# Localization and Lie-Rinehart algebras in deformation quantization

Hamilton Menezes de Araujo

Univeristé de Haute Alsace
IRIMAS, Département de Mathématiques

# Thèse

pour obtenir le grade de
**Docteur en Mathématiques**

présentée par

Hamilton Araujo

# Localization and Lie-Rinehart algebras in deformation quantization

*Thèse dirigée par* Martin Bordemann
soutenue le 18/02/2021 devant le jury composé de:

| | | | |
|---|---|---|---|
| M | Stefan Waldmann | Université de Würzburg | (Raporteur) |
| M | Camille Laurent-Gengoux | Université de Lorraine | (Raporteur) |
| M | Friedrich Wagemann | Université de Nantes | (Examinateur) |
| M | Daniel Panazzolo | Univeristé de Haute Alsace | (Examinateur) |
| M | Abdenacer Makhlouf | Univeristé de Haute Alsace | (Examinateur) |
| M | Martin Bordemann | Univeristé de Haute Alsace | (Directeur) |

If I have seen farther than others, it is by standing upon the shoulders of giants.
*Isaac Newton*

# Acknowledgments

I must write this part in French, this is a way to thank this beautiful country, as well as this beautiful language and make it clear that I did not learn only math during my thesis.

## Remerciements

Premièrement, je voudrais remercier Dieu pour la vie et pour l'opportunité de faire cette thèse.

Ensuite, je tiens à dédier ce travail en guise de remerciement à ma mère, Margarida Menezes, qui m'a tout fourni pour que je puisse être qui je suis aujourd'hui et *in memorian* a mon père, Djacir Araujo, qui illumine toujours ma mémoire avec ses beaux souvenirs. Également, je dedie ce travaille à ma tante Luiza Menezes et mon oncle Otavio Freire pour leur précieux soutien.

Je tiens aussi à remercier Célia Attabary pour être celle qui m'a apporté plus d'attention, d'affection et pour partager énormément de bons moments avec moi.

Ah les amis... Je tiens à dire que sans vous je n'arriverai pas à ce moment-là. Merci beaucoup à Allan Junior, Danielson Filho, David Nobre, Felipe Fernandes, Joel Alencar, Kelly Rodrigues et Tiago Silva pour les discussions et pour les bons moments qu'on a partagé ensemble, ça m'a formé comme étudiant et comme personne dès la licence. Également, je voudrais également remercier Diego Silva, Danuso Rocha, Rosa Tayane, Emanoel Ferreira et Geraldo Herbert pour votre aide et pour l'amitié pendant le master.

Tout ça pour arriver à Mulhouse. Sans l'aide d'Icaro Chaves et d'Aninda Chatterjje ma vie ici aurait été une autre, sûrement moins intéressante. Merci pour votre amitié et pour avoir partagé des très joyeux moments ensemble et avec "Les cacahuètes for the nigths" dont je nomme: Thomas Jacumin, Otavio Perez, Lucas David, Sara Boussaid, Valentina Felisatti et Isabelle Hoffmann.

Pour la partie académique, je tiens a dire que j'ai eu beaucoup de chance comme élève et dernièrement comme thésard. Merci beaucoup a vous les mestres, dont Regis Alves mon premiere encadrant, Edson Sampaio mom superviseur au Master et également à Angelo Papa, Stalio Rodrigues, Janio Kleo, Alexandre Fernandes, Lev Birbrair et Daniel Cibotaru, car vous avez d'une certaine manière marqué ma vie académique.

Je voudrais également remercier Benedikt Hurle, Otmann Yaklhef et Viviane Kuhn et tous les autres membres du Département de Mathématiques de l'Université de Haute Alsace à Mulhouse pour votre accueil.

Je tiens à remercier spécialement mon superviseur de thèse Martin Bordemann. Merci pour votre bienveillance, pour votre gentillesse et pour votre aide dans tous les sens. Je crois d'être chanceux d'avoir eu l'opportunité de partager ce travail avec vous et pouvoir profiter de votre connaissance et expérience en mathématiques.

Également, je voudrais remercier Stefan Waldmann et Camille Laurent-Gengoux d'avoir participé au Jury de soutenance comme rapporteurs et pour votre contribution pour ce travail se concrétiser. Également, un grand merci à Friedrich Wagemann, Daniel Panazzolo et Abdenacer Makhlouf d'avoir accepté de faire partie du Jury de soutenance et pour votre aide pendant toute la durée de ma thèse.

Finalement, je voudrais remercier Sylvie Paycha pour m'avoir donné l'opportunité de présenter la première partie de ma thèse à Potsdam.

# Contents

# Version Française

## 1 Introduction

La thèse contient deux parties qui sont liées à la quantification par déformation, une théorie qui utilise la déformation algébrique des algèbres de fonctions commutatives pour décrire certains aspects de la mécanique quantique des physiciens.

La première partie de la thèse contient une discussion de la localisation analytique des algèbres de fonctions en quantification par déformation, i.e. les fonctions ne sont définies que sur une partie ouverte de la variété, comparée avec la localisation non commutative à la Ore des algèbres de fonctions déformées. L'exemple le plus élémentaire d'une localisation est le passage de l'anneau des entiers relatifs aux nombres rationnels, les fractions, où certains nombres sont rendus inversibles.

Nous décrivons d'abord le cadre de la localisation algébrique : il y a un procédé général qui n'est pas très explicite, et il y a la construction d'Ore (voir [36]) qui est beaucoup plus concrète, mais est plus particulière parce que l'ensemble multiplicatif, qui constitueront l'ensemble des futurs dénominateurs, doit respecter une certaine condition, les conditions d'Ore.

Ensuite, nous regardons deux exemples élémentaires, celui des fonctions définies sur une partie ouverte et celui des germes de fonctions autour d'un point donné. Dans le premier cas on obtient l'équivalence entre l'approche analytique et l'approche algébrique. L'outil principal de la démonstration se base sur les travaux analytiques de Whitney, Malgrange et surtout sur le livre de J.-C.Tougeron. Pour les germes, on obtient également une certaine équivalence, mais le choix le plus naturel échoue, et on doit légèrement modifier l'ensemble multiplicatif. A la fin on discute un cadre plus général, et algébrique, qui permet de formuler la question suivante : « Est-ce que localisation et déformation commutent ? ». On donne également un exemple non Ore.

D'aute part, le lien de la deuxième partie avec la quantification par déformation est le plus élémentaire, celui des opérateurs différentiels, comme par exemple les opérateurs de Schrödinger, en mécanique quantique. Dans cette partie on ne discute pas les propriétés analytiques de ces opérateurs, comme par exemple le spectre, mais on cherche à décrire la multiplication de deux opérateurs différentiels sur une variété différentielle où on n'a plus de coordonnés globales, mais –dans beaucoup de situations géométriques intéressantes– on doit utiliser des dérivées covariantes itérées.

Par conséquent, il est intéressant de savoir comment la courbure et la torsion de la connection, qui en effet donnent ces dérivées et qui fournissent beaucoup d'invariants de la variété, entrent dans les formules de multiplication. Nous avons donc choisi le cadre algébrique, notamment des algèbres de Lie-Rinehart (Rinehart [42] et Huebschmann [19]) qui généralisent les algèbres de Lie de tous les champs de vecteurs et permettent d'utiliser des méthodes purement algébriques qui ne sont pas utilisées en géométrie différentielle usuelle.

De sorte que, on a réussi à donner une description très explicite de « l'algébroïde des chemins » de M.Kapranov (voir [23]) en termes des dérivées covariantes itérées, de maniéré que courbure et torsion apparaissent dans une application canonique de l'algébroïde de Kapranov dans l'algèbre de Lie-Rinehart en question.

En autre, cette construction permet de décrire l'enveloppante de l'algèbre de Lie-Rinehart, d'ailleurs analogue de l'algèbre des opérateurs différentiels, comme quotient d'une algèbre plus grande –dont la multiplication est très explicite – modulo un idéal. En fait, la construction est –pour parler géométriquement – '« tensorielle » et consiste en une « symétrisation » perturbée par des termes de courbure et torsion. Dans plusieurs cas particuliers, la multiplication est calculable en termes d'une factorisation des algèbres enveloppante des algèbres de Lie ce qui est un problème connu en théorie de Lie.

## 2 Localisation

L'objectif dans un premier temps (Chapitre 1) est de fixer la notation que nous utiliserons tout au long du texte et d'introduire localisation d'un point de vue purement algébrique. Nous avons essentiellement divisé ce chapitre en deux parties, dans la section 1.2 nous décrirons comment la localisation peut être définie pour les $K$-algèbres et nous chercherons les principales propriétés des constructions. Ensuite, dans la section 1.3, nous traiterons la localisation dans un contexte général, c'est-à-dire pour les algèbres $K$ qui ne sont pas nécessairement commutatives.

La proposition suivante résume bien le premier cas de localisation.

**Proposition 1.2.10 :** Soit $R$ une $K$-algébre commutaive et $S \subset R$ un sous ensemble multiplicatif.be a multiplicative subset. Alors, ce qui suit est vrai :

a. $\eta_{(R,S)}(S) \subset U(R_S)$, c'est-à-dire, l'homomorphisme $\eta_{(R,S)}$ envoie des éléments de $S$ aux éléments inversibles de $R_S$. En plus, pour $K$-algèbre $R$ unitaire et commutatif, doté d'un sous-ensemble multiplicatif $S \subset R$, le paire $(R_S, \eta_{(R,S)})$ est **universel**. [1]

b. Chaque élément de $R_S$ s'écrit comme une fraction $\eta(r)\eta(s)^{-1}$, où $r \in R$ et $s \in S$.

1. Voir la remarque (1.2.11).

c. $\ker(\eta_{(R,S)}) = \{r \in R \mid rs = 0 \text{ for some } s \in S\}$.

Ensuite, la proposition suivante généralise localisation pour toutes les $K$-algebres, pas forcement commutatives. C'est important de remarquer le traitement via la théorie de catégories.

**Proposition 1.3.3 :** Il y a une adjonction de foncteurs

$$K\mathbf{AlgMS} \underset{\mathcal{U}}{\overset{\mathcal{L}}{\rightleftarrows}} K\mathbf{Alg}$$

où $\mathcal{L}$ est l'adjointe à gauche au foncteur $\mathcal{U}$ ci-dessus de sorte que chaque composante $\eta_{(R,S)}$ de l'unité $\eta : I_{K\mathbf{AlgMS}} \dot{\to} \mathcal{U}\mathcal{L}$ de l'adjonction satisfait à la propriété universelle *a.* de la proposition (1.3.1) précédente dans le cas général non commutatif. Nous nous référons à $\mathcal{L}$ en tant que **foncteur localisation**. Pour un paire $(R,S)$ donné $K\mathbf{AlgMS}$ nous désignons par $R_S$ la $K$-algèbre $\mathcal{L}(R,S)$ donné par le foncteur $\mathcal{L}$, et pour $\eta_{(R,S)} : R \to R_S$ la composante de l'unité de l'adjonction.

Donc $\eta_{(R,U(R))} : R \to R_{U(R)}$ est un isomorphisme, l'inverse étant la composante $\epsilon_R$ de la counité $\epsilon : \mathcal{L}\mathcal{U} \dot{\to} I_{K\mathbf{Alg}}$ de l'adjonction. De plus, chaque élément de la $K$-algèbre $R_S$ est une somme finie de produits de la forme $(\eta = \eta_{(R,S)})$

$$\eta(r_1)\big(\eta(s_1)\big)^{-1} \cdots \eta(r_N)\big(\eta(s_N)\big)^{-1}.$$

où $r_1, \ldots, r_N \in R$ et $s_1, \ldots, s_N \in S$. Des termes comme ceux-ci peuvent être appelés «multifractions». Notez que, $r_1$ ou $s_N$ peut être égale à l'unité de $R$.

Nous terminons ce première chapitre avec un théorème de la localisation de Ore.

**Théorème 1.3.8 :** Soit $R$ une $K$-algèbre unitaire et $S \subset R$ un sous-ensemble multiplicative. Alors, ce qui suit est vrai :

1. La $K$-algèbre $R$ une $K$-algèbre de fractions à droite $\check{R}_S$ en ce qui concerne le sous-ensemble multiplicatif $S$ si et seulement si $S$ est un ensemble de dénominateurs à droite.
2. Si tel est le cas, chacune de ces paires $(\check{R}_S, \check{\eta})$ est universel au sens du diagramme (1.3.2) est chaque $\check{R}_S$ est isomorphe à l'algèbre canonique localisée $R_S$ de la Proposition 1.3.3.
3. Chaque $\check{R}_S$ est isomorphe a l'ensemble quotient $RS^{-1} := (R \times S)/\sim$ en ce qui concerne la relation binaire suivante $\sim$ en $R \times S$

$$(r_1, s_1) \sim (r_2, s_2) \;\Leftrightarrow\; \exists b_1, b_2 \in R \text{ tel que}$$

$$s_1 b_1 = s_2 b_2 \in S \text{ and } r_1 b_1 = r_2 b_2 \in R \tag{2.1}$$

qui est une relation d'équivalence généralisant la relation (1.2.1).

Ensuite, nous réservons un chapitre dont l'objectif est d'explorer un objet important dans la quantification par déformation, les produits star, qui nous donnera une structure non commutative dans l'algèbre des fonctions lisses définies sur une variété.

Néanmoins, depuis l'article fondateur de Bayen, Flato, Frønsdal, Lichnerowicz et Sternheimer en 1978, voir [2], la quantification par déformation est devenue un vaste domaine de recherche qui couvre plusieurs théories algébriques comme la théorie de la déformation formelle des algèbres associatives, ainsi que des théories géométriques comme la théorie des variétés symplectiques et de Poisson, et des théories physiques comme la théorie des cordes et la théorie de la jauge non commutative.

Dans ce cas, la multiplication associative non commutative des opérateurs en mécanique quantique est considérée comme une déformation associative formelle de la multiplication ponctuelle de l'algèbre des symboles de ces opérateurs. Pour les variétés de Poisson, les travaux de Kontsevich [28] sont assez importants pour garantir l'existence de certaines constructions de cette manière.

Par contre, nous n'approfondirons pas cette belle théorie, nous explorerons essentiellement la multiplication déformée, le produit star, qui sera le symbole de calcul des opérateurs différentiels en termes de séries de puissances formelles. Plus précisément,

**Definition 2.1.4 :** Un produit star $*$ sur une variété $X$ est une opération associative bilinéaire $\mathcal{C}^\infty(X)[[\lambda]] \times \mathcal{C}^\infty(X)[[\lambda]] \to \mathcal{C}^\infty(X)[[\lambda]]$ satisfaisant les caractéristiques suivantes pour tous $f, g \in \mathcal{C}^\infty(X)$ :

(i.) $1 * f = f * 1 = f$,

(ii.) $f * g = f \cdot g + \mathcal{O}(\lambda)$,

(iii.) $f * g = \sum_{k=0}^{\infty} C_k(f,g)\lambda^k$,

(iv) $f * g - g * f = \lambda\{f,g\} + O(\lambda^2)$ si une structure de Poisson est donné.

avec les opérateurs bilinéaires $C_k : \mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(X) \to \mathscr{C}^\infty(X)$. Nous supposons que tous les $C_k$ sont des opérateurs bidifferentiels.

Nous pouvons donc regarder les deux cas suivantes.

**Localisation analytique :** Notez que chaque produit star $*$ peut être localisé analytiquement à un produit star $*_\Omega$ défini sur $\mathcal{C}^\infty(\Omega)[[\lambda]]$ par la localisation de tous les opérateurs bidifferentiels $C_k$ à $C_k|_\Omega$. Ensuite on peux considérer

**Localisation algébrique :** Soit $(X, \pi)$ une variété de Poisson, soit $* = \sum_{k=0}^{\infty} \lambda^k C_k$ un produit étoile en $(X, \pi)$ et soit $\Omega \subset X$ un ensemble ouverte fixée. On définie $K = \mathbb{K}[[\lambda]]$ et on considéré la $K$-algèbre

$$R = \big(\mathcal{C}^\infty(X)[[\lambda]], *\big). \tag{2.2}$$

De plus, comme le produit star $*$ n'implique que des opérateurs bidifferentiels, il se limite à un produit étoile $*_\Omega$ sur les séries entières formelles $\phi \in R_\Omega := \mathcal{C}^\infty(\Omega, \mathbb{K})[[\lambda]]$ tel que $\big(R_\Omega, *_\Omega\big)$ est aussi une $K$-algèbre.

Il s'ensuit que l'application de restriction $\eta_\Omega = \eta : R \to R_\Omega : f \mapsto f|_\Omega$ est un morphisme de $K$-algèbres unitaires. Nous définissons le sous-ensemble suivant $S_\Omega \subset R$ :

$$S_\Omega := \{g \in R \mid \forall\, x \in \Omega :\ g_0(x) \neq 0\} \tag{2.3}$$

Il est clair que la fonction constante 1 est en $S$, et pour toute $g, h \in S$ nous avons $(g*h)_0(x) = g_0(x)h_0(x) \neq 0$ (pour tous $x \in X$) d'où *$S$ est un sous-ensemble multiplicatif de la $K$-algèbre unitaire $R$.*

Nous pouvons maintenant considérer la localisation non commutative de $R$ par rapport à $S$ et la comparer avec la $K$-algèbre unitaire $R_\Omega$.

Le principal résultat de la première partie de cette thèse est de répondre à la question suivante :

Quelle est la relation entre $R_{S_\Omega}$, la localisation algébrique au sens du chapitre 1, et $R_\Omega$ la localisation analytique que nous décrivons ci-dessus. Ces algèbres sont-elles isomorphiques ?

Ainsi, le théorème 3.1.1 suivant, qui est le plus important de la première partie de la thèse, montre que la localisation pour les produits étoile peut être démontrée en recherchant les propriétés de la définition 1.3.4.

**Théorème 3.1.1 :** En utilisant les notations précédemment fixées, nous obtenons pour tout ensemble ouvert $\Omega \subset X$ :

1. $(R_\Omega, *_\Omega)$ avec le morphisme de restriction $\eta$ constitue une $K$-algèbre de fractions à droite pour $(R, S)$.
2. La conséquence immédiate est que $S$ est un bon ensemble des dénominateurs.
3. Cela implique en particulier que la localisation algébrique $RS^{-1}$ de $R$ en ce qui concerne $S$ est isomorphe à la localisation concrète $R_\Omega$ comme des $K$-algèbres unitaires.

## 3 Algèbres de Lie-Rinehart et connections

Il existe une formule explicite bien connue pour la multiplication de deux opérateurs différentiels dans tout ensemble ouvert de $\mathbb{R}^n$ en fonction de leurs symboles, au moyen des coordonnées globales $x$ et des coordonnées "conjuguées" supplémentaires p. Sur une variété différentiable équipé d'une connexion $\nabla$ dans le fibré tangent, tout opérateur différentiel peut être paramétré par un champ tenseur symétrique associé à une dérivée covariante itérée. Le produit de deux opérateurs différentiels peut également être écrit sous cette forme, mais la forme explicite contiendra des termes de courbure et de torsion compliqués qui, en général, semblent n'être connus que par l'application (inverse de la) exponentielle de $\nabla$ et le transport parallèle. Le problème est fortement lié à la difficulté de trouver des formules explicites pour les produits star sur les fibres cotangents : ces produits star ont été traités il y

a longtemps par Fedosov, Bordemann, Neumaier, Waldman et Pflaum (voir [5], [4] et [14]), où l'existence et la classification posent des questions avait été résolu.

Dans cette deuxième partie de la thèse, nous souhaitons exprimer plus explicitement les termes de courbure et de torsion apparaissant dans le produit de l'opérateur différentiel. Nous avons choisi approche algébrique qui semble fonctionner pour les anneaux commutatifs généraux $K$, $A$ et tout morphisme $K \to A$ à condition que $K$ et donc $A$ contienne les nombres rationnels.

Dans un premier temps, l'algèbre de Lie de tous les champs vectoriels d'une variété forme algèbre de Lie-Rinehart $L$ sur l'algèbre unitaire commutative réelle $A$ de toutes les fonctions lisses a valeur réelle de la variété. En fait, l'algèbre de tous les opérateurs différentiels est isomorphe à l'algèbre enveloppante dite universelle $\mathcal{U}(L,A)$ de $L$, c'est pourquoi nous souhaitons décrire ces algèbres en général. La $A$ linéarité (par opposition à la simple $R$ linéarité) peut être traduite en géométrie comme "fibre-par-fibre" ou "tensorielle".

Notamment, le problème d'algèbre enveloppante, définie dans la section 4.2.2 peux être abordé – algébriquement – comme suit : Soit $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ une algèbre de Lie-Rinehart sur $A$ (voir [42]), existe-t-il une $K$-algèbre unital et associatif $\big(\mathcal{U}(L,A),\blacklozenge,\mathbf{1}\big)$ équipé de deux applications $K$-linéaires $\iota_L : L \to \mathcal{U}(L,A)$ et $\iota_A : A \to \mathcal{U}(L,A)$ satisfaisant pour tous $x \in L$ et $a \in A$

$$\begin{aligned}
\iota_L : L \to \mathcal{U}(L,A)^- \qquad & \text{morphisme de } K-\text{algbres de Lie} && (3.1)\\
\iota_A : A \to \mathcal{U}(L,A) \qquad & \text{morphisme de } K-\text{algbres unitaires} && (3.2)\\
\iota_A(a) \blacklozenge \iota_L(x) \quad &= \quad \iota_L(ax) \quad \text{et} && (3.3)\\
\iota_L(x) \blacklozenge \iota_A(a) - \iota_A(a) \blacklozenge \iota_L(x) \quad &= \quad \iota_A\big(\rho_x(a)\big), && (3.4)
\end{aligned}$$

de telle sorte que pour toute $k$-algèbre unitaire associative donnée $B$ et toute application $K$-linéaire $\theta : L \to B$ et $j : A \to B$ satisfaisant aux conditions analogues à $\iota_L$ et $\iota_A$ il existe un morphisme unique $\tilde{\theta} : \mathcal{U}(L,A) \to B$ de telle sorte que $\tilde{\theta} \circ \iota_L = \theta$ et $\tilde{\theta} \circ \iota_A = j$ ?

Nous n'utilisons pas la construction de Rinehart de $\mathcal{U}(L,A)$, mais plutôt la construction de Huebschmann (voir [19]) car elle se rapproche beaucoup plus des constructions traitées dans cette thèse.

Ensuite, pour un $A$-module ancré donné $(L,\rho)$ (voir définition 4.5) et un $A$-module $W$, une ***connection***, ou encore une *dérive covariante* $\nabla^W = \nabla$, c'est une application $K$-linéaire

$$\nabla : L \otimes W \to W,$$

écrite comme $\nabla(x \otimes w) =: \nabla_x(w)$ de sorte que pour tous $a \in A$, $x \in L$, et $w \in W$

$$\nabla_{ax}(w) = a\nabla_x(w) \quad \text{et} \quad \nabla_x(aw) = \rho_x(a)w + a\nabla_x(w). \tag{3.5}$$

Les deux conditions mentionnées ci-dessus, pour la définition de $\nabla$, sont connues comme « les axiomes de Kozul».

Nous appellerons le quadruple (voir 5.1.1) $(L,\rho,W,\nabla^W)$ un *A-module W avec connexion paramétrée par le A-module ancré* $(L,\rho)$. Par la suite, les calcules avec des dérivées covariantes itérées sont traitées dans le Chapitre 5 notamment sur l'algèbre libre unitaire associative $\mathsf{T}_A(L)$ itérée.

Pour cette partie, on peut souligner le théorème suivant, qui est liée au produit

$$b \diamond_\nabla b' := \sum_{(b)} b^{(1)} \nabla_{b^{(2)}}(b') \tag{3.6}$$

sur l'algèbre tensorielle $\mathsf{T}_A(L)$ et exprimé en termes de la connection en $\mathsf{T}_A(L)$ en utilisant la notation de Sweedler pour la comultiplication Shuffle :

**Théorème 5.2.1 :** Soit $(L,\rho)$ un $A$-module ancré, $W$ un $A$-module, et soit $\nabla'$ une connection en $W$. En suite, soit aussi $\nabla$ une connection en $L$, paramétré par $(L,\rho)$, et soit $\nabla^A$ la connection en $A$ induite par l'application d'ancré $\rho$, et $\nabla$, $\nabla'$ et $\nabla^A$ désignent également les dérivés covariants itérés tels que définis en (5.1.18).

Nous avons les propriétés suivantes pour tous $b,b',b'' \in \mathsf{T}_A(L)$ et $w \in W$ :

$$\begin{aligned}
\nabla'_b\big(\nabla'_{b'}(w)\big) &= \nabla'_{b\diamond_\nabla b'}(w), && (3.7)\\
\nabla_b(b'b'') &= \sum_{(b)} \big(\nabla_{b^{(1)}}(b')\big)\big(\nabla_{b^{(2)}}(b'')\big) && (3.8)\\
\big[\mathsf{deg},\nabla_b\big] &= 0 && (3.9)\\
\Delta_{sh}\big(\nabla_b(b')\big) = &= \sum_{(b)(b')} \nabla_{b^{(1)}}(b'^{(1)}) \otimes_A \nabla_{b^{(2)}}(b'^{(2)}), && (3.10)\\
\Delta_{sh}(b\diamond_\nabla b') &= \sum_{(b)(b')} \big(b^{(1)}\diamond_\nabla b'^{(1)}\big)\otimes_A\big(b^{(2)}\diamond_\nabla b'^{(2)}\big), && (3.11)\\
(b\diamond_\nabla b')\diamond_\nabla b'' &= b\diamond_\nabla (b'\diamond_\nabla b''), && (3.12)\\
b\diamond_\nabla \mathbf{1} &= b \;=\; \mathbf{1}\diamond_\nabla b, && (3.13)\\
\epsilon(b\diamond_\nabla b') &= \nabla^A_b\big(\epsilon(b')\big) = \epsilon\Big(b\diamond_\nabla\big(\epsilon(b')\mathbf{1}\big)\Big). && (3.14)
\end{aligned}$$

En particulière, $\big(\mathsf{T}_A(L),\mathbf{1},\diamond,\Delta_{sh},\epsilon\big)$ est une bialgèbre de Rinehart sur $A$, voir section 4.3 pour les définitions.

Pour la suite, l'algèbre enveloppante $\mathcal{U}(L,A)$ sera un quotient de $\mathsf{T}_A(L)$ : l'idéal bilatéral $\mathcal{J}(L,A)$ pour la seule multiplication $\mathbb{R}$-linéaire (que nous devons moduler) est également un coidéal par rapport à la comultiplication $A$-linéaire qui peut être explicitement décrite.

D'autre part, la partie primitive de $\mathsf{T}_A(L)$ deviendra importante : il s'agit d'une algèbre de Lie-Rinehart sur $A$ isomorphe à l'algèbre de Lie des chemins de M.Kapranov ([23], 2007). Il existe un morphisme canonique $Z$ des algèbres de Lie-Rinehart de la partie primitive à $L$ dont le noyau $\mathcal{P}_0(L,A)$ porte une représentation en $L$ égale à

la $A$-algèbre de Lie de l'holonomie infinitésimale. Il existe une équation de récursion pour $Z$ en termes de courbure et de torsion. Le coideal $J(L,A)$ est engendré $A$-linéairement par le noyau $\mathcal{P}_0(L,A)$

La relation avec l'algebroïde de Lie des chemins de (voir Section 4.5.1) est contenu dans le prochain théorème.

**Théorème 5.3.2 :** L'algebre de Lie-Rinehart $\big(\mathcal{P}_\nabla(L,A),\rho^\nabla,[\ ,\ ]^\nabla,A\big)$ est une algèbre libre de Lie-Rinehart, et pour tout morphisme de modules ancrés $\theta : (L,\rho) \to (L',\rho')$ donné, où $\big(L',\rho',[\![\ ,\ ]\!]'\big)$ est une algèbre de Lie-Rinehart sur $A$, le morphisme induit des algèbres de Lie-Rinehart $\bar\theta : \mathcal{P}_\nabla(L,A) \to L'$ peut être calculé par la récurrence suivante pour tous $x \in L$ and $\xi \in \mathcal{P}_\nabla(L,A)$

$$\bar\theta(x) = \theta(x) \ \text{ et } \ \bar\theta\big([x,\xi]\big) = [\![\theta(x),\bar\theta(\xi)]\!]' - \bar\theta\left(\nabla_x(\xi)\right) + \theta\left(\nabla_\xi(x)\right). \tag{3.15}$$

D'où l'algèbre de Lie-Rinehart $\mathcal{P}_\nabla(L,A)$ est isomorphe à l'algébroïde de Lie de Kapranov $\mathcal{P}(L,A)$ engendré parr $(L,\rho)$. En particulier, pour deux connexions différentes $\nabla$, $\nabla'$ les deux algebres de Lie-Rinehart $\mathcal{P}_\nabla(L,A)$ et $\mathcal{P}_{\nabla'}(L,A)$ sont isomorphes.

Par aieurs, la définition suivante est importante pour la suite.

**Definition 5.4.1 :** Nous définissons la *torsion* $\mathsf{Tor} = \mathsf{Tor}^\nabla$ et la *courbure* $\mathsf{R}' = \mathsf{R}^{\nabla'}$ comme des applications linéaires $L \otimes_K L \to L$ et $(L \otimes_K L) \otimes_K V \to V$ de manière bien connue. Pour tous $x,y \in L$ et $v \in V$,

$$\mathsf{Tor}(x,y) := \nabla_x(y) - \nabla_y(x) - [\![x,y]\!], \tag{3.16}$$

$$\mathsf{R}'(x,y)(v) := \nabla'_x\big(\nabla'_y(v)\big) - \nabla'_y\big(\nabla'_x(v)\big) - \nabla'_{[\![x,y]\!]}(v) \tag{3.17}$$

Dans le cadre de cette définition on peux par exemple faire un lien entre courbure et torsion et les applications $Z$ et $H$ comme montre le théorème suivant :

**Théorème 5.4.4 :** Soit $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ une algèbre de Lie-Rinehart sur $A$. Soit $\nabla$ une connection sur $L$ paramétré par $L$, et soit $V$ an $A$-module donné muni d'une connexion $\nabla'$ sur $L$. Ensuite, nous avons ce qui suit :

1. Il existe des récursions explicites simultanées en termes de courbure et de torsion pour les applications $Z$ et $H$ : pour tout $v \in V$, $x \in L$, et $\xi \in \mathcal{P}_\nabla(L,A)$ dont le $A$-module est identifie avec la $A$-algèbre de Lie libre sur $L$, le bracket $[\ ,\ ]$ étant le crochet libre $A$-bilinéaire

$$Z(x) = x \quad \text{and} \quad H'_x = 0, \tag{3.18}$$

$$Z\big([x,\xi]\big) = (\nabla_x Z)(\xi) + H_\xi(x) - \mathsf{Tor}\big(x,Z(\xi)\big), \tag{3.19}$$

$$H'_{[x,\xi]}(v) = \big(\nabla_x H'\big)_\xi(v) + R'\big(x,Z(\xi)\big)(v). \tag{3.20}$$

2. En particulier, nous obtenons pour $x_1, x_2, x_3 \in L$ :

$$\begin{aligned} Z([x_1,x_2]) &= -\mathrm{Tor}(x_1,x_2), && (3.21)\\ Z\left(\left[x_1,[x_2,x_3]\right]\right) &= -\big(\nabla_{x_1}\mathrm{Tor}\big)(x_2,x_3) + \mathrm{Tor}\big(x_1,\mathrm{Tor}(x_2,x_3)\big) + \\ &+ R(x_2,x_3)(x_1), && (3.22)\\ H'_{[x_1,x_2]}(v) &= R'(x_1,x_2)(v), && (3.23)\\ H'_{\left[x_1,[x_2,x_3]\right]}(v) &= \big(\nabla'_{x_1}R'\big)(x_2,x_3)(v) - R'\big(x_1,\mathrm{Tor}(x_2,x_3)\big)(v). && (3.24) \end{aligned}$$

Finalement, on considére une algèbre de Lie-Rinehart $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ sur $A$. Soit $\nabla$ une connection en $L$ sur $L$. On rappelle la bialgèbre de Rinehart $\big(\mathsf{T}_A(L),\diamond = \diamond_\nabla, \mathbf{1}, \Delta_{sh}, \epsilon\big)$. Rappelons aussi l'algebroïde de Lie $\mathcal{P}_\nabla(L,A)$ (dont le sous-jacent $A$-module est l'algèbre le Lie libre $\mathcal{L}_A(L) =: \mathfrak{g}$), les applications $Z$ (voir équation (5.4.5)) et $H$ (voir équation (5.4.6)), et le noyau de $Z$, $\mathcal{P}^0_\nabla(L,A)$ (voir équation (5.4.7)). En outre, ce dernier est un idéal de Lie-Rinehart $\mathcal{P}_\nabla(L,A)$ dont nous avions montré qu'il était isomorphe au $A$-sousmodule

$$\mathfrak{h} := \mathcal{L}_A^{\geqslant 2}(L) = \bigoplus_{n=2}^{\infty} \mathcal{L}_A^n(L) \tag{3.25}$$

muni du crochet $A$-bilinéaire

$$\forall\ \zeta,\zeta' \in \mathfrak{h}: \quad [\zeta,\zeta']^{(\nabla)} = \big[\zeta - Z(\zeta), \zeta' - Z(\zeta')\big] + H_\zeta(\zeta') - H_{\zeta'}(\zeta), \tag{3.26}$$

(voir équation (5.4.9) et Théorème 5.4.4) *iv.*). Le crochet de Lie (6.2.2) peut être considéré comme une déformation du crochet de Lie libre restreint à $\mathfrak{h}$. On définit l'application bilinéaire suivante pour chaque $\zeta \in \mathfrak{h}$ et $b \in \mathsf{T}_A(L)$ :

$$\zeta \rhd b := \mathfrak{D}_\zeta(b) := \big(\zeta - Z(\zeta)\big) \diamond b = \zeta b - Z(\zeta)b + H_\zeta(b) \tag{3.27}$$

et le suivante $K$-submodule de $\mathsf{T}_A(L)$

$$\mathcal{J}_\nabla(L,A) := \mathfrak{h} \rhd \mathsf{T}_A(L) := \mathrm{Span}\{\zeta \rhd b \mid \zeta \in \mathfrak{h},\ b \in \mathsf{T}_A(L)\}. \tag{3.28}$$

Soit $\Upsilon : \mathsf{S}_A(L) \to \mathsf{T}_A(L)$ le morphisme de symétrisation usuel. On obtitein –a l'aide des considerations coalbebriques tres tecniques - la decomposition d'$A$-modules suivant (voir proposition 6.2.2).

$$\mathsf{T}_A(L) = \Upsilon\left(\mathsf{S}_A(L)\right) \oplus \mathcal{J}_\nabla(L,A).$$

On clésique par

$$P_\nabla : \mathsf{T}A_(L) \to \mathsf{S}_A(L)$$

projection $A$-lineaire canonique qui est l'inverse de $\Upsilon$ (restreite à $\Upsilon(\mathsf{S}_A(L))$) et dont le noyau est égal au coidéal $\mathcal{J}_\nabla(L,A)$. De plus, on définit

$$\mathcal{J}_0(L,A) = \text{ l'ideal et coidal de } \mathsf{T}_A(L) \text{ engendr par } [x,y] \tag{3.29}$$

$$P_0 : \mathsf{T}_A(L) \to \mathsf{T}_A(L) \tag{3.30}$$

la projection de bialgèbres qui envoie les mots noncommutatifs sur des mots commutatifs (dont le noyau vaut $\mathcal{J}_0(L,A)$).

Ensuite, soit $D_0 : \mathsf{T}A(L) \to \mathsf{T}A(L)$ tel que

$$D_0(\mathbf{1}) = 0,\ \ D_0(x) = 0,\ \ D_0(x_1 \cdots x_n) = \sum_{r=2}^{n} [x_1 \cdots x_{r-1}, x_r] x_{r+1} \cdots x_n, \tag{3.31}$$

codérivation de $\mathsf{T}_A(L)$ dont l'image est $\mathcal{J}_0(L,A)$, et soit

$$\tilde{D} : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$$

l'application $A$-linéaire qie s'annule sur $\Upsilon(\mathsf{S}_A(L))$ et vaut l'inverse de la restriction de $D_0$ à $\mathcal{J}_0(L,A)$. Il y a une formule explicite de $\tilde{D}$ (voir 6.2.31). En fin, on définit les deux applications $A$-linéaires $D_Z, D_H : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$ suivantes ($\forall b \in \mathsf{T}_A(L)$)

$$D_Z(b) \ := \ \sum_{(b)} \Big( Z\big((\pi_{\mathfrak{h}} \circ \tilde{e}_D)\big(b^{(1)}\big)\big)\Big) b^{(2)}, \tag{3.32}$$

$$D_H(b) \ := \ \sum_{(b)} H_{\tilde{e}_D(b^{(1)})}\big(b^{(2)}\big). \tag{3.33}$$

qui sont tout les deux des codérivations $A$-linéaires de $\mathsf{T}_A(L)$ on $\tilde{e}_D : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$ et l'idempotent de Dynkin modifie, i. e.

$$\tilde{e}_D(1) = 0, \tilde{e}_D(x) = x, \tilde{e}_D(x_1 \cdots x_n) = [x_1, [x_2, \cdots, [x_{n-1}, x_n]]]$$

Le résultat final de la deuxième partie de la thèse est le suivant :

**Theoreme 6.2.6 et 6.2.7 :** $\forall \gamma_1, \gamma_2 \in \mathsf{S}_A(L)$

1.
$$\mathcal{U}(A,L) \cong \mathsf{T}_A(L)/\mathsf{S}_A(L) \cong \mathcal{J}_\nabla(L,A)$$
(isomorphisme de $C^3$-coalgèbres sur $A$)

2.
$$P_\nabla := P_0 \circ \sum_{r=0}^{\infty} \big((-D_Z + D_H) \circ \tilde{D}\big)^{\circ r}$$
et le produit $K$-bilinéaire $\blacklozenge$ sur $\mathsf{S}_A(L)$ vaut

3.

$$\gamma_1 \blacklozenge \gamma_2 = P_\nabla \left( \sum_{(\gamma_1)} \Upsilon\left(\gamma_1^{(1)}\right) \nabla_{\Upsilon\left(\gamma_2^{(2)}\right)} \left(\Upsilon(\gamma_2)\right) \right)$$

# Introduction

This thesis is written in two parts that have a connection in the theory of deformation quantization. We prefer to introduce the main branches of this work also in two parts.

## Part I: Localization

The first part of this thesis is devoted to localization. In commutative algebra, it means a universal construction where a set of chosen elements in a given commutative ring is made invertible: the outcome is called a ring of fractions. The classical example is the well-known passage from the integers to the field of rational numbers. It is a very important tool in algebraic and analytical geometry. In differential geometry, however, localization is rather used in the analytic sense, i.e. the passage from globally defined smooth functions to those which are only defined on an open subset. It follows from the classical works by Whitney, Malgrange (see [33]) and Tougeron that these analytical localizations are often isomorphic to certain algebraic localizations in the smooth (or even $\mathcal{C}^k$, $k \in \mathbb{N}$) case.

Based on old work by ø.Ore in the 1930's localization can be transferred to noncommutative algebras: it turns out that there is a completely general construction which is in some situations not very practical: on the other hand if there is an additional condition on the set of potential denominators, the famous right (or left) Ore condition, the construction shares almost all properties of the commutative localization.

In this work we should like to study noncommutative localization of algebras arising in deformation quantization. In this theory, founded by [2] in 1978, formal associative deformations of the algebra of all smooth complex valued functions on a Poisson manifold, so-called star products, are studied aiming at an interpretation of the noncommutative multiplication of operators used in quantum mechanics. It is well-known that the first order commutator of such a deformation always gives rise to a Poisson bracket, but it is highly non-trivial to show that every Poisson bracket arises as a first order commutator of a deformation: this latter result is the famous Kontsevich formality Theorem, see [28].

We consider star products given by formal power series of bidifferential operators (as almost every-one): these multiplications immediately define star products of locally defined functions by suitable 'restrictions'.

We first show that this analytical localization is isomorphic to the commutative algebraic localization with respect to the set of all those formal power series of smooth functions whose zeroth order term does nowhere vanish on the given open set. As a by-product we have the result that this multiplicative set satisfies the right Ore condition.

In a similar way we can show that the set of all analytical germs of a star product algebra at a given point of the manifold is isomorphic the noncommutative localization of the complement of the maximal ideal of all those formal power series of functions whose term of order zero vanishes at the point.

We also sketch a more algebraic framework to compare the commutative localization of bidifferential operators giving rise to a deformation of the localized algebra and the noncommutative localization of the deformed algebra by a rather natural mutliplicative set: here the question 'Does localization commutes with deformation' arises.

The first part of the thesis is organized as follows: in the Chapter 1 we recall some basic concepts of the commutative algebra of smooth function algebras and (non)commutative localization following Tougeron's book [52] and Lam's very nice text-book [29]. Next, in the Chapter 2 we explore a bit Deformation Quantization, basically concerning Star products

In the following, in the Chapter 3 we show the first localization result concerning open sets and we prove a similar result for germs.

## Part II: Lie-Rinehart algebras and connections

The second part of this thesis cover several objects, notably Lie-Rinehart algebras as well as their universal enveloping algebras and connections. Of course, the choice of these subjects was not by chance.

Fist of all, there is the following problem in differential geometry: whereas the multiplication of differential operators on an open set of $\mathbb{R}^n$ is relatively easy to describe thanks to the existence of global co-ordinates, it is less evident for differential operators on a smooth manifold: of course, in every coordinate chart it looks like the easy multiplication in $\mathbb{R}^n$, but coordinate changes are quite involved, and very often the differential operators are formulated in terms of connections related for instance to a (pseudo)riemannian metric on the manifold.

On the other hand, since composition of differential operators again gives a differential operator, the interesting question is how this is reflected on global symbols. One would expect formulas involving complicated curvature and torsion terms, and we are convinced that it would be useful to get some precise description of that.

The problem is also related to deformation quantization, see e.g. [11], [5], [4] and [15] where each symbol gives rise to a fibrewise polynomial function on the

cotangent bundle of the underlying manifold, and this bundle is well-known to be a symplectic manifold. We have decided to choose another more algebraic approach towards a solution of the problem: it is well-known that the algebra of all differential operators (for simplicity we have restricted our attention to those acting on smooth functions on the manifold) is generated as an associative algebra with unit by the all the multiplications by smooth functions and the Lie derivatives along all the vector fields on the manifold. The latter carries the algebraic structure of a *Lie-Rinehart algebra*, see [42], [19] which had recently been studied extensively in particular in differential geometry.

A good algebraic object which shares all these features is the *universal enveloping algebra of a Lie-Rinehart algebra*, already defined by G.Rinehart [42] which is isomorphic to the differential operator algebra for the Lie-Rinehart algebra of all vector fields in differential geometry, and which is a kind of analog of the classical universal enveloping algebra of a Lie algebra, see [12].

As an additional feature we equip the Lie-Rinehart algebra with a connection in the form of an algebraic version of a covariant derivative $\nabla$ well-known in differential geometry and *study its iterations*. For conceptual reasons we investigate this for the more generalized *anchored modules* which do no longer carry Lie brackets: their importance had been recognized by M.Kapranov [23] for the development of his path algebroid.

In the Chapters 4 and 5 we discuss the basics topics and recall some results about Lie Rinehart algebras and covariant derivatives.

Next, in the Chapter 6 we have come to the following results:

1. For any anchored module $(L,\rho)$ with connection $\nabla$ in the $A$-module $L$ the free algebra $\mathsf{T}_A(L)$ is equipped with a simple 'smashed product-like' $K$-bilinear multiplication $\diamond$ deforming the $A$-bilinear free multiplication which is explicitly expressed in terms of the iterated covariant derivative and the $A$-linear shuffle comultiplication.
2. If $\mathbb{Q} \subset K$ (and hence $\mathbb{Q} \subset A$): the primitive part of $\mathsf{T}_A(L)$, $\mathcal{P}_\nabla(L,A)$ which is isomorphic to the free $A$-linear Lie algebra over the $A$-module $L$ carries a $K$-bilinear Lie bracket deforming the free bracket, and an anchor morphism such that it carries the structure of a Lie-Rinehart algebra over $A$: this Lie-Rinehart algebra is isomorphic to Kapranov's path algebroid. The Lie bracket is explicitly expressed in terms iterated covariant derivatives. The above algebra $\big(\mathsf{T}_A(L),\diamond\big)$ is isomorphic to the universal enveloping algebra of its primitive part.
3. As M.Kapranov has already remarked, there is a canonical Lie-Rinehart algebra morphism $Z$ from the primitive part $\mathcal{P}_\nabla(L,A)$ to $L$ induced by the identity map $L \to L$. We construct an explicit recursion of $Z$ in terms of torsion and curvature of the connection and their covariant derivatives. $\mathcal{P}_\nabla(L,A)$ decom-

poses into the direct sum of $L$ and $\mathcal{P}_\nabla(L,A)$, the kernel of $Z$. The restriction of the Lie-Rinehart bracket to this kernel is $A$-bilinear. It parametrizes the well-known infinitesimal affine holonomy transformations.

4. Finally, the 2-sided ideal generated by $\mathcal{P}^0_\nabla(L,A)$, $\mathcal{J}_\nabla(L,A)$, in $\big(\mathsf{T}_A(L),\diamond\big)$ can be described in a rather explicit and $A$-linear way, and the quotient algebra is isomorphic to the universal enveloping algebra of $L$. The key point here is the well-known and innocent fact that $\mathcal{J}_\nabla(L,A)$ is a coideal of the free algebra with respect to the shuffle comultiplication and 'deforms' the usual ideal which is the kernel of the symmetrization map $P_0 : \mathsf{T}_A(L) \to \mathsf{S}_A(L)$. The final multiplication formula for two symmetric symbols $\gamma, \gamma' \in \mathsf{S}_A(L)$ is of the following form

$$\gamma \blacklozenge \gamma' \quad := \quad = \left( P_0 \circ \sum_{r=0}^{\infty} \big( (-D_Z + D_H) \circ \tilde{D} \big)^{\circ r} \right) \left( \sum_{(\gamma)} \Upsilon\big(\gamma^{(1)}\big) \otimes_A \nabla_{\Upsilon(\gamma^{(2)})}\big(\Upsilon(\gamma')\big) \right).$$

see eqn (6.2.51). Here $\Upsilon : \mathsf{S}_A(L) \to \mathsf{T}_A(L)$ is the usual symmetrization embedding, Sweedler's notation refers to the shuffle comultiplication, $\nabla$ denotes the iterated covariant derivative, and the information on curvature and torsion is contained in the terms $D_Z$ and $D_H$ which are extensions of the above maps $Z$ and $H$ as 'left ordered' coderivations by means of the Dynkin idempotent, and $\tilde{D}$ is an explicit $A$-linear endomorphism containing precise combinatorial information in terms of convolutions. The geometric series in the formula recalls the features of homological perturbation theory although there is no grading with a 'true sign' or a 'differential'.

## Appendix

An important part of the thesis is the appendix where we collect important definitions, results and proofs that are needed for all the subjects of the main part of the thesis but for organization and didactic reasons were placed separated at the end of this work.

# Part I.

# Localization

# 1. Algebraic localization

Contents

The aim of this introductory chapter is to fix the notation that we will utilize along the text and introduce localization from a purely algebraic point of view. We basically divided this chapter into two parts, in the section 1.2 we will describe how localization can be defined for $K$-algebras and look for the main properties of the constructions. After that, in the section 1.3, we will treat localization in a general context, i. e. for $K$-algebras that are not necessarily commutative.

## 1.1 Preliminary

We can talk about localization in very basic context, for instance if we consider integral domains. The basic idea is to formally invert nonzero elements to obtain a quotient field. Let us recall some facts about it.

**Definition 1.1.1** (Integral domain) An integral domain $R$ is a commutative ring such that does not have zero divisors, that is, if $xy = 0$ then $x = 0$ or $y = 0$, for $x, y \in R$.

Let $R$ be a integral domain and let us consider the set $S = R\backslash\{0_R\}$. In this case lets consider the follow relation in the set $R \times S$ :

$$(r,s) \sim (r',s') \quad \text{if an only if} \quad rs' = sr' \tag{1.1.1}$$

where $r, r' \in R$ and $s, s' \in S$.

Notice that, obviously $(r,s) \sim (r,s)$ and if $(r,s) \sim (r',s')$ we obtain $(r',s') \sim (r,s)$, for all $r, r' \in R$ and $s, s' \in S$. Moreover, if $(r,s) \sim (r',s')$ and $(r',s') \sim (r'',s'')$ we have

$$\begin{aligned} rs' = sr' \quad &\overset{\times s''}{\Longrightarrow} \quad rs's'' = \overbrace{sr's''} \\ &\qquad\qquad\qquad\quad \Rightarrow \quad rs'' = sr'' \\ r's'' = s'r'' \quad &\underset{\times s}{\Longrightarrow} \quad \overbrace{r's''s} = s'r''s \end{aligned} \tag{1.1.2}$$

and then $(r,s) \sim (r'',s'')$. It shows that $\sim$ in fact define an equivalence relation. Hence we can consider the equivalence classes $\frac{r}{s} := \overline{(r,s)} = \{(r',s'); (r',s') \sim (r,s)\}$ in the quotient set $R_S := \dfrac{R \times S}{\sim}$.

**Definition 1.1.2** (Field of Fractions) The field of fractions of an integral domain is the smallest field in which it can be embedded.

Following, we observe that there is an application $\eta : R \to R_S$ which maps $r \longmapsto \frac{r}{1}$. In fact, this function transforms elements of $S$ in invertible elements in $R_S$. Notice that, we have $1_{R_S} = \frac{1_R}{1_R}, 0_{R_S} = \frac{0_R}{s}$ and $\left(\frac{s}{1}\right)^{-1} = \frac{1}{s}$.

In that first approach of localization we use the fact that in $R$ worth the cancellation law to prove the reflexivity propriety of $\sim$. Of course, all this facts are very easy to be checked and well known to all. Even so, we decided to describe it with the objective that in the next steps we can perceive the difference in the definition of the equivalence relation. We will see that this definition will become more and more general.

**Example 1.1.3** If we localize the ring of integers $\mathbb{Z}$ we obtain the field of rational numbers $\mathbb{Q}$.

The quotient $\mathbb{Z}_{\mathbb{Z}\setminus\{0\}}$ that we obtain is isomorphic to the well-know field of rational numbers.

**Example 1.1.4** If we take the ring of real polynomials in one variable $\mathbb{R}[x]$ we will obtain the set of rational polynomials $\mathbb{R}(x)$.

## 1.2 The commutative case

We will define a similar relation in a more general context. In this way, from now -if we do not specify- $K$ will denote a fixed commutative associative unital ring, such that $1 = 1_K$ and $0 = 0_K$. We will consider $0 \neq 1$ to avoid trivial cases.

**Definition 1.2.1** (Modules) Let $K$ be a commutative associative unital ring. A ***left* K-*module*** $M$ -or a left module over $K$- is an abelian group $(M, +)$ with an additional operation

$$\begin{array}{ccc} K \times M & \to & M \\ (k, x) & \longmapsto & kx \end{array}$$

such that satisfies the following properties for all $x, y \in M$ and $k, k'$ in $K$:

(a) $1x = 1$
(b) $(kk')x = k(k'x)$
(c) $(k + k')x = kx + k'x$
(d) $k(x + y) = kx + ky$

Of course, we can also define right modules defining $xk$ instead of $xk$ making analogous statements. We will use the word **module**, without specifying if its left or right module, when is the case of left modules.

**Definition 1.2.2** (Ideals) Let $M$ be a $K$-module. A subset $I \subset M$ is called a **left ideal** of $M$ if $(I, +)$ is an additive subgroup of $(M, +)$ and absorbs multiplication from the left by elements of $M$, i. e. $\forall m \in M$ and $x \in I$ we have $mx \in I$.

**Definition 1.2.3** (Algebras) A $K$-**Algebra** $A$ is a $K$-module equipped with an additional binary operation

$$\begin{array}{ccc} A \times A & \to & A \\ (x,y) & \longmapsto & x \cdot y \end{array}$$

such that

(a) $(x+y)\cdot z = x\cdot z + y\cdot z$
(b) $z\cdot(x+y) = z\cdot x + z\cdot y$
(c) $(k_1)x\cdot(k_2)y = (k_1k_2)(x\cdot y)$

for all $x,y,z \in A$ and $k_1,k_2 \in K$.

To simplify, we will write $xy$ instead of $x\cdot y$. All the $K$-algebras are supposed to be associative and unital. We shall include unital $K$-algebras isomorphic to $\{0\}$ for which $1=0$. In order to avoid exaggerated notations we shall not write $1_R$, $1_K$, $0_R$ or $0_K$, but simply 1 and 0 where the precise interpretation should be clear from the context.

For more definitions around those fundamental algebraic facts see [7] and [9].

### 1.2.1 Localization for commutative $K$-algebras

In this section we will consider commutative $K$-algebras. Of course, the previous case that we saw in the Section 1.1 is included in that one.

That way, the set that we want to formally invert will be more general. Let us consider the following.

**Definition 1.2.4** (Multiplicative subset) If $R$ is a $K$-algebra, a subset $S$ of $R$ is called **multiplicative subset** if $0 \notin S, 1 \in S$ and for all $s,s' \in S$ we have $ss' \in S$.

We can also say that $S$ is closed for multiplications.

**Remark 1.2.5** Notice that $S$ is non empty set because $1 \in S$. Note also that if we allow $0 \in S$ the multiplicative condition will always work for all the subsets of $R$. So those conditions avoid trivial cases.

**Example 1.2.6** Let $R$ be a $K$-algebra. Let us fix $x \in R$ an arbitrary nonzero element. The set $S = \{x^n; n \in \mathbb{N}\}$ is a multiplicative subset or $R$.

**Example 1.2.7** Let $R$ be a $K$-algebra. Let $\mathfrak{p}$ be an ideal of $R$. The set $S := R\backslash\mathfrak{p} \subset R$ is a multiplicative subset if and only if $\mathfrak{p}$ is prime.

Let $R$ be a commutative $K$-algebra and $S \subset R$ a multiplicative subset, then the following binary relation $\sim$ on $R \times S$ defined by

$$(r_1,s_1) \sim (r_2,s_2) \quad \text{if and only if} \quad \exists\, s \in S :\ r_1 s_2 s = r_2 s_1 s \tag{1.2.1}$$

is an equivalence relation.

The reflexivity and the symmetry of this relation are completely trivial to check. However, the transitivity is also not difficult but we decide to emphasizing to realize the difference from the integral domain case. For instance in the following equation (1.2.2) we can't cancel elements. More explicity, for $(r_1,s_1),(r_2,s_2)$ and $(r_3,s_3)$ in $R\times S$ such that $(r_1,s_1)\sim(r_2,s_2)$ and $(r_2,s_2)\sim(r_3,s_3)$ follows this two equations

$$\begin{array}{rcl} r_1s_2s = r_2s_1s & \overset{\times s_3s'}{\Longrightarrow} & r_1s_2s(s_3s') = r_2s_1s(s_3s') \\ r_2s_3s' = r_3s_2s' & \underset{\times s_1s}{\Longrightarrow} & r_2s_3s'(s_1s) = r_3s_2s'(s_1s) \end{array} \tag{1.2.2}$$

by which we obtain

$$r_1s_3(ss's_2) = r_3s_1(ss's_2) \tag{1.2.3}$$

It shows that $\exists s_0 := ss's_2 \in S$ such that $r_1s_3s_0 = r_3s_1s_0$ and then $(r_1,s_1)\sim(r_3,s_3)$. In fact, there is always elements that remain multiplying $r_1s_3$ and $r_3s_1$ in the equation (1.2.3). It justifies the change in the definition of this equivalence relation.

Finally, we can consider the quotient set given by

$$R_S := \frac{R\times S}{\sim}.$$

We will see that $R_S$ is $K$- algebra, called **quotient ring**, their elements are equivalence classes that we will denote by

$$\frac{r}{s} := \overline{(r,s)} = \left\{(r',s')\in(R,S);\exists s_0\in S, rs's_0 = r'ss_0\right\} \tag{1.2.4}$$

For this, it is sufficient shows that the addition and multiplication rules, defined in the natural way, precisely $\frac{r}{s}+\frac{r'}{s'} := \frac{r+r'}{s+s'}$ and $\frac{r}{s}\cdot\frac{r'}{s'} := \frac{rr'}{ss'}$, are well defined. Indeed, a very simple verification shows that this definitions independent of the chosen classes.

**Definition 1.2.8** ($K$- algebra morphism) If $A$ and $B$ are $K$-algebras a function $f:A\to B$ is $K$-algebra morphism if

(a) $f(kx) = kf(x)$
(b) $f(x+y) = f(x)+f(y)$
(c) $f(xy) = f(x)f(y)$

forall $k\in K$ and $x,y\in A$.

The property of making the elements of $S$ invertible can be translated in the following definition.

**Definition 1.2.9** If $R$ and $R'$ are $K$-algebras, a $K$-algebra morphism $\phi : R \to R'$ is called $S$-**inverting** if $\phi(S) \subset U(R')$, where $U(R')$ denote the group of invertible elements of $R'$.

In the above construction for each pair $(R, S)$, where $R$ is $K$-algebra and $S \subset R$ is a multiplicative subset, we obtained a $K$-algebra morphism, called ***numerator morphism***,

$$\eta_{(R,S)} = \eta : R \to R_S \quad \text{given by} \tag{1.2.5}$$
$$r \mapsto \frac{r}{1}$$

which in particular defines the $K$-algebra structure of $R_S$.

Actually, $\eta_{(R,S)}$ is $S$-inverting. For that it is sufficient to check that the image of elements in $S$ by $\eta$ have the form $\frac{s}{1}$ and $\frac{s}{1} \cdot \frac{1}{s} = 1_{R_S}$.

Finally, the construction described above for a $K$-algebra and a multiplicative subset can be resume in the following proposition.

**Proposition 1.2.10** *Let $R$ be a commutative $K$-algebra and $S \subset R$ be a multiplicative subset. Then the following is true:*

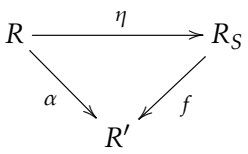
a. $\eta_{(R,S)}(S) \subset U(R_S)$, that is, the homomorphism $\eta_{(R,S)}$ sends elements of $S$ to invertible elements of $R_S$. Moreover, for any commutative unital $K$-algebra $R$ equipped with a multiplicative subset $S \subset R$, the pair $(R_S, \eta_{(R,S)})$ is **universal**. [1]

b. Every element of $R_S$ is written as a fraction $\eta(r)\eta(s)^{-1}$, for some $r \in R$ and $s \in S$.

c. $\ker(\eta_{(R,S)}) = \{r \in R \mid rs = 0 \text{ for some } s \in S\}$.

See We shall give a more categorical description in the section 1.3.2.

**Remark 1.2.11** In the Proposition (1.2.10) the **universality** of the pair $(R_S, \eta_{(R,S)})$ means the following. For any morphism of commutative unital $K$-algebras $\alpha : R \to R'$ mapping $S$ into the group of invertibles $U(R')$ uniquely factorizes, i.e. the following diagram commutes, were $f$ is a morphism of unital $K$-algebras determined by $\alpha$.




(1.2.6)

1. See remark (1.2.11).

In other words, for each $\eta_{(R,S)}$ and given an $S$-inverting morphism $\alpha : R \to R'$ exists a uniquely determined morphism of unital $K$-algebras $f : R_S \to R'$ such that $\alpha = f \circ \eta$.

### 1.2.2 Examples

**Example 1.2.12** Let us consider $R = \mathbb{Z}$ and $S$ the subset of all the powers of 10, precisely $S = \{10^k; k \in \mathbb{N}\}$.

Observe that $0 \notin S$ and $1 \in S$. It is also easy to see that $S$ is a multiplicative subset of $R$. In this case, it is clear that $R_S$ is isomorphic to the set of rational numbers that have a power of 10 is the denominator. Notice that $R_S \subsetneq \mathbb{Q}$ as for example $\frac{2}{3} \notin R_S$.

**Example 1.2.13** Let us set $R = \mathcal{C}^0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}; f \text{ is continuous}\}$ and fix an open set $\mathcal{U} \subset \mathbb{R}^n$. Now consider the subset $S = \{f \in R; f(x) \neq 0, \forall x \in \mathcal{U}\}$. Furthermore, we can consider the $K$-algebra $\mathcal{C}^0(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{R}; f \text{ is continuous}\}$. Then we can show that $R_S \cong \mathcal{C}^\infty(\mathcal{U})$.

*Proof.* In the sense of the Proposition 1.2.10, the $K$-algebra $R_S$ that we obtain, is given by the formal fractions $\frac{f}{g} = \{(f', g') \mid \exists h \in S; fg'h = hf'g\}$.

Let us define the following map considering equivalence classes in $R_S$:

$$\frac{f}{g} \overset{\psi}{\longmapsto} \left( x \overset{\phi}{\mapsto} \frac{f(x)}{g(x)}, x \in \mathcal{U} \right) \tag{1.2.7}$$

First of all, if $\frac{f'}{g'} = \frac{f}{g}$, there is $h \in S$ such that $f(x)g'(x)h(x) = f'(x)g(x)h(x)$ which implies $\frac{f'(x)}{g'(x)} = \frac{f(x)}{g(x)}, \forall x \in \mathcal{U}$. It shows that $\psi$ is independent of the chosen element in the class.

Injectivity: Consider $\frac{f}{g}$ in $R_S$ and suppose that $\phi(\frac{f}{g}) = 0$. It implies that $\frac{f(x)}{g(x)} = 0, \forall x \in \mathcal{U}$ and then we have $f(x) = 0$ in $\mathcal{U}$. But for $\frac{f}{g} = 0_{R_S} = \frac{0}{1}$ we should find $h \in S$ such that $fh = 0$.

In the appendix B we will describe a very important construction that works basically around construct a $\mathcal{C}^\infty$ function in $\mathbb{R}^n$, called *fonction aplatisseur* (in french), that is strictly positive in some open set and zero outside this set.

Take $\alpha$ as in the Lemma B.2.1, i. e. exists $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha(x) \in [0,1], \alpha(x) = 0$ for all $x \notin \mathcal{U}$ and $\alpha(x) > 0$ for all $x \in \mathcal{U}$.

It is clear that is the function $h$ that we want. Indeed $\alpha \in S$ and $f(x)\alpha(x) = 0$ in $\mathcal{U}$ because $f(x) = 0$ in $\mathcal{U}$ on the other hand $f(x)\alpha(x) = 0$ in $\mathbb{R}^n \backslash \mathcal{U}$ since that $\alpha(x) = 0$ for $x \notin \mathcal{U}$.

Surjectivity: Consider $\phi \in \mathcal{C}^0(\mathcal{U})$. According to the $\mathcal{C}^0$-version of the Lemma B.2.1 there is a continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha(x) = 0, x \notin \mathcal{U}$ and $\alpha(x) > 0, x \in \mathcal{U}$ such that the following function,

$$\overline{\phi}(x) = \begin{cases} \phi(x)\alpha(x) & \text{if } x \in \mathcal{U} \\ 0 & \text{if } x \notin \mathcal{U}. \end{cases} \tag{1.2.8}$$

is continuous in $\mathbb{R}^n$.

It follows that $\psi\left(\frac{\overline{\phi}}{\alpha}\right)(x) = \frac{\phi(x)\alpha(x)}{\alpha(x)} = \phi(x)$. It finishes the proof.

□

The example 1.2.13 is also related to the following references: [40] and [13].

## 1.3 The noncommutative case

The procedure that we explore in the previous chapter, for each commutative $K$-algebra and a multiplicative subset $S \subset R$, give us a universal $K$- algebra $R_S$ and a $K$- algebra morphism $\eta : R \to R_S$ with the following properties:

(1.2.10).(a) $\eta_{(R,S)}(S) \subset U(R_S)$ and the universality for the pair $(R_S, \eta_{(R,S)})$ is **universal** in the sense that any morphism of commutative unital $K$-algebras $\alpha : R \to R'$ mapping $S$ into the group of invertibles $U(R')$ uniquely factorizes, i.e. the diagram

$$\begin{array}{ccc} R & \xrightarrow{\eta} & R_S \\ & \searrow^{\alpha} & \downarrow f \\ & & R' \end{array} \tag{1.3.1}$$

commutes.

(1.2.10).(b) Every element in $R - S$ has the form $\eta(r)\eta(s)^{-1}$ where $r \in R$ and $s \in S$.

(1.2.10).(c) $\ker(\eta) = \{r \in R : rs = 0 \text{ for some } s \in S\}$ (an ideal in $R$)

In commutative algebra, localization provides one of the most powerful tools for proving theorems. Thus, in studying noncommutative rings, it is natural to ask how much of the localization machinery can be made to work in the noncommutative case.

### 1.3.1 Existence of localization

We shall begin with a quite easy and completely general statement. From here we will consider $K$- algebras not necessarily commutative.

**Proposition 1.3.1** *Let $R$ be a $K$-algebra and $S \subset R$ a multiplicative set as above. There is an $S$- inverting morphism from $\eta$ to some $K$- algebra, denoted $R_S$, with the following universal property: for any $S$- inverting morphism $f : R \to R'$ there is a unique $K$- algebra morphism $\overline{f} : R_S \to R'$ such that $f = \overline{f} \circ \eta$. In other words, the diagram below commutes where $R'$ is an arbitrary $K$- algebra.*

$$\begin{array}{ccc} R & \xrightarrow{\eta} & R_S \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & R' \end{array} \tag{1.3.2}$$

The proof of this fact can be found in [29, p. 289]. We will present this proposition in a different way. For the moment, let us consider the following example.

**Example 1.3.2** Consider $R = M_2(K)$ where $K$ is a nonzero ring and $e_{ij} \in R$ denote the matrix where in the position $ij$ is $1_K$ and in the other positions is $0_K$. Let S be the multiplicative set $\{Id, e_{11}\}$. Notice that the kernel of the map $\eta : R \to R_S$ is an ideal in $R$. In the other hand we can show that this ideal have the form $M_2(\mathcal{U})$ where $\mathcal{U}$ is an ideal in $K$. But $E_{22}E_{11} = 0$ that implies $E_{22} \in \ker(\eta)$ and it follow that $1 \in \mathcal{U} \Leftrightarrow \mathcal{U} = K$. Therefore, $\eta$ is the zero map and $R_S = (0)$.

This example, for instance, shows that we cannot predict the nature of $R_S$. In general, it is difficult to prove things about $R_S$, because the universal map $\eta : R \to R_S$ may no longer have the properties (1.2.10).(b) and (1.2.10).(c) and also there is no easy description for the kernel of $\eta$.

Another problem, that we will treat latter, concerns the elements in $R_S$ the are sums of words in $\eta(r)\eta(s)^{-1}$ like

$$\eta(r)\eta(s)^{-1}\eta(r') + \eta(s')^{-1}\eta(r'')\eta(s'')^{-1}$$

where $r, r', r'' \in R$ and $s, s's'' \in S$.

### 1.3.2 Categories and Localization

The objective of this section is to present another proposition, consequently another proof, that is more useful then the Proposition 1.3.1. For that propose, we will utilize some category language. Of course, for that, we will not go deeper in this beautiful and powerful theory. We use the classic book [32] about categories.

Let us consider this two categories with their respective objects and morphisms.

| Category : | $K$**AlgMS** | $K$**Alg** |
|---|---|---|
| Objects : | $(R,S)$ ; $S \subset R$ mult. subset | $R$ |
| Morphisms : | $\phi : (R,S) \to (R',S')$ ; $\phi(S) \subset S'$ | $\phi : R \to R'$ |

In this context, $K$**Alg** is the category of the unital $K$-algebras. The objects are associative unital $K$-algebras and the morphisms are naturally $K$-algebra morphisms. Moreover, let $K$**AlgMS** be the category witch the objects are all pairs $(R,S)$ of associative unital $K$-algebras $R$ with a muliplicative subset $S \subset R$ where the morphisms $(R,S) \to (R',S')$ are morphisms of unital $K$-algebras $R \to R'$ mapping $S$ into $S'$.

### The Localization Functor

First of all, since any morphism of unital $K$-algebras maps the group of invertible elements in the group of invertible elements there is an obvious functor

$$\mathcal{U} : K\mathbf{Alg} \to K\mathbf{AlgMS} \quad \text{given by}$$

$$R \xrightarrow{\mathcal{U}} \big(R, U(R)\big).$$

The localization that we already describe can be seen as a functor between the categories that we describe above. For *commutative $K$-algebras*, the Proposition (1.3.1, a.), gives rise to a functor

$$\mathcal{L} : K\mathbf{AlgMS} \to K\mathbf{Alg} \quad \text{given by}$$

$$(R,S) \xrightarrow{\mathcal{L}} R_S$$

where $R_S$ is the quotient algebra that we already know from the Chapter 2.

It is not hard to see that $\mathcal{L}$ is a *left adjoint* of the functor $\mathcal{U}$, see e.g. [32, p.79, Ch.IV] for definitions: the unit of the adjunction gives back the canonical numerator morphism $\eta$, and the counit is an isomorphism since localization w.r.t. the group of all invertible elements is isomorphic to the original algebra.

Finally, the following statement is another version of the Proposition 1.3.1.

