega_0 \setminus \overline{\Omega_1}} = -\Lambda b|_{\Omega_0 \setminus \overline{\Omega_1}}$  we also obtain the continuity of the normal derivative across  $\Gamma$ . The uniqueness follows from the uniqueness of (5.3).  $\square$

This physical problem exhibits multiple boundary layers when  $M \gg 1$ , see Figure 5.3, but before let us first simplify the problem.

### 5.1.2 Main setting

If the previous model, even if the domain  $\Omega \subset \mathbb{R}^2$ , all the geometry comes from a three-dimensional problem, for example for all integration by parts in a circular domain must use  $\nabla \cdot (w) = \partial_r w_r + \frac{1}{r} w_r + \frac{1}{r} \partial_{\theta} w_{\theta}$ . Thus, for simplicity, we choose to study a fully two-dimensional

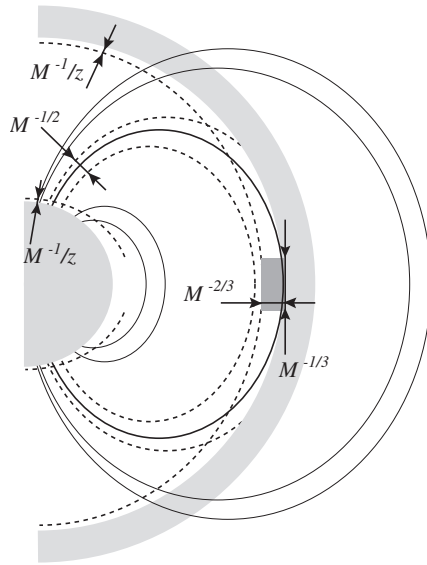


Figure 5.3 – The expected boundary layers.

model

$$\begin{aligned} M(B \cdot \nabla + a(s, z))b + (\Delta - c(s, z))v &= 0 \\ M(B \cdot \nabla - a(s, z))v + (\Delta - c(s, z))b &= 0 \end{aligned} \quad (5.4)$$

where  $a \geq a_0 > 0$ ,  $c \geq c_0 > 0$  are regular functions, representing arbitrarily the zeroth order coupling.

The only difference in this two-dimensional model, is that  $B$  now satisfies  $B = \nabla^\perp A = -\nabla \phi$  (instead of  $B = \nabla A \times (e_\phi s^{-1})$ ). So the dipolar field is

$$\begin{aligned} A &= \frac{s}{r^2} \\ \phi &= \frac{z}{r^2} \\ B &= \left( \frac{2sz}{r^4}, \frac{z^2 - s^2}{r^4} \right) \end{aligned}$$

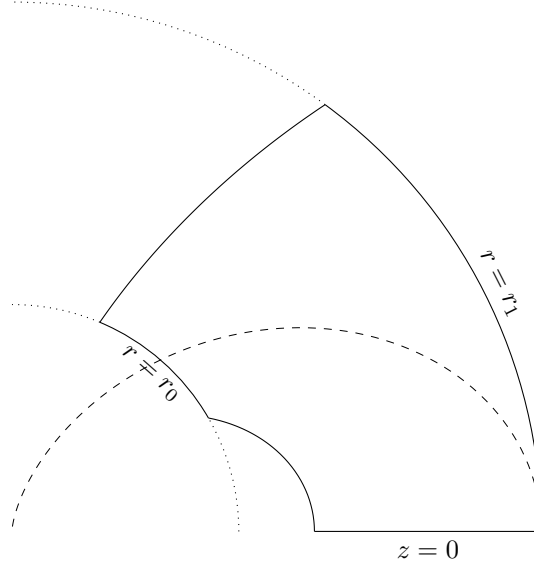
instead of the formulas (2.1) of [22].

In order to focus on the shear layer we will consider the equation on the domain defined by  $r \in (r_0, r_1)$  and  $A \in (A_0, A_1)$  where  $A_0$  and  $A_1$  are chosen such that  $A^* \in (A_0, A_1)$ . This choice allows us to avoid the equatorial boundary layer near  $r = r_0, z = 0$  and simplify the analysis, since iso- $A$  surfaces are characteristics of the interior term.

Lastly thank to symmetries we can limit ourselves to  $z > 0$ .

With a source term  $\mathbf{f}$ , the system we will consider is, inside  $\Omega$ ,

$$\begin{aligned} M(B \cdot \nabla + a)b + (\Delta - c)v &= Mf_v \\ M(B \cdot \nabla - a)v + (\Delta - c)b &= Mf_b \end{aligned} \quad (5.5)$$

Figure 5.4 – Physical domain  $\Omega$ .

with homogeneous boundary conditions

$$\begin{aligned}
 v|_{r=r_1} &= 0, v|_{r=r_0} = 0 \\
 b|_{r=r_1} &= 0, (\partial_r b + \Lambda b)|_{r=r_0} = 0 \\
 \partial_z v|_{z=0} &= 0, b|_{z=0} = 0 \\
 v|_{A=A_0} &= 0, b|_{A=A_0} = 0 \\
 v|_{A=A_1} &= 0, b|_{A=A_1} = 0.
 \end{aligned} \tag{5.6}$$

Note that, if, instead of the variables  $(s, z)$ , we use the variables  $(A, \phi)$  we obtain (using only  $\nabla^\perp A = -\nabla\phi$ )

$$\begin{aligned}
 M(|B|^2 \partial_\phi + a(s, z)) b + (|B|^2 \Delta_{A, \phi} - c(s, z)) v &= M f_v \\
 M(|B|^2 \partial_\phi - a(s, z)) v + (|B|^2 \Delta_{A, \phi} - c(s, z)) b &= M f_b
 \end{aligned} \tag{5.7}$$

so the dipole problem maps exactly to the constant  $B = B e_z$  problem up to a change of  $a$  and  $c$ .

As a consequence, the equation we will consider in all generality will be

$$\boxed{
 \begin{aligned}
 M(-\partial_z + a(s, z)) b + (\Delta - c(s, z)) v &= M f_v \\
 M(-\partial_z - a(s, z)) v + (\Delta - c(s, z)) b &= M f_b.
 \end{aligned}
 } \tag{5.8}$$

Introducing the Alfvén variables  $V_\pm = v \pm b$  these equations write also as

$$\begin{aligned}
 M(\partial_z V_+ - a V_-) - (\Delta - c) V_+ &= M F_+ \\
 M(\partial_z V_- - a V_+) + (\Delta - c) V_- &= M F_-
 \end{aligned} \tag{5.9}$$

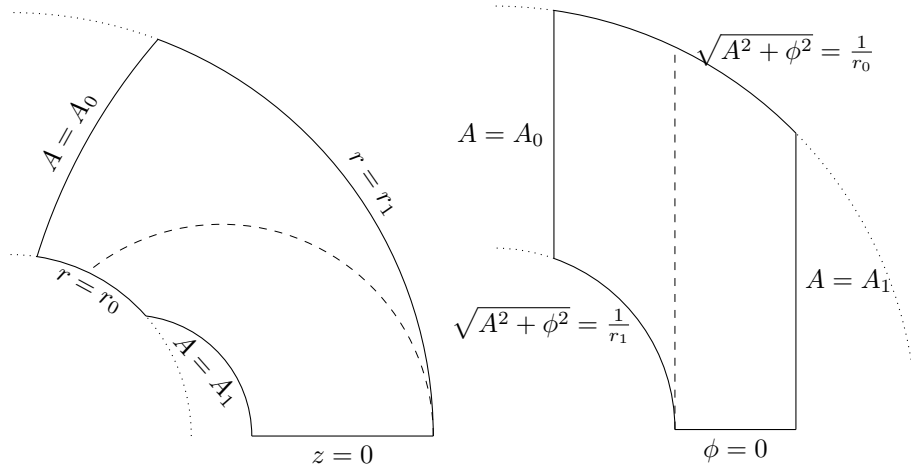


Figure 5.5 – Reparametrization of the domain

with boundary conditions

$$\begin{aligned}
 V_{+|r=r_1} &= V_{-|r=r_1} = 0 \\
 V_{+|r=r_1} + V_{-|r=r_1} &= 0, \quad (\partial_r (V_+ - V_-) + \Lambda(V_+ - V_-))|_{r=r_1} = 0 \\
 (\partial_z V_+ + \partial_z V_-)|_{z=0} &= 0, \quad (V_+ - V_-)|_{z=0} = 0 \\
 V_{+|A=A_0} &= V_{-|A=A_0} = V_{+|A=A_1} = V_{-|A=A_1} = 0.
 \end{aligned} \tag{5.10}$$

Moreover, we will also change the nature of the transparent boundary condition: instead of taking for  $\Lambda$  the Dirichlet to Neumann operator we will simply take  $-\Lambda$  to be a positive constant.

*Remark 5.1.4.* This simplification keep the main properties of the Dirichlet to Neumann operator, and many results remain the same. The only difference is in the construction of the Hartmann layers when tangential frequency goes to  $+\infty$ . This simplification will be discussed in Subsection 5.3.3.

*Remark 5.1.5.* This system of equation is very close to the Proudman problem studied in Chapter 3. Let us emphasis some differences

- First of all, one of the main difficulties of the Proudman problem was the asymmetry between  $v$  and  $\psi$ , as the equivalent of Alfvén variable  $V_{\pm}$  would be the non-local  $w_{\pm} = v \pm (-\Delta)^{\frac{1}{2}}\psi$ , leading to difficulties when considering boundary conditions. Consequently, we can expect easier estimates.
- The geometrical coupling  $a$  seems not that important but in fact is of importance when considering for example the shear layer, see Section 5.4.
- More importantly the boundary conditions are a crucial point of this model. Indeed even for  $a = 0$  the problem is non-obvious as  $V_{\pm}$  are linked through the boundary conditions.

## 5.2 Preliminary results

### 5.2.1 Existence, uniqueness and stability estimates

As usual, we start by deriving estimates for the main problem.

**Proposition 5.2.1.** *For any  $\mathbf{f} \in H^{-1} \times H^{-1}$  there exists only one solution of (5.8), (5.6) satisfying*

$$\|v\|_{H^1}^2 + \|b\|_{H^1}^2 \leq CM (\|f_b\|_{H^{-1}}^2 + \|f_v\|_{H^{-1}}^2).$$

More over if  $\mathbf{f} \in L^2 \times L^2$  then

$$\|v\|_{L^2}^2 + \|b\|_{L^2}^2 \leq C (\|f_b\|_{L^2}^2 + \|f_v\|_{L^2}^2).$$

*Proof.* Let  $E$  be the space of  $(u, v) \in H^1 \times H^1$  satisfying  $v|_{r=r_0} = v|_{r=r_1} = b|_{r=r_1} = 0$ .

The equation can be rewritten as: find  $(b, v)$  such that for all  $(\omega, \beta) \in E$  we have

$$M \int_{\Omega} (\partial_z b \omega + \partial_z v \beta + a(v\beta - b\omega)) + \int_{\Omega} (\nabla v \cdot \nabla \omega + cv\omega + \nabla b \cdot \nabla \beta + cb\beta) - \int_{r=r_0} \Lambda b \beta = M \int_{\Omega} (f_v \omega + f_b \beta)$$

From there, as  $-\Lambda > 0$ , the existence follows and with  $(\omega, \beta) = (v, b)$  we obtain the claimed  $H^1$  estimate.

The idea to obtain the  $L^2$  estimate is to take a test function like  $(\omega, b) = (b, v)$ . Note that, contrary to Proudman problem, there is no longer difficulties emerging from the asymmetry in regularity between  $v$  and  $\psi$ , as both  $v$  and  $b$  are in  $H^1$ . Unfortunately, because of the Robin condition on  $b$ , it does not match the boundary conditions.

We can easily recover the estimate on  $v$ , with  $(\omega, \beta) = (0, ve^{h(z)})$  we obtain

$$M \int_{\Omega} \left( 0 + \partial_z v v e^{h(z)} + a v^2 e^{h(z)} \right) + \int_{\Omega} \left( \nabla b \cdot \nabla (v e^{h(z)}) + c b v e^{h(z)} \right) = M \int_{\Omega} f_b v e^{h(z)}$$

and from there

$$M \int_{\Omega} \left| a - \frac{h'}{2} \right| |v|^2 e^h \leq \underbrace{\left| \int_{\Omega} \nabla b \cdot \nabla v e^h \right|}_{\leq C_h M \|\mathbf{f}\|_{H^{-1}}} + \underbrace{\left| \int_{\Omega} (\partial_z b h' + c b) v e^h \right|}_{\leq C_h \sqrt{M} \sqrt{\int |v|^2 e^h}} + M C_h \sqrt{\int f_b^2} \sqrt{\int |v|^2 e^h}$$

so with  $h(z) = \lambda z$  with  $\lambda$  large enough (but independent of  $M$ ) we obtain the claimed  $L^2$  inequality on  $v$ .

To obtain the same on  $b$  we reiterate this method but in Alfvén variables. This will allow to obtain bounds on  $b + v$ , which is sufficient as we already have bound on  $v$ . With  $V_{\pm} = e^{\lambda z} \tilde{V}_{\pm}$ , we have

$$\begin{aligned} M(\partial_z \tilde{V}_+ + \lambda \tilde{V}_+ - a \tilde{V}_-) - (\Delta + 2\lambda \partial_z + \lambda^2 - c) \tilde{V}_+ &= M \tilde{F}_+ \\ M(\partial_z \tilde{V}_- + \lambda \tilde{V}_- - a \tilde{V}_+) + (\Delta + 2\lambda \partial_z + \lambda^2 - c) \tilde{V}_- &= M \tilde{F}_- \end{aligned}$$

so with  $\Omega_R = \Omega \cap \{r \leq R\}$ , for  $r_0 \leq R < r_1$  we obtain dropping the  $\tilde{\cdot}$  and taking, from a formal point of view, as test-function in the first equation  $V_+ \mathbf{1}_{\Omega_R}$

$$(M - 2\lambda) \int_{\Omega_R} \frac{1}{2} \partial_z (V_+^2) + \int_{\Omega_R} ((\lambda M + c - \lambda^2) V_+^2 - a M V_+ V_-) + \int_{\Omega_R} |\nabla V_+|^2 - \int_{\Gamma_R} \partial_r V_+ V_+ \\ = M \int_{\Omega_R} F_+ V_+$$

$$(M - 2\lambda) \int_{\Gamma_R} \frac{1}{2} |V_+|^2 + \int_{\Omega_R} ((\lambda M + c - \lambda^2) V_+^2 + a M V_+ (V_+ - 2v)) \\ + \int_{\Omega_R} |\nabla V_+|^2 - \partial_R \int_{\Gamma_R} |V_+|^2 + \frac{1}{R} \int_{\Gamma_R} |V_+|^2 = M \int_{\Omega_R} F_+ V_+$$

with  $E^+(R) = \int_{\Omega_R} |V_+|^2$  (all the previous computations can be justified by computing  $\partial_R E_+$  in a weak sense) we thus obtain

$$(M - 2\lambda + R_{\max}^{-1}) \partial_R E^+ + ((\lambda + a_{\min} - \eta) M + c - \lambda^2) E^+ - \partial_R^2 E^+ \leq M \frac{1}{4\eta} \|F\|_{L^2} - \int_{\Omega_R} |\nabla V_+|^2$$

i.e

$$\alpha_1 \partial_R E^+ + \alpha_0 E^+ - M^{-1} \partial_R^2 E^+ = G$$

where  $\|G\|_{L^\infty} \leq C$  and  $\alpha_1, \alpha_0$  far from 0 uniformly in  $M$  for  $\lambda$  large enough (but independent of  $M$ ).

More over we know that  $E^+(r_0) = 0$ ,  $E^+$  is increasing and  $\partial_R E^+(r_1) = \int_{\Gamma_{r_1}} |b|^2 = O(M)$ .

From there, for example by integrating with respect to  $R$ , we can conclude that  $E_+(r_1) = O_{M \rightarrow \infty}(1)$ , leading to the  $L^2$  stability.  $\square$

Without surprise we will consider an approximate problem

$$M(-\partial_z + a(s, z)) b^{app} + (\Delta - c(s, z)) v^{app} = M(f_v + r_v) \\ M(-\partial_z - a(s, z)) v^{app} + (\Delta - c(s, z)) b^{app} = M(f_b + r_b) \quad (5.11)$$

the energy estimate being also a stability estimate: if  $\mathbf{u}^{app}$  satisfies the approximate equation 5.11 with the exact boundary conditions (5.11) then

$$\|v - v^{app}\|_{H^1}^2 + \|b - b^{app}\|_{H^1}^2 \leq CM (\|r_v\|_{H^{-1}}^2 + \|r_b\|_{H^{-1}}^2) \\ \|v - v^{app}\|_{L^2}^2 + \|b - b^{app}\|_{L^2}^2 \leq C (\|r_v\|_{L^2}^2 + \|r_b\|_{L^2}^2) \quad (5.12)$$

From there, and following once more Dalibard and Saint-Raymond [17], we search an approximate solution such that by denoting  $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1$  the error term:

**Definition 5.2.2.** A function  $\mathbf{u}^{app} = (v^{app}, b^{app}) \in H^1 \times H^1$  is said to be an approximate solution of (5.8)-(5.6) if it is a solution of (5.11) with the exact boundary conditions (5.6), such that  $\mathbf{r} = (r_v, r_b) = \mathbf{r}^0 + \mathbf{r}^1$  is an acceptable remainder, meaning it satisfies

$$\lim_{M \rightarrow \infty} \frac{\|\mathbf{r}_0\|_{L^2} + M \|\mathbf{r}_1\|_{H^{-1}}}{M \|\mathbf{u} - \mathbf{u}^{app}\|_{L^2} + \|\mathbf{u} - \mathbf{u}^{app}\|_{H^1}} = 0.$$



## 5.2.2 Expected boundary layers

As mentioned in the introduction, we expect three kind of boundary layers, see Figure 5.3:

- The Hartmann layers of size  $M^{-1}/z$  near the shells  $r = r_0$  and  $r = r_1$ . These comes from the scaling  $\partial_z = \cos(\theta)d_r \simeq E\partial_r^2$ . It is to be noted that these layers degenerate near  $z = 0$ , and that near the conductive shell, we expect a smaller amplitude, as we have Robin boundary conditions.
- Near the line  $A = A^*$ , due to the discontinuity of the limit solution, a shear layer with a scaling  $M^{\frac{1}{2}}$  similar to Shercliff layers. This layer comes from the scaling  $\partial_z \simeq E\partial_x^2$ .
- Near  $r = r_1, z = 0$  these layers degenerate into the equatorial Hartmann layer of size  $M^{-\frac{1}{3}} \times M^{-\frac{2}{3}}$ .

We will thus start, in Section 5.3, with a quick study of the non-degenerate Hartmann layers, showing how to treat the Robin condition using such operator. Then, in Section 5.4, we will analyse how these shears layers behave when  $A = A^*$ , i.e when a change of the boundary conditions occurs.

### Construction of an approximate solution

If we intend to construct an approximate solution on the whole domain the steps would be

- The construction of the Hartmann layer and the associated boundary layer compatibility condition. These layer of size  $(Mz)^{-1}$  degenerate close to the equator and Proposition 5.3.2 provides their construction under an hypothesis of non-degeneracy.
- Using the compatibility condition the construction of an interior term. This rises no problem, but the term we obtain is discontinuous at  $A = A^*$ , i.e,  $x = 0$ .
- This discontinuity is lifted through a shear layer of size  $M^{-\frac{1}{2}}$ .
- The boundary layer effective condition create a degeneracy of the boundary conditions for the shear layer, leading to a singularity. This is the main objective of Section 5.4.
- Lastly an equatorial layer lifting this discontinuity must be constructed. It seems that adapting tools from Chapter 4 for this context is sufficient to achieve such purpose.

## 5.3 Hartmann layers

### 5.3.1 Hartmann boundary layer operator

Let us first recall the definition of the weighted spaces we already used in Chapter 3 in order to solve boundary layer equations:

**Definition 5.3.1.** *Let  $\lambda > 0, s_0 \in \mathbb{N}$  and  $k \in \mathbb{N}$ . For  $(\theta, \zeta) \mapsto u(\theta, \zeta)$  a function from  $(0, 1) \times \mathbb{R}_+$  we define*

$$\|u\|_{\mathcal{H}_\lambda^{s_0, k}}^2 = \sum_{0 \leq i \leq s_0} \int_0^1 \int_0^\infty |\partial_\zeta^k \partial_\theta^i u|^2 e^{2\eta\lambda}.$$

Then, we have already seen that for  $\tilde{s}_0 \leq s_0$ ,  $\tilde{k} \leq k$ ,  $\tilde{\lambda} \leq \lambda$ ,  $\mathcal{H}_\lambda^{s_0, k} \hookrightarrow \mathcal{H}_{\tilde{\lambda}}^{\tilde{s}_0, \tilde{k}}$ , and more importantly the technical Lemmas A.5.1, A.5.2 allowing us to write for example the scaling, for  $s \leq s_0$

$$\left\| u\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^s} \leq \varepsilon^{\frac{1}{2}-s} C \|u\|_{\mathcal{H}_\lambda^{s_0, s_0}}.$$

The Hartmann layers are very similar to the Ekman layers. In fact we have the following boundary layer operator

**Proposition 5.3.2** (Hartmann boundary layer operator). *Let  $s_0 = 2$ ,  $V, B \in H^2(\Gamma_0)$ , and let  $\mathbf{f}$  with be with exponential decay of rate  $\eta < 1$ , i.e*

$$\mathbf{f} \in \mathcal{H}_\eta^{2,0} \times \mathcal{H}_\eta^{2,0}. \quad (5.13)$$

Suppose that the following compatibility is satisfied

$$V(\theta) + B(\theta) = \int_0^\infty (1 - e^{-\zeta})(f_v + f_b)(\theta, \zeta) d\zeta. \quad (5.14)$$

And, in order to avoid degeneracy, let us suppose that  $f$

$$\begin{aligned} V(\theta) - B(\theta) &= O(\cos(\theta)^2) \\ \mathbf{f} &= O(\cos(\theta)^2). \end{aligned} \quad (5.15)$$

There exist  $\mathbf{u}^{bl} = \mathbf{u}\left(\theta, \frac{r-r_0}{\delta_0(\theta)}\right)$  solution of the approximate equation (5.11) with source term  $\mathbf{f}^{bl} = M\mathbf{f}\left(\theta, \frac{r-r_0}{\delta_0(\theta)}\right)$ .

$$\begin{aligned} (v^{bl}, b^{bl})_{r=r_0} &= (V, B) \\ (v^{bl}, b^{bl})_{r=r_1} &= (0, 0) \end{aligned}$$

and the estimates, for all  $\tilde{\eta} < \eta$

$$\begin{aligned} \|r_v\|_{\mathcal{H}_\eta^{0,0}} + \|r_b\|_{\mathcal{H}_\eta^{0,0}} &\leq C \\ \|v\|_{\mathcal{H}_\eta^{1,1}} + \|b\|_{\mathcal{H}_\eta^{1,1}} &\leq C. \end{aligned}$$

leading to

$$\begin{aligned} \|r_v^{bl}\|_{L^2} + \|r_b^{bl}\|_{L^2} &\leq CM^{-\frac{1}{2}} \\ \|v^{bl}\|_{H^s} + \|b^{bl}\|_{H^s} &\leq CM^{s-\frac{1}{2}} \text{ for } s \in \{0, 1\}. \end{aligned}$$

*Remark 5.3.1.* If  $|\theta - \frac{\pi}{2}| \geq c > 0$ , then we can replace  $s_0 = 2$  by  $s_0 \geq 2$ . In this case we need  $V, B$  to be in  $H^{s_0}(\Gamma_0)$  and  $\mathbf{f} \in \mathcal{H}_\eta^{s_0, s_0-2}$ . And, we obtain  $v, b \in \mathcal{H}_\eta^{s_0, s_0}$ ,  $r_v, r_b \in \mathcal{H}_\eta^{s_0-2, s_0-2}$ , the loss of derivative coming from the neglect of  $\partial_\theta^2$ .

*Remark 5.3.2.* The condition (5.15) is far from optimal. Indeed, not only we can make a trade-off between power of  $\cos(\theta)$  and power of  $M$ , but, moreover, the area where  $\cos(\theta)^{-1} \gg 1$  is small. So if we are interested in the global  $L^2$  estimate of  $r^{bl}$  we must take this into account.

*Proof.* Let us choose the polar coordinates  $(r, \theta)$ , and as for Ekman layers start with  $\mathbf{f} = 0$ . Then the equation is

$$\begin{pmatrix} \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2 - c & -M(\cos(\theta)\partial_r + r^{-1}\sin(\theta)\partial_\theta + a) \\ -M(\cos(\theta)\partial_r + r^{-1}\sin(\theta)\partial_\theta - a) & \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2 - c \end{pmatrix} \begin{pmatrix} v^{bl} \\ b^{bl} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

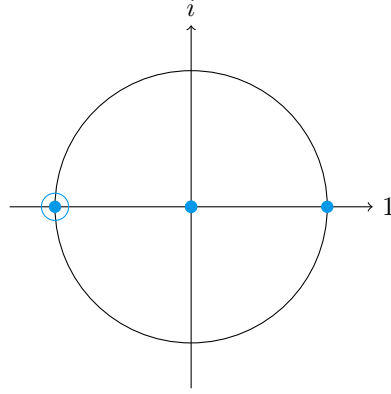


Figure 5.6 – The modes of the Hartmann boundary layer operator.

so the, keeping only the higher order in  $\partial_r$ , we obtain as a characteristic manifold

$$\begin{vmatrix} -\xi^2 & -M \cos(\theta) i \xi \\ -M \cos(\theta) i \xi & -\xi^2 \end{vmatrix} = \xi^4 - M \cos(\theta)^2 \xi = 0.$$

Therefore, the scaling is

$$r = r_0 + \underbrace{\frac{1}{M \cos(\theta)}}_{\delta_0} \zeta.$$

And the boundary layer equation is

$$\begin{aligned} -\partial_\zeta b + \partial_\zeta^2 v &= 0 \\ -\partial_\zeta v + \partial_\zeta^2 b &= 0. \end{aligned} \tag{5.16}$$

of corresponding characteristic polynomial

$$\lambda^4 - \lambda^2 = 0.$$

There are 4 roots:  $-1, +1$  and  $0$  as a double root. The only admissible mode is  $\lambda = -1$  leading to

$$\begin{aligned} v(\zeta) &= \alpha e^{-\zeta} \\ b(\zeta) &= -\alpha e^{-\zeta} \end{aligned}$$

for a  $\alpha \in \mathbb{R}$ . From there we deduce the compatibility condition

$$V + B = 0.$$

We can now estimate the remainders. By symmetry, it is sufficient to estimate  $r_v$ .

Thanks to the simple form of  $v$  and  $b$  we have the explicit value of  $r_v(\theta, \zeta)$ .

