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Rationality and Information Value: Four Essays in Microeconomics

Morgan Patty

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Morgan Patty. Rationality and Information Value: Four Essays in Microeconomics. Economics and Finance. Université Paris sciences et lettres, 2021. English. NNT : 2021UPSLD028 . tel-03631486

HAL Id: tel-03631486

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THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à Université Paris Dauphine - PSL

RATIONALITY AND INFORMATION VALUE
FOUR ESSAYS IN MICROECONOMICS

Soutenue par

Morgan Patty

Le 8 Décembre 2021

École doctorale n°ED 543

SDOSE

Spécialité

Economie

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It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle — they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.

An Introduction to Mathematics
Alfred North Whitehead

Acknowledgements

This thesis has been a path full of discoveries, encounters and would not have been possible without the support of many persons.

I am especially grateful to my thesis supervisor Sidartha Gordon. Sidartha helped me in so many ways that I cannot list them all. Being his co-author is a challenging and an exciting task and I have learnt a lot all through the writing of the third chapter. Sidartha has made crucial remarks on each chapter of this thesis. His precise readings have been precious. Thank you Sidartha.

I cannot fail to thank the jury for accepting to take the time to review this work: Françoise Forges, Itzhak Gilboa, Alessandro Ispano, Matias Nuñez, Marion Oury and Peter Vida.

I am particularly in debt with Alessandro Ispano and Marion Oury who have followed me in my monitoring committee. Your helping hand along these years has been invaluable. I would like to thank also Françoise Forges for her important comments concerning the two first chapters.

I am thankful to Charlotte Scouflaire for co-writing the fourth chapter. Discovering electricity markets has been a fantastic journey. Charlotte taught me so much about this subject. Thank you Charlotte for your unending support.

I am also grateful to people that I have met in seminars, conferences, or by e-mails and who have helped me to improve this work.

Je remercie également l'école doctorale, notamment Eve Caroli et Chantal Charlier d'abord, Elise Huilery et Laëtitia Charlier ensuite, qui s'assurent du bon déroulement des thèses. Je remercie aussi l'ensemble des membres du LEDa, le laboratoire qui m'a accueilli.

Cette thèse n'aurait pas non plus été possible sans le soutien sans faille de mes collègues du bureau C104 Droite. Merci à Charlotte et Christian d'avoir toujours été là pour moi. Merci à Dio pour m'avoir partagé tant de lectures et d'idées. Merci à Alexis pour ta bonne humeur. Merci à Doriane et Maroua pour tous nos moments passés ensemble et votre soutien. Merci évidemment à Sandra et Sarah qui m'ont montré la voie. Merci à vous pour votre aide et vos relectures. Je remercie aussi Mustapha, Matteo. Merci à Charles-Thierry pour son sens du partage inégalé.

Je dois beaucoup à Noémie pour ses relectures patientes et nombreuses, merci pour ton appui indéfectible. Merci également à Leslie pour tes encouragements. Je remercie Arnold pour avoir animé tant de débats.

Merci bien sûr à Yohan, Aimée et Eléonore, cette dernière année n'aurait pas été la même sans vous. Merci à Julien, Fatou, Adèle, Cécile, Cédric, Benoit, Paul, Arthur, Michele, Eléonore, et Emy ainsi qu'à tous les doctorants et anciens doctorants que je pourrais oublier.

Je dois remercier également tous mes amis. Parmi eux, Arthur a joué un rôle prépondérant tout au long de ce périple, merci pour tout Arthur. Merci à Romain également, ta compréhension du monde m'est précieuse.

Enfin, je voudrais remercier ma famille. Cette thèse est aussi la leur. Le goût du débat, la soif d'apprendre, je vous les dois. Merci à mes parents pour tout ce que vous avez fait pour moi, le mot "dette" n'est pas un grand mot cette fois-ci. Merci à mon grand-père pour tout ce que tu m'as transmis. Merci à mon frère. Merci à toi.

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Introduction

Adam Smith asserts that balance in society emerges, not despite, but thanks to individual will-
ingnesses [Smith, 1776]. It can be summed up by the following statement: if we all maximize our
utility, we achieve optimality, that is, the best collective situation. This idea has been developed
by Léon Walras in his General Equilibrium Theory [Walras, 1874] (and later by Kenneth Ar-
row and Gérard Debreu [Arrow and Debreu, 1954] who have formalized the perfect competition
framework) which provides the powerful conceptualization considering price as the only infor-
mation needed by people to achieve the best outcome possible. As stated by Friedrich Hayek,
economic agents need not know why prices go up or go down: what matters to achieve optimality
is whether a good is more or less costly [Hayek, 1945]. In the wake of Walras, Vilfredo Pareto
defines optimality as a situation where no agent can be better off without hurting another agent's
welfare [Pareto, 1894]. As Smith predicted, the Walras' equilibrium is Pareto optimal [Arrow,
1951].

By contrast, the Nash equilibrium does not correspond to Pareto optimality while it relies only
on individual incentives like in the Smith's view: the Nash equilibrium requires the agents to
be rational, i.e. they maximize their utility [Nash, 1950, 1951]. Generally, they differ because
agents are not atomistic in Game Theory as they are in perfect competition and thus they can
be strategic. The most famous example of such a discrepancy is the Prisoners' dilemma where
the unique Nash equilibrium is the only outcome not being optimal in the sense of Pareto. As
well, the underlying mechanisms of pricing are still fiercely debated and despite Friedrich Hayek's
assertion, prices do not convey *all* information.

Aiming to describe real-life phenomena, Economics has integrated additional elements within
its framework. These elements can be considered at a collective level or at an individual one.
As an example of such a collective concept, *externalities* are benefits (or costs) due to the
activity of an agent which influence other agents' payoffs. If not taken into account, externalities
prevent optimality. Taxing externalities or creating property rights for externalities restores
optimality according to the celebrated Pigovian tax [Pigou, 1920] and Coase theorem [Coase,
1960]. As an example of an individual feature releasing the standard framework, John Maynard
Keynes considers the possibility that workers do not accept decreases in nominal wages while a
rational worker *should* only care about real wages. According to Keynes, such rigidities prevent
adjustments towards equilibrium and may explain involuntary unemployment [Keynes, 1936].
The impact of this hypothesis goes beyond theories on macroeconomic aggregates since it is
clear that releasing the agents' rationality assumption helps understanding phenomena which

remain unexplained within the classical framework. In this perspective, Herbert Simon pledges to reconcile what occurs in reality and rationality in his paper originating literature about limited rationality [Simon, 1955]. In a word, he states that:

The task is to replace the global rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in the kinds of environments in which such organisms exist.

The general idea of this thesis is to analyze economic problems where agents face or choose limited rationality. In [Chapter 1](#), we introduce a new dominance relation between strict and weak dominances, characterized by a notion of limited rationality. In [Chapter 2](#), we discuss the inconsistent outcomes of standard rationality in infinite games and propose an alternative dominance relation. In [Chapter 3](#), we assume an investment game where agents have complementarities in actions and where information is costly. We show it does not necessarily lead to complementarities in information acquisition: substitutabilities may exist, i.e., some agents may decide to restrain their access to information when other agents increase their information purchase. In [Chapter 4](#), we state under which conditions information disclosure may be welfare improving or damaging in a Cournot game.

In the following section, we discuss how economists have attempted to include real life features in their analysis. Then, we point out that the goal of predictability has influenced Economics and Game Theory. Afterwards, we turn on the meaning of consistency and remark a discrepancy between this notion and the two previous ones. Next, we study the role of information. Finally, we give a detailed outline of this thesis.

1 A realistic science

Turning back to [Simon \[1955\]](#), he clarifies what he means by this idea of limited rationality by adding that:

The problem can be approached initially either by inquiring into the properties of the choosing organism, or by inquiring into the environment of choice.

Economics has mainly focused on the first approach. It is also our choice in [Chapter 3](#) to restrict the agents' ability to process information by adding a cost to information acquisition. The second approach, less adopted in economics¹ is the approach we adopt in [Chapter 2](#). Specifically, we do not model agents interacting with an environment. Rather, we restrict the possibility for the agents to eliminate strategies if the dominating strategy is not "strong" enough to make the elimination effective with respect to other strategies. This idea leads us to the definition of the *top* condition ([TD3](#)). Briefly, a strategy cannot top dominate another strategy if there are available strategies that may eliminate the dominating strategy. To the best of our knowledge, no dominance relation has been defined in such a way that other available strategies (of the

¹The reason we can imagine is that the first approach is more compatible with the view that in the end, only individual choices are relevant whereas dealing with the so called environment may introduce other forms of agent beyond individuals.

same player) affect the dominance relation between two strategies. Let us take a detour: *perfect* equilibria are affected by the addition of irrelevant alternatives, i.e. adding dominated strategies to the game affects whether an equilibrium is perfect or not. Additionally, [Kohlberg and Mertens \[1986\]](#) argue that *stability* does not have to require such a criterion. Again, this idea that “environment” may play a role is present here, and can be translated in Game Theory. Yet, a distinction might be made. If a dominated strategy is added to the strategy set, it is also reasonable to think that it should not change agents’ choices unless they have beliefs that depend on strategy set. In this case, it could be admitted that only undominated strategies can affect choices. However, this is still an open question.

Going back to [Simon \[1955\]](#)’s work, there may be a simple way to disentangle the problem raised above: exogenizing aspects assigned to the environment and endogenizing elements attributed to individuals. However, [Simon \[1955\]](#) vigorously rejects this adaptation. Indeed, he argues that there is a fine line between what is outside and inside an individual with this luminous example:

For example, the maximum speed at which an organism can move establishes a boundary on the set of its available behavior alternatives.

Here, we see that not only what is called the environment influence individuals, but individuals also impact the environment. In [Chapter 4](#), we adopt a similar point of view, by allowing agents to choose the information design by voting according to their preferences.

With the objective to make economic theory more realistic, [Simon \[1955\]](#) puts forward simplified utility functions. It is the approach we use in [Chapter 3](#), following the formalization of [Szkup and Trevino \[2015\]](#). This can be seen as the second aspect of limited rationality we insert in our investment game. Utility functions are state dependent. However, even if the state is a continuous variable, we assume that agents only value whether the investment is successful or not.

Testing whether an action is good or not can be also a way of releasing the rationality assumption of “direct” maximization. Paradoxically, our standard view that agents maximize their utility according to their beliefs might be *too* simple. For example, [Simon \[1955\]](#) asserts that once an agent has chosen an action, he then “explores”² the opponent’s alternatives. Implicitly the idea of trembles (see our discussion on perfectness below) is already present here. In a similar way, we model in [Chapter 1](#) an agent who does explore alternatives, but his own alternatives. Additionally, when considering an alternative, the agent believes his opponent may detect it and thus react optimally.

[Simon \[1955\]](#) explains also that order dependence and multiplicity are an issue when prediction is required (see [Chapters 1](#) and [2](#) for detailed discussions on this subject). However, [Simon \[1955\]](#) remarks additionally that it might not be a problem when choices are sequential and that a satisfaction threshold can be defined. It is a way to understand our discussion about strict dominance and the standard view of rationality in [Chapter 2](#). When a “top” strategy – in another context we would say a focal point – does not exist, an agent may revise the level of rationality he requires for himself. Therefore, rather than optimizing, he might “just” be satisfied with any

²Between “ ” in [Simon \[1955\]](#).

level above the threshold he can consider. The opposition between satisficing and optimizing is further developed in [Simon \[1956\]](#). In this paper, the author frankly departs from the standard model of rationality and shows how simple rules might better fit agents' behavior.

[Simon \[1955\]](#)'s ideas have been well-known for decades. Yet, the literature applying his principles is quite recent. For instance, [Caplin et al. \[2011\]](#) proposes a model of costly search where the level of satisfaction can be endogenously determined. Choice literature is based on fundamental axioms marking the consistency of the rational agent. For instance, the *Weak Axiom of Revealed Preference (WARP)* of [Samuelson \[1938\]](#) which states that if an alternative x is chosen when y is present in a given alternative set X , then y cannot be chosen from any alternatives set containing x . It is immediate that such an axiom rejects any environment effect. An other complementary axiom is the *Independence of Irrelevant Alternatives (IIA)*. IIA states that removing the alternatives y from X does not affect the choice of the agent when facing $X \setminus \{y\}$. That is, the choice made by an agent when facing a given set of alternatives cannot be modified when only a subset (containing the previous choice) of the previous set is considered. Again, here, environment is disregarded.

[Kalai et al. \[2002\]](#)'s paper is motivated by the fact that IIA is violated in real life. They introduce the idea of multiple rationales (*rationalization by multiple rationales (RMR)*). In their paper, each alternatives set is endowed with one choice relation, which can differ with respect to the alternative set considered. They explain that "the [decision maker] has in mind a partition [...] and he applies one ordering to each cell in the partition." In the perturbation we introduce in [Chapter 1](#), the agent also considers strategy subsets (of his complete strategy set) and applies maximization of his utility considering only this subset. Our agent is cautious in the sense that he considers all strategy subsets and checks whether his chosen strategy is optimal or not. Besides, [Kalai et al. \[2002\]](#) exemplify RMR with the (u, v) procedure, where u and v are two functions over the alternative set. This example goes further than the previous description in the sense that the choice of the rationale used is partly endogenized. Indeed, the decision maker uses (maximizes) the function u while u exceeds a certain given threshold v^* and when u goes below v^* , he maximizes with function v . Now, assume that function u is a standard utility function and that instead of requiring an exogenous threshold v^* , we simply require that u is maximized. Additionally, if u is not maximized, function v is maximized with any choice. Furthermore, assume the following situation adapted from [\[Dufwenberg and Stegeman, 2002, Example 5\]](#): let an agent i facing the alternative $X = (0, 1)$ and whose utility is $U_i(x) = x$. In this example, the procedure (u, v) we have just described would choose any x . It is also the outcome of our procedure in [Chapter 2](#).

In this same literature, [Manzini and Mariotti \[2007\]](#) introduce the notion of *Rational Shortlist Method (RSM)*. Broadly speaking, RSM selects alternatives using several ordered rationales (i.e. selection rule): the first rationale makes a first selection among the alternatives, the second one makes a second selection among the previous selected subset and so on. Therefore, in contrast with [Kalai et al. \[2002\]](#), the situation is not such that there are several possible rationales and only one is picked to make the choice. Instead, several rationales are used in a given order to make the choice. Interestingly, RSM cannot be characterized by WARP, but by a weak form of WARP and by the Expansion property. This *weak WARP* is stated as follows: if an alternative

x is preferred over alternative y in pairwise comparison and in a large set, then y is not preferred over x in a subset of the large set. In other words we can say that the menu effects (when they exist) are monotonic. Again, our procedure in [Chapter 2](#) would verify such a property in the situation adapted from [[Dufwenberg and Stegeman, 2002](#), Example 5] (and the Expansion property as well).

[Tversky and Kahneman \[1991\]](#) introduce the notion of *reference point* in decision theory. Interestingly, about this matter, industrial organization and game theory seem to have been in advance with respect to decision theory. Indeed, if we think of conjectural variation theory (see [Section 1.8.3](#)) and perfectness (see below), the idea of *reference* is quite old in Economics (explicitly in the former case and implicitly in the latter case). The basic point is to say that agents compare what they can get to an initial point or a default option. In industrial organization, this reference point can just be the current situation. In game theory, it can be an equilibrium point. In decision theory, it can be either the initial situation (or decision) of the agent or the first item of a search results. Beyond this first aspect, the reference also influences the agent's assessment. That is, an alternative will not be evaluated in the same way whether the reference is the point A or the point B .

[Salant and Rubinstein \[2008\]](#) generalize the previous approach by introducing *framing* effects. They define a *frame* as information unrelated to the alternatives choice but which influences the choice. This definition seems contradictory in the sense that if the choice is influenced by some information, then the choice is obviously related to this piece of information. What we have to understand is that a *frame* is any object that would not alter a rational agent's choices, but which modifies those of an agent with limited rationality.

[Masatlioglu et al. \[2012\]](#) and [Eliaz and Spiegelger \[2011\]](#) are interested in consumer choice subject to limited attention. Again, these authors argue that it is not realistic to assume that consumers are able to assess all the alternatives they face. [Masatlioglu et al. \[2012\]](#) extends the traditional model of revealed preferences to limited agents, by introducing notably *attention filters*. *Attention filters* applied to alternative sets limit the set of alternatives to which agents pay attention. In the same vein, [Eliaz and Spiegelger \[2011\]](#) show that firms can influence the *consideration set* – i.e., the set of alternatives considered – consumers use when making their choice. By contrast, [Masatlioglu and Nakajima \[2013\]](#) analyze the consumer side by including the notion of dynamic search. That is, the consideration set can be modified after evaluating the alternatives (present initially). Finally, [Manzini and Mariotti \[2014\]](#) model an agent who makes “choice errors” due to the “agent's failure to consider all feasible alternatives”. Thus, they define a *random choice rule* which assigns to each alternative a probability to be chosen. In words, it represents for each given alternative the probability that this given alternative is in the consideration set and that alternatives which are strictly preferred to the given alternative are not in the consideration set. Interestingly, one property characterizing random choice rules, namely the *I-Asymmetry*, states that if removing an alternative a from the menu increases the probability that b is picked, then the converse cannot occur. As argued by the authors, such a property exhibits “consideration errors” and not “utility errors”. Then, it is clear that if a random choice rule is found, agents can be said inattentive to their available alternatives. The message of this literature is positive. Indeed, even if rationality assumptions are released, predictions are still possible and enable us

to better understand agents' behavior.

2 A predictive science

As stated above, the most famous example of the discrepancy between Pareto optimality and Nash equilibrium is the Prisoners' dilemma. Prisoners have a dominant strategy *defect* which strictly dominates the other strategy *cooperate*. Consequently, the only Nash equilibrium is the outcome where both agents play *defect* while both players would be better off if they both played *cooperate*. In other words, the strategy *cooperate* is not "credible", and Game Theory predicts that only the Nash equilibrium can be played. Here, prediction is easy. However, in general, it is not the case. Therefore, affirming that an equilibrium or playing a strategy is "credible" or not is obviously at heart of debates in Economics and Game Theory, and often depends on the story built around the game. Many solution concepts try to answer these questions. For instance, to frame our discussion, let us recall that [Fudenberg and Tirole, 1991, p.49] state that:

The starting point of iterated strict dominance is the observation that a rational player will never play a strictly dominated strategy. The starting point of rationalizability is the complementary question: what are all the strategies that a rational player could play? The answer is that a rational player will use only those strategies that are best responses to some beliefs he might have about the strategies of his opponents.

In two-player games, *iterated elimination of strictly dominated strategies (IESDS)* is equivalent to the concept of *rationalizability* [Bernheim, 1984; Pearce, 1984]. Then, playing an iteratively strictly undominated strategy is further justified by the fact that a rational player maximizes his utility and only plays best responses. For instance, Tan and Werlang [1988] show that common knowledge of *Bayesian rationality* (and two other assumptions) imply that agents play only iteratively strictly undominated strategies. Common knowledge applied to rationality is in fact the following reasoning applied *ad infinitum*: if i believes that j is rational and that i believes that j believes that i is rational, then i believes that j plays only best responses and believes that j believes that i plays only best responses and so on.

Though, IESDS never rules out any Nash equilibrium. Still, as stated above, some equilibria seem unreasonable. Such examples could be equilibria which do not survive *iterated elimination of weakly dominated strategies (IEWDS)*. Like IESDS, IEWDS has been well-known for decades [Luce and Raiffa, 1957]. The advantage of IEWDS with respect to IESDS is its more important predictive power. As disadvantages, IEWDS faces inconsistencies issues as order independence. Furthermore, Samuelson [1992] notes that common knowledge of weak dominance is not related to the outcome of IEWDS.

As an answer to the Samuelson [1992]'s remark, Brandenburger [1992]; Stahl [1995]; Brandenburger et al. [2008] rationalize iterated admissibility (i.e., the elimination of all weakly dominated strategies at each step of the elimination procedure). Their solutions can be summed up as follows: it is very likely that the inadmissible strategies will not be played. This result and the way both papers formalize the knowledge and beliefs of players appear very reasonable. Bran-

denburger [1992]; Stahl [1995]; Brandenburger et al. [2008]’s formalizations resolve the inclusion exclusion challenge in an elegant way by assuming lexicographic beliefs. When a strategy is eliminated, it is infinitely less likely to be played with respect to remaining strategies, but still infinitely more likely to be played than previously eliminated strategies. By contrast, in Chapter 2 we will work with *memorylessness* iterated elimination procedure. A procedure is said *memoryless* if it does not keep in memory the eliminated strategies when eliminating further strategies and when considering the final outcome of the process. As well, in Chapter 1, eliminated strategies play no role in further eliminations.

Linking this discussion about iterated elimination procedures and equilibria refinements, in Chapter 1, we consider certain perturbations of the game occurring with probability $\epsilon > 0$. This is obviously in the spirit of the refinement literature started by Selten [1975] with the idea of *perfectness*. *Perfectness* is a refinement of Nash equilibrium which considers that in an extensive-form game, if an agent observes a deviation from the equilibrium, he should not respond with a move which is not a best response. It gives the notion of *subgame perfect* equilibrium, eliminating all Nash equilibria not fulfilling the criterion described above. In finite games, it corresponds to the backward induction solution implemented by Kuhn [1953] (see [Osborne and Rubinstein, 1994, p.98-99] for the equivalence and the existence results). In normal form games, an equilibrium is said *trembling-hand perfect* if it is still an equilibrium when each agent makes the hypothesis that opponents’ mistakes (when playing their strategy) can occur with small probabilities³. Our notion of *ϵ -local rationality* points out the link between perfectness and the conjectural variation theory. Conjectural variation theory assumes that each agent has conjectures about his opponents’ reactions if he moves. Here, we assume that each agent can have *mind tremble*, i.e. he considers other strategies with respect to his reference strategy. Player i conjectures that j best responds to his mind trembles (i.e. i believes that j may observe his mind tremble with probability ϵ). A strategy is locally rational if there is a rational conjecture system such that no mind tremble can make it dominated.

Contrary to Selten [1975], we do not assume that some strategies are played by mistake. Rather, we postulate a perturbation and the reaction of the agent observing the perturbation is “rationalized”. In a sense, in Chapter 1, we are closer to the idea of *properness* [Myerson, 1978]. In perfectness notion of Selten [1975], any error can occur. By contrast, Myerson [1978] attempts to “rationalize” mistakes (in the sense of minimizing the losses) by assuming that the likeliest mistakes are the least costly.

Pursuing this literature, Hamilton and Slutsky [2005] point out that when the idea of trembling-hand perfectness is made more consistent, the refinement vanishes except in one special case and quite every Nash equilibrium becomes *perfect*. The authors assume that each agent does not only

³The interested reader will remark that both views are present in Section A.9. They can be grouped in the *perception* game in the following sense: perception mistakes induce virtual subgames where the opponent j plays optimally according to his perception. By contrast, in the *deviation* game, the notion of mistakes is less central. That is, we use also a forward induction argument. Like the Simon [1955]’s idea and more recently the signaling games literature [Cho and Kreps, 1987], the player “tests” deviations. The deviation is required to be *credible*. If not, the observer of a deviation considers the deviation will not *actually* occur. Therefore, he may not best respond. Instead, if the deviation is credible, he best responds because the deviation increases the deviator’s payoff. Thus, the deviation occurs.

consider the possibility that opponents may make mistakes (like in trembling-hand perfectness), but that additionally he may make mistakes as well. Consequently, each agent has conjectures about his own mistakes, and mistakes are naturally strategy dependent⁴. This opens the door to many potential beliefs, which in fact will sustain even non perfect equilibria (in the [Selten \[1975\]](#)'s sense). In [Chapter 1](#), we do not consider i 's strategies when player i observes a deviation. In fact, if i observes a deviation and "believes" what he sees, he will choose a best response to the deviation (like in the notion of perfectness). It is exactly what we induce when we make our [Assumption R](#)⁵ in [Section 1.8.1](#). Thus, considering own observations is superficial and can be easily skipped.

Importantly, we do not consider the case where the two agents observe the other's deviation. From a theoretical perspective, [Wärneryd \[2014\]](#) forcefully argues that reciprocal observations are not technically formalizable. Indeed, [Wärneryd \[2014\]](#) shows that there is a logical inconsistency if one believes that two players might observe each other's strategies in a standard game where strategies are *just* the action planned by each player before the observation of the opponent's strategy. Simply, the idea is that observing the opponent's strategy adds information to each player that has not been planned by the players since their strategy is composed of only one element: what the player wishes to play in any case. Then, either observing the opponent's strategy does not change anything (i.e., strategies are played whatever the observation is) and this new specification is useless, or it may change the final outcome of the game (i.e., the game played after the observation), but this observation has to be specified in the strategies of the players. In the latter case, it necessarily leads to strategies with an infinite number of elements. Indeed, the player i should choose one strategy and strategies conditionally on what he observes. Since j does the same, i should also specify what he does given what j may answer to his strategy. Player j doing the same, we see that this process repeats itself *ad infinitum*.

In words of [Wärneryd \[2014\]](#), a strategy is the *complete contingent plans of action* the agent schedules. If there is one observer, there is no need for too sophisticated complete contingent plans of action with respect to the standard one. To be clear, we do have (implicitly) a more complete contingent plan of actions, where the (classical) strategy is chosen, and additionally, where the strategy is chosen for each strategy chosen by the opponents, once no strategic uncertainty is left (the remaining uncertainty is due to randomization if mixed strategies are played). With two observers, one strategy of an opponent is still uncertain and the game has not be played yet by all the other opponents: it still requires to be strategic. That is why, only observation by one opponent is compatible with our standard view of games.

⁴Simply, assume a player i whose strategy set is equal to $S_i = \{s'_i, s''_i\}$. If i considers his own mistakes, it is obvious that when playing s'_i , he can only assume that he will, by mistake, play s''_i and similarly when playing s''_i . Then, mistakes are necessarily strategy dependent.

⁵If we have real-life situation in mind, we should consider continuous timing. That is, if agents deviate, the probability that both deviate at the same time is near zero. Besides, once a player has deviated, he should not consider further deviation except the optimal reaction of the opponent (see [Section 1.8.3](#)).

3 A consistent science

In choice literature, consistency is defined essentially as *binariness*, i.e., there is a preference relation comparing all alternatives two by two whose outcomes are seen as a maximizing solution. According to Mas-Colell et al. [1995], binariness and rationality are merged into one since a preference relation is said *rational* if complete and transitive. Sen [1990] recognizes the consistency and the maximizing self interest requirements as the two main approaches of rationality. Yet, consistency “is much too permissive”. One criticism is that a behavior can be consistent if an agent always does the opposite of what would require his self interest. Then, it is direct that there is no correspondence between consistency and maximization. Still, in Economics, maximizing self interest has been “made [...] clearly binary and more typically an ordering (and often seen as being numerically representable)” [Sen, 1990]. Thus, it may explain why maximizing self interest is no more than a special case of consistency. Sen [1990] sees WARP as reconciliating both approaches, by requiring consistency of the choice and by assimilating revealed preference to maximization. We also show in Chapters 1 and 2 that consistency issues are not necessarily incompatible with the goal of predictability. However, as pointed out in the previous sections, our approach might be nearer real life concerns.

Sen [1990] evokes Simon’s hypothesis that people do not “actually maximize any utility”. He defines satisficing as the fact that an agent has a “target level of achievement” which, once it has been reached will not lead to any further research: the agent is satisfied with the outcome he has reached. This should implicitly reflect the fact that in real world, finding the optimal solution is costly (it takes times, energy or even it requires acquiring skills). This is how economic theory has interpreted bounded rationality. An other interpretation is that preferences are incomplete in real world: satisficing would be then maximizing under a release of a rationality hypothesis. Finally, Sen [1990] expresses another interpretation of Simon’s hypothesis: even if the preferences are complete and the agent can discern at no cost which outcomes are the “best”, the agent may still settle for a lower level of utility. Obviously, this interpretation is far different from maximization. However there is still a problem of existence (if the target level is too high, no outcome is retained). Then, we should pledge for the determination of endogenous target level, as implied by our definition of our dominance relation in Chapter 2. If environment matters, we should be interested in how agents perceive it and what they know about it. Now, let us see the role played by information.

4 The role of information in Economics

One common outcome of our study is that information does not necessarily improve welfare. Obviously, this is not new. From a general equilibrium point of view, information asymmetries prevent from full appropriation. If a monopoly is able to perfectly scrutinize the consumers, all the surplus is extracted. However, no one can say that an absence of information is like the presence of (negative) externalities. A (negative) externality affects the payoff simply because it exists. Information affects the payoff because it modifies the equilibrium. Therefore, we can say that there is always over production at the equilibrium with negative externalities, while the welfare is not necessarily improved when more information is acquired or when information be-

comes public. It is only if there is perfect competition in other dimensions that more information (or less asymmetry) is welfare improving.

[Hayek \[1945\]](#) states that price conveys information held by the individuals. However, [Hayek \[1945\]](#) does not reject centralized processes from the outset. Indeed, he states that:

[This question] depends on whether we are more likely to succeed in putting at the disposal of a single central authority all the knowledge [...] initially dispersed among many different individuals, or in conveying to the individuals such additional knowledge as they need in order to enable them to fit their plans in with those of others.

Even if [Hayek \[1945\]](#)'s point is that it is difficult to provide feedback on information from the field, he also clearly mentions that centralization might be superior to decentralization. Externalities are a good example showing why markets may fail to reach the first best optimum. Uncertainty might also imply such a consequence. In the concrete case of pollution, [Weitzman \[1974\]](#) shows whether price setting or quantity setting by a central planner is the second-best optimum when there is uncertainty on both supply and demand functions. In an investment problem with two firms, [Bolton and Farrell \[1990\]](#) compare centralization and decentralization. The framework is a natural monopoly where centralization can help mitigating delay and miscoordination (that is, each firm has to pay a (privately known) sunk cost to enter into the market in a continuous time setting). Results in [Weitzman \[1974\]](#) and [Bolton and Farrell \[1990\]](#) are driven by the cost and the size of errors (with respect to the first-best optimum). Costly errors generally involve that the best scheme is quantity setting in [Weitzman \[1974\]](#) and decentralization in [Bolton and Farrell \[1990\]](#). Intuitively, large errors entail that individual preferences are very uncertain. Consequently, information should be transmitted from individual agents through the price (free in the quantity setting scheme) or private information (from the firms) matters in the natural monopoly game. In [Chapter 3](#), our results in terms of information acquisition are also driven by the (investment) cost. Intuitively, if information acquisition of some agents increases (or decreases) the size of the most costly error⁶, then an individual increases (or decreases) his information acquisition.

From [Crawford and Sobel \[1982\]](#) (cheap talk) to [Kamenica and Gentzkow \[2011\]](#) (Bayesian persuasion), many papers have treated the question of the influence of information disclosure at no cost. This literature differs from the literature on signaling. Indeed, signaling implies a costly information transmission whereas cheap talk literature assumes that information transmission does not affect payoffs directly (but affects them only through equilibrium modifications). In the simplest framework, in cheap talk, a sender who has private information can choose to disclose it to a receiver who chooses an action (only the receiver chooses an action). Both agents have a state and action dependent utility. The sender and the receiver have partially aligned interests, in the sense that the ideal action (at a given state of the world) for both players is not the same. In this case, information transmission is necessarily distorted at the equilibrium. Furthermore, if the receiver does not take into account the information disclosed by the sender (and then plays

⁶There are only two errors: not investing when investment is profitable and investing when investment is not profitable.

according to his prior), the equilibrium is said babbling. There might be more informative equilibria, but in general, the babbling equilibrium cannot be eliminated. By contrast, in Bayesian persuasion, the sender is allowed to commit to a disclosure rule. Therefore, it is more likely that there is information transmission. Obviously, real-life situations where commitment is possible are reduced. However, we can apply this framework to Capacity Remuneration Mechanisms (CRMs) in [Chapter 4](#). The public authority has information about the state of the world and can choose to disclose or not, in order to maximize welfare.

As we saw above, literature about bounded rationality comes from the seminal paper of [Simon \[1955\]](#) and has been emphasized more recently by [Rubinstein \[1998\]](#); [Sims \[2003\]](#); [Woodford \[2009\]](#). It has given a framework to situations where it is obvious that all information cannot be obtained or processed or gathered by agents. This originates a literature on costly information acquisition. To be precise, two kinds of model can be distinguished, those with a limited given information available to agents (that they may pay or not) and those where agents pay for every unit of information they wish to acquire. In the latter case, information becomes a good like any other. However, this good is valuable because it can help agents to make better decision in incomplete information settings. The way these problems are solved is generally the same, agents can in a first stage, acquire information and in the second stage they act given the information acquired. As usual, backward induction is used, second stage is solved given information of the first stage and then first stage is solved, given Bayesian Nash Equilibria. Things are more complex when agents know that other agents have acquired information as well and that actions of others affect their payoff. The question of the influence of other agents' information acquisition is crucial but is not straightforward. We show in [Chapter 3](#) that even when agents have complementarities in actions, they may have substitutabilities in information acquisition, which reveals a miscoordination problem.

As stated above, the Prisoner's Dilemma is an interesting situation to understand how the notions of optimality and equilibrium differ. In a more prosaic way, the Prisoner's Dilemma helps to understand the fundamental notion of *incentives*. The idea that agents are able to commit or not is central in Economics. If not taken into account, incentives can damage welfare⁷ and leads to miscoordination. The idea is that even if players agree that the outcome (*cooperate, cooperate*) is a better outcome than (*defect, defect*), incentives necessarily lead players to defect. Indeed, even if the opponent plays *cooperate*, a player would be better off by defecting. Game theory adds the view that agents should not care only about their own incentives (and the others' incentives) but also about the effect of their behavior on other agents' incentives. By contrast, limited rationality states that behaviors should be deemed as simple as possible. Reconciliating these two aspects is not an easy task but this thesis tries to contribute with this respect.

5 Outline

In [Chapter 1](#), we are interested in the problem of order dependence. Order dependence issues are interesting because they precisely emphasize the relation between prediction and consistency.

⁷In the precise example of the Prisoners' dilemma, incentives are well taken into account by the prosecutor since he *designs* the game in the way his preferred outcome is generated.

Indeed, when a solution concept exhibits multiplicity due to order dependence, the use of this concept seems difficult from a practical point of view. Thus, order dependent concepts such that IEWDS appear to be less legitimate than order independent ones like IESDS. Nevertheless, IESDS has a weak predictive power. For instance, IESDS never eliminates any Nash equilibrium. In this context, we introduce a new dominance relation called *root* dominance and the associated *iterated elimination of root dominated strategies (IERDS)* whose outcome is more predictive than IESDS and is order independent in finite games. The idea of root dominance is to require, in addition to weak dominance, strict dominance on a specific profile set. We call this set the *Best Reply Set* to the dominating strategy on each profile containing the dominating strategy where opponents play a best response to the given profile. Order independence is in fact explained by the following additional consistency requirement. We show that the procedure IERDS is immutable: eliminated strategies remain dominated in the final outcomes of the procedure. The point is quite simple: since strict dominance is required for each profile where a best response is played, and that *all* best responses cannot be eliminated, the dominance relation remains through the elimination path. Besides, we know that the question of rationality has been an important issue for IEWDS. Finding which maximizing behavior corresponds to a given dominance relation can also be seen as a consistency criterion. That is why we introduce a notion of *ϵ -local rationality* which characterizes root undominance. This rationality concept is inspired, on the one hand by the refinement literature initiated by Selten [1975], and, on the other hand by the idea that real-life agents use consideration sets, i.e. they cannot compare all their available alternatives at the same time. That is, we introduce a well known element of limited rationality to justify our concept. Additionally, we argue that this approach can be linked to the conjectural variation theory (see Section 1.8.3). Though, we add a rationality assumption that seems to be missing in the conjectural variation framework.

In Chapter 2, we go further in releasing the standard rationality framework. In fact, we allow players not to *maximize* their utility, but we obtain a consistent behavior. As we saw, consistency and maximization may appear to be the two faces of the same coin but their relation is more ambiguous. For instance, IESDS, implied by maximizing behavior, is order dependent in infinite games (i.e. games where the number of strategies is infinite) [Dufwenberg and Stegeman, 2002]. In this perspective, we propose a new dominance relation called *top* dominance and the associated *iterated elimination of top dominated strategies (IETDS)* whose outcome is IESDS and is order independent. As expected, IETDS can be less predictive than IESDS. Yet, IETDS can also be more predictive than IESDS according to the considered game. Top dominance relies on root dominance. However, since we consider infinite games, two important changes operate: first, the Best Reply Set is turned into a *Better Reply Set* to ensure its non emptiness when maximizing utility is not possible. Second, we add the *top* condition to keep the order independence result. The top condition requires that the dominating strategy does not only strictly dominate the dominated strategy on the Better Reply Set, but also all other available strategies. Therefore, the dominance relation between two strategies is affected by the composition of the strategy set. That is, beyond the obvious dependence to strategy sets of opponents, even in one-player games, the dominance relation is altered by the presence of other strategies. Therefore, we obtain a concept that is order independent (then consistent) and which may depend on the environment

of the players in the sense of [Simon \[1955\]](#). An additional consistency requirement that we respect is that IETDS does not generate *spurious Nash equilibria* [[Dufwenberg and Stegeman, 2002](#)]. That is, a profile which is not a Nash equilibrium of a given initial game does not become a Nash equilibrium once IETDS is applied to the given game. This result is an improvement with respect to the procedure IESDS* developed by [Chen et al. \[2007\]](#), which is order independent even in infinite games, but generates spurious Nash equilibria. Broadly speaking, IESDS* is IESDS augmented with the additional feature that previously eliminated strategies at a given step may eliminate other strategies at further steps. This explains why the order of elimination does not matter since technically no dominance relation never disappears. In contrast, IETDS is *memoryless order independent*, i.e. no eliminated strategy is needed to eliminate additional strategies during the procedure.

In [Chapter 3](#), we investigate another part of limited rationality by adding a cost to information acquisition. In fact, our framework is near [Szkup and Trevino \[2015\]](#)'s one. In a word, [Szkup and Trevino \[2015\]](#) introduces information acquisition in a global game. Global games are coordination games (i.e., agent's actions are complement) where two actions are available to players. Utilities being state dependent, there are states where either one action or the other is a dominant strategy. In [Szkup and Trevino \[2015\]](#) information is generated through a normal signal, i.e. agents receive a signal which represents the true state of the world with a normally distributed noise. As an essential difference with [Szkup and Trevino \[2015\]](#), we consider binary signals for the sake of realism. That is, in our game, agents decide whether they should invest in an asset profitable only if the state of the world and/or the number of agents investing is high enough. Agents can acquire information about the state of the world through the binary signal at an increasing and convex cost. A first intuition may lead to think that the complementary nature of agents' actions should lead them to adopt similar information acquisition behaviors, as in the beauty contest of [Hellwig and Veldkamp \[2009\]](#). Nonetheless, we find that complementarities in actions do not necessarily translate into complementarities in information acquisition. This result is already present in [Szkup and Trevino \[2015\]](#)'s framework but out of the equilibrium. Here, we find that substitutabilities may emerge even at the equilibrium. Therefore, our framework helps to generate a broader range of situation.

In [Chapter 4](#), by contrast with [Chapter 3](#), the problem we analyze is not about individual information acquisition, but about public information disclosure. As established above, more information is not necessarily better for society as a whole. Then, a public authority might strategically choose to retain information in order to maximize welfare. Adapting the [Roy et al. \[2019\]](#)'s framework, we propose to study an empirical application of the problem of public Bayesian persuasion [[Kamenica and Gentzkow, 2011](#)]. Bayesian persuasion models situations where an agent (the sender) has private information about the state of the world and may reveal it to other agents (the receivers) according to a disclosure rule that he can choose and to which he can be committed. The utility of the sender depends on the state of the world and the receivers' actions. Therefore, the disclosure rule depends on the actions played by the receiver at each state of the world. In fact, in the simple setting of binary states of the world we use, the optimal disclosure rule can be deduced from the utility of the sender when no information is disclosed, that is, when receivers play according to their (common) prior about the state of the world.

If this function is convex, then there is full disclosure. Instead, if it is concave, it is optimal for the sender to retain all information. In the perspective of our application, we consider only public disclosure, in contrast with [Eliaz and Forges \[2015\]](#). We study the problem of a public authority which has a superior information with respect to buyers in a capacity market. *Capacity Remuneration Mechanisms (CRMs)* are mechanisms designed by electricity markets regulators to value the contribution of existing capacities in terms of Security of Supply. Physics requires instant supply and demand balance, and electricity is quite non storable. Black outs being very costly, the role of any electricity plant can be understood as a positive externality. Indeed, by their presence (and their capacity), the risk that consumption of electricity exceeds production (and thus a black out occurs) is reduced. We consider a Cournot game in line with the existing capacity markets which are quantity-based. Before the game is played, the authority sends a signal about the state of the world, according to its optimal disclosure rule. We apply our results to 2010 German data. We find that if Germany wanted to apply such a quantity-based CRM, Germany would have chosen a full disclosure rule in order to maximize welfare. Besides, we consider alternative disclosure rules based on buyers or sellers' preferences. We analyze also the case where sellers vote for their preferred design: it is a case where agents can influence institutions and shape their environment. With respect to previous chapters, agents are not embedded with limited rationality. However, they can voluntarily restrain their access to information.

Bibliography

- Arrow, K. J. (1951). An Extension of the Basic Theorems of Classical Welfare Economics. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 507–532.
- Arrow, K. J. and Debreu, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica*, 22(3):265–290.
- Bernheim, B. D. (1984). Rationalizable Strategic Behavior. *Econometrica*, 52(4):1007–1028.
- Bolton, P. and Farrell, J. (1990). Decentralization, Duplication, and Delay. *Journal of Political Economy*, 98(4):803–826.
- Brandenburger, A. (1992). Lexicographic Probabilities and Iterated Admissibility. In Dasgupta, P., Gale, D., Hart, O. D., and Maskin, E. S., editors, *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, pages 282–290. Cambridge, MA: MIT Press.
- Brandenburger, A., Friedenberg, A., and Keisler, H. J. (2008). Admissibility in Games. *Econometrica*, 76(2):307–352.
- Caplin, A., Dean, M., and Martin, D. (2011). Search and Satisficing. *American Economic Review*, 101(7):2899–2922.
- Chen, Y. C., Long, N. V., and Luo, X. (2007). Iterated Strict Dominance in General Games. *Games and Economic Behavior*, 61(2):299–315.
- Cho, I.-K. and Kreps, D. M. (1987). Signaling Games and Stable Equilibria. *Quarterly Journal of Economics*, 116(2):1133–1165.
- Coase, R. H. (1960). The Problem of Social Cost. *Journal of Law and Economics*, 3:1–44.
- Crawford, V. P. and Sobel, J. (1982). Strategic Information Transmission. *Econometrica*, 50(6):1431–1451.
- Dufwenberg, M. and Stegeman, M. (2002). Existence and Uniqueness of Maximal Reductions under Iterated Strict Dominance. *Econometrica*, 70(5):2007–2023.
- Eliaz, K. and Forges, F. (2015). Information Disclosure to Cournot Duopolists. *Economics Letters*, 126:167–170.
- Eliaz, K. and Spiegler, R. (2011). Consideration Sets and Competitive Marketing. *Review of Economic Studies*, 78(1):235–262.
- Fudenberg, D. and Tirole, J. (1991). *Game Theory*. MIT Press.
- Hamilton, J. and Slutsky, S. (2005). Equilibrium with Strategy-Dependent Trembles. *International Journal of Game Theory*, 33(4):461–465.
- Hayek, F. A. (1945). The Use of Knowledge in Society. *American Economic Review*, 35(4):519–530.

- Hellwig, C. and Veldkamp, L. (2009). Knowing What Others Know: Coordination Motives in Information Acquisition. *The Review of Economic Studies*, 76(1):223–251.
- Kalai, G., Rubinstein, A., and Spiegler, R. (2002). Rationalizing Choice Functions By Multiple Rationales. *Econometrica*, 70(6):2481–2488.
- Kamenica, E. and Gentzkow, M. (2011). Bayesian Persuasion. *American Economic Review*, 101(6):2590–2615.
- Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*. Macmillan.
- Kohlberg, E. and Mertens, J.-F. (1986). On the Strategic Stability of Equilibria. *Econometrica*, 54(5):1003–1037.
- Kuhn, H. W. (1953). Extensive Games and the Problem of Information. In Kuhn, H. W. and Tucker, A. W., editors, *Contributions to the Theory of Games, Volume II*, chapter 11, pages 193–216. Princeton University Press.
- Luce, R. D. and Raiffa, H. (1957). *Games and Decisions: Introduction and Critical Survey*. New York: Wiley.
- Manzini, P. and Mariotti, M. (2007). Sequentially Rationalizable Choice. *American Economic Review*, 97(5):1824–1839.
- Manzini, P. and Mariotti, M. (2014). Stochastic Choice and Consideration Sets. *Econometrica*, 82(3):1153–1176.
- Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press.
- Masatlioglu, Y. and Nakajima, D. (2013). Choice by Iterative Search. *Theoretical Economics*, 8(3):701–728.
- Masatlioglu, Y., Nakajima, D., and Ozbay, E. Y. (2012). Revealed Attention. *American Economic Review*, 102(5):2183–2205.
- Myerson, R. B. (1978). Refinements of the Nash Equilibrium Concept. *International Journal of Game Theory*, 7(2):73–80.
- Nash, J. F. (1950). Equilibrium Points in N-Person Games. *Proceedings of the National Academy of Sciences*, 36:48–49.
- Nash, J. F. (1951). Non-Cooperative Games. *Annals of Mathematics*, 54(2):286–295.
- Osborne, M. J. and Rubinstein, A. (1994). *A Course in Game Theory*. MIT Press.
- Pareto, V. (1894). Il Massimo di Utilità dato dalla Libera Concorrenza. *Giornale degli Economisti*, pages 48–66.
- Pearce, D. G. (1984). Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52(4):1029–1050.

- Pigou, A. C. (1920). *The Economics of Welfare*. Macmillan.
- Roy, J., Silvers, R., and Sun, C.-J. (2019). Majoritarian Preference, Utilitarian Welfare and Public Information in Cournot Oligopoly. *Games and Economic Behavior*, 116:269–288.
- Rubinstein, A. (1998). *Modeling Bounded Rationality*. MIT Press.
- Salant, Y. and Rubinstein, A. (2008). (A, f): Choice with Frames. *Review of Economic Studies*, 75(4):1287–1296.
- Samuelson, L. (1992). Dominated Strategies and Common Knowledge. *Games and Economic Behavior*, 4(2):284–313.
- Samuelson, P. A. (1938). A Note on the Pure Theory of Consumer’s Behaviour. *Economica*, 5(17):61–71.
- Selten, R. (1975). Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games. *International Journal of Game Theory*, 4(1):25–55.
- Sen, A. (1990). Rational Behaviour. In Eatwell, J., Milgate, M., and Newman, P., editors, *Utility and Probability*, pages 198–216. Palgrave Macmillan, London.
- Simon, H. A. (1955). A Behavioral Model of Rational Choice. *Quarterly Journal of Economics*, 69(1):99–118.
- Simon, H. A. (1956). Rational Choice and the Structure of the Environment. *Psychological Review*, 63:129–138.
- Sims, C. A. (2003). Implications of Rational Inattention. *Journal of Monetary Economics*, 50(3):665–690.
- Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations*. London.
- Stahl, D. O. (1995). Lexicographic Rationalizability and Iterated Admissibility. *Economics Letters*, 47(2):155–159.
- Szkup, M. and Trevino, I. (2015). Information Acquisition in Global Games of Regime Change. *Journal of Economic Theory*, 160:387–428.
- Tan, T. C. C. and Werlang, S. R. (1988). The Bayesian Foundations of Solution Concepts of Games. *Journal of Economic Theory*, 45(2):370–391.
- Tversky, A. and Kahneman, D. (1991). Loss Aversion in Riskless Choice: A Reference-Dependent Model. *Quarterly Journal of Economics*, 106(4):1039–1061.
- Walras, L. (1874). *Éléments d’Economie Politique Pure, ou Théorie de la Richesse Sociale*. Guillaumin.
- Wärneryd, K. (2014). Observable Strategies, Commitments, and Contracts. *CESifo Working Papers*, (5089).

Weitzman, M. L. (1974). Prices vs. Quantities. *Review of Economic Studies*, 41(4):477–491.

Woodford, M. (2009). Information-Constrained State-Dependent Pricing. *Journal of Monetary Economics*, 56(S):100–124.

Chapter 1

Root Dominance

Abstract

Root dominance is an intermediate dominance relation between weak and strict dominances. In addition to weak dominance, root dominance requires strict dominance on all profiles where an opponent plays a best response to the dominating strategy. The iterated elimination of root dominated strategies (IERDS) outcome refines the iterated elimination of strictly dominated strategies (IESDS) outcome, and IERDS is an order independent procedure in finite games, contrary to the iterated elimination of weakly dominated strategies (IEWDS). In addition, IERDS does not face the inconsistency that we call mutability. That is, IERDS does not alter the dominance relation between two strategies like IEWDS does. Finally, we introduce a rationality concept which corresponds to root undominated strategies. This rationality concept is induced by perturbations of the game such that a player believes that the strategies he considers might be observable by his opponent. We discuss the links between our concept and other concepts established in various literatures such as the conjectural variations theory.

1.1 Introduction

1.1.1 Motivating example

Assume two agents who have coordination incentives but also have strong egocentric biases. That is, each agent is indifferent between, on the one hand, coordinating on his least preferred action with the other agent and, on the other hand, miscoordinating but choosing his preferred action. This situation can be represented in the following game which can be seen as a modified version of the battle of the sexes (BoS) where best responses payoffs are underlined¹:

¹Remark that utility functions can be denoted: $U_i(A_i) = 2 + \mathbf{1}_{A_j} - \mathbf{1}_{O_j}$, $U_i(B_i) = 1 + \mathbf{1}_{B_j} - \mathbf{1}_{O_j}$, $U_i(O_i) = 0 + \mathbf{1}_{O_j}$; and in a symmetric way for player j : $U_j(B_j) = 2 + \mathbf{1}_{B_i} - \mathbf{1}_{O_i}$, $U_j(A_j) = 1 + \mathbf{1}_{A_i} - \mathbf{1}_{O_i}$, $U_j(O_j) = 0 + \mathbf{1}_{O_i}$.

	j 's Strategy		
i 's Strategy	A_j	B_j	O_j
A_i	(<u>3</u> ,2)	(<u>2</u> ,2)	(<u>1</u> ,0)
B_i	(1, <u>1</u>)	(<u>2</u> , <u>3</u>)	(0,0)
O_i	(0,0)	(0, <u>1</u>)	(<u>1</u> , <u>1</u>)

Figure 1.1: Modified Version of the Battle of the Sexes

Remark first that no strategy is strictly dominated. Thus, the iterated elimination of strictly dominated strategies (IESDS) does not eliminate any strategy. In contrast, both outside options O_i and O_j are weakly dominated (respectively by A_i and B_j). As well, B_i and A_j are weakly dominated². However, as noted by Samuelson [1992]:

It is well known that the order in which dominated strategies are eliminated can affect the outcome of the [iterated elimination of weakly dominated strategies (IEWDS)].

In other words, IEWDS is order dependent (see also Marx and Swinkels [1997]; Hillas and Samet [2020]). Here, it is the case since IEWDS always eliminates outside options O_i and O_j but only sometimes A_j and/or B_i . It is striking that no iterated elimination procedure based on a dominance relation³ can both provide a unique outcome when applied to this game and still eliminate some strategies. Particularly, it is remarkable that even the Nash equilibrium (O_i, O_j) cannot be ruled out while we could intuitively think that players “should” try to coordinate on better outcomes. In this paper, we introduce a new dominance relation named *root dominance* and an associated order independent iterated elimination procedure the *iterated elimination of root dominated strategies (IERDS)* such that IERDS eliminates both O_i and O_j and stops there. Root dominance requires weak dominance and strict dominance on *all* the profiles where the opponent best responds to the dominating strategy. In our version of the Battle of the Sexes, j best responds to A_i by playing A_j or B_j . At these two profiles, A_i strictly payoff dominates O_i . Therefore, A_i root dominates O_i . On the contrary, playing A_i does not yield a strictly higher payoff than playing B_i when j plays B_j . Thus, A_i does not root dominate B_i and B_i is never eliminated by IERDS.

1.1.2 Elimination procedures based on dominance relations

Iterated elimination of strictly dominated strategies (IESDS) is one of the most basic tools of game theory. It is among the least vulnerable solution concepts when analysts eliminate strategies to predict the outcome of a situation. Notably, it is equivalent to the concept of rationalizability in two-player games (see Bernheim [1984]; Pearce [1984]) and when a

²In addition to the outside options, this is the main difference with the standard BoS.

³See Definition 1 for the precise definition.

game is dominance solvable⁴, it reinforces the use of the Nash equilibrium as a solution concept, like in the Cournot duopoly. Remarkably, for instance, IESDS is essential to understand why there is a unique equilibrium in global games (see [Carlsson and van Damme \[1993\]](#)). However, the conceptual robustness of IESDS necessarily reduces its use when precise predictions are required. Instead, iterated elimination of weakly dominated strategies (IEWDS) outcome is a refinement of IESDS outcome. IEWDS has been largely applied in different strands of the economic literature such that the voting literature (see [Moulin \[1979\]](#)). Additionally, a certain order of IEWDS is equivalent to the backward induction solution⁵ (see [Moulin \[1986, p.84\]](#)). Though, IEWDS may go sometimes “too far” in the selection. As an example, it may eliminate the only Nash equilibrium in certain games such that the Bertrand duopoly. Furthermore, inconsistencies of IEWDS refrain its use as a solution concept. In particular, order dependence⁶ of IEWDS (and therefore the multiplicity of final outcomes) prevents firm forecasts. However, attempts to justify the use of IEWDS have been made. Among this literature, [Marx and Swinkels \[1997\]](#) shows that IEWDS is *payoffs* order independent in games with *transference of decisionmaker indifference (TDI)*⁷, and define in association, the *nice weak domination*⁸. Nevertheless, the order independence result is limited to payoffs (and does not apply to strategies)⁹, while in the context of decision theory, [Kahneman and Tversky \[1979\]](#) show that payoffs may not determine entirely the preferences. Then, from both theoretical and practical points of view, payoffs independence might not be considered as strong a result as strategies order independence. Alternatively, we propose in this paper a dominance relation and an associated procedure whose outcome refines the IESDS outcome and is (payoff and strategies) order independent in every finite game.

1.1.3 Outline

We introduce in this paper a new kind of iterated elimination procedure based on a new dominance relation called *root dominance*. Root dominance is a stronger relation than weak dominance and weaker than strict dominance. That is, root dominance requires weak dominance and the strict payoff dominance on a specific profile set: the *best reply set* to the dominating strategy. Note that this last property depends essentially on the dominating strategy, which is, to the best of our knowledge, a novelty. We introduce also

⁴Dominance solvability means that IESDS outcome is a unique profile.

⁵It is true in games where, if a player is indifferent between two terminal nodes, it implies that all players are indifferent at these same terminal nodes. [Moulin \[1986\]](#) calls this assumption the one-to-one assumption.

⁶It means that different applications of the procedure may lead to different final outcomes. See [Section 1.2](#) for definitions. The problem of order independence of procedures has given a rich literature (see for instance [Gilboa et al. \[1990\]](#); [Apt \[2005, 2011\]](#); [Luo et al. \[2020\]](#); [Hillas and Samet \[2020\]](#)).

⁷A game exhibits *TDI* when, if one agent is indifferent between two strategies at a given opponents’ profile, every player is indifferent between the two profiles formed by either one or the other strategy of the first player, and the given opponents’ profile.

⁸A strategy s'_i of player i is said *nice weakly dominated* by strategy s''_i if, in addition to weak dominance, everywhere where i is indifferent between s'_i and s''_i , i ’s opponents are also indifferent between i playing s'_i and s''_i .

⁹See [Section A.4](#) to distinguish our notion of order independence and the [Marx and Swinkels \[1997\]](#)’s one.

a new iterated elimination procedure, whose order independence property is not limited to payoffs, but concerns strategies as well.

In the next section, we establish a simplified framework with only pure strategies. In [Section 1.3](#), we define the notion of root dominance and our iterated elimination procedure IERDS. Additionally, we illustrate them with some examples. In [Section 1.4](#), we show the technical lemmas and the order independence result. We make a succinct literature review about iterated elimination procedures in [Section 1.5](#). Then, in [Section 1.6](#), we present the mutability issue, notably faced by IEWDS, and show that IERDS is immutable. In [Section 1.7](#), we extend our concepts to a framework with mixed strategies and show that our results hold true. We introduce our rationality concepts in [Section 1.8](#) and we compare them specifically to the concepts in conjectural variation theory concepts. Finally, we conclude in [Section 1.9](#).

1.2 Framework with pure strategies

We denote $\Gamma = \{I, S, U\}$ a finite game with I the set of players, $S = \prod_{i \in I} S_i$, S_i being the finite strategy set of player $i \in I$ (we consider only pure strategies), and U the vector of utility functions of each player i where $U_i : S \rightarrow \mathbb{R}$. We denote $S_{-i} = \prod_{j \in I \setminus \{i\}} S_j$ the strategy profiles set of i 's opponents. Finally, we denote $s \in S$ a strategy profile, and $s_{-i} \in S_{-i}$ the strategy profile of the opponents of $i \in I$ such that when i plays s_i , $s = (s_i, s_{-i})$.

Here, we define the main notion that motivates this paper, namely *order independence*. Before, we define a process associated with any dominance relation: A *process* iteratively eliminates *some* dominated strategies *at the step they are eliminated* with a *specific order*, and ends when there is no dominated strategy anymore. Then, a *procedure* associated with a game is the class of all *processes* applied to the game.

Now, we can state what we mean by order independence when we study a precise game:

Definition 1. *A procedure associated with a dominance relation and a game is said order independent for this game if all processes have the same final (strategies) outcome.*

Importantly, the final outcome of a process contains the payoffs and the strategies. Again, this feature distinguishes ourselves from [Marx and Swinkels \[1997\]](#) who look only at payoffs to define order independence¹⁰. Finally, we define *order independence* for the class of games we study, namely the finite games:

Definition 2. *A procedure associated with a dominance relation is said order independent if it is order independent for every finite game.*

In the next section, we define formally the *sequence of games*¹¹ associated with a process, which further specifies the kind of order independence we consider. Importantly, except

¹⁰Obviously, both notions often coincide, but it is not always the case.

¹¹See [Definition 5](#) below for the formal definition.

explicit mention, we consider that a procedure is order independent if and only if any number (but zero) of strategies can be eliminated at each step of the processes run by the procedure and all processes have the same final outcome.

1.3 Root dominance

In this section, we define our dominance relation as well as our iterated elimination procedure.

1.3.1 The dominance relation

To establish the dominance relations in this section, we first redefine a standard notion of game theory, the *Best Reply Set to a strategy*:

Definition 3. *The Best Reply Set to $s_i'' \in S_i$, denoted $\mathbf{b}(s_i'')$, is the set of all strategy profiles $s^* \in S$ such that:*
 $s_i^* = s_i''$, and, if $S_{-i} \neq \emptyset$:

$$\exists j \in I \setminus \{i\}, s_j^* \in \arg \max_{s_j \in S_j} U_j(s_j, s_{-j}^*) \quad (\text{OM})$$

The Best Reply Set is simply the set of all profiles which contain s_i'' and where at least one i 's opponent best responds to the profile (OM). If there is no opponent or their strategy sets are empty, the Best Reply Set is simply the strategy s_i'' .

Now, we define our dominance relation, namely *root dominance*:

Definition 4. *A strategy $s_i' \in S_i$ is said root dominated by the strategy $s_i'' \in S_i$, (denoted $s_i'' \succ s_i'$), if:*

$$\forall s_{-i} \in S_{-i} : U_i(s_i'', s_{-i}) \geq U_i(s_i', s_{-i}) \quad (\text{RD1})$$

$$\forall s_{-i}^* \text{ such that } s^* \in \mathbf{b}(s_i'') : U_i(s_i'', s_{-i}^*) > U_i(s_i', s_{-i}^*) \quad (\text{RD2})$$

RD1 and RD2 are *inadmissibility* conditions, i.e., they ensure that root dominated strategies are weakly dominated. Precisely, RD1 states that s_i' is *very weakly dominated* by s_i'' . There is *very weak dominance* if a strategy always pays off at least as much as another strategy (see Marx and Swinkels [1997] for a formal definition). Therefore, either the former strategy (weakly) dominates the latter, or they are equivalent. RD2 states that s_i'' is strictly preferred to s_i' if the opponents play a profile in $\mathbf{b}(s_i'')$. Additionally, we will denote respectively the strict and the weak dominance relation:

$$s_i'' \underset{S}{\succ} s_i' \text{ and } s_i'' \underset{W}{\succ} s_i'$$

1.3.2 Finite sequence of games

Since we are interested in defining iterated elimination procedures and comparing them to IEWDS and IESDS, we formally define the *sequence of games* that will be used in this section, in association with the dominance relation we have defined above:

Definition 5. A sequence of games associated with a game Γ is:

$$\{\Gamma^\lambda\}_{\lambda \leq \Lambda} \equiv \{\Gamma^0 \equiv \Gamma, \dots, \Gamma^\lambda, \dots, \Gamma^\Lambda\}$$

with $\lambda \in \llbracket 0, \Lambda \rrbracket$ such that:

- $\forall \lambda \in \llbracket 0, \Lambda \rrbracket$, $\Gamma^\lambda = \{I, S^\lambda, U\}$, with $S^\lambda = \prod_{i \in I} S_i^\lambda$, S_i^λ being the strategy set of player $i \in I$, I the unchanged set of players of Γ , and U the vector of utility functions of each player i (whose domain is restricted), $U_i : S^\lambda \rightarrow \mathbb{R}$,
- $\forall \lambda \in \llbracket 1, \Lambda \rrbracket$, Γ^λ is a restriction of $\Gamma^{\lambda-1}$, i.e., $S^\lambda = \prod_{i \in I} S_i^{\lambda-1} \setminus \underline{S_i^{\lambda-1}}$ where for each player i , $\underline{S_i^{\lambda-1}}$ is an arbitrary (possibly empty) set of strategies in $S_i^{\lambda-1}$ dominated in $\Gamma^{\lambda-1}$, but such that for at least one player $i \in I$, $\underline{S_i^{\lambda-1}}$ is non empty.
- $\underline{S^\lambda} = \prod_{i \in I} \emptyset$ if and only if $\lambda = \Lambda$.