**Proposition 1.3.3** *There is an adjunction of functors*

$$K\mathbf{AlgMS} \underset{\mathcal{U}}{\overset{\mathcal{L}}{\rightleftarrows}} K\mathbf{Alg}$$

*where $\mathcal{L}$ is the left adjoint to the above functor $\mathcal{U}$ such that each component $\eta_{(R,S)}$ of the unit $\eta : I_{K\mathbf{AlgMS}} \dot{\to} \mathcal{U}\mathcal{L}$ of the adjunction satisfies the universal property a. of the previous Proposition (1.3.1) in the general noncommutative case. We refer to $\mathcal{L}$ as a* ***localization functor****. For a given $(R,S)$ in $K$***AlgMS** *we denote by $R_S$ the $K$-algebra $\mathcal{L}(R,S)$ given by the functor $\mathcal{L}$, and by $\eta_{(R,S)} : R \to R_S$ the component of the unit of the adjunction.*

*Then $\eta_{(R,U(R))} : R \to R_{U(R)}$ is an isomorphism, the inverse being the component $\epsilon_R$ of the counit $\epsilon : \mathcal{L}\mathcal{U} \dot{\to} I_{K\mathbf{Alg}}$ of the adjunction. Moreover, every element of the $K$-algebra $R_S$ is a finite sum of products of the form ($\eta = \eta_{(R,S)}$)*

$$\eta(r_1)\big(\eta(s_1)\big)^{-1}\cdots\eta(r_N)\big(\eta(s_N)\big)^{-1} \tag{1.3.3}$$

*(which may be called 'multifractions') with $r_1,\ldots,r_N \in R$ and $s_1,\ldots,s_N \in S$ (note that $r_1$ or $s_N$ may be equal to the unit of $R$).*

*Proof.* The idea of the proof given in [29, Prop.(9.2), p.289] is as follows: there is a natural surjective morphism of unital $K$-algebras $\hat{\epsilon}_R$ from the free $K$-algebra generated by the $K$-module $R$, $T_K R$, to $R$ which provides us with a natural categorical presentation of $R$ 'by generators and relations': this morphism is given by the $R$-component of the counit $\hat{\epsilon}$ of the well-known adjunction

$$K\mathbf{Mod} \quad \underset{\mathcal{O}}{\overset{\mathcal{T}_K}{\rightleftarrows}} \quad K\mathbf{Alg}$$

where $\mathcal{O}$ is the forgetful functor and $\mathcal{T}_K$ the free algebra functor. Let $\kappa(R) \subset T_K R$ denote the kernel of $\hat{\epsilon}_R$. The next step is to add to the generating $K$-module $R$ the free $K$-module $KS$ with basis $S$, and to consider the two-sided ideal $\kappa(R,S)$ in the free algebra $T_K(R \oplus KS)$ generated by $\kappa(R)$ and by the subsets

$$\{(s,0)\otimes(0,s) - \mathbf{1}_T \mid s \in S\} \text{ and } \{(0,s)\otimes(s,0) - \mathbf{1}_T \mid s \in S\}$$

of $T_K\big(R \oplus KS\big)$ where the multiplication $\otimes$ and the unit $\mathbf{1}_T$ are taken in the free algebra $T_K\big(R \oplus KS\big)$. The localized algebra $\mathcal{L}(R,S) = R_S$ is then defined by

$$R_S = T_K\big(R \oplus KS\big)/\kappa(R,S),$$

and the 'numerator morphism' $\eta_{(R,S)} : R \to R_S$ is simply the canonical injection of $R$ into $T_K R \subset T_K\big(R \oplus KS\big)$ followed by the obvious projection. It follows that for every $s \in S$ its image $\eta_{(R,S)}(s)$ has an inverse by construction. The verification that this leads to a well-defined functor $\mathcal{L}$ which is a left adjoint to the functor $\mathcal{U}$ is lengthy, but straight-forward.

Otherwise, we transfer all the details of this proof for the Appendix C.1 □

In fact, the preceding construction shows that the functor $\mathcal{L}$ provides us with an abstract universal numerator map $\eta_{(R,S)}$ which is $S$-*inverting* in the sense that every $\eta_{(R,S)}(s)$, $s \in S$, is invertible in $R_S$ and a natural isomorphism $\epsilon_R$ of an algebra with its localization w.r.t. its group of units.

However, the construction by generators and relations renders the localized algebra $R_S$ quite implicit and not always computable.

Moreover, even for multiplicative subsets $S \subset R$ not containing 0 it may happen that the localized algebra $R_S$ is trivial as example 1.3.2 shows.

This can never happen in the commutative case since the equation $\frac{1}{0} = \frac{0}{0}$ is equivalent to the fact that $0 \in S$. This shows the lack of control over the kernel of the 'numerator morphism' $\eta_{(R,S)}$.

There is another reason that we will discuss in the following section.

### 1.3.3 Ore sets

The presentation of elements of $R_S$ in terms of sums of '*multifractions*' as equation (1.3.3) shows is quite clumsy, and on would prefer simple right or left fractions.

In order to motivate some conditions on $S$ in the following Definition we look at the multifractions which span the localized $K$-algebra $R_S$, see eqn (C.1.1): it may be desirable to transform a multifraction in a simple right fraction, and a partial step may consist in transforming a left fraction $\big(\eta(s)\big)^{-1}\eta(r)$ (with $r \in R$ and $s \in S$) directly into a right fraction $\eta(r')\big(\eta(s')\big)^{-1}$ (for some $r' \in R$ and $s' \in S$) which implies that every multifraction is equal to a right fraction by applying this step a finite number of times. This above condition implies the equation $\eta(rs') = \eta(sr')$ and thus motivates the stronger condition that for any pair $(r,s) \in R \times S$ there is a pair $(r',s') \in R \times S$ such that $rs' = sr'$, and this the well-known *right Ore conditions*.

**Definition 1.3.4** Let $R$ be an associative unital $K$-algebra, and $S \subset R$ be a multiplicative subset. A $K$-algebra $\check{R}_S$ equipped with a morphism of unital $K$-algebras $\check{\eta}_{(R,S)} = \check{\eta} : R \to \check{R}_S$ is said to be a **right $K$-algebra of fractions of** $(R,S)$ if the following conditions are satisfied:

a. $\check{\eta}_{(R,S)}$ is $S$-inverting,
b. Every element of $\check{R}_S$ is of the form $\check{\eta}(r)\big(\check{\eta}(s)\big)^{-1}$ for some $r \in R$ and $s \in S$;
c. $\ker(\check{\eta}) = \{r \in R \mid rs = 0, \text{for some } s \in S\} =: I_{(R,S)} =: I$.

**Definition 1.3.5** Let $R$ be an associative unital $K$-algebra, and $S \subset R$ be a multiplicative subset, $S$ is called a **right denominator set** if it satisfies the following two properties:

a. For all $r \in R$ and $s \in S$ we have $rS \cap sR \neq \varnothing$ ($S$ **right permutable** or **right Ore set**), i.e. there are $r' \in R$ and $s' \in S$ such that $rs' = sr'$.

b. For all $r \in R$ and for all $s' \in S$: if $s'r = 0$ then there is $s \in S$ such that $rs = 0$ ($S$ **right reversible**).

**Remark 1.3.6** In case that $R$ is commutative every multiplicative subset is a right denominator set of $R$.

**Example 1.3.7** The group of all invertible elements $\mathcal{U}(R)$ of any unital $K$-algebra is a right denominator set.

In fact, if $x \in R$ and $s \in \mathcal{U}(R)$ we have $s(s^{-1}r) = r$ with implies $sR \cap r\mathcal{U}(R) \neq \varnothing$. And also, if $xr = 0$ for $x \in \mathcal{U}(R)$ and $r \in R$ we have

$$xrx^{-1} = 0 \Rightarrow x^{-1}xrx^{-1} = 0 \Rightarrow rx^{-1} = 0.$$

The next Theorem shows that such a right algebra of fractions exists if and only if $S$ is a right denominator set. In other words, it shows that the definitions 1.3.4 and 1.3.5 are equivalents. See also [29, Thm (10.6), p.300]:

**Theorem 1.3.8** *Let $R$ be a unital $K$-algebra and $S \subset R$ be a multiplicative subset. Then the following is true:*

1. *The $K$-algebra $R$ has a right $K$-algebra of fractions $\check{R}_S$ with respect to the multiplicative subset $S$ if and only if $S$ is a right denominator set.*
2. *If this is the case each such pair $(\check{R}_S, \check{\eta})$ is universal in the sense of diagram (1.3.2) and each $\check{R}_S$ is isomorphic to the canonical localized algebra $R_S$ of Proposition 1.3.3.*
3. *Each $\check{R}_S$ is isomorphic to the quotient set $RS^{-1} := (R \times S)/\sim$ with respect to the following binary relation $\sim$ on $R \times S$*

$$(r_1, s_1) \sim (r_2, s_2) \;\Leftrightarrow\; \exists b_1, b_2 \in R \text{ such that}$$

$$s_1 b_1 = s_2 b_2 \in S \text{ and } r_1 b_1 = r_2 b_2 \in R \tag{1.3.4}$$

*which is an equivalence relation generalizing relation (1.2.1).*

*Proof.* The proof of this theorem is quite involved and can be found in [38, p.244, Thm. 25.3] [2] and in the appendix C.1. □

**Remark 1.3.9** Let us state some commentaries about the proof.

- In the part (1.) the verification of the implication "$(i.) \implies (ii.)$" in Definition 1.3.4 is straight-forward. The converse implication is much more involved: the traditional -and difficult- way, that was originally set up by Øystein Ore, [36], consists of a concrete construction of the $K$-algebra $RS^{-1}$ upon using the above

2. We are indebted to Alberto Eduque for having pointed out this reference.

relation described in (3.) equation (1.3.4), which reflects the idea of creating 'common denominators, and defining and verifying the canonical $K$-algebra structure (1.3.5) on the quotient set $R \times S/\sim$ by hand which is elementary, but extremely tedious. Moreover, even the fact that the above relation is transitive requires some work. We refer to Lam's book [29, p.300-302] for some of the details.

- There is a different, and more elaborate, way to prove the part (1.) and the rest of this Theorem. The proof is presented in [38, p.244, Thm. 25.3] and [29, p.302].
- Actually, it is instructive to look first at the equivalence relations created by an arbitrary $S$-inverting morphism of unital $K$-algebras $\alpha : R \to R'$, the classes being defined by the fibres of the map

$$p_\alpha : R \times S \to R' \text{ given by } p_\alpha(r,s) = \alpha(r)\big(\alpha(s)\big)^{-1},$$

  which is already very close to relation (1.3.4): thanks to the fact that the right fractions $\alpha(r)\big(\alpha(s)\big)^{-1}$ form a $K$-subalgebra of $R'$ (here the Ore axiom is needed) it creates an algebra structure on the quotient set isomorphic to the aforementioned subalgebra of $R'$ whence there is no need of tedious verifications of identities of algebraic structures.
- The central point then is to construct a unital $K$-algebra $R'$ and an $S$-inverting morphism $\alpha : R \to R'$ whose kernel is minimal, hence *equal to* $I_{(R,S)}$ which finally shows that the above algebra $RS^{-1}$ exists and does everything it should do. For this construction, the following trick is used: after 'regularizing' $R$ by passing to the factor algebra $\overline{R} = R/I_{(R,S)}$ (where the image multiplicative set $\overline{S}$ does no longer contain right or left divisors of zero) one looks at the endomorphism algebra of the injective hull $E$ of the right $\overline{R}$-module $\overline{R}$. Every left multiplication with elements of $\overline{R}$ can nonuniquely be extended to $E$, and the extensions of left multiplications with elements of $\overline{S}$ turn out to be invertible (here the Ore axiom is needed). $R'$ will then be given by the subalgebra generated by all extensions of left multiplications and the inverses of left multiplications with elements of $\overline{S}$ modulo the two-sided ideal of all $\overline{R}$-linear maps $E \to E$ vanishing on $\overline{R}$: this will resolve the ambiguity of extension, and $\overline{R}$ injects in $R'$, the injection being $\overline{S}$-inverting.

**Remark 1.3.10** Moreover, $RS^{-1}$ carries a canonical unital $K$-algebra structure, i.e. addition and multiplication on equivalence classes $r_1 s_1^{-1}$ and $r_2 s_2^{-1}$ (with $r_1, r_2 \in R$ and $s_1, s_2 \in S$) is given by

$$r_1 s_1^{-1} + r_2 s_2^{-1} = (r_1 c_1 + r_2 c_2) s^{-1}, \quad \text{and} \quad (r_1 s_1^{-1})(r_2 s_2^{-1}) = (r_1 r')(s_2 s')^{-1} \tag{1.3.5}$$

where we have written $s_1 c_1 = s_2 c_2 = s \in S$ (with $c_1 \in S$ and $c_2 \in R$) and $r_2 s' = s_1 r'$

(with $s' \in S$ and $r' \in R$) using the right Ore property. The numerator morphism $\eta_I : R \to RS^{-1}$ is given by $\eta_I(r) = r1^{-1}$ for all $r \in R$.

### 1.3.4 Examples

**Example 1.3.11** A *Noetherian ring* is a ring which every ascending chain of right ideals stabilizes. Precisely, for $I_1, I_2, \cdots, I_n, \cdots$ a sequence of right ideals which

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

there is an $r \in \mathbb{N}$ such that $I_r = I_{r+1} = \cdots$.

That being said, in any noncommutative domain (no nontrivial zero divisors) which is *right noetherian* the subset of nonzero elements is always a right denominator set. This fact is shown in [29, p.304, Cor. (10.23)] and [6, p.14, Beisp. 2.3 b)]).

Two particular cases of the example 1.3.11: Every universal enveloping algebra over a finite-dimensional Lie algebra (over a field of characteristic zero) and for the Weyl-algebra generated by $\mathbb{K}^n$.

**Example 1.3.12** For the free algebra $R = T_K V$ generated by a vector space $V$ of dimension $\geqslant 2$ (which is well-known to be isomorphic to the universal enveloping algebra of the free Lie algebra generated by $V$) the multiplicative subset of all nonzero elements is neither a right nor a left denominator set: for two linearly independent elements $v$ and $w$ in $V$ we clearly have $vR \cap wR = \{0\}$.

Hence the above statement about universal enveloping algebras does no longer apply to infinite-dimensional Lie algebras like the free Lie algebra generated by $V$.

**Example 1.3.13** The inverse images of right denominator subsets are in general no right denominator subsets as the example of the natural homomorphism $T_K V \to S_K V$ of the free to the free commutative algebra generated by $V$ shows: as $S_K V$ is a commutative domain, the subset $S = S_K V \backslash \{0\}$ is a right denominator set whereas its inverse image $T_K V \backslash \{0\}$ is not.

On the other hand, every homomorphic image of a right (or left) denominator set clearly is again a right (or left) denominator set. However, there may be subsets of right (or left) denominator sets which are no longer right (or left) denominator sets, as we shall see later.

# 2. Star products and localization

Contents

The aim of this chapter is to explore an important object in deformation quantization, called Star Products, that will give us a noncommutative structure in the algebra of the smooth functions defined in a manifold.

## 2.1 Preliminary

Since the seminal article by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in 1978, see [2], deformation quantization has become a large research area which cover several algebraic theories like the formal deformation theory of associative algebras and as well as geometric theories like the theory os symplectic and Poisson manifolds, and of physical theories like string theory and noncommutaive gauge theory.

In this case, the noncommutative associative multiplication of operators in quantum mechanics is considered as a formal associative deformation of the pointwise multiplication of the algebra of symbols of these operators. For Poisson manifolds, the work of Kontsevich [28] is quite important to guarantees the existence os some constructions in this way.

Nevertheless, we will not go deep in this beautiful theory, basically we will explore the deformed multiplication, the star product, that will be the symbol calculus of differential operators in terms of formal power series.

We had as basis for some concepts and definitions some excellent textbooks [30], [1], [14] and [55].

### 2.1.1 Multidifferential operators in $\mathbb{R}^n$

Let $M$ be an $n$-dimensional manifold. Let $(U, \phi = (x_1, \cdots, x_n))$ be a chart. Recalling that a multi-index $I = (i_1, \cdots, i_n)$ is an element of $\mathbb{N}^n$ with $|N| := i_1 + \cdots + i_n$ we can denote by

$$\partial_I := \frac{\partial^{i_1 + \cdots in}}{(\partial x_1)^{i_1} \cdots (\partial x_n)^{i_n}} \tag{2.1.1}$$

the usual abbreviation for iterated partial derivatives.

**Definition 2.1.1** A *differential operator* $D$ of order $N$ is a $\mathbb{K}$-linear map $D : \mathcal{C}^\infty(M, \mathbb{K}) \to \mathcal{C}^\infty(M, \mathbb{K})$ such that in each chart $(U, \phi = (x_1, \cdots, x_n))$ the operator takes the local form

$$D(f)|_U = \sum_{I \in \mathbb{N}^n; |I| \leqslant N} D^I \partial_I (f|_U) \tag{2.1.2}$$

where $f \in \mathcal{C}^\infty(M, \mathbb{K})$ and for each multi-index $I$ the function $D^I : U \to \mathbb{K}$.

More generally, a *multidifferential operator* of rank $r$, or a $r$-differential operator, is a $\mathbb{K}$-$r$-multilinear map $D : \mathcal{C}^\infty(M, \mathbb{K}) \times \cdots \mathcal{C}^\infty(M, \mathbb{K}) \to \mathcal{C}^\infty(M, \mathbb{K})$ such that there is

an integer $N$ such that in each chart $(U, \phi = (x_1, \cdots, x_n))$ the operator takes the local form

$$D(f_1, \cdots, f_r)|_U = \sum_{I_1, \cdots, I_r \in \mathbb{N}^n; |I_1|, \cdots |I_r| \leqslant N} D^{I_1, \cdots, I_r} \partial_{I_1}(f_1|_U) \cdots \partial_{I_r}(f_r|_r) \tag{2.1.3}$$

where $f_1, \cdots, f_r \in \mathcal{C}^\infty(M, \mathbb{K})$ and for each $r$-tuple of multi-indicis $(I_1, \cdots, I_r)$ the function $D^{I_1, \cdots, I_r} : U \to \mathbb{K}$ is $\mathcal{C}^\infty$.

### 2.1.2 Formal Power Series

We will start with some basic definitions about formal power series that latter will allows us to define Star Products. For more details and proofs see the book [45].

Let $K$ be a ring and $R$ be a $K$-algebra. First of all we can consider the $K$-algebra of the formal power series with coefficients in $K$, more precisely

$$K[[\lambda]] = \left\{ a = \sum_{i=0}^{\infty} \lambda^i a_i, a_i \in K, \forall i \right\}.$$

We shall write a map $a : \mathbb{N} \to K$ in the form of a formal power series with coefficients in $K$ to represent $a = \sum_{i=0}^{\infty} \lambda^i a_i$, where $a_i = a(i)$, $i \in \mathbb{N}$ is called the rth component of $a$ and the symbol $\lambda$ is called the formal parameter.

Similarly, we can consider the $K$-algebra $R[[\lambda]]$ of the formal power series with coefficients in $R$. These two $K$-algebras have structure of abelian groups. For instance, if $a, b \in K[[\lambda]]$ we have

$$a + b = \left( \sum_{i=0}^{\infty} \lambda^i a_i \right) + \left( \sum_{i=0}^{\infty} \lambda^i b_i \right) = \sum_{i=0}^{\infty} \lambda^i (a_i + b_i)$$

Furthermore, $R[[\lambda]]$ carries the $K[[\lambda]]$- module structure given by

$$\left( \sum_{i=0}^{\infty} \lambda^i \alpha_i \right) \left( \sum_{i=0}^{\infty} \lambda^i a_i \right) = \sum_{i=0}^{\infty} \lambda^i b_i$$

where $b_i = \sum_{k=0}^{i} \alpha_k b_{i-k} \in K, \alpha_i \in K$ and $b_i \in R, \forall i$.

An element of $a \in R[[\lambda]]$ can be written uniquely as $a = \sum_{i=0}^{\infty} a_i \lambda^i$ with $a_i \in R$, and for a given $a \in R[[\lambda]]$ and $i \in \mathbb{N}$ we shall always write $a_i \in R$ for the $i$th component of $a$ as a formal power series. We also note that for two $K$-algebras $R, R'$ we have $\mathrm{Hom}(R[[\lambda]], R'[[\lambda]]) \cong \mathrm{Hom}(R, R')[[\lambda]]$.

### 2.1.3 Formal Deformations of Associative Algebras

Let $(\mathcal{A}_0, \mu_0)$ be an associative algebra with unit over a commutative ring $K$.

**Definition 2.1.2** A formal associative deformation of the associative algebra $(\mathcal{A}_0, \mu_0)$ is given by a sequence of $K$-bilinear maps $\mu_1, \mu_2, \cdots : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ such that
(1)

$$\sum_{s=0}^{r} (\mu_s(\mu_{r-s}(a,b),c) - \mu_s(a,\mu_{r-s}(b,c))) = 0$$

for all $r \in \mathbb{N}$ and $a, b, c \in \mathcal{A}_0$.
(2) $\mu_r(1,a) = 0 = \mu_r(a,1)$ for all $r \leqslant 1$ and $a \in \mathcal{A}_0$.

**Proposition 2.1.3** *The space $\mathcal{A} = \mathcal{A}_0[[\lambda]]$ equipped with the $K[[\lambda]]$-bilinear multiplication $\mu := \sum_{r=0}^{\infty} \lambda^r \mu_r$, i. e.*

$$\mu(a,b) := \sum_{r=0}^{\infty} \lambda^r \sum_{s+t+u=0}^{r} \mu_s(a_t, b_u)$$

*for all $a = \sum_{t=0}^{\infty} \lambda^t\, a_t$ and $b = \sum_{u=0}^{\infty} \lambda^u\, b_u$ in $\mathcal{A}$, is an associative algebra over the algebra $K[[\lambda]]$.*

### 2.1.4 Star Products

The following definition, introduced in [2] by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer given us the notion of formal Star Products.

**Definition 2.1.4** (Star products) A (formal) star product $*$ on a manifold $X$ is a bilinear continous associative operation $\mathcal{C}^\infty(X)[[\lambda]] \times \mathcal{C}^\infty(X)[[\lambda]] \to \mathcal{C}^\infty(X)[[\lambda]]$ satisfying the following properties for all $f, g \in \mathcal{C}^\infty(X)$:
(i.) $1 * f = f * 1 = f$,
(ii.) $f * g = f \cdot g + \mathcal{O}(\lambda)$,
(iii.) $f * g = \sum_{k=0}^{\infty} C_k(f,g)\lambda^k$,
with bilinear operators $C_k : \mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(X) \to \mathscr{C}^\infty(X)$. We assume that all $C_k$ are bidifferential operators. It is called natural if every $C_k$ is a differential operator of order $k$.

**Example 2.1.5** The following well-known explicit star-product $*$ on $\mathbb{R}^2$ with coordinates $(x,p)$ will be used in the sequel:

$$f * g = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial x^k} \tag{2.1.4}$$

for any two functions $f, g \in \mathcal{C}^\infty(\mathbb{R}^2)$.

In the physics literature $\lambda$ corresponds to $(-\mathbf{i}\hbar)$. Moreover, for functions polynomial in the 'momenta' $p$ it is obvious that the above series converge, and for $\lambda = 1$ one obtains the usual formula for the symbol calculus of multiplication of differential operators on the real line (where partial derivatives are always brought to the right and replaced by the new variable $p$).

We mention the following facts although they are not necessary for the main subject of this paper:
We define the star commutator for $a, b \in \mathscr{C}^\infty(X)[[\lambda]]$ by $[a,b]_\star = a\star b - b\star a$. As usual, the star commutator satisfies the Leibniz-identity, i.e. $[a, b\star c]_\star = [a,b]_\star \star c + b\star[a,c]_\star$, and the Jacobi-identity and thus defines the structure of a non-commutative Poisson algebra. Also the adjoint action is a derivation of $\mathscr{C}^\infty(X)[[\lambda]]$ for all $a \in \mathscr{C}^\infty(X)[[\lambda]]$. From this it can easily be deduced that the first order term of a star product defines a Poisson bracket as follows

$$\{f,g\} = \frac{1}{2}(C_1(f,g) - C_1(g,f)) = \frac{1}{2\lambda}[f,g]|_{\lambda=0} \text{ for } f,g \in \mathscr{C}^\infty(X). \tag{2.1.5}$$

For $\mathscr{C}^\infty(X)$ it is well-known that every Poisson bracket comes from a unique *Poisson structure* $\pi$ which is a smooth bivector field $\pi$, i.e. a smooth section in $\Lambda^2 TX$ satisfying the identity $[\pi,\pi]_S = 0$ where $[\,,\,]_S$ denotes the Schouten bracket, see e.g. [55, p.84-87]: the relation is $\{f,g\} = \pi(df,dg)$. The very difficult converse problem whether the Poisson bracket associated to any given Poisson structure $\pi$ arises as the first order commutator of a star-product had been solved by M. Kontsevich, see [28].
We also note that two star products $\star$, $\star'$ are called *equivalent* if there exists a formal power series of differential operators $T = \mathrm{id} + \sum_{k=1}^\infty \lambda^k T_k$, with $T(1) = 1$ such that $T(f)\star T(g) = T(f\star' g)$ for all $f,g \in \mathscr{C}^\infty(X)[[\lambda]]$. The operator $T$ in the above definition is always invertible and indeed, given a star product $\star$, $f\star' g := T^{-1}(T(f)\star T(g))$ always gives a new equivalent star product. Two equivalent star products clearly give rise to the same Poisson bracket.

## 2.2 Noncommutative localization of smooth star-products on open subsets

The aim of this section is to relate localization with Star Products.

### 2.2.1 Analytic and algebraic localization

#### Analytic localization

Note that every star-product $*$ can be analytically localized to an associative star-product $*_U$ defined on $\mathcal{C}^\infty(U)[[\lambda]]$ by the localization of all the bidifferential operators $C_k$ to $C_k|_U$.

### Algebraic localization

Let $(X,\pi)$ be a Poisson manifold, let $* = \sum_{k=0}^{\infty} \lambda^k C_k$ be a star-product on $(X,\pi)$, and let $\Omega \subset X$ be a fixed open set. We set $K = \mathbb{K}[[\lambda]]$, and consider the $K$-algebra

$$R = \big(\mathcal{C}^\infty(X)[[\lambda]], *\big). \tag{2.2.1}$$

Moreover, since the star-product $*$ only involves bidifferential operators, it restricts to a star-product $*_\Omega$ on formal power-series $\phi \in R_\Omega := \mathcal{C}^\infty(\Omega,\mathbb{K})[[\lambda]]$ such that $\big(R_\Omega, *_\Omega\big)$ is also a $K$-algebra.

It follows that the restriction map $\eta_\Omega = \eta : R \to R_\Omega : f \mapsto f|_\Omega$ is a morphism of unital $K$-algebras. We define the following subset $S \subset R$:

$$S_\Omega := \{g \in R \mid \forall\, x \in \Omega :\ g_0(x) \neq 0\} \tag{2.2.2}$$

Clearly, the constant function 1 is in $S$, and for any $g,h \in S$ we have $(g * h)_0(x) = g_0(x)h_0(x) \neq 0$ (for all $x \in X$) whence $S$ *is a multiplicative subset of the unital $K$-algebra $R$.*

We can now consider the noncommutative localization of $R$ with respect to $S$ and compare it with the unital $K$-algebra $R_\Omega$

The main result from the frist part of this thesis is to anser the following **question:**

What is the relation between $R_{S_\Omega}$, the algebraic localization in the sense of the Chapter 1, and $R_\Omega$ analytic localization the we describe above. Are these algebras isomorphic?

## 2.3 Germs

Let $(X,\pi)$ again be a Poisson manifold, and let $* = \sum_{l=0}^{\infty} \lambda^l C_l$ be a bidifferential star-product. Let $K = \mathbb{K}[[\lambda]]$, and we denote the unital $K$-algebra $\big(\mathcal{C}^\infty(X,\mathbb{K})[[\lambda]], *\big)$ by $R$. For any open set $U \subset X$ let $R_U$ denote the unital $K$-algebra $\big(\mathcal{C}^\infty(U,\mathbb{K})[[\lambda]], *_U\big)$, where $*_U$ denotes the obvious action of the bidifferential operators in $*$ to the local functions in $\mathcal{C}^\infty(U,\mathbb{K})$. We write $R_X = R$. For any two open sets with $U \supset V$, denote by $\eta_V^U : R_U \to R_V$ be the restriction morphism where we write $\eta_U$ for $\eta_U^X$. Clearly, for $U \supset V \supset W$ one has the categorical identities $\eta_W^V \circ \eta_V^U = \eta_W^U$ and $\eta_U^U = \mathrm{id}_U$. Denoting by $\underline{X}$ the topology of $X$ it is readily checked that the family $\big(R_U\big)_{U\in\underline{X}}$ with the restriction morphisms $\eta_V^U$ defines a *sheaf of $K$-algebras over $X$*, see e.g. the book [**?** ] for definitions.

Let $x_0$ a fixed point in $X$, and let $\underline{X}_{x_0} \subset \underline{X}$ the set of all open sets containing $x_0$. We recall the definition of the *stalk at* $x_0$, $R_{x_0}$ of the sheaf $\big(R_U\big)_{U\in\underline{X}}$ whose elements are called *germs at* $x_0$: it is defined as the inductive limit (or colimit, see [**?** ]) $\lim_{U\in\underline{X}_{x_0}} R_U$. In order to perform computations we recall the more down-to-earth

definition: let $\tilde{R}_{x_0}$ be the disjoint union of all the $R_U$, i.e. the set of all pairs $(U,f)$ where $U$ is an open set containing $x_0$ and $f \in \mathcal{C}^\infty(U,\mathbb{K})[[\lambda]]$. Define an addition $+$ and a multiplication $*$ on these pairs by

$$(U,f)+(V,g) := \big(U\cap V, \eta^U_{U\cap V}(f)+\eta^V_{U\cap V}(g)\big) \quad \text{and}$$

$$(U,f)*(V,g) := \big(U\cap V, \eta^U_{U\cap V}(f)*_{U\cap V}\eta^V_{U\cap V}(g)\big),$$

and it is easily checked that the addition is associative and commutative, that the multiplication is associative, and that there is the distributive law. Furthermore, the sum of $(U,f)$ and $(V,0)$ equals $\big(U\cap V,\eta^U_{U\cap V}(f)\big)$ which is equal to $(U,f)*(V,1) = (V,1)*(U,f)$. Next the binary relation $\sim_{x_0}$ defined by

$$(U,f)\sim_{x_0}(V,g) \text{ iff } \exists\, W\in \underline{X}_{x_0} \text{ with } W\subset U\cap V:\ \eta^U_W(f)=\eta^V_W(g)$$

turns out to be an equivalence relation. Denoting by $R_{x_0}$ the quotient set $\tilde{R}_{x_0}/\sim_{x_0}$ and by $\eta^U_{x_0}: R_U \to R_{x_0}$ the restriction of the canonical projection $\tilde{R}_{x_0}\to R_{x_0}$ to $R_U\subset \tilde{R}_{x_0}$ (where $\eta^X_{x_0}$ will be shortened by $\eta_{x_0}: R\to R_{x_0}$) it is easy to see that the above addition and multiplication passes to the quotient, that all the zero elements $(U,0)$ are equivalent as are all the unit elements $(U,1)$, and that this defines the structure of a unital associative $K$-algebra denoted by $\big(R_{x_0},*_{x_0}\big)$ on the quotient set such that all maps $\eta^U_{x_0}:\big(R_U,*_U\big)\to\big(R_{x_0},*_{x_0}\big)$ are morphisms of unital $K$-algebras. Note the following equations for all open sets $U\supset V$:

$$\eta^V_{x_0}\circ\eta^U_V = \eta^U_{x_0}. \tag{2.3.1}$$

Define the following subset $S=S(x_0)$ and $I=I_{x_0}$ of $R$:

$$S=S(x_0)=\{g\in R\mid g_0(x_0)\neq 0\} \quad \text{and} \quad I=I_{x_0}=\{g\in R\mid g_0(x_0)=0\}. \tag{2.3.2}$$

It is easy to see that $S=R\backslash I$, that $S$ is a *multiplicative subset of* $R$, and that $I_{x_0}$ is a *maximal ideal of* $R$ (the quotient $R/I$ is isomorphic to the quotient $K/(\lambda K)\cong\mathbb{K}$ which is a field).

## 2.4 Commutative Localization of Star-Products

Let $A$ be a commutative associative unital $K$-algebra. We recall briefly the well-known algebraic definition of an (algebraic) multidifferential operator where we follow the book [55, p.566-578]:
We shall write unadorned tensor products $\otimes$ short for $\otimes_K$. A multidifferential operator $D$ of rank $p$ on $A$ is a $K$-linear map $D: A^{\otimes p}\to A$ satisfying certain properties:

We denote by $L_a : A \to A$, $L_a(b) = ab$ for $a, b \in A$ the left multiplication, and similarly for each integer $1 \leqslant i \leqslant p$ the map $L_a^i : A^{\otimes p} \to A^{\otimes p}, L_a^i(a_1 \otimes \cdots \otimes a_i \otimes a_p) = a_1 \otimes \cdots \otimes (aa_i) \otimes \cdots \otimes a_p$. Further we denote by $\mathsf{k} = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ a multi-index and by $e_i \in \mathbb{Z}^p$ the multi-index which is 1 in the $i$-th position and zero otherwise. We shall use the partial ordering $\prec$ on $\mathbb{Z}^p$ defined by $\mathsf{k} \prec \mathsf{l}$ iff for all $1 \leqslant i \leqslant p$ we have $k_i \leqslant l_i$.

**Definition 2.4.1** We define the left $A$-module of $p$-multidifferential operators

$$\mathrm{DiffOp}^{\mathsf{k}}(A, \ldots, A; A)$$

on $A$ of order $\mathsf{k} \in \mathbb{N}^p$ inductively by $\mathrm{DiffOp}^{\mathsf{k}}(A, \ldots, A; A) = \{0\}$ if there exists $k_i < 0$ and

$$\mathrm{DiffOp}^{\mathsf{k}}(A, \ldots, A; A) = \tag{2.4.1}$$

$$\left\{ D \in \mathrm{Hom}_K(A^{\otimes n}, A) \mid \forall a \in A \; \forall 1 \leqslant i \leqslant p : \; L_a \circ D - D \circ L_a^i \in \mathrm{DiffOp}^{\mathsf{k}-e_i}(A, \ldots, A; A) \right\}$$

for $\mathsf{k} \in \mathbb{N}^p$. Furthermore, we set $\mathrm{DiffOp}^{(p)}(A; A) = \bigcup_{\mathsf{k} \in \mathbb{Z}^p} \mathrm{DiffOp}^{\mathsf{k}}(A, \ldots, A; A)$.

Since clearly $\mathsf{k} \prec \mathsf{l}$ implies that $\mathrm{DiffOp}^{\mathsf{k}}(A, \ldots, A; A) \subset \mathrm{DiffOp}^{\mathsf{l}}(A, \ldots, A; A)$ there is the well-known result that each $A$-module of $p$-multidifferential operators is exhaustively filtered by the abelian group $\mathbb{Z}^p$.

Furthermore, for $A = \mathscr{C}^\infty(X)$ for a manifold $X$ this algebraic definition is well-known to coincide with the analytic definition, see e.g. [55, p. 575, Satz A.5.2.] which means that in local charts a (algebraically defined) differential operator looks as in equation (2.4.1).

Returning to general $A$ there is a well-known procedure to localize multidifferential operators:

**Proposition 2.4.2** *Let $S_0 \subset A$ be a multiplicative subset, let $A_{S_0}$ be the ordinary commutative localization of $A$ w.r.t. $S_0$, and let $\eta_{(A,S_0)} = \eta : A \to A_{S_0}$ be the numerator morphism. Let $D \in \mathrm{DiffOp}^{(p)}(A; A)$ a multidifferential operator of rank $p$.*

*Then there exists a unique multidifferential operator $D_S \in \mathrm{DiffOp}^{(p)}(A_{S_0}; A_{S_0})$ of rank $p$ such that $\eta \circ D = D_S \circ \eta^{\otimes p}$.*
*Furthermore, given another multidifferential operator $D' \in \mathrm{DiffOp}(A; A)$ we have $(D \circ_i D')_S = D_S \circ_i D'_S$ for each integer $1 \leqslant i \leqslant p$.*

*Proof.* This follows from the similar statement for differential operators, see e.g. [53, Prop.3.3]. The second part follows from the uniqueness of the localization. □

Observe now that the Definition 2.1.4 of star-products can be generalized to any commutative associative unital $K$-algebra $A$ whence the significance 'bidifferential'

for the $K$-bilinear maps $C_k : A \times A \to A$ is now given by the algebraic Definition 2.4.1.

**Proposition 2.4.3** *Given a differential star product $\star = \sum_{i=0}^{\infty} \lambda^i C_i$ with bidifferential operators $C_i$ on an algebra $R = A[[\lambda]]$ and a multiplicative set $S_0 \subset A$ there exists a unique star product $\star_{S_0}$ on $A_{S_0}[[\lambda]]$ such that the canonical $K$-linear map $A[[\lambda]] \to A_{S_0}[[\lambda]]$ induced by the numerator morphism $\eta : A \to A_{S_0}$ is a morphism of unital $K$-algebras.*

*Proof.* This follows from the previous proposition by considering the localization of the bidifferential operators $C_i$. It remains associative since the localization is compatible with composition. □

# 3. Results

Contents

In this chapter we want to present the main results of the localization part.

## 3.1 Non commutative localization for smooth Star Products on open subsets

This first theorem shows that localization for star products can be demonstrated looking for the properties of the Definition 1.3.4.

**Theorem 3.1.1** *Using the previously fixed notations we get for any open set $\Omega \subset X$:*

1. *$(R_\Omega, *_\Omega)$ together with the restriction morphism $\eta$ consitutes a right $K$-algebra of fractions for $(R, S)$.*
2. *As an immediate consequence we have that $S$ is a right denominator set.*
3. *This implies in particular that the algebraic localization $RS^{-1}$ of $R$ with respect to $S$ is isomorphic to the concrete localization $R_\Omega$ as unital $K$-algebras.*

*Proof.* **1.** We have to check properties $(i.a.)$, $(i.b.)$, and $(i.c.)$ of Definition 1.3.4:

• **[Property (i.a.)]** For this property we need to show that *"$\eta$ is $S$-inverting"*. Indeed, this is a classical reasoning from deformation quantization which we shall repeat for the convenience of the reader. Let $g \in S$ and $\gamma = \eta(g)$ its restriction to $\Omega$. Take $\psi \in R_\Omega$ and try to solve the equation $\gamma *_\Omega \psi = 1$. At order $k = 0$ we get the condition $\gamma_0\psi_0 = 1$, but since $\gamma_0(x) \neq 0$ for all $x \in \Omega$ the function $x \mapsto \psi_0(x) := \gamma_0(x)^{-1}$ is well-defined and smooth in $\mathcal{C}^\infty(\Omega, \mathbb{K})$.

Suppose by induction that the functions $\psi_0, \ldots, \psi_k \in \mathcal{C}^\infty(\Omega, \mathbb{K})$ have already been found in order to satisfy equation $\gamma *_\Omega \psi = 1$ up to order $k$. At order $k+1 \geqslant 1$ the condition reads

$$\begin{aligned} 0 = \left(\gamma *_\Omega \psi\right)_{k+1} &= \sum_{\substack{l,p,q=0 \\ l+p+q=k+1}}^{k+1} C_l(\gamma_p, \psi_q) \quad = \\ &= \gamma_0\psi_{k+1} + F_{k+1}(\psi_0, \ldots, \psi_k, \gamma_0, \ldots, \gamma_{k+1}) \end{aligned} \tag{3.1.1}$$

where the term starting with $F_{k+1}$ denotes the difference $\left(\gamma *_\Omega \psi\right)_{k+1} - \gamma_0\psi_{k+1}$ which obviously does not contain $\psi_{k+1}$.

Again, since $\gamma_0$ is nowhere zero on $\Omega$ the function $\psi_{k+1}$ can be computed from this equation by multiplying with $x \mapsto \gamma_0(x)^{-1}$. Hence there is a solution $\psi \in R_\Omega$ of equation $\gamma *_\Omega \psi = 1$. In a completely analogous way there is a solution $\psi' \in R_\Omega$ of the equation $\psi' *_\Omega \gamma = 1$. By associativity of $*_\Omega$ we get $\psi = \psi'$ as the unique inverse of $\gamma$ in the unital $K$-algebra $R_\Omega$.

• **[Property (i.b.)]** *"Every $\phi \in R_\Omega$ is equal to $\eta(f) *_\Omega \eta(g)^{*_\Omega -1}$ for some $f \in R$ and $g \in S$"*:

The main idea is to transfer the proof of Lemme 6.1 of Jean-Claude Tougerons's book to the non-commutative situation. Let $\phi = \sum_{i=0}^{\infty} \lambda^i \phi_i \in R_\Omega$. We then fix the following data which we get thanks to the fact that $X$ and therefore each open set $\Omega$ is a second countable locally compact topological space: there is a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ of $X$, a sequence of open sets $(W_n)_{n\in\mathbb{N}}$, and a sequence of smooth functions $(g_n)_{n\in\mathbb{N}} : X \to \mathbb{R}$ such that

$$\bigcup_{n\in\mathbb{N}} K_n = \Omega,$$

and

$$\forall\, n \in \mathbb{N} : K_n \subset W_n \subset \overline{W_n} \subset K_{n+1}^\circ \text{ and } g_n(x) = \begin{cases} 1 & \text{if } x \in W_n, \\ 0 & \text{if } x \notin K_{n+1}, \\ y \in [0,1] & \text{else.} \end{cases}.$$

We denote by $\gamma_j$ the restriction $\eta(g_j)$ of $g_j$ to $\Omega$ for each nonnegative integer $j$. The idea is to define the denominator function $g$ as a (non formal!) converging sum $g = \sum_{j=0}^{\infty} \epsilon_j g_j$. Choose a sequence $(\epsilon_j)_{j\in\mathbb{N}}$ of strictly positive real numbers such that

$$\forall\, j \in \mathbb{N} : \ \epsilon_j p_{K_{j+1},j}(g_j) < \frac{1}{2^j} \quad \text{and} \quad \forall\, i \leqslant j \in \mathbb{N} : \ \epsilon_j \sum_{l=0}^{i} p_{K_{j+1},j}\big(C_l(\phi_{i-l}, g_j)\big) < \frac{1}{2^j}$$

(For the definition of the seminorms $p_{K,m}$ see Appendix A.4) which is possible since for each nonnegative integer $j$ there are only finitely many seminorms and functions involved. For all nonnegative integers $i, j, N$ we define the functions $g_{(N)} \in \mathcal{C}^\infty(X, \mathbb{K})$, and $\psi_{ij}, \psi_{(i,N)} \in \mathcal{C}^\infty(\Omega, \mathbb{K})$:

$$g_{(N)} := \sum_{j=0}^{N} \epsilon_j g_j, \quad \psi_{ij} := \sum_{l=0}^{i} C_l\big(\phi_{i-l}, \gamma_j\big), \quad \psi_{(i,N)} := \sum_{j=0}^{N} \epsilon_j \psi_{ij} = \sum_{l=0}^{i} C_l\big(\phi_{i-l}, \gamma_{(N)}\big),$$

and since $\operatorname{supp}(g_j) \subset K_{j+1} \subset \Omega$, hence $\operatorname{supp}(g_{(N)}) \subset K_{N+1} \subset \Omega$, there are unique functions $f_{ij} \in \mathcal{C}^\infty(X, \mathbb{K})$ such that

$$f_{ij}(x) := \begin{cases} \psi_{ij}(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}, \quad \text{hence } \eta(f_{ij}) = \psi_{ij} \quad \text{and} \quad \operatorname{supp}(f_{ij}) \subset K_{j+1}.$$

For each nonnegative integer $N$ we set $f_{(i,N)} := \sum_{j=0}^{N} \epsilon_j f_{ij} \in \mathcal{C}^\infty(X, \mathbb{K})$ with $\operatorname{supp}(f_{(i,N)}) \subset K_{N+1}$. Clearly, $\eta(f_{(i,N)}) = \phi_{(i,N)}$.

We shall now prove that both sequences $(g_{(N)})_{N\in\mathbb{N}}$, and for each nonnegative integer $i$, $(f_{(i,N)})_{N\in\mathbb{N}}$ are Cauchy sequences in the complete metric space $\mathcal{C}^\infty(X,\mathbb{K})$. First, it is obvious that for any two compact subsets $K,K'$ and nonnegative integers $N,N'$ we always have for all $f\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$

$$\text{if } K\subset K' \text{ and } m\leqslant m' \text{ then } \ p_{K,m}(f)\leqslant p_{K',m'}(f). \tag{3.1.2}$$

Fix a nonnegative integer $i$. Let $\epsilon\in\mathbb{R}$, $\epsilon>0$, $K\subset X$ a compact subset, and $m\in\mathbb{N}$. Then there is a nonnegative integer $N_0$ such that

$$\frac{1}{2^{N_0}}<\epsilon,\quad m\leqslant N_0,\quad \text{and}\ \ i\leqslant N_0$$

Then for all nonnegative integers $N,p$ with $N\geqslant N_0$ we get (since for all $j\in\mathbb{N}$ such that $N+1\leqslant j$ we have $m\leqslant N_0\leqslant N\leqslant j$ and $i\leqslant N$, and $\operatorname{supp}(f_{i,j})\subset K_{j+1}^\circ\subset K_{j+1}$)

$$\begin{aligned}
p_{K,m}\big(f_{(i,N+p)}-f_{(i,N)}\big) &= p_{K,m}\left(\sum_{j=N+1}^{N+p}\epsilon_j f_{i,j}\right)\leqslant\sum_{j=N+1}^{N+p}\epsilon_j p_{K,m}\big(f_{i,j}\big)=\sum_{j=N+1}^{N+p}\epsilon_j p_{K\cap K_{j+1},m}\big(\psi_{ij}\big)\\
&\leqslant \sum_{j=N+1}^{N+p}\epsilon_j p_{K_{j+1},j}\left(\sum_{l=0}^{i}C_l(\phi_{i-l},g_j)\right)\leqslant\sum_{j=N+1}^{N+p}\epsilon_j\sum_{l=0}^{i}p_{K_{j+1},j}\left(C_l(\phi_{i-l},g_j)\right)\\
&< \sum_{j=N+1}^{N+p}\frac{1}{2^j}=\frac{1}{2^N}\left(1-\frac{1}{2^p}\right)<\frac{1}{2^N}\leqslant\frac{1}{2^{N_0}}<\epsilon.
\end{aligned}$$

It follows that for each $i\in\mathbb{N}$ the sequence $(f_{(i,N)})_{N\in\mathbb{N}}$ is a Cauchy sequence in the locally convex vector space $\mathcal{C}^\infty(X,\mathbb{K})$ hence converges to a smooth function $f_i=\sum_{j=0}^\infty\epsilon_j f_{i,j}$. Replacing in the above reasoning the function $\phi_0$ by the constant function 1 on $\Omega$ it follows that the sequence $(g_{(N)})_{N\in\mathbb{N}}$ converges to a smooth function $g:X\to\mathbb{R}$. Now let $x\in\Omega$. Then there is a nonnegative integer $j_0$ such that $x\in K_{j_0}$. It follows from the nonnegativity and the definition of all the $g_j$ and from the strict positivity of $\epsilon_j$ that

$$g(x)=\sum_{j=0}^{\infty}\epsilon_j g_j(x)\geqslant\epsilon_{j_0}g_{j_0}(x)=\epsilon_{j_0}>0$$

showing that $g$ takes strictly positive values on $\Omega$ whence $g\in S$.

Now let $x\notin\Omega$. Then for any $v\in T_xX$ with $h(v,v)\leqslant 1$ we have that

$$\forall\, m\in\mathbb{N}:\ \ (D^m g_{(N)})(v)=\sum_{j=0}^{N}\epsilon_j(D^m g_j)(v)=0$$

because each $g_j$ has compact support in $K_{j+1} \subset \Omega$. Since $g_{(N)} \to g$ for $N \to \infty$ it follows by the continuity of differential operators and evaluation functionals that $D^m g_{(N)}(v) \to D^m g(v)$, and hence

$$\forall\, x \in X\backslash\Omega,\ \forall\, m \in \mathbb{N},\ \forall\, v \in T_xX,\ h(v,v) \leqslant 1 :\ \ (D^m g)(v) = 0, \tag{3.1.3}$$

and in a completely analogous manner

$$\forall\, x \in X\backslash\Omega,\ \forall\, m \in \mathbb{N},\ \forall\, v \in T_xX,\ h(v,v) \leqslant 1 :\ \ (D^m f_i)(v) = 0.$$

Hence the infinite jets of all the functions $g$ and $f_i$, $i \in \mathbb{N}$, vanish outside the open subset $\Omega$. J.-C. Tougeron calls the function $g$ *fonction aplatisseur* for the family $(\phi_i)_{i\in\mathbb{N}}$ in case $C_l = 0$ for $l \geqslant 1$.

Now we get

$$\big(\phi *_U \eta(g_{(N)})\big)_i = \sum_{l=0}^{i} C_l\big(\phi_{i-l}, \eta(g_{(N)})\big) = \psi_{(i,N)} = \eta(f_{(i,N)}).$$

Since the restriction map $\eta : \mathcal{C}^\infty(X,\mathbb{K}) \to \mathcal{C}^\infty(\Omega,\mathbb{K})$ is continuous (where the Fréchet topology on $\mathcal{C}^\infty(\Omega,\mathbb{K})$ is induced by those seminorms $p_{K,m}$ where $K \subset \Omega$) as are the bidifferential operators $C_l$ we can pass to the limit $N \to \infty$ in the above equation and get

$$\phi *_\Omega \eta(g) = \sum_{i=0}^{\infty} \lambda^i \big(\phi *_\Omega \eta(g)\big)_i = \sum_{i=0}^{\infty} \lambda^i \eta(f_i) =: \eta(f).$$

Since $g \in S$ it follows that $\eta(g)$ is invertible in $R_\Omega$ by property $(i.a)$ of Definition 1.3.4, and the preceding equation implies $\phi = \eta(f) *_\Omega \eta(g)^{*_\Omega -1}$ thus proving property $(i.b)$ of Definition 1.3.4.

• **[Property (i.c.)]** "*The kernel of $\eta$ is equal to the space of functions $f \in R$ such that there is $g \in S$ with $f * g = 0$*":

Clearly if there is $f \in R$ and $g \in S$ such that $f * g = 0$ then $\eta(f) *_\Omega \eta(g) = 0$, and since $\eta(g)$ is invertible in $R_\Omega$ we have $\eta(f) = 0$.

Conversely, if $f \in R$ such that $\eta(f) = 0$, then for all integers $i \in \mathbb{N}$ and for all $x \in \Omega$ we have $f(x) = 0$. Hence the infinite jet of each $f_i$ vanishes at each point $x \in \Omega$ since $\Omega$ is open. Take the *fonction aplatisseur* $g \in S$ constructed in the preceding part of the proof for $\phi_0 = 1, \phi_i = 0$ for all $i \geqslant 1$. Then we get $\forall\, x \in X$ :

$$(f * g)_i(x) = \sum_{l=0}^{i} C_l(f_{i-l}, g)(x) =$$

$$= \begin{cases} 0 & \text{if } x \in \Omega \text{ since every jet of each} f_i \text{ vanishes in } \Omega, \\ 0 & \text{if } x \notin \Omega \text{ since every jet of } g \text{ vanishes outside of } \Omega, \end{cases}$$

where we have used equation (3.1.3) for the second alternative of the above statement. This proves part **1.** of the Theorem.

Statements **2.** and **3.** are immediate consequences of **1.** and Theorem 1.3.8. □

**Remark 3.1.2** For zero Poisson structure and trivial deformation $C_l = 0$ for all $l \geqslant 1$ the above result specializes to the classical result that algebraic and analytic localisation with respect to an open subset $\Omega \subset X$ are isomorphic for the commutative $\mathbb{K}$-algebra $\mathcal{C}^\infty(X, \mathbb{K})$.

Moreover, since for any closed set $F \subset X$ Tougeron's above construction gives us a smooth function $g : X \to \mathbb{R}$ which is nowhere zero on the open set $\Omega = X \backslash F$ and zero outside $\Omega$, hence on $F$, one gets the well-known result that the Zariski topology on $X$ induced by the commutative $\mathbb{K}$-algebra $\mathcal{C}^\infty(X, \mathbb{K})$ coincides with the usual manifold topology because each set $Z(I)$ is closed by continuity of all the functions in the ideal $I$, and conversely every closed set $F$ is the zero set $Z(gA)$ of the ideal $gA$ (where $A = \mathcal{C}^\infty(X, \mathbb{K})$).

## 3.2 Noncommutative germs for smooth star products

**Theorem 3.2.1** *Using the previously fixed notations we get for any point $x_0 \in X$:*

1. *$(R_{x_0}, *_{x_0})$ together with the morphism $\eta_{x_0} : R \to R_{x_0}$ consitutes a right $K$-algebra of fractions for $(R, S(x_0))$.*
2. *As an immediate consequence we have that $S(x_0)$ is a right denominator set.*
3. ***This implies in particular that the algebraic localization $RS^{-1}$ of $R$ with respect to $S = S(x_0)$ is isomorphic to the concrete stalk $R_{x_0}$ as unital $K$-algebras.***

*Proof.* **1.** Once again, we have to check properties $(i.a.)$, $(i.b.)$, and $(i.c.)$ of Definition 1.3.4:

- *"$\eta_{x_0}$ is S-inverting"* (property $(i.a.)$):

Indeed, let $g \in S(x_0)$. Since $g_0(x_0) \neq 0$ there is an open neighbourhood $U$ of $x_0$ such that $g_0(y) \neq 0$ for all $y \in U$. Hence the restriction $\eta_U(g)$ is invertible in $(R_U, *_U)$ by Theorem 3.1.1.

Using eqn (2.3.1) we see that $\eta_{x_0}(g) = \eta^U_{x_0}\big(\eta_U(g)\big)$, and the r.h.s. is invertible in $R_{x_0}$ as the image of an invertible element $\eta_U(g)$ in $R_U$ with respect to the morphism of unital $K$-algebras $\eta^U_{x_0}$.

- *"Every $\phi \in R_{x_0}$ is equal to $\eta_{x_0}(f) *_{x_0} \eta_{x_0}(g)^{*_{x_0}{}^{-1}}$ for some $f \in R$ and $g \in S(x_0)$"* (property $(i.b.)$):

Indeed, let $\phi \in R_{x_0}$. By definition of $R_{x_0}$ as a quotient set there is an open neighbourhood $U$ of $x_0$ and an element $\psi \in R_U$ with $\eta^U_{x_0}(U, \psi) = \phi$.

According to the preceding Theorem 3.1.1 there are elements $f, g \in R$ with $g_0(y) \neq 0$ for all $y \in U$ such that $\eta_U(f) = \psi *_U \eta_U(g)$. In particular, $g_0(x_0) \neq 0$, hence $g \in S(x_0)$. Applying $\eta^U_{x_0}$ to the preceding equation we get (upon using eqn (2.3.1))

$$\eta_{x_0}(f) = \eta^U_{x_0}\big(\eta_U(f)\big) = \Big(\eta^U_{x_0}(\psi)\Big) *_{x_0} \Big(\eta^U_{x_0}\big(\eta_U(g)\big)\Big) = \phi *_{x_0} \big(\eta_{x_0}(g)\big)$$

proving the result since $g \in S(x_0)$ and $\eta_{x_0}(g)$ is invertible in the unital $K$-algebra $(R_{x_0}, *_{x_0})$.

• *The kernel of $\eta_{x_0}$ is equal to the space of functions $f \in R$ such that there is $g \in S(x_0)$ with $f * g = 0$* (property $(i.c)$):

Indeed, given $f \in R$ with $\eta_{x_0}(f) = 0$ then there is an open neighbourhood $W$ of $x_0$ such that $\eta_W(f) = \eta_W(0) = 0$. By the preceding Theorem 3.1.1 there is an element $g \in S_W \subset S(x_0)$ (which can be chosen to be a *fonction aplatisseur*) such that $f * g = 0$. This proves **1.** of the Theorem.

**2.** and **3.** are immediate consequences of part **1.** and Theorem 1.3.8. □

# Part II.

# Lie-Rinehart algebras and connections

# 4. Preliminary

Contents

## 4.1 Lie-Rinehart algebras

In this Section $K$ is always a fixed commutative associative unital ring. All modules are considered over $K$, and the symbol $\otimes$ is short for $\otimes_K$. Moreover, let $A$ be a commutative associative unital $K$-algebra in the sense that $K \to A$ is a morphism of unital commutative associative rings.

### 4.1.1 Lie algebras

First let us recall the definition of Lie algebras.

**Definition 4.1.1** [Lie algebras] A Lie algebra over $K$ is an $K$-module $L$, together with a $K$-bilinear map

$$\begin{array}{ccc} L \times L & \to & L \\ (x,y) & \longmapsto & [x,y] \end{array}$$

called Lie bracket, satisfying the following properties:

(L1) $[x,x] = 0,\ \forall x \in L$;
(L2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,\ \forall x,y,z \in L$ (Jacobi identity).

We can denote a Lie algebra by $\big(L, [\ ,\ ]\big)$.

**Remark 4.1.2** The condition $[x,x] = 0\ \forall x \in L$, in the definition (4.1.1, L1), is equivalent to $[x,y] = -[y,x]$ for all $x,y \in L$. In fact, $0 = [x+y, x+y] = [x,x]+[x,y]+[y,x]+[y,y] = [x,y]+[y,x] \implies [x,y] = -[y,x]$.

**Example 4.1.3** (Derivations) Let $A$ be a $K$- algebra and let us consider the set

$$\mathrm{Der}_K(A) = \{f \in \mathrm{Hom}_K(A,A); f \text{ is derivation}\}.$$

To recall, a derivation $f : A \to A$ is a $K$- linear map such that $f(aa') = f(a)a' + af(a')$ for all $a, a' \in A$. Of course this is a subalgebra of the set the $K$-linear maps from $A$ to $A$, precisely $\mathrm{Hom}_K(A,A)$.

Then we can define the bracket $[f,g] = f \circ g - g \circ f$ for $f,g \in \mathrm{Hom}_K(A,A)$. Moreover, if we consider $f,g \in \mathrm{Der}_K(A)$ we obtain

$$\begin{aligned} [f,g](ab) &= f(g(ab)) - g(f(ab)) = \\ &= f[g(a)b + ag(b)] - g[f(a)b + af(b)] = \\ &= f(g(a))b + g(a)f(b) + f(a)g(b) + \\ &+ af(g(b)) - g(f(a))b - f(a)g(b) - g(a)f(b) - ag(f(b)) = \\ &= [f(g(a)) - g(f(a))]b + a[f(g(b)) - g(f(b))] = \\ &= [f,g](a)b + a[f,g](b). \end{aligned}$$

It shows that $[f,g]$ is a derivation. In other words $(\mathrm{Der}_K(A), [\ ,\ ])$ is a Lie algebra.

**Example 4.1.4** More general, if we consider an associative $K$-algebra $A$ and define the bracket $[x,y] = xy - yx$, for $x,y \in A$, called frequently the commutator, is not difficult to check that $(A,[\ ,\ ])$ is a Lie algebra.

The Jacobi identity follows from the associativity. Indeed, it follows that, for $x,y,z \in A$ we have:

$$\begin{aligned} [x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= [x,yz]-[x,zy]+[y,zx]-[y,xz]+[z,xy]-[z,yx] = \\ = xyz-yzx-xzy+zyx+yzx-zxy &- yxz+xzy+zxy-xyz-zyx+yxz = 0 \end{aligned}$$

### 4.1.2 Lie Rinehart algebras

We shall now recall the notion of a *Lie-Rinehart algebra over A* as defined in G.Rinehart's article, see [42]. Another rather good account of most of the material is J. Huebschmann's classical article [19].

**Definition 4.1.5** A **Lie-Rinehart algebra over** $A$, $(L,\rho,[\![\ ,\ ]\!],A)$ is defined by the following data:

1. $L$ is an left $A$-module.
2. $\rho : L \to \mathrm{Der}_K(A,A)$ is an $A$-linear map called the *anchor morphism*. We shall write $\rho(x)(a)$ in the more common way $\rho_x(a)$ for all $x \in L$ and $a \in A$.
3. $(L,[\![\ ,\ ]\!])$ is a $K$-Lie algebra.
4. $\rho : L \to \mathrm{Der}_K(A,A)$ is a morphism of $K$-Lie algebras, i.e.

$$\forall\ x,y \in L: \quad \rho_{[\![x,y]\!]} = \rho_x \circ \rho_y - \rho_y \circ \rho_x.$$

5. For all $a \in a$, $x,y \in L$:

$$[\![x,ay]\!] = \rho_x(a)y + a[\![x,y]\!].$$

**Definition 4.1.6** A map $\Phi : (L,\rho,[\![\ ,\ ]\!],A) \to (L',\rho',[\![\ ,\ ]\!]',A)$, considered over the same commutative algebra $A$, is called a *morphism of Lie Rinehart algebras*, if $\Phi : L \to L'$ is $A$-linear map, such that for all $x,y \in L$,

$$\Phi\big([\![x,y]\!]\big) = [\![\Phi(x),\Phi(y)]\!]' \quad \text{and} \quad \rho'_{\Phi(x)} = \rho_x. \tag{4.1.1}$$

One thus obtains a category $\mathbf{LieRin}_{A|K}$ whose objects consist of all the Lie-Rinehart algebras over $A$ and whose morphisms are defined as above in eqn (4.1.1).

**Definition 4.1.7** Recall that this definition can be weakened in two ways:

1. A *Lie derivation algebra over A* is defined by the first four conditions $i)$, $ii)$, $iii)$, and $iv)$ in Definition 4.1.5 above with the modification that $L$ carries no $A$-module structure, and that the anchor morphism $\rho$ is just $K$-linear.

2. More importantly, an *anchored A-module* is defined by just the first two conditions $i)$ and $ii)$ of Definition 4.1.5 above without specifying a Lie bracket. Defining morphisms in the appropriate way using eqn (4.1.1) one gets categories $\mathbf{LieDer}_{A|K}$ and $A\mathbf{ModAnc}_K$.

**Remark 4.1.8** This first case of the definition 4.1.7 occurs in differential geometry in the case of a Lie algebra action on a smooth manifod $X$.

**Example 4.1.9** We mention the following natural classes of examples of Lie-Rinehart algebras:

— Choosing $A = K$ we get $\mathrm{Der}_K(K,K) = \{0\}$, hence a Lie-Rinehart algebra in this case has zero anchor morphism and reduces to an ordinary Lie algebra over $K$.

— For arbitrary $A$ the $A$-module $L = \mathrm{Der}_K(A,A)$ always yields a Lie-Rinehart algebra with the identity map as anchor morphism: in differential geometry, given a smooth manifold $X$ with $A = \mathcal{C}^\infty(X,\mathbb{K})$, this example just describes the space of all vector fields on $X$.

**Definition 4.1.10** A *Lie-Rinehart ideal* $\mathfrak{i} \subset L$ of a given Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ is an ideal of the $K$-Lie algebra $\big(L,[\![\ ,\ ]\!]\big)$ which in addition an $A$-submodule on which the anchor morphism vanishes.

The following Lemma is elementary, but is quite important. The proof is entirely straight-forward.

**Lemma 4.1.11** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra.*

1. *Let $\mathfrak{i} \subset L$ be a Lie-Rinehart ideal. Then the restriction of the Lie-bracket $[\![\ ,\ ]\!]$ to $\mathfrak{i}$ is $A$-bilinear, and the quotient $A$-module $L/\mathfrak{i}$ carries a canonically induced Lie-Rinehart structure.*
2. *Let $\phi : \big(L,\rho,[\![\ ,\ ]\!],A\big) \to \big(L',\rho',[\![\ ,\ ]\!]',A\big)$ be a morphism of Lie-Rinehart algebras over $A$.*
   *Then the kernel*
   $$\mathfrak{h} := \mathrm{Ker}(\phi) \subset L$$
   *is a Lie-Rinehart ideal.*

**Remark 4.1.12** Note that this applies to the particular case

$$\rho : \big(L,\rho,[\![\ ,\ ]\!],A\big) \to \big(\mathrm{Der}_K(A,A),\mathrm{id},[\ ,\ ],A\big),$$

i.e. the anchor morphism is always a morphism of Lie-Rinehart algebras whence the kernel of the anchor morphism is an ideal of $L$ with $A$-linear Lie bracket.

Besides that, for later use we mention that by the induction functor $A\otimes$ (see Appendix A.2.1) every Lie derivation algebra over $A$, $\big(L,[\ ,\ ],\rho,A\big)$ can be promoted to a Lie-Rinehart algebra $(A\otimes L,{}^A\rho,{}^A[\![\ ,\ ]\!]_\rho,A)$ given us a functor

$$\mathbf{LieDerAlg}_{A|K}\longrightarrow\mathbf{LieRinAlg}_{A|K}$$

which we shall refer $A\odot L$ by setting

$${}^A\big[\![a\otimes x,a'\otimes x']\!\big]_\rho:=(a\rho_x(a'))\otimes x'-(a'\rho_{x'}(a))\otimes x+(aa')\otimes[x,x']$$

$$\text{and}\quad {}^A\rho_{a\otimes x}(a'):=a\big(\rho_x(a')\big),\tag{4.1.2}$$

and a morphism $\Phi$ of Lie derivation algebras will be mapped as usual to $\mathrm{id}_A\otimes\Phi$.

We mention that this defines a functor $L\to A\odot L$ (whose underlying $A$-module is the relatively free $A$-module $A\otimes L$) from $\mathbf{LieDer}_{A|K}$ to $\mathbf{LieRin}_{\mathbf{A|K}}$, and there is the obvious adjunction of functors

$$\mathbf{LieDerAlg}_{A|K}\ \underset{\mathsf{Res}}{\overset{A\odot}{\rightleftarrows}}\ \mathbf{LieRinAlg}_{A|K}$$

Note that for any Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ the counit of the adjunction is just the module multiplication map $\mu_L:A\odot L\to L$ given by equation (A.2.1, appendix A.2) which is a morphism of Lie-Rinehart algebra whence its kernel $\mathfrak{h}(L,A)$, see equation (A.2.3, appendix A.2), is a Lie-Rinehart ideal.

In fact, on the $A$-module $A\otimes L$ there is also the trivial $A$-bilinear bracket ${}^A[\ ,\ ]$ given by

$${}^A[a\otimes x,a'\otimes x']:=(aa')\otimes[x,x'].\tag{4.1.3}$$

## 4.2 Universal Enveloping Algebras

The central idea of the enveloping algebra construction is an universal process to produce an associative algebra with 'envelope' the original algebra and also have some analogous structure given by the map that give this 'inclusion'.

First, in case $A=K$ the categories of Lie-Rinehart algebras and ordinary Lie algebras are isomorphic.

### 4.2.1 Universal Enveloping Algebras of Lie algebras

Recall that for any ordinary $K$-Lie algebra $\big(\mathfrak{g},[\ ,\ ]\big)$ there is the notion of *Universal Enveloping Algebra*, $\mathsf{U}_K(\mathfrak{g})$.

There is the following problem of universals: given a Lie algebra $\big(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}}\big)$ over $K$, is there an associative unital $K$-algebra $\mathsf{U}_K(\mathfrak{g})$ equipped with a $K$-linear map $i_{\mathfrak{g}} = i : \mathfrak{g} \to \mathsf{U}_K(\mathfrak{g})$ satisfying

$$i\big([x,y]_{\mathfrak{g}}\big) = i(x)i(y) - i(y)i(x) \text{ for all } x,y \in \mathfrak{g} \tag{4.2.1}$$

such that for any unital associative $K$-algebra $B$ and any $K$-linear map $\theta : \mathfrak{g} \to B$ satisfying $\theta\big([x,y]_{\mathfrak{g}}\big) = \theta(x)\theta(y) - \theta(y)\theta(x)$ for all $x,y \in \mathfrak{g}$ there is a unique morphism of unital $K$-algebras $\bar{\theta} : \mathsf{U}_{\mathbf{K}}(\mathfrak{g}) \to B$ satisfying $\bar{\theta} \circ i = \theta$ ?

In other words, the $K$-linear map $i_{\mathfrak{g}}$ is such that given a $K$- linear map $\theta : \mathfrak{g} \to B$, with the same property as the equation 4.2.1, exists a $K$-linear map $\bar{\theta} : \mathsf{U}_{\mathbf{K}}(\mathfrak{g}) \to B$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i_{\mathfrak{g}}} & \mathsf{U}_{\mathbf{K}}\big(\mathfrak{g}\big) \\ & \searrow^{\theta} & \downarrow{\scriptstyle \bar{\theta}} \\ & & B \end{array} \tag{4.2.2}$$

The positive answer to this question can be rephrased in more categorical terms that the obvious *commutator functor* $(\ )^- : \mathbf{AssAlg_K} \to \mathbf{LieAlg_K}$ from all unital associative $\mathbf{K}$-algebras to all $\mathbf{K}$-Lie algebras has a left adjoint:

$$\mathbf{LieAlg_K} \underset{(\ )^-}{\overset{\mathsf{U}}{\rightleftarrows}} \mathbf{AssAlg_K} \tag{4.2.3}$$

Here to any associative algebra $B$ over $\mathbf{K}$ the Lie algebra $B^-$ is associated where the Lie bracket on the $\mathbf{K}$-module $B^- = B$ is just the commutator $[b,b'] := bb' - b'b$ for all $b,b' \in B$.

We shall recall the usual construction, see e.g. the books by H.Cartan and S.Eilenberg, [12, p.266-270], and by Bourbaki [8, Ch.I, p.22], for more details about this construction see the Appendix A.5.

### 4.2.2 Universal Enveloping Algebras of Lie-Rinehart algebras

For Lie-Rinehart algebras G.Rinehart formulated and solved the according universal problem in his thesis [42, p.197-198]: given a Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ over $A$, is there an associative unital $K$-algebra $\big(\mathcal{U}(L,A),\blacklozenge,\mathbf{1}\big)$ equipped with two $K$-linear maps $\iota_L : L \to \mathcal{U}(L,A)$ and $\iota_A : A \to \mathcal{U}(L,A)$ satisfying for all $x \in L$ and $a \in A$

$$\iota_L : L \to \mathcal{U}(L,A)^- \qquad \text{morphism of } K\text{−Lie algebras} \tag{4.2.4}$$

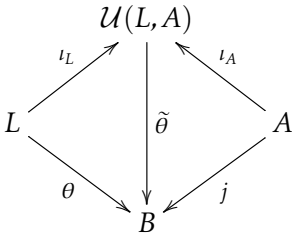
$$\iota_A : A \to \mathcal{U}(L,A) \qquad \text{morphism of unital } K\text{−algebras} \tag{4.2.5}$$

$$\iota_A(a) \blacklozenge \iota_L(x) = \iota_L(ax) \quad \text{and} \tag{4.2.6}$$

$$\iota_L(x) \blacklozenge \iota_A(a) - \iota_A(a) \blacklozenge \iota_L(x) = \iota_A\big(\rho_x(a)\big), \tag{4.2.7}$$

such that for any given associative unital $K$-algebra $B$ and any given $K$-linear maps $\theta : L \to B$ and $j : A \to B$ satisfying the conditions analogous to $\iota_L$ and $\iota_A$ there is a unique morphism $\tilde{\theta} : \mathcal{U}(L,A) \to B$ such that $\tilde{\theta} \circ \iota_L = \theta$ and $\tilde{\theta} \circ \iota_A = j$?

In othe words, the $K$-linear maps $\iota_A$ and $\iota_L$ are such that for any given associative unital $K$-algebra $B$ and any given $K$-linear maps $\theta : L \to B$ and $j : A \to B$ satisfying the conditions analogous to $\iota_L$ and $\iota_A$ exists $\widetilde{\theta} : \mathcal{U}(L,A) \to B$ such that both sides of the following diagram commutes?