More precisely, with  $u^{bl} = u \left( \theta, \underbrace{\frac{r-r_0}{\delta_0}}_{\zeta} \right)$  we have the Jacobian

$$\begin{pmatrix} \partial_r u^{bl} \\ \partial_\theta u^{bl} \end{pmatrix} = \begin{pmatrix} \frac{1}{\delta_0} & 0 \\ -\frac{\partial_\theta \delta_0}{\delta_0} \zeta & 1 \end{pmatrix} \begin{pmatrix} \partial_\zeta u \\ \partial_\theta u \end{pmatrix}.$$

And, as in our case  $-\frac{\partial_\theta \delta_0}{\delta_0} = \tan(\theta)$ , we obtain,

$$\begin{aligned} M |r_v| \leq C & \left( M \cos(\theta) |\partial_\zeta v| + |\partial_\theta^2 v| + \frac{1}{\cos(\theta)} \zeta |\partial_\zeta \partial_\theta v| + \frac{1}{\cos(\theta)^2} (\zeta |\partial_\zeta v| + \zeta^2 |\partial_\zeta^2 v|) \right. \\ & \left. + M \zeta \frac{1}{\cos(\theta)} |\partial_\zeta b| + M |\partial_\theta b| + M |a| |b| \right). \end{aligned} \quad (5.17)$$

We thus obtain the claimed estimate for  $\theta$  far from  $\frac{\pi}{2}$ . For  $\theta$  close to  $\frac{\pi}{2}$ , as  $b$  and  $v$  behave like  $\pm(V-B)e^{-\zeta}$ , the condition to have  $r_v \in \mathcal{H}_\eta^{0,2}$  and of norm  $O(1)$  in this space is

$$\left\| \left( \frac{1}{M} \left( \partial_\theta^2 + \frac{1}{\cos(\theta)} \partial_\theta + \frac{1}{\cos(\theta)^2} \right) + \left( \partial_\theta + \frac{1}{\cos(\theta)} \right) \right) (V-B) \right\|_{L^2} \leq C$$

which is indeed the case thanks to the condition (5.15).

For a source term  $\mathbf{f} \neq 0$  the same computation applies. The only point where one must be careful is the rescaling as

$$\begin{pmatrix} \partial_r^2 & -M \cos(\theta) \partial_r \\ -M \cos(\theta) M \partial_r & \partial_r^2 \end{pmatrix} \begin{pmatrix} v^{bl} \\ b^{bl} \end{pmatrix} = \begin{pmatrix} M f_v^{bl} \\ M f_b^{bl} \end{pmatrix}$$

becomes

$$\begin{pmatrix} \partial_\zeta^2 & -\partial_\zeta \\ -\partial_\zeta & \partial_\zeta^2 \end{pmatrix} \begin{pmatrix} v \\ b \end{pmatrix} = \frac{1}{\cos(\theta)^2} \begin{pmatrix} M f_v^{bl} \\ M f_b^{bl} \end{pmatrix}$$

so in order avoid degeneracy when  $\theta \rightarrow \frac{\pi}{2}$  the condition (5.15) is used once more.

Lastly, for the estimate on  $\mathbf{u}^{bl}, \mathbf{r}^{bl}$ , terms like  $C \int_0^{\frac{\pi}{2}} \int_{r_0}^{r_1} |\cos(\theta)^\beta e^{-\lambda M(r-r_0)\cos(\theta)}|^2 r dr d\theta$  must be estimated near  $\theta = \frac{\pi}{2}$ . But thanks to (5.15), all the powers  $\beta$  are  $\geq 0$ , so we recover  $O(M^{-1})$ .  $\square$

*Remark 5.3.3.* The main difference with the Ekman layers of Chapter 3 is that all traces are Dirichlet traces. As a consequence, the size of the boundary layer does not appears in the compatibility condition (5.14).

*Remark 5.3.4.* The same operator exists for  $r = r_1$  with the only difference being that (5.14) is replaced by

$$V - B = \int_0^\infty (1 - e^{-\zeta})(f_v - f_b)(\zeta) d\zeta.$$

Note that both conditions can be rewritten naturally in terms of the Alfvén variables  $V_\pm$ .

### 5.3.2 Expansion without shear layers

Since the interior equation is simply

$$\begin{aligned}(-\partial_z + a)b^{int} &= f_v \\ (-\partial_z - a)v^{int} &= f_b\end{aligned}$$

there is no difficulty to construct an interior operator.

*Remark 5.3.5.* In the case where we have Dirichlet condition for  $b$  on both shells, and  $a = 0$ , we can conclude that the geometry has no effect on the limiting flow. Indeed the previous operator is sufficient to conclude that the limit of the solution  $(v, b)$  of

$$\begin{aligned}M\partial_z b + (\Delta - c)v &= 0 \\ M\partial_z v + (\Delta - c)b &= 0\end{aligned}$$

inside

$$\Omega = \{(x, z); \gamma_0(x) < z < \gamma_1(x)\}$$

and with boundary condition

$$(v, b)|_{z=\gamma_i} = (v_i, b_i)$$

is

$$(\bar{v}, \bar{b}) = \left( \frac{v_0 + v_1}{2} + \frac{b_0 - b_1}{2}, \frac{b_0 + b_1}{2} + \frac{v_0 - v_1}{2} \right).$$

And this is true no matter  $\gamma_0, \gamma_1$ , as long as they are smooth.

The only non-obvious step is how to work with the Robin condition. In fact, effect of scale mixing comes from this Robin condition on  $b$ . In fact, as  $\partial_r b_{\Gamma_0}^{bl}$  is of the same size as  $\frac{1}{\delta_0} b_{\Gamma_0}^{bl}$ , the only way to lift a condition  $\partial_r b + \lambda b = 0$  is to lift the Neumann trace  $\partial_n b$ , leaving a Dirichlet error one order smaller. This error is then lifted by the next order boundary layer operator.

More precisely, as the boundary layer solution are in a one dimensional affine space, we can lift any  $k$  boundary conditions as long as  $k - 1$  compatibility are given. These compatibility conditions comes from the explicit solutions of the boundary layer equation, combined with the boundary layer scaling.

For example, we have the exact same Proposition as Proposition 5.3.2 if we replace the compatibility condition (5.14) by

$$V + \frac{1}{M \cos(\theta)} N = \int_0^\infty (f_v(\zeta) + e^{-\zeta}(f_v + f_b)(\zeta)) d\zeta \quad (5.18)$$

and the boundary conditions by

$$\begin{aligned}(v^{bl}, \partial_r b^{bl})_{r=r_0} &= (V, N) \\ (v^{bl}, b^{bl}) &= (0, 0) \text{ for } r > \frac{2r_0 + r_1}{3}.\end{aligned}$$

*Remark 5.3.6.* We can as easily construct a boundary layer operator that lift directly the Robin condition, but we will not do so. Indeed, such operator would mix the power of  $M$  at each step, for example the asymptotic expansion with such operator look like

$$v = \sum_k M^{-k} v^{in,k} + \sum_k \left( \alpha_k(\theta) + \frac{\beta_k(\theta)}{M} \right) e^{-M(r-r_0) \cos(\theta_0)} + \sum_k \gamma_k e^{-M(r_1-r) \cos(\theta_1)} + \dots$$

which is correct but betrays the idea of asymptotic development where orders are ordered.

**Proposition 5.3.3.** *Let us  $a, c \in C^\infty(\Omega)$ ,  $c > 0$ ,  $a > 0$  with  $a|_{z=0} = 0$ . Let  $v_0, v_1 \in C_c^\infty(\Gamma_0), C_c^\infty(\Gamma_1)$   $\Lambda \geq 0$ .*

*Let  $\mathbf{u} = (v, b) \in H^1(\Omega) \times H^1(\Omega)$  be the unique solution of (5.8) with boundary conditions (5.6). Then when  $M \rightarrow \infty$  we have*

$$\lim_{M \rightarrow \infty} \|(v, b) - (\bar{v}, \bar{b})\|_{L^2} = 0$$

with

$$\bar{v}(x, z) = \begin{cases} e^{-\int_{r_0(x)}^z a(x, \tau) d\tau} v_0(x) & \text{for } |x| < r_0 \\ e^{\int_z^{r_1(x)} a(x, \tau) d\tau} v_1(x) & \text{for } |x| > r_0 \end{cases}$$

and

$$\bar{b}(x, z) = \begin{cases} e^{-\int_{r_0(x)}^z a(x, \tau) d\tau} (v_0 - v_1)(x) & \text{for } |x| < r_0 \\ 0 & \text{for } |x| > r_0 \end{cases}$$

In fact, for all  $K \in \mathbb{N}$  we have an asymptotic expansion

$$\begin{aligned} \mathbf{u} &= \sum_{k=0}^K M^{-K} \mathbf{u}^{int, K} (x, z) + \sum_{k=0}^K M^{-K} \mathbf{u}^{int, K, t} (\theta, M \cos(\theta)(r_1 - r)) \\ &+ \sum_{k=0}^K M^{-K} \mathbf{u}^{int, K, b} (\theta, M \cos(\theta)(r - r_0)) + O_{L^2} (M^{-(K+1)}). \end{aligned}$$

*Remark 5.3.7.* A similar result holds for  $\gamma_0, \gamma_1 \in C^\infty(\mathbb{T})$ ,  $\gamma_0 - \gamma_1 > C$  and  $\Omega = \{(x, z); \gamma_0(x) < z < \gamma_1(x)\}$ .

*Sketch of proof.* We have all the operators we need, represented in Figures 5.7, 5.8, 5.9.

- The interior operator take a source term  $\mathbf{r}^{i, k}$ , two boundary conditions  $v_1^{i, k} - b_1^{i, k}, v_0^{i, k}$  and construct an approximate solution up to a remainder  $M^{-1} \mathbf{r}^{i, k+1}$ .
- The top boundary layer operator take a source term  $\mathbf{r}^{t, k}$  and two boundary conditions  $v_1^{k, t}, b_1^{k, t}$  satisfying condition (5.14), and construct an approximate solution up to a remainder  $M^{-1} \mathbf{r}^{t, k+1}$ .
- The bottom boundary layer operator take a source term  $\mathbf{r}^{b, k}$  and two boundary conditions  $v_0^{b, k}, n_0^{b, k}$  satisfying condition (5.18), and construct an approximate solution up to a remainder  $M^{-1} \mathbf{r}^{t, k+1}$ , and with a remainder for the Robin conditions  $\partial_n b - \Lambda b = n_0^{b, k+1}$ . More precisely, the idea is to lift at each step, in boundary layer variables

$$\partial_n b^{k+1} = \Lambda b^k. \quad (5.19)$$

And at each step we distribute the value of the parameters such that all conditions are satisfied, as we have the exact number of degree of freedom.

As all terms are  $C^\infty$ , there is no problem with the loss of regularity. At the step  $k$  of the construction the errors are

- For the interior remainder

$$\mathbf{r}^{i, k+1} = O_{L^2}(M^{-K-1})$$

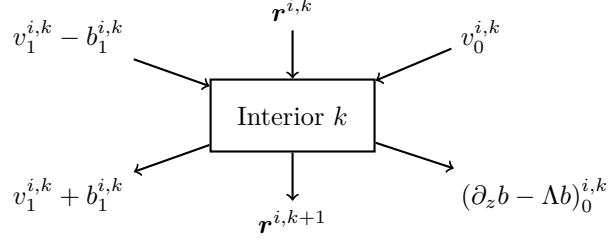
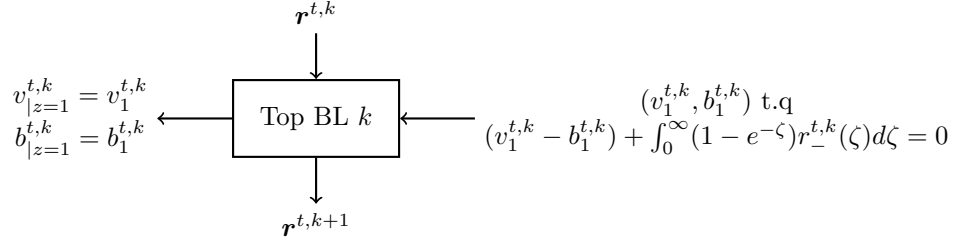


Figure 5.7 – Interior operator if there are no shear layers.

Figure 5.8 – Top boundary layer operator  $k$ 

- For the boundary layer remainders

$$\mathbf{r}^{b,k+1}, \mathbf{r}^{t,k+1} = O_{L^2}(M^{-K-1-\frac{1}{2}})$$

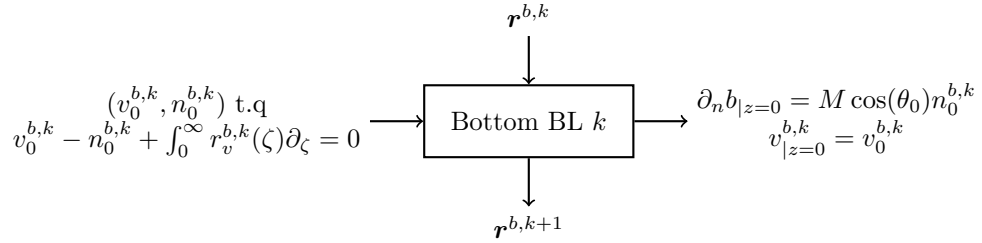
- For the boundary conditions

$$(\partial_n b - \Lambda b)_{k+1} = O_{H^2}(M^{-K-1})$$

which can be lifted with an arbitrary function, corresponding to a remainder

$$\mathbf{r}^{l,k} = O_{L^2}(M^{-K}).$$

So continuing the construction up to the order  $K + 1$ , we obtain the claimed expansion.  $\square$

Figure 5.9 – Bottom boundary layer operator  $k$

### 5.3.3 Remarks on the transparent condition

Instead of the transparent condition, which we will denote  $(\partial_r b + DtoNb)|_{\Gamma_0} = 0$  of the equation

$$(-\Delta + c)b = 0$$

we choose for simplification a constant Robin condition. In a way, it correspond to the replacement of  $\Delta b$  by  $r^{-1}\partial_r(r\partial_r b) + \lambda^2 b$  for some  $\lambda$ .

The main consequences of such simplification are the following

- The  $DtoN$  is non-local while the multiplication by  $\Lambda$  is localized. This is not a major problem, as in the physical case  $DtoN$  is localized in frequency, i.e for each mode  $e^{in\theta}$  we can solve the problem with a fixed  $\Lambda_n$ .
- More importantly the corresponding  $\Lambda_n$  goes to  $\infty$  when  $n \rightarrow +\infty$ . This is more problematic, but one can hope that such mode are not created by the boundary conditions.

In order to do explicit computation without the spherical geometry, let us consider a bottom conductive shell in the half space  $z < \gamma_0(x) = 0$  and a top insulating shell in the half-space  $z > \gamma_1(x) = 1$ . We will also take  $a = 0$ ,  $c = 1$ .

Then the equation is inside the domain  $\mathbb{T} \times [0, 1]$ ,

$$\begin{aligned} M\partial_z b - (\Delta - 1)v &= 0 \\ M\partial_z v - (\Delta - 1)b &= 0. \end{aligned} \tag{5.20}$$

If we suppose that the conductive half plane has the same conductivity than the fluid the boundary conditions are

$$\begin{aligned} v|_{z=0} &= v_b, v|_{z=1} = v_t \\ b|_{z=1} &= 0 \\ [b]_{z=0} &= 0, [\partial_z b]_{z=0} = 0 \end{aligned} \tag{5.21}$$

where  $b$  satisfy inside the domain  $\mathbb{T} \times (-\infty, 0)$

$$(\Delta - 1)b = 0. \tag{5.22}$$

The natural framework is to use Fourier series in the  $x$  variable.

Inside the conductive half plane

$$\partial_z^2 \widehat{b} - (k^2 + 1)\widehat{b} = 0$$

so

$$\widehat{b}(k, z) = e^{-\sqrt{1+k^2}z} \widehat{b}_0$$

and in particular

$$\partial_z \widehat{b}(k, 0) = -\sqrt{1+k^2} \widehat{b}(k, 0).$$

*Remark 5.3.8.* We see that if  $k \gg 1$  we have what can be considered as a boundary layer inside the conductor, of size  $\frac{1}{\sqrt{1+k^2}}$ .

In Alfvén variables with  $v_{\pm} = v \pm b$  the equation is

$$\mp M\partial_z \widehat{v}_{\pm} + \partial_z^2 \widehat{v}_{\pm} - (k^2 + 1)\widehat{v}_{\pm} = 0$$



We obtain for  $v_{\pm}$  two modes:

- an interior one

$$\lambda_i^{\pm} = \frac{1}{2} \left( \pm M \mp \sqrt{M^2 + (1 + k^2)} \right) = \mp \frac{1 + k^2}{4} + O_{M \rightarrow \infty, |k| \ll M} \left( \frac{1}{M} \right)$$

- and a boundary layer one

$$\lambda_{bl}^{\pm} = \frac{1}{2} \left( \pm M \pm \sqrt{M^2 + (1 + k^2)} \right) = \pm M + O_{M \rightarrow \infty, |k| \ll M}(1).$$

*Remark 5.3.9.* The so-called interior terms are in fact, at high frequencies, boundary layer terms too.

With  $\widehat{v}_+ = c_i^+ e^{\lambda_i^+ z} + c_{bl}^+ e^{\lambda_{bl}^+ (z-1)}$ ,  $\widehat{v}_- = c_i^- e^{\lambda_i^- z} + c_{bl}^- e^{\lambda_{bl}^- z}$  the boundary conditions are

$$\underbrace{\begin{pmatrix} 1 & e^{-\lambda_{bl}^+} & 1 & 1 \\ e^{\lambda_i^+} & 1 & e^{\lambda_i^-} & e^{\lambda_{bl}^-} \\ \lambda_i^+ + \sqrt{1+k^2} & e^{-\lambda_{bl}^+} (\lambda_{bl}^+ + \sqrt{1+k^2}) & -\lambda_i^- - \sqrt{1+k^2} & -\lambda_{bl}^- - \sqrt{1+k^2} \\ e^{\lambda_i^+} & 1 & -e^{\lambda_i^-} & -e^{\lambda_{bl}^-} \end{pmatrix}}_{\begin{pmatrix} 1 & 0 & 1 & 1 \\ e^{\lambda_i^+} & 1 & e^{\lambda_i^-} & 0 \\ \lambda_i^+ + \sqrt{1+k^2} & 0 & -\lambda_i^- - \sqrt{1+k^2} & -\lambda_{bl}^- - \sqrt{1+k^2} \\ e^{\lambda_i^+} & 1 & -e^{\lambda_i^-} & 0 \end{pmatrix} + o(M^{-\infty})} \begin{pmatrix} c_i^+ \\ c_{bl}^+ \\ c_i^- \\ c_{bl}^- \end{pmatrix} = \begin{pmatrix} 2\widehat{v}_b(k) \\ 2\widehat{v}_t(k) \\ 0 \\ 0 \end{pmatrix}$$

The first order is

$$\begin{aligned} c_i^+ &= 2\widehat{v}_b - \widehat{v}_t \\ c_i^- &= \widehat{v}_t \\ c_{bl}^+ &= -2\widehat{v}_b + 2\widehat{v}_t \\ c_{bl}^- &= \frac{1}{M} \left( 2\sqrt{1+k^2}(-\widehat{v}_b + \widehat{v}_t) \right). \end{aligned}$$

And an explicit calculation seems to show that if  $v^b, v^t$  are bounded in  $H^3$ , uniformly in  $M$ , then the limit in  $L^2$  is

$$(\bar{v}, \bar{b}) = \left( v_b, \frac{v_b + v_t}{2} \right)$$

which is the same as in the case where the bottom boundary condition is a Robin condition.

But, unfortunately, when dealing with the whole problem, the shear layers create naturally high frequencies (the inverse of the boundary layer size), so such heuristic is no longer valid. However, from a numerical point of view [22] it seems that the construction of the boundary layers does not change too much, as there is no equatorial boundary layer on the conductive shell.

## 5.4 Toy model for the shear layer

### 5.4.1 Setting

From the analysis of the Ekman equatorial layer in Chapter 3, it appears that the shear layers are not due directly by the geometrical degeneracy of the problem, but by the discontinuity

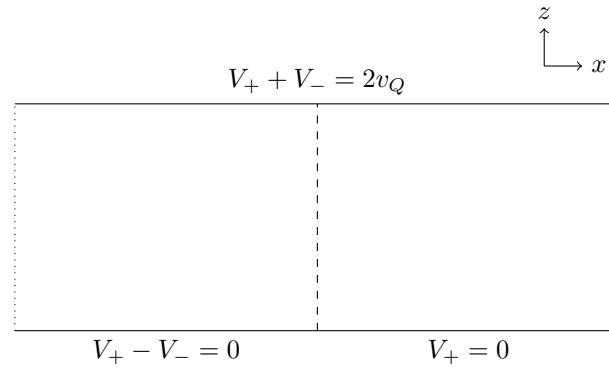


Figure 5.10 – Domain and boundary conditions.

arising from the change of boundary conditions. As a consequence, in order to investigate this phenomenon without being perturbed by the degeneracy of Hartmann layer, let us consider (5.8) on the domain  $\mathbb{T} \times (0, 1)$ , with boundary conditions

$$v|_{z=1} = v_Q \quad b|_{z=1} = 0 \quad v|_{z=0} = 0 \quad \begin{cases} b|_{z=0} = 0 & \text{for } x < 0 \\ (\partial_n b + \Lambda b)|_{z=0} = 0 & \text{for } x > 0. \end{cases} \quad (5.23)$$

*Remark 5.4.1.* As noted by Dormy, Jault and Soward [22], this approximation could be formally justified a posteriori. Indeed, for a shear layer of size  $\delta \ll 1$  in the  $x$  variable, the domain  $z > (-x)^\alpha \mathbf{1}_{x < 0}$  becomes  $z > \delta^\alpha (-x)^\alpha \mathbf{1}_{x < 0}$  after rescaling. Thus, for the shear layers (and not the equatorial layer) and Dirichlet traces, both geometry are very alike.

From the study of the first order of the Hartmann boundary layer operator, the effective boundary condition are

- For  $z = 1$ , the conductive shell, we obtain

$$(V_+ + V_-)|_{z=1} = 2v_Q$$

- For  $z = 0$ ,  $x > 0$ , the insulating shell, we obtain the Hartmann condition

$$V_+|_{z=0, x > 0} = 0.$$

- For  $z = 0$ ,  $x < 0$  the symmetry conditions read as  $b = 0$  i.e

$$(V_+ - V_-)|_{z=0, x < 0} = 0.$$

The exact problem we are interested in is

$$\begin{aligned} M(\partial_z V_+ - aV_-) - (\Delta - c)V_+ &= 0 \\ M(\partial_z V_- - aV_+) + (\Delta - c)V_- &= 0 \end{aligned} \quad (5.24)$$

with the boundary conditions of Figure 5.10. where the perturbation  $v_Q$  is smooth.

*Remark 5.4.2.* In order to remain close to the physical problem, we choose to consider the problem with no source term and non-homogeneous boundary conditions. What follows can

be adapted for non-homogeneous source term. Indeed the only oversimplification would be to choose boundary conditions in such a way that no singularity appears at  $(0, 0)$ . But as we will see this is not the case if  $v_Q(0) \neq 0$ .

The goal is to construct an approximate solution, in the sense of Definition 5.2.2. First, if  $v_Q$  is zero near  $x = 0$ , the usual construction works

**Proposition 5.4.1** (Interior operator without shear layer). *If  $v_Q(0) = 0$ ,  $v'_Q(0) = 0$ , then there exists  $V_+, V_-$  approximate solution of (5.24) satisfying the boundary conditions*

$$\begin{aligned} (V_+ + V_-)|_{z=0} &= 2v_Q \\ (V_+ - V_-)|_{z=0, x < 0} &= 0 \quad (V_+)|_{z=0, x > 0} = 0. \end{aligned} \quad (5.25)$$

*Proof.* If we drop all lower order terms the equations we obtain is

$$\begin{aligned} \partial_z V_+ - a(x, z)V_- &= 0 \\ \partial_z V_- - a(x, z)V_+ &= 0. \end{aligned}$$

From there the solution is straightforward

$$\begin{aligned} V_+ &= \alpha(x)e^{-\int_z^1 a(x, \zeta)d\zeta} + \beta(x)e^{-\int_0^z a(x, \zeta)d\zeta} \\ V_- &= \alpha(x)e^{-\int_z^1 a(x, \zeta)d\zeta} - \beta(x)e^{-\int_0^z a(x, \zeta)d\zeta} \end{aligned}$$

and this is not surprising that we have the right number of degree of freedom on the boundary conditions, leading to

$$\begin{cases} 2\alpha(x) = 2v_Q(x) \\ 2\beta(x) = 0 & \text{for } x < 0 \\ \beta(x) = -\alpha(x)e^{-\int_0^1 a(x, \zeta)d\zeta} & \text{for } x > 0. \end{cases}$$

This explicit expression,  $V_{\pm} = v_Q(x)e^{-\int_z^1 a(x, \zeta)d\zeta} \mp \mathbf{1}_{x > 0}(v_Q(x)e^{-\int_0^1 a(x, \zeta)d\zeta})$ , leads to remainders

$$\begin{aligned} r_{\pm} &= (\Delta - b)V_{\pm} \\ &= O_{L^2}(1) \pm \left( \delta'_{x=0}v_Q(0) + 2\delta_{x=0} \left( v'_Q(0) - \int_0^1 a(0, \zeta)d\zeta v_Q(0) \right) \right) e^{-\int_0^1 a(0, \zeta)d\zeta}. \end{aligned}$$

So, as long as  $v_Q(0) = 0$  and  $v'_Q(0) = o(1)$  we obtain an acceptable remainder in the sense of 5.2.2.  $\square$

From there one way would be to distinguish the first two order of the Taylor expansion of  $v_Q$  near 0, i.e write  $v_Q(x) = v_Q(0) + xv'_Q(0) + \widetilde{v}_Q(x)$ . But a more robust way is to separate the scales through judicious cut-off

$$v_Q(x) = v_Q(x)\chi\left(\frac{x}{\delta}\right) + v_Q(x)\left(1 - \chi\left(\frac{x}{\delta}\right)\right).$$

where  $\chi(x) = 1$  for  $|x| < \frac{1}{3}$  and is equal to 0 for  $|x| > \frac{2}{3}$ . In this case an additional error term is created, of size  $\delta^{-2}$  in  $L^\infty$  norm, and localized in  $|x| \in \left[\frac{\delta}{3}, \frac{2\delta}{3}\right]$

With such cut-off,  $\Delta_{x,z}$  becomes  $M^{-2\beta}\partial_x^2 + \partial_z^2$ . Thus, the balance is  $\beta = \frac{1}{2}$ , and the corresponding scaling

$$x = M^{\frac{1}{2}}s.$$

### 5.4.2 Study of the shear layer equation

Let us study the equation (5.26) with boundary condition (5.27) near  $x = 0$ . The previous scaling  $(x, z) \rightarrow (xE^{-\frac{1}{2}}, z) = (s, z)$  leads to

$$M (\partial_z V_{\pm} - a(0, z)V_{\mp} \mp \partial_s^2 V_{\pm}) + \left( M(a(0, z) - a(M^{-\frac{1}{2}}s, z))V_{\mp} \pm (\partial_z^2 + b(M^{-\frac{1}{2}}s, z))V_{\pm} \right) = 0.$$

Therefore the boundary layer equation we obtain

$$\partial_z V_{\pm} - aV_{\mp} \mp \partial_s^2 V_{\pm} = 0. \quad (5.26)$$

#### Existence and uniqueness

**Proposition 5.4.2.** *For any  $\kappa \in C_c^\infty(\mathbb{R})$  there exist a solution of (5.26)*

$$\partial_z V_{\pm} - aV_{\mp} \mp \partial_s^2 V_{\pm} = 0.$$

with boundary condition

$$\begin{aligned} (V_+ + V_-)|_{z=0} &= \kappa \\ (V_+ - V_-)|_{z=0, x < 0} &= 0 \quad (V_+)|_{z=0, x > 0} = 0. \end{aligned} \quad (5.27)$$

More over this solution is unique, and satisfies

$$\iint_{\mathbb{R} \times (0,1)} (|\partial_s V_+|^2 + |\partial_s V_-|^2 + |V_+|^2 + |V_-|^2) dz ds \leq C.$$

*Proof.* The first step is to consider, thanks to a well-chosen lifting, an equation with homogeneous boundary conditions and a source term  $\mathbf{f}$  localized near  $z = 1$ .