The sequence of games starts from the original game Γ , and then restricts the strategy set by eliminating *some* (i.e. at least one but not necessarily all) dominated strategies at each step of the sequence. The sequence ends if and only if no more strategy is dominated. Then, we can define the *iterated elimination of root dominated strategies (IERDS)* as the procedure that iteratively eliminates *some* root dominated strategies *at the step they are eliminated* and ends when there is no root dominated strategy anymore. As explained above, the procedure can lead to several processes, each one associated to a sequence of games.

Let us study how root dominance and IERDS work in finite games through the next example:

	j's Strategy	
i's Strategy	L	R
T	(<u>4</u> ,2)	(1,1)
B	(2, <u>2</u>)	(<u>4</u> , <u>2</u>)
O	(2,2)	(2, <u>2</u>)

$\xrightarrow{\text{IERDS}}$

	j's S.
i's S.	L
T	(<u>4</u> , <u>2</u>)

Figure 1.2: Game with a Unique Prediction

O is not root dominated by B nor by T . Even if O is (very) weakly dominated by B (RD1 is thus respected), we see that both (B, L) and (B, R) are in $\mathbf{b}(B)$, and since $U_i(B, L) = U_i(Z, L)$, there is no root dominance since it requires strict dominance on all profiles in $\mathbf{b}(B)$ (RD2). T does not either, because of RD1. Indeed, there is no (very)

weak dominance since $U_i(T, R) < U_i(Z, R)$. However **RD2** is checked since $\mathbf{b}(T) = (T, L)$ and $U_i(T, L) > U_i(Z, L)$. Concerning player j , L root dominates R . Actually, L (very) weakly dominates R and $U_j(T, L) > U_j(T, R)$ while $\mathbf{b}(L) = (T, L)$. After eliminating R , we see that both B and O are root dominated since T strictly dominates them. Finally, IERDS selects (T, L) like IEWDS.

Before focusing ourselves on the results, we make a semantical precision: we say that s'_i is *eliminated* by s''_i at step λ of a sequence of games if:

$$s''_i \succ s'_i, \text{ and } s''_i \in S_i^{\lambda+1}, \text{ and } s'_i \in S_i^\lambda \setminus S_i^{\lambda+1}.$$

Obviously, s'_i is *eliminated* by s''_i only if it is root dominated by s''_i , but the converse is not necessarily true in a given process. The reason is that both s'_i and s''_i , or only s''_i or neither of them might be eliminated at a given step. However, for the case of root dominance and IERDS, the distinction between domination and elimination is only made to ease the establishment of the next results. That is, a root dominated strategy always has an undominated dominator in finite games, and then, for each root dominated strategy, one can find a strategy that *eliminates* it. We formally prove this statement below in [Lemma 2](#).

1.4 Order independence result

1.4.1 Technical results

Lemma 1. $\forall i \in I, \forall s_i \in S_i, \mathbf{b}(s_i) \neq \emptyset$

Proof. By [Definition 3](#), it is straightforward that $\mathbf{b}(s_i)$ is never empty for any finite game. Indeed, either there is no opponent (or equivalently opponents' strategy sets are empty) and then $\mathbf{b}(s_i) = s_i$. Otherwise, since the game is finite, each player has (at least) a best response to each strategy profiles of his opponents. ■

Now, we state that root dominance forms a strict partial order:

Proposition 1. *With respect to a fixed game, root dominance induces a strict partial order on the strategy set of any player $i \in I$: it is a binary relation such that irreflexivity, asymmetry and transitivity hold.*

Proof. Root dominance is irreflexive: by [Lemma 1](#), $\mathbf{b}(s''_i) \neq \emptyset$, and it is not possible to have $U_i(s_i, s_{-i}) > U_i(s_i, s_{-i})$ for any profile $s_{-i} \in S_{-i}$. Then, **RD2** cannot be respected. Root dominance is transitive: assume $s''_i \succ s'_i$ and $s'''_i \succ s''_i$. Here, we have to prove that $s'''_i \succ s'_i$. First, it is straightforward that **RD1** is respected. Second, since $s'''_i \succ s''_i$, we know that $U_i(s'''_i, s_{-i}) > U_i(s''_i, s_{-i})$ for each strategy profile s_{-i} contained in $\mathbf{b}(s'''_i)$. Since $s''_i \succ s'_i$, $U_i(s''_i, s_{-i}) \geq U_i(s'_i, s_{-i})$ for each strategy profile s_{-i} in S_{-i} , and thus for each strategy profile s_{-i} contained in $\mathbf{b}(s'''_i)$. Therefore, $U_i(s'''_i, s_{-i}) > U_i(s''_i, s_{-i}) \geq U_i(s'_i, s_{-i})$

for each strategy profile s_{-i} contained in $\mathbf{b}(s_i''')$ and **RD2** is respected. Finally, irreflexivity and transitivity together imply asymmetry. ■

Lemma 2. *If $s'_i \in S_i$ is root dominated, there is (at least) one strategy $s''_i \in S_i$ that may eliminate it, i.e., s''_i is not root dominated by any strategy in S_i and s''_i root dominates s'_i .*

Proof. Since the number of strategies is finite, the number of strategies root dominating s'_i is necessarily finite. Let us denote it m and denote $g(s'_i)$ the set of these strategies. Then, (at most) $m - 1$ of these strategies are root dominated. Otherwise, it means that the m^{th} strategy is root dominated by an other strategy outside $g(s'_i)$ ¹². By transitivity of root dominance, it means that the latter strategy also root dominates s'_i , contradicting the fact that the number of strategies root dominating s'_i is m . If less than $m - 1$ strategies are root dominated, we do have that there is (at least) one strategy that is not root dominated by an other strategy and which root dominates s'_i . ■

The next lemma establishes that the set $\mathbf{b}(s_i)$ never expands as we progress through the steps of IERDS:

Lemma 3. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall i \in I, \forall s_i \in S_i^{\lambda+1},$

$$\mathbf{b}^{\lambda+1}(s_i) \subseteq \mathbf{b}^\lambda(s_i).$$

Proof. Assume there exists a profile $s' \equiv \prod_{k \in I} s'_k \in \mathbf{b}^{\lambda+1}(s_i) \setminus \mathbf{b}^\lambda(s_i)$. Since $s' \notin \mathbf{b}^\lambda(s_i)$ but $s' \in \mathbf{b}^{\lambda+1}(s_i)$, we know that there is no best response in s' at λ but also that (at least) one player j best responds with the strategy s'_j to s'_{-j} at $\lambda + 1$. Thus, we assume that there is (at least) one player $j \neq i$ with a best response $s''_j \in S_j^\lambda$ to s'_{-j} , eliminated at step $\lambda + 1$ such that:

$$U_j(s''_j, s'_{-j}) > U_j(s'_j, s'_{-j}).$$

Since s''_j is root dominated, then by **Lemma 2** s''_j is root dominated by (at least) an uneliminated strategy s'''_j , present at step $\lambda + 1$. Since s''_j is a best response for j to the profile s'_{-j} , we necessarily have $U_j(s''_j, s'_{-j}) = U_j(s'''_j, s'_{-j}) > U_j(s'_j, s'_{-j})$. Therefore, at step $\lambda + 1$, player j still wants to deviate from s'_j to s'''_j . It contradicts the hypothesis that s'_j is a best response for j at step $\lambda + 1$ and finally it contradicts that $s' \subseteq \mathbf{b}^{\lambda+1}(s_i)$. ■

This property would not be true if, for instance, we considered only profiles where each opponent plays a best response. Clearly, either these profiles could not exist, or they could be eliminated (see **Section A.3**), inducing new profiles in $\mathbf{b}(s_i)$ where a “new” maximal payoff would be obtained.

¹²By **Proposition 1**, root dominance is asymmetric and transitive. Then, there is at least one strategy (the m^{th} here) that is not root dominated by a strategy in $g(s'_i)$. Indeed, if each strategy is root dominated by a strategy in $g(s'_i)$, one can find a contradiction with asymmetry and transitivity.

Now we establish that the relation of root dominance between two strategies is maintained through the steps of IERDS:

Lemma 4. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 2 \rrbracket, \forall i \in I, \forall s'_i, s''_i \in S_i^{\lambda+1}$, if $s''_i \succ s'_i$ in Γ^λ , then $s''_i \succ s'_i$ in $\Gamma^{\lambda+1}$.

Proof. Assume $s''_i \succ s'_i$ in Γ^λ . It is straightforward that **RD1** is still verified in $\Gamma^{\lambda+1}$. By **Lemma 3**, we know that for any strategy s_i , $\mathbf{b}^{\lambda+1}(s_i) \subseteq \mathbf{b}^\lambda(s_i)$. Therefore **RD2**, is still verified as well. ■

Note that $\mathbf{b}^\lambda(s''_i)$ being not empty for each λ by **Lemma 1**, there is still a profile such s''_i strictly payoff dominates s'_i . Besides, remark that we consider only $\lambda \in \llbracket 0, \Lambda - 2 \rrbracket$ for a given sequence because in Γ^Λ no strategy is root dominated.

Now, we define a notion introduced by **Apt [2011]**, namely the *hereditariness* of a dominance relation. *Hereditariness* is useful to establish order independence of the procedure associated with the dominance relation which verifies it. Denote $c(\Gamma)$, the Γ -choice, i.e. the set of strategies in S which are not dominated in Γ (given a dominance relation). *Hereditariness* means that no strategy previously dominated becomes non dominated after one step of a process:

Definition 6. A dominance relation is said to verify *hereditariness* if $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket$,

$$\Gamma^\lambda, \Gamma^{\lambda+1} \in \{\Gamma^\lambda\}_{\lambda \leq \Lambda} \Rightarrow c(\Gamma^{\lambda+1}) \subseteq c(\Gamma^\lambda).$$

Note that *hereditariness* is called *1-Monotonicity** in **Luo et al. [2020]**. Here, we verify that root dominance is hereditary:

Lemma 5. *Root dominance verifies hereditariness. It is also equivalent to the following statement: $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 2 \rrbracket, \forall i \in I, \forall s'_i \in S_i^{\lambda+1}$, if $s''_i \in S_i^\lambda$ root dominates s'_i in Γ^λ , then s'_i is still root dominated in $\Gamma^{\lambda+1}$.*

Proof. First, if $s''_i \in S_i^{\lambda+1}$, by **Lemma 4**, the result is immediate, i.e. **RD1** and **RD2** are still respected. Second, if $s''_i \notin S_i^{\lambda+1}$, then by **Lemma 2** there is (at least) a strategy s'''_i that eliminates s''_i . By **Proposition 1**, each strategy that root dominates s''_i root dominates s'_i as well in Γ^λ . Thus, there is still (at least) one strategy that root dominates s'_i in $\Gamma^{\lambda+1}$. ■

By **Apt [2011, Theorem 1]**, we know that hereditariness implies, in finite games, order independence of the procedure associated with the dominance relation. The following result, a corollary of **Lemma 5**, illustrates with another perspective why our elimination procedure is order independent. In the words of **Dufwenberg and Stegeman [2002]; Luo et al. [2020]**, each root dominated strategy has an undominated root dominator, i.e. each root dominated strategy at a point of a sequence of games will be deleted by the end of the sequence:

Proposition 2. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall i \in I, \forall s_i \in S_i^\lambda$ root dominated in Γ^λ ,
 $s_i \notin \Gamma^\Lambda$.

Proof. The proof is made by applying an induction reasoning on [Lemma 5](#). Assume a process of IERDS applied to the game Γ , and the associated sequence of games $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$. Assume s_i is root dominated at step $\lambda - 1$. By the definition of the sequences of games, s_i is eliminated, and s_i is not in Γ^Λ . Now assume the property that *a root dominated strategy s_i at $\Lambda - \mu$ is not in Γ^Λ for a given $\mu \in \llbracket 2, \Lambda - 1 \rrbracket$* is true. Let us show it is true for $\mu + 1$. Thus, assume that s_i is root dominated at $\Lambda - (\mu + 1)$. Either s_i is eliminated at this step and we have the result, or, it is not eliminated. In this latter case, by [Lemma 5](#), s_i is root dominated at $\Lambda - \mu$, and therefore, we have the result by the induction hypothesis. We have shown that a strategy root dominated at $\Lambda - (\mu + 1)$ was deleted by the end of the sequence. Thus, by induction, it is true for each $\mu \in \llbracket 2, \Lambda \rrbracket$. Since we did not need any assumption on the process used to construct our initial sequence, this result is true for any process. ■

1.4.2 Main result

Theorem 1. *IERDS is order independent in finite games.*

Proof. By [Lemma 5](#) and [Apt \[2011, Theorem 1\]](#), the result is immediate. ■

1.5 Related literature about other elimination procedures

In an unifying framework gathering weak and strict dominances, [Hillas and Samet \[2020\]](#) eliminate *flaws*, i.e., strategy profiles rather than strategies. A flaw deletion occurs if playing the given flaw implies that an agent plays a dominated strategy. If flaws elimination is used, then weak and strict dominance are order independent in finite games ([Hillas and Samet \[2020, Proposition 1\]](#)). Therefore, weak dominance rationality seems to be as legitimate as strict dominance rationality if iterated elimination of flaws is considered¹³. Nevertheless, the purpose in [Hillas and Samet \[2020\]](#) is mainly to rationalize the use of weak dominance. Moreover, the iterative elimination of flaws does not actually eliminate the profiles or strategies from the original game that is considered. Rather, eliminated profiles or strategies are seen as not playable by the agents, but they may be used in order to justify further flaws deletions.

In the same vein, [Asheim and Dufwenberg \[2003\]](#) refine the notion of *permissibility* of [Dekel and Fudenberg \[1990\]](#)¹⁴ with *full permissibility sets* and the associated *iterated*

¹³Interestingly, in finite games, the outcome of IEWDS is contained in the outcome of weak flaws elimination and the outcome of IESDS is equal to the outcome of strict flaws elimination. Then, one may wonder whether the order independence of IESDS in finite games may only be due to the fact that IESDS is incidentally equivalent to strict flaws elimination.

¹⁴A strategy is permissible if, after one round of elimination of all the weakly dominated strategies, it survives to the iterated elimination of strictly dominated strategies.

elimination of choice sets under full admissible consistency (IECFA). IECFA considers strategy subsets (and not strategies like in IESDS or IEWDS). Roughly speaking, IECFA eliminates weakly dominated strategies, and then keeps a strategy subset of the first player if there is at least a surviving opponent's subset such that considering only the profiles contained in this opponent's subset, the strategies in the subset of the first player are the only undominated strategies (i.e. not weakly dominated strategies). The outcome of IECFA is made of subsets. All these subsets can correspond to a belief about a surviving opponent's subset, but the beliefs do not have to be consistent between players (like in rationalizability and contrary to Nash equilibrium). IECFA is order independent by definition. Indeed, like the Dekel-Fudenberg (DF) procedure of [Dekel and Fudenberg \[1990\]](#), each eliminable strategy (subset) is eliminated at each step. Nevertheless, the outcome still exhibits multiplicity.

An other procedure based on beliefs is the *reasoning-based expected utility procedure* (RBEU) of [Cubitt and Sugden \[2011\]](#). RBEU is an iterated procedure in which strategies are *accumulated*¹⁵ if there is no belief such that another strategy gives a strictly higher payoff to the player (the strategy is dominant). If a player's strategy is accumulated, the procedure allows only opponents' beliefs which allocate a strictly positive probability to the occurrence of this strategy. Strategies which are always strictly dominated for these beliefs are deleted and so on. It is immediate by its definition that RBEU deletes (at least) as many strategies as IESDS in finite games. Thus, RBEU refines IESDS. Moreover, it is order independent in finite games, contrary to IEWDS. However, RBEU refines strictly IESDS if and only if there is (at least) a dominant strategy, a quite huge requirement.

1.6 The mutability problem

Now, we define the second consistency requirement we are concerned with, namely *immutability*. Note that we call *mutability* what [Cubitt and Sugden \[2011\]](#) call “undercutting problems” and what [Hillas and Samet \[2020\]](#) call “inconsistency”. [Samuelson \[1992\]](#) contrasts iterated admissibility (i.e. IEWDS) and common knowledge of admissibility by emphasizing this inconsistency with the following words:

The difference in these two outcomes reflects the fact that once a strategy [...] is eliminated by iterated admissibility, it cannot return even if the reason for its elimination has been purged.

First, we introduce the notion of *virtual* domination:

Definition 7. *A strategy eliminated by a process is said virtually dominated if, added to the final outcome of the process, it is a dominated strategy.*

Definition 8. *A procedure is immutable (for a given game) if in each process associated to it (for this given game), all eliminated strategies are virtually dominated.*

¹⁵Briefly, accumulated strategies are the undeletable strategies.

Table 1.1 summarizes the inconsistencies associated to the procedures we have mentioned above. Now, we study in details these “inconsistencies” of IEWDS through various versions

Inconsistencies	Definitions	Procedures
Order Dependence	The order of elimination affects the final outcome	IEWDS
Mutability	A strategy may be virtually not dominated whereas it is dominated at a previous step	IEWDS, IECFA, DF

Table 1.1: Inconsistencies of Elimination Procedures in Finite Games

of an example taken in [Hillas and Samet \[2020\]](#). Note that such remarks had been already formulated in [Samuelson \[1992\]](#) for instance. We compare IERDS to the solution of [Hillas and Samet \[2020\]](#) to deal with these inconsistency problems, namely the *flaws elimination* or also called *deletion of inferior profiles*¹⁶. Following [Stalnaker \[1994\]](#), [Hillas and Samet \[2020\]](#) propose to eliminate profiles (rather than strategies) such that if they were played, it would mean that a (weakly) dominated strategy is effectively played. We illustrate mutability with the following example, such that $\xrightarrow{\text{IEWDS}}$ means that a process of IEWDS is run (and gives the final outcome when cells color is blank), and the cells in blue indicates eliminated strategies but which are non virtually dominated after the process has been terminated:

	j 's Strategy	
i 's Strategy	L	R
T	(<u>2</u> , <u>1</u>)	(3,0)
B	(<u>2</u> ,0)	(2, <u>1</u>)

 $\xrightarrow{\text{IEWDS}}$

	j 's Strat.
i 's Strat.	L
T	(<u>2</u> , <u>1</u>)
B	(<u>2</u> ,0)

Figure 1.3: [Hillas and Samet \[2020\]](#)'s Game with IEWDS Mutability

The game of [Figure 1.3](#) has one pure Nash equilibrium (T, L) . T weakly dominates B . If B is eliminated, then R is strictly dominated and the surviving outcome is (T, L) , the pure Nash equilibrium. However, as mentioned by [Hillas and Samet \[2020\]](#), this iterated deletion is inconsistent. Indeed, if R is eliminated, then B is not weakly dominated anymore and then (T, L) should not be the only surviving outcome. Weak flaws elimination of [Hillas and Samet \[2020\]](#) deletes profiles (B, R) (because if this profile is played, it means that strategy R is played, implying that B is weakly dominated), and then (T, R) (after eliminating (B, R) , R is strictly dominated by L). Outcomes (T, L) and (B, L) are sur-

¹⁶See also [Bonanno and Tsakas \[2018\]](#) who study the properties of the so-called *iterated deletion of inferior profiles (IDIP)* in a framework with ordinal utilities.

living. Thus, there is no mutability. IERDS deletes no strategy. Now, we slightly modify the payoffs matrix in a way that yields order dependence of IEWDS outcome:

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(<u>2</u> , <u>1</u>)	(<u>3</u> , <u>1</u>)
<i>B</i>	(<u>2</u> ,0)	(<u>2</u> , <u>1</u>)

 $\xrightarrow{\text{IEWDS}}$

	<i>j</i> 's Strategy	
<i>i</i> 's Strat.	<i>L</i>	<i>R</i>
<i>T</i>	(<u>2</u> , <u>1</u>)	(<u>3</u> , <u>1</u>)

 or

	<i>j</i> 's Strategy	
<i>i</i> 's Strat.	<i>L</i>	<i>R</i>
<i>T</i>	(<u>2</u> , <u>1</u>)	(<u>3</u> , <u>1</u>)

Figure 1.4: Modified Game with IEWDS Order Dependence (and Mutability)

In this modified version of the game, a second pure Nash equilibrium appears: (T, R) . T weakly dominates B and R weakly dominates L . If B is eliminated, then L is not dominated and both Nash equilibria survive. On the opposite, if L is eliminated, so is B , and the only surviving outcome is (T, R) (note that only the outcome (T, R) can be achieved as well by eliminating B and L at the same step). There, IEWDS is order dependent. It may also generate mutability. Indeed, if the final outcome is (T, R) , then the strategy L is not virtually dominated. Hillas and Samet [2020]'s deletion procedure eliminates both profiles (B, R) and (B, L) , letting the two Nash equilibria survive. IERDS deletes no strategy.

Now, we focus on the last example of this section:

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(<u>2</u> ,0)	(<u>3</u> , <u>1</u>)
<i>B</i>	(<u>2</u> , <u>1</u>)	(2,0)

 $\xrightarrow{\text{IEWDS}}$

	<i>j</i> 's Strat.
<i>i</i> 's Strat.	<i>R</i>
<i>T</i>	(<u>3</u> , <u>1</u>)

Figure 1.5: Modified Game with a Unique Prediction for IEWDS (and IERDS)

In this last version of the game, there are two Nash equilibria: (B, L) and (T, R) . T weakly dominates B . If B is eliminated, then, R dominates L and the only outcome is (T, R) . There, IEWDS is not mutable. Indeed, since R is played and thus uneliminated, T does weakly dominate B . It is order independent as well. Moreover, it predicts a unique outcome whereas the Hillas and Samet [2020]'s procedure eliminates only the profile (B, R) , letting the two Nash equilibria survive. To compare IERDS to another procedure, notice that RBEU of Cubitt and Sugden [2011] accumulates the strategy T , but then stops¹⁷. IERDS deletes B , leading to the unique outcome (T, R) .

To sum up, in these various examples, when weak dominance is mutable or order dependent, our elimination procedure deletes less strategy than IEWDS. When IEWDS is

¹⁷Even if T is played with a strictly positive probability, for all j 's beliefs where B is played with a higher probability, L is optimal and cannot be deleted.

both non mutable and order independent, our elimination procedure predicts the same outcome as IEWDS (see [Section A.5](#) for an attempt to generalize this discussion in two-player games), being more predictive than weak flaws elimination of [Hillas and Samet \[2020\]](#) or RBEU of [Cubitt and Sugden \[2011\]](#).

Now, we state the result of this section, i.e., the immutability of IERDS. With the help of *sequences of games*, we recall what is immutability: a procedure is immutable if there is no process associated with it such that at the end of the sequence of game, there is no strategy $s'_i \in S_i^0 \setminus S^\Lambda$ which is not dominated in the game formed by Γ^Λ and the strategy s'_i , i.e. the game Γ' (with the same players and utilities as Γ^Λ) and the strategy set $S^\Lambda \cup s'_i$. Now, we can state that there is no mutability in any sequence of games generated by IERDS:

Theorem 2. *IERDS is immutable in finite games.*

Proof is relegated to [Section A.1](#).

1.7 Mixed root dominance

Consider the mixed extension of a game Γ and denote Σ the set $\prod_{i \in I} \Sigma_i \equiv \prod_{i \in I} \Delta(S_i)$ the set of all (mixed) strategies. Thus, $\sigma_i \in \Sigma_i$ is a mixed strategy if it is a probability distribution over the set S_i of pure strategies. As in the pure strategy case, we denote Σ_{-i} the set $\prod_{j \in I \setminus \{i\}} \Sigma_j \equiv \prod_{j \in I \setminus \{i\}} \Delta(S_j)$, the strategy profiles set of i 's opponents. Let $\sigma_i(s_i)$ be the probability that s_i is effectively used when σ_i is played and denote $R_{\sigma_i} = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$ the support of σ_i ¹⁸. We apply the definition of a Best Reply Set to mixed strategies in the same way as in the pure strategy case:

Definition 9. *The Best Reply Set to $\sigma''_i \in \Sigma_i$, denoted $\mathbf{b}(\sigma''_i)$, is the set of all strategy profiles $\sigma^* \in \Sigma$ such that:*
 $\sigma_i^* = \sigma''_i$, and, if $S_{-i} \neq \emptyset$:

$$\exists j \in I \setminus \{i\}, \sigma_j^* \in \arg \max_{\sigma_j \in \Sigma_j} U_j(\sigma_j, \sigma_{-j}^*) \quad (\text{OM}') \tag{OM'}$$

Now, we extend the notion of Best Reply Set to strategy subsets:

Definition 10. *For any strategy subset $\bar{S}_i \subset S_i$, we denote $\mathbf{b}(\bar{S}_i) = \bigcup_{\sigma_i \in \Delta \bar{S}_i} \mathbf{b}(\sigma_i)$ the Best Reply Set to the strategy subset \bar{S}_i .*

Note that if the subset is a singleton, [Definitions 3](#) and [10](#) obviously coincide. Importantly, in order to define *mixed root dominance*, we will use the Best Reply Set to the strategy subset formed by the support of the mixed strategy:

¹⁸Note that this definition of the support cannot be weakened by allowing e.g. a continuous distribution as a support. We clarify this point below.

Definition 11. A strategy $s'_i \in S_i$ is said root dominated by the mixed strategy $\sigma''_i \in \Sigma_i$ whose support is $R_{\sigma''_i}$, if:

$$\forall s_{-i} \in S_{-i}: \quad U_i(\sigma''_i, s_{-i}) \geq U_i(s'_i, s_{-i}) \quad (\text{RD1}')$$

$$\forall \sigma_{-i}^* \text{ such that } \sigma^* \in \mathbf{b}(R_{\sigma''_i}): U_i(\sigma''_i, \sigma_{-i}^*) > U_i(s'_i, \sigma_{-i}^*) \quad (\text{RD2}')$$

Definition 11 is in fact a generalization of Definition 4. Besides, if the Best Reply Set to $\sigma''_i \in \Sigma_i$ was defined such that it contained only the best responses to σ''_i , root dominance would lack hereditariness. Assume a mixed strategy $\sigma''_i \in \Sigma_i$ composed of two pure strategies in S_i , s''_i and s'_i such that $\sigma''_i \succ_S s'_i$. Then, it is immediate that s''_i strictly dominates s'_i . Here, the point is that the mixed strategy is not necessary to establish that s'_i is strictly dominated: that is, even if σ''_i is eliminated, s''_i still strictly dominates s'_i ¹⁹. Concerning root dominance, the fact is that mixing does not affect only the payoffs, it affects also the set of best responses. In order to keep hereditariness, all the best responses to strategies contained in $\Delta(R_{\sigma''_i})$ have to be considered. We can see it with the next example, where we use directly Definition 4 to define root dominance by mixed strategies and not Definition 11:

	j's Strategy		
i's Strat.	L	C	R
T	(4,0)	(4,0)	(0,0)
M	(4,0)	(4,0)	(4,0)
B	(0,0)	(4,2)	(8,1)

with dominance relations:

R	$\succ L$
σ_R	$\succ L, R$
C	$\not\succeq L, R$

Figure 1.6: Order Dependence Issue with Definition 4 applied to Mixed Strategies

Strategy C weakly dominates both L and R . However, it does not root dominate them, player i best responding to C with the three strategies T , M and B . Now, look at any mixing σ_R of C and R . Then, i 's best response is only B . Instead, for any mixing σ_L of C and L , i 's best responses are T and M .

Thus, if $\mathbf{b}(\sigma_R) = (\sigma_R, B)$, σ_R root dominates R . Besides, R root dominates L . On the opposite, if $\mathbf{b}(\sigma_L) = \{(\sigma_L, T), (\sigma_L, M)\}$, σ_L does not root dominates L . Therefore, both L and R are root dominated but only R root dominates L . Consequently, if R is eliminated before L , L cannot be eliminated at any further step, showing that the procedure would be order dependent. When we apply Definition 11, the dominance relation is modified such that:

$$\sigma_R \not\succeq R \text{ and } \sigma_R \not\succeq L$$

Then, order dependence disappears and only strategy L is eliminated.

The next result states that mixed strict dominance implies mixed root dominance:

¹⁹A different property but implying similar consequences is established for root dominance in Lemma 7.

Lemma 6. $\sigma_i'' \succ_S s_i \Rightarrow \sigma_i'' \succ s_i'$.

The proof is straightforward since strict dominance implies trivially both RD1' and RD2'.

The next example shows how mixed IERDS behaves with respect to pure IERDS. It presents the final outcome associated to each procedure. Assume a Bertrand duopoly where the marginal cost is zero, the market size equal to 1 and admit as classically that when both firms set the same price, the market is equally shared. Then we have following payoffs matrix:

		j's Strategy				
i's S.	0	1	2	3	4	
0	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	
1	(0,0)	(0.5,0.5)	(1,0)	(1,0)	(1,0)	
2	(0,0)	(0,1)	(1,1)	(2,0)	(2,0)	
3	(0,0)	(0,1)	(0,2)	(1.5,1.5)	(3,0)	
4	(0,0)	(0,1)	(0,2)	(0,3)	(2,2)	

→
pure IERDS

		j's Strategy			
i's S.	1	2	3	4	
1	(0.5,0.5)	(1,0)	(1,0)	(1,0)	
2	(0,1)	(1,1)	(2,0)	(2,0)	
3	(0,1)	(0,2)	(1.5,1.5)	(3,0)	
4	(0,1)	(0,2)	(0,3)	(2,2)	

Figure 1.7: Symmetric Discrete Bertrand Game after Pure IERDS

Once the strategies 0 are eliminated, no strategy is any longer root dominated by a pure one. However, one can find a mixture of strategies 1, and 3 that root dominates the strategy 4 (it is enough to have a weight higher than $\frac{2}{3}$ for strategy 3 and a strictly positive weight for 1). After elimination of the strategies 4, some mixtures of strategies 1 and 2 can strictly dominates the strategy 3 (it is enough to have a weight higher than $\frac{1}{2}$ for strategy 2 and a strictly positive weight for 1). Once strategies 3 are eliminated, we can show that strategies 2 are root dominated by 1.

		j's Strategy				
i's S.	0	1	2	3	4	
0	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	
1	(0,0)	(0.5,0.5)	(1,0)	(1,0)	(1,0)	
2	(0,0)	(0,1)	(1,1)	(2,0)	(2,0)	
3	(0,0)	(0,1)	(0,2)	(1.5,1.5)	(3,0)	
4	(0,0)	(0,1)	(0,2)	(0,3)	(2,2)	

→
mixed IERDS

		j's S.
i's S.	1	1
1	(0.5,0.5)	

Figure 1.8: Symmetric Discrete Bertrand Game after Mixed IERDS

The next result demonstrates that a strategy $s_i' \in S_i$ root dominated by a mixed strategy whose support contains s_i' is also root dominated by another strategy whose support does not contain s_i' :

Lemma 7. *If $s_i' \in S_i$ is root dominated by $\sigma_i'' \in \Sigma_i$ such that $R_{\sigma_i''} = (\check{S}_i \cup s_i')$ (with $\check{S}_i \subset S_i$ which contains at least one strategy different from s_i'), then s_i' and σ_i'' are root dominated by $\sigma_i''' \in \Sigma_i$ such that $R_{\sigma_i'''} = \check{S}_i$.*

Proof. Assume $s'_i \in S_i$ is root dominated by $\sigma''_i \in \Sigma_i$. Thus, σ''_i weakly dominates s'_i . Then, we can construct $\sigma'''_i \in \Sigma_i$ such that the weight of each pure strategy forming σ'''_i is proportionally the same as in σ''_i when s'_i is removed from the support. It is clear that σ'''_i weakly dominates σ''_i (and s'_i). Indeed, the average payoff is (weakly) increased when s'_i is removed, since the payoff to i of s'_i against any profile is below the average payoff of σ''_i . Furthermore, we know that σ''_i strictly payoff dominates s'_i on $\mathbf{b}(R_{\sigma''_i})$. For the same reason, it is clear that σ'''_i strictly payoff dominates σ''_i (and s'_i) on $\mathbf{b}(R_{\sigma''_i})$. By construction, $\mathbf{b}(R_{\sigma'''_i}) \subset \mathbf{b}(R_{\sigma''_i})$. Therefore, σ''_i (and s'_i) are root dominated by σ'''_i . ■

This result allows us to keep the result of [Lemma 2](#), and then to show order independence of mixed IERDS:

Theorem 3. *Mixed IERDS is order independent.*

Proof. By adding [Lemma 7](#), all results of [Section 1.4](#) hold true when we apply the mixed framework. See [Section A.7](#) for more details. ■

Finally, we can establish the next statement:

Theorem 4. *Mixed IERDS refines mixed IESDS.*

This result is a direct implication of [Lemma 6](#) and [Theorem 3](#).

To be clear this result means that in some games, mixed IERDS refines strictly mixed IESDS and in other games, they have the same outcome.

In the next section, we will simply write IERDS for mixed IERDS.

1.8 Rationality concepts

First, we recall that in the standard framework, in a two-player game, a strategy s_i^* is rational if and only if there is a strategy for j such that σ_i^* maximizes the utility of i :

Definition 12. *A strategy s_i^* is rational if:*

$$\begin{aligned} \exists \sigma_j \in \Sigma_j, \text{ such that } \forall \sigma_i \in \Sigma_i, \\ U_i(s_i^*, \sigma_j) \geq U_i(\sigma_i, \sigma_j) \end{aligned}$$

By [Pearce \[1984, Lemma 3\]](#), in a two-player game, a strategy is not rational if and only if it is strictly dominated. In the remaining of the paper, we will restrict ourselves to two-player games as well. Furthermore, [Pearce \[1984, Lemma 4\]](#) shows that a strategy is weakly dominated if and only if it is not a best response to any totally mixed profile. That is, beliefs are said *cautious*, i.e., players believe that opponents play only full support strategies). This cautiousness is justified by the fact that players may not exclude totally the possibility that opponents can play any strategy. Yet, this view is apparently in contradiction with the belief that weakly dominated strategies should not be played.

Indeed, admissibility requires that agents consider possible that opponents play all their strategies with positive probability. It means that each agent believes that his opponents will play non admissible strategies. This is emphasized by Samuelson [1992] as the third issue with IEWDS:

The process appears initially to call for agents to assume that opponents may play any of their strategies but subsequently to assume that opponents will certainly not play some strategies.

This problem is known as the *inclusion-exclusion challenge* (see Barelli and Galanis [2013]) and has opened a rich literature attempting to reconcile weak dominance rationality with consistency.

Brandenburger [1992]; Stahl [1995]; Brandenburger et al. [2008] propose to use the *lexicographic probability system* introduced in Blume et al. [1991] to characterize weak dominance rationality. In a word, it is assumed players believe that when a strategy is eliminated, it is infinitely less likely to be played with respect to remaining strategies, but still infinitely more likely to be played than previously eliminated strategies. Therefore, the inclusion-exclusion challenge is solved in an elegant way: a weakly dominated strategy is unlikely to be played and at the same time not totally unlikely if “necessary”. In contrast with the view defended in Samuelson [1992], Brandenburger et al. [2008] state that: “A player is rational if he optimizes and also rules nothing out.” Alternatively, Barelli and Galanis [2013] introduce the notion of *event-rationality* which allows two levels of beliefs. A first which is standard, and a second one used in case of equivalence between two strategies. When there is equivalence, a player can break ties by using opponents’ strategies deemed unlikely. Therefore, again, even dominated strategies are never totally excluded of the players’ “thoughts”. The rationality concepts we introduce do consider thought experiments but contrasts with the option proposed in Barelli and Galanis [2013]: our experiments assume a certain sense of rationality about the opponent’s play at the second level of belief.

In the next subsection, we will assume that some perturbations of the game can occur with probability $\epsilon > 0$. Considering ruling out “unreasonable” Nash equilibria in extensive-form games, Selten [1975] formalizes this idea with the notion of *perfect equilibria*, which are Nash equilibria robust to the possibility that agents may deviate (by mistakes). Additionally, Fudenberg et al. [1988] introduce the idea that payoffs knowledge might not be complete, i.e., agents are unsure about their own payoffs and others’ payoffs. Therefore, the authors introduce forward induction in the reasoning: the deviation is not necessarily a mistake but might be a “signal”. The DF procedure of Dekel and Fudenberg [1990] is the outcome of such games where agents are uncertain about payoffs. Besides, Börgers [1994] shows that the DF procedure can also be the result of *approximate* common knowledge of weak dominance rationality (i.e. each player believes that his opponents play strategies with full support). That is, Börgers [1994] assumes that weak dominance rationality is common knowledge with probability p . When p converges to 1, agents plays

only strategies which remain after the DF procedure.

The kind of perturbations we introduce does not consider such payoff uncertainties. Rather, we are closer to Selten [1975]’s idea that a player may observe “mistakes” and react optimally. We also relate to Hamilton and Slutsky [2005] who study the possibility that an agent takes into account his own errors. More precisely, we consider simultaneous games where an agent can generate reactions by his own thoughts. We suppose that despite having a “reference” strategy, a player may alternatively consider some strategy subsets with probability ϵ . If so, the opponent reacts optimally (in a naive way) to this strategy subset. Thus, the reference strategy is “tested” against such mind trembles. “Mind trembles” can be seen as potential trembles which will be realized only if they are profitable. For instance, assume a poker player who sets a reference strategy before the game starts. However, he knows that during the play he may be tempted to adopt another strategy with probability ϵ : this is a mind tremble. Now, if the reference strategy is not optimal when he believes that the opponent can detect this tremble and react optimally, the tremble should be realized and, in fact the reference strategy never played in such a game.

Therefore, we assume a framework with *conjectural variations* (see our discussion below in Section 1.8.3). Then, we introduce the concept of *local ϵ -rationality* which selects the strategies maximizing i ’s utility when i forms conjectures about j ’s reactions to mind trembles, those occurring with probability ϵ . With respect to the usual conjectural variation framework, two differences operate: (i) an actual deviation is not required but a mind tremble is enough to generate the opponent’s reaction, and (ii) reaction is said *rational*, i.e., agent i conjectures that j will play a best response to the mind tremble.

In Section A.9 we propose two others perturbations that lead to two additional characterization of root undominance by rationality.

1.8.1 Characterization of root undominance by rationality

In order to characterize root undominance, we define in this subsection a new rationality concept called *local ϵ -rationality*. For this purpose, we introduce first a *conjectural system* C_{ij} for player i about strategies of player j when a perturbation occurs (with probability ϵ). We say that player i has a *mind tremble* when he thinks of a strategy subset $\check{S}_i \subset S_i$ whereas he has a *reference* strategy $\sigma_i \in \Sigma_i$. Finally, for each strategy subset $\check{S}_i \subset S_i$, i believes that j will play a certain strategy s_j with probability $C_{ij}(\check{S}_i, s_j)$ if i has a mind tremble towards \check{S}_i .

We define C_{ij} as a mapping from the tuple formed by the product of the power set $\mathcal{P}(S_i)$ ²⁰ of S_i and j ’s strategy set S_j to $[0, 1]$. Our conjectural system is naturally reminiscent of the conjectural variation theory (see our discussion below in Section 1.8.3), except that we consider strategy subsets.

²⁰The power set of S_i is the set containing all the subsets of S_i .

Definition 13. A conjectural system C_{ij} is the mapping $C_{ij} := \mathcal{P}(S_i) \times S_j \rightarrow [0, 1]$ which associates any i 's strategy subset with a pure strategy for j to a probability, that is, it satisfies: $\forall \check{S}_i \in \mathcal{P}(S_i), \sum_{s_j \in S_j} C_{ij}(\check{S}_i, s_j) = 1$.

According to i 's belief, $C_{ij}(\check{S}_i, s_j)$ is the probability that j will play s_j if he ‘‘observes’’ that i thinks of a strategy whose support is \check{S}_i . We denote \mathcal{C}_{ij} the set of all conjectural systems of i about j .

Before going further, we define an additional notion which will be useful below, namely, the *expected ϵ -perturbed utility*:

Definition 14. The *expected ϵ -perturbed utility* of i from playing σ_i when j plays σ_j in the game without perturbation and $\hat{\sigma}_j$ in the game with perturbation is:

$$V_i^\epsilon(\sigma_i, \sigma_j, \hat{\sigma}_j) \equiv (1 - \epsilon) \mathbb{E}[U_i(\sigma_i, \sigma_j)] + \epsilon \mathbb{E}[U_i(\sigma_i, \hat{\sigma}_j)]$$

Simply, the ϵ -perturbed utility formalizes the expected utility when player i has the belief that j plays σ_j with probability $1 - \epsilon$ and $\hat{\sigma}_j$ with probability ϵ . In the remainder of the paper, we will see σ_j as the ‘‘normal’’ or standard belief (the belief when no exogenous event occurs), and we will assume that $\hat{\sigma}_j$ is played when a perturbation occurs. We emphasize that it does not mean that an extensive form game is played. Rather, the thoughts of i (about his own play) influence his beliefs about j 's actions with probability ϵ .

For a given mixed strategy $\sigma_i \in \Sigma_i$ of player i , we recall that we denote $\sigma_i(s_i)$ the probability that s_i to be drawn when σ_i is chosen. Now, we introduce a new rationality concept in association to a conjectural system:

Definition 15. A strategy $s_i \in S_i$ is *locally ϵ -rational* if:

$$\begin{aligned} &\exists \sigma_j \in \Sigma_j, \exists C_{ij} \in \mathcal{C}_{ij}, \text{ such that } \forall \sigma_i \in \Sigma_i, \text{ if we set:} \\ &\sigma_j^* \text{ with } \sigma_j^*(s_j) \equiv C_{ij}(R_{\sigma_i}, s_j) \text{ then we have,} \\ &V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \end{aligned}$$

From now, distinctly from the conjectural variation theory, we assume that the conjectures are *rational* (see our discussion below in [Section 1.8.3](#)), i.e., $C_{ij}(\check{S}_i, s_j)$ cannot be strictly positive unless $s_j \in \mathbf{b}(\check{S}_i)$:

Assumption R. C_{ij} is a *rational conjectural system* (with \mathcal{R}_{ij} the set of such rational conjectural systems), i.e.:

$$\forall (\check{S}_i, s_j) \in \mathcal{P}(S_i) \times S_j, C_{ij}(\check{S}_i, s_j) > 0 \Rightarrow s_j \in \mathbf{b}(\check{S}_i)$$

Now, we can re-write our definition:

Definition 16. Under [Assumption R](#), a strategy $s_i \in S_i$ is locally ϵ -rational if:

$$\begin{aligned} & \exists \sigma_j \in \Sigma_j, \exists C_{ij} \in \mathcal{R}_{ij}, \text{ such that } \forall \sigma_i \in \Sigma_i, \text{ if we set:} \\ & \sigma_j^* \text{ with } \sigma_j^*(s_j) \equiv C_{ij}(R_{\sigma_i}, s_j) \text{ then we have:} \\ & V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \end{aligned}$$

Under [Assumption R](#), a strategy $s_i \in S_i$ is locally ϵ -rational if there is a belief σ_j and a rational conjectural system C_{ij} such that the expected utility of s_i is larger than any tested $\sigma_i \in \Sigma_i$. That is, s_i is optimal if i believes that j plays σ_j with probability $1 - \epsilon$ and reacts optimally to the tested σ_i with probability ϵ .

Naturally, we can establish the following result that simplifies the previous definition:

Lemma 8. Under [Assumption R](#), a strategy $s_i \in S_i$ is locally ϵ -rational if and only if it verifies:

$$\exists \sigma_j \in \Sigma_j, \text{ such that } \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \mathbf{b}(R_{\sigma_i}), \quad V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \quad (1.1)$$

Proof. Assume s_i is locally ϵ -rational. Then, there is $\sigma_j \in \Sigma_j$ and a rational conjectural system C_{ij} against which s_i (weakly) payoff dominates all other strategies in Σ_i . That is, if we compare s_i to any $\sigma_i \in \Sigma_i$, we use with probability $1 - \epsilon$ the strategy σ_j and with probability ϵ the strategy σ_j^* such that $\sigma_j^*(s_j) \equiv C_{ij}(R_{\sigma_i}, s_j)$. By [Assumption R](#), we know that all s_j are in $\mathbf{b}(R_{\sigma_i})$. Therefore, σ_j^* is in $\mathbf{b}(R_{\sigma_i})$. Finally, we can write that:

$$\begin{aligned} & \exists \tilde{\sigma}_j \in \Sigma_j, \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \mathbf{b}(R_{\sigma_i}) \text{ such that:} \\ & V_i^\epsilon(s_i, \tilde{\sigma}_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \tilde{\sigma}_j, \sigma_j^*) \end{aligned} \quad (1.1)$$

Conversely, assume the above [Equation \(1.1\)](#). If this is true we can construct a rational conjectural system C_{ij} by using the hyperplane theorem. Assume a strategy subset $\check{S}_i \in \mathcal{P}(S_i)$. Consider the vectors $\vec{V}_i^\epsilon(\sigma_i, \check{S}_i) = \{V_i^\epsilon(\sigma_i, s_j, s_j^*)\}_{s_j \in S_j, s_j^* \in \mathbf{b}(\check{S}_i)}$ ²¹ for each $\sigma_i \in S_i \cup \Delta(\check{S}_i)$. Simply, these vectors are such that each component $l+m$ is the payoff i can obtain when playing σ_i and when j plays the pure strategy $s_j^l \in S_j$ with probability $1 - \epsilon$ and the pure strategy $s_j^{*m} \in \mathbf{b}(\check{S}_i)$ with probability ϵ . We denote $Y(s_i, \check{S}_i)$ the set of such vectors. Besides, we can construct the following set X . If k is equal to $\#(S_j) \times \#(\mathbf{b}(\check{S}_i))$ ²², then X is the set $\{x \in \mathbb{R}^k \mid x > \vec{V}_i^\epsilon(s_i)\}$, that is the set of all payoffs that strictly dominate s_i payoffs. Both X and $Y(s_i, \check{S}_i)$ are convex sets. By [Equation \(1.1\)](#), these sets are disjoint. Then, we can apply the separating hyperplane theorem which states that there is a vector in \mathbb{R}^k , $\pi \geq 0$ with $\pi \neq 0$ and such that:

$$\forall y \in Y(s_i, \check{S}_i), \forall x \in X, \pi \cdot y \leq \pi \cdot \vec{V}_i^\epsilon(s_i) \leq \pi \cdot x$$

²¹Note that we make a slight abuse of notation here: we consider $s_j^* \subset \mathbf{b}(R_{\sigma_i})$ if $\exists \sigma_j \subset \mathbf{b}(R_{\sigma_i})$ and $s_j \in R_{\sigma_i}$. For technical reasons, we consider only pure strategies but all mixed strategies in the Best Reply Set are well present through the pure strategies that support them.

²²We denote $\#(S_i)$ the number of elements in the set S_i . According to the above footnote, $\#(\mathbf{b}(\check{S}_i))$ is well finite.

It directly implies that $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i)$, $\pi \cdot \left(\vec{V}_i^\epsilon(s_i) - \vec{V}_i^\epsilon(\sigma_i) \right) \geq 0$.

Especially, there is such a vector $\tilde{\pi}$ such that $\forall l \in [[1, \#(S_j)]]$, $\tilde{\pi}(l) = (1 - \epsilon) \times \tilde{\sigma}_j(\sigma_j^l)$, since the hypothesis that [Equation \(1.1\)](#) is verified implies (by continuity of V_i^ϵ in ϵ) that:

$$\forall \sigma_i \in \Sigma_i, U_i(s_i, \tilde{\sigma}_j) \geq U_i(\sigma_i, \tilde{\sigma}_j)$$

Thus we can have $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i)$, $\tilde{\pi} \cdot \left(\vec{V}_i^\epsilon(s_i) - \vec{V}_i^\epsilon(\sigma_i) \right) \geq 0$ when $\epsilon \rightarrow 0^+$.

Now, we can start constructing the rational conjectural system C_{ij} by setting

$$\forall s_j^{*m} \subset \mathbf{b}(\check{S}_i), C_{ij}(\check{S}_i, s_j^{*m}) \equiv \pi(\#(S_j) + m)$$

It is clear that it is rational since $s_j^{*m} \subset \mathbf{b}(\check{S}_i)$. We can apply all the previous reasoning to each $\check{S}_i \in \mathcal{P}(S_i)$ with $\forall \check{S}_i \in \mathcal{P}(S_i), \forall l \in [[1, \#(S_j)]]$, $\tilde{\pi}(l) = (1 - \epsilon) \times \tilde{\sigma}_j(\sigma_j^l)$. Finally, we obtain a full rational conjectural system and we can write that:

$$\begin{aligned} & \exists \tilde{\sigma}_j \in \Sigma_j, \exists C_{ij} \in \mathcal{R}_{ij}, \text{ such that } \forall \sigma_i \in \Sigma_i, \text{ if we set:} \\ & \sigma_j^* \text{ with } \sigma_j^*(s_j) \equiv C_{ij}(R_{\sigma_i}, s_j) \text{ then we have:} \\ & V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \end{aligned}$$

■

Now, we can state the main result of this section, the characterization of root undominance by local ϵ -rationality:

Theorem 5. *Under [Assumption R](#), a strategy $s_i \in S_i$ is locally ϵ -rational when $\epsilon \rightarrow 0^+$ if and only if it is root undominated.*

Proof. Assume [Equation \(1.1\)](#) for s_i and by contrapositive that s_i is root dominated. Therefore $\exists \sigma_i'' \in \Sigma_i$ such that $\forall \sigma_j \in \Sigma_j, U_i(\sigma_i'', \sigma_j) \geq U_i(s_i, \sigma_j)$ ([RD1'](#)), and $\forall \sigma_j^* \in \mathbf{b}(R_{\sigma_i''}), U_i(\sigma_i'', \sigma_j^*) > U_i(s_i, \sigma_j^*)$ ([RD2'](#)). Then, clearly:

$$\begin{aligned} & \forall \epsilon > 0, \forall \sigma_j \in \Sigma_j, \forall \sigma_j^* \subset \mathbf{b}(R_{\sigma_i''}) \\ & V_i^\epsilon(\sigma_i'', \sigma_j, \sigma_j^*) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \end{aligned}$$

It is an immediate contradiction with [Equation \(1.1\)](#). By [Lemma 8](#), s_i is not locally ϵ -rational.

Conversely, assume by contrapositive that s_i is not locally ϵ -rational. Then, we can write that there is (at least) a $\sigma_i'' \in \Sigma_i$ such that:

$$\begin{aligned} & \forall \sigma_j \in \Sigma_j, \forall \sigma_j^* \subset \mathbf{b}(R_{\sigma_i''}), \\ & V_i^\epsilon(\sigma_i'', \sigma_j, \sigma_j^*) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \end{aligned}$$

To see why **RD2'** is respected, we can see that the previous expression implies that $\forall \sigma_j^* \in \mathbf{b}(R_{\sigma_i})$, then:

$$V_i^\epsilon(\sigma_i, \sigma_j^*, \sigma_j^*) > V_i^\epsilon(s_i, \sigma_j^*, \sigma_j^*) \quad (\Leftrightarrow U_i(\sigma_i, \sigma_j^*) > U_i(s_i, \sigma_j^*))$$

Now (with **RD2'** being respected), when $\epsilon \rightarrow 0^+$, **RD1'** is respected by continuity of V_i^ϵ in ϵ . Otherwise, we could find a $\sigma_j \in \Sigma_j$ such that $\forall \sigma_j^* \subset \mathbf{b}(R_{\sigma_i''})$, $V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i'', \sigma_j, \sigma_j^*)$. ■

Remark that the expression $\epsilon \rightarrow 0^+$ implies that for a given game, $\exists \bar{\epsilon} > 0$ such that $\forall \epsilon < \bar{\epsilon}$, there is equivalence between root undominance and local ϵ -rationality.

1.8.2 Other notions of rationality

In this section we introduce different but close notions of rationality with respect to the one introduced in the previous subsection. It will help us to understand what local ϵ -rationality is and is not. As well, it will be useful in the following subsections. We distinguish local ϵ -rationality from global ϵ -rationality and self-local ϵ -rationality. Global ϵ -rationality induces the belief that the strategy support of the strategy actually played is observed by the opponent with probability ϵ . Instead, self-local ϵ -rationality is such that i believes that j observes the strategy support of the reference strategy. We can summarize these differences in [Table 1.2](#). Two main differences appear: first, what the

Rationality	Type of Utility Maximization	Observed Strategy
Global	Ex-Ante (Conjectural Variation)	Played Strategy
Self-local	Ex-Post	Reference Strategy
Local	Projected	Targeted Strategy

Table 1.2: Rationalities

agent conjectures his opponent may observe if he detects the agent's thoughts. Second, which kind of utility is maximized for each rationality concept. Self-local ϵ -rationality is an ex post concept because once the opponent believes the agent is committed to a given strategy, the agent can still decide to move ex post. By contrast, the global concept is ex ante since the strategy since once the agent is committed to an action, he cannot move. Finally, local ϵ -rationality correspond to a *projected* utility maximization. That is, even if the agent might play his reference strategy, testing it against other strategies induces beliefs such that this reference strategy was not available anymore. Alternatively, we can remark that our three notions of rationality can be interpreted and distinguished with the conjectures about the opponent's speed of adjustment. Self-local rationality corresponds to the case where the agent conjectures that his opponent his stickier and is not able to adjust his strategies. Then, opponents best respond to the reference (considered initially) strategy. Global rationality corresponds to the case where the opponent adjusts perfectly

and then always best responds. Local rationality is such that the agent conjectures that his opponent anticipates the adjustment, even if no move is finally made. That is why the opponent only best responds to the targeted strategy.

Now, we define our two additional concepts:

Definition 17. *A strategy $s_i \in S_i$ is globally ϵ -rational if and only if:*

$$\begin{aligned} &\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \mathbf{b}(s_i), \forall \sigma_i \in \Sigma_i, \exists \sigma_j^{**} \subset \mathbf{b}(R_{\sigma_i}) \text{ such that:} \\ &V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^{**}) \end{aligned}$$

Global rationality of s_i means that s_i may maximize the ex-ante utility of i , given that whatever the strategy chosen by i , j reacts optimally to it with probability ϵ .

Definition 18. *A strategy $s_i \in S_i$ is self-local ϵ -rational if and only if:*

$$\begin{aligned} &\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \mathbf{b}(s_i), \text{ such that } \forall \sigma_i \in \Sigma_i, \\ &V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \end{aligned}$$

Self-local rationality of s_i is the converse of local ϵ -rationality of s_i in terms of reference point. That is, when i considers the strategy s_i , he believes that j reacts optimally to s_i with probability ϵ . The strategy is self-local rational if there is a belief satisfying this condition such that no move increases the i 's payoff. In other words, s_i may maximize i 's ex post utility given that s_i is the reference point to which j best responds with probability ϵ .

Besides, note that these three notions of rationality are obviously a refinement of the classical one (see [Definition 12](#)) when $\epsilon \rightarrow 0^+$:

Fact 1. *A strategy is globally/self-locally/locally ϵ -rational when $\epsilon \rightarrow 0^+$ only if it is rational.*

1.8.3 Links with the conjectural variation theory

Since [Bowley \[1924, p.38\]](#) has introduced the idea of conjectural variations, the industrial organization literature has largely been fueled by this theory²³ which considers that a market situation can remain stable if the conjectures all firms have on their opponents refrain all of them from deviating. Contrary to the Cournot approach, the equilibrium does not emerge from a *tatônnement*, but is postulated ex ante. The interest is to understand why competitors may not deviate from a situation far from a Cournot equilibrium. For instance, [Sweezy \[1939\]](#) introduces the kinked demand by arguing that firms react differently when they face opponents' downward or upward price moves. Nevertheless, conjectures can be insane and consequently sustain an infinite number of conjectural

²³See e.g. [Figuères et al. \[2004\]](#) for a review. Besides, for a recent contribution of this theory to public economics, see [McGinty \[2021\]](#).

variation equilibria. That is why several authors had tried to *rationalize* the agents' conjectures. Notably, they stated that the conjectured reactions should be "optimal" in a certain sense (see for example Hahn [1978]; Laitner [1980]; Ulph [1983]). Mainly, the conjecture of player i should be such that he expects that his opponent j maximizes his utility given j 's conjectures (i.e. j anticipates the reaction of i after his own deviation responding to i 's deviation), once he has attained the new "statu quo". Yet, these attempts have been showed to miss their mark. Strikingly, Makowski [1987] notices two main problems with the concepts developed in the papers cited above. The first one is that the reaction of the opponent is optimal with respect to the new "statu quo", and not from the initial equilibrium. In other words, an agent does not conjecture that an opponent who faces his deviation will best respond to the deviation, simply that once he has moved to the new equilibrium, he does not wish to move (but the move is not *rationalized*). Alternatively, Makowski [1987] proposes to consider this type of conjecture with best responses to the deviation with the notion of *only slightly more rational, rational conjecture equilibrium* or SMR-RCE. However, he points out himself another flaw: conjectures are not time consistent. That is, when player i maximizes his utility, he considers his potential deviation followed by the reaction of his opponent j . And j maximizes his utility by considering also that his potential deviation will be followed by the reaction of his opponent i . In words of Makowski [1987], i expects that the game ends at time $t = 2$ (after j 's response to his deviation), but conjectures as well that j expects that the game ends at time $t = 3$ (after i 's response to j 's response). In fact, we can simply observe that there is no reason that the process stops at any given time. Undeniably, with SMR-RCE, i does not consider he can deviate from the new "statu quo" he will establish by deviating a first time (whereas he may naturally want to deviate if he has a better response to the new "statu quo"). This criticism might seem severe, since many concepts²⁴ assume an end in the reasoning process when a deviation is tested. However, this criticism generally vanishes when the agents react by playing best responses, ending *de facto* the reasoning process of the deviator once an equilibrium is reached (if it exists of course). If each reaction is conjecture dependent, the next reaction is conjecture dependent as well. If a reaction is not based on arbitrary conjectures, but solely on optimality, then the reasoning process may terminate immediately.

Clearly, the beliefs assumed under global ϵ -rationality have the flavor of an " ϵ -rational conjectural variation". The previous discussion shows the trouble with two players "behaving in the same way"²⁵. That is, if the deviator i believes that j will react optimally, there could be a difficulty if j believed that i will best respond in turn. This problem is technically solved when ϵ converges to 0, since it becomes obvious that i should not move (ex-ante) in reaction to the conjectured response of j which can only occur with a small probability. The meaning of such a theory when ϵ moves away from zero is an open

²⁴Simply, think of the ones introduced in this paper and other as the intuitive criterion of Cho and Kreps [1987] (see our discussion below in Section 1.8.5).

²⁵Assuming asymmetry seems justifiable since the deviator decides alone to deviate and then, introduce asymmetry *de facto*.

question. We attempt to give some answers in [Section A.10](#).

Before this, how to situate local ϵ -rationality in this framework? Local ϵ -rationality seems to be the converse of a ϵ -rational conjectural variation theory. ϵ -Rational conjectural variation could be stated (partially) as follows: if player i deviates, j will react optimally with probability ϵ . Now, local ϵ -rationality states that: whether player i deviates or not, j will react optimally to the deviation with probability ϵ . Thus, why would j reacts to a deviation that may not appear? Why would it be more reasonable? We attempt to answer these questions in the next subsections.

1.8.4 Observability of actions

[Hamilton and Slutsky \[1990\]](#) consider a duopoly where firms can choose the timing of their action before playing the actual game. That is, a firm can decide to move at the first period. In this case, if the competitor does not move first as well, the game is a Stackelberg duopoly (i.e. the follower observes the action at the first period). Otherwise, the game is simultaneous (e.g., it becomes a Cournot game if the considered variable is quantity). Several types of equilibria appear according to the parameters and the considered variable: either equilibria with a leader and a follower or simultaneous equilibria. In the second configuration, there are cases where being a leader is suboptimal, and both firms wait the second period to move, and by contrast, cases where being a follower is suboptimal, and both firms plays at the first period. Let us focus on the latter case, the most classical one.