(4.2.8)

### The comma category

We have already discussed the following categories

| Category | Objects | Morphisms |
|---|---|---|
| $\mathbf{AssAlg}_K$ | Associative unital $K$-algebras B | $\alpha : B \to B'$ s. t. $\alpha(1_B) = 1_{B'}$ |
| $\mathbf{LieAlg}_K$ | $K$-Lie Algebras $(L,[,])$ | $\phi : L \to L'$ $\phi([x,y]) = [\phi(x),\phi(y)]'$ |
| $\mathbf{LieRin}_{A\|K}$ | Lie-Rinehart Algebras $(L,\rho,[\![\ ,\ ]\!],A)$ | $\Phi : (L,\rho,[\![,]\!],A) \to (L',\rho',[\![,]\!]',A)$ s. t. $\Phi : L \to L'$ is $A$-linear map and for all $x,y \in L$, $\Phi\big([\![x,y]\!]\big) = [\![\Phi(x),\Phi(y)]\!]'$ and $\rho'_{\Phi(x)} = \rho_x$. |

The idea in the following is to see the universal construction as a left adjoint functor from $\mathbf{LieRin}_{A|K}$ to some analogue of the above category $\mathbf{AssAlg}_K$ of all associative unital $K$-algebras but which must incorporate the commutative algebra $A$.

In that sense, this category can be found in a quite simple way by taking $A \downarrow \mathbf{AssAlg}_K$ denoting the *category of all associative $K$-algebras over $A$*, see e.g. [50, p.88] or [51, p.459] which is nothing but the comma category of all associative $K$-algebras under $A$ in the sense of Mac Lane, cf. [32, p.45, eqn (1)]. This category can be described as following:

Objects: Pairs $(j,B,\diamond,\mathbf{1})$ where $(B,\diamond,\mathbf{1})$ is an associative unital $K$-algebra, and $j : A \to B$ is a (not necessarily injective) morphism of unital $K$-algebras.

Morphisms: $(j,B,\diamond,\mathbf{1}) \to (j',B',\diamond',\mathbf{1}')$ is a morphism $\Phi : B \to B'$ of unital algebras intertwining $j$ with $j'$ that means $\Phi \circ j = j'$.

**Remark 4.2.1** Note that in general the image $j(A)$ is a commutative subalgebra of $B$ which is in general NOT in the centre of $B$. An object $(j,B)$ in that category will be considered as a left $A$-module in the natural manner $ab := j(a)b$ for all $a \in A$ and $b \in B$.

Next, in order to get the analogue of the commutator functor we consider first two $K$-Lie algebras over $K$: the $K$-Lie algebra $\mathrm{Der}(A,A)$ of all derivations of $A \to A$ with respect to the commutator bracket of linear maps, and the $K$-Lie algebra $B^-$ of $B$ with the commutator bracket. In the direct sum $\mathrm{Der}(A,A) \times B^-$ (where the Lie bracket is just componentwise) consider the following submodule

$$B_A^- := \Big\{ (D,b) \in \mathrm{Der}(A,A) \times B^- \;\Big|\; \forall\, a \in A :\; j\big(D(a)\big) = \big[b, j(a)\big] \Big\}, \tag{4.2.9}$$

which is analogous to the construction of the algebraic version of infinitesimal gauge transformations in [19, p. 72]. Note that for $A = K$ this construction is isomorphic to the ordinary $B^-$, and in case $j$ is injective $B_A^-$ is isomorphic to the Lie normalizer of $j(A)$ in $B$.

**Proposition 4.2.2** *The association $B \to B_A^-$ is a covariant functor from the category $A \downarrow \mathbf{AssAlg}_K$ to the category $\mathbf{LieRinAlg}_{A|K}$ with respect to the following definitions where $(j,B)$ and $(j',B')$ are two associative unital $K$-algebras under $A$, $a \in A$, $D, D_1, D_2 \in \mathrm{Der}_K(A,A)$, $b, b_1, b_2 \in B$ such that $(D,b),(D_1,b_1),(D_2,b_2) \in B_A^-$, and $\Phi : B \to B'$ is a morphism in $A \downarrow \mathbf{AssAlg}_K$:*

$$\big[\!\big[(D_1,b_1),(D_2,b_2)\big]\!\big] := \big(D_1 \circ D_2 - D_2 \circ D_1, [b_1,b_2]\big), \tag{4.2.10}$$

$$\rho_{(D,b)}(a) := D(a), \tag{4.2.11}$$

$$a(D,b) := \big(aD, j(a)b\big), \tag{4.2.12}$$

$$\Phi_A^-\big(D,b\big) := \big(D, \Phi(b)\big). \tag{4.2.13}$$

*Proof.* Note that $B_A^-$ is easily checked to be a $K$-Lie subalgebra of $\mathrm{Der}(A,A) \times B^-$ which proves antisymmetry and the Jacobi identity of the bracket (4.2.10), as well as the fact that the map (4.2.11) is a morphism of Lie algebras since it is the restriction of the projection on the first factor to a subalgebra. The fact that the subagebra $B_A^-$ of $\mathrm{Der}(A,A) \times B^-$ is stable by left multiplication by $a \in A$ as indicated in (4.2.12) follows from the commutativity of $A$, and (4.2.13) is a straight-forward computation. □

### The construction of universal enveloping algebras of Lie-Rinehart algebras

The universal enveloping algebra of a Lie-Rinehart algebra will now be constructed in several steps for which Huebschmann's paper [19, p.63-66] has been a crucial in-

spiration. Let us fix a Lie-Rinehart algebra over $A$, $\big(L,\rho,[\![\ ,\ ]\!],A\big)$.

We shall not recall Rinehart's construction of $\mathcal{U}(L,A)$, but rather Huebschmann's construction (see [19]) because it comes much closer to the constructions we shall do later on.

**(1).** Consider the universal enveloping algebra $\mathsf{U}_K(L)$ of the $K$-Lie algebra $\big(L,[\![\ ,\ ]\!]\big)$. Recall that $\mathsf{U}_K(L)$ is an associative $K$-algebra with unit element $\mathbf{1}$, see Appendix A.5 and 4.2.1.

Moreover, $\mathsf{U}_K(L)$ carries the well-known structure of a *cocommutative Hopf algebra over* $K$, $\big(\mathsf{U}_K(L),\mu,\mathbf{1},\Delta,\epsilon,S\big)$ where $\Delta:\mathsf{U}_K(L)\to\mathsf{U}_K(L)\otimes_K\mathsf{U}_K(L)$ is the coassociative cocommutative comultiplication, $\epsilon:\mathsf{U}_K(L)\to K$ is the counit, and $S:\mathsf{U}_K(L)\to\mathsf{U}_K(L)$ the antipode.

We shall use Sweedler's notation $\Delta(u)=\sum_{(u)}u^{(1)}\otimes u^{(2)}$ for computations involving the comultiplication. Thanks to the universal property of $\mathsf{U}_K(L)$ the morphism $\rho:L\to\mathrm{Der}_K(A,A)^-$ of $K$-Lie algebras extends (via the unit of the adjunction $i_L:L\to\mathsf{U}_K(L)$) to a morphism of associative unital $K$-algebras, $\bar\rho:\mathsf{U}_K(L)\to\mathrm{Hom}_K(A,A)$. Since $L$ acts by derivations on $A$ there is the Leibniz rule (for all $u\in\mathsf{U}_K(L)$, $a,a'\in A$)

$$\bar\rho_u(aa')=\sum_{(u)}\bar\rho_{u^{(1)}}(a)\bar\rho_{u^{(2)}}(a'). \tag{4.2.14}$$

In other words, $A$ becomes a $\mathsf{U}_K(L)$-module algebra, see e.g. [49, p.153-154] for definitions.

**(2).** Consider the tensor product $A\otimes_K\mathsf{U}_K(L)=A\otimes\mathsf{U}_K(L)$. First there is the trivial factor-wise multiplication (for all $a,a'\in A$ and all $u,u'\in\mathsf{U}_K(L)$)

$$(a\otimes u)(a'\otimes u'):=(aa')\otimes(uu') \tag{4.2.15}$$

on $A\otimes\mathsf{U}_K(L)$ by which $A\otimes\mathsf{U}_K(L)$ is an $A$-module and an associative $A$-algebra. In a similar way, recall the trivial $A$-bilinear Lie bracket ${}^A[\![\ ,\ ]\!]$ (4.1.3) on $A\otimes L$. The universal property for both enveloping algebras $\mathsf{U}_K(L)$ of the $K$-Lie algebra $\big(L,[\![\ ,\ ]\!]\big)$ and $\mathsf{U}_A(A\otimes L)$ of the $A$-Lie algebra $\big(A\otimes L,{}^A[\![\ ,\ ]\!]\big)$ shows the existence of a canonical isomorphism of unital associative $A$-algebras:

$$A\otimes\mathsf{U}_K(L)\ \ \cong\ \ \mathsf{U}_A(A\otimes L). \tag{4.2.16}$$

The second more important multiplication $\diamond$ on $A\otimes\mathsf{U}_K(L)$ takes into account the anchor morphism $\rho$ and is given in Huebschmann's paper [19, p.63-66]: it is only $K$-bilinear and given by the the *smashed product* $A\odot\mathsf{U}_K(L)$ of $\mathsf{U}_K(L)$ with $A$ (see e.g. [24, p. 207, Example 3], where it is called crossed product): here the $K$-module $A\odot\mathsf{U}_K(L):=A\otimes\mathsf{U}_K(L)$ is equipped with the multiplication $\diamond$ (for all $a,a'\in A$ and $u,v\in\mathsf{U}_K(L)$)

$$(a\otimes u)\diamond(a'\otimes v)=\sum_{(u)}\Big(a\big(\bar\rho_{u^{(1)}}(a')\big)\Big)\otimes\big(u^{(2)}v\big) \tag{4.2.17}$$

which is associative and unital for $1 \otimes \mathbf{1}$ which is fairly easy to check upon using coassociativity of $\Delta$ and (4.2.14). Clearly, there are $A$-linear maps $\hat{\imath}_A : A \to A \odot \mathsf{U}_K(L)$ and $\hat{\imath}_{A\otimes L} : A \otimes L \to A \odot \mathsf{U}_K(L)$ given by (for all $a \in A$ and $x \in L$)

$$\hat{\imath}_A(a) := a \otimes_K \mathbf{1} \quad \text{and} \quad \hat{\imath}_{A\otimes L}(a \otimes x) = a \otimes i_L(x) \tag{4.2.18}$$

where $\hat{\imath}_A$ is a morphism of $K$-algebras satisfying

$$\hat{\imath}_A(a') \diamond \hat{\imath}_{A\otimes L}(a \otimes x) = \hat{\imath}_{A\otimes L}((a'a) \otimes x), \tag{4.2.19}$$

and we have for all $a, a' \in A$ and $x, x' \in L$:

$$\begin{aligned} \big[\hat{\imath}_{A\otimes L}(a \otimes x), \hat{\imath}_{A\otimes L}(a' \otimes x')\big]_\diamond \;&:= \\ = \hat{\imath}_{A\otimes L}(a \otimes x) \diamond \hat{\imath}_{A\otimes L}(a' \otimes x') \;&- \; \hat{\imath}_{A\otimes L}(a' \otimes x') \diamond \hat{\imath}_{A\otimes L}(a \otimes x) = \\ = \hat{\imath}_{A\otimes L}\big((a\rho_x(a')) \otimes x' \;&- \; (a'\rho_{x'}(a)) \otimes x + (aa') \otimes [x, x']\big) \end{aligned} \tag{4.2.20}$$

**(3).** The associative unital $K$-algebra $\big(A \odot \mathsf{U}_K(L), \diamond, 1 \otimes \mathbf{1}\big)$ will still be too big, and we need to divide out a two-sided ideal: first note that the $K$-bilinear Lie bracket ${}^A[\![\ ,\ ]\!]_\rho$ on $A \otimes L$ defined by eqn (4.1.2) occurs in the preceding equation (4.2.20) whence we can write

$$\big[\hat{\imath}_{A\otimes L}(a \otimes x), \hat{\imath}_{A\otimes L}(a' \otimes x')\big]_\diamond = \hat{\imath}_{A\otimes L}\big({}^A[\![a \otimes x, a' \otimes x']\!]_\rho\big). \tag{4.2.21}$$

Recall from Section 4.1.2 that the kernel of the module multiplication $\mu_L : A \otimes L \to L$, see equation (A.2.1), $\mathfrak{h}(L, A)$, is a Lie-Rinehart ideal. It is easy to see that the bracket ${}^A[\![\ ,\ ]\!]_\rho$ can be expressed as follows for all $\xi = \sum_{i=1}^M a_i \otimes x_i, \xi' = \sum_{j=1}^N a'_j \otimes x'_j \in A \otimes L$

$${}^A[\![\xi, \xi']\!]_\rho = \sum_{j=1}^N \big(\hat{\rho}_\xi(a'_j)\big) \otimes x'_j - \sum_{i=1}^M \big(\hat{\rho}_{\xi'}(a_i)\big) \otimes x_i + {}^A[\![\xi, \xi']\!], \tag{4.2.22}$$

whence the bracket ${}^A[\![\eta, \eta']\!]_\rho$ of two elements $\eta, \eta' \in \mathfrak{h}(L, A)$ reduces to the $A$-linear bracket (4.1.3),

$${}^A[\![\eta, \eta']\!]_\rho = {}^A[\![\eta, \eta']\!] \tag{4.2.23}$$

whence $\mathfrak{h}(L, A)$ is a $A$-Lie subalgebra (and in general NOT an ideal) of the $A$-Lie algebra $A \otimes L$ equipped with the simple $A$-bilinear bracket ${}^A[\![\ ,\ ]\!]$. Moreover, by the definition (4.2.17) of the multiplication $\diamond$ we get (since $\hat{\rho}_\eta = 0$ for all $\eta \in \mathfrak{h}(L, A)$) the following for all $\eta \in \mathfrak{h}(L, A)$, $a \in A$, and $u \in \mathsf{U}_K(L)$:

$$\hat{\imath}_{A\otimes L}(\eta) \diamond (a \otimes u) = \hat{\imath}_{A\otimes L}(\eta)(a \otimes u). \tag{4.2.24}$$

Note that this implies that

$$\mathsf{I}(L, A) := \hat{\imath}_{A\otimes L}\big(\mathfrak{h}(L, A)\big)\big(A \otimes \mathsf{U}_K(L)\big) = \hat{\imath}_{A\otimes L}\big(\mathfrak{h}(L, A)\big) \diamond \big(A \otimes \mathsf{U}_K(L)\big). \tag{4.2.25}$$

It is straight-forward to see that $\mathsf{I}(L,A)$ is an $A$-submodule of the $A$-module $A\otimes \mathsf{U}_K(L)$ and a two-sided ideal of the unital associative $K$-algebra $\big(A\odot\mathsf{U}_K(L),\diamond,1\otimes\mathbf{1}\big)$: indeed, eqn (4.2.25) implies at once that $\mathsf{I}(L,A)$ is a right ideal of $\big(A\odot\mathsf{U}_K(L),\diamond,1\otimes\mathbf{1}\big)$. By induction over the filtration degree of $u\in\mathsf{U}_K(L)$ we shall show that for each $a\in A$ and $\eta\in\mathfrak{h}(L,A)$ there is a positive integer $N$, elements $b'_1,\dots,b'_N\in A\odot\mathsf{U}_K(L)$ and elements $\eta'_1,\dots,\eta'_N\in\mathfrak{h}(L,A)$ such that

$$(a\otimes u)\diamond\hat{\imath}_{A\otimes L}(\eta)=\sum_{i=1}^{N}\hat{\imath}_{A\otimes L}(\eta'_i)\diamond b'_i,$$

which of course shows that $\mathsf{I}(L,A)$ is a left ideal. Indeed, for $u$ a $K$-multiple of $\mathbf{1}$ (degree 0) this is clear from eqn (4.2.19), and for $u=i_L(x)$, $x\in L$, (degree 1) this follows from eqs (4.2.20) and (4.2.21) and the fact that $\mathfrak{h}(L,A)$ is an ideal of the $K$-Lie algebra $\big(A\otimes L,{}^A[\![\ ,\ ]\!]_\rho\big)$. For the induction step $n\to n+1$ we may take $a\otimes(i_L(x)u)$ –where $A\in A$, $x\in L$, and $u\in\mathsf{U}_K(L)$ is of filtration degree $n$–upon noting that $a\otimes(i_L(x)u)=(1\otimes i_L(x))\diamond(a\otimes u)-(\rho_x(a)\otimes u)$ and using the induction hypothesis and the case $n=1$.

The universal enveloping algebra of the Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ is then defined by the quotient

$$\mathcal{U}(L,A):=\frac{A\odot\mathsf{U}_K(L)}{\mathsf{I}(L,A)}\tag{4.2.26}$$

which is an associative $K$-algebra with respect to the multiplication $\blacklozenge$ induced by $\diamond$: denoting by $\Pi_L:A\otimes\mathsf{U}_K(L)\to\mathcal{U}(L,A)$ the canonical projection we define for any $b,b'\in A\odot\mathsf{U}_K(L)$ and $a\in A$

$$\Pi_L(b)\blacklozenge\Pi_L(b'):=\Pi_L(b\diamond b')\quad\text{and}\quad\iota_A:A\to\mathcal{U}(L,A):a\mapsto\Pi_L\big(\hat{\imath}_A(a)\big).\tag{4.2.27}$$

**(4).** Finally, let $\phi:\big(L,\rho,[\![\ ,\ ]\!],A\big)\to\big(L',\rho',[\![\ ,\ ]\!]',A\big)$ a morphism of Lie-Rinehart algebras. Since $L\to\mathsf{U}_K(L)$ is a functor, the $A$-linear map $\mathrm{id}_A\otimes\mathsf{U}_K(\phi):A\otimes\mathsf{U}_K(L)\to A\otimes\mathsf{U}_K(L')$ is a morphism of $A$-algebras with respect to the $A$-bilinear multiplication (4.2.15) mapping the unit element $1_A\otimes_K\mathbf{1}$ to the unit element $1_A\otimes_K\mathbf{1}'$. Moreover, thanks to this fact and to the equation (A.2.3) it follows that $\mathrm{id}_A\otimes\mathsf{U}_K(\phi)$ maps the ideal $\mathsf{I}(L,A)$ into the ideal $\mathsf{I}(L',A)$ and hence passes to the quotients to induce a well-defined $A$-linear map $\mathcal{U}(\phi):\mathcal{U}(L,A)\to\mathcal{U}(L',A)$.

After this construction we can state the following.

**Proposition 4.2.3** *The rule associating to each Lie-Rinehart algebra $\big(L,[\![\ ,\ ]\!],\rho,A\big)$ over $A$ the pair $\big(\iota_A,\mathcal{U}(L,A)\big)$ and to each morphism $\phi:\big(L,[\![\ ,\ ]\!],\rho,A\big)\to\big(L',[\![\ ,\ ]\!]',\rho',A\big)$ of Lie-Rinehart algebras the $A$-linear map $\mathcal{U}(\phi):\mathcal{U}(L,A)\to\mathcal{U}(L',A)$ defines a covariant*

*functor* $\mathcal{U} : \mathbf{LieRinAlg}_{A|K} \to A \downarrow \mathbf{AssAlg}_K$.
*Moreover* $\mathcal{U}$ *is a left adjoint functor to the above commutator functor* $(\ )_A^-$*:*

$$\mathbf{LieRinAlg}_{A|K} \underset{(\ )_A^-}{\overset{\mathcal{U}}{\rightleftarrows}} A \downarrow \mathbf{AssAlg}_K$$

**Remark 4.2.4** The relation to the aforementioned universal property of $\mathcal{U}(L,A)$ is given as follows: the first morphism $\iota_A : A \to \mathcal{U}(L,A)$ of $K$-algebras had been part of the definition of the category $A \downarrow \mathbf{AssAlg}_K$ and was defined in the equation (4.2.27), and the morphism $\iota_L : L \to \big(\mathcal{U}(L,A), \blacklozenge\big)^-$ is defined by (for all $x \in L$)

$$\iota_L(x) = \Pi_L\big(\hat{\imath}_{A\otimes L}(1 \otimes x)\big) = \Pi_L\big(1 \otimes i_L(x)\big), \tag{4.2.28}$$

and it is easy to see using the reasoning after the equation (A.2.6) in the particular case $\Phi = \mathrm{id}_{\mathcal{U}(L,A)}$ in the proof of the preceding Proposition 4.2.3 that both $\iota_A$ and $\iota_L$ satisfy (4.2.6). It is immediate that $\iota_L$ is the second component of the unit of the adjunction.

Moreover, the associative unital $K$-algebra $\mathcal{U}(L,A)$ is generated by all the $\iota_A(a)$, $a \in A$, and all the $\iota_L(x)$, $x \in L$.

## 4.3 Rinehart bialgebras

In the category of associative unital $K$-algebras over $A$ there is no immediate monoidal or tensor structure: for the usual tensor product over $K$ there would in general not be a morphism of the reference algebra $A$ to the tensor product, and for the tensor product over $A$ there is no longer a multiplication. Actually, there are four possibilities to define a multiplications regarding the two possibilities in each component.

Thus, there is a useful intermediate solution going back at least to M.E.Sweedler [50] and M.Takeuchi [51], and has frequently been used since, see [46], [23], [34] and others.

Let $(j, B, \diamond, \mathbf{1})$ and $(j', B', \diamond', \mathbf{1}'))$ be associative unital $K$-algebras over $A$ and let us denote by

$${}_\bullet B \otimes_A {}_\bullet B'$$

the $A$-tensor product of the two left $A$-modules $B$ and $B'$ where

$$ab = j(a) \diamond b \ , \ \ ab' = j'(a) \diamond' b'$$

for all $a \in A, b \in B, b' \in B'$.

Note that the right multiplication by $j(a) \in A$ on the first factor, $r_a^{(1)}$, and the right multiplication by $j'(a) \in A$ on the second factor, $r_a^{(2)}$, of ${}_\bullet B \otimes_A {}_\bullet B'$ are well-defined left $A$-linear maps ${}_\bullet B \otimes_A {}_\bullet B' \to {}_\bullet B \otimes_A {}_\bullet B'$, hence define

$$B \times_A B' := \bigcap_{a \in A} \mathrm{Ker}\big(r_a^{(1)} - r_a^{(2)}\big) \subset {}_\bullet B \otimes_A {}_\bullet B'. \tag{4.3.1}$$

The $A$-submodule $B \times_A B'$ of the left $A$-module ${}_\bullet B \otimes_A {}_\bullet B'$ that we define above carries a natural associative unital multiplication $\diamond^\times$ induced by the multiplications on $B$ and on $B'$: express $B \times_A B'$ as a quotient of $K$-tensor products in the following way (see Appendix A.2.1 equations (A.2.6) and (A.2.7) for the notation):

$$B \times_A B' \cong \frac{\Big\{\beta \in B \otimes B' \mid \forall\, a \in A:\ \beta \diamond^{[2]} \big(j(a) \otimes \mathbf{1}' - \mathbf{1} \otimes j'(a)\big) \in \mathcal{K}\big({}_\bullet B, {}_\bullet B'\big)\Big\}}{\mathcal{K}\big({}_\bullet B, {}_\bullet B'\big)} \tag{4.3.2}$$

Observe that, the denominator is a two-sided ideal in the numerator which is a subalgebra of the associative unital $K$-algebra $\big(B \otimes B', \diamond^{[2]}, \mathbf{1} \otimes \mathbf{1}'\big)$.

Moreover the $K$-linear map $j \times_A j' : A \to B \times_A B'$ given by $a \mapsto j(a) \otimes_A \mathbf{1}' = \mathbf{1} \otimes_A j'(a)$ is a well-defined morphism of unital algebras. It follows that $\big(j \times_A j', B \times_A B', \diamond^\times, \mathbf{1} \otimes_A \mathbf{1}'\big)$ is again an associative unital $K$-algebra over $A$. This association can be extended to a bifunctor in $A \downarrow \mathbf{AssAlg}_K$: note that for two morphisms $\Phi : B_1 \to B_2$ and $\Psi : B_1' \to B_2'$ of unital associative $K$-algebras over $A$ one clearly has

$$(\Phi \otimes_A \Psi)\big(B_1 \times_A B_1'\big) \subset B_2 \times_A B_2', \tag{4.3.3}$$

and the restriction of $\Phi \otimes_A \Psi$ to $B_1 \times_A B_1'$, denoted by $\Phi \times_A \Psi$, thus gives a morphism of unital associative $K$-algebras over $A$. However, Takeuchi has remarked in [51] that this bifunctor is in general NOT a monoidal structure.

In the following, an associative unital $K$-algebra over $A$, $(j, B, \diamond, \mathbf{1})$, together with the structure of an $A$-linear coassociative cocommutative counitary coaugmented coalgebra $\big({}_\bullet B, \Delta, \epsilon, \mathbf{1}\big)$ is called a **Rinehart bialgebra** by Moerdijk and Mrčun [34] and [23] if an only if for all $b, b' \in B$ there is the following compatibility of the $A$-linear comultiplication $\Delta$ with the merely $K$-bilinear mutiplication $\diamond$:

$$\begin{gathered} \Delta(B) \subset B \times_A B \subset {}_\bullet B \otimes_A {}_\bullet B, \\ \Delta(b \diamond b') = \Delta(b) \diamond^\times \Delta(b'), \ \text{ and } \ \epsilon(b \diamond b') = \epsilon\big(b \diamond j(\epsilon(b'))\big) \end{gathered} \tag{4.3.4}$$

where the first inclusion makes the right hand side of the second identity well-defined.

The class of all Rinehart bialgebras, with respect to $A$ and $K$, constitutes the object class of a category $\mathbf{RinBiAlg}_{A|K}$ whose morphisms are morphisms of unital $K$-algebras over $A$ (in particular $A$-linear with respect to the left $A$-module structure) and in addition morphisms of counital coalgebras.

Observe that, for each such morphism $\Phi : B \to B'$ the $A$-linear map $\Phi \otimes \Phi$ maps the submodule $B \times_A B$ of ${}_\bullet B \otimes_A {}_\bullet B$ into the submodule $B' \times_A B'$ of ${}_\bullet B' \otimes_A {}_\bullet B'$. Note that the multiplication of a Rinehart bialgebra is $A$-linear in its left argument:

$$(ab) \diamond b' = \big(j(a) \diamond b\big) \diamond b' = j(a) \diamond \big(b \diamond b'\big) = a(b \diamond b'). \tag{4.3.5}$$

Furthermore, it is well-known (see e.g. [34]) that for any Lie-Rinehart algebra $\big(L, \rho, [\![\ ,\ ]\!], A\big)$ its universal enveloping algebra $\mathcal{U}(L,A)$ canonically carries the structure of an $A$-linear coassociative cocommutative counitary coaugmented coalgebra $(\Delta, \epsilon, \mathbf{1})$:

Indeed, it is not hard to see that the $A$-linear map $\delta : L \to {}_\bullet\mathcal{U}(L,A) \otimes_A {}_\bullet\mathcal{U}(L,A)$ given by $\delta(x) = \iota_L(x) \otimes_A \mathbf{1} + \mathbf{1} \otimes_A \iota_L(x)$ takes its values in the $A$-submodule ${}_\bullet\mathcal{U}(L,A) \times_A {}_\bullet\mathcal{U}(L,A)$, is a morphism of $K$-Lie algebras and satisfies the identities (4.2.6) and (4.2.7). By the universal property, the induced map $\Delta = \bar{\delta} : \mathcal{U}(L,A) \to {}_\bullet\mathcal{U}(L,A) \times_A {}_\bullet\mathcal{U}(L,A) \subset {}_\bullet\mathcal{U}(L,A) \otimes_A {}_\bullet\mathcal{U}(L,A)$ is a morphism of unital algebras over $A$ satisfying coassociativity. Defining $\epsilon : \mathcal{U}(L,A) \to A$ by $u \mapsto \bar{\rho}_u(\mathbf{1})$ the rest of the axioms is clear. Moreover each morphism $\mathcal{U}(\phi)$ of universal enveloping algebras is readily seen to preserve in addition the coalgebra structures whence the functor $\mathcal{U}$ corestricts to the subcategory $\mathbf{RinBiAlg}_{A|K}$ of $A \downarrow \mathbf{AssAlg}_K$.

Alternatively, using Huebschmann's construction in Section 4.2.2, one may take the $K$-linear coalgebra structure $\Delta_{\mathsf{U}_K(L)}$ and $\epsilon_{\mathsf{U}_K(L)}$ of the universal enveloping algebra of the $K$-Lie algebra $L$, extend this in a natural $A$-linear way to $A \otimes \mathsf{U}_K(L)$ and observe that the ideal $\mathsf{I}(A,L)$, compare eqn (4.2.25), is a coideal in the coalgebra $A \otimes \mathsf{U}_K(L)$ since $\hat{\iota}_{A \otimes L}\big(\mathfrak{h}(L,A)\big)$ is in the primitive part of $A \otimes \mathsf{U}_K(L)$ whence the comultiplication and the counit pass to the quotient.

Finally is easy to see that the primitive part of each Rinehart bialgebra is naturally equipped with the structure of a Lie-Rinehart algebra: the Lie bracket of two primitive elements is given by their commutator, and the anchor map by

$$x \mapsto (a \mapsto \epsilon(x \diamond j(a))).$$

Furthermore, Moerdijk and Mrčun [34] proved the analog of the *Cartier-Milnor-Moore Theorem*: viewing $\mathcal{U}$ as a functor from $\mathbf{LieRinAlg}_{A|K}$ to the category $\mathbf{RinBiAlg}_{A|K}$ it has a right adjoint, which is the passage to the primitive part. In the particular case where $K$ is a field containing the rationals they show that this adjunction restricts to an equivalence for $A$-projective Lie Rinehart algebras and complete graded projective Rinehart bialgebras, see [34, Thm 4.1, Cor 4.2].

Similarly to the notion of a Lie-Rinehart ideal as we already define in the Section 4.1.2 we can define the notion of a *Rinehart ideal* $\mathcal{J}$ of a Rinehart bialgebra $\big(j, B, \diamond, \mathbf{1}, \Delta, \epsilon\big)$ over $A$ as a *two-sided ideal* of the unital associative $K$-algebra $\big(B, \diamond, \mathbf{1}\big)$. In particular, it makes a left $A$-module via left multiplication with $j(a)$) and a *coideal* of the counital coaugmented $A$-coalgebra $\big(B, \Delta, \epsilon, \mathbf{1}\big)$. The following Lemma is straight-forward:

**Lemma 4.3.1** *Let $\big(j, B, \diamond, \mathbf{1}, \Delta, \epsilon\big)$ be a Rinehart bialgebra over $A$.*

1. *Let $\mathcal{J} \subset B$ be a Rinehart ideal. Then the quotient $A$-module $B/\mathcal{J}$ carries a canonical structure of a Rinehart bialgebra over $A$ such that the canonical projection $B \to B/\mathcal{J}$ is a morphism of Rinehart bialgebras.*
2. *Let $\big(j', B', \diamond', \mathbf{1}', \Delta', \epsilon'\big)$ be another Rinehart bialgebra over $A$, and $\Phi : B \to B'$ a **surjective** morphism of Rinehart bialgebras. Then the kernel of $\Phi$ is a Rinehart ideal of $B$.*

The following Proposition relates Lie-Rinehart ideals and Rinehart ideals and will be important in the sequel for the description of universal enveloping algebras:

**Proposition 4.3.2** *Let $\big(L, \rho, [\![\ ,\ ]\!], A\big)$ be a Lie-Rinehart algebra over $A$, let $\mathfrak{i} \subset L$ be a Lie-Rinehart ideal. Let $\mathcal{J}_\mathfrak{i} \subset \mathcal{U}(L,A)$ denote the following $A$-submodule of the universal enveloping algebra $\mathcal{U}(L,A)$:*

$$\mathcal{J}_\mathfrak{i} := \mathrm{Span}\big\{\iota_L(\eta) \diamond u \in \mathcal{U}(L,A) \mid \eta \in \mathfrak{i}, u \in \mathcal{U}(L,A)\big\}. \tag{4.3.6}$$

1. *Then $\mathcal{J}_\mathfrak{i}$ is a Rinehart ideal of $\big(\iota_A, \mathcal{U}(L,A), \diamond, \mathbf{1}, \Delta, \epsilon\big)$.*
2. *Let $\pi : L \to L/\mathfrak{i}$ be the canonical morphism of Lie-Rinehart algebras. Suppose that the $A$-submodule $\mathfrak{i}$ has a complementary $A$-submodule $\mathfrak{m}$ in $L$, i.e. $L = \mathfrak{i} \oplus \mathfrak{m}$. Then the induced morphism $\mathcal{U}(\pi) : \mathcal{U}(L,A) \to \mathcal{U}\big(L/\mathfrak{i}, A\big)$ descends to an isomorphism of Rinehart bialgebras over $A$:*

$$\mathcal{U}(L,A)/\mathcal{J}_\mathfrak{i} \ \cong \mathcal{U}\big(L/\mathfrak{i}, A\big). \tag{4.3.7}$$

*Proof. i.)* First note that for all $x \in L$, $\eta \in \mathfrak{i}$, $a \in A$, and $u \in \mathcal{U}(L,A)$ we have

$$\iota_L(\eta) \diamond (au) = \iota_L(\eta) \diamond \iota_A(a) \diamond u \overset{(4.2.7)}{=} \iota_A(a) \diamond \iota_L(\eta) \diamond u + \iota_A\big(\rho_\eta(a)\big) \diamond u = a\big(\iota_L(\eta) \diamond u\big) + 0 \tag{4.3.8}$$

since $\mathfrak{i}$ is a Lie-Rinehart ideal, and all the maps $L^\diamond_\eta : u \mapsto \iota_L(\eta) \diamond u$ are $A$-linear. Next thanks to the fact that all the elements $\iota_L(x)$, $x \in L$ are primitive elements of the coalgebra $\big(\mathcal{U}(L,A), \Delta, \epsilon\big)$ and to the compatibility of $\diamond$ with the comutiplication, see eqn (4.3.4), we can infer that each map $L^\diamond_\eta$ with $\eta \in \mathfrak{i}$ is an $A$-linear coderivation whence $\mathcal{J}_\mathfrak{i}$ is a coideal as a sum of images of coderivations. Moreover

$$\epsilon\big(\iota_L(\eta) \diamond u\big) \overset{(4.3.4)}{=} \epsilon\Big(\iota_L(\eta) \diamond \iota_A\big(\epsilon(u)\big)\Big) \overset{(4.3.8)}{=} \big(\epsilon(u)\big)\epsilon\big(\iota_L(\eta)\big) = 0.$$

Next, $\mathcal{J}_\mathfrak{i}$ clearly is a right ideal of the $K$-algebra $\big(\mathcal{U}(L,A), \diamond, \mathbf{1}\big)$, and since it is generated by all the $\iota_L(x)$, $x \in L$, we have

$$\iota_L(x) \diamond \iota_L(\eta) \diamond u = [\iota_L(x), \iota_L(\eta)]_\diamond \diamond u + \iota_L(\eta) \diamond \iota_L(x) \diamond u \overset{(4.2.4)}{=} \iota_L\big([\![x, \eta]\!]\big) \diamond u + \iota_L(\eta) \diamond \iota_L(x) \diamond u$$

and both terms on the right hand side are in $\mathcal{J}_{\mathfrak{i}}$ since $\mathfrak{i}$ is a Lie-Rinehart ideal. By induction over the number of generators and by eqn (4.3.8) it follows that $\mathcal{J}_{\mathfrak{i}}$ is a two-sided ideal of $\big(\mathcal{U}(L,A),\diamond,\mathbf{1}\big)$ and therefore a Rinehart ideal.
*ii.*) For the sake of simplicity we set $L' = L/\mathfrak{i}$. The morphism $\mathcal{U}(\pi)$ clearly is a morphism of Rinehart algebras $\mathcal{U}(L,A) \to \mathcal{U}\big(L',A\big)$, and since by construction $\mathcal{U}(\pi)\circ\iota_L = \iota'_{L'}\circ\pi$ and $\mathcal{U}(\pi)\circ\iota_A = \iota'_A$ (where $\iota'_A$ and $\iota'_{L'}$ denote the corresponding canonical maps for $\mathcal{U}(L',A)$) it follows that $\mathcal{U}(\pi)$ is surjective, since $\pi$ is surjective, and both universal enveloping algebras are generated by $\iota_A(a)$, $\iota_L(x)$, and $\iota'_A(a)$, $\iota'_{L'}(\pi(x))$, respectively, for all $a\in A$ and $x\in L$.

Moreover for every $\eta\in\mathfrak{i}$ the element $\mathcal{U}(\pi)\big(\iota_L(\eta)\big)\big) = \iota'_{L'}\big(\pi(\eta)\big) = 0$, and since $\mathcal{U}(\pi)$ is a morphism of unital associative $K$-algebras it follows that $\mathcal{U}(\pi)\big(\mathcal{J}_{\mathfrak{i}}\big) = \{0\}$. Hence $\mathcal{U}(\pi)$ descends to a well-defined surjective morphism of Rinehart bialgebras $\Phi : \mathcal{U}(L,A)/\mathcal{J}_{\mathfrak{i}} \to \mathcal{U}(L',A)$.

In order to define an inverse map of $\Phi$ we need the fact that $\mathfrak{i}$ is complemented: note that the restriction of the projection $\pi : L \to L'$ to the submodule $\mathfrak{m}$ of $L$ is an $A$-linear bijection, and let $\alpha : L' \to \mathfrak{m}$ denote its inverse. Let $B$ denote the Rinehart bialgebra $\mathcal{U}(L,A)/\mathcal{J}_{\mathfrak{i}}$ and $\blacklozenge$ its multiplication, and let $j : A \to B$ be the map $a \mapsto \iota_A(a) \bmod \mathcal{J}_{\mathfrak{i}}$. Define

$$\theta : L' \to B : x' \mapsto \theta(x') := \iota_L\big(\alpha(x')\big) \bmod \mathcal{J}_{\mathfrak{i}}.$$

Clearly $\theta$ is $A$-linear since $\alpha$, $\iota_L$ and the projection modulo $\mathcal{J}_{\mathfrak{i}}$ are $A$-linear, and $j$ is a morphism of associative unital $K$-algebras. Moreover for all $x'_1, x'_2 \in L'$ we get

$$\begin{aligned}\big[\theta(x'_1),\theta(x'_2)\big]_{\blacklozenge} - \theta\big([\![x'_1,x'_2]\!]'\big) &= \Big[\iota_L\big(\alpha(x'_1)\big),\iota_L\big(\alpha(x'_2)\big)\Big]_{\diamond} - \iota_L\Big(\alpha\big([\![x'_1,x'_2]\!]'\big)\Big) \bmod \mathcal{J}_{\mathfrak{i}} \\ &= \iota_L\Big(\big[\![\alpha(x'_1),\alpha(x'_2)]\!\big] - \alpha\big([\![x'_1,x'_2]\!]'\big)\Big) \bmod \mathcal{J}_{\mathfrak{i}} = 0\end{aligned}$$

since the term $\big[\![\alpha(x'_1),\alpha(x'_2)]\!\big] - \alpha\big([\![x'_1,x'_2]\!]'\big)$ is contained in $\mathfrak{i} = \mathrm{Ker}(\pi)$ as an application of $\pi$ to it readily shows. Likewise, for all $x'\in L'$ and $a\in A$ we compute

$$\big[\theta(x'),j(a)\big]_{\blacklozenge} = \Big[\iota_L\big(\alpha(x')\big),\iota_A(a)\Big]_{\diamond} \bmod \mathcal{J}_{\mathfrak{i}} = \iota_A\big(\rho_{\alpha(x')}(a)\big) \bmod \mathcal{J}_{\mathfrak{i}} = j\big(\rho'_{x'}(a)\big).$$

It follows that $j$ and $\theta$ satisfy the properties (4.2.5), (4.2.4), (4.2.7), and (4.2.6). By the universal property of $\mathcal{U}(L',A)$ there is a morphism $\overline{\theta} =: \Psi : \mathcal{U}(L',A) \to B$ of assoicative $K$-algebras over $A$ such that $\Psi\circ\iota'_{L'} = \theta$ and $\Psi\circ\iota'_A = j$. In order to show that $\Phi\circ\Psi = \mathrm{id}_{\mathcal{U}(L',A)}$ and $\Psi\circ\Phi = \mathrm{id}_{\mathcal{U}(L,A)}$ one observes that these are identities of $K$-algebra morphisms, and it thus suffices to check them on generators: clearly $(\Psi\circ\Phi)\big(j(a)\big) = j(a)$ and $(\Phi\circ\Psi)\big(\iota'_A(a)\big) = \iota'_A(a)$ for all $a\in A$, and for all $x\in L$

$$(\Psi\circ\Phi)\Big(\iota_L(x) \bmod \mathcal{J}_{\mathfrak{i}}\Big) = \Psi\Big(\iota'_{L'}\big(\pi(x)\big)\Big) = \theta\big(\pi(x)\big) = \iota_L\Big(\alpha\big(\pi(x)\big)\Big) \bmod \mathcal{J}_{\mathfrak{i}} = \iota_L(x) \bmod \mathcal{J}_{\mathfrak{i}}$$

since the term $x - \alpha\big(\pi(x)\big)$ is in $\mathfrak{i}$. On the other hand for all $x' \in L'$

$$(\Phi \circ \Psi)\big(\iota'_{L'}(x')\big) = \Phi\big(\theta(x')\big) = \Phi\Big(\iota_L\big(\alpha(x')\big) \bmod \mathcal{J}_{\mathfrak{i}}\Big) = \iota'_{L'}\Big(\pi\big(\alpha(x')\big)\Big) = \iota'_{L'}(x').$$

This shows that $\Psi$ is the inverse of $\Phi$ which ends the proof. □

## 4.4 Differential Operators

Given two $A$-modules $P$ and $Q$, recall the algebraic definition of the left $A$-module of all differential operators $D : P \to Q$:
First, the $K$-module of all differential operators of order 0, $\mathbf{D}^0_{A|K}(P,Q)$ is defined to be equal to $\mathrm{Hom}_A(P,Q)$. By induction, for every nonnegative integer $k$ define $\mathbf{D}^{k+1}_{A|K}(P,Q)$ to be the $K$-module of those $K$-linear maps $D : P \to Q$ such that

$$\forall\, a \in A : \quad p \mapsto D(ap) - aD(p) \ \text{ is a differential operator of order } k. \tag{4.4.1}$$

It is well-known (see Krasil'chik, Vinogradov; Jet Nestruev; Lunts et al.) that $\mathbf{D}^k_{A|K}(P,Q) \subset \mathbf{D}^{k+1}_{A|K}(P,Q)$ for all nonnegative integer $k$, and that each $\mathbf{D}^k_{A|K}(P,Q)$ is a left $A$-submodule of the left $A$-module $\mathrm{Hom}_K(P,Q)$. Moreover the composition of differential operators $P \to Q$ and $Q \to V$ of order $k$ and $l$, respectively, is a differential operator of order $k+l$ from $P$ to $V$. We write $\mathbf{D}_{A|K}(P,Q)$ for the union of all the $\mathbf{D}^k_{A|K}(P,Q)$ and speak of the left $A$-module of all differential operators $P \to Q$. For $P = Q$, the left $A$-module $\mathbf{D}_{A|K}(P,P)$ is an associative unital $K$-algebra by means of composition of $K$-linear maps.

It is well-known, see as for example Stefan Waldmann book [55], that for the particular case $A = \mathcal{C}^\infty(X,\mathbb{K})$, $P = \Gamma^\infty(X,E)$, and $Q = \Gamma^\infty(X,E')$ (where $X$ is a smooth manifold and $E, E'$ are smooth vector bundles over $X$) that the algebraic definition coincides with the analytic definition of a differential operator by iterated partial derivatives in coordinates.

Returning to general algebras $A$ over $K$, the case $P = A = Q$ is interesting: here $A$ injects into $\mathrm{Hom}_A(A,A) = \mathbf{D}^0_{A|K}(A,A) \subset \mathbf{D}_{A|K}(A,A)$ in the natural way via $j : a \mapsto (a' \mapsto aa')$ whence $\big(j, \mathbf{D}^0_{A|K}(A,A)\big)$ is an associative unital $K$-algebra over $A$. Since derivations are obviously differential operators of order 1 the anchor morphism $\rho$ of any Lie-Rinehart algebra $\big(L, \rho, [\![\ ,\ ]\!], A\big)$ provides us with a canonical morphism, also called $\rho : L \to \mathbf{D}_{A|K}(A,A)^-_A \cong \mathrm{Der}_K(A,A)$, of Lie-Rinehart algebra, and thus by the universal property a morphism of associative unital $K$-algebras over $A$

$$\bar{\rho} : \mathcal{U}(L,A) \to \mathbf{D}_{A|K}(A,A). \tag{4.4.2}$$

In the particular case where $L = \mathrm{Der}_K(A,A)$ with the identity as the anchor morphism the above morphism (4.4.2) is an isomorphism provided $L = \mathrm{Der}_K(A,A)$ is a finitely generated projective left $A$-module: this is the case for the Lie algebroid consisting of all vector fields on a smooth manifold.

## 4.5 Anchored $A$-modules and Free Lie-Rinehart algebras

### 4.5.1 Anchored $A$-modules

We have already mentioned the notion of an anchored $A$-module, notion appearing in [23]. It consists in a left $A$-module $M$ equipped with an $A$-linear map

$$\begin{array}{ccc} \rho : M & \longrightarrow & \mathrm{Der}_K(A,A) \\ x & \longmapsto & \rho_x \end{array}$$

These modules occur in differential geometry as the smooth section spaces of so-called *anchored vector bundles* $E$ over a manifold $X$ which are equipped with a morphism of vector bundles also called $\rho : E \to TX$.

In that sense, they form a category $A\mathbf{ModAnc}_K$ whose morphisms are $A$-module morphisms $\phi : L \to M$, where $(L,\rho)$ and $(M,\rho')$ are anchored $A$-modules, intertwining the anchor morphisms, precisly

$$\rho'_{\phi(x)} = \rho_x \tag{4.5.1}$$

There is an obvious forgetful functor from the category of all Lie-Rinehart algebras over $A$, $\mathbf{LieRinAlg_{A|K}}$, to $A\mathbf{ModAnc}_K$ by just forgetting the Lie bracket $[\![\ ,\ ]\!]$:

$$\mathbf{LieRinAlg}_{A|K} \overset{\mathsf{Forget}}{\longrightarrow} A\mathbf{ModAnc}_K \tag{4.5.2}$$

### 4.5.2 Free Lie-Rinehart algebras

In this Section we shall briefly describe a left adjoint functor $\mathcal{P}$ to this functor, the *free Lie-Rinehart algebra generated by the anchored A-module M* or the *path Lie algebroid* according to Kapranov's construction in [23]:

**Theorem 4.5.1** (M.Kapranov 2007) *The forgetful functor from the category of all Lie-Rinehart algebras over A to the category of all anchored modules over A has a left adjoint $\mathcal{P}$:*

$$A\mathbf{ModAnc}_K \quad \underset{\mathsf{Forget}}{\overset{\mathcal{P}}{\rightleftarrows}} \quad \mathbf{LieRinAlg}_{A|K} \tag{4.5.3}$$

*Proof.* We sketch a different proof à la Huebschmann: Take an anchored $A$-module $(M,\rho)$ and consider first the free Lie algebra $\mathsf{Lie}_K(M)$ generated by the $K$-module $M$, see Appendix A.6.2, with the canonical map $i_M : M \to \mathsf{Lie}_K(M)$, the unit of the adjunction

$$K\mathbf{Mod} \quad \underset{\mathsf{Forget}}{\overset{\mathsf{Lie_K}}{\rightleftarrows}} \quad \mathbf{LieAlg}_K \qquad \text{(see also eq.A.6.12)} \tag{4.5.4}$$

Then there is a unique morphism of $K$-Lie algebras $\bar{\rho} : \mathsf{Lie}_K(M) \to \mathrm{Der}_K(A,A)$ induced by the anchor morphism $\rho$, i.e. $\bar{\rho} \circ i_M = \rho$.

Clearly $\big(\mathsf{Lie}_K(M), [\ ,\ ], \bar{\rho}, A\big)$ will be a Lie derivation algebra over $A$. It follows from Section 4.1.2 that $\big(A \otimes \mathsf{Lie}_K(M), {}^A[\ ,\ ]_{\bar{\rho}}, \widehat{\bar{\rho}}, A\big)$ will be a Lie-Rinehart algebra where $\widehat{\bar{\rho}}$ is the anchor morphism induced by the anchor morphism $\bar{\rho}$ in eqn (4.1.2) (with $\rho$ replaced by $\bar{\rho}$), and ${}^A[\ ,\ ]_{\bar{\rho}}$ is the Lie bracket in eqn (4.1.2).

Denote by $\hat{\mathfrak{B}}_M : A \otimes M \to A \otimes \mathsf{Lie}_K(M)$ the $A$-linear map $\mathrm{id}_A \otimes i_M$. Recall the kernel $\mathfrak{h}(M,A)$ of the multiplication map $\mu_M : A \otimes M \to M$, see Appendix A.2.1 equation A.2.2.

Clearly, by definition $\widehat{\bar{\rho}}$ vanishes on $\hat{\mathfrak{B}}_M\big(\mathfrak{h}(M,A)\big)$. Define the following $K$-submodule $\mathsf{J}(M,A)$ of $A \otimes \mathsf{Lie}_K(M)$ by

$$\begin{aligned} \mathsf{J}(M,A) \;&=\; \hat{\mathfrak{B}}_M\big(\mathfrak{h}(M,A)\big) + \qquad\qquad (4.5.5)\\ &+\; K\mathrm{Span}\Big\{ {}^A\Big[\hat{\mathfrak{B}}_M(\xi_1), {}^A\big[\hat{\mathfrak{B}}_M(\xi_2), \ldots, {}^A[\hat{\mathfrak{B}}_M(\xi_{n-1}), \hat{\mathfrak{B}}_M(\xi_n)]_{\bar{\rho}} \cdots ]_{\bar{\rho}}\Big]_{\bar{\rho}} \Big| \\ &\qquad n \in \mathbb{N},\ n \geqslant 2,\ \xi_1, \ldots, \xi_n \in A \otimes M,\ \ \exists\, i \in \mathbb{N},\ 1 \leqslant i \leqslant n :\ \xi_i \in \mathfrak{h}(M,A)\Big\}. \end{aligned}$$

It is not hard to check using the Jacobi identity that $\mathsf{J}(M,A)$ is the ideal of the $K$-Lie algebra $\big(A \otimes \mathsf{Lie}_K(M), {}^A[\ ,\ ]_{\bar{\rho}}\big)$ generated by the $K$-submodule $\hat{\mathfrak{B}}_M\big(\mathfrak{h}(M,A)\big)$. Note that the anchor morphism $\widehat{\bar{\rho}}$ vanishes on $\mathsf{J}(M,A)$ because it vanishes on $\hat{\mathfrak{B}}_M\big(\mathfrak{h}(M,A)\big)$.

Next, $\mathsf{J}(M,A)$ is an $A$-submodule of $A \otimes \mathsf{Lie}_K(M)$: indeed this is true in degree $n = 1$ since $\mathfrak{h}(M,A)$ is an $A$-submodule of $A \otimes M$; and for elements of degree $n \geqslant 2$ in eqn (4.5.5) being of the form ${}^A\big[\hat{\mathfrak{B}}_M(\xi), \zeta\big]_{\bar{\rho}}$ with $\xi \in A \otimes M$ and $\zeta \in A \otimes \mathsf{Lie}_K(M)$ such that $\xi \in \mathfrak{h}(M,A)$ or $\zeta \in \mathsf{J}(M,A)$ there is the trivial identity for all $a \in A$,

$$a^A\big[\hat{\mathfrak{B}}_M(\xi), \zeta\big]_{\bar{\rho}} = {}^A\big[\hat{\mathfrak{B}}_M(a\xi), \zeta\big]_{\bar{\rho}} + \hat{\mathfrak{B}}_M\left(\widehat{\bar{\rho}}_\zeta(a)\xi\right).$$

If $\xi \in \mathfrak{h}(M,A)$ the r.h.s. of the preceding equation clearly is in $\mathsf{J}(M,A)$, and if $\zeta \in \mathsf{J}(M,A)$ then $\widehat{\bar{\rho}}_\zeta(a) = 0$ and the r.h.s. is also in $\mathsf{J}(M,A)$. It follows that $\mathsf{J}(M,\rho)$ is a Lie-Rinehart ideal of $A \otimes \mathsf{Lie}_K(M)$ whence the projection on the quotient $A$-module

$$\Pi_M : A \otimes \mathsf{Lie}_K(M) \to \mathcal{P}(M,A) := \frac{A \otimes \mathsf{Lie}_K(M)}{\mathsf{J}(M,A)} \qquad (4.5.6)$$

naturally equips $\mathcal{P}(M,A)$ with the structure of a Lie-Rinehart algebra

$$\big(\mathcal{P}(M,A), [\ ,\ ]_{M,\rho}, \tilde{\rho}, A\big)$$

over $A$ such that $\Pi_M$ is a morphism of Lie-Rinehart algebras, see e.g. Lemma 4.1.11.

This defines the functor $\mathcal{P}$ on objects, and for a morphism $\phi : (M,\rho,A) \to (M',\rho',A)$ of anchored modules over $A$ it is easy to see that the $A$-linear map $\mathrm{id}_A \otimes \mathsf{Lie}_K(\phi)$

defines a morphism $A \otimes \mathsf{Lie}_K(M) \to A \otimes \mathsf{Lie}_K(M')$ of Lie-Rinehart algebras over $A$ (where the fact that $\phi$ intertwines the anchor morphisms plays a crucial role) mapping the ideal $\mathsf{J}(M,A)$ to the ideal $\mathsf{J}(M',A)$, hence $\mathrm{id}_A \otimes \mathsf{Lie}_K(\phi)$ passes to the quotient to define a morphism $\mathcal{P}(\phi) : \mathcal{P}(M,A) \to \mathcal{P}(M',A)$ of Lie-Rinehart algebras. The functor properties of $\mathcal{P}$ are immediate. We shall write $\check{\mathfrak{s}}_M : M \to \mathcal{P}(M,A)$ for the $A$-linear map $\Pi_M \circ \hat{\mathfrak{s}}_M$.

In order to see that $\mathcal{P}$ is a left adjoint to the forgetful functor, fix an anchored $A$-module $(M,\rho,A)$ and a Lie-Rinehart algebra $\big(L',\rho',[\![\ ,\ ]\!]',A\big)$ over $A$. The adjugant $\varphi_{M,L'}$ (see Appendix A.1) will be the composition of the morphism $\Phi : \mathcal{P}(M,A) \to L'$ of Lie-Rinehart algebras with $\check{\mathfrak{s}}_M$, $\Phi \circ \check{\mathfrak{s}}_M$, and its inverse goes as follows: to every $A$-linear map $\theta : M \to L'$ associate its $K$-Lie algebra morphism $\bar{\theta} : \mathsf{Lie}_K(M) \to L'$. Then the $A$-linear map $\mu_{L'} \circ \big(\mathrm{id}_A \otimes \bar{\theta}\big)$ from $A \otimes \mathsf{Lie}_K(M)$ to $L'$ clearly vanishes on the ideal $\mathsf{J}(M,A)$ and thus passes to the quotient to define a morphism of Lie-Rinehart algebras $\check{\theta} : \mathcal{P}(M,A) \to L'$. By using the fact that $\check{\mathfrak{s}}_M(M)$ generates $\mathcal{P}(M,A)$ it is easily seen that the two preceding maps are inverses, their naturality being obvious. $M \to \check{\mathfrak{s}}_M$ will be the unit, and $\check{\mathrm{id}}_{L'} : \mathcal{P}(L',A) \to L'$ the counit of the adjunction which will become rather important in the sequel. □

# 5. Connections

Contents

## 5.1 Covariant derivatives

### 5.1.1 Basic definitions

**Definition 5.1.1** For a given anchored $A$-module $(L,\rho)$ and a given $A$-module $W$, a ***connection***, or a *covariant derivative* $\nabla^W = \nabla$, is a $K$-linear map

$$\nabla : L \otimes W \to W,$$

written $\nabla(x \otimes w) =: \nabla_x(w)$ such that for all $a \in A$, $x \in L$, and $w \in W$

$$\nabla_{ax}(w) = a\nabla_x(w) \quad \text{and} \quad \nabla_x(aw) = \rho_x(a)w + a\nabla_x(w). \tag{5.1.1}$$

We shall call the quadruple $(L,\rho,W,\nabla^W)$ an *$A$-module $W$ with connection parametrized by (or along) the anchored $A$-module* $(L,\rho)$.

**Remark 5.1.2** About the above Definition 5.1.1 we have:
— Sometimes the conditions in the equation 5.1.1 are called *Koszul's axioms*.
— For each element $w \in w$ there is a canonical $A$-linear map

$$\nabla(w) : L \to W : \; x \mapsto \nabla_x(w). \tag{5.1.2}$$

**Definition 5.1.3** A *morphism of $A$-modules with connection parametrized by the anchored $A$-module $L$*, $(L,\rho,W,\nabla)$ to $(L,\rho,W',\nabla')$ is an $A$-linear map $\Phi : W \to W'$ satisfying $\Phi\big(\nabla_x(w)\big) = \nabla'_x\big(\Phi(w)\big)$ for all $x \in L$ and $w \in W$. In this case $\Phi$ is also called *connection preserving*.

The difference of two connections $\nabla' - \nabla$, with is an $A$-linear map $L \otimes_A W \to W$, is a connection. Furthermore, the sum of any connection with any $A$-linear map is again a connection.

**Example 5.1.4** For each $x \in L$ the $K$-linear map $v \mapsto \nabla_x(v)$ clearly is a differential operator of order 1, see Section 4.4 for more details.

**Example 5.1.5** A first canonical example of a connection parametrized by a given anchored $A$-module $(L,\rho)$ is $A$ seen as an $A$-module in the usual way: here the connection $\nabla^A$ is simply given by the anchor morphism

$$\nabla^A_x(a) := \rho_x(a) \tag{5.1.3}$$

for all $x \in L$ and $a \in A$.

**Example 5.1.6** More generally, let $P$ be a $K$-module, and let $A \otimes P$ be the relatively free $A$-module generated by the $K$-module $P$, see Appendix A.2.1. Then it is easy to check that the following map $\nabla^0$ from $L \otimes_K (A \otimes P)$ to $A \otimes P$ is a covariant derivative

in $A \otimes P$ parametrised by $(L, \rho)$, called the *canonical covariant derivative* (for all $a \in A$ and $p \in P$):

$$\nabla^0_x(a \otimes p) := \big(\rho_x(a)\big) \otimes p. \tag{5.1.4}$$

We can easily go from relatively free $A$-modules to relatively projective modules, see again Appendix A.2.1 for details, where connections always exist:

**Proposition 5.1.7** *Let $(L, \rho)$ be an anchored $A$-module and let $W$ be a relatively projective $A$-module. Pick any section $\sigma : W \to A \otimes_K W$ (see Appendix A.2.1 for details). Then the map $\nabla^\sigma : L \otimes_K W \to W$ defined by (for all $x \in L$ and $w \in W$)*

$$\nabla^\sigma_x(w) := \big(\mu_W \circ \nabla^0_x \circ \sigma\big)(w) \tag{5.1.5}$$

*is a covariant derivative in $W$ parametrised by $(L, \rho)$.*

The following result of transferring connections to tensor products and Hom-spaces is very important for the sequel and well-known in differential geometry:

**Proposition 5.1.8** *Let $(L, \rho)$ be an anchored $A$-module, and let $(V, \nabla^V)$, $(W, \nabla^W)$, $(Y, \nabla^Y)$, and $(Z, \nabla^Z)$ be $A$-modules with connection parametrized by the anchored $A$-module $L$. Then we have the following:*

1. *There is a unique connection $\nabla^{V \otimes_A W}$ in the $A$-module $V \otimes_A W$ parametrized by the anchored $A$-module $L$ such that for all elements $x \in L$, $v \in V$ and $w \in W$ the following holds:*

$$\nabla^{V \otimes_A W}_x(v \otimes_A w) = \big(\nabla^V_x(v)\big) \otimes_A w + v \otimes_A \big(\nabla^V_x(w)\big). \tag{5.1.6}$$

2. *There is a connection $\nabla^{\mathrm{H}(V,W)}$ in the $A$-module $\mathrm{Hom}_A(V, W)$ defined for any $A$-linear map $\varphi : V \to W$ and any $v \in V$ by*

$$\big(\nabla^{\mathrm{H}(V,W)}_x(\varphi)\big)(v) := \nabla^W_x\big(\varphi(v)\big) - \varphi\big(\nabla^V_x(v)\big). \tag{5.1.7}$$

   *Hence $\varphi$ is connection preserving iff $\nabla^{\mathrm{H}(V,W)}_x(\varphi) = 0$ for all $x \in L$. Moreover there is a canonical $A$-linear map $\nabla(\varphi) : L \otimes_A V \to W$ defined by*

$$\nabla(\varphi)\big(x \otimes_A v\big) := \nabla^{\mathrm{H}(V,W)}_x(\varphi)\big(v\big) \tag{5.1.8}$$

   *for all $x \in L$ and $v \in V$.*

3. *For any $\varphi \in \mathrm{Hom}_A(V, W)$ and $\psi \in \mathrm{Hom}_A(W, Y)$ we have for all $x \in L$*

$$\nabla^{\mathrm{H}(V,Y)}_x(\psi \circ \varphi) = \big(\nabla^{\mathrm{H}(W,Y)}_x(\psi)\big) \circ \varphi + \psi \circ \big(\nabla^{\mathrm{H}(V,W)}_x(\varphi)\big). \tag{5.1.9}$$

4. *For any* $\varphi \in \mathrm{Hom}_A(V,W)$ *and* $\chi \in \mathrm{Hom}_A(Y,Z)$ *we have for all* $x \in L$

$$\nabla_x^{\mathrm{H}(V\otimes_A Y, W\otimes_A Z)}(\varphi\otimes_A \chi) = \big(\nabla_x^{\mathrm{H}_A(V,W)}(\varphi)\big)\otimes_A \chi + \varphi\otimes_A \big(\nabla_x^{\mathrm{H}_A(Y,Z)}(\chi)\big) \quad (5.1.10)$$

**Remark 5.1.9** The preceding Proposition 5.1.8 has the following categorical interpretation: consider the class of all pairs $(V,\nabla)$ where $V$ is a left $A$-module (where $A$ is a fixed associative commutative unital $K$-algebra, and $K$ is a fixed associative commutative unital ring containing the rationals) and $\nabla$ is a covariant derivative with respect to an anchored $A$-module $(L,\rho)$. It becomes a category $A$-**modConn**$L$ by declaring a morphism $(V,\nabla)\to(W,\nabla')$ to be an $A$-linear map $\Phi$ which is connection preserving, i.e. $\nabla'_x\circ\Phi = \Phi\circ\nabla_x$ for all $x\in L$. This category is clearly symmetric monoidal when equipped with the usual tensor product $\otimes_A$ over $A$ of $A$-modules (and covariant derivative according to eqn (5.1.6)) and unit object $A$ (with covariant derivative the anchor map), and it is a routine-check that the associator, left and right unitors $\lambda$ and $\rho$, and the braiding $\gamma$ are all connection preserving. Next the symmetric monoidal category $\mathcal{C}$ =$A$-**modConn**$L$ is closed, i.e. the usual natural isomorphism from the category of all $A$-modules,

$$\mathrm{Hom}_A(V\otimes_A W, X) \cong \mathrm{Hom}_A\big(V, \mathrm{Hom}_A(W,X)\big),$$

restricts to a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\big((V\otimes_A W, \nabla^{V\otimes_A W}), (X,\nabla^X)\big) \cong \mathrm{Hom}_{\mathcal{C}}\big((V,\nabla^V), (\mathrm{Hom}_A(W,X), \nabla^{\mathrm{H}(W,X)})\big), \quad (5.1.11)$$

whence the usual Hom-space $\mathrm{Hom}_A(W,X)$ with its covariant derivative $\nabla^{\mathrm{H}(W,X)}$ (see eqn(5.1.7)) will become an internal Hom-object in that closed symmetric monoidal category $\mathcal{C}$.

### 5.1.2 Iterated covariant derivatives on $\mathsf{T}_A(L)$

Now fix an anchored $A$-module $(L,\rho)$ (where again $A$ is an associative commutative unital $K$-algebra, $K$ being an associative commutative unital ring containing the rational numbers as a subring), and fix a connection $\nabla = \nabla^L$ in the $A$-module $L$ parametrized by the anchored $A$-module $(L,\rho)$. Let $(W,\nabla^W = \nabla')$ be another $A$-module with connection parametrized by the anchored $A$-module $(L,\rho)$.

We shall first extend $\nabla$ in a standard 'derivational' manner to a connection

$$L\otimes_K \mathsf{T}_A(L) \to \mathsf{T}_A(L),$$

also written $\nabla(x\otimes_K b) =: \nabla_x^{\mathsf{T}_A(L)}(b) = \nabla_x(b)$ for all $x\in L$ and $b\in\mathsf{T}_A(L)$, i.e.

$$\nabla_x(\mathbf{1}) := 0, \text{ and } \nabla_x(x_1\cdots x_n) := \sum_{r=1}^{n} x_1\cdots x_{r-1}\big(\nabla_x(x_r)\big)x_{r+1}\cdots x_n \quad (5.1.12)$$

for all integers $n \geqslant 1$ and $x, x_1, \ldots, x_n \in V$. This is a well-defined connection thanks to an iteration of the first statement eqn (5.1.6) of Proposition 5.1.8. It is immediate from the definition that $\nabla_x$ preserves tensor degree, i.e. for all $x \in L$

$$\nabla_x \circ \mathsf{deg} = \mathsf{deg} \circ \nabla_x, \tag{5.1.13}$$

and that it is a *derivation of the unital algebra* $\mathsf{T}_A(L)$, *seen as a* $K$*-algebra* (!), i.e. for all $b, b' \in \mathsf{T}_A(L)$ and $x \in L$

$$\nabla_x(bb') = \big(\nabla_x(b)\big)b' + b\big(\nabla_x(b')\big). \tag{5.1.14}$$

In other words, the free multiplication $\mu : \mathsf{T}_A(L) \otimes_A \mathsf{T}_A(L) \to \mathsf{T}_A(L)$ is connection preserving. It follows from the preceding Proposition 5.1.8, iv), that the induced covariant derivative $\nabla_x^{[2]} = \nabla_x^{\mathsf{T}_A(L)\otimes_A \mathsf{T}_A(L)}$ is a derivation of the associative algebra $\mathsf{T}_A(L) \otimes_A \mathsf{T}_A(L) =: \mathsf{T}_A(L)^{[2]}$ since $\mu \otimes_A \mu$ is connection preserving as well as the usual middle-four-interchange permution. Clearly this generalizes in the obvious way to any $k$-fold tensor product of $\mathsf{T}_A(L)$ with itself.

Next, it is not hard to see that the shuffle-comultiplication $\Delta_{sh}$ is connection preserving, i.e. $\nabla_x$ is compatible with the comultiplication $\Delta_{sh}$ in the following way for all $x \in L$, $b \in \mathsf{T}_A(L)$:

$$\Delta_{sh}\big(\nabla_x(b)\big) = \nabla_x^{[2]}\big(\Delta_{sh}(b)\big), \tag{5.1.15}$$

or in Sweedler's notation for all $b \in \mathsf{T}_A(L)$:

$$\begin{aligned} \sum_{\left(\nabla_x^{\mathsf{T}_A(L)}(b)\right)} & \left(\nabla_x^{\mathsf{T}_A(L)}(b)\right)^{(1)} \otimes_A \left(\nabla_x^{\mathsf{T}_A(L)}(b)\right)^{(2)} \\ &= \sum_{(b)} \big(\nabla_x^{\mathsf{T}_A(L)}(b^{(1)})\big) \otimes_A b^{(2)} + \sum_{(b)} b^{(1)} \otimes_A \big(\nabla_x^{\mathsf{T}_A(L)}(b^{(2)})\big). \end{aligned} \tag{5.1.16}$$

*Proof.* Indeed, this is easily seen by induction over the tensorial degree of $b$: since $\nabla_x(\mathbf{1}) = 0$, for all $b \in \mathsf{T}_A(L)$ and $x \in L$ we can infer from formula eqn (5.1.6) that

$$\nabla_x^{[2]}(b \otimes_A \mathbf{1}) = \Big(\nabla_x(b)\Big) \otimes_A \mathbf{1}, \quad \nabla_x^{[2]}(\mathbf{1} \otimes_A b) = \mathbf{1} \otimes_A \Big(\nabla_x(b)\Big). \tag{5.1.17}$$

For any $a \in A$ we have

$$\nabla_x^{[2]}\big(\Delta_{sh}(a\mathbf{1})\big) = \nabla_x^{[2]}\big(a(\mathbf{1} \otimes_A \mathbf{1})\big) = \rho_x(a)\big(\Delta_{sh}(\mathbf{1})\big) = \Delta_{sh}\big(\nabla_x(a\mathbf{1})\big),$$

and for all $y \in L$

$$\nabla_x^{[2]}\big(\Delta_{sh}(y)\big) = \nabla_x^{[2]}\big(y \otimes_A \mathbf{1} + \mathbf{1} \otimes_A y\big) \overset{(5.1.17)}{=} (\nabla_x(y)) \otimes_A \mathbf{1} + \mathbf{1} \otimes_A (\nabla_x(y)) = \Delta_{sh}\big(\nabla_x(y)\big).$$

Hence the equation (5.1.15) is satisfied for tensorial degree 0 and 1.

Suppose that the equation (5.1.15) holds up to tensor degree equal to $n$, let $b$ be of tensor degree $n$ and $y \in L$. We compute (using the induction hypothesis and the fact that $\nabla_x^{[2]}$ is a derivation of $\mathsf{T}_A(L) \otimes_A \mathsf{T}_A(L)$)

$$\begin{aligned}
\Delta_{sh}\big(\nabla_x(yb)\big) &= \Delta_{sh}\Big(\big(\nabla_x(y)\big)b\Big) + \Delta_{sh}\Big(y\big(\nabla_x(b)\big)\Big) \\
&= \Big(\big(\nabla_x(y)\big) \otimes_A \mathbf{1} + \mathbf{1} \otimes_A \big(\nabla_x(y)\big)\Big)\big(\Delta_{sh}(b)\big) \;+\; \big(y \otimes_A \mathbf{1} + \mathbf{1} \otimes_A y\big)\Big(\Delta_{sh}\big(\nabla_x(b)\big)\Big) \\
&= \Big(\nabla_x^{[2]}\big(\Delta_{sh}(y)\big)\Big)\big(\Delta_{sh}(b)\big) \;+\; \big(\Delta_{sh}(y)\big)\Big(\nabla_x^{[2]}\big(\Delta_{sh}(b)\big)\Big) = \nabla_x^{[2]}\big(\Delta_{sh}(yb)\big)
\end{aligned}$$

which shows the induction, and equation 5.1.15 holds. □

Observe that $\nabla_x^{\mathsf{T}_A(L)}$ is *NOT a coderivation of the coalgebra* $(\mathsf{T}_A(L), \Delta_{sh})$ since it is in general not $A$-linear! The equation 5.1.15 is an equality of $K$-linear maps,

$$\mathsf{Res}(\Delta_{sh}) \circ \nabla_x^{\mathsf{T}_A(L)} = \nabla_x^{\mathsf{T}_A(L) \otimes_A \mathsf{T}_A(L)} \circ \mathsf{Res}(\Delta_{sh}),$$

(where we have used the forgetful functor Res, see Appendix A.2.1), but the covariant derivative $\nabla_x^{\mathsf{T}_A(L) \otimes_A \mathsf{T}_A(L)}$ is NOT equal to an expression like $\nabla_x^{\mathsf{T}_A(L)} \otimes_A \mathrm{id} + \mathrm{id} \otimes_A \nabla_x^{\mathsf{T}_A(L)}$ for which the tensor product $\otimes_A$ for the corresponding maps would be ill-defined.