Using exactly the same proof as for Theorem 4.2.1 for the equatorial Ekman equation, we obtain existence in the space

$$\|\mathbf{V}\|_{E_0}^2 = \iint_{\mathbb{R} \times (0,1)} \left( |\partial_s V_+|^2 + |\partial_s V_-|^2 + \left| \frac{V_+}{|s|+1} \right|^2 + \left| \frac{V_-}{|s|+1} \right|^2 \right) dz ds < +\infty$$

as long as, for any solution of the *dual equation*

$$-\partial_z W_{\pm} - a(0, z)W_{\mp} \mp \partial_s^2 W_{\pm} = F_{\pm} \quad (5.28)$$

with the *dual boundary conditions* of Figure 5.11

$$\begin{aligned} (W_+ + W_-)|_{z=0} &= 0 \\ (W_+ - W_-)|_{z=0, x < 0} &= 0 \quad (W_-)|_{z=0, x < 0} = 0 \end{aligned} \quad (5.29)$$

we have the *observability inequality*

$$\|F_{\pm}\|_{L^2} \geq C (\|\mathbf{W}\|_{E_0} + \|\partial_z \mathbf{W}\|_{E'_0}).$$

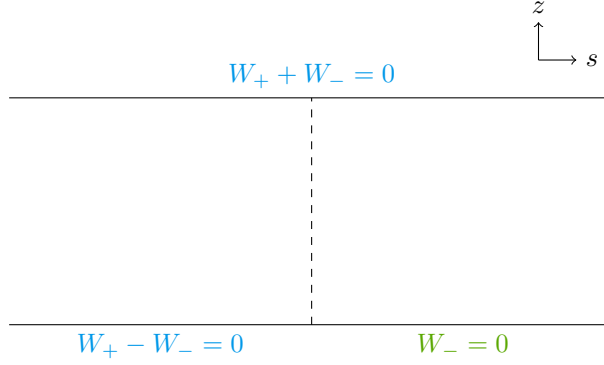


Figure 5.11 – Dual boundary conditions.

Thanks to the equation (5.28), it is enough to prove that

$$\|F_{\pm}\|_{E'_0} \geq C\|\mathbf{W}\|_{E_0}$$

and, multiplying equation (5.28) by  $(W_+, W_-)$ , we have indeed

$$\begin{aligned} \int (F_+ W_+ + F_- W_-) &= \int (-\partial_z W_+ W_+ - a W_- W_+ - \partial_s^2 W_+ W_+) \\ &\quad + \int (+\partial_z W_- W_- + a W_+ W_- - \partial_s^2 W_- W_-) \\ &= \int (|\partial_s W_+|^2 + |\partial_s W_-|^2) + \frac{1}{2} \int \partial_z (|W_-|^2 - |W_+|^2) \\ &= \int (|\partial_s W_+|^2 + |\partial_s W_-|^2) + \underbrace{\frac{1}{2} \left( 0 + \int_{z=0, x>0} |W_+|^2 \right)}_{\geq 0}. \end{aligned}$$

In order to obtain the  $L^2$  integrability, for  $a \geq a_0 > 0$  we use the formal computation

$$\begin{aligned} \iint a(|V_+|^2 + |V_-|^2) ds dz &= \left[ \int (V_+ V_-)(s, \cdot) ds \right]_{z=0}^{z=1} + \iint (f_+ V_- + f_- V_+) ds dz \\ &\leq 0 + \sqrt{\iint \frac{1}{a} (|f_+|^2 + |f_-|^2) ds dz} \sqrt{\iint a(|V_+|^2 + |V_-|^2) ds dz} \end{aligned}$$

which can be justified by computing in a weak sense (in the spirit of the proof of Theorem 4.3.1)

$$\frac{d}{dZ} \int_{z=Z} V_+ V_-(s, Z) ds.$$

Or, alternatively without the hypothesis on the sign of  $a$ , we can use the same methods as in Subsection 3.5.5: thanks to Cacciopoli inequalities, interior regularity in  $s$  (far from  $s = 0$ ) to have regularity on  $z$  and then Poincaré inequality as the domain is bounded in the  $z$  direction.

From this integrability, we recover the uniqueness, using the same methods as in the Theorem 4.3.1 of Chapter 4.

□

*Remark 5.4.3.* The dual boundary conditions are formally the ones needed to have no boundary terms when integrating by parts in order to obtain the weak formulation.

For example, here,

$$\begin{aligned} \int (\partial_z V_+ W_+ - \partial_z V_- W_-) &= \int_{|z|=1} (V_+ W_+ - V_- W_-) - \int_{|z=0} (V_+ W_+ - V_- W_-) \\ &\quad - \int (-V_+ \partial_z W_+ + V_- \partial_z W_-) \end{aligned}$$

and the condition

$$V_+ W_+ - V_- W_- = 0$$

- for  $V_+ - V_-$  is equivalent to  $W_+ + W_- = 0$ ,
- for  $V_+ - V_-$  is equivalent to  $W_+ - W_- = 0$ ,
- for  $V_+ = 0$  is equivalent to  $W_- = 0$ .

### Regularity

Taking inspiration from Section 4.2.5, we see that we can obtain elliptic regularity far from the boundaries. More precisely we recover regularity with respect to  $s$  thanks to Cacciopoli inequalities, and we use the equation expressing  $\partial_z V_\pm$  from  $V_\mp$  and  $\partial_s^2 V_\pm$  to recover regularity with respect to  $z$ .

In fact, we can recover regularity far from  $(s, z) = (0, 0)$ . In order to quantify the regularity let us define, for  $k \in \mathbb{N}$  and  $\omega \subset \Omega$  let

$$\|u\|_{E_k(\omega)}^2 = \int_{\omega} (|\partial_z^k u|^2 + |\partial_y^{k+1} u|^2).$$

For example, the energy space is  $E_0$ .

**Proposition 5.4.3** (Regularity of the solution). *The solution  $\mathbf{V}$  of Proposition 5.4.2 is in fact in  $E_1(\Omega \setminus \{|(s, z)| \leq \epsilon\})$  for all  $\epsilon > 0$ .*

*More precisely there exist a constant  $\Xi \in \mathbb{R}$  and  $V_+^{reg} \in E_1(\Omega)$  such that*

$$\begin{aligned} V_- &\in E_1(\Omega) \\ V_+ &= \underbrace{V_+^{reg}}_{\in E_1(\Omega)} + \underbrace{\Xi \int_{\frac{s}{\sqrt{z}}}^{\infty} e^{-\frac{\tau^2}{4}} \chi^\dagger(s - \sqrt{z}\tau) d\tau}_{\in E_0(\Omega) \setminus E_1(\Omega)}. \end{aligned}$$

*Remark 5.4.4.* Such behavior, that the solution are in higher regularity space modulo a finite dimensional space of singular function is to be expected. Indeed, for elliptic equation with change of boundary conditions such phenomenon appears, see the work of Grisvard [41], [43].

*Proof.* The idea is to look at each equation as a forward or backward heat equation with a smooth source term.

---

<sup>†</sup> $\chi$  is any cut-off function such that  $\chi(0) = 1$ .

From  $V^+ \in L^2$ , there exist  $z_0 \in (0, 1)$  such that  $V_+(\cdot, z_0)$  is well defined and  $L^2$ . Then, we solve the equation

$$\partial_z V_+ + \partial_s^2 V_+ = F(s, z)$$

where  $F_+ = -aV_-$  is such that

$$\|F_+\|_{E_0(\Omega)}^2 = \int |F|^2 + |\partial_s F|^2 < +\infty.$$

For  $z > z_0$  thanks to the smoothing properties of the heat equation, or the explicit expression

$$\widehat{V}_+(\xi, z) = e^{-(z-z_0)\xi^2} \widehat{V}_{+|z=z_0} - \int_{z_0}^z e^{-\xi^2(z-\tau)} \widehat{F}_+(\xi, \tau) d\tau$$

we recover that

$$V_+ \in E_1((z_0 + \epsilon, 1) \times \mathbb{R}) \forall \epsilon > 0.$$

And

$$V_{+|z=1} \in H^2(\mathbb{R}).$$

For  $z < 1$  thanks to the smoothing of the backward heat equation

$$\begin{aligned} \partial_z V_- + \partial_s^2 V_- &= F_- \in E_0(\Omega) \\ V_-(z=1) &\in H^2(\mathbb{R}) \end{aligned}$$

so, similarly we have

$$V_- \in E_1((0, 1) \times \mathbb{R}).$$

And

$$V_{-|z=0} \in H^2(\mathbb{R}).$$

*Remark 5.4.5.* We can continue the bootstrapping in order to obtain  $C^\infty$  regularity far from  $(0, 0)$ . The idea is that, at each step the regularity on  $V_\pm$  is used to obtain more regularity on the source term  $F_\pm$  and  $V_{-|z=1}$ .

The singularity of  $V_+$  is due to its initial condition

$$V_{+|z=0} = \mathbf{1}_{s<0} V_{-|z=0}$$

which can be discontinuous at  $s = 0$ .

Let us decompose by linearity  $V_+$  into  $V_+ = V_+^{reg} + V_+^{sing}$  where

$$\begin{aligned} \partial_z V_+^{reg} - \partial_s^2 V_+^{reg} &= aV_- \\ V_+^{reg}(s, z=0) &= (V_-(s, 0) - \chi(s)V_-(0, 0)) \mathbf{1}_{s<0} \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} \partial_z V_+^{sing} - \partial_s^2 V_+^{sing} &= 0 \\ V_+^{sing}(s, z=0) &= \chi(s)V_-(0, 0) \mathbf{1}_{s<0}. \end{aligned} \tag{5.31}$$

where  $\chi$  is a cut-off function such that  $\chi(0) = 1$ . Note that for  $\chi = 1$  we recover the  $\text{erf}(sz^{-\frac{1}{2}})$  but that for an unbounded domain in  $s$  we must take  $\chi$  decreasing at infinity. But the behavior of the function near  $(s, z) = (0, 0)$  is nevertheless the same.

Since  $V_-(s, 0) - \chi(s)V_-(0, 0) = 0$ ,  $(V_-(s, 0) - \chi(s)V_-(0, 0)) \mathbf{1}_{s<0}$  is  $H^1$ , and we recover the claimed regularity on  $V_+^{sing}$ .

And the expression of  $V_+^{sing}$  is explicit from the heat kernel.  $\square$

*Remark 5.4.6.* For the case  $\kappa = 1$  we recover as singular part

$$\frac{\Xi}{2} \operatorname{erf}\left(\frac{s}{2\sqrt{z}}\right)$$

which is exactly the singularity constructed by Dormy, Jault and Soward [22]. The main point is the visible self-similar scaling

$$\frac{s}{\sqrt{z}}.$$

### 5.4.3 About the singularity at $(0, 0)$

#### How to compute $\Xi$ ?

A question that emerges from the previous Proposition is to find  $\Xi$ , or more generally  $V_{-|z=0}$ , as  $\Xi$  is a linear function of  $V_{-|z=0}(0)$ .

Note that we can formally write the linear form  $\kappa \mapsto \Xi$ . Indeed, for  $\phi \in C_c^\infty(\mathbb{R})$  let  $\mathbf{W}^\phi$  be the solution of the dual equation (5.28) with inhomogeneous boundary conditions (5.29)

$$\begin{aligned} (W_+ + W_-)|_{z=0} &= 0 \\ (W_+ - W_-)|_{z=0, x<0} &= \phi \quad (W_-)|_{z=0, x>0} = \phi. \end{aligned}$$

Then the weak formulation leads to

$$\int_{-\infty}^{\infty} \kappa(s) W_+^\phi(s, 1) ds = - \int_{-\infty}^{\infty} \phi(s) V_-(s, 0) ds.$$

So with  $\phi \rightarrow \delta_0$  we recover  $V_-(0, 0)$ . Unfortunately, such method requires additional information on the dual problem, which is exactly as difficult as the main problem.

In their paper [22] Dormy, Jault and Soward proposed a numerical scheme in order to compute the solution and notably  $V_-(s, 0)$  for  $a = 0$ . Let us comment the method they used.

Let  $K_t$  be the heat kernel at time  $t$  i.e

$$K_t u(s) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{(s-\tau)^2}{4t}} u(\tau) d\tau.$$

Solving the heat equation on  $V_+$  from 0 to 1, followed by the backward heat equation on  $V_-$  from 1 to 0, we obtain  $V_-$  as a fixed point:

- At  $z = 0$ ,  $V_+(s, 0) = \mathbf{1}_{s<0} V_-(s, 0)$ .
- Solving the heat equation (as  $a = 0$ ) there is no coupling we obtain

$$V_+(s, 1) = K_1 * (\mathbf{1}_{s<0} V_-(s, 0))$$

- At  $z = 1$  we have  $V_-(s, 1) = \kappa - V_+(s, 1) = \kappa - K_1 * (\mathbf{1}_{s<0} V_-(s, 0))$ .
- Solving the backward heat equation from 1 to 0 we recover

$$\begin{aligned} V_-(s, 0) &= K_1 * (\kappa - K_1 * (\mathbf{1}_{s<0} V_-(s, 0))) \\ &= K_1 * \kappa - K_2 * (\mathbf{1}_{s<0} V_-(s, 0)). \end{aligned} \tag{5.32}$$



So it is necessary that  $V_{-|s=0}$  satisfy (5.32), and it is also a sufficient condition as this construction from such  $V_{-|s=0}$  leads to a correct solution.

Formally the equation (5.32) reads as

$$v_- = f - (K_2 * (gv_-))$$

with  $g = \mathbf{1}_{s<0}$ .

The scheme proposed by [22] in order to solve such integral equation is to write

$$v_- = \sum_k v_-^k \quad (5.33)$$

where  $v_-^0$  and

$$v_-^{k+1} = - \underbrace{(K_2 * (\mathbf{1}_{s<0} v_-^k))}_{G(v_-^k)}. \quad (5.34)$$

However, the operator  $G$  is not a contraction, and the numerical convergence rely on the positivity of  $G$ , as  $(-G)^k$  is of constant sign  $(-1)^k$ , thus creating an alternating series.

In fact

$$\begin{aligned} G : L^2 &\rightarrow L^2 \\ u &\mapsto (K_2 * (\mathbf{1}_{s<0} u)) \end{aligned}$$

is a linear operator, bounded (and of norm  $\leq 1$ ) as

$$\|G(u)\|_{L^2} = \|K_2 * (ug)\|_{L^2} \leq \|ug\|_{L^2} \leq \|u\|_{H^2}.$$

We can express explicitly  $\widehat{G}$  as  $e^{-2\xi}$  times the Hilbert transform to obtain a full spectral description of  $G$ . But here, as we only want  $(1 + G)^{-1}$ , it is enough to notice that for  $\lambda$ , with

$$\tilde{\lambda} = \max_{\rho \in [0,1]} |\lambda - \rho| = \left| \lambda - \frac{1}{2} \right| + \frac{1}{2} > |\lambda|$$

we have

$$\|(G - \lambda)u\|^2 = \int \left( |\lambda - e^{-2\xi^2}|^2 |\widehat{ug}|^2 + |\lambda|^2 u \widehat{(1-g)} + 2\lambda(\lambda - e^{-2\xi^2}) \Re(\widehat{ug} \overline{\widehat{(1-g)}}) \right) \leq 2\tilde{\lambda}^2 \|u\|_{L^2}^2.$$

As a consequence rewriting

$$(1 + G) = (1 + \lambda) \left( 1 - \frac{\lambda - G}{1 + \lambda} \right)$$

we obtain that the alternative scheme

$$\begin{aligned} v_-^0 &= \frac{1}{1 + \lambda} f \\ v_-^{k+1} &= \frac{\lambda - G}{1 + \lambda} v_-^k \end{aligned} \quad (5.35)$$

converges with geometric speed, as long as,  $\sqrt{2\tilde{\lambda}} < 1 + \lambda$ . In this case,

$$v_- = \sum_{k=0}^{\infty} v_-^k$$

is well defined in  $L^2$  and is indeed a solution of (5.32). Note that from  $v_- \in L^2$  and the equation

$$v_- = f - \underbrace{K_2 * (gv_-)}_{\text{smooth}}$$

we recover that  $v_-$  has the same regularity than  $f$ .

*Remark 5.4.7.* In practice two values of  $\lambda$  are of particular interest:  $\lambda = \frac{1}{2}$  as it correspond to the best rate of convergence, and  $\lambda = 1$  as it is sign preserving. The scheme (5.34) correspond to  $\lambda = 0$ .

In order to illustrate these methods, the Figure 5.12 represent the solution obtained after a few iterations, and Figure 5.13 the norms of  $v_-^k$  with respect to  $k$ .

### Link with the equatorial layer: some heuristics

The fact that  $V_+ \notin H^1$  was to be expected, as even the equation (5.26) has a change of boundary conditions. But in order to be an approximate it need to be in  $H^1$ . This is the same problem as for the discontinuity at  $x = 0$ .

The term we neglected was, in shear layer variables,  $\partial_z^2 V_+$ . The equation we need to consider is thus

$$\partial_z V_+^{sing} - \partial_s^2 V_+^{sing} - M^{-1} \partial_z^2 V_+^{sing} = 0 \quad (5.36)$$

with boundary condition at  $z = 0$ ,

$$V_{+|z=0}^{sing} = \underbrace{\chi}_{\in C^\infty} \mathbf{1}_{s < 0}.$$

The scaling is then

$$\begin{aligned} s &= M^{-\frac{1}{2}} \sigma \\ z &= M^{-1} \zeta. \end{aligned} \quad (5.37)$$

Which is an equatorial like scaling. There would remain to decompose  $V_{+|z=0}$  as  $V_{+|z=0} \chi(sE^{-\beta}) + V_{+|z=0} (1 - \chi(sE^{-\beta}))$  with a right  $\beta$  and  $\chi$  a cut-off localized near 0, so that the first part create an equatorial layer and the second one a non singular shear layer.

## 5.5 Conclusion and perspectives

In this chapter we have constructed various boundary layers for the main MHD problem, notably the well-known Hartmann layers and a shear layer of size  $M^{-\frac{1}{2}}$ . For this shear layer, the change of effective boundary condition create a singularity. This singularity is expected to be lifted by an equatorial layer.

### Link with the equatorial layer

The scaling of the equatorial layer is dictated by the combination of two balances:

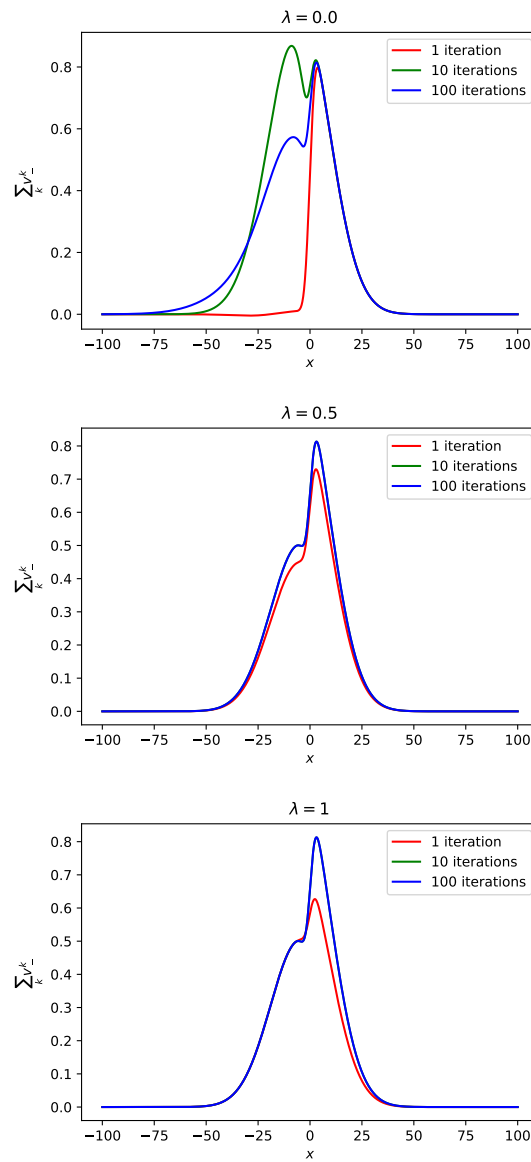


Figure 5.12 – First terms of the series expansion of  $v_-$ . Note that  $k$  iterations correspond to  $\sum_{k=0}^n v_k$  i.e  $n + 1$  terms in the series (5.33).

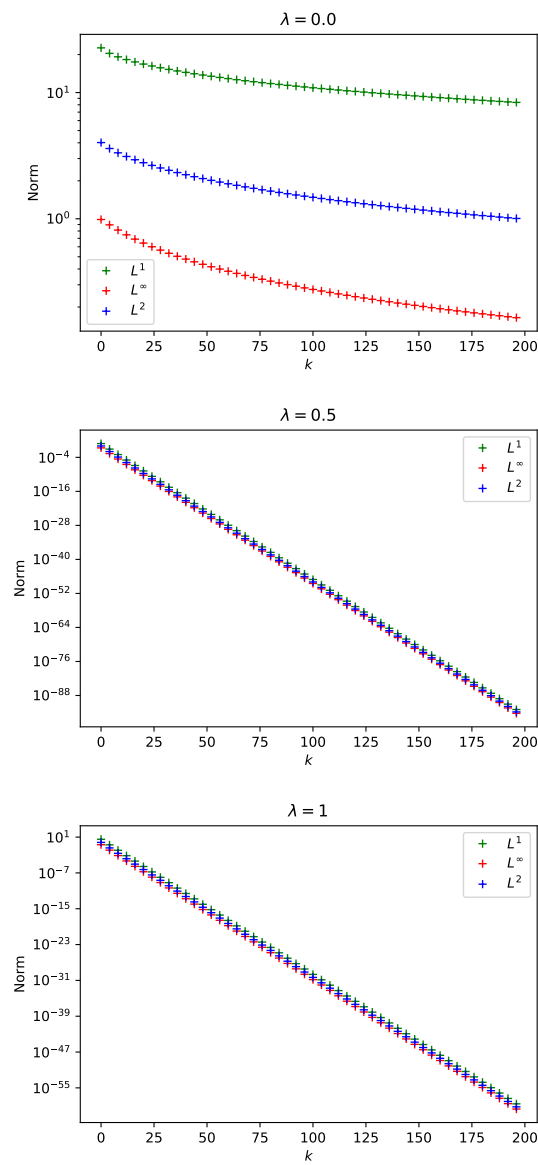


Figure 5.13 – Norms of  $v^k$  with respect to  $k$ . For  $\lambda = 0$  we have a subgeometric decay, while for  $\lambda = \frac{1}{2}, 1$  we observe a decay of ratio  $\tilde{\lambda}$ .

- The parabolic scaling  $\frac{x E^{-\frac{1}{2}}}{\sqrt{z}}$ .
- And the scaling of the geometric degeneracy, for example, for a sphere  $z = \sqrt{x}$  leading to the  $E^{\frac{2}{3}} \times E^{\frac{1}{3}}$ .

This is the same for the Proudman problem when we use both the scaling  $x = (\frac{E}{k})^{\frac{1}{3}} \simeq (Ez)^{\frac{1}{3}}$  of the  $E^{\frac{1}{3}}$  boundary layer and the geometry  $x \sim (-x)^\alpha \mathbf{1}_{x < 0}$ . But here, in the MHD problem, we are clearly able to see the influence of the equatorial region: it is a well-identified singularity, preventing the approximate solution to be in the space  $H^1(\Omega)$ .

So the role of the equatorial layer is not only to lift the boundary conditions in a very small area, but more importantly, it is to lift such singularity (as the shear layers were lifting a discontinuity), in order to have the approximate solution in the energy space.

*Remark 5.5.1.* If we were to write down the equatorial layer equation, we will obtain a very close to the equatorial Ekman equation

$$\begin{aligned} \partial_z v + z \partial_y v - \partial_y^2 b &= 0 \\ \partial_z b + z \partial_y b - \partial_y^2 v &= 0 \end{aligned} \tag{5.38}$$

on the domain

$$y > 0 \quad z > 0.$$

But, this equation is much more simpler than the one studied in Chapter (4) as it can be explicitly diagonalized into two independent parabolic equations

$$\begin{aligned} \partial_z V_+ + z \partial_y V_+ - \partial_y^2 V_+ &= 0 \\ \partial_z V_- + z \partial_y V_- + \partial_y^2 V_- &= 0. \end{aligned} \tag{5.39}$$

Thus it is likely that in this case we can justify the full approximation.

### Perspectives

This MHD toy-model proved itself to retain most characteristics of the Proudman model with two main differences

- Contrary to the Ekman boundary layer compatibility condition, the Hartmann boundary layer compatibility condition does not depends of the size of said layer. If it does not prevents degeneracy, as we still have  $\frac{\partial \theta \delta}{\delta} \rightarrow \infty$ , this has multiple consequences: for example we no longer have the singularity in the effective boundary condition, (so there is no equivalent of the  $E^{\frac{1}{3+\alpha}}$  layer).
- From a technical point of view, the bilaplacian is replaced by a laplacian. This allows a full use of the symmetries of the problem, as demonstrated by the diagonalization of the equatorial equation,  $v$  and  $b$  playing symmetric roles (which was not the case for  $v$  and  $\psi$  in the Proudman problem).

This model therefore seems to be an excellent choice for a complete study of the following three behaviors: degenerate layers, shear layers and equatorial layer, as well as their respective interactions. Indeed, the tools used for the Proudman problem can be adapted without further difficulty (and even with simplification, see previous points).

# Sign-changing solutions to the stationary Burgers equation with transverse viscosity

## Outline of the current chapter

<b>6.1 Introduction</b>	<b>161</b>
<b>6.2 Well-posedness results for the linear problem</b>	<b>166</b>
<b>6.3 Fixed-point scheme for the nonlinear problem</b>	<b>171</b>
<b>6.4 Local uniqueness of solutions to the nonlinear problem</b>	<b>173</b>
<b>6.5 Strategy of proof of Conjecture 6.2.6</b>	<b>173</b>
<b>6.6 Appendix: Technical lemmas</b>	<b>186</b>

Anne-Laure Dalibard, Frédéric Marbach and Jean Rax

## 6.1 Introduction

The goal of this work is to prove the existence and uniqueness of sign-changing solutions for a stationary Burgers equation with transverse viscosity. More precisely, let us consider the equation

$$u\partial_x u - \partial_{yy} u = -1, \quad (6.1)$$

in the domain  $\Omega := (x_0, x_1) \times (0, H)$ , where  $x_0 < x_1$  and  $H > 0$  are real parameters. Then, for all fixed  $\bar{y} \in (0, H)$ , the function

$$\bar{u} := \frac{1}{2}y(\bar{y} - y) \quad (6.2)$$

is an exact solution to (6.1), that changes sign at  $y = \bar{y}$ . Our purpose is to construct a solution to (6.1) in the vicinity of  $\bar{u}$ , by perturbing the lateral boundary data  $\bar{u}|_{x=x_0}$  and  $\bar{u}|_{x=x_1}$ .

### 6.1.1 Motivation from recirculation problems in fluid mechanics

Our motivation stems from fluid mechanics. Indeed, the stationary Prandtl equation, which describes the behavior of a fluid with small viscosity in the vicinity of a wall, reads

$$\begin{aligned} u\partial_x u + v\partial_y u - \partial_{yy} u &= -\partial_x p_E, \\ u|_{y=0} = v|_{y=0} &= 0, \\ \lim_{y \rightarrow \infty} u(x, y) &= u_E(x), \end{aligned} \tag{6.3}$$

where  $u_E(x)$  (resp.  $p_E(x)$ ) is the trace of an outer Euler flow (resp. pressure) on the wall, and satisfies  $u_E \partial_x u_E = -\partial_x p_E$ .