In this context, the idea of global ϵ -rationality can be thought in the following way. Even if, at the equilibrium, firms play simultaneously at the first period, one firm may tremble²⁶ and become a follower. Then, if a firm has several Cournot strategies, it will choose the one that maximizes its payoff taking into account that it might be a leader with probability ϵ . Therefore, global ϵ -rationality can be thought as a trembling-hand refinement, motivated by the ex-post rationality of the trembling agent. The link with local ϵ -rationality appears when the situation is more constrained: assume an incumbent with a given strategy. However, this incumbent fears an entry. Besides, it has another strategy that is strictly better than his current strategy if a potential entrant best responds to this deviation and is equivalent otherwise. It is clear that this deviation can be anticipated by the entrant, making the deviation of the incumbent perfectly rational. It is what [Hamilton and Slutsky \[1990\]](#) may mean when they state:

Of course, if the existing firms had sufficient postentry flexibility, then entrants will not react to current choices but to their perception of postentry behavior.

This example shows in a simple way how a firm can be incentivized to change its strategy if the entrant's perception about the actual situation is accurate enough. Here, the entrant reacts to the postentry behavior. Therefore, the "observed" strategy is not the reference

²⁶In the context of a duopoly, the idea of tremble seems quite natural since real life contingencies often delay decision making processes.

strategy but the targeted one, since it is what the entrant anticipates. It does correspond to our local ϵ -rationality concept.

1.8.5 Further ideas

Two remarks have to be made. The first one is that among the three notions of rationality we have developed so far, only one leads to an order independent iterated elimination procedure (the proof of this observation is left to the reader but we give some elements of understanding below). How can we explain this lack of consistency?

With respect to signaling games, self-local rationality seems to be linked to the *intuitive criterion* of [Cho and Kreps \[1987\]](#). Indeed, under self-local rationality, a strategy is not played when it is not a best response if a best response to this strategy is played with probability ϵ . Then, the reference point is the potentially dominated strategy. In the intuitive criterion, the reference point is the tested equilibrium. Broadly, in a signaling game with two types of agents (the senders) and a principal (the receiver), an equilibrium fails the *intuitive test* with respect to a deviation if (i) this deviation from the initial equilibrium is never profitable for one type, and if (ii) the other type prefers the new equilibrium when the receiver best responds to the deviation. Let us be clear: this criterion might be said global in a sense since we first look at an equilibrium (where everybody best responds) and check if a deviation is profitable (where *only* the receiver best responds). Nevertheless, what interests us in this story is the response of the other type. Indeed, the intuitive criterion *forgets* the optimal reaction of the type for who the deviation is never profitable. That is, the intuitive criterion assumes this type still best responds to the initial equilibrium whereas the deviation leads to another equilibrium. In this sense, the intuitive criterion is self-local. This point had notably been made by [Mailath \[1988\]](#) and led to the notion of *undefeated equilibrium* in [Mailath et al. \[1993\]](#). In fact, this logic is reminiscent of the *E2* equilibrium in [Wilson \[1977\]](#). Loosely speaking, an equilibrium is said *E2* if there is no profitable deviation for a player in the following sense: after the opponents' "optimal" reaction²⁷ to the deviation, the deviation is still profitable, with respect to the initial equilibrium. Since all actions were optimal at the initial equilibrium, and are still optimal when the deviation is tested, we can see the *E2* equilibrium as a global concept, while the intuitive criterion is well self-local.

In the pure strategy case, global rationality can be stated as follows: player i never wants to play strategy $s_i \in S_i$ once j plays best response to i 's strategies with probability ϵ and i can find another strategy that yields strictly more. However, if the strategy s_i is deemed unplayable, the reason of the elimination may vanish immediately since i requires a best response to s_i to be played²⁸. This reasoning similar with self-local rationality. When a player checks whether he should eliminate a strategy, he should not fear losing

²⁷The paper is applied to insurances: thus, the "optimal" reaction is to withdraw insurance policies which reward a negative profit.

²⁸This directly shows why an iterated elimination procedure based on global and self-local rationality would be order dependent: these rationality concepts lack hereditariness.

the payoff if he plays the eliminated strategy, but rather see what he gets if he plays the eliminating strategy. In a word, the situation at the deviation (i.e. by playing the eliminating strategy) should be checked, not the others. In our view, the agent should test a deviation such that this deviation *works*²⁹ and not such that the reference point still exists. Indeed, if a deviation is tested and works, it is not necessary anymore to keep in mind the reference point³⁰. Finally, when an agent tests a deviation, he *should* anticipate that his opponent will also test it and react accordingly, whatever the true action of the agent.

The second remark is that IERDS is not the only order independent procedure. Additionally, it might not be the only procedure whose dominance is grounded on the existence of a set of profiles which will survive each process (not all the profiles survive but at least one profile does and the set does not expand, letting the required property on the set untouched by each process). In the spirit of RBEU of [Cubitt and Sugden \[2011\]](#), one may think for example that payoff domination on all the profiles where an opponent plays a dominant strategy (if it exists) will survive to any reasonable elimination procedure, and it would be enough. Then, our condition [RD2](#) could be weakened by adding this possibility. This question is still open and might be the object of further research.

1.9 Conclusion

In this paper, we introduce a new dominance relation named *root dominance* between weak and strict dominances. It requires weak dominance and an additional condition based on the *Best Response Set* to the dominating strategy. We associate to this dominance relation an *iterated elimination procedure* named *IERDS*. The main result of this paper is that IERDS is an order independent procedure in finite games and refines IESDS. Additionally, we show that IERDS does not face the inconsistency named *mutability*. Mutability concerns especially IEWDS but also other existing elimination procedure. In a word, mutability means that an eliminated (and thus dominated) strategy in a *process* is finally not dominated at the end of the process. Finally, we introduce new rationality concepts such that our rational strategies correspond to root undominated strategies. Furthermore, we establish a link between our rationality concepts and a *rational* kind of conjectural variations theory, a framework well-known in industrial organization literature and public economics.

²⁹We mean by *works* that the agent gets a strictly higher payoff by deviating rather than playing the reference strategy.

³⁰This point is reminiscent of the idea of *memorylessness* developed in [Chapter 2](#).

Bibliography

- Apt, K. R. (2005). Order Independence and Rationalizability. In *Proceedings of the Tenth Conference on the Theoretical Aspects of Rationality and Knowledge*, pages 22–38.
- Apt, K. R. (2011). Direct Proofs of Order Independence. *Economics Bulletin*, 31(1):106–115.
- Asheim, G. B. and Dufwenberg, M. (2003). Admissibility and Common Belief. *Games and Economic Behavior*, 42(2):208–234.
- Baliga, S. and Morris, S. (2002). Co-ordination, Spillovers, and Cheap Talk. *Journal of Economic Theory*, 105(2):450–468.
- Barelli, P. and Galanis, S. (2013). Admissibility and Event-Rationality. *Games and Economic Behavior*, 77(1):21–40.
- Bernheim, B. D. (1984). Rationalizable Strategic Behavior. *Econometrica*, 52(4):1007–1028.
- Blume, L., Brandenburger, A., and Dekel, E. (1991). Lexicographic Probabilities and Choice Under Uncertainty. *Econometrica*, 59(1):61–79.
- Bonanno, G. and Tsakas, E. (2018). Common Belief of Weak-Dominance Rationality in Strategic-Form Games: A Qualitative Analysis. *Games and Economic Behavior*, 112:231–241.
- Börgers, T. (1993). Pure Strategy Dominance. *Econometrica*, 61(2):423–430.
- Börgers, T. (1994). Weak Dominance and Approximate Common Knowledge. *Journal of Economic Theory*, 64(1):265–276.
- Bowley, A. (1924). *The Mathematical Groundwork of Economics*. Oxford University Press.
- Brandenburger, A. (1992). Lexicographic Probabilities and Iterated Admissibility. In Dasgupta, P., Gale, D., Hart, O. D., and Maskin, E. S., editors, *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, pages 282–290. Cambridge, MA: MIT Press.
- Brandenburger, A., Friedenberg, A., and Keisler, H. J. (2008). Admissibility in Games. *Econometrica*, 76(2):307–352.
- Carlsson, H. and van Damme, E. (1993). Global Games and Equilibrium Selection. *Econometrica*, 61(5):989–1018.
- Cho, I.-K. and Kreps, D. M. (1987). Signaling Games and Stable Equilibria. *Quarterly Journal of Economics*, 116(2):1133–1165.

- Cubitt, R. P. and Sugden, R. (2011). The Reasoning-Based Expected Utility Procedure. *Games and Economic Behavior*, 71(2):328–338.
- Dekel, E. and Fudenberg, D. (1990). Rational Behavior with Payoff Uncertainty. *Journal of Economic Theory*, 52(2):243–267.
- Dufwenberg, M. and Stegeman, M. (2002). Existence and Uniqueness of Maximal Reductions under Iterated Strict Dominance. *Econometrica*, 70(5):2007–2023.
- Farrell, J. and Rabin, M. (1996). Cheap Talk. *Journal of Economic Perspectives*, 10(3):103–118.
- Figuières, C., Jean-Marie, A., Quérou, N., and Tidball, M. (2004). *Theory of Conjectural Variations*, volume 2.
- Fudenberg, D., Kreps, D. M., and Levine, D. K. (1988). On the Robustness of Equilibrium Refinements. *Journal of Economic Theory*, 44(2):354–380.
- Gilboa, I., Kalai, E., and Zemel, E. (1990). On the Order of Eliminating Dominated Strategies. *Operations Research Letters*, 9(2):85–89.
- Hahn, F. (1978). On Non-Walrasian Equilibria. *Review of Economic Studies*, 45(1):1–17.
- Hamilton, J. and Slutsky, S. (1990). Endogenous timing in duopoly games: Stackelberg or Cournot equilibria. *Games and Economic Behavior*, 2(1):29–46.
- Hamilton, J. and Slutsky, S. (2005). Equilibrium with Strategy-Dependent Trembles. *International Journal of Game Theory*, 33(4):461–465.
- Hillas, J. and Samet, D. (2020). Dominance Rationality: A Unified Approach. *Games and Economic Behavior*, 119:189–196.
- Kahneman, D. and Tversky, A. (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, 47(2):263–292.
- Kohlberg, E. and Mertens, J.-F. (1986). On the Strategic Stability of Equilibria. *Econometrica*, 54(5):1003–1037.
- Laitner, J. (1980). “Rational” Duopoly Equilibria. *Quarterly Journal of Economics*, 95(4):641–662.
- Luo, X., Qian, X., and Qu, C. (2020). Iterated Elimination Procedures. *Economic Theory*, 70(2):437–465.
- Mailath, G. J. (1988). A Reformulation of a Criticism of the Intuitive Criterion and Forward Induction.
- Mailath, G. J., Okuno-Fujiwara, M., and Postlewaite, A. (1993). Belief-Based Refinements in Signalling Games. *Journal of Economic Theory*, 60(2):241–276.

- Makowski, L. (1987). Are ‘Rational Conjectures’ Rational? *Journal of Industrial Economics*, 36(1):35–47.
- Marx, L. M. and Swinkels, J. M. (1997). Order Independence for Iterated Weak Dominance. *Games and Economic Behavior*, 18(2):219–245.
- McGinty, M. (2021). Rational conjectures and evolutionary beliefs in public goods games. *Journal of Public Economic Theory*, forthcoming.
- Moulin, H. (1979). Dominance Solvable Voting Schemes. *Econometrica*, 47(6):1337–1351.
- Moulin, H. (1986). *Game Theory for the Social Sciences*. NYU Press.
- Pearce, D. G. (1984). Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52(4):1029–1050.
- Rubinstein, A. (1991). Comments on the Interpretation of Game Theory. *Econometrica*, 59(4):909–924.
- Samuelson, L. (1992). Dominated Strategies and Common Knowledge. *Games and Economic Behavior*, 4(2):284–313.
- Selten, R. (1975). Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games. *International Journal of Game Theory*, 4(1):25–55.
- Stahl, D. O. (1995). Lexicographic Rationalizability and Iterated Admissibility. *Economics Letters*, 47(2):155–159.
- Stalnaker, R. (1994). On the Evaluation of Solution Concepts. *Theory and Decision*, 37(1):49–73.
- Sweezy, P. M. (1939). Demand under Conditions of Oligopoly. *Journal of Political Economy*, 47(4):568–573.
- Ulph, D. (1983). Rational Conjectures in the Theory of Oligopoly. *International Journal of Industrial Organization*, 1:131–154.
- Wilson, C. (1977). A Model of Insurance Markets with Incomplete Information. *Journal of Economic Theory*, 16(2):167–207.

Appendices

Appendix A

Appendices to Root Dominance

A.1 Omitted proofs

Theorem 2. *IERDS is immutable in finite games.*

Proof. Assume there is a strategy $s'_i \in S_i$ eliminated through IERDS. Assume a given process of IERDS and the sequence of games associated $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$. Consider the game formed by Γ^Λ and the strategy s'_i , i.e. the game Γ' (with the same players and utilities as Γ^Λ) and the strategy set $S^\Lambda \cup s'_i$. We reason by induction.

Stage 1: Assume s'_i has been eliminated by s''_i at step $\Lambda - 1$. Suppose also by contradiction that s'_i is not root dominated in Γ' . By [Lemma 2](#), $s''_i \in \Gamma^\Lambda \subset \Gamma'$. We repeat the same arguments as in the proof of [Lemma 4](#): comparing s''_i and s'_i , it can be verified that [RD1](#) is still respected. By [Lemma 3](#), we know that $\mathbf{b}^\Lambda(s''_i) \subseteq \mathbf{b}^{\Lambda-1}(s''_i)$. Additionally, $\mathbf{b}^\Lambda(s'_i)$ cannot be empty by [Lemma 1](#). Therefore, [RD2](#) is still satisfied and s'_i is not root dominated in Γ' .

Stage $\mu + 1$: Now assume the property that a root dominated strategy s'_i at $\Lambda - \mu$ is root dominated in Γ' for a given $\mu \in \llbracket 2, \Lambda - 1 \rrbracket$ is true. Let us show it is true for $\mu + 1$. Assume the sequence of games $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$ is such that the considered s'_i is eliminated at $\Lambda - (\mu + 1)$.

One can construct a sequence of games $\{\tilde{\Gamma}^\lambda\}_{\lambda \leq \tilde{\Lambda}}$ identical to the previous one until step $\Lambda - (\mu + 1)$, but s'_i is not eliminated at $\Lambda - (\mu + 1)$. By [Lemma 5](#), s'_i is still root dominated in the latter sequence $\{\tilde{\Gamma}^\lambda\}_{\lambda \leq \tilde{\Lambda}}$ at the step $\Lambda - \mu$.

There are two cases: either (i) the strategy s''_i which eliminates s'_i in the first sequence $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$ is in $S^\Lambda (= \tilde{S}^{\tilde{\Lambda}}$ by [Theorem 1](#)), and thus is never eliminated; or (ii) the strategy s''_i is eliminated at a further step of the sequence $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$.

In the former case (i), it is straightforward to show that s'_i is root dominated in Γ' by repeating the arguments used in the first stage of our induction reasoning.

In the latter case (ii), we know by the induction hypothesis that s''_i is root dominated in the game Γ'' (where Γ'' is analogous to Γ' with the same players and utilities as Γ^Λ and the strategy set $S^\Lambda \cup s''_i$). Clearly, if s''_i is root dominated by a strategy s'''_i in Γ'' , since s''_i always very weakly dominates s'_i , s'_i is also root dominated by s'''_i in Γ' (since Γ' and Γ'' are the same games but either s'_i or s''_i is added to $\Gamma^\Lambda = \tilde{\Gamma}^\Lambda$).

Conclusion: We have shown that a strategy eliminated at $\Lambda - (\mu + 1)$ is still virtually root dominated at the end of the sequence. Thus, by induction, it is true for each $\mu \in \llbracket 2, \Lambda \rrbracket$. Since we did not need any assumption on the process used to construct our initial sequence, this result is true for any process. ■

A.2 Additional results

The next result states that no finite game becomes empty through IERDS:

Fact 2. $S^0 \neq \emptyset \Rightarrow \forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, S^\Lambda \neq \emptyset$.

Proof. By [Proposition 1](#) and by the finiteness of the games, it is clear that for each strategy set, there is at least one undominated strategy that can never be eliminated. ■

IERDS satisfies the Individual Independence of the Irrelevant Alternatives (IIIA) as formulated by [Gilboa et al. \[1990\]](#), i.e. the addition of one i 's strategy does not affect the dominance relation between i 's strategies:

Proposition 3. *Assume Γ and Γ' two games such that $N = N', S_{-i} = S'_{-i}, U = U'$, and $S'_i = S_i \cup s^*_i$. Then:*

$$s''_i \succ s'_i \text{ in } \Gamma \Rightarrow s''_i \succ s'_i \text{ in } \Gamma'.$$

Proof. Adding s^*_i does not affect the payoff of i when playing s'_i and s''_i . As well it does not affect the profiles in $\mathbf{b}(s''_i)$. Thus, if all conditions of [Definition 4](#) are checked in Γ , it is also the case in Γ' . ■

Nevertheless, we cannot use the main result of [Gilboa et al. \[1990\]](#) that states the order independence of *hereditary* dominance relations which are partial orders and respect IIIA. Indeed, root dominance is not *hereditary* in their sense:

Definition 19. *Assume Γ and Γ' such that $N = N', U = U', S' \subset S$. If $S \subset S'$, then the well defined dominance relation \succ is said *hereditary* if:*

$$\forall s''_i, s'_i \in S_i, s''_i \succ s'_i \text{ in } \Gamma' \Rightarrow s''_i \succ s'_i \text{ in } \Gamma.$$

The next example shows why \succ is not hereditary. We show in green the Nash equilibria. In the game below, IERDS eliminates B , then $K1$ and finally L :

		<i>k</i> 's Strategy				
		<i>j</i> 's Strategy		<i>j</i> 's Strategy		
<i>i</i>	<i>L</i>	<i>R</i>	<i>i</i>	<i>L</i>	<i>R</i>	→
<i>T</i>	(3, 1, 1)	(3, 0, 1)	<i>T</i>	(3, 0, 2)	(3, 1, 2)	→
<i>B</i>	(2, 1, 1)	(3, 0, 1)	<i>B</i>	(2, 0, 0)	(2, 0, 0)	
<i>K1</i>				<i>K2</i>		
				<i>k</i>		
				<i>i</i>	<i>R</i>	
				<i>T</i>	(3, 1, 2)	
				<i>K2</i>		

Table A.1: Games where hereditariness fails

However if we arbitrarily suppress the strategy L of the game, then no elimination can be made with IETDS. Therefore, root dominance is not hereditary, since B is not root dominated by T in the following “subgame”:

		<i>k</i> 's Strategy	
		<i>j</i>	<i>j</i>
<i>i</i>	<i>R</i>	<i>i</i>	<i>R</i>
<i>T</i>	(3, 0, 1)	<i>T</i>	(3, 1, 2)
<i>B</i>	(3, 0, 1)	<i>B</i>	(2, 0, 0)
<i>K1</i>		<i>K2</i>	

Table A.2: Subset of the previous Game with no Possible Elimination

A.3 Best reply set

Now we show with the next example (we show in green the Nash equilibria) why the definition of the Best Reply Set requires to consider all profiles where (at least) one opponent best responds and not only where *all* the opponents mutually best respond (beyond the obvious problem of existence with more than two players):

k 's Strategy

		j 's Strategy		
i		L	C	R
T		$(3, 3, 1)$	$(3, 3, 1)$	$(3, 2, 1)$
M		$(3, 2, 1)$	$(3, 1, 0)$	$(3, 1, 0)$
B		$(3, 2, 1)$	$(3, 1, 0)$	$(3, 1, 2)$

$K1$

		j 's Strategy		
i		L	C	R
T		$(3, 3, 1)$	$(3, 3, 1)$	$(3, 2, 2)$
M		$(3, 0, 2)$	$(3, 2, 0)$	$(4, 2, 1)$
B		$(3, 0, 2)$	$(3, 2, 0)$	$(3, 2, 1)$

$K2$

		j 's Strategy		
i		L	C	R
T		$(3, 3, 1)$	$(3, 3, 1)$	$(3, 2, 1)$
M		$(3, 2, 0)$	$(3, 1, 2)$	$(3, 0, 0)$
B		$(3, 2, 0)$	$(3, 1, 2)$	$(3, 0, 0)$

$K3$

Table A.3: Best Reply Set with a Three Players Game

In this game, if the Best Reply Set definition was modified, M would root dominate B and T . Indeed, the only profile where j and k mutually best respond to M is $(M, R, K2)$. At this profile M is strictly better than the two other strategies. Since M weakly dominates the two other strategies, it would be done. However, it can be easily seen that C would also root dominate R (all mutual best responses to C are indeed parts of the Nash equilibria). Then the order of elimination would matter. What is important here is that at $(M, C, K2)$, k strictly wants to deviate, making the profile unchecked with a modified version of the Best Reply Set.

A.4 Are inadmissible strategies playable?

Despite the inconsistencies of IEWDS, one may still assert that weakly dominated strategies *should* not be played. For instance, [Kohlberg and Mertens \[1986, p. 1014\]](#) justify admissibility as a criterion of strategic stability with the following reasoning: assume a pure strategy two-player game with player i having one strategy s_i'' which weakly dominates s_i' and additionally, such that if i is indifferent between s_i'' and s_i' , j is also indifferent at these profiles (it is the TDI condition of [Marx and Swinkels \[1997\]](#)). Now, the game has the next extensive form (see [Figure A.1](#)): first, i is asked to choose between (s_i', s_i'') and all of his other strategies. Second, j chooses his strategy. Finally, there is a third stage only if i has chosen (s_i', s_i'') at the first step and if s_i' and s_i'' do not give the same payoffs (i.e. if j has chosen a strategy among the strategies \succsim where both players are not indifferent with respect to the choice of i between s_i' and s_i''). [Kohlberg and Mertens \[1986\]](#) claim that in this form of game, s_i' is never played. It is true. However, [Kohlberg and Mertens \[1986\]](#) do not consider the games with payoffs such that s_i'' is never played either. Let us see the behavior of j if i has chosen the couple (s_i', s_i'') rather than another strategy at the

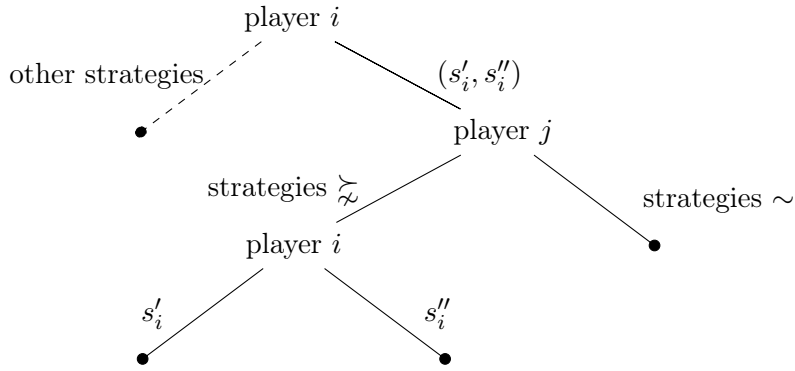


Figure A.1: Extensive-Form Game: s_i'' Weakly Dominates s_i' but might Never be Played in the third stage

first stage. Then, j necessarily plays a best response to s_i' or s_i'' . Assume j chooses a best response s_j^* to s_i' . Then, either i is not indifferent and will necessarily choose s_i'' (since s_i'' weakly dominates s_i' , if i is not indifferent, he strictly prefers s_i''), making the choice of j suboptimal if it is not a best response to s_i'' too, or i is indifferent. In this latter case, by assumption (the TDI condition), j is also indifferent. However, if s_j^* is not a best response to s_i'' , then j could have obtained a strictly higher payoff by deviating towards a best response to s_i'' . Therefore, in this part of the game, j always plays a best response to s_i'' . Then, i may play s_i'' at the third stage only if players are not indifferent at (at least) one profile where j best responds to s_i'' . If there is (at least) one best response of j to s_i'' such that i and j are indifferent between s_i' and s_i'' , then s_i'' might be never played. The idea of [Kohlberg and Mertens \[1986\]](#) is that a strategy is inadmissible if it is never played in such an extensive-form game. Nevertheless, this criterion cannot characterize inadmissibility since an admissible strategy might never be played either (if j always plays a strategy in \sim), according to the considered game. We claim that one possibility is to choose a more cautious criterion: s_i' is dominated by s_i'' if and only if s_i'' is always played in this part of the game. Precisely, we should require that s_i'' is played with probability 1 in the third stage when i chooses the couple (s_i', s_i'') at the first stage. In this case, s_i'' should strictly payoff dominate s_i' where j best responds to s_i'' . It is exactly our second condition of dominance. Note that the reasoning we have just made does require weak dominance, like our notion of dominance does.

Besides, remark that root dominance differs from the notion of nice weak dominance introduced by [Marx and Swinkels \[1997\]](#) since nice weak dominance is equivalent to weak dominance in games where the TDI condition is respected. Thus, in all the games we have considered, s_i' is *nice weakly dominated* by s_i'' . One can see why the iterated elimination of nicely weakly dominated strategies is payoff order independent in such games with the two following examples:

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(2,2)	(3,1)
<i>B</i>	(2,2)	(2,1)

 $\xrightarrow{\text{IEWDS}}$

	<i>j</i> 's Strat.
<i>i</i> 's Strat.	<i>L</i>
<i>T</i>	(2,2)
<i>B</i>	(2,2)

or

	<i>j</i> 's Strat.
<i>i</i> 's Strat.	<i>L</i>
<i>T</i>	(2,2)

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(2,2)	(3,3)
<i>B</i>	(2,2)	(2,3)

 $\xrightarrow{\text{IEWDS}}$

	<i>j</i> 's Strat.
<i>i</i> 's Strat.	<i>R</i>
<i>T</i>	(3,3)

Figure A.2: IEWDS applied to Games with the TDI Condition

In the top game, j best responds to T by playing L . At this profile, i is indifferent between T and B . With respect to our previous remarks, it might be problematic. Indeed, here, the order of deletion of IEWDS matters: the outcome of IEWDS is either (T, L) or $(\Delta(T, B), L)$. Nevertheless, thanks to the TDI condition, it does not affect the payoffs. Again, in this paper, we consider such an outcome of IEWDS as an example of order dependence. Now, in the bottom game, j best responds to T by playing R . There, i is not indifferent, and the order does not matter, the outcome of IEWDS always being (T, R) . One can remark that the TDI condition does not matter either in this game. Indeed, whatever the payoff of j at the profile (B, L) , IEWDS would still be order independent. Naturally, we depart from the notion of nice weak dominance since root dominance requires payoff dominance at the profiles where j best responds to T .

A.5 Weak dominance in 2×2 games

Assume the following general form for a 2×2 game where T weakly dominates B (i.e. $a > c$):

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(a, α)	(a, β)
<i>B</i>	(c, γ)	(a, δ)

Table A.4: Game with Weak Dominance

With respect to the value of α, β, γ and δ , there are 9 possible configurations that we gather in subsets according to their properties. The three configurations (i) where $\alpha > \beta$ are order independent and immutable. The special configuration (i') where $\alpha = \beta$ and $\gamma = \delta$ is order independent and immutable as well. The configurations (ii) with $\alpha < \beta$

and $\gamma > \delta$ is order independent but is mutable¹. Finally, other configurations (*iii*) are order dependent and mutable (those either with $\alpha = \beta$ and $\gamma \neq \delta$ or $\alpha < \beta$ and $\gamma \leq \delta$). Configurations (*i*) correspond to cases where T root dominates B . All other configurations are such that $(T, R) \in \mathbf{b}(T)$, and therefore T does not root dominate B . Note that (*ii*) differs from (*iii*) also because j does not have (weakly) dominated strategy in (*ii*). In this game, except for configuration (*i'*), both root dominance solvability and consistency of IEWDS correspond to cases where the selected Nash equilibrium is strict (but not necessarily Pareto-dominant), i.e. no player has a payoff-equivalent unilateral deviation. Other cases are such that no Nash equilibrium is strict. The case (*i'*) is such that IERDS eliminates no strategy. In contrast, IEWDS eliminates B and that is all. Again, the configuration (*i'*) is special. However, it shows that IERDS fails to delete some strategies which are virtually dominated in the IEWDS outcome. Thus, IERDS is not the “maximal” immutable elimination procedure.

	j 's Strategy	
i 's Strategy	L	R
T	(a, α)	(a, α)
B	(c, α)	(a, α)

Table A.5: Game with configuration (*i'*): IEWDS eliminates B

A.6 Pareto dominance and pre-play communication

Note that our procedure does not lead to the selection of the Pareto dominant equilibrium. Even if the Pareto dominant strictly dominates another equilibrium, the latter may still be selected instead as it is shown with this example:

	j 's Strategy	
i 's Strategy	L	R
T	$(2, 1)$	$(1, 2)$
B	$(2, 3)$	$(0, 0)$

Table A.6: Game with a Pareto Dominated Unique Prediction

However, if we define a strict Nash equilibrium as an equilibrium where each player best responds to the profile and this best response is unique (i.e. no strategy is payoff equivalent

¹Note that it can be compared to [Samuelson \[1992, Example 8\]](#) which shows that common knowledge of admissibility may not exist. Note that if in addition to these specifications, we assume that $\beta = \delta$, this game respects the *transference of decisionmaker indifference (TDI)* condition of [Marx and Swinkels \[1997\]](#) which ensures the *outcome* order independence of IEWDS in finite games (i.e. any order of elimination leads to the same payoffs). Therefore, it shows that *nice weak dominance* (which is equivalent to weak dominance in the class of finite TDI games) may exhibit mutability.

at this profile), we can easily show that IERDS never eliminates this kind of equilibrium:

Fact 3. *IERDS does not eliminate strict Nash equilibria.*

The proof is immediate since if a profile is a strict Nash equilibrium, then all strategies of the profile cannot be iteratively weakly dominated. Note that IEWDS does not eliminate strict Nash equilibria by the same argument.

Remark that if we invoke the notion of *self signaling*², (T, R) is the only equilibrium such that both agents play a self signaling action. Briefly, in a two-players game with pre-play communication, an action is said *self signaling* if the action the sender announces is a strict best response if his opponent plays a best response to this action; if he plays another action, he strictly prefers that the opponent plays another strategy. Therefore there should not be an incentive to deviate for the sender once he thinks his opponent trusts him³. It is not surprising that a root dominating strategy enables a strongly believed commitment since it is an undominated strategy⁴. At (B, L) , only j can self signal, while i cannot even *self commit* (*self commitment* requires only that the action announced is a strict best response if the opponent plays a best response). Moreover, even if (B, L) is the Pareto dominant profile, action L is not a *Stackelberg* action (i.e. the unique preferred action if the opponent always plays a best response) because T is also a best response to L , and at (T, L) , j wants to deviate.

A.7 Proof of mixed IERDS order independence result

First, it is obvious that [Lemma 1](#) still applies. Now, we state that mixed root dominance forms also a strict partial order:

Proposition 4. *Mixed root dominance is a strict partial order: it is a binary relation such that irreflexivity, asymmetry and transitivity hold.*

The proof is analogous to the pure strategy case:

Proof. Root dominance is irreflexive: by [Lemma 1](#), $\forall \sigma_i, \in \Sigma_i \mathbf{b}(R_{\sigma_i}) \neq \emptyset$, and it is not possible to have $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i})$ for any profile $\sigma_{-i} \in \Sigma_{-i}$. Then, [RD2'](#) cannot be respected. Root dominance is transitive: assume $\sigma_i'' \succ \sigma_i'$ and $\sigma_i''' \succ \sigma_i''$. Here, we have

²*Self signaling* is described in [Farrell and Rabin \[1996\]](#) for pre-play communication, see [Baliga and Morris \[2002\]](#) for formal definitions

³In other words, if the sender announces something that he always wants to be believed (whatever it is true or false), his commitment is weak, he cannot self signal. Here, conditions to have T and R self signaling are: $U_i(T, R) > U_i(B, R)$ and $U_i(B, R) < U_i(B, L)$ for agent i (conditionally to j best responding at (T, R)) and symmetrically $U_j(T, R) > U_j(T, L)$ and $U_j(T, L) < U_j(B, L)$ for agent i (conditionally to i best responding at (T, R)).

⁴Note that if we modified the payoff of i such that i earns -1 when j plays L , there would still be root dominance but T would not be self signaling. Then, there is obviously not equivalence between the two concepts.

to prove that $\sigma_i''' \succ \sigma_i'$. First, it is straightforward that **RD1'** is respected. Second, since $\sigma_i''' \succ \sigma_i''$, we know that $U_i(\sigma_i''', \sigma_{-i}) > U_i(\sigma_i'', \sigma_{-i})$ for each strategy profile σ_{-i} contained in $\mathbf{b}(R_{\sigma_i''})$. Since $\sigma_i'' \succ \sigma_i'$, $U_i(\sigma_i'', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$ for each strategy profile σ_{-i} in Σ_{-i} , and thus for each strategy profile σ_{-i} contained in $\mathbf{b}(R_{\sigma_i''})$. Therefore, $U_i(\sigma_i''', \sigma_{-i}) > U_i(\sigma_i'', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$ for each strategy profile σ_{-i} contained in $\mathbf{b}(R_{\sigma_i''})$ and **RD2'** is respected. Finally, irreflexivity and transitivity together imply asymmetry. ■

Lemma 9. *If $s_i' \in S_i$ is root dominated, there is (at least) one strategy $\sigma_i'' \in \Sigma_i$ that may eliminate it, i.e. a strategy σ_i'' whose no strategy in the support is root dominated by another strategy and root dominates s_i' .*

Proof. Since the number of pure strategies is finite, the number of pure strategies contained in all the supports containing (mixed) strategies root dominating s_i' is necessarily finite. Let us denote it m and denote $g(s_i')$ the set of these strategies. Then, (at most) $m - 1$ of these strategies are root dominated. Otherwise, it means that the m^{th} strategy, named s_i'' , is root dominated by an other strategy whose support contains (at least) one pure strategy outside $g(s_i')$. By transitivity of root dominance, it means that the latter strategy also root dominates s_i' , contradicting the fact that the number of pure strategies contained in all the supports containing (mixed) strategies root dominating s_i' is m . Thus, we have established that at least s_i'' is not root dominated. Additionally, by [Lemma 7](#), we know that the $m - 1$ strategies root dominated are root dominated by strategies $\tilde{\sigma}_i \in \Sigma_i$ whose supports do not contain them. Again, by transitivity, the support of these strategies $\tilde{\sigma}_i$ is necessarily contained in $g(s_i')$. Therefore, s_i'' root dominates each of these strategies: otherwise, either one of these strategies is not root dominated and there is a contradiction, or it is dominated by a strategy whose support is outside $g(s_i')$, a contradiction. Finally, s_i'' root dominates s_i' . We can make the same reasoning when more than one pure strategy is not root dominated and the proof is done. ■

It is straightforward to show that $\mathbf{b}(R_{\sigma_i})$ never expands as we progress through the steps of mixed IERDS thanks to the previous result. Finally, all remaining results are written in the same until the hereditariness result, and we get the order independence result.

A.8 Burned money

Root dominance fails to be as predictive as IEWDS or Iterated Elimination of Choice sets under Full Admissible consistency (IECFA) of [Asheim and Dufwenberg \[2003\]](#) if we study the battle of sexes game with a burning option for one player (see for instance [Rubinstein \[1991, p.920\]](#)). If payoffs are as described in the payoffs matrix of [Table A.7](#), we only delete one strategy for each agent, eliminating one Nash equilibrium. This result is not completely satisfying since we preserve a strategy where money is burnt and the

equilibrium deleted is the one where the second agent has the maximal payoff. With our procedure, there is no mixed equilibrium anymore.

	<i>j</i> 's Strategy			
<i>i</i> 's Strat.	<i>LL</i>	<i>LR</i>	<i>RL</i>	<i>RR</i>
<i>NU</i>	(3, 1)	(3, 1)	(0, 0)	(0, 0)
<i>ND</i>	(0, 0)	(0, 0)	(1, 3)	(1, 3)
<i>BU</i>	(2, 1)	(-1, 0)	(2, 1)	(-1, 0)
<i>BD</i>	(-1, 0)	(0, 3)	(-1, 0)	(0, 3)

→

	<i>j</i> 's Strategy		
<i>i</i> 's Strat.	<i>LL</i>	<i>LR</i>	<i>RL</i>
<i>NU</i>	(3, 1)	(3, 1)	(0, 0)
<i>ND</i>	(0, 0)	(0, 0)	(1, 3)
<i>BU</i>	(2, 1)	(-1, 0)	(2, 1)

Table A.7: Burned Money in Rubinstein [1991]

(*BD*) is root dominated by (*NU*). This deletion is necessary to eliminate (*RR*) (by (*RL*)). No further elimination is possible. However, if we allow a mixed extension of the game, mixed strategies where *BU* is more used than *NU* root dominate *ND*. Then *LL* root dominates *RL*. Finally we end the procedure by eliminating *BU*, and we get the two Nash equilibria favoring *i*:

	<i>j</i> 's Strategy	
<i>i</i> 's Strat.	<i>LL</i>	<i>LR</i>
<i>NU</i>	(3, 1)	(3, 1)

Table A.8: Final outcome of Burned Money after mixed IETDS

A.9 Additional concepts characterizing root dominance

In this part, we introduce two additional rationality concepts which can characterize root dominance. For this purpose, we introduce two new types of games where the perception of player *j* is perturbed with probability ϵ .

In the first configuration, the *hesitation game*, we suppose that despite having a “reference” strategy (unobserved by the opponent), a player may alternatively consider some strategy subsets. If so, the opponent reacts optimally (in a naive way) to this strategy subset. Thus, the reference strategy is “tested” against such mind trembles. If the reference strategy is not optimal when he believes that the opponent can detect this tremble and react optimally, the tremble should be realized and, in fact the reference strategy never played in such a game.

In the second configuration, named *deviation game*, the opponent observes both the reference strategy and the strategy subset from which a potential deviation is picked. Closer to the spirit of Fudenberg et al. [1988], *j* analyses whether the deviation is sustainable before reacting optimally. That is, we assume *j* plays a best response to the deviation if

and only if it the deviation is deemed *credible* with respect to the reference strategy. We summarize j 's beliefs in [Figure A.3](#).

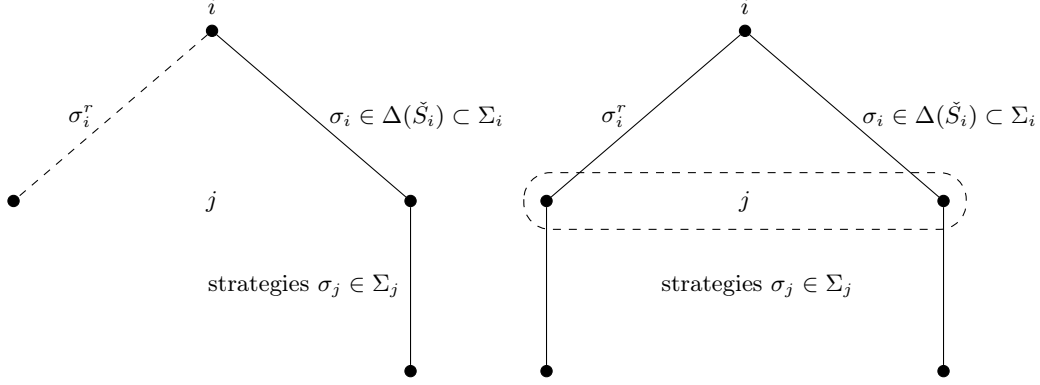


Figure A.3: Beliefs of j if a Perturbation occurs in Hesitation (left) and Deviation (right) Games

A.9.1 Hesitation games

Assume each player believes that with probability ϵ he may “hesitate”. That is, if he has chosen a reference strategy $\sigma_i^r \in \Sigma_i$, he may think to choose other (mixed) strategies supported by any strategy subset $\check{S}_i \subset S_i$. Additionally, assume that this thought is observable by the opponent j and that j believes that i will actually play a strategy in $\Delta(\check{S}_i)$. At this point, i can substitute a strategy in $\Delta(\check{S}_i)$ for σ_i^r . If with such a perturbation, σ_i^r does not maximize i 's utility, then σ_i^r should not be played.

First, we define two concepts which assume restrictions on the available strategies:

Definition 20. A restricted game $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$ is a simultaneous game such that player i chooses a strategy $\sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i)$ where $\check{S}_i \subset S_i$, and such that it is common knowledge that player j believes with probability 1 that $\Sigma_i = \Delta(\check{S}_i)$.

A restricted game $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$ is a game where the strategy set is $\sigma_i^r \cup \Delta(\check{S}_i) \times S_j$ but player j believes that the strategy set is $\Delta(\check{S}_i) \times \Sigma_j$. Now, we can define the ϵ -hesitation game, whose name indicates that players might *hesitate* with probability ϵ :

Definition 21. An ϵ -hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$ for player i and strategy σ_i^r is a game where:

1. Player i chooses the strategy $\sigma_i \in \Sigma_i$, and player j chooses a strategy in Σ_j with probability $1 - \epsilon$,
2. With probability ϵ , i and j play a restricted game $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$.

That is, we assume the perception of the game by player j is restricted to $\Delta(\check{S}_i) \times \Sigma_j$ with probability ϵ . Clearly, the perception can be false since i is allowed to choose the strategy σ_i^r . However, we assume that j almost guesses the thought of i with probability ϵ , since j perceives (at least) partially where the attention of i is.

Furthermore, we can make a link with the idea of deviation and its observation. Indeed, an ϵ -hesitation game formalizes the reasoning process of player i when:

1. The “usual” strategy of player i is σ_i^r ,
2. Player i thinks about a deviation to any other strategy contained in $\Delta(\check{S}_i)$,
3. Opponent j observes with probability ϵ that i is thinking to choose a strategy in $\Delta(\check{S}_i)$.

The consequence of step 3 is that i believes that j will choose a best response to $\Delta(\check{S}_i)$ with probability ϵ .

More concretely, the reasoning is the following. When i thinks about whether a strategy σ_i^r is “playable”, he takes it as a reference point. Then, he wonders whether he may want to deviate. For this purpose, he considers all strategy subsets \check{S}_i . For each one, he believes that j will react optimally with probability ϵ . Finally, he checks if he would want to deviate from σ_i^r in all cases verifying this belief. If there is a deviation that yields strictly more, player i never chooses σ_i^r to avoid to pay the cost c when facing the restricted game. One could remark that the behavior of j seems too “naive”. In the next subsection we introduce a second kind of perturbation that tackles this issue.

Now, we define the best response of an ϵ -hesitation game:

Definition 22. Consider an ϵ -hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$. A strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the ϵ -hesitation game if:

$$\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \in \mathbf{b}(\check{S}_i), \forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i), V_i^\epsilon(\sigma_i^*, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \quad (\text{H-BR})$$

Finally, we introduce the concept of ϵ -hesitation dominance which formalizes the dominance relation when we consider the expected ϵ -perturbed utility, and such that the dominating strategy is “observed” by the opponent:

Definition 23. A strategy $s_i \in S_i$ is ϵ -hesitation dominated by $\sigma_i \in \Sigma_i$ if:

$$\begin{aligned} \forall \sigma_j \in \Sigma_j, \forall \sigma_j^* \in \mathbf{b}(R_{\sigma_i}), \\ V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \end{aligned}$$

In words, σ_i ϵ -hesitation dominates s_i when j reacts optimally to σ_i with probability ϵ . Naturally, this concept seems similar to root dominance. Indeed, [Lemma 12](#) shows their equivalence when $\epsilon \rightarrow 0^+$.

Lemma 10. A strategy $s_i \in S_i$ is ϵ -hesitation dominated if and only if it is a never best response of (at least) one ϵ -hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$.

Proof. Assume a strategy s_i is ϵ -hesitation dominated by $\sigma_i \in \Sigma_i$. It means that if R_{σ_i} is observed with probability ϵ , then the utility from playing σ_i is strictly higher than from playing s_i . Therefore, s_i is never best response of the ϵ -hesitation game $\widehat{\Gamma}^\epsilon(s_i, R_{\sigma_i})$. Now,

by contrapositive, assume that s_i is not ϵ -hesitation dominated by any $\sigma_i \in \Sigma_i$ and let us show it is a best response to a belief for i when i plays a given ϵ -hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Consider the vectors $\vec{V}_i^\epsilon(s_i, \check{S}_i) = \{V_i^\epsilon(\sigma_i, s_j, s_j^*)\}_{s_j \in S_j, s_j^* \subset \mathbf{b}(\check{S}_i)}$ for each $\sigma_i \in s_i \cup \Delta(\check{S}_i)$. Simply, these vectors are such that each component $l+m$ is the payoff i can obtain when playing σ_i and when j plays the pure strategy $s_j^l \in S_j$ with probability $1 - \epsilon$ and the pure strategy $s_j^{*m} \subset \mathbf{b}(\check{S}_i)$ with probability ϵ . We denote $Y(s_i, \check{S}_i)$ the set of such vectors. Besides, we can construct the following set X . If k is equal to $\sharp(S_j) \times \sharp(\mathbf{b}(\check{S}_i))$ ⁵, then X is the set $\{x \in \mathbb{R}^k \mid x > \vec{V}_i^\epsilon(s_i)\}$, that is the set of all payoffs that strictly dominate s_i payoffs. Both X and $Y(s_i, \check{S}_i)$ are convex sets. Since s_i is not ϵ -hesitation dominated, these sets are disjoint. Then, we can apply the separating hyperplane theorem which states that there is a vector in \mathbb{R}^k , $\pi \geq 0$ with $\pi \neq 0$ and such that:

$$\forall y \in Y(s_i, \check{S}_i), \forall x \in X, \pi \cdot y \leq \pi \cdot \vec{V}_i^\epsilon(s_i) \leq \pi \cdot x$$

It directly implies that $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i)$, $\pi \cdot (\vec{V}_i^\epsilon(s_i) - \vec{V}_i^\epsilon(\sigma_i)) \geq 0$.

Now, remark that this is true for every hesitation game and finally we get the result. ■

Conversely, a strategy $s_i \in S_i$ being ϵ -hesitation undominated is a best response in all ϵ -hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Though, it does not mean that s_i necessarily verifies [Equation \(1.1\)](#):

$$\exists \sigma_j \in \Sigma_j, \text{ such that } \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \mathbf{b}(R_{\sigma_i}), \quad V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \quad (1.1)$$

Here, we stress the fact that the strategy $\sigma_j \in \Sigma_j$ is not necessarily the same for all the hesitation games when it is stated that a strategy is a best response in all hesitation games. The equivalence holds only when $\epsilon \rightarrow 0^+$:

Lemma 11. *A strategy $s_i \in S_i$ is a best response to all ϵ -hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \rightarrow 0^+$ if and only if it verifies [Equation \(1.1\)](#) when $\epsilon \rightarrow 0^+$.*

Proof. Set $\epsilon \rightarrow 0^+$. Assume that s_i is a best response in all ϵ -hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Then,

$$\forall \sigma_i \in \Sigma_i, \exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \mathbf{b}(R_{\sigma_i}), \text{ such that } V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)$$

By continuity of V_i^ϵ in parameter ϵ , it is immediate that we have:

$$\forall \sigma_i \in \Sigma_i, \exists \sigma_j \in \Sigma_j \text{ such that } \mathbb{E}[U_i(s_i, \sigma_j)] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j)]$$

By [Pearce \[1984, Lemma 3\]](#), the previous equation is equivalent to:

$$\exists \sigma_j \in \Sigma_j \text{ such that } \forall \sigma_i \in \Sigma_i, \mathbb{E}[U_i(s_i, \sigma_j)] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j)]$$

This last equation is well equivalent to [Equation \(1.1\)](#) when $\epsilon \rightarrow 0^+$. The same reasoning as above can be applied to show the converse part of this result. ■

⁵We denote $\sharp(S_i)$ the number of elements in the set S_i .

Now, we state the equivalence between hesitation dominance when the perturbation occurs with an infinitesimal probability and root dominance:

Lemma 12. *A strategy is ϵ -hesitation dominated when $\epsilon \rightarrow 0^+$ if and only if it is root dominated.*

Proof. The “if” part is straightforward. Indeed, assume that $s_i \in S_i$ is root dominated by $\sigma_i \in \Sigma_i$. First, **RD1'** and **RD2'** imply that σ_i weakly dominates s_i . Thus, $\forall \sigma_j \in \Sigma_j$, $U_i(\sigma_i, \sigma_j) \geq U_i(s_i, \sigma_j)$. Second, **RD2'** states that for each best response to a strategy in the support of σ_i , the expected payoff from playing σ_i is strictly higher. Therefore, $\forall \sigma_j^*(\sigma_i) \subset \mathbf{b}(R_{\sigma_i})$, we have $U_i(\sigma_i, \sigma_j^*(\sigma_i)) > U_i(s_i, \sigma_j^*(\sigma_i))$. Then, for any $\epsilon > 0$, and $\forall \sigma_j \in \Sigma_j, \forall \sigma_j^*(\sigma_i) \subset \mathbf{b}(R_{\sigma_i})$:

$$V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*(\sigma_i)) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*(\sigma_i))$$

For the “only if” part, assume that s_i is ϵ -hesitation dominated by σ_i but root undominated by σ_i . Undomination means that either (i) there is a $\sigma_j \in \Sigma_j$ such that $\mathbb{E}[U_i(s_i, \sigma_j)] > \mathbb{E}[U_i(\sigma_i, \sigma_j)]$ or (ii) there is a $\sigma_j^{**} \subset \mathbf{b}(R_{\sigma_i})$ such that $\mathbb{E}[U_i(s_i, \sigma_j^{**})] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j^{**})]$. About (i), we remark that V_i^ϵ is continuous in the parameter ϵ . Then, it is not possible to have simultaneously $\mathbb{E}[U_i(s_i, \sigma_j)] > \mathbb{E}[U_i(\sigma_i, \sigma_j)]$ and $\forall \sigma_j^*(\sigma_i) \subset \mathbf{b}(R_{\sigma_i}), V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*(\sigma_i)) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*(\sigma_i))$ when $\epsilon \rightarrow 0^+$. Besides, the hypothesis (ii) directly implies that $V_i^\epsilon(s_i, \sigma_j^{**}, \sigma_j^{**}) \geq V_i^\epsilon(\sigma_i, \sigma_j^{**}, \sigma_j^{**})$. In both cases, there is a contradiction with the hypothesis of 0^+ -perturbed dominance. ■

Finally, we can state the first main result of this section, namely the equivalence between root dominance of s_i and rationality when considering all the ϵ -hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ associated to Γ :

Theorem 6. *A strategy $s_i \in S_i$ is root dominated if and only if it is a never best response of (at least) one ϵ -hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \rightarrow 0^+$.*

Proof. The result is immediate by **Lemmas 10** and **12**. ■

Thus, if player i believes that his opponent j may have his perception of the game altered by the alternatives he considers when testing strategies, he never plays root dominated strategies.

In the different context of ordinal preferences, **Börgers [1993]** characterizes non rationality by weak dominance against every j 's strategy subset (but weak dominance is not required to be made by the same strategy). Here, in contrast, player i does not restrict the game with respect to j 's strategies, but with respect to his own strategies (and then j reacts optimally to these restrictions with probability ϵ). Furthermore, it is the notion of rationality that we test against strategy subsets and not the dominance relation since the requirements of **RD1'** and **RD2'** are with respect to the whole game.

Besides, we can write the alternative characterization of root undominance:

Corollary 1. *A strategy $s_i \in S_i$ is root undominated if and only if it verifies [Equation \(1.1\)](#) when $\epsilon \rightarrow 0^+$.*

Proof. The result is immediate by combining [Lemma 11](#) and [Theorem 6](#). ■

A.9.2 Deviation games

Here, we introduce our second perturbation of the game. This perturbation is such that each player believes that the opponent may observe both his “reference” strategy and the support of strategies from which a deviation might be picked by the player contemplating alternatives. In this case we will say the game is turned into a pseudo extensive form game:

Definition 24. *A pseudo extensive form game $\check{\Gamma}(\sigma_i^r, \check{S}_i)$ is a game where i chooses a strategy in $\sigma_i^r \cup \Delta(\check{S}_i)$, where \check{S}_i is a subset of S_i . Strategy σ_i is the reference strategy of i , and \check{S}_i is the support of any strategy towards which i wants to deviate. Player j observes this information perfectly, then forms beliefs, and plays accordingly.*

Definition 25. *An ϵ -deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$ for strategy $\sigma_i^r \in \Sigma_i$ is a game where:*

1. *Player i chooses the strategy $\sigma_i^r \in \Sigma_i$,*
2. *Player i chooses a deviation subset $\check{S}_i \subset S_i$,*
3. *Player i plays any strategy in $\sigma_i^r \cup \Delta(\check{S}_i)$,*
4. *With probability ϵ , the previous steps form the first stage of a pseudo extensive form game,*
5. *Player j chooses a strategy in the second stage.*

When player j faces a deviation, we assume that his only concern is whether the deviation is credible according to all available information. If the deviation is *credible*, player j should react optimally. Otherwise, he can have any belief. This last assumption does not imply that j believes that i has lied, or the observation is not accurate (we assume it is not possible), but rather than a non credible deviation is meaningless for j . In other words, it is as if i said some thoughtless things that do not impact real decisions. In this case, the deviation is disregarded. Now, what do we mean by *credible*? Following [Baliga and Morris \[2002\]](#) and their notion of *self signaling* strategies for games with pre-play communication (see [Section A.6](#) for more details), we now introduce the notion of *self improving* strategy subset:

Definition 26. *A strategy subset $\check{S}_i \subset S_i$ is self improving with respect to $\sigma_i \in \Sigma_i$ if $\forall \sigma_j^* \in \mathbf{b}(\check{S}_i), \exists \sigma_i''$ with $R_{\sigma_i''} = \check{S}_i$:*

$$U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i, \sigma_j^*)$$

In words, \check{S}_i is self improving with respect to σ_i if for all best responses to \check{S}_i , there is a strategy whose support is \check{S}_i which yields a strictly higher payoff than σ_i . Remark that

if the subset \check{S}_i is reduced to a singleton $\{s_i''\}$, then we have the same condition as in [RD2](#). Furthermore, if it is the same strategy σ_i'' which strictly dominates σ_i , then we have the same condition as in [RD2'](#). Since we only consider two-player games, this is always verified thanks to [Pearce \[1984, Lemma 3\]](#). Thus, we can equivalently write the following definition:

Definition 27. A strategy $\sigma_i'' \in \Sigma_i$ is self improving with respect to $\sigma_i \in \Sigma_i$ if $\forall \sigma_j^* \subset b(R_{\sigma_i''})$:

$$U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i, \sigma_j^*)$$

Instead, a strategy σ_i'' is *self signaling* when it is a best response itself (to the best response(s) played by j)⁶. Then, this requirement is stronger and seems to be more attractive when j assesses the credibility of the deviation. However, we have to recall that the chosen strategy matters both when no deviation is observed (with probability $1 - \epsilon$) and when there is deviation (with probability ϵ). Therefore, it may be natural that player j does not “expect” player i to maximize his utility when seeing the deviation, since observation by j will be made unknown for i . Conversely, player j cannot expect that i has played the deviation if whatever the optimal response he makes to this deviation, i 's payoff is not increased. That is why we define *credibility* in the following way:

Definition 28. A deviation $\check{S}_i \in S_i$ from $\sigma_i^r \in \Sigma_i$ in a pseudo extensive form game $\check{\Gamma}(\sigma_i^r, \check{S}_i)$ is credible if there is a strategy $\sigma_i'' \in \Sigma_i$ such that:

- $R_{\sigma_i''} = \check{S}_i$,
- And, σ_i'' is self improving with respect to σ_i^r .

Thus, the deviation is credible if there is a strategy σ_i'' whose support is \check{S}_i , and if played in the first stage of an extensive form game, pays off strictly more than σ_i^r (given that j would react optimally to σ_i'').

With the next assumption, we will restrict the beliefs of player j when facing a pseudo extensive form game $\check{\Gamma}(\sigma_i, \check{S}_i)$. We assume that when the deviation is credible, the belief of j that i has played a strategy whose support is contained in \check{S}_i is 1. In any other case, any belief is allowed. We note β_i^j the vector which contains all the elements $\beta_i^j[\check{S}_i]$ and that represents j 's assessment of the probability that i plays a strategy contained in \check{S}_i . Now, we formalize the assumption described just above:

Assumption C. Player j , when observing a deviation $\check{S}_i \in S_i$ from $\sigma_i \in \Sigma_i$ has the following beliefs:

- Either \check{S}_i is credible in which case $\beta_i^j[\check{S}_i | \check{S}_i \text{ is “observed”}] = 1$,

⁶Of course, we make a slight abuse here because there is no notion of Best Response Set in [Baliga and Morris \[2002\]](#) and we take into account strategy subsets. However, since [Baliga and Morris \[2002\]](#) consider only pure strategies, the comparison would be relevant in their framework.

- Or \check{S}_i is not credible in which case cells in the vector β_i^j [\check{S}_i is “observed”] can take any value.

Now, we define the best response of an ϵ -deviation game:

Definition 29. Consider an ϵ -deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$. A strategy $\sigma_i^* \in s_i \cup \Delta(\check{S}_i)$ is a best response of the ϵ -deviation game if:

$$\exists \sigma_j \in \Sigma_j, \exists \hat{\sigma}_j \in \Sigma_j, \forall \sigma_i \in \Delta(\check{S}_i): \quad V_i^\epsilon(\sigma_i^*, \sigma_j, \hat{\sigma}_j) \geq V_i^\epsilon(\sigma_i, \sigma_j, \hat{\sigma}_j)$$

Naturally, a best response for i in the ϵ -deviation game is a strategy which maximizes i 's utility when j plays σ_j with probability $1-\epsilon$, and $\hat{\sigma}_j$ with probability ϵ . Obviously, without further restriction, any standard best response is a best response of the ϵ -deviation game (think simply to cases where $\hat{\sigma}_j = \sigma_j$). When using [Assumption C](#), we can re-write the above definition in the following way:

Lemma 13. Consider an ϵ -deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$. Under [Assumption C](#), a strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the ϵ -deviation game if and only if either:

- The deviation \check{S}_i from σ_i is credible,
- And,

$$\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \mathbf{b}(\check{S}_i), \forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i): V_i^\epsilon(\sigma_i^*, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \quad (\text{cD-BR})$$

Or,

- The deviation \check{S}_i from σ_i is not credible,
- And, $\exists \sigma_j \in \Sigma_j$ such that $\forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i): U_i(\sigma_i^*, \sigma_j) \geq U_i(\sigma_i, \sigma_j)$.

In words, [Lemma 13](#) means that if the deviation is credible, a best response of the ϵ -deviation game $\check{\Gamma}^\epsilon(s_i, \check{S}_i)$ is a best response to a game where j reacts optimally to \check{S}_i with probability ϵ . Instead, if the deviation is not credible, a best response is simply a best response according to the standard definition (see [Definition 12](#) above) applied to $\sigma_i^r \cup \Delta(\check{S}_i)$. Remark that a best response response of a ϵ -hesitation game is also a best response of the linked ϵ -deviation game when the deviation is credible:

Lemma 14. Consider an ϵ -deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$. Under [Assumption C](#), if the deviation is credible, a strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the ϵ -deviation game if and only if it is a best response of the ϵ -hesitation game $\hat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$.

Proof. The proof is immediate since [Equation \(H-BR\)](#) and [Equation \(cD-BR\)](#) are equivalent. ■

Besides, notice that the deviation credibility does not imply that the reference strategy σ_i^r is a never best response of the ϵ -deviation game if the deviation is credible. Now, we can show that the previous result can be applied to the reference strategy even if the deviation is not credible:

Lemma 15. *Under [Assumption C](#), a strategy $\sigma_i^r \in \Sigma_i$ is a best response of an ϵ -deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$ if and only if it is a best response of the associated ϵ -hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$.*

Proof. First, when the deviation is credible, [Lemma 14](#) applies. Now assume the deviation is not credible. The “if” part is straightforward. Indeed, a best response in the ϵ -hesitation game is with respect to a belief with probability $1 - \epsilon$ and to a belief that a best response to \check{S}_i is played with probability ϵ . Then, when the deviation \check{S}_i from s_i is not credible, any belief can be sustained, among which the one inducing that σ_i^r is a best response of the ϵ -hesitation game. Conversely, assume σ_i^r is a best response to the considered ϵ -deviation game. If the deviation is not credible, it means that there is no strategy σ_i'' whose support is \check{S}_i and is self improving with respect to σ_i^r , i.e. checking $\forall \sigma_j^* \in \mathbf{b}(\check{S}_i)$, $U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i^r, \sigma_j^*)$. Thus, no strategy strictly dominates σ_i^r when we restrict attention to $\mathbf{b}(\check{S}_i)$. Therefore, since it is a two-player game, by [Pearce \[1984, Lemma 3\]](#), σ_i^r is a best response to at least one strategy $\sigma_j^* \in \mathbf{b}(\check{S}_i)$. Since σ_i^r is also a best response to another strategy σ_j (potentially outside $\mathbf{b}(\check{S}_i)$) by [Lemma 13](#), σ_i^r is a best response to (σ_j, σ_j^*) in the ϵ -hesitation game. ■

In fact, any best response of the ϵ -hesitation game is also a best response of the ϵ -deviation game. However, the converse is not true and the result only holds for the reference strategy σ_i^r or when the deviation is credible.