We resume the above considerations in the following

**Proposition 5.1.10** *All the $A$-linear maps of the free algebra $\mathsf{T}_A(L)$ over the $A$-module $L$, the free multiplication, $\mu : \mathsf{T}_A(L)^{[2]} \to \mathsf{T}_A(L)$, the shuffle comultiplication $\Delta_{sh} : \mathsf{T}_A(L) \to \mathsf{T}_A(L)^{[2]}$, the counit $\epsilon : \mathsf{T}_A(L) \to A$, the unit $\mathbf{1} : A \to \mathsf{T}_A(L)$, and the antipode $S : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$ are connection preserving.*

Now we shall extend the aforementioned 'action' of $L$ on $W$ or on $\mathsf{T}_A(L)$ via the covariant derivatives $\nabla'$ and $\nabla$ to a $K$-linear map, also written $\nabla' : \mathsf{T}(V) \otimes W \to W$ by the *classical rule of iterated covariant derivatives in differential geometry*, see as for example [25, p.124-125].

In order to get an idea, recall that –according to the definition of covariant derivatives (5.1.1)– for fixed $w \in W$ the map $\nabla'(w) : L \to W : x \mapsto \nabla'_x(w)$ is $A$-linear. Hence according to Proposition 5.1.8, eqn (5.1.7), we can compute its covariant derivative $\nabla_{x_1}^{H(L,W)}\big(\nabla'(w)\big)$, and the map $(x_1, x_2) \mapsto \Big(\nabla_{x_1}^{H(L,W)}\big(\nabla'(w)\big)\Big)(x_2)$ can be considered as an $A$-linear map $L \otimes_A L \to W$, see eqn (5.1.8) of Proposition 5.1.8. We can thus set for all $x_1, x_2 \in L$ and $w \in W$

$$\nabla'^2_{x_1 x_2}(w) = \nabla'_{x_1 x_2}(w) := \Big(\nabla'^{\mathrm{H}(L,W)}_{x_1}\big(\nabla'(w)\big)\Big)(x_2) \overset{(5.1.7)}{=} \nabla'_{x_1}\big(\nabla'_{x_2}(w)\big) - \nabla'_{\nabla_{x_1}(x_2)}(w).$$

The map $x_1 x_2 \mapsto \nabla'^2_{x_1 x_2}(w)$ is thus a well-defined $A$-linear map from $\mathsf{T}^2_A(L)$ to $W$. It is thus possible to recursively define for each $w \in W$ a sequence $\big(\nabla'^n(w)\big)_{n \in \mathbb{N}}$ of

$A$-linear maps $\mathsf{T}^n_A L \to W$, the $n$th iterated covariant derivative as follows: for all $w \in W$, all $x \in L$, and all $b \in \mathsf{T}^n_A(L)$

$$\nabla'^0_{\mathbf{1}}(w) := w, \text{ and } \nabla'^{(n+1)}_{xb}(w) := \nabla'_x\big(\nabla'^n_b(w)\big) - \nabla'^n_{\nabla_x(b)}(w), \tag{5.1.18}$$

and this gives the usual recursion formula for iterated covariant derivatives by setting $x = x_1 \in L$ and $b = x_2 \cdots x_{n+1} \in L^{\otimes n}$:

$$\nabla'^{(n+1)}_{x_1 x_2 \cdots x_{n+1}}(w) = \nabla'_{x_1}\big(\nabla'^n_{x_2 \cdots x_{n+1}}(w)\big) - \sum_{r=2}^{n+1} \nabla'^n_{x_2 \cdots x_{r-1}\left(\nabla_{x_1}(x_r)\right) x_{r+1} \cdots x_{n+1}}(w). \tag{5.1.19}$$

In the following we shall use the simpler notation $\nabla'_b(w)$ instead of $\nabla'^n_b(w)$, and $\nabla'(w)$ for the $A$-linear map $\mathsf{T}_A L \to W$. Replacing $W$ by $L$ or $\mathsf{T}_A(L)$ and $\nabla'$ by $\nabla$ we get analogous maps $\nabla : \mathsf{T}_A(L) \otimes_K L \to L$ and $\mathsf{T}_A(L) \otimes_K \mathsf{T}_A(L) \to \mathsf{T}_A(L)$.

There is the following analog of Proposition 5.1.8 which is a *Leibniz formula of iterated covariant derivatives*:

**Proposition 5.1.11** *Let $V, W, X$ three $A$-modules equipped with connections $\nabla^V, \nabla^W$, and $\nabla^X$, respectively, with respect to an anchored $A$-module $(L, \rho)$ with connection $\nabla$ over $A$, and let $\nu : V \otimes_A W \to X$ be an $A$-linear map which is connection preserving. Then for all $b \in \mathsf{T}_A(L)$, $v \in V$, and $w \in w$ we have the following iterated Leibniz formula*

$$\nabla^X_b\big(\nu(v \otimes_A w)\big) = \sum_{(b)} \nu\Big(\nabla^V_{b^{(1)}}(v) \otimes_A \nabla^W_{b^{(2)}}(w)\Big) \tag{5.1.20}$$

*where the right hand side of the preceding equation (5.1.20) makes sense as the tensor product over $A$ of the two $A$-linear maps $\nabla^V(v) : \mathsf{T}_A(V) \to V$ and $\nabla^W(w) : \mathsf{T}_A(V) \to W$ composed with the $A$-linear shuffle comultiplication $\Delta_{sh} : \mathsf{T}_A(V) \to \mathsf{T}_A(V) \otimes_A \mathsf{T}_A(V)$. We get the following four particular cases*

1. *For $V = A$ and $\nu$ being the module multiplication $A \otimes_A W \to W$ we get for all $a \in A$ and $w \in W$*

$$\nabla'_b(aw) \quad = \quad \sum_{(b)} \nabla_{b^{(1)}}(a) \nabla'_{b^{(2)}}(w). \tag{5.1.21}$$

2. *For $V, W$ arbitrary, and $X = V \otimes_A W$ (where $\nu$ is the identity map), and $\nabla^X$ the obvious connection in the tensor product (5.1.6) we get for all $b \in \mathsf{T}_A(L)$, $v \in V$, and $w \in W$:*

$$\nabla^{V \otimes_A W}_b(v \otimes_A w) = \sum_{(b)} \nabla^V_{b^{(1)}}(v) \otimes_A \nabla^W_{b^{(2)}}(w). \tag{5.1.22}$$

3. *Let $\varphi \in \mathrm{Hom}_A(V, W)$ and $\psi \in \mathrm{Hom}_A(W, X)$. Then we get*

$$\nabla^{\mathrm{H}(V,X)}_b(\psi \circ \varphi) = \sum_{(b)} \nabla^{\mathrm{H}(W,X)}_{b^{(1)}}(\psi) \circ \nabla^{\mathrm{H}(V,W)}_{b^{(2)}}(\varphi). \tag{5.1.23}$$

4. *Let $\varphi \in \mathrm{Hom}_A(V,W)$ and $v \in V$. Then*

$$\nabla^W_b\big(\varphi(v)\big) = \sum_{(b)} \big(\nabla^{\mathrm{H}(V,W)}_{b^{(1)}}(\varphi)\big)\big(\nabla^V_{b^{(2)}}(v)\big). \tag{5.1.24}$$

*In particular, if $\varphi$ is connection preserving, then for all $b \in \mathsf{T}_A L$ we have*

$$\nabla^{\mathrm{H}(V,W)}_b(\varphi) = \epsilon(b)\varphi, \text{ hence } \nabla^W_b \circ \varphi = \varphi \circ \nabla^V_b. \tag{5.1.25}$$

*(equation of $K$-linear maps).*

*Proof.* See Appendix C.2. □

We shall now show that the iterated covariant derivative $\nabla'$ leads to an action of $\mathsf{T}_A(L)$ on $W$, but with a modified (only $K$-bilinear) multiplication:

**Proposition 5.1.12** *Let $(L,\rho)$ be an anchored $A$-module, and let $W$ be an $A$-module, and let $\nabla'$ be connection in $W$, $\nabla$ be a connection in $L$ parametrized by $(L,\rho)$, and let $\nabla$ and $\nabla'$ also denote the iterated covariant derivatives as defined above in (5.1.18).*
*Then for any $b, b' \in \mathsf{T}(V)$ and $w \in W$ we get the two formulas*

$$\nabla'_{bb'}(w) = \Big(\nabla'^{\mathrm{H}(\mathsf{T}_A L,W)}_b\big(\nabla'(w)\big)\Big)(b'), \tag{5.1.26}$$

$$\nabla'_b\big(\nabla'_{b'}(w)\big) = \sum_{(b)} \nabla'_{b^{(1)}\left(\nabla_{b^{(2)}}(b')\right)}(w). \tag{5.1.27}$$

*Proof.* The first statement is proved by induction on the tensor degree of $b$, the cases of degree 0, $b = a\mathbf{1}$ (where $a \in A$), and of degree 1, $b = y \in L$, being obvious and direct consequences of the definition, equation (5.1.18).

Supposing equation (5.1.26) to be true up to tensor degree $n$ of $b$, we compute the equation replacing $b$ by $xb$ where $x \in L$, $b, b' \in \mathsf{T}(V)$, and upon writing $\nabla'^H$ for $\nabla'^{\mathrm{H}(\mathsf{T}_A L,W)}$

$$\begin{aligned}
\nabla'_{xbb'}(w) &= \nabla'_x\big(\nabla'_{bb'}(w)\big) - \nabla'_{(\nabla_x(b))b'}(w) - \nabla'_{b(\nabla_x(b'))}(w) \\
&= \nabla'_x\Big(\big(\nabla'^H_b\big(\nabla'(w)\big)\big)(b')\Big) - \Big(\nabla'^H_b\big(\nabla'(w)\big)\Big)\big(\nabla_x(b')\big) - \Big(\nabla'^H_{\nabla_x b}\big(\nabla'(w)\big)\Big)(b') \\
&= \Big(\nabla'^H_x\Big(\nabla'^H_b\big(\nabla'(w)\big)\Big)\Big)(b') - \Big(\nabla'^H_{\nabla_x b}\big(\nabla'(w)\big)\Big)(b') = \Big(\nabla'^H_{xb}\big(\nabla'(w)\big)\Big)(b')
\end{aligned}$$

which proves the induction and eqn (5.1.26).

Next, we compute

$$\nabla'_b\big(\nabla'_{b'}(w)\big) = \nabla'_b\big(\big(\nabla'(w)\big)(b')\big) \overset{(5.1.24)}{=} \sum_{(b)} \Big(\nabla'^{\mathrm{H}(\mathsf{T}_A(L),W)}_{b^{(1)}}\big(\nabla'(w)\big)\Big)\big(\nabla_{b^{(2)}}(b')\big) \overset{(5.1.26)}{=} \sum_{(b)} \nabla'_{b^{(1)}\left(\nabla_{b^{(2)}}(b')\right)}(w)$$

which proves the equation (5.1.27). □

## 5.2 Rinehart bialgebra structure on $\mathsf{T}_A(L)$

The preceding Proposition 5.1.12 suggests the following multiplication $\diamond_\nabla = \diamond$ on $\mathsf{T}_A(L)$ defined by

$$b \diamond_\nabla b' := \sum_{(b)} b^{(1)} \nabla_{b^{(2)}}(b') \tag{5.2.1}$$

for all $b, b' \in \mathsf{T}_A(L)$ which we shall call the $\nabla$-*deformed multiplication of* $\mathsf{T}_A(L)$ because obviously

$$b \diamond_\nabla b' = bb' + \text{ terms of strictly lower tensor degree.} \tag{5.2.2}$$

Notice that the multiplication $\diamond_\nabla$ is only $K$-bilinear. Then, we get the following:

**Theorem 5.2.1** *Let $(L, \rho)$ be an anchored $A$-module, let $W$ be an $A$-module, and let $\nabla'$ be connection in $W$, $\nabla$ be a connection in $L$ parametrized by $(L, \rho)$, let $\nabla^A$ denote the connection in $A$ induced by the anchor map $\rho$, and let $\nabla$, $\nabla'$ and $\nabla^A$ also denote the iterated covariant derivatives as defined above in (5.1.18).*

*We have the following properties for all $b, b', b'' \in \mathsf{T}_A(L)$ and $w \in W$:*

$$\nabla'_b\big(\nabla'_{b'}(w)\big) = \nabla'_{b \diamond_\nabla b'}(w), \tag{5.2.3}$$

$$\nabla_b(b'b'') = \sum_{(b)} \big(\nabla_{b^{(1)}}(b')\big)\big(\nabla_{b^{(2)}}(b'')\big) \tag{5.2.4}$$

$$\big[\mathsf{deg}, \nabla_b\big] = 0 \tag{5.2.5}$$

$$\Delta_{sh}\big(\nabla_b(b')\big) = \sum_{(b)(b')} \nabla_{b^{(1)}}(b'^{(1)}) \otimes_A \nabla_{b^{(2)}}(b'^{(2)}), \tag{5.2.6}$$

$$\Delta_{sh}(b \diamond_\nabla b') = \sum_{(b)(b')} \big(b^{(1)} \diamond_\nabla b'^{(1)}\big) \otimes_A \big(b^{(2)} \diamond_\nabla b'^{(2)}\big), \tag{5.2.7}$$

$$(b \diamond_\nabla b') \diamond_\nabla b'' = b \diamond_\nabla (b' \diamond_\nabla b''), \tag{5.2.8}$$

$$b \diamond_\nabla \mathbf{1} = b = \mathbf{1} \diamond_\nabla b, \tag{5.2.9}$$

$$\epsilon(b \diamond_\nabla b') = \nabla^A_b\big(\epsilon(b')\big) = \epsilon\Big(b \diamond_\nabla \big(\epsilon(b')\mathbf{1}\big)\Big). \tag{5.2.10}$$

*In particular, $\big(\mathsf{T}_A(L), \mathbf{1}, \diamond, \Delta_{sh}, \epsilon\big)$ is a Rinehart bialgebra over $A$, see Section 4.3 for definitions.*

*Proof.* The first equation, number (5.2.3), is just a reformulation of the equation (5.1.27) of Proposition 5.1.12 using the $\nabla$-deformed multiplication described in the equation (5.2.1).

The iterated derivation rule (5.2.4) is a direct consequence Proposition 5.1.11 upon setting $\nu = \mu$.

The homogeneity equation (5.2.5) is shown by induction over the tensor degree of $b$: again, since $\nabla_{\mathbf{1}} = \mathrm{id}$ this is true in degree 0, and in degree 1 ($b = x \in v$) this

follows from eqn (5.1.13). Suppose by induction that the asserted equation (5.2.5) is true for all $b \in \mathsf{T}(V)$ whose tensor degree is less or equal than $n$, and consider the element $xb$ for any $x \in V$. We compute (using the fact that $\nabla_x(b)$ is of degree $\leqslant n$ according to eqn (5.1.13))

$$\begin{aligned}\Big[\nabla_{xb}, \mathsf{deg}\Big] = \Big[\nabla_x \circ \nabla_b - \nabla_{\nabla_x(b)}, \mathsf{deg}\Big] \quad &= \quad \Big[\nabla_x, \mathsf{deg}\Big] \circ \nabla_b + \nabla_x \circ \Big[\nabla_b, \mathsf{deg}\Big] - \Big[\nabla_{\nabla_x(b)}, \mathsf{deg}\Big] \\ &\overset{\text{Ind.Hyp.}}{=} \quad 0 + 0 - 0 = 0\end{aligned}$$

proving the induction.

In the following, the formula (5.2.6) follows from the fact that the shuffle comutiplication $\Delta_{sh}$ is connection preserving (see Proposition 5.1.10), from statement eqn (5.1.25), and from formula (5.1.22) of the Proposition 5.1.11.

The equation (5.2.7) can be shown directly upon using eqs (5.2.1) and (5.2.6):

$$\begin{aligned}\Delta_{sh}(b \diamond_\nabla b') \quad &= \quad \sum_{(b)} \Delta_{sh}\big(b^{(1)} \nabla_{b^{(2)}}(b')\big) \overset{(5.2.6)}{=} \sum_{(b)(b')} \big((b^{(1)}) \nabla_{b^{(3)}}(b'^{(1)})\big) \otimes_A \big((b^{(2)}) \nabla_{b^{(4)}}(b'^{(2)})\big) \\ &\overset{(\Delta_{sh}\ \text{cocomm.})}{=} \quad \sum_{(b)(b')} \big((b^{(1)}) \nabla_{b^{(2)}}(b'^{(1)})\big) \otimes_A \big((b^{(3)}) \nabla_{b^{(4)}}(b'^{(2)})\big) \\ &= \quad \sum_{(b)(b')} \big(b^{(1)} \diamond_\nabla b'^{(1)}\big) \otimes_A \big(b^{(2)} \diamond_\nabla b'^{(2)}\big).\end{aligned}$$

Next, the equation (5.2.8) encodes the associativity of the $\nabla$-deformed multiplication $\diamond_\nabla$: this can also be shown directly:

$$\begin{aligned} b \diamond_\nabla (b' \diamond_\nabla b'') \quad &= \\ = \sum_{(b)(b')} (b^{(1)}) \nabla_{b^{(2)}}\big((b'^{(1)}) \nabla_{b'^{(2)}}(b'')\big) \quad &\overset{(5.2.4)}{=} \quad \sum_{(b)(b')} (b^{(1)}) \big(\nabla_{b^{(2)}}(b'^{(1)})\big) \big(\nabla_{b^{(3)}}\big(\nabla_{b'^{(2)}}(b'')\big)\big) = \\ \overset{(5.2.1),(5.2.3)}{=} \sum_{(b)(b')} \big(b^{(1)} \diamond_\nabla b'^{(1)}\big) \big(\nabla_{b^{(2)} \diamond_\nabla b'^{(2)}}(b'')\big) \quad &\overset{(5.2.7)}{=} \quad \sum_{(b)(b')} \big((b \diamond_\nabla b')^{(1)}\big) \big(\nabla_{(b \diamond_\nabla b')^{(2)}}(b'')\big) = \\ &\overset{(5.2.1)}{=} \quad (b \diamond_\nabla b') \diamond_\nabla b''.\end{aligned}$$

In order to prove the equation (5.2.9) we note that trivially $\mathbf{1} \diamond_\nabla b = b$ from the definition (5.2.1). On the other hand, since the unit $\mathbf{1}$ is connection preserving by Proposition 5.1.10 it follows by the equation (5.1.25) of Proposition 5.1.11 that

$$\nabla_b(\mathbf{1}) = \epsilon(b)\mathbf{1}, \quad \text{hence} \quad b \diamond_\nabla \mathbf{1} = \sum_{(b)} b^{(1)} \nabla_{b^{(2)}}(\mathbf{1}) = \sum_{(b)} b^{(1)} \epsilon(b^{(2)}) \mathbf{1} = b\mathbf{1} = b$$

which proves (5.2.9).

Finally, in order to show the last equation, number (5.2.10), we compute for all $b, b' \in \mathsf{T}(V)$

$$\begin{aligned}\epsilon(b * \diamond_\nabla b') = \sum_{(b)} \epsilon\big(b^{(1)}\big)\nabla_{b^{(2)}}(b')\big) &= \sum_{(b)} \epsilon(b^{(1)})\epsilon\big(\nabla_{b^{(2)}}(b')\big) = \\ = \epsilon\big(\nabla_{\sum_{(b)}\epsilon(b^{(1)})b^{(2)}}(b')\big) &= \epsilon\big(\nabla_b(b')\big) \\ = \nabla^A_b\big(\epsilon(b')\big) &= \epsilon\Big(\nabla^A_b\big(\epsilon(b')\mathbf{1}\big)\Big) = \epsilon\Big(\nabla_b\big(\epsilon(b')\mathbf{1}\big)\Big).\end{aligned}$$

where the fact that $\epsilon$ is also connection preserving by Proposition 5.1.10 and equation (5.1.25) has been used. Replacing in the above equation $b'$ by its projection $\epsilon(b')\mathbf{1}$ is easily seen to give the same result, whence statement (5.2.10) is shown.

Finally, the preceding statements (5.2.8), (5.2.9), the fact that $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\big)$ is a coassociative cocommutative counitary coaugmented coalgebra over $A$, statements (5.2.7) and (5.2.10) show almost all the properties of a Rinehart bialgebra over $A$, see Section 4.3, eqn (4.3.4).

It remains to show that the image of the shuffle comultiplication $\Delta_{sh}$ is in the $A$-submodule $\mathsf{T}_A(L) \times_A \mathsf{T}_A(L)$, see eqn (4.3.1): let $b \in \mathsf{T}_A(L)$ and $a \in A$. We compute – upon using coassoacitivity and cocommutativity of $\Delta_{sh}$:

$$\begin{aligned} r^{(2)}_a\big(\Delta_{sh}(b)\big) &= \sum_{(b)} b^{(1)} \otimes_A \big(b^{(2)} \diamond a\big) = \sum_{(b)} b^{(1)} \otimes_A \big(b^{(2)}\nabla_{b^{(3)}}(a)\big) = \sum_{(b)} b^{(1)} \otimes_A \big(\nabla_{b^{(3)}}(a)b^{(2)}\big) \\ &= \sum_{(b)} b^{(1)}\big(\nabla_{b^{(3)}}(a)\big) \otimes_A b^{(2)} = \sum_{(b)} b^{(1)}\big(\nabla_{b^{(2)}}(a)\big) \otimes_A b^{(3)} = \sum_{(b)} \big(b^{(1)} \diamond a\big)\big) \otimes_A b^{(2)} \\ &= r^{(1)}_a\big(\Delta_{sh}(b)\big) \end{aligned}$$

□

The particular case $A = K$ is simple, but important for the sequel: here any anchored module $K$-module $L$ has vanishing anchor morphism, and the Rinehart bialgebra $T_K(L)$ is an ordinary bialgebra over $K$. Any connection $\nabla$ is thus an arbitrary $K$-bilinear map $L \otimes L \to L$. It comes with no surprise that the bialgebra structures on $T_K(L)$ with the free multiplication and the multiplication $\diamond_\nabla$ are isomorphic:

**Proposition 5.2.2** *The $K$-linear map $\Phi : T_K(L) \to T_K(L)$ defined by $\Phi(\mathbf{1}) = \mathbf{1}$ and for all $N \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_N \in L$ by*

$$\Phi(x_1 \cdots x_N) = x_1 \diamond_\nabla \cdots \diamond_\nabla x_N. \tag{5.2.11}$$

*provides an isomorphism of $K$-bialgebras $\big(T_K(L), \cdot, \mathbf{1}, \Delta_{sh}, \epsilon\big) \to \big(T_K(L), \diamond_\nabla, \mathbf{1}, \Delta_{sh}, \epsilon\big)$.*

*Proof.* Since $(T_K(L), \cdot, \mathbf{1})$ is a free algebra over the $K$-module $L$, the map $\Phi$ obviously is a well-defined morphism of unital associative algebras $\big(T_K(L), \cdot, \mathbf{1}\big) \to \big(T_K(L), \diamond_\nabla, \mathbf{1}\big)$ induced by the identity map on the generating module $L$. Since the product on the right hand side of eqn (5.2.11) is equal to the free product $x_1 \cdots x_N$ plus terms of lower tensor degree, a simple filtration argument shows that $\Phi$ is bijective.

The identities $\epsilon \circ \Phi = \epsilon$ and $(\Phi \otimes \Phi) \circ \Delta_{sh} = \Delta_{sh} \circ \Phi$ are identities of algebra morphisms $\big(\mathsf{T}_K(L), \mu\big) \to K$ and $\big(\mathsf{T}_K(L), \mu\big) \to \big(\mathsf{T}_K(L) \otimes \mathsf{T}_K(L), \diamond_\nabla^{[2]}\big)$, respectively (see eqs (5.2.7) and (5.2.10)), and are obviously satisfied on generators in $L$, hence they hold everywhere thanks to the freeness of $\mathsf{T}_K(L)$. □

Returning to the general case with a priori different $A$ and $K$ we shall discuss the relation between $\mathsf{T}_K(L)$ and $\mathsf{T}_A(L)$ in the presence of a connection $\nabla : L \otimes L \to L$ in the anchored $A$-module $(L, \rho)$: first, more generally, let $(V, \nabla^V)$ and $(W, \nabla^W)$ two $A$-modules with connection along $(L, \rho)$. Denoting by $\tilde\nabla^V$ and $\tilde\nabla^W$ the same maps $\nabla^V$ and $\nabla^W$, respectively, considered just as $K$-bilinear (one could have used the restriction functor Res, see Appendix A.2.1 we obviously get for all $v \in V$ and $w \in W$ (thanks to eqn (5.1.6)

$$\pi_{V,W}\Big(\tilde\nabla^V_x(v) \otimes w + v \otimes \tilde\nabla^w_x(w)\Big) = \nabla_x^{V \otimes_A W}\big(v \otimes_A w\big), \tag{5.2.12}$$

where $\pi_{V,W} : V \otimes W \to V \otimes_A W$ is the canonical projection, see equation A.2.7.

Applying this to $V = W = L$, $\nabla^V = \nabla^W = \nabla$ in an iterative way (where $\tilde\nabla$ also denotes the induced iterated connection in $\mathsf{T}_K(L)$) and writing $\pi : \mathsf{T}_K(L) \to \mathsf{T}_A(L)$ for the $K$-linear canonical projection, see also Appendix A.6.1, an easy induction over tensorial degree yields the following for all $\tilde b, \tilde b' \in \mathsf{T}_K(L)$

$$\pi\big(\tilde\nabla_{\tilde b}(\tilde b')\big) = \nabla_{\pi(\tilde b)}\big(\pi(\tilde b')\big) \quad \text{hence} \quad \pi\big(\tilde b \tilde\diamond \tilde b'\big) = \pi(\tilde b) \diamond_\nabla \pi(\tilde b). \tag{5.2.13}$$

where we have written $\tilde\diamond = \diamond_{\tilde\nabla}$ for the multiplication in $\mathsf{T}_K$ according to Proposition 5.2.2.

## 5.3 The primitive part $\mathcal{P}_\nabla(L, A)$ of $\mathsf{T}_A(L)$ and the path Lie algebroid

Let $(L, \rho)$ be an anchored module over $A$, let $\nabla$ be a connection in the $A$-module $L$ along $(L, \rho)$. We denote by the same symbol $\nabla$ the iterated covariant derivative in $T_A(L)$. We will suppose in this section that $\mathbb{Q} \subset K$.

Recall that the primitive part of the counital coaugmented $A$-coalgebra $\mathsf{T}_A(L)$ is equal to $\mathcal{L}_A(L)$, the $A$-Lie subalgebra of the free $A$-algebra $\mathsf{T}_A(L)$ generated by the $A$-submodule $L$, see Appendix A.6.2, which in turn is isomorphic to the free $A$-Lie algebra over the $A$-module $L$.

We can conclude that the statement 5.2.7 of the preceding Theorem 5.2.1 directly implies the following

**Proposition 5.3.1** *The primitive part of the Rinehart bialgebra $\big(T_A(L), \diamond_\nabla = \diamond, \mathbf{1}, \Delta_{sh}, \epsilon\big)$ carries the structure of a Lie-Rinehart algebra over $A$ where the $K$-bilinear Lie bracket $[\ ,\ ]^\nabla$ and the anchor morphism $\rho^\nabla$ are explicitly given as follows for all $\xi, \xi' \in \mathcal{P}_\nabla(L,A)$ and $a \in A$:*

$$[\![\xi, \xi']\!]^\nabla := \xi \diamond \xi' - \xi' \diamond \xi = [\xi, \xi'] + \nabla_\xi(\xi') - \nabla_{\xi'}(\xi), \quad \text{and} \quad \rho^\nabla_\xi(a) = \nabla^A_\xi(a). \tag{5.3.1}$$

*Moreover, note that for any $A$-module with connection along $L$, $(V, \nabla')$ there is the formula*

$$\forall\ \xi_1, \xi_2 \in \mathcal{P}^\nabla(L,A): \qquad \nabla'_{[\![\xi_1,\xi_2]\!]^\nabla} = \Big[\nabla'_{\xi_1}, \nabla'_{\xi_2}\Big]. \tag{5.3.2}$$

*We shall write $\mathcal{P}_\nabla(L,A)$ for the $A$-module $\mathcal{L}_A(L)$ equipped with the above Lie-Rinehart structure.*

Indeed this follows from the multiplication formula (5.2.1) and the fact that

$$\Delta_{sh}(\xi) = \sum_{(\xi)} \xi^{(1)} \otimes_A \xi^{(2)} = \xi \otimes_A \mathbf{1} + \mathbf{1} \otimes_A \xi$$

for each primitive element in $T_A(L)$.

Moreover, according to the Leibniz formula (5.1.21) for the case $W = A$, it follows that for each $\xi \in \mathcal{P}_\nabla(L,A)$ the map $\nabla^A_\xi : A \to A$ is a derivation, and the commutator of $K$-linear maps, $\big[\nabla^A_\xi, \nabla^A_\eta\big]$, clearly equals $\nabla^A_{[\xi,\eta]^\nabla}$ (as well as for $A$ replaced by $V$) according to eqn (5.2.3) which shows (5.3.1).

Note that the underlying $A$-module of $\mathcal{P}_\nabla(L,A)$, $\mathcal{L}_A(L)$, is graded by tensor degree, i.e. $\mathcal{P}_\nabla(L,A) = \oplus_{n=1}^\infty \mathcal{P}_\nabla(L,A)^n$, but the Lie bracket $[\ ,\ ]^\nabla$ does no longer preserve the grading, but is merely filtration preserving.

The relation to Kapranov's path Lie algebroid (see Section 4.5.1) is contained in the following

**Theorem 5.3.2** *The Lie-Rinehart algebra $\big(\mathcal{P}_\nabla(L,A), \rho^\nabla, [\ ,\ ]^\nabla, A\big)$ is a free Lie-Rinehart algebra, and given any morphism of anchored modules $\theta : (L,\rho) \to (L',\rho')$ where $\big(L', \rho', [\![\ ,\ ]\!]'\big)$ is any Lie-Rinehart algebra over $A$, the induced morphism of Lie-Rinehart algebras $\bar\theta : \mathcal{P}_\nabla(L,A) \to L'$ can be computed by the following recursion for all $x \in L$ and $\xi \in \mathcal{P}_\nabla(L,A)$*

$$\bar\theta(x) = \theta(x) \ \text{ and } \ \bar\theta\big([x,\xi]\big) = [\![\theta(x), \bar\theta(\xi)]\!]' - \bar\theta\left(\nabla_x(\xi)\right) + \theta\left(\nabla_\xi(x)\right). \tag{5.3.3}$$

*Hence the Lie-Rinehart algebra $\mathcal{P}_\nabla(L,A)$ is isomorphic to Kapranov's path Lie algebroid $\mathcal{P}(L,A)$ generated by $(L,\rho)$. In particular, for two different connections $\nabla, \nabla'$ the two Lie-Rinehart algebras $\mathcal{P}_\nabla(L,A)$ and $\mathcal{P}_{\nabla'}(L,A)$ are isomorphic.*

*Proof.* We shall show the universal property for the Lie-Rinehart algebra $\mathcal{P}_\nabla(L,A)$, i.e. that for any given Lie-Rinehart algebra $(L', \rho', [\![\ ,\ ]\!]', A)$ and every morphism

$\theta : (L,\rho) \to (L',\rho')$ of anchored $A$-modules there is a unique induced morphism $\tilde{\theta}$ of Lie-Rinehart algebras $\mathcal{P}_\nabla(L,A) \to L'$, and every morphism of Lie-Rinehart algebras $\mathcal{P}_\nabla(L,A) \to L'$ is of that form.

Indeed, take the free Lie algebra over $K$ generated by the $K$-module $L$, $\mathsf{Lie}_K(L)$. Since $\mathbb{Q} \subset K$ we can take the Lie subalgebra $\mathcal{L}_K(L) \subset \mathsf{T}_K(L)$ generated by the $K$-module $L$ which is equal to the primitive part of the coalgebra $\big(\mathsf{T}_K(L), \tilde{\mathbf{1}}, \tilde{\Delta}_{sh}, \tilde{\epsilon}\big)$ and isomorphic to $\mathsf{Lie}_K(L)$, see Appendix A.6.2.

There is a unique morphism of $K$-Lie algebras $\check{\theta} : \big(\mathcal{L}_K(L), [\ ,\ ]^\sim\big) \to \big(L', [\![\ ,\ ]\!]'\big)$ such that for all $x \in L$ we have $\check{\theta}(x) = \theta(x)$.

Observe now that the restriction of the above bialgebra morphism

$$\Phi : \big(\mathsf{T}_K(L), \tilde{\mu}, \tilde{\mathbf{1}}, \tilde{\Delta}_{sh}, \tilde{\epsilon}\big) \to \big(\mathsf{T}_K(L), \tilde{\diamond}, \tilde{\mathbf{1}}, \tilde{\Delta}_{sh}, \tilde{\epsilon}\big),$$

see Proposition 5.2.2, to $\mathcal{L}_K(L)$ is an isomorphism of $K$-Lie algebras $\big(\mathcal{L}_K(L), [\ ,\ ]^\sim\big) \to \big(\mathcal{L}_K(L), [\ ,\ ]^{\tilde{\nabla}}\big)$ (where we have written $[\ ,\ ]^{\tilde{\nabla}}$ for the Lie bracket (5.3.1) with $A$ replaced by $K$ and $\nabla$ by $\tilde{\nabla}$, see the paragraph at the end of Section 5.1.2) since $\Phi$ is an isomorphism of coaugmented coalgebras and preserves primitive elements. It follows that the $K$-linear map

$$\hat{\theta} = \check{\theta} \circ \Phi^{-1} : \Big(\mathcal{L}_K(L), [\ ,\ ]^{\tilde{\nabla}}\Big) \to \big(L', [\![\ ,\ ]\!]'\big) \tag{5.3.4}$$

is a morphism of $K$-Lie algebras (where we have used the same symbol $\Phi$ for the restriction of $\Phi$ to $\mathcal{L}_K(L)$).

Recall that the canonical $K$-algebra morphism $\pi : \big(\mathsf{T}_K(L), \tilde{\mu}\big) \to \big(\mathsf{T}_A(L), \mu\big)$, see equation (A.6.10), is also a $K$-algebra morphism $\big(\mathsf{T}_K(L), \tilde{\diamond}\big) \to \big(\mathsf{T}_A(L), \diamond_\nabla\big)$, see the discussion at the end of Section 5.1.2.

Thanks to the equation A.6.11 $\pi$ maps primitive elements onto primitive elements, and the restriction of $\pi$ to $\mathcal{L}_K(L)$ (which we continue to write $\pi$) is thus a morphism of $K$-Lie algebras from $\big(\mathcal{L}_K(L), [\ ,\ ]^{\tilde{\nabla}}\big)$ to $\big(\mathcal{P}_\nabla(L,A), [\ ,\ ]^\nabla\big)$.
We shall now show by induction on the tensor degree $n \in \mathbb{N}\backslash\{0\}$ that the above map $\hat{\theta}$ descends to an $A$-linear map: more precisely, for each positive integer $n$ there exists an $A$-linear map $\bar{\theta}_n : \mathcal{P}_\nabla(L,A)^n \to L'$ such that for all $\tilde{\xi} \in \mathcal{L}^n(L,A)$

$$\bar{\theta}_n\big(\pi(\tilde{\xi})\big) = \hat{\theta}(\tilde{\xi}) \ \text{ and } \ \rho'_{\bar{\theta}_n(\pi(\tilde{\xi}))} = \nabla^A_{\pi(\tilde{\xi})} \Big(= \rho^\nabla_{\pi(\tilde{\xi})}\Big). \tag{5.3.5}$$

Indeed, this is obvious for $n = 1$ by setting $\bar{\theta}_1 = \theta$ from the definition of a morphism of anchored modules.

Suppose that the induction hypothesis is true up to rank $n \geqslant 1$.

**Induction step** ($n \to n+1$): Let $\tilde{\xi} \in \mathcal{L}^n_K(L)$ be a left ordered multiple commutator

$$\tilde{\xi} = \Big[x_1, \big[x_2, [x_3, \ldots, [x_i, \ldots, [x_{n-1}, x_n]^\sim]]^\sim \cdots\big]^\sim\Big]^\sim$$

with $x_1,\dots,x_n$ if $n \geqslant 2$ and if $n=1$, $\tilde{\xi} = x_1$. For a chosen integer $1 \leqslant i \leqslant n$ and $a \in A$ we shall write $\tilde{\xi}_{(i)}(a)$ for the above multiple commutator in which $x_i$ is replaced by $ax_i$.

It is not hard to see that the intersection of the kernel of the projection $\pi$ (which is graded by the positive integers) with each $\mathcal{L}^n_K(L)$, $n \geqslant 2$ an integer, is spanned over $K$ by all elements of the form $\tilde{\xi}_{(i)}(a) - \tilde{\xi}_{(j)}(a)$ for all $1 \leqslant i < j \leqslant n$.

In the following, choose any $x \in L$. Thanks to the fact that $\hat{\theta}$ is a morphism of $K$-Lie algebras, thanks to the form of the Lie bracket $[\ ,\ ]^{\tilde{\nabla}}$ (see eqn (5.3.1)) for $\nabla$ replaced by $\tilde{\nabla}$) we get upon using the induction hypothesis (5.3.5) and formula (5.2.13) and upon writing $\xi = \pi(\tilde{\xi})$, $x = \pi(x)$, $\bar{\theta}_1(x) = \theta(x)$:

$$\begin{aligned}\hat{\theta}\big([x,\tilde{\xi}]^\sim\big) &= [\![\theta(x), \hat{\theta}(\tilde{\xi})]\!]' - \hat{\theta}\left(\tilde{\nabla}_x(\tilde{\xi})\right) + \hat{\theta}\left(\tilde{\nabla}_{\tilde{\xi}}(x)\right)\\ &= [\![\theta(x), \bar{\theta}_n(\xi)]\!]' - \bar{\theta}_n\left(\nabla_x(\xi)\right) + \theta\left(\nabla_\xi(x)\right). \qquad (5.3.6)\end{aligned}$$

Replacing $\tilde{\xi}$ by $\tilde{\xi}_{(i)}(a)$ (for any $a \in A$) we get upon using the evident equation $\pi\left(\tilde{\xi}_{(i)}(a)\right) = a\pi(\tilde{\xi}) = a\xi$ and the $A$-linearity of $\bar{\theta}_n$ thanks to the induction hypothesis:

$$\begin{aligned}\hat{\theta}\big([x,\tilde{\xi}_{(i)}(a)]^\sim\big) &= [\![\theta(x), a\bar{\theta}_n(\xi)]\!]' - \bar{\theta}_n\left(\nabla_x(a\xi)\right) + a\theta\left(\nabla_\xi(x)\right)\\ &= a[\![\theta(x), \bar{\theta}_n(\xi)]\!]' + \rho'_{\theta(x)}(a)\bar{\theta}_n(\xi) - a\bar{\theta}_n\left(\nabla_x(\xi)\right) - \rho_x(a)\bar{\theta}_n(\xi) + a\theta\left(\nabla_\xi(x)\right)\\ &= a\hat{\theta}\big([x,\tilde{\xi}]^\sim\big) \qquad (5.3.7)\end{aligned}$$

and (here we use the second part of the induction hypothesis (5.3.5))

$$\begin{aligned}\hat{\theta}\big([ax,\tilde{\xi}]^\sim\big) &= [\![a\theta(x), \bar{\theta}_n(\xi)]\!]' - a\bar{\theta}_n\left(\nabla_x(\xi)\right) + \theta\left(\nabla_\xi(ax)\right)\\ &= a[\![\theta(x), \bar{\theta}_n(\xi)]\!]' - \rho'_{\bar{\theta}_n(\xi)}(a)\theta(x) - a\bar{\theta}_n\left(\nabla_x(\xi)\right) + a\theta\left(\nabla_\xi(x)\right) + \nabla^A_\xi(a)\theta(x)\\ &= a\hat{\theta}\big([x,\tilde{\xi}]^\sim\big). \qquad (5.3.8)\end{aligned}$$

Both equations (5.3.7) and (5.3.8) give the same result independently of the integer $i$ which implies that $\hat{\theta}$ vanishes on the kernel of $\pi$ in degree $n+1$ and descends to an $A$-linear map $\bar{\theta}_{n+1} : \mathcal{P}_\nabla(L,A)^{n+1} \to L'$ satisfying $\hat{\theta}|_{\mathcal{L}^{n+1}(L,K)} = \bar{\theta}_{n+1} \circ \pi|_{\mathcal{L}^{n+1}(L,K)}$. In particular it follows that $\hat{\theta}\big([x,\tilde{\xi}]^\sim\big) = \bar{\theta}\big([x,\xi]\big)$.

It remains to show the second part of the induction, equation (5.3.5): again using eqn (5.3.6) we can write for all $a \in A$

$$\begin{aligned}\rho'_{\bar{\theta}_{n+1}([x,\xi])}(a) &= \rho'_{\hat{\theta}([x,\tilde{\xi}]^\sim)}(a) = \rho'_{[\![\theta(x),\bar{\theta}_n(\xi)]\!]'}(a) - \rho'_{\bar{\theta}_n(\nabla_x(\xi))}(a) + \rho'_{\theta\left(\nabla_\xi(x)\right)}(a)\\ &= \left[\rho'_{\theta(x)}, \rho'_{\bar{\theta}_n(\xi)}\right](a) - \rho'_{\bar{\theta}_n(\nabla_x(\xi))}(a) + \rho'_{\theta\left(\nabla_\xi(x)\right)}(a)\\ &\overset{(5.3.5)}{=} \left[\nabla^A_x, \nabla^A_\xi\right](a) - \nabla^A_{\nabla_x(\xi)}(a) + \nabla^A_{\nabla_\xi(x)}(a) = \nabla^A_{[x,\xi]}(a), \qquad (5.3.9)\end{aligned}$$

which proves the induction.

Let $\bar\theta : \mathcal{P}_\nabla(L,A) \to L'$ be the $A$-linear map defined on each component $\mathcal{P}_\nabla(L,A)^n$ by the above $A$-linear map $\bar\theta_n$. We clearly have $\hat\theta = \bar\theta \circ \pi$. Next, the second part of the induction hypothesis (5.3.5) implies that $\bar\theta$ intertwines anchor morphisms. It remains to show that $\bar\theta : \big(\mathcal{P}_\nabla(L,A), [\,,\,]^\nabla\big) \to \big(L', [\![\,,\,]\!]'\big)$ is a morphism of $K$-Lie algebras: let $\xi_1, \xi_2 \in \mathcal{P}_\nabla(L,A)$, and by the surjectivity of the restriction of $\pi$ to positive degrees there are $\tilde\xi_1, \tilde\xi_2 \in \mathcal{L}_K(L)$ such that $\pi(\tilde\xi_1) = \xi_1$ and $\pi(\tilde\xi_2) = \xi_2$. Recall that $\hat\theta : \big(\mathcal{L}_K(L), [\,,\,]^{\tilde\nabla}\big) \to \big(L', [\![\,,\,]\!]'\big)$ is a morphism of $K$-Lie algebras by construction, and that $\pi : \big(\mathcal{L}_K(L), [\,,\,]^{\tilde\nabla}\big) \to \big(\mathcal{P}_\nabla(L,A), [\,,\,]^\nabla\big)$ is a morphism of $K$-Lie algebras. We get

$$\begin{aligned}
\bar\theta\Big([\xi_1,\xi_2]^\nabla\Big) &= \bar\theta\Big(\big[\pi(\tilde\xi_1),\pi(\tilde\xi_2)\big]^\nabla\Big) = \bar\theta\Big(\pi\big([\tilde\xi_1,\tilde\xi_2]^{\tilde\nabla}\big)\Big) = \hat\theta\big([\tilde\xi_1,\tilde\xi_2]^{\tilde\nabla}\big) = \\
&= [\![\hat\theta(\tilde\xi_1),\hat\theta(\tilde\xi_2)]\!]' = [\![\bar\theta\big(\pi(\tilde\xi_1)\big),\bar\theta\big(\pi(\tilde\xi_2)\big)]\!]' = \\
&= [\![\bar\theta(\xi_1),\bar\theta(\xi_2)]\!]' \qquad (5.3.10)
\end{aligned}$$

The recursion equation (5.3.3) is a simple particular case of the preceding equation (5.3.10) for $x = \xi_1$ and $\xi = \xi_2$ showing that $\bar\theta$ is a morphism of $K$-Lie algebras which shows existence of the induced morphism $\bar\theta$. Again by the recursion equation (5.3.3) it is clear that the higher degree terms $\bar\theta_n$ for $n \geqslant 2$ uniquely depend on $\bar\theta_1 = \theta$ which makes the assignment $\theta \mapsto \bar\theta$ a map.

Clearly if $\theta, \chi : (L,\rho) \to \big(L', \rho', [\![\,,\,]\!]', A\big)$ are morphisms of anchored $A$-modules, then $\bar\theta = \bar\chi$ implies $\theta(x) = \chi(x)$ for all $x \in L$ and $\theta \mapsto \bar\theta$ is thus injective. Finally, note that every morphism of Lie-Rinehart algebras $\Theta : \mathcal{P}_\nabla(L,A) \to L'$ satisfies the recursion relation (5.3.3), is hence uniquely determined by its restriction $\theta = \Theta|_L : L \to L'$, and hence equal to $\bar\theta$ which shows the surjectivity of $\theta \mapsto \bar\theta$, and the universal property.

Since $\mathcal{P}(L,A)$, $\mathcal{P}_\nabla(L,A)$ and $\mathcal{P}_{\nabla'}(L,A)$ are all universal objects the rest of the Theorem follows. □

In the same way as above we show the following

**Theorem 5.3.3** *Let $(L,\rho,\nabla)$ be an anchored $A$-module with connection. Then the Rinehart bialgebra $\big(\mathsf{T}_A(L), \diamond_\nabla, \mathbf{1}, \Delta_{sh}, \epsilon\big)$ is isomorphic to the free unitary associative $K$-algebra over $A$ generated by the anchored $A$-module $(L,\rho)$: given any unital associative $K$-algebra over $A$, $\big(j', B', \diamond', \mathbf{1}'\big)$, and a morphism $\theta : (L,\rho) \to B'^{-}_{A}$ of anchored $A$-modules there is a unique morphism $\Theta : \big(\iota_A, \mathsf{T}_A(L), \diamond_\nabla, \mathbf{1}\big) \to \big(B', \diamond', \mathbf{1}'\big)$ of unital associative $K$-algebras over $A$ with $\Theta(x) = \theta(x)$ for all $x \in L$.*
*It follows that $\big(\mathsf{T}_A(L), \diamond_\nabla, \mathbf{1}, \Delta_{sh}, \mathbf{1}\big)$ is isomorphic (as a Rinehart bialgebra) to the universal enveloping algebra $\mathcal{U}\big(\mathcal{P}_\nabla(L,A), A\big)$ of the Lie-Rinehart algebra $\mathcal{P}_\nabla(L,A)$.*

*Proof.* The construction of $\Theta$ goes along the same lines as the proof of the preceding Theorem 5.3.2: note first that the morphism $\theta : (L,\rho) \to B'^{-}_{A}$ of anchored $A$-modules means that there exists a unique $A$-linear map $\vartheta : L \to B'$ satisfying for all $x \in L$

$$\theta(x) = \big(\rho_x, \vartheta(x)\big) \text{ such that } \forall\, a \in a : \; j'(\rho_x(a)) = \vartheta(x) \diamond' j'(a) - j'(a) \diamond' \vartheta(x) \quad (5.3.11)$$

see eqn (4.2.9) for details.

Next we define a morphism of unital $K$-algebras $\check{\vartheta} : \big(\mathsf{T}_K(L), \tilde{\mu}, \tilde{\mathbf{1}}\big) \to \big(B', \diamond', \mathbf{1}'\big)$ thanks to the freeness of $\mathsf{T}_K(L)$, use the $K$-linear isomorphism $\Phi : \big(\mathsf{T}_K(L), \tilde{\mu}, \tilde{\mathbf{1}}\big) \to \big(\mathsf{T}_K(L), \diamond_{\tilde{\nabla}}, \tilde{\mathbf{1}}\big)$, see eqn (5.2.11), to get a morphism of unital $K$-algebras $\hat{\vartheta} = \check{\vartheta} \circ \Phi^{-1} : \big(\mathsf{T}_K(L), \diamond_{\tilde{\nabla}}, \tilde{\mathbf{1}}\big) \to \big(B', \diamond', \mathbf{1}'\big)$. By induction over the tensor degree $n \in \mathbf{N}$ we show the existence of $A$-linear maps $\bar{\vartheta}_n$ from $\mathsf{T}^n_A(L) \to B'$ satisfying for all $a \in A$, $x \in L$, and $\tilde{b} \in \mathsf{T}^n_K(L)$

$$\bar{\vartheta}_0(a\mathbf{1}) = j'(a), \; \bar{\vartheta}_1(x) = \vartheta(x), \text{ and } \bar{\vartheta}_n\big(\pi(\tilde{b})\big) = \hat{\vartheta}(\tilde{b}).$$

Replacing in the proof of Thm 5.3.2 the multiple commutator $\tilde{\xi}$ by the product $\tilde{b} = x_1 \cdot_K \cdots \cdot_K x_n$ where $\cdot_K$ is the free multiplication in $\mathsf{T}_K(L)$, and $x_1, \ldots, x_n \in L$, and for each $a \in A$ and each integer $1 \leqslant i \leqslant n$ replacing $\tilde{\xi}_{(i)}(a)$ by $\tilde{b}_{(i)}(a)$ (the factor $x_i$ goes to $(ax_i)$) and using the morphism equation

$$\hat{\vartheta}(x \cdot_K \tilde{b}) \quad = \quad \vartheta(x) \diamond' \hat{\vartheta}(\tilde{b}) - \hat{\vartheta}\big(\tilde{\nabla}_x(\tilde{b})\big) \quad (5.3.12)$$

we get by induction, equation (5.3.11) and eqn (5.2.13) that

$$\hat{\vartheta}\big((ax) \cdot_K \tilde{b}\big) = j'(a) \diamond' \hat{\vartheta}(x \cdot_K \tilde{b}) = \hat{\vartheta}\big(x \cdot_K \tilde{b}_{(i)}(a)\big)$$

which shows that $\hat{\vartheta}$ vanishes on the kernel of $\pi$ in positive degrees and descends to an $A$-linear map $\bar{\vartheta}$ implying the induction and the unique existence of the morphism $\bar{\vartheta} : \mathsf{T}_A(L) \to B'$ of unital $K$-algebras over $A$. Universality is shown in a similar way as in Theorem 5.3.2.

Finally, given any morphism $\chi : \mathcal{P}_\nabla(L,A) \to B'^{-}_{A}$ of Lie-Rinehart algebras, it is uniquely determined by its restriction $\theta$ to $L$ (which is a morphism of anchored $A$-modules), and the preceding construction $\bar{\vartheta} : \mathsf{T}_A(L) \to B'$ will do the job whence $\big(\iota_A, \mathsf{T}_A(L), \diamond_\nabla, \mathbf{1}\big)$ is isomorphic to $\mathcal{U}\big(\mathcal{P}_\nabla(L,A), A\big)$. □

In the following section we shall suppose that $\mathbb{Q} \subset K$.

## 5.4 Path Lie algebroid for Lie-Rinehart algebras with connections

We shall apply the results of the preceding Section to the following important particular case where the anchored $A$-module is a Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ over $A$. Let $\nabla$ be a connection in $L$ along $L$, let $V$ be an arbitrary $A$-module equipped with a connection $\nabla'$ along $L$.

Let us recall two well-known classical quantities which are of fundamental importance in differential geometry.

**Definition 5.4.1** We define the *torsion* $\mathsf{Tor} = \mathsf{Tor}^{\nabla}$ and the *curvature* $\mathsf{R}' = \mathsf{R}^{\nabla'}$ as linear maps from $L \otimes_K L \to L$ and $(L \otimes_K L) \otimes_K V \to V$ in the following well-known way: for all $x, y \in L$ and $v \in V$

$$\mathsf{Tor}(x,y) := \nabla_x(y) - \nabla_y(x) - [\![x,y]\!], \tag{5.4.1}$$

$$\mathsf{R}'(x,y)(v) := \nabla'_x\big(\nabla'_y(v)\big) - \nabla'_y\big(\nabla'_x(v)\big) - \nabla'_{[\![x,y]\!]}(v) \tag{5.4.2}$$

**Remark 5.4.2** Both quantities have the following properties which are well-known in differential geometry. See e.g. [25, p.133-135] and [48, p.59-61].

The following result is quite lengthy to check then we will omit its proof.

**Proposition 5.4.3** *With the above definitions and notations: both torsion and curvature descend to well-defined $A$-linear maps $L \otimes_A L \to L$ and $(L \otimes_A L) \otimes_A V \to V$, respectively, which we shall denote in the usual way by $x \otimes_A y \mapsto \mathsf{Tor}^{\nabla}(x,y)$ and by $x \otimes_A y \otimes_A v \mapsto \mathsf{R}^{\nabla'}(x,y)(v)$, respectively. They satisfy the following* ***Bianchi identities*** *where the symbol $\mathfrak{S}_{(x,y,z)}F(x,y,z)$ of a map of three arguments $(x,y,z) \mapsto F(x,y,z)$ denotes the cyclic sum $F(x,y,z) + F(y,z,x) + F(z,x,y)$:*

$$\mathfrak{S}_{(x,y,z)}\Big(\mathsf{R}^{\nabla}(x,y)(z) - \big(\nabla_x(\mathsf{Tor}^{\nabla})\big)(y,z) - \mathsf{Tor}^{\nabla}\big(\mathsf{Tor}^{\nabla}(x,y),z\big)\Big) = 0, \tag{5.4.3}$$

$$\mathfrak{S}_{(x,y,z)}\Big(\big(\nabla'_x(\mathsf{R}^{\nabla'})\big)(y,z)(v) + \mathsf{R}^{\nabla'}\big(\mathsf{Tor}^{\nabla}(x,y),z\big)(v)\Big) = 0. \tag{5.4.4}$$

Here $\nabla_x(\mathsf{Tor}^{\nabla})$ or $\nabla'_x(\mathsf{R}^{\nabla'})$ denote the covariant derivatives of the $A$-linear maps $\mathsf{Tor}^{\nabla} : L \otimes_a L \to L$ and $\mathsf{R}^{\nabla'} : (L \otimes_A L) \otimes_A W \to W$.

We shall now describe an important particular case of Theorem 5.3.2 where the Lie-Rinehart algebra $L'$ is equal to $L$ (in categorical terms this corresponds to the counit of the adjunction (4.5.3)): we shall denote the morphism of Lie-Rinehart

algebras $\big(\mathcal{P}_\nabla(L,A),\rho^\nabla,[\ ,\ ]^\nabla,A\big) \to \big(L,\rho,[\![\ ,\ ]\!],A\big)$ induced by the identity map $L \to L$ by $Z = Z_\nabla$. It satisfies for all $x \in L$, $\xi,\xi' \in \mathcal{P}_\nabla(L,A)$, and $a \in A$

$$Z(x) = x, \quad Z\left([\![\xi,\xi']\!]^\nabla\right) = [\![Z(\xi),Z(\xi')]\!], \quad \rho_{Z(\xi)}(a) = \nabla^A_\xi(a). \tag{5.4.5}$$

It follows at once that for each $\xi \in \mathcal{P}_\nabla(L,A)$ the $K$-linear map

$$H'_\xi : V \to V : v \mapsto H'_\xi(v) = \nabla'_{\xi - Z(\xi)}(v) \tag{5.4.6}$$

is $A$-bilinear from $\mathcal{P}_\nabla(L,A) \times V$ to $V$: indeed, this is clear for the index argument, and for all $a \in A$ we have

$$\begin{aligned} H'_\xi(av) = \nabla'_\xi(av) - \nabla'_{Z(\xi)}(av) \qquad = \\ = \nabla^A_\xi(a)v + a\nabla'_\xi(v) - \rho_{Z(\xi)}(a)v - a\nabla'_{Z(\xi)}(v) \quad \overset{(5.4.5)}{=} \quad aH'_\xi(v). \end{aligned}$$

In particular, this holds for $V = L$ and $\nabla' = \nabla$ in which case we write $H_\xi$. We shall use the same symbol $H_\xi$ for the $A$-linear derivation of the free algebra $\mathsf{T}_A(L)$ equal to $H_\xi$ on generators. It is straight-forward to see that $H_\xi$ is also a coderivation of $\big(\mathsf{T}_A(L),\Delta_{sh},\epsilon\big)$ whence it preserves the primitive part $\mathcal{P}_\nabla(L,A)$ and is a derivation for the free $A$-bilinear Lie bracket $[\ ,\ ]$.

Next, define

$$\mathcal{P}^0_\nabla(L,A) \quad := \quad \mathrm{Ker}(Z_\nabla) \tag{5.4.7}$$

which is a Lie-Rinehart ideal, and since the $A$-linear map can be seen as a projection $\mathcal{P}_\nabla(L,A) \to \mathcal{P}_\nabla(L,A)$ onto $L$ with kernel $\mathcal{P}^0_\nabla(L,A)$ we have the direct decomposition

$$\mathcal{P}_\nabla(L,A) = \mathcal{P}^0_\nabla(L,A) \oplus L. \tag{5.4.8}$$

The following skew-symmetric $A$-bilinear bracket $\mathcal{P}_\nabla(L,A) \times \mathcal{P}_\nabla(L,A) \to \mathcal{P}_\nabla(L,A)$ will be of interest: for all $\xi,\xi' \in \mathcal{P}_\nabla(L,A)$ set

$$[\xi,\xi']^{(\nabla)} := \big[\xi - Z(\xi),\xi' - Z(\xi')\big] + H_\xi(\xi') - H_{\xi'}(\xi). \tag{5.4.9}$$

We provide a fairly explicit description of $Z$ and $H$ in the following Theorem.

**Theorem 5.4.4** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ be a Lie-Rinehart algebra over $A$. Let $\nabla$ be a connection in $L$ along $L$, and let $V$ be an arbitrary $A$-module equipped with a connection $\nabla'$ along $L$. Then we have the following:*

1. *There are simultaneous explicit recursions in terms of curvature and torsion for the maps $Z$ and $H$: for all $v \in V$, $x \in L$, and $\xi \in \mathcal{P}_\nabla(L,A)$ whose underlying $A$-module*

*is identified with free A-Lie algebra over L, the bracket [ , ] being the A-bilinear free bracket:*

$$Z(x) = x \quad \text{and} \quad H'_x = 0, \tag{5.4.10}$$
$$Z\big([x,\xi]\big) = (\nabla_x Z)(\xi) + H_\xi(x) - \mathrm{Tor}\big(x, Z(\xi)\big), \tag{5.4.11}$$
$$H'_{[x,\xi]}(v) = \big(\nabla_x H'\big)_\xi(v) + R'\big(x, Z(\xi)\big)(v). \tag{5.4.12}$$

2. *In particular we get for $x_1, x_2, x_3 \in L$:*

$$Z([x_1,x_2]) = -\mathrm{Tor}(x_1,x_2), \tag{5.4.13}$$
$$Z\big([x_1,[x_2,x_3]]\big) = -\big(\nabla_{x_1}\mathrm{Tor}\big)(x_2,x_3) + \mathrm{Tor}\big(x_1,\mathrm{Tor}(x_2,x_3)\big) + R(x_2,x_3)(x_1), \tag{5.4.14}$$
$$H'_{[x_1,x_2]}(v) = R'(x_1,x_2)(v), \tag{5.4.15}$$
$$H'_{[x_1,[x_2,x_3]]}(v) = \big(\nabla'_{x_1}R'\big)(x_2,x_3)(v) - R'\big(x_1,\mathrm{Tor}(x_2,x_3)\big)(v). \tag{5.4.16}$$

3. *We have the following two identities for the bracket that we define in the equation (5.4.9):*

$$\big[H'_\xi, H'_{\xi'}\big] - H'_{[\xi,\xi']^{(\nabla)}} = 0, \tag{5.4.17}$$
$$H_\xi\big(Z(\xi')\big) - H_{\xi'}\big(Z(\xi)\big) - Z\Big([\xi,\xi']^{(\nabla)}\Big) = 0. \tag{5.4.18}$$

4. *For all $\eta, \eta' \in \mathcal{P}^0_\nabla(L,A)$ the Lie-Rinehart bracket $[\![\eta,\eta']\!]^\nabla$ coincides with the above A-bilinear bracket $[\eta,\eta']^{(\nabla)}$ in (5.4.9).*
*It follows that $[\,,\,]^{(\nabla)}$ is an A-bilinear Lie bracket, and the A-Lie algebra $\big(\mathcal{P}_\nabla(L,A), [\,,\,]^{(\nabla)}\big)$ is isomorphic to the semidirect product of the A-Lie algebra $\mathcal{P}^0_\nabla(L,A)$ with its module L (by eqn (5.4.17)) which is an abelian ideal for this structure.*
5. *The A-submodule $\mathcal{L}_A^{\geqslant 2}(L) := \oplus_{n=2}^\infty \mathcal{L}_A^n(L)$ is a Lie subalgebra of $\big(\mathcal{P}_\nabla(L,A), [\,,\,]^{(\nabla)}\big)$ which is isomorphic to the subalgebra $\mathcal{P}^0_\nabla(L,A)$ via the isomorphism of A-Lie algebras*

$$\mathcal{P}_\nabla(L,A) \to \mathcal{P}_\nabla(L,A) : \xi \mapsto \xi - Z\big(\xi^{\geqslant 2}\big). \tag{5.4.19}$$

*where $\xi^{\geqslant 2}$ denotes the projection onto $\mathcal{L}_A^{\geqslant 2}(L)$ with kernel L.*

*Proof.* 1.) The initial conditions (5.4.10) are clear from the definitions of $Z$ and of $H'$ (see (5.4.6)). Moreover, since $Z$ is a morphism of Lie algebras we get for all $v \in V$, $x \in L$ and $\xi \in \mathcal{P}_\nabla(L,A)$ using the initial conditions $Z(y) = y$ and $H_y = 0$ for all $y \in L$ and eqn (5.3.2) in a straight-forward computation:

$$\begin{aligned} Z\big([x,\xi]\big) &= Z\big([x,\xi]^\nabla\big) - Z\left(\nabla_x(\xi)\right) + Z\left(\nabla_\xi(x)\right) \\ &= \big[\!\big[x, Z(\xi)\big]\!\big] - \nabla_x\big(Z(\xi)\big) + \nabla_{Z(\xi)}(x) + \nabla_x\big(Z(\xi)\big) - Z\left(\nabla_x(\xi)\right) + \nabla_{\xi - Z(\xi)}(x) \\ &= -\mathrm{Tor}\big(x, Z(\xi)\big) + \left(\nabla_x Z\right)(\xi) + H_\xi(x) \end{aligned}$$

proving (5.4.11), and

$$\begin{aligned} H'_{[x,\xi]}(v) &= H'_{[x,\xi]^\nabla}(v) - H'_{\nabla_x(\xi)}(v) + H'_{\nabla_\xi(x)}(v) \\ &= \nabla'_{[x,\xi]^\nabla}(v) - \nabla_{Z\left([x,\xi]^\nabla\right)}(v)' - H'_{\nabla_x(\xi])}(v) + 0 \\ &= \left[\nabla'_x, \nabla'_\xi\right](v) - \nabla'_{[\![x,Z(\xi)]\!]}(v) - H'_{\nabla_x(\xi)}(v) \\ &= \left[\nabla'_x, H'_\xi\right](v) + \left[\nabla'_x, \nabla'_{Z(\xi)}\right](v) - \nabla'_{[\![x,Z(\xi)]\!]}(v) - H'_{\nabla_x(\xi)}(v) \\ &= \left(\nabla'_x H'\right)_\xi(v) + R'\big(x, Z(\xi)\big)(v), \end{aligned}$$

proving eqn (5.4.12).

2.) Is not hard to see that the equations (5.4.13), (5.4.14), (5.4.15), and (5.4.16) are simple consequences of the recursion equations (5.4.11) and (5.4.12).

3.) Clearly, since $Z(x) = x \ \ \forall\ x \in L$ each $\xi - Z(\xi)$ is an element of $\mathrm{Ker}(Z)$, and we get

$$\begin{aligned} \left[\!\left[\xi - Z(\xi), \xi' - Z(\xi')\right]\!\right]^\nabla &= \left[\xi - Z(\xi), \xi' - Z(\xi')\right] + H_\xi\big(\xi' - Z(\xi')\big) - H_{\xi'}\big(\xi - Z(\xi)\big) \\ &= [\xi, \xi']^{(\nabla)} - H_\xi\big(Z(\xi')\big) + H_{\xi'}\big(Z(\xi)\big). \end{aligned} \tag{5.4.20}$$

Applying $Z$ to both sides of eqn (5.4.20) and using the morphism property (5.4.5) of $Z$ we get the cocycle identity (5.4.18) since $H_\xi\big(Z(\xi')\big) \in L$.

Moreover, applying $\nabla'$ to both sides of the equation (5.4.20) and using equation (5.3.2) we get the equation (5.4.17) since $H'_y = 0$ for all $y \in L$.

4.) By the direct decomposition (5.4.8) it suffices to express the bracket $[\ ,\ ]^{(\nabla)}$ on elements $\xi = x + \eta$, $\xi' = x' + \eta'$ for $x, x' \in L$ and $\eta, \eta' \in \mathcal{P}^0_\nabla(L, A)$: an easy computation using the definitions (5.3.1), (5.4.9), and the fact that $Z(x) = x$ for all $x \in L$ and $Z(\eta) = 0$ for all $\eta \in \mathcal{P}^0_\nabla(L, A)$ gives

$$[\eta, \eta']^{(\nabla)} = [\![\eta, \eta']\!]^\nabla, \quad [\eta, x']^{(\nabla)} = H_\eta(x'), \quad \text{and} \quad [x, x']^{(\nabla)} = 0. \tag{5.4.21}$$

The Jacobi identity for the bracket $[\ ,\ ]^{(\nabla)}$ trivially holds if two of the three elements are in $L$, it is also clear if the three elements are in $\mathcal{P}^0_\nabla(L, A)$ (because it holds for the Lie bracket $[\![\ ,\ ]\!]^\nabla$), and in case two elements are in $\mathcal{P}^0_\nabla(L, A)$ and one in $L$ the Jacobi identity follows from the representation identity (5.4.17).

The semidirect product structure is now clear from the concrete brackets (5.4.21).

5.) It is clear that the $A$-linear map (5.4.19) (which we call $T$) is invertible with inverse $T^{-1}(\xi) = \xi + Z(\xi^{\geqslant 2})$. It clearly maps $\mathcal{L}_A^{\geqslant 2}(L)$ into $\mathcal{P}_\nabla(L, A)$. On the other hand if an element $\eta \in \mathcal{P}^0_\nabla(L, A)$ is written as a sum $\zeta + x$ with $\zeta \in \mathcal{L}_A^{\geqslant 2}(L)$ and $x \in L$, then $0 = Z(\eta) = Z(\zeta) + x$ whence $x = -Z(\zeta)$, and the restriction of $T$ to $\mathcal{L}_A^{\geqslant 2}(L)$ is

an isomorphism onto $\mathcal{P}^0_\nabla(L,A)$. Next we compute

$$\begin{aligned} T\big([\xi,\xi']^{(\nabla)}\big) &= [\xi,\xi']^{(\nabla)} - Z\big([\xi,\xi']^{(\nabla)}\big) \\ &\overset{(5.4.18)}{=} [\xi - Z(\xi), \xi' - Z(\xi')] + H_\xi(\xi') - H_{\xi'}(\xi) - H_\xi\big(Z(\xi')\big) + H_{\xi'}\big(Z(\xi)\big) \\ \big[T(\xi),T(\xi')\big]^{(\nabla)} &= \big[\xi - Z(\xi), \xi' - Z(\xi')\big]^{(\nabla)} \\ &= \big[\xi - Z(\xi), \xi' - Z(\xi')\big] + H_\xi\big(\xi' - Z(\xi')\big) - H_{\xi'}\big(\xi - Z(\xi)\big) \end{aligned}$$

since $Z\big(\xi - Z(\xi)\big) = 0$ and $H_x = 0$, and the two preceding equations show that $T$ is an isomorphism of $A$-Lie algebras. This shows that $\mathcal{L}_A^{\geqslant 2}(L)$ is a Lie subalgebra of $\big(\mathcal{P}_\nabla(L,A),[\ ,\ ]^{(\nabla)}\big)$ which can also be seen directly by the definition (5.4.9). □

**Remark 5.4.5** Note that the $A$-linear map $Z$ can be seen as a 1-cocycle of the Lie algebra $\big(\mathcal{P}_\nabla(L,A),[\ ,\ ]^{(\nabla)}\big)$ with values in the module $L$ (via the map $H$, see eqn (5.4.17)).

Besides that, under the hypotheses of Theorem 5.4.4, we can define the following $A$-submodule of $\mathrm{Hom}_A(V,V)$

$$\mathsf{Hol}'_\nabla(V) := \big\{H'_\xi \;\big|\; \xi \in \mathcal{P}_\nabla(L,A)\big\} \;\subset\; \mathrm{Hom}_A(V,V). \tag{5.4.22}$$

This is related to the usual *infinitesimal linear holonomy Lie algebra* in differential geometry, see e.g. [25, p.152-153, Thm.9.2] following the work by Nijenhuis (1953), [35], and Ozeki (1956), [37] which describes an infinitesimal version of parallel transport around closed loops:

**Proposition 5.4.6** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over $A$ with connection $\nabla$ in $L$ along $L$. Let $V$ be an $A$-module with connection $\nabla'$ along $L$.*

*Then $\mathsf{Hol}'_\nabla(V)$ is an $A$-Lie subalgebra of the $A$-Lie algebra $\mathrm{Hom}_A(V,V)$ equipped with the commutator or $A$-linear maps.*

*Moreover*

$$\mathsf{Hol}'_\nabla(V) = A\text{–Span}\big\{\big(\nabla'_b(R')\big)(x,y) \;\big|\; b \in \mathsf{T}_A(L),\ x,y \in L\big\} \tag{5.4.23}$$

*whence the connection $\nabla'$ restricts to the $A$-submodule $\mathsf{Hol}'_\nabla(V)$ of $\mathrm{Hom}_A(V,V)$.*

*Proof.* The first statement follows at once from eqn (5.4.17) and the fact $H_\xi = H_{\xi - Z(\xi)}$ whence $\eta = \xi - Z(\xi) \in \mathcal{P}^0_\nabla(L,A)$. The second statement is proved by induction over the tensor degree of $\xi$ using the recursion (5.4.12).