As long as  $u$  remains positive, (6.3) can be seen as a nonlocal, nonlinear diffusion type equation, the variable  $x$  being the evolution variable. Using this point of view, Oleinik (see e.g. [58, Theorem 2.1.1]) proved the local well-posedness of a solution of (6.3) when the equation (6.3) is supplemented with a boundary data  $u|_{x=0} = u_0$ , where  $u_0(y) > 0$  for  $y > 0$  and such that  $u_0'(0) > 0$ . Let us mention that such positive solutions exist globally when  $\partial_x p_E \leq 0$ , but are only local when  $\partial_x p_E > 0$ . More precisely, when  $\partial_x p_E = 1$  for instance, for a large class of boundary data  $u_0$ , there exists  $x^* > 0$  such that  $\lim_{x \rightarrow x^*} u_y(x, 0) = 0$ . Furthermore, the solution may develop a singularity at  $x = x^*$ , known as Goldstein singularity. The point  $x^*$  is called the separation point: intuitively, if the solution of Prandtl exists beyond  $x^*$ , then it must have a negative sign close to the boundary (and therefore change sign). We refer to the seminal works of [38] and Stewartson [71] for formal computations on this problem. A first mathematical statement describing separation was given by Weinan E in [24] in a joint work with Luis Caffarelli, but the complete proof was never published. Recently, the first author and Nader Masmoudi gave a complete description of the formation of the Goldstein singularity [15]. A preprint of Weiming Shen, Yue Wang and Zhifei Zhang indicates that this singularity holds for a large class of initial data [66].

Because of this singularity, it is actually unclear that the Prandtl system is a relevant physical model in the vicinity of the separation point  $x^*$ , because the normal velocity  $v$  becomes unbounded at  $x = x^*$ . Consequently, more refined models, such as the triple deck model (see citeLagree for a presentation of the triple deck), were designed specifically to replace the Prandtl system with a more intricate boundary layer model in the vicinity of the separation point. However, beyond the separation point, i.e. for  $x > x^*$ , it is expected that the Prandtl system becomes valid again, but with a changing sign solution.

To the best of our knowledge, there are no results on the well-posedness of (6.3) when the solution  $u$  is allowed to change sign. Such solutions are called “recirculating solutions”, and the zone where  $u < 0$  is called a recirculation bubble, the usual convention being that  $u_E(x) > 0$ , so that the flow is going forward far from the boundary.

A family of explicit self-similar solutions, called Falkner-Skan profiles, are known, and are given by

$$u(x, y) = x^m f'(\zeta), \tag{6.4}$$

$$v(x, y) = -y^{-1} \zeta f(\zeta) - \frac{m-1}{m+1} y^{-1} \zeta^2 f'(\zeta), \tag{6.5}$$

where  $\zeta := (\frac{m+1}{2})^{\frac{1}{2}} y x^{\frac{m-1}{2}}$  is the self-similarity variable,  $m$  is a real parameter and  $f$  is the solution to the Falkner-Skan equation

$$f''' + f f'' + \beta(1 - (f')^2) = 0, \tag{6.6}$$

where  $\beta = \frac{2m}{m+1}$ , subject to the boundary conditions  $f(0) = f'(0) = 0$  and  $f'(+\infty) = 1$ . Such flows correspond to an outer Euler velocity field  $u_E(x) = x^m$ .

For some particular values of  $m$  (or, equivalently,  $\beta$ ), these formulas provide physical solutions to (6.3) which exhibit recirculation (see [9]). But the question of solving (6.3) in the vicinity of such flows is an open problem. Apart from the usual difficulties of the Prandtl equation (nonlocality of the transport term  $v\partial_y u = -\partial_y u \int_0^y \partial_x u$ , nonlinearity), the issue here lies in the fact that in the zone where  $u > 0$ , (6.3) can be considered as a (forward) heat equation, whereas in the recirculation zone  $u < 0$ , (6.3) becomes a backward heat equation. As a consequence, the intuition is that (6.3) must be supplemented with a boundary condition at  $x = x_0$  in the zone where  $u > 0$ , and at  $x = x_1$  in the zone where  $u < 0$ . Of course the exact sets on which these conditions are prescribed depend on the solution  $u$  itself, and the line  $\{u = 0\}$  can be seen as a type of free boundary.

In order to focus on the difficulties associated with this change of sign, we have chosen in the present work to consider the toy-model (6.1), which differs from (6.3) through the lack of the nonlinear transport term  $v\partial_y u$  and through the boundedness of the domain. This allows us to put aside the specific problems associated with the term  $v\partial_y u$  (nonlocality, loss of derivative). Concerning the boundedness of the domain  $\Omega$ , we emphasize that our results could most certainly be adapted to a domain of the type  $(x_0, x_1) \times (0, +\infty)$ , which would be closer to the Prandtl setting (6.3). In the latter case, there are essentially two issues that we do not address here. The first one is the construction of a special solution  $\bar{u} \in L^\infty((x_0, x_1) \times (0, +\infty))$ , such that  $\bar{u}(x, y) \rightarrow u_\infty(x)$  as  $y \rightarrow \infty$ . It is then quite natural to look for  $\bar{u}$  as a self-similar solution, in the spirit of Falkner-Skan flows. We are led to study a nonlinear 1d ODE; we do not expect this step to give rise to any substantial difficulty. The second issue about the boundedness of  $\Omega$  lies in the fact that we repeatedly use the Poincaré inequality in the vertical variable in the proofs of the present paper. When switching to a non-bounded domain, we will probably need to replace the Poincaré inequality with some variants, such as the Hardy inequality, and thus to adapt the functional spaces used here.

### 6.1.2 Statement of the main result

We must now specify the boundary conditions for equation (6.1). At  $y = 0$  and  $y = H$ , we simply choose Dirichlet boundary conditions, namely

$$u|_{y=0} = 0, \quad u|_{y=H} = \bar{u}|_{y=H}. \quad (6.7)$$

At  $x = x_0$  and  $x = x_1$ , we perturb the trace of  $\bar{u}$ . Let  $\delta_0, \delta_1 \in C^1([0, H])$  vanishing at both endpoints, i.e. such that  $\delta_i(0) = \delta_i(H) = 0$  for  $i \in \{0, 1\}$ . We also define the sets

$$\begin{aligned} \Sigma_0 &:= \{(x_0, y); \bar{u}(x_0, y) + \delta_0(y) > 0\}, \\ \Sigma_1 &:= \{(x_1, y); \bar{u}(x_1, y) + \delta_1(y) < 0\}. \end{aligned} \quad (6.8)$$

Following the above discussion, we endow (6.1) with the lateral boundary conditions

$$u|_{\Sigma_i} = \bar{u}(x_i, y) + \delta_i(y), \quad \text{for } i \in \{0, 1\}. \quad (6.9)$$

When  $\delta_0$  and  $\delta_1$  are small enough in  $C^1([0, H])$ , there exist  $y_0, y_1 \in (0, H)$  such that  $\Sigma_0 = \{x_0\} \times (y_0, H]$  and  $\Sigma_1 = \{x_1\} \times (0, y_1)$  (see Lemma 6.6.1 in the Appendix). However, we do not require that the critical points satisfy  $y_0 = \bar{y}$  or  $y_1 = \bar{y}$  (see e.g. Fig. 6.1 where  $y_0 < \bar{y}$  and  $y_1 > \bar{y}$ ).



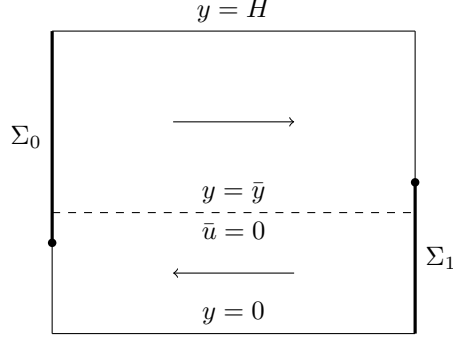


Figure 6.1 – Geometrical setting of the problem

Let us now introduce the functional spaces we will be working with. For  $k \in \mathbb{N}$ , we set

$$X^k := \left\{ u \in L^2(\Omega); \partial_x^j u \in L^2((x_0, x_1); H^1(0, H)), 0 \leq j \leq k \right\},$$

$$\|u\|_{X^k} := \sum_{j=0}^k \|\partial_x^j u\|_{L^2(\Omega)} + \|\partial_y \partial_x^j u\|_{L^2(\Omega)}. \quad (6.10)$$

We also introduce  $X_0^k$  given by replacing  $H^1(0, H)$  with  $H_0^1(0, H)$  in the definition of  $X^k$ .

Our main result is the following local well-posedness result, which holds up to the validity of a conjecture concerning the existence of strong solutions to a linearized problem.

**Theorem 6.1.1.** *Assume that Conjecture 6.2.6 holds (see page 168). There exist constants  $C, c, \eta > 0$ , depending only on  $\Omega$ , such that the following statement holds. Let  $\delta_0, \delta_1 \in C^4([0, H])$ , vanishing at both endpoints, satisfying the following compatibility conditions*

$$\|\delta_i\|_{\bar{u}}^2 := \int_{\Sigma_i} |\bar{u} + \delta_i| \delta_i^2 + \int_{\Sigma_i} \frac{(\delta_i'')^2}{|\bar{u} + \delta_i|} + \int_{\Sigma_i} \frac{1}{|\bar{u} + \delta_i|} \left( \partial_y^2 \frac{\delta_i''}{\bar{u} + \delta_i} - \left( \frac{\delta_i''}{\bar{u} + \delta_i} \right)^2 \right)^2 < +\infty \quad (6.11)$$

and  $\Delta_{1,i} \in \mathcal{C}([0, H])$  with  $\Delta_{1,0}(H) = \Delta_{1,1}(0) = 0$ , where

$$\Delta_{1,i} = \frac{1}{\mathfrak{u}(x_i, \cdot)} [\delta_i'' + f(x_i, \cdot) - b(x_i, \cdot) \delta_i].$$

Then, under the smallness condition  $\|\delta_i\|_{C^1} + \|\delta_i\|_{\bar{u}} \leq c$ , there exists a strong solution  $u$  in the class  $X_0^2$  to equation (6.1) endowed with the boundary conditions (6.7) and (6.9), which satisfies the stability estimate

$$\|u - \bar{u}\|_{X^2} \leq C(\|\delta_0\| + \|\delta_1\|). \quad (6.12)$$

Moreover, this solution is unique in a neighborhood of  $\bar{u}$  in  $X_0^2$  of size  $\eta$ .

*Remark 6.1.1.* Theorem 6.1.1 is a (conditional) well-posedness result in the vicinity of the quadratic flow  $\bar{u}$  defined in (6.2). This allows to fix ideas and draw pictures (as in Fig. 6.1). However, it is very likely that the theorem (and the proofs we develop) still hold with minor adjustments when  $\bar{u}$  is a smooth function (depending on both  $x$  and  $y$ ) with  $\|\bar{u}_x\|_{L^\infty}$  small enough.

### 6.1.3 Scheme of proof

We will prove existence and uniqueness of solutions separately. We consider here strong solutions, so that uniqueness is fairly easy to prove (see Section 6.4).

Concerning the existence, a first natural idea would be to linearize equation (6.1) around the solution  $\bar{u}$ , and to use a fixed point theorem. However, this strategy fails. Indeed, since  $\bar{u}_x = 0$ , the linearized equation would read

$$\bar{u}\partial_x u - \partial_{yy}u = f, \quad (6.13a)$$

$$u|_{y=0} = u|_{y=H} = 0, \quad (6.13b)$$

$$u|_{\Sigma_i} = \delta_i. \quad (6.13c)$$

There are several issues with this linearized equation. First, equation (6.13a) can be considered as a degenerate elliptic equation (see [28]). As a consequence, we can supply (6.13a) with a boundary condition at  $x = x_0$  (resp.  $x = x_1$ ) on the zone  $\{\bar{u}(x_0, y) > 0\}$  (resp.  $\{\bar{u}(x_1, y) < 0\}$ ). But, except for particular choices of  $\delta_0$  (resp.  $\delta_1$ ), this set differs a priori from  $\Sigma_0$  (resp.  $\Sigma_1$ ). As a consequence, the boundary condition (6.13c) should be modified, but then there might be an error in the boundary conditions. Second, if we perform a fixed point argument, the right-hand side  $f$  in (6.13a) should be  $f = -u\partial_x u$ . Therefore we need to have regularity estimates on  $\partial_x u$  in order to close the fixed point argument, but solutions of the linearized equation do not satisfy such estimates. This phenomenon corresponds to a “loss of derivative”.

Hence we will rather construct solutions of (6.1) through an iterative scheme, which does not rely on the linearized equation (6.13a). In a way, the issues stemming from the linearized equation (6.13a) come from the following fact: in equation (6.1), the geometry of the problem is dictated by the line where the whole solution  $u$  changes sign. On the contrary, in (6.13a), the geometry of the problem follows the cancellation of  $\bar{u}$ . Keeping this in mind, we will construct a sequence  $(u_n)_{n \in \mathbb{N}}$  solving the following iterative scheme

$$\begin{cases} (\bar{u} + u_n)\partial_x u_{n+1} - \partial_{yy}u_{n+1} = 0, \\ u_{n+1}|_{y=0} = u_{n+1}|_{y=H} = 0, \\ u_{n+1}|_{\Sigma_i} = \delta_i. \end{cases} \quad (6.14)$$

We are able to prove the existence of  $u_n$  in  $X^0$ . However, the existence of a solution in  $X^k$  for  $k \geq 1$  (and its uniqueness) is harder to prove, in spite of the fact that we are able to obtain a priori estimates in  $X^2$  on the solution if we know that it belongs to this space. Therefore, unfortunately, our theorem only holds under a certain conjecture (see Sections 6.2 and 6.5 for more details). Under this conjecture, we will prove that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges strongly in  $X^1$ , and we will set  $u := \lim_{n \rightarrow \infty} (\bar{u} + u_n)$ . Although the equation for  $u_n$  bears a strong resemblance with (6.13a), it does not share any of the issues mentioned above. This scheme is similar to the one used to construct solutions of quasilinear symmetric hyperbolic systems, see for instance [5, Section 4.3].

The plan of this work is as follows. In Section 6.2, we study the well-posedness of linear problems which are generalized versions of (6.14). Section 6.3 is devoted to the proof of convergence of the iterative scheme which proves the existence of a solution to the nonlinear problem. Section 6.4 is devoted to the proof of the uniqueness of the solution to the nonlinear problem. Eventually, Section 6.5 gives details on a possible strategy to prove Conjecture 6.2.6.

## 6.2 Well-posedness results for the linear problem

We start by introducing notations and regularity assumptions which will be implicit throughout this section. Let  $\mathfrak{u} \in X^2$  be a reference flow. We define the associated inflow lateral boundaries at  $x = x_i$  for  $i \in \{0, 1\}$  by

$$\Sigma_i := \{(x_i, y); (-1)^i \mathfrak{u}(x_i, y) > 0\}. \quad (6.15)$$

We also introduce an associated weighted space  $L_{\mathfrak{u}}^2(\Sigma_i)$  for boundary data defined by the norm

$$\|\delta_i\|_{L_{\mathfrak{u}}^2(\Sigma_i)}^2 := \int_{\Sigma_i} |\mathfrak{u}| \delta_i^2. \quad (6.16)$$

Let  $b \in L^\infty(\Omega)$ ,  $f \in L^2(\Omega)$  and  $\delta_i \in L_{\mathfrak{u}}^2(\Sigma_i)$ . These regularity assumptions, along with  $\mathfrak{u} \in X^2$ , will be implicit in the sequel. We consider the following linear problem, which will be denoted by  $\mathcal{P}[\mathfrak{u}, b, f, \delta]$  when we will need to refer to it with different inputs:

$$\begin{cases} \mathfrak{u} \partial_x u + bu - \partial_{yy} u = f, \\ u|_{y=0} = u|_{y=H} = 0, \\ u|_{\Sigma_i} = \delta_i. \end{cases} \quad (6.17)$$

This system can be viewed as a degenerate elliptic system. Fichera studies such systems in [28] and proves existence and uniqueness results concerning weak solutions under various assumptions, the key point being the choice of the regions where boundary conditions must be imposed depending on the properties of the degenerate elliptic operator. His proofs mostly rely on maximum principles and the construction of signed super or sub-solutions. Here, we focus on solutions to (6.17) with more regularity, which are not covered by Fichera's work.

**Definition 6.2.1** (Implicit constants). *In this section, most results and estimates hold up to a smallness assumption on some norm of the coefficients in (6.17). We will use the notation  $\|\cdot\| \ll 1$  to denote the existence of a constant  $c > 0$ , depending only on the geometry of  $\Omega$  (and not on any other parameter), such that, if  $\|\cdot\| \leq c$ , the result holds. Similarly we will write  $A \lesssim B$  when there exists  $C > 0$  depending only on the geometry of  $\Omega$  such that  $A \leq CB$ .*

**Lemma 6.2.2.** *For  $\phi \in X^1$ , there holds*

$$\|\phi\|_{L^\infty} \lesssim \|\phi\|_{X^1}. \quad (6.18)$$

*Proof.* By Sobolev embeddings, one has  $\|\phi\|_{L^\infty} \lesssim \|\phi\|_{L_x^\infty(H_y^1)} \lesssim \|\phi\|_{H_x^1(H_y^1)} \lesssim \|\phi\|_{X^1}$ .  $\square$

**Definition 6.2.3** (Weak solution). *We say that  $u \in L^2(\Omega)$  is a weak solution to (6.17) when, for all  $v \in H^1(\Omega)$  vanishing on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$  with  $\partial_{yy} v \in L^2(\Omega)$ , the following weak formulation holds*

$$\int_{\Omega} u (-\mathfrak{u} \partial_x v - \mathfrak{u}_x v + bv - \partial_{yy} v) = \int_{\Omega} f v - \int_{\Sigma_1} \mathfrak{u} \delta_1 v + \int_{\Sigma_0} \mathfrak{u} \delta_0 v. \quad (6.19)$$

**Definition 6.2.4** (Strong solution). *We say that  $u \in H^1(\Omega)$  is a strong solution to (6.17) when  $u_{yy} \in L^2(\Omega)$ , the partial differential equation (6.17) holds in  $L^2(\Omega)$  and the four trace equalities hold in  $L^2$  of the four parts of the rectangular boundary of  $\partial\Omega$ .*

**Proposition 6.2.5** (Existence in  $X^0$ ). *When  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} \ll 1$ , there exists a weak solution  $u \in X_0^0$  to (6.17) which moreover satisfies the estimate*

$$\|u\|_{X^0}^2 \lesssim \|f\|_{L^2}^2 + \sum_{i \in \{0,1\}} \|\delta_i\|_{L^2_\mathfrak{u}(\Sigma_i)}^2. \quad (6.20)$$

*Proof.* We proceed by viscous regularization in the horizontal direction. For small  $\varepsilon > 0$ , we consider the vanishing viscosity problem

$$\begin{cases} \mathfrak{u}\partial_x u^\varepsilon + bu^\varepsilon - \varepsilon\partial_{xx}u^\varepsilon - \partial_{yy}u^\varepsilon = f, \\ u^\varepsilon|_{y=0} = u^\varepsilon|_{y=H} = 0, \\ (\mathfrak{u}u^\varepsilon - \varepsilon\partial_x u^\varepsilon)|_{\Sigma_i} = \mathfrak{u}\delta_i, \\ (\partial_x u^\varepsilon)|_{\Sigma_i^c} = 0, \end{cases} \quad (6.21)$$

where  $\Sigma_i^c$  denotes  $(\{x_i\} \times [0, H]) \setminus \Sigma_i$ . For each fixed  $\varepsilon > 0$ , this is an elliptic problem set in a rectangle with mixed Dirichlet, Neumann and Robin boundary conditions, which are inhomogeneous and with variable coefficients.

Let  $V := X_0^0 \cap H^1(\Omega)$ . Hence  $V$  is a Hilbert space for the usual scalar product in  $H^1(\Omega)$ . The variational formulation associated with (6.21) is

$$\forall v \in V, \quad a^\varepsilon(u^\varepsilon, v) = \ell(v), \quad (6.22)$$

where we define

$$a^\varepsilon(u, v) := \int_\Omega \partial_y u \partial_y v + \varepsilon \int_\Omega \partial_x u \partial_x v + \int_\Omega u((b - \mathfrak{u}_x)v - \mathfrak{u}\partial_x v) + \int_{\Sigma_1^c} \mathfrak{u}uv - \int_{\Sigma_0^c} \mathfrak{u}uv, \quad (6.23)$$

$$\ell(v) := \int_\Omega fv - \int_{\Sigma_1} \mathfrak{u}\delta_1 v + \int_{\Sigma_0} \mathfrak{u}\delta_0 v. \quad (6.24)$$

It is clear that  $a^\varepsilon$  is a continuous bilinear form on  $V$  and that  $\ell$  is a continuous linear form on  $V$ . Moreover, for all  $u \in V$ ,

$$\begin{aligned} a^\varepsilon(u, u) &= \varepsilon \int_\Omega (\partial_x u)^2 + \int_\Omega (\partial_y u)^2 + \int_\Omega \left(b - \frac{1}{2}\mathfrak{u}_x\right) u^2 + \frac{1}{2} \sum_{i \in \{0,1\}} \int_{\Sigma_i \cup \Sigma_i^c} |\mathfrak{u}| u^2 \\ &\geq \varepsilon \int_\Omega (\partial_x u)^2 + \frac{1}{2} \int_\Omega (\partial_y u)^2 + \frac{1}{2} \sum_{i \in \{0,1\}} \int_{\Sigma_i \cup \Sigma_i^c} |\mathfrak{u}| u^2, \end{aligned} \quad (6.25)$$

thanks to Poincaré's inequality in the vertical direction, provided that  $\|b\|_{L^\infty} + \|\mathfrak{u}_x\|_{L^\infty} \ll 1$ . This proves that  $a^\varepsilon$  is  $V$ -coercive for each fixed  $\varepsilon > 0$ . Thanks to Lax-Milgram's theorem, there exists a unique  $u^\varepsilon \in V$  such that (6.22) holds.

Moreover, by Cauchy-Schwarz, for any  $\gamma > 0$ , one has the estimate

$$\ell(u^\varepsilon) \leq \gamma \|u^\varepsilon\|_{L^2}^2 + \frac{1}{\gamma} \|f\|_{L^2}^2 + \frac{1}{2} \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}| \delta_i^2 + \frac{1}{2} \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}| (u^\varepsilon)^2. \quad (6.26)$$

Using again Poincaré's inequality in the vertical direction and choosing  $\gamma$  small enough, we

conclude

$$\|u^\varepsilon\|_{X^0}^2 + \varepsilon \|\partial_x u^\varepsilon\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 + \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}| \delta_i^2. \quad (6.27)$$

Hence, we can extract a subsequence which converges weakly in  $X^0$  to some limit  $u \in X_0^0$ . Moreover, this limit satisfies the estimate (6.20).

Let us check that  $u$  is a weak solution to (6.17). Let  $v \in H^1(\Omega)$  vanishing on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$  with  $\partial_{yy}v \in L^2(\Omega)$ . Hence  $v \in V$  and we can plug  $v$  in (6.22) and integrate by parts to obtain

$$\int_{\Omega} u^\varepsilon ((b - \mathfrak{u}_x)v - \mathfrak{u}\partial_x v - \partial_{yy}v) + \int_{\Omega} \varepsilon \partial_x u^\varepsilon \partial_x v = \int_{\Omega} f v - \int_{\Sigma_1} \mathfrak{u} \delta_1 v + \int_{\Sigma_0} \mathfrak{u} \delta_0 v. \quad (6.28)$$

Using the fact that  $u^\varepsilon$  converges weakly to  $u$  in  $L^2(\Omega)$  and that  $\varepsilon \partial_x u^\varepsilon$  converges strongly to 0 in  $L^2(\Omega)$  (thanks to (6.27)), we conclude that the weak formulation (6.19) holds.  $\square$

**Conjecture 6.2.6** (Existence in  $X^1$  for smooth data). *When  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} \ll 1$ , if  $\mathfrak{u}, b, f \in C^\infty(\bar{\Omega})$ ,  $\delta_i \in C^\infty(\bar{\Sigma}_i)$  with  $\delta_0(H) = \delta_1(0) = 0$  and satisfy the compatibility conditions*

$$\Delta_{1,i} \in L_{\mathfrak{u}}^2(\Sigma_i), \quad (6.29)$$

where

$$\Delta_{1,i} := \frac{1}{\mathfrak{u}(x_i, \cdot)} [\delta_i'' + f(x_i, \cdot) - b(x_i, \cdot) \delta_i], \quad (6.30)$$

there exists a strong solution  $u \in X_0^1$  to the linear problem (6.17).

We give details concerning a possible strategy to prove Conjecture 6.2.6 in Section 6.5.

**Lemma 6.2.7** (Uniqueness of the weak solution for smooth coefficients). *Assume that Conjecture 6.2.6 holds,  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} \ll 1$  and  $\mathfrak{u}, b \in C^\infty(\Omega)$ . Then, for all  $f \in L^2(\Omega)$  and  $\delta_i \in L_{\mathfrak{u}}^2(\Sigma_i)$  there exists at most one weak solution  $u \in L^2(\Omega)$  to (6.17).*

*Proof.* Let  $u_1, u_2 \in L^2(\Omega)$  be two weak solutions to (6.17). Then, for all  $v \in H^1(\Omega)$  vanishing on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$  with  $\partial_{yy}v \in L^2(\Omega)$ , the weak formulations (6.19) give

$$\int_{\Omega} (u_1 - u_2) (-\mathfrak{u}\partial_x v + (b - \mathfrak{u}_x)v - \partial_{yy}v) = 0. \quad (6.31)$$

Let  $h \in C_c^\infty(\Omega)$ . We apply Conjecture 6.2.6 to obtain the existence of  $v_h \in X_0^1$ , a weak solution to the adjoint problem with a force term  $h$  (which is compactly supported in  $\Omega$ ) and null boundary data,  $\mathcal{P}[-\mathfrak{u}, b - \mathfrak{u}_x, h, 0]$ , i.e.  $v_h$  solves  $-\mathfrak{u}\partial_x v_h + (b - \mathfrak{u}_x)v_h - \partial_{yy}v_h = h$  and moreover vanishes on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$ . Since  $v_h$  is a strong solution,  $\partial_{yy}v_h \in L^2(\Omega)$  and we can use it as a test function in (6.31). Hence, for every  $h \in C_c^\infty(\Omega)$

$$\int_{\Omega} (u_1 - u_2) h = 0. \quad (6.32)$$

Thus  $u_1$  and  $u_2$  are equal.  $\square$

**Lemma 6.2.8** (Estimate in  $X^1$  for smooth data). *Assume that Conjecture 6.2.6 holds. Under the same assumptions as in Conjecture 6.2.6, when  $\|b_x\|_{L^2} \ll 1$ , the unique strong solution*

$u \in X_0^1$  to (6.17) satisfies the estimate

$$\|u\|_{X^1}^2 \lesssim \|f\|_{H_x^1(L_y^2)}^2 + \sum_{i \in \{0,1\}} \left( \|\delta_i\|_{L_{\mathfrak{u}}^2(\Sigma_i)}^2 + \|\Delta_{1,i}\|_{L_{\mathfrak{u}}^2(\Sigma_i)}^2 \right). \quad (6.33)$$

*Proof.* Let  $u \in X_0^1$  be the strong solution to (6.17) constructed in Conjecture 6.2.6, which is unique thanks to Lemma 6.2.7. Let  $w \in H^1(\Omega)$  vanishing on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$  with  $\partial_{yy}w \in L^2(\Omega)$ . Then  $w_x \in L^2(\Omega)$  and, since  $u$  is a strong solution to (6.17), there holds

$$- \int_{\Omega} (\mathfrak{u}\partial_x u + bu - \partial_{yy}u)w_x = - \int_{\Omega} fw_x. \quad (6.34)$$

Integrating by parts and using  $u|_{\Sigma_i} = \delta_i$  yields

$$\int_{\Omega} (\partial_x u)(-\mathfrak{u}\partial_x w - \mathfrak{u}_x w + (b + \mathfrak{u}_x)w - \partial_{yy}w) = \int_{\Omega} (f_x - b_x u)w - \int_{\Sigma_1} \mathfrak{u}\Delta_{1,1}w + \int_{\Sigma_0} \mathfrak{u}\Delta_{1,0}w. \quad (6.35)$$

Hence  $\partial_x u$  is a weak solution to the problem  $\mathcal{P}[\mathfrak{u}, b + \mathfrak{u}_x, f_x - b_x u, \Delta_1]$ , where  $f_x - b_x u \in L^2(\Omega)$ . Since  $\mathfrak{u}$  and  $b + \mathfrak{u}_x$  are smooth, Lemma 6.2.7 applies and  $\partial_x u$  is also the solution given by Proposition 6.2.5 to this problem, for which (6.20) holds, which proves

$$\|\partial_x u\|_{X^0}^2 \lesssim \|f_x - b_x u\|_{L^2}^2 + \sum_{i \in \{0,1\}} \|\Delta_{1,i}\|_{L_{\mathfrak{u}}^2(\Sigma_i)}^2. \quad (6.36)$$

Moreover  $\|b_x u\|_{L^2} \leq \|b_x\|_{L^2} \|u\|_{L^\infty} \lesssim \|b_x\|_{L^2} \|u\|_{X^1}$  thanks to Lemma 6.2.2, which concludes the proof of (6.33) in the regime  $\|b_x\|_{L^2} \ll 1$ .  $\square$

**Corollary 6.2.9** (Existence in  $X^1$ ). *Assume that Conjecture 6.2.6 holds. When  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} \ll 1$ , for all  $\mathfrak{u} \in X^2$ ,  $b \in L^\infty(\Omega) \cap H_x^1(L_y^2)$  such that  $\|b_x\|_{L^2} \ll 1$ ,  $f \in H_x^1(L_y^2)$  and  $\delta_i \in C^0(\overline{\Sigma}_i)$  satisfying  $\delta_0(H) = \delta_1(0) = 0$  and (6.29), there exists a strong solution  $u \in X_0^1$  to (6.17) which moreover satisfies the estimate (6.33).*

*Proof.* Let  $c > 0$  be given by Conjecture 6.2.6 and Lemma 6.2.8 such that, when  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} + \|b_x\|_{L^2} \leq c$ , both results hold. Let  $\mathfrak{u} \in X^2$  and  $b \in L^\infty(\Omega) \cap H_x^1(L_y^2)$  be such that  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} + \|b_x\|_{L^2} \leq c/2$ . Let  $f \in H_x^1(L_y^2)$  and  $\delta_i \in C^0(\overline{\Sigma}_i)$  with  $\delta_0(H) = \delta_1(0) = 0$  and  $\Delta_{1,i} \in L_{\mathfrak{u}}^2(\Sigma_i)$ .