Now, we can state the second main result of this section, still considering only two-player games:

Theorem 7. *Under [Assumption C](#), a strategy $s_i \in S_i$ is root dominated if and only if it is a never best response in (at least) one ϵ -deviation game $\check{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \rightarrow 0^+$.*

Proof. The result is immediate by [Lemma 15](#) and [Theorem 6](#). ■

[Theorem 7](#) establishes that a strategy $s_i \in S_i$ is root dominated if it is never optimal in (at least) one 0^+ -deviation game. That is, if i thinks about deviations from a reference strategy and believes that these thoughts can be observed with an infinitesimal probability, he never plays root dominated strategies.

Root Undominated $s_i \in S_i$ is equivalent to be:	
- Locally 0^+ -rational under Assumption R	(Theorem 5)
- Best Response of all 0^+ -hesitation games	(Theorem 6)
- Best Response of all 0^+ -deviation games under Assumption C	(Theorem 7)

Figure A.4: Summary of the Results

A.10 Rationality when ϵ moves away from 0

Now, let us examine the implications of such concepts on games outcomes when ϵ is far from 0. By contrast with the statement of [Fact 1](#), our concepts of rationality do not refine the standard definition of rationality (see [Definition 12](#)) in this case: they are unnested. This might be seen as theoretical weakness. However, it can still be of interest in situations where experimental studies results differ from game theory predictions. The most famous example is the discrepancy between them in the prisoners' dilemma. In the dilemma, the strictly dominated strategy "cooperate" would never be rational under our concepts. Thus, the cooperation outcome would never emerge. Though, it is not because the strategy is dominated, it is because the strategy "cooperate" of *both* players is dominated. When ϵ is high enough, in the case where only one player has a strictly dominated strategy, a strictly dominated strategy can be globally rational, and the dominant strategy not globally rational as the following example shows. Global rationality may generate Pareto improvement with respect to the Nash outcome:

	<i>j</i> 's Strategy	
<i>i</i> 's Strategy	<i>L</i>	<i>R</i>
<i>T</i>	(4, 1)	(2, 2)
<i>B</i>	(3, 3)	(1, 1)

Table A.9: Pareto efficiency of Global Rationality for intermediate values of ϵ

If ϵ is high enough, but not too high, *T* is not globally rational since the payoff of (*T*, *R*) is below the payoff of (*B*, *L*) (both profiles where *j* best responds), and both *j*'s strategies are globally rational. Then, an iterated elimination of non globally rational strategies would generate the outcome (*B*, *L*). However, notice that if ϵ is very high, *L* is not globally rational anymore. That is, a consistency problem appears when players falsely firmly believe that the opponent best responds to his strategy. Additionally, it could lead to the Pareto worst outcome (*B*, *R*).

Chapter 2

Top Dominance

Abstract

To deal with issues of inconsistency faced by iterated elimination of weakly or strictly dominated strategies (IEWDS or IESDS), we propose a new elimination procedure. Our procedure, named iterated elimination of top dominated strategies (IETDS), is based on the new notion of top dominance. It is more consistent than IESDS in a certain sense. Top dominance is more stringent than weak dominance (and may be more stringent than strict dominance in infinite games): it requires weak dominance and strict payoff domination of the strategy on a specific profiles set. Furthermore, it requires that the dominating strategy to be not weakly dominated. Contrary to IESDS, IETDS can reduce the set of Nash equilibria (whilst never eliminating strict Nash equilibria) without the problems of order dependence and spurious Nash equilibria encountered by IEWDS and IESDS.

2.1 Introduction

It is often argued that iterated elimination of weakly dominated strategies (IEWDS) is not an entirely satisfying procedure of strategies elimination because of problems such as the order dependence of the final outcome of a game (see [Samuelson \[1992\]](#); [Hillas and Samet \[2020\]](#) and [Figure 2.1](#) to summarize the inconsistencies of elimination procedures). In contrast, iterated elimination of strictly dominated strategies (IESDS)¹ is more restrictive, yet more consistent (i.e. outcome order independent in *finite games*). However, we know that IESDS fails to be consistent in *infinite games*² (see [Dufwenberg and Stegeman \[2002\]](#)). Solutions to deal with this problem have been twofold.

One is to keep IEWDS and IESDS and restrict the class of games in which it is consistent. [Luo et al. \[2020\]](#) have defined the class of *closed under dominance** (CD^*) games for this purpose³. However, the notion of CD^* games has to be associated with a dominance relation, i.e. a game may be CD^* under strict dominance but not under weak dominance. Roughly, CD^* requires that every dominated strategy at every step of an elimination procedure is dominated by a strategy at the end of the procedure (i.e. any dominated strategy at a given step will necessarily be eliminated by the end of the procedure, or in other words, dominance does not vanish). For instance, [[Luo et al., 2020](#), Theorem 4] establishes that compact and own-uppersemicontinuous (COUSC) games are CD^* under IESDS.

A second way has been paved by [Milgrom and Roberts \[1990\]](#) through their notion of *serially undominated strategies*. Characterizing serially undominated strategies is similar to IESDS but surviving strategies have to be undominated even in previous sequences of deletion – in short in the original game. Procedures respecting this condition are named *global* in [Apt \[2011\]](#). [Apt \[2005\]](#) studied the properties of rationalizability under this specification (in the same idea as rationalizability in [Bernheim \[1984\]](#)), and found an independent order result with weak assumptions. Symmetrically, [Chen et al. \[2007\]](#) defined $IESDS^*$ as the elimination of strictly dominated strategies such that the number of elimination steps may be uncountable⁴, and strategies may be eliminated even by

¹We define an iterated elimination procedure as an elimination rule of strategies which have a certain property *at the step they are eliminated* (see [Dufwenberg and Stegeman \[2002\]](#); [Hillas and Samet \[2020\]](#) for equivalent definitions). However, other kinds of iterated elimination procedures exist, as we see in the discussion on global procedures below. Besides, we define strict dominance such that a strategy pays strictly less than another strategy at each opponents' profile. There is weak dominance if a strategy pays less than another strategy and pays strictly less (at least) at one opponents' profile.

²To the best of our knowledge, there is no precise definition of infinite games, the concept being generally defined in a negative way. In this paper, we call *infinite games*, games whose strategy sets have an infinite number of elements, in a similar way as [[Myerson and Reny, 2020](#), p.495]. Besides, this paper is not about repeated games.

³[Luo et al. \[2020\]](#) refine the concept of games *closed under dominance* (CD) described in [Dufwenberg and Stegeman \[2002\]](#) who formalized for game theory the notion of *bounded* mechanism defined by [Jackson \[1992\]](#). In CD games, each removed strategy is eliminated by an undominated strategy *at the elimination step*.

⁴Countable IESDS may not lead to a maximal reduction in the sense of [Dufwenberg and Stegeman \[2002\]](#). Additionally, [Chen et al. \[2007\]](#) epistemically justify this possibility by the fact that common knowledge of rationality may not be obtained by procedures with countable steps as established by [Lipman \[1994\]](#).

already eliminated strategies. This last feature makes the procedure order independent but does not prevent *spurious Nash equilibria*⁵, as well as final outcomes of games with empty strategy sets. Furthermore if Apt [2005]; Chen et al. [2007] procedures are *iterated*, they are not *memoryless* in the sense that a strategy may be undominated in the game at the step it is eliminated⁶.

In this paper we propose an iterated procedure of strategies elimination that has the property of being order independent, even in infinite games. Moreover, no spurious Nash equilibrium is generated, even when best responses do not exist, in contrast to what occurs in Apt [2005]; Chen et al. [2007]. We base our procedure on the new notion of *top dominance*. Roughly, an agent’s strategy is top dominated by another if the latter is undominated and (weakly) dominates the former and, at all the profiles contained in the *Better Reply Set* to the dominating strategy (see Definition 30 below) – where either (see Equation (OM) below) (at least) one agent’s opponent best responds to the profile, or (see Equation (AS) below) all opponents do not have an available best response to the profile – we have two conditions: the player plays a best response and the dominating strategy is strictly better than the dominated one.

Figure 2.1 summarizes the inconsistencies associated with the three elimination procedures we have mentioned above. We define them as well.

Inconsistencies	Definitions	Procedures
Order Dependence	The order of elimination affects the final outcome	IEWDS in finite Games, IESDS in infinite Games
Spurious Nash	A profile which is not a Nash in the initial game is a Nash of the final outcome	IEWDS, IESDS and IESDS* in infinite Games

Figure 2.1: Inconsistencies of Elimination Procedures

In the next section, we establish the framework of this paper and show with some examples which issues we propose to solve. In Section 2.3, we define the notion of top dominance and our iterated elimination procedure. Additionally we give insights about them with some examples. In Section 2.4, we show the technical lemmas and the main results of this paper. Section 2.5 explores in detail some questions about IESDS. Finally, we conclude in Section 2.6.

⁵A spurious Nash equilibrium is created by a procedure when a profile which was not a Nash equilibrium at the start of the procedure becomes a Nash equilibrium at the end. It means that the set of Nash equilibria may expand.

⁶We propose a refinement of order independence named *memoryless order independence* (see below Section 2.4.2) to distinguish the property of “usual” procedures with respect to these global procedures.

2.2 Framework and illustrative examples

2.2.1 Framework

We denote $\Gamma = \{I, S, U\}$ a game with I the set of players, $S = \prod_{i \in I} S_i$, S_i being the strategy set of player $i \in I$ (we consider first only pure strategies), and U the vector of utility functions of each player i where $U_i : S \rightarrow \mathbb{R}$. We denote $s \in S$ a strategy profile, and s_{-i} the strategy profile of the opponents of $i \in I$ such that when i plays s_i , $s = (s_i, s_{-i})$. Finally we denote $S_{-i} = \prod_{j \in I \setminus \{i\}} S_j$ the set of opponents' strategy profiles.

For the sake of clarity, we define IESDS and IEWDS in the standard following way: given a game, IESDS (IEWDS) is the procedure that iteratively eliminates *some* strictly (weakly) dominated strategies *at the step they are eliminated* and ends when there is no (weakly) dominated strategy anymore. A procedure associated to a game is a class of *processes* which are all the applications of a procedure to the game with a specific order. Each process leads to a *sequence of games*⁷ starting from the original game and representing the game at each step of the process. If all sequences of games end with the same final outcome (after applying the procedure to a given game), the procedure is said *order independent for this game*. If it is true for all games, the procedure is said *order independent*. Besides, the procedure creates *spurious Nash equilibria* if a Nash equilibrium at the end of a sequence of games is not a Nash equilibrium of the original game.

2.2.2 Examples

Among other issues, we identify two polar cases with which IESDS fails to deal: namely, on the one hand, the non-existence of a Nash equilibrium and, on the other hand, the presence of all strategies in the Nash equilibria. For the former case, the point is that IESDS may eliminate all strategies or create spurious Nash equilibria⁸. In the latter case, by definition of IESDS, there is no deletion of strategies which belong to Nash equilibria, and IESDS may seem too restrictive. Our goal is to create a consistent and predictive⁹ procedure in infinite games. Thus, we want ours to handle these problems differently from the way IESDS deals with them. We illustrate this discussion with two simple examples. The first is described in [Dufwenberg and Stegeman, 2002, Example 5] (with the Nash equilibria showed in green):

Example 1. Let $\Gamma = \{I, S, U\}$ be a game with $I = \{1\}$, $S = (0, 1)$, $U_1(s_1) = s_1$. Every strategy $s_1 \in S_1$ is strictly dominated, and can be eliminated by IESDS, i.e. there are processes of IESDS that end with an empty set¹⁰. Γ has no Nash equilibrium. However, if

⁷See Definition 32 below for the formal definition.

⁸Even if there is a Nash equilibrium, it may play no role in the procedure, leading to equivalent problems as without Nash equilibrium, see for instance [Dufwenberg and Stegeman, 2002, Example 1].

⁹We call a procedure more predictive than another if the former deletes more strategies than the latter.

¹⁰Remark that rationalizability and cautiousness introduced by Pearce [1984]) may both lead to eliminate all

a process of IESDS eliminates all strategies but one¹¹, this strategy is a Nash equilibrium (since it is the only remaining strategy).

Example 1 shows directly that IESDS may create order dependence and spurious Nash equilibria.

i 's Strategy	j 's Strategy											
	...	$(s+1)_j$	s_j	$(s-1)_j$...	7_j	6_j	5_j	4_j	3_j	2_j	1_j
\vdots			\vdots	\vdots					\vdots	\vdots	\vdots	\vdots
$(s+1)_i$		$s+1$	s	$s-1$					\vdots	\vdots	\vdots	\vdots
s_i		$s-1$	s	$s-1$					\vdots	\vdots	\vdots	\vdots
$(s-1)_i$		$s-1$	$s-2$	$s-1$					\vdots	\vdots	\vdots	\vdots
\vdots						\ddots			\vdots	\vdots	\vdots	\vdots
4_i	1	2	3	4	3	2	1
3_i	1	1	1	2	3	2	1
2_i	1	1	1	1	1	2	1
1_i	1	1	1	1	1	1	1

Figure 2.2: Payoffs Matrix of **Example 2**

Example 2. Let $\Gamma = \{I, S, U\}$ be a game with $I = \{i, j\}$. Players share common payoffs presented in **Figure 2.2**.

Every strategy $s_i \in S_i$ is weakly dominated, and can be eliminated by IEWDS. Γ has an infinity of Nash equilibria. However, if IEWDS eliminates all strategies but 1_i , an infinity of spurious Nash equilibria is created. IESDS does not meet these problems here, but it does not eliminate any strategy.

Additionally, these two examples show that the transitivity of dominance relations (i.e. if one strategy dominates another, the latter is also dominated by all strategies that dominate the former) might be the heart of the problem. It appears that what seems to be a desirable property in finite games creates issues in infinite games. Indeed, transitivity of strict and weak dominance may let the procedure eliminate all the strategies if there is no Nash equilibrium, leading to a lack of rationality (see **Section 2.5** for further discussions on IESDS.).

strategies as well.

¹¹Since all strategies are strictly dominated, a process of IESDS may apply strict dominance to every strategy but one. Indeed, since this strategy is the only remaining, it is not dominated anymore. However, note that it would be eliminated by IESDS*.

2.3 Top dominance and the procedure IETDS

2.3.1 Top dominance

In this section, we first define the *Better Reply Set to a strategy*, which is a profile set checked for special purposes in the domination relation we establish below. Finally, we propose a new procedure of elimination of strategies.

Definition 30. *The Better Reply Set to $s_i'' \in S_i$, denoted $\mathbf{B}(s_i'')$, is the set of all strategy profiles such that: $s_i^* = s_i''$, and,*

$$\text{either } \exists j \in I \setminus \{i\}, s_j^* \in \arg \max_{s_j \in S_j} U_j(s_j, s_{-j}^*) \quad (\text{OM})$$

$$\text{or } \forall j \in I \setminus \{i\}, \arg \max_{s_j \in S_j} U_j(s_j, s_{-j}^*) = \{\emptyset\} \quad (\text{AS})$$

In words, the Better Reply Set to a given strategy s_i'' of player i regroups all the profiles containing s_i'' such that either (at least) one agent (different from i) best replies to the profile (OM), or no opponent can best respond to the profile (AS). We see below with Lemma 16 of Section 2.4.1 that $\mathbf{B}(s_i'')$ cannot be empty.

Now, we can define top dominance:

Definition 31. *A strategy $s_i' \in S_i$ is said top dominated by the strategy $s_i'' \in S_i$, if:*

$$\forall s_{-i} \in S_{-i} : U_i(s_i'', s_{-i}) \geq U_i(s_i', s_{-i}) \quad (\text{TD1})$$

$$\forall s_{-i}^* \text{ such that } \forall s^* \in \mathbf{B}(s_i'') : U_i(s_i'', s_{-i}^*) > U_i(s_i', s_{-i}^*) \quad (\text{TD2})$$

$$\forall s_i \in S_i,$$

$$\text{either (i) } \forall s_{-i} \in S_{-i} : U_i(s_i'', s_{-i}) \geq U_i(s_i, s_{-i}),$$

$$\text{or (ii) } s_{-i}^* \text{ such that } \forall s^* \in \mathbf{B}(s_i'') : U_i(s_i'', s_{-i}^*) > U_i(s_i, s_{-i}^*) \quad (\text{TD3})$$

TD1 and TD2 are *inadmissibility* conditions, i.e. they ensure that top dominated strategies are weakly dominated. Precisely, TD1 states that s_i' is *very weakly dominated*¹² by s_i'' . TD2 states that s_i' pays off strictly less than s_i'' if the opponents play a profile of $\mathbf{B}(s_i'')$.

TD3 is the *top* condition. It states that any other strategy of i is either (i) very weakly dominated by s_i'' , or (ii) gives a strictly lower payoff than s_i'' if the opponents play a profile of $\mathbf{B}(s_i'')$.

¹²There is *very weak dominance* if a strategy always pays off at least as another strategy (see Marx and Swinkels [1997]). Therefore, either the former strategy (weakly) dominates the latter, either they are equivalent.

2.3.2 Procedure IETDS

Now, we rely on the framework of Luo et al. [2020] whose *iterated elimination procedures* are (transfinite) sequences of choice problems (see also Lipman [1994]; Apt [2005]), such that each element is a restriction of its predecessors. Especially, the use of *ordinal numbers* allows us to use uncountable sequences as it is necessary with infinite sets games. *Ordinal numbers* are useful to deal with infinities issues. Ordinal are of three types: either 0, or a successor, or a limit. Briefly, we start at 0, then, all natural numbers (except 0) are successors, and right after all these natural numbers, there is ω , the first limit ordinal (in short, ω is a limit ordinal because there is no ordinal λ such that $\omega = \lambda + 1$, in contrast with natural numbers). $\omega + 1$ is the successor to ω and the same process continues until reaching $\omega + \omega$ and so on. Thus, by denoting $0, 1, \dots, \lambda, \lambda + 1, \dots, \Lambda$ ordinal numbers, we define the *sequence of games associated with a game*:

Definition 32. A sequence of games associated with a game Γ is:

$$\{\Gamma^\lambda\}_{\lambda \leq \Lambda} \equiv \{\Gamma^0 \equiv \Gamma, \dots, \Gamma^\lambda, \dots, \Gamma^\Lambda\}$$

with $\lambda \in \llbracket 0, \Lambda \rrbracket$ such that:

- $\forall \lambda \in \llbracket 0, \Lambda \rrbracket$, $\Gamma^\lambda = \{I, S^\lambda, U\}$, with $S^\lambda = \prod_{i \in I} S_i^\lambda$, S_i^λ being the strategy set of player $i \in I$, the unchanged set of players of Γ , and U the vector of utility functions of each player i (whose domain is restricted), $U_i : S^\lambda \rightarrow \mathbb{R}$,
- $\forall \lambda \in \llbracket 1, \Lambda \rrbracket$, Γ^λ is a restriction of $\Gamma^{\lambda-1}$, i.e. if λ is a successor ordinal, $S^\lambda = S^{\lambda-1} \setminus \underline{S}^{\lambda-1}$ where $\underline{S}^{\lambda-1}$ is an arbitrary set of strategies in $\prod_{i \in I} S_i^{\lambda-1}$ top dominated in $\Gamma^{\lambda-1}$, and if λ is a limit ordinal, $S^\lambda = \bigcap_{\lambda' < \lambda} S^{\lambda'}$,
- $\underline{S}^\lambda = \prod_{i \in I} \emptyset$ if and only if $\lambda = \Lambda$.

Finally, we call *iterated elimination of top dominated strategies (IETDS)* the procedure that iteratively eliminates top dominated strategies of a game Γ through a sequence of games. A sequence of games associated with Γ and IETDS is a sequence that starts at Γ and which at each step, deletes (at least) one top dominated strategy, and ends at step Λ where no strategy is top dominated. At each step λ , S_i^λ is the set of all i 's strategies that have survived all the previous steps of IETDS.

In [Example 1](#), IETDS eliminates no strategy, “outperforming” all the previously mentioned procedures. In [Example 2](#), IETDS eliminates no strategy like IESDS, while IEWDS may eliminate all strategies. The next example is a modified version of [Example 2](#) that shows how IETDS might be more flexible IESDS:

Example 3. Let $\Gamma = \{I, S, U\}$ be a game with $I = \{i, j\}$, and the following common payoffs matrix (compared to [Example 2](#), we put 1 everywhere outside the diagonal of the matrix):

	j 's Strategy							
i 's Strategy	...	7_j	6_j	5_j	4_j	3_j	2_j	1_j
\vdots				\ddots	\vdots	\vdots	\vdots	\vdots
4_i	...	1	1	1	<u>4</u>	1	1	1
3_i	...	1	1	1	1	<u>3</u>	1	1
2_i	...	1	1	1	1	1	<u>2</u>	1
1_i	...	1	1	1	1	1	1	<u>1</u>

Figure 2.3: Payoffs Matrix of Example 3

Only 1_i and 1_j are weakly dominated and are always eliminated by IEWDS (and cautiousness). IESDS (and rationalizability) eliminates no strategy. IETDS gives the same outcome as IEWDS.

Let us study how top dominance and IETDS work in finite games with next examples (with the best response underlined):

	j 's Strategy	
i 's Strategy	L	R
T	(<u>4</u> ,2)	(3, <u>1</u>)
B	(3, <u>2</u>)	(<u>4</u> , <u>2</u>)
Z	(2, <u>2</u>)	(2, <u>2</u>)

Figure 2.4: Game with a Unique Prediction

Z is not top dominated by B . Even if Z is strictly dominated, we see that L and R are in $\mathbf{B}(B)$, and since $U_i(B, L) < U_i(T, L)$, there is no top dominance. However, T top dominates Z , and L top dominates R . The remaining strategies are T and B for i and L for j . Finally IETDS selects (T, L) as IEWDS.

However, this consistency may seem too restrictive since in a slightly modified version of the previous game (see Figure 2.5), we have no prediction anymore and no elimination.

	j 's Strategy	
i 's Strategy	L	R
T	(<u>4</u> , <u>1</u>)	(3, <u>1</u>)
B	(3, <u>2</u>)	(<u>4</u> , <u>2</u>)
Z	(2, <u>2</u>)	(2, <u>2</u>)

Figure 2.5: Game with No Prediction

Besides, we would need a slight modification of the definition of top dominance to allow a

mixed strategy to top dominate the strategy Z in the game of [Figure 2.5](#). See [Section B.3](#) for the definition of mixed top dominance and the results.

2.4 Results

2.4.1 Technical results

Denote by $B^\lambda(s_i)$ the Best Reply Set to s_i when the step of a sequence is λ . The set $B(s_i)$ never becomes empty with IETDS:

Lemma 16. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda \rrbracket, \forall i \in I, \forall s_i \in S_i^\lambda,$

$$B^\lambda(s_i) \neq \emptyset.$$

Proof. Assume $B^\lambda(s_i) = \emptyset$. By negating both conditions in [Definition 30](#), it means that all profiles s^* containing s_i'' verify the two following conditions:

$$\begin{aligned} \nexists j \in I \setminus \{i\}, s_j^* \in \arg \max_{s_j \in S_j^\lambda} U_j(s_j, s_{-j}^*) \\ \text{and } \exists j \in I \setminus \{i\}, \exists s_j^{**} \in \arg \max_{s_j \in S_j^\lambda} U_j(s_j, s_{-j}^*) \end{aligned}$$

However, if one profile containing s_i'' verifies these two conditions, it is immediate that another profile containing s_i'' exists and does not verify the first condition. \blacksquare

The next lemma shows that a strategy s'_j best response to a profile s_{-j}^* containing a top dominating strategy s_i'' , cannot be top dominated if the profile s_{-j}^* remains. The first stage of the proof relies on the simple fact that if s'_j is top dominated by s_j''' , then s_j''' is also a best response to the same profile s_{-j}^* . And symmetrically, if s_i'' is a top dominating strategy it is also a best response to both profiles with $s_{-i,-j}^*$ and respectively with s'_j or s_j''' . Then, if s_j''' was top dominating s'_j , it should strictly pay off more than s'_j on s_{-j}^* . However, it would immediately contradict the hypothesis that s'_j is a best response to s_{-j}^* . The next stages are straightforward.

Lemma 17. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall \mu \in \llbracket 1, \Lambda - \lambda \rrbracket, \forall i \in I, \forall s_i \in S_i^{\lambda+\mu},$ if s_i'' top dominates s'_i at step λ , and if s'_j is a best response to a profile s_{-j}^* containing s_i'' at step λ and s_{-j}^* is still present at step $\lambda + \mu$, s'_j is not top dominated at step $\lambda + \mu$.

Proof. We prove this statement by induction. At the first stage, the statement is equivalent to the following one: If s_i'' top dominates s'_i , and if s'_j is a best response to a profile containing s_i'' , s'_j is not top dominated.

Stage 1: Assume s'_j is a best response to a profile s_{-j}^* with $s_i^* = s_i''$ (and we add $s_j^* = s'_j$). Naturally, $s^* \in B(s_i'')$. Furthermore, assume that s_j''' top dominates s'_j . It is immediate by [TD1](#) that the profile $s^{**} \equiv (s_{-j}^*, s_j''')$ is also in $B(s_i'')$ (s_j''' is a best response, otherwise,

s_j''' would not very weakly dominate s_j' and TD1 would not be checked). Additionally, assume that s_i'' top dominates s_i' . Then, by TD3 (it is true for either one or the other condition required):

$$s_i'' \in \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}^*) \text{ and } s_i'' \in \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}^{**})$$

Thus, $s^{**} \in \mathbf{B}(s_j''')$. However, since, s_j' is a best response to s_{-j}^* , it is also a best response to s_{-j}^{**} (since $s_{-j}^* = s_{-j}^{**}$) and we have $U_j(s_{-j}^{**}, s_j''') = U_j(s_{-j}^{**}, s_j')$. Besides, the hypothesis of top domination of s_j' by s_j''' requires that $U_j(s_{-j}^{**}, s_j''') > U_j(s_{-j}^{**}, s_j')$ (TD2). Therefore, TD2 is not respected for j , and we cannot have simultaneously the domination of s_j' and s_j'' , if s_j' is a best response to a profile containing s_j'' .

Stage $\mu + 1$: Assume the property is true for a given $\mu \in \llbracket 2, \Lambda - \lambda - 1 \rrbracket$. Thus, no best response is top dominated and eliminated at step $\lambda + \mu$. Besides, while s_{-j}^* is present, it is still true that:

$$s_i'' \in \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}^*) \text{ and } s_i'' \in \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}^{**}),$$

for all $s^{**} \equiv (s_{-j}^*, s_j''')$ with $s_j'' \neq s_j'''$ a best response to s_{-j}^* . Thus, $\forall s_j''', s^{**} \in \mathbf{B}(s_j''')$. Again, s_j' cannot be top dominated since no s_j''' strictly dominates it at s_{-j}^* , and other strategies of j rewards strictly less (and then do not very weakly dominates s_j'). Finally the property is true for $\mu + 1$ and by induction for any $\mu \in \llbracket 2, \Lambda - \lambda \rrbracket$. ■

In order to simplify notations, recall that for a given sequence of games and $\forall \mu \in \llbracket 0, \Lambda - \lambda \rrbracket$, we have that $S_i^\lambda \cap S_i^{\lambda+\mu} = S_i^{\lambda+\mu}$. The next result shows that the set $\mathbf{B}(s_i)$ never expands as we progress through the steps of IETDS:

Lemma 18. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}$, $\forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket$, $\forall \mu \in \llbracket 1, \Lambda - \lambda \rrbracket$, $\forall i \in I$, if $s_i \in S^{\lambda+\mu}$ top dominates a strategy at step λ , then:

$$\mathbf{B}^{\lambda+\mu}(s_i) \subseteq \mathbf{B}^\lambda(s_i).$$

Proof. Assume $\mu \in \llbracket 1, \Lambda - \lambda \rrbracket$ and by contradiction: $\mathbf{B}^{\lambda+\mu}(s_i) \not\subseteq \mathbf{B}^\lambda(s_i)$. Take a profile $s' \equiv \prod_{k \in I} s'_k \in \mathbf{B}^{\lambda+\mu}(s_i) \setminus \mathbf{B}^\lambda(s_i)$.

Since $s' \notin \mathbf{B}^\lambda(s_i)$ but $s' \in \mathbf{B}^{\lambda+\mu}(s_i)$, we know that (at least) one player j has an available best response to s'_{-j} at λ but not with s_j' and either (*Case a*) best responds at $\lambda + \mu$ with the strategy s_j' , or (*Case b*) does not have a best response anymore at $\lambda + \mu$.

Case a: We assume that there is (at least) one player $j \neq i$ with a strategy $s_j'' \in S_j$ contained in a profile $s'' \in \mathbf{B}^\lambda(s_i)$, eliminated at a step between λ and $\lambda + \mu$ such that:

$$U_j(s_j'', s'_{-j}) > U_j(s_j', s'_{-j}) \text{ and } s_j'' \in \arg \max_{s_j \in S_j^\lambda} U_j(s_j, s'_{-j})$$

Since s_j'' is a best response, it is not top dominated by [Lemma 17](#) and thus, it cannot be eliminated at any step. Therefore, either the profile disappears (one strategy s_k' of a player $k \neq j$ can be eliminated if it is not a best response), or s_j'' still dominates s_j' on this profile. In both cases, it contradicts the hypothesis that $s' \subseteq \mathbf{B}^{\lambda+\mu}(s_i)$.

Case b: Now assume there is a strategy $s_l' \in s'$, such that l has an available best response to s'_{-l} at step λ (different from s_l'). However, assume at step $\lambda + \mu$, there is no best response to s'_{-l} anymore. Again, by [Lemma 17](#), no such an elimination is possible.

Finally we have shown that $\mathbf{B}^{\lambda+\mu}(s_i) \subseteq \mathbf{B}^\lambda(s_i)$ for all $\mu \in \llbracket 2, \Lambda - \lambda \rrbracket$. ■

The following lemma states that a top dominating strategy is never eliminated with IETDS:

Lemma 19. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda \rrbracket, \forall \mu \in \llbracket 0, \Lambda - \lambda \rrbracket, \forall i \in I, \forall s_i \in S_i^\lambda, \forall s_i'', s_i''' \in S_i^{\lambda+\mu},$

$$s_i'' \text{ top dominates } s_i' \text{ in } \Gamma^\lambda \Rightarrow \nexists s_i''' \in S_i^{\lambda+\mu}, s_i''' \text{ top dominates } s_i'' \text{ in } \Gamma^{\lambda+\mu}.$$

Proof. By [TD3](#) we know that either s_i'' strictly dominates other strategies of S_i if we restrict attention to opponents' profiles in $\mathbf{B}^\lambda(s_i'')$, or if there is equality with an other strategy (for at least one opponents' profile), then s_i'' weakly dominates or is equivalent to the latter for all opponents' profiles. By [Lemma 16](#), we know that $\mathbf{B}^\lambda(s_i'')$ is never reduced to an empty set. Besides, by [Lemma 18](#), $\mathbf{B}^{\lambda+\mu}(s_i) \subseteq \mathbf{B}^\lambda(s_i)$ for all $\mu \in \llbracket 2, \Lambda - \lambda - 1 \rrbracket$. Therefore, at every step of the sequence, either s_i'' pays off strictly more than other strategies in (at least) one profile of $\mathbf{B}^{\lambda+\mu}(s_i'')$, or if it pays off the same as another strategy in one profile of $\mathbf{B}^{\lambda+\mu}(s_i'')$, the latter is equivalent or weakly dominated by s_i'' . Thus, s_i'' is still undominated (i.e. not weakly dominated) at every step of the sequence. ■

Now we establish that the relation of top dominance between two strategies is maintained through IETDS:

Lemma 20. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall \mu \in \llbracket 0, \Lambda - \lambda - 1 \rrbracket, \forall i \in I, \forall s_i' \in S_i^{\lambda+\mu}, \forall s_i'' \in S_i^\lambda,$
if s_i'' top dominates s_i' in Γ^λ , then s_i'' top dominates s_i' in $\Gamma^{\lambda+\mu}$.

Proof. First, note the result is immediate if $\mu = 0$. Second, note that by [Lemma 19](#), s_i'' is still in $\Gamma^{\lambda+\mu}$ and our statement always makes sense. Now, assume $\mu \in \llbracket 1, \Lambda - \lambda - 1 \rrbracket$ and assume s_i'' top dominates s_i' in Γ^λ . It is straightforward that [TD1](#) is still verified at each μ . By [Lemma 18](#), we know that $\mathbf{B}^{\lambda+\mu}(s_i'') \subseteq \mathbf{B}^\lambda(s_i'')$. Additionally, $\mathbf{B}(s_i'')$ cannot be empty by [Lemma 16](#). Therefore [TD2](#), and [TD3](#) are still verified as well. ■

We end this section by linking top dominance to the property of *Monotonicity** of [Luo et al. \[2020\]](#) which ensures outcome order independence. Denote $c(\Gamma)$, the Γ -choice, i.e.

the set of strategies in S which are not top dominated in Γ . The following property states that no strategy previously top dominated becomes non top dominated through IETDS:

Definition 33. A dominance relation is said to verify *Monotonicity**, if $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall \mu \in \llbracket 0, \Lambda - \lambda - 1 \rrbracket,$

$$\Gamma^\lambda, \Gamma^{\lambda+\mu} \in \{\Gamma^\lambda\}_{\lambda \leq \Lambda} \Rightarrow c(\Gamma^{\lambda+\mu}) \subseteq c(\Gamma^\lambda).$$

Lemma 21. *Top dominance verifies Monotonicity*.*

Proof. Assume $\exists \{\Gamma^\lambda\}_{\lambda \leq \Lambda}$ such that $\exists \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \exists s_i \in c(\Gamma^{\lambda+1}) \setminus c(\Gamma^\lambda)$. That is, s_i is top dominated in Γ^λ but is not top dominated in $\Gamma^{\lambda+1}$. By combining [Lemmas 19](#) and [20](#), the contradiction is immediate. ■

By [[Luo et al., 2020](#), Theorem 2], we know that *Monotonicity** implies outcome order independence.

2.4.2 Main results

An iterated elimination of strategies is not order independent if starting by the elimination of strategies of one given player rather than another (or by one strategy of a given player rather than another strategy) modifies the final outcome of the elimination procedure.

Theorem 8. *IETDS is order independent.*

Proof. By [Lemma 21](#) and [[Luo et al., 2020](#), Theorem 2], the result is immediate. ■

A spurious Nash equilibrium is created if a strategy profile $s \in S$ is not a Nash equilibrium of Γ but is a Nash equilibrium of Γ^Λ . We denote $\mathbf{N}(\Gamma)$ the set of Nash equilibria of a game Γ . We show that no spurious Nash equilibrium is created by IETDS:

Theorem 9. $\forall s \in S^\Lambda, s \in \mathbf{N}(\Gamma^\Lambda) \Rightarrow s \in \mathbf{N}(\Gamma)$

Proof. Assume that IETDS leads to the selection of a spurious Nash equilibrium due to a sequence of games. Then, we have a profile $s^* \in S \cap S^\Lambda$ such that:

$$\begin{aligned} \forall k \in I, s_k^* &\in \arg \max_{s_k \in S_k^\Lambda} U_k(s_k, s_{-k}^*) \\ \exists i \in I, s_i^* &\notin \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}^*) \end{aligned}$$

Clearly, all strategies $s'_i \in S_i$ such that $U_i(s'_i, s_{-i}^*) > U_i(s_i^*, s_{-i}^*)$ have been eliminated (there is necessarily (at least) one such strategy s'_i , otherwise s_i^* would be a best response in Γ). However, for all of these strategies, there is (at least) a strategy s''_i that top dominates them. By [Lemma 19](#), this strategy is not eliminated. Then, by weak domination of eliminated strategies, we have $U_i(s''_i, s_{-i}^*) > U_i(s_i^*, s_{-i}^*)$. It contradicts the hypothesis that $s^* \in \mathbf{N}(\Gamma^\Lambda)$. ■

The non existence of spurious Nash equilibria may seem of little matter. However, a striking fact is that even global procedures cannot prevent their existence whereas all strategies can still be used to eliminate strategies (see [Chen et al., 2007, Example 4]). Therefore, it may seem surprising that a strategy can be used to eliminate another strategy but not to effectively play the game and break the spurious equilibrium. Our procedure does not exhibit such queries. We distinguish also IETDS from these global procedures with the following notion:

Definition 34. *A procedure is said memoryless order independent if starting the procedure at every step of a sequence of games without knowing the previous steps of the sequence does not change the outcome of the game.*

Theorem 10. *IETDS is memoryless order independent.*

Proof. Once a strategy has been eliminated by IETDS, it is not used anymore in the elimination procedure. Then by Theorem 8, the result is immediate. ■

2.5 Discussion about strict dominance

Gilboa et al. [1990] gives sufficient conditions for the outcome order independence of a dominance relation in finite games, and proved the outcome order independence of strict dominance in finite games. Börgers [1993] refines the concept of strict dominance with *pure strategy dominance*, applied to a framework with ordinal utility functions where the eliminated strategy is not necessarily dominated by only one strategy. Loosely speaking, a strategy is eliminated if for each strategy subset of the opponent there is a strategy (not necessarily the same strategy at each profile as in strict dominance) that weakly dominates the former. Iterated elimination of such dominated strategies would be more predictive than IESDS (if IESDS was applied to the same framework), and still outcome order independent in finite games. Marx and Swinkels [1997] refine IESDS by eliminating strategies that are *nicely* weakly dominated. Nice weak dominance requires weak dominance and if there is indifference at some profiles, it requires that all players are also indifferent at these profiles. However, Marx and Swinkels [1997]’s procedure is only payoffs order independent (and not strategies order independent). When we consider infinite games, as pointed out by Dufwenberg and Stegeman [2002], IESDS (and consequently each procedure whose outcome refines IESDS outcome) fails to be order independent in infinite games. Moreover, it may predict *irrational* outcomes. In Example 1, strict dominance may predict that all strategies are dominated (we can always find a larger x in $(0, 1)$ that gives a strictly larger payoff), and then that the agent “should not play the game” (if it is how we may understand the emptiness of the strategy set) while it gives to him a strictly positive payoff for every strategy! Then, the problem with strict dominance is not only order dependence, it may also be *rationality*¹³. Interestingly, this example highlights that

¹³Of course, we mean by *rationality* something different from the formal statement that a rational player should maximize his utility, but rather as the more general economic assumption that agents “will choose ‘more’ rather

IESDS is sensitive to addition of new strategies. The strategy 0.4 may not be eliminated by IESDS in [Example 1](#). However, if we add the strategy 1 to the previous game, 0.4 would surely be eliminated. Top dominance states that, for instance, 0.5 never dominates 0.4 except if all strategies above 0.5 are not in the game (like IESDS here). This leads our procedure to select the same strategy as IESDS when it selects a strategy, and to select all strategies when IESDS may eliminate all strategies. If we denote $B = (0, 1]$ and $B' = (0, 1)$, this discussion can be linked to the following concern raised by [Herzberger \[1973\]](#):

According to classical views of rationality, there will be a rational choice from [...] set B , but no rational choice will be possible from its subset B' .

Even if we try to avoid order dependence with a more restrictive definition like in [Fudenberg and Tirole \[1991\]](#), where all dominated strategies at one round of elimination are all deleted (or if IESDS is defined as a global procedure as in [Chen et al. \[2007\]](#)), we still eliminate all the available strategies. Thus, order independence does not necessarily prevent *irrationality*.

In another example of [Dufwenberg and Stegeman \[2002\]](#), where the agents 1 and 2 have respectively strategies s_1 and s_2 in \mathbb{R}_+ , and a common payoff $u_i = \max\{s_1, (1 - s_1 - s_2)\}/(1 + s_1)$, the elimination is not order independent with IESDS. In this game, the unique Nash equilibrium is $(0, 0)$. No strategy for 2 is strictly dominated but every strategy $s_2 > 0$ is weakly dominated by $s_2 = 0$. On the contrary $s_1 = 0$ does not dominate any strategy. However, it can be seen that every $s_1 > 0$ is strictly dominated by a larger s_1 ¹⁴. The order of elimination matters since if only one strategy $s_1 > 0$ remains, no strategy can be eliminated anymore by IESDS. Worse, as pointed out by [Dufwenberg and Stegeman \[2002\]](#), this elimination creates spurious equilibria. With IETDS, the player 2's strategies may be reduced immediately to the singleton $\{0\}$. Then, the only best reply of the player 1 is 0 leading to the unique Nash equilibrium.

2.6 Conclusion

In this paper, we propose a new dominance relation, *top dominance*, and its associated elimination procedure, *IETDS*. The main idea behind top dominance is that players eliminate strategies only if there is a strategy that “dominates” them and which may not be dominated as well. The *Better Reply Set* is constructed around this idea: each player considers the moves of his opponents if they are either a best response, or if no best response is available. We show that IETDS is outcome order independent in any game. Furthermore, it does not generate spurious Nash equilibria and it is *memoryless order independent*. We mainly compare IETDS with IEWDS and IESDS. Prediction with IEWDS seems to be

than ‘less’” (see [\[Buchanan and Tullock, 1962, p.17\]](#)).

¹⁴ $\frac{s_1}{1+s_1}$ converges to 1 as s_1 goes to infinity and $\frac{1-s_1}{1+s_1} < 1$ for $s_1 > 0$.

a challenging task since the same game may generate several outcomes. IESDS does not perform better in infinite games. We show through simple examples that IETDS can be as predictive as IEWDS when IEWDS generates an order independent outcome (so when IEWDS generates an understandable outcome in fact). We show also that IETDS can predict a unique Nash equilibrium where IESDS generates spurious Nash equilibria.

Bibliography

- Apt, K. R. (2005). Order Independence and Rationalizability. In *Proceedings of the Tenth Conference on the Theoretical Aspects of Rationality and Knowledge*, pages 22–38.
- Apt, K. R. (2011). Direct Proofs of Order Independence. *Economics Bulletin*, 31(1):106–115.
- Bernheim, B. D. (1984). Rationalizable Strategic Behavior. *Econometrica*, 52(4):1007–1028.
- Börgers, T. (1993). Pure Strategy Dominance. *Econometrica*, 61(2):423–430.
- Buchanan, J. M. and Tullock, G. (1962). *The Calculus of Consent: Logical Foundations of Constitutional Democracy*. University of Michigan Press.
- Chen, Y. C., Long, N. V., and Luo, X. (2007). Iterated Strict Dominance in General Games. *Games and Economic Behavior*, 61(2):299–315.
- Dufwenberg, M. and Stegeman, M. (2002). Existence and Uniqueness of Maximal Reductions under Iterated Strict Dominance. *Econometrica*, 70(5):2007–2023.
- Fudenberg, D. and Tirole, J. (1991). *Game Theory*. MIT Press.
- Gilboa, I., Kalai, E., and Zemel, E. (1990). On the Order of Eliminating Dominated Strategies. *Operations Research Letters*, 9(2):85–89.
- Herzberger, H. G. (1973). Ordinal Preference and Rational Choice. *Econometrica*, 41(2):187–237.
- Hillas, J. and Samet, D. (2020). Dominance Rationality: A Unified Approach. *Games and Economic Behavior*, 119:189–196.
- Jackson, M. O. (1992). Implementation in Undominated Strategies: A Look at Bounded Mechanisms. *Review of Economic Studies*, 59(4):757–775.
- Lipman, B. L. (1994). A Note on the Implications of Common Knowledge of Rationality. *Games and Economic Behavior*, 6(1):114–129.
- Luce, R. D. and Raiffa, H. (1957). *Games and Decisions: Introduction and Critical Survey*. New York: Wiley.
- Luo, X., Qian, X., and Qu, C. (2020). Iterated Elimination Procedures. *Economic Theory*, 70(2):437–465.
- Marx, L. M. and Swinkels, J. M. (1997). Order Independence for Iterated Weak Dominance. *Games and Economic Behavior*, 18(2):219–245.
- Milgrom, P. and Roberts, J. (1990). Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities. *Econometrica*, 58(6):1255–1277.

- Myerson, R. B. and Reny, P. J. (2020). Perfect Conditional ϵ -Equilibria of Multi-Stage Games With Infinite Sets of Signals and Actions. *Econometrica*, 88(2):495–531.
- Nash, J. F. (1950). The Bargaining Problem. *Econometrica*, 18(2):155–162.
- Pearce, D. G. (1984). Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52(4):1029–1050.
- Samuelson, L. (1992). Dominated Strategies and Common Knowledge. *Games and Economic Behavior*, 4(2):284–313.

Appendices

Appendix B

Appendices to Top Dominance

B.1 Additional results

The next result states that no game becomes empty through IETDS:

Proposition 5. $S^0 \neq \emptyset \Rightarrow \forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, S^\Lambda \neq \emptyset$.

Proof. By [Lemma 19](#), we know that if a strategy top dominates another, it is never top dominated (thus never eliminated). If no strategy top dominates another, then $\Lambda = 0$ and by hypothesis $S^0 \neq \emptyset$. ■

The following result may show why our elimination procedure is order independent. In the words of [Dufwenberg and Stegeman \[2002\]](#); [Luo et al. \[2020\]](#), each top dominated strategy has an undominated top dominator, i.e. each top dominated strategy at a point of a sequence of games will be deleted by the end of the sequence:

Proposition 6. $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}, \forall \lambda \in \llbracket 0, \Lambda - 1 \rrbracket, \forall i \in I, \forall s_i \in S_i^\lambda$ top dominated in Γ^λ ,

$$s_i \notin \Gamma^\Lambda$$

Proof. By combining [Lemmas 19](#) and [20](#), the result is immediate. ■

IETDS satisfies a weak form of the independence of irrelevant alternatives¹, the irrelevance of weakly dominated acts², i.e. the addition of weakly dominated strategies (by the top dominating strategy studied) does not affect the dominance relation:

Proposition 7. Assume Γ and Γ' such that $N = N', S_{-i} = S'_{-i}, U = U', S_i = S'_i \cup s_i^*$, Then, s''_i top dominates s'_i in $\Gamma \Rightarrow s''_i$ top dominates s'_i in Γ' if s_i^* is weakly dominated by s''_i .

¹See [Nash \[1950\]](#); [Gilboa et al. \[1990\]](#).

²See [Luce and Raiffa, 1957](#), p.288, Axiom 6].

Proof. If s_i^* is weakly dominated by s_i'' , then $\forall s_{-i} \in \prod_{j \in I \setminus \{i\}} S_j$, $U_i(s_i'', s_{-i}) \geq U_i(s_i^*, s_{-i})$, especially for profiles $s \in \mathbf{B}(s_i'')$. Thus, if all conditions of [Definition 31](#) are checked in Γ , it is also the case in Γ' . ■

B.2 Why is [TD3](#) needed?

		k 's Strategy							
		j 's Strategy							
i 's Strategy		...	7_j	6_j	5_j	4_j	3_j	2_j	1_j
\vdots									
4_i		...	4	4	4	4	4	4	4
3_i		...	3	3	3	3	3	3	3
2_i		...	2	2	2	2	2	2	2
1_i		...	1	1	1	1	1	1	1
		1_k							
		j 's Strategy							
i 's Strategy		...	7_j	6_j	5_j	4_j	3_j	2_j	1_j
\vdots				...	\vdots	\vdots	\vdots	\vdots	\vdots
4_i		...	1	1	1	4	1	1	1
3_i		...	1	1	1	1	3	1	1
2_i		...	1	1	1	1	1	2	1
1_i		...	1	1	1	1	1	1	1
		0_k							

Figure B.1: [TD3](#) with a Three-Player Game

In the above game, k has two strategies: 1_k and 0_k which always reward 1 and 0 respectively. If k plays 0_k , then i and j play as in the game of [Example 3](#). If k plays 1_k , then only i 's choice matters and both i and j get $U(s_i, s_j) = s_i$ (s_i being the strategy chosen by i). Here, only 0_k is top dominated. Once it is eliminated, nothing else happens (as in [Example 1](#)). Now, if we modify the definition of top dominance and we do not require that the top dominating strategy rewards a (weakly) higher payoff than all other strategies on the Better Reply Set (i.e. we release [TD3](#)) but only rewards a strictly higher payoff on (at least) one profile of the Better Reply Set, 2_i would top dominate 1_i (consistently with what happens in [Example 3](#)). So if we start by eliminating i 's strategies, 1_i would be eliminated. In contrast, if we start by eliminating k 's strategies, we would first eliminate

0_k , preventing 1_i elimination. Therefore IETDS would not be order independent.

B.3 Mixed top dominance

Consider the mixed extension of a game Γ and denote Σ the set $\prod_{i \in I} \Sigma_i \equiv \prod_{i \in I} \Delta(S_i)$ the set of all (mixed) strategies. Thus, $\sigma_i \in \Sigma_i$ is a mixed strategy if it is a probability distribution over the set S_i of pure strategies. As in the pure strategy case, we denote Σ_{-i} the set $\prod_{j \in I \setminus \{i\}} \Sigma_j \equiv \prod_{j \in I \setminus \{i\}} \Delta(S_j)$, the strategy profiles set of i 's opponents. Let $\sigma_i(s_i)$ be the probability that s_i is effectively used when σ_i is played and denote $R_{\sigma_i} = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$ the support of σ_i ³. We apply the definition of a Better Reply Set to mixed strategies in the same way as in the pure strategy case:

Definition 35. *The Better Reply Set to $\sigma_i'' \in \Sigma_i$, denoted $\mathbf{B}(\sigma_i'')$, is the set of all strategy profiles $\sigma^* \in \Sigma$ such that:*

$\sigma_i^ = \sigma_i''$, and, if $S_{-i} \neq \emptyset$:*

$$\exists j \in I \setminus \{i\}, \sigma_j^* \in \arg \max_{\sigma_j \in \Sigma_j} U_j(\sigma_j, \sigma_{-j}^*) \quad (\text{OM}')$$

$$\text{or } \forall j \in I \setminus \{i\}, \arg \max_{\sigma_j \in \Sigma_j} U_j(\sigma_j, \sigma_{-j}^*) = \{\emptyset\} \quad (\text{AS}')$$

Now, we extend the notion of Better Reply Set to strategy subsets:

Definition 36. *For any strategy subset $\bar{S}_i \subset S_i$, we denote $\mathbf{B}(\bar{S}_i) = \bigcup_{\sigma_i \in \Delta \bar{S}_i} \mathbf{B}(\sigma_i)$ the Better Reply Set to the strategy subset \bar{S}_i .*

Note that if the subset is a singleton, [Definitions 30](#) and [36](#) obviously coincide. Importantly, in order to define *mixed top dominance*, we will use the Better Reply Set to the strategy subset formed by the support of the mixed strategy:

Definition 37. *For any strategy subset $\bar{S}_i \subset S_i$, we denote $\mathbf{B}(\bar{S}_i) = \bigcup_{\sigma_i \in \Delta \bar{S}_i} \mathbf{B}(\sigma_i)$ the Better Reply Set to the strategy subset \bar{S}_i .*

Note that if the subset is a singleton, both definitions of Better Reply Sets obviously coincide. Remark also that we directly apply the definition of a Better Reply Set to a pure strategies to mixed strategies. Besides, in order to define *mixed top dominance*, we will use the Better Reply Set to the strategy subset formed by the support of the mixed strategy:

Definition 38. *A strategy $s'_i \in S_i$ is said top dominated by the mixed strategy $\sigma_i'' \in \Sigma_i$ whose support is $R_{\sigma_i''}$, if:*

³Note that this definition of the support cannot be weakened by allowing e.g. a continuous distribution as a support. We clarify this point below.

$$\forall s_{-i} \in S_{-i} : \quad U_i(\sigma_i'', s_{-i}) \geq U_i(s'_i, s_{-i}) \quad (\text{TD1}')$$

$$\forall \sigma_{-i}^* \text{ such that } \sigma^* \in \mathbf{B}(R_{\sigma_i''}) : \quad U_i(\sigma_i'', \sigma_{-i}^*) > U_i(s'_i, \sigma_{-i}^*) \quad (\text{TD2}')$$

$$\forall s_i \in S_i \setminus R_{\sigma_i},$$

$$\text{either (i) } \forall s_{-i} \in S_{-i} : \quad U_i(\sigma_i'', s_{-i}) \geq U_i(s_i, s_{-i})$$

$$\text{or (ii) } \forall \sigma_{-i}^* \text{ such that } \sigma^* \in \mathbf{B}(R_{\sigma_i''}) : \quad U_i(\sigma_i'', \sigma_{-i}^*) > U_i(s_i, \sigma_{-i}^*) \quad (\text{TD3}')$$

We use the same definition as the pure top dominance case except that we consider the Better Reply Set of the support and that **TD3** is checked only for strategies outside the support of the mixed strategy. This definition is in fact a generalization of the pure version, given that a strategy cannot be top dominated by itself, and given that **TD3** is verified if we compare a strategy to itself. Note that we require that the support of a mixed strategy must be finite, since the probabilities are strictly positive.

The next result states that mixed top dominance refines strict dominance in finite games:

Theorem 11. $\sigma_i'' \in \Sigma_i$ strictly dominates $s'_i \in S_i$ in a finite game:

$$\Rightarrow \exists \sigma_i \in \Sigma_i \text{ such that } \sigma_i \text{ top dominates } s'_i.$$

Proof. Set $\text{card} = \sharp(S_i)$, the cardinal of S_i . If $\text{card} = 2$, set $\sigma_i = \sigma_i''$ and the result is straightforward. Now assume $\text{card} > 2$ (but still a finite number).

Set $\sigma_i \in \Sigma_i$ such that $R_{\sigma_i} = S_i \setminus s'_i$. For each strategy s_i in $S_i \setminus s'_i$ which does not very weakly dominates σ_i'' , set:

$$d_{\sigma_i'' \rightarrow s_i} = \max_{s_{-i} \in S_{-i}} (U_i(\sigma_i'', s_{-i}) - U_i(s_i, s_{-i})) > 0, \text{ and set:}$$

$$d_{\sigma_i'' \rightarrow s'_i} = \min_{s_{-i} \in S_{-i}} (U_i(\sigma_i'', s_{-i}) - U_i(s'_i, s_{-i})) > 0$$

Denote σ_i the probabilities such that $\sigma_i(s_i) < \frac{d_{\sigma_i'' \rightarrow s'_i}}{\text{card} \times d_{\sigma_i'' \rightarrow s_i}}$ and any arbitrary positive probability for strategies which very weakly dominate σ_i'' . Therefore, the payoff difference between the two strategies is such that:

$$\begin{aligned} U_i(\sigma_i) - U_i(s'_i) &= U_i(\sigma_i) - U_i(\sigma_i'') + U_i(\sigma_i'') - U_i(s'_i) > \\ &\quad - \sum_{s_i \in S_i \setminus \{s'_i\}} \sigma_i(s_i) \times d_{\sigma_i'' \rightarrow s_i} + d_{\sigma_i'' \rightarrow s'_i} > 0 \end{aligned}$$

It means we have constructed a mixed strategy that strictly dominates s'_i . Consequently **TD1'** and **TD2'** are verified. Since every strategy but s'_i composed the mixed strategy, **TD3'** is trivially verified. ■

Chapter 3

Complementarities in Information Acquisition

Co-authored with Sidartha Gordon

Abstract

In games with complementarities in actions, i.e. an increase in the action of a player induces that other players should increase their actions as well, complementarities in information acquisition usually emerge. [Hellwig and Veldkamp \[2009\]](#) show that in this kind of games, if an agent increases the precision of his information acquisition, other agents do as well (the inheritance result). In a global game, [Szkup and Trevino \[2015\]](#) exhibit that this inheritance result is not robust when an individual does not play the equilibrium action. We introduce a coordination game with private information acquisition to show that the inheritance result may fail in a symmetric equilibrium. Complementarities and substitutabilities emerge depending on the parameters of the game. The frontier between these two areas corresponds to the situation where the strategy profile of the other players does not affect the value of information.

3.1 Introduction

In games where actions can be ordered, actions are said complements, if the player i should increase his action to get a higher utility, when a player j increases her action. If the opposite mechanism appears, the actions of i and j are said substitutes.

Then, what happens when agents are allowed to acquire information? Do complementarities in information acquisition emerge when there are complementarities in action, i.e. should players whose actions are complements acquire more information when their opponents act this way? More precisely, if other agents acquire more information, is an individual player always encouraged to acquire more information as well (what we call the inheritance result).

The inheritance result can be defined in other words as the transmission of complementarities from actions to information acquisition. It is a well known result established by Hellwig and Veldkamp [2009] in a beauty contest with a continuum of players. In their paper, the utility of each agent is represented by a quadratic loss¹:

$$EL(a_i, a, \theta) = E[(1 - r)(a_i - \theta)^2 + r(a_i - a)^2]$$

Then, when agents exhibit complementarities in actions (i.e. $r > 0$) (and in the state, i.e. $r < 1$), acquiring more information is always advantageous when the precision of the others' signals increase.

However, few changes to the original model may revert the result. In Jiménez-Martínez [2014], a two-player case (more a strategic interaction between the other agent's action and the state in the player's utility) shows that the inheritance result does not hold when the atomicity vanishes. Especially, the variance of the other players' action affects the value of information and when the actions are said highly complements, substitutabilities in information acquisition may appear.

As well, Szkup and Trevino [2015] illustrate in a global game that even with a continuum of agents, the inheritance result is shaky. In their paper, agents can make a risky investment. Investment is successful if the state of the world and the proportion of agents investing are high enough². In Szkup and Trevino [2015], agents coordinate on a threshold θ^* (a state of the world) above which the investment is always successful after getting a costly signal on the state of the world (and always unsuccessful otherwise). If all the players but an individual i increase the precision of their signal, i may not be encouraged to increase his precision as well. However, this counter example occurs in very specific cases. The individual considered should have a very low precision while the other agents have a very high precision. Nevertheless, this result show that even if there are complementarities in actions, certain forces may reverse the incentive to mimic the action of other players in terms of information³.

We propose to test if the result in Szkup and Trevino [2015] is due to technical assumptions (e.g. the normal distribution of the state of the world and the signals) so to establish a frontier between the games with and without inheritance. For this purpose, we introduce an information game where the agents are able to know with a certain error if the state of the world is above or below a certain threshold.

¹ a_i, a, θ representing respectively the action of the individual i , the aggregate action and the state of the world.

²Other configurations are possible in global games, see Morris and Shin [2003] for an extensive survey.

³Szkup and Trevino [2015] explain well the difference between beauty contest and global games. If other players use more information in a beauty contest, they will be closer to the state of the world, which incentivizes to acquire more information for any individual player. Since being closed to the fundamental and the others' action matters, both forces work together. On the contrary, in global games, the actions of other players affect the individual through the threshold θ^* . If the threshold is non monotonic in the aggregate precision, more information for other players might result in less information for the individual one.

Our model is close from the model proposed by [Yang \[2015\]](#) who studies a coordination game under uncertainty with rationally inattentive agents. In that way, agents can reduce uncertainty by acquiring costly information. The cost of information is linked to the reduction of uncertainty (proportional to the reduction of entropy in his paper). The information acquisition technology is said flexible, in opposition to noisy signals in global games which are rigid structures (see [Szkup and Trevino \[2015\]](#)). Indeed, agents are allowed to specify on each state of the world, the amount of information they want to acquire. Each state of the world will imply a specific signal structure, what is flexible with respect to global games, where the signal structure is designed for all the states of the world. Thanks to this flexibility, agents can coordinate on more equilibria than in global games (which exhibit uniqueness under certain conditions), and especially, they can reach the Pareto best equilibrium when information becomes infinitely cheap. Because of the entropic form of the cost function, the cost paid by an agent to acquire information depends on the probability of success of the risky action. A change in the precision chosen by the agents may affect this probability. In turn, an individual agent has to change the level of his own precision. Furthermore, every equilibrium has different properties since the ex ante uncertainty is modified for each of them. More certainty leads to more precise signals because there are less costly, but this certainty comes from the strategy of the agents, and not from the distribution of the states. Information should have value when information make earn money, and information should be costly when there is uncertainty. However, in [Yang \[2015\]](#) value and cost of information are similar, because uncertainty is proportional to the probability of earning money. With entropy, it is not possible to disentangle the information value problem and the information cost problem. Since agents' strategies shape this cost, information may be costed differently for the same distribution of the states of the world. As we want to analyze the precision problem, we do not use entropy to cost information. More recently, [Yang \[2018\]](#) seeks the same kind of questions as ours in a general setting. However, information cost is fixed and the value of information is set as the difference between the payoff when the individual acquires information and when he does not (holding constant the acquisition of the other agents). Thus, this framework does not answer to questions about the amount of information agents acquire. In the next section, we introduce the model. Then we establish the equilibria. In the fourth section, we make the analysis in terms of complementarities or substitutabilities of information acquisition. And finally we conclude.