**The inclusion** "$\subset$": We show that each $H'_\xi$, $\xi \in \mathcal{P}_\nabla(L,A)$, is a covariant derivative of the curvature tensor $R'$: indeed if $\xi = x \in L$ then $H'_x = 0$, and we are done choosing $b = 0, x = 0 = y$, and if $\xi = [x,y]$ $(x,y \in L)$ then $H'_{[x,y]} = R'(x,y)$, so $b = \mathbf{1}$ and the given $x,y \in L$ will do.

Suppose that for each $\xi \in \mathcal{P}_\nabla(L,A)$ of degree $\leqslant n$ there is a nonnegative integer $N$, elements $b_i \in \mathsf{T}_A(L)$, $x_i, y_i \in L$ for each integer $1 \leqslant i \leqslant N$ (all dependent on $\xi$) such that $H'_\xi = \sum_{i=1}^N \left(\nabla'_{b_i}(R')\right)(x_i,y_i)$. Let $z \in L$ and consider $\nabla_z(\xi)$ which also has degree $\leqslant n$. Likewise there is a nonnegative integer $N'$, elements $b'_j \in \mathsf{T}_A(L)$, $x'_j, y'_j \in L$ for each integer $1 \leqslant j \leqslant N'$ (all dependent on $\xi$) such that $H'_{\nabla_z(\xi)} = \sum_{j=1}^{N'} \left(\nabla'_{b'_j}(R')\right)(x'_j,y'_j)$. we get –upon using (5.4.12)–

$$\begin{aligned}
H'_{[z,\xi]} &= \nabla'_z(H'_\xi) - H'_{\nabla_z(\xi)} + R'\big(z,Z(\xi)\big) \\
&= \sum_{i=1}^N \Big( \left(\nabla'_{zb_i}(R')\right)(x_i,y_i) + \left(\nabla'_{\nabla_z(b_i)}(R')\right)(x_i,y_i) + \left(\nabla'_{b_i}(R')\right)\big(\nabla_z(x_i),y_i\big) \\
&\quad + \left(\nabla'_{b_i}(R')\right)\big(x_i,\nabla_z(y_i)\big)\Big) - \sum_{j=1}^{N'} \left(\nabla'_{b'_j}(R')\right)(x'_j,y'_j) \;+\; R'\big(z,Z(\xi)\big)
\end{aligned}$$

proving the induction.

**The inclusion** "$\supset$": We show by induction over the tensor degree of $b \in \mathsf{T}_A(L)$ that for all $x,y \in L$ there is $\xi \in \mathcal{P}_\nabla(L,A)$ with $\big(\nabla'_b(R')\big)(x,y) = H'_\xi$. The cases $n=1$ and $n=2$ are as in the preceding inclusion. Let $z \in L$. Then we write by definition of the iterated covariant derivative and there by the induction hypothesis there are $\xi, \xi' \in \mathcal{P}_\nabla(L,A)$ of degree at most $n$ such that

$$\begin{aligned}
\big(\nabla'_{zb}(R')\big)(x,y) &= \nabla'_z\big(\big(\nabla'_b(R')\big)(x,y)\big) - \big(\nabla'_b(R')\big)\big(\nabla_z(x_i),y_i\big) - \big(\nabla'_b(R')\big)\big(x_i,\nabla_z(y_i)\big) \\
&\quad - \left(\nabla'_{\nabla_z(b)}(R')\right)(x,y) \\
&= \nabla'_z\big(H'\big)_\xi + H'_{\nabla_z(\xi)} + H'_{\xi'} \overset{(5.4.12)}{=} H'_{[z,\xi]} - H'_{[z,Z(\xi)]} + H'_{\xi''}
\end{aligned}$$

with $\xi'' = \nabla_z(\xi) + \xi'$ proving the induction which obviously implies the last statement. □

**Remark 5.4.7** Note that in general $\mathsf{Hol}'_\nabla(V)$ is no longer a finitely generated projective module even if $V$ and $L$ are although it always carries a connection: as a counterexample take a 2-dimensional smooth manifold $X$ embedded in $\mathbb{R}^3$ where $L = V = \Gamma^\infty(X,TX)$ is given by the Lie algebroid of all smooth vector fields and $\nabla$ the Levi-Civita connection (see e.g. [25]) of the induced Riemannian metric where the embedding is chosen in such a way that there are open sets of $X$ where the curvature vanishes (here the localization of $\mathsf{Hol}'_\nabla(V)$ vanishes) and others where it does not (where the localization of $\mathsf{Hol}'_\nabla(V)$ is non trivial): here $\mathsf{Hol}'_\nabla(V)$ can no longer be regarded as the smooth section space of a regular constant rank subbundle of $\mathrm{Hom}(TX,TX)$.

# 6. Results evolving connections and Lie-Rinehart algebras

Contents

## 6.1 Particular cases and flat extensions of Lie-Rinehart algebras with connection

Consider a Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ over $A$. Let $\nabla$ be a connection in $L$ along $L$. Let $V$ be an $A$-module with connection $\nabla'$ along $L$.

The following particular cases for connections $\nabla$ in the $A$-module $L$ are important for the sequel:

**Definition 6.1.1** Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over $A$ with connection $\nabla$ in $L$ along $L$. Let $V$ be an $A$-module with connection $\nabla'$ along $L$.

1. The connection $\nabla'$ is called **flat** if and only if $R'(x,y)(v)=0$ for all $x,y\in L$ and $v\in V$.
2. The connection $\nabla$ is called **CRCT** if and only if $\nabla_x(R)=0$ and $\nabla_x(\mathrm{Tor})=0$ for all $x\in L$. The abbreviation means *constant curvature and constant torsion*.
3. The connection is called **FCT** if and only if $R=0$ and $\nabla_x(\mathrm{Tor})=0$ for all $x\in L$. The abbreviation means *flat constant torsion*.

The above notions all come from differential geometry: flatness of connections is of course well-known, moreover CRCT connection are related to manifolds which locally look like *reductive homogeneous spaces*, see e.g. [26], equipped with their canonical invariant connections.

Here the particular case of *constant curvature and vanishing torsion*, the so-called *locally symmetric spaces*, see e.g. [26] or [16] is very important. The case of a flat constant torsion connection is typically given for any *Lie group* where the connection is defined to be zero on all left invariant vector fields.

We mention some simple well-known properties of flat connections:

**Proposition 6.1.2** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over $A$ with connection $\nabla$ in $L$ and let $V$ be an $A$-module equipped with a **flat** connection $\nabla'$ along $L$.*
*Then the map $H'$ (see (5.4.6)) vanishes whence $\nabla'_\xi=\nabla'_{Z(\xi)}$ for all $\xi\in\mathcal{P}_\nabla(L,A)$.*
*Moreover if the connection $\nabla$ is flat then $H=0$, and the recursion equation (5.4.11) simplifies to*

$$Z\big([x,\xi]\big)=(\nabla_xZ)(\xi)-\mathrm{Tor}\big(x,Z(\xi)\big). \tag{6.1.1}$$

*In that case the restriction $[\ ,\ ]^0$ of the Lie bracket $[\ ,\ ]^\nabla$ to the Lie-Rinehart ideal $\mathcal{P}^0_\nabla(L,A)$ coincides with the free Lie bracket whence it is a $A$-Lie-subalgebra of the free Lie algebra $\mathcal{L}_A(L)$.*

*Proof.* This clearly follows by induction from eqs (5.4.10), (5.4.12), and (5.4.9). □

Before we turn to the other cases of Definition 6.1.1 it is advantageous to consider the following construction of a *flat extension*:

The $A$-linear maps $Z = Z_\nabla$ and $H$ (see (5.4.5) and (5.4.6) together give an $A$-linear map of $\mathcal{P}_\nabla(L,A)$ into $L \oplus \mathrm{Hom}_A(L,L)$. One may wonder whether there is a Lie-Rinehart structure on this $A$-module extending in some way the Lie-Rinehart structure on $L$: this is indeed the case as the following Theorem shows:

**Theorem 6.1.3** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ be a Lie-Rinehart algebra over $A$ and $\nabla$ a connection in $L$ along $L$.*

1. *The following $K$-bilinear map $[\![\ ,\ ]\!]^H$ on the $A$-module $L\oplus\mathrm{Hom}_A(L,L)$ and the $A$-linear map $\rho^H$ define the structure of a Lie-Rinehart algebra over $A$ on $L\oplus\mathrm{Hom}_A(L,L)$: for all $x,y\in L$, $a\in A$ and $\phi,\psi\in\mathrm{Hom}_A(L,L)$*

$$\begin{aligned}\big[\!\big[(x,\phi),(y,\psi)\big]\!\big]^H &:= \Big([[x,y]],R(x,y)+\nabla_x(\psi)-\nabla_y(\phi)+[\phi,\psi]\Big), &(6.1.2)\\ \rho^H_{(x,\phi)}(a) &:= \rho_x(a). &(6.1.3)\end{aligned}$$

   *Here the bracket $[\phi,\psi]$ denotes the commutator of $A$-linear maps. Moreover the projection on the first factor is a morphism of Lie-Rinehart algebras over $A$.*

2. *The map $Y:\mathcal{P}_\nabla(L,A)\to L\oplus\mathrm{Hom}_A(L,L)$ given by*

$$Y(\xi)=\big(Z(\xi),H_\xi\big) \tag{6.1.4}$$

   *for all $\xi\in\mathcal{P}_\nabla(L,A)$ is a morphism of Lie-Rinehart algebras.*

3. *The following map $\nabla^H$ is a **flat** connection on $L\oplus\mathrm{Hom}_A(L,L)$ along $L\oplus\mathrm{Hom}_A(L,L)$ for all $x,y\in L$ and $\phi,\psi\in\mathrm{Hom}_A(L,L)$:*

$$\nabla^H_{(x,\phi)}(y,\psi) := \Big(\nabla_x(y)+\phi(y),\nabla_x(\psi)+[\phi,\psi]\Big) \tag{6.1.5}$$

   *whose torsion (and covariant derivative there of) is given by*

$$\mathrm{Tor}^H\big((x,\phi),(y,\psi)\big) = \Big(\mathrm{Tor}(x,y)+\phi(y)-\psi(x),-R(x,y)+[\phi,\psi]\Big). \tag{6.1.6}$$

$$\begin{aligned}\Big(\nabla^H_{(x,\phi)}\big(\mathrm{Tor}^H\big)\Big)\big((y,\psi),(z,\chi)\big) &= \Big(\big(\nabla_x(\mathrm{Tor})\big)(y,z)+\phi\big(\mathrm{Tor}(y,z)\big)-\mathrm{Tor}\big(\phi(y),z\big)\\ &- \mathrm{Tor}\big(y,\phi(z)\big),-\big(\nabla_x(R)\big)(y,z)\\ &- [\phi,R(y,z)]+R\big(\phi(y),z\big)+R\big(y,\phi(z)\big)\Big) \qquad (6.1.7)\end{aligned}$$

   *for all $(x,\phi),(y,\psi),(z\chi)\in L\oplus\mathrm{Hom}_A(L,L)$*

*Proof.* 1.) The proof of the Jacobi identity of the bracket (6.1.2) is long, but straightforward and uses the two Bianchi identities (5.4.3) and (5.4.4), and the fact that the $\rho^H$ is an anchor morphism is not hard to see.

2.) $Y$ is clearly $A$-linear, intertwines anchor maps, and we compute for all $\xi, \xi' \in \mathcal{P}_\nabla(L,A)$

$$Y\left([\xi,\xi']^\nabla\right) = \left(Z\left([\xi,\xi']^\nabla\right), H_{[\xi,\xi']^\nabla}\right) = \left(\left[\!\left[Z(\xi), Z(\xi')\right]\!\right], \nabla_{[\xi,\xi']^\nabla} - \nabla_{Z\left([\xi,\xi']^\nabla\right)}\right)$$

and

$$\begin{aligned}\nabla_{[\xi,\xi']^\nabla} - \nabla_{Z\left([\xi,\xi']^\nabla\right)} &= \left[\nabla_\xi, \nabla_{\xi'}\right] - \nabla_{[\![Z(\xi),Z(\xi')]\!]} = \left[H_\xi + \nabla_{Z(\xi)}, H_{\xi'} + \nabla_{Z(\xi')}\right] - \nabla_{[\![Z(\xi),Z(\xi')]\!]} \\ &= \left[H_\xi, H_{\xi'}\right] + \nabla_{Z(\xi)}\left(H_{\xi'}\right) - \nabla_{Z(\xi')}\left(H_\xi\right) + R\big(Z(\xi), Z(\xi')\big)\end{aligned}$$

whence

$$Y\left([\xi,\xi']^\nabla\right) = \left[\!\left[\left(Z(\xi), H_\xi\right), \left(Z(\xi'), H_{\xi'}\right)\right]\!\right]^H = \left[\!\left[Y(\xi), Y(\xi')\right]\!\right]^H. \tag{6.1.8}$$

3.) Note that $\nabla^H_{(x,\phi)}$ preserves the $A$-submodule $L$ and the $A$-submodule $\mathrm{Hom}_A(L,L)$ separately, and the latter is induced by the former. It suffices to compute vanishing curvature for the component $\nabla^{(1)}$ of $\nabla^H$ acting on $L$: for all $x,y,z \in L$ and $\phi,\psi \in \mathrm{Hom}_A(L,L)$

$$\begin{aligned}&\left[\nabla^{(1)}_{(x,\phi)}, \nabla^{(1)}_{(y,\psi)}\right](z) - \nabla^{(1)}_{[\![(x,\phi),(y,\psi)]\!]}(z) \\ &= \left[\nabla_x + \phi, \nabla_y + \psi\right](z) - \nabla_{[\![x,y]\!]}(z) - R(x,y)(z) - \left(\nabla_x(\psi)\right)(z) + \left(\nabla_y(\phi)\right)(z) - [\phi,\psi](z) = 0.\end{aligned}$$

The computation for the torsion of $\nabla^H$ and its covariant derivative is lengthy, but straight-forward. □

**Remark 6.1.4** Note that the extended Lie bracket $[\![\ ,\ ]\!]^H$ (6.1.2) is motivated by a simple geometric construction: given a smooth manifold $X$, consider the vector fields on its tangent bundle $TX$ whose Lie bracket with the Euler field vanishes. In the presence of a connection $\nabla$ in the tangent bundle they can be written as sums of horizontal lifts of vector fields $x$ on $X$ and those vertical vector fields $\phi$ on $TX$ which are linear along the fibres. Their Lie brackets correspond to (6.1.2) up to signs due to the definition of the Lie bracket: the Lie bracket of 'matrix vector fields' is minus the matrix vector field corresponding to the matrix commutator.

For the **CRCT** case we have the following statement.

**Proposition 6.1.5** *Let $\left(L,\rho,[\![\ ,\ ]\!],A\right)$ a Lie-Rinehart algebra over $A$ with **CRCT** connection $\nabla$ in $L$ along $L$.*

*Then the maps $Z$ and $H$ (see (5.4.6)) are covariantly constant, i.e. $\nabla_y(Z) = 0$ and $\left(\nabla_y(H)\right)_\xi = 0$ for all $y \in L$ and $\xi \in \mathcal{P}_\nabla(L,A)$.*

*The recursions for $Z$ and $H$ simplify in the following way for all $x,z \in L$ and $\xi \in \mathcal{P}_\nabla(L,A)$: $Z(x) = x$ and $H_x = 0$,*

$$Z\big([x,\xi]\big) = H_\xi(x) - \mathrm{Tor}\big(x, Z(\xi)\big), \tag{6.1.9}$$
$$H_{[x,\xi]}(z) = R\big(x, Z(\xi)\big)(z). \tag{6.1.10}$$

*It follows that the A-Lie algebra* $\mathsf{Hol}_\nabla(L)$ *is spanned by the values of the curvature tensor.*

*Finally, the A-linear map* $Y$*, see equation (6.1.4), is also a morphism of the A-module* $\mathcal{P}_\nabla(L,A) = \mathcal{L}_A(L)$*, equipped with the free A-bilinear Lie bracket* $[\ ,\ ]$*, onto the A-module* $\mathfrak{g} := L \oplus \mathsf{Hol}_\nabla(L)$ *equipped with the A-bilinear Lie bracket (for all* $x, y \in L$ *and* $\phi, \psi \in \mathsf{Hol}_\nabla(L)$*)*

$$\big[(x,\phi),(y,\psi)\big]_\mathfrak{g} := \big(-\mathrm{Tor}(x,y) - \phi(y) + \psi(x), R(x,y) - [\phi,\psi]\big) \tag{6.1.11}$$

*Proof.* We prove the equations $\big(\nabla_y(Z)\big)(\xi) = 0$ and $\big(\nabla_y(H)\big)_\xi = 0$ by induction over the tensor degree of $\xi \in \mathcal{P}_\nabla(L,A)$. We first compute the followig expressions for all $x, y, z \in L$ and $\xi$ in $\mathcal{P}_\nabla(L,A)$ in a lengthy, but straight-forward manner upon using the equations (5.4.11) and (5.4.12):

$$\begin{aligned}
\big(\nabla_y(Z)\big)\big([x,\xi]\big) &= \big(\nabla_{yx}Z\big)(\xi) - \mathrm{Tor}\Big(x, \big(\nabla_y(Z)\big)(\xi)\Big) + \big(\nabla_y(H)\big)_\xi(x) - \big(\nabla_y(\mathrm{Tor})\big)\big(x, Z(\xi)\big),\\
\big(\nabla_y(H)\big)_{[x,\xi]}(z) &= \big(\nabla_{yx}(H)\big)_\xi(z) + R\Big(x, \big(\nabla_y(Z)\big)(\xi)\big)\Big)(z) - \big(\nabla_y R\big)\big(x, Z(\xi)\big)(z).
\end{aligned}$$

The equations $\big(\nabla_y(Z)\big)(\xi) = 0$ and $\big(\nabla_y(H)\big)_\xi = 0$ are obvious true for $\xi$ of degree 1 since the restriction of $Z$ to $L$ is the identity map and the restriction of $H$ (in its index argument) to $L$ vanishes. Suppose that both equations are satisfied for all $\xi$ of degree less or equal that $n$. For the induction step observe that the right hand sides of the above equations for $\big(\nabla_y(Z)\big)\big([x,\xi]\big)$ and $\big(\nabla_y(H)\big)_{[x,\xi]}(z)$ only depend on $\nabla_x(Z)(\xi')$ and $(\nabla_x(H))_{\xi'}$ for $\xi'$ of degree $\leqslant n$ whence we can use induction and the CRCT-property to conclude that the left hand sides vanish which proves the induction step.

The two simplified recursion equations (6.1.9) and (6.1.10) are now obvious consequences of (5.4.11) and (5.4.12). The statement about the infinitesimal holonomy Lie algebra $\mathsf{Hol}_\nabla(L)$ then becomes clear by eqn (5.4.23).

Finally, it is clear that the image of $Y$ is equal to $L \oplus \mathsf{Hol}_\nabla(L)$ (since $Z$ is surjective on $L$ and $H$ maps $\mathrm{Ker}(Z)$ surjectively on $\mathsf{Hol}_\nabla(L)$.

For all $\xi, \xi' \in \mathcal{P}_\nabla(L,A)$, if we utilize the equation (6.1.8) and that $\nabla_x(Y) = \big(\nabla_x(Z), \nabla_x(H)\big) =$

$(0,0)\big)$ we can obtain

$$\begin{aligned} Y\big([\xi,\xi']\big) &= Y\big([\xi,\xi']^{\nabla}\big) - Y\big(\nabla_{\xi}(\xi')\big) + Y\big(\nabla_{\xi'}(\xi)\big) = \big[\!\big[Y(\xi),Y(\xi')\big]\!\big]^{H} - \nabla_{\xi}\big(Y(\xi')\big) + \nabla_{\xi'}\big(Y(\xi)\big) \\ &= \Big(\big[\!\big[Z(\xi),Z(\xi')\big]\!\big] - \nabla_{Z(\xi)}\big(Z(\xi')\big) - H_{\xi}\big(Z(\xi')\big) + \nabla_{Z(\xi')}\big(Z(\xi)\big) + H_{\xi'}\big(Z(\xi)\big), \\ &\quad \big[H_{\xi},H_{\xi'}\big] + \nabla_{Z(\xi)}\big(H_{\xi'}\big) - \nabla_{Z(\xi')}\big(H_{\xi}\big) + R\big(Z(\xi),Z(\xi')\big) \\ &\quad -\nabla_{\xi}\big(H_{\xi'}\big) + \nabla_{\xi'}\big(H_{\xi}\big)\Big) \\ &= \Big(-\mathrm{Tor}\big(Z(\xi),Z(\xi')\big) - H_{\xi}\big(Z(\xi')\big) + H_{\xi'}\big(Z(\xi)\big), R\big(Z(\xi),Z(\xi')\big) - \big[H_{\xi},H_{\xi'}\big]\Big). \end{aligned} \tag{6.1.12}$$

It remains to show that the antisymmetric $A$-bilinear bracket (6.1.11) satisfies the Jacobi identity when restricted to the $A$-module $L\oplus \mathsf{Hol}_{\nabla}(L)$. Actually, this fact follows from an iteration of the defining CRCT-equations $\nabla_x(\mathrm{Tor})=0$ and $(\nabla_x R)=0$ implying for all $v,w,x,y\in L$

$$\begin{aligned} R(v,w)\big(\mathrm{Tor}(x,y)\big) &= \mathrm{Tor}\big(R(v,w)(x),y\big) + \mathrm{Tor}\big(x,R(v,w)(y)\big) \\ [R(v,w),R(x,y)] &= R\big(R(v,w)(x),y\big) + R\big(x,R(v,w)(y)\big) \end{aligned}$$

hence for $x,y,z,v,w,v',w',v'',w''\in L$ and $\phi=R(v,w),\psi=R(v',w'),\chi=R(v'',w'')\in \mathsf{Hol}_{\nabla}(L)$ the Jacobi identity for $[\ ,\ ]_{\mathfrak{g}}$ follows the above derivational identities and the Bianchi identities (5.4.3) and (5.4.4). □

**Remark 6.1.6** Note that this case and the reductively decomposed Lie algebra $\mathfrak{g}$ is well-known from the 1950's, see the work by Konstant and Yamaguti [1], see [56], and occurs for reductive homogeneous spaces, see for instance [26].

The following **FCT** case will be very important in the sequel and is an easy consequence of the preceding Proposition 6.1.5:

**Corollary 6.1.7** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over $A$ with **FCT** connection $\nabla$ in $L$ along $L$.*

*Then the map $H$ (see (5.4.6)) vanishes and $Z$ is covariantly constant. It can be computed explicitly for all $x,x_1,\ldots,x_n\in L$ for all integers $n\geqslant 2$: $Z(x)=x$ and*

$$Z\Big(\Big[x_1,\big[x_2,\ldots,[x_{n-1},x_n]\cdots\big]\Big]\Big) = (-1)^{n-1}\mathrm{Tor}\Big(x_1,\mathrm{Tor}\big(x_2,\ldots,\mathrm{Tor}(x_{n-1},x_n)\cdots\big)\Big). \tag{6.1.13}$$

*Moreover, the $A$-bilinear map $-\mathrm{Tor}(\ ,\ )$ is a Lie bracket on $L$, and the $A$-linear map $Z$ is a surjective morphism of $A$-Lie algebras from the $A$-Lie algebra $\mathcal{L}_A(L)$ (equipped with the free Lie bracket $[\ ,\ ]$) to the $A$-Lie algebra $\big(L,-\mathrm{Tor}(\ ,\ )\big)$. It follows that the Lie-Rinehart*

1. I thank S.Benayadi and F.Wagemann for referring me to Yamaguti's work.

*ideal $\mathcal{P}^0_\nabla(L,A)$ is not only an $A$-Lie-subalgebra, but an ideal with respect to the free Lie bracket of the free Lie algebra $\mathcal{L}_A(L)$.*

**Example 6.1.8** We have already come across two examples of Lie-Rinehart algebras with **FCT** connections:

*i.)* Let $\big(L,\rho,[\ ,\ ],A\big)$ an arbitrary Lie derivation algebra (for instance a Lie-Rinehart algebra), and consider the induced Lie-Rinehart algebra $A\odot L$, more precisely $\big(A\otimes L,{}^A[\ ,\ ]_\rho,A\big)$, see eqn (4.1.2).
Then the $K$-bilinear map $\nabla^0$ on $A\odot L$ which can be deduced from (5.1.4), namely (for all $a,a'\in A$ and $x,y\in L$)

$$\nabla^0_{a\otimes x}\big(a'\otimes y\big) := \big(a\rho_x(a')\big)\otimes y. \tag{6.1.14}$$

is seen to be an FCT-connection on $A\odot L$ in a straight-forward way. We shall call $\nabla^0$ the *canonical connection in $A\otimes L$*. The covariantly constant torsion of this connection, $\mathrm{Tor}^0$, is given by

$$\mathrm{Tor}^0 = -{}^A[\ ,\ ] \tag{6.1.15}$$

which is minus times the $A$-bilinear Lie bracket (4.1.3).

*i.)* Let $(L,\rho)$ be an anchored $A$-module and $\nabla$ a connection in $L$ along $L$. Consider the path Lie algebroid $\mathcal{P}_\nabla(L,A)$. Then the iterated covariant derivative $\nabla$ defines an FCT connection in $\mathcal{P}_\nabla(L,A)$ along $\mathcal{P}_\nabla(L,A)$: in fact this follows immediately from eqs (5.3.1) and (5.3.2) from which we can deduce that the covariantly constant torsion equals minus the free $A$-bilinear Lie bracket, for all $\xi,\xi'\in\mathcal{P}_\nabla(L,A)$

$$\mathrm{Tor}(\xi,\xi') = -[\xi,\xi']. \tag{6.1.16}$$

## 6.2 Multiplication of Universal Enveloping Algebras of Lie-Rinehart algebras

In this Section we shall suppose that $\mathbb{Q}\subset K$. Moreover we shall often write $\mathfrak{g}$ for the $A$-module $\mathcal{L}_A(L)\subset \mathsf{T}_A(L)$.

### 6.2.1 The Rinehart ideal $\mathcal{J}_\nabla(L,A)$

Consider a Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ over $A$. Let $\nabla$ be a connection in $L$ along $L$. Recall the Rinehart bialgebra $\big(\mathsf{T}_A(L),\diamond=\diamond_\nabla,\mathbf{1},\Delta_{sh},\epsilon\big)$. Recall furthermore the path Lie algebroid $\mathcal{P}_\nabla(L,A)$ (whose underlying $A$-module is the free Lie algebra $\mathcal{L}_A(L)=:\mathfrak{g}$), the maps $Z$ (see equation (5.4.5)) and $H$ (see equation (5.4.6)), and the kernel of $Z$, $\mathcal{P}^0_\nabla(L,A)$ (see equation (5.4.7)). Moreover the latter is a Lie-Rinehart

ideal of $\mathcal{P}_\nabla(L,A)$ and hence a $K$-Lie subalgebra of $\big(\mathsf{T}_A(L),\diamond\big)$ which we had shown to be isomorphic to the $A$-submodule

$$\mathfrak{h} := \mathcal{L}_A^{\geqslant 2}(L) = \bigoplus_{n=2}^{\infty} \mathcal{L}_A^n(L) \tag{6.2.1}$$

(by means of the $A$-linear map $\zeta \mapsto \zeta - Z(\zeta)$) equipped with the $A$-bilinear Lie bracket

$$\forall\ \zeta,\zeta' \in \mathfrak{h}: \quad [\zeta,\zeta']^{(\nabla)} = \big[\zeta - Z(\zeta), \zeta' - Z(\zeta')\big] + H_\zeta(\zeta') - H_{\zeta'}(\zeta), \tag{6.2.2}$$

(see eqn (5.4.9) and Theorem 5.4.4) *iv.*). The Lie bracket (6.2.2) can be seen as a deformation of the free Lie bracket restricted to $\mathfrak{h}$. Define the following bilinear map for each $\zeta \in \mathfrak{h}$ and $b \in \mathsf{T}_A(L)$:

$$\zeta \rhd b := \mathfrak{D}_\zeta(b) := \big(\zeta - Z(\zeta)\big) \diamond b = \zeta b - Z(\zeta)b + H_\zeta(b) \tag{6.2.3}$$

and the following $K$-submodule of $\mathsf{T}_A(L)$

$$\mathcal{J}_\nabla(L,A) := \mathfrak{h} \rhd \mathsf{T}_A(L) := \mathrm{Span}\{\zeta \rhd b \mid \zeta \in \mathfrak{h},\ b \in \mathsf{T}_A(L)\}. \tag{6.2.4}$$

**Proposition 6.2.1** *With the above notations:*

1. *The map (6.2.3) is $A$-bilinear and defines an $A$-linear representation of the $A$-Lie algebra $\big(\mathfrak{h}, [\ ,\ ]^{(\nabla)}\big)$ on $\mathsf{T}_A(L)$ by coderivations of $(\mathsf{T}_A(L), \Delta_{sh}, \epsilon)$.*
2. *The $K$-submodule $\mathcal{J}_\nabla(L,A)$ of $\mathsf{T}_A(L)$ is a Rinehart ideal of $\mathsf{T}_A(L)$, hence an $A$-submodule, a two-sided ideal of the $K$-algebra $\big(\mathsf{T}_A(L), \diamond, \mathbf{1}\big)$ and a coideal of the $A$-coalgebra $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\big)$.*

*Proof.* In the following let $\zeta,\zeta' \in \mathfrak{h}$ and set $\eta = \zeta - Z(\zeta)$, $\eta' = \zeta' - Z(\zeta')$ which are two elements in $\mathrm{Ker}(Z) = \mathcal{P}_\nabla^0(L,A)$.
*i.*): Equation (6.2.3) is well-defined since $\nabla_\eta = H_\eta = H_\zeta$, see eqn (5.4.6). It follows that $\rhd$ is $A$-bilinear. We compute

$$\begin{aligned}
\Delta_{sh}\big(\mathfrak{D}_\zeta(b)\big) = \Delta_{sh}(\zeta \rhd b) = \Delta_{sh}\big((\eta \diamond b\big) &\overset{(5.2.7)}{=} \sum_{(b)} \big(\eta \diamond b^{(1)}\big) \otimes_A b^{(2)} + \sum_{(b)} b^{(1)} \otimes_A \big(\eta \diamond b^{(2)}\big) \\
&= \big(\mathfrak{D}_\zeta \otimes_A \mathrm{id} + \mathrm{id} \otimes_A \mathfrak{D}_\zeta\big)\big(\Delta_{sh}(b)\big)
\end{aligned}$$

whence each $\mathfrak{D}_\zeta$ is an $A$-linear coderivation of $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\big)$. Moreover since $[\eta,\eta']^{(\nabla)} = [\zeta,\zeta']^{(\nabla)} - Z\big([\zeta,\zeta']^{(\nabla)}\big)$, see Theorem 5.4.4, *v.*) and eqn (5.4.19) we get

$$\begin{aligned}
\big[\mathfrak{D}_\zeta, \mathfrak{D}_{\zeta'}\big](b) &= \eta \diamond \eta' \diamond b - \eta' \diamond \eta \diamond b = [\![\eta,\eta']\!]^\nabla \diamond b = [\eta,\eta']^{(\nabla)} \diamond b \\
&= \Big(\big([\zeta,\zeta']^{(\nabla)} - Z\big([\zeta,\zeta']^{(\nabla)}\big)\big)\Big) \diamond b = \mathfrak{D}_{[\zeta,\zeta']^{(\nabla)}}(b),
\end{aligned}$$

showing that $\mathfrak{D}$ gives a representation of the $A$-Lie algebra $\left(\mathfrak{h}, [\ ,\ ]^{(\nabla)}\right)$ on the $A$-module $\mathsf{T}_A(L)$.
*ii.*): Clearly, $\mathcal{J}_\nabla(L,A)$ is given by finite sums sums of elements of the form $\zeta \rhd b$ with $\zeta \in \mathfrak{h}$ and $b \in \mathsf{T}_A(L)$, and thanks to the $A$-bilinearity of $\rhd$ it is an $A$-submodule of $\mathsf{T}_A(L)$. Moreover it is a sum of the images of the coderivations $\mathfrak{D}_\zeta$, $\zeta \in \mathfrak{h}$, and therefore a coideal of the $A$-coalgebra $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\right)$.
In order to prove that $\mathcal{J}_\nabla(L,A)$ is a two-sided ideal w.r.t. the multiplication $\diamond$ we proceed as follows: firstly, since $\zeta \rhd b = \eta \diamond b$ we get for all $b' \in \mathsf{T}_A(L)$ that $(\zeta \rhd b) \diamond b' = \eta \diamond b \diamond b' = \zeta \rhd (b \diamond b')$ whence $\mathcal{J}_\nabla(L,A)$ is a right ideal with respect to $\diamond$. In order to show that is a left ideal we use induction of the tensor degree of $b'$. Indeed for $b' = a\mathbf{1}$ (for some $a \in A$) this is evident. Let $x' \in L$. Then $[\![x,\eta]\!]^\nabla \in \mathcal{P}^0_\nabla(L,A)$ since $\mathcal{P}^0_\nabla(L,A) = \mathrm{Ker}(Z)$ is a Lie-Rinehart ideal. We can uniquely write it as $[\![x,\eta]\!]^\nabla = \hat{\zeta} - Z(\hat{\zeta})$ where $\hat{\zeta} \in \mathfrak{h}$ is the canonical projection of $[\![x,\eta]\!]^\nabla$ onto $\mathfrak{h} = \mathcal{L}_A^{\geqslant 2}$ along $L$. Therefore

$$x' \diamond (\zeta \rhd b) = x' \diamond \eta \diamond b = [\![x',\eta]\!]^\nabla \diamond b + \eta \diamond x' \diamond b = \hat{\zeta} \rhd b + \zeta \rhd (x' \diamond b) \in \mathcal{J}_\nabla(L,A).$$

Suppose by induction that $b' \diamond (\zeta \rhd b)$ is in $\mathcal{J}_\nabla(L,A)$ for all $b'$ of tensor degree $\leqslant n$. Let $x' \in L$, and to show the induction step we compute

$$(x'b') \diamond (\zeta \rhd b) = x' \diamond b' \diamond (\eta \diamond b) - \nabla_{x'}(b') \diamond (\zeta \rhd b)$$

and both terms on the right hand side are in $\mathcal{J}_\nabla(L,A)$ by the case $n = 1$ and by the induction hypothesis (note that the tensor degree of $\nabla_{x'}(b')$ is $\leqslant n$). This proves the induction, whence $\mathcal{J}_\nabla(L,A)$ is a two-sided ideal to the $K$-algebra $\left(\mathsf{T}_A(L), \diamond, \mathbf{1}\right)$.

□

For later use we mention that the 'coderivational' action $\mathfrak{D}$ of the $A$-Lie algebra $\left(\mathfrak{h}, [\ ,\ ]^{(\nabla)}\right)$ on the coalgebra $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon\right)$ canonically induces a module action (also denoted by $\mathfrak{D}$) of its *universal enveloping algebra* $\mathsf{U}_A(\mathfrak{h})$ on $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon\right)$ in the usual sense: let $u = \zeta_1 \cdots \zeta_n \in \mathsf{U}_A(\mathfrak{h})$ (multiplication in $\mathsf{U}_A(\mathfrak{h})$) with $n$ a strictly positive integer, $\zeta_1, \ldots, \zeta_n \in \mathfrak{h}$, and $b \in \mathsf{T}_A(L)$ then

$$\mathfrak{D}(u \otimes_A b) := \mathfrak{D}_u(b) := u \rhd b := \zeta_1 \rhd \left(\zeta_2 \rhd \cdots (\zeta_n \rhd b) \cdots\right), \tag{6.2.5}$$

and upon using the $A$-Hopf algebra structure, $\left(\mathsf{U}_A(\mathfrak{h}), \mu_U, \mathbf{1}_U, \Delta_U, \epsilon_U, S_U\right)$ and the fact that the primitive elements act as coderivations

$$\Delta_{sh}(u \rhd b) = \sum_{(u)} \sum_{(b)} \left(u^{(1)} \rhd b^{(1)}\right) \otimes_A \left(u^{(2)} \rhd b^{(2)}\right) \tag{6.2.6}$$

whence $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon\right)$ becomes a *module coalgebra over the Hopf algebra* $\left(\mathsf{U}_A(\mathfrak{h}), \mu_U, \mathbf{1}_U, \Delta_U, \epsilon_U, S_U\right)$.

We shall be interested in the quotient algebra $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A)$ (which –as we shall see later– turns out to be isomorphic to the universal enveloping algebra $\mathcal{U}(L,A)$). But we first need a more practicable description of the ideal $\mathcal{J}_\nabla(L,A)$. In order to do this we have to make a detour to symmetric algebras since they are much simpler to handle: consider the symmetric algebra $\mathsf{S}_A(\mathfrak{g}) = \mathsf{S}_A(\mathcal{L}_A(L))$. Thanks to the direct decomposition $\mathfrak{g} = \mathfrak{h} \oplus L$ as $A$-submodules of $\mathfrak{g}$ we have the natural morphisms of Hopf algebras

$$I : \mathsf{S}_A(L) \hookrightarrow \mathsf{S}_A(\mathfrak{g}) \quad \text{and} \quad P : \mathsf{S}_A(\mathfrak{g}) \twoheadrightarrow \mathsf{S}_A(L) \tag{6.2.7}$$

induced by the injection $L \hookrightarrow \mathfrak{g}$ and by the projection $\mathfrak{g} \twoheadrightarrow L$ along $\mathfrak{h}$, respectively.

Note that $P \circ I = \mathrm{id}_{\mathsf{S}_A(L)}$ whence $I \circ P$ is an idempotent Hopf algebra map. The kernel of $P$ clearly is the ideal and coideal

$$\mathfrak{I}(L,A) := A\text{-Span}\left\{\zeta \bullet \beta \;\middle|\; \zeta \in \mathfrak{h},\ \beta \in \mathsf{S}_A(\mathfrak{g})\right\} \tag{6.2.8}$$

and we thus have the direct decomposition

$$\mathsf{S}_A(\mathfrak{g}) = *I\left(\mathsf{S}_A(\mathfrak{g})\right) \ \oplus\ \mathfrak{I}(L,A). \tag{6.2.9}$$

To make contact to $\mathsf{T}_A(L)$, recall first the *symmetrization map* $\omega : \mathsf{S}_A(\mathcal{L}_A(L)) \to \mathsf{T}_A(L)$ given by $\omega(\mathbf{1}_S) = \mathbf{1}$ and for all strictly positive integers $n$ and $\xi_1, \ldots, \xi_n \in \mathcal{L}_A(L)$:

$$\omega : \xi_1 \bullet \cdots \bullet \xi_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \cdots \xi_{\sigma(n)}. \tag{6.2.10}$$

It is well-known that $\omega$ defines an $A$-linear isomorphism of coalgebras from $\left(\mathsf{S}_A(\mathfrak{g}), \Delta_S, \epsilon_S, \mathbf{1}_S\right)$ to $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\right)$, the inverse being the convolution exponential $e^{\tilde{\star} e^{(1)}}$, see the convolution table (6.2.12) and the definition of the Eulerian idempotent $e^{(1)}$, eqn (A.6.5). We therefore have the injective morphism of $C^3$-coalgebras $\Upsilon : \mathsf{S}_A(L) \to \mathsf{T}_A(L)$ given by

$$\Upsilon = \omega \circ I \tag{6.2.11}$$

In order to simplify the combinatorial notations, for later use in the sequel, we shall introduce the following list of *convolutions* (see Appendix A.3 for definitions) on the Hom-spaces $\mathrm{Hom}_A(C,B)$ depending on the $A$-coalgebra $C$ and the $A$-algebra $B$ and some of their actions on the Hom-spaces $\mathrm{Hom}_A(C,V)$ where the $A$-module $V$

is a left $B$-module which:

| convolution | algebra $B$ | $B-$module $V$ | coalgebra $C$ |
|---|---|---|---|
| $\star$ | $\big(\mathsf{T}_A(L),\mu,\mathbf{1}\big)$ | none | $\big(\mathsf{T}_A(L),\Delta_{sh},\epsilon\big)$ |
| $*$ | $\big(\mathsf{T}_A(L),\mu,\mathbf{1}\big)$ | none | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $\tilde{*}$ | $\big(\mathsf{S}_A(\mathfrak{g}),\bullet,\mathbf{1}_S\big)$ | none | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $\tilde{\star}$ | $\big(\mathsf{S}_A(\mathfrak{g}),\bullet,\mathbf{1}_S\big)$ | none | $\big(\mathsf{T}_A(L),\Delta_{sh},\epsilon\big)$ |
| $\hat{*}$ | $\big(\mathsf{U}_A(\mathfrak{h}),\mu_U,\mathbf{1}_U\big)$ | none | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $*'$ | $\big(\mathrm{Hom}_A(\mathfrak{h},\mathfrak{h}),\circ,\mathrm{id}_{\mathfrak{h}}\big)$ | none | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $*''$ | $\big(\mathrm{Hom}_A(\mathfrak{h},\mathfrak{h}),\circ,\mathrm{id}_{\mathfrak{h}}\big)$ | $\mathfrak{h}$ | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $\overset{\triangleright}{*}$ | $\big(\mathsf{U}_A(\mathfrak{h}),\mu_U,\mathbf{1}_U\big)$ | $\mathsf{T}_A(L)$ | $\big(\mathsf{S}_A(\mathfrak{g}),\Delta_S,\epsilon_S\big)$ |
| $\overset{\triangleright}{\star}$ | $\big(\mathsf{U}_A(\mathfrak{h}),\mu_U,\mathbf{1}_U\big)$ | $\mathsf{T}_A(L)$ | $\big(\mathsf{T}_A(L),\Delta_{sh},\epsilon\big)$ |

(6.2.12)

Recall that the first six convolutions are associative $A$-bilinear multiplications on the corresponding Hom-spaces $\mathrm{Hom}_A(C,B)$, and the last three describe the left module actions of these associative $A$-algebras on the Hom spaces $\mathrm{Hom}_A(C,V)$.

Next, we shall need several projection maps followed by injections in the sequel:

| | | | | | | kernel |
|---|---|---|---|---|---|---|
| $p_L:$ | $\mathsf{T}_A(L)$ | $\twoheadrightarrow$ | $L$ | $\hookrightarrow$ | $\mathsf{T}_A(L)$ | $A\mathbf{1}\oplus_{n=2}^{\infty}\mathsf{T}_A^n(L)$ |
| $q:$ | $\mathsf{S}_A(\mathfrak{g})$ | $\twoheadrightarrow$ | $\mathfrak{g}$ | $\hookrightarrow$ | $\mathsf{T}_A(L)$ | $A\mathbf{1}\oplus_{n=2}^{\infty}\mathsf{S}_A^n(\mathfrak{g})$ |
| $q_L:$ | $\mathsf{S}_A(\mathfrak{g})$ | $\twoheadrightarrow$ | $L$ | $\hookrightarrow$ | $\mathsf{T}_A(L)$ | $A\mathbf{1}\oplus\mathfrak{h}\oplus\oplus_{n=2}^{\infty}\mathsf{S}_A^n(\mathfrak{g})$ |
| $q_{\mathfrak{h}}:$ | $\mathsf{S}_A(\mathfrak{g})$ | $\twoheadrightarrow$ | $\mathfrak{h}$ | $\hookrightarrow$ | $\mathsf{T}_A(L)$ | $A\mathbf{1}\oplus L\oplus\oplus_{n=2}^{\infty}\mathsf{S}_A^n(\mathfrak{g})$ |
| $r_{\mathfrak{h}}:$ | $\mathsf{S}_A(\mathfrak{g})$ | $\twoheadrightarrow$ | $\mathfrak{h}$ | $\hookrightarrow$ | $\mathsf{U}_A(\mathfrak{h})$ | $A\mathbf{1}\oplus L\oplus\oplus_{n=2}^{\infty}\mathsf{S}_A^n(\mathfrak{g})$ |

(6.2.13)

where of course $q=q_L+q_{\mathfrak{h}}$.

Recall first that the symmetrization map $\omega$ (see (A.6.21)) has the form of a convolution exponential, see eqn (A.6.22). We shall need another more refined isomorphism of coalgebras $\Theta_\nabla:\mathsf{S}_A(\mathfrak{g})\to\mathsf{T}_A(L)$ for which we make the following ansatz:

$$\Theta_\nabla := e^{\tilde{*}r_{\mathfrak{h}}}\ \overset{\triangleright}{*}\ e^{*q_L}. \tag{6.2.14}$$

Before we prove that this is an isomorphism of $C^3$-coalgebras recall the isomorphism of $A$-modules

$$\mathrm{Hom}_A\left(\mathsf{S}_A(\mathfrak{g}),\mathfrak{g}\right)\to\mathrm{Coder}_A\left(\mathsf{S}_A(\mathfrak{g}),\mathsf{S}_A(\mathfrak{g})\right):d\mapsto d\tilde{*}\mathrm{id}_{\mathsf{S}_A(\mathfrak{g})} \tag{6.2.15}$$

(see e.g. eqn (A.3.7)) with inverse $D\to D\tilde{*}S_{\mathsf{S}_A(\mathfrak{g})}$ by means of the antipode of $\mathsf{S}_A(\mathfrak{g})$.

Define

$$\mathrm{Coder}_A^{\mathfrak{h}}\left(\mathsf{S}_A(\mathfrak{g}),\mathsf{S}_A(\mathfrak{g})\right):=\left\{d\tilde{*}\mathrm{id}_{\mathsf{S}_A(\mathfrak{g})}\ \middle|\ d\in\mathrm{Hom}_A\left(\mathsf{S}_A(\mathfrak{g}),\mathfrak{h}\right)\right\}. \tag{6.2.16}$$

We have the following

**Proposition 6.2.2** *With the above notations we have the following:*

1. *The $A$-linear map $\Theta_\nabla$ is an isomorphism of $C^3$-coalgebras.*
2. *The following map is an injective morphism of $A$-modules:*

$$\mathrm{Hom}_A(\mathsf{T}_A(L),\mathfrak{h}) \to \mathrm{Coder}_A(\mathsf{T}_A(L),\mathsf{T}_A(L)) : \chi \mapsto \chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)} \tag{6.2.17}$$

   *We shall call its image* $\mathrm{Coder}^{\mathfrak{h}}_A(\mathsf{T}_A(L),\mathsf{T}_A(L))$.
3. *There is an explicit $A$-linear isomorphism $\mathcal{E} : \mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}),\mathfrak{h}\big) \to \mathrm{Hom}_A(\mathsf{T}_A(L),\mathfrak{h})$ inducing the equality $\Theta_\nabla \circ \mathrm{Coder}^{\mathfrak{h}}_A(\mathsf{S}_A(\mathfrak{g}),\mathsf{S}_A(\mathfrak{g})) \circ \Theta_\nabla^{-1} = \mathrm{Coder}^{\mathfrak{h}}_A(\mathsf{T}_A(L),\mathsf{T}_A(L))$.*
4. *We have the isomorphisms*

$$\Theta_\nabla(\mathcal{I}(L,A)) = \mathcal{J}_\nabla(L,A) \quad \text{and} \quad \Theta_\nabla \circ I = \Upsilon, \ \text{hence} \ \mathsf{T}_A(L) = \Upsilon(\mathsf{S}_A(L)) \oplus \mathcal{J}_\nabla(L,A). \tag{6.2.18}$$

5. *The quotient Rinehart bialgebra $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A)$ is isomorphic to the universal enveloping algebra $\mathcal{U}(L,A)$.*

*Proof. i.)* By the usual convolution exponential form, see equation (A.3.4), it follows that $\Theta_{\mathfrak{h}} : \mathsf{S}_A(\mathfrak{g}) \to \mathsf{U}_A(\mathfrak{h})$ and $\Theta_L : \mathsf{S}_A(\mathfrak{g}) \to \mathsf{T}_A(L)$ given by

$$\Theta_{\mathfrak{h}} := e^{\hat{*} r_{\mathfrak{h}}} \quad \text{and} \quad \Theta_L := e^{* q_L} \tag{6.2.19}$$

are morphisms of $C^3$-coalgebras over $A$. Upon using this fact, eqn (6.2.6) and the cocommutativity of $\Delta_S$ we compute

$$\begin{aligned}
\Delta_{sh} \circ \Theta_\nabla &= \Delta_{sh} \circ \mathfrak{D} \circ \big(\Theta_{\mathfrak{h}} \otimes_A \Theta_L\big) \circ \Delta_S \\
&= \big(\mathfrak{D} \otimes_A \mathfrak{D}\big) \circ \big(\mathrm{id}_{\mathsf{U}(\mathfrak{h})} \otimes_A \tau \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \big(\Delta_{\mathsf{U}(\mathfrak{h})} \otimes_A \Delta_{sh}\big) \circ \big(\Theta_{\mathfrak{h}} \otimes_A \Theta_L\big) \circ \Delta_S \\
&= \big(\mathfrak{D} \otimes_A \mathfrak{D}\big) \circ \big(\Theta_{\mathfrak{h}} \otimes_A \Theta_L \otimes_A \Theta_{\mathfrak{h}} \otimes_A \Theta_L\big) \circ \big(\Delta_S \otimes_A \Delta_S\big) \circ \Delta_S \\
&= \big(\Theta_\nabla \otimes_A \Theta_\nabla\big) \circ \Delta_S
\end{aligned}$$

showing that $\Theta_\nabla$ is a morphism of coalgebras. By a similar computation it becomes clear that $\Theta_\nabla$ intertwines counits and maps $\mathbf{1}_S$ to $\mathbf{1}$ whence it is a morphism of $C^3$-coalgebras. Since both coalgebras are cofree, see Appendix A.6.1, it suffices to show that the restriction of $\Theta_\nabla$ to the primitive part $\mathfrak{g}$ of $\mathsf{S}_A(\mathfrak{g})$ induces a $A$-linear isomorphism onto the primitive part of $\mathsf{T}_A(L)$ which is $\mathcal{L}_A(L) = \mathfrak{g}$: indeed, let $\xi \in \mathfrak{g}$, and decompose it $\xi = \zeta + x$ with $\zeta \in \mathfrak{h}$ and $x \in L$. Since $\Delta_S(\xi) = \xi \otimes_A \mathbf{1}_S + \mathbf{1}_S \otimes_A \xi$, since $\Theta_{\mathfrak{h}}$ projects away $x$ and $\Theta_L$ projects away $\zeta$ then

$$\Theta_\nabla(\xi) = \Theta_{\mathfrak{h}}(\zeta) \triangleright \mathbf{1} + \mathbf{1}_{\mathsf{U}(\mathfrak{h})} \triangleright \Theta_L(x) = \zeta \triangleright \mathbf{1} + x = \zeta - Z(\zeta) + x$$

and this is an isomorphism $\mathfrak{g} \to \mathfrak{g}$ according to eqn (5.4.19).

*ii.*) Using the fact that $\chi$ takes values in the primitive part $\mathfrak{h}$ of $\mathsf{U}_A(\mathfrak{h})$, eqn (6.2.6), the fact that $\Delta_{sh}$ is cocommutative and coassociative, and the fact that $\mathbf{1}_{\mathsf{U}(\mathfrak{h})} \triangleright b = b$ for all $b \in \mathsf{T}_A(L)$ we get

$$\begin{aligned}
\Delta_{sh} \circ \big(\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\big) &= \Delta_{sh} \circ \mathfrak{D} \circ \big(\chi \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \Delta_{sh} \\
&= \big(\mathfrak{D} \otimes_A \mathfrak{D}\big) \circ \big(\mathrm{id}_{\mathsf{U}(\mathfrak{h})} \otimes_A \tau \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \big(\Delta_{\mathsf{U}(\mathfrak{h})} \otimes_A \Delta_{sh}\big) \circ \big(\chi \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \Delta_{sh} \\
&= \big(\mathfrak{D} \otimes_A \mathfrak{D}\big) \circ \big(\chi \otimes_A \mathrm{id}_{\mathsf{T}_A(L)} \otimes_A \mathbf{1}_{\mathsf{U}(\mathfrak{h})} \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \big(\Delta_{sh} \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \Delta_{sh} \\
&\quad + \big(\mathfrak{D} \otimes_A \mathfrak{D}\big) \circ \big(\mathbf{1}_{\mathsf{U}(\mathfrak{h})} \otimes_A \mathrm{id}_{\mathsf{T}_A(L)} \otimes_A \chi \otimes_A \mathrm{id}_{\mathsf{T}_A(L)}\big) \circ \big(\mathrm{id}_{\mathsf{T}_A(L)} \otimes_A \Delta_{sh}\big) \circ \Delta_{sh} \\
&= \Big(\big(\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\big) \otimes_A \mathrm{id}_{\mathsf{T}_A(L)} + \mathrm{id}_{\mathsf{T}_A(L)} \otimes_A \big(\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\big)\Big) \circ \Delta_{sh}
\end{aligned}$$

whence $\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}$ is always a coderivation of $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon\big)$.

Next, suppose that for some $\chi \in \mathrm{Hom}_A\big(\mathsf{T}_A(L), \mathfrak{h}\big)$ we have $\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)} = 0$. We shall prove by induction over the tensor degree of $b \in \mathsf{T}_A(L)$ that then $\chi(b) = 0$ for all $b \in \mathsf{T}_A(L)$ implying $\chi = 0$ which proves the desired injectivity: indeed for $b = \mathbf{1}$ we have

$$0 = \Big(\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\Big)(\mathbf{1}) = \chi(\mathbf{1}) \triangleright \mathbf{1} = \chi(\mathbf{1}) - Z\big(\chi(\mathbf{1})\big).$$

Since $\chi(\mathbf{1}) \in \mathfrak{h}$ and $Z\big(\chi(\mathbf{1})\big) \in L$ it follows that $\chi(\mathbf{1}) = 0$. Suppose that $\chi(b) = 0$ for all $b \in \mathsf{T}_A(L)$ of tensor degree $\leqslant n$. Then for all $x \in L$ the element $xb$ is of tensor degree $n+1$, (and every such element is a linear combination of this kind of elements), and we get

$$0 = \Big(\chi \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\Big)(xb) = \sum_{(b)} \chi\big(xb^{(1)}\big) \triangleright b^{(2)} + \sum_{(b)} \chi\big(b^{(1)}\big) \triangleright \big(xb^{(2)}\big).$$

By induction the second sum has to vanish since all the arguments $b^{(1)}$ of $\chi$ are of tensor degree $\leqslant n$, and in the first sum the only surviving term of the comultiplication $\Delta_{sh}(b)$ is $b \otimes_A \mathbf{1}$ yielding

$$0 = \big(\chi(xb)\big) \triangleright \mathbf{1} = \chi(xb) - Z\big(\chi(xb)\big)$$

implying $\chi(xb) = 0$ which proves the induction.

*iii.*) In order to get an idea, we take any $d \in \mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}), \mathfrak{h}\big)$ and compute (setting $\overline{d} = d \tilde{*} \mathrm{id}_{\mathsf{S}_A(\mathfrak{g})}$ and using the fact that it is a coderivation and eqn (A.3.6))

$$\begin{aligned}
\Theta_\nabla \circ \overline{d} &= \mathfrak{D} \circ \big(\Theta_{\mathfrak{h}} \otimes_A \Theta_L\big) \circ \Delta_S \circ \overline{d} \\
&= \mathfrak{D} \circ \Big(\big(\Theta_{\mathfrak{h}} \circ \overline{d}\big) \otimes_A \Theta_L\Big) \circ \Delta_S + \mathfrak{D} \circ \Big(\Theta_{\mathfrak{h}} \otimes_A \big(\Theta_L \circ \overline{d}\big) \circ \Delta_S
\end{aligned}$$

Using the definition of $\Theta_L$ in terms of the projection $q_L$, see eqs (6.2.19), (6.2.13), and (6.2.14) we see –thanks to the derivation property (A.3.6)– that the second term

vanishes since the projection $q_L$ in $\Theta_L$ kills the values of the map $d$. Next, we compute the term $\Theta_{\mathfrak{h}} \circ \overline{d}$: recall first the following well-known *derivative of the exponential map*: Let $\left(\mathcal{B}, \blacksquare, \mathbf{1}_{\mathcal{B}}, (F_{(n)}B)_{n\in\mathbb{Z}}\right)$ be complete filtered associative unital $A$-algebra, let $z \in F_{(-1)}B$, and $D : \mathcal{B} \to \mathcal{B}$ a filtration preserving derivation, then –upon writing $\mathrm{ad}_{\blacksquare z'} : z'' \mapsto z' \blacksquare z'' - z'' \blacksquare z'$ for any $z', z'' \in \mathcal{B}$– we have

$$D\left(e^{\blacksquare z}\right) = \left(\frac{e^{\mathrm{ad}_{\blacksquare z}} - \mathrm{id}_{\mathcal{B}}}{\mathrm{ad}_{\blacksquare z}}\big(D(z)\big)\right) \blacksquare e^{\blacksquare z}. \tag{6.2.20}$$

We get –using (A.3.6) and setting $\mathcal{B} = \mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}), \mathsf{U}_A(\mathfrak{h})\big)$, $\blacksquare = \hat{*}$

$$\Theta_{\mathfrak{h}} \circ \overline{d} \quad = \quad e^{\hat{*} r_{\mathfrak{h}}} \circ \overline{d} = \left(\frac{e^{\mathrm{ad}_{\hat{*} r_{\mathfrak{h}}}} - \mathrm{id}_{\mathcal{B}}}{\mathrm{ad}_{\hat{*} r_{\mathfrak{h}}}}\big(r_{\mathfrak{h}} \circ \overline{d}\big)\right) \hat{*}\, e^{\hat{*} r_{\mathfrak{h}}}.$$

Observe now that $r_{\mathfrak{h}} \circ \overline{d} = d$ (where the values of $d$ are seen as elements of $\mathsf{U}_A(\mathfrak{h})$). Moreover, let $\psi : \mathsf{S}_A(\mathfrak{g}) \to \mathfrak{h}$ any $A$-linear map then for any $\beta \in \mathsf{S}_A(\mathfrak{g})$ thanks to the cocommutativity of $\Delta_S$:

$$\begin{aligned}
\Big(\mathrm{ad}_{\hat{*} r_{\mathfrak{h}}}(\psi)\Big)(\beta) &= \sum_{(\beta)} \Big(r_{\mathfrak{h}}(\beta^{(1)})\psi(\beta^{(2)}) - \psi(\beta^{(2)}) r_{\mathfrak{h}}(\beta^{(1)})\Big) = \sum_{(\beta)} \big[r_{\mathfrak{h}}(\beta^{(1)}), \psi(\beta^{(2)}\big]^{(\nabla)} \\
&= \sum_{(\beta)} \mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}(\beta^{(1)})}\Big(\psi(\beta^{(2)}\Big) = \Big(\mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}} *'' \psi\Big)(\beta)
\end{aligned}$$

where we have written $\mathrm{ad}^{(\nabla)}$ for the adjoint representation of the $A$-Lie algebra $\big(\mathfrak{h}, [\ ,\ ]^{(\nabla)}\big)$, i.e. $\mathrm{ad}^{(\nabla)}_{\zeta}(\zeta') = [\zeta, \zeta']^{(\nabla)}$ for any $\zeta, \zeta' \in \mathfrak{h}$ and used the convolution action $*''$, see the table (6.2.12). It follows that the resulting map again takes its values in the primitive part $\mathfrak{h}$ of $\mathsf{U}_A(\mathfrak{h})$, and using an easy induction we can finally write (see the table (6.2.12)))

$$\Theta_{\mathfrak{h}} \circ \overline{d} = \left(\frac{e^{*' \mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}}} - \mathrm{id}_{\mathfrak{h}} \epsilon_S}{*' \mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}}} *'' d\right) \hat{*}\, e^{\hat{*} r_{\mathfrak{h}}} =: E(d) \hat{*}\, e^{\hat{*} r_{\mathfrak{h}}}.$$

Clearly the above $A$-linear map $E$ is an invertible endomorphism of $\mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}), \mathfrak{h}\big)$ because the zeroth order term of the series is the identity map and the higher order terms lower the degree. It follows that

$$\begin{aligned}
\big(\Theta_\nabla \circ \overline{d}\big)(\beta) &= \Big(\mathfrak{D} \circ \Big(\big(\Theta_{\mathfrak{h}} \circ \overline{d}\big) \otimes_A \Theta_L\Big) \circ \Delta_S\Big)(\beta) = \sum_{(\beta)} \Big(\big(E(d)(\beta^{(1)})\big)\big(e^{\hat{*} r_{\mathfrak{h}}}(\beta^{(2)})\big)\Big) \rhd \big(e^{* q_L}(\beta^{(3)})\big) \\
&= \sum_{(\beta)} \big(E(d)(\beta^{(1)})\big) \rhd \Big(\big(e^{\hat{*} r_{\mathfrak{h}}}(\beta^{(2)})\big) \rhd \big(e^{* q_L}(\beta^{(3)})\big)\Big) = \sum_{(\beta)} \big(E(d)(\beta^{(1)})\big) \rhd \big(\Theta_\nabla(\beta^{(2)})\big).
\end{aligned}$$

If we define the linear isomorphism $\mathcal{E} : \mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}), \mathfrak{h}\big) \to \mathrm{Hom}_A\big(\mathsf{T}_A(L), \mathfrak{h}\big)$ by

$$\mathcal{E}(d) := E(d) \circ \Theta_\nabla^{-1} = \left( \frac{e^{*' \mathrm{ad}^{(\nabla)}_{r_\mathfrak{h}}} - \mathrm{id}_\mathfrak{h} \epsilon_S}{*' \mathrm{ad}^{(\nabla)}_{r_\mathfrak{h}}} *'' d \right) \circ \Theta_\nabla^{-1} \tag{6.2.21}$$

we finally get

$$\Theta_\nabla \circ \overline{d} \circ \Theta_\nabla^{-1} = \big(\mathcal{E}(d)\big) \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)} \tag{6.2.22}$$

which proves the statement.

*iv.*) An arbitrary element of the ideal and coideal $\mathcal{I}(L,A)$ is a linear combination of elements of the form $\zeta \bullet \beta$ with $\beta \in \mathsf{S}_A(\mathfrak{g})$ and $\zeta \in \mathfrak{h}$. Observe that the map $\beta \mapsto \zeta \bullet \beta$ is a coderivation of the form $\zeta \epsilon_S \tilde{*} \mathrm{id}_{\mathsf{S}_A(\mathfrak{g})} = \overline{\zeta \epsilon_S}$. By part *iii.*) we have for all $\beta \in \mathsf{S}_A(\mathfrak{g})$ upon setting $b = \Theta_\nabla(\beta) \in \mathsf{T}_A(L)$:

$$\Theta_\nabla(\zeta \bullet \beta) = \big(\Theta_\nabla \circ \overline{\zeta \epsilon_S}\big)(\beta) = \Big(\big(\mathcal{E}(\zeta \epsilon_S)\big) \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\Big)\big(\Theta_\nabla(\beta)\big) = \sum_{(b)} \big(\mathcal{E}(\zeta \epsilon_S)\big)(b^{(1)}) \triangleright b^{(2)}$$

and the last term is a finite sum of terms of the form $\zeta' \triangleright b'$ with $\zeta' \in \mathfrak{h}$ and $b' \in \mathsf{T}_A(L)$ since the maps $\mathcal{E}(\zeta \epsilon_S)$ takes its values in $\mathfrak{h}$. Hence $\Theta_\nabla(\zeta \bullet \beta)$ is an element of the coideal $\mathcal{J}_\nabla(L,A)$ proving the inclusion $\Theta_\nabla\big(\mathcal{I}(L,A)\big) \subset \mathcal{J}_\nabla(L,A)$.

On the other hand, an arbitry element of the coideal $\mathcal{J}_\nabla(L,A)$ is a finite sum of elements of the form $\zeta \triangleright b$ with $\zeta \in \mathfrak{h}$ and $b \in \mathsf{T}_A(L)$. Clearly, according to part *ii.*) the coderivation $b \mapsto \zeta \triangleright b$ is of the form $(\zeta \epsilon) \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}$. Let $d_\zeta \in \mathrm{Hom}_A\big(\mathsf{S}_A(\mathfrak{g}), \mathfrak{h}\big)$ the unique $A$-linear map such that $\mathcal{E}(d_\zeta) = \zeta \epsilon$. Then acccording to part *iii.*) we get for all $b \in \mathsf{T}_A(L)$ upon setting $\beta = \Theta_\nabla^{-1}(b) \in \mathsf{S}_A(\mathfrak{g})$:

$$\zeta \triangleright b = \big((\zeta \epsilon) \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}\big)(b) = \Theta_\nabla\big(\overline{d_\zeta}(\beta)\big) = \sum_{(\beta)} \Theta_\nabla\big(d_\zeta(b^{(1)}) \bullet \beta^{(2)}\big)$$

and the right hand side is in $\Theta_\nabla\big(\mathcal{I}(L,A)\big)$ since the values of the map $d_\zeta$ are elements of $\mathfrak{h}$. This proves the other inclusion $\Theta_\nabla\big(\mathcal{I}(L,A)\big) \supset \mathcal{J}_\nabla(L,A)$ and the first equality in (6.2.18).

Next, using the fact that $I : \mathsf{S}_A(L) \to \mathsf{S}_A(\mathfrak{g})$ is a morphism of $C^3$-coalgebras and eqn (A.3.5) we get

$$\Theta_\nabla \circ I = \mathfrak{D} \circ \big(\Theta_\mathfrak{h} \otimes_A \Theta_L\big) \circ \Delta_S \circ I = \mathfrak{D} \circ \big((\Theta_\mathfrak{h} \circ I) \otimes_A (\Theta_L \circ I)\big) \circ \Delta_{\mathsf{S}_A(L)}.$$

We have (writing $\hat{*}_L$ for the convolution with algebra $\mathsf{U}_A(L)$ and coalgebra $\mathsf{S}_A(L)$)

$$\Theta_\mathfrak{h} \circ I = e^{\hat{*} q_\mathfrak{h}} \circ I = e^{\hat{*}_L (q_\mathfrak{h} \circ I)} = e^{\hat{*}_L 0} = \mathbf{1}_{\mathsf{U}_A(\mathfrak{h})} \epsilon_{\mathsf{S}_A(L)} = \big(\mathbf{1}_{\mathsf{U}_A(\mathfrak{h})} \epsilon_S\big) \circ I$$

and (writing $*_L$ for the convolution with algebra $\mathsf{T}_A(L)$ and coalgebra $\mathsf{S}_A(L)$)

$$\Theta_L \circ I = e^{* q_L} \circ I = e^{*_L (q_L \circ I)} = e^{*_L (q \circ I)} = e^{* q} \circ I \overset{(A.6.22)}{=} \omega \circ I,$$

whence

$$\begin{aligned}\Theta_\nabla \circ i &= \mathfrak{D} \circ \Big( \big( (\mathbf{1}_{\mathsf{U}_A(\mathfrak{h})} \epsilon_S) \circ i \big) \otimes_A (\omega \circ i) \Big) \circ \Delta_{\mathsf{S}_A(L)} = \mathfrak{D} \circ \Big( \big( \mathbf{1}_{\mathsf{U}_A(\mathfrak{h})} \epsilon_S \big) \otimes_A \omega \Big) \circ \Delta_S \circ i \\ &= \omega \circ i = \Upsilon\end{aligned}$$

which shows the second part of statement (6.2.18).
Finally, the direct decomposition $\mathsf{T}_A(L) = \Upsilon\big(\mathsf{S}_A(L)\big) \oplus \mathcal{J}_\nabla(L,A)$ is a direct consequence of the fact that $\Theta_\nabla$ is an isomorphism, the direct decomposition (6.2.9), and the two preceding isomorphisms. *v.*) This follows at once from the second statement of Proposition 4.3.2 upon noting that the Rinehart bialgebra $\big(\mathsf{T}_A(L), \diamond, \mathbf{1}, \Delta_{sh}, \epsilon\big)$ is isomorphic to $\mathcal{U}\big(\mathcal{P}_\nabla(L,A), A\big)$ (see the last statement of Theorem **??**), and that the Rinehart ideal $\mathcal{J}_\nabla(L,A)$ is generated by the kernel of the surjective Lie-Rinehart algebra morphism $Z$ and is complemented by $L$ according to *iv.*). □

There is the following Corollary which may be of interest:

**Corollary 6.2.3** *The following $A$-linear map is an isomorphism of left $\mathsf{U}_A(\mathfrak{h})$-modules:*

$$\Xi : \mathsf{U}_A(\mathfrak{h}) \otimes_A \mathsf{S}_A(L) \to \mathsf{T}_A(L) : u \otimes_A \gamma \mapsto u \triangleright (\Upsilon(\gamma)). \tag{6.2.23}$$

### 6.2.2 The coderivation $D_\nabla$

In order to describe the projection $\Pi_\nabla : \mathsf{T}_A(L) \to \mathsf{S}_A(L) \hookrightarrow \mathsf{T}_A(L)$ modulo the Rinehart ideal $\mathcal{J}_\nabla(L,A)$ a straight-forward description would be to use the *inverse of the isomorphism* $\Theta_\nabla$, see eqn (6.2.14) and then use the 'easy' projection $\mathsf{S}_A(\mathfrak{g}) \to \mathsf{S}_A(L)$ along the 'easy' ideal $\mathcal{I}(L,A)$, followed by the symmetrization injection $\Upsilon : \mathsf{S}_A(L) \to \mathsf{T}_A(L)$. The problem is that the inverse of $\Theta_\nabla$ does not seem to be very explicit. We have therefore chosen another description of $\mathcal{J}_\nabla(L,A)$ –which is the sum of the images of many coderivations, i.e. of the $\mathfrak{D}_\zeta$, $\zeta \in \mathfrak{h}$– in terms of the image of **only one coderivation** $D_\nabla$ which we describe in this Section. The fact that $D_\nabla$ is a coderivation will allow us to use the 'pull-through-formulas' (6.2.22) to check certain properties on the 'easier' symmetric algebra $\mathsf{S}_A(\mathfrak{g})$.