We construct a sequence  $\mathfrak{u}^n, b^n, f^n \in C^\infty(\Omega)$  and  $\delta_i^n \in C^\infty(\Sigma_i)$  with  $\|\mathfrak{u}_x^n\|_{L^\infty} + \|b^n\|_{L^\infty} + \|b_x^n\|_{L^2} \leq c$  and  $\int_{\Sigma_i} |\mathfrak{u}^n|(\Delta_{1,i}^n)^2 \leq \int_{\Sigma_i} |\mathfrak{u}|\Delta_{1,i}^2$ , such that  $\mathfrak{u}^n \rightarrow \mathfrak{u}$  in  $X^2$  (in particular  $\mathfrak{u}_x^n \rightarrow \mathfrak{u}_x$  in  $L^\infty(\Omega)$ ),  $b^n \rightarrow b$  in  $L^\infty(\Omega)$ ,  $f^n \rightarrow f$  in  $L^2(\Omega)$  and  $\delta_i^n \rightarrow \delta_i$  in  $L^2(\Sigma_i)$ . We choose  $\mathfrak{u}^n$  such that  $\Sigma_i^n = \Sigma_i$  for all  $n \in \mathbb{N}$ .

*Regularization of the problem data.* To this aim, we denote by  $R^n$  a sequence of regularization operators from  $L^2$  to  $C^\infty$  (e.g. defined by convolution with a kernel), for functions defined either on  $\overline{\Omega}$  (in two variables) or on  $\overline{\Sigma}_i$  (in one variable). We design this sequence such that it is uniformly continuous from each Sobolev space to itself. We set  $\mathfrak{u}^n := R^n[\mathfrak{u}]$ ,  $b^n := R^n[b]$ ,  $f^n := R^n[f]$ . Moreover, we construct  $\delta_i^n$  such that

$$\Delta_{1,i}^n = |\mathfrak{u}^n|^{-\frac{1}{2}} R^n \left[ |\mathfrak{u}|^{\frac{1}{2}} \Delta_{1,i} \right]. \quad (6.37)$$

This means that  $\delta_i^n$  are defined implicitly as the solutions to

$$(\delta_i^n)'' - b^n(x_i, \cdot)\delta_i^n = |\mathfrak{u}^n|^{\frac{1}{2}}R^n \left[ |\mathfrak{u}|^{\frac{1}{2}}\Delta_{1,i} \right] - f^n(x_i, \cdot), \quad (6.38)$$

with  $\delta_0(H) = \delta_1(0) = 0$  and  $\delta_0'(y_0) = \delta_1'(y_1) = 0$  where  $\Sigma_0 = (y_0, H]$  and  $\Sigma_1 = (0, y_1)$ . A consequence of (6.37) is that

$$\int_{\Sigma_i} |\mathfrak{u}^n|(\Delta_{1,i}^n)^2 = \int_{\Sigma_i} \left( R^n \left[ |\mathfrak{u}|^{\frac{1}{2}}\Delta_{1,i} \right] \right)^2 \leq C_R \int_{\Sigma_i} |\mathfrak{u}|\Delta_{1,i}^2. \quad (6.39)$$

One also has

$$\|f^n\|_{H_x^1(L_y^2)} \leq C_R \|f\|_{H_x^1(L_y^2)}. \quad (6.40)$$

*Convergence of the boundary data.* Let  $g_i^n$  denote the right-hand side of (6.38). One has  $g_i^n \in C^\infty(\overline{\Sigma}_i)$  and  $g_i^n \rightarrow |\mathfrak{u}|\Delta_{1,i} - f(x_i, \cdot)$  in  $L^2(\Sigma_i)$ . Thanks to Poincaré's inequality, for  $\|b^n\|_{L^\infty} \ll 1$ , problem (6.38) has a unique solution  $\delta_i^n \in H^1(\Sigma_i)$  with the given boundary conditions. Moreover, this solution depends continuously on the source term and on the zero-order coefficient, so that  $\delta_i^n \rightarrow \delta_i$  in  $L^2(\Sigma_i)$ .

*Convergence of the sequence of approximate solutions.* For each  $n \in \mathbb{N}$ , we apply Lemma 6.2.8 to obtain the existence of  $u^n \in X_0^1$ , strong solution to the regularized problem. In particular, there exists  $C > 0$  given by Lemma 6.2.8 such that

$$\begin{aligned} \|u^n\|_{X^1}^2 &\leq C \|f^n\|_{H_x^1(L_y^2)}^2 + C \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}^n| ((\delta_i^n)^2 + (\Delta_{1,i}^n)^2) \\ &\leq CC_R \|f\|_{H_x^1(L_y^2)}^2 + CC_R \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}|\Delta_{1,i}^2 + C \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}^n|(\delta_i^n)^2, \end{aligned} \quad (6.41)$$

thanks to (6.39) and (6.40). Since  $\delta_i^n \rightarrow \delta_i$  in  $L^2(\Sigma_i)$  and  $\mathfrak{u}^n \rightarrow \mathfrak{u}$  in  $L^\infty(\Omega)$ , the right-hand side of (6.41) is bounded. Thus, the sequence  $u^n$  is uniformly bounded in  $X_0^1$  and we can extract a sub-sequence weakly converging to some  $u \in X_0^1$  which satisfies (6.33). For each fixed  $v \in H^1(\Omega)$  vanishing on  $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$  with  $\partial_{yy}v \in L^2(\Omega)$ , for all  $n \in \mathbb{N}$ , the weak formulation (6.19) reads

$$\int_{\Omega} u^n (-\mathfrak{u}^n \partial_x v - \mathfrak{u}_x^n v + b^n v - \partial_{yy}v) = \int_{\Omega} f^n v - \int_{\Sigma_1} \mathfrak{u}^n \delta_1^n v + \int_{\Sigma_0} \mathfrak{u}^n \delta_0^n v. \quad (6.42)$$

The weak convergence of  $u^n$  in  $L^2(\Omega)$ , the strong convergence of  $\mathfrak{u}^n$  and  $\mathfrak{u}_x^n$  in  $L^\infty(\Omega)$  and the strong convergence of  $\delta_i^n$  in  $L^2(\Sigma_i)$  imply that  $u$  satisfies the weak formulation (6.19).  $\square$

**Lemma 6.2.10** (Uniqueness of the weak solution). *Assume that Conjecture 6.2.6 holds. Then, when  $b \in L^\infty(\Omega) \cap H_x^1(L_y^2)$  and in the regime  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} + \|b_x\|_{L^2} + \|\mathfrak{u}_{xx}\|_{L^2} \ll 1$ , there exists at most one weak solution  $u \in L^2(\Omega)$  to (6.17).*

*Proof.* The proof is exactly identical to that of the uniqueness result for smooth coefficients Lemma 6.2.7, except that we obtain the existence of a solution to the adjoint problem from Corollary 6.2.9 instead of Conjecture 6.2.6. Applying Corollary 6.2.9 requires the assumption  $\|(b - \mathfrak{u}_x)_x\|_{L^2} \ll 1$ , which motivates the chosen regime.  $\square$

**Corollary 6.2.11** (Existence in  $X^2$ ). *Assume that Conjecture 6.2.6 holds. Then, when  $b \in L^\infty(\Omega) \cap H_x^2(L_y^2)$  with  $\|b_{xx}\|_{L^2} \leq 1$ ,  $f \in H_x^2(L_y^2)$  and  $\delta_i \in C^0(\overline{\Sigma}_i)$  satisfying  $\|\mathfrak{u}_x\|_{L^\infty} + \|b\|_{L^\infty} +$*

$\|b_x\|_{L^2} + \|\mathfrak{u}_{xx}\|_{L^2} \ll 1$ ,  $\Delta_{1,i} \in C^0(\overline{\Sigma}_i)$  with  $\Delta_{1,0}(H) = \Delta_{1,1}(0) = 0$  and such that the compatibility conditions

$$\Delta_{2,i} \in L^2_{\mathfrak{u}}(\Sigma_i) \quad (6.43)$$

are satisfied, where

$$\Delta_{2,i} := \frac{1}{\mathfrak{u}(x_i, \cdot)} [\Delta''_{1,i} + f_x(x_i, \cdot) - b_x(x_i, \cdot)\delta_i - (b + \mathfrak{u}_x)(x_i, \cdot)\Delta_{1,i}], \quad (6.44)$$

there exists a strong solution  $u \in X_0^2$  to (6.17), which moreover satisfies the estimate

$$\|u\|_{X^2}^2 \lesssim \|f\|_{H_x^2(L_y^2)}^2 + \sum_{i \in \{0,1\}} \int_{\Sigma_i} |\mathfrak{u}_i| (\delta_i^2 + \Delta_{1,i}^2 + \Delta_{2,i}^2). \quad (6.45)$$

*Proof.* First, thanks to Corollary 6.2.9, there exists a solution  $u$  with regularity  $X^1$  which satisfies (6.33) and it is unique thanks to (6.2.10). Moreover,  $\partial_x u \in X_0^0$  is a weak solution to the problem  $\mathcal{P}[\mathfrak{u}, b + \mathfrak{u}_x, f_x - b_x u, \Delta_1]$ . This solution is unique thanks to Lemma 6.2.10 because  $b + \mathfrak{u}_x \in L^\infty(\Omega) \cap H_x^1(L_y^2)$ ,  $\|\mathfrak{u}_x\|_{L^\infty} + \|\mathfrak{u}_{xx}\|_{L^2} \ll 1$  and  $\|b + \mathfrak{u}_x\|_{L^\infty} + \|(b + \mathfrak{u}_x)_x\|_{L^2} \ll 1$ . Hence  $\partial_x u$  is also the solution given by Corollary 6.2.9 to the problem  $\mathcal{P}[\mathfrak{u}, b + \mathfrak{u}_x, f_x - b_x u, \Delta_1]$ . Thanks to (6.33), it satisfies the estimate

$$\|\partial_x u\|_{X^1}^2 \lesssim \|f_x - b_x u\|_{H_x^1(L_y^2)}^2 + \sum_{i \in \{0,1\}} \left( \|\Delta_{1,i}\|_{L^2_{\mathfrak{u}}(\Sigma_i)}^2 + \|\Delta_{2,i}\|_{L^2_{\mathfrak{u}}(\Sigma_i)}^2 \right). \quad (6.46)$$

Moreover,

$$\begin{aligned} \|b_x u\|_{H_x^1(L_y^2)} &\lesssim \|b_x u\|_{L^2} + \|b_{xx} u\|_{L^2} + \|b_x \partial_x u\|_{L^2} \\ &\lesssim \|b_x\|_{L^\infty} \|u\|_{X^0} + \|b_{xx}\|_{L^2} \|u\|_{L^\infty} + \|b_x\|_{L^\infty} \|\partial_x u\|_{L^2} \\ &\lesssim (\|b_x\|_{L^\infty} + \|b_{xx}\|_{L^2}) \|u\|_{X^1}, \end{aligned} \quad (6.47)$$

using Lemma 6.2.2. This concludes the proof of (6.45) using the assumption  $\|b_{xx}\|_{L^2} \leq 1$ . (If  $b_{xx}$  is larger in  $L^2(\Omega)$ , one can still obtain (6.45) up to multiplying the right-hand side by  $1 + \|b_{xx}\|_{L^2}^2$ ).  $\square$

*Remark 6.2.1.* It is likely that one can iterate the construction and prove the existence of solutions with regularity  $X^k$  for  $k > 2$ , provided that one assumes enough regularity on the data and enough compatibility conditions.

## 6.3 Fixed-point scheme for the nonlinear problem

In this section, we prove the existence statement of Theorem 6.1.1 thanks to a nonlinear fixed-point scheme well-suited to the geometrical structure of the nonlinear problem, which avoids a “loss of derivative” in the horizontal direction, as sketched out in Section 6.1.3.

Let  $c > 0$  small enough to be chosen later. Let  $\delta_0, \delta_1 \in C^4([0, H])$  with  $\|\delta_i\| \leq c$ , for the norm defined in (6.11) which corresponds to compatibility conditions, and vanishing at both endpoints.

Let  $\chi \in C^\infty(\mathbb{R}, [0, 1])$ , identically equal to one on  $[-\frac{1}{3}, \frac{1}{3}]$  and compactly supported in  $[-\frac{1}{2}, \frac{1}{2}]$ .



We define the initialization profile of our iterative scheme as

$$u_0(x, y) := \delta_0(y)\chi\left(\frac{x-x_0}{x_1-x_0}\right) + \delta_1(y)\chi\left(\frac{x_1-x}{x_1-x_0}\right). \quad (6.48)$$

Hence, there exists  $C_\chi > 0$  such that  $u_0 \in X_0^2$  and  $\|u_0\|_{X^2} \leq C_\chi(\|\delta_0\|_{H^1(0,H)} + \|\delta_1\|_{H^1(0,H)}) \leq 2cC_\chi$ .

We define the sequence  $(u_n)_{n \in \mathbb{N}}$  by induction on  $n$  where  $u_{n+1}$  is the solution to (6.14) as explained in Section 6.1.3. At each step, we intend to apply Corollary 6.2.11. Therefore, let  $C_2, c_2 > 0$  be given by Corollary 6.2.11 (i.e. for which  $\|\mathbf{u}_x\|_{L^\infty} + \|\mathbf{u}_{xx}\|_{L^2} \leq c_2$  implies the existence of a solution  $u \in X_0^2$  with an energy estimate (6.45) holding up to the constant  $C_2$ ). We apply the proposition with  $\mathbf{u} = \bar{u} + u_n$ ,  $u = u_{n+1}$  and  $b = f = 0$ . From the boundary conditions in  $x_0, x_1$  of (6.14), we infer that  $u_n|_{\Sigma_i} = \delta_i$ , so that  $\mathbf{u}_i = \bar{u} + \delta_i$  is independent of  $n$  and therefore the compatibility conditions (6.29) and (6.43) are satisfied once and for all, from the assumption (6.11).

Let us prove by induction that  $\|\partial_x u_n\|_{L^\infty} + \|\partial_{xx} u_n\|_{L^2} \leq c_2$  and  $\|u_n\|_{L^\infty} \leq \|\bar{u}\|_{L^\infty}$ . This is true for  $n = 0$  up to choosing  $c$  small enough. Applying estimate (6.45), we obtain

$$\|u_{n+1}\|_{X^2}^2 \leq 2C_2 c^2. \quad (6.49)$$

Hence the bounds propagate if  $(2C_2)^{\frac{1}{2}}c \leq \min\{\frac{c_2}{2}, \|\bar{u}\|_{L^\infty}\}$ . We deduce that the sequence  $u_n$  is uniformly bounded in  $X^2$ .

We now turn to the convergence of the sequence. Let  $w_n := u_{n+1} - u_n \in X_0^2$ . Then  $w_n$  is the strong solution to

$$\begin{cases} (\bar{u} + u_n)\partial_x w_n - \partial_{yy} w_n = -w_{n-1}\partial_x u_n, \\ (w_n)|_{y=0} = (w_n)|_{y=H} = 0, \\ (w_n)|_{\Sigma_i} = 0. \end{cases} \quad (6.50)$$

The  $\Delta_{1,i}$  associated with this problem (6.50) are null because the boundary data is null and the trace of the force also. From Corollary 6.2.9 and estimate (6.33), we obtain, under the condition  $\|\partial_x u_n\|_{L^\infty} \leq c_1$  (which is uniformly satisfied provided that  $c$  is small enough, and where  $c_1$  is given by Corollary 6.2.9),

$$\begin{aligned} \|w_n\|_{X^1}^2 &\leq C_1 (\|w_{n-1}\partial_x u_n\|_{L^2}^2 + \|\partial_x(w_{n-1}\partial_x u_n)\|_{L^2}^2) \\ &\leq C_1 \left( \|\partial_x u_n\|_{L^\infty}^2 \|w_{n-1}\|_{H_x^1(L_y^2)}^2 + \|\partial_{xx} u_n\|_{L^2}^2 \|w_{n-1}\|_{L^\infty}^2 \right) \\ &\leq C_1 C_\Omega^2 \|u_n\|_{X^2}^2 \|w_{n-1}\|_{X^1}^2 \\ &\leq (2C_1 C_2 C_\Omega^2 c^2) \|w_{n-1}\|_{X^1}^2. \end{aligned} \quad (6.51)$$

Hence, the sequence  $\|w_n\|_{X^1}$  decays geometrically provided that  $c$  is small enough. This guarantees that  $u_n$  converges strongly in  $X^1$  towards some limit  $u_\infty \in X_0^1$ . Moreover, thanks to the uniform bound in  $X^2$ , one also has  $u_\infty \in X_0^2$  with  $\|u_\infty\|_{X^2} \lesssim \|\delta_0\| + \|\delta_1\|$ .

We define  $u := \bar{u} + u_\infty$ . The strong convergence of  $u_n$  to  $u_\infty$  in  $X^1$  allows to pass to the limit in the weak formulation of the partial differential equation because  $u_n \partial_x u_{n+1} \rightarrow u_\infty \partial_x u_\infty$  in  $L^2(\Omega)$ . Hence,  $u \in X^2$  is a strong solution to (6.1) with boundary conditions (6.7) and (6.9), and it satisfies the stability estimate (6.12).

## 6.4 Local uniqueness of solutions to the nonlinear problem

We prove the local uniqueness statement in Theorem 6.1.1.

The argument is straightforward: if two strong solutions  $u_1$  and  $u_2$  exist which are close enough to the reference solution  $\bar{u}$ , their difference  $w := u_1 - u_2$  is the solution to a degenerate elliptic linear equation with null source term and boundary values, so it vanishes identically. Moreover, up to choosing  $c > 0$  small enough (i.e.  $2Cc < \eta$ ), the solution  $u$  of which we prove the existence is indeed in the ball of radius  $\eta > 0$  within which we prove the uniqueness, thanks to the smallness assumption  $\|\delta_i\| \leq c$  and the stability estimate (6.12).

Let  $c_1 > 0$  be given by Lemma 6.2.10 and  $C_\Omega > 0$  as in Lemma 6.2.2. Let us prove that the uniqueness holds within a ball of radius  $\eta := c_1/(2C_\Omega) > 0$ .

Let  $u_1, u_2 \in X^2$  be two solutions to (6.1) with boundary conditions (6.7) and (6.9) satisfying  $\|u_1 - \bar{u}\|_{X^2} \leq \eta$  and  $\|u_2 - \bar{u}\|_{X^2} \leq \eta$ . Then  $w := u_1 - u_2 \in X_0^2$  and solves the system

$$\begin{cases} u_1 \partial_x w + (\partial_x u_2) w - \partial_y^2 w = 0, \\ w|_{y=0} = w|_{y=H} = 0, \\ w|_{\Sigma_i} = 0. \end{cases} \quad (6.52)$$

According to Lemma 6.2.10 (applied with  $\mathbf{u} = u_1$ ,  $w = \partial_x u_2$ ,  $f = 0$  and  $\delta_i = 0$ ),  $w$  is null as soon as  $\|\partial_x u_1\|_{L^\infty} + \|\partial_x u_2\|_{L^\infty} \leq c_1$ . By the Sobolev embedding Lemma 6.2.2,  $\|\partial_x u_1\|_{L^\infty} \leq C_\Omega \|u_1 - \bar{u}\|_{X^2} \leq \eta C_\Omega$  and the same bound holds for  $\|\partial_x u_2\|_{L^\infty}$ . This concludes the proof of the uniqueness within the ball of radius  $\eta$  in  $X^2$ .

## 6.5 Strategy of proof of Conjecture 6.2.6

In this section, we have gathered some steps towards the proof of Conjecture 6.2.6. We regularize problem (6.17) by viscosity, and we consider throughout this section the problem

$$\begin{cases} \mathbf{u} \partial_x u^\varepsilon + b u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon - \partial_{yy} u^\varepsilon = f, \\ u^\varepsilon|_{y=0} = u^\varepsilon|_{y=H} = 0, \\ u^\varepsilon|_{x=x_i} = \tilde{\delta}_i, \end{cases} \quad (6.53)$$

where  $\tilde{\delta}_i \in C^\infty([0, H])$  is a smooth extension of  $\delta_i \in C^\infty(\bar{\Sigma}_i)$ . We assume that  $\mathbf{u}, b, f \in C^\infty(\Omega)$ .

Let us lift the boundary data by setting  $u_l := \chi_0(x) \tilde{\delta}_0 + \chi_1(x) \tilde{\delta}_1$ , with  $\chi_0, \chi_1 \in C^\infty([x_0, x_1])$  and  $\chi_i \equiv 1$  in a neighbourhood of  $x = x_i$ ,  $\chi_i(x) = 0$  if  $|x - x_i| \geq |x_0 - x_1|/2$ . Then  $\tilde{u}^\varepsilon := u^\varepsilon - u_l$  is a solution of (6.53) with homogeneous Dirichlet boundary conditions (thanks to  $\tilde{\delta}_0(H) = \delta_0(H) = 0$  and  $\tilde{\delta}_1(0) = \delta_1(0) = 0$ ) and with source term  $\tilde{f}^\varepsilon := f - \mathbf{u} \partial_x u_l + b u_l - \varepsilon \partial_{xx} u_l - \partial_{yy} u_l$ . Note that  $\tilde{f}^\varepsilon$  is bounded in  $W^{k_0, \infty}(\Omega)$  uniformly in  $\varepsilon$ . Hence, without loss of generality, we can always assume that  $\tilde{\delta}_i = 0$ , which allows us to look for a solution in  $H_0^1(\Omega)$ . From now on, we drop the tildas, and consider equation (6.53) with  $\delta_i = 0$ .

Note that if  $\|b\|_\infty$  and  $\|\mathbf{u}_x\|_\infty$  are small enough, then existence and uniqueness of a solution  $u^\varepsilon \in H_0^1(\Omega)$  follow easily from the Lax-Milgram Lemma and the Poincaré inequality. This solution satisfies, provided  $\|\mathbf{u}_x\|_{L^\infty} \leq c_0$ ,  $\|b\|_\infty \leq c_0$ ,

$$\|\partial_y u^\varepsilon\|_{L^2}^2 + \varepsilon \|\partial_x u^\varepsilon\|_{L^2}^2 \leq C_0 \|f\|_{L^2}^2. \quad (6.54)$$

The goal of the present section is to derive higher regularity bounds on the sequence  $(u^\varepsilon)_{\varepsilon > 0}$ , in order to give a partial proof of the validity of Conjecture 6.2.6. Note first that if  $f \in L^2(\Omega)$ , then the results on elliptic equations in corner domains (see for instance [42, 43]) ensure that  $u^\varepsilon \in H^2(\Omega)$ . Moreover, if  $f$  is more regular, say  $H^\infty(\Omega)$ , then it can be easily proved that  $u^\varepsilon \in H^\infty(\Omega)$  by induction. These arguments legitimate all the following integrations by parts, where we attempt to obtain estimates independent on  $\varepsilon$ .

### 6.5.1 Difficulties with a mixed boundary conditions strategy

Instead of working with (6.53), it would also be natural to adopt the following scheme, with mixed boundary conditions

$$\begin{cases} \mathfrak{u} \partial_x \tilde{u}^\varepsilon + b \tilde{u}^\varepsilon - \varepsilon \partial_{xx} \tilde{u}^\varepsilon - \partial_{yy} \tilde{u}^\varepsilon = f, \\ \tilde{u}^\varepsilon|_{y=0} = \tilde{u}^\varepsilon|_{y=H} = 0, \\ \tilde{u}^\varepsilon|_{\Sigma_i} = \tilde{\delta}_i, \\ \partial_x \tilde{u}^\varepsilon|_{\{x=x_i\} \setminus \Sigma_i} = 0. \end{cases} \quad (6.55)$$

The Neumann boundary condition on the sets  $\{x = x_i, (-1)^i \mathfrak{u}(x_i, y) < 0\}$  is a kind of transparent boundary condition, whose purpose is to avoid the boundary layers which will occur in the vicinity of these sets with the approximation scheme (6.53) (see paragraph 6.5.5 below). It can be checked that the solution of (6.55) also satisfies the estimate (6.54). It is then natural, keeping in mind the  $X^1$  a priori estimates of section 6.2, to look for estimates on  $\partial_x \tilde{u}^\varepsilon$ . If  $\tilde{u}^\varepsilon \in H^2(\Omega)$ , then it is legitimate to differentiate equation (6.55) with respect to  $x$  and to perform energy estimate. The reader can then verify that if the compatibility condition (6.29) is satisfied, the family  $(\tilde{u}^\varepsilon)_{\varepsilon > 0}$  is uniformly bounded in  $X^1$ . Passing to the limit, we would prove Conjecture 1.