3.2 Model

In our model, as in [Szkup and Trevino \[2015\]](#), we assume a continuum of agents with two possible actions: investing I and not investing NI . Investing costs $0 < T < 1$ and rewards 1 if it is successful (0 either). Not investing always rewards 0.

Therefore, the **payoff** for any individual i can be written:

$$\begin{array}{l} \text{If } I \\ \text{If } NI \end{array} \quad \begin{cases} \Pi_i = 1 - T & \text{if the investment is successful.} \\ \Pi_i = -T & \text{if the investment is unsuccessful.} \end{cases}$$

$$\begin{cases} \Pi_i = 0 & \text{if the investment is unsuccessful.} \\ \Pi_i = 0 & \text{if the investment is successful.} \end{cases}$$

However, the action NI when the investment is the good action is a mistake and the opportunity cost (i.e. what would have been earned by playing the good action) of such an action in this situation is $1 - T$. Then, we can define the two errors:

Definition 39. *Errors are of two distinct types.*

- *Type-I error: Play the action Invest while the investment is not successful. The associated loss is $-T$.*
- *Type-II error: Play the action Not Invest while investment is successful. The associated loss is $-(1 - T)$.*

Whether the investment is successful depends positively on the state of the nature and depends positively on the proportion of agents investing. We define the mass condition, i.e. the condition such that the investment is successful:

$$\theta + p \geq 1$$

where θ is the state of the world which has a continuous proper cumulative distribution function \mathbb{F}_θ with a positive density f_θ between 0 and 1⁴, and, p is the proportion of agents who have chosen to invest.

3.2.1 Signal

In order to maximize their profits, the agents may observe the realization of a binary costly signal. Every individual i is allowed to choose a threshold ξ_i and observe if the state of the world θ is greater than ξ_i . However, this signal is exact only with probability $(1 - \epsilon_i)$. The cost of the signal is $C_i(\epsilon_i)$ and is decreasing in ϵ_i , which is the error of the signal chosen by the agent (i.e. the probability that the signal is untrue). To sum up, the

⁴The case in which the prior has an improper distribution, such that the uniform distribution on the real line is in fact a limiting case where neither complementarities nor substitutabilities may appear because the cumulative distribution function is a constant. See below the analysis for further developments.

agent i observes:

$$\begin{aligned} \text{With probability } (1 - \epsilon_i), \quad & X_i(\xi_i, \epsilon_i) = \begin{cases} 1 & \text{if } \theta \geq \xi_i \\ 0 & \text{if } \theta < \xi_i \end{cases} \\ \text{With probability } \epsilon_i, \quad & X_i(\xi_i, \epsilon_i) = \begin{cases} 1 & \text{if } \theta < \xi_i \\ 0 & \text{if } \theta \geq \xi_i \end{cases} \end{aligned}$$

There is no cost on the threshold chosen by the agent. In other words, choosing a threshold or another does not affect the cost paid by the agent. In this sense, this cost function is similar to the one displayed in [Morris and Yang \[2016\]](#). However, their cost function depends on the maximal slope of the agent's action function. In our case, the slope of the action function is infinite. It means that the agent has a discontinuous action around the threshold, i.e. the probability to choose the action I shifts up or down radically. It does correspond to the specific case where the cost is cheap in [Morris and Yang \[2016\]](#). Indeed, in our framework, distinguishing nearby states is assumed to be costless. On the contrary, and differently from [Morris and Yang \[2016\]](#), we assume that the level of maximal precision is costly.

In order to answer our question, we formalize the definition of complementarities and substitutabilities in information acquisition:

Definition 40. *Information acquisitions are said complements (substitutes) for an individual player with respect to his opponents at a given equilibrium if a slight change in the information cost of the opponents lead the individual player to acquire less (more) information when the opponents acquire less information.*

3.2.2 Stages

The game is a two-stage game. The players first choose their precision $(1 - \epsilon_i)$ and their threshold ξ_i . Then, they observe a binary signal and choose whether to invest according to their expected profit, given the signal. Thus, each agent i sets a decision rule $a_i(\epsilon_i, \xi_i)$ such that $a_i(\epsilon_i, \xi_i | X_i) = \mathbb{P}(I)$ for $X_i = 0, 1$, with $\mathbb{P}(I)$ the probability to play action I when receiving each signal.

3.3 Equilibria characterization

Let $B_i(\epsilon_i, \xi_i, a_i; \Gamma, a_{-i})$ the expected payoff by choosing an error ϵ_i , a threshold ξ_i and a decision rule a_i for a given distribution Γ of error choices and threshold choices made by the opponents, and given the decision rules a_{-i} chosen by the opponents, i.e.,

$$B_i(\epsilon_i, \xi_i, a_i; \Gamma, a_{-i}) \equiv \mathbb{E}(\Pi_i | \epsilon_i, \xi_i, a_i; \Gamma, a_{-i})$$

We denote $B_i(\epsilon_i, \xi_i, a_i; \Gamma, a_{-i} | X_i)$ is the expected payoff conditional on the value of X_i .

Let $A_i^*(\epsilon_i, \xi_i)$ the set of decision rules $a_i^*(\epsilon_i, \xi_i)$ such that each $a_i^*(\epsilon_i, \xi_i|X_i)$ maximizes the expected payoff of agent i when choosing ϵ_i, ξ_i and facing X_i , i.e., checking for $X_i = 0, 1$:

$$B_i(\epsilon_i, \xi_i, a_i^*(\epsilon_i, \xi_i); \Gamma, a_{-i}|X_i) \geq B_i(\epsilon_i, \xi_i, a_i(\epsilon_i, \xi_i); \Gamma, a_{-i}|X_i)$$

Therefore, $A_i^*(\epsilon_i, \xi_i)$ is the set of the i 's decision rules such that for each $a_i^*(\epsilon_i, \xi_i)$ in $A_i^*(\epsilon_i, \xi_i)$, $a_i^*(\epsilon_i, \xi_i|X_i)$ is optimal for each $X_i = 0, 1$.

Definition 41. A *Perfect Bayesian Nash Equilibrium* is a set of error choices $\{\epsilon_i^*, i \in [0, 1]\}$, with a set of threshold choices $\{\xi_i^*, i \in [0, 1]\}$ together with a set of optimal decision rules for the second period $\{a_i^*(\epsilon_i^*, \xi_i^*) \in A_i^*(\epsilon_i^*, \xi_i^*), i \in [0, 1]\}$. The distribution of error choices and threshold choices $\Gamma = [\Gamma_\epsilon, \Gamma_\xi]$ (with \mathbb{S} the support of the error choices) is such that the following holds :

Each investor i 's choice (ϵ_i^*, ξ_i^*) is optimal, given Γ and a_{-i} :

$$\forall i, \forall \epsilon_i \in \mathbb{S}, \forall \xi_i \in \mathbb{R} :$$

$$B_i(\epsilon_i^*, \xi_i^*, a_i^*(\epsilon_i^*, \xi_i^*); \Gamma, a_{-i}) - C_i(\epsilon_i^*) \geq B_i(\epsilon_i, \xi_i, a_i^*(\epsilon_i, \xi_i); \Gamma, a_{-i}) - C_i(\epsilon_i)$$

Proposition 8. Suppose that $\mathbb{F}_\theta(0) > \left(\frac{\mathbb{F}_\theta(0) + \mathbb{F}_\theta(1)}{2}\right)^2$ holds. Then for any agent i , a strategy (ξ_i, ϵ_i, a_i) such that $\epsilon_i < 1/2$ and $a_i(0) = I$ and $a_i(1) = NI$ is never a best response regardless of what the other agents play.

Proof. Consider an arbitrary profile $(\xi_{-i}, \epsilon_{-i}, a_{-i})$ of strategies of all agents other than agent i .

Let $\xi_i \in \mathbb{R}$ and $\epsilon_i \in (0, 1/2)$. Let p_- be the fraction of agents (different from i) who play I conditional on $\theta < \xi_i$, and similarly let p_+ be the fraction of agents (different from i) who play I conditional on $\theta > \xi_i$, according to the profile $(\xi_{-i}, \epsilon_{-i}, a_{-i})$.

Note that if $p_- \leq p_+$, it is clear that a strategy such that $a_i(0) = I$ and $a_i(1) = NI$ cannot possibly be optimal for agent i , since $X_i = 1$ signals both a higher state and higher aggregate investment than $X_i = 0$. So let us restrict attention to the case where $p_+ < p_-$.

We distinguish three cases. If either $\xi_i + p_- \leq 1$ or $\xi_i + p_+ \geq 1$, we can easily show that playing $\epsilon_i < 1/2$, $a_i(0) = I$ and $a_i(1) = NI$ is not optimal

- Suppose first that $\xi < \epsilon^*(\xi)$. Then, investing is successful if and only if $\theta \geq 1 - \epsilon^*(\xi)$. Thus, conditional on observing the signal $X_i = 0$, investment is successful if and only if the signal is in fact incorrect and this inequality holds, since $\theta + p < 1$ when the signal is correct. The expected payoff of playing I , conditional on $X_i = 0$ equals

$$\frac{\epsilon(1 - \mathbb{F}_\theta(1 - \epsilon^*(\xi)))}{(1 - \epsilon^*(\xi)) \mathbb{F}_\theta(\xi) + \epsilon^*(\xi)(1 - \mathbb{F}_\theta(\xi))} - T \geq 0.$$

The inequality is an implication of the optimality of investing, upon receiving the signal $X_i = 0$. But then, the expected payoff of playing I upon receiving $X_i = 1$ would be equal to:

$$\begin{aligned}
 \frac{(1-\epsilon)(1-\mathbb{F}_\theta(1-\epsilon^*(\xi)))}{\epsilon^*(\xi)\mathbb{F}_\theta(\xi)+(1-\epsilon^*(\xi))(1-\mathbb{F}_\theta(\xi))} - T &= \frac{(1-\mathbb{F}_\theta(1-\epsilon^*(\xi)))}{\frac{\epsilon^*(\xi)}{(1-\epsilon)}\mathbb{F}_\theta(\xi)+(1-\mathbb{F}_\theta(\xi))} - T \\
 &> \frac{(1-\mathbb{F}_\theta(1-\epsilon^*(\xi)))}{\frac{1-\epsilon^*(\xi)}{\epsilon}\mathbb{F}_\theta(\xi)+(1-\mathbb{F}_\theta(\xi))} - T \\
 &= \frac{\epsilon(1-\mathbb{F}_\theta(1-\epsilon^*(\xi)))}{(1-\epsilon^*(\xi))\mathbb{F}_\theta(\xi)+\epsilon^*(\xi)(1-\mathbb{F}_\theta(\xi))} - T \\
 &\geq 0.
 \end{aligned}$$

This contradicts the optimality of not investing upon receiving the signal $X_i = 1$. Thus, there is no equilibrium where $\xi < \epsilon(\xi)$ and agents invest when they receive the signal $X_i = 0$ and do not invest when they receive $X_i = 1$.

- If $1 - \epsilon^*(\xi) < \xi$, investing is unsuccessful if and only if $\theta \leq \epsilon^*(\xi)$. Thus, conditional on observing the signal $X_i = 1$, playing I yields a payoff of $-T$ if and only if the signal is in fact incorrect and this inequality holds, since $\theta + p > 1$ when the signal is correct. The expected payoff of playing I , conditional on $X_i = 1$ equals

$$1 - \frac{\epsilon(\mathbb{F}_\theta(\epsilon^*(\xi)))}{\epsilon^*(\xi)\mathbb{F}_\theta(\xi)+(1-\epsilon^*(\xi))(1-\mathbb{F}_\theta(\xi))} - T \leq 0.$$

The inequality is an implication of the optimality of not investing, upon receiving the signal $X_i = 1$. But then, the expected payoff of playing I upon receiving $X_i = 0$ would equal to:

$$\begin{aligned}
 1 - \frac{(1-\epsilon)(\mathbb{F}_\theta(\epsilon^*(\xi)))}{(1-\epsilon^*(\xi))\mathbb{F}_\theta(\xi)+\epsilon^*(\xi)(1-\mathbb{F}_\theta(\xi))} - T \\
 < 1 - \frac{\epsilon(\mathbb{F}_\theta(\epsilon^*(\xi)))}{\epsilon^*(\xi)\mathbb{F}_\theta(\xi)+(1-\epsilon^*(\xi))(1-\mathbb{F}_\theta(\xi))} - T \leq 0
 \end{aligned}$$

This contradicts the optimality of investing upon receiving the signal $X_i = 0$. Thus, there is no equilibrium where $1 - \epsilon^*(\xi) < \xi$, and agents invest when they receive the signal $X_i = 0$ and do not invest when they receive $X_i = 1$.

Consider the last remaining cases where $1 - p_- < \xi_i < 1 - p_+$.

Conditional on observing $X_i = 0$, the expected payoff of playing I equals

$$\frac{(1-\varepsilon_i)(\mathbb{F}_\theta(\xi_i)-\mathbb{F}_\theta(1-p_-))+\varepsilon_i(1-\mathbb{F}_\theta(1-p_+))}{(1-\varepsilon_i)\mathbb{F}_\theta(\xi_i)+\varepsilon_i(1-\mathbb{F}_\theta(\xi_i))} - T.$$

Conditional on observing $X_i = 1$, the expected payoff of playing I equals

$$\frac{\varepsilon_i(\mathbb{F}_\theta(\xi_i)-\mathbb{F}_\theta(1-p_-))+\varepsilon_i(1-\mathbb{F}_\theta(1-p_+))}{\mathbb{F}_\theta(\xi_i)+(1-\varepsilon_i)(1-\mathbb{F}_\theta(\xi_i))} - T.$$

The result we would like to prove is that if the first expression is positive, then the second one is strictly positive. This requires showing that the function

$$G(x) = \frac{(1-x)(\mathbb{F}_\theta(\xi_i)-\mathbb{F}_\theta(1-p_-))+x(1-\mathbb{F}_\theta(1-p_+))}{(1-x)\mathbb{F}_\theta(\xi_i)+x(1-\mathbb{F}_\theta(\xi_i))}$$

is strictly increasing. This is the case iff

$$(1 - \mathbb{F}_\theta(1 - p_+) - \mathbb{F}_\theta) \mathbb{F}_\theta(1 - p_-) - (\mathbb{F}_\theta(1 - p_+) - \mathbb{F}_\theta(\xi_i)) (\mathbb{F}_\theta(\xi_i) - \mathbb{F}_\theta(1 - p_-)) > 0.$$

The smallest possible value of the left hand-side is reached when

$$\mathbb{F}_\theta(\xi_i) = \frac{1}{2} (\mathbb{F}_\theta(1 - p_-) + \mathbb{F}_\theta(1 - p_+)).$$

A sufficient condition for the inequality to hold is then that:

$$\mathbb{F}_\theta(1 - p_-) > \left(\frac{(\mathbb{F}_\theta(1 - p_-) + \mathbb{F}_\theta(1 - p_+))}{2} \right)^2.$$

Equivalently:

$$\mathbb{F}_\theta(1 - p_+) < 2\sqrt{\mathbb{F}_\theta(1 - p_-)} - \mathbb{F}_\theta(1 - p_-).$$

The corresponding area is the area below the curve, with $y = \mathbb{F}_\theta(1 - p_+)$ and $x = \mathbb{F}_\theta(1 - p_-)$.

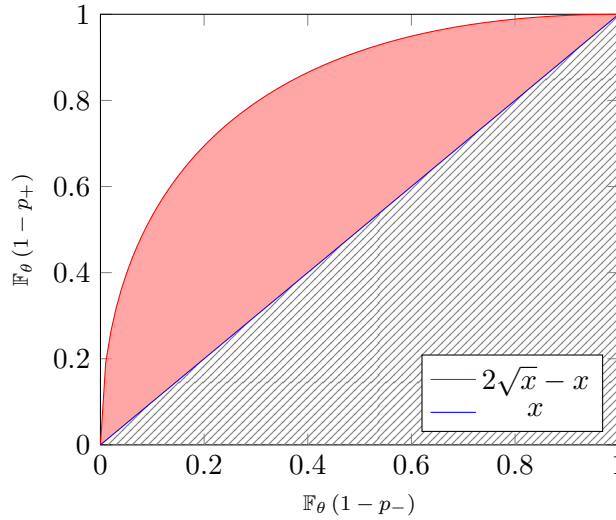


Figure 3.1: Set (in red) of the $\mathbb{F}_\theta(1 - p_+)$ with respect to $\mathbb{F}_\theta(1 - p_-)$ such that there is no equilibrium with $a_i(0) = I$ and $a_i(1) = NI \forall i$

Since $0 \leq 1 - p_- \leq 1 - p_+ \leq 1$, a sufficient condition for this to hold is that

$$\mathbb{F}_\theta(0) > \left(\frac{(\mathbb{F}_\theta(0) + \mathbb{F}_\theta(1))}{2} \right)^2,$$

which is the desired conclusion. ■

Remark 1. *The inequality says that in the above graph, the point $(\mathbb{F}_\theta(0), \mathbb{F}_\theta(1))$ lies below the curve. Intuitively, it requires that not too much probability mass is concentrated in the interval $[0, 1]$.*

Remark 2. *The sufficient condition we state cannot be weakened, in the following sense. Suppose that \mathbb{F}_θ is uniform in the interval $[-\delta, 1 + \delta]$. If $\delta > 0$ is small enough, it is possible to find a value of $T > 0$ and a symmetric cost function C such that the profile where all agents chose the same error $\varepsilon \in (0, 1/2)$, the same threshold $\xi = 1/2$ and play $a_i(0) = I$ and $a_i(1) = NI$ is an equilibrium.*

Proposition 8 shows under which conditions no equilibrium where the agents do the opposite to what their signal indicates is possible. Therefore, there are three kind of pure equilibria in incomplete information: the ones (i) where the agents always invest, the ones (ii) where the agents never invest, and the ones (iii) where the agents follow the signal. (i) may happen if $1 - \mathbb{F}_\theta(0) > T \Leftrightarrow \mathbb{F}_\theta(0) < 1 - T$. (ii) may happen if $1 - \mathbb{F}_\theta(1) < T \Leftrightarrow \mathbb{F}_\theta(1) > 1 - T$.

Possible equilibria:

1. (i) and (iii) if $\mathbb{F}_\theta(1) < 1 - T \Leftrightarrow T < 1 - \mathbb{F}_\theta(1)$,
2. (i), (ii) and (iii) if $\mathbb{F}_\theta(0) < 1 - T < \mathbb{F}_\theta(1) \Leftrightarrow 1 - \mathbb{F}_\theta(1) < T < 1 - \mathbb{F}_\theta(0)$,
3. (ii) and (iii) if $1 - T < \mathbb{F}_\theta(0) \Leftrightarrow 1 - \mathbb{F}_\theta(0) < T$.

Now, we make two assumptions to frame the role of information acquisition in this game, with ϵ_m the minimal error such that the error cost is null:

Assumption A1. (i) if $\epsilon_m \leq \epsilon_i \leq \frac{1}{2}$, then $C_i(\epsilon_i) = 0$, (ii) if $0 < \epsilon_i < \epsilon_m < \frac{1}{2}$, then $C_i'(\epsilon_i) < 0$, $C_i''(\epsilon_i) > 0$, and (iii) if $\epsilon_i = 0$, then $C_i(0) = \infty$ for every agent i

Assumption A1 is crucial to get a clear shift in the equilibrium when there is a change in the mean error. Then, it ensures that substitutability in information acquisition may emerge. However, if the cost function is assumed to be concave, the opposite results would emerge⁵.

Now, we can focus on equilibrium where there is only information acquisition. We give an explicit value to ϵ_m :

Assumption A2. *In the case where:*

$$\frac{\mathbb{F}_\theta(1) - (1 - T)}{T} < \mathbb{F}_\theta(0) < 1 - T < \mathbb{F}_\theta(1) < \frac{\mathbb{F}_\theta(0)}{1 - T} \quad (\text{Information seeking cases})$$

$$\text{Set } \epsilon_{00} = \frac{\mathbb{F}_\theta(0) - \mathbb{F}_\theta(1)(1 - T)}{\mathbb{F}_\theta(0) + ((1 - T) - 2\mathbb{F}_\theta(1))(1 - T)}, \quad \epsilon_1 = \frac{(1 - T) + T\mathbb{F}_\theta(0) - \mathbb{F}_\theta(1)}{(1 - T) + 2T\mathbb{F}_\theta(0) - \mathbb{F}_\theta(1)}.$$

Assume $\epsilon_m = \min(\epsilon_{00}, \epsilon_1)$, for every agent i

Assumption A2 is necessary to be sure that every agent is willing to get information under certain conditions and to be sure that every solution is interior in terms of error.

⁵The linear case would give corner solutions in information acquisition (so no marginal complementarities or substitutabilities may appear) as well as mixed solutions. All of these mixed solutions would have the same threshold (determined by the parameters of the game). Then, any complementarities or substitutabilities would be driven by the threshold, determined exogeneously.

How ϵ_0 and ϵ_1 are established is let to [Section 3.5](#). [Assumption A2](#) ensures that without need of coordination, investing when the signal indicates that the state of the world is above 0 is dominant, and not investing is dominant when the signal indicates that the state is below 1. In other words, there is never need for coordination to follow the signal (or coordination can never be sufficient).

Lemma 22. $\forall \xi_l, \xi_h \in [1, +\infty) \times (1, +\infty)$ $\xi_l < \xi_h$, any individual i prefers strictly ξ_l to ξ_h . The opposite is true in $(-\infty, 0) \times (-\infty, 0]$. Finally, if the thresholds choice is limited to $[1, +\infty)$ (resp. $(-\infty, 0]$) the equilibrium threshold is 1 (resp. 0).

Proof. Assume $\xi_i > 1$. If $\mathbb{F}_\theta(\xi) < \frac{\mathbb{F}_\theta(0)}{1-T}$, then the individual i follows his signal. Since he earns money only when the state of the world is above the threshold, it is immediate that by decreasing the threshold ξ_i , i increases the probability to invest and thus increases his benefit. In other words, it reduces the area where the agent does not invest while it would have been profitable (Type-II error). The same mechanism appears in the negative area. Conversely, if $\mathbb{F}_\theta(\xi) \geq \frac{\mathbb{F}_\theta(0)}{1-T}$, there is no dominant action when the signal is low. Then, in this case, the agent may always (or sometimes) invest when the state of the world is below his threshold. However, what would happen if the threshold ξ_i was decreased? Again, the space where the agent can invest without being mistaken would increase (in the high signal case). In addition (and differently from above), the space where the agent takes the risk to lose money (Type-I error) diminishes. It leads to the conclusion that 1 is preferred to greater thresholds, and 0 preferred to lower thresholds. ■

Lemma 23. Under [Assumptions A1](#) and [A2](#), all equilibria are such that all agents (*i*) acquire information, and (*ii*) choose a threshold belonging to the interval $[0, 1]$.

Proof. [Assumption A2](#) ensures agents are willing to acquire information and that following the signal is a dominant action between 0 and 1. By [Lemma 22](#), we know that 0 is preferred to any lower threshold and that 1 is preferred to any higher threshold. Thus, no information acquisition, nor choosing a threshold outside $[0, 1]$ is profitable. ■

Similarly as [Yang \[2015\]](#), the Pareto Best situation (i.e. the situation where all the agents choose the threshold 0 is an equilibrium) when perfect information is available. In our case, under imperfect information, the set of equilibrium's thresholds is restricted:

Proposition 9. Under [Assumption A2](#), for any set of pairs (ϵ, ξ) chosen by every agent i , there is a state of the world $\theta^* \in [\mathbb{E}_i(\epsilon_i), 1 - \mathbb{E}_i(\epsilon_i)]$ above which the investment is successful, and below which the investment is not successful.

Proof. First note that the investment is always successful if $\theta > 1$, and never successful if $\theta \leq 0$. Then, set the following function:

$$p(\theta, \xi, \epsilon) = m(\theta) + n(\theta) = \int_0^\theta (1 - \epsilon_i) d\xi_i + \int_\theta^1 \epsilon_i d\xi_i$$

p is the proportion of agents investing when the state of the world is θ for any given vector (ϵ, ξ) composed of the choices made by the agents. m represents the agents whose the threshold is below θ and who get a correct signal. n represents the agents whose the threshold is above θ but who get an incorrect signal. By [Assumptions A1](#) and [A2](#), these two types of agents will invest. Therefore, the investment is successful if $p(\theta) > 1 - \theta$. Note that, by [Lemma 23](#), $p(0, \xi, \epsilon) = \mathbb{E}_i(\epsilon_i)$ and $p(1, \xi, \epsilon) = 1 - \mathbb{E}_i(\epsilon_i)$. Indeed, by [Assumption A2](#), agents invest when they receive a high signal and do not invest when they receive a low signal if they have chosen a threshold in $[0, 1]$. Then, it is sufficient to show that p is non decreasing in θ to prove the statement. By [Assumption A2](#), $\epsilon_i < \frac{1}{2}$, so for all i , $1 - \epsilon_i > \epsilon_i$. Since the number of agents whose the threshold is below θ does not decrease with θ , it is immediate that p is non decreasing as well. ■

3.3.1 Determination of the error at equilibrium

Assume that θ^* is fixed at a certain value such that no investment is successful below, and all the investment are successful above. Then, all the agents choose the threshold θ^* :

Proposition 10. *For every agent i with any error $\epsilon_i < \frac{1}{2}$, $\xi_i = \theta^*$.*

Proof. Suppose $\xi_i \geq \theta^*$. Then three regions can be distinguished for each agent with respect to the distribution of θ : $-\infty < \theta < \theta^*$, $\theta^* < \theta < \xi_i$ and $\xi_i < \theta < \infty$. Then, the signal drawn by the agent is correct with probability $1 - \epsilon_i > \frac{1}{2}$. In the first area, the agent earns 0, and in the third area, he earns $1 - T$. However, he makes a Type-II error in the second area, and thus loses $1 - T$. When the signal is incorrect (it occurs with probability $\epsilon_i < \frac{1}{2}$), the opposite situation emerges : the agent loses $-T$ in the first area (Type-I error) and $1 - T$ in the third one. In the second area, the signal wrongly tells to invest but as $\theta^* < \theta$, investing is actually successful.

θ	$-\infty < \theta < \theta^*$	$\theta^* < \theta < \xi_i$	$\xi_i < \theta < \infty$
Signal is right [$1 - \epsilon_i > \frac{1}{2}$]	0	$-(1 - T)$	$(1 - T)$
Signal is wrong [$\epsilon_i < \frac{1}{2}$]	$-T$	$(1 - T)$	$-(1 - T)$

As $1 - \epsilon_i > \epsilon_i$ it is then straightforward that ξ_i should be minimized at the equilibrium, i.e. $\xi_i = \theta^*$. The result is similar when it is assumed $\xi_i \leq \theta^*$. Then, at the equilibrium $\xi_i = \theta^*$. ■

The set of the possible θ^* , and then the set of possible equilibria under incomplete information depends dramatically on the value of the parameter T .

This is due to the fact that at any interior equilibrium (i.e. $\epsilon < \frac{1}{2}$), the first-order condition on the error ϵ is respected. Then the partial derivative of the equilibrium error ϵ^* with respect to the threshold θ^* can be deduced. For any θ^* , we know that (giving that at equilibrium, all the agents choose the threshold θ^* , and follow what their signal indicates):

$$\frac{\partial B_i(\epsilon_i, \theta^*)}{\partial \epsilon} = -[(1-T)(1 - \mathbb{F}_\theta(\theta^*)) + T \mathbb{F}_\theta(\theta^*)] = -[(1-T) + (2T-1) \mathbb{F}_\theta(\theta^*)]$$

By the optimality of the investor's choice, we get as the First Order Condition:

$$\frac{\partial B_i(\epsilon_i^*, \theta^*)}{\partial \epsilon} = C'_i(\epsilon_i^*) \quad (\text{FOC})$$

Now, we determine all the equilibria where the agents acquire information. We use **FOC** to build the error function with the threshold θ^* as argument.

We note $R_{0,i}(\theta, \epsilon) \mapsto -(1-T) + (1-2T) \mathbb{F}_\theta(\theta) - C'_i(\epsilon)$ a well-defined function of $\mathbb{R} \times [0, \frac{1}{2}] \rightarrow \mathbb{R}$.

Then, any first-order condition respects: $R_{0,i}(\theta^*, \epsilon_i^*) = -(1-T) + (1-2T) \mathbb{F}_\theta(\theta^*) - C'_i(\epsilon_i^*) = 0$. Convexity of the cost function and implicit differentiation give:

$$\frac{d \epsilon_i^*}{d \theta^*} = \frac{(1-2T) f_\theta(\theta^*)}{C''_i(\epsilon_i^*)} > (<) 0 \quad \text{if } T < (>) \frac{1}{2} \quad (3.1)$$

To simplify, assume first that the agents have the same cost function.

We note $R_{1,i}(x) \mapsto -(1-T) + (1-2T) \mathbb{F}_\theta(x) - C'_i(x)$ a function of $[0, \frac{1}{2}] \rightarrow \mathbb{R}$. By **Assumption A1**, $R_{1,i}(0) = +\infty$ and $R_{1,i}(\frac{1}{2}) = -(1-T) + (1-2T) \mathbb{F}_\theta(\frac{1}{2}) < 0$.

To ensure that $R_{1,i}$ is a decreasing function and then, that there is a unique point e_1 such that $R_{1,i}(e_1) = 0$, we can assume that $\forall \epsilon \in [0, \frac{1}{2})$, $C''_i(\epsilon) > (1-2T) \bar{f}_\theta$ where \bar{f}_θ is the mode of the density function f_θ . Note that it is already the case if $T > \frac{1}{2}$.

Assumption A3. $C''_i(\epsilon_i) > (1-2T) \bar{f}_\theta$ if $T < \frac{1}{2}$.

Assumption A3 ensures that $\frac{d \epsilon_i^*}{d \theta^*} < 1$ and then a single cross property of the thresholds and the associated errors.

Under this hypothesis, by continuity of $R_{1,i}$, there is a unique e_1 such that $R_{1,i}(e_1) = 0$. It is immediate that the point (e_1, e_1) is an equilibrium since the first order condition is binding and that no deviation from the threshold e_1 is profitable since above the investment is successful and below the investment is unsuccessful. Now, we can relax the hypothesis of homogeneity, and set $E_1 = \mathbb{E}_i(e_1)$.

Lemma 24. *At any equilibrium, no agent chooses a threshold below E_1 .*

Proof. Assume all the agents choose a threshold $\xi_u < E_1$. The optimal error is then $\epsilon^*(\xi_u) < (>) E_1$ if $T < (>) \frac{1}{2}$. To be an equilibrium, $(\xi_u, \epsilon^*(\xi_u))$ should be such that no investment is successful below ξ_u , and all the investment are successful above ξ_u .

If $T > \frac{1}{2}$, we get $\epsilon^*(\xi_u) > E_1 > \xi_u$ since $\frac{\partial \epsilon_i^*}{\partial \theta^*} < 0$. There is a profitable deviation for any agent i by setting $\xi_i = \epsilon^*$, since under the state of the world $\theta = \epsilon^*$ the investment can not be successful and then (ξ_u, ϵ^*) is not an equilibrium.

If $T < \frac{1}{2}$, $\epsilon^*(\xi_u)$ is below E_1 . Then, we already know that $\frac{d\epsilon_i^*}{d\theta^*} < 1$ because we assume that $\forall x \in [0, \frac{1}{2}]$, $C_i''(x) > (1 - 2T)\bar{f}_\theta$. Thus, $\xi_u < \epsilon^*(\xi_u)$. Again, there is a profitable deviation for all the agents since investments are not successful between ξ_u and $\epsilon^*(\xi_u)$.

Then, no threshold under E_1 is chosen at any equilibrium. ■

We note $R_{2,i}(x) \mapsto -(1 - T) + (1 - 2T)\mathbb{F}_\theta(1 - x) - C_i'(x)$ a function of $[0, \frac{1}{2}] \rightarrow \mathbb{R}$. By hypothesis, $R_{2,i}(0) = +\infty$ and $R_{2,i}(\frac{1}{2}) = -(1 - T) + (1 - 2T)\mathbb{F}_\theta(\frac{1}{2}) < 0$. By assuming that $C_i''(x) > (2T - 1)\bar{f}_\theta$, we get that the point e_2 defined by $R_{2,i}(e_2) = 0$ is unique. We note $E_2 = \mathbb{E}_i(e_2)$

Lemma 25. *At any equilibrium, no agent chooses a threshold above $1 - E_2$.*

The proof is analogous to the proof of [Lemma 24](#).

Proposition 11. *Under [Assumption A3](#), equilibria can be characterized as the ordered pairs (θ^*, ϵ^*) such that $E_1 \leq \theta^* \leq 1 - E_2$ and such that $R_{0,i}(\theta^*, \epsilon^*) = 0$ for all the agents $i \in [0, 1]$.*

Proof. [Lemmas 24](#) and [25](#) show that no threshold outside the area $[E_1, e_2]$ can be chosen at any equilibrium. [Assumption A3](#) guarantees that $|\frac{d\theta^*}{d\epsilon^*}| > 1$. Thus, for any $\theta^* > E_1$, $\theta^* > \epsilon^*(\theta^*)$. As well, for any $\theta^* < 1 - e_2$, $\theta^* < 1 - \epsilon^*(\theta^*)$ since even if $\epsilon^*(\theta^*)$ increases (so $1 - \epsilon^*(\theta^*)$ decreases) when θ^* decreases, θ^* decreases faster. So, there is no individual profitable deviation from these thresholds. Since $R_{0,i}(\theta^*, \epsilon^*) = 0$ for all of these pairs, it is immediate by the convexity of error cost that the error is optimal for each θ^* considered. Therefore, these pairs are the only possible equilibria since no other threshold might be considered by [Lemmas 24](#) and [25](#), and since there is just one optimal error for each equilibrium threshold. ■

[Assumption A3](#) ensures that *all* the thresholds in the interval are candidates to be part of an equilibrium, but there would still be existence of several equilibria without this assumption.

Lemma 26. *Assume that the cost function increases (decreases) for all error levels ϵ such that $0 \leq \epsilon < \epsilon_m$ for a non negligible part of the agents (and remains the same for the others). Then, E_1 and E_2 increase (decrease), shifting up (down) the lowest threshold, and shifting down (up) the highest threshold.*

Proof. The increasing cost function implies that $|C'_i(\epsilon)|$ increases for all ϵ such that $0 \leq \epsilon < \epsilon_m$. For the agents concerned, it implies that e_1 is modified. We recall that $R_{1,i}(x) \mapsto -(1-T) + (1-2T)\mathbb{F}_\theta(x) - C'_i(x)$ is null at e_1 . If $|C'_i(e_1)|$ is shift up, then $R_{1,i}(e_1)$ becomes positive. Then, e_1 has to be increased to decrease the cost derivative. If $T \geq \frac{1}{2}$, it is clear that e_1 has to be increased to get again $R_{1,i}(e_1) = 0$, since the term $(1-2T)\mathbb{F}_\theta(x)$ decreases (or is null). However, if $T < \frac{1}{2}$, e_1 increasing makes the term $(1-2T)\mathbb{F}_\theta(x)$ increase. **Assumption A3** ensures that $C''_i(x) > (1-2T)f_\theta(x)$ if $T < \frac{1}{2}$. Thus, any increase of x makes decrease more the cost derivative than the term $(1-2T)\mathbb{F}_\theta(x)$. We have shown that all e_1 are shift up (or remain the same). It is straightforward that E_1 increases as well. The proof is analogous for E_2 . ■

3.3.2 Representation of the equilibria

By 3.1, we can deduce the second derivative of $\epsilon^*(\theta^*)$:

$$\frac{d\epsilon_i^*}{d\theta^*} = \frac{(1-2T)f_\theta(\theta^*)}{C''_i(\epsilon_i^*)} \Rightarrow \frac{d^2\epsilon_i^*}{d\theta^{*2}} = \frac{(1-2T)f'_\theta(\theta^*)}{C''_i(\epsilon_i^*)} \quad (3.2)$$

For instance, if the states are distributed according to a normal law centered in $\frac{1}{2}$, if $T < \frac{1}{2}$, the optimal error will be first convex and then concave. This is due to the fact that the region where misinvesting is more costly diminishing fastly (due to the increase of the density f), the incentive to have a high precision decreases more and more when the threshold moves to the right. When the threshold is above $\frac{1}{2}$, the density f decreasing, the error becomes concave, since the reduction of the region where misinvesting is more costly becomes slower.

On the contrary, if $T > \frac{1}{2}$, as the error decreases, concavity reflects that the error should first decrease fastly because the region where misinvesting is more costly grows up. Convexity comes then symmetrically to what happens with $T < \frac{1}{2}$.

Figure 3.2 represents the best reply (the optimal error) of an individual given θ^* and that he chooses the threshold θ^* . The equilibria sets are the points on the thick part of $\epsilon^*(\theta)$. Note that in the situation where the agents are identical, the best replies function is a straight line between E_1 and E_2 .

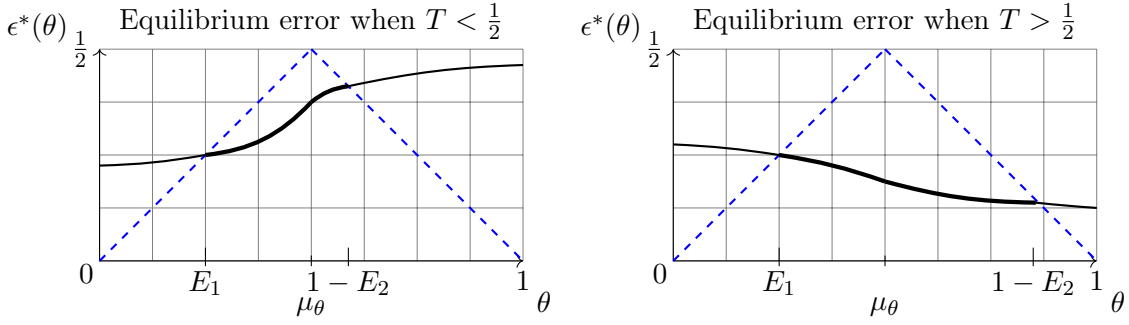


Figure 3.2: Graphical examples - Equilibrium error for given thresholds

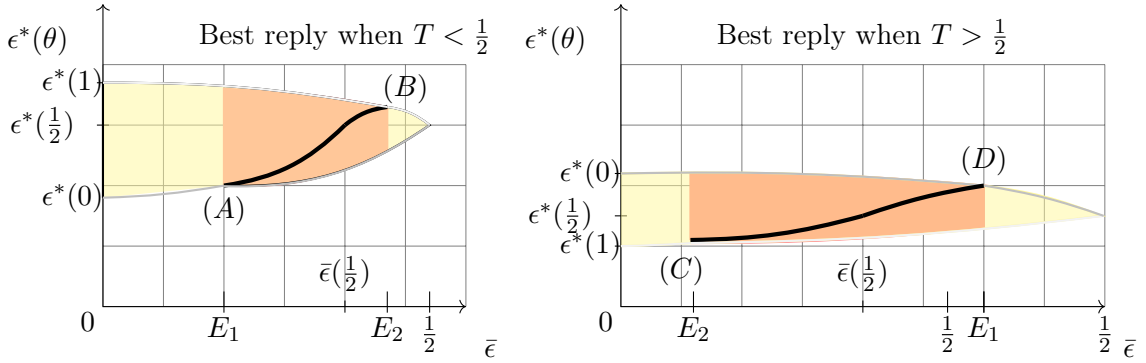


Figure 3.3: Errors best replies for given overall errors

Figure 3.3 shows the set of best responses to the mean error chosen by the opponents. Note that we indicate on the y-axis the error chosen by the individual with respect to θ^* . The dark gray line represents $\epsilon^*(\bar{\epsilon})$ and the light gray line represents $\epsilon^*(1 - \bar{\epsilon})$, such that $\bar{\epsilon}$ and $1 - \bar{\epsilon}$ are the threshold θ^* . However, we do represent the correspondence between a given overall error and the best reply of the individual. In the red area, we find all the errors that the individual is willing to select given the overall error, between $\min(E_1, E_2)$ and $\max(E_1, E_2)$. This interval represents the only possible overall errors at the equilibrium. As we know, from Proposition 11, at the equilibrium, each agent associates an error to a given threshold. As a consequence, each threshold is associated to an overall error. Thus, for each agent we can associate on the thick dark line his error and the overall error at the equilibrium. In fact, the red area represents all the best replies if the threshold and the overall error are not compatible at the equilibrium (i.e. there exists a non negligible part of the agents whose the error is not optimal). Out of the interval $[\min(E_1, E_2), \max(E_1, E_2)]$ (i.e. such that the overall error is not compatible with the equilibrium), the individual would be willing to select any error in the yellow area. Note that if the overall error is hypothetically set to $\frac{1}{2}$, and that all the agents carry on following their signal, the only best reply is to choose $\epsilon^*(\frac{1}{2})$. On the contrary, if the error is set to 0, i.e. all agents but the individual have a perfect information, the set of possible errors chosen by the individual is comprised between $\min(\epsilon^*(0), \epsilon^*(1))$ and $\max(\epsilon^*(0), \epsilon^*(1))$. Note that the points $(E_1, \epsilon^*(E_1))$ and $(E_2, \epsilon^*(1 - E_2))$ are respectively

the only intersection between $\epsilon^*(\bar{\epsilon})$ and $\bar{\epsilon}$ (points A and D) and between $\epsilon^*(1 - \bar{\epsilon})$ and $\bar{\epsilon}$ (points B and C).

For any game, we can set $\underline{\epsilon}^* = \min_{\theta^* \in (-\infty, +\infty)}(\epsilon(\theta^*)) = \begin{cases} \lim_{\theta^* \rightarrow -\infty} \epsilon(\theta^*) = C'_\epsilon{}^{-1}(-(1 - T)) & \text{if } T < \frac{1}{2} \\ \lim_{\theta^* \rightarrow +\infty} \epsilon(\theta^*) = C'_\epsilon{}^{-1}(-T) & \text{if } T > \frac{1}{2} \end{cases}$

3.3.3 Welfare ranking of the equilibria

The next result confirms the fact that the equilibria in complementarities games are Pareto-ranked. The best equilibrium, i.e. the one with the largest payoff for the agents is the left extremal equilibrium with $\theta^* = E_1$:

Proposition 12. *The profit of each agent decreases when θ^* increases. Then, the Pareto rank of the equilibria decreases with θ^* , the equilibrium with threshold E_1 for all agents being the Pareto best equilibrium and the equilibrium with threshold $1 - E_2$ for all agents being the Pareto worst equilibrium.*

By using the envelope theorem, the proof is direct: $\frac{dB_i}{d\theta^*} = \frac{\partial B_i}{\partial \theta^*} = -f_\theta(\theta^*)[(1 - T) + (2T - 1)\epsilon_i] < 0$.

3.4 Analysis of the inheritance result and the value of information

We study in this section the Pareto best (and worst) equilibria with the distribution of errors and thresholds Γ^* composed of vectors $[\epsilon_i = \epsilon_i^*(E_1), \xi_i = E_1] \forall i$ ($[\epsilon_i = \epsilon_i^*(1 - E_2), \xi_i = 1 - E_2] \forall i$). We analyze the effect of the increase of the error of all but one agent j . We recall that by [FOC](#):

$$\left(\frac{\partial B_j(\epsilon_j, \theta^*)}{\partial \epsilon} \right) - [(1 - T) + (2T - 1)\mathbb{F}_\theta(\theta^*)] = C'_j(\epsilon_j^*) \quad (\text{FOC}_j)$$

At the Pareto best equilibrium, we have $\mathbb{E}_i(\epsilon_i(\theta^*)) = \theta^* = E_1$ and at the Pareto worst equilibrium, we have $\mathbb{E}_i(\epsilon^*(\theta^*)) = 1 - \theta^* = E_2$.

3.4.1 Equilibria with information acquisition

Under [Assumptions A1](#) to [A3](#), we already know also that $\frac{d\epsilon_j^*}{d\theta^*} > (<)0$ if $T < (>)\frac{1}{2}$ by [3.1](#).

We know by [Lemma 26](#) that an increase of the overall error will shift up E_1 and E_2 . Therefore, at the Pareto best (worst) equilibrium, we can write: $\frac{\partial \theta^*}{\partial \mathbb{E}_i(\epsilon_i)} > (<)0$.

Combining these facts, we can sum up the effects of an increasing overall error on the incentives of an individual j in [Table 3.1](#). The value of information decreases when j is incentivized to increase his error. From [Table 3.1](#), we can deduce how evolves this value

	$T < \frac{1}{2}$	$T > \frac{1}{2}$
Pareto Best Equilibria	$\frac{d\epsilon_j^*}{d\theta^*} > 0$ and $\frac{\partial\theta^*}{\partial\mathbb{E}_i(\epsilon_i)} > 0$	$\frac{d\epsilon_j^*}{d\theta^*} < 0$ and $\frac{\partial\theta^*}{\partial\mathbb{E}_i(\epsilon_i)} > 0$
Pareto Worst Equilibria	$\frac{d\epsilon_j^*}{d\theta^*} > 0$ and $\frac{\partial\theta^*}{\partial\mathbb{E}_i(\epsilon_i)} < 0$	$\frac{d\epsilon_j^*}{d\theta^*} < 0$ and $\frac{\partial\theta^*}{\partial\mathbb{E}_i(\epsilon_i)} < 0$

Table 3.1: Effect of an increasing overall error on the Value of Information for the Pareto Best and Pareto Worst Equilibria

for j when the overall error is increasing, through the channel of θ^* . Indeed, at Pareto best and worst equilibria, the overall error affects the threshold θ^* (overall error effect on θ^*). Then, the threshold affects the information acquisition of j (threshold effect on information acquisition). If both effects are in the same direction, we get complementarities. Otherwise, we obtain substitutabilities. In the white cells, the global effect is positive (i.e. there are complementarities in information acquisition) while in the grey cells, the global effect is negative (i.e. there are substitutabilities in information acquisition).

Pareto best equilibria

Suppose that we are at the equilibrium, and suppose that the error cost $C_i(\epsilon)$ increases for all agents but j . By Lemma 26, we know that E_1 increases (so θ^* increases). Thus, all the agents shift their error up and change their threshold as well for the new E_1 , higher than the previous one⁶.

If $T > \frac{1}{2}$ (which means that we have a Type-I error, i.e. investing when it should not be is more costly), the term in Equation (FOCj), $-(2T - 1)\mathbb{F}_\theta(\theta^*)$ decreases⁷ and the individual j is encouraged to increase its precision (acting unlike others). Indeed, the absolute value of $\frac{\partial B_j}{\partial \epsilon_j}$ increases and by the convexity of the cost function, FOCj gives that the optimal error has decreased. This is due to the increasing shift of the threshold: the probability that investing is successful has decreased. Therefore, the most costly Type-I error is more likely: the agent has to increase his precision to counterbalance this effect, and we observe substitutabilities in information acquisition.

If $T < \frac{1}{2}$, the opposite mechanism appears, i.e. the individual j 's information acquisition will mimic the behavior of his opponents. Under Assumption A3, if the cost function increases, E_1 increases as well as in the previous case. However, now, the incentives of j have changed, since the derivative's sign of the optimal error is reversed. Complementarities in information acquisition emerge.

Pareto worst equilibria

⁶It is possible that agents coordinate on a even higher threshold (but not on a lower threshold), but we focus on the Pareto best equilibrium.

⁷The positive density between 0 and 1 of the distribution of θ ensures that the cumulative distribution function is modified when θ^* moves. If the density is not positive, the game is at the frontier.

However, things are not so simple. Indeed, at the Pareto worst equilibria, where all the agents choose the threshold $1 - E_2$, symmetric results appear: substitutabilities emerge when $T < \frac{1}{2}$ and complementarities when $T > \frac{1}{2}$. It is clear that the Pareto worst equilibrium has a different nature. Whereas in the Pareto best equilibrium, the individual could bet that his opponent would maximize the probability of success, in the Pareto worst equilibrium, the individual can bet that at least a fraction of his opponents will wrongly invest. Paradoxically, more errors make the investment more likely to succeed. Thus, this is not coordination that matters for the individual, but accounting on the other players' errors. Therefore, the individual does not mind about coordination, but about how much his opponents might be wrong. When $T < \frac{1}{2}$, the more the opponents are likely to make a mistake, the more investing can be successful: there is an incentive to acquire more information. When $T > \frac{1}{2}$, the agent does not want to invest wrongly. The fact that the opponents increase their error ensures the individual that the investment is less likely to fail and enables him to reduce his precision. In this equilibrium, increasing the error makes decrease the threshold: (there is an incentive to coordinate on couples with a high error and a low threshold to avoid the costly Type-I error.)

Finally, at the Pareto best and worst equilibria, when the mean precision and the likelihood of the most costly error are increasing together, there are complementarities (and otherwise substitutabilities). Indeed, when the most costly error is more likely, the individual agent wants to increase his precision. *De facto*, if the mean precision and this likelihood increase together, the individual should act as his opponents.

Frontier case

If $T = \frac{1}{2}$, we see that $\frac{\partial B_j}{\partial \epsilon_j} = -(1 - T)$. In this case, the choice about the error made by the individual j does not depend on the precision of the other agents since the marginal benefit is a constant. This is clearly a frontier point between complementarities and substitutabilities since whatever the information acquisition of the other player is, it does not influence the one of the individual while their actions are complements. Here, the two errors have the same cost. Therefore, the individual j knows, that in any case, if he follows the available signal, making the incorrect decision will always cost the same. Then, whatever the threshold is, nothing may change in the agent's incentives for information acquisition.

This case at the frontier between complementarity and substitutability shows how information acquisition may not be influenced by the other players' information acquisition, even if the threshold is shift up or down. Since costs are equal, the size of the most costly error is unchanged. That is why, no effect on an individual's information acquisition is possible.

Finally, in this game, the three cases emerge according to the parameter T . On one hand, for the Pareto best (worst) equilibrium, when $T > (<) \frac{1}{2}$, the second term in the marginal benefit is positive. Then, if the mean error increases, the individual j should counterbalance this effect by increasing his precision. On the other hand, when $T < (>) \frac{1}{2}$,

the second term becomes negative, and information acquisitions become complements. At the frontier, neither complementarities nor substitutabilities may appear.

The value of information

The value of information in [Szkup and Trevino \[2015\]](#) is maximal when Type-I and Type-II errors are equally likely ex ante. The intuition is that increasing precision is more valuable when it enables players to reduce simultaneously the two errors that may happen. Nevertheless, as the authors explain, the cost of both errors is not present in the value of information as in our model, because of the equilibrium condition in global games and because of the normality of the distribution. However, these costs affect the determination of the equilibrium. Then, the value of T still has an indirect impact on the value of information⁸. Finally, it results that information is more valuable when costs are equal (i.e. $T = \frac{1}{2}$). On the contrary, in our setup, the information value is monotonic in θ^* (see [Table 3.1](#)), according to the value of T (increasing if $T > \frac{1}{2}$, decreasing if $T < \frac{1}{2}$, constant if $T = \frac{1}{2}$ by [3.1](#)). Note that the determination of E_1 is influenced by the parameter T but also by the distribution of θ . In the case where the distribution is centered in $\frac{1}{2}$, E_1 increases with T , but E_2 decreases with T (see [Section 3.5](#)). Thus, information is not always more valuable when the errors' cost is equal or when errors are equally likely: for a fixed distribution of θ , there is a monotonicity of the information value in T . In [Szkup and Trevino \[2015\]](#), information has more value when the threshold θ^* is close from the mean of the distribution. Again, it is different in our model, and there is no monotonicity in the distance between the threshold θ^* and the mean.

The size of the errors (the likelihood of each error) drives our results. Contrary to [Szkup and Trevino \[2015\]](#), where information is more valuable when the sizes are equal, information has more value, here, when the most costly error is the most likely.

At the Pareto best equilibrium, agents coordinate on the most optimistic equilibrium: likelihood of success is maximal. Thus, when $T < \frac{1}{2}$, Type-II error is costly: being aligned with the state of the world matters less than being aligned with other agents since the individual does not want to miss successful investments, what relies on coordination with other agents. When $T > \frac{1}{2}$, Type-I error is costly. Being aligned with the state of the world matters more than being aligned with other agents because what matters is not being wrong when investment fails.

At the Pareto worst equilibrium, on the contrary, agents coordinate on the most pessimistic equilibrium. Taking into account the errors of their opponents, θ^* is decreasing when the overall error is increasing. Agents miscoordinate in information when Type-II error is costly. Indeed, the objective still being not to miss successful investments, and the errors of the opponents increasing this possibility, an individual is willing to decrease his own error. This time, successful investments depend positively on the overall error. Sym-

⁸Furthermore, one can note that if the payoff function was different in [Szkup and Trevino \[2015\]](#), for example investing successfully rewards $1 - T$ and investing unsuccessfully costs αT with $\alpha > 0$ (and not investing rewards 0), the parameter T would be present in the expression of the value of information.

metrically, setting the Type-I error more costly, make emerge complementarities. Being wrong when investments fail is less likely when more agents take incorrect decisions.

Interestingly, the results follow also this simple rule: when information has a high (low) value, complementarities (substituabilities) in information may emerge. When information has value, agents acquire a lot of information. It happens when the costly error occurs for a large range of states of the world (so the agents are willing to avoid it by acquiring a large amount of information).

In beauty contest games, it is valuable to be close from others' action, and more valuable when they acquire a lot of information since it makes them closer from the true state of the world. Here, if the likelihood of the most costly error increases with the mean precision, the willingness to avoid the most costly error increases with the mean precision as well. Naturally, complementarities emerge. However, substituabilities can appear as well, contrary to beauty contests.

Note that the Pareto best equilibrium maximizes the probability to receive an incorrect signal $X_i = 0$ and the Pareto worst maximizes the probability to receive an incorrect signal $X_i = 1$. If incorrect, these signals leads to make the most costly error respectively when $T < \frac{1}{2}$ and $T > \frac{1}{2}$. By coordinating on the Pareto best (worst) equilibrium when $T < \frac{1}{2}$ ($T > \frac{1}{2}$), the agents show that they are willing to have a high precision. Indeed, the threshold decision makes the errors made by the agents the most costly. This choice makes agents coordinate in information as well. In the opposite equilibria, this coordination vanishes. It is remarkable in this game that Pareto optimality is not linked with high information acquisition (see [Table 3.2](#)). It is not because information might be harmful, but because information can be less valuable at this point than at others.

	Coordination on high information acquisition	Coordination on low information acquisition
Coordination on low threshold	Pareto Best ($T < \frac{1}{2}$)	Pareto Best ($T > \frac{1}{2}$)
Coordination on high threshold	Pareto Worst ($T > \frac{1}{2}$)	Pareto Worst ($T < \frac{1}{2}$)

Table 3.2: Complementarity and Substability for the Pareto Best and Pareto Worst Equilibria with respect to the Value of Information

Effects of the distribution of the states of the world

In [Table 3.3](#), we summarize complementarities in information acquisition at the Pareto best and worst equilibria, with respect to the distribution of the states of the world and the value of T (i.e. $T < (>) \frac{1}{2}$). We recall that if an increase of the overall error implies a decrease in the size of the most costly error, then there are complementarities. In the

$\mathbb{F}_\theta(1 - E_2) < \frac{1}{2}$ (Good Quality Projects)	$\mathbb{F}_\theta(E_1) < \frac{1}{2} < \mathbb{F}_\theta(1 - E_2)$ (Intermediate Quality Projects)	$\frac{1}{2} < \mathbb{F}_\theta(E_1)$ (Bad Quality Projects)
$E_1 \nearrow \Rightarrow$ Type-II \searrow $E_2 \nearrow \Rightarrow$ Type-II \nearrow	$E_1 \nearrow \Rightarrow$ Type-II \searrow $E_2 \nearrow \Rightarrow$ Type-I \searrow	$E_1 \nearrow \Rightarrow$ Type-I \nearrow $E_2 \nearrow \Rightarrow$ Type-I \searrow
Type-II error is the most likely: complementarities if the Type-II error is the most costly (i.e. $T < \frac{1}{2}$) only at the Pareto best equilibrium [and when the Type-II error is the least costly (i.e. $T > \frac{1}{2}$) at the Pareto worst equilibrium]	Complementarities if the most likely error is also the most costly at both equilibria ($T < \frac{1}{2}$ at best and $T > \frac{1}{2}$ at worst equilibrium)	Type-I error is the most likely: complementarities if the Type-I is the most costly (i.e. $T > \frac{1}{2}$) only at the Pareto worst equilibrium [and when the Type-I error is the least costly (i.e. $T < \frac{1}{2}$) at the Pareto best equilibrium]
Type-II error is the most likely: substitutabilities if the Type-II error is the more costly (i.e. $T > \frac{1}{2}$) at the Pareto worst equilibrium [and when the Type-II error is the least costly (i.e. $T > \frac{1}{2}$) at the Pareto best equilibrium]	Substitutabilities if the most likely error is the least costly at both equilibria ($T > \frac{1}{2}$ at best and $T < \frac{1}{2}$ at worst equilibrium)	Type-I error is the most likely: substitutabilities if the Type-I is the more costly (i.e. $T < \frac{1}{2}$) at the Pareto best equilibrium [and when the Type-I error is the least costly (i.e. $T < \frac{1}{2}$) at the Pareto worst equilibrium]

Table 3.3: Effect of a moving overall error on Complementarities in Information Acquisition for the Pareto Best and Pareto Worst Equilibria with respect to the distribution of the states of the world

case where many projects have intermediate quality, players exhibit complementarities if the most likely error is also the most costly. In these cases, any increase of the overall error reduce the likelihood of the most costly error. Thus, any individual j is incentivized to increase his error as well. (At these equilibria, coordination matters more since incorrect decision in information acquisition is much more costly than in the other equilibria.) On the contrary, substitutabilities may appear if the most likely error is not the most costly. (Indeed, in this case, coordination is less important since acquiring too much (or not enough) information is less detrimental. Intuitively, if the other players' information acquisition affects less the profit of the individual j , substitutabilities may emerge.) When many projects are such that their value is out of intermediate values, things change. Indeed, one error is always the most likely: Type-II for good projects, Type-I for bad projects. Thus, results are similar with respect to the intermediate case, when the Type-II is the most costly for the Pareto best equilibrium, and when the Type-I is the most costly for the Pareto worst equilibrium. However, when the Type-II (Type-I) error is the most costly for the good (bad) projects at the Pareto worst (best) equilibrium, there are

3.4.2 Equilibria without information acquisition

From now, the framework is modified: some (or all) agents may not acquire information. Assume that we are at an equilibrium where all the agents invest but one (e.g. j)⁹. Therefore, $\theta^* = 0$ and j choose the optimal error $\epsilon_j^*(0)$ and there is no information acquisition from the other agents (since signals would be useless since the investment decision has already been made). The situation is optimal from a welfare point of view. Obviously, players make incorrect decisions (only Type-I error) but they all have a positive gain (since in this case $1 - \mathbb{F}_\theta(0) > T$). The agent j , benefits also from this equilibrium since the probability of a successful investment is maximal. Nevertheless, j can increase his profit by reducing the probability of a Type-I error (but has to make Type-II errors as a counterpart).

In the case where $T < \frac{1}{2}$, the Type-II error is the most costly. Then, j acquires a large amount of information since the size of the Type-II error is maximal. Now, if a non negligible part of the players starts to acquire information, the equilibrium is changed (let us say that the errors' costs is reduced for at least a part of the players), and θ^* is shift up. The size of the (most costly) Type-II error is reduced, and j is incentivized to increase his error, acting in the opposite way. In the case where $T > \frac{1}{2}$, the opposite situation happens and complementarities in information acquisition emerge.

Assume now that we are at an equilibrium such that no agent invests but one, j , who acquires information and invests when he receives a high signal¹⁰. Symmetrically to what happens in the previous case, there are complementarities when $T < \frac{1}{2}$, and substitutabilities when $T > \frac{1}{2}$.

	Coordination on full information acquisition		Coordination on no information acquisition	
Coordination on threshold 0	Pareto Best ($T < \frac{1}{2}$)	Pareto Best ($T > \frac{1}{2}$)	Pareto Best ($T < \frac{1}{2}$)	Pareto Best ($T > \frac{1}{2}$)
Coordination on threshold 1	Pareto Worst ($T > \frac{1}{2}$)	Pareto Worst ($T < \frac{1}{2}$)	Pareto Worst ($T > \frac{1}{2}$)	Pareto Worst ($T < \frac{1}{2}$)

Table 3.4: Complementarity and Substutability for the Pareto Best and Pareto Worst Equilibria at poles of information acquisition

Firstly, note that coordination on threshold 0 (or 1) is possible (it depends on the value of $\mathbb{F}_\theta(0)$, $\mathbb{F}_\theta(1)$ and T) when agents do (not) acquire information.

We remark that when the players do not acquire information, results are reversed in terms of information complementarities with respect to the equilibria we have found previously where all the agents acquired information. From 0, shifting up the threshold makes the

⁹We assume that [Assumption A2](#) is relaxed. This equilibrium exists if $\mathbb{F}_\theta(0) < 1 - T$ and cost functions are such that [Assumption A2](#) is respected only for j .