#### The trivial case: $[\![\ ,\ ]\!] = 0$, $\rho = 0$, $\nabla = 0$

We shall first treat the *trivial case* where Lie-Rinehart bracket, anchor map, and connection are vanishing: here the maps $H$ and the restriction of $Z$ to $\mathfrak{h}$ vanish, whence the bracket $[\ ,\ ]^{(\nabla)}$ on $\mathfrak{h}$ reduces to the free Lie bracket, and $\mathfrak{h}$ is a Lie subalgebra (in fact a Lie ideal) of the free Lie algebra $\mathfrak{g} = \mathcal{L}_A(L)$ equipped with its usual bracket. The action $\triangleright$ of $\mathfrak{h}$ on $\mathsf{T}_A(L)$ is just left multiplication, i.e. $\zeta \triangleright b = \zeta b$ for all $\zeta \in \mathfrak{h}$ and $b \in \mathsf{T}_(L)$. Define the well-known symmetrization map $\Pi_0 : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$

defined for all integers $n \geqslant 1$ and $x_1, \ldots, x_n \in L$ by

$$\Pi_0(\mathbf{1}) = \mathbf{1}, \quad \text{and} \quad \Pi_0(x_1 \cdots x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}. \tag{6.2.24}$$

which is involutive, i.e. $\Pi_0 \circ \Pi_0 = \Pi_0$. In convolution notation this is equal to

$$\Pi_0 = e^{\star p_L} \tag{6.2.25}$$

see the tables (6.2.12) and (6.2.13) for the notation, whence $\Pi_0$ is a morphism of $C^3$-coalgebras over $A$ according to (A.3.4). Next define the coideal

$$\mathcal{J}_0(L, A) = A\text{-Span}\{\zeta b \mid \zeta \in \mathfrak{h},\ b \in \mathsf{T}_A(L)\} \tag{6.2.26}$$

It is easily seen by direct inspection of the definitions and by elementary combinatorics that $\mathcal{J}_0(L, A)$ is a two-sided ideal of the free algebra $\mathsf{T}_A(L)$, and that defining the complementary projection $Q_0 := \mathrm{id}_{\mathsf{T}_A(L)} - \Pi_0$

$$\mathrm{Ker}(\Pi_0) = \mathcal{J}_0(L, A) = \mathrm{Im}(Q_0) \quad \text{and} \quad \mathrm{Im}(\Pi_0) = \Upsilon\left(\mathsf{S}_A(L)\right) = \mathrm{Ker}(Q_0). \tag{6.2.27}$$

Define the two projections $\pi_L : \mathfrak{g} \twoheadrightarrow L \hookrightarrow \mathfrak{g}$ with kernel $\mathfrak{h}$ and $\pi_{\mathfrak{h}} : \mathfrak{g} \twoheadrightarrow \mathfrak{h} \hookrightarrow \mathfrak{g}$ with kernel $L$. Clearly, $\pi_L + \pi_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{g}}$.

Recall the 'modified Dynkin idempotent' $\tilde{e}_D$, see eqn (A.6.17). We define the following coderivation of $\left(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\right)$ (where we have not explicitly written the corestriction of $\tilde{e}_D$ to $\mathfrak{g}$):

$$D_0 := \left(\pi_{\mathfrak{h}} \circ \tilde{e}_D\right) \star \mathrm{id}_{\mathsf{T}_A(L)}. \tag{6.2.28}$$

Note that in our trivial case the usual convolution $\star$ is equal to $\overset{\triangleright}{\star}$ whence $D_0$ is a coderivation. There is the following very simple formula for $D_0$: using $\pi_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{g}} - \pi_L$, the obvious identity $\pi_L \circ \tilde{e}_D = p_L$, see the table (6.2.12), and the Von Waldenfels formula (A.6.18) we get

$$D_0 = \tilde{e}_D \star \mathrm{id}_{\mathsf{T}_A(L)} - p_L \star \mathrm{id}_{\mathsf{T}_A(L)} = \mathsf{Deg} \star S \star \mathrm{id}_{\mathsf{T}_A(L)} - p_L \star \mathrm{id}_{\mathsf{T}_A(L)} = \mathsf{Deg} - p_L \star \mathrm{id}_{\mathsf{T}_A(L)} \tag{6.2.29}$$

since by definition the antipode $S$ is the convolution inverse to the identity map. This computation immediately gives us the explicit formula for all $x, x_1, \ldots, x_n \in L$ where the integer $n$ is $\geqslant 2$:

$$D_0(\mathbf{1}) = 0, \ \ D_0(x) = 0, \ \ D_0(x_1 \cdots x_n) = \sum_{r=2}^{n} [x_1 \cdots x_{r-1}, x_r] x_{r+1} \cdots x_n. \tag{6.2.30}$$

For each $n \in \mathbb{N}$ define the following $A$-linear map $\tilde{D}^{(n)} : \mathsf{T}_A^n(L) \to \mathsf{T}_A^n(L)$ by $\tilde{D}^{(0)} = 0$, $\tilde{D}^{(1)} = 0$, and for each $n \geqslant 2$

$$\tilde{D}^{(n)} = \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)! \left(p_L^{\star k} \star \mathrm{id}_{\mathsf{T}_A(L)}\right)\Big|_{\mathsf{T}_A^n(L)} - \frac{1}{n!} \left(\sum_{k=0}^{n-2} \frac{1}{n-k}\right) p_L^{\star n}|_{\mathsf{T}_A^n(L)} \tag{6.2.31}$$

and let $\tilde{D} : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$ be the direct sum of the $\tilde{D}^{(n)}$, i.e. $\tilde{D}|_{\mathsf{T}^n_A(L)} = \tilde{D}^{(n)}$ for all $n \in \mathbb{N}$.

**Lemma 6.2.4** *The coderivation $D_0$ and the $A$-linear map $\tilde{D}$ have the following properties:*

1. *$D_0$ preserves each submodule $\mathsf{T}^n_A(L)$, $n \in \mathbb{N}$.*
2. *Both $D_0$ and $\tilde{D}$ vanish on $\Upsilon(\mathsf{S}_A(L))$ and preserve the ideal and coideal $\mathcal{J}_0(L,A)$*
3. *The restriction of $D_0$ to $\mathcal{J}_0(L,A)$ is invertible, and its inverse is equal to the restriction of $\tilde{D}$ to $\mathcal{J}_0(L,A)$:*

$$\forall\, b \in \mathcal{J}_0(L,A): \quad D_0\big(\tilde{D}(b)\big) = b = \tilde{D}\big(D_0(b)\big). \tag{6.2.32}$$

4. *$\mathrm{Im}(D_0) = \mathcal{J}_0(L,A) = \mathrm{Im}(\tilde{D})$ and $\mathrm{Ker}(D_0) = \Upsilon(\mathsf{S}_A(L)) = \mathrm{Ker}(\tilde{D})$.*

*Proof.* 1.) is obvious from eqn (6.2.30).

Before going on it is useful to show some preparatory combinatorial identities 'in convolutional disguise': in this proof only, in order to avoid clumsy computations we shall write $p$ for $p_L$, id for $\mathrm{id}_{\mathsf{T}_A(L)}$, and $\mathrm{id}_n$ for $\mathrm{id}_{\mathsf{T}^n_A(L)}$. We set $E := p \star \mathrm{id}$, and for each nonnegative integer $k$: $E^{(k)} := p^{\star k} \star \mathrm{id}$ (i.e. $E^{(0)} :=$ id, and of course $E^{(1)} = E$). Observe that $E$ is a coderivation according to (A.3.7). It is not a derivation, but we rather have –using $p(bc) = p(b)\epsilon(c) + \epsilon(b)p(c)$ for all $b, c \in \mathsf{T}_A(L)$:

$$\begin{aligned} E(bc) &= \sum_{(b)(c)} p\left(b^{(1)}c^{(1)}\right) b^{(2)}c^{(2)} = \sum_{(b)(c)} \left(p(b^{(1)})\epsilon(c^{(1)}) + \epsilon(b^{(1)})p(c^{(1)})\right) b^{(2)}c^{(2)} \\ &= \sum_{(b)} p(b^{(1)})b^{(2)}c + \sum_{(c)} p(c^{(1)})bc^{(2)} = E(b)c + bE(c) + \sum_{(c)} \big[p(c^{(1)}), b\big]c^{(2)}. \end{aligned} \tag{6.2.33}$$

Clearly, $p \circ E = p = E \circ p$. Since $E$ is a coderivation we get for all integers $k \geqslant 1$

$$p^{\star k} \circ E \overset{(A.3.6)}{=} \sum_{i=0}^{k-1} p^{\star i} \star (p \circ E) \star p^{\star k-1-i} = k p^{\star k}. \tag{6.2.34}$$

On the other hand we get –using (6.2.33)–

$$\begin{aligned} \left(E \circ p^{\star(k+1)}\right)(b) &= \sum_{(b)} E\left(p^{\star k}\big(b^{(1)}\big)p\big(b^{(2)}\big)\right) \\ &\overset{(6.2.33)}{=} \sum_{(b)} E\big(p^{\star k}\big(b^{(1)}\big)\big)p\big(b^{(2)}\big) + \sum_{(b)} p^{\star k}\big(b^{(1)}\big)p\big(b^{(2)}\big) + \sum_{(b)} \big[p\big(b^{(2)}\big), p^{\star k}\big(b^{(1)}\big)\big] \\ &= \left(\big(E \circ p^{\star k}\big) \star p + p^{\star k} \star p + 0\right)(b) \end{aligned}$$

which allows to show by induction over $k$ the identity

$$E \circ p^{\star k} = k p^{\star k} \overset{(6.2.34)}{=} p^{\star k} \circ E. \tag{6.2.35}$$

Moreover, since $E$ is a coderivation we get

$$\begin{aligned} E^{(k)} \circ E &= \left(p^{\star k} \star \mathrm{id}\right) \circ E \overset{(A.3.6)}{=} (p^{\star k} \circ E) \star \mathrm{id} + p^{\star k} \star E = k p^{\star k} \star \mathrm{id} + p^{\star (k+1)} \star \mathrm{id} \\ &= k E^{(k)} + E^{(k+1)} \end{aligned} \tag{6.2.36}$$

which allows to show by induction over all integer $k \geqslant 1$ that

$$E^{(k)} = (E - (k-1)\mathrm{id}) \circ (E - (k-2)\mathrm{id}) \circ \cdots (E - \mathrm{id}) \circ E.$$

This proves in particular that all the maps $E^{(k)}$, $E^{(l)}$, and $p^{\star m}$ commute for arbitrary nonnegative integers $k, l, m$ since each $E^{(k)}$ is a composition polynomial of $E$. Moreover, eqn (6.2.35) shows that

$$\Pi_0 \circ E = p \star e^{\star p} = E \circ \Pi_0, \quad \text{hence} \quad \Pi_0 \circ E^{(k)} = p^{\star k} \star e^{\star p} = E^{(k)} \circ \Pi_0 \tag{6.2.37}$$

since $\left(p^{\star k} \star \mathrm{id}\right) \circ \Pi_0 = p^{\star k} \star \Pi_0$ because $\Pi_0$ is a morphism of $C^3$-coalgebras and $p \circ \Pi_0 = p$. Next, since Deg is a derivation and coderivation we get using $\mathsf{Deg} \circ p = p = p \circ \mathsf{Deg}$

$$\mathsf{Deg} \circ p^{\star k} = k p^{\star k} = p^{\star k} \circ \mathsf{Deg}, \quad \text{hence} \quad \mathsf{Deg} \circ \Pi_0 = p \star e^{\star p} = \Pi_0 \circ \mathsf{Deg}. \tag{6.2.38}$$

To sum up, all the maps $E^{(k)}, p^{\star l}, \mathsf{Deg}$ and $\Pi_0$ commute and satisfy the preceding identities.
*ii.*) We compute

$$\Pi_0 \circ D_0 = \Pi_0 \circ \left(\mathsf{Deg} - E\right) \overset{(6.2.37),(6.2.38)}{=} \left(\mathsf{Deg} - E\right) \circ \Pi_0 = p \star e^{\star p} - p \star e^{\star p} = 0.$$

This shows that $D_0$ vanishes on $\Upsilon(\mathsf{S}_A(L))$ and preserves $\mathcal{J}_0(L, A)$ because it commutes with $\mathrm{id} - \Pi_0$.
Next we compute for fixed integer $n \geqslant 2$ recalling that $\Pi_0$ is a morphism of $C^3$-coalgebras and eqn (A.3.5) and $p \circ \Pi_0 = p$

$$\begin{aligned} \tilde{D} \circ \Pi_0 &= \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)! \left(p^{\star k} \star \mathrm{id}\right) \circ \Pi_0 \Big|_{\mathsf{T}^n_A(L)} - \frac{1}{n!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n} \circ \Pi_0 |_{\mathsf{T}^n_A(L)} \\ &= \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)! \left(p^{\star k} \star \Pi_0\right) \Big|_{\mathsf{T}^n_A(L)} - \frac{1}{n!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n} |_{\mathsf{T}^n_A(L)} \\ &= \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)! \left( p^{\star k} \star \frac{p^{\star (n-k)}}{(n-k)!} \right) \Big|_{\mathsf{T}^n_A(L)} - \frac{1}{n!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n} |_{\mathsf{T}^n_A(L)} = 0, \end{aligned}$$

and of course also $\Pi_0 \circ \tilde{D} = 0$ thanks to eqn (6.2.37). Therefore $\tilde{D}$ vanishes on $\Upsilon(\mathrm{S}_A(L))$ and preserves $\mathcal{J}_0(L,A)$ because it commutes with the projection $Q_0 = \mathrm{id} - \Pi_0$.
*iii.*) From the preceding identities it is clear that the maps $D_0$ and $\tilde{D}$ commute. For fixed degree $n \geqslant 2$ we compute observing that $\mathsf{Deg}|_{\mathsf{T}^n_A(L)} = n\mathrm{id}_n$:

$$
\begin{aligned}
\tilde{D} \circ D_0|_{\mathsf{T}^n_A(L)} &= \tilde{D} \circ \big(n\mathrm{id} - E\big)|_{\mathsf{T}^n_A(L)} \\
&= \frac{n}{n!} \sum_{k=0}^{n-2} (n-k-1)!\, E^{(k)}\Big|_{\mathsf{T}^n_A(L)} - \frac{n}{n!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n}|_{\mathsf{T}^n_A(L)} \\
&\quad - \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)!\, E^{(k)} \circ E\Big|_{\mathsf{T}^n_A(L)} + \frac{1}{n!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n} \circ E|_{\mathsf{T}^n_A(L)} \\
&\overset{(6.2.36)(6.2.34)}{=} \sum_{k=0}^{n-2} \frac{(n-k-1)!}{(n-1)!}\, E^{(k)}\Big|_{\mathsf{T}^n_A(L)} - \frac{1}{(n-1)!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n}|_{\mathsf{T}^n_A(L)} \\
&\quad - \frac{1}{n!} \sum_{k=0}^{n-2} (n-k-1)! \left( kE^{(k)} + E^{(k+1)} \right)\Big|_{\mathsf{T}^n_A(L)} \\
&\quad + \frac{1}{(n-1)!} \left( \sum_{k=0}^{n-2} \frac{1}{n-k} \right) p^{\star n}|_{\mathsf{T}^n_A(L)} \\
&= \mathrm{id}_n + \sum_{k=1}^{n-2} \frac{(n-k-1)!}{n!} (n-k)\, E^{(k)}\Big|_{\mathsf{T}^n_A(L)} - \sum_{k=1}^{n-2} \frac{(n-k)!}{n!}\, E^{(k)}\Big|_{\mathsf{T}^n_A(L)} \\
&\quad - \frac{1}{n!} E^{(n-1)}\Big|_{\mathsf{T}^n_A(L)} \\
&= \mathrm{id}_n - \frac{1}{n!} E^{(n-1)}\Big|_{\mathsf{T}^n_A(L)} = \mathrm{id}_n - \frac{1}{n!} p^{\star n}\Big|_{\mathsf{T}^n_A(L)} = Q_0|_{\mathsf{T}^n_A(L)}
\end{aligned}
$$

because obviously $E^{(n-1)}\big|_{\mathsf{T}^n_A(L)} = p^{\star(n-1)} \star p\big|_{\mathsf{T}^n_A(L)} = p^{\star n}|_{\mathsf{T}^n_A(L)}$ which proves the statement.
*iv.*) This is an immediate consequence of the preceding statement. □

### The general case

Let us return to the case of a general Lie-Rinehart algebra $\big(L, \rho, [\![\ ,\ ]\!], A\big)$ over $A$, and $\nabla$ a connection in $L$ along $L$. We can see this as a deformation of the trivial case, for instance by introducing a parameter $t \in K$ and observing that $\big(L, t\rho, t[\![\ ,\ ]\!], A, t\nabla\big)$ is a Lie-Rinehart algebra over $A$ for all values of $t$: this would be a sort of interpolation from the trivial case ($t = 0$) to the general case ($t = 1$).

In order to find a description of the ideal and coideal $\mathcal{J}_\nabla(L,A)$ as the image of a single coderivation as in the trivial case we generalize eqn (6.2.28) in the obvious way and define

$$D_\nabla := \left(\pi_{\mathfrak{h}} \circ \tilde{e}_D\right) \overset{\triangleright}{\star} \mathrm{id}_{\mathsf{T}_A(L)}. \tag{6.2.39}$$

We define the $A$-linear maps $D_Z$ and $D_H$ from $\mathsf{T}_A(L)$ to itself in the following way $\forall\ b \in \mathsf{T}_A(L)$:

$$D_Z(b) := \sum_{(b)} \Big( Z\big((\pi_{\mathfrak{h}} \circ \tilde{e}_D)\big(b^{(1)}\big)\big)\Big) b^{(2)}, \tag{6.2.40}$$

$$D_H(b) := \sum_{(b)} H_{\tilde{e}_D(b^{(1)})}\big(b^{(2)}\big). \tag{6.2.41}$$

We have the following

**Theorem 6.2.5** *With the above notations:*

1. *We have the decomposition*

$$D_\nabla = D_0 - D_Z + D_H, \tag{6.2.42}$$

*and $D_\nabla$, $D_Z$ and $D_H$ are coderivations of $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon\big)$. $D_Z$ and $D_H$ are decreasing the tensor degree by at least one.*

2. *We have*

$$\mathrm{Ker}\big(D_\nabla\big) = \Upsilon\big(\mathsf{S}_A(L)\big) \quad \text{and} \quad \mathrm{Im}\big(D_\nabla\big) = \mathcal{J}_\nabla(L,A). \tag{6.2.43}$$

*Moreover, the restriction of $D_\nabla$ to $\mathcal{J}_0(L,A)$ induces an $A$-linear isomorphism of the $A$-submodule $\mathcal{J}_0(L,A)$ with the $A$-submodule $\mathcal{J}_\nabla(L,A)$.*

*Proof. i.)* The decomposition (6.2.42) follows directly from the definition of $D_\nabla$ (6.2.41) and of $\triangleright$, see eqn (6.2.3). It is clear that $D_0$ and $D_\nabla$ are coderivations, see Prop. 6.2.2, *ii*). Moreover, $D_Z$ is of the form $\chi \star \mathrm{id}_{\mathsf{T}_A(L)}$ with $\chi(b) = Z\big((\pi_{\mathfrak{h}} \circ \tilde{e}_D)\big(b\big) \in L \subset \mathfrak{g}$, hence a coderivation according to eqn (A.3.7). It follows that $D_H$ is a coderivation. Furthermore, for given $b \in \mathsf{T}_A(L)$ consider $D_Z(b)$ and $D_H(b)$: by the definitions (6.2.40) and (6.2.41) the argument $b$ is split into two parts $b^{(1)}$ and $b^{(2)}$ by the shuffle comultiplication, and in the surviving terms $b^{(1)}$ is of degree at least 2 (since it is projected to $\mathfrak{h}$) whence the degree of the 'rest' $b^{(2)}$ is decreased by at least two: the result is decreased by at least two by $D_H$ (since $H_\zeta$ preserves degrees) and by at least one by $D_Z$ since the values of $Z$ are of degree one. *ii.)* We have

$$\tilde{e}_D \circ \Pi_0 = p_L,$$

which can be shown by direct combinatorics from the definitions (6.2.24) and (**??**) or using eqs (6.2.25), (A.6.18), (6.2.38), and $S \circ p_L = -p_L$:

$$\tilde{e}_D \circ \Pi_0 = \big(\mathsf{Deg} \star S\big) \circ \Pi_0 = \big(\mathsf{Deg} \circ \Pi_0\big) \star \big(S \circ \Pi_0\big) = p_L \star e^{\star p_L} \star e^{-\star p_L} = p_L.$$

This implies $\pi_{\mathfrak{h}} \circ \tilde{e}_D \circ \Pi_0 = \pi_{\mathfrak{h}} \circ p_L = 0$, and since $D_\nabla$ contains the combination $\pi_{\mathfrak{h}} \circ \tilde{e}_D$ it follows that $D_\nabla$ vanishes on the image of $\Upsilon$.
Let $b \in \mathsf{T}_A(L)$. Then

$$D_\nabla(b) = \sum_{(b)} \Big( (\pi_{\mathfrak{h}} \circ \tilde{e}_D)\big(b^{(1)}\big) \Big) \triangleright b^{(2)}$$

which clearly is a finite linear combination of terms of the form $\zeta \triangleright b'$ with $\zeta \in \mathfrak{h}$ and $b' \in \mathsf{T}_A(L)$ showing that the image of $D_\nabla$ is contained in $\mathcal{J}_\nabla(L,A)$.
On the other hand, to show the other inclusion we take a detour to the symmetric algebra $\mathsf{S}_A(\mathfrak{g})$:let $c \in \mathcal{J}_\nabla(L,A)$. Since $D_\nabla$ obviously is a coderivation in the particular $A$-module $\mathrm{Coder}^{\mathfrak{h}}_A(\mathsf{T}_A(L),\mathsf{T}_A(L))$, see Proposition 6.2.2 *ii.*), the corresponding coderivation $\Theta_\nabla^{-1} \circ D_\nabla \circ \Theta_\nabla = \hat{D}$ of $\mathsf{S}_A(\mathfrak{g})$ is in the $A$-module $\mathrm{Coder}^{\mathfrak{h}}_A(\mathsf{S}_A(\mathfrak{g}),\mathsf{S}_A(\mathfrak{g}))$, see eqs (6.2.14), (6.2.16) and Proposition 6.2.2 *iii.*) for definitions, and hence of the form

$$\Theta_\nabla^{-1} \circ D_\nabla \circ \Theta_\nabla = \hat{D} = \hat{d} \tilde{*} \mathrm{id}_{\mathsf{S}_A(\mathfrak{g})}$$

where $\hat{d} : \mathsf{S}_A(\mathfrak{g}) \to \mathfrak{h}$ is the $A$-linear map defined by

$$\hat{d} = \mathcal{E}^{-1}(\pi_{\mathfrak{h}} \circ \tilde{e}_D) = \frac{*' \mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}}}{e^{*' \mathrm{ad}^{(\nabla)}_{r_{\mathfrak{h}}}} - \mathrm{id}_{\mathfrak{h}} \epsilon_S} *'' (\pi_{\mathfrak{h}} \circ \tilde{e}_D \circ \Theta_\nabla),$$

see eqs (6.2.21) and (6.2.22) for definitions. We compute the zeroth and first Taylor coefficients $\hat{d}_0$ and $\hat{d}_1$ of $\hat{d}$: clearly $\hat{d}_0(\mathbf{1}_S) = \hat{d}(\mathbf{1}_S) = 0$ since $\tilde{e}_D(\mathbf{1}) = 0$, and for any $\xi \in \mathfrak{g}$ of tensor degree $n \geqslant 1$ which we write $\xi = \zeta + x$ with $\zeta \in \mathfrak{h}$ and $x \in L$, we get –upon using $\Theta_\nabla(\xi) = \zeta - Z(\zeta) + x$ and $\Delta_{sh}(\xi) = \xi \otimes_A \mathbf{1} + \mathbf{1} \otimes_A \xi$–

$$\hat{d}_1(\xi) = \hat{d}(\xi) = \pi_{\mathfrak{h}}\Big(\tilde{e}_D\big(\Theta_\nabla(\xi)\big)\Big) \overset{(A.6.18)}{=} \pi_{\mathfrak{h}}(n\zeta - Z(\zeta) + x) = n\zeta = n\pi_{\mathfrak{h}}(\xi).$$

Let $\hat{D}_1$ denote the coderivation $\hat{d}_1 \tilde{*} \mathrm{id}_{\mathsf{S}_A(\mathfrak{g})}$, and let $\xi_1, \ldots, \xi_m \in \mathfrak{g}$ of tensor degree $n_1, \ldots, n_m$, respectively. Then

$$\hat{D}_1(\xi_1 \bullet \cdots \bullet \xi_k) = \sum_{i=1}^{m} n_i\big(\pi_{\mathfrak{h}}(\xi_i)\big) \bullet \xi_1 \bullet \cdots \bullet \xi_{i-1} \bullet \xi_{i+1} \bullet \cdots \bullet \xi_m$$

and $\hat{D}_1$ is also a derivation of the commutative untial algebra $\big(\mathsf{S}_A(\mathfrak{g}), \bullet, \mathbf{1}_S\big)$ and for any $x_1, \ldots, x_k \in L$ and $\zeta_1, \ldots, \zeta_l \in \mathfrak{h}$ of degree $n_1, \ldots, n_l \geqslant 2$, respectively, we thus get

$$\hat{D}_1(x_1 \bullet \cdots \bullet x_k \bullet \zeta_1 \bullet \cdots \bullet \zeta_l) = (n_1 + \cdots + n_l) x_1 \bullet \cdots \bullet x_k \bullet \zeta_1 \bullet \cdots \bullet \zeta_l.$$

For $l \geqslant 1$ the sum $(n_1 + \cdots + n_l)$ is strictly positive, hence invertible in $\mathbb{Q}$, and this shows that the restriction of $\hat{D}_1$ to the ideal and coideal $\mathcal{I}(L,A)$ induces an invertible map $\mathcal{I}(L,A) \to \mathcal{I}(L,A)$ since it is spanned by elements $x_1 \bullet \cdots \bullet x_k \bullet \zeta_1 \bullet \cdots \bullet \zeta_l$ with $l \geqslant 1$. Since the higher order Taylor coefficients $\hat{d}_k = \hat{d}|_{\mathsf{S}_A(\mathfrak{g})}$ for $k \geqslant 2$ lead to coderivations $\hat{d}_k {}^{*}\!\mathrm{id}_{\mathsf{S}^k_A(\mathfrak{g})}$ which strictly **lower the symmetric degree** and leave the ideal and coideal $\mathcal{I}(L,A)$ invariant it follows that the restriction of the coderivation $\hat{D}$ to $\mathcal{I}(L,A)$ induces an invertible map $\mathcal{I}(L,A) \to \mathcal{I}(L,A)$:

$$\hat{D}|_{\mathcal{I}(L,A)} : \mathcal{I}(L,A) \to \mathcal{I}(L,A) \quad \text{is invertible.} \tag{6.2.44}$$

Now let $c \in \mathcal{J}_\nabla(L,A)$. We apply the inverse $\Theta_\nabla^{-1}$ of the map $\Theta_\nabla : \mathsf{S}_A(\mathfrak{g}) \to \mathsf{T}_A(L)$ to $c$ and get an element $\beta = \Theta_\nabla^{-1}(c)$ of the ideal and coideal $\mathcal{I}(L,A)$ of $\mathsf{S}_A(\mathfrak{g})$, see eqn (6.2.8) and Proposition 6.2.2 *iv.*) eqn (6.2.18). By the preceding reasoning there is a unique element $\beta' \in \mathcal{I}(L,A)$ such that $\beta = \hat{D}(\beta')$ and therefore

$$c = \Theta_\nabla\big(\hat{D}(\beta')\big) = D_\nabla\big(\Theta_\nabla(\beta')\big).$$

This shows that every $c \in \mathcal{J}_\nabla(L,A)$ is in the image of $D_\nabla$ which proves the second statement of eqn (6.2.43).
Conjugating the coderivation $\hat{D}$ of the preceding part with $\Theta_\nabla$ gives us the following statement analogous to (6.2.44):

$$D_\nabla|_{\mathcal{J}_\nabla(L,A)} : \mathcal{J}_\nabla(L,A) \to \mathcal{J}_\nabla(L,A) \quad \text{is invertible.} \tag{6.2.45}$$

Thanks to the direct decomposition $\mathsf{T}_A(L) = \mathcal{J}_\nabla(L,A) \oplus \Upsilon\left(\mathsf{S}_A(L)\right)$, see eqn (6.2.18) of Proposition 6.2.2, we can thus conclude that

$$\mathrm{Ker}(D_\nabla) = \Upsilon\left(\mathsf{S}_A(L)\right),$$

and this implies that the restriction of $D_\nabla$ to $\mathcal{J}_0(L,A)$ is still surjective on $\mathcal{J}_\nabla(L,A)$ and has vanishing kernel which proves the last statement of *ii.*). □

### 6.2.3 The projection modulo $\mathcal{J}_\nabla(L,A)$ and the multiplication formula

The description of the Rinehart ideal $\mathcal{J}_\nabla(L,A)$ in the last Section will allow us to give a fairly explicit formula for the $A$-linear projection defined as the $A$-linear map

$$\Pi_\nabla : \mathsf{T}_A(L) \to \mathsf{T}_A(L) : b \mapsto \begin{cases} b & \text{if } b \in \Upsilon\left(\mathsf{S}_A(L)\right), \\ 0 & \text{if } b \in \mathcal{J}_\nabla(L,A). \end{cases} \tag{6.2.46}$$

which obviously is an idempotent map whose image is $\Upsilon\left(\mathsf{S}_A(L)\right)$ and whose kernel is the Rinehart ideal $\mathcal{J}_\nabla(L,A)$. We are seeking a description of $\Pi_\nabla$ as a sort of *deformation of the symmetrization projection* $\Pi_0$ which has the same image as $\Pi_\nabla$, but the Rinheart ideal $\mathcal{J}_0(L,A)$ as its kernel.
Inspired by *homological perturbation theory* we have come to the following

**Theorem 6.2.6** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over A, and let $\nabla$ be a connection in the A-module L along L. With the above conventions and notations, the projection $\Pi_\nabla : \mathsf{T}_A(L) \to \mathsf{T}_A(L)$, see eqn (6.2.46), has the following properties:*

1. *$\Pi_\nabla$ is morphism of $C^3$-coalgebras over A.*
2. *There is the following explicit formula for $\Pi_\nabla$ in terms of the maps Z, H, and $\tilde{D}$:*

$$\Pi_\nabla = \Pi_0 \circ \sum_{r=0}^{\infty} (-1)^r \big((-D_Z + D_H)\circ \tilde{D}\big)^{\circ r} = \Pi_0 \circ \big(\mathrm{id}_{\mathsf{T}_A(L)} + (-D_Z + D_H)\circ \tilde{D}\big)^{-1} \tag{6.2.47}$$

*Proof.* *i.*) We have to compare the two $A$-linear maps $\Delta_{sh}\circ\Pi_\nabla$ and $\big(\Pi_\nabla \otimes_A \Pi_\nabla\big)\circ \Delta_{sh}$. Since the symmetrization embedding $\Upsilon : \mathsf{S}_A(L) \to \mathsf{T}_A(L)$ is a morphism of $C^3$-coalgebras over $A$ we get for all $\gamma \in \mathsf{S}_A(L)$

$$\begin{aligned}\Delta_{sh}\Big(\Pi_\nabla\big(\Upsilon(\gamma)\big)\Big) &= \Delta_{sh}\big(\Upsilon(\gamma)\big) = \sum_{(\gamma)} \Upsilon\big(\gamma^{(1)}\big)\otimes_A \Upsilon\big(\gamma^{(2)}\big) = \sum_{(\gamma)} \Pi_\nabla\big(\Upsilon\big(\gamma^{(1)}\big)\big)\otimes_A \Pi_\nabla\big(\Upsilon\big(\gamma^{(2)}\big)\big)\\ &= \big(\Pi_\nabla\otimes_A\Pi_\nabla\big)\Big(\Delta_{sh}\big(\Upsilon(\gamma)\big)\Big),\end{aligned}$$

and for all $c \in \mathcal{J}_\nabla(L,A)$ we have –since $\mathcal{J}_\nabla(L,A)$ is a coideal:

$$\Delta_{sh}\big(\Pi_\nabla(c)\big) = \Delta_{sh}(0) = 0 \ \text{ and } \ \big(\Pi_\nabla\otimes_A\Pi_\nabla\big)\big(\Delta_{sh}(c)\big) = \sum_{(c)} \Pi_\nabla\big(c^{(1)}\big)\otimes_A \Pi_\nabla\big(c^{(2)}\big) = 0$$

because in the sum $\sum_{(c)}(c^{(1)}\otimes_A (c^{(2)}$ either $c^{(1)}$ or $c^{(2)}$ can be chosen to be in the coideal $\mathcal{J}_\nabla(L,A) = \mathrm{Ker}(\Pi_\nabla)$. Hence $\Pi_\nabla$ is a morphism of $C^3$-coalgebras since $\mathcal{J}_\nabla(L,A)$ is in the kernel of $\epsilon$.
*ii*). First of all the infinite series on the right hand side of eqn (6.2.47) is well-defined since the maps $D_Z$ and $D_H$ strictly decrease the tensor degree. We have to check the right hand side of eqn (6.2.47) on elements $\Upsilon(\gamma)$, $\gamma \in \mathsf{S}_A(L)$ and on elements $c$ of the coideal $\mathcal{J}_\nabla(L,A)$. Since $\tilde{D}\circ\Upsilon = 0$, see statement *ii.*) of Lemma 6.2.4, it follows that

$$\left(\Pi_0 \circ \sum_{r=0}^{\infty} (-1)^r \big((-D_Z + D_H)\circ \tilde{D}\big)^{\circ r}\right)\big(\Upsilon(\gamma)\big) = \Pi_0\big(\Upsilon(\gamma)\big) = \Upsilon(\gamma).$$

Since $D_\nabla$ maps $\mathcal{J}_0(L,A)$ to $\mathcal{J}_\nabla(L,A)$ and $\tilde{D}$ induces a bijection of $\mathcal{J}_0(L,A)$ being the inverse of the restriction of $D_0$ to $\mathcal{J}_0(L,A)$, see eqn (6.2.43) and eqn (6.2.32) in statement *iii.*) of Lemma 6.2.4, we have that for any $c \in \mathcal{J}_\nabla(L,A)$ there exists a $c' \in \mathcal{J}_0(L,A)$ such that

$$c = D_\nabla\big(\tilde{D}(c')\big) = (D_0 - D_Z + D_H)\big(\tilde{D}(c')\big) = \big(\mathrm{id}_{\mathsf{T}_A(L)} + (-D_Z + D_H)\circ\tilde{D}\big)(c'),$$

hence for each nonnegative integer $r$

$$\big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ r}(c) \quad = \quad \big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ r}(c')+\big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ(r+1)}(c')$$

implying that only the term $r=0$ survives in the alternating sum on the right hand side of eqn (6.2.47) giving

$$\left(\Pi_0\circ\sum_{r=0}^{\infty}(-1)^r\big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ r}\right)\big(c\big)=\Pi_0(c')=0$$

showing that formula (6.2.47) is correct. □

We shall now come to an 'explicit description' of the multiplication $\blacklozenge$ in the universal enveloping algebra $\mathcal{U}(L,A)$ of the Lie-Rinehart algebra $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ by means of a connection $\nabla$ in $L$ along $L$: we have seen that the universal enveloping algebra is isomorphic as a Rinehart bialgebra to the quotient $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A)$, see Proposition 6.2.2, *v.*). Let $P_0:\mathsf{T}_A(L)\to\mathsf{S}_A(L)$ denote the usual canonical Hopf algebra morphism given by (for all positive integers $n$ and $x_1,\ldots,x_n\in L$)

$$P_0(\mathbf{1})=\mathbf{1}_{S_A(L)}\quad\text{and}\quad P_0(x_1\cdots x_n)=x_1\bullet\cdots\bullet x_n. \tag{6.2.48}$$

Note that

$$P_0\circ\Upsilon=\mathrm{id}_{\mathsf{S}_A(L)}\quad\text{and}\quad\Pi_0=\Upsilon\circ P_0,\quad\text{hence}\quad P_0\circ\Pi_0=P_0. \tag{6.2.49}$$

We define the projection $P_\nabla:\mathsf{T}_A(L)\to\mathsf{S}_A(L)$ by

$$P_\nabla:=P_0\circ\sum_{r=0}^{\infty}\big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ r}=P_0\circ\Pi_\nabla \tag{6.2.50}$$

and a $K$-bilinear multiplication $\blacklozenge$ on $\mathsf{S}_A(L)$ for all $\gamma,\gamma'\in\mathsf{S}_A(L)$

$$\begin{aligned}\gamma\blacklozenge\gamma' \quad &:= \quad P_\nabla\big(\Upsilon(\gamma)\diamond\Upsilon(\gamma')\big)= \\ &= \quad \left(P_0\circ\sum_{r=0}^{\infty}\big((-D_Z+D_H)\circ\tilde{D}\big)^{\circ r}\right)\left(\sum_{(\gamma)}\Upsilon\big(\gamma^{(1)}\big)\otimes_A\nabla_{\Upsilon(\gamma^{(2)})}\big(\Upsilon(\gamma')\big)\right).\end{aligned} \tag{6.2.51}$$

As an $A$-module the quotient $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A)$ is isomorphic to the symmetric algebra $\mathsf{S}_A(L)$. We have the following

**Theorem 6.2.7** *Let $\big(L,\rho,[\![\ ,\ ]\!],A\big)$ a Lie-Rinehart algebra over $A$, and let $\nabla$ be a connection in the $A$-module $L$ along $L$. With the above conventions and notations we have:*
*The quintuple $\big(\mathsf{S}_A(L),\blacklozenge,\mathbf{1}_{S_A(L)},\Delta_{S_A(L)},\epsilon_{S_A(L)}\big)$ is a Rinehart bialgebra over $A|K$ which is isomorphic –as a Rinehart bialgebra– to the universal enveloping algebra $\mathcal{U}(A,L)$ of $L$.*

*Proof.* Note first that $\Upsilon \circ P_\nabla = \Pi_\nabla$. Moreover

$$\Upsilon \circ P_\nabla \circ \Upsilon = \Pi_\nabla \circ \Upsilon = \Upsilon, \quad \text{hence} \quad P_\nabla \circ \Upsilon = \mathrm{id}_{\mathsf{S}_A(L)}$$

according to the definition (6.2.46) of $\Pi_\nabla$. It follows that $P_\nabla$ is surjective and that the kernel of $P_\nabla$ is equal to the Rinehart ideal $\mathcal{J}_\nabla(L,A)$. Next the equation $P_\nabla = P_0 \circ \Pi_\nabla$, see eqn (6.2.50), shows that $P_\nabla$ is a morphism of $C^3$-coalgebras over $A$ from $\big(\mathsf{T}_A(L), \Delta_{sh}, \epsilon, \mathbf{1}\big)$ to $\big(\mathsf{S}_A(L), \Delta_{\mathsf{S}_A(L)}, \epsilon_{\mathsf{S}_A(L)}, \mathbf{1}_{\mathsf{S}_A(L)}\big)$ because $P_0$ and $\Pi_\nabla$ are morphisms of $C^3$-coalgebras over $A$, see the first statement of Theorem 6.2.6. In particular we have $P_\nabla(a\mathbf{1}) = a\mathbf{1}_{\mathsf{S}_A(L)}$ for all $a \in A$ whence $P_\nabla$ preserves unit element and intertwines the injections of $A$. The natural injection $A \to \mathsf{S}_A(L)$ is an algebra morphism for $\diamond$ since $\Upsilon(a) \diamond \Upsilon(a') = \Upsilon(aa')$, and the map $(-D_Z + D_H) \circ \tilde{D}$ clearly vanishes on the image of $\Upsilon$. Next since for any $b \in \mathsf{T}_A(L)$ the difference $b - \Upsilon\big(P_\nabla(b)\big)$ is annihilated by $P_\nabla$, hence contained in the Rinehart ideal $\mathcal{J}_\nabla(L,A)$, it follows from the definition (6.2.51) of the multiplication $\bullet$ that for all $b, b' \in \mathsf{T}_A(L)$

$$P_\nabla(b \diamond b') = P_\nabla\Big(\big(\Upsilon\big(P_\nabla(b)\big)\big) \diamond \big(\Upsilon\big(P_\nabla(b')\big)\big)\Big) = \big(P_\nabla(b)\big) \bullet \big(P_\nabla(b')\big)$$

proving that $P_\nabla$ is a morphism of unital $K$-algebras and showing the associativity of $\bullet$. It therefore is a morphism of unital associative $K$-algebras over $A$. This implies that $P_\nabla$ maps $\mathsf{T}_A(L) \times_A \mathsf{T}_A(L)$ onto $\mathsf{S}_A(L) \times_A \mathsf{S}_A(L)$ whence properties (4.3.4) become clear for $\big(\mathsf{S}_A(L), \bullet, \mathbf{1}_{S_A(L)}, \Delta_{S_A(L)}, \epsilon_{S_A(L)}\big)$ which thus is a Rinehart bialgebra over $A|K$, and $P_\nabla$ is a morphism of Rinehart bialgebras. Since $P_\nabla$ vanishes on the Rinehart ideal $\mathcal{J}_\nabla(L,A)$ and since $P_\nabla \circ \Upsilon = \mathrm{id}_{\mathsf{S}_A(L)}$ its follows that $P_\nabla$ descends to an isomorphism of Rinehart bialgebras $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A) \to \mathsf{S}_A(L)$. It had been shown before that the quotient $\mathsf{T}_A(L)/\mathcal{J}_\nabla(L,A)$ is isomorphic to the universal enveloping algebra $\mathcal{U}(L,A)$, see Propostion 6.2.2, *v.*). This proves the Theorem. □

# Appendix

# A. Some basic topics

Contents

## A.1 Adjunction of functors (Category theory)

The aim of this part of the appendix is to fix the notation in which we follow almost exclusively Mac Lane's book [32, p.79,Chap.IV].

The following definition is important for understand adjunctions.

**Definition A.1.1** Given two categories $\mathbf{C}$, $\mathbf{D}$, and functors $\mathsf{F} : \mathbf{C} \to \mathbf{D}$, $\mathsf{G} : \mathbf{D} \to \mathbf{C}$, hence a diagram

$$\mathbf{C} \underset{\mathsf{G}}{\overset{\mathsf{F}}{\rightleftarrows}} \mathbf{D}$$

with a natural isomorphism

$$\varphi_{c,d} : \mathrm{Hom}_{\mathbf{D}}\big(\mathsf{F}(c), d\big) \to \mathrm{Hom}_{\mathbf{C}}\big(c, \mathsf{G}(d)\big)$$

for any object $c$ in $\mathbf{C}$ and any object $d$ in $\mathbf{D}$ is called an adjunction of functors, with $\mathsf{F}$ the *left adjoint functor of* $\mathsf{G}$ (or $\mathsf{G}$ the *right adjoint functor of* $\mathsf{F}$), see e. g. [22], and the *adjugant* $\varphi_{c,d}$.

The importance is that, left adjoint functors (and right adjoint functors) to a given functor are unique up to natural isomorphism if they exist.

Moreover, with any adjunction there are two important natural morphisms, the *unit of the adjunction*, sometimes called $\eta : \mathsf{I}_{\mathbf{C}} \dot{\longrightarrow} \mathsf{GF}$ and the *counit*, sometimes written as $\epsilon : \mathsf{FG} \dot{\longrightarrow} \mathsf{I}_{\mathbf{D}}$, where $\eta_c = \varphi_{c,\mathsf{F}(c)}(\mathrm{id}_{\mathsf{F}(c)})$ and $\epsilon_d = \varphi^{-1}_{\mathsf{G}(d),d}(\mathrm{id}_{\mathsf{G}(d)})$.

Furthermore, for each object $c$ in $\mathbf{C}$ the pair $\big(\mathsf{F}(c), \eta_c$ is universal to $\mathsf{G}$. In this thesis, we mostly deal with the situation where $\mathsf{G}$ is some 'forgetful' functor, and $\mathsf{F}$ creates 'free objects'. We shall often denote by $\chi \mapsto \bar{\chi}$ the inverse of the adjugant, $\varphi^{-1}_{c,d}$ which describes the 'induced map' of the 'simpler map' $\chi$. The adjugant itself, $\varphi_{c,d}$, is often someting like 'restriction to generators'. The unit is 'insertion of generators' and the counit plays the rôle of some 'natural presentation by a free object'.

## A.2 Basic algebra

In this appendix we will suppose that $K$ is always a fixed commutative associative unital ring. Recall that $K = \{0\}$ iff $1 = 0$. We shall frequently assume that $K$ contains the field of all rational numbers $\mathbb{Q}$ as a unital subring, and we shall indicate it at every instant when it is really needed. All modules are considered over $K$, and the symbol $\otimes$ is short for $\otimes_K$. In view of Schauenburg's *Strengesatz* (see [47, p.264, Cor.4.4] we can, but shall not always assume that it is associative.

Moreover, let $A$ be a commutative associative unital $K$-algebra in the sense that $\iota : K \to A$ is a (not necessarily injective) morphism of unital commutative associative

rings. Its unit will sometimes be denoted by $1_A$ and identified with the unit 1 of $K$. Recall that any associative commutative unital ring $A$ can naturally be viewed as an algebra over the ring of all integers.

In several of the following Sections which are not dealing with the relative $(A,K)$-situation we shall use the fixed commutative associative unital ring $\mathbf{K}$ which may play the rôle of $A$ or of $K$. Then all modules will be over $\mathbf{K}$ and $\otimes$ is short for $\otimes_{\mathbf{K}}$.

### A.2.1 $A$-Modules and $K$-Modules

Let $K\mathbf{Mod}$ and $A\mathbf{Mod}$ denote the categories of all $K$-modules and $A$-modules, respectively. Recall the well-known *restriction functor* $\mathsf{Res} : A\mathbf{Mod} \to K\mathbf{Mod}$ which considers $A$-modules as $K$-modules. It has a left adjoint, the *induction functor* $A\otimes$, which transforms every $K$-module $E$ in the tensor product $A\otimes E$ which thus becomes a left $A$-module in a natural way and maps each $K$-linear map $\phi$ to the $A$-linear map $\mathrm{id}_A \otimes \phi$:

$$K\mathbf{Mod} \quad \underset{\mathsf{Res}}{\overset{A\otimes}{\rightleftarrows}} \quad A\mathbf{Mod}$$

The induced module $A \otimes E$ is also called the *relatively free $A$-module generated by the $K$-module $E$*, see also [32, p.196]. **Definition:** In an $A$-module isomorphic to an $A$-module $A \otimes E$ is called a *relatively free module*. The notation is a generalization of a *free $A$-module over a set $S$* where $A$ is an associative commutative unital ring: $A$ is an algebra over $K = \mathbb{Z}$ in a natural way, and it is easy to see that the free $A$-module generated by $S$ is isomorphic to the relatively free module $A \otimes_{\mathbb{Z}} (\mathbb{Z}S)$ for the case $K = \mathbb{Z}$ where $\mathbb{Z}S$ is the free $\mathbb{Z}$-module generated by $S$.

Note that the unit of the adjunction $\mathrm{I}_{K\mathbf{mod}} \stackrel{\cdot}{\longrightarrow} \mathsf{Res}(A\otimes\ )$ is given by the natural $K$-linear map $E \to A \otimes E$ given by $x \mapsto 1_A \otimes_K x$, for each element $x$ in the $K$-module $E$ whereas the counit of the adjunction, $(A\otimes\ )\mathsf{Res} \stackrel{\cdot}{\longrightarrow} \mathrm{I}_{A\mathbf{mod}}$, is just the module multiplication (we sloppily write $W$ for $\mathsf{Res}(W)$)

$$\mu_W : A \otimes W \to W : a \otimes w \mapsto \mu_W(a \otimes w) = aw \tag{A.2.1}$$

for any $A$-module $W$, $a \in A$ and $w \in W$ which is always surjective. Quite often we shall use the kernel of $\mu_W$, denoted by $\mathfrak{h}(W,A) \subset A \otimes W$: from the identities $a \otimes w = a \otimes w - 1 \otimes (aw) + 1 \otimes (aw)$ and $\mu_w(1 \otimes w) = w$ we can infer

$$\mathfrak{h}(W,A) = \left\{ \sum_{i=1}^{N} \big((a_i' a_i) \otimes w_i - a_i' \otimes (a_i w_i)\big) \;\middle|\; N \in \mathbb{N}\backslash\{0\},\ a_i', a_i \in A,\ w_i \in W \ \ \forall\, i \in \mathbb{N},\ 1 \leqslant i \leqslant N \right\}. \tag{A.2.2}$$

Note also that the naturality of $\mu$ entails that for any $A$-linear map $\Phi : W \to W'$ the $A$-linear map $\mathrm{id}_A \otimes \phi : A \otimes W \to A \otimes W'$ maps kernels to kernels, hence

$$(\mathrm{id}_A \otimes \phi)\big(\mathfrak{h}(W,A)\big) \subset \mathfrak{h}(W',A). \tag{A.2.3}$$

Furthermore, recall the notion of a *relatively projective A-module*: here any commutative diagram (where $f : P \to N$ is $A$-linear, $\pi : M \to N$ is $A$-linear and surjective, and $s : M \leftarrow N$ is $K$-linear such that $\pi \circ s = \mathrm{id}_N$)

$$\begin{array}{ccccc} P & & & & \\ & \searrow & & & \\ M & \underset{\leftarrow}{\rightarrow} & N & \to \{0\} & \end{array} \qquad \begin{array}{c} \text{can be completed to} \\ \text{the commutative diagram} \end{array} \qquad \begin{array}{ccccc} P & & & & \\ \downarrow & \searrow & & & \\ M & \to & N & \to \{0\} & \end{array} \tag{A.2.4}$$

where $\hat{f} : P \to M$ is $A$-linear. Obviously, any projective $A$-module is relatively projective (the converse statement is obviously true in the important particular case where $K$ is a field). The following Lemma is no doubt well-known and of some practical use:

**Lemma A.2.1** *An $A$-module $W$ is relatively projective if and only if there is a* ***section****, i.e. an $A$-linear map $\sigma_W = \sigma : W \to A \otimes W$ such that $\mu_W \circ \sigma_W = \mathrm{id}_W$. In that case we have*

$$A \otimes W = \sigma_W(W) \oplus \mathfrak{h}(W, A). \tag{A.2.5}$$

Indeed, if $W$ is relatively projective, then the particular case $P = W = N$, $M = A \otimes W$, $W \to W$ the identity map, $\pi = \mu_W$, and $s(w) = 1 \otimes w$ for all $w \in W$ of the left diagram of (A.2.4) shows the existence of $\sigma$. Conversely, if a section $\sigma_P : P \to A \otimes P$ exists, and if we are given the diagram on the left of (A.2.4), then this diagam can be tensored by $A$ over $K$: in the resulting commutative diagram the map $\mathrm{id}_A \otimes \pi$ is still surjective (since $A \otimes$ is a right exact functor), and $\mathrm{id}_A \otimes s$ is now $A$-linear, whence the $A$-linear map $\hat{F} = (\mathrm{id}_A \otimes s) \circ (\mathrm{id}_A \otimes f) : A \otimes P \to A \otimes M$ completes the tensored diagram. It is not hard to see that $\hat{f} = \mu_P \circ \hat{F} \circ \sigma_P$ then completes the original diagram whence $P$ is relatively projective.

Actually, each relatively free $A$-module $A \otimes E$ is relatively projective (choose the section $\sigma_0 : a \otimes x \mapsto a \otimes (1_A \otimes x)$), and Lemma A.2.1 shows that every relatively projective $A$-module is isomorphic to a direct summand of a relatively free $A$-module.

Conversely, any direct summand $W$ of a relatively free $A$-module $F$ is relatively projective where a section $\sigma_W$ is given by the composition $(\mathrm{id}_A \otimes \pi_W) \circ \sigma_0 \circ i_W$ with the obvious inclusion $i_W : W \to F$ and projection $\pi_W : F \to W$.

Recall that both categories $A$**mod** and $K$**mod** are well-known to be *closed symmetric monoidal categories*, see e.g. [32, p.255], by means of the corresponding tensor products $\otimes_A$ over $A$ and $\otimes$ over $K$. The restriction functor Res (which we abbreviate by G in this paragraph) is known to be symmetric monoidal, the morphism $G_0 : K \to \mathsf{G}(A)$ being $k \mapsto k1_A$ and the natural morphism $\mathsf{G}_2 : \otimes(\mathsf{G} \times \mathsf{G}) \dot{\to} \mathsf{G}\otimes_A$ (see [32, p.255, eqn (1)] for definitions) being the canonical $K$-linear morphism $\mathsf{G}_{2\ V,W} =: \pi_{V,W} : \mathsf{G}(V) \otimes \mathsf{G}(W) \to \mathsf{G}(V \otimes_A W)$ induced by the biadditive middle-$K$-associative map $(v, w) \mapsto v \otimes_A w$ for any $A$-modules $V, W$ and any $v \in V$ and $w \in W$. The kernel of $\pi_{V,W}$ is given by the $K$-submodule $\mathcal{K}(V, W) \subset \mathsf{G}(V) \otimes \mathsf{G}(W)$ defined

by

$$\mathrm{Ker}(\pi_{V,W}) =: \mathcal{K}(V,W) := K\mathrm{span}\{(ax)\otimes y - x\otimes(ay) \mid x\in V, y\in W, \text{and } a\in A\}. \tag{A.2.6}$$

Indeed, it is clear that $\mathcal{K}(V,W)$ is contained in the kernel of $\pi_{V,W}$ thus $\pi_{V,W}$ passes to the quotient to define a $K$-linear map $\big(\mathsf{G}(V)\otimes_K \mathsf{G}(W)\big)/\mathcal{K}(V,W) \to \mathsf{G}(V\otimes_A W)$. On the other hand, the map $V\times W \to \big(\mathsf{G}(V)\otimes_K \mathsf{G}(W)\big)/\mathcal{K}(V,W)$ sending the pair $(x,y)$ to $(x\otimes_K y)$ modulo $\mathcal{K}(V,W)$ is clearly biadditive and middle-associative, hence -by the universal property of the tensor product $\otimes_A$- defines a unique map of $K$-modules $\mathsf{G}(V)\otimes_A \mathsf{G}(W) \to \big(\mathsf{G}(V\otimes_K W)\big)/\mathcal{K}(V,W)$, and it is a routine check that the two preceding $K$-linear maps are inverses to each other. Both $\mathsf{G}(V)\otimes_A \mathsf{G}(W)$ and its $K$-submodule $\mathcal{K}(V,W)$ are $A$-$A$-bimodules in a natural way, and the $A$-$A$-bimodule structure on the quotient $(V\otimes_K W)/\mathcal{K}(V,W)$ is automatically symmetric and reduces to a left (or right) $A$-module structure. Hence we shall often use the map $\pi_{V,W}$ (in more sloppy notation omitting the restriction functor)

$$\pi_{V,W} : V\otimes W \to V\otimes_A W \;\cong\; (V\otimes W)/\mathcal{K}(V,W) \tag{A.2.7}$$

to parametrize the tensor product over $A$, $V\otimes_A W$, by the 'easier' tensor product over $K$, $V\otimes W$.

Next, let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras over $A$. Then both $\mathcal{A}\otimes\mathcal{B}$ and $\mathcal{A}\otimes_A\mathcal{B}$ are associative algebras over $K$ in a natural way, and it is straight-forward to check that the $K$-linear map $\pi_{\mathcal{A},\mathcal{B}} : \mathcal{A}\otimes\mathcal{B} \to \mathcal{A}\otimes_A\mathcal{B}$ is a morphism of associative algebras over $K$.

## A.3 Coalgebras

In this Section we will utilize as standard reference the Sweedler's book [49] in which **K** mostly is a field. For more general rings we recommend the Appendix of [39] and [8].

Recall that a *coassociative counitary coaugmented coalgebra* $(C,\Delta,\epsilon,\mathbf{1})$ over **K** consists of a **K**-module $C$, $K$-linear maps $\Delta : C\to C\otimes C$ (comultiplication) and $\epsilon : C\to \mathbf{K}$ (counit), and an element $\mathbf{1}\in C$ satisfying the usual identities $(\Delta\otimes \mathrm{id}_C)\circ\Delta = (\mathrm{id}_C\otimes\Delta)\circ\Delta$, $(\epsilon\otimes\mathrm{id}_C)\circ\Delta = \mathrm{id}_C = (\mathrm{id}_C\otimes\epsilon)\circ\Delta$, $\epsilon(\mathbf{1})=1$, and $\Delta(\mathbf{1})=\mathbf{1}\otimes\mathbf{1}$. We refer to them as $C^3$-coalgebras.

As usual, we use *Sweedler's notation* $\Delta(c)=\sum_{(c)} c^{(1)}\otimes c^{(2)}$ where the sum is finite, is in general not unique (and does not have to be) and the constituents symbolized by $c^{(1)}$ and $c^{(2)}$ are in $C$. We have the direct decomposition $C = \mathbf{K1}\oplus C^+$ where $C^+$ denotes the kernel of the counit. *Morphisms* of this category of coalgebras are **K**-linear maps intertwining comultiplications, counits and coaugmentations in the appropriate way. We shall sometimes call this category $\mathbf{C^3}$-$\mathbf{Coalg_K}$.

Recall that a *coderivation along a coalgebra morphism* $\phi : C \to C'$ is a $\mathbf{K}$-linear map $D : C \to C'$ satisfying $\Delta' \circ D = (D \otimes \phi + \phi \otimes D) \circ \Delta$. Note that for any coderivation $\epsilon' \circ D = 0$, but we do NOT necessarily have and demand that $D(\mathbf{1}) = 0$. We shall speak of coderivations in the particular case $C = C'$ and $\phi = \mathrm{id}_C$.

Recall that a *coideal* $I \subset C$ is a $\mathbf{K}$-submodule such that $I \subset C^+$ and $\Delta(I) \subset \mathrm{Im}(I \otimes C) + \mathrm{Im}(C \otimes I)$. It follows that on the factor module $C/I$ there exists a well-defined comultiplication, counit and coaugmentation making the canonical projection $\pi : C \to C/I$ a morphism of coalgebras. Conversely, kernels of surjective (!) morphisms of coalgebras of this category are always coideals.

Moreover, images of coderivations are always coideals. Recall that the *primitive part of* $C$, $\mathrm{Prim}(C) \subset C$ is the $\mathbf{K}$-submodule of all $c \in C$ such that $\Delta(c) = c \otimes \mathbf{1} + \mathbf{1} \otimes c$.

Recall that every $C^3$-coalgebra $C$ over $\mathbf{K}$ is equipped with an (ascending) filtration $\left(Q'_{(n)}C\right)_{n\in\mathbb{Z}}$ where $Q'_{(n)}C := \{0\}$ whenever $n \leqslant -1$, $Q'_{(0)}C = \mathbf{K}\mathbf{1}$, and

$$Q'_{(n+1)}C := \left\{ c \in C \;\middle|\; \Delta(c) - c \otimes \mathbf{1} - \mathbf{1} \otimes c \;\in\; \mathbf{K}\mathbf{1} \otimes \mathbf{1} + \sum_{r=1}^{n} \mathrm{Im}\left(Q'_{(r)}C \otimes Q'_{(n+1-r)}C\right) \right\} \tag{A.3.1}$$

Comultiplication and counit are clearly filtration preserving. The $C^3$-coalgebra is called *connected* iff this filtration is exhaustive, i.e. $\bigcup_{n\in\mathbb{Z}} Q'_{(n)}C = C$, see [39] for a similar definition. This filtration is related to the well-known coradical filtration in case $\mathbf{K}$ is a field, see [49, p.185-191]). Returning to the case of a general ring $\mathbf{K}$, note that every morphism of $C^3$-coalgebras preserves the above filtrations (A.3.1). In particular, it is easy to see that every homomorphic image of a connected $C^3$-coalgebra is again connected.

Next a very important tool is *convolution*: for an arbitrary $C^3$-coalgebra $\left(C, \Delta, \epsilon, \mathbf{1}_C\right)$ over $\mathbf{K}$ and an arbitrary associative unital algebra $(B, \mu_B, \mathbf{1}_B)$ over $\mathbf{K}$ let $\star$ denote the following $\mathbf{K}$-bilinear operation on the Hom-space $\mathrm{Hom}_{\mathbf{K}}(C, B)$: for any $\varphi, \varphi' \in \mathrm{Hom}_{\mathbf{K}}(C, B)$

$$\varphi \star \varphi' = \mu_B \circ \left(\varphi \otimes \varphi'\right) \circ \Delta_C, \quad \text{or} \quad \forall\, c: \;\; \left(\varphi \star \varphi'\right)(c) = \sum_{(c)} \varphi(c^{(1)})\varphi'(c^{(2)}). \tag{A.3.2}$$

It is well-known that the convolution multiplication $\star$ equips the Hom-space $\mathrm{Hom}_{\mathbf{K}}(C, B)$ with an associative multiplication with unit $\mathbf{1}_B\epsilon$. Moreover, in case $(C, \Delta, \epsilon, \mathbf{1}_C)$ is connected and if the algebra $(B, \mu_B, \mathbf{1}_B)$ is equipped with the *trivial filtration* defined by $F_nB = \{0\}$ if $n \leqslant -1$, and $F_nB = B$ if $n \geqslant 0$, then the Hom-space $\mathrm{Hom}_{\mathbf{K}}(C, B)$ equipped with its canonical filtration and convolution is a complete filtered associative unital algebra over $\mathbf{K}$.

Note that a $\mathbf{K}$-linear map $\varphi : C \to B$ is strictly filtration decreasing if and only if $\varphi(\mathbf{1}_C) = 0$; and for those maps any *convolution power series* $\sum_{r=0}^{\infty} \alpha_r \varphi^{\star r}$, $\alpha_r \in \mathbf{K}$, automatically converges to a well-defined $\mathbf{K}$-linear map $C \to B$.

Consider now a *bialgebra* $\big(B, \mu_B, \mathbf{1}_B, \Delta_B, \epsilon_B\big)$ over $\mathbf{K}$: there are both the structure of an asscociative unital algebra and a $C^3$-coalgebra such that $\mu_B : B \otimes B \to B$ is a morphism of counital coalgebras. The following important relation appears in an article by J.Helmstetter [17]: there is the following explicit natural bijection (where $C$ is a connected **cocommutative**)

$$\mathrm{Hom}_{\mathbf{K}}\big(C^+, \mathsf{Prim}(B)\big) \to \mathrm{Hom}_{\mathbf{C^3\text{-}Coalg_K}}(C, B) \tag{A.3.3}$$

given by

$$\varphi \mapsto e^{*\varphi} \quad \text{with the inverse} \quad \Phi \mapsto \ln_*(\Phi) := \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} (\Phi - \mathbf{1}_B \epsilon_C)^{*(r+1)}. \tag{A.3.4}$$

Note further that for any morphism of $C^3$-coalgebras $\Psi : \big(\hat{C}, \Delta_{\hat{C}}, \epsilon_{\hat{C}}, \mathbf{1}_{\hat{C}}\big) \to \big(C, \Delta_C, \epsilon_C, \mathbf{1}_C\big)$ right composition is a morphism of convolution algebras $\big(\mathrm{Hom}_A(C, B), *\big) \to \big(\mathrm{Hom}_A(\hat{C}, B), \hat{*}\big)$, i.e. for any $\varphi, \varphi' \in \mathrm{Hom}_A(C, B)$

$$(\varphi * \varphi') \circ \Psi = (\varphi \circ \Psi) \hat{*} (\varphi' \circ \Psi). \tag{A.3.5}$$

Similarly left composition with morphisms of unital associative algebras are also morphism of convolution algebras. Likewise, for any coderivation $D : \hat{C} \to C$ along $\Psi$ right composition with $D$ is a derivation of convolution algebras, i.e.

$$(\varphi * \varphi') \circ D = (\varphi \circ D) \hat{*} (\varphi' \circ \Psi) + (\varphi \circ \Psi) \hat{*} (\varphi' \circ D). \tag{A.3.6}$$

and likewise for left composition with derivations along algebra morphisms.

Finally, note that for any bialgebra $\big(B, \mu_B, \mathbf{1}_B, \Delta_B, \epsilon_B\big)$ over $\mathbf{K}$ whose comultiplication is **cocommutative** and any $\mathbf{K}$-linear map $\chi : B \to \mathsf{Prim}(B)$ the convolutions

$$\chi \star \mathrm{id}_B \quad \text{and} \quad \mathrm{id}_B \star \chi \tag{A.3.7}$$

are always coderivations of $\big(B, \Delta_B, \epsilon_B\big)$.

## A.4 Fréchet topology

Recall the definition of the seminorms $p_{K,N} : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$ (where $K \subset \mathbb{R}^n$ is a compact set and $N$ is a nonnegative integer)

$$p_{K,N}(f) := \max\{|D^\beta f(x)| \mid x \in K,\ |\beta| \leqslant N\} \tag{A.4.1}$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ is a multi-index, $|\beta| := \beta_1 + \cdots + \beta_n$, and

$$D^\beta := \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_1}\right)^{\beta_n}.$$

$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$ is known to be a locally convex topological vector space which is complete in the sense that every Cauchy sequence converges, see e.g. [43, p.33]. It is obvious that for any two compact subsets $K,K'$ and nonnegative integers $N,N'$ we always have for all $f\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$

$$\text{if } K\subset K' \text{ and } N\leqslant N' \text{ then } \ p_{K,N}(f)\leqslant p_{K',N'}(f) \tag{A.4.2}$$

## A.5 Universal enveloping algebras of Lie algebras

We shall recall the usual construction, see e.g. the books by H.Cartan and S.Eilenberg, [12, p.266-270], and by Bourbaki [8, Ch.I, p.22].

There is the following problem of universals: given a Lie algebra $\big(\mathfrak{g},[\ ,\ ]_\mathfrak{g}\big)$ over $\mathbf{K}$, is there an associative unital $\mathbf{K}$-algebra $\mathsf{U}_\mathbf{K}(\mathfrak{g})$ equipped with a $\mathbf{K}$-linear map $i_\mathfrak{g}=i:\mathfrak{g}\to\mathsf{U}_\mathbf{K}(\mathfrak{g})$ satisfying $i\big([x,y]_\mathfrak{g}\big)=i(x)i(y)-i(y)i(x)$ for all $x,y\in\mathfrak{g}$ such that for any unital associative $\mathbf{K}$-algebra $B$ and any $\mathbf{K}$-linear map $\theta:\mathfrak{g}\to B$ satisfying $\theta\big([x,y]_\mathfrak{g}\big)=\theta(x)\theta(y)-\theta(y)\theta(x)$ for all $x,y\in\mathfrak{g}$ there is a unique morphism of unital $\mathbf{K}$-algebras $\bar\theta:\mathsf{U}_\mathbf{K}(\mathfrak{g})\to B$ satisfying $\bar\theta\circ i=\theta$?

The positive answer to this question can be rephrased in more categorical terms that the obvious *commutator functor* $(\ )^-:\mathbf{AssAlg_K}\to\mathbf{LieAlg_K}$ from all unital associative $\mathbf{K}$-algebras to all $\mathbf{K}$-Lie algebras has a left adjoint:

$$\mathbf{LieAlg_K}\ \underset{(\ )^-}{\overset{\mathsf{U}}{\rightleftarrows}}\ \mathbf{AssAlg_K} \tag{A.5.1}$$

In this case, to any associative algebra $B$ over $\mathbf{K}$ the Lie algebra $B^-$ is associated where the Lie bracket on the $\mathbf{K}$-module $B^-=B$ is just the commutator $[b,b']:=bb'-b'b$ for all $b,b'\in B$. The map $\theta\mapsto\bar\theta$ is the inverse of the adjugant, and the natural morphism $i:\mathfrak{g}\to\mathsf{U}_\mathbf{K}(\mathfrak{g})$ is the unit of the adjunction. The standard construction of $\mathsf{U}_\mathbf{K}(\mathfrak{g})$ is given by the free algebra over the $\mathbf{K}$-module $\mathfrak{g}$, $\mathsf{T}_\mathbf{K}(\mathfrak{g})$, modulo the two-sided ideal $\mathcal{I}_\mathbf{K}(\mathfrak{g})$ generated by the set of all elements of the form $xy-yx-[x,y]_\mathfrak{g}$, $x,y\in\mathfrak{g}$ (warning: in [12] $\mathsf{U}_\mathbf{K}(\mathfrak{g})$ is denoted by $\mathfrak{g}^e$ and the ideal $\mathcal{I}_\mathbf{K}(\mathfrak{g})$ by $U(\mathfrak{g})$...). Morphisms of $\mathbf{K}$-Lie algebras are first lifted to algebra morphisms of the corresponding free algebras where they map the first ideal to the second and thus descend to morphisms of universal enveloping algebras.

Note that the $\mathbf{K}$-linear map $i:\mathfrak{g}\to\mathsf{U}_\mathbf{K}(\mathfrak{g})$ is not necessarily injective. It is classical that $i$ is always injective in case $\mathfrak{g}$ is a free $\mathbf{K}$-module, for instance if $\mathbf{K}$ is a field, thanks to to the Poincaré-Birkhoff-Witt Theorem, see e.g. [12, p.271-274], where a basis of $\mathsf{U}_\mathbf{K}(\mathfrak{g})$ is constructed. Moreover, $i$ is known to be injective in the other important particular case $\mathbb{Q}\subset\mathbf{K}$, see e.g. the Appendix of [39] or Theorem A.5.1.

Moreover, it is well-known that for any $\mathbf{K}$-module $V$ the natural morphism of Lie algebras $\mathsf{Lie}_{\mathbf{K}}(V) \to \mathsf{T}_{\mathbf{K}}(V)^-$ induces the isomorphism

$$\mathsf{U}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(V)\big) \cong \mathsf{T}_{\mathbf{K}}(V) \tag{A.5.2}$$

of unitary associative algebras which can easily be seen by the universal properties of $\mathsf{Lie}_{\mathbf{K}}(V)$ and $\mathsf{T}_{\mathbf{K}}(V)$.

Next, returning to a general Lie algebra $\mathfrak{g}$ recall that $\mathsf{U}_{\mathbf{K}}(\mathfrak{g})$ carries a natural Hopf algebra structure $\big(\mathsf{U}_{\mathbf{K}}(\mathfrak{g}), \mu_U, \mathbf{1}, \Delta, \epsilon, S\big)$ where the cocommutative comultiplication $\Delta$ and the counit $\epsilon$ are induced by the diagonal morphism $\delta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g} : x \to (x, x)$ of Lie algebras, i.e. $\Delta = \mathsf{U}_{\mathbf{K}}(\delta)$, and the zero map $\mathfrak{g} \to \{0\}$, i.e. $\epsilon = \mathsf{U}_{\mathbf{K}}(0)$, respectively. This can also be seen by the fact that the ideal $\mathcal{I}_{\mathbf{K}}(\mathfrak{g})$ of the free algebra is a coideal stable by the antipode $S$ of the free algebra: this implies that each universal enveloping algebra is a connected cocommutative $C^3$-coalgebra because it is a homomorphic image of the connected cocommutative $C^3$-coalgebra $\mathsf{T}_{\mathbf{K}}(V)$ with its structure, see Section A.6.1.