However, this strategy fails. Indeed, looking at the results of Grisvard on elliptic equations in polygonal domains [41], we see that in general,  $\tilde{u}^\varepsilon$  does not belong to  $H^2(\Omega)$ , because of the sharp transition between Dirichlet and Neumann boundary conditions. More precisely, let  $y_i \in (0, H)$  be such that  $\mathfrak{u}(x_i, y_i) = 0$ , and  $(-1)^i \mathfrak{u}(x_i, y) < 0$  for  $y \in (0, y_i)$ ,  $(-1)^i \mathfrak{u}(x_i, y) > 0$  for  $y \in (y_i, H)$ . In other words,  $\Sigma_0 = \{(x_0, y), y \in (y_0, H)\}$  and  $\Sigma_1 = \{(x_1, y), y \in (0, y_1)\}$ . Then Theorem 4.4.3.7 in [41] implies that

$$\tilde{u}^\varepsilon \in u_{\text{sing}, 0}^\varepsilon + u_{\text{sing}, 1}^\varepsilon + H^2(\Omega),$$

and

$$u_{\text{sing}, i}^\varepsilon = C_{\varepsilon, i} r_{\varepsilon, i}^{1/2} \sin\left(\frac{\theta_{\varepsilon, i}}{2}\right)$$

in the vicinity of  $(x_i, y_i)$ , where

$$r_{\varepsilon, i} = \left( \frac{(x - x_i)^2}{\varepsilon} + (y - y_i)^2 \right)^{1/2},$$

$$\tan \theta_{\varepsilon, i} = \frac{x - x_i}{\sqrt{\varepsilon}(y - y_i)}.$$

The functions  $u_{\text{sing}, i}^\varepsilon$  are smooth outside a neighborhood of  $(x_i, y_i)$ , but their singularity in  $r_{\varepsilon, i}^{1/2}$  prevents  $\tilde{u}^\varepsilon$  from belonging to  $H^2$ .

Unfortunately, it seems difficult to control the dependency with respect to  $\varepsilon$  of the constants  $C_{\varepsilon, i}$ , and to prove that they vanish in the limit  $\varepsilon \rightarrow 0$ , for instance. Therefore, we rather chose to work with the Dirichlet problem (6.53), and to deal with the boundary layers therein.

### 6.5.2 Vertically weighted vertical regularity estimates

In order not to burden the presentation, we will present the vertical regularity estimates (and the subsequent estimates on the trace of  $\partial_x u^\varepsilon$ ) in the slightly simplified setting where  $b = 0$ . The extension to an arbitrary  $b$  (smooth and small in  $L^\infty$  norm) is easy and left to the reader.

#### First-order estimate

We multiply (6.53) by  $-\mathfrak{u}^2 \partial_{yy} u^\varepsilon$  to obtain

$$\int \mathfrak{u}^2 (\partial_{yy} u^\varepsilon)^2 - \int \mathfrak{u}^3 \partial_x u^\varepsilon \partial_{yy} u^\varepsilon + \varepsilon \int \mathfrak{u}^2 \partial_{xx} u^\varepsilon \partial_{yy} u^\varepsilon = - \int f \mathfrak{u}^2 \partial_{yy} u^\varepsilon \quad (6.56)$$

The second term is

$$\begin{aligned} - \int \mathfrak{u}^3 \partial_x u^\varepsilon \partial_{yy} u^\varepsilon &= 3 \int \mathfrak{u}^2 \mathfrak{u}_y \partial_x u^\varepsilon \partial_y u^\varepsilon + \int \mathfrak{u}^3 \partial_{xy} u^\varepsilon \partial_y u^\varepsilon \\ &= 3 \int \mathfrak{u} \mathfrak{u}_y (f + \varepsilon \partial_{xx} u^\varepsilon + \partial_{yy} u^\varepsilon) \partial_y u^\varepsilon - \frac{3}{2} \int \mathfrak{u}^2 \mathfrak{u}_x (\partial_y u^\varepsilon)^2 \\ &= -3\varepsilon \int (\mathfrak{u}_x \mathfrak{u}_y + \mathfrak{u} \mathfrak{u}_{xy}) \partial_x u^\varepsilon \partial_y u^\varepsilon + \frac{3}{2} \varepsilon \int (\mathfrak{u} \mathfrak{u}_{yy} + \mathfrak{u}_y^2) (\partial_x u^\varepsilon)^2 \\ &\quad + 3 \int \mathfrak{u} \mathfrak{u}_y (f + \partial_{yy} u^\varepsilon) \partial_y u^\varepsilon - \frac{3}{2} \int \mathfrak{u}^2 \mathfrak{u}_x (\partial_y u^\varepsilon)^2. \end{aligned} \quad (6.57)$$

The third term is

$$\begin{aligned} \varepsilon \int \mathfrak{u}^2 \partial_{xx} u^\varepsilon \partial_{yy} u^\varepsilon &= -2\varepsilon \int \mathfrak{u} \mathfrak{u}_x \partial_x u^\varepsilon \partial_{yy} u^\varepsilon - \varepsilon \int \mathfrak{u}^2 \partial_x u^\varepsilon \partial_{xyy} u^\varepsilon \\ &= -2\varepsilon \int \mathfrak{u} \mathfrak{u}_x \partial_x u^\varepsilon \partial_{yy} u^\varepsilon + \varepsilon \int \mathfrak{u}^2 (\partial_{xy} u^\varepsilon)^2 + 2\varepsilon \int \mathfrak{u} \mathfrak{u}_y \partial_x u^\varepsilon \partial_{xy} u^\varepsilon \\ &= -2\varepsilon \int \mathfrak{u} \mathfrak{u}_x \partial_x u^\varepsilon \partial_{yy} u^\varepsilon + \varepsilon \int \mathfrak{u}^2 (\partial_{xy} u^\varepsilon)^2 - \varepsilon \int (\partial_x u^\varepsilon)^2 (\mathfrak{u} \mathfrak{u}_{yy} + \mathfrak{u}_y^2). \end{aligned} \quad (6.58)$$

Hence, when  $\mathfrak{u}$ ,  $\mathfrak{u}_x$ ,  $\mathfrak{u}_y$ ,  $\mathfrak{u}_{yy}$  and  $\mathfrak{u}_{xy}$  are bounded in  $L^\infty$ , we obtain

$$\|\mathfrak{u} \partial_{yy} u^\varepsilon\|_{L^2}^2 + \varepsilon \|\mathfrak{u} \partial_{xy} u^\varepsilon\|_{L^2}^2 \leq C_1 \|f\|^2. \quad (6.59)$$

#### Higher-order estimates

**Lemma 6.5.1** (Commutator formula). *For every  $l \in \mathbb{N}$ , there exists  $(l+1)$  functionals  $\alpha_{l,0}, \dots, \alpha_{l,l} : C^l([0, H]; \mathbb{R}) \rightarrow C^0([0, H]; \mathbb{R})$  such that, for every weight  $\chi \in C^l([0, H]; \mathbb{R})$  and every test function  $\phi \in C^\infty([0, H]; \mathbb{R})$ , there holds*

$$\chi^{l+1} \partial_y^l \phi = \sum_{l'=0}^l \alpha_{l,l'}[\chi] \chi^{l'} \partial_y^{l'} (\chi \phi). \quad (6.60)$$

Moreover, there exists  $C_l > 0$  such that, for all  $\chi \in C^l([0, H])$ ,

$$\sup_{0 \leq l' \leq l} \|\alpha_{l,l'}[\chi]\|_{L^\infty} \leq C_l \left( \sup_{0 \leq i \leq l} \|\chi^{i-1} \partial_y^i \chi\|_{L^\infty} \right)^l. \quad (6.61)$$

More precisely, for  $0 \leq l' \leq l$ , the function  $\alpha_{l,l'}[\chi]$  depends polynomially on the functions  $\chi^{i-1} \partial_y^i \chi$  for  $0 \leq i \leq l$ , with a total degree (in  $\chi$ ) of  $l - l'$ .

*Proof.* Let  $\chi, \phi \in C^\infty$ . For every  $l \in \mathbb{N}$ , the Leibniz formula yields

$$\frac{1}{l!} \chi^l \partial_y^l (\chi \phi) = \sum_{l'=0}^l c_{l-l'} \frac{1}{l'!} \chi^{l'+1} \partial_y^{l'} \phi, \quad (6.62)$$

where we introduce, for  $i \in \mathbb{N}$ ,

$$c_i := \frac{1}{i!} \chi^{i-1} \partial_y^i \chi, \quad (6.63)$$

with, in particular,  $c_0 \equiv 1$ . Letting similarly, for  $0 \leq i \leq l$ ,  $a_i := \chi^i \partial_y^i (\chi \phi) / i!$  and  $b_i := \chi^{i+1} (\partial_y^i \phi) / i!$ , one obtains the equality  $a(y) = C(y)b(y)$ , where  $C(y)$  is the lower triangular matrix with coefficients  $C_{i,j}(y) := c_{i-j}(y) \delta_{i \geq j}$ . Since  $c_0 \equiv 1$ ,  $\det C \equiv 1$  and  $C$  is invertible. Then, (6.60) and (6.61) are respectively consequences of the invertibility of  $C$  and of the explicit computation of  $C^{-1}$  from the cofactor matrix.  $\square$

We proceed by induction on  $k \in \mathbb{N}^*$ . We differentiate (6.53) with respect to  $y$ ,  $k$  times, we multiply by  $\mathfrak{u}^{2k} \partial_y^k u^\varepsilon$  and we integrate on the whole domain. Hence

$$- \int \mathfrak{u}^{2k} \partial_y^{k+2} u^\varepsilon \partial_y^k u^\varepsilon + \int \mathfrak{u}^{2k} \partial_y^k (\mathfrak{u} \partial_x u^\varepsilon) \partial_y^k u^\varepsilon - \varepsilon \int \mathfrak{u}^{2k} (\partial_{xx} \partial_y^k u^\varepsilon) \partial_y^k u^\varepsilon = \int \mathfrak{u}^{2k} (\partial_y^k f) \partial_y^k u^\varepsilon. \quad (6.64)$$

The right-hand side is estimated by Cauchy-Schwarz as

$$\int \mathfrak{u}^{2k} (\partial_y^k f) \partial_y^k u^\varepsilon \leq \frac{1}{2} \|\mathfrak{u}^{k+1} \partial_y^k f\|_{L^2}^2 + \frac{1}{2} \|\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon\|_{L^2}^2. \quad (6.65)$$

We move on to the left-hand side. Integrations by parts in  $y$  in the first term yield

$$- \int \mathfrak{u}^{2k} \partial_y^{k+2} u^\varepsilon \partial_y^k u^\varepsilon = \int \mathfrak{u}^{2k} (\partial_y^{k+1} u^\varepsilon)^2 - k(2k-1) \int \mathfrak{u}^{2k-2} (\partial_y^k u^\varepsilon)^2, \quad (6.66)$$

where the boundary terms in  $y = 0$  are null because  $\mathfrak{u}$  vanishes on  $y = 0$ . Concerning the boundary terms in  $y = H$ , by induction we get estimates on  $\partial_y^k u^\varepsilon|_{y=H}$ . Up to a lifting of these boundary condition before performing the estimates above, with a lifting compactly supported in the zone where  $\mathfrak{u} > 0$ , we can always assume that these boundary terms are zero, without affecting the rest of the estimates.

Integrations by parts in  $x$  in the third term and the homogeneous boundary conditions at  $x = x_0$  and  $x = x_1$  for  $u^\varepsilon$  yield

$$\begin{aligned} -\varepsilon \int \mathfrak{u}^{2k} (\partial_{xx} \partial_y^k u^\varepsilon) \partial_y^k u^\varepsilon &= \varepsilon \int \mathfrak{u}^{2k} (\partial_x \partial_y^k u^\varepsilon)^2 + 2k\varepsilon \int \mathfrak{u}^{2k-1} \mathfrak{u}_x (\partial_x \partial_y^k u^\varepsilon) (\partial_y^k u^\varepsilon) \\ &\geq \varepsilon \int \mathfrak{u}^{2k} (\partial_x \partial_y^k u^\varepsilon)^2 - k\varepsilon \|\mathfrak{u} \mathfrak{u}_x\|_{L^\infty} \left( c \int \mathfrak{u}^{2k} (\partial_x \partial_y^k u^\varepsilon)^2 + c^{-1} \int \mathfrak{u}^{2k-2} (\partial_y^k u^\varepsilon)^2 \right) \\ &\geq \frac{2\varepsilon}{3} \int \mathfrak{u}^{2k} (\partial_x \partial_y^k u^\varepsilon)^2 - 3\varepsilon k^2 \|\mathfrak{u} \mathfrak{u}_x\|_{L^\infty}^2 \int \mathfrak{u}^{2k-2} (\partial_y^k u^\varepsilon)^2. \end{aligned} \quad (6.67)$$

The second term is the trickiest. When all  $\partial_y$  derivatives hit the  $\partial_x u^\varepsilon$  term of the product,

a single integration by parts in  $x$  and the homogeneous boundary conditions at  $x = x_0, x_1$  yield

$$\begin{aligned} \int \mathfrak{u}^{2k+1} (\partial_x \partial_y^k u^\varepsilon) \partial_y^k u^\varepsilon &= -\frac{2k+1}{2} \int \mathfrak{u}^{2k} \mathfrak{u}_x (\partial_y^k u^\varepsilon)^2 \\ &\geq -\frac{2k+1}{2} \|\mathfrak{u}^2 \mathfrak{u}_x\|_{L^\infty} \int \mathfrak{u}^{2k-2} (\partial_y^k u^\varepsilon)^2. \end{aligned} \quad (6.68)$$

We compute the remaining terms using the Leibniz formula and using the commutator lemma for  $\chi = \mathfrak{u}$  and  $\phi = \partial_x u^\varepsilon$ .

$$\begin{aligned} I_2 &= \sum_{j=0}^{k-1} \binom{k}{j} \int (\partial_y^{k-j} \mathfrak{u}) (\partial_x \partial_y^j u^\varepsilon) (\partial_y^k u^\varepsilon) \mathfrak{u}^{2k} \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \int (\mathfrak{u}^{k-j} \partial_y^{k-j} \mathfrak{u}) (\mathfrak{u}^{j+1} \partial_x \partial_y^j u^\varepsilon) (\partial_y^k u^\varepsilon) \mathfrak{u}^{k-1} \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \int (\mathfrak{u}^{k-j} \partial_y^{k-j} \mathfrak{u}) \left( \sum_{l=0}^j \alpha_{j,l} [\mathfrak{u}] \mathfrak{u}^l \partial_y^l (\mathfrak{u} \partial_x u^\varepsilon) \right) (\partial_y^k u^\varepsilon) \mathfrak{u}^{k-1}. \end{aligned} \quad (6.69)$$

Plugging (6.53) to compute  $\partial_y^l (\mathfrak{u} \partial_x u^\varepsilon)$  yields

$$I_2 = \sum_{j=0}^{k-1} \sum_{l=0}^j \binom{k}{j} \int (\mathfrak{u}^{k-j-1} \partial_y^{k-j} \mathfrak{u}) \alpha_{j,l} [\mathfrak{u}] \mathfrak{u}^{l+1} (\partial_y^l f + \partial_y^{l+2} u^\varepsilon + \varepsilon \partial_{xx} \partial_y^l u^\varepsilon) (\partial_y^k u^\varepsilon) \mathfrak{u}^{k-1}. \quad (6.70)$$

With obvious notations, we split  $I_2$  in three parts:  $I_{2,f}$ ,  $I_{2,\partial_{yy}}$  and  $I_{2,\partial_{xx}}$ . One obtains

$$|I_{2,f}| \leq C_k \left( \sup_{0 \leq i \leq k} \|\mathfrak{u}^{i-1} \partial_y^i \mathfrak{u}\|_{L^\infty} \right)^k \left( \|\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon\|_{L^2}^2 + \sum_{i=0}^{k-1} \|\mathfrak{u}^{i+1} \partial_y^i f\|_{L^2}^2 \right). \quad (6.71)$$

In  $I_{2,\partial_{yy}}$ , we bound carefully the term corresponding to  $j = l = k - 1$ , and we bound roughly the others. This yields

$$\begin{aligned} |I_{2,\partial_{yy}}| &\leq \frac{1}{2} \|\mathfrak{u}^k \partial_y^{k+1} u^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|\mathfrak{u}_y\|_{L^\infty}^2 \|\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon\|_{L^2}^2 \\ &\quad + C_k \left( \sup_{0 \leq i \leq k} \|\mathfrak{u}^{i-1} \partial_y^i \mathfrak{u}\|_{L^\infty} \right)^k \left( \|u^\varepsilon\|_{L^2}^2 + \sum_{i=1}^{k-1} \|\mathfrak{u}^{i-1} \partial_y^i u^\varepsilon\|_{L^2}^2 \right). \end{aligned} \quad (6.72)$$

The  $I_{2,\partial_{xx}}$  term requires an integration by parts in  $x$ . The boundary terms in  $x = x_0$  and  $x = x_1$  vanish thanks to the homogeneous boundary conditions satisfied by  $u^\varepsilon$  and its  $y$  derivatives. For

$0 \leq l \leq j \leq k-1$ , one has

$$\begin{aligned}
& \varepsilon \int (\mathfrak{u}^{k-j-1} \partial_y^{k-j} \mathfrak{u}) \alpha_{j,l}[\mathfrak{u}] \mathfrak{u}^{l+1} (\partial_{xx} \partial_y^l u^\varepsilon) (\partial_y^k u^\varepsilon) \mathfrak{u}^{k-1} \\
&= -\varepsilon \int (\mathfrak{u}^{k-j-1} \partial_y^{k-j} \mathfrak{u}) \alpha_{j,l}[\mathfrak{u}] (\mathfrak{u}^l \partial_x \partial_y^l u^\varepsilon) (\mathfrak{u}^k \partial_x \partial_y^k u^\varepsilon) \\
&\quad - \varepsilon(l+k) \int (\mathfrak{u}^{k-j-1} \partial_y^{k-j} \mathfrak{u}) \alpha_{j,l}[\mathfrak{u}] \mathfrak{u}_x (\mathfrak{u}^l \partial_x \partial_y^l u^\varepsilon) (\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon) \\
&\quad - \varepsilon \int \mathfrak{u} \partial_x (\alpha_{j,l}[\mathfrak{u}] \mathfrak{u}^{k-j-1} \partial_y^{k-j} \mathfrak{u}) (\mathfrak{u}^l \partial_x \partial_y^l u^\varepsilon) (\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon).
\end{aligned} \tag{6.73}$$

We conclude that there exists  $C_k > 0$  such that

$$|I_{2, \partial_{xx}}| \leq \frac{\varepsilon}{3} \|\mathfrak{u}^k \partial_x \partial_y^k u^\varepsilon\|_{L^2}^2 + \varepsilon C_k A^k \left( \|\mathfrak{u}^{k-1} \partial_y^k u^\varepsilon\|_{L^2}^2 + \sum_{i=0}^{k-1} \|\mathfrak{u}^i \partial_x \partial_y^i u^\varepsilon\|_{L^2}^2 \right), \tag{6.74}$$

where we introduce

$$A := \|\mathfrak{u}_x\|_{L^\infty} + \sup_{0 \leq i \leq k} \|\mathfrak{u}^{i-1} \partial_y^i \mathfrak{u}\|_{L^\infty} + \|\mathfrak{u} \partial_x (\mathfrak{u}^{i-1} \partial_y^i \mathfrak{u})\|_{L^\infty}. \tag{6.75}$$

Plugging the estimates (6.65), (6.66), (6.67), (6.68), (6.71), (6.72), (6.74) in the energy equality (6.64) yields by induction on  $k \in \mathbb{N}^*$ ,

$$\|\mathfrak{u}^k \partial_y^{k+1} u^\varepsilon\|_{L^2}^2 + \varepsilon \|\mathfrak{u}^k \partial_x \partial_y^k u^\varepsilon\|_{L^2}^2 \leq C_k(A) \left( \|f\|_{L^2}^2 + \sum_{i=0}^k \|\mathfrak{u}^{i+1} \partial_y^i f\|_{L^2}^2 \right), \tag{6.76}$$

for some constant  $C_k(A)$  independent on  $f, \varepsilon, u^\varepsilon$ , but depending on  $\mathfrak{u}$  through  $A$ .

### 6.5.3 General strategy

In order to derive uniform  $X^1$  estimates on the sequence  $(u^\varepsilon)_{\varepsilon>0}$ , a natural idea would be to try and derive bounds on  $\partial_x u^\varepsilon$ . However this strategy, if used without precaution, is bound to fail. Indeed, as  $\varepsilon$  vanishes, we expect boundary layers to form in the vicinity of  $\{x = x_i\} \setminus \Sigma_i$ . More precisely, let us look for instance at the behavior of  $u^\varepsilon$  far from the line of cancellation of  $\mathfrak{u}$ , in order to simplify the discussion. In such regions we have

$$-\varepsilon \partial_{xx} u^\varepsilon + \mathfrak{u} \partial_x u^\varepsilon = g_\varepsilon,$$

and  $g_\varepsilon$  is locally bounded in  $L^2$ , according to the preceding paragraph. We then classically observe that boundary layers occur close to  $x = x_0$  in the region where  $\mathfrak{u} < 0$ , and close to  $x = x_1$  in the region where  $\mathfrak{u} > 0$ . More precisely, close to  $x = x_0$ , we expect that

$$u^\varepsilon \simeq \int_{x_0}^x \frac{f}{\mathfrak{u}}$$

in the region where  $\mathfrak{u} > 0$ , and

$$u^\varepsilon \simeq \int^x \frac{f}{\mathfrak{u}} + B(y) \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right)$$

in the region where  $u < 0$ . This heuristics leads to

$$\partial_x u^\varepsilon \sim \frac{u}{\varepsilon} B(y) \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x u(x', y) dx'\right)$$

close to  $\{x = x_0, u < 0\}$ . Hence we do not expect  $\partial_x u^\varepsilon$  to be bounded in  $L^2$  uniformly in  $\varepsilon$ .

However, if we subtract this boundary layer part from  $\partial_x u^\varepsilon$ , we can hope that the resulting function is bounded in  $L^2$ . This is the core of our strategy to prove Conjecture 6.2.6. More precisely, the goal is to construct a boundary layer corrector  $W^\varepsilon$  such that  $\partial_x u^\varepsilon - W^\varepsilon$  vanishes on  $\{x = x_i\} \setminus \Sigma_i$  (except possibly in a neighborhood of size  $\varepsilon^\alpha$  of the endpoints of  $\Sigma_i$ ), and such that

$$-\varepsilon \partial_{xx} W^\varepsilon - \partial_{yy} W^\varepsilon + u \partial_x W^\varepsilon + u_x W^\varepsilon = O(1) \text{ in } L^2(\Omega).$$

Indeed, if we are able to construct such a corrector, then  $v^\varepsilon := \partial_x u^\varepsilon - W^\varepsilon$  satisfies

$$\begin{aligned} -\varepsilon \partial_{xx} v^\varepsilon - \partial_{yy} v^\varepsilon + u \partial_x v^\varepsilon + u_x v^\varepsilon &= O(1), \\ v_{y=0}^\varepsilon = v_{y=H}^\varepsilon &= 0, \\ v_{\{x=x_i\} \setminus \Sigma_i}^\varepsilon &= 0. \end{aligned}$$

In the system above, we have assumed that the corrector lifts the trace of  $\partial_x v^\varepsilon$  on the whole domain  $\{x = x_i\} \setminus \Sigma_i$ , which will not quite be the case.

A classical energy estimate would then prove that

$$\varepsilon \int (\partial_x v^\varepsilon)^2 + \int (\partial_y v^\varepsilon)^2 \leq C.$$

Since the boundary layer part is exponentially small outside a small set whose size depends on  $\varepsilon$ , we infer that  $\partial_x \partial_y u^\varepsilon$  is bounded in  $L^2((x_0 + \delta, x_1 - \delta) \times (0, H))$  for all  $\delta > 0$  (with a bound independent of  $\delta$  provided  $\varepsilon$  is small enough). Hence  $u^\varepsilon \rightharpoonup u$  in  $X^1$ , and  $u$  is a solution of (6.17).

This strategy is not quite complete in the present paper. Indeed, because of the degeneracy of the vertical regularity estimates, we have a bad control of the traces of  $\partial_x u^\varepsilon$  as we approach the points where  $u(x_i, \cdot)$  vanishes. Consequently, we are only able to lift the trace of  $\partial_x u^\varepsilon$  outside a neighborhood of size  $\varepsilon^\alpha$  of such points. This prevents us from using directly the energy estimate above, which must be modified. An option is to construct a suitable weight  $\rho^\varepsilon$ , as we will explain in paragraph 6.5.6.

#### 6.5.4 Estimates on the traces of $\partial_x u^\varepsilon$

Let us now go back to equation (6.53). We define  $V^\varepsilon(X, y) := \varepsilon \partial_x u^\varepsilon(x_0 + \varepsilon X, y)$ , for  $X \in (0, (x_1 - x_0)\varepsilon^{-1})$ ,  $y \in (0, H)$ . According to (6.54), we have

$$\|V^\varepsilon\|_{L^2((0, (x_1 - x_0)\varepsilon^{-1}) \times (0, H))} \leq C_0 \|f\|_{L^2(\Omega)}.$$

The estimates of the previous paragraph also entail that

$$\|u(x_0 + \varepsilon X, y)^k \partial_y^k V^\varepsilon\|_{L^2((0, (x_1 - x_0)\varepsilon^{-1}) \times (0, H))} \leq C.$$

Furthermore, note that

$$-\varepsilon \partial_x \left[ \partial_x u^\varepsilon \exp\left(-\frac{1}{\varepsilon} \int_0^x u(x', y) dx'\right) \right] = (f + \partial_{yy} u^\varepsilon - b u^\varepsilon) \exp\left(-\frac{1}{\varepsilon} \int_0^x u(x', y) dx'\right).$$



Therefore

$$\partial_X \left[ V^\varepsilon \exp \left( - \int_0^X \mathfrak{u}(x_0 + \varepsilon X', y) dX' \right) \right] = G^\varepsilon(X', y),$$

where the function  $G^\varepsilon$  is such that

$$\|\mathfrak{u}^{k+1}(x_0 + \varepsilon X, y) \partial_y^k G^\varepsilon\|_{L^2((0,1) \times (0,H))} \leq C_k,$$

and the constant  $C_k$  does not depend on  $\varepsilon$ . Standard trace estimates immediately entail that

$$\|\mathfrak{u}(x_0, y)^{k+1} \partial_y^k V|_{X=0}\|_{L^2(0,H)} \leq C_k$$

Going back to the original variables, we infer that

$$\|\mathfrak{u}(x_0, y)^{k+1} \partial_y^k \partial_x u|_{x=x_0}^\varepsilon\|_{L^2(0,H)} \leq \frac{C_k}{\varepsilon}. \quad (6.77)$$

*Remark 6.5.1.* This estimate is certainly sub-optimal on the sets  $\Sigma_i$ , where we expect  $\partial_x u^\varepsilon$  to be bounded. However, we will not need to lift the trace of  $\partial_x u^\varepsilon$  on  $\Sigma_i$ , and therefore we do not look for finer estimates.

### 6.5.5 Lifting of boundary layers

As in the previous paragraph, we focus on the boundary layers close to  $x = x_0$ , and leave the ones close to  $x = x_1$  to the reader.

Let us set

$$B^\varepsilon(y) := \varepsilon \partial_x u^\varepsilon(x_0, y).$$

The estimates of the previous paragraph ensure that for all  $k \in \mathbb{N}$ , there exists a constant  $C_k$  such that

$$\|\mathfrak{u}(x_0, y)^{k+1} \partial_y^k B^\varepsilon\|_{L^2(0,H)} \leq C_k. \quad (6.78)$$

We will lift this trace thanks to a boundary layer corrector. In order to have a small enough remainder in the equation, we will need to construct several orders of correctors. We first explain how the main order corrector is constructed, and then how the iteration works.