¹⁰This equilibrium exists if $1 - T < \mathbb{F}_\theta(1)$ and cost functions are such that [Assumption A2](#) is respected only for j .

information value decreasing (increasing) when $T < \frac{1}{2}$ ($T > \frac{1}{2}$) while $\int \epsilon_i^*(\theta^*) \, di > \theta^*$, i.e. the mean error is above θ^* (and some agents still invest whatever the signal they may receive). Nevertheless, there is still a kind of complementarities at these equilibria with substitutabilities. Indeed, when some agents start to acquire information, the mean error is reduced. However, as illustrated in Figure 3.7 for the case where all the agents invest but one, the error made when the investment is successful (Type-II error) is increasing. In both cases, there are substitutabilities when the size of the most costly error is decreasing. But in both cases, agents make more often this error (because they did not make it when they did not acquire information). Thus, while the error of the individual moves in opposite direction to his opponents' ones, they still increase simultaneously the level of the error of the most costly one. That is why, there are still complementarities in the sense that, considering the most costly error, the agents has the information acquisition incentives.

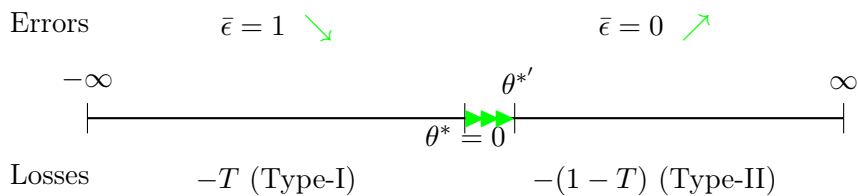


Figure 3.5: Effect of agents starting to acquire information on the equilibrium where all the agents always invest

The fact that some agents do not invest blindly as they used to do is detrimental for an individual j who was already acquiring information. Indeed, θ^* increases, and so decreases the likelihood of a successful investment. However, the individual was not benefiting from the errors of the other players, only of their lack of access to a value for money information and their optimism. Indeed, when $\theta > 0$, they invested successfully. By not acquiring information, they reduced the potential Type-II error likelihood at 0. Nevertheless, for j , the size of Type-II error is maximal, and if $T < \frac{1}{2}$, information acquisition is maximal. Thus, by increasing their precision, the other agents reduce the incentive j has to acquire information. Contrary to what happens in $1 - E_2$, error of other players is not the key element but their optimism (which can be seen as the guarantee that they will make Type-I errors). When their optimism (or naivety) disappears, mechanisms are reversed and j may start to benefit from the increased precision of his opponents. Symmetric results are found when considering $T > \frac{1}{2}$.

On the contrary, when agents never invest (i.e. $\theta^* = 1$), an increasing precision results in an increase of the investments' likelihood. Agents who do not acquire information never make Type-I error but make Type-II errors (pessimism). Paradoxically, the increased precision enables more investments because information acquisition of other agents guarantees the existence of the Type-I errors. Thus, when $T < \frac{1}{2}$ ($T > \frac{1}{2}$), the information value is increasing (decreasing) in the other players' precision and there are complementarities (substitutabilities) in information acquisition. To sum up, firstly, precision shifts

down the threshold (because more precision implies more Type-I errors) ; but secondly, once all the agents have acquired information, only a reduced precision makes increase the likelihood of Type-I errors.

Finally, coming from a situation where agents did not acquire information, the transition to the situation where all the agents acquire information is such that, actually, agents acquiring information makes one error (the one agents did not make by always choosing the same action) more likely (whereas the overall error has been reduced). It explains why the results are reversed with respect to what we found in [Section 3.4.1](#). All the results are summed up in the right hand side of [Table 3.4](#).

3.4.3 Hypothetical full information equilibria

Assume now that agents can acquire perfect information at no cost (relaxing [Assumption A1](#)). Then, any equilibrium where all the agents choose the same threshold (belonging to $[0, 1]$) is possible. Let us focus on the case where $\theta^* = 0$, i.e. all the agents chooses the threshold 0. An agent j with limited attention¹¹, would acquire a large amount of information if $T < \frac{1}{2}$ and less if $T > \frac{1}{2}$. If information becomes costly for a non negligible part of the players (other than j), θ^* will be shift up. If $T < \frac{1}{2}$ ($T > \frac{1}{2}$), the information value is reduced (increased), and j has an incentive to reduce (increase) his precision and there are complementarities (substitutabilities) in information acquisition. Contrary to what we found in [Section 3.4.2](#), the effect of the increasing mean error is unambiguous: it increases both errors simultaneously. Thus, only the size of the most costly error matters (as in [Section 3.4.1](#)) and that is why we find again the same results. All the results are summed up in the left hand side of [Table 3.4](#).

The role of information acquisition is enlightened through these special cases (with no information and with full information). Indeed, the fact that agents can acquire information, or not, is rarely studied. However, it is clear that our main analysis (in [Section 3.4.1](#)) depends on the fact that all the agents acquire information at the equilibrium.

3.5 Discussion and conclusion

Our paper is related to a literature that investigates the effect of information in risky markets. With a market design approach, [Banner and Heinemann \[2005\]](#) seeks the best information disclosure policy from a central bank perspective in a currency attack model. The public authority decides the optimal level of private information, according to the prior distribution of the states of the world. Especially, the results depend on the fact that the mean of the distribution is below (pessimism) or above (optimism) the regime change cutoff (of the infinitely precise private information case). When there is prior pessimism, the ex ante probability of a currency crisis is high and only precise information can reverse the beliefs of the agents (when the fundamental is high). On the contrary,

¹¹Such that [Assumptions A1](#) and [A2](#) are respected for j .

when there is prior optimism, the public authority should commit to an intermediate level of information disclosure. Indeed, agents coordinate on the currency attack if their posterior is sufficiently bad (about the state of the world). Therefore, even in cases where the fundamental is low, if information is not precise, the posterior can hardly diminish and less attacks are made. Symmetrically, if the central authority could set the precision in our model as in [Bannier and Heinemann \[2005\]](#), the public authority would be willing to make θ^* as low as possible. Thus, it would increase the precision as much as possible in case of optimism (as we have described), and would decrease it as much as possible (limited by the bound on precision that prevents equilibria without information acquisition) in case of pessimism (they study only the case where $T < \frac{1}{2}$, see the left hand side of [Figure 3.6](#) as comparison)¹². Our notion of optimism is different from the one developed by [Bannier and Heinemann \[2005\]](#), since we derive whether there is optimism or not, after seeing the selected equilibrium. In their framework, optimism is defined ex ante by the parameters of the game and the equilibrium is unique. Nevertheless, we show that the intuitions found in their paper, are validated by the endogenous information acquisition we add. Private agents act as would impose a public authority in a certain respect. These results remain when $T > \frac{1}{2}$. However, if T takes a high value, decreasing precision might be dangerous since if it decreases θ^* (and reduces the size of the most costly error), it generates also more errors made by the agents. Having a high level of error can be detrimental when agents have a budget constraint (or equivalently have a limited access to credit). Thus, the public authority may not be willing to have a too high level of error to avoid too much bankruptcy.

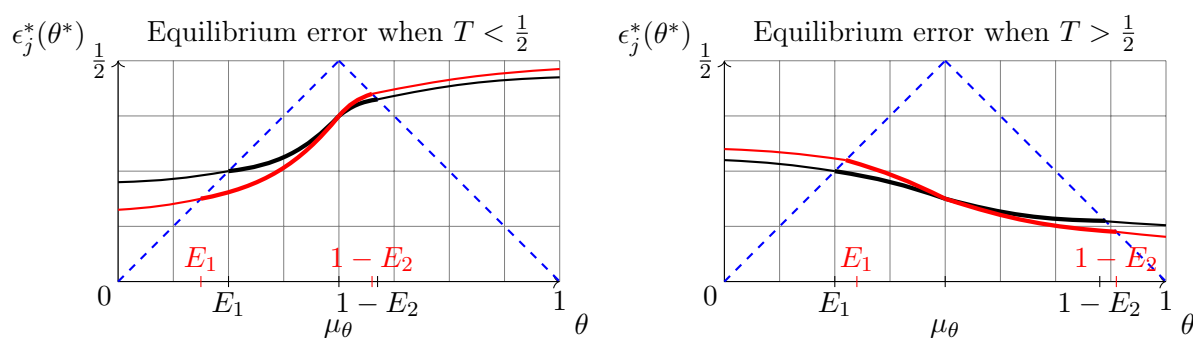


Figure 3.6: Graphical examples - Equilibrium error for given thresholds: The effect of a decreased variance on the equilibria (in red)

[Bannier and Heinemann \[2005\]](#) study the possibility for the public authority to decrease or increase the variance of the distribution of the states of the world. As explained above, when there is pessimism, the central bank is willing to reverse the agents' view. Thus, it should increase the variance of the distribution to reduce the probability of an attack, because with a high variance, the fundamental may be sufficiently high to

¹²Note that our results are similar since our model is an investment game while theirs is an attack game and the incentives of the public authority are reversed.

prevent an attack. Conversely, when the attack is not likely, reducing the variance avoid the possibility that agents get a bad posterior. In their paper, the public authority's objective is clear: minimizing the occurrence of an attack on the currency. However, in this article, the public authority may have two objectives (that may be contradictory): maximizing the likelihood of successful projects, or minimizing the errors made by agents (see [Figure 3.6](#)). If our public authority is willing to maximize the likelihood of successful projects, it faces also asymmetric incentives, either agents are optimistic or pessimistic. As in [Bannier and Heinemann \[2005\]](#), if agents are optimistic, reducing the variance will reduce the quantity of bad projects (such that $\theta < 0$), and increase the probability of successful projects. Furthermore, E_1 is decreasing when the variance decreases (when $T < \frac{1}{2}$), confirming that the public authority should reduce the variance as much as possible. On the contrary, when $T > \frac{1}{2}$, E_1 is increasing. However, from the first order condition, one can see that if the absolute derivative of the cost function is decreasing, then, $\mathbb{F}_\theta(E_1)$ has to decrease. If the agents are pessimistic, the effects are reversed. Indeed, less variance is detrimental since it reduces the quantity of good projects. $1 - E_2$ is decreasing with less variance when $T < \frac{1}{2}$, but as described above, the likelihood of successful projects is diminished (since $C'(E_2)$ is reduced, $\mathbb{F}_\theta(1 - E_2)$ has to increase when $T < \frac{1}{2}$ from the first order condition). Now, if the public authority is willing to reduce the errors chosen by the agents, it contradicts the objective of maximizing the likelihood of success when $T > \frac{1}{2}$. Decreasing the variance leads to more successful projects but also to a higher overall error when agents are optimistic. It reduces the probability of successful projects and the errors when agents are pessimistic. Again, having a high level of error can be detrimental and the public authority may prioritize a policy that would reduce the error at the expense of the unconditional likelihood of success.

[Iachan and Nenov \[2015\]](#) explore the same kind of questions that we have analyzed. However, in their article, there is no endogenous information acquisition. They mainly analyze the effect of a decrease of the precision signal in a global game on the regime change cutoff in a global game. In global games, agents receive a imprecise signal on the value of the fundamental. The signal is composed of the value of the fundamental plus an error term multiplied by a coefficient that gives the level of precision of the signal. In their paper, there is no endogenous information acquisition as in [Szkup and Trevino \[2015\]](#), but their results are more direct about how a change of precision influence the equilibrium of a global game. Indeed, the threshold θ^* is monotonic with respect to the signal precision. Their payoff function is sensitive to the fundamental. If the payoff function when the risky action is successful is more sensitive than when the risky action fails, then the threshold decreases (increases) when the precision of the signal increases (decreases). In this case, it follows that more precision leads to less successful attacks. They confirm the result found in [Heinemann and Illing \[2002\]](#) who make a case for more transparency in a situation with uniform distributions where the payoff was fundamental-sensitive only in case of successful attack. [Heinemann and Illing \[2002\]](#) seek also the effect of information precision on the equilibrium in a global game. In their specific example,

the strategic and the regime change thresholds are increasing in the signal noise. It is due to the fact that being more often wrong will be more costly when the signal is too high (with respect to the true value of the fundamental) rather than too low (indeed the payoff is more fundamental sensitive when the state of the world is low). Thus, agents should increase their threshold (and attack more often) in order to not fall too often in the situation where they do not attack while the attack is successful (Type-II error). The cutoff moves naturally to equalize the incentive to attack and not to attack at this point. Thus, it would be difficult to clearly understand what would be the incentives of an individual player able to acquire information. However, it can be said that the rewards of a successful attack around the threshold decrease when the threshold increase (because the payoff of a successful attack decreases with the value of the fundamental). Then, it can be inferred that the information value around the threshold has decreased as well. In this case, there would be only complementarities in information acquisition. [Iachan and Nenov \[2015\]](#) exhibit a frontier in the direction's change of the regime change cutoff, but around this frontier, there are complementarities in information acquisition. Indeed, as a reaction to the increase of the noise, the threshold is shifted (up or down) to lower the risk to face the error that is the most costly when the fundamental takes extreme values. Since this risk is lowered, the information value decreases as well and substitutabilities in information acquisition can not emerge. Interestingly, when the payoff is never sensitive to the fundamental (as in our model), the regime change threshold does not move with the noise of the signal. Therefore, as in our case with $T = \frac{1}{2}$, there is neither complementarities nor substitutabilities in information acquisition.

The advantage we have on the global game approach is that the equilibrium threshold is not determined by an indifference condition. Because of this payoff indifference condition, agents facing a decrease in the precision always choose to lower the risk to have the worst loss. Naturally, an individual acquiring information is incentivized to reduce his information as well. [Szkup and Trevino \[2015\]](#) make emerge substitutabilities in specific cases where the monotonicity of the regime change cutoff with overall information precision disappears.

In [Szkup and Trevino \[2015\]](#), the action cutoff is positioned such that the most costly error is the most avoided¹³. In [Figure 3.7](#), $T > \frac{1}{2}$ and the regime change cutoff increases with the precision of the players. However, for an individual j with a low initial precision, this increase leads to a decrease in his own precision (substitutabilities in information acquisition). The authors explain that it is due to the fact that the (Type-I) error's likelihood (that is the most feared by j) has increased ($\theta^* < \theta^{*'}).$ Thus, his strategic cutoff increases more than the increase of the regime change cutoff ($|x_j^* - \theta^*| < |x_j^{*'} - \theta^{*'}|$). [Figure 3.7](#) is helpful to understand the differences between our models. x_j^* is positioned far from θ^* to avoid to receive signals higher than x_j^* while the state of the world is below

¹³The action cutoff is such that, when receiving the signal with the value of the action cutoff, the agent should be indifferent between investing and not investing. Thus, if an error is more costly, the action cutoff is positioned away from the regime change cutoff to decrease the likelihood of the most costly error.

θ^* . Thus, the agent makes Type-II errors when his error is small and when the state of the world lies between θ^* and x_j^* . In the example described in [Szkup and Trevino \[2015\]](#), the information value is said to be decreasing because the likelihood of Type-I error has increased while it was already the error that was the most feared. The fact that the size of two errors is more equalized should increase the value of information according to what we have described above (information is more valuable when the two errors are more equally likely¹⁴). However, in fact, we see that the crucial criterion for the individual is that, since the size of the Type-II error due to the spread $|x_j^* - \theta^*|$ is larger, he cares less of this error.

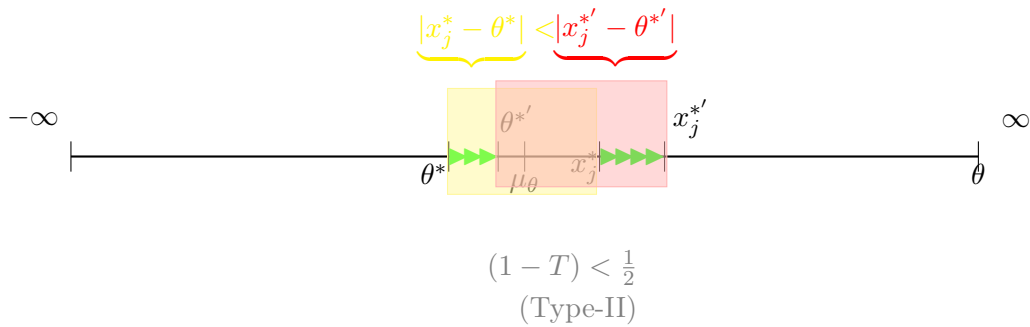


Figure 3.7: Strategic and regime change cutoffs in [Szkup and Trevino \[2015\]](#): a case with substitutabilities in information acquisition: overall precision increasing leads an individual to acquire less information

In this case, the ex ante likelihood of making both type of errors is more equalized, but the strategic cutoff is moved away from the regime change cutoff. The rigidity of the equilibrium condition of the global games explains their results. We remark as well that without assumption on the minimal endowment of information (which ensures the unicity of the equilibrium), there would be more cases such that substitutabilities appear.

One fundamental difference between our results is that substitutabilities may only appear when the most costly error's size and the precision increase together in [Szkup and Trevino \[2015\]](#), while in our case this always implies complementarities in information acquisition. As a constant, when $T = \frac{1}{2}$, there are never substitutabilities. In their setting, initial information of the individual has to be very low while the initial information of other players has to be high to make emerge substitutabilities. In our case, substitutabilities can emerge in any symmetric or asymmetric situation. Our results are reversed when the individual has a large amount of information and the other players have low information. Indeed, substitutabilities are generally associated with an individual acquiring few information, both in our model and in [Szkup and Trevino \[2015\]](#). Nevertheless, our special case where the mean precision is low and where some agents can acquire information show that substitutabilities can emerge even if agents are endowed with a large amount of

¹⁴Information value is maximal when the ex ante likelihoods of making both types of errors are equal, and the action cutoff should be equal to the regime change cutoff.

information.

We have showed with a simple setup that the inheritance result of [Hellwig and Veldkamp \[2009\]](#) was shaky, without need of technical assumptions as normal distributions, and in large regions of the games. Contrary to [Szkup and Trevino \[2015\]](#), we prove that substitutability can emerge at a symmetric equilibrium (i.e. where all the agents have the same cost function). This is due to the fact that an individual may be willing to compensate an increase in the threshold, if this increase is due to an increase of the mean error. Through the three kinds of equilibria we have studied, we have seen that the game makes emerge complementarities and substitutabilities at the Pareto best and worst equilibria with respect to the parameter T . The choice of the individual is always linked to the size of the most costly error. We have seen that complementarities emerge in the case when there is relatively more information acquisition (for the individual j) and where all the agents acquire (imperfect or perfect) information. When some agents do not acquire information (and always play the same action), these results are reversed. In fact, in these special cases, substitutabilities are accompanied with complementarities in the error's level of the most costly error (i.e. agents makes together more (or less) the most costly error).

Bibliography

- Bannier, C. E. and Heinemann, F. (2005). Optimal Transparency and Risk-Taking to Avoid Currency Crises. *Journal of Institutional and Theoretical Economics*, 161(3):374.
- Heinemann, F. and Illing, G. (2002). Speculative Attacks: Unique Equilibrium and Transparency. *Journal of International Economics*, 58(2):429–450.
- Hellwig, C. and Veldkamp, L. (2009). Knowing What Others Know: Coordination Motives in Information Acquisition. *The Review of Economic Studies*, 76(1):223–251.
- Iachan, F. S. and Nenov, P. T. (2015). Information Quality and Crises in Regime-Change Games. *Journal of Economic Theory*, 158(PB):739–768.
- Jiménez-Martínez, A. (2014). Information Acquisition Interactions in Two-Player Quadratic Games. *International Journal of Game Theory*, 43(2):455–485.
- Morris, S. and Shin, H. S. (2003). Global Games: Theory and Applications. In M. Dewatripont, L. Hansen, & S. T., editor, *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*, volume 35, pages 56–114. Econometric Society Monographs.
- Morris, S. and Yang, M. (2016). Coordination and Continuous Choice.
- Szkup, M. and Trevino, I. (2015). Information Acquisition in Global Games of Regime Change. *Journal of Economic Theory*, 160:387–428.
- Yang, L. L. (2018). Relevant Decision Problems and Value of Information.
- Yang, M. (2015). Coordination with Flexible Information Acquisition. *Journal of Economic Theory*, 158:721–738.

Appendices

Appendix C

Appendices to Complementarities in Information Acquisition

C.1 Determination of maximal errors

Suppose $0 < \xi \leq 1$, i.e. all the agents choose a threshold between 0 and 1. The payoff from investing when the signal is high becomes:

$$\begin{aligned} & \mathbb{P}(\theta + p > 1 \cap \theta > \xi | X_i = 1) + \mathbb{P}(\theta + p > 1 \cap \theta < \xi | X_i = 1) \\ = & \frac{\mathbb{P}(\theta > 1 \cap \theta > \xi \cap X_i = 1)}{\mathbb{P}(X_i = 1)} + \frac{\mathbb{P}(p > 1 - \theta \cap \theta > \xi \cap X_i = 1)}{\mathbb{P}(X_i = 1)} + \frac{\mathbb{P}(p > 1 - \theta \cap \theta < \xi \cap X_i = 1)}{\mathbb{P}(X_i = 1)} \\ = & \frac{(1 - \epsilon)(1 - \mathbb{F}_\theta(1))}{\mathbb{P}(X_i = 1)} + \frac{(1 - \epsilon) \left[\int_\xi^1 \int_{1-\theta}^1 \bar{g}_{P,\Theta}(p, \theta | \theta < \xi) dp d\theta \right]}{\mathbb{P}(X_i = 1)} + \frac{\epsilon \left[\int_0^\xi \int_{1-\theta}^1 \underline{g}_{P,\Theta}(p, \theta | \theta < \xi) dp d\theta \right]}{\mathbb{P}(X_i = 1)} \end{aligned}$$

Then it is dominant to invest when:

$$\frac{(1 - \epsilon)(1 - \mathbb{F}_\theta(1))}{\mathbb{P}(X_i = 1)} > T$$

And, it is dominant not to invest when:

$$\frac{(1 - \epsilon)(1 - \mathbb{F}_\theta(\xi))}{\mathbb{P}(X_i = 1)} + \frac{\epsilon(\mathbb{F}_\theta(\xi) - \mathbb{F}_\theta(0))}{\mathbb{P}(X_i = 1)} < T$$

Few computations give that investing is dominant when:

$$\begin{aligned} \mathbb{F}_\theta(\xi) & > \frac{(1 - \epsilon)(\mathbb{F}_\theta(1) - (1 - T))}{(1 - 2\epsilon)T} \\ \Leftrightarrow & \begin{cases} \text{if } \mathbb{F}_\theta(\xi) > 1 - \frac{1 - \mathbb{F}_\theta(1)}{T} = \xi_1, & \epsilon < \frac{(1 - \mathbb{F}_\theta(\xi))(1 - T) - (\mathbb{F}_\theta(1) - \mathbb{F}_\theta(\xi))}{(1 - 2\mathbb{F}_\theta(\xi))(1 - T) + \mathbb{F}_\theta(1) - 2(\mathbb{F}_\theta(1) - \mathbb{F}_\theta(\xi))} = \epsilon_1 \\ \text{if } \mathbb{F}_\theta(\xi) \leq 1 - \frac{1 - \mathbb{F}_\theta(1)}{T}, & \nexists \epsilon \text{ s.t. investing is dominant} \end{cases} \end{aligned}$$

Similarly, not investing is dominant when:

$$\begin{aligned} \mathbb{F}_\theta(\xi) &> \frac{(1-\epsilon)(1-T) - \epsilon \mathbb{F}_\theta(0)}{(1-2\epsilon)(1-T)} \\ \Leftrightarrow \begin{cases} \text{if } \mathbb{F}_\theta(\xi) < \frac{1}{2} + \frac{\mathbb{F}_\theta(0)}{2(1-T)} = \xi_0, & \epsilon > \frac{(1-\mathbb{F}_\theta(\xi))(1-T)}{(1-2\mathbb{F}_\theta(\xi))(1-T) + \mathbb{F}_\theta(0)} = \epsilon_0 \\ \text{if } \mathbb{F}_\theta(\xi) \geq \frac{1}{2} + \frac{\mathbb{F}_\theta(0)}{2(1-T)}, & \nexists \epsilon \text{ s.t. not investing is dominant} \end{cases} \end{aligned}$$

Note that ϵ_1 is decreasing in $\mathbb{F}_\theta(1) - \mathbb{F}_\theta(\xi)$ when $1 - T < \mathbb{F}_\theta(1)$. Intuitively the error should decrease when the threshold moves far from 1, since the area which depends on the coordination becomes larger. However, this fact contradicts the intuition that when the uncertainty decreases (since the threshold is closer from the middle of the distribution) the error can increase. One can remark that when $1 - T > \mathbb{F}_\theta(1)$, $\xi_1 < 0$, $\epsilon_1 > \frac{1}{2}$, i.e. it is still dominant to invest.

Again, $\epsilon_1 < \epsilon_0$. The relation between ξ_0 and ξ_1 is more ambiguous. In most distributions, we should get that $\xi_1 < \xi_0$, but both cases are workable.

If the hypothesis (especially $\mathbb{F}_\theta(0) < 1 - T$) is respected we see that ϵ_0 is strictly above $\frac{1}{2}$, i.e. it is never dominant not to invest when the signal is high. Indeed, it can be checked in this case that $2(1 - \mathbb{F}_\theta(\xi))(1 - T) > (1 - 2\mathbb{F}_\theta(\xi))(1 - T) + \mathbb{F}_\theta(0)$.

When the signal is low, we find the same result for the non-investment decision. We get that it is dominant not to invest when:

$$\begin{aligned} \mathbb{F}_\theta(\xi) &< \frac{\mathbb{F}_\theta(0)(1-\epsilon) - \epsilon(1-T)}{(1-2\epsilon)(1-T)} \\ \Leftrightarrow \begin{cases} \text{if } \mathbb{F}_\theta(\xi) < \frac{\mathbb{F}_\theta(0)}{1-T} = \xi_{00}, & \epsilon < \frac{\mathbb{F}_\theta(0) - \mathbb{F}_\theta(\xi)(1-T)}{(1-2\mathbb{F}_\theta(\xi))(1-T) + \mathbb{F}_\theta(0)} = \epsilon_{00} \\ \text{if } \mathbb{F}_\theta(\xi) \geq \frac{\mathbb{F}_\theta(0)}{1-T}, & \nexists \epsilon \text{ s.t. not investing is dominant} \end{cases} \end{aligned}$$

However there is a slight change and it is dominant to invest when:

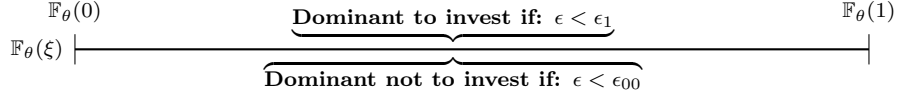
$$\begin{aligned} \mathbb{F}_\theta(\xi) &< \frac{\epsilon(1-T - \mathbb{F}_\theta(1))}{(1-2\epsilon)T} \\ \Leftrightarrow \begin{cases} \text{if } \mathbb{F}_\theta(\xi) > \frac{\mathbb{F}_\theta(1) - (1-T)}{2T} = \xi_{01}, & \epsilon > \frac{T\mathbb{F}_\theta(\xi)}{2T\mathbb{F}_\theta(\xi) + (1-T) - \mathbb{F}_\theta(1)} = \epsilon_{01} \\ \text{if } \mathbb{F}_\theta(\xi) \leq \frac{\mathbb{F}_\theta(1) - (1-T)}{2T}, & \nexists \epsilon \text{ s.t. investing is dominant} \end{cases} \end{aligned}$$

One can check that $\epsilon_{01} > \epsilon_{00}$ but again nothing can be said about the two cutoffs.

If the hypothesis (especially $1 - T < \mathbb{F}_\theta(1)$) is respected we see that ϵ_{01} is strictly above $\frac{1}{2}$, i.e. it is never dominant to invest when the signal is low.

We get that $\xi_1 < \mathbb{F}_\theta(0)$ and $\xi_{00} > \mathbb{F}_\theta(1)$.

Dominance regions when $0 < \xi \leq 1$:



C.2 Effect of an increasing T on the equilibrium

Now, let us study the effect of T on the value of information. From [Equation \(FOCj\)](#), we see that T will affect differently the determination of ϵ_j , whether the Type-I or the Type-II error is the most likely, i.e. whether $\mathbb{F}_\theta(\theta^*)$ is below or above $\frac{1}{2}$. Indeed, the left hand side of [Equation \(FOCj\)](#) derivative with respect with T is $1 - 2\mathbb{F}_\theta(\theta^*)$. It is positive (negative) if $\mathbb{F}_\theta(\theta^*) < (>)\frac{1}{2}$, i.e. if the Type-II (Type-I) error is the most likely. Thus, the marginal benefit is less (more) negative when T is increasing and $\mathbb{F}_\theta(\theta^*) < (>)\frac{1}{2}$. It means that ϵ_j^* is decreasing (increasing) in T if the Type-II (Type-I) error is the most likely. Intuitively, if the Type-II error is the most likely, T increasing makes the most likely error relatively less costly with respect the other error. Therefore, at a fixed θ^* , if T is increasing, being precise matters less since the expected cost of being wrong (i.e. the cost of each error multiplied by their respective likelihood) is diminishing. On the contrary, the value of information is increasing with T if the Type-I error is the most likely.

As an illustration, if we assume that the distribution is centered at $\frac{1}{2}$, the Type-II error is the most likely at the Pareto best equilibrium (since $\theta^* < \frac{1}{2}$), and the Type-I error is the most likely at the Pareto worst equilibrium (since $\theta^* > \frac{1}{2}$).

$\mathbb{F}_\theta(1 - e_2) < \frac{1}{2}$	$\mathbb{F}_\theta(e_1) < \frac{1}{2} < \mathbb{F}_\theta(1 - e_2)$	$\frac{1}{2} < \mathbb{F}_\theta(e_1)$
$\frac{d e_1}{d T} > 0$ and $\frac{d e_2}{d T} > 0$	$\frac{d e_1}{d T} > 0$ and $\frac{d e_2}{d T} < 0$	$\frac{d e_1}{d T} < 0$ and $\frac{d e_2}{d T} < 0$
$\frac{d \epsilon_j}{d T} > 0 \forall \theta^* \in [e_1, 1 - e_2]$	$\frac{d \theta^*}{d T} > 0$ at $\theta^* = e_1, 1 - e_2$	$\frac{d \epsilon_j}{d T} < 0 \forall \theta^* \in [e_1, 1 - e_2]$

Table C.1: Effect of an increasing T on the Value of Information for the Pareto Best and Pareto Worst Equilibria

On the left hand side of [Table C.1](#), T increasing reduces the size of the possible thresholds. If many projects are good, errors unambiguously increase. Agents want to avoid mainly the Type-II error (i.e. not to miss successful investments). As its relative cost is decreasing, agents are willing to acquire less information.

In the middle case, the possible thresholds are shifted up (increase or decrease of the size of the possible thresholds). Intuitively, a higher T reduces the possibility to coordinate on a good equilibrium because the willingness to pay for information is reduced (because investments are less rewarding).

On the contrary, on the right hand side, this size increases. It means that when there are lot of bad projects, an increase of cost T , there are more ways (lower and higher) to coordinate because more information is acquired. Indeed, agents want to avoid mainly

the Type-I error. As its relative cost is increasing, agents are willing to acquire more information.

$\mathbb{F}_\theta(1 - e_2) < \frac{1}{2}$	$\mathbb{F}_\theta(e_1) < \frac{1}{2} < \mathbb{F}_\theta(1 - e_2)$	$\frac{1}{2} < \mathbb{F}_\theta(e_1)$
$\frac{de_1}{dT} > 0 \Rightarrow$ Type-II \searrow	$\frac{de_1}{dT} > 0 \Rightarrow$ Type-II \searrow	$\frac{de_1}{dT} < 0 \Rightarrow$ Type-I \searrow
$\frac{de_2}{dT} > 0 \Rightarrow$ Type-II \nearrow	$\frac{de_2}{dT} < 0 \Rightarrow$ Type-I \nearrow	$\frac{de_2}{dT} < 0 \Rightarrow$ Type-I \nearrow

Table C.2: Effect of an increase of T on the most likely error for the Pareto Best and Pareto Worst Equilibria

Naturally, the size of the interval of the possible thresholds has implications on the size of the errors. If we assume that the increase of T affects only a fraction of the agents, it will still affect the untouched agents, by moving down or up θ^* . Combining [Tables C.1](#) and [3.3](#), we can write [Table C.2](#). In the cases with complementarities, T increasing will push all the errors up at the Pareto best equilibrium, and will push all the errors down at the Pareto worst equilibrium. In the cases with substitutabilities, the agents affected by the increase see their incentives reversed. Their over optimism or pessimism is reinforced. Since there are substitutabilities, other agents still increase their error (or decrease at the Pareto worst equilibrium).

In comparison with Proposition 6 in [Yang \[2018\]](#), we find opposite results about the effect of a change of the investment cost. The result claims that, in a global game, when the investment cost increases (decreases) and that no investing (investing) is a dominant action, the information value decreases. The intuition is that when there is a default action, increasing the value of this action with respect to any other (or decreasing the value of any other action) will make the agents more willing to keep this default action. In our case, the cost of the most costly error increases, making an individual more willing to acquire information. As a result, information value increases in our game but the information value decreases in the sense of [Yang \[2018\]](#)¹.

¹In [Yang \[2018\]](#), the value of information for an individual player is the payoff's difference between the equilibrium with information acquisition and without, holding constant the strategy of the opponents acquiring information.

Chapter 4

Bayesian Persuasion in Capacity Market Designs

Co-authored with Charlotte Scouflaire

Abstract

In electricity markets, the public authority can design capacity markets (CMs) to bring market outcomes in line with social optimality. By pricing capacity as a separate good from electricity, CMs stabilize and complement the revenues of plants so to generate sufficient security of supply (i.e., having enough capacity available to avoid rationing) at potentially lower cost for end consumers.

In decentralized forward capacity markets, consumers and / or retailers are required to secure capacity and information about likely capacity requirements is a crucial issue. The –welfare optimizing– public authority thus decides whether to make this information binding (*ex ante* requirements as in the PJM capacity market design of 1999) or to base capacity demand on realized market outcomes (*ex post* requirements as in the current French CM design). Adapting a Cournot oligopoly model from [Roy et al. \[2019\]](#), we analyze a capacity market where homogeneous buyers aim to comply with their capacity obligation under uncertainty regarding future realized demand. We thus consider a Cournot oligopoly in which the capacity buyers (mainly load-serving entities) are profit-maximizing agents engaging in strategic behavior. Heterogeneous capacity owners bid their valuation of capacity. As information disclosure reduces uncertainty on the one hand, but may also decrease precautionary capacity buying by load-serving entities on the other, we seek the preferred level of information precision, i.e., *ex ante* or *ex post* disclosure of capacity requirements. Counter-intuitively, the welfare-maximizing level can be lower than full precision of information. The model also highlights possible disagreements between capacity buyers and capacity owners in terms of preferred design, considering that dissemination of public information might affect their surpluses in different ways. In addition, when public consultations are organized, capacity owners are able to individually express their

preferences. This may lead to divergence between the result of majority voting and aggregated profit maximization as not all agents are uniformly affected by different modes of disclosure. The particular case where the public authority chooses majority-voting as a decision rule is investigated in detail. Using German data from 2010, model parameters are set to mimic several plausible capacity market designs. Results suggest that *ex ante* requirements are empirically likely to be favored in the particular setting analyzed.

4.1 Introduction

The features which make electricity so peculiar compared to other goods are the non-storability combined with demand inelasticity. Indeed, no economically viable solution has yet emerged to store electricity on a large scale. It follows that demand and supply require a continuous balance to avoid system failure (black-outs). Unfortunately, this task is impaired by the low predictability of demand. End consumers are rarely informed of the state of the wholesale market and thus do not receive any kind of price or scarcity signal to adjust their consumption to real-time conditions¹. In this framework, the peak demand is binding, and available capacity units play a crucial role in keeping the system up and running. In a free market, this means that those units need full cost recovery - at least in expectations. If not, adequacy² problems appear. Indeed, the Security of Supply (SoS) is supposedly a byproduct of the energy market, but some stakeholders cast doubt on the capacity of an energy-only remuneration to ensure a rate of return high enough to trigger investments. Observers even mention plants being unable to cover their fixed costs through the energy market [Joskow, 2006; Cramton and Stoft, 2006]. Such early retirements could endanger the system stability. In that case, the most mainstream option to make sure enough capacity will be available is to implement a capacity market as a complement to the energy market –other designs exist but are not considered in this chapter.

In reality, consumers have little incentive (or way for that matter) to disclose their willingness to pay for SoS, mainly due to the non-excludability of SoS³. Therefore, they cannot be discriminated against their willingness to pay for reliability. Their theoretical willingness to pay for the SoS is thus estimated by the Value of Lost Load (VoLL)⁴. They are commonly assumed to be willing to avoid black-outs in an equal manner, and retailers

¹See Joskow and Tirole [2006] for a discussion about the causes of end consumers' inelasticity.

²NERC [2017] defines adequacy as "the ability of the electric system to supply the aggregate electrical demand and energy requirements of the end-use customers at all times, taking into account scheduled and reasonably expected unscheduled outages of system elements". We will alternatively use the Security of supply (SoS) and Reliability for similar meaning.

³Imbalance between supply and demand can provoke system failure and rationing independently of individual preferences. Electricity rationing is seen here as the unplanned forced reduction of consumption.

⁴The Value of Lost Load is defined as the average value placed by end consumers on losing power in an average rolling blackout by Cramton et al. [2013]. Besides, it should be noted that the VoLL is challenging to estimate and probably non-linear as well. Security of supply is also valued at the margin by the cost of new capacity. At equilibrium, willingness to pay for additional capacity and cost of new capacity are equal.

are enjoined to get enough capacity to cover their aggregated peak demands on behalf of their portfolios. The underlying reasoning is the following: the level of available capacity has to surpass demand at any point in time to avoid rationing. This implies that a piece of SoS is embedded in each capacity unit.

Thus, capacity markets standardize capacity with respect to the SoS through the notion of "firm capacity"⁵ and then pay units for their insurance value (against black-out). This involves a structural change in remuneration compared to Energy Only Markets (EOM). Instead of solely depending on the volatile hourly energy prices, capacity owners⁶ can additionally benefit from a (yearly⁷) fixed remuneration contingent on the size of their unit. This covers at least part of the fixed costs, reducing the risk associated with the participation in the energy market, simultaneously decreasing the required return on investment. The cost incurred by the contractual obligations such as availability and outage management should remain low because they are already incentivized by the normal functioning of the energy market [Stoft, 2002]. Thus, the capacity remuneration is often seen as a complement to the energy revenues. Without strategic bidding, capacity owners should be willing to recover at least their participation constraint to the energy market (b_i) being $b_i = \max(-\mathbb{E}(\pi_i); 0)$. It is, however, complicated to know whether capacity owners actually bid their competitive price on the capacity market⁸. Even if they do so, the participation constraint highly depends on each owner's expectations, costs, as well as the cost of meeting the contractual requirements. In real-world conditions, the very sensitive nature of the information embedded in each bid causes the order book to remain undisclosed or anonymous. This chapter will thus empirically focus on competitive bidding from the capacity owners, although the theoretical model allows for a wider variety of strategies.

Under capacity remuneration, the demand-side (mainly load-serving entities - LSEs)⁹ benefits from additional SoS and, the supply-side gets extra revenues. This is Pareto-improving as long as the marginal cost of procurement (cost of the marginal unit) is lower than the marginal cost of a black-out. In this sense, capacity markets may make both kinds of actors better off, but the market design affects them in different ways. More specifically, this analysis focuses on decentralized capacity markets.

Where most existing capacity markets centrally procure capacity on behalf of the LSEs (or end consumers) who pay for their own market share –e.g., the United Kingdom (the UK) or the Eastern US systems–, decentralized capacity markets shift this burden directly on the

⁵Firm capacity is the share of the installed capacity that is expected to be available at peak.

⁶Capacity owners are generally, but not limited to, the electricity generators.

⁷Most common current capacity contract length for existing capacity.

⁸Indeed, great attention is granted to market power mitigation in all existing capacity markets: according to Teirilä [2017], current strategies are quite effective.

⁹Load Serving Entities (LSEs) are generally the electricity retailers that supply the end consumers. More formally, according to NERC [2017], an LSE is an entity that "secures energy and Transmission Service (and related Interconnected Operations Services) to serve the electrical demand and energy requirements of its end-use customers".

LSEs – e.g. France or former MISO¹⁰–. As recent designs tend to procure capacity several years ahead, the definition of capacity requirements becomes a key feature of decentralized capacity markets. Again, two types arise: (i) *ex ante* requirements are determined by the system operator¹¹ and announced in advance to obliged parties (LSEs). This leaves no uncertainty for them (former MISO, former PJM), except maybe if their market shares change. In (ii) *ex post* requirements, the obligation depends directly on the realized market conditions (France). Therefore, LSEs do not know the exact amount they should buy at the moment the auction takes place: there is uncertainty not only on their market shares but also on total volume to be acquired. If their level of procurement is not sufficient, they may be penalized by the public authority for having endangered the SoS (i.e. their peak demand has surpassed the amount of their capacity certificates). This leaves a high uncertainty. Table 4.1 presents empirical examples of existing and past capacity market designs with respect to the two features discussed above. For clarifications, Chapter J provides a thorough discussion on design-related information precision and discusses the demand-side specification in the respective capacity markets of the UK and France. This chapter focuses on actors’ preferences with respect to this last design feature: requirement definition.

		Requirement	
		<i>ex ante</i>	<i>ex post</i>
Market Structure	Centralized	UK (2017)	-
	Decentralized	Former MISO (2009), Former PJM (1999)	France (2017)

Table 4.1: Market Structures and Requirements in CRMs

An institutional perspective on market-based CRMs illustrates the key role of uncertainty in CRM designs over time. Along with the SoS objective, CRMs seek to replace scarcity pricing¹² when deemed inefficient. This is why the first CRMs were implemented on a short term basis in the 1990s. In the England and Wales Pool, the payments resulted from an *ex post* settlement based on the energy market clearing (half-hourly)¹³. In the PJM¹⁴, the LSEs were obliged to secure their share of total capacity obligation on a daily basis. Both types of markets (price-based and quantity-based) ended up with the same flaws: high price volatility and confirmed market power expression. All short-term mechanisms were abandoned by the end of 2000s to the profit of extended contract periods

¹⁰MISO (Midcontinent Independent System Operator) has a broad footprint that spreads from Canadian provinces to southern US states such as Mississippi and Louisiana.

¹¹The system operator is the public structure in charge of the operability and the distribution of electricity.

¹²Cramton et al. [2013] explain that scarcity periods are, in fact, market failures and prices during these periods do not represent the meeting of supply and demand.

¹³Refer to Newbery [1998] for further details on the England and Wales Pool

¹⁴PJM (Pennsylvania, New Jersey, Maryland) ran a decentralized capacity market. Refer to Bowring [2008] for further details on the former PJM capacity market.

and forward procurement of capacity (PJM, ISO-NE, Colombia and later the UK). Both changes aimed to reduce the uncertainty on the capacity market outcomes, but also on the level of SoS achieved. The increased predictability of future profits was supposed to improve the investment climate as well. On the demand-side, the shift of decentralised markets to central procurement lowered more the risk borne by the LSEs as the main cost of capacity would be known 3 to 4 years ahead. Only the split of such costs between LSEs would keep a slight uncertainty for being based on their market shares at peak. Until recently, the LSEs did not seem to be given an active role in capacity markets: even decentralized markets such as the former PJM centrally determined the amount of capacity to be procured¹⁵. LSEs' liability was restricted to their market share irrespective of the actual Loss of Load Probability (LoLP)¹⁶ achieved. The situation in MISO (2009-2012) corresponded to a slightly different liability: LSEs did self-estimate their peak demands, but as long as their good faith was not questioned *ex ante*, they did not face penalties even if their forecasts were proven wrong *ex post*¹⁷. In this sense, the choices made in France, with a decentralized market and *ex post* requirement settlement raise questions around the role of the risk borne by LSEs. While *ex ante* requirements have historically dominated implemented designs, the technical and economic improvements of Demand Response (DR) give an opportunity for *ex post* requirements to demonstrate their advantages. Indeed, in a decentralized CRM combined with *ex post* requirements, LSEs have to procure capacity based on their realized peak demand (which implies that no explicit capacity target is set). This is claimed to reduce over procurement [RTE, 2014] but also to foster DR, which is becoming an essential SoS levy. The main difference with the previously described regulations (former PJM and MISO) is that the accuracy of LSEs' prediction matters since they fully benefit from their decisions¹⁸.

The importance of transparency is illustrated by continuous discussions about the level of available information, raising concerns on the best information structure. For instance, the European Federation for Energy Trading (EFET) advocates for more transparency of the French mechanism [EFET, 2016], meaning that a remaining margin for additional information disclosure has been identified. Elseways, the market monitoring of ISO-NE¹⁹ has been advising for reduced transparency under the belief that it increases the market power of the supply-side [Patton et al., 2017]. This chapter focuses on the informational value of peak demand forecasts for capacity procurement in decentralized capacity markets. To the best of our knowledge, it is the first study isolating the effect of requirements in capacity markets with the buyers' side taken as the strategic side. We ask questions about the actors' preferences in terms of design, focusing on the uncertainty aspect of the problem, and, letting the market organization for further research.

¹⁵Each LSE had to procure its share of the centrally forecasted peak demand [Bowring, 2008].

¹⁶The LoLP measures the probability that demand out passes supply.

¹⁷As long as the error ranged within acceptable boundaries specific to the methodology used.

¹⁸When the capacity cost is dealt depending on LSEs market share at peak (and not realized demand at peak), the individual actions of the LSEs are diluted which is more prone to free-riding.

¹⁹ISO-NE: Independent System Operator of New England.

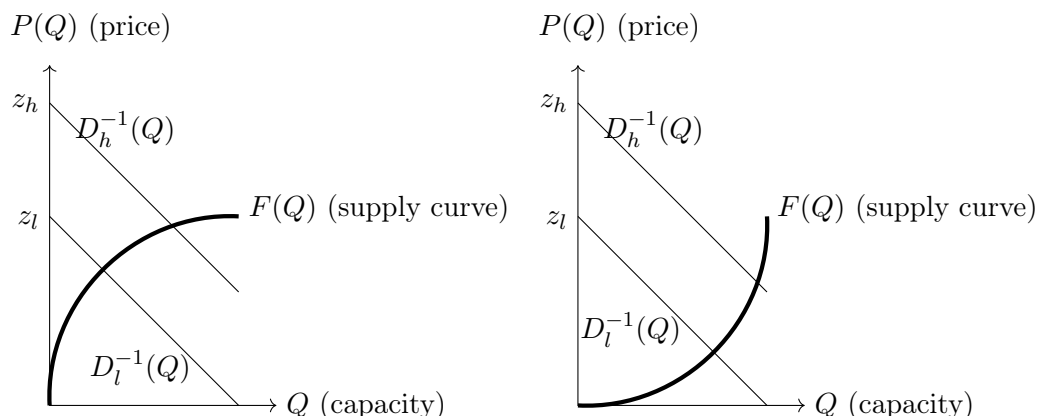


Figure 4.1: Graphical example - Uncertain linear marginal value curve with two possible states h and l (two possible intercepts z_h and z_l), and a concave supply curve (left) or a convex supply curve (right)

The chapter is organized as follow: [Section 4.2](#) establishes a brief literature review. [Section 4.3](#) develops the theoretical model used to analyze the preferences in terms of information precision in capacity markets. We adapt the Cournot oligopoly model of [Roy et al. \[2019\]](#) to a Cournot oligopsony²⁰ under the hypothesis of atomicity on the supply-side, i.e. all capacity owners are small agents, and capacity owners have heterogeneous valuations of their good. On the demand-side, homogeneous LSEs have linear marginal values²¹ (see [Figure 4.1](#)). We assume there are two states of the world, in which the level of the capacity buyers' marginal value differs (see [Figure 4.1](#)). The public authority can reduce the uncertainty about the state of the world, by making the signal binding: with *ex ante* requirements, the capacity buyers are unaffected by the uncertainty around the states of the world. Indeed, the signal indicates to the capacity buyers what is their true marginal value. Precision is the key element of the design: increased precision reduces the uncertainty on the requirements (see [Chapter J](#) for the link between demand uncertainty and capacity requirements). Thus, it impacts the quantities and the price of capacity units traded in the market by affecting the beliefs of the capacity buyers (their posterior). Results show that preferences about information disclosure depend on the shape of the supply and marginal value curves and that they may diverge among market participants. In a context where the public authority does not perfectly know the surpluses, it resorts to a public consultation to aggregate the expressed preferences following its objective function.

The public authority can be mistaken by collecting preferences if she uses the majority-winning criterion as a proxy for the profit-maximizing one. When capacity owners are heterogeneous, the majority-winning criterion can differ from the profit-maximizing one.

²⁰An oligopsony is equivalent to an oligopoly where buyers represent the strategic side.

²¹The marginal value is analogous to the marginal cost for an oligopsony i.e., it is the willingness to pay an additional unity for the good (capacity unit). In oligopsonies, the price is lower than the marginal value for a given quantity, since the buyers can exert market power.

However, [Section 4.4](#) shows that this situation is empirically unlikely to occur. Indeed, the fourth section illustrates the implications through a case study. The model is parametrized according to the former German situation (2010) as well as possible capacity market designs. In this case, disagreement between the majority-winning criterion and the profit-maximizing one is quite unlikely. Indeed, it only appears locally when the shape of the supply curve (all other things being equal) changes such that a lower information precision starts being preferred by capacity owners: it only represents a transition where the preference of the majority changes before the profit-maximizing one. Similarly, extrapolating the case study results to real-life situations indicates that reduced information precision would not be welfare-maximizing in an adequate market. However, the need for new entrants, who bid significantly higher than existing units, might generate such a situation. The last section concludes the analysis.

As simplifications are inherent to modeling, we do not consider some important elements of the design. For example, the timing aspects of a capacity market do not enter in the framework of our static model. For instance, a more complete setting may be needed when considering questions about the degree of forwardness and contract duration. Besides, performance obligation and assessment or even the market eligibility of different technologies are considered to be directly accounted for in the bidding behavior of the capacity owners: the supply-side behavior is only restricted by the hypothesis of continuity. Similarly, all SoS considerations (cost of black-out, VoLL etc.) are implicitly embedded in the model parameters and thus little discussed.

All in all, we use a simple oligopsony model to investigate the design process of CRMs and the informativeness of public consultations for the public authority. We focus on decentralized capacity markets and investigate the actors' preferences with respect to requirement definitions as if it was the only design question left unanswered. Alternative forms of CRMs are disregarded.

4.2 Literature review

4.2.1 Electricity markets: Relevant literature

As discussed in [Section 4.1](#), potential market failures endangering SoS have been identified and thoroughly discussed in the literature for some time. A good synthesis of the discussion can be found in [Cramton et al. \[2013\]](#). While the usual taxonomy does discriminate for the degree of centralization [[Henriot and Glachant, 2013](#); [Meulman and Méray, 2012](#)], the timing of requirement assessment (*ex ante* vs. *ex post*) is often blended in the definition. Indeed, [Woodhouse \[2016\]](#) rightfully argues that decentralized CRMs reduce the risk of over procurement and generate a more flexible framework for new SoS products. In this discussion, it is, however, implicitly assumed that *ex post* requirements are inherent to decentralization. In turn, [Parsons and De Sisternes \[2016\]](#) note that uncertainty about extreme events (which are hard to apprehend since they are rare) leads to

ex ante disagreements between LSEs and generators. Naturally, *ex post* disagreements about the underlying causes of scarcity during these events subsist. Therefore, conflicts emerge about the responsibilities of each actor during rare events. Finally, the question of the optimal level of SoS is left unsolved. However, when designing a CRM, the public authority brings out the debate, and forces market actors to solve this conflict *ex ante*, justifying the existence of the CRM. [Parsons and De Sisternes \[2016\]](#) conclude that a CRM should help reducing the *ex ante* uncertainty. Through this channel, the risk is reduced for both LSEs and generators. The link with *ex ante* requirements is obvious: actors settle their disagreement on the SoS and agree on a given level of capacity. With respect to *ex post* requirements, CRMs leave part of the disagreement unsettled because they only set the liabilities, not firm objectives.

A broad strand of theoretical literature investigates the relationship between the optimal level of SoS and demand characteristics (uncertainty, elasticity). It reveals that full demand coverage is socially inefficient. For instance, [Chao \[1983\]](#) considers an electricity market under supply and demand uncertainties. The stochastic demand leads the utility (the vertically integrated electricity supplier) to adopt a technology mix to minimize its costs. Then, the paper establishes an optimal capacity structure and discusses reliability criteria, giving the best level of SoS from a welfare point of view. In the case of a convex outage cost²², the optimal mix depends naturally on the probability and the expected amplitude of a blackout (demand-side), and the cost and the availability of each technology (supply-side). [Chao and Wilson \[1987\]](#) are interested in how the SoS can be ensured by reducing the end demand during scarcity periods. For this purpose, they set a similar framework, but they add priority service (i.e. the fact to discriminate end consumers, by selling different levels of reliability to the different types of end consumers who self-select with menus). Priority service transmits information about the distribution of the willingness to pay to the utility. Therefore, the optimal level of capacity can be measured directly with the optimal priority charges. They show that reducing the demand of the less willing to pay end consumers in times of scarcity and spot prices are equivalent from an optimality point of view. [Chao et al. \[1988\]](#) look at the market structure impact on surpluses. They note that most of the efficiency gains are realized with few priority service classes (i.e. end consumers are pooled in two or three categories), end consumer surplus and welfare then keeping on increasing with the number of classes.

The role of consumption choices (consumption patterns) in the SoS problem has thus long been identified. The literature also highlights the optimality of respecting demand-side preferences in terms of reliability. However, the proposed solutions to approach optimality have long been little applicable in reality. Only recent technical evolutions (smart meters) open the way for accurate market segmentation based on reliability preferences. In the meanwhile, public authorities often exogenously set a reliability target as a proxy for the optimal level of SoS, which can be further enforced via a CRM. The motivations and consequences of such a choice have been investigated in the literature.

²²Outage cost is the sum of the rationing cost and the foregone consumers' surplus.

In the same strand of the literature, [Joskow and Tirole \[2006, 2007\]](#) set a more realistic framework with price-insensitive and price-sensitive end consumers. Interested in retailers' behaviors, [Joskow and Tirole \[2006\]](#) study the competition between LSEs for end consumers. Assuming competitive markets, they focus on potential suboptimality because of information withholding about the profile of end consumers (due to the presence of several LSEs). With this paper, they open interesting debates on the regulation of the LSEs and the demand of end consumers that are still unsolved. Expanding this analysis in [Joskow and Tirole \[2007\]](#), they study the effect of the price cap and capacity payment together. They argue that setting a price cap is useful to decrease the market power of the supply-side but creates a missing money problem. In line with [Chao and Wilson \[1987\]](#), [Joskow and Tirole \[2007\]](#) find that rationing price-insensitive consumers may be optimal. The authors show that the Ramsay optimum may be attained with capacity payments when all consumers pay for capacity. However, the market power of the supply-side in the electricity market can undermine this result.

Considering two interconnected areas, [Crampes and Salant \[2018\]](#) investigate the social gains from trading between the two regions and further regard the case of capacity credits. They assume that only a reduction in the occurrence probability of blackouts matters in the optimal capacity programs resolution (and that is why some plants are only built to deal with capacity problems). [Crampes and Salant \[2018\]](#) explain that financial incentives for peak supply or installed capacity may generally solve missing money problems.

As highlighted in the aforementioned papers, the elasticity of the demand for electricity plays a key role in determining the optimal level of SoS. The increasing technical and economic feasibility of Demand Response (DR) asserts voluntary rationing as a levy towards a better understanding of end consumers' preferences with respect to SoS and increased demand elasticity. For instance, [Lambin \[2018\]](#) analyses the effects of heterogeneous DR units (ordered with respect to the opportunity cost) on an electricity market with a price cap. DR units ranged below, and above the price cap are found to have distinct effects. Then, an optimal payment structure for DR units in a CRM can be derived, where distinct payments are recommended: DR units below the price cap should receive a more advantageous remuneration.

Finally, the complementarities of the electricity and capacity markets are repeatedly highlighted by different stakeholders either to discuss the need for CRM or even to caution against the risk of complex strategic behavior. This is why scholarship often analyses capacity markets jointly with electricity markets [[Joskow and Tirole, 2007](#); [Teirilä, 2017](#); [Astier and Lambin, 2019](#)]. A step further, [Teirilä \[2017\]](#) specifically focuses on the bidding strategies that can emerge in both markets when capacity owners acknowledge the complementarities between them. The present analysis will, however, consider the capacity market in isolation from the electricity market to favor intuition. Some strategic interactions between the two markets are implicitly accounted for thanks to the reduced assumptions on the bidding behavior of the supply-side.

While CRMs are broadly discussed in the literature, decentralized capacity markets have received limited attention, although the strategic dynamics might significantly differ. The uncertainty on capacity requirements is often discussed, but rarely investigated because most CRMs rely on *ex ante* requirements. In the continuity of this literature, we specifically focus on requirements uncertainty. With the implementation of *ex post* requirements, the risk associated with the peak consumption unpredictability is partially shifted onto the demand-side. In a context where capacity markets may be increasingly decentralized in the near future to further involve the demand-side (and other new products) [Woodhouse, 2016], the question of design-related risk shifting and actors preferences with this respect deserves reflection.

4.2.2 An adaptation of Roy et al. [2019]

The majority-winning procedure used in Roy et al. [2019] is an interesting way to materialize the discussions between market participants and the public authority about the design-related information precision (*ex ante* or *ex post* requirements). Indeed, in the context of implementing a brand new capacity market or redesigning one, having a view on the interests of the different stakeholders is essential. Both groups, LSEs, and generators, are usually consulted in the process and are probably to defend their own interest rather than the welfare-maximizing design. In the electricity sector, the heterogeneity of production technologies makes the discussion on the individual preferences of capacity owners highly relevant. Thus, only looking at the profit-maximizing design may not be sufficient to fully understand the preferences of the capacity owners. On the contrary, retailers can be seen rather homogeneous actors competing on quantity in a (decentralized) capacity market.

To match our interest in the demand-side strategic behavior under uncertainty and actors' preferences in terms of information, we adapt an existing model from Roy et al. [2019]. They are interested in the preferred information disclosure when homogeneous producers competing *à la Cournot* face uncertainty about the state of the world (their constant marginal cost can take two equiprobable values, high or low). The non-linear demand is made of a continuum of heterogeneous consumers with different willingnesses to pay. An omniscient central authority sends a public signal (with value high or low) on this marginal cost with perfect or imperfect precision. Perfect precision means that producers are certain that the marginal cost is the one that the signal indicates. Consumers' and producers' preference about the precision of the signal is characterized by the shape of the demand curve. The notion of majority-winning precision is introduced in this context, and represents the precision preferred by most of the consumers, in contrast to the surplus, which is the traditional criterion. Majority-winning notion enables the authors to show under which conditions there is a disagreement amongst consumers and gives a way to select the best precision when consumers are considered. Consumers can be divided into three groups: (a) the ones who will always consume (because their willingness to pay

is high enough)²³, (b) the ones who may consume, according to the level of production (because their willingness to pay is intermediary), (c) the ones who will never consume (because their willingness to pay is too low). Consumers in group (a) are only concerned by the effect of precision on the mean price of the good (the utility being higher if the price is lower). On the contrary, consumers in group (b) are also affected by the fact that they may *not* consume the good if the level of production is too low. Therefore, a disagreement between consumers may appear if the precision minimizing the mean price reduces the ability to consume for the group (b). On the other side, precision may be detrimental to producers when the competition effect (i.e. the effect of information used by competitors on profit) is too high with respect to the alignment effect (i.e. the effect of knowing better the state of the world on the profit).

In this chapter, we try to understand the consequences of the various designs by investigating how different groups of interest might benefit from one or the other structure of information. Adapting the model from Roy et al. [2019], we explore the preferences of each actor in terms of design in a framework of Cournot competition and uncertainty on the level of demand with varying numbers of LSEs. We generalize the approach of Roy et al. [2019] by considering that the *ex ante* probability of the states of the world can take any value. Considering an oligopsony, our marginal value (the equivalent of the marginal cost) is sloped, enabling the demand-side parametrization to influence the results.

4.3 Model

4.3.1 Description of the model

We adapt the model from Roy et al. [2019] to CRMs, by switching the role of buyers and sellers, buyers becoming the strategic side:

The game is composed of:

- Two States of the World (SoW) $\in \{l, h\}$ affecting the buyers' side, changing the value of: $Z \in \{z_l, z_h\}$, with $z_l < z_h$, and we note μ_0 the common prior belief about the value of $\mathbb{P}(Z = z_h)$;
- n homogeneous buyers, giving the aggregated state-contingent marginal value: $D_{SoW}^{-1}(Q) = z_{SoW} - aQ$;
- Heterogeneous owners, each owner i bidding its capacity unit^a at level b_i , giving the following continuous increasing price function: $F(k) = b_k$;
- A signal about Z , sent by the public authority, observed by all the buyers: $S \in \{s_l, s_h\}$, of precision $\lambda \in [\frac{1}{2}, 1]$. The signal formed by the authority is

²³In Roy et al. [2019], the equilibrium is unique for each level of precision. Thus, the minimal and maximal levels of production are known and allow to define these three groups.

correct with probability λ , i.e.: $\mathbb{P}(Z = z_{SoW} \cap S = s_{SoW}) = \lambda \mathbb{P}(Z = z_{SoW})$.