Turn to the case $\mathbb{Q} \subset \mathbf{K}$: there is the well-known symmetrization map $\omega = \omega_{\mathfrak{g}} : \mathsf{S}_{\mathbf{K}}(\mathfrak{g}) \to \mathsf{U}_{\mathbf{K}}(\mathfrak{g})$ defined by (for all $n \in \mathbb{N}\backslash\{0\}$, $x_1, \dots, x_n \in \mathfrak{g}$)

$$\omega(\mathbf{1}) = \mathbf{1}, \quad \omega\big(x_1 \bullet \cdots \bullet x_n\big) = \frac{1}{n!} \sum_{\sigma \in S_n} i\big(x_{\sigma(1)}\big) \cdots i\big(x_{\sigma(n)}\big). \tag{A.5.3}$$

Then, writing $s_{\mathfrak{g}}$ for the $\mathbf{K}$-linear map $\mathsf{S}_{\mathbf{K}}(\mathfrak{g}) \twoheadrightarrow \mathfrak{g} \xrightarrow{i} \mathsf{U}_{\mathbf{K}}(\mathfrak{g})$ (which has kernel $\mathbf{K}\mathbf{1} \oplus \mathrm{Ker}(i) \oplus \oplus_{n=2}^{\infty} \mathsf{S}^n(\mathfrak{g})$) it is easy to see that $\omega$ can be written as a convolution exponential $\omega = e^{*s_{\mathfrak{g}}}$ (with the comultiplication in $\mathsf{S}_{\mathbf{K}}(\mathfrak{g})$ and the multiplication in $\mathsf{U}_{\mathbf{K}}(\mathfrak{g})$) whence it is a morphism of $C^3$-coalgebras, see eqn (A.3.4). It turns out that this is always an isomorphism of $C^3$-coalgebras:

**Theorem A.5.1** *Let $\mathbb{Q} \subset \mathbf{K}$. Then the collection of all symmetrization maps $\omega_{\mathfrak{g}}$ defines a natural isomorphism $\omega : \mathsf{S} \dot{\to} \mathsf{U}$ where both functors are seen as functors from the category of $\mathbf{K}$-Lie algebras to the category of all cocommutative connected $C^3$-coalgebras over $\mathbf{K}$. In particular the insertion maps $i$ are always injective.*

*Proof.* It is well know that the collection of all the symmetrization maps forms a natural transformation $\mathsf{S} \dot{\to} \mathsf{U}$. It remains to show that each $\omega_{\mathfrak{g}}$ is an isomorphism

In that sense, we have already seen that the symmetrization map $\omega$ is an isomorphism for free Lie algebras, see Proposition A.6.2 upon using the isomorphism (A.5.2). We shall prove the statement of the Theorem by a detour to free (Lie) algebras.

Let $\big(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}\big)$ be a $\mathbf{K}$-Lie algebra, consider the free Lie algebra generated by the $K$-module $\mathfrak{g}$, $\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$, and consider the natural counit map $\epsilon_{\mathfrak{g}} : \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g}) \to \mathfrak{g}$ of the adjunction (A.6.12) which is a surjective morphism of $\mathbf{K}$-Lie algebras restricting to

the identiy map on $\mathfrak{g} \subset \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$ the natural inclusion being given by the unit of the adjunction (A.6.12). Let $\mathfrak{k} = \mathfrak{k}_{\mathfrak{g}} \subset \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$ be the kernel of $\epsilon_{\mathfrak{g}}$. It follows that there is the direct decomposition

$$\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g}) = \mathfrak{k} \oplus \mathfrak{g} \tag{A.5.4}$$

where $\mathfrak{k}$ is an ideal of the Lie algebra $\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$. Thanks to Proposition A.6.1 we can and shall henceforth identify the free Lie algebra with the Lie subalgebra $\mathcal{L}_{\mathbf{K}}(\mathfrak{g})$ of the free algebra $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ which is isomorphic to the universal enveloping algebra $\mathsf{U}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$, see (A.5.2).

Consider the right ideal $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ of $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$: since $\mathfrak{k}$ is an ideal of the free Lie algebra and since the free algebra is generated by $\mathfrak{g}$ it follows that $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ is a two-sided ideal of the associative unital algebra $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$. Moreover since left multiplications are coderivations and images of coderivations are coideals it follows that $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ is a coideal of $\big(\mathsf{T}_{\mathbf{K}}(\mathfrak{g}), \Delta_{sh}, \epsilon, \mathbf{1}\big)$.

We shall first show that the quotient algebra $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ is isomorphic to to the universal enveloping algebra $\mathsf{U}_{\mathbf{K}}(\mathfrak{g})$ by showing the universal property. Indeed, let $B$ be any unital associative $\mathbf{K}$-algebra, and let $\theta : \mathfrak{g} \to B^-$ be a morphism of $\mathbf{K}$-Lie algebras. Observe that the morphism $\theta \circ \epsilon_{\mathfrak{g}} : \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g}) \to B^-$ is equal to the Lie algebra morphism $\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g}) \to B^-$ induced by the $\mathbf{K}$-linear map $\theta$ and the universal property of the free Lie algebra since both morphisms restrict to the same $\mathbf{K}$-linear map $\theta$ on $\mathfrak{g}$. This is also a simple consequence of the naturality of the counit and works for any adjunction of functors. Let $\tilde{\theta} : \mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$ the morphism of unital associative algebras induced by the $\mathbf{K}$-linear map $\theta$. Since $\tilde{\theta}$ coincides with the unital algebra morphism $\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$ induced by the Lie algebra morphism $\theta \circ \epsilon_{\mathfrak{g}}$ (upon using again the isomorphism (A.5.2)) it follows that $\tilde{\theta}$ vanishes on $\mathfrak{k}$, the kernel of $\epsilon_{\mathfrak{g}}$, and more generally, on each multiple of $\mathfrak{k}$ whence it vanishes on the ideal $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ and thus passes to the quotient to define a morphism of unital algebras $\overline{\theta} : \mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$. The map $\theta \mapsto \overline{\theta}$ clearly is injective which can immediately be seen by restricting to generators. It is also surjective: let $\Theta : \mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$ be any morphism of unital associative algebras. By composing with the natural projection $\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to \mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ the resulting algebra morphism $\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$ restricted to $\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g}) \cong \mathcal{L}_{\mathbf{K}}(\mathfrak{g})$ vanishes on $\mathfrak{k}$ whence there is a unique Lie algebra morphism $\vartheta : \mathfrak{g} \to B^-$ such that $\Theta|_{\mathcal{L}_{\mathbf{K}}(\mathfrak{g})} = \vartheta \circ \epsilon_{\mathfrak{g}}$. Clearly the induced morphism $\overline{\vartheta} : \mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to B$ coincides with $\Theta$ on $\mathcal{L}_{\mathbf{K}}(\mathfrak{g})$ and hence on the generating submodule $\mathfrak{g}$, whence $\Theta = \overline{\vartheta}$ proving surjectivity. Hence the quotient algebra $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ satisfies the universal property and is thus isomorphic to the universal enveloping algebra of $\mathfrak{g}$.

Next, thanks to the decomposition (A.5.4), we have the canonical isomorphism $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big) \cong \mathsf{S}_{\mathbf{K}}(\mathfrak{k}) \otimes \mathsf{S}_{\mathbf{K}}(\mathfrak{g})$. The kernel of the projection $\mathsf{S}(\epsilon_{\mathfrak{g}}) : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big) \to \mathsf{S}_{\mathbf{K}}(\mathfrak{g})$ is thus the ideal and coideal $\mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$ of the commutative and cocommutative bialgebra $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$, and there is the direct decomposition $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big) = \mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big) \oplus I\big(\mathsf{S}_{\mathbf{K}}(\mathfrak{g})\big)$ where $I : \mathsf{S}_{\mathbf{K}}(\mathfrak{g}) \to \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$ denotes the natural in-

jection of bialgebras induced by the inclusion $\mathfrak{g} \to \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$. We shall show that the symmetrization isomorphism $\omega : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big) \to \mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ satisfies

$$\omega\Big(\mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)\Big) = \mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g}). \tag{A.5.5}$$

Since $\omega$ is a bijection this will imply that the subalgebra $I\big(\mathsf{S}_{\mathbf{K}}(\mathfrak{g})\big)$ –which is a complementary submodule to $\mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$ in $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$– will bijectively be mapped onto a submodule $\Upsilon\big(\mathsf{S}_{\mathbf{K}}(\mathfrak{g})\big)$ of $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ which is complementary to $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$. Passing to the quotient shows that the restriction of $\omega$ to $I\big(\mathsf{S}_{\mathbf{K}}(\mathfrak{g})\big)$ followed by the projection $\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to \mathsf{T}_{\mathbf{K}}(\mathfrak{g})/\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ will give the symmetrization $\omega_{\mathfrak{g}}$ which thus is a bijection. In order to show the statement (A.5.5), it is easy to see that $\omega$ maps $\mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$ into the ideal and coideal $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ because the value of $\omega$ of a commutative word containing at least one element $\eta$ of $\mathfrak{k}$ will be a linear combination of noncommutative words each containing $\eta$ but not necessarily at the beginning. By iterated commutators with $\eta$ –which create new elements of $\mathfrak{k}$ since it is an Lie ideal– is is seen that each such word is an element of $\mathfrak{k}\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$.

The inverse inclusion is a bit more involved: let $\eta \in \mathfrak{k}$ and $b \in \mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ we want to show that $\eta b$ is a linear combination of terms $\omega(\eta' \bullet \beta)$ with $\eta' \in \mathfrak{k}$ and $\beta' \in \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$. In order to avoid too concrete combinatorics we shall argue with the coalgebra structures: the left multiplication $L_\eta$ with the primitive element $\eta$ is a coderivation $\mathsf{T}_{\mathbf{K}}(\mathfrak{g}) \to \mathsf{T}_{\mathbf{K}}(\mathfrak{g})$ which can be written in the convolution form $\eta\epsilon \star \mathrm{id}_{\mathsf{T}_{\mathbf{K}}(\mathfrak{g})}$ as can be seen immediately, where the convolution $\star$ is w.r.t. the multiplication and comultiplication of the bialgebra $\mathsf{T}_{\mathbf{K}}(\mathfrak{g})$. It follows that the $\mathbf{K}$-linear map $\omega^{-1} \circ L_\eta \circ \omega$ is a coderivation of the coalgebra $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$ which is always of the convolution form $\overline{d} = d \bar{*} \mathrm{id}_{\mathsf{S}_{\mathbf{K}}(\mathsf{Lie}_K(\mathfrak{g}))}$ where $d$ is a $\mathbf{K}$-linear map $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big) \to \mathsf{Lie}_K(\mathfrak{g})\big)$ and the convolution $\bar{*}$ is w.r.t. the multiplication and comultiplication of the bialgebra $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_K(\mathfrak{g})\big)$. Suppose that the values of $d$ lie in the ideal $\mathfrak{k}$ of $\mathsf{Lie}_K(\mathfrak{g})$. Recall the classical formula for the derivative of the exponential map: Let $\big(\mathcal{B}, \blacksquare, \mathbf{1}_{\mathcal{B}}, (F_{(n)}B)_{n\in\mathbb{Z}}\big)$ be complete filtered associative unital $A$-algebra, let $z \in F_{(-1)}B$, and $D : \mathcal{B} \to \mathcal{B}$ a filtration preserving derivation, then –upon writing $\mathrm{ad}_{\blacksquare z'} : z'' \mapsto z' \blacksquare z'' - z'' \blacksquare z'$ for any $z', z'' \in \mathcal{B}$– we have

$$D\left(e^{\blacksquare z}\right) = \left(\frac{e^{\mathrm{ad}_{\blacksquare z}} - \mathrm{id}_{\mathcal{B}}}{\mathrm{ad}_{\blacksquare z}}\big(D(z)\big)\right) \blacksquare e^{\blacksquare z}. \tag{A.5.6}$$

It follows that, upon setting $\mathcal{B} = \big(\mathrm{Hom}_{\mathbf{K}}\left(\mathsf{S}_{\mathbf{K}}(\mathsf{Lie}_K(\mathfrak{g})), \mathsf{T}_{\mathbf{K}}(\mathfrak{g})\right), *\big)$ and $D(\varphi) = \varphi \circ \overline{d}$,

$$\omega \circ \overline{d} \quad = \quad e^{*q} \circ \overline{d} \overset{(A.5.6)}{=} \left(\frac{e^{\mathrm{ad}_{*q}} - \mathrm{id}_{\mathcal{B}}}{\mathrm{ad}_{*q}}\big(d\big)\right) * e^{*q}.$$

since $q \circ \overline{d} = d$ viewed as a map into $\mathfrak{k} \subset \mathsf{Lie}_K(\mathfrak{g}) \subset \mathsf{T}_{\mathbf{K}}(\mathfrak{g})$.

Moreover, let $\psi : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big) \to \mathfrak{k}$ any $\mathbf{K}$-linear map then for any $\beta \in \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$ thanks to the cocommutativity of $\Delta_S$:

$$\begin{aligned}\big(\mathrm{ad}_{*q}(\psi)\big)(\beta) &= \sum_{(\beta)}\Big(q(\beta^{(1)})\psi(\beta^{(2)}) - \psi(\beta^{(2)})q(\beta^{(1)})\Big) = \sum_{(\beta)}\big[q(\beta^{(1)}), \psi(\beta^{(2)})\big] \\ &= \sum_{(\beta)}\mathrm{ad}_{q(\beta^{(1)})}\Big(\psi(\beta^{(2)})\Big) = \big(\mathrm{ad}_q *'' \psi\big)(\beta)\end{aligned}$$

where we have written ad for the adjoint representation of the free $\mathbf{K}$-Lie algebra, i.e. $\mathrm{ad}_\zeta(\zeta') = [\zeta, \zeta']$ for any $\zeta, \zeta' \in \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$ and used the convolution action $*''$ of the convolution algebra $\Big(\mathrm{Hom}_{\mathbf{K}}\Big(\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big), \mathrm{Hom}_{\mathbf{K}}\big(\mathfrak{k}, \mathfrak{k}\big)\Big), *'\Big)$ on the module $\mathrm{Hom}_{\mathbf{K}}\Big(\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big), \mathfrak{k}\Big)$ which is well-defined since $\mathfrak{k}$ is an ideal of $\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$ hence stable by all the linear maps $\mathrm{ad}_\zeta$, $\zeta \in \mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})$. By an easy induction we can finally write

$$\omega \circ \overline{d} = \left(\frac{e^{*'\mathrm{ad}_q} - \mathrm{id}_{\mathfrak{k}}\epsilon_S}{*'\mathrm{ad}_q} *'' d\right) * \omega =: E(d) * \omega = \Big(\big(E(d) \circ \omega^{-1}\big) \star \mathrm{id}_{\mathsf{T}_{\mathbf{K}}(\mathfrak{g})}\Big) \circ \omega$$

Clearly the above $\mathbf{K}$-linear map $E$ is an invertible endomorphism of $\mathrm{Hom}_{\mathbf{K}}\Big(\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big), \mathfrak{k}\Big)$ because the zeroth order term of the series is the identity map and the higher order terms lower the degree. It follows that there is a unique solution $d : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big) \to \mathfrak{k}$ of the equation $E(d) = \eta\epsilon_S$ showing that for all $\beta \in \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)$

$$\eta\omega(\beta) = L_\eta\big(\omega(\beta)\big) = \omega\big(\overline{d}(\beta)\big) = \omega\left(\sum_{(\beta)} d(\beta^{(1)}) \bullet \beta^{(2)}\right) \in \omega\Big(\mathfrak{k} \bullet \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(\mathfrak{g})\big)\Big)$$

which proves the inclusion $\supset$ of eqn (A.5.5) and hence the Theorem. □

It follows that that the primitive part of the universal enveloping algebra is isomorphic to the Lie algebra in this case. Moreover, note that a posteriori it becomes clear that the inverse of $\omega$ can written in the convolution exponential form $e^{\star\chi}$ (see the proof of Proposition A.6.2) where $\chi : \mathsf{U}(\mathfrak{g}) \to \mathfrak{g}$ is given by the composition of the *Eulerian idempotent* $e^{(1)}_{\mathfrak{g}}$ of $\mathsf{U}_{\mathbf{K}}(\mathfrak{g})$ –defined as for the free algebra in eqn (A.6.5) but which can be done for the more general universal enveloping algebras since all the ingredients (identity map, counit, cocommutative comultiplication) are also there– followed by the injection into $\mathfrak{g} \subset \mathsf{S}_{\mathbf{K}}(\mathfrak{g})$ which is now well-defined thanks to the preceding Theorem. Note however that the very nice (modified) Dynkin idempotent in general makes no sense for universal enveloping algebra.

There is a Baker-Campbell-Hausdorff type convolution formula for universal enveloping algebras in case $\mathbb{Q} \subset K$, see e.g. [3].

## A.6 Free algebras

### A.6.1 Free (Symmetric) Algebras

Recall some standard material on the tensor algebra over a fixed $\mathbf{K}$-module $V$: It is defined by $\mathsf{T}_{\mathbf{K}}(V) = \mathsf{T}(V) = \oplus_{r=0}^{\infty} V^{\otimes r}$ with $V^{\otimes 0} := \mathbf{K} =: \mathbf{K1}$, $V^{\otimes 1} := V$, and for each integer $r \geqslant 2$ the symbol $V^{\otimes r}$ means the $r$-times iterated tensor product of $V$ with itself, see [21, p.139-141].

Recall that $\mathsf{T}(V)$ carries an associative multiplication $\mu$ written $\mu(b \otimes b') =: bb'$ defined by the tensor product by means of which $\big(\mathsf{T}(V), \mu, \mathbf{1}\big)$ is a unital associative algebra which is a *free unital algebra over* $V$, i.e. for any given associative unital algebra $\big(B, \mu_B, \mathbf{1}_B\big)$ there is a natural bijection $\phi \mapsto \bar{\phi}$ of the set $\mathrm{Hom}_K(V, B)$ of all linear maps from $V$ to the underlying $K$-module of $B$ to the set $\mathrm{Hom}_{\mathrm{Alg}}\big(\mathsf{T}(V), B\big)$ of all morphisms of unital algebras by means of the well-known formula 'on generators'

$$\bar{\phi}(\mathbf{1}) = \mathbf{1}_A, \;\; \bar{\phi}(x_1 \cdots x_n) := \big(\phi(x_1)\big) \cdots \big(\phi(x_n)\big)$$

for all integers $n \geqslant 1$, and $x_1, \ldots, x_n \in V$ and where the multiplication on the right hand side is in $B$. In other words $V \to \mathsf{T}(V)$ defines a functor $\mathbf{K}\mathsf{mod}$ to $\mathsf{AssAlg}_{\mathbf{K}}$ which is left adjoint to the obvious forgetful functor:

$$\mathbf{KMod} \quad \underset{\text{Forget}}{\overset{\mathsf{T}_{\mathbf{K}}}{\rightleftarrows}} \quad \mathbf{AssAlg}_{\mathbf{K}} \tag{A.6.1}$$

Recall furthermore that $\mathsf{T}(V)$ carries a comultiplication $\Delta_{sh} : \mathsf{T}(V) \to \mathsf{T}(V) \otimes \mathsf{T}(V)$ which is defined to be the morphism of associative algebras defined on generators $x \in V$ by $\Delta_{sh}(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$. Define for all integers $1 \leqslant r \leqslant n-1$ the subset of all *shuffle multiplications* in the usual way by

$$\mathrm{Sh}(r, n-r) := \big\{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(r), \;\; \sigma(r+1) < \cdots < \sigma(n)\big\}. \tag{A.6.2}$$

It is now easy to see by induction the following expression for $\Delta_{sh}$: $\Delta_{sh}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\Delta_{sh}(x_1) = x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_1$ for all $x_1 \in V$, and for all integers $n \geqslant 2$ and $x_1 \ldots, x_n \in V$ we get

$$\begin{aligned} \Delta_{sh}(x_1 \cdots x_n) = {} & (x_1 \cdots x_n) \otimes \mathbf{1} + \mathbf{1} \otimes (x_1 \cdots x_n) \\ & + \sum_{r=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(r, n-r)} \big(x_{\sigma(1)} \cdots x_{\sigma(r)}\big) \otimes \big(x_{\sigma(r+1)} \cdots x_{\sigma(n)}\big) \end{aligned} \tag{A.6.3}$$

It easy to check on generators that $\Delta_{sh}$ is cocommutative and coassociative. In the text we shall use Sweedler's notation (see Appendix A.3) to avoid the above clumsy expression (A.6.3). Next, the projection map $\epsilon : \mathsf{T}(V) \to \mathbf{K}$ which is defined to vanish on the *augmentation ideal* $\mathsf{T}^+(V) := \oplus_{r=1}^{\infty} V^{\otimes\, r}$ and satisfies $\epsilon(\lambda \mathbf{1}) := \lambda$ for

all $\lambda \in \mathbf{K}$ is a *counit* of the coalgebra $\big(\mathsf{T}(V), \Delta_{sh}\big)$, i.e. we get hence the quintuple $\big(\mathsf{T}(V), \mu, \mathbf{1}, \Delta_{sh}, \epsilon\big)$ is a cocommutative bialgebra.

In addition to that, we mention the *antipode* $S : \mathsf{T}(V) \to \mathsf{T}(V)$ which is a $\mathbf{K}$-linear map defined by

$$S(\mathbf{1}) = \mathbf{1}, \quad \text{and} \quad S(x_1 x_2 \cdots x_k) = (-1)^k x_k x_{k-1} \cdots x_2 x_1. \tag{A.6.4}$$

Note that the canonical filtration $\Big(Q'_{(n)}\big(\mathsf{T}_{\mathbf{K}}(V)\big)\Big)_{n\in\mathbb{N}}$ of the $C^3$ coalgebra

$$\big(\mathsf{T}_{\mathbf{K}}(V), \Delta_{sh}, \epsilon, \mathbf{1}\big), \text{ see equation A.3.1,}$$

is exhaustive since for all nonnegative integers $n$ we have $\oplus_{r=0}^{n} \mathsf{T}^r_{\mathbf{K}}(V) \subset Q'_{(n)}\big(\mathsf{T}_{\mathbf{K}}(V)\big)$. It follows that the $C^3$ coalgebra $\big(\mathsf{T}_{\mathbf{K}}(V), \Delta_{sh}, \epsilon, \mathbf{1}\big)$ is always connected.

Next, we shall very often perform *induction arguments with respect to the tensor degree*: recall that the *degree derivation* $\mathsf{Deg} : \mathsf{T}(V) \to \mathsf{T}(V)$ is defined in the obvious way by $\mathsf{Deg}(b) = nb$ for any nonnegative integer $n$ and $b \in V^{\otimes n}$. It is quite useful in case $\mathbb{Q} \subset \mathbf{K}$. It is obvious that Deg is both a derivation of $\mu$ and a coderivation of $\Delta_{sh}$ as can be seen directly.

Clearly, all elements of $\mathsf{T}(V)$ of tensor degree zero are of the form $k\mathbf{1}$ for some $k \in \mathbf{K}$, all elements of tensor degree 1 are of the form $x \in V$, and every element of tensor degree less or equal than $n+1$ for some nonnegative integer $n$ is a $\mathbf{K}$-linear combination of elements of the form $xb$ or $b'x$ where $x \in V$ and $b, b' \in \mathsf{T}(V)$ having tensor degree less or equal than $n$. We shall not repeat these characterizations each time we are using it.

Note that, in case $\mathbb{Q} \subset \mathbf{K}$, there is a convolution logarithm of the identity map $\mathrm{id}_{\mathsf{T}_{\mathbf{K}}(V)}$ the so-called *Eulerian idempotent*

$$e^{(1)} := \ln_\star\big(\mathrm{id}_{\mathsf{T}_{\mathbf{K}}(V)}\big) := \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} \big(\mathrm{id}_{\mathsf{T}_{\mathbf{K}}(V)} - \mathbf{1}\epsilon\big)^{*(r+1)} \tag{A.6.5}$$

which is well-defined because $\mathsf{T}_{\mathbf{K}}(V)$ is a connected cocommutative coalgebra. The above equation (A.6.5) is a particular case of formula (A.3.4). The following formulas

$$\Delta_{sh} \circ e^{(1)} = e^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes e^{(1)} \quad \text{et} \quad e^{(1)} \circ e^{(1)} = e^{(1)} \tag{A.6.6}$$

can be shown: the first by showing an exponentiated version, and the second by a straight-forward computation using the first identity. These equations imply that $e^{(1)}$ is a projection onto the primitive part of the $C^3$-coalgebra $\big(\mathsf{T}_{\mathbf{K}}(V), \Delta_{sh}, \epsilon, \mathbf{1}\big)$. This primitive part will be shown to be isomorphic to the free Lie algebra generated by $V$, see Proposition A.6.1 of the Appendix A.6.2.

Recall the *symmetric algebra generated by the* $\mathbf{K}$*-module* $V$, $\mathsf{S}_{\mathbf{K}}(V)$: it is defined to be the quotient of the free algebra $\mathsf{T}_{\mathbf{K}}(V)$ modulo the two-sided ideal $\mathcal{J}_0(V) =$

$\mathcal{J}_0(V,\mathbf{K})$ of $\mathsf{T}_{\mathbf{K}}(V)$ generated by all the elements of the form $xy - yx$ with $x,y \in V$. This makes the $\mathsf{S}_{\mathbf{K}}(V)$ a commutative associative unital $\mathbf{K}$-algebra such that the natural projection $P_0 : \mathsf{T}_{\mathbf{K}}(V) \to \mathsf{S}_{\mathbf{K}}(V)$ is a morphism of unital algebras. We shall denote the resulting commutative multiplication by $\bullet$. Note that the grading of the free algebra induces a grading $\mathsf{S}_{\mathbf{K}}(V) = \oplus_{n=0}^{\infty}\mathsf{S}^n_{\mathbf{K}}(V)$. Since $\mathcal{J}_0(V)$ is also easily seen to be a coideal it follows that $\mathsf{S}_{\mathbf{K}}(V)$ carries a canonical comultiplication $\Delta_S$, a counit $\epsilon_S$, and an antipode $S$ such that the coalgebra $\mathsf{S}_{\mathbf{K}}(V)$ is a connected cocommutative $C^3$-coalgebra and such that $P_0$ is a map of $C^3$-coalgebras. As in the case of the free algebra $\mathsf{T}_{\mathbf{K}}(V)$, the assignment $V \to \mathsf{S}_{\mathbf{K}}(V)$ is a functor from the category of all $\mathbf{K}$-modules to the category of all unital commutative associative $\mathbf{K}$-algebras where the action on $\mathbf{K}$-linear maps $\phi : V \to W$ is defined as the induced map of $\mathsf{T}\phi$ which sends the ideal $\mathcal{J}_0(V)$ to the ideal $\mathcal{J}_0(W)$. This functor is easily seen to be left adjoint functor for the obvious forgetful functor from the category of all unital commutative associative $\mathbf{K}$-algebras to the category of all $\mathbf{K}$-modules, hence similar to diagram (A.6.1) where the category on the right is specialized to commutative algebras.

The coalgebra structure on $\mathsf{S}_{\mathbf{K}}(V)$ is particularly important: note that the morphisms $\mathsf{S}_{\mathbf{K}}\phi$ are also morphisms of $C^3$-coalgebras. For the case $\mathbf{Q} \subset \mathbf{K}$ –which will be the most important for us– it turns out that $\mathsf{S}_{\mathbf{K}}(V)$ is *cofree* in the sense that the functor $\mathsf{S}$ defines a **right adjoint functor** for the particular forgetful functor $(\,)^+$ from the category of all cocommutative connected $C^3$-coalgebras over $\mathbf{K}$, $C^3\mathbf{CoalgCC}_{\mathbf{K}}$, to the category of all $\mathbf{K}$-modules assigning to each coalgebra $C$ the $\mathbf{K}$-submodule $C^+ = \mathrm{Ker}(\epsilon) \subset C$:

$$C^3\mathbf{CoalgCC}_{\mathbf{K}} \quad \underset{\mathsf{S}_{\mathbf{K}}}{\overset{(\,)^+}{\rightleftarrows}} \quad \mathbf{K}\text{-}\mathbf{mod} \tag{A.6.7}$$

meaning that for a any given connected cocommutative $C^3$-coalgebra $C$ and a $\mathbf{K}$-module $W$ the two following Hom-spaces are naturally isomorphic:

$$\mathrm{Hom}_{\mathbf{K}\text{-}\mathbf{mod}}\big(C^+, W\big) \cong \mathrm{Hom}_{C^3\mathbf{CoalgCC}_{\mathbf{K}}}\big(C, \mathsf{S}_{\mathbf{K}}(W)\big)$$

The isomorphism from the left Hom-space to the right one is again given by a convolution exponential

$$\varphi \mapsto e^{\tilde{*}\varphi} \tag{A.6.8}$$

where $\varphi : C^+ \to W$ is $\mathbf{K}$-linear and $\tilde{*}$ denotes the convolution with respect to the multiplication $\bullet$ in $\mathsf{S}_{\mathbf{K}}(W)$ and the comultiplication $\Delta_C$ of $C$ which of course is a particular case of formula (A.3.4). The restriction of $\varphi$ to the submodule $\mathsf{S}^r(V)$ is called the $r$th *Taylor coefficient* of $\varphi$. We also note the following well-known fact that the map

$$\mathrm{Hom}_{\mathbf{K}}\big(\mathsf{S}_{\mathbf{K}}(V), V\big) \to \mathrm{Coder}_{\mathbf{K}}\big(\mathsf{S}_{\mathbf{K}}(V), \mathsf{S}_{\mathbf{K}}(V)\big) : d \mapsto \overline{d} = d \,\tilde{*}\, \mathrm{id}_{\mathsf{S}_{\mathbf{K}}(V)} \tag{A.6.9}$$

induces an isomorphism of **K**-modules where the latter **K**-module denotes the set of all coderivations of the coalgebra $\big(\mathsf{S}_{\mathbf{K}}(V), \Delta_S, \epsilon_S\big)$. The inverse map is just the composition of the coderivation with the natural projection $pr_V : \mathsf{S}_{\mathbf{K}}(V) \to V$ (the counit of the above adjunction (A.6.7)).

Finally, we need a *natural comparison between* $\mathsf{T}_A(V)$ *and* $\mathsf{T}_K(V)$ for a given $A$-module $V$: in order to distinguish the structures we shall denote by $\mu$, $\mathbf{1}$, $\Delta_{sh}$, and $\epsilon$ the usual $A$-linear bialgebra structure of $\mathsf{T}_A(V)$, and by $\tilde{\mu}$, $\tilde{\mathbf{1}}$, $\tilde{\Delta}_{sh}$, and $\tilde{\epsilon}$ the corresponding $K$-linear bialgebra structure of $\mathsf{T}_K(V)$. Note that the natural morphism $\pi_{V,W} : V \otimes W \to V \otimes_A W$, see Appendix A.2.1 where $V, W$ are $A$-modules canonically extends to a natural morphism of unital $K$-algebras

$$\pi : \big(\mathsf{T}_K(V), \tilde{\mu}, \tilde{\mathbf{1}}\big) \to \big(\mathsf{T}_A(V), \mu, \mathbf{1}\big) \tag{A.6.10}$$

where of course $\pi(\lambda\tilde{\mathbf{1}}) = \iota(\lambda)\mathbf{1}$ where $\iota : K \to A$ denotes the given morphism of unital rings. Then there is the relation

$$\pi_{\mathsf{T}_A(V),\mathsf{T}_A(V)} \circ \big(\pi \otimes \pi\big) \circ \tilde{\Delta}_{sh} = \Delta_{sh} \circ \pi \quad \text{and} \quad \iota \circ \tilde{\epsilon} = \epsilon \circ \pi \tag{A.6.11}$$

which is clear by their definitions.

Note also that the restriction of $\pi$ to $\mathsf{T}^+(V)$ is surjective, the kernel being the two-sided ideal generated by $\mathcal{K}(L, L)$, see eqn (A.2.6).

### A.6.2 Free Lie Algebras

A good introduction to this topic is Bourbaki's book [8] or Reutenauer's book [41]. Let $\mathbf{LieAlg}_{\mathbf{K}}$ be the category of all **K**-Lie algebras, and let Forget be the obvious forgetful functor from this category to the category **K-mod** by omitting the Lie bracket. There is a well-known functor $\mathsf{Lie}_{\mathbf{K}}$ from **K-mod** to $\mathbf{LieAlg}_{\mathbf{K}}$ which is a left adjoint for Forget:

$$\mathbf{K\text{-}mod} \underset{\text{Forget}}{\overset{\mathsf{Lie}_{\mathbf{K}}}{\rightleftarrows}} \mathbf{LieAlg}_{\mathbf{K}} \tag{A.6.12}$$

For any **K**-module $V$ the **K**-Lie algebra $\mathsf{Lie}_{\mathbf{K}}(V)$ is called the *free* **K***-Lie algebra generated by* $V$. We shall briefly recall its definition, see also [8] since there seem to exist misleading definitions in some text-books.

Let BPT denote the set of all *binary planar trees*, see e.g. [31, p.597], with grafting as nonassociative binary multiplication. It parametrizes bracketings in nonassociative algebras such as $((ab)c)d$ or $a((bc)d)$. It can also be seen as a subset of the free associative monoid (without unit for simplicity) generated by two generators $t, p$: look at the nonassociative operation $u \odot v = uvp$ and take the nonassociative monoid generated by $t$ under this operation, see Jacobson's contribution [20, p. 122-123] in [18]. In any case BPT is a nonassociative monoid (without unit) graded by

the degree in $t$, $\mathrm{BPT} = \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathrm{BPT}_n$. Then the free $\mathbf{K}$-module $\mathbf{K}\mathrm{BPT}$ generated by the set BPT is a nonassociative algebra over $\mathbf{K}$ graded by the positive integers. For any $\mathbf{K}$-module $V$ the $\mathbf{K}$-module

$$\mathsf{Mag}_{\mathbf{K}}(V) = \oplus_{n=1}^{\infty} \mathbf{K}\mathrm{BPT}_n \otimes V^{\otimes n} \tag{A.6.13}$$

is a nonassociative algebra in a natural way. Moreover it is free in the sense that for any given nonassociative algebra $(B, \blacksquare)$ and any $\mathbf{K}$-linear map $\theta : V \to B$ there is a unique induced morphism $\bar{\theta} : \mathsf{Mag}_{\mathbf{K}}(V) \to B$ of nonassociative algebras such that $\bar{\theta}(t \otimes x) = \theta(x)$ for all $x \in V$. The free Lie algebra $\mathsf{Lie}_K(V)$ is defined to be the quotient of $\mathsf{Mag}_{\mathbf{K}}(V)$ by the two-sided ideal generated by the set of all $\{xx \mid x \in V\}$ and the set $\{(xy)z + (yz)x + (zx)y \mid x, y, z \in V\}$. The Lie bracket is the induced nonassociative multiplication. Note that the unit of the adjunction, the natural insertion of generators $i_V : V \to \mathsf{Lie}_{\mathbf{K}}(V)$ is still injective since the ideal is graded and contained in degrees $\geqslant 2$. This also implies that the free Lie algebra is graded by the positive integers. It is easy to see –upon using the Jacobi identity– that a system of non independent generators of the $K$-module $\mathsf{Lie}_{\mathbf{K}}(V)$ is given by the followng iterated left ordered commutators:

$$x,\ \ldots,\ \Big[x_1, \big[x_2, [x_3, \ldots, [x_{n-1}, x_n]\cdots]\big]\Big] \tag{A.6.14}$$

where $n \geqslant 2$ is a positive integer, and $x, x_1, \ldots, x_n \in V$. Though nonunique, these generators serve to express the unique morphism of Lie algebras $\bar{\theta} : \mathsf{Lie}_{\mathbf{K}}(V) \to (\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ induced by an arbitrary $\mathbf{K}$-linear map $\theta : V \to \mathfrak{g}$ (with $x_1, \ldots, x_n \in V$)

$$\overline{\theta}\Big(\Big[x_1, \big[x_2, [x_3, \ldots, [x_{n-1}, x_n]\cdots]\big]\Big]\Big) \quad = \quad \Big[\theta(x_1), \big[\theta(x_2), [\theta(x_3), \ldots, [\theta(x_{n-1}), \theta(x_n)]_{\mathfrak{g}}]_{\mathfrak{g}} \ldots\big]_{\mathfrak{g}}\Big]_{\mathfrak{g}}. \tag{A.6.15}$$

Next, recall that the free algebra $\mathsf{T}_{\mathbf{K}}(V)$ is a Lie algebra over $\mathbf{K}$ with respect to the commutator of the associative multiplication $\mu$, denoted by $\mathsf{T}_{\mathbf{K}}(V)^-$. There is thus a unique morphism $\mathsf{Lie}_{\mathbf{K}}(V) \to \mathsf{T}_{\mathbf{K}}(V)^-$ of $\mathbf{K}$-Lie algebras induced by the injection $V \to \mathsf{T}_{\mathbf{K}}(V)$ whose image is the Lie subalgebra of $\mathsf{T}_{\mathbf{K}}(V)^-$ generated by $V$, $\mathcal{L}_{\mathbf{K}}(V)$. Note that in general this morphism is NOT injective in higher degrees.

However, in case $\mathbb{Q} \subset \mathbf{K}$ as a subring, and this we shall suppose for the rest of this appendix part, this is the case (see [20, p.167-174], [31], [41] for fields of characteric 0).

For the convenience of the reader we shall give the indication of the proof:

**Proposition A.6.1** *Let $\mathbb{Q} \subset \mathbf{K}$ as a subring, and let $V$ be a $\mathbf{K}$-module. Then*

$$\begin{aligned} \mathsf{Lie}_{\mathbf{K}}(V) \cong \mathcal{L}_{\mathbf{K}}(V) \quad &= \quad \mathsf{Prim}\big(\mathsf{T}(V)\big) := \\ &= \quad \big\{b \in \mathsf{T}(V) \mid \Delta_{sh}(b) = b \otimes \mathbf{1} + \mathbf{1} \otimes b\big\}. \end{aligned} \tag{A.6.16}$$

*Proof.* Firstly it is clear by the Jacobi identity that the $\mathbf{K}$-module $\mathcal{L}_{\mathbf{K}}(V)$ is spanned by left ordered commutators of elements of $V$ in $\mathsf{T}_{\mathbf{K}}(V)$, see eqn (A.6.14). This easily implies the inclusion $\mathcal{L}_{\mathbf{K}}(V) \subset \mathsf{Prim}\big(\mathsf{T}(V)\big)$. For the converse inclusion and for the rest of the proof the important thing is the existence of the following $\mathbf{K}$-linear map $\tilde{e}_D : \mathsf{T}_{\mathbf{K}}(V) \to \mathsf{T}_{\mathbf{K}}(V)$ whose image is equal to $\mathcal{L}_{\mathbf{K}}(V)$ defined by (for all $x \in V$, $n \in \mathbb{N}$, $n \geqslant 2$, $x_1, \ldots, x_n \in V$)

$$\tilde{e}_D(\mathbf{1}) := 0, \;\; \tilde{e}_D(x) = x, \;\; \tilde{e}_D(x_1 \cdots x_n) = \Big[x_1, \big[x_2, [x_3, \ldots, [x_{n-1}, x_n] \cdots]\big]\Big]. \tag{A.6.17}$$

The following *Von Waldenfels convolution formula* (see [54] and [41, p.20-21, Lemma 1.5]) is easy to check by induction over the tensor degre as well as the stated consequence

$$\tilde{e}_D = \mathsf{Deg} \star S \quad \text{implying} \quad \forall\, \xi \in \mathsf{Prim}\big(\mathsf{T}(V)\big) : \;\; \tilde{e}_D(\xi) = \mathsf{Deg}(\xi). \tag{A.6.18}$$

where $\star$ is convolution with respect to the free multiplication and the shuffle comultiplication of $\mathsf{T}_{\mathbf{K}}(V)$ and $S$ is the antipode of $\mathsf{T}_{\mathbf{K}}(V)$. Since the homogeneous components of every primitive element are obviously primitive and have strictly positive tensor degree (which is invertible in $\mathbb{Q} \subset \mathbf{K}$) the other inclusion is clear showing the last equality in eqn (A.6.16). As has been stated above there is a natural morphism of $\mathbf{K}$-Lie algebras $\mathsf{Lie}_{\mathbf{K}}(V) \to \mathcal{L}_{\mathbf{K}}(V)$ induced by the identity map of $V$.

It remains to show that $\mathsf{Lie}_{\mathbf{K}}(V)$ has the universal property because this implies that the above natural map is an isomorphism: indeed, let $\big(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}\big)$ an arbitrary $\mathbf{K}$-Lie algebra, and $\theta : V \to \mathfrak{g}$ an arbitrary $\mathbf{K}$-linear map. Then the following $\mathbf{K}$-linear map $\check{\theta} : \mathsf{T}_{\mathbf{K}}(V) \to \mathfrak{g}$ written in the following way for any integer $n \geqslant 2$, and $x, x_1, \ldots, x_n \in V$: $\check{\theta}(\mathbf{1}) := 0$, $\check{\theta}(x) := \theta(x)$, and

$$\check{\theta}(x_1 \cdots x_n) := \Big[\theta(x_1), \big[\theta(x_2), [\theta(x_3), \ldots, [\theta(x_{n-1}), \theta(x_n)]_{\mathfrak{g}}]_{\mathfrak{g}} \ldots\big]_{\mathfrak{g}}\Big]_{\mathfrak{g}}. \tag{A.6.19}$$

is well-defined by the universal property of the tensor product. We shall show by induction over the tensor degree that for all $b \in \mathsf{T}_{\mathbf{K}}(V)$

$$\check{\theta}\big(\tilde{e}_D(b)\big) = \check{\theta}\big(\mathsf{Deg}(b)\big). \tag{A.6.20}$$

Indeed, this is clear for $b$ of degree 0 or 1. In order to do the induction we can take $b$ of degree $n$ and $x \in V$, and use first the obvious equation $\tilde{e}_D(xb) = \big[x, \tilde{e}_D(b)\big]$ whence

$$\check{\theta}\big(\tilde{e}_D(xb)\big) = \check{\theta}\big(x\tilde{e}_D(b)\big) - \check{\theta}\big(\tilde{e}_D(b)x\big) = \big[\theta(x), \check{\theta}\big(\tilde{e}_D(b)\big)\big]_{\mathfrak{g}} - \check{\theta}\big(\tilde{e}_D(b)x\big) = n\check{\theta}(xb) - \check{\theta}\big(\tilde{e}_D(b)x\big)$$

by the induction hypothesis. To compute the term $-\check{\theta}\big(\tilde{e}_D(b)x\big)$ we can assume that $b$ is of the form $b = y_1 \cdots y_n$ (for $y_1, \ldots, y_n \in V$) and use the adjoint representation of $\mathfrak{g}$, i.e. $\mathrm{ad}^{\mathfrak{g}}_{\zeta}(\zeta') = [\zeta, \zeta']_{\mathfrak{g}}$ for all $\zeta, \zeta' \in \mathfrak{g}$ to write

$$\check{\theta}(y_1 \cdots y_n x) = \Big(\mathrm{ad}^{\mathfrak{g}}_{\theta(y_1)} \circ \cdots \circ \mathrm{ad}^{\mathfrak{g}}_{\theta(y_n)}\Big)\big(\theta(x)\big)$$

and thanks to the Jacobi identity, i.e. $\big[\mathrm{ad}^{\mathfrak{g}}_{\zeta}, \mathrm{ad}^{\mathfrak{g}}_{\zeta'}\big] = \mathrm{ad}^{\mathfrak{g}}_{[\zeta,\zeta']_{\mathfrak{g}}}$ we get

$$\begin{aligned}
-\check{\theta}\big(\tilde{e}_D(b)x\big) &= -\Big[\mathrm{ad}^{\mathfrak{g}}_{\theta(y_1)}, \big[\mathrm{ad}^{\mathfrak{g}}_{\theta(y_2)}, \ldots, [\mathrm{ad}^{\mathfrak{g}}_{\theta(y_{n-1})}, \mathrm{ad}^{\mathfrak{g}}_{\theta(y_n)}]\ldots\big]\Big]\big(\theta(x)\big) \\
&= -\mathrm{ad}^{\mathfrak{g}}_{\left[\theta(y_1), \left[\theta(y_2), \ldots, [\theta(y_{n-1}), \theta(y_n)]_{\mathfrak{g}}\right]_{\mathfrak{g}}\ldots\right]_{\mathfrak{g}}}\big(\theta(x)\big) \\
&= \Big[\theta(x), \big[\theta(y_1), \big[\theta(y_2), \ldots, [\theta(y_{n-1}), \theta(y_n)]_{\mathfrak{g}}\big]_{\mathfrak{g}}\ldots\big]_{\mathfrak{g}}\big]_{\mathfrak{g}}\Big]_{\mathfrak{g}} = \check{\theta}(xb)
\end{aligned}$$

whence $\check{\theta}\big(\tilde{e}_D(xb)\big) = n\check{\theta}(xb) + \check{\theta}(xb) = (n+1)\check{\theta}(xb)$ proving the induction.

Defining now the $\mathbf{K}$-linear map $\tilde{\theta} : \mathcal{L}_{\mathbf{K}}(V) \to \mathfrak{g}$ on each homogeneous element $\xi$ of positive degree $n$ by $\tilde{\theta}(\xi) = \frac{1}{n}\check{\theta}(\xi)$ we see –upon using eqn (A.6.20)– that this obviously well-defined map satisfies the above identity (A.6.15) with $\bar{\theta}$ replaced by $\tilde{\theta}$. Finally, by an easy induction over the tensor degree of $\xi \in \mathcal{L}_{\mathbf{K}}(V)$ it can be shown that for all $\xi, \xi' \in \mathcal{L}_A$ (which we alwys can write as left-ordered commutators) the morphism identity $\big[\tilde{\theta}(\xi), \tilde{\theta}(\xi')\big]_{\mathfrak{g}} = \tilde{\theta}\big([\xi, \xi']\big)$ holds. This and the fact that $\mathcal{L}_{\mathbf{K}}(V)$ in generated by $V$ shows the universal property, and in particular the isomorphism with the Lie algebra $\mathsf{Lie}_{\mathbf{K}}(V)$. □

Recall that the normalized version of $\tilde{e}_D$, $e_D$, which is defined by $e_D(x_1 \cdots x_n) = \frac{1}{n}\tilde{e}_D(x_1 \cdots x_n)$ for all strictly positive integers $n$ is an idempotent map (one of the many *Lie idempotents*) and sometimes known under the name of *Dynkin idempotent*.

Recall that for the case $\mathbb{Q} \subset \mathbf{K}$ and any $\mathbf{K}$-module $V$ there is the symmetrization map $\omega : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big) \to \mathsf{T}_{\mathbf{K}}(V)$ given by (for all $n \in \mathbb{N}\backslash\{0\}$, $\xi_1, \ldots, \xi_n \in \mathsf{Lie}(V)$)

$$\omega(\mathbf{1}) = \mathbf{1}, \quad \omega\big(\xi_1 \bullet \cdots \bullet \xi_n\big) := \frac{1}{n!}\sum_{\sigma \in S_n} \big(i(\xi_{\sigma(1)})\big) \cdots \big(i(\xi_{\sigma(1)}) \tag{A.6.21}$$

where $i : \mathsf{Lie}_{\mathbf{K}}(V) \to \mathsf{T}_{\mathbf{K}}(V)$ denotes the natural morphism which is injective in our case, see Proposition A.6.1. There is the well-known

**Proposition A.6.2** *Let $\mathbb{Q} \subset \mathbf{K}$, and let $V$ be a $\mathbf{K}$-module. Then the symmetrization map $\omega : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big) \to \mathsf{T}_{\mathbf{K}}(V)$, see eqn (A.6.21) is an isomorphism of $C^3$-coalgebras.*

*Proof.* Let $q : \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big) \to \mathsf{T}_{\mathbf{K}}(V)$ the $\mathbf{K}$-linear map consisting of the projection $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big) \to \mathsf{Lie}(V)$ (with kernel equal to $\mathbf{K}\mathbf{1} \oplus \oplus_{r=2}^{\infty}\mathsf{S}^r_{\mathbf{K}}\big(\mathsf{Lie}(V)\big)$) followed by the injection $i$ (see also the second map of eqn (6.2.13) ), and let $*$ denote the convolution

with respect to the comultiplication in $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big)$ and the free multiplication in $\mathsf{T}_{\mathbf{K}}(V)$. Then clearly $\omega$ is equal to the following convolution exponential

$$\omega = e^{*q} \tag{A.6.22}$$

and hence a morphism of $C^3$-coalgebras. In order to construct an inverse map recall the Eulerian idempotent $e^{(1)} : \mathsf{T}_{\mathbf{K}}(V) \to \mathsf{T}_{\mathbf{K}}(V)$ (see eqn (A.6.5)), which projects onto the primitive part of $\mathsf{T}_{\mathbf{K}}(V)$, hence the Lie subalgebra $\mathcal{L}_{\mathbf{K}}(V)$ which is isomorphic to the free Lie algebra, see Proposition A.6.1. Let $\chi : \mathsf{T}_{\mathbf{K}}(V) \to \mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}_{\mathbf{K}}(V)\big)$ be equal to the corestriction of $e^{(1)}$ to $\mathcal{L}_{\mathbf{K}}(V)$ followed by the injection to $\mathsf{S}_{\mathbf{K}}\big(\mathsf{Lie}(V)\big)$, then the convolution exponential $e^{\tilde{\star}\chi}$, see the convolution table (6.2.12), is readily checked to be an inverse of $\omega$. □

## A.7 Differential Geometry versus Algebra

Let $X$ be a differentiable manifold (always supposed to be Hausdorff and second countable). For the sake of laziness we shall assume that $X$ is connected. Let $\mathbb{K}$ denote the field $\mathbb{R}$ of all real numbers or the field $\mathbb{C}$ of all complex numbers. Let $A$ be the $\mathbb{K}$-algebra of all smooth real-valued functions $X \to \mathbb{K}$, $A = \mathcal{C}^\infty(X, \mathbb{K})$. Here the constant functions 1 and 0 are the only idempotents of $A$ thanks to the fact that $X$ is connected. We first set $K = \mathbb{K}$. Next, let $\mathbb{K}\mathbf{VB}_X$ the category of all vector bundles over $X$ where the object class consists of all $\mathbb{K}$-vector bundles $(E, \tau, X)$ over $X$ ($\tau : E \to X$ denoting the bundle projection; we shall often write just $E, E'$), where a morphism $\Psi : (E, \tau, X) \to (E', \tau', X)$ is a smooth fibrewise linear map $\Psi : E \to E'$ such that $\tau' \circ \Psi = \tau$.

Recall that for any two vector bundles $E, E'$ over $X$ there is the fibrewise direct sum $E \oplus E'$ and the fibrewise tensor product $E \otimes E'$ which can be seen as a coproduct structure and a symmetric closed monoidal structure on the category $\mathbb{K}\mathbf{VB}_X$ where the unit object is $X$ (seen as the trivial vector bundle $X \times \{0\}$) for the fibrewise direct sum, and the trivial bundle $X \times \mathbb{K}$ for the fibrewise tensor product. These structures are compatible in the usual 'distributive manner' explicited in the so-called *distributive monoidal categories*, see e.g. [**?** ] for more details.

Recall that the category of all $R$-modules (where $R$ is some fixed unital ring) is also distributive monoidal with respect to the usual direct sum $\oplus$ and tensor product $\otimes$.

There is a well-known functor $\Gamma$ from $\mathbb{K}\mathbf{VB}_X$ to $A$-**mod** associating to each vector bundle $(E, \tau, X)$ over $X$ its $A$-module $\Gamma(E) := \Gamma^\infty(X, E)$ of all smooth sections, i.e. smooth maps $\varphi : X \to E$ with $\tau \circ \varphi = \mathrm{id}_X$, and to each morphism $\phi : E \to E'$ of vector bundles over $X$ the composition $\varphi \mapsto \phi \circ \varphi$. This clearly is a monoidal functor (for both structures), see [32, p.255] for details, where the natural map $\Gamma_2 : \Gamma(E) \otimes_A \Gamma(E') \to \Gamma(E \otimes E')$ mapping two sections to their fibrewise tensor product is an isomorphism (the same being true for direct sums). According to the *Serre-*

*Swan Theorem* (see e.g. [10, p.154, Lemma (14.4)]) all the $A$-modules $\Gamma(E)$ are finitely generated and projective.

On the other hand, if we restrict the category $A$-**mod** to the full subcategory $A$-$\mathbf{mod}_{\mathrm{fgp}}$, where the objects are *finitely generated projective $A$-modules*, then there is a functor $\mathsf{B}$ from $A$-$\mathbf{mod}_{\mathrm{fgp}}$ to $\mathbb{K}\mathbf{VB}_X$: Let $V$ be a finitely generated projective $A$-module, and fix a set $\epsilon_1,\ldots,\epsilon_M \in V$ of generators. Furthermore, for each $x \in X$ let $\delta_x : A \to \mathbb{K} : a \mapsto a(x)$ be the usual evaluation ('delta') functional and $I_x \subset A$ its kernel is a maximal ideal of $A$ (and closed w.r.t. the usual Fréchet topology on $A$). The field $\mathbb{K}$ becomes an $A$-module by means of $\delta_x$ isomorphic to $A/I_x \cong \mathbb{K} \otimes_A A$. Denote by $\mathsf{B}(V)_x = V/(I_x V) \cong \mathbb{K} \otimes_A V$ which is a finite-dimensional $\mathbb{K}$-vector space of dimension $n_x \leqslant M$ (which we call the *rank of $V$ at $x$*), and write –by abuse of notation– $\delta_x : V \to \mathsf{B}(V)_x$ for the canonical map $v \mapsto 1_{\mathbb{K}} \otimes_A v$ for which we shall also write $\delta_x(v) =: v(x)$. By elementary linear algebra we can assume that there are positive integers $1 \leqslant i_1 < \cdots < i_{n_x} \leqslant M$ such that the elements $\epsilon_{i_1}(x),\ldots,\epsilon_{i_{n_x}}(x)$ form a basis of $\mathsf{B}(V)_x$. Consider the set given by

$$\mathsf{B}(V) := \bigcup_{x\in X} \left( \{x\} \times \frac{V}{I_x V} \right) \tag{A.7.1}$$

(disjoint union) together with the obvious projection $\tau : \mathsf{B}(V) \to X$ induced by the first factor projection.

Moreover, since each $A$-linear map $\Phi : V \to W$ induces a well-defined canonical $\mathbb{K}$-linear map $\mathsf{B}(\Phi)_x : \mathsf{B}(V)_x \to \mathsf{B}(W)_x$ there is a fibrewise linear set map $\mathsf{B}(\Phi) : \mathsf{B}(V) \to \mathsf{B}(W)$. It remains to show how to construct the locally trivial vector bundle structure on $\mathsf{B}(V)$: this is traditionally be done by first passing to the algebra and module of all germs at a point $x$ and applying the theorem that all finitely generated projective modules over a local ommutative ring are free, see e.g. [21, p.413, Thm.7.5], and then going back to the localized modules.

We sketch a more elementary reasoning [1]: For each open subset $U$ of $X$ consider the local function algebra $A_U := \mathcal{C}^\infty(U,\mathbb{K})$ with the obvious restriction map $r_U : A \to A_U : f \mapsto \big(y \mapsto f(y)\big)$ for all $y \in U$ which is a morphism of unital $\mathbb{K}$-algebras, and write $V_U$ for the 'localized' module $A_U \otimes_A V$. Since $V$ is finitely generated and projective there is another finitely generated (and a posteriori projective) $A$-module $V'$ such that the direct sum $V \oplus V'$ is isomorphic to a free module $A^N$ for some nonnegative integer $N$. By choosing generators $\epsilon'_1,\ldots,\epsilon'_{M'}$ of $V'$ we get by an entirely analogous construction a choice $1 \leqslant j_1 < \cdots j_{n'_x} \leqslant M'$ of integers such that $\epsilon'_{j_1}(x),\ldots,\epsilon'_{j_{n'_x}}(x)$ form a base of the $\mathbb{K}$-vector space $\mathsf{B}(V')_x$ whence –thanks to the obvious decomposition $\mathbb{K}^N \cong \mathsf{B}(V)_x \oplus \mathsf{B}(V')_x$ resulting from $V \oplus V' \cong A^N$– we have the rank equality $n_x + n'_x = N$. Upon considering all the elements of $V$ and of $V'$,

1. I would like to thank S.Waldmann for communicating this nice short argument.

in particular the generators $\epsilon_1,\ldots,\epsilon_N$ and $\epsilon'_1,\ldots,\epsilon_{M'}$ as $\mathbb{K}^N$-valued smooth functions on $X$ and upon applying the usual arguments of minors we can infer that neither the rank of $V$ nor the rank of $V'$ can locally fall, and since they always sum up to $N$ they have to be locally constant. Hence for each $x \in X$ there is an open neighbourhood $U_x$ such that the restrictions $r_{U_x}(\epsilon_{i_1}),\ldots,r_{U_x}(\epsilon_{i_{n_x}})$ form a base for the localized module $V_{U_x}$ which is hence a free module of rank $n_x$. Since $X$ is connected and since the rank of a free module over any commutative ring is well-known to be an invariant (see e.g. [21, p.412, Prop.7.18]), it follows that there is a nonnegative integer $n$ for all $x \in X$ such that $n_x = n$. This allows to construct local trivializations of $\big(\mathsf{B}(V),\tau,X\big)$ as set maps, and the change of these trivializations on the overlap $U$ of two domains will be a $GL(n,\mathbb{K})$-valued smooth map (composed out of elements of $A_U$) giving rise to the usual *cocycle of transition functions* from which the entire differentiable bundle structure can uniquely be constructed, see e.g. [27, p.51-52, Theorem]. This makes the functor $\mathsf{B}$ well-defined on objects, as well as on morphisms (upon passing to local representatives).

It is not hard to see that the diagram of functors

$$\mathbb{K}\mathbf{VB}_{\mathbf{X}} \quad \underset{\mathsf{B}}{\overset{\Gamma}{\rightleftarrows}} \quad A\mathbf{mod}_{\mathrm{fgp}} \tag{A.7.2}$$

is an *equivalence of categories* preserving the distributive symmetric closed monoidal structures.

# B. Tougeron's Lemma

Contents

## B.1 Preliminary

We will present in this part of the appendix some constructions around the Tougeron's lemma.

First of all, the objective is construct a smooth real function defined in $\mathbb{R}^n$ that is constant igual to 1 in $\overline{B}_{r-\varepsilon}(0)$, 0 outside $\overline{B}_r(0)$ and this function assume values in $[0,1]$.

Let us consider the following function $\rho : \mathbb{R} \to \mathbb{R}$ given by

$$\rho(t) := \begin{cases} e^{\frac{-1}{t}} & \text{if t} \geqslant 0 \\ 0 & \text{if t} \leqslant 0 \end{cases}$$

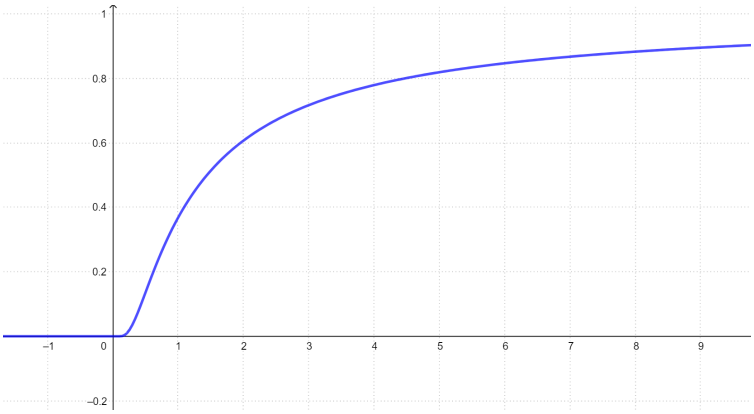



Figure B.1. – Graph of $\rho$

Of course, $\rho$ is $\mathcal{C}^\infty$ in $\mathbb{R}\backslash\{0\}$ and $\frac{d^k\rho}{dt^k}(t) = 0$ for $t < 0$. If $t > 0$ we have $\frac{d\rho}{dt}(t) = (e^{\frac{-1}{t}})' = e^{\frac{-1}{t}}\left(\frac{-1}{t}\right)' = \frac{1}{t^2}e^{\frac{-1}{t}}$ and $\frac{d^2\rho}{dt^2}(t) = \left(\frac{-2}{t^3} + \frac{1}{t^4}\right)e^{\frac{-1}{t}}$.

Actually, we can proof by recurrence that

$$\frac{d^k\rho}{dt^k}(t) := \begin{cases} p_k(\frac{1}{t})e^{\frac{-1}{t}} & \text{if t} > 0 \\ 0 & \text{if t} < 0 \end{cases}$$

where $p_k$ is a polynomial of degree $2k$.

It is not dificult to proof that $\frac{d^k\rho}{dt^k}$ always exists in $\mathbb{R}$ and is equal to 0 in $t = 0$, it means that $\rho$ is $\mathcal{C}^\infty$. Observe also that $\rho$ is positive for $t > 0$ and strictly increasing and limitated by $\lim_{t\to+\infty} e^{\frac{-1}{t}} = 1$.

Afterward, les us take two real numbers $0 < a < b$. Then the function $\rho_{ab} : \mathbb{R} \to \mathbb{R}$

$$\rho_{ab} := \rho(t-a)\rho(b-t)$$

it is clearly a $\mathcal{C}^\infty$-function such that assumes values in $[0,1] \subset \mathbb{R}$. Also note that $\rho_{ab}(t) = 0$ for $t \leqslant a$ and $t \geqslant b$ and also $0 < \rho_{ab}(t) < 1$ for $a < t < b$.

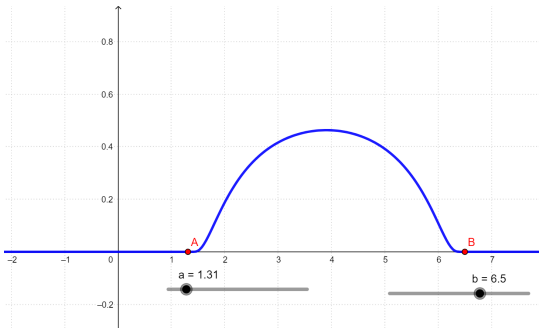



Figure B.2. – Graph of $\rho_{ab}$ for $a = 1,31$ and $b = 6,5$

Next let us define $\phi_{ab} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{ab}(t) := \frac{\int_{-\infty}^{t} \rho_{ab}(\tau)d\tau}{\int_{-\infty}^{+\infty} \rho_{ab}(\tau)d\tau}.$$

But from $0 < \int_{-\infty}^{+\infty} \rho_{ab}(\tau)d\tau = \int_a^b \rho_{ab}(\tau)d\tau < \infty$ follow that $\phi_{ab}$ is increasing and $\mathcal{C}^\infty$ such that

$$\phi_{ab}(t) \in \begin{cases} \{0\} & \text{if } t \leqslant a \\ [0,1] & \text{if } a \leqslant t \leqslant b \\ \{1\} & \text{if } t \geqslant b. \end{cases}$$

Finally, let us take $\epsilon$ and $r$ two positive real numbers such that $0 < r - \epsilon < r$. We then define the function $\psi_{r,\epsilon} : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi_{r,\epsilon}(x) := \phi_{(r-\epsilon)^2 r^2}(r^2 + (r-\epsilon)^2 - ||x||^2)$$

where $||x||^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $x = (x_1, x_2, \cdots, x_n)$. This function is also $\mathcal{C}^\infty$ and

$$\psi_{r,\epsilon}(x) = \begin{cases} 1 & \text{if } ||x|| \leqslant r - \epsilon \\ y \in [0,1] & \text{if } r - \epsilon \leqslant ||x|| \leqslant r \\ 0 & \text{if } ||x|| \geqslant r. \end{cases}$$

The following theorem can be found in Rudin's book, see [44][p.147, Thm 6.20].

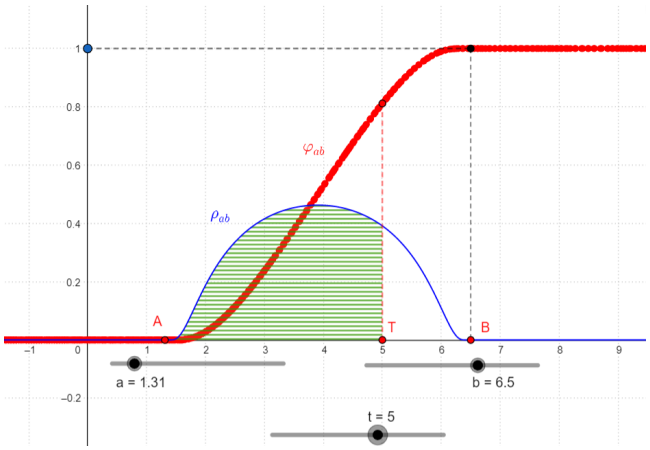



Figure B.3. – Graph of $\varphi_{ab}$ in red

**Theorem B.1.1** *Soit $\Omega \subset \mathbb{R}^n$ une partie ouverte nonvide et $(U_\alpha)_{\alpha\in\mathfrak{S}}$ une famille d'ouverts telle que l'on a $\bigcup_{\alpha\in\mathfrak{S}} U_\alpha = \Omega$. Alors*

1. *Il existe une suite $(B_i)_{i\in\mathbb{N}\setminus\{0\}}$ de boules fermées de centre $s_i \in \Omega$ et de rayon $r_i > 0$, i.e. $B_i = B_{r_i}(s_i)$ telles que*

$$\forall\, i \in \mathbb{N}\setminus\{0\}\ \ \exists\, \alpha_i \in \mathfrak{S} \text{ tels que } B_i \subset U_{\alpha_i} \quad \text{et} \quad \bigcup_{i\in\mathbb{N}\setminus\{0\}} U_{\frac{r_i}{2}}(s_i) = \bigcup_{i\in\mathbb{N}\setminus\{0\}} B_i = \Omega$$

   *En particulier, $\Omega$ est une réunion dénombrable de parties compactes de $\mathbb{R}^n$.*

2. *Il existe une suite $(\psi_i)_{i\in\mathbb{N}\setminus\{0\}}$ de fonctions de classe $C^\infty$ de $\mathbb{R}^n \to \mathbb{R}$ telle que*

   a) $\forall\, i \in \mathbb{N}\setminus\{0\}, \forall\, x \in \mathbb{R}^n :\ 0 \leqslant \psi_i(x) \leqslant 1$ *et* $\operatorname{supp}(\psi_i) \subset B_i \subset U_{\alpha_i}$,

   b) $\forall\, x \in \Omega\ \exists\, i \in \mathbb{N}\setminus\{0\}\ \forall\, j \in \mathbb{N}\setminus\{0\} : x \in U_{\frac{r_i}{2}}(s_i)$ *et si* $j \geqslant i+1$ *alors* $U_{\frac{r_i}{2}}(s_i) \cap \operatorname{supp}(\psi_j) = \varnothing$. *En particulier, cela veut dire que la famille des supports des $\psi_i$ est localement finie.*

   c) *La somme $\sum_{i=1}^{\infty} \psi_i$ est une fonction numérique bien définie et l'on a*

$$\sum_{i=1}^{\infty} \psi_i(x) = \begin{cases} 1 & \text{si } x \in \Omega, \\ 0 & \text{si } x \notin \Omega. \end{cases}$$

   d) *Pour toute partie compacte $K \subset \Omega$ il existe un entier strictement positif $m$ et une partie ouverte $W \supset K$ tels que*

$$\forall\, x \in W :\quad \psi_1(x) + \cdots + \psi_m(x) = 1.$$

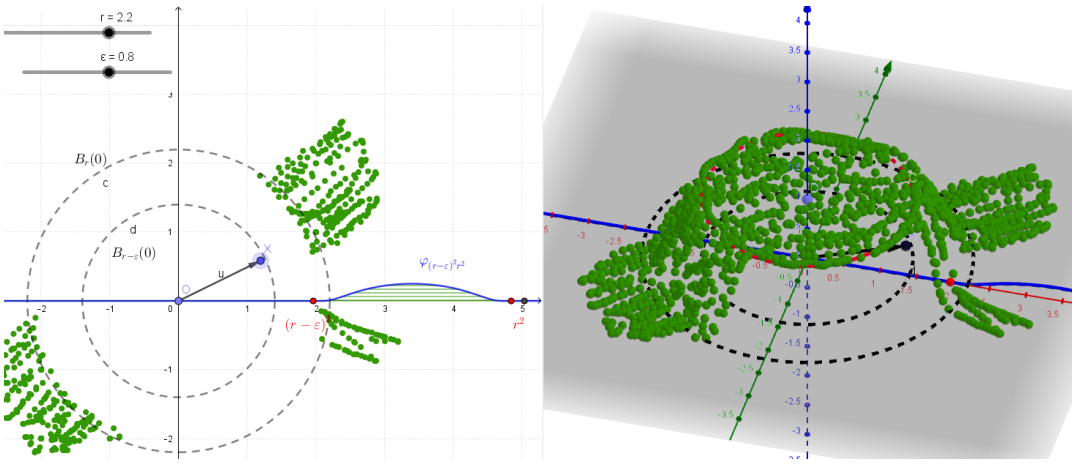



Figure B.4. – Example in $\mathbb{R}^2$: points in the surface of $\psi_{r,\epsilon}$ in the form $(x, y, \psi_{r,\epsilon}(x, y))$

*Proof.* **1.** Consider $S \subset \Omega$ a countable dense subset, as for example $S = \Omega \cap \mathbb{Q}^n$. If we take $s \in S$ as $\Omega = \bigcup_{\alpha \in L} U_\alpha$ then $\exists \overline{r} \in \mathbb{R}; B_r(s) \subset \Omega$ and $B_{\overline{r}}(s) \cap U_\alpha \neq \varnothing$, for some $\alpha \in L$. Notice that, $U_\alpha$ is open and then we can take $r \in \mathbb{Q}$ such that $B_r(s) \subset U_\alpha$. In addition, as the center of this balls are in $S$, this is a countable number of balls, so let us call then $B_i = B_{r_i}(s_i)$ and $V_i = B_{\frac{r_i}{2}}(s_i)$.