- Main order boundary layer corrector:

Let us define, for  $x$  close to  $x_0$ ,  $y < y_0$ ,

$$V_0^\varepsilon(x, y) := \chi_1 \left( \frac{x - x_0}{\varepsilon^\beta} \right) \chi_2 \left( \frac{y - y_0}{\varepsilon^\alpha} \right) \frac{B^\varepsilon(y)}{\varepsilon} \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right),$$

where  $\chi_1, \chi_2 \in \mathcal{C}^\infty(\mathbb{R})$ , with  $\chi_2(Y) = 0$  for  $Y \geq -1$ ,  $\chi_2(Y) = 1$  for  $Y \leq -2$ , and  $\chi_1(X) = 1$  for  $X \leq 1$ ,  $\chi_1(X) = 0$  for  $X \geq 2$ . The parameters  $\alpha, \beta > 0$  will be chosen later on. Let us merely mention that we require that  $\alpha < \beta < 1 - \alpha$ , so that  $\alpha < 1/2$ .

We then plug  $V_0^\varepsilon$  into the equation satisfied by  $v^\varepsilon := \partial_x u^\varepsilon$ . We have

$$\begin{aligned}
R_0^\varepsilon &:= -\varepsilon \partial_{xx} V_0^\varepsilon - \partial_{yy} V_0^\varepsilon + \mathfrak{u} \partial_x V_0^\varepsilon + \mathfrak{u}_x V_0^\varepsilon \\
&= -\frac{1}{\varepsilon^{1+\beta}} \chi_1' \left( \frac{x-x_0}{\varepsilon^\beta} \right) \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon(y) \mathfrak{u}(x, y) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
&\quad - \frac{1}{\varepsilon^{2\beta}} \chi_1'' \left( \frac{x-x_0}{\varepsilon^\beta} \right) \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon(y) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
&\quad - \frac{1}{\varepsilon} \chi_1 \left( \frac{x-x_0}{\varepsilon^\beta} \right) \partial_{yy} \left[ \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon \right] \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
&\quad - \frac{2}{\varepsilon^2} \chi_1 \left( \frac{x-x_0}{\varepsilon^\beta} \right) \partial_y \left[ \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon \right] \left( \int_{x_0}^x \mathfrak{u}_y(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
&\quad - \frac{1}{\varepsilon^2} \chi_1 \left( \frac{x-x_0}{\varepsilon^\beta} \right) \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon \left( \int_{x_0}^x \mathfrak{u}_{yy}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
&\quad - \frac{1}{\varepsilon^3} \chi_1 \left( \frac{x-x_0}{\varepsilon^\beta} \right) \chi_2 \left( \frac{y-y_0}{\varepsilon^\alpha} \right) B^\varepsilon \left( \int_{x_0}^x \mathfrak{u}_y(x', y) dx' \right)^2 \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right).
\end{aligned}$$

Let us now assume that  $\mathfrak{u}_y(x_0, y_0) > 0$ . In a neighborhood of  $(x_0, y_0)$ , we have, since  $\mathfrak{u}$  is smooth,

$$\mathfrak{u}(x, y) = \mathfrak{u}_y(y - y_0) + \mathfrak{u}_x(x_0, y_0)(x - x_0) + O(|x - x_0|^2 + |y - y_0|^2).$$

If  $(x, y)$  belongs to the support of  $\chi_1((x - x_0)/\varepsilon^\beta) \chi_2((y - y_0)/\varepsilon^\alpha)$ , then

$$|x - x_0| \leq 2\varepsilon^\beta \ll \varepsilon^\alpha \leq |y - y_0|.$$

Consequently, there exists a positive constant  $C$  such that for  $(x, y)$  as above,

$$\mathfrak{u}(x, y) \leq C(y - y_0).$$

We infer then that

$$\int_{x_0}^x \mathfrak{u}(x', y) dx' \leq C(y - y_0)(x - x_0). \quad (6.79)$$

On the other hand, since  $\mathfrak{u}$  is smooth,

$$\left| \int_{x_0}^x \partial_y^k \mathfrak{u}(x', y) dy \right| \leq C|x - x_0|.$$

In order to estimate the error associated with  $V_0^\varepsilon$  and higher order correctors, we will rely on the following Lemma:

**Lemma 6.5.2.** *Let  $\mathfrak{u} \in \mathcal{C}^2(\Omega)$  such that  $\mathfrak{u}_y(x_0, y_0) > 0$ .*

- For all  $k \in \mathbb{N}$ , if  $|y - y_0| \geq \varepsilon^\alpha$ ,

$$\left\| (x - x_0)^k \chi_1 \left( \frac{x - x_0}{\varepsilon^\beta} \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \right\|_{L_x^2((x_0, x_1))} \leq C \left( \frac{\varepsilon}{|y - y_0|} \right)^{k + \frac{1}{2}}.$$

- For all  $m \in \mathbb{N}$ , for all  $l \geq 0$ ,

$$\left\| |y - y_0|^{-l} \partial_y^m \left( B^\varepsilon(y) \chi_2 \left( \frac{y - y_0}{\varepsilon^\alpha} \right) \right) \right\|_{L_y^2(0, H)} \leq C \varepsilon^{-(l+m+1)\alpha}.$$

- If  $x - x_0 \in [\varepsilon^\beta, 2\varepsilon^\beta]$ , and  $y - y_0 \leq -\varepsilon^\alpha$ ,

$$\exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right) \leq \exp(-C\varepsilon^{\beta+\alpha-1}).$$

We postpone the proof of Lemma 6.5.2 to the end of this subsection.

Using our choice of  $\alpha$  and  $\beta$ , the third property of Lemma 6.5.2 ensures that the terms involving derivatives of  $\chi_1$  are  $o(\varepsilon^N)$  for all  $N$ . Let us explain how to estimate one of the terms in  $R_0^\varepsilon$  (we shall leave the other ones to the reader, since they follow a similar pattern.) We have, using the first property,

$$\begin{aligned} \left\| \frac{1}{\varepsilon} \chi_1\left(\frac{x-x_0}{\varepsilon^\beta}\right) \chi_2\left(\frac{y-y_0}{\varepsilon^\alpha}\right) \partial_{yy} B^\varepsilon \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right) \right\|_{L_x^2((x_0, x_1))} \\ \lesssim \varepsilon^{-1/2} \chi_2\left(\frac{y-y_0}{\varepsilon^\alpha}\right) |y-y_0| L^{-1/2} |\partial_{yy} B^\varepsilon|. \end{aligned}$$

The second property then implies

$$\begin{aligned} \left\| \frac{1}{\varepsilon} \chi_1\left(\frac{x-x_0}{\varepsilon^\beta}\right) \chi_2\left(\frac{y-y_0}{\varepsilon^\alpha}\right) \partial_{yy} B^\varepsilon \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right) \right\|_{L^2(\Omega)} \\ \lesssim \varepsilon^{-\frac{1}{2} - \frac{7\alpha}{2}}. \end{aligned}$$

It follows that

$$\|R_0^\varepsilon\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2} - \frac{5\alpha}{2}}.$$

Therefore the sole construction of  $R_0^\varepsilon$  is not sufficient to have a remainder that is  $O(1)$  in  $L^2$ , and we must construct higher order correctors.

• In order to construct higher order correctors, the idea is the following: for every  $j \geq 1$ , we construct  $V_j^\varepsilon$  such that  $V_j^\varepsilon$  is an approximate solution of

$$\begin{aligned} -\varepsilon \partial_{xx} V_j^\varepsilon + \mathfrak{u} \partial_x V_j^\varepsilon + \mathfrak{u}_x V_j^\varepsilon &= -R_{j-1}^\varepsilon, \\ V_j^\varepsilon|_{x=x_0, y < y_0} &= 0. \end{aligned}$$

The error generated by  $\partial_{yy} V_j^\varepsilon$  will then go into the next error term, namely  $R_j^\varepsilon$ , since it is expected to have a less singular behavior than  $-\varepsilon \partial_{xx}$  within this scaling. We will use the following result:

**Lemma 6.5.3.** *Let  $\zeta_1, \zeta_2 \in C^\infty(\mathbb{R})$  with  $\text{Supp} \zeta_1 \subset [-2, 2]$  and  $\text{Supp} \zeta_2 \subset (-\infty, -1]$ . Let  $a \in C^\infty(\Omega)$ ,  $P \in \mathbb{R}[X]$ ,  $N > 0$ , and define*

$$P^\varepsilon := \varepsilon^{-N} a(x, y) \zeta_1\left(\frac{x-x_0}{\varepsilon^\alpha}\right) \zeta_2\left(\frac{y-y_0}{\varepsilon^\beta}\right) P\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right) \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right).$$

Then the following properties hold:

- $\|R^\varepsilon\|_{L^2} \lesssim \varepsilon^{-N + \frac{1-\alpha}{2}}$ ;
- *Setting*

$$Q^\varepsilon := \varepsilon^{1-N} \frac{a}{\mathfrak{u}^2} \zeta_1\left(\frac{x-x_0}{\varepsilon^\alpha}\right) \zeta_2\left(\frac{y-y_0}{\varepsilon^\beta}\right) Q\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right) \exp\left(\frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx'\right),$$

where  $Q$  is the polynomial such that  $Q'' + Q' = -P$  and  $Q(0) = 0$ , we have

$$-\varepsilon \partial_{xx} Q^\varepsilon - \partial_{yy} Q^\varepsilon + \mathfrak{u} \partial_x Q^\varepsilon + \mathfrak{u}_x Q^\varepsilon = P^\varepsilon + S^\varepsilon + o(1),$$

where the new remainder  $S^\varepsilon$  is a linear combination of terms of the form

$$\varepsilon^{1-N-4\alpha} \tilde{a}(x, y) \zeta_1 \left( \frac{x-x_0}{\varepsilon^\alpha} \right) \tilde{\zeta}_2 \left( \frac{y-y_0}{\varepsilon^\beta} \right) T \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right),$$

with  $\tilde{a} \in C^\infty(\Omega)$ ,  $\tilde{\zeta}_2 \in C^\infty(\mathbb{R})$  satisfying the same assumptions as  $\zeta_2$ , and  $T \in \mathbb{R}[X]$ .

- As a consequence,

$$\|S^\varepsilon\|_{L^2} \lesssim \varepsilon^{1-N+\frac{1-9\alpha}{2}}.$$

It follows that if  $\alpha < 1/4$ , we gain a factor  $\varepsilon^{1-4\alpha}$  with each iteration. More precisely, we apply at each step the above Lemma with  $P^\varepsilon = R_j^\varepsilon$ , and we set  $V_{j+1}^\varepsilon = Q^\varepsilon$ ,  $R_{j+1}^\varepsilon = S^\varepsilon$ . In particular, after  $k$  iterations, the remainder is of order  $\varepsilon^{k(1-4\alpha)-\frac{1}{2}-\frac{5\alpha}{2}}$ . Therefore, as soon as

$$k \geq \frac{1+5\alpha}{2(1-4\alpha)},$$

the remainder is  $O(1)$ . For instance, if  $\alpha < 1/13$ , we have an admissible remainder after just one iteration.

We now prove Lemmas 6.5.2 and 6.5.3.

*Proof of Lemma 6.5.2.* • We recall that there exists a positive constant  $C$  such that for  $x$  in the support of  $\chi_1((\cdot - x_0)/\varepsilon^\beta)$  and  $y$  such that  $y - y_0 \leq -\varepsilon^\alpha$ , we have

$$\int_{x_0}^x \mathfrak{u}(x', y) dx' \leq C(y - y_0)(x - x_0).$$

Therefore

$$\begin{aligned} & \left\| (x - x_0)^k \chi_1 \left( \frac{x - x_0}{\varepsilon^\beta} \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \right\|_{L^2_{\frac{x}{\varepsilon}}((x_0, x_1))}^2 \\ & \leq \int_{x_0}^{x_1} (x - x_0)^{2k} \exp \left( -\frac{C}{\varepsilon} |x - x_0| |y - y_0| \right) dx \\ & \leq C \left( \frac{\varepsilon}{|y - y_0|} \right)^{2k+1}. \end{aligned}$$

- We use estimate (6.78). We have

$$\begin{aligned}
& \left\| |y - y_0|^{-l} \partial_y^m \left( B^\varepsilon(y) \chi_2 \left( \frac{y - y_0}{\varepsilon^\alpha} \right) \right) \right\|_{L_y^2(0, H)} \\
& \leq \varepsilon^{-\alpha l} \sum_{k=0}^m \varepsilon^{-\alpha(m-k)} \left\| \partial_y^k B^\varepsilon \mathbf{1}_{|y-y_0| \geq \varepsilon^\alpha} \right\|_{L_y^2(0, H)} \\
& \leq \varepsilon^{-\alpha l} \sum_{k=0}^m \varepsilon^{-\alpha(m-k)} \varepsilon^{-\alpha(k+1)} \left\| \mathfrak{u}(x_0, y)^{k+1} \partial_y^k B^\varepsilon \right\|_{L_y^2(0, H)} \\
& \leq \varepsilon^{-\alpha(l+m+1)}.
\end{aligned}$$

- Let  $x \in [x_0 + \varepsilon^\beta, x_0 + 2\varepsilon^\beta]$ ,  $y \leq y_0 - \varepsilon^\alpha$ . Then, using (6.79), we infer that

$$\begin{aligned}
\exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) & \leq \exp \left( \frac{C}{\varepsilon} |x - x_0| |y - y_0| \right) \\
& \leq \exp(-C\varepsilon^{-1+\beta+\alpha}).
\end{aligned}$$

This concludes the proof of the estimates. □

*Proof of Lemma 6.5.3.* The estimate on  $P^\varepsilon$  follows easily from a variant of Lemma 6.5.2. We focus on the second property of the Lemma. We have

$$\begin{aligned}
& (-\varepsilon \partial_{xx} + \mathfrak{u} \partial_x) \left[ Q \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \right] \\
& = -\frac{\mathfrak{u}^2}{\varepsilon} (Q'' + Q') \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
& \quad + \mathfrak{u}_x (Q - Q') \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
& = \frac{\mathfrak{u}^2}{\varepsilon} P \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \\
& \quad + \mathfrak{u}_x (Q - Q') \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right).
\end{aligned}$$

As a consequence, using Lemma 6.5.2, we have

$$\begin{aligned}
& (-\varepsilon \partial_{xx} + \mathfrak{u} \partial_x) Q^\varepsilon \\
& = P^\varepsilon \\
& \quad + \varepsilon^{1-N} (-\varepsilon \partial_{xx} + \mathfrak{u} \partial_x) \left( \frac{a}{\mathfrak{u}^2} \right) \zeta_1 \left( \frac{x - x_0}{\varepsilon^\alpha} \right) \zeta_2 \left( \frac{y - y_0}{\varepsilon^\beta} \right) Q \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u} \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u} \right) \\
& \quad - 2\varepsilon^{2-N} \partial_x \left( \frac{a}{\mathfrak{u}^2} \right) \zeta_1 \left( \frac{x - x_0}{\varepsilon^\alpha} \right) \zeta_2 \left( \frac{y - y_0}{\varepsilon^\beta} \right) \partial_x \left[ Q \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u} \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u} \right) \right] \\
& \quad + o(1).
\end{aligned}$$

We observe that each term of the right-hand side (except for  $P^\varepsilon$ ) can be written as

$$\varepsilon^{k-N-4\alpha} \tilde{a}(x, y) \tilde{\zeta}_2 \left( \frac{y - y_0}{\varepsilon^\beta} \right) T \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right) \exp \left( \frac{1}{\varepsilon} \int_{x_0}^x \mathfrak{u}(x', y) dx' \right)$$

with  $k \geq 1$ ,  $\tilde{a}$  a smooth function,  $\tilde{\zeta}_2 \in \mathcal{C}^\infty(\mathbb{R})$  satisfying the same assumptions as  $\zeta_2$ , and  $T \in \mathbb{R}[X]$ .

The same is true for  $\partial_{yy} Q^\varepsilon$  and for  $\mathfrak{u}_x Q^\varepsilon$ . The estimate on the new remainder  $S^\varepsilon$  is a consequence of Lemma 6.5.2.  $\square$

### 6.5.6 Design of a weight function

Following the steps outlined in the previous paragraph, we construct a corrector  $W^\varepsilon$  with the following properties:

- $-\varepsilon \partial_{xx} W^\varepsilon + \mathfrak{u} \partial_x W^\varepsilon + \mathfrak{u}_x W^\varepsilon - \partial_{yy} W^\varepsilon = O(1)$  in  $L^2(\Omega)$ ;
- $W^\varepsilon$  is supported in  $|x - x_0| \leq \varepsilon^\beta$ ,  $|y - y_0| \geq \varepsilon^\alpha$ ;
- $W^\varepsilon(x_0, y) = -\partial_x u^\varepsilon(x_0, y)$  if  $y \leq y_0 - 2\varepsilon^\alpha$ .

Of course the same scheme can be adapted to the other degeneracy points of  $\mathfrak{u}$  on the boundary, namely near  $(x_1, y_1)$  and  $(x_0, 0)$ . As a consequence, setting  $v^\varepsilon := \partial_x u^\varepsilon - W^\varepsilon$ , we have

$$-\varepsilon \partial_{xx} v^\varepsilon + \mathfrak{u} \partial_x v^\varepsilon + \mathfrak{u}_x v^\varepsilon - \partial_{yy} v^\varepsilon = f_x + O(1) =: g$$

and  $v^\varepsilon|_{y=0} = v^\varepsilon|_{y=H} = 0$ .

Let  $\rho^\varepsilon$  be a smooth, non-negative weight function. Multiplying the above equation by  $v^\varepsilon \rho^\varepsilon$  and integrating by parts, we have

$$\begin{aligned} & \varepsilon \int_{\Omega} (\partial_x v^\varepsilon)^2 \rho^\varepsilon + \int_{\Omega} (\partial_y v^\varepsilon)^2 \rho^\varepsilon + \frac{1}{2} \int_{\Omega} \mathfrak{u}_x (v^\varepsilon)^2 \rho^\varepsilon \\ & - \frac{1}{2} \int_{\Omega} (v^\varepsilon)^2 [\varepsilon \partial_{xx} \rho^\varepsilon + \partial_{yy} \rho^\varepsilon + \mathfrak{u} \partial_x \rho^\varepsilon] \\ & - \varepsilon \left[ \int_0^H \partial_x v^\varepsilon v^\varepsilon \rho^\varepsilon \right]_{x=x_0}^{x=x_1} + \frac{1}{2} \left[ \int_0^H \mathfrak{u} (v^\varepsilon)^2 \rho^\varepsilon \right]_{x=x_0}^{x=x_1} + \frac{\varepsilon}{2} \left[ \int_0^H (v^\varepsilon)^2 \partial_x \rho^\varepsilon \right]_{x=x_0}^{x=x_1} \\ & \leq C \|v^\varepsilon\|_{L^2} \end{aligned}$$

Here are some requirements that we wish to make on the weight function  $\rho^\varepsilon$ :

- $\rho^\varepsilon \equiv 1$  except in small neighborhoods (whose size should vanish with  $\varepsilon$ ) of the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_0, 0)$ ;
- In small neighborhoods of the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_0, 0)$ , we require that

$$\varepsilon \partial_{xx} \rho^\varepsilon + \partial_{yy} \rho^\varepsilon + \mathfrak{u} \partial_x \rho^\varepsilon \leq 0;$$

- Let us now look at the boundary terms. We distinguish between  $\Sigma_i$  and  $\{x = x_i\} \setminus \Sigma_i$ .
  - Close to the sets  $\Sigma_0$ ,  $\Sigma_1$ , we have  $W^\varepsilon = 0$ , and therefore  $v^\varepsilon = \partial_x u^\varepsilon$ . As a consequence, on these sets, using the equation satisfied by  $u^\varepsilon$ , we have

$$-\varepsilon \partial_{xx} v^\varepsilon = f(x_i) + \delta_i'' - \mathfrak{u} v^\varepsilon.$$

Let us look for instance at the boundary term on  $\Sigma_1$ . We have, assuming that  $\rho^\varepsilon \equiv 1$  in a vicinity of  $\Sigma_1$ ,

$$\begin{aligned} & -\varepsilon \int_{\Sigma_1} \partial_x v^\varepsilon v^\varepsilon + \frac{1}{2} \int_{\Sigma_1} \mathfrak{u}(x_1)(v^\varepsilon)^2 \\ &= \int_{\Sigma_1} (f(x_i) + \delta_i'') v^\varepsilon - \frac{1}{2} \int_{\Sigma_1} \mathfrak{u}(x_1)(v^\varepsilon)^2. \end{aligned}$$

Using the compatibility condition (6.29) together with the definition of  $\Sigma_1$ , we infer that there exists a constant  $C$ , independent of  $\varepsilon$ , such that

$$-\varepsilon \int_{\Sigma_1} \partial_x v^\varepsilon v^\varepsilon + \frac{1}{2} \int_{\Sigma_1} \mathfrak{u}(x_1)(v^\varepsilon)^2 \geq -C.$$

- We now address the sets  $\{x = x_i\} \setminus \Sigma_i$ . On these sets, if  $|y - y_i| \geq 2\varepsilon^\alpha$ , by construction  $v^\varepsilon$  vanishes, and therefore we have no constraint on  $\rho^\varepsilon$ . Therefore we only need to look at the trace of these terms close to the cancellation points of  $\mathfrak{u}$ . If we are able to construct a weight function  $\rho^\varepsilon$  with the properties stated above and such that the boundary terms in a small vicinity of  $(x_i, y_i)$  and  $(x_0, 0)$  are non-negative, then we obtain a uniform  $X^1$  bound on the sequence  $u^\varepsilon$  and Conjecture 6.2.6 is proved.

Therefore the proof of Conjecture 6.2.6 relies on the construction of a suitable weight function.

## 6.6 Appendix: Technical lemmas

**Lemma 6.6.1.** *There exists  $c > 0$  such that, if  $\delta_0, \delta_1 \in C^1([0, H])$  satisfy  $\delta_i(0) = \delta_i(H) = 0$  and  $\|\delta_i\|_{C^1} \leq c$ , then there exists  $y_0, y_1 \in (0, H)$  such that  $\Sigma_0 = \{x_0\} \times (y_0, H]$  and  $\Sigma_1 = \{x_1\} \times (0, y_1)$ .*

*Proof.* From the definition of  $\bar{u}$  in (6.2) and of  $\Sigma_i$  in (6.8), for  $\delta_i = 0$ , one has  $\Sigma_0(\bar{u}) = \{x_0\} \times (\bar{y}, H]$  and  $\Sigma_1(\bar{u}) = \{x_1\} \times (0, \bar{y})$ . Let  $\eta > 0$  to be chosen below.

- For  $\eta$  small enough, there exists  $c_\eta > 0$  such that  $\bar{u}(y) \geq c_\eta y$  with for  $y \in [0, \eta]$ . Hence  $(\bar{u} + \delta_i)(y) > 0$  for  $y \in (0, \eta]$  when  $\delta_i(0) = 0$  and  $\delta_i$  is small enough in  $C^1$ .
- For  $\eta$  small enough, there exists  $c_\eta > 0$  such that  $\bar{u}'(y) \geq c_\eta$  for  $|y - \bar{y}| \leq \eta$ . Hence, if  $\delta_i$  is small enough in  $C^1$ ,  $\bar{u} + \delta_i$  is strictly increasing on  $[\bar{y} - \eta, \bar{y} + \eta]$  and there exists  $y_i$  such that  $\bar{u}(y_i) + \delta_i(y_i) = 0$  and  $(\bar{u}(y) + \delta_i(y))(y - y_i) > 0$  for  $y \in [\bar{y} - \eta, \bar{y} + \eta]$  with  $y \neq y_i$ .
- For any  $\eta > 0$ , there exists  $c_\eta > 0$  such that  $|\bar{u}| \geq c_\eta$  on  $[\eta, \bar{y} - \beta] \cup [\bar{y} + \eta, H]$ . Hence, for  $\delta_i$  small enough in  $C^0$ ,  $\bar{u} + \delta_i$  has the same sign as  $\bar{u}$  on  $[\eta, \bar{y} - \beta] \cup [\bar{y} + \eta, H]$ .

Combining these regions concludes the proof of Lemma 6.6.1. □

# Appendix **A**

## Some basic methods and tools

### Outline of the current chapter

<b>A.1 How to solve a linear ODE ?</b>	<b>187</b>
<b>A.2 Duhamel formula</b>	<b>188</b>
<b>A.3 Laplace transform</b>	<b>189</b>
<b>A.4 Weighted Sobolev spaces</b>	<b>189</b>
<b>A.5 Scalings, cut-offs and scaling</b>	<b>190</b>

The goal of this appendix is to show on the simplest example possible some tools used in the rest of the thesis. As a consequence, we will study specific examples rather than prove generic results.

### A.1 How to solve a linear ODE ?

When working with non-degenerate boundary layers, a sometimes easy but important step is to compute the dimension of the kernel of each mode, i.e to compute the number of degrees of freedom we have in order to lift boundary conditions. Therefore, in simple cases, we search the number of initial conditions we can set on an ODE to have a solution with decay at infinity.

In this section, we will show very quickly from three different points of view how to solve what is maybe the simplest ODE

$$\begin{aligned} \partial_z^2 w - \alpha^2 w &= f \\ w(0) = w_0 \quad w'(0) &= w_1. \end{aligned} \tag{A.1}$$

Our goal is to find conditions on  $w_0$  and  $w_1$  such that the solution  $w$  is decaying at infinity. For  $f = 0$  we see that there are two modes as  $w = \frac{1}{2\alpha} ((\alpha w_0 + w_1)e^{\alpha t} + (\alpha w_0 - w_1)e^{-\alpha t})$ , so if  $w_0, w_1$  are chosen arbitrarily, there is no decaying solution. More precisely, we already see the need to consider a linear subspace of initial condition.

**Proposition A.1.1.** *Let  $\alpha > 0$  and  $\beta \in (-\alpha, \alpha)$ . Let  $g \in C_0^\infty(\mathbb{R}^+)$  and  $f(z) = g(z)e^{\beta z}$ . Let  $w_0, w_1 \in \mathbb{R}$ .*



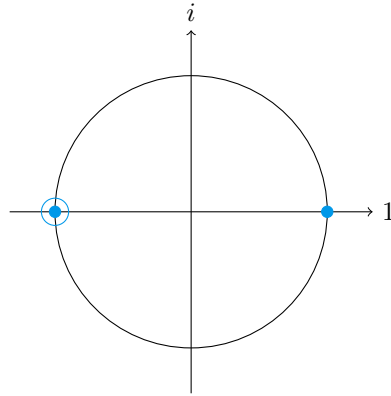


Figure A.1 – The modes of Equation (A.1).

Then there exists a (unique) solution of (A.1) such that  $w(z)e^{\gamma z}$  and all its derivatives are bounded, for all  $\gamma < \beta$  if and only if

$$\alpha w_0 + w_1 = - \int_0^\infty f(\zeta) e^{-\alpha \zeta} d\zeta. \quad (\text{A.2})$$

*Remark A.1.1.* In fact we can state almost as easily the same result for an ODE of any order, the key idea being that we have one compatibility condition for each root of the characteristic polynomial with real part greater than  $\beta$ .

## A.2 Duhamel formula

The first method is simply to write down the solution of the equation using the method of variation of parameters.

With  $W = \begin{pmatrix} w \\ w' \end{pmatrix}$  we obtain

$$W'(z) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_M W(z) + \underbrace{\begin{pmatrix} 0 \\ f(z) \end{pmatrix}}_{B(z)}$$

$$W(0) = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

so

$$W(z) = e^{Mz} \left( W_0 + \int_0^z e^{-M\zeta} B(\zeta) d\zeta \right).$$

We then decompose  $\mathbb{C}^2$  in the two eigenspaces of  $M$ ,  $\mathbb{C}^2 = E_{-1} \oplus E_1$ , the condition coming from it is indeed (A.2).

If we establish the pro and cons of this method, then the main problem is the lack its robustness.