- We note $\mu_h(S) = \mathbb{P}(Z = z_h|S)$, the posterior belief after receiving a signal.

^aThe size of the unit can be considered as small enough to give a continuous approximation. For instance, [Joskow and Tirole \[2007\]](#) considers the supply-side of plants' building as "a continuum of investment opportunities".

Thus, $\mathbb{P}(z_h \cap s_h) = \lambda\mu_0$ and $\mathbb{P}(z_h \cap s_l) = (1 - \lambda)\mu_0$.

Consequently, the buyers have the following beliefs when the signal is high or low:

$$\mu_h(s_h) = \mathbb{P}(z_h|s_h) = \frac{\lambda\mu_0}{\lambda\mu_0 + (1 - \lambda)(1 - \mu_0)}, \quad \mu_h(s_l) = \mathbb{P}(z_h|s_l) = \frac{(1 - \lambda)\mu_0}{(1 - \lambda)\mu_0 + \lambda(1 - \mu_0)}$$

Thus, the buyers have the following conditional expectations about Z :

$$\begin{aligned} \forall \lambda, \quad \mathbb{E}(Z|s_h; \lambda) &= \mu_h(s_h)z_h + (1 - \mu_h(s_l))z_l \\ &\geq \mathbb{E}(Z|s_l; \lambda) = \mu_h(s_l)z_h + (1 - \mu_h(s_l))z_l \end{aligned}$$

signal preciseness	expected state of the world	requirements
$\lambda = 1$	z_h if s_h z_l if s_l	Fully regulated <i>ex ante</i> requirements
$1 > \lambda > \frac{1}{2}$	$z_h > \mathbb{E}(Z s_h; \lambda) > z_l$ if s_h $z_h > \mathbb{E}(Z s_l; \lambda) > z_l$ if s_l	Partial hedging
$\lambda = \frac{1}{2}$	$\mu_0 z_h + (1 - \mu_0)z_l$ in any case	<i>ex post</i> requirements without hedging

Table 4.2: How requirements are stylized in the model

[Table 4.2](#) interprets the precision of the signal as the type of requirement in place in CRMs. In the case of *ex ante* requirements definition, λ naturally equals one as the forecast is fully informative (binding): if the central authority indicates that the state of the world will be high, then the buyers know they do have to procure accordingly. With *ex post* requirements definition, however, the signal precision is reduced. Even though the central authority expects the state of the world to be high, it can still be wrong with probability $1 - \lambda$.

Preferences

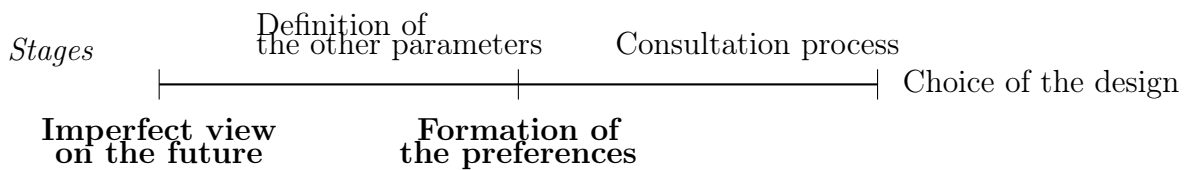
The public authority can stand on the four following criteria to select the precision: either maximize the buyers' surplus, or maximize the owners' profit, or maximize the sum of them, or follow a separate objective function:

- Buyers' surplus: $\max_{\lambda} S(Q) = \max_{\lambda} \int_0^Q D^{-1}(x) - F(Q) \, dx$.
- Owners' surplus: $\max_{\lambda} \Pi(Q) = \max_{\lambda} \int_0^Q F(Q) - F(x) \, dx$.

- Welfare: $\max_{\lambda} W(Q) = \max_{\lambda} (S(Q) + \Pi(Q))$.
- An example of public authorities' objective function: the owners' Majority rule.

The solution concept of the game is a Bayesian Cournot equilibrium: after receiving the signal, all the buyers equalize their expected marginal value with their marginal expenditure (see [Pindyck and Rubinfeld \[2013\]](#), pp. 382-4), giving the expected price and quantities at the equilibrium.

In a context of uncertainty with respect to the level of future peak demand of electricity –that conditions the need for capacity–, the public authority can design the capacity market so that the capacity buyers face the consequences of their own electricity demand uncertainty (*ex post* requirements $\lambda = \frac{1}{2}$) or make the adequacy forecast binding (*ex ante* requirements $\lambda = 1$). For having a limited knowledge on the market, the public authority seeks to know market actors' preferences in terms of requirements (λ) through a public consultation. To do so, she first defines other features of the capacity market such as the level of non compliance penalty (affecting the slope a of the marginal value), and the two possible states of the world (z_l and z_h) so market actors can built their preferences in terms of requirements. Such preferences are then collected through a public consultation so the final design can be chosen:



In this chapter, we focus on the formation of the preferences and the consultation process, the parameters being given. However, [Section 4.4](#) allows to test the influence of the level of the different parameters empirically, and we establish in this section several results about the sensitivity of preferences to the level of parameter a .

4.3.2 Empirical ground and model hypotheses

This section seeks to link the model assumptions to their empirical meaning. The following considerations are not necessary to understand the model resolution and the main results that are developed in the next section. However, they will be useful in understanding the empirical interpretations provided.

In terms of general hypotheses, the good is homogeneous –as in real life–: each unit of certified capacity confers the same amount of SoS. Therefore, all units of capacity are equivalent in the remainder of the chapter. In addition, the market participants are considered as risk-neutral, meaning that they seek to maximize their surpluses on the capacity market. This is a common hypothesis in the literature (see [Scouflaire \[2019\]](#)). If those two assumptions are rather standard in this class of models, the remainder of

the section will further discuss the advantages and limits of the choices made in terms of hypothesis compared to actual market conditions.

The supply-side is parsimoniously described in the model. Two main simplifications are made: the heterogeneous capacity owners are (i) atomic and (ii) unaffected by the signal. They thus bid their valuation of capacity in a continuous supply curve where real-life conditions rather lead to step-wise supply curves in capacity markets due to the lumpiness of power plants. The underlying assumption is that the size of the steps is considered small enough to defend continuous approximation. This approach is discussed by [Chao \[1983\]](#), who mentions that continuity simplifies but also approximates well the discrete analysis of the optimal capacity structure problem. Similarly, [Joskow and Tirole \[2007\]](#); [Crampes and Salant \[2018\]](#) explicitly consider continuous capacity supply curves. In addition, capacity owners' bids can be strategic to the limit that the strategy does not account for the signal produced by the public authority. Although this already embeds a broad range of market power expressions, it obviously excludes others that are likely to be observed in practice (e.g., pivotal buyers). In sum, the two simplifications reduce the complexity of the resolution without losing too much in accuracy.

The buyers' side is portrayed through a linear and price elastic aggregated marginal value of capacity units. In practice, the marginal value of capacity is effectively considered as decreasing [[Cramton and Stoft, 2007](#)] as the probability of shortage decreases with the amount of quantity procured. Empirically, these considerations translate into the construction of centralized demand curves in capacity markets. Although they are often piece-wise linear (kinked)²⁴, assuming a linear marginal value of capacity in our framework should not affect equilibrium conditions since they are locally defined. The comparison with centralized mechanisms is facilitated by the assumption that capacity buyers are homogeneous. They thus all have the same marginal value, which is a share of the aggregated one. When their number is high enough, perfect competition arises, which should yield similar outcomes (optimality) than a centralized mechanism run by a benevolent central authority. Any reference to centralized mechanisms in the interpretation of results will refer to this parallel.

The constant slope of the marginal value curve a is assumed exogenous and unaffected by the realization of the state of the world. a is the decrease (respectively increase) of the marginal utility an LSE derives from an additional (resp. less) unit of capacity. It thus embeds the opportunity cost of not buying the additional capacity unit. This can include the non-compliance penalty as defined by the regulator (explicit penalty), the cost of voluntary rationing (Demand Response (DR)) or any direct or indirect costs an LSE can suffer from not buying enough capacity. The unitary implicit penalty resulting from

²⁴In systems with a centralized mechanism, the aggregated inverse demand curves are piecewise linear and decreasing on \mathbb{R}_+ . See [Chapter J](#) for a description of requirements in the UK and France and <https://www.aeso.ca/assets/Uploads/Working-Group-Demand-Curve-Presentation-Final-07282017.pdf> accessed on April 5th, 2019, for a discussion on centralized demand curves considerations in capacity markets.

a linear marginal value is increasing in quantity²⁵. Reasoning locally, a will represent the slope of least-cost solution between the explicit penalty and DR. When drawing a parallel with [Chao et al. \[1988\]](#), a low a would suggest that the retailer is able to discriminate its end consumers according to their willingness to pay either via priority service classes or real-time pricing. This interpretation relates to the discussion on the advantages of *ex post* requirements in terms of end consumers' involvement.

Besides, price elasticity being decreasing in a , the larger a is, the more constrained the buyers are to be close from a certain target²⁶.

For instance, one can define the under procurement as the fact to buy too few quantities with respect to the certainty case. If the slope a is high, the surplus loss is large. Indeed, for a given interval of quantity, the corresponding price interval will spread out when a is increased. In a word, the higher the a is, the higher the cost of under procurement is. About the over procurement (the fact to buy too many quantities with respect to the certainty case), the reasoning is the same. A high a implies that the extra capacities bought may be very expensive with respect to their true marginal value. However, if under and over procurement costs increase with a , they are not equal. For instance, the over procurement cost is bounded by the marginal value of each extra capacity unit (which is at least non-negative), while the under procurement cost can be infinite in theory (e.g. when a is infinite). Then, the adequate choice of the penalty is driven by the public authority's priority that might either be to limit the cost of the mechanism, or to maximize the amount of capacity procured or more probably a mix of the two, meaning that the a should take intermediate values in reality. This is why discussions on capacity market designs insist on the importance of a well-designed penalty [[Mastropietro et al., 2016](#)].

As discussed in [Section 4.2](#), the issue of optimal SoS emanates from the stochastic nature of the electrical demand. The public authority naturally resorts to forecasting to estimate the optimal level of capacity needed so to meet a given reliability criterion. This is common to all power systems. However, in *ex ante* requirements, this forecast is binding, and its accuracy does not matter for the buyers. On the contrary, it is only indicative in the *ex post* case. In other words, the forecasts cannot be accurate in real life, and the public authority is likely to do her best to send a signal as close to the state of the world as possible. However, it can limit the exposure of the capacity buyers to the forecast errors through the capacity market design. In this sense, the model considers the capacity forecast as a signal with respect to the capacity requirements, not the actual level of capacity needs during the delivery year. The precision of the signal, λ reflects the

²⁵For instance, in France, over procurement might be rewarded through *ex post* trading, meaning that over procurement is valued. In addition, the penalty is piece-wise linear, while a linear marginal value implies that deviations from optimal quantities have a convex cost.

²⁶The demand depending on Z as well, increasing a does not make the buyers necessarily buy more quantities in the model. However, the decision to buy more or fewer quantities will affect more their surplus than before. Thus, a may have a key role in the buyers' preferences in terms of information design.

market design that makes capacity requirements forecasts more or less informative for the capacity buyers. All market participants will form preferences with this respect, knowing all other parameters. In other words, a is known, meaning that the cost of DR is known, and/or the level of the penalty has already been decided. Similarly, the two possible SoW have been announced, and the supply curve is known since it is unaffected by the SoW nor the signal. This last point is the most difficult to verify empirically.

In the model, the uncertainty on future capacity needs is represented via two states of the world affecting the intercept of the marginal value curve. Thus, we assume that the state of the world does change the aggregated demand (and consequently the demand of each buyer) but that it does not affect the willingness to pay for an additional unit²⁷. This formalization is similar to Crampes and Salant [2018] who assume a stochastic demand and a welfare loss function, which is convex in the spread between the realized consumption and the installed capacity²⁸.

Ultimately, the majority-winning rule implies that each unit (or even fraction of unit) can build a separate preference. This assumption does not seem too far from reality in the current market challenges. Each technology participates in the capacity market under different rationales, and the increasingly local nature of SoS needs further shifts the focus on unit-based remuneration. Indeed, competition has led to a decrease in cross-subsidies between the units of a given operator. In other words, profit maximization is increasingly sought at the unit level rather than at the utility or plant level. For instance, the Drax power station in the UK has been abandoning coal-fired generation for profitability – and regulatory – reasons. It has converted from coal firing to biomass firing several (but not all) of its units. It is also proposing to convert up to two of them into gas-fired generating units²⁹. In addition, utilities tend to separate the operation of the different technologies³⁰. While plant operators used to speak on behalf of their generation portfolio, it seems coherent to develop an individual approach where the individual preferences of capacity owners matter and are expressed through public consultations. The objective of the public authority with this respect will be further discussed in Section 4.3.7.

4.3.3 Equilibrium

Let q_j be the capacity demanded by the buyer j and $Q = \sum_{i=1}^n q_i$ the total bought capacity. We consider only the symmetric equilibria. Let $Q^*(\mu_h)$ and $q^*(\mu_h)$ be the equilibrium quantities when the buyers have the belief that $\mathbb{P}(Z = z_h) = \mu_h$. Since we only consider symmetric equilibria: $Q^*(\mu_h) = \sum_{j=1}^n q^*(\mu_h) = nq^*(\mu_h)$. If there is no

²⁷Equivalently, it implies that the state of the world does not impact how buyers can reduce their electricity peak demand.

²⁸The loss function considered in this chapter would be convex as well since we assume a linear marginal value, with a the constant degree of convexity.

²⁹See <http://repower.drax.com/> accessed on September 25th, 2019.

³⁰For instance, German utilities, E.On and RWE have announced their split-up with renewables and grid operations on the one hand and conventional (thermal) generation on the other. See <https://www.group.rwe/en/the-group/history> accessed on September 25th, 2019.

ambiguity, we note for the sake of simplicity $Q^* = Q^*(\mu_h)$ and $q^* = q^*(\mu_h)$. The buyers maximize their surplus:

$$\max_{q_j} S(q_j) = \max_{q_j} \int_0^{q_j} D_j^{-1}(x) - F(Q) dx = \max_{q_j} \int_0^{q_j} \mathbb{E}(Z|s) - na x - F(Q) dx$$

Few computations available in [Chapter D](#) give the following symmetric First Order Condition [sFOC](#):

$$\underbrace{\mathbb{E}(Z|s) - na \times q^*}_{\text{Marginal Value } (V(q^*,n))} = \underbrace{\left(\frac{\partial F(nq^*)}{\partial q} q^* + F(nq^*) \right)}_{\text{Marginal Cost } (C(q^*,n))} \quad (\text{sFOC})$$

With few additional computations (available in [Chapter D](#)), we obtain the next equilibrium condition with respect to μ_h , with $\zeta = z_h - z_l$:

$$\frac{d q^*}{d \mu_h} = \frac{\zeta}{L'(Q^*)}$$

With $L_i(q^*, n) = \frac{\partial F(nq^*)}{\partial q} q^* + F(nq^*) + na \times q^*$ and $L'(Q^*) = Q^* F''(Q^*) + (n+1)F'(Q^*) + na$ ³¹.

Finally, relations between aggregated quantities and precision are written:

$$\frac{d Q^*}{d \mu_h} = \frac{n\zeta}{L'(Q^*)} \quad (\text{Quantity Derivative})$$

To ensure that the previous expressions are well-defined and that higher beliefs lead to buy a higher amount of quantities (i.e. $\frac{d Q^*}{d \mu_h} > 0$), we make the [Assumption A4](#):

Assumption A4. $L'(Q) > 0$.

Note that [Assumption A4](#) is respected for any convex increasing function F .

Based on the contraction approach used in [Gaudet and Salant \[1991\]](#), the existence of a unique equilibrium in a Cournot oligopsony may be established. [Assumption A4](#) ensures both that the buyers' surplus is strictly concave³² in the quantity (then strictly quasi-concave) and the uniqueness of the symmetric equilibrium:

Theorem 12. *Under [Assumption A4](#), there exists a unique symmetric equilibrium to the Cournot oligopsony with n buyers.*

The interested reader can refer to [Chapter E](#) for all the proofs. We begin by stating some properties of the equilibrium:

³¹Note that $Q^* F''(Q^*) + (n+1)F'(Q^*) + na = nq^* F''(nq^*) + (n+1)F'(nq^*) + na = \frac{\partial \mathbb{E}(L_i(q^*, n))}{\partial q}$.

³²Concavity is in fact implied by $F'(Q) > -\frac{Q F''(Q) + na}{2n}$ which is not binding. Note that this is always true under [Assumption A4](#).

Lemma 27. Under [Assumption A4](#), for $s = s_h$ (resp. $s = s_l$), the equilibrium quantities increase (resp. decrease) with the belief μ_h . Furthermore, $\mathbb{E}(Q^*)$ is increasing (resp. decreasing) with respect to λ if $L(Q)$ is concave (resp. $L(Q)$ is convex). Equivalently, $\mathbb{E}(Q^*)$ is increasing (resp. decreasing) if $QF'''(Q) + (n+2)F''(Q) < 0$ (resp. $QF'''(Q) + (n+2)F''(Q) < 0$).

It is a well-known result in persuasion's literature (see [Kamenica and Gentzkow \[2011\]](#)) that if the public authority's objective function is convex (resp. concave) in the belief μ_h , then the public authority should disclose the maximal (minimal) amount of information. The intuition behind this result is that if the objective function is convex in the belief, the fact that the agents have extreme beliefs will maximize this objective function. Thus, the public authority should disclose information to polarize these beliefs. On the contrary, if the objective function is concave, it is better that the agents have intermediate beliefs, so the public authority limits its information disclosure. In our case, if the public authority is willing to maximize the mean quantity, she should disclose information if and only if her utility is convex in μ_h , i.e. if $\frac{\partial^2 Q^*}{\partial \mu_h^2} > 0$. Then, it corresponds to the case where $\left(\frac{dQ^*}{d\mu_h}\right)' > 0 \Leftrightarrow L''(Q^*) < 0$.

The proof of [Lemma 27](#) can be applied to any other function of μ_h : price, surplus, profit and welfare. It means that for any criterion, we just need to show that this criterion is either convex or concave in μ_h to determine the optimal information disclosure according to this criterion.

A critical feature of [Lemma 27](#) is that the parameter a does not influence the impact of information on quantities. It does affect the quantities at the equilibrium, but because of the linearity of the marginal value, any change in the demand does not change how average quantities evolve with information.

Now, we can study how precision influences the mean price. Using [Quantity Derivative](#):

$$\frac{\partial P^*(\mu_h)}{\partial \mu_h} = \frac{\partial F(Q^*(\mu_h))}{\partial Q^*(\mu_h)} \frac{dQ^*(\mu_h)}{d\mu_h} = F'(Q^*) \times \frac{n\zeta}{L'(Q^*)} \quad (\text{Price Derivative})$$

As for the quantities, the price is increasing in information if the partial derivative of the price with respect to the belief μ_h is increasing. Then, $\frac{\partial \mathbb{E}(P)}{\partial \lambda} > 0$ if $\frac{F'(Q)}{L'(Q)}$ is increasing.

To characterize the influence of information on prices, it is useful to note $\eta(Q, n) = \frac{QF'(Q)}{n} = \frac{F(Q)}{ne(Q)}$. η is the price divided by the product of the number of buyers and the elasticity of the price function ($e(Q) = \frac{\Delta Q}{\Delta F(Q)} \times \frac{F(Q)}{Q} = \frac{F(Q)}{QF'(Q)}$). At the equilibrium, η is simply the markdown (the ability of the oligopsonists to obtain a price below the competitive one). In other words, η represents the degree of market power of the competitors. Its shape (with respect to the supply function) around the equilibrium affects the preferences of both kinds of agents.

Aditionnaly, we note the first and second derivative:

$$\eta'(Q) = QF''(Q) + F'(Q) = L'(Q) - nF'(Q) - an$$

$$\eta''(Q) = QF'''(Q) + 2F''(Q) = L''(Q) - nF''(Q)$$

We state the results with expressions containing L and η . L allows us to derive quite direct results and make links with the paper of Roy et al. [2019]. However, η is helpful to disentangle how the equilibrium and the preferences in terms of information disclosure are affected by market power on the one hand, and the implicit penalty represented by the parameter a on the other hand. To complete the analysis, we provide expressions with respect to the parameter a .

We can now state express the precision effect on the mean price at the equilibrium:

Lemma 28. $\mathbb{E}(P^*)$ is increasing (resp. decreasing) with respect to λ if the price is convex in the belief μ_h , i.e. if $\frac{F''(Q)}{F'(Q)} > \frac{L''(Q)}{L'(Q)}$ (resp. $\frac{F''(Q)}{F'(Q)} < \frac{L''(Q)}{L'(Q)}$).

$\mathbb{E}(P)$ is increasing (resp. decreasing) if:

$$\eta''(Q) < (>) \frac{F''(Q)}{F'(Q)} (\eta'(Q) + an) \equiv \rho$$

The parameter a impacts the result. When a is high, the situation becomes close from perfect competition (so the markdown has less weight in sFOC), and only the convexity of the price function matters in this case (see below Proposition 18).

Corollary 2. $\mathbb{E}(P^*)$ is increasing (resp. decreasing) when:

$$\begin{cases} \text{if } F'' > 0, & a > (<) \frac{F'(Q)\eta''(Q)}{nF''(Q)} - \frac{\eta'(Q)}{n} \\ \text{if } F'' < 0, & a < (>) \frac{F'(Q)\eta''(Q)}{nF''(Q)} - \frac{\eta'(Q)}{n} \end{cases}$$

When the supply curve F is convex, increasing (decreasing) parameter a increases (decreases) unambiguously the span (of functions F) where the mean price increases with information. Conversely, when the supply curve F is concave, increasing (decreasing) parameter a decreases (increases) unambiguously the span where the mean price increases with information.

In other words, it is more likely that the mean price increases (decreases) with the (explicit or implicit) penalty if F is convex (concave).

For capacity owners, higher expected prices imply higher individual profits and then higher aggregated profit, as we state in Lemma 29:

Lemma 29. If $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$, then individual profits are non decreasing and aggregated profit increases with the precision λ of the signal.

This result drives the discussion about the preferences –the profit-maximizing precision and the majority-winning precision (Sections 4.3.5 and 4.3.7). The proof enlightens about the conditions under which a consensus will be established amongst capacity owners.

Now, we have all the tools to derive the preferences of the actors in terms of information design. We start by using the traditional criteria and end with the majority-winning precision.

4.3.4 Buyers' surplus maximization

The theoretical monopsony case helps to emphasize the existence of detrimental effects of competition on the preference for information.

4.3.4.1 Monopsony

For any total demanded quantity Q , the buyers' surplus is:

$$S(Q) = \int_0^Q D^{-1}(x) - P(Q)dx = \int_0^Q \mathbb{E}(Z|S) - ax - F(Q)dx$$

Few computations give that the derivative with respect to μ_h when there is just one buyer is:

$$\begin{aligned} \frac{\partial S(Q^*(\mu_h))}{\partial \mu_h} &= \frac{d Q^*(\mu_h)}{d \mu_h} \underbrace{(\mu_h \zeta + z_l - a Q^*(\mu_h) - F(Q^*(\mu_h)) - F'(Q^*(\mu_h)) Q^*(\mu_h))}_{=0 \text{ if sFOC}} + Q^*(\mu_h) \zeta \\ \Rightarrow \frac{\partial S(Q^*(\mu_h))}{\partial \mu_h} &= Q_h^*(\mu_h) \zeta \quad \text{(Monopsony's Surplus Derivative)} \end{aligned}$$

Then, $\frac{\partial^2 S(Q^*(\mu_h))}{\partial \mu_h^2} = \frac{n \zeta^2}{L'(Q_h^*(\mu_h))} > 0$ and the following result is straightforward:

Proposition 13. *With a monopsony, i.e. $n = 1$, the buyer's surplus increases with information precision (increases in λ). The monopsony shall always prefer $\lambda = 1$.*

As expected, when the market is unified (i.e. one buyer strategically chooses how to bid with respect to its own marginal value), the buyer's side sees its surplus increase with precision. In this case, information can not hurt: in the absence of competition, additional information always makes profit surplus more effective. This result shows that however the markdown evolves, the monopsony will always be able to use its market power more efficiently with additional information. Nevertheless, this result vanishes for several buyers: in this case, more information can dramatically alter the cost of buying capacity units.

4.3.4.2 Oligopsony

The presence of other buyers reduces the benefits of increased signal precision on the surplus. It also generates an ambiguity on the sign of the derivative of the surplus with

respect to the precision. The **Monopsony's Surplus Derivative** simplifies thanks to **sFOC** for one buyer (for each signal). However, with several buyers ($n > 1$): $\mu_h \zeta + z_l - aQ^*(\mu_h) - F(Q^*(\mu_h)) - F'(Q^*(\mu_h))Q^*(\mu_h) < 0$. By dividing the total surplus by n , and by writing the surplus as a function of individual quantities ($q^* = q^*(\mu_h) = \frac{Q^*(\mu_h)}{n}$): the expected surplus for any buyer j at the equilibrium is:

$$S_j(q^*) = q^* \left(\mu_h \zeta + z_l - \frac{1}{2} a n q^* - F(nq^*) \right)$$

By differentiating with respect to μ_h :

$$\begin{aligned} \frac{\partial S_j(q^*)}{\partial \mu_h} &= \frac{d q^*}{d \mu_h} (\mu_h \zeta + z_l - a n q^* - F(nq^*) - n q^* F'(nq^*)) + q^* \zeta \\ &= - \frac{\partial q^*}{\partial \mu_h} ((n-1)q^* F'(nq^*)) + q^* \zeta \\ &= q^* \zeta \left(1 - (n-1) \frac{F'(nq^*)}{L'(nq^*)} \right) \end{aligned}$$

Proposition 14. *If $n > 1$, $\mathbb{E}(S(Q^*))$ is increasing (resp. decreasing) with respect to λ if the surplus is convex in the belief μ_h , i.e. if $(1 - \frac{(n-1)F'(Q)}{L'(Q)})Q$ is increasing (resp. decreasing). Then, in the first case, the surplus maximizer precision is $\lambda_S = 1$, and in the second case $\lambda_S = \frac{1}{2}$. As well, the surplus is increasing (resp. decreasing) with information precision if:*

$$\eta''(Q) > (<) \rho - \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)QF'(Q)} \equiv \rho$$

Corollary 3. *When (i) $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) \leq 0$ the surplus is always increasing with information precision.*

Now, note:

$$a_{1s} = - \frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

and

$$a_{2s} = \frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

When (ii) $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) > 0$, the surplus is increasing (resp. decreasing) with information precision if:

$$\left\{ \begin{array}{l} \text{in case where } a_{2s} < 0 \\ \text{in case where } a_{2s} > 0 \text{ and } a_{1s} < 0, \text{ if } a > (<) a_{2s} \\ \text{in case where } a_{1s} > 0, \text{ if } a < a_{1s} \text{ or } a > a_{2s}, (\text{ if } a_{1s} < a < a_{2s}) \end{array} \right.$$

Increasing parameter a increases unambiguously the span (of η'') where the surplus is increasing with information when $n = 2$ and $F''(Q) > 0$, or when $n > 3$ and $F''(Q) < 0$. When $n = 3$, increasing a increases the preference for information.

In [Proposition 14](#), η is linked to the price function and the penalty represented by the parameter a . If η' increases faster than the price derivative, and that a is high, information is preferred to enjoy η and to avoid the (implicit or explicit) penalty. On the contrary, if buying capacities at the high state is expensive, and that the penalty is low, being aligned with the state of the world matters less. Naturally, increasing the penalty (i.e. fixing a higher a in the model) can reverse the result.

According to [Corollary 3](#), whatever the impact of precision on mean prices, an increase in the implicit penalty a does not always lead to an increased preference for precision. Naturally, [Corollary 3](#) shows that an infinite penalty would make the buyers willing to have the maximal precision, whatever the supply curve is ($a > a_{2s}$ is a sufficient condition). This corresponds to the inelastic marginal value case, where capacity buyers have to procure the required amount. On the contrary, when $a_{1s} > 0$ if a is between a_{1s} and a_{2s} , an increase of a may lead to a preference for no information. To form an intuition on this case, take the case of perfect competition with $\mu_0 = \frac{1}{2}$. If the slope is horizontal ($a = 0$), the buyers prefer full information. Indeed, without information, the gain in the high state (price is lowered) is equal to the loss in the low state. However, there is a cost of under procurement (because valuable capacities in the high states are not bought) that is never compensated. If the slope is increased, now, the price is not equal to $\mathbb{E}(Z)$ anymore. Especially, if the supply curve is increasingly convex, the expected price may grow with information in a dramatic way, leading to less preference for information.

Note that $\underline{\rho}$ will always be lower than ρ when $L'(Q) > 0$, $\forall n \geq 2$ ³³. Thus, a situation where buyers prefer full information while the expected price increases with information can emerge (when $\underline{\rho} < \eta'' < \rho$). By being better aligned with the states of the world, buyers gain enough surplus compared to the increase of the mean price.

4.3.5 Profit maximization

After studying the buyers' side, we look at the usual preferences criterion for owners: the profit maximization. Let λ_P be the signal precision that maximizes the owners' surplus $\mathbb{E}(\Pi(Q)|\lambda)$. When the price increases, every (price-taker) owner prefers the maximal precision as stated by [Lemma 29](#). It then maximizes the profit for all of them individually and it is natural to expect that $\lambda_P = 1$ at the aggregate level as well. In addition, the profit may also increase if the mean price decreases but the quantities purchased increase. Then, we need to characterize the profit maximization precision. The profit is expressed

³³However, for the resolution of the model, [Assumption A4](#) does not need to be verified for all $n \geq 2$, but only for the given level of competition, i.e. the number of buyers n . If $L'(Q, n = 2) < 0$, the second term in $\underline{\rho}$ may be negative for $n > 3$. If F is convex, it can not be negative, so $\underline{\rho} < \rho$. On the contrary, if F is concave, there are cases where $\underline{\rho} > \rho$. Then, buyers may prefer no information while prices are maximal with no information. It might correspond to a situation where η decreases and where buying in the high state is too costly with respect to the marginal value.

as follow (with Q the aggregate quantities sold to the buyers):

$$\Pi(Q) = \int_0^Q F(Q) - F(x)dx$$

The derivative of the expected profit can be written³⁴:

$$\frac{\partial \mathbb{E}(\Pi(Q^*(\mu_h)))}{\partial \mu_h} = \frac{n\zeta Q^*(\mu_h)F'(Q^*(\mu_h))}{L'(Q^*(\mu_h))}$$

We now characterize the necessary condition to an increasing profit with respect to the precision:

Lemma 30. $\mathbb{E}(\Pi(Q)|\lambda)$ is increasing (resp. decreasing) in λ if the profit is convex in the belief μ_h , i.e. if $\frac{QF'(Q)}{L'(Q)}$ is an increasing (resp. decreasing) function. Then in the first case, the profit maximizing precision is $\lambda_P = 1$, and in the second case $\lambda_P = \frac{1}{2}$. As well, the profit is increasing (resp. decreasing) with information precision if:

$$\eta''(Q) < (>) \rho(Q) + \underbrace{\frac{L'(Q)}{Q}}_{>0} \equiv \bar{\rho}$$

Note also that the sufficient condition in [Lemma 27](#) to observe an increase in quantities with respect to precision is sufficient as well to observe an increase in profit. Indeed, few computations show that the previous equation is equivalent to:

$$\eta''(Q) < (>) - nF''(Q) + \underbrace{\frac{\eta'(Q)L'(Q)}{QF'(Q)}}_{>0}$$

We also confirm the result of [Lemma 29](#): higher expected price unambiguously increases the aggregated profit. However, it is not a necessary condition.

Similarly as the buyers's point of view, owners do not systematically increase their preference for information with the slope of the marginal value. However increasing the penalty (i.e. increasing a) favors preference for information for the owners for all increasing power function:

Corollary 4. Note $a_P = \frac{QF'(Q)L''(Q)}{n\eta'(Q)} - \frac{\eta'(Q)}{n} - F'(Q)$. The profit is increasing (resp. decreasing) with information precision if:

$$\begin{cases} \text{In case where } \eta'(Q) > 0, \text{ if } a > (<) a_P \\ \text{In case where } \eta'(Q) < 0, \text{ if } a < (>) a_P \end{cases}$$

³⁴Cf. proof of [Lemma 30](#).

When the supply curve F is convex, increasing parameter a increases unambiguously the span (of η'') where the owners prefer information. Otherwise, increasing parameter a decreases the span (of η'') where the owners prefer information if $\eta'(Q) = QF''(Q) + F'(Q) < 0$ ³⁵.

It seems straightforward since any increase in the penalty decreases the relative weight of the markdown in the decision of the buyers. However, we show with [Corollary 2](#) that the mean price may decrease when the penalty increases for a concave price function. Moreover, as we see below in [Section 4.3.9](#), perfect competition does not imply that owners prefer precision. Thus, it should be noticed that even if the mean price may decrease, the better alignment of the demand with the state of the world (implied by information and by the higher penalty) will increase the aggregated profit for high a in most of the cases.

Now, after studying the preferences from separated buyers and owners sides, we turn on the characterization of the preferred precision from a welfare point of view.

4.3.6 Economic welfare

The economic welfare is the sum of the buyers' surplus and the owners' profit:

$$\mathbb{E}(W) = \int_0^Q D^{-1}(x) - F(x)dx = \int_0^Q \mathbb{E}(Z|S) - ax - F(x)dx$$

The derivative with respect to μ_h can be found:

$$\frac{\partial \mathbb{E}(W(Q^*(\mu_h)))}{\partial \mu_h} = n\zeta Q^*(\mu_h) \left(1 + \frac{F'(Q^*(\mu_h))}{L'(Q^*(\mu_h))} \right)$$

Consequently, the economic welfare is increasing in the precision if $x \mapsto x \left(1 + \frac{F'(x)}{L'(x)} \right)$ is an increasing function.

Proposition 15. *Whenever $n \geq 1$, if $Q \left(1 + \frac{F'(Q)}{L'(Q)} \right)$ is increasing (resp. decreasing), then the welfare is increasing (resp. decreasing) with information precision. As well, the welfare is increasing (resp. decreasing) with information precision if:*

$$\eta''(Q) < (>) \rho(Q) + \underbrace{\frac{L'(Q)(L'(Q) + F'(Q))}{QF'(Q)}}_{>0} \equiv \bar{\rho}$$

[Lemma 28](#) and [Proposition 15](#) imply that if the mean price increases with the precision, the welfare increases as well. By [Lemma 29](#), when the mean price increases with information, the profit increases as well. Due to the non-strategic role of the supply-side, the welfare

³⁵This condition is not verified for any increasing power functions. Note that $QF''(Q) + F'(Q) = 0$ for the function $F(Q) = \log(Q)$.

becomes naturally higher as well. Even if the mean price decreases, a better alignment of the demand with the state of the world may also increase the welfare.

Besides, information also can reduce the welfare. In this case, profit is also reduced with information ($\frac{\partial W}{\partial \lambda} < 0 \Rightarrow \frac{\partial \Pi}{\partial \lambda} < 0$). What buyers earn (thanks to the average increase of markdown) does not compensate for the losses of the owners' side.

Concerning the parameter a , similar qualitative results as in [Corollary 4](#) appear. Indeed, an increase of a may result in preference for information from a welfare point of view:

Corollary 5. *When (i) $\eta'(Q)^2 + 4QF'(Q)L''(Q) \leq 0$, the welfare is always increasing with information precision*

Now, note

$$a_{1w} = -\frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

and

$$a_{2w} = \frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

When (ii) $\eta'(Q)^2 + 4QF'(Q)L''(Q) > 0$ the welfare is increasing (resp. decreasing) if:

$$\begin{cases} \text{in case where } a_{2w} < 0 \\ \text{in case where } a_{2w} > 0 \text{ and } a_{1w} < 0, \text{ if } a > (<)a_{2w} \\ \text{in case where } a_{1w} > 0, \text{ if } a > a_{2w} \text{ or } a < a_{1w}, (\text{ if } a_{1w} < a < a_{2w}) \end{cases}$$

When the supply curve F is convex, increasing parameter a increases unambiguously the span (of η'') where the welfare increases with information. Otherwise, increasing parameter a decreases the span (of η'') where the welfare increases with information if $\eta'(Q) = QF''(Q) + F'(Q) < 0$.

We see that if a is set high enough, the welfare criterion will favor information precision. Generally, a preference for a low level of information from a welfare perspective is a way to limit the (buyers') market power. Indeed, if the supply curve is (convex and) increasing enough, we can see that precision does not affect much the level of quantity bought, but that the price can be moved dramatically. In this case, limiting market power is done via limiting information precision. Naturally, it implies that taking the risk of not being aligned with the state of the world and increasing the probability of rationing if the state of the world is high. However, according to [Corollary 5](#), resorting to a lower degree of information might not be necessary if the penalty is set high enough. ³⁶

³⁶If the parameter a remains low despite a high explicit penalty, resorting to DR might be cheaper than paying the penalty. In this situation, the CM might be able to unleash significant amounts of untapped DR. This explanation directly relates to the discussion about the advantages to make bear (to create) the risk on buyers by setting *ex post* requirements.

Finally, we can order the different thresholds which shape the preferences considering the different surpluses:

$$\underline{\rho} < \rho < \bar{\rho} < \bar{\bar{\rho}}$$

Note that there is an agreement between buyers and owners for maximal precision in the interval $(\underline{\rho}, \bar{\rho})$.

Nevertheless, we consider heterogeneous capacity owners in this model (as it is the case in real-life). Therefore, preferences exhibited by the aggregate profit criterion may not reflect the preferences of *all* the individual owners³⁷. In the next section, we determine under which conditions such a disagreement emerges and propose an alternative to the profit maximization criterion.

4.3.7 Public authority's objective functions

In the economic literature, the public authority is often described as seeking welfare maximization. This approach is nuanced by institutional economists who argue that a wide range of factors such as history and culture affects the objective function of the public authority, which can then differ from welfare maximization. In the context of capacity markets, [Newbery \[2016\]](#) warns against a tendency of the central authority to over procure capacity, as a way to avoid the political consequences of a black-out. This is one example among many that welfare maximization might not be its only objective. In this sense, when a public authority is willing to create a capacity market, it has to define the aims of this capacity market, accounting for the characteristics of the country, its historical background, as well as its own view about the future.

These various aims can be summarized in an objective function of the public authority. Here, we present simplified examples of objective functions:

- Quota rule: it is the decision process based on the approval of a given proportion (50%, 66%, 90%... for example) of voters. For instance, [Nitzan and Paroush \[1984\]](#) determine the optimal number of voters needed to approve a proposition for each situation. In their framework, the voters are homogeneous and have a given probability to vote for the incorrect decision and there is a bias in favor of statu quo. In our case, the capacity owners are heterogeneous and know their favorite precision. The statu quo may be to give perfect information. Then, the public authority may choose the minimal necessary proportion of capacity owners choosing the same $\lambda_C \neq 1$ to modify the design. For instance, a stringent quota rule could require a share close to unanimity to choose *ex post* requirements as a design. This would give greater weight to peak generators which might be more concerned about their own survival than aggregate profit maximization if the capacity price hardly covers their

³⁷The assumption of homogeneity of the buyers' side makes that no disagreement can emerge among buyers. However, it seems that differences in the buyers' side matter less than differences between capacity owners, due to the structural heterogeneity of the generators that rely on different technologies.

profitability gap. We know that the mean price can be decreasing in information precision. Naturally, in this case, profit-maximizing owners would prefer *ex post* requirements. Nevertheless, *ex ante* requirements enable more owners to sell and generates more certainty (since the level of required quantities is known *ex ante*). These two elements may result in a higher level of SoS. Paradoxically, the peak generators' profits may still be larger with *ex ante* requirements (see below the frame *majority-winning criterion*). These considerations may lead the public authority to overweight the peak generators' (dis-)agreement with *ex ante* requirements in the decision process.

- **Technologies:** the objective of the public authority might be biased towards given technologies. This is a recurrent criticism made to CMs. In this case, the public authority might only account for the preferences of the subset of actors operating such technology(ies). For instance, some authors argue that capacity markets should favor flexible technologies [Buck et al., 2015]. Indeed, those are direly needed to meet the SoS objective on a shorter timescale. Similarly, the capacity market could be designed with the side objective to meet the emission targets. The public authority could then favor green technologies.
- **Maximize Quantities:** this objective is coherent with a public authority that would fear the political consequences of a black-out more than the welfare consequences of such a choice. Alternatively, it is also a way to favor capacity owners.
- **Minimize Prices:** this objective function would be in line with greater importance of the retailers' surplus in the public authority's considerations. Limiting the cost of the measure is also a way to protect end consumers.

Among those examples, some objective functions clearly require further knowledge about individual actors' preferences. The most common tool to uncover those is public consultations. The preferences expressed during the consultation can then be aggregated in different ways to match the public authority's objective.

As an illustration, we study a particular case of the quota rule: the majority rule (i.e. the proportion of voters needed to change the design should be at least 50%). However, we discuss as well the implications of a more binding quota rule (i.e. with a quota higher than 50%). We make the analysis for the special case where $\mu_0 = \frac{1}{2}$. We consider in the rest of the chapter that each capacity owner has the same weight in the selection process.

Majority-winning criterion

In a discrete world, the majority-winning precision λ_C can be defined as the precision of public information that maximizes the number of capacity owners satisfied with respect to their bid. By contrast to the previous condition of maximizing the owners'

surplus, this equates satisfying the greater number of them. The majority-winning procedure gives new insights about the preference of capacity owners with respect to the market design.

In the simplified framework, where each owner bids one unit, each owner has the same weight in the preference of the majority. As stated below, when the mean price increases with information (so the profit increases for each owner by [Lemma 29](#)), they all agree, and the majority-winning precision will be 1.

Though, if the mean price decreases with information precision (i.e. $\frac{F'(Q)}{L'(Q)}$ decreases or $\eta''(Q) > \rho(Q)$), owners may disagree about the optimal precision. In this case, the majority-winning precision depends on the distribution of the bids.

When $\mu_0 = \frac{1}{2}$ consider $\lambda = 1$ and $\lambda = \frac{1}{2}$ when $\eta''(Q) > \rho(Q)$. First, recall that:

$$F(Q^*(0)) < F(Q^*(\mu_0)) < F(Q^*(1))$$

Let $\hat{b} \in [F(Q^*(0)); F(Q^*(\mu_0))]$ be the bid for which a owner is indifferent between the two information structures:

$$\underbrace{F(Q^*(\mu_0)) - \hat{b}}_{\text{Individual profit when } \lambda = \frac{1}{2}} = \underbrace{\mu_0(F(Q^*(1)) - \hat{b})}_{\text{Individual profit when } \lambda = 1} \Rightarrow \hat{b} = \frac{F(Q^*(\mu_0)) - \mu_0 F(Q^*(1))}{1 - \mu_0}$$

On the one hand, owners (*i*) with a bid below than \hat{b} will prefer the lowest precision since their expected profit is higher when $\lambda = \frac{1}{2}$. Indeed, as mentioned in the proof of [Lemma 29](#), the ones with a bid below $F(Q^*(0))$ always sell in both states of the world. Thus, they want to maximize the mean price. The ones whose the bid is higher than $F(Q^*(0))$ will sell, by definition, only if the price is higher than $F(Q^*(0))$. Their bid is such that:

$$F(Q^*(\mu_0)) - \hat{b} > \mu_0(F(Q^*(1)) - \hat{b})$$

On the other hand, owners (*ii*) bidding higher than \hat{b} (but below $F(Q^*(1))$) are such that their bids respect:

$$F(Q^*(\mu_0)) - \hat{b} < \mu_0(F(Q^*(1)) - \hat{b})$$

Thus, even if they do not sell their capacities in both states of the world, these owners prefer the highest precision.

Note that if $\mu_0 \geq \frac{1}{2}$, the owners prefer either $\lambda = \frac{1}{2}$, either $\lambda = 1$. The reason is that $\lambda = 1$ maximizes: $\mathbb{P}(s = S_h) \times (F(\mu_h(s_h)) - b)$, since $\mathbb{P}(s = S_h)$ increases with λ in this case ($\frac{\partial \mathbb{P}(s=S_h)}{\partial \lambda} = 2\mu_0 - 1 > 0$).

In our context, Q is the number of owners that are able to sell their capacity (bid under the clearing price). Thus, if we define $m(Q)$ as the difference between the size

of these two groups of owners (*i*) and (*ii*), we get the λ_C that corresponds to the majority rule, with respect to the positivity of L . Let:

$$m(Q) = \underbrace{\int_{\hat{b}}^{F(Q^*(1))} F^{-1'}(b_i) db_i}_{\text{Preference for } \lambda=1} - \underbrace{\int_0^{\hat{b}} F^{-1'}(b_i) db_i}_{\text{Preference for } \lambda = \frac{1}{2}} = Q^*(1) - 2F^{-1}(\hat{b})$$

Here, if $m(Q) > 0$ (resp. $m(Q) < 0$), the preferred precision is 1 (resp. $\frac{1}{2}$).

This result can be extrapolated to any value of $\mu_0 \geq \frac{1}{2}$. However, if $\mu_0 < \frac{1}{2}$, the probability of getting a high signal decreases when the precision increases. It means that a part of the owners who prefer 1 to $\frac{1}{2}$ will prefer another λ . As well for a part of the owners who prefer $\frac{1}{2}$ to 1 (see [Chapter E](#)).

We can now state the next result:

Lemma 31. *When $\mu_0 = \frac{1}{2}$, whenever $n \geq 1$, if (1) $\frac{F'(Q)}{L'(Q)}$ increases, $\lambda_C = 1$. If (2) $\frac{F'(Q)}{L'(Q)}$ decreases, $\lambda_C = 1$ when (2i) $m(Q) > 0$ and $\lambda_C = \frac{1}{2}$ when (2ii) $m(Q) < 0$.*

Owners' preferences can be summed up in the next proposition derived directly from [Lemma 30](#) and from [Lemma 31](#) since $\left(\frac{F'(Q)}{L'(Q)}\right)' < \left(\frac{QF'(Q)}{L'(Q)}\right)'$:

Proposition 16. *When $\mu_0 = \frac{1}{2}$, whenever $n \geq 1$, if (1) $\frac{F'(Q)}{L'(Q)}$ increases ($\eta''(Q) < \rho(Q)$), $\lambda_C = \lambda_P = 1$. If (2) $\frac{F'(Q)}{L'(Q)}$ decreases but $\frac{QF'(Q)}{L'(Q)}$ increases ($\rho(Q) < \eta''(Q) < \bar{\rho}(Q)$), $\lambda_C = \lambda_P = 1$ if $L(Q) > 0$ and $\lambda_P = 1 > \lambda_C = \frac{1}{2}$ if $L(Q) < 0$. If (3) $\frac{QF'(Q)}{L'(Q)}$ decreases ($\bar{\rho}(Q) < \eta''(Q)$), $\lambda_C = 1 > \lambda_P = \frac{1}{2}$ if $m(Q) > 0$ and $\lambda_P = \lambda_C = \frac{1}{2}$ if $m(Q) < 0$.*

In other words, if the price increases with information precision, all the owners will prefer full information: both the profit-maximizing and the majority-winning criteria are require the maximal precision. When the price decreases with information precision ($\frac{F'(Q)}{L'(Q)}$ decreases), but the profit is nonetheless increasing in λ then again, the majority and profit maximization criteria agree towards full information if $m(Q) > 0$. They would however disagree if $m(Q) < 0$ ($\lambda_P = 1 > \lambda_C = \frac{1}{2}$). When the profit decreases with information precision, i.e. η'' is relatively high enough, the lowest precision will always maximize the profit ($\lambda_P = \frac{1}{2}$) while the majority-winning criterion will depend on the relative size of the owner groups preferring each level of information: $\lambda_C = 1$ if $m(Q) > 0$ or $\lambda_C = \frac{1}{2}$ if $m(Q) < 0$.

4.3.8 Model results

As highlighted by the previous sections, preferences in terms of information precision depend very much on the curvature of the supply curve. Summarizing [Lemmas 27](#) and [28](#),

there are six distinct cases concerning the impact of information on equilibrium prices and quantities.

Proposition 17. *We note six different cases such that:*

If (a) $F'' > 0$	If (b) $F'' < 0$
(a1) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$.	(b1i) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$.
(a2i) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$.	(b1ii) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$.
(a2ii) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$.	(b2) $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$.

η''	$-nF''$	ρ	
if $F'' > 0$	$\partial Q > 0$ $\partial P > 0$	$\partial Q < 0$ $\partial P > 0$	$\partial Q < 0$ $\partial P < 0$
	(a1)	(a2i)	(a2ii)
η''	ρ	$-nF''$	
if $F'' < 0$	$\partial Q > 0$ $\partial P > 0$	$\partial Q > 0$ $\partial P < 0$	$\partial Q < 0$ $\partial P < 0$
	(b1i)	(b1ii)	(b2)

Output Derivatives w.r.t. λ

We make the following statements:

If (a) $F'' > 0$, then $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0 \Rightarrow \frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0 \Rightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$.

If (b) $F'' < 0$, then $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0 \Rightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0 \Rightarrow \frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$.

Unambiguous satisfaction of owners with precision (a1, b1i)

When the mean price and the mean quantities are both increasing with precision (a1, b1i), the profit-maximizing and the majority-winning preferences will be aligned. Intuitively, a higher price boosts the profit of owners already selling and higher quantities enables new owners to enter the market. When F is convex, the case (a1) will correspond to intervals where the curve becomes increasingly linear ($F'' \rightarrow 0^+$). The situation (b1i) is, however, unlikely for a concave supply curve because it would require F'' to strongly decrease. Indeed, under the model assumption, F is increasing so F'' cannot strongly decrease on a large interval. For instance, the square root or logarithm function would never lead to (b1i).

Unambiguous satisfaction of buyers with precision (a2ii, b2)

Buyers gain surplus when both quantities and prices decrease with precision (according to Proposition 14 since $\underline{\rho} < \rho$). This may be at the expense of the aggregated profits (and the welfare) since capacity owners would then sell fewer quantities at a

lower mean price. When F is convex, this corresponds to a strong increase in F'' while if it is concave, such a situation would appear when F becomes flat ($F'' \rightarrow 0^-$).

Ambiguous preferences (a2i, b1ii)

In (a2i, b1ii), equilibrium outputs (price and quantities) move in opposite directions when precision increases: the preferences will then depend on the relative weight of the two effects. For example, when buyers are better informed, the concavity of the supply curve may result in a large decrease of the price when the state of the world is low and in a large increase of the quantities sold when the state of the world is high: the aggregated profit may increase when mean quantities increase enough. These cases correspond to situations where F'' does not move sharply.

After giving some intuition about the preferences by distinguishing convex and concave cases, we characterize the preferences. Investigating four different criteria of optimal information disclosure, we highlight the existence of four distinct thresholds for the second derivative of η :

$$\rho = \frac{F''}{F'}(\eta' + an) \quad \bar{\rho} = \rho + \frac{L'}{Q} \quad \underline{\rho} = \bar{\rho} - \frac{L'(L' - (n-1)F')}{(n-1)QF'} \quad \bar{\bar{\rho}} = \bar{\rho} + \frac{L'(L'+F')}{QF'}$$

As noted above, few computations show that $\underline{\rho} < \rho$ under [Assumption A4](#) for all n .

$\eta''(Q)$		$\underline{\rho}$	ρ	$\bar{\rho}$	$\bar{\bar{\rho}}$
Welfare		$\lambda_W = 1$	$\lambda_W = 1$	$\lambda_W = 1$	$\lambda_W = 1$
Buyers' Surplus		$\lambda_S = \frac{1}{2}$	$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = 1$
Owners' Profit		$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = \frac{1}{2}$
Majority-Winning	if $m(Q) > 0$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$
	if $m(Q) < 0$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = \frac{1}{2}$	$\lambda_C = \frac{1}{2}$

Table 4.3: Maximizing precision for $n > 1$

[Table 4.3](#) sums up the previous results and gives an insight about the disagreement between buyers and owners and, amongst heterogeneous owners. It only represents the buyers' surplus for $n > 1$ since the monopsony always prefers full information. Full information is preferred by both the buyers' side and the owners' side (in terms of surplus) when the relative increase of η derivative is approximately the same as the relative increase of the price derivative (i.e. $\frac{\eta''(Q)}{\eta'(Q)} \simeq \frac{F''(Q)}{F'(Q)}$). When Q is large enough, it becomes $\frac{F'''(Q)}{F''(Q)} \simeq \frac{F''(Q)}{F'(Q)}$ ³⁸. Then the supply curve should be such that derivatives have relatively the same behavior until the order 3. However, conflicts arise between buyers and owners when this is not true anymore. For a rather slowly increasing (or decreasing) η ($\eta''(Q)$ small) with respect to the supply function, no information will maximize the buyers' surplus while the welfare, as well as all owners' preference criteria, will be higher under full information. Recalling that the mean quantities increase with λ when $\eta'' < -nF''$, we deduce in this case that Q_h increases faster than Q_l decreases. Recalling also that the

³⁸It implies that the price function is K-bilipschitz with K around 1 up to the order 3.

marginal gain to increase the quantity is everywhere the same, the gain from getting more quantities in the high state (and getting less in the low state) is offset by paying more in the high state (with a price decrease in the low state not large enough). Because of the presence of competitors, when η does not increase rapidly enough, the buyers are worse off with information³⁹. Unsurprisingly, we find the owners' profit on the other side of the spectrum. Indeed, in this example, an increase in the mean quantities makes the profit higher. $\eta'' < \rho$ correspond to cases (a1), (a2i) and (b1i). As noted above, (b1i) is unrealistic. It means that in the concave case, we do not expect disagreement from the buyers' side (cases (b1ii) and (b2) are such that buyers prefer information). In addition, we do not expect that all the criteria will be in favor of information. From what we have described above, we deduce that, in the convex case, disagreement from the buyers is likely when the supply curve becomes linear rapidly (roughly case (a1)).

On the contrary, when η is increasing quickly (with respect to the supply function), no information is chosen by owners as a whole. Thus, the more F'' increases, the more the owners might be disadvantaged by precision. It corresponds to situations where the supply curve increases rapidly (and is convex) (roughly case (a2ii)) or where the supply curve becomes rapidly flat (and is concave) (roughly case (b2)).

For the welfare-maximizing criterion to choose no information, an even steeper change is needed. The welfare is more influenced by the owners than the buyers because of the strategic behavior of the latter. However, at the moment of designing the capacity remuneration scheme, the supply curve is not known yet and the system operator has to decide for the precision of information. The welfare cannot be deduced either, and the solution found is often to ask the stakeholders about their preferences directly. This is where the majority-winning criterion takes all its purpose.

When considering the majority-winning criterion, conflicts can arise more often than in the surplus cases. In practice, in CRMs, the uncertainty on the intercept is unlikely to be big enough for the group of owners preferring full information to overweight the group of owners, which are certain to sell under both states of the world. In other words, if the public authority puts all the owners in a room and makes them vote for their preferred level of information, it is more likely to be "no information" than under the usual profit-maximizing criterion. It follows that, alone, a public consultation might not be fully trustworthy and need to be completed by a good knowledge of both the economics at stake and the system specificities in order to knowingly weight each actor's view and advice.

Interestingly, buyers and owners can agree when profits are considered, but they may disagree when majority-winning is the decision-making process and vice versa.

³⁹To understand why, imagine the same equilibrium with a high signal (i.e. the same couple (price, quantity)) with two different supply curves and assume an increase in the precision: sFOC indicates that quantities will increase more in the case of the softest curve since the price derivative is lower. Under competition, this increase may be high enough (because all the buyers increase their demand together) to offset the fact that the supply curve increases at a lower rate and may lead to a higher price with the softest curve.

4.3.9 Large number of buyers

When we move away from an oligopsony with few buyers (as n increases), results simplify. First, let us study the effect of n on the equilibria quantities:

Lemma 32. *Total quantities at the high and low equilibria are increasing concave functions with respect to n .*

The fact that quantities increase with the number of buyers is in line with the existing literature on oligopsonies (see Okuguchi [2000]). The proof uses sFOC with $Q = nq$.

Individual markdown decreases with respect to n . When n increases, the share of each individual buyer decreases and an increase of the market price (by increasing marginally an individual consumption) affects less the buyers than if they buy individually large amounts of quantities. One can notice that the derivative goes to zero when n becomes large enough, meaning that in a competitive state, additional buyers stop increasing the aggregated demand. Indeed, concavity emerges naturally since even if a new buyer has the same incentives to meet the needed demand as any previous entering agent, price increasing (and then the net incentives decrease), the new equilibria shift upward more slowly when n increases.

This may seem coherent with the fact that buyers are more incentivized to get capacities when they are numerous. Indeed, the decreasing marginal value curve represents the decreasing willingness to pay for additional units of capacity. The fact that all buyers highly value the first capacity units means that buyers are individually ready to get more capacities (relative to their market share) for a large n . In this case, more aggregated capacities are sold.

When n is large enough, the effect of information on equilibria outputs simplifies:

Lemma 33. *For any thrice differentiable supply curve, it exists $N \in \mathbb{N}$ such that $\forall n > N$: If (a) $F'' > 0$, then (a2i): $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$. If (b) $F'' < 0$, then (b1ii): $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$ and $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$.*

When n is large enough, with respect to the curvature of the bid function, outputs can move only in one direction. Results become more intuitive: mean prices and mean quantities move in opposite ways with information precision. If the supply curve is concave, more information increases the quantities dramatically in the high state (with limited prices increase) and slightly lowers the quantities when the state of the world is low (with prices dropping a lot). The reverse mechanism appears when the supply curve is convex.

Following Lemma 33 and the results of the previous subsections, we can derive the preferences of the agents when n is large enough as well as the optimal precision when considering welfare:

Proposition 18. *When n is large enough, preferences can be characterized with respect to the convexity of the bid function:*

F''		$-\frac{F'(F'+a)}{aQ}$	0	$\frac{F'+a}{Q}$	
Buyers' Surplus		$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = \frac{1}{2}$
Owners' Profit		$\lambda_P = \frac{1}{2}$	$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = 1$
Majority-Winning	if $m(Q) > 0$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$
	if $m(Q) < 0$	$\lambda_C = \frac{1}{2}$	$\lambda_C = \frac{1}{2}$	$\lambda_C = 1$	$\lambda_C = 1$
Welfare		$\lambda_W = 1$			

Table 4.4: Maximizing precision under perfect competition

When the market power vanishes, information can not hurt the welfare when this competition effect is null. Then, decreasing the precision of information reveals that the policy maker favors one side or that the market is not competitive enough.

As discussed in [Section 4.1](#), it is decentralization that introduces the issue of competition in this market. In this sense, the perfect competition case is of special interest when considering capacity markets. For not having strategic behaviors on the buyer-side, there is an obvious relationship between the perfection competition case and the centralized capacity market case where a benevolent central planner directly procures the capacity. If her objective function is the welfare and there is no asymmetry of information, then the two situations are equivalent.

4.4 Case study: model parametrization

In a simple framework, we have shown that the welfare-maximizing design can be difficult to derive from public consultations. Indeed, it might be aligned with the profit-maximizing design but due to capacity owners' heterogeneity, the majority-winning design might not even be profit-maximizing. In addition, *ex post* requirements are never consensual: their implementation is equivalent to favoring one or the other type of actors. The model results are sensitive to the level of parameters, meaning that generalizing the results to actual situations is not straightforward. Apprehending the empirical likelihood of each situation requires to investigate plausible situations. Indeed, the theoretical model has been kept rather general, with limited assumptions on the parameters. Such implicit consideration of market characteristics makes the model handy and adaptable but limits the intuitiveness of the results.

With this respect, German data from 2010 is used to parametrize the model and understand to what extent specific assumptions on parameters might affect preferences on information precision. Using data from 2010 presents a double advantage: (i) it sets aside discussions on the interactions between renewables integration and security of supply, (ii) it also prevents from over-interpreting the results in terms of policy recommendations. Indeed, reasoning in a hypothetical framework -Germany never did implement a capacity

market- allows testing a broader scope of hypotheses than would a real-life capacity market allow. Mainly, the data derived from existing markets would already account for the implemented design, be it with *ex ante* or *ex post* requirements, and, as such, be biased in some way. We thus build up a hypothetical consultation process where preferences are assessed under different levels of demand, penalty (or cost of voluntary peak shaving), and competition. Plants are assumed to have only two sources of income: energy and capacity revenues. Thus, they bid their profit gap on the capacity market as in a competitive environment [Keles et al., 2016; Iychettira et al., 2014; Hach and Spinler, 2016]. Capacity buyers being the strategic side of the model, we test different values for the slope, the intercept, or the degree of competition. As in the majority-winning case, we restrict the prior to an equiprobability of the states of the world ($\mu_0 = \frac{1}{2}$). The objective is to mimic different market conditions, in a way different designs proposed in a public would, to bring out how the actors preferences in terms of requirement definition are affected. The reader uninterested on the assumptions around parametrization can go directly to the results in Section 4.4.3.