In the other hand, fixed $x \in \Omega$ we obtain a $\alpha \in L$ such that $x \in U_\alpha \subset \Omega$. As $U_\alpha$ is open there is a $\delta > 0$ , that we can suppose rational, such that $B_\delta(x) \subset U_\alpha$. Moreover, exists $s \in S$ such that $d(x, s) < \frac{\delta}{4}$, because $S$ is dense. Here $d$ is the Euclidean distance in $\mathbb{R}^n$. Let be $z \in B_{\frac{\delta}{2}}(s)$. Then,

$$d(x, z) \leqslant d(x, s) + d(s, z) < \frac{\delta}{4} + \frac{\delta}{2} = \frac{3\delta}{4} < \delta \quad \text{and}$$

$$x \in B_{\frac{\delta}{4}}(s) \subset B_{\frac{\delta}{2}}(s) \subset B_\delta(x) \subset \mathrm{U}_\alpha .$$

Consequently, $B_{\frac{\delta}{2}}(s) \subset \mathrm{U}_\alpha$ and then $B_{\frac{\delta}{2}}(s)$ is some $B_i$, and moro $x$ is in some $V_i$, it shows that

$$\bigcup_{i \in \mathbb{N}} V_i = \bigcup_{i \in \mathbb{N}} B_i = \Omega.$$

**2.** Let us consider a sequence of functions $(\Phi_i)_{i \in \mathbb{N}}$ where for each $i \in \mathbb{N}$ we have $\Phi_i : \mathbb{R}^n \to \mathbb{R}$ such that $\forall x \in \mathbb{R}^n, 0 \leqslant \Phi_i(x) \leqslant 1$, $\{x \in \mathbb{R}^n; \phi(x) \neq 0\} = \mathrm{supp}(\Phi) \subset B_i$ and $\Phi_i(x) = 1, \forall x \in V_i$. From the above construction is sufficient to take the function $\Phi_i : x \mapsto \psi_{r_i, \frac{r_i}{2}}(x - s_i)$.

Now let us define the following sequence of functions

$$\psi_1 = \Phi_1 \quad \text{and} \quad \forall i \geqslant 2, \psi_i(x) = (1-\Phi_1(x))(1-\Phi_2(x))\cdots(1-\Phi_{i-1}(x))\Phi_i$$

We obtained that each $\psi_i$ is $\mathcal{C}^\infty$, $0 \leqslant \psi_i(x) \leqslant 1, \forall x \in \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$

$$\operatorname{supp}(\psi_i) \subset \operatorname{supp}(\Phi_i) \subset B_i \quad \text{and}$$

$$\forall j; 1 \leqslant j \leqslant i-1 \text{ and } \forall x \in V_1 \cup V_2 \cup \cdots \cup V_j \Rightarrow \psi_i(x) = 0$$

because $\Phi_j(x) = 1 \Rightarrow (1-\Phi_j(x) = 0), \forall x \in V_j$. It shows **(a)** and **(b)**.

**(d)** Let $K \subset \Omega$ be a compact subset. As the family $(V_i)_{i\in\mathbb{N}\setminus\{0\}}$ covers $K$ so there is a strictly positive integer $m$ such that $K \subset V_1 \cup \cdots \cup V_m := W$. The equation

$$\psi_1 + \cdots + \psi_i = 1 - (1-\phi_1)\cdots(1-\phi_i) \quad \big(\ \forall i \in \mathbb{N}\setminus\{0\}\ \big) \tag{B.1.1}$$

shows that for all $x \in W$ it comes that $\psi_1(x) + \cdots + \psi_m(x) = 1$. □

**Corollary B.1.2** *Soit $\Omega \subset \mathbb{R}^n$ un ouvert non vide, et $(K_n)_{n\in\mathbb{N}}$ une suite de parties compactes de $\Omega$ telle que*

$$\bigcup_{n\in\mathbb{N}} K_n = \Omega \quad \text{et} \quad \forall\, n \in \mathbb{N}:\ K_n \subset K_{n+1}^\circ \text{ (l'intérieur de } K_{n+1}).$$

*Alors il existe une suite $(W_n)_{n\in\mathbb{N}}$ de parties ouvertes de $\Omega$ et une suite $(\alpha_n)_{n\in\mathbb{N}}$ de fonctions de classe $\mathcal{C}^\infty$ à valeurs réelles sur $\mathbb{R}^n$ telle que*

$$\forall\, n \in \mathbb{N}: K_n \subset W_n \subset \overline{W_n} \subset K_{n+1}^\circ \ \text{ et } \ \alpha_n(x) = \begin{cases} 1 & \text{si } x \in W_n, \\ 0 & \text{si } x \notin K_{n+1}, \\ y \in [0,1] & \text{autrement.} \end{cases}$$

## B.2 Tougereon's Lemma

The following Lemma is quite important for the proof of the Theorem 3.1.1 and was taken from Jean-Claude Tougeron's book [52, p.113, Lemme 6.1].

**Lemma B.2.1** *Let $\Omega$ be an open set of $\mathbb{R}^n$, and $(\phi_i)_{i\in\mathbb{N}}$ a sequence of smooth functions $\Omega \to \mathbb{K}$. Then there is a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that*

1. *$\alpha$ takes only values between 0 and 1. Moreover $\alpha(x) = 0$ for all $x \notin \Omega$, and $\alpha(x) > 0$ for all $x \in \Omega$.*
2. *For each nonnegative integer $i$ the function $\phi_i' : \mathbb{R}^n \to \mathbb{K}$ defined by*

$$\phi_i'(x) := \begin{cases} \phi_i(x)\alpha(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

*is smooth.*

*Proof.* Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets of $\mathbb{R}^n$ such that

$$\forall\, n\in\mathbb{N}:\ K_n\subset K_{n+1}^{\circ}\quad\text{and}\quad\bigcup_{n\in\mathbb{N}}K_n=\Omega,$$

let $(W_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of $\Omega$, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions $\mathbb{R}^n\to\mathbb{R}$ such that

$$\forall\, n\in\mathbb{N}: K_n\subset W_n\subset\overline{W_n}\subset K_{n+1}^{\circ}\ \text{ and }\ \alpha_n(x)=\begin{cases}1 & \text{si } x\in W_n,\\ 0 & \text{si } x\notin K_{n+1},\\ y\in[0,1] & \text{autrement.}\end{cases}$$

Recall the definition of the seminorms $p_{K,N}:\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$ (where $K\subset\mathbb{R}^n$ is a compact set and $N$ is a nonnegative integer)

$$p_{K,N}(f):=\max\{|D^\beta f(x)|\mid x\in K,\ |\beta|\leqslant N\}\tag{B.2.1}$$

where $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}^n$ is a multi-index, $|\beta|:=\beta_1+\cdots+\beta_n$, and

$$D^\beta:=\left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_1}\right)^{\beta_n}.$$

$\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$ is known to be a locally convex topological vector space which is complete in the sense that every Cauchy sequence converges, see e.g. [43, p.33].

It is obvious that for any two compact subsets $K,K'$ and nonnegative integers $N,N'$ we always have for all $f\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{K})$

$$\text{if } K\subset K' \text{ and } N\leqslant N' \text{ then }\ p_{K,N}(f)\leqslant p_{K',N'}(f)\tag{B.2.2}$$

Choose a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of strictly positive real numbers such that

$$\forall\, j\in\mathbb{N}:\ p_{K_{j+1},j}(\epsilon_j\alpha_j)<\frac{1}{2^j}\quad\text{and}\quad\forall\, i\leqslant j\in\mathbb{N}:\ p_{K_{j+1},j}(\epsilon_j\alpha_j\phi_i)<\frac{1}{2^j}$$

which is possible since for each nonnegative integer $j$ there are only finitely many seminorms involved. For each nonnegative integer $N$ set

$$\alpha_{(N)}:=\sum_{j=0}^{N}\epsilon_j\alpha_j.$$

Clearly $\alpha_{(N)}$ is smooth, has nonnegative real values, and has its support in $K_{N+1}^{\circ}\subset K_{N+1}\subset\Omega$. Fix $i\in\mathbb{N}$. Then for each nonnegative integer $j$ the function $\phi_{i,j}$ defined by

$$\forall\, x\in\Omega:\ \phi_{i,j}(x)=\phi_i(x)\alpha_j(x)$$

is clearly smooth in $\Omega$ and has compact support contained in the support of $\alpha_j$, hence in $K_{j+1}^{\circ} \subset K_{j+1} \subset \Omega$. It trivially follows that each $\phi_{i,j}$ can be considered as the restriction to $\Omega$ of a smooth function $\phi'_{i,j} : \mathbb{R}^n \to \mathbb{K}$ (having compact support in $K_{j+1}^{\circ} \subset K_{j+1} \subset \Omega$) which is defined to be zero outside of $\Omega$.

In the same spirit we can extend the smooth function $x \mapsto \phi_i(x)\alpha_{(N)}(x)$ to the smooth function

$$\phi'_{i(N)} = \sum_{j=0}^{N} \epsilon_j \phi'_{i,j}.$$

having compact support in $K_{N+1}^{\circ} \subset K_{N+1} \subset \Omega$.
Fix a nonnegative integer $i$. Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $K \subset \mathbb{R}^n$ a compact subset, and $m \in \mathbb{N}$. Then there is a nonnegative integer $N_0$ such that

$$\frac{1}{2^{N_0}} < \epsilon, \quad m \leqslant N_0, \quad \text{and} \quad i \leqslant N_0$$

Then for all nonnegative integers $N, p$ with $N \geqslant N_0$ we get (since for all $j \in_{\mathbb{N}}$ such that $N+1 \leqslant j$ we have $m \leqslant N_0 \leqslant N \leqslant j$ and $i \leqslant N$, and $\operatorname{supp}(\phi'_{i,j}) \subset K_{j+1}^{\circ} \subset K_{j+1}$)

$$\begin{aligned} p_{K,m}\big(\phi'_{i(N+p)} - \phi'_{i(N)}\big) &= p_{K,m}\left(\sum_{j=N+1}^{N+p} \epsilon_j \phi'_{i,j}\right) \leqslant \sum_{j=N+1}^{N+p} \epsilon_j p_{K,m}\big(\phi'_{i,j}\big) = \sum_{j=N+1}^{N+p} \epsilon_j p_{K\cap K_{j+1},m}\big(\phi_i \alpha_j\big) \\ &\leqslant \sum_{j=N+1}^{N+p} \epsilon_j p_{K_{j+1},j}\big(\phi_i \alpha_j\big) \\ &< \sum_{j=N+1}^{N+p} \frac{1}{2^j} = \frac{1}{2^N}\left(1 - \frac{1}{2^p}\right) < \frac{1}{2^N} \leqslant \frac{1}{2^{N_0}} < \epsilon. \end{aligned}$$

It follows that for each $i \in \mathbb{N}$ the sequence $(\phi'_{i(N)})_{N\in\mathbb{N}}$ is a Cauchy sequence in the locally convex vector space $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{K})$ hence converges to a smooth function $\phi'_i = \sum_{j=0}^{\infty} \epsilon_j \phi'_{i,j}$. Replacing in the above reasoning the function $\phi_i$ by the constant function 1 on $\Omega$ it follows that the sequence $(\alpha_{(N)})_{N\in\mathbb{N}}$ converges to a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$.

Now let $x \in \Omega$. Then there is a nonnegative integer $j_0$ such that $x \in K_{j_0}$. It follows from the nonnegativity of all the $\alpha_j$ that

$$\alpha(x) = \sum_{j=0}^{\infty} \epsilon_j \alpha_j(x) \geqslant \epsilon_{j_0} \alpha_{j_0}(x) = \epsilon_{j_0} > 0$$

showing that $\alpha$ takes strictly positive values on $\Omega$. Now let $x \notin \Omega$. Then for any

multi-index $\beta \in \mathbb{N}^n$ we have that

$$\forall\, N \in \mathbb{N}: \;\; (D^\beta \phi'_{i(N)})(x) = \sum_{j=0}^{N} \epsilon_j (D^\beta \phi'_{i,j})(x) = 0$$

because each $\phi'_{i,j}$ has compact support in $\Omega$.

Since $\phi'_{i(N)} \to \phi'_i$ for $N \to \infty$ it follows by the continuity of differential operators that $D^\beta \phi'_{i(N)} \to D^\beta \phi'_i$, and hence $D^\beta \phi'_{i(N)}(x) \to D^\beta \phi'_i(x)$ for all $x \in \mathbb{R}^n$ by the continuity of the delta functional $\delta_x$. It follows that

$$\forall\, x \in \mathbb{R}^n \backslash \Omega,\; \forall\, \beta \in \mathbb{N}^n: \;\; (D^\beta \phi'_i)(x) = 0,$$

and in a completely analogous manner

$$\forall\, x \in \mathbb{R}^n \backslash \Omega,\; \forall\, \beta \in \mathbb{N}^n: \;\; (D^\beta \alpha)(x) = 0,$$

which proves the Lemma. □

# C. Proof of some results

Contents

In this last part of the appendix we will give the proof of some important results that are more technical and flee a bit from the main object of the thesis.

## C.1 Results from the Part I

### Proposition 1.3.3

There is an adjunction of functors

$$K\mathbf{AlgMS} \underset{\mathcal{L}}{\overset{\mathcal{U}}{\rightleftarrows}} K\mathbf{Alg}$$

where $\mathcal{L}$ is the left adjoint to the above functor $\mathcal{U}$ such that each component $\eta_{(R,S)}$ of the unit $\eta : I_{K\mathbf{AlgMS}} \dot{\to} \mathcal{U}\mathcal{L}$ of the adjunction satisfies the universal property *a.* of the previous Proposition (1.3.1) in the general noncommutative case. We refer to $\mathcal{L}$ as a **localization functor**.
For a given $(R,S)$ in $K\mathbf{AlgMS}$ we denote by $R_S$ the $K$-algebra $\mathcal{L}(R,S)$ given by the functor $\mathcal{L}$, and by $\eta_{(R,S)} : R \to R_S$ the component of the unit of the adjunction. Then $\eta_{(R,U(R))} : R \to R_{U(R)}$ is an isomorphism, the inverse being the component $\epsilon_R$ of the counit $\epsilon : \mathcal{L}\mathcal{U} \dot{\to} I_{K\mathbf{Alg}}$ of the adjunction. Moreover, every element of the $K$-algebra $R_S$ is a finite sum of products of the form ($\eta = \eta_{(R,S)}$)

$$\eta(r_1)\big(\eta(s_1)\big)^{-1} \cdots \eta(r_N)\big(\eta(s_N)\big)^{-1} \tag{C.1.1}$$

(which may be called 'multifractions') with $r_1, \ldots, r_N \in R$ and $s_1, \ldots, s_N \in S$ (note that $r_1$ or $s_N$ may be equal to the unit of $R$).

*Proof.* Recall first the following functorial presentation of a unital $K$-algebra by 'generators and relations': for a given unital $K$-algebra $R$ (with unit $1_R$) there is a natural surjective algebra homomorphism $\hat{\epsilon}_R : T_K(R) \to R$ where $T_K(R)$ is the free associative unital $K$-algebra (or tensor algebra) generated by the $K$-module $R$. Note that the natural morphism $\hat{\epsilon}_R$ is just theh $R$-component of the counit $\hat{\epsilon}$ of the adjunction given by the functor $T_K$ from the category $K\mathbf{mod}$ of all $K$-modules to the category $K\mathbf{Alg}$ which is a left adjoint of the obvious forgetful functor $K\mathbf{Alg}$ to $K\mathbf{mod}$. The morphism $\hat{\epsilon}_R$ is determined by defining it to be the identity on the generating module $R$. The kernel $\kappa(R) \subset T_K(R)$ of $\hat{\epsilon}_R$ is a canonical 2-sided ideal in the free algebra $T_K(R)$ (containing for instance $r \otimes r' - rr'$, $r, r' \in R$, and $1_T - 1_R$) for which $T_K\Phi(\kappa(R)) \subset \kappa(R')$ for any morphism of unital $K$-algebras $\Phi : R \to R'$. Hence $R$ is canonically presented by the '$K$-module of generators $R$' and by the 'ideal of relations $\kappa(R)$'.

Next, for any object $(R,S)$ in $K\mathbf{AlgMS}$ let $KS$ denote the free (!) $K$-module having basis $S$, and consider the free $K$-algebra $T_K\big(R \oplus KS\big)$ generated by the $K$-module

$R \oplus KS$. The natural K-linear injection $i_R : R \to R \oplus KS$ given by $i_R(r) = (r, 0)$ defines a natural injection $T_K i_R : T_K R \to T_K\big(R \oplus KS\big)$. Let $\kappa(R,S)$ be the two-sided ideal in $T_K\big(R \oplus KS\big)$ generated by $T_K i_R\big(\kappa(R)\big)$ and by the subsets $\{(s,0) \otimes (0,s) - \mathbf{1}_T \mid s \in S\}$ and $\{(0,s) \otimes (s,0) - \mathbf{1}_T \mid s \in S\}$ of $T_K\big(R \oplus KS\big)$ where the multiplication $\otimes$ and the unit $\mathbf{1}_T$ are taken in the free algebra $T_K\big(R \oplus KS\big)$. Define the *localized K-algebra with respect to S*, $\mathcal{L}(R,S) =: R_S$, by the factor algebra $R_S := T_K\big(R \oplus KS\big)/\kappa(R,S)$. Since a morphism $\Phi : (R,S) \to (R',S')$ in $K$**AlgMS** clearly maps $R$ to $R'$ and $KS$ to $KS'$, the induced algebra morphism $T_K\big(R \oplus KS\big) \to T_K\big(R' \oplus KS'\big)$ maps $\kappa(R,S)$ to $\kappa(R',S')$, and induces hence a morphism $\mathcal{L}\Phi : R_S \to R'_{S'}$ of unital $K$-algebras. It is readily checked that $\mathcal{L}$ is a covariant functor $K$**AlgMS** $\to K$**Alg**. Denoting by $\pi_{(R,S)} : T_K\big(R \oplus KS\big) \to R_S$ the canonical projection we observe that –by construction– for every $s \in S$ the image $\pi_{(R,S)}(s,0) \in R_S$ has the inverse $\pi_{(R,S)}(0,s)$ and is thus an invertible element of $R_S$.

Furthermore, for any $(R,S)$ in $K$**AlgMS** there is a canonical map $\eta_{(R,S)} : R \to R_S$ determined by the diagram

$$\eta_{(R,S)} \circ \hat{\epsilon}_R = \pi_{(R,S)} \circ T_K i_R, \quad \text{hence } \forall\, r \in R : \ \eta_{(R,S)}(r) = \pi_{(R,S)}(r,0), \tag{C.1.2}$$

which is a well-defined morphism of $K$-algebras since the right hand side of this equation vanishes on the kernel $\kappa(R)$ of $\hat{\epsilon}_R$ thanks to $T_K i_R(\kappa(R)) \subset \kappa(R,S) = \ker(\pi_{(R,S)})$. It follows that for any $s \in S$ we have $\eta_{(R,S)}(s) = \pi_{(R,S)}(s,0)$ which is invertible in $R_S$, hence $\eta_{(R,S)}$ defines a morphism $(R,S) \to \big(R_S, U(R_S)\big) = \mathcal{UL}(R,S)$ in the category $K$**AlgMS**.

Moreover, for any $R$ in $K$**Alg** we consider the canonical $K$-linear map $j_R : R \oplus KU(R) \to R$ given by $j_R\big(r, \sum_{n=1}^N \lambda_n s_n\big) = r + \sum_{n=1}^N \lambda_n s_n^{-1}$ for any $r \in R$, $\lambda_1, \ldots, \lambda_n \in K$, and $s_1, \ldots, s_n \in U(R)$, and its induced morphism of $K$-algebras $T_K j_R : T_K(R \oplus KU(R)) \to T_K R$. Note that $j_R \circ i_R = \mathrm{id}_R$. There is a canonical $K$-linear map $\epsilon_R : R_{U(R)} \to R$ defined by the diagram

$$\epsilon_R \circ \pi_{(R,U(R))} = \hat{\epsilon}_R \circ T_K j_R.$$

This is a well-defined morphism of $K$-algebras since the right hand side $\hat{\epsilon}_R \circ T_K j_R$ vanishes on the generators of the ideal $\kappa(R,U(R))$: this is clear for $T_K i_R(\kappa(R))$, and $(s,0) \otimes (0,s) - 1_T$ is first mapped to $(s \otimes s^{-1}) - 1_T$ by $T_K j_R$, and then clearly annihilated by $\hat{\epsilon}_R$.

It is readily seen that the collection $\eta$ of all the maps $\eta_{(R,S)}$ defines a natural transformation $I_{K\mathbf{AlgMS}} \dot{\to} \mathcal{UL}$, and the collection $\epsilon$ of all the maps $\epsilon_R$ defines a natural transformation $\mathcal{LU} \dot{\to} I_{K\mathbf{Alg}}$. Moreover, the identity $j_R \circ i_R = \mathrm{id}_R$ immediately shows the identity

$$\epsilon_R \circ \eta_{(R,U(R))} = \mathrm{id}_R$$

by a combination of the above two diagrams. This implies the categorical equation $(\mathcal{U}\epsilon_R) \circ (\eta_{\mathcal{U}R}) = \mathrm{id}_{\mathcal{U}R}$ which is the first equation of eqn (8) in [32, p.82], and if $R$ is

replaced by some localized algebra $R_S$ w.r.t. some multiplicative $S \subset R$ in the above equation we get the categorical equation $\epsilon_{\mathcal{L}(R,S)} \circ \mathcal{L}\eta_{(R,S)} = \mathrm{id}_{\mathcal{L}(\mathcal{R},\mathcal{S})}$ which is the second equation of equation (8) of [32, p.82]. It follows now from [32, p.83, Thm 2.(v)] that $\mathcal{L}$ is a left adjoint of $\mathcal{U}$ with unit $\eta$ and counit $\epsilon$.

Finally, for any $r \in R$ and $s \in U(R)$ we have $(i_R \circ j_R)(r,s) = (r + s^{-1}, 0)$: since $\pi_{(R,S)}(0,s)$ is the inverse of $\pi_{(R,S)}(s,0)$ as is $\pi_{(R,S)}(s^{-1},0)$ we have $(0,s) - (s^{-1},0) \in \ker(\pi_{(R,S)}) = \kappa(R,S)$ showing that $\eta_{(R,U(R))} \circ \epsilon_R = \mathrm{id}_{R_{U(R)}}$ by a combination of the above first and second diagram. It follows in addition that $\epsilon_R$ is a natural isomorphism $R_{U(R)} \to R$ with inverse $\eta_{(R,U(R))}$. This proves the the first statement of the Proposition since units and counits of adjunctions are automatically universal. In order to prove formula (C.1.1) we observe that each element of $R_S$ is a finite sum of elements of images (under $\pi_{(R,S)}$) of words in $T_K(R \oplus KS)$ consisting of letters of the form $(r,0)$ or $(0,s)$ with $r \in R$ and $s \in S$. We clearly have (writing $\pi = \pi_{(R,S)}$) for all $r, r' \in R$: $\pi\big((r,0) \otimes (r',0)\big) = \pi(r,0)\pi(r',0) = \pi(rr',0)$ (since $T_K i_R(\kappa(R)) \subset \kappa(R,S)$). Moreover, for all $s, s' \in S$: $\pi\big((0,s) \otimes (0,s')\big) = \pi(0,s)\pi(0,s') = \big(\pi(s,0)\big)^{-1}\big(\pi(s,0)\big)^{-1} = \big(\pi(s',0)\pi(s,0)\big)^{-1} = (\pi(s's,0))^{-1} = \pi(0,s's)$, which shows that it suffices to take words where the generators $(r,0)$ and $(0,s)$ are alternating. Now $\pi(r,0) = \eta(r)$ and $\pi(0,s) = \big(\eta(s)\big)^{-1}$ which proves formula (C.1.1). □

### Theorem 1.3.8

Let $R$ be a unital $K$-algebra and $S \subset R$ be a multiplicative subset. Then the following is true:

1. The $K$-algebra $R$ has a right $K$-algebra of fractions $\check{R}_S$ with respect to the multiplicative subset $S$ if and only if $S$ is a right denominator set.
2. If this is the case each such pair $(\check{R}_S, \check{\eta})$ is universal in the sense of diagram (1.3.2) and each $\check{R}_S$ is isomorphic to the canonical localized algebra $R_S$ of Proposition 1.3.3.
3. Each $\check{R}_S$ is isomorphic to the quotient set $RS^{-1} := (R \times S)/\sim$ with respect to the following binary relation $\sim$ on $R \times S$

$$(r_1,s_1) \sim (r_2,s_2) \;\Leftrightarrow\; \exists b_1, b_2 \in R \text{ such that } s_1 b_1 = s_2 b_2 \in S \text{ and } r_1 b_1 = r_2 b_2 \in R \tag{C.1.3}$$

which is an equivalence relation generalizing relation (1.2.1). Moreover, $RS^{-1}$ carries a canonical unital $K$-algebra structure, i.e. addition and multiplication on equivalence classes $r_1 s_1^{-1}$ and $r_2 s_2^{-1}$ (with $r_1, r_2 \in R$ and $s_1, s_2 \in S$) is given by

$$r_1 s_1^{-1} + r_2 s_2^{-1} = (r_1 c_1 + r_2 c_2) s^{-1}, \quad \text{and} \quad (r_1 s_1^{-1})(r_2 s_2^{-1}) = (r_1 r')(s_2 s')^{-1} \tag{C.1.4}$$

where we have written $s_1c_1 = s_2c_2 = s \in S$ (with $c_1 \in S$ and $c_2 \in R$) and $r_2s' = s_1r'$ (with $s' \in S$ and $r' \in R$) using the right Ore property.The numerator morphism $\eta_I : R \to RS^{-1}$ is given by $\eta_I(r) = r1^{-1}$ for all $r \in R$.

*Proof.* We shall write **RP** for the property 'right permutable' $(ii.a)$ and **RR** for the property 'right reversible' $(ii.b)$:

**1.** "$(i.) \implies (ii.)$".

Indeed, in order to see Property $(a.)$ of Definition 1.3.4, given $r \in R$ and $s \in S$ we can write the element $\big(\tilde{\eta}(s)\big)^{-1}\tilde{\eta}(r)$ in the form $\tilde{\eta}(r'')\big(\tilde{\eta}(s'')\big)^{-1}$ for some $r'' \in R$ and $s'' \in S$ according to Property $(i.b)$ of Definition 1.3.4. So we have $\tilde{\eta}(r)\tilde{\eta}(s'') = \tilde{\eta}(s)\tilde{\eta}(r'')$ implying $\tilde{\eta}(rs'' - sr'') = 0$, and therefore –according to Property $(c.)$ of Definition 1.3.4– it follows that there is $s''' \in S$ such that $(rs'' - sr'')s''' = 0$ whence $r(s''s''') = s(r''s''')$ which shows $(ii.a)$.

In order to see Property $(b.)$ of Definition 1.3.4, given $r \in R$ and $s' \in S$ with $s'r = 0$, then $0 = \tilde{\eta}(s')\tilde{\eta}(r)$, hence $0 = \tilde{\eta}(r)$ since $\tilde{\eta}(s')$ is invertible. By Property $(i.c)$ of Definition 1.3.4 there is $s \in S$ with $rs = 0$, proving $(ii.b)$. ∇

By the preceding implication we have seen that the fact that $S$ is a right denominator set is necessary for the two other statements in the Theorem. Before we prove the converse implication "$(i.) \impliedby (ii.)$" of part **1.** and the two other parts we shall first look at general $S$-inverting morphisms $R \to R'$ and come to an embedding statement from which the rest of the Theorem will easily follow. We shall proceed in several steps:

We suppose that $S$ is a right denominator set.

**I. Suppose that $R'$ is another unital $K$-algebra and $\alpha : R \to R'$ is an $S$-inverting morphism of unital $K$-algebras**. We shall denote by $J_\alpha = J$ the kernel of $\alpha$, a two-sided ideal of $R$. Clearly, if $r \in I_{(R,S)}$ then there is $s \in S$ such that $rs = 0$, hence $0 = \alpha(r)\alpha(s)$, hence $0 = \alpha(r)$ whence we always have the inclusion

$$I_{(R,S)} \subset J_\alpha. \tag{C.1.5}$$

Consider the map

$$p_\alpha : R \times S \to R' : (r,s) \mapsto \alpha(r)\big(\alpha(s)\big)^{-1}.$$

**I.1** *The equivalence relation $\sim_J$ on $R \times S$ given by $(r_1,s_1) \sim_J (r_2,s_2)$ iff $p_\alpha(r_1,s_1) = p_\alpha(r_2,s_2)$ only depends on the two sided ideal $J_\alpha \subset R$ and is equivalent to*

$$(r_1,s_1) \sim_J (r_2,s_2) \iff \exists\, c_1,c_2 \in R :\ s_1c_1 = s_2c_2 \in S \text{ and } r_1c_1 - r_2c_2 \in J_\alpha. \tag{C.1.6}$$

*We shall denote the equivalence class of $(r,s) \in R \times S$ by the 'right $J_\alpha$-fraction $(rs^{-1})_J$'.*

Indeed, $\sim_J$ is an equivalence relation by definition whose classes are the fibres of $p_\alpha$. Suppose first that $(r_1,s_1) \sim_J (r_2,s_2)$. Then $\alpha(r_1) = \alpha(r_2)\big(\alpha(s_2)\big)^{-1}\alpha(s_1)$. An

application of **RP** on the pair $(s_1, s_2) \in R \times S$ yields the existence of a pair $(r', s') \in R \times S$ with $s_1 s' = s_2 r'$ which is in $S$ since $s_1 s'$ is. Applying $\alpha$ gives $\big(\alpha(s_2)\big)^{-1}\alpha(s_1) = \alpha(r')\big(\alpha(s')\big)^{-1}$, and therefore

$$\alpha(r_1 s') = \alpha(r_2 r') \implies s_1 s' = s_2 r' \in S \text{ and } r_1 s' - r_2 r' \in J_\alpha$$

implying the existence of $c_1 = s'$ and $c_2 = r'$ such that the condition of the r.h.s. of eqn C.1.6 is satisfied. In order to prove the other implication note first the following important property of 'multiplying numerator and denominator by the same element from the right': let $(r, s) \in R \times S$ and suppose that there is $c \in R$ such that $sc \in S$. It follows that $\alpha(sc)$ is invertible in $R'$ whence

$$\alpha(r)\big(\alpha(s)\big)^{-1}\alpha(sc) = \alpha(r)\big(\alpha(s)\big)^{-1}\alpha(s)\alpha(c) = \alpha(rc) \implies \alpha(r)\big(\alpha(s)\big)^{-1} = \alpha(rc)\big(\alpha(sc)\big)^{-1}. \tag{C.1.7}$$

Suppose now that there are $c_1, c_2 \in R$ such that the r.h.s. of eqn C.1.6 holds. By the preceding equation (C.1.7) we get

$$\alpha(r_1)\big(\alpha(s_1)\big)^{-1} = \alpha(r_1 c_1)\big(\alpha(s_1 c_1)\big)^{-1} = \alpha(r_2 c_2)\big(\alpha(s_2 c_2)\big)^{-1} = \alpha(r_2)\big(\alpha(s_2)\big)^{-1}$$

implying that $(r_1, s_1) \sim_J (r_2, s_2)$. Thanks to eqn (C.1.6) the relation $\sim_J$ only depends on $J_\alpha$. ∇

The following reasoning will be used quite often:

**I.1a** *Let $\zeta : R \to R_1$ be another $S$ inverting morphism of unital $K$-algebras such that each element of $R_1$ is equal to a right fraction $\zeta(r)\big(\zeta(s)\big)^{-1}$. If $f, g : R_1 \to R'$ are morphisms of unital $K$-algebras such that $\alpha = f \circ \zeta$ and $\alpha = g \circ \zeta$ then $f = g$.*

Indeed we get for any $(r, s) \in R \times S$ the following equation proving $f = g$:

$$f\Big(\zeta(r)\big(\zeta(s)\big)^{-1}\Big) = (f \circ \zeta)(r)\big((f \circ \zeta)(s)\big)^{-1} =$$

$$= \alpha(r)\big(\alpha(s)\big)^{-1} = (g \circ \zeta)(r)\big((g \circ \zeta)(s)\big)^{-1} = g\Big(\zeta(r)\big(\zeta(s)\big)^{-1}\Big)$$

**I.2** *The quotient set $RS_J^{-1} := (R \times S)/\sim_J$ carries a unique structure of a unital $K$-algebra (only depending on $J_\alpha$) isomorphic to the subalgebra $\alpha(R)\alpha(S)^{-1} := \{\alpha(r)\alpha(s)^{-1} \mid (r, s) \in R \times S\}$ of $R'$ by the injective morphism of unital $K$-algebras $f_\alpha : RS_J^{-1} \to R'$ induced by the map $p_\alpha$. Moreover, there is an $S$-inverting morphism $\eta_J : R \to RS_J^{-1}$ of unital $K$-algebras –only depending on $J_\alpha$– such that $\alpha = f_\alpha \circ \eta_J$. The morphism $\eta_J$ satisfies properties (i.a.) and (i.b.) of Definition 1.3.4 having kernel* $\ker(\eta_J) = J_\alpha$.
*Any other morphism of unital $K$-algebras $\phi : RS_J^{-1} \to R'$ satisfying $\alpha = \phi \circ \eta_J$ is equal to $f_\alpha$.*

Indeed, the map $p_\alpha$ descends to a set-theoretical injection $f_\alpha$ of the quotient set $RS_J^{-1}$ onto the subset $\alpha(R)\alpha(S)^{-1}$ of $R'$ by definition of the equivalence relation $\sim_J$.

Let $(r_1,s_1)$ and $(r_2,s_2)$ be two elements in $R\times S$. Applying **RP** on the pair $(s_1,s_2)$ we get $r'\in R$ and $s'\in S$ such that $s_1s'=s_2r'=:s\in S$, and the application of **RP** on the pair $(r_2,s_1)$ gives us $(r'',s'')\in R\times S$ such that $r_2s''=s_1r''$. We compute

$$\begin{aligned} f_\alpha\big((r_1s_1^{-1})_J\big)+f_\alpha\big((r_2s_2^{-1})_J\big) &= p_\alpha(r_1,s_1)+p_\alpha(r_2,s_2)=\alpha(r_1)\big(\alpha(s_1)\big)^{-1}+\alpha(r_2)\big(\alpha(s_2)\big)^{-1}\\ &= \alpha(r_1s'+r_2r')\big(\alpha(s)\big)^{-1}=f_\alpha\Big(\big((r_1s'+r_2r')s^{-1}\big)_J\Big), \end{aligned}$$

showing that the subset $\alpha(R)\alpha(S)^{-1}$ of $R'$ is closed under addition, and using the inverse of the corestriction of $f_\alpha$ to the r.h.s. of the above equation we see that the formula

$$(r_1s_1^{-1})_J+(r_2s_2^{-1})_J:=\big((r_1s'+r_2r')s^{-1}\big)_J \tag{C.1.8}$$

equips $RS_J^{-1}$ with a well-defined addition (only depending on $J_\alpha$) such that $f_\alpha: RS_J^{-1}\to R'$ is an additive map. Similarly, replacing addition in $R'$ by multiplication in the above reasoning we show that the subset $\alpha(R)\alpha(S)^{-1}$ of $R'$ is a unital $K$-subalgebra, and that the following multiplication

$$(r_1s_1^{-1})_J(r_2s_2^{-1})_J:=\big((r_1r'')(s_2s'')^{-1}\big)_J \tag{C.1.9}$$

(which only depends on $J_\alpha$) together with the above addition (C.1.8) equips $RS_J^{-1}$ with the structure of a unital $K$-algebra (with zero element $(01^{-1})_J$ and unit element $(11^{-1})_J$) isomorphic to the subalgebra $\alpha(R)\alpha(S)^{-1}$ via $f_\alpha$ which turns out to be a morphism of unital $K$-algebras $RS_J^{-1}\to R'$. Moreover, since $f_\alpha\big((s1^{-1})_J\big)=\alpha(s)$ and $f_\alpha\big((1s^{-1})_J\big)=\big(\alpha(s)\big)^{-1}$ we can infer from the injectivity of $f_\alpha$ that $(1s^{-1})_J\in RS_S^{-1}$ is the inverse of $(s1^{-1})_J$ for each $s\in S$. Note that addition (C.1.8) and multiplication (C.1.9) are already very similar to the –still to be proved– formulas (**??**) in statement **3.** of the Theorem.

The map $\eta_J$ is defined by $\eta_J(r):=(r1^{-1})_J$ for all $r\in R$. It follows that $f_\alpha\big(\eta_J(r)\big)=p_\alpha(r,1)=\alpha(r)$. Since $f_\alpha$ is an injective morphism of unital $K$-algebras it follows that $\eta_J$ is a morphism of unital $K$-algebras whose kernel is equal to to $J_\alpha$, the kernel of $\alpha$. Since the inverse of $\eta_J(s)=(s1^{-1})_J$ has been computed to be $(1s^{-1})_J$ it follows that $\eta_J$ is $S$-inverting (property $(i.a)$ of Definition **??** for the pair $\big(RS_J^{-1},\eta_J\big)$). Moreover, observing that in formula (C.1.9) for the particular case $s_1=1$ we can choose $s''=1$ and $r''=r_2$ getting

$$(r_11^{-1})_J(r_2s_2^{-1})_J=\big((r_1r_2)s_2^{-1}\big)_J \implies (rs^{-1})_J=(r1^{-1})_J(1s^{-1})_J=\eta_J(r)\big(\eta_J(s)\big)^{-1},$$

proving property $(i.b)$ of Definition 1.3.4 for the pair $\big(RS_J^{-1},\eta_J\big)$. The final statement follows from **I.1a**. ∇

**I.3.** *Let $J_1\subset J_2\subset R$ be two two-sided ideals of $R$ which both are kernels of $S$-inverting*

*unital K-algebra morphisms. Then there is a unique morphism* $\theta_{21} : RS_{J_1}^{-1} \to RS_{J_2}^{-1}$ *of unital K-algebras such that* $\eta_{J_2} = \theta_{21} \circ \eta_{J_1}$.
Indeed, since $J_1 \subset J_2$ for any two pairs $(r_1, s_1), (r_2, s_2) \in R \times S$ we have the implication $(r_1, s_1) \sim_{J_1} (r_2, s_2) \Longrightarrow (r_1, s_1) \sim_{J_2} (r_2, s_2)$. This gives a canonical map $\theta_{21} : (RS^{-1})_{J_1} \to (RS^{-1})_{J_2}$ on quotient sets sending the class $(rs^{-1})_{J_1}$ to the 'bigger' class $(rs^{-1})_{J_2}$ clearly satisfying $\eta_{J_2} = \theta_{21} \circ \eta_{J_1}$. Upon using formulas (C.1.8) and (C.1.9) it is easy to see that $\theta_{21}$ is a morphism of unital $K$-algebras which is unique by **I.1a**.

$\nabla$

**II.** The preceding considerations in **I.** work well for those two-sided ideals in $R$ which are kernels of $S$-inverting morphisms. According to eqn (C.1.5) those ideals all contain $I_{(R,S)}$ of which we shall now describe the structure.
**II.1.** *$I_{(R,S)}$ is a two-sided ideal of the K-algebra R which is contained in the kernel $I_\alpha$ of any S-inverting morphism $\alpha : R \to R'$* (see also [29, Exercise 0., p.317]):

Indeed, $I$ is obviously closed by left multiplications with any element of $R$. Let $r_1, r_2 \in I$, then there are $s_1, s_2 \in S$ with $r_1 s_1 = 0$ and $r_2 s_2 = 0$. Clearly $(r_1 + r_2)s_1 = r_2 s_1$. Then **RP** on $(s_1, s_2)$ yields the existence of $(r', s') \in R \times S$ with $s_1 s' = s_2 r'$, whence $(r_1 + r_2)s_1 s' = r_2 s_1 s' = r_2 s_2 r' = 0$, hence $r_1 + r_2 \in I$. Thirdly, let $r \in I$ and $\hat{r} \in R$. There is $s \in S$ with $rs = 0$. **RP** on the pair $(\hat{r}, s)$ yields the pair $(r'', s'') \in R \times S$ such that $\hat{r}s'' = sr''$. It follows that $r\hat{r}s'' = rsr'' = 0$ whence $r\hat{r} \in I$ proving that $I$ is also a right ideal.

$\nabla$

We shall denote by $\overline{R}$ the factor algebra $R/I_{(R,S)}$, by $\pi : R \to \overline{R}$ the canonical projection, and by $\overline{S} = \pi(S)$.
**II.2.** *The subset $\overline{S}$ of the factor algebra $\overline{R}$ is a multiplicative subset satisfying **RP** and does not contain right or left divisors of zero which implies **RR*** (see also [29, Exercise 1., p.317]):

Indeed, since $\pi$ is a surjective morphism of unital $K$-algebras it is immediate that the subset $\overline{S}$ is multiplicative and satisfies **RP**. Next, if there are $r \in R$ and $s \in S$ with $0 = \pi(r)\pi(s) = \pi(rs)$ then $rs \in I$. Hence there is $\hat{s} \in S$ with $0 = rs\hat{s}$, whence $r \in I$ and $\pi(r) = 0$. On the other hand if there are $r \in R$ and $s' \in S$ with $0 = \pi(s')\pi(r) = \pi(s'r)$ then $s'r \in I$ whence there is $s'' \in S$ with $s'rs'' = 0$, and according to **RR** there is $s''' \in S$ such that $0 = rs''s'''$, and it follows that $r \in I$, hence $\pi(r) = 0$.

$\nabla$

**III.** In this principal part of the proof we shall show the existence of a unital $K$-algebra $R'$ together with an $\overline{S}$-inverting injection $\overline{R} \to R'$. This will imply the key result that the important two-sided ideal $I_{(R,S)}$ is also the kernel of an $S$-inverting morphism $R \to R'$.
$\overline{R}$ is a right $\overline{R}$-module by means of its multiplication. Let $E$ be a right $\overline{R}$-module which is an *injective hull* of the right $\overline{R}$-module $\overline{R}$: recall that this means that $\overline{R}$ is a right $\overline{R}$-submodule of $E$, that $E$ is *injective* in the sense that for each right $\overline{R}$-module

$M$ and $\overline{R}$-submodule $M' \subset M$ each morphism of right $\overline{R}$-modules $\zeta : M' \to E$ can –in general not uniquely– be extended to a morphism $\tilde{\zeta} : M \to E$ of right $\overline{R}$-modules, and that $\overline{R} \subset E$ is a *large submodule* in the sense that any submodule $N$ of $E$ such that $N \cap \overline{R} = \{0\}$ has to be trivial, $N = \{0\}$. For each $a \in \overline{R}$ let $l_a : \overline{R} \to \overline{R}$ denote the left multiplication $l_a(a') = aa'$ (for all $a' \in \overline{R}$) which obviously is a morphism of right $\overline{R}$-modules. Since $E$ is injective every $l_a : \overline{R} \to \overline{R} \subset E$ has at least one –in general nonunique– extension $\tilde{l}_a : E \to E$ as a morphism of right $\overline{R}$-modules. Then the following holds:

**III.1.** $\{e \in E \mid \exists\ t \in \overline{S} :\ et = 0\} = \{0\}$. Indeed, let $N$ denote the left hand side of this equation, and let $e_1, e_2 \in N$. There are $t_1, t_2 \in \overline{S}$ such that $e_1 t_1 = 0$ and $e_2 t_2 = 0$. Clearly $(e_1 + e_2)t_1 = e_2 t_1$. Then **RP** on $(t_1, t_2)$ yields the existence of $(a', t') \in \overline{R} \times \overline{S}$ with $t_1 t' = t_2 a'$, whence $(e_1 + e_2)t_1 t' = e_2 t_1 t' = e_2 t_2 a' = 0$, hence $e_1 + e_2 \in N$. Let $e \in N$ and $a \in \overline{R}$. Then there is $t \in \overline{S}$ with $et = 0$. An application of **RP** to the pair $(a, t)$ yields $(a'', t'') \in \overline{R} \times \overline{S}$ with $at'' = ta''$ whence $eat'' = eta'' = 0$ which shows $ea \in N$, and $N$ is a right $\overline{R}$-submodule of $E$. Since $\overline{S}$ does not contain any right divisor of zero, $N \cap \overline{R} = \{0\}$, hence $N = \{0\}$ because $\overline{R}$ is a large submodule of $E$.

$\nabla$

**III.2.** *For each $t \in \overline{S}$: every extension $\tilde{l}_t : E \to E$ of $l_t$ is an invertible morphism of right $\overline{R}$-modules.* Indeed, let $K_t$ denote the kernel of $\tilde{l}_t$. It clearly is a right $\overline{R}$-submodule of $E$, and $K_t \cap \overline{R} = \{a \in \overline{R} \mid 0 = \tilde{l}_t(a) = ta\}$, but the latter vanishes according to **II.2.**. Since $\overline{R}$ is large, $K_t$ vanishes, and $\tilde{l}_t$ is injective. Let $M_t \subset E$ be the image of $\tilde{l}_t$. Then the corestriction of $\tilde{l}_t$ to $M_t$ has an inverse $\chi_t : M_t \to E$ which is a surjective morphism of right $\overline{R}$-modules. Since $E$ is injective, there is an extension $\tilde{\chi}_t : E \to E$ of $\chi_t$ as a surjective morphism of right $\overline{R}$-modules. It follows that $\tilde{\chi}_t \circ \tilde{l}_t = \mathrm{id}_E$ whence the morphism $P_t = \tilde{l}_t \circ \tilde{\chi}_t$ is a projection, i.e. $P_t \circ P_t = P_t$, onto $M_t$. This also follows from injectivity, see [29, Prop.(3.4)(2), p.61]. Let $Q_t$ be the kernel of $P_t$. Then there is the direct sum of right $\overline{R}$-modules $E = M_t \oplus Q_t$. Let $a \in Q_t \cap \overline{R}$. Applying **RP** to the pair $(a, t)$ we get a pair $(a', t') \in \overline{R} \times \overline{S}$ such that $at' = ta'$. Since $Q_t \cap \overline{R}$ is a right $\overline{R}$-submodule of $E$ it follows that $at' \in Q_t \cap \overline{R}$, but $at' = ta' = l_t(a') \in M_t$, whence $at' \in \overline{R} \cap Q_t \cap M_t = \{0\}$ thanks to the above direct sum. Hence $at' = 0$, and by **II.2.** it follows $a = 0$ whence $Q_t \cap \overline{R} = \{0\}$, and finally $Q_t = \{0\}$ because $\overline{R}$ was a large submodule of $E$. Hence $M_t = E$, hence $\tilde{l}_t$ is surjective and therefore invertible.

$\nabla$

The remaining problem in the preceding paragraph is the possible **nonuniqueness of the extensions of left multiplications of $\overline{R}$ to $E$**: two such extensions may a priori differ by a morphism of right $\overline{R}$-modules $E \to E$ vanishing on $\overline{R}$. In order to cope with that we consider the following subsets of the $K$-algebra $\mathfrak{H} := \mathrm{Hom}_{\overline{R}}(E, E)$ of all right $\overline{R}$-module morphisms $E \to E$:

$$\begin{aligned} \mathfrak{B} &:= \{\varphi \in \mathfrak{H} \mid \exists\ a \in \overline{R} : \varphi(a') = aa' = l_a(a')\ \forall\ a' \in \overline{R}\}, \\ \mathfrak{I} &:= \{\varphi \in \mathfrak{H} \mid \varphi(a') = 0\ \forall\ a' \in \overline{R}\}, \end{aligned}$$

$\mathfrak{K}$ := unital $K-$subalgebra of $\mathfrak{H}$ generated by $\mathfrak{B}$ and by $\{\varphi \in \mathfrak{H} \mid \exists\, t \in \overline{R}:\ \varphi = \tilde{l}_t^{-1}\}$.

By definition, $\mathfrak{B}$ is a $K$-subalgebra of $\mathfrak{H}$ consisting of all the extensions of left multiplications, and $\mathfrak{I}$ –which obviously is a left ideal in $\mathfrak{H}$– is contained in $\mathfrak{B}$ and hence in $\mathfrak{K}$. $\mathfrak{I}$ 'measures' the nonuniqueness of the extensions. We can now show the desired embedding:

**III.3.** *$\mathfrak{I}$ is a two-sided ideal of $\mathfrak{K}$, and the prescription $\beta : \overline{R} \to R' := \mathfrak{K}/\mathfrak{I}$ defined by $a \mapsto \tilde{l}_a$ mod $\mathfrak{I}$ is a well-defined injective $\overline{S}$-inverting morphism of unital $K$-algebras.*

Indeed, it remains to show that $\mathfrak{I}$ is a right ideal in $\mathfrak{K}$: let $\psi \in \mathfrak{I}$, $\varphi \in \mathfrak{B}$ (restricting to $l_{\hat{a}}$ for some $\hat{a} \in \overline{R}$ on the submodule $\overline{R}$ of $E$), $t \in \overline{S}$, and $a \in \mathfrak{K}$. Then $\psi\big(\varphi(a)\big) = \psi(\hat{a}a) = 0$, whence $\psi \circ \varphi \in \mathfrak{I}$. Moreover apply **RP** to the pair $(a,t)$ in order to get a pair $(a',t') \in \overline{R} \times \overline{S}$ with $at' = ta'$. Then

$$\Big(\psi\big(\tilde{l}_t^{-1}(a)\big)\Big)t' = \psi\big(\tilde{l}_t^{-1}(at')\big) = \psi\big(\tilde{l}_t^{-1}(ta')\big) = \psi\Big(\tilde{l}_t^{-1}\big(\tilde{l}_t(a')\big)\Big) = \psi(a') = 0$$

which implies –according to **III.1.**– that $\big(\psi \circ \tilde{l}_t^{-1}\big)(a) = 0$, and since the above reasoning works for any $a \in \overline{R}$ we have $\psi \circ \tilde{l}_t^{-1} \in \mathfrak{I}$ for all $t \in \overline{R}$. Since $\mathfrak{K}$ is generated by $\varphi \in \mathfrak{B}$ and all $\tilde{l}_t^{-1}$ it follows that $\mathfrak{I}$ is a two-sided ideal in $\mathfrak{K}$, and that hence the factor algebra $R'$ is well-defined. Let $\varpi : \mathfrak{K} \to R'$ the canonical projection. Since two extensions $\tilde{l}_a$ and $\tilde{l}'_a$ of the left multiplication $l_a$, $a \in \overline{R}$, differ by an element of $\mathfrak{I}$ it follows that the map $\beta : \overline{R} \to R'$ is well defined satisfying $\beta(a) = \varpi(\tilde{l}_a)$. For any $a_1, a_2 \in \overline{R}$ we have

$$\beta(a_1 + a_2) = \varpi\big(\tilde{l}_{a_1+a_2}\big) = \varpi\big(\tilde{l}_{a_1}\big) + \varpi\big(\tilde{l}_{a_2}\big) + \varpi\big(\tilde{l}_{a_1+a_2} - \tilde{l}_{a_1} - \tilde{l}_{a_2}\big) = \beta(a_1) + \beta(a_2) + 0$$

since the restriction of $\tilde{l}_{a_1+a_2} - \tilde{l}_{a_1} - \tilde{l}_{a_2}$ to $\overline{R}$ clearly vanishes whence this map belongs to $\mathfrak{I}$. Likewise it is shown that $\beta(a_1a_2) = \beta(a_1)\beta(a_2)$ upon noting that $\tilde{l}_{a_1a_2} - \tilde{l}_{a_1} \circ \tilde{l}_{a_2}$ belongs to $\mathfrak{I}$. It follows that $\beta$ is a morphism of unital $K$-algebras. Let $\check{a} \in \ker(\beta)$. Then $\tilde{l}_{\check{a}} \in \mathfrak{I}$ whence in particular $0 = \tilde{l}_{\check{a}}(1_{\overline{R}}) = \check{a}$ showing that $\beta$ is injective. Finally, let $t \in \overline{S}$ and choose an extension $\tilde{l}_t$. According to **III.2.** there is an inverse $\tilde{l}_t^{-1}$ of the map $\tilde{l}_t$ which by construction is in $\mathfrak{K}$. If $\tau := \varpi\big(\tilde{l}_t^{-1}\big)$ then

$$\beta(t)\tau = \varpi\big(\tilde{l}_t\big)\varpi\big(\tilde{l}_t^{-1}\big) = \varpi\big(\tilde{l}_t \circ \tilde{l}_t^{-1}\big) = \varpi\big(\mathrm{id}_E\big) = 1_{R'}$$

and likewise $\tau\beta(t) = 1_{R'}$ showing that each $\beta(t)$ is invertible in R'. ∇

**III.4.** *There is an $S$-inverting morphism of unital $K$-algebras $\alpha : R \to R'$ whose kernel equals $I_{(R,S)}$. It follows that the kernel of $\eta_I$ is equal to $I = I_{(R,S)}$.*

Indeed, the composition $\alpha = \beta \circ \pi$ does the job since the kernel of $\pi$ equals $I_{(R,S)}$ and $\beta$ is injective. ∇

We can now define the following: according to **III.4** the unital $K$-algebra $(RS^{-1})_I$ together with the $S$-inverting morphism $\eta_I : R \to (RS^{-1})_I$ having kernel $I = I_{(R,S)}$

is well-defined. We shall call it $RS^{-1}$. It clearly satisfies all three properties $(i.a.)$, $(i.b.)$, and $(i.c.)$ of Definition 1.3.4 according to **I.2** and **III.4** whence it constitutes (together with $\eta_I$) a right $K$-algebra of fractions which proves the more difficult implication "$(i.) \Longleftarrow (ii.)$" in Definition 1.3.4, and therefore part **1** of the Theorem is proved.

The relation $\sim$ (see eqn (1.3.4)) in part **3** of the Theorem is equal to $\sim_I$: indeed, if two pairs $(r_1, s_1)$ and $(r_2, s_2)$ are equivalent w.r.t. $\sim_I$ then there are $c_1, c_2 \in R$ with $s_1c_1 = s_2c_2 \in S$ and $r_1c_1 - r_2c_2 \in I$, hence there is $s \in S$ with $r_1c_1s = r_2c_2s$, and of course $s_1c_1s = s_2c_2s \in S$ whence they are equivalent w.r.t. $\sim$ (upon setting $b_1 = c_1s, b_2 = c_2s$). The other implication is trivial since $0 \in I$. This proves part **3.** of the Theorem except for the universality statement. We shall abbreviate the equivalence classes by $(rs^{-1})_I$ by $rs^{-1}$.

Let $\alpha : R \to R'$ any $S$-inverting morphism of unital $K$-algebras with kernel $J_\alpha = J$. By **I.2** $\alpha$ factorizes as $\alpha = f_\alpha \circ \eta_J$ (with unique morphism $f_\alpha : RS_J^{-1} \to R'$), and thanks to **I.3** (setting $J_1 = I$ and $J_2 = J$) there is a unique morphism of unital $K$-algebras $\theta_{21} : RS^{-1} \to RS_J^{-1}$ with $\eta_J = \theta_{21} \circ \eta_I$ whence $\alpha = (f_\alpha \circ \theta_{21}) \circ \eta_I$. The uniqueness of composition $\phi := (f_\alpha \circ \theta_{21})$ in the preceding equation follows from **I.1a**. Hence the pair $(RS^{-1}, \eta_I)$ is universal in the sense of diagram (1.2.6) proving **3.** for $RS^{-1}$.

Finally, let $(\check{R}_S, \check{\eta})$ be a right $K$-algebra of fractions of $R$. By hypothesis $\check{\eta}$ is $S$-inverting. Thanks to **I.2** there is a unique morphism of unital $K$-algebras $f_{\check{\eta}} : RS^{-1} \to \check{R}_S$ satisfying $\check{\eta} = f_{\check{\eta}} \circ \eta_I$. Since by hypothesis $\check{R}_S$ is equal to its subalgebra $\check{\eta}(R)\big(\check{\eta}(S)\big)^{-1}$ it follows that $f_{\check{\eta}}$ is surjective, and the fact that the kernel of both $\check{\eta}$ and $\eta_I$ is equal to $I$ shows that $f_{\check{\eta}}$ is an isomorphism, thus proving universality of the pair $(\check{R}_S, \check{\eta})$ and therefore the first statement in part **2.** of the Theorem.
Now, both pairs $(RS^{-1}, \eta_I)$ and $(R_S, \eta)$ are universal with respect to diagram (1.2.6) by the preceding reasoning and thanks to Proposition 1.3.3. It follows that there are two unique morphisms of unital $K$-algebra, $\phi : RS^{-1} \to R_S$ and $\psi : R_S \to RS^{-1}$ satisfying $\eta = \phi \circ \eta_I$ and $\eta_I = \psi \circ \eta$. Hence the morphism of unital $K$-algebras $\phi \circ \psi : R_S \to R_S$ is the identity on the image of $\eta$, and using the fact that each element of $R_S$ is a sum of multifractions (see (C.1.1)) it follows that $\phi \circ \psi = \mathrm{id}_{R_S}$. Likewise the morphism of unital $K$-algebras $\psi \circ \phi : RS^{-1} \to RS^{-1}$ is the identity on the image of $\eta_I$, and since every element of $RS^{-1}$ is a right fraction it follows that $\psi \circ \phi = \mathrm{id}_{RS^{-1}}$ proving that $RS^{-1}$ is isomorphic to $R_S$ which finishes the proof of the Theorem. □

## C.2 Results from the Part II

### Proof of Proposition 5.1.11

*Proof.* Eqn (5.1.20) is proved by induction on the tensorial degree of $b$, the case of degree 0 being obvious, and the case of degree 1 follows from the fact that $\nu$ intertwines the connection on $X$ and the connection $\nabla^{V\otimes_A W}$ on $V\otimes_A W$, and the property (5.1.6) of the latter. Let $b\in \mathsf{T}_A(L)$ have tensor degree less or equal than $n$ and $x\in L$, then for all $v\in V$ and $w\in W$ we get –since $\nu$ and the shuffle comultiplication in $\mathsf{T}_A(L)$ are connection preserving–

$$
\begin{aligned}
\nabla^X_{xb}\big(\nu(v\otimes_A w)\big) &= \nabla^X_x\Big(\nabla^X_b\big(\nu(v\otimes_A w)\big)\Big) - \nabla^X_{\nabla_x(b)}\big(\nu(v\otimes_A w)\big)\\
&\overset{\text{Ind.hyp.}}{=} \sum_{(b)}\nabla^X_x\Big(\nu\big(\nabla^V_{b^{(1)}}(v)\otimes_A \nabla^W_{b^{(2)}}(w)\big)\Big)\\
&\quad - \sum_{(\nabla_x(b))}\nu\big(\nabla^V_{(\nabla_x(b))^{(1)}}(v)\otimes_A \nabla^W_{(\nabla_x(b))^{(2)}}(w)\big)\\
&\overset{n=1,(5.1.6)}{=} \sum_{(b)}\nu\Big(\big(\nabla^V_x(\nabla^V_{b^{(1)}}(v))\big)\otimes_A\big(\nabla^W_{b^{(2)}}(w)\big)\Big)\\
&\quad + \sum_{(b)}\nu\Big(\big(\nabla^V_{b^{(1)}}(v)\big)\otimes_A\big(\nabla^W_x(\nabla^W_{b^{(2)}}(w))\big)\Big)\\
&\qquad -\sum_{(b)}\nu\big(\nabla^V_{\nabla_x(b^{(1)})}(v)\otimes_A\nabla^W_{(b)^{(2)}}(w)\big) - \sum_{(b)}\nu\big(\nabla^V_{b^{(1)}}(v)\otimes_A\nabla^W_{\nabla_x(b^{(2)})}(w)\big)\\
&= \sum_{(b)}\nu\Big(\big(\nabla^V_{xb^{(1)}}(v)\big)\big)\otimes_A\big(\nabla^W_{b^{(2)}}(w)\big)\Big) + \sum_{(b)}\nu\Big(\big(\nabla^V_{b^{(1)}}(v)\big)\otimes_A\big(\nabla^W_{xb^{(2)}}(w)\big)\big)\Big)\\
&= \sum_{(xb)}\nu\Big(\big(\nabla^V_{(xb)^{(1)}}(v)\big)\big)\otimes_A\big(\nabla^W_{(xb)^{(2)}}(w)\big)\Big),
\end{aligned}
$$

proving the induction and hence the stated equation (5.1.20).
Moreover the eqs (5.1.21) and (5.1.22) are immediate consequences. In order to prove eqn (5.1.23) we have to replace $V$ by the $A$-module $\mathrm{Hom}_A(V,W)$ with connection $\nabla^{\mathrm{H}(V,W)}$, $W$ by the $A$-module $\mathrm{Hom}_A(W,X)$ with connection $\nabla^{\mathrm{H}(W,X)}$, and $X$ by the $A$-module $\mathrm{Hom}_A(V,X)$ with connection $\nabla^{\mathrm{H}(V,X)}$. Equation (5.1.7) and (5.1.9) show that the composition $\circ : \mathrm{Hom}_A(W,X)\otimes_A \mathrm{Hom}_A(V,W)\to\mathrm{Hom}_A(V,X)$ is $A$-bilinear and connection preserving. Finally the evaluation map can be seen as an $A$-linear map $\mathrm{Hom}_A(V,W)\otimes_A V\to W$, and the main statement (5.1.20) will give statement (5.1.24). By an easy induction over the tensorial degree of $b'$ with $\epsilon(b')=0$ it is shown that if $\varphi$ is connection preserving then $\nabla^{\mathrm{H}(V,W)}_{b'}(\varphi)=0$. Then statement (5.1.24) will imply the last statement eqn (5.1.25) since $b^{(1)}$ can be replaced by $\epsilon(b^{(1)})\mathbf{1}$ and the result follows thanks to $b=\sum_{(b)}\epsilon(b^{(1)})b^{(2)}$. □

# Bibliography


[1] André, M. (1974). *Homologie des algèbres commutatives*. Springer Verlag, Berlin. (Cited page 20.)

[2] Bayen, F., M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer (1978). Deformation theory and quantization. *Annals of Physics (N.Y.) 11*, 61–151. (Cited pages xiv, xxiii, 20, and 22.)

[3] Bordemann, M. (2015). An unabelian version of the Voronov higher bracket construction. *Georgian Math. J. 22*, 189–214. (Cited page 122.)

[4] Bordemann, M., N. Neumaier, M. Pflaum, and S. Waldmann (2003). On representations of star product algebras over cotangent spaces on hermitian line bundle. *J. Funct. Analysis 199*, 1–47. (Cited pages xvi and xxiv.)

[5] Bordemann, M., N. Neumaier, and S. Waldmann (1998). Homogeneous fedosov star products on cotangent bundles i: Weyl and standard ordering with differential operator representation. *Comm. Math. Phys. 108*, 363–396. (Cited pages xvi and xxiv.)

[6] Borho, W., P. Gabriel, and R. Rentschler (1973). *Primideale in Einhüllenden auflösbarer Lie-Algebren*. Springer, Heidelberg. (Cited page 17.)

[7] Bourbaki, N. *Elements of mathmatiques. Algebra I. Chapters 1-3*. (Cited page 6.)

[8] Bourbaki, N. *Groupes et algèbres de Lie; Chapitres 4, 5 et 6*. Springer-Verlag Berlin Heidelberg. (Cited pages 44, 115, 118, and 126.)

[9] Bourbaki, N. (1989). *Commutative Algebra*. Springer Verlag. (Cited page 6.)

[10] Bröcker, T. and K. Jänich (1973). *Einführung in die Differentialtopologie*. Springer, Berlin. (Cited page 131.)

[11] Cahen, M. and S. Gutt (1982). Regular $\star$-representations of Lie algebras. *Lett. Math. Phys. 6*, 395–404. (Cited page xxiv.)

[12] Cartan, H. and S. Eilenberg (1956). *Homological Algebra*. Princeton University Press, Princeton, N.J. (Cited pages xxv, 44, and 118.)

[13] Farinati, M. A. and A. Solotar (1998). Localization on (necessarily) topological coalgebras and cohomology. *Journal of Algebra 208*, 575–603. (Cited page 10.)

[14] Fedosov, B. (1996). *Deformation Quantization and Index Theory*. Akademie Verlag, Berlin. (Cited pages xvi and 20.)

[15] Fedosov, B. (2001). Pseudodifferential operators and deformation quantization. *Birkhäuser, Basel*, 1–47. (Cited page xxiv.)

[16] Helgason, S. (1978). *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press, New York. (Cited page 84.)

[17] Helmstetter, J. (1989). Série de Hausdorff d'une algébre de Lie et projections canoniques dans l'algèbre enveloppante. *J. Algebra*, 170–199. (Cited page 117.)

[18] Herstein, I. (1965). *Some Aspects of Ring Theory*. Lectures given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.), held in Varenna (Como), Italy, Reprint Springer Verlag, Berlin. (Cited page 126.)

[19] Huebschmann, J. (1990). Poisson cohomology and quantization. *J.reine angew. Math. 408*, 57–113. (Cited pages xii, xvi, xxv, 41, 46, and 47.)

[20] Jacobson, N. (1965). Representation theory of Jordan algebras; some aspects of ring theory. *Reprint Springer Verlag, Berlin*, 117–200. (Cited pages 126 and 127.)

[21] Jacobson, N. (1980). *Basic Algebra II*. 1 st ed., Freeman and Co., San Francisco, Ca. (Cited pages 123, 131, and 132.)

[22] Kan, D. (1958). Adjoint functors. *Trans. Am. Math. Soc. 87*, 294–329. (Cited page 112.)

[23] Kapranov, M. (2007). Free Lie algebroids and the space of paths. *Selecta Math. 13*, 277–319. (Cited pages xii, xvii, xxv, 50, 51, and 56.)

[24] Kassel, C. (1995). *Quantum groups*. Springer Verlag, Berlin. (Cited page 47.)

[25] Kobayashi, S. and K. Nomizu (1963). *Fondations of Differential Geometry, Vol I*. Insterscience Publishers, Wiley and Sons. (Cited pages 64, 76, 80, and 81.)

[26] Kobayashi, S. and K. Nomizu (1969). *Foundations of Differential Geometry, Vol II*. Wiley, New York. (Cited pages 84 and 88.)

[27] Kolář, I., P. Michor, and J. Slovák (1993). *Natural Operations in Differential Geometry*. Springer, Berlin. (Cited page 132.)

[28] Kontsevich, M. (2003). Deformation quantization of Poisson manifolds. *Lett. Math. Phys. 66*(3), 157–216. (Cited pages xiv, xxiii, 20, and 23.)

[29] Lam, T. Y. (1998). *Lectures on Modules and Ring*. Springer. (Cited pages xxiv, 11, 13, 15, 16, 17, 150, and 151.)

[30] Laurent-Gengoux, C.; Pichereau, A. V. P. (2012). *Poisson Structures*. Grundlehren der mathematischen Wissenschaften, Springer Heidelberg. (Cited page 20.)

[31] Loday, J.-L. (2012). *Algebraic Operads*. Springer Verlag, Berlin. (Cited pages 126 and 127.)

[32] Mac Lane, S. (1998). *Categories for the working mathematician* (Second ed.), Volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. (Cited pages 11, 12, 45, 112, 113, 114, 130, 145, and 146.)

[33] Malgrange, B. (1967). *Ideals of differentiable functions*. Tata Institute of Fundamental Research, Bombay; University Press, London. (Cited page xxiii.)

[34] Moerdijk, I. and J. Mrčun (2010). On the universal enveloping algebra of a Lie algebroid. *Proc. Amer. Math. Soc. 138*, 3135–3145. (Cited pages 50, 51, and 52.)

[35] Nijenhuis, A. (1953,1954*). On the holonomy groups of linear connections. *Kon. Nederl. Akad. Wetenschappen Amsterdam, Indag. Math. 15,16**, 233–249, 17–25*. (Cited page 80.)

[36] Ore, O. (1931). Linear equations in non-commutative fields. *Ann. of Math. 32*, 463–477. (Cited pages xi and 15.)

[37] Oseki, H. (1956). Infinitesimal holonomy groups of bundle connections. *Nagoya Math.J. 10*, 105–123. (Cited page 80.)

[38] Passman, D. S. (1991). *A Course in Ring Theory*. Wadsworth and Brooks/Cole Advanced Books and Software. Pacific Grove, CA. (Cited pages 15 and 16.)

[39] Quillen, D. (1969). Rational homotopy theory. *Ann. of Math. (2) 90*, 205–295. (Cited pages 115, 116, and 118.)

[40] Requejo, B. and J. Sancho (1994). Localization in the rings of continous functions. *Topology and its Applications 57*, 87–93. (Cited page 10.)

[41] Reutenauer, C. (1993). *Free Lie Algebras*. Clarendon Press, Oxford. (Cited pages 126, 127, and 128.)

[42] Rinehart, G. (1963). Differential forms on general commutative algebras. *Trans. Am. Math. Soc. 108*, 195–222. (Cited pages xii, xvi, xxv, 41, and 44.)

[43] Rudin, W. (1973). *Functional Analysis*. Tata - McGraw-Hill Publ., New Delhi. (Cited pages 118 and 139.)

[44] Rudin, W. (1991). *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill. (Cited page 135.)

[45] Ruiz, J. M. (1993). *The Basic Theory of Power Series*. Vieweg Verlag, Braunschweig. (Cited page 21.)

[46] Schauenburg, P. (1998). Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules. *Appl. Categ. Structures 6*, 193–222. (Cited page 50.)

[47] Schauenburg, P. (2001). Turning monoidal categories into strict ones. *New York J. Math 7*, 257–265. (Cited page 112.)

[48] Straumann, N. (1987). *Allgemeine Relativitätstheorie und relativistische Astrophysik*. Springer, Berlin. (Cited page 76.)

[49] Sweedler, M. E. (1969). *Hopf algebras*. Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York. (Cited pages 47, 115, and 116.)

[50] Sweedler, M. E. (1974). *Groups of Simple Algebras*. I.H.E.S. Publ. (Cited pages 45 and 50.)

[51] Takeuchi, M. (1977). Groups of algebras over $A \otimes \bar{A}$. *J. Math. Soc. Japan 29*, 459–492. (Cited pages 45, 50, and 51.)

[52] Tougeron, J. C. (1972). *Idéaux de fonctions différentiables*. Springer-Verlag, Berlin-New York. (Cited pages xxiv and 138.)

[53] Vezzosi, G. (1997). Localization of differential operators and of higher order de rham complexes. *Rendiconti del Seminario Matematico della Università di Padova. 97*, 113–1332. (Cited page 26.)

[54] Von Waldenfels, W. (1966). Zur charakterisierung liescher elemente in freien algebren. *Archiv der Mathematik 17*, 44–48. (Cited page 128.)

[55] Waldmann, S. (2007). *Poissongeometrie und Deformationsquantisierung*. Springer, Heidelberg. (Cited pages 20, 23, 25, 26, and 55.)

[56] Yamaguti, K. (1955/58). On the Rie triple system and its generalization. *J. Sci. Hiroshima Univ. Ser. 21*, 155–160. (Cited page 88.)