+ Simple and explicit for ODE with constant coefficients.

- Difficult to adapt in other situations.

For an arbitrary ODE  $P(\partial_z)w = f$ , since  $M$  is the companion matrix of  $P$  we obtain the announced result.

### A.3 Laplace transform

A perhaps more suited method is the use of the Laplace transform (see for example [51]).

$$p^2 \tilde{w}(p) - pw_0 - w_1 - \alpha^2 \tilde{w}(p) = \tilde{f}(p)$$

so

$$\tilde{w}(p) = \frac{\tilde{f}(p) + pw_0 + w_1}{p^2 - \alpha^2}.$$

So  $\tilde{w}$  is a meromorphic function with two poles of at  $-\alpha$  and  $\alpha$ , both of order 1. And for all  $\mu > 1$ ,  $\tilde{p}$  is holomorphic on  $\Re(p) > \mu$  and with the necessary estimates on the lines  $\Re(p) = \sigma > p$ . In order to have the same result past the line  $\Re(p) = \mu = 1$  a necessary and sufficient condition is

$$(\tilde{f}(p) + pw_0 + w_1)(p = \alpha) = 0$$

i.e the condition (A.2)

$$\int_0^\infty f(s)e^{-\alpha s} ds + \alpha w_0 + w_1 = 0.$$

Indeed, if this condition is fulfilled,  $\tilde{w}$  has no pole at  $p = \alpha$ .

For a generic ODE we obtain

$$\tilde{w}(p) = \frac{\tilde{f}(p) + Q(p)}{P(p)}$$

so we have a meromorphic function with  $d$  poles  $\mu_1, \dots, \mu_d$  (with multiplicity). And in order to be holomorphic for  $\Re(p) > \beta$  we must cancel out the  $r - d$  poles in the half-space  $\Re(p) \geq \beta$ .

This is no surprise that the pros and cons of this method are the same as the ones of Fourier transform.

- + Leads to the Sobolev norms.
- + Can be adapted for PDE with constant coefficients.
- Difficult to adapt for variable coefficients.

### A.4 Weighted Sobolev spaces

The previous method hints to the existence of natural spaces for these problems. In fact as in Subsection 3.5.3, for  $\lambda > 0$  and  $s \in \mathbb{N}$  let us define

$$\|f\|_{\mathcal{H}_\lambda^s(\mathbb{R}^+)}^2 = \int_0^\infty |\partial_\zeta^s f|^2 e^{2\lambda\zeta} d\zeta$$

and the associated space

$$\mathcal{H}_\lambda^s = \overline{C_c^\infty((0, \infty))}^{\mathcal{H}_\lambda^s}.$$

These methods retain all the advantages of elliptic methods.

- + Leads directly to the Sobolev norms.
- + Can be directly adapted for PDE.
- + Allow variable coefficients.
- Can be tricky for odd orders

## A.5 Scalings, cut-offs and scaling

In this thesis we often use the scaling of the boundary layer solutions to derive the size of their Sobolev norms with respect to the boundary layer size. The main idea comes from the following naive computation in a one dimensional setting. For  $s \in \mathbb{N}$

$$\int_0^\infty \left| \partial_z \left( f \left( \frac{z}{\varepsilon} \right) \right) \right|^2 dz = \varepsilon^{1-2s} \int_0^\infty |\partial_\zeta (f(\zeta))|^2 d\zeta.$$

When the boundary layer equation leads to a solution  $f(\sigma, \zeta)$ , the corresponding term in the asymptotic expansion is often  $f(\sigma, \frac{z}{\delta(\sigma)})\chi(\sigma, z)$ , where  $\delta(\sigma)$  is the boundary layer size and  $\chi$  a cut-off function (in  $z$ ).

Note that a cut-off of the same scaling as the function does not radically change its norms. If  $\chi \in C^\infty(\Omega)$  then for any  $s$  there exists a constant  $C_{s,\chi}$  such that

$$\|f\chi\|_{H^s(\Omega)} \leq C_{s,\chi} \|f\|_{H^s(\Omega)}$$

as all derivatives of  $\chi$  are in  $L^\infty$ .

The effect of such a cut-off is less obvious when multiple scales are involved. In particular, for the boundary layer variables we often choose unbounded domain. In this case the inhomogeneous and homogeneous Sobolev spaces are quite different. We stress that in the following lemmas the Sobolev spaces are *inhomogeneous* Sobolev spaces. In particular we have  $H^{s'} \hookrightarrow H^s$  for  $s' \geq s$  and more precisely  $L^2 \hookrightarrow H^s$  for  $s < 0$ .

Moreover we will only consider the case  $s \in \mathbb{Z}$  and the tangential variable  $\sigma \in \mathbb{T}$ . We can replace the torus by any compact set, and the case  $s \notin \mathbb{Z}$  follows directly from interpolation.

Let us first show that knowing the norms in the boundary layer variables and the size of said boundary layer is sufficient to have the norms in the main variables.

**Lemma A.5.1.** *Let  $\tilde{\delta} \in C^\infty(\sigma)$  be a smooth function, bounded, and far from 0:  $0 < \delta_0 < \tilde{\delta}(\sigma) < \delta_1 \forall \sigma \in \mathbb{T}$ .*

*Let  $f \in C^\infty(\mathbb{T} \times (0, +\infty))$  and for  $\varepsilon > 0$ ,  $\delta = \varepsilon\tilde{\delta}$ . We define the rescaled  $f$*

$$f_\delta(\sigma, Z) = f \left( \sigma, \frac{Z}{\delta(\sigma)} \right).$$

*Then for  $s \geq 0$ , with any  $\eta > 0$  and cut-off  $\chi$  such that  $\chi(z) = 0$  for  $z > 1$ , when  $\varepsilon \rightarrow 0$*

$$\begin{aligned} \|f_\delta\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f e^{\eta\zeta}\|_{H^s \varepsilon^{\frac{1}{2}-s}} \\ \|f_\delta \chi\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f\|_{H^s \varepsilon^{\frac{1}{2}-s}}. \end{aligned}$$

*And for  $s < 0$*

$$\begin{aligned} \|f_\delta\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f e^{\eta\zeta}\|_{H^s \varepsilon^{\frac{1}{2}-s}} \\ \|f_\delta\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f\|_{H^s \varepsilon^{\frac{1}{2}}}. \end{aligned}$$

*Remark A.5.1.* These hypotheses are far from optimal, the important idea is to have enough decay in  $z$  to have a control of the low frequency of  $f$  in  $z$ . For example instead of  $e^{\eta z}$  any weight increasing faster than any polynomial is sufficient, or for the  $L^2$  norm,  $s = 0$ , there is no need of such weight.

*Proof.* For  $s \geq 0$  this follows from a direct computation,

$$\begin{aligned} & \int_{\mathbb{T}} \int_0^\infty \sum_{k_\sigma + k_z \leq s} \left| \partial_\sigma^{k_\sigma} \partial_z^{k_z} f \left( \sigma, \frac{Z}{\delta(\sigma)} \right) \right|^2 dZ d\sigma \\ &= \int_{\mathbb{T}} \int_0^\infty \sum_{k_\sigma + k_z \leq s} \left| \partial_\sigma^{k_\sigma} \left( \frac{1}{\tilde{\delta}(\sigma)^{k_z}} \varepsilon^{-k_z} \partial_2^{k_z} f \left( \sigma, \frac{\zeta}{\tilde{\delta}(\sigma)} \right) \right) \right|^2 \varepsilon d\zeta d\sigma \\ &\leq \varepsilon^{1-2s} \int_{\mathbb{T}} \int_0^\infty \sum_{k_\sigma + k_z \leq s} \left| \partial_\sigma^{k_\sigma} \left( \frac{1}{\tilde{\delta}(\sigma)^{k_z}} \partial_2^{k_z} f \left( \sigma, \frac{\zeta}{\tilde{\delta}(\sigma)} \right) \right) \right|^2 d\zeta d\sigma \\ &\leq \varepsilon^{1-2s} \int_{\mathbb{T}} \int_0^\infty \mathcal{C}(\tilde{\delta})(\sigma) (1 + \zeta^{2s}) \sum_{k_\sigma + k_z \leq s, k_1 \leq k_\sigma} \left| \partial_1^{k_1} \partial_2^{k_2} f \left( \sigma, \frac{\zeta}{\tilde{\delta}(\sigma)} \right) \right|^2 d\zeta d\sigma \end{aligned}$$

where  $\mathcal{C}(\tilde{\delta})$  is a smooth function, bounded as long as all derivatives up to  $\partial_\sigma^s$  of  $\tilde{\delta}$  are bounded and  $\tilde{\delta}(\sigma) > \delta_0 > 0$ . And after the change of variables  $\zeta \mapsto \tilde{\delta}\zeta$ , and as  $(1 + \zeta^s)e^{-\eta\zeta}$  and all its derivatives are bounded, we obtain

$$\|f_\delta\|_{H^s}^2 \leq C_{\tilde{\delta}} \varepsilon^{1-2s} \|f e^{\eta\zeta}\|_{H^s}^2.$$

In the case of a cut-off function  $\chi(Z)$  the same computation leads to

$$\begin{aligned} \|f_\delta \chi\|_{H^s}^2 &\leq \varepsilon \int_{\mathbb{T}} \int_0^{\frac{1}{\varepsilon}} \mathcal{C}(\tilde{\delta}(\sigma)) \sum_{k_\sigma + k_2 \leq s, k_1 \leq k_\sigma} \left| \varepsilon^{-k_1} \zeta^{k_2} \partial_1^{k_1} \partial_2^{k_2} f \left( \sigma, \frac{\zeta}{\tilde{\delta}(\sigma)} \right) \right|^2 d\zeta d\sigma \\ &\leq C \sum_{k_\sigma + k_2 \leq s, k_1 \leq k_\sigma} \varepsilon^{1-2k_1-2k_2} \int_{\mathbb{T}} \int_0^{\frac{1}{\varepsilon}} \left| \partial_1^{k_1} \partial_2^{k_2} f \left( \sigma, \frac{\zeta}{\tilde{\delta}(\sigma)} \right) \right|^2 d\zeta d\sigma \leq C \varepsilon^{1-2s} \|f\|_{H^s}^2 \end{aligned}$$

*Remark A.5.2.* Note that the same exact computation leads to  $\|g_{\frac{1}{\varepsilon}}\|_{H^s} \leq \frac{C}{\sqrt{\varepsilon}} \|g\|_{H^s}$  when  $\varepsilon \rightarrow 0$ . The power  $-\frac{1}{2}$ , instead of  $s - \frac{1}{2}$ , is due to the fact that these Sobolev spaces are nonhomogeneous with respect to the second variable, so we cannot hope to have a better scaling than the  $L^2$  one.

For  $s < 0$  we use the definition of the norm

$$\|f_\delta\|_{H^s} = \sup_{g \in H_0^{-s}} \frac{\int f_\delta g}{\|g\|_{H^{-s}}}.$$

Unfortunately, noticing  $\int f_\delta g = \delta \int f g_{\frac{1}{\varepsilon}}$  is not enough to conclude as the  $L^2$  norm of  $g_{\frac{1}{\varepsilon}}$  scales like  $\varepsilon^{-\frac{1}{2}}$  (see the previous remark).

But with the exponential decay

$$\left| \int f_\delta g \right| = \left| \int \delta f e^{\frac{\eta}{2}z} g_{\frac{1}{\varepsilon}} e^{-\frac{\eta}{2}z} \right| \leq C_{\tilde{\delta}} \varepsilon \|F e^{\frac{\eta}{2}z}\|_{L^2} \|\partial_z^{-s} (g_{\frac{1}{\varepsilon}}) e^{-\frac{\eta}{2}z}\|_{L^2}.$$

where  $F$  such that  $\partial_z^{-s} F = f$  is obtained by integrating with respect to  $Z$  from  $+\infty$ , in particular  $\|F e^{\frac{\eta}{2} z}\|_{L^2} \leq \|f e^{\eta z}\|_{H^s}$ . And

$$\begin{aligned} \|\partial_z^{-s} \left(g_{\frac{1}{3}}\right) e^{-\frac{\eta}{2} z}\|_{L^2}^2 &\leq C_{\delta} \varepsilon^{-s-\frac{1}{2}} \|\partial_z^{-s} (g)\|_{L^2}^2 \\ &\leq C_{\delta} \varepsilon^{-s-\frac{1}{2}} \|g\|_{H^{-s}}^2 \end{aligned}$$

so we finally obtain  $\forall g \in H_0^{-s}$

$$\left| \int f_{\delta} g \right| \leq C_{\delta} \varepsilon^{1-s-\frac{1}{2}} \|g\|_{H^{-s}}^2.$$

*Remark A.5.3.* With  $f(\zeta) = (1 + \zeta)^{-\alpha}$ ,  $\alpha > \frac{1}{2}$  we see that the decay is necessary as  $\|f_{\delta} \chi\|_{H^s}$  scales like  $\varepsilon^{\alpha}$  for  $s < 0$ . This is due to the  $L^2$  part of the  $H^{-s}$  norm.  $\square$

Another recurrent technical step is to show that by cutting-off the solution of the boundary layer variable we do not introduce a too large error. Even if the remainders introduced by such cut-off depends on the exact equation, in fact it is sufficient to estimate  $\|u \kappa\|_{H^s}$  for  $\kappa$  a cut-off function being 0 near the boundary.

As an example, let us consider the equation

$$-\varepsilon \partial_z^2 u + u = r \text{ on } (0, 1).$$

With  $u$  an exact solution near  $z = 0$  and  $\chi$  a cut-off such that  $\chi(z) = \begin{cases} 1 & \text{for } z < \frac{1}{3} \\ 0 & \text{for } z > \frac{2}{3} \end{cases}$  then  $u \chi$  is a solution with additional remainders terms

$$-\varepsilon \partial_z^2 (u \chi) + (u \chi) = r \chi + \underbrace{2\varepsilon \partial_z u \partial_z \chi + \varepsilon u \partial_z^2 \chi}_{\text{additional terms}}$$

so we need to estimate  $\partial_z u \partial_z \chi$  and  $u \partial_z^2 \chi$ . The main point is that  $\partial_z \chi$  and  $\partial_z^2 \chi$  are being supported inside  $(\frac{1}{3}, \frac{2}{3})$ .

In order to estimate such terms we have the following lemma

**Lemma A.5.2.** *Let  $\tilde{\delta} \in C^{\infty}(\sigma)$  be a smooth function, bounded, and far from 0:  $0 < \delta_0 < \tilde{\delta}(\sigma) < \delta_1 \forall \sigma \in \mathbb{T}$ .*

*Let  $f \in C^{\infty}$  and let us define, with*

$$f_{\delta}(\sigma, Z) = f\left(\sigma, \frac{Z}{\delta(\sigma)}\right).$$

*Let  $\kappa \in C_c^{\infty}$  be a cut-off function such that*

$$\kappa(\sigma, Z) = 0 \text{ for } Z < 1.$$

*Then for  $s \geq 0$ , when  $\varepsilon \rightarrow 0$  for any  $\eta > 0$*

$$\begin{aligned} \|f_{\delta} \kappa\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f\|_{H^s} \varepsilon^{\frac{1}{2}-s} \\ \|f_{\delta} \kappa\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f e^{\eta \zeta}\|_{H^s} \varepsilon^k \quad \forall k > 0. \end{aligned}$$

And for  $s < 0$

$$\begin{aligned} \|f_\delta \kappa\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f\|_{H^s} \varepsilon^{\frac{1}{2}} \\ \|f_\delta \kappa\|_{H^s(\mathbb{T} \times (0, +\infty))} &\leq C \|f e^{\eta \zeta}\|_{H^s} \varepsilon^k \quad \forall k > 0. \end{aligned}$$

*Remark A.5.4.* In fact the estimates with exponential weight are true even if  $\kappa$  has a non-compact support.

*Proof.* For  $s > 0$ , as in the previous lemma, we see that, thanks to an explicit computation and the fact that  $\kappa$  and all its derivatives are 0 for  $z < 1$ , we only need to estimate terms like

$$\varepsilon^{1-2k_1} \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\frac{M}{\varepsilon}} C(\tilde{\delta}, \kappa)(\sigma) \left| \zeta^{k_2} \partial_\zeta^{k_1} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right|^2 d\zeta d\sigma \leq C_{\tilde{\delta}, \kappa} \varepsilon^{1-2k_1} \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\frac{M}{\varepsilon}} \left| \zeta^{k_2} \partial_\zeta^{k_1} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right|^2 d\zeta d\sigma.$$

In the case where  $f$  has exponential decay, we can simply write (for  $\varepsilon$  small enough)

$$\begin{aligned} \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\infty} \left| \zeta^{2k_2} \partial_\zeta^{k_1} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right|^2 d\zeta d\sigma &\leq \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\infty} \zeta^{k_2} e^{-2\eta \zeta} \left| e^{\eta \zeta} \partial_\zeta^{k_1} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right|^2 d\zeta d\sigma \\ &\leq \varepsilon^{-2k_2} e^{-\frac{2\eta}{\varepsilon}} \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\infty} \left| e^{\eta \zeta} \partial_\zeta^{k_1} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right|^2 d\zeta d\sigma \\ &\leq C \varepsilon^{-2k_2} e^{-\frac{2\eta}{\varepsilon}} \int_{\mathbb{T}} \int_{\frac{1}{\varepsilon}}^{\infty} \left| \partial_\zeta^{k_1} \left( e^{\eta \zeta} f \left( \sigma, \frac{\zeta}{\tilde{\delta}} \right) \right) \right|^2 d\zeta d\sigma \\ &\leq C e^{-\frac{\eta}{\varepsilon}} \|f e^{\eta \zeta}\|_{H^s}^2. \end{aligned}$$

The case where we only know that  $f \in H^s$  is the same as before. And once more the  $s < 0$  are done by duality.  $\square$

*Remark A.5.5.* This is where we see the importance to control at least the  $L^2$  norm (in fact a much stronger weighted norm) of solutions. For example, for  $s \in \mathbb{N}$  let us consider the homogeneous space  $\dot{H}_s((0, +\infty))$ , defined by

$$\|f\|_{\dot{H}_s}^2 = \int_0^\infty \left( |\partial_\zeta^s u|^2 + \frac{|u|^2}{1 + \zeta^{2s}} \right) d\zeta$$

then we have for any  $\alpha < s + \frac{1}{2}$  the function  $\phi_\alpha : \zeta \mapsto (1 + \zeta)^\alpha$  is in  $\dot{H}^s$ . Then with  $\kappa$  a cut-off with support inside  $(1, M)$ .

$$\left\| \phi_\alpha \left( \frac{\cdot}{\varepsilon} \right) \kappa \right\|_{L^2}^2 \simeq C \varepsilon \int_{\frac{1}{\varepsilon}}^{\frac{M}{\varepsilon}} \zeta^{2\alpha} d\zeta \simeq C \varepsilon^{-2\alpha} M^{2\alpha+1}$$

which is by no mean small.



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# Contents

<b>Abstract</b>	<b>ix</b>
<b>Contents</b>	<b>xi</b>
<b>1 Présentation générale (en français)</b>	<b>1</b>
1.1 Mécanique des fluides et petits paramètres . . . . .	1
1.2 Perturbations singulières et régulières . . . . .	2
1.2.1 Premier exemple de développement asymptotique . . . . .	2
1.2.2 Une perturbation singulière . . . . .	5
1.2.3 Résumé . . . . .	7
1.3 Équations elliptiques . . . . .	7
1.4 Les couches limites: pourquoi et comment ? . . . . .	8
1.4.1 Perturbation régulière . . . . .	8
1.4.2 Un petit exemple de perturbation singulière . . . . .	9
1.4.3 Bref historique . . . . .	10
1.4.4 Couches limites et simulations numériques . . . . .	11
1.5 Organisation du manuscrit . . . . .	12
<b>2 Introduction</b>	<b>13</b>
2.1 Notations . . . . .	13
2.1.1 Fonctions . . . . .	13
2.1.2 Smallness . . . . .	15
2.1.3 Some indications . . . . .	16
2.2 Context of the thesis . . . . .	17
2.2.1 Quick historical context . . . . .	17
2.2.2 Objective and methods . . . . .	20
2.2.3 Proudman problem and equatorial Ekman boundary layer . . . . .	21
2.2.4 About shear layers in Magnetohydrodynamics . . . . .	25
2.2.5 The problem of recirculation for Prandtl layers . . . . .	29
2.3 About degenerate elliptic equations . . . . .	32
2.4 The one dimensional Proudman problem . . . . .	33
2.4.1 Energy estimate and stability . . . . .	34
2.4.2 Boundary layer ansatz, boundary layer operator . . . . .	37
<b>3 Proudman problem</b>	<b>47</b>
3.1 Physical modelization . . . . .	47
3.1.1 Context . . . . .	47
3.1.2 Non-dimensional parameters and orders of magnitudes . . . . .	48

3.1.3	Taylor-Proudman theorem and Ekman layer . . . . .	50
3.1.4	Proudman problem . . . . .	51
3.1.5	Reformulation of the Proudman problem . . . . .	52
3.1.6	Boundary layers . . . . .	53
3.2	Preliminary results on the Proudman problem . . . . .	54
3.2.1	Mathematical setting . . . . .	54
3.2.2	Existence, energy estimate and weak convergence . . . . .	55
3.2.3	Stability estimate, approximate solutions . . . . .	58
3.2.4	Some bibliographical remarks . . . . .	58
3.3	Overall view of the construction . . . . .	59
3.4	The Ekman layers for two disks . . . . .	61
3.5	The shear layers . . . . .	69
3.5.1	About the validity of Ekman layers . . . . .	69
3.5.2	Objective . . . . .	70
3.5.3	Stewartson $2/7$ and $1/4$ layers . . . . .	72
3.5.4	Interior terms and connection between operators . . . . .	81
3.5.5	Stewartson $\frac{1}{3}$ layer . . . . .	84
3.5.6	How to put the pieces together ? . . . . .	97
3.6	Formal remarks on the equatorial problem . . . . .	101
3.7	Conclusion and perspectives . . . . .	105
<b>4</b>	<b>Mathematical study of the equatorial Ekman boundary layer</b>	<b>109</b>
4.1	Introduction . . . . .	109
4.2	Existence and properties of weak solutions . . . . .	111
4.2.1	Statement of the result . . . . .	112
4.2.2	Remarks on nonhomogeneous boundary conditions . . . . .	113
4.2.3	Duality principle (proof of (i)) . . . . .	114
4.2.4	Boundary conditions (proof of (ii)) . . . . .	116
4.2.5	Caccioppoli type inequality and interior regularity . . . . .	117
4.3	Uniqueness and transparent boundary conditions . . . . .	119
4.3.1	Uniqueness . . . . .	119
4.3.2	Transparent boundary conditions . . . . .	121
4.3.3	The case of the half plane . . . . .	124
4.4	Appendix . . . . .	125
4.4.1	Physical derivation . . . . .	125
4.4.2	Duality argument . . . . .	127
	Acknowledgements . . . . .	128
<b>5</b>	<b>MHD shear layer</b>	<b>129</b>
5.1	Introduction . . . . .	129
5.1.1	Physical derivation and orders of magnitude . . . . .	129
5.1.2	Main setting . . . . .	134
5.2	Preliminary results . . . . .	138
5.2.1	Existence, uniqueness and stability estimates . . . . .	138
5.2.2	Expected boundary layers . . . . .	140
5.3	Hartmann layers . . . . .	140
5.3.1	Hartmann boundary layer operator . . . . .	140
5.3.2	Expansion without shear layers . . . . .	144
5.3.3	Remarks on the transparent condition . . . . .	147

5.4	Toy model for the shear layer . . . . .	148
5.4.1	Setting . . . . .	148
5.4.2	Study of the shear layer equation . . . . .	151
5.4.3	About the singularity at $(0, 0)$ . . . . .	155
5.5	Conclusion and perspectives . . . . .	157
<b>6</b>	<b>Sign-changing solutions to the stationary Burgers equation with transverse viscosity</b>	<b>161</b>
6.1	Introduction . . . . .	161
6.1.1	Motivation from recirculation problems in fluid mechanics . . . . .	162
6.1.2	Statement of the main result . . . . .	163
6.1.3	Scheme of proof . . . . .	165
6.2	Well-posedness results for the linear problem . . . . .	166
6.3	Fixed-point scheme for the nonlinear problem . . . . .	171
6.4	Local uniqueness of solutions to the nonlinear problem . . . . .	173
6.5	Strategy of proof of Conjecture 6.2.6 . . . . .	173
6.5.1	Difficulties with a mixed boundary conditions strategy . . . . .	174
6.5.2	Vertically weighted vertical regularity estimates . . . . .	175
6.5.3	General strategy . . . . .	178
6.5.4	Estimates on the traces of $\partial_x u^\varepsilon$ . . . . .	179
6.5.5	Lifting of boundary layers . . . . .	180
6.5.6	Design of a weight function . . . . .	185
6.6	Appendix: Technical lemmas . . . . .	186
<b>A</b>	<b>Some basic methods and tools</b>	<b>187</b>
A.1	How to solve a linear ODE ? . . . . .	187
A.2	Duhamel formula . . . . .	188
A.3	Laplace transform . . . . .	189
A.4	Weighted Sobolev spaces . . . . .	189
A.5	Scalings, cut-offs and scaling . . . . .	190
	<b>Bibliography</b>	<b>195</b>
	<b>Contents</b>	<b>201</b>







## Couches limites fluides et équations elliptiques dégénérées

### Abstract

This thesis is devoted to the mathematical analysis of several boundary layer models in fluid dynamics. The starting point of our study is the Proudman problem, which describes the behavior of an incompressible highly rotating fluid between two spheres. In the low Rossby and low Ekman number limit, multiple nested boundary layers appear. They had been described, at a formal level, in previous works. The goal of the first part of this manuscript is to study rigorously several of them (in particular the equatorial Ekman layer) and to propose strategies of proof to justify the complete asymptotic expansion. In a second part we consider an MHD model, within which shear layers take place. They bear a strong resemblance to the boundary layers studied in the first part.

Eventually, the last part is dedicated to a stationary Burgers equation with transverse viscosity, for which we construct sign-changing solutions. Our long term goal is to construct solutions of the Prandtl system with a recirculation bubble, for which the present system play the role of a toy-model.

Remarkably, several of the systems studied in the present manuscript belong to the class of degenerate elliptic equations.

**Keywords:** boundary layers, fluid dynamics, multiscale analysis, degenerate elliptic equations, magnetohydrodynamics

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### Résumé

Cette thèse est consacrée à l'analyse mathématique de plusieurs modèles de couches limites en mécanique des fluides.

Le point de départ de l'étude est le problème de Proudman qui modélise le comportement d'un fluide incompressible en rotation rapide entre deux sphères. Dans la limite où le nombre de Rossby et le nombre d'Ekman sont petits, de nombreuses couches limites imbriquées apparaissent, et avaient été décrites formellement dans des travaux antérieurs. Le but de la première partie de cette thèse est d'étudier rigoureusement plusieurs d'entre elles (en particulier la couche d'Ekman équatoriale) et de donner des pistes pour justifier le développement asymptotique.

Dans une seconde partie on s'intéresse à un modèle issu de la magnétohydrodynamique, au sein duquel apparaît une couche de cisaillement très similaire aux couches limites étudiées dans la première partie.

Enfin, en vue de construire des solutions stationnaires du système de Prandtl possédant des bulles de recirculation, on étudie dans la dernière partie une équation de type Burgers stationnaire avec viscosité transverse, pour laquelle on construit des solutions changeant de signe.

De façon remarquable plusieurs des systèmes étudiés entrent dans le cadre des équations elliptiques dégénérées.

**Mots clés :** couches limites, mécanique des fluides, analyse multi-échelles, équations elliptiques dégénérées, magnétohydrodynamique

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