4.4.1 Supply-side assumptions

The model allows for all types of supply-side behavior as long as the supply curve is continuous and unaffected by the signal. For tractability purposes, this case study will consider a competitive situation on the supply-side⁴⁰.

In this context, capacity owners bid their participation constraint, which can be approximated by the profitability gap in a market that already provides the right availability incentives. This complies with the zero profit competitive paradigm and allows building a supply curve with transparent assumptions. In an exercise similar to Joskow [2007], we build up an estimation of the profit gap. However, two main elements differ from Joskow's methodology. First, capacity owners are assumed to be remunerated only for their energy produced and capacity, disregarding ancillary, and balancing revenues. Second, the use of representative units (base, intermediate, and peaking units) is too restrictive for our purpose. We thus use He et al. [2013]'s "Dual Exponential" estimation of the merit order⁴¹ to generate a continuous merit order. The day-ahead market revenues are computed based on 2010 hourly price data from EPEX Spot and abstracting from non-linearities: ramping costs are disregarded. A unit is considered as producing and earning money each time the price is higher than its marginal cost. To account for fixed operation and maintenance costs (O&M), the producers as represented by the "Dual Exponential" are

⁴⁰Although CRMs commonly mitigate supply-side market power, the absence of strategic behavior from capacity owners remains a strong assumption (which is not required by the theoretical model).

⁴¹Based on publicly available data such as EEX (European Energy Exchange) hourly prices for 2010, available generation capacity, outages, historical production, wind and solar production, vertical load, consumption, and exports, they empirically estimate the German merit order from 2010 considering that prices are function of a normalized load. The vertical load is increased with exports and nuclear and lignite unavailabilities. The maximum value of the resulting "equivalent transmission grid load" is then used to normalize the data set. Comparing three specifications, they find the "Dual Exponential" to perform better. See He et al. [2013] for further details.

assigned a technology based on installed capacity in Germany ordered according to their merit. Figure G.1 summarizes the main computations. For instance, the first 26% of the merit order are assumed to reflect renewable units bidding. This is because geothermal, wind and solar made up to 26% of installed capacity in 2010 according to the German Federal Ministry for Economic Affairs and Energy - BMWi. Then follows hydropower, nuclear, coal, biomass, gas and oil-fired power plants. Once each unit corresponds to a category, fixed O&M are assigned and deduced from the revenues to derive a net rent (see Table 4.5). Renewables are assumed to bear no fixed costs because subsidies ensure their profitability. Similarly, hydropower is known to bid at opportunity cost and will effectively recover its full costs: a zero fixed cost is assumed for those two technologies by simplification. The resulting net rent for each unit (fixed costs being allocated on a fuel basis) is represented in Figure G.1. When considering the day-ahead market as the only source of revenues, more than 20% of the installed capacity is not profitable⁴².

Considering that the market participants bid their profit gap on the capacity market, their bid is deducted from the net rent: it is the maximum between zero and the opposite of the net rent. Those bids are then ordered from the lowest to the highest, and the corresponding capacity is derated⁴³ with a fuel-specific coefficient (see Table G.1). Abstracting from already profitable units as well as possible new entrants (investments), the resulting distribution is approximated on three intervals with a cubic spline function to obtain a thrice differentiable monotonic supply curve⁴⁴. The resulting spline approximation is represented along the estimated bidding curve in Figure 4.2.

Fuel	Derating	Fixed O&M (€/kW)
Nuclear	0.8524	82.1
Coal	0.8758	30
Gas	0.9293	20
Oil	0.8804	15

Table 4.5: Cost assumptions for Germany in 2010 based on Villavicencio [2017] and the EMR delivery body

4.4.2 Demand-side assumptions

In the model, the behavior of the demand-side embedded in the linear marginal value results from 5 parameters: the slope (a), the signal (s_h or s_l), the precision of the signal (λ), the uncertainty ($z_h - z_l$) and the level of competition on the market (n). As discussed, model parameters are exogenous as they are affected by the capacity market design. Mainly, the level of the intercept depends on the capacity target. In addition, the higher

⁴²It is noticeable that the alternative specifications proposed by He et al. [2013] would lead to similar results.

⁴³Using the UK capacity market deratings as in <https://www.emrdeliverybody.com/Lists/Latest%20News/Attachments/114/Capacity%20Market%20Auction%20Guidelines%20July%207%202017.pdf> last accessed on February 4th, 2019.

⁴⁴The spline coefficients are summed up in Table G.1.

the non-compliance penalty, the greater the slope (a) of the marginal value. In the absence of a capacity market, or even a reliability target –as in Germany–, none of the parameters are straightforward so several cases will be discussed. Several capacity market designs are compared to assess the sensitivity of preferences regarding requirement definition. They differ in terms of non-compliance penalty level, competition level, quantity target, and degree of uncertainty. Some of those elements are calibrated –to some extent– to the German situation in 2010, others are extrapolated from existing CRMs.

Various sources are used to select possible capacity targets and thus derive the corresponding intercepts. Indeed, the use of past data provides the advantage of knowing the risk realization (peak demand) and eases the estimation of potential targets in the absence of official reliability criterion. For instance, the European Network of Transmission System Operators’s (ENTSO-e) transparency platform provides hourly demand for Germany with a 91% coverage: the maximum hourly demand over 2010 (around 79.9 GW) is used as a minimum capacity requirement in this case study. Scaling it up to cover 100% of the German demand leads to a peak demand of roughly 87.1 GW. This leads in the second interval. In turn, the IEA considers that a reasonable reserve margin⁴⁵ is between 15 to 20% (see Figure G.2 from International Energy Agency [2010]). The two extremes of what the IEA considers a "reasonable" interval provides the two additional points⁴⁶ to test the sensitivity of the preferences to the level of procurement (i.e., the local shape of the supply curve). The set of capacity targets considered are $Q_1^t = 79.9$, $Q_2^t = 87.1$, $Q_3^t = 100.1$ and $Q_4^t = 104.5$ as displayed in Figure 4.2.

With respect to uncertainty, we consider three situations: (i) no uncertainty, meaning that $z_h = z_l = aQ_i^t + F(Q_i^t)$, (ii) 1% uncertainty where $z_h = F(1.01Q_i^t) - 1.01aQ_i^t$ and $z_l = F(0.99Q_i^t) - 0.99aQ_i^t$, (iii) 2% uncertainty with $z_h = F(1.02Q_i^t) - 1.02aQ_i^t$ and $z_l = F(0.98Q_i^t) - 0.98aQ_i^t$. As a benchmark for such levels, RTE [2015] presents the official forecasts of total obligation for delivery years 2016 to 2020. The high and low scenarios deviate from the baseline by less than 3% (see Figure G.3). This is considered as an upper bound bearing. For the sake of simplicity, the corresponding situations will always be called through the corresponding target (Q_i^t).

With respect to the slope (a), centralized demand curves do provide a useful benchmark although they lie on a different rationale. Indeed, in a centralized market, a sloped demand curve reflects the decreasing probability of black-out as the reserve margin increases. In decentralized capacity markets, LSEs will confront the cost of capacity to their alternative options. Those differ very much from one design to the other: paying the non compliance penalty is commonly considered as the cost of the option, alternatively, one can incentivize changes in load pattern to reduce its portfolio’s peak demand as it has been experienced

⁴⁵According to the Energy Information Administration (EIA), the "reserve margin is $\frac{Capacity - Demand}{Demand}$, where "capacity" is the expected maximum available supply and "demand" is expected peak demand" <https://www.eia.gov/>.

⁴⁶87.1 GW increased by respectively 15% and 20% leads to capacity targets of 100.1 GW and 104.5 GW.

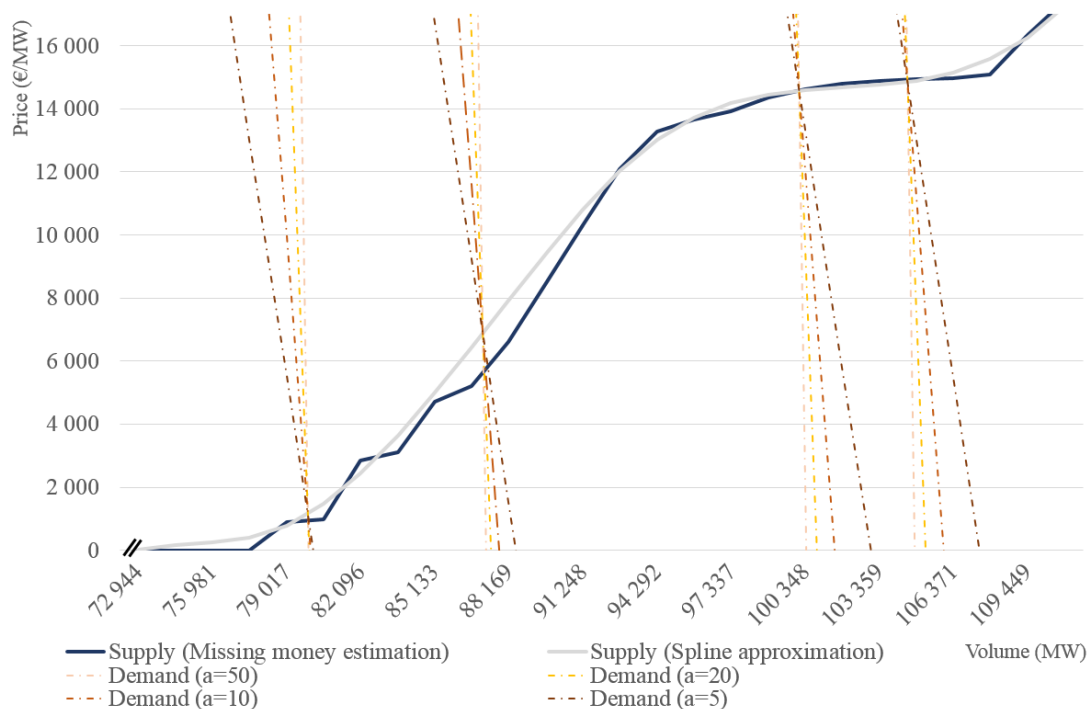


Figure 4.2: Estimated supply and demand curves (no uncertainty) for Germany 2010

in France⁴⁷. However, the continuous discussions with respect to the careful definition of penalties⁴⁸ suggest that they would indeed be designed so private opportunity costs converge towards social opportunity costs. In this case, the aggregated demand curve of a decentralized mechanism could equate a centrally built linear demand curve. We thus turn towards existing centralized mechanisms to get an idea of what could be considered as credible slopes for the aggregated demand. The 2022-2023 demand curve for the UK is piecewise linear, the first segment decreases by around 17£/MW/Year for each additional megawatt procured [DECC, 2014]. On the second segment, the slope is almost twice as steep. Similarly, the NYISO demand curve for the New York Control Area also has a slope around -30\$/MW/Year once converted to comparable values. More constrained areas such as New York City, Long Island, and the "Lower Hudson Valley Zone" (G-J Locality) have much steeper slopes to reflect the high opportunity cost [NYISO, 2018]. Increasing the range to account for possible changes in load pattern, a range of slopes between 5 and 50€/MW/Year decrease in price for each additional megawatt procured is considered. As an example, Figure 4.2 represents the various targets considered.

As for the degree of competition, European Commission [2014] states that four major retailers supply the German electricity consumers but households were able to choose

⁴⁷For instance: Grand Lyon (metropolitan area) has investigated process changes in water treatment plants to locate electricity-intensive tasks out of peak hours.

⁴⁸For instance, setting a penalty high enough would increase the slope and reduce the market power problem, however, setting the penalty too high can result too risky, and lead to buyers exiting the market which reduces the competition.

between no less than 65 different retailers in 2011 while the number of retailers operating in the German market was above a thousand (all categories of consumers included). For the sake of the exercise, we will test the preferences of the buyers for respectively four and sixty-five capacity buyers as well as the competitive state.

In the process of designing a capacity market, it is common to organize public consultations to gather insights from the actors. The different combinations of the previously defined parameters can describe distinct market structures, providing the stakeholders with some context to build their preferences. For instance, a steep slope ($a = 50$) suggests either a high penalty for under procurement (capacity market characteristic), or expensive demand response (system characteristics). Similarly, perfect competition is a proxy for a centralized procurement from a benevolent public authority (minimization of the deadweight loss). The intercept reflects the preferred level of reliability. In turn, this determines which owners will be affected by the uncertainty.

4.4.3 Results

The previously defined parameters can be combined to mimic market designs with different features and system specifics: they provide a realistic set up to the model. Each of the twelve resulting designs (three levels of competition and four levels of opportunity cost) represents a credible situation to put into public consultation so to collect preferences with respect to requirement definition. For instance, perfect competition is highly unlikely in the electricity sector, but centralized capacity markets act on behalf of the consumer in a non-strategic way -as if perfectly competitive-. It is thus a good proxy for a centralized mechanism. As discussed in [Section 4.3.9](#), perfect information is always welfare-maximizing in perfect competition: a centralized mechanism with *ex post* procurement would not make sense unless the public authority's objective function is not welfare. The preferences under perfect competition are thus displayed as a benchmark so to measure how market power affects preferences as well.

To analyze the actors' preferences, we name the possible sets of preferences with respect to the actors disagreeing with perfect information –*ex ante* requirements (see [Table 4.6⁴⁹](#)). Consequently, the set of preferences called "Buyer-side" is characterized by all actors but the demand-side preferring full information. "-" pictures the absence of disagreement: all actors prefer full information. Similarly, the "Majority" sees individual capacity owners preferring low information while all aggregated surpluses are all maximized by full information. With this respect, remind that in the capacity market, the size of uncertainty is relatively small compared to the size of the market. It follows that the preferences of the capacity buyers who are certain to sell in any case will dominate the majority-winning case. In the "Owner-side" set, only capacity owners do disagree with full information while in the "Owner and welfare" set, only capacity buyers prefer additional information.

⁴⁹Note that only the thresholds differ in perfect information, but the sets of preferences are the same: we thus use the same name.

$\eta''(Q)$		$\underline{\rho}$	ρ	$\bar{\rho}$	$\bar{\bar{\rho}}$	
Welfare		$\lambda_W = 1$	$\lambda_W = 1$	$\lambda_W = 1$	$\lambda_W = 1$	$\lambda_W = \frac{1}{2}$
Buyers' Surplus		$\lambda_S = \frac{1}{2}$	$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = 1$
Owners' Profit		$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = \frac{1}{2}$	$\lambda_P = \frac{1}{2}$
Majority-Winning	$m(Q) < 0$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = \frac{1}{2}$	$\lambda_C = \frac{1}{2}$	$\lambda_C = \frac{1}{2}$
Naming: sets of preferences		Buyer-side	-	Majority	Owner-side	Owners and welfare
Disagree with information						

Table 4.6: Maximizing precision for $n > 1$ and $m(Q) < 0$: Naming the sets of preferences

The size of the uncertainty ($z_h - z_l$) does not affect much preferences since information becomes increasingly valuable for all parties as it increases. It only affects the equilibrium outcomes (see the frame *Accounting for uncertainty* and [Chapter H](#) for a discussion on the effects of uncertainty). For the sake of simplicity, [Table 4.7](#) abstracts from uncertainty ($z_h = z_l$) to depict the preferences of the actors under each of the predefined settings. Synthesizing the information in [Table 4.7](#), it is clear that *ex ante* requirements would probably emerge from a "global" consultation. Indeed, public consultations are often conducted for all design features at once i.e. stakeholder are to be consulted on requirement definition at the same time as on other design features such as the non-compliance penalty of the buyers (a), the level of requirement and sometimes even the obligated actors. In this sense, the buyer-side prefers information unless the implicit penalty (a) is low enough. Similarly, the supply-side only prefers *ex post* payments in the limited situation where the supply curve becomes flat⁵⁰ (Q_3^t). If the capacity target has not been announced, they would not take the chance to formulate such preference. The same reasoning applies to the capacity buyers, who are even less likely to benefit from reduced information.

On the contrary, if the bulk of the capacity market has already been defined, as in the model framework, meaning that the opportunity cost and the level of centralization are known as well as the reliability targets, actors can build firm preferences with respect to the requirement definition.

When competition decreases (lower n), capacity buyers are able to use their market power and information becomes more and more valuable as the size of the capacity buyers increases (n decreases). This implies that their preference for a lower degree of information ($\lambda = \frac{1}{2}$) becomes increasingly constrained all things equal: situations where all actors prefer additional information become more probable as n decreases ([Table 4.7](#)). Namely, a plays a role in preferences: a high slope makes the buyers willing to acquire information. The reason is that when the markdown is low, and when the penalty is high, being aligned with the state of the world matters. On the contrary, when a decreases, the implicit penalty decreases as well, so for a reduced market power (high n), the convexity of the price dominates any other effect in the buyers' preferences similarly as in the perfect

⁵⁰Or alternatively when it increases rapidly, which is not the case here.

		$Q_1^t = 79.9$ GW	$Q_2^t = 87.1$ GW	$Q_3^t = 100.1$ GW	$Q_4^t = 104.5$ GW
$n = \infty$	$a = 50$	-	-	Owner-side	-
	$a = 20$	-	-	Owner-side	-
	$a = 10$	Buyer-side	-	Owner-side	-
	$a = 5$	Buyer-side	-	Owner-side	-
$n = 65$	$a = 50$	-	-	Owner-side	-
	$a = 20$	-	-	Owner-side	-
	$a = 10$	Buyer-side	-	Owner-side	-
	$a = 5$	Buyer-side	-	Owner-side	-
$n = 4$	$a = 50$	-	-	Owner-side	-
	$a = 20$	-	-	Owner-side	-
	$a = 10$	-	-	Owner-side	-
	$a = 5$	-	-	Owner-side	-

Table 4.7: Sensitivity of the form of disagreement to the slope, the competitive equilibrium and the competition

competition case. This explains the antagonism of preferences for the high competition case with a gentle slope in Q_1^t .

In this framework, the role of the implicit penalty and competition appears limited in terms of preference. However, both have a significant effect on surplus allocation and welfare. As the implicit penalty increases, deviations from the competitive equilibrium become more and more costly, reducing the structural under procurement (see [Chapter H](#) for a discussion on structural under procurement due to the imperfect competition). As well, the higher the number of buyers, the lower the quantities bought by each of them are, and finally, the lower their individual market power is. In this sense, the total welfare increases with both n and a by way of a reduction of the deadweight loss.

Accounting for uncertainty

[Figure I.1](#) to [Figure I.16](#) give further insights on the preferences in [Table 4.7](#) by representing the differences in mean quantities and mean prices in full information compared to the no information case ($\lambda = \frac{1}{2}$). When the represented difference is positive, the mean price (resp. quantity) increases with information.

[Figure I.1](#) and [Figure I.2](#) illustrate the demand-side disagreement at Q_1^t : capacity buyers use information to reduce the mean quantities procured. However, information leads to an increase in mean prices as in the situation ($a2i$). Unsurprisingly, when the penalty is low, additional information becomes detrimental to their surplus –as the mean price increases with information anyways.

In comparison, when $n = 4$ (always at Q_1^t , see [Figures I.9](#) and [I.10](#)), the mean price still increases with information. The effect of information on the mean quantity highly depends on demand parameters: when a is high, situation $(a1)$ arises, but a low (a) leads to $(a2i)$. [Table 4.7](#) indicates that information improves the situation of capacity buyers even though the mean quantity might slightly increase in information (when the penalty (a) is high enough). Compared to the case where $n = 65$, the increase of the mean price in information is decreasing more when (a) decreases. It means that when (a) decreases, the market power in the case with information do increase. It is sufficient to make buyers prefer information (with respect to the situation where $n = 65$). Thus, the penalty may have a different impact on the equilibrium outcomes (and the preferences), according to the market structure (i.e. level of competition). As the mean price increases in precision with both market structures, capacity owners prefer information.

At Q_2^t however ([Figures I.3](#), [I.4](#), [I.11](#) and [I.12](#)), both the mean price and the mean quantity increase with information (case $(a1)$) whatever the level of competition. Again, capacity owners always prefer information in this situation. With respect to capacity buyers, they prefer information. Contrary to Q_1^t , the increase of the mean price in information increases when (a) decreases. Thus, the effect of more information does not always multiply the negative effect of a lower penalty on the mean prices and quantities (as the structural under procurement increases when (a) decreases).

At Q_3^t , capacity buyers prefer information as it allows an important decrease in the mean price while the mean quantity decreases ($n = 4$, [Figures I.13](#) and [I.14](#)) (case $(b2)$) or even increases slightly ($n = 65$, [Figures I.5](#) and [I.6](#)) (case $(b1ii)$). This situation creates a clear loss for capacity owners who then prefer no information.

At Q_4^t ([Figures I.7](#), [I.8](#), [I.15](#) and [I.16](#)), the mean quantities are decreasing in information while prices do increase (case $(a2i)$). Better alignment with the state of the world is enough to make capacity buyers gain from information in this situation. Naturally, capacity owners are satisfied with the resulting increase of the mean price.

4.4.4 Discussion

In our framework, disagreement between profit-maximizing and majority-winning criteria is improbable. Indeed, if the price increases with information precision (so the majority criterion selects full information), then the profit is likely to increase as well since the marginal effect on quantity is much lower than the marginal increase price. When such a disagreement exists locally, it corresponds to an unstable situation, given the uncertainty. Indeed, the "Majority" type of disagreement is characterized by a price decreasing in information precision while the profit is still increasing. However, it is only natural that the variation of profit closely follows the variation of price when information pre-

cision increases. It follows that between the situation where no party disagrees (profit and price increasing in information) and the "Owner-side" type of disagreement, only a small interval will allow price and profits to evolve in opposite directions when information increases. For instance, when $n = 65$ and $a = 5$, disagreement between capacity owners appears when the demand curve crosses the supply curve in the rather small interval $Q^t \in [101940, 101960]$. This represents less than 0.2% of the total capacity to be procured at this point. In comparison, the French TSO, RTE, considers a 2% uncertainty between its high and low scenario [RTE, 2015]. Consequently, the "Majority" type of disagreement is empirically more of a buffer zone between capacity owners preferring full information towards a preference for reduced information as the markdown becomes increasingly convex (or less concave) –compared to the convexity of the supply function–.

Finally, the "Owner and welfare" type of disagreement does not appear in the case study. It is also an intriguing result from the model: how could a central planner prefer low information as a design? Once again, the answer lies in the market power. From a welfare perspective, it might be preferable to have lower information in order to prevent the oligopsony from abusing its market power by buying too little capacity even from a welfare perspective. More precisely, for a central (welfare-maximizing) authority to prefer a lower precision, the supply curve behavior needs to be radically changed. In the convex case, the supply function should be increasing more than exponentially. In this situation, a small change in quantities allows for great variations in prices, which greatly benefits the oligopsony under perfect information: the informational gains of the capacity buyers are lower than the respective losses of the capacity owners. However, this situation is unlikely to happen with a competitive behavior from the existing capacity owners as pictured here. Indeed, the curve of stacked profit gaps is rather smooth because they already recover at least their marginal cost from the energy market. Nevertheless, the supply curve estimated in Section 4.4.1 abstracts from possible new entrants who would need to recover their investment costs in addition to fixed ones: if investments were to be necessary to comply with the reliability target, the end of the competitive supply curve might become very steep since those units would also need to recover investment annuity costs. Only in this case, or in a market with specific supply-side strategic behavior could the "Owner and welfare" type of disagreement emerge.

However, those results remain indicative as not all the assumptions are empirically verified (or verifiable). Mainly, the shape of the supply curve is deemed independent of the signal, which reduces the range of possible strategic behaviors. In addition, the only uncertainty in the model regards the level of required capacity, meaning that all remaining information is public and common to all actors (shape of the supply curve, the elasticity of the marginal value and the number of market participants). Those assumptions are obviously little realistic: when a public authority wants to design a capacity market, she does not benefit from such information yet. Only the first market clearings will convey accurate information on the supply and demand. In the meanwhile, it still needs to decide what design to implement. For all those reasons, a theoretical case study is preferred to

an actual empirical application based on publicly available auction data. This chapter provides insights on actors' preferences in terms of requirement definition under different options with respect to the other design features. This echoes the stakeholders' response to the British public consultation on the electricity market reform in 2011 [DECC, 2011]: it is complex to discuss implementation features without a view on the global design.

4.5 Conclusion

In this chapter, we investigate information disclosure in capacity remuneration mechanisms (CRMs). We argue that the precision of the available information is embedded in the design itself. By tailoring CRMs in different ways, the public authority controls the amount and accuracy of the information revealed by the market and consequently affects the agents' forecast errors on their optimal level of capacity needed. By mostly choosing forward capacity markets, public authorities around the world limit the ability of LSEs to predict their future capacity demand efficiently. To hedge for the lead time related uncertainties, the capacity requirements are more or less precisely stated at the moment of procurement. The UK or eastern US type of centralized procurement leaves no doubt on the aggregated level of demand: the demand curve presented at the auction is the legal capacity requirement. The state of the world is known, the requirements -and thus procurements- are either high or low. On the contrary, the former MISO or former PJM as well as the French mechanism leave the LSEs estimate and secure their load in a decentralized way. Where *ex ante* requirements in a perfect competition framework would resume to a centralized procurement, *ex post* requirements penalize forecast errors. This represents an uncertainty around the state of the world that cannot be fully hedged. In other words, the French mechanism does not allow full information disclosure as its counterparts do. If the centralized versus decentralized discussion is recurrent in the literature, this CRM feature is rarely disentangled from the timing of requirements (*ex ante* vs. *ex post*) which does affect risk sharing and incentives [CIGRE, 2016].

Symmetrically to Roy et al. [2019], we model heterogeneous price taker capacity owners and homogeneous buyers competing *a la Cournot* under uncertainty on their level of capacity obligation. We investigate both the effect of a decentralized capacity mechanism under imperfect competition on the equilibrium and the actors preferences in terms of information disclosure (requirement definition). The reference model is adapted to capacity markets by considering an oligopsony – instead of an oligopoly. By introducing a decreasing slope for the buyers' marginal value –instead of a constant marginal cost–, we can analyze the slope of the marginal value curve in terms of implicit penalty. It represents the cost of not procuring enough capacity like the legal penalty or the ability to release Demand Response (DR) in the short term. Whereas Roy et al. [2019] was focused on preferences, we add results that deal with the industrial organization. For this purpose, we use the function $\eta(Q, n)$ introduced to enhance the influence of the market structure and asymmetric incentives on the equilibrium outcomes. It is equivalent to the

markdown at the equilibrium. Naturally, results indicate that the aggregated quantities procured increase with the number of buyers. As a consequence, the decentralized capacity mechanism will tend to under procure capacity compared to a benevolent central planner (perfect competition case). In practice, this means that if the parameters of the market (i.e. the penalty and the reward) are not adjusted accordingly, both price and quantities will be unnecessarily low compared to the optimal situation from the welfare point of view.

In line with Roy et al. [2019], we consider 3 decision criteria: the capacity buyers' surplus and capacity owners' profit as well as the majority criterion. We add the welfare criterion since it is usually the objective of the public authority to maximize the welfare. The majority criterion mimics the preferences that could be expressed by market participants during the consultations organized previous to the regulation change. Indeed, regulation should seek welfare maximization, and real-life imperfect information makes actors' preferences a good proxy. For homogeneous buyers, their individual preference will always be aligned with the surplus maximizing criterion. However, heterogeneous capacity owners' preferences might differ at the individual level expressed in a consultation process. The latter indeed gains relevance with the growing competition in electricity markets and the raising concern about plants' profitability: each plant is now individually required to be profitable and cross-unit subsidies abandoned. In this context, reasoning in survival terms brings out interesting considerations. Consistently with the diversity of capacity market designs in terms of requirement definition, the model introduces cases where buyers and owners agree as well as the reverse. As in the reference model, we have found that the preferences about the information disclosure depend mainly on the supply curve. Following the economic intuition, results show that full information is mostly preferred. Nonetheless, conflicts between the categories of agents may emerge. We highlight the preferences in terms of precision for each group under the assumption that the public authority might be influenced by one or the other depending on its utility (mission of public interest) as much as the efficiency with which they make their preferences heard. Antagonism amongst capacity owners might emerge when the interests of the owners guaranteed to sell in any case differ from the interest of the owners affected by the uncertainty. When the sum of people in the first group is much higher than the sum of people in the second group, the aggregation of private preferences may not be profit-maximizing in theory. Besides, even without considering market power abuse from the supply-side, we have found that information can be detrimental to the welfare. In addition, letting the owners strategically use information would probably increase the number of cases where the welfare decreases with precision.

In terms of CRMs, this chapter sheds light on the market power of the demand-side. In this simple framework, without risk aversion, we show that *ex post* requirements might be preferred, even by the buyers' side (whose the surplus is state-contingent). However, in this model, a consensus for no information is never generated. Thus, retaining information is always detrimental to at least one side. We show that no information can be welfare

improving when the markdown increases quickly with information (relative to the price function). A case study is built based on German data (2010) to develop a better understanding of the model implications. For tractability reasons, capacity owners are assumed to bid competitively their profitability gap. Then, various market structures in terms of demand level, slope or competition are applied against the supply curve to determine the actors' preferences under different circumstances. Results show that information becomes more valuable for LSEs as their number decreases. Indeed, as the number of capacity buyers and the implicit penalty decrease, the markdown mostly drives the equilibrium decisions and consensus towards full information becomes more probable. Naturally, with respect to equilibrium quantities, under procurement tends to increase as the number of capacity buyers decreases. A reduction in the implicit penalty further increases market power. When the market power is important, the level of under procurement can be such that SoS would be endangered under any structure of information. This result sheds light on the importance of setting an adequate penalty. Indeed, the implicit penalty drives not only the preferences, but also the market power and the optimality of market outcomes from a welfare perspective.

Fortunately for policy makers, results suggest the efficiency of consultations as a way to uncover actors' preferences: the majority type disagreement appears empirically unlikely. In addition, the welfare type of disagreement would not naturally occur in real capacity markets. Indeed, for information to be detrimental to welfare, the curvature of the supply function has to be radically modified. This situation is improbable in capacity markets if only existing units compete. However when new units are needed, the supply curve can be quickly increasing at its end. Indeed, they might need to recover much more from the capacity market.

Setting aside the risks related to market power abuse from the demand-side, decentralizing markets with *ex post* requirements are often presented as a way to increase system reliability at a limited cost due to the accurate capacity cost allocation [RTE, 2014; Woodhouse, 2016]. By making capacity buyers accountable for what they buy, the classical "skin in the game" argument may play an important role. This argument is empirically nuanced by the actual hedging LSEs benefit from: demand uncertainty is known to average out with aggregation, but end consumers have their hands on their load pattern. If LSEs are not able to better forecast their peak demand than the central authority, then shifting uncertainty on LSEs becomes only risk sharing, especially if their load management possibilities are limited. In this framework, further research on LSEs risk aversion is needed. Mainly, two types of risk aversion can be identified: (i) the probability of being wrong is indeed accounted for in this model as it relates to the value of information. It only affects $\mathbb{E}(Z|S)$. On the contrary, (ii) the relative cost of being wrong does affect the slope of the marginal value a that is considered as exogenous in our framework. If LSEs are able to accurately forecast their peak demand by way of DR, then accurate cost allocation as in *ex post* decentralized mechanisms does provide accurate incentive to lower peak consumption. This discussion relates to Crew and Kleidorfer [1976] consideration:

Rationing costs occur with stochastic demand whenever capacity is exceeded. These occur because it is not possible costlessly to rank consumers according to their willingness to pay and because there are added costs to the utility in actually performing the operation of rationing. In addition, there may be added production costs [...] of running to the limit of total capacity. There is no reason to suppose that such costs will on the average decrease as the amount by which demand exceeds capacity increases. Indeed, it is likely that such costs will increase at an increasing rate, because as the deficit of capacity increases the utility is more likely to cut off consumers with higher valuation.

Until recently, this statement was true and counting on DR was for dreamers. This is why former decentralized mechanisms as MISO or PJM relied on *ex ante* requirements (second-best capacity cost allocation). However, technological improvements allow DR units to be valued as SoS tools. Implementing real time metering at large scale would partially solve the two problems mentioned by [Crew and Kleidorfer \[1976\]](#) as it means inexpensive ranking of consumers and reduced costs of performing the operation of rationing. In decentralized mechanisms with *ex post* requirements, DR can be valued for what it is in the demand-side. However, this requires that each actor is accountable for its consumption pattern with respect to SoS objective (accurate cost allocation and real-time metering). A critical limit to this incentive is that it represents a rather short term solution: because of the interrelations with the electricity markets, peak shaving tends to push units out of the market in the long run, which only delays SoS concerns (until the residual load is rather flat).

Bibliography

- Astier, N. and X. Lambin (2019). Ensuring Capacity Adequacy in Liberalised Electricity Markets. *The Energy Journal* 40(3), 227–242.
- Bowring, J. E. (2008). The Evolution of PJM’s Capacity Market. In *Competitive Electricity Markets*, pp. 363–386. Elsevier.
- Buck, M., M. Hogan, and C. Redl (2015). The Market Design Initiative and Path Dependency: Smart Retirement of Old, High-Carbon, Inflexible Capacity as a Prerequisite for a Successful Market Design.
- Chao, H.-p. (1983). Peak Load Pricing and Capacity Planning with Demand and Supply Uncertainty. *The Bell Journal of Economics* 14(1), 179–190.
- Chao, H.-p., S. Oren, S. Smith, and R. Wilson (1988). Priority Service: Market Structure and Competition. *The Energy Journal* 9, 77–104.
- Chao, H.-p. and R. Wilson (1987). Priority Service: Pricing, Investment, and Market Organization. *The American Economic Review* 77(5), 899–916.
- Chen, B. (2017). Modelling Electricity Demand in Smart Grids: Data, Trends and Use Cases. In *Individual Electricity Consumers, Data, Packages and Methods*.
- CIGRE (2016). Capacity Mechanisms: Needs, Solutions and State of Affairs. Technical Report February.
- Crampes, C. and D. Salant (2018). A Multi-Regional Model of Electric Resource Adequacy. *TSE Working Paper 18-877*(January).
- Cramton, P., A. Ockenfels, and S. Stoft (2013). Capacity Market Fundamentals. *Economics of Energy & Environmental Policy* 2(2).
- Cramton, P. and S. Stoft (2006). The Convergence of Market Designs for Adequate Generating Capacity with Special Attention to the CAISO’s Resource Adequacy Problem.
- Cramton, P. and S. Stoft (2007). Columbia Firm Energy Market. In *System Sciences, 2007. HICSS 2007. 40th Annual Hawaii International Conference on Systems Science (HICSS-40 2007)*.
- Crew, M. A. and P. Kleidorfer (1976). Peak Load Pricing with a Diverse Technology. *The Bell Journal of Economics* 7(1), 207–231.
- DECC (2011). Planning our Electric Future: a White Paper for Secure, Affordable and Low-Carbon Electricity. Technical report.
- DECC (2014). Background on Setting Capacity Market Parameters. Technical report.
- EFET (2016). RTE consultation on the evolution of the Capacity Mechanism rules. Technical report.

- European Commission (2014). Germany - 2014 Country Report. Technical report.
- Gaudet, G. and S. W. Salant (1991). Uniqueness of Cournot Equilibrium: New Results from Old Methods. *Review of Economic Studies* 58(2), 399–404.
- Hach, D. and S. Spinler (2016). Capacity Payment Impact on Gas-Fired Generation Investments under Rising Renewable Feed-In — A Real Options Analysis. *Energy Economics* 53, 270–280.
- He, Y., M. Hildmann, F. Herzog, and G. Andersson (2013, aug). Modeling the Merit Order Curve of the European Energy Exchange Power Market in Germany. *IEEE Transactions on Power Systems* 28(3), 3155–3164.
- Henriot, A. and J.-M. Glachant (2013). Melting-Pots and Salad Bowls: The Current Debate on Electricity Market Design for Integration of Intermittent RES. *Utilities Policy* 27, 57–64.
- International Energy Agency (2010). Challenges in Electricity – a Focus on Europe.
- Iychettira, K. K., P. C. Bhagwat, J. C. Richstein, and L. De Vries (2014). Interaction between Security of Supply and Investment into Renewable Energy in the Netherlands and Germany. In *USAEE*.
- Joskow, P. L. (2006). Markets for Power in the United States: An Interim Assessment. *The Energy Journal* 27(1).
- Joskow, P. L. (2007). Competitive Electricity Markets and Investment in New Generating Capacity. In D. Helm (Ed.), *The New Energy Paradigm*, Chapter 4. Oxford University Press.
- Joskow, P. L. and J. Tirole (2006). Retail Electricity Competition. *The RAND Journal of Economics* 37(4), 799–815.
- Joskow, P. L. and J. Tirole (2007). Reliability and Competitive Electricity Markets. *The RAND Journal of Economics* 38(1), 60–84.
- Kamenica, E. and M. Gentzkow (2011). Bayesian Persuasion. *American Economic Review* 101(6), 2590–2615.
- Keles, D., A. Bublitz, F. Zimmermann, M. Genoese, and W. Fichtner (2016). Analysis of Design Options for the Electricity Market: The German Case. *Applied Energy* 183, 884–901.
- Lambin, X. (2018). The Integration of Demand Response in Capacity Mechanisms.
- Mastropietro, P., I. Herrero, P. Rodilla, and C. Batlle (2016). A Model-Based Analysis on the Impact of Explicit Penalty Schemes in Capacity Mechanisms. *Applied Energy* (168).
- Meulman, L. and N. Méray (2012). Capacity Mechanisms in Northwest Europe.

- NERC (2017). Glossary of Terms Used in NERC Reliability Standards. Technical report.
- Newbery, D. (1998). The Regulator's Review of the English Electricity Pool. *Utilities Policy* 7, 129–141.
- Newbery, D. (2016). Missing Money and Missing Markets: Reliability, Capacity Auctions and Interconnectors. *Energy Policy* 94, 401–410.
- Nitzan, S. and J. Paroush (1984). Are Qualified Majority Rules Special? *Public Choice* 42(3), 257–272.
- NYISO (2018). ICAP / UCAP Translation of Demand Curve - Summer 2018 Capability Period. Technical report.
- Okuguchi, K. (2000). Comparative Statics for Oligopoly, Oligopsony and Oligopsonistic Oligopoly. *Journal of Economic Research* 5, 173–187.
- Parsons, J. E. and F. J. De Sisternes (2016). The Impact of Uncertainty on the Need and Design of Capacity Remuneration Mechanisms in Low-Carbon Power Systems. *MIT Center for Energy and Environmental Policy Research CEEPR Working Paper Series WP-2016-004* (February).
- Patton, D. B., P. LeeVanSchaick, and J. Chen (2017). 2016 Assessment of the ISO New England Electricity Markets: External Market Monitor for ISO-NE. Technical report.
- Pindyck, R. S. and D. L. Rubbinfield (2013). *Microeconomics Eight Edition* (Pearson ed.).
- Roy, J., R. Silvers, and C.-J. Sun (2019). Majoritarian Preference, Utilitarian Welfare and Public Information in Cournot Oligopoly. *Games and Economic Behavior* 116, 269–288.
- RTE (2014). French Capacity Market: Report Accompanying the Draft Rules. Technical report.
- RTE (2015). Generation Adequacy Report on the Electricity Supply-Demand Balance in France. Technical report.
- Scouflaire, C. (2019). *Capacity Remuneration Mechanisms: An Assessment of their Performance and Implications for Market Design*. Ph. D. thesis, Paris Sciences et Lettres.
- Stoft, S. (2002). *Power System Economics: Designing Markets for Electricity*.
- Teirilä, J. (2017). Market Power in the Capacity Market? The Case of Ireland. *Cambridge University EPRG Working Paper 1712* (July).
- Villavicencio, M. (2017). *Analyzing the Optimal Development of Electricity Storage in Electricity Markets*. Ph. D. thesis, Paris Sciences et Lettres.

Woodhouse, S. (2016). Decentralized Reliability Options: Market Based Capacity Arrangements. In *Future of Utilities - Utilities of the Future: How Technological Innovations in Distributed Energy Resources Will Reshape the Electric Power Sector*, pp. 231–246. Elsevier Inc.

Appendices

Appendix D

Equilibrium Conditions

Let q_j be the capacity demanded by the buyer j and $Q = \sum_{i=1}^n q_i$ the total bought capacity. The individual buyers' surplus is naturally defined as the surface between their individual demand curve and the price: $S(q_j) = \int_0^{q_j} D_j^{-1}(x) - F(Q)dx = \int_0^{q_j} \mathbb{E}(Z|s) - na x - F(Q)dx$.

Maximizing with respect to q_i , with \bar{Q} being the quantity bought by all the other buyers ($Q = \bar{Q} + q_j$), we derive the First Order Conditions **FOC**:

$$\begin{aligned} \frac{\partial S(q_j)}{\partial q_j} = 0 &\Leftrightarrow \mathbb{E}(Z|s) - na \times q_j - \frac{\partial F(q_j + \bar{Q})}{\partial q_j} q_j - F(q_j + \bar{Q}) = 0 \\ &\Leftrightarrow \mathbb{E}(Z|s) = \left(\frac{\partial F(q_j + \bar{Q})}{\partial q_j} q_j + F(q_j + \bar{Q}) + na \times q_j \right) \quad (\text{FOC}) \end{aligned}$$

Let $Q^*(\mu_h)$ and $q^*(\mu_h)$ be the equilibrium quantities when the buyers have the belief that $\mathbb{P}(Z = z_h) = \mu_h$. Since we only consider symmetric equilibria: $Q^*(\mu_h) = \sum_{j=1}^n q^*(\mu_h) = nq^*(\mu_h)$. If there is no ambiguity, we note for the sake of simplicity $Q^* = Q^*(\mu_h)$ and $q^* = q^*(\mu_h)$. **FOC** becomes:

$$\underbrace{\mathbb{E}(Z|s) - na \times q^*}_{\text{Marginal Value } (V(q^*,n))} = \underbrace{\left(\frac{\partial F(nq^*)}{\partial q} q^* + F(nq^*) \right)}_{\text{Marginal Cost } (C(q^*,n))} \quad (\text{sFOC})$$

We implicitly differentiate **sFOC** with respect to the precision λ :

$$\frac{dq^*}{d\mu_h} = \frac{\frac{\partial \mathbb{E}(Z|s)}{\partial \mu_h}}{\frac{\partial \left(\frac{\partial F(nq^*)}{\partial q} q^* + F(nq^*) + na \right)}{\partial q^*}}$$

And we note the following function:

$$L_i(q^*, n) = \frac{\partial F(nq^*)}{\partial q} q^* + F(nq^*) + na \times q^*$$

We obtain the next equilibrium condition with respect to μ_h , with $\zeta = z_h - z_l$:

$$\frac{d q^*}{d \mu_h} = \frac{\zeta}{L'(Q^*)}$$

Appendix E

Proofs

Theorem 12. Under [Assumption A4](#), there exists a unique symmetric equilibrium to the Cournot Oligopsony with n buyers.

Proof. The First Order Condition [FOC](#) for any buyer j gives:

$$\frac{\partial S(q_j)}{\partial q_j} = 0 \Leftrightarrow \mathbb{E}(Z|S) - an \times q_j - F(Q) - \frac{\partial F(Q)}{\partial q_j} q_j = 0$$

Assume a function $g(x_i, X) = \mathbb{E}(Z|S) - anx_i - F(X) - F'(X)x_i$, defined in $\mathbb{R} \times [0, \xi]$ where the bound ξ is such that $F(\xi) = z_h + \epsilon$ with $\epsilon > 0$ and bounded.

Then a Cournot equilibrium must satisfy:

$$g(q_i, X) \leq 0, \quad q_i \geq 0, \quad g_i(q_i, X)q_i = 0 \quad \forall i = 1 \dots n \quad (\text{Optimality})$$

$$X = \sum_{i=1}^n q_i \quad (\text{Feasibility})$$

Note that g is continuous and twice-continuously differentiable. Furthermore g is decreasing in x_i , since the partial derivative in x_i is bounded away from 0: $\frac{\partial g(x_i, X)}{\partial x_i} = -(an + F'(X)) < 0$. Then, there exists a unique $x_i(X)$ such that $g(x_i(X), X) = 0$. The implicit function theorem is then applicable and $x_i(X)$ must be unique for each $X \in [0, \xi]$. Since this is true for a neighborhood of each X , $x_i(X)$ must be a continuous function. Finally, the Optimality condition has a unique solution:

$$q_i(X) = \max[0, x_i(X)] \quad \forall X \in [0, \xi]$$

We must show that the Feasibility condition has at least one solution. One can check that $q_i(X)$ and then $Q(X) = \sum_{i=1}^n q_i(X)$ are continuous on $[0, \xi]$. By definition, $q_i(X) \geq 0 \forall X \in [0, \xi]$ and $q_i(\xi) = 0$. Then, by continuity, and since $Q(0) \geq 0$ and $Q(\xi) = 0$, there is at least one $X \in [0, \xi]$ such that the Feasibility condition is met: $X = Q(X)$. Then a solution to both conditions is such that: $\bar{X} = Q(\bar{X})$ with $Q(\bar{X}) = \sum_{i=1}^n q_i(\bar{X})$.

Set $q_i = \max[0, \varphi(Q)]$. By the implicit function theorem on $g(q_i, Q) = 0$, we can write when $q_i > 0$ ¹:

$$\frac{d q_i}{d Q} = -\frac{F''(Q)q_i + F'(Q)}{an + F'(Q)}$$

Since $Q = \sum_{i=1}^n q_i$,

$$Q' = \sum_{i=1}^n \frac{q_i}{d Q}$$

$$Q' = -\frac{F''(Q)Q + nF'(Q)}{an + F'(Q)} < 1 \text{ if and only if } F''(Q)Q > -[(n+1)F'(Q) + an]$$

As already stated, $Q \geq 0$, and there is exactly one Cournot equilibrium such that $\bar{X} = Q(\bar{X})$ under [Assumption A4](#). ■

Lemma 27. Under [Assumption A4](#), for $s = s_h$ (resp. $s = s_l$), the equilibrium quantities increase (resp. decrease) with the belief μ_h . Furthermore, $\mathbb{E}(Q^*)$ is increasing (resp. decreasing) with respect to λ if $L(Q)$ is concave (resp. $L(Q)$ is convex). Equivalently, $\mathbb{E}(Q^*)$ is increasing (resp. decreasing) if $QF'''(Q) + (n+2)F''(Q) < 0$ (resp. $QF'''(Q) + (n+2)F''(Q) < 0$).

Proof. Assume the objective function of the public authority is the mean quantity. Under which condition should the public authority disclose information to maximize its objective function? We want to establish that if $(Q^*(\mu_h))$ is convex, $\mathbb{E}(Q^*)$ is maximal when $\lambda = 1$. When $\lambda = 1$, information being perfect, buyers believes with probability 1 that the state of the world is either h , either l . We want to show that $\forall(\mu_h(s_l), \mu_h(s_h))$ with $\mu_h(s_h) \in [\mu_0, 1]$, and $\mu_h(s_l) \in (0, \mu_0]$ (with $\bar{\lambda} < 1$ the precision giving $\mu_h(s_l), \mu_h(s_h)$):

$$\begin{aligned} \mathbb{P}(S = s_h | \lambda = 1)Q^*(1) + \mathbb{P}(S = s_l | \lambda = 1)Q^*(0) &> \mathbb{P}(S = s_h | \lambda = \bar{\lambda})Q^*(\mu_h(s_h)) \\ &+ \mathbb{P}(S = s_l | \lambda = \bar{\lambda})Q^*(\mu_h(s_l)) \\ \Leftrightarrow \mu_0 Q^*(1) + (1 - \mu_0)Q^*(0) &> \frac{\bar{\lambda}\mu_0}{\mu_h(s_h)}Q^*(\mu_h(s_h)) + \frac{(1 - \bar{\lambda})\mu_0}{\mu_h(s_l)}Q^*(\mu_h(s_l)) \end{aligned}$$

We know, by convexity of $Q^*(\mu_h)$, that:

$$Q^*(1) - Q^*(0) > \frac{Q^*(\mu_h(s_h)) - Q^*(0)}{\mu_h(s_h)} > \frac{Q^*(\mu_h(s_l)) - Q^*(0)}{\mu_h(s_l)}$$

Then, $\forall \alpha \in [0, 1]$,

$$Q^*(1) - Q^*(0) > \alpha \frac{Q^*(\mu_h(s_h)) - Q^*(0)}{\mu_h(s_h)} + (1 - \alpha) \frac{Q^*(\mu_h(s_l)) - Q^*(0)}{\mu_h(s_l)}$$

¹One can easily remark that for any $z_h, z_l > 0$, if $Q = 0$, then $x_i(Q) > 0$, which shows that 0 is not an equilibrium quantity.

Recalling that $\bar{\lambda} \frac{\mu_0}{\mu_h(s_h)} + (1 - \bar{\lambda}) \frac{\mu_0}{\mu_h(s_l)} = 1$, we can write:

$$\begin{aligned} \mu_0(Q^*(1) - Q^*(0)) &> \alpha \frac{\mu_0}{\mu_h(s_h)} (Q^*(\mu_h(s_h)) - Q^*(0)) + \\ &\quad (1 - \alpha) \frac{\mu_0}{\mu_h(s_l)} (Q^*(\mu_h(s_l)) - Q^*(0)) \\ \Leftrightarrow \mu_0(Q^*(1) - Q^*(0)) &> \bar{\lambda} \frac{\mu_0}{\mu_h(s_h)} Q^*(\mu_h(s_h)) + (1 - \bar{\lambda}) \frac{\mu_0}{\mu_h(s_l)} Q^*(\mu_h(s_l)) - Q^*(0) \\ \Leftrightarrow \mu_0 Q^*(1) + (1 - \mu_0) Q^*(0) &> \bar{\lambda} \frac{\mu_0}{\mu_h(s_h)} Q^*(\mu_h(s_h)) + (1 - \bar{\lambda}) \frac{\mu_0}{\mu_h(s_l)} Q^*(\mu_h(s_l)) \end{aligned}$$

If the function $Q^*(\mu_h)$ is concave, analog computations show that the mean quantity is maximal when $\lambda = \frac{1}{2}$. \blacksquare

Lemma 28. $\mathbb{E}(P^*)$ is increasing (resp. decreasing) with respect to λ if the price is convex in the belief μ_h , i.e. if $\frac{F''(Q)}{F'(Q)} > \frac{L''(Q)}{L'(Q)}$ (resp. $\frac{F''(Q)}{F'(Q)} < \frac{L''(Q)}{L'(Q)}$).

$\mathbb{E}(P)$ is increasing (resp. decreasing) if:

$$\eta''(Q) < (>) \frac{F''(Q)}{F'(Q)} (\eta'(Q) + an) \equiv \rho$$

Proof. We want to show that $\frac{F''(Q)}{F'(Q)} > (<) \frac{L''(Q)}{L'(Q)} \Leftrightarrow \eta''(Q) < (>) \rho$. Note that:

$$\begin{aligned} \frac{F''(Q)}{F'(Q)} > \frac{L''(Q)}{L'(Q)} \\ \Leftrightarrow QF'''(Q) + (n+2)F''(Q) &< (QF''(Q) + (n+1)F'(Q) + an) \frac{F''(Q)}{F'(Q)} \\ \Leftrightarrow QF'''(Q) + 2F''(Q) &< (QF''(Q) + F'(Q) + an) \frac{F''(Q)}{F'(Q)} \\ \Leftrightarrow \eta''(Q) &< \frac{F''(Q)}{F'(Q)} (\eta'(Q) + an) \end{aligned}$$

Corollary 2. $\mathbb{E}(P^*)$ is increasing (resp. decreasing) if:

$$\begin{cases} \text{if } F'' > 0, a > (<) \frac{F'(Q)\eta''(Q)}{nF'''(Q)} - \frac{\eta'(Q)}{n} \\ \text{if } F'' < 0, a < (>) \frac{F'(Q)\eta''(Q)}{nF'''(Q)} - \frac{\eta'(Q)}{n} \end{cases}$$

Proof. From **Lemma 28**, $\mathbb{E}(P^*)$ is increasing if:

$$\begin{aligned} \eta''(Q) &< \frac{F''(Q)}{F'(Q)} (\eta'(Q) + an) \\ \Leftrightarrow anF'''(Q) &> F'(Q)\eta''(Q) + \eta'(Q)F''(Q) \\ \Leftrightarrow \begin{cases} \text{if } F'' > 0, a > \frac{F'(Q)\eta''(Q)}{nF'''(Q)} - \frac{\eta'(Q)}{n} \\ \text{if } F'' < 0, a < \frac{F'(Q)\eta''(Q)}{nF'''(Q)} - \frac{\eta'(Q)}{n} \end{cases} \end{aligned}$$

Lemma 29. If $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$, then individual profits are non decreasing and aggregated profit increases with the precision λ of the signal.

Proof. We note $Q = Q^*$ for simplicity. Assume $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$. For owners with bid under $F(Q(\mu_h = 0))$, i.e. the ones that always sell, the expected profit naturally increases as well with respect to λ . Thus, they prefer $\lambda = 1$. Note also that owners with bids above $F(Q(\mu_h = 1))$, i.e. the ones that never sell, do not see their situation evolve.

Consider now that the precision is set to a given $\tilde{\lambda} \in [\frac{1}{2}, 1)$, giving two beliefs according to the signal received: $\mu_h(s_h)$ and $\mu_h(s_l)$. Owners with bids above $F(Q(\mu_h(s_h)))$ but below $F(Q(1))$, do not sell when the precision is set at $\tilde{\lambda}$ whilst they would sell if λ was set to 1 when the state of the world is high. All of them see their expected profit increase with respect to λ .

It is still true for owners whose bid is above $F(Q(\mu_h(s_l)))$ (but below $F(Q(\mu_h(s_h)))$). They sell only if the signal is high. Then, if there is an increase in $F(Q(\mu_h(s_h)))$ their individual profits increase. We show in the next step why they do not prefer a decrease in λ (i.e. the possibility to sell in both cases).

As well, for owners with bids included in $[F(Q(\mu_h = 0)), F(Q(\mu_h(s_l)))]$, an ambiguity may arise because their quantity sold may be reduced when λ increases. Initially, they sell in both cases. However, if $\lambda = 1$, they do not sell anymore when the signal is low. Their expected profit with bid b_i in first case is:

$$\begin{aligned} \Pi(\tilde{\lambda}) &= \mathbb{P}(S = s_h | \lambda = \tilde{\lambda})F(Q(\mu_h(s_h))) + \mathbb{P}(S = s_l | \lambda = \tilde{\lambda})F(Q(\mu_h(s_l))) - b_i \\ &= \frac{\tilde{\lambda}\mu_0}{\mu_h(s_h)}F(Q(\mu_h(s_h))) + \frac{(1 - \tilde{\lambda})\mu_0}{\mu_h(s_l)}F(Q(\mu_h(s_l))) - b_i \end{aligned}$$

Their expected profit in second case is:

$$\Pi(\lambda = 1) = \mu_0[F(Q(1)) - b_i]$$

By assumption, we know that $\mathbb{E}(P(\tilde{\lambda})) < \mathbb{E}(P(\lambda = 1))$ for any $\tilde{\lambda} \in [\frac{1}{2}, 1)$.

Then, since $F(Q(0)) < b_i$ for any owner i considered here:

$$\begin{aligned} &\mathbb{E}(P(\tilde{\lambda})) < \mathbb{E}(P(\lambda = 1)) \\ \Leftrightarrow &\frac{\tilde{\lambda}\mu_0}{\mu_h(s_h)}F(Q(\mu_h(s_h))) + \frac{(1 - \tilde{\lambda})\mu_0}{\mu_h(s_l)}F(Q(\mu_h(s_l))) < \mu_0F(Q(1)) + (1 - \mu_0)F(Q(0)) \\ \Leftrightarrow &\frac{\tilde{\lambda}\mu_0}{\mu_h(s_h)}F(Q(\mu_h(s_h))) + \frac{(1 - \tilde{\lambda})\mu_0}{\mu_h(s_l)}F(Q(\mu_h(s_l))) - (1 - \mu_0)F(Q(0)) - \mu_0b_i < \\ &\mu_0F(Q(1)) - \mu_0b_i \\ \Rightarrow &\frac{\tilde{\lambda}\mu_0}{\mu_h(s_h)}F(Q(\mu_h(s_h))) + \frac{(1 - \tilde{\lambda})\mu_0}{\mu_h(s_l)}F(Q(\mu_h(s_l))) - b_i < \mu_0[F(Q(1)) - b_i] \\ \Leftrightarrow &\Pi(\tilde{\lambda}) < \Pi(\lambda = 1) \end{aligned}$$

Therefore, each individual profit increases or stays null when information gains precision. Immediately, the aggregated profit increases with the precision since it is the sum of the individual profits and since there are more owners selling their capacity even if the mean quantities may decrease with precision. ■

Proposition 13. With a monopsony, i.e. $n = 1$, the buyers' surplus increases with information precision (increases in λ). The monopsony shall always prefer $\lambda = 1$.

Proof. We first write the expected surplus:

$$\begin{aligned} S(Q_S) &= \int_0^{Q_S} \mathbb{E}(Z|S) - ax - F(Q_S) dx \\ \Rightarrow S(Q^*(\mu_h)) &= Q^*(\mu_h) \left(\mu_h \zeta + z_l - \frac{1}{2} a Q^*(\mu_h) - F(Q^*(\mu_h)) \right) \end{aligned}$$

Since $\mathbb{E}(Z|S) = \mu_h z_h + (1 - \mu_h) z_l = \mu_h \zeta + z_l$ and the derivative of $-\frac{1}{2} a x^2$ is $-ax$.

Then, we can deduce the derivative with respect to μ_h when there is just one buyer:

$$\begin{aligned} \frac{\partial S(Q^*(\mu_h))}{\partial \mu_h} &= \frac{\partial Q^*(\mu_h)}{\partial \mu_h} \underbrace{(\mu_h \zeta + z_l - a Q^*(\mu_h) - F(Q^*(\mu_h)) - F'(Q^*(\mu_h)) Q^*(\mu_h))}_{=0 \text{ if sFOC}} \\ &\quad + Q^*(\mu_h) \zeta \\ \Rightarrow \frac{\partial S(Q^*(\mu_h))}{\partial \mu_h} &= Q_h^*(\mu_h) \zeta \quad (\text{Monopsony's Surplus Derivative}) \end{aligned}$$

Then, $\frac{\partial^2 S(Q^*(\mu_h))}{\partial \mu_h^2} = \frac{n \zeta^2}{L'(Q_h^*(\mu_h))} > 0$ ■

Proposition 14. If $n > 1$, $\mathbb{E}(S(Q^*))$ is increasing (resp. decreasing) with respect to λ if the surplus is convex in the belief μ_h , i.e. if $(1 - \frac{(n-1)F'(Q)}{L'(Q)})Q$ is increasing (resp. decreasing). Then, in the first case, the surplus maximizer precision is $\lambda_S = 1$, and in the second case $\lambda_S = \frac{1}{2}$. As well, the surplus is increasing (resp. decreasing) with information precision if:

$$\eta''(Q) > (<) \rho - \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)QF'(Q)} \equiv \underline{\rho}$$

Proof. By dividing the total surplus (see the proof of [Proposition 13](#)) by n , and by writing the surplus as a function of individual quantities $(q = q^*(\mu_h) = \frac{Q^*(\mu_h)}{n})$: the expected surplus for any buyer j at the equilibrium is:

$$S_j(q) = q \left(\mu_h \zeta + z_l - \frac{1}{2} a n q - F(nq) \right)$$

By differentiating with respect to μ_h :

$$\begin{aligned}\frac{\partial S_j(q)}{\partial \mu_h} &= \frac{dq}{d\mu_h} (\mu_h \zeta + z_l - anq - F(nq) - nqF'(nq)) + q\zeta \\ &= -\frac{dq}{d\mu_h} ((n-1)qF'(nq)) + q\zeta \\ &= q\zeta \left(1 - (n-1) \frac{F'(nq)}{L'(nq)} \right)\end{aligned}$$

Differentiating up to the order two:

$$\begin{aligned}\frac{\partial^2 S_j(q)}{\partial \mu_h^2} &= \frac{\zeta}{d\mu_h} \left(\left(1 - (n-1) \frac{F'(nq)}{L'(nq)} \right) - (n-1)nq \left(\frac{F''(nq)L'(nq) - F'(nq)L''(nq)}{L'(nq)^2} \right) \right) \\ &= \frac{\zeta}{L'(Q)^3} (L'(Q)(L'(Q) - (n-1)F'(Q)) - (n-1)Q(F''(Q)L'(Q) - F'(Q)L''(Q)))\end{aligned}$$

This second derivative is positive if and only if, $\forall Q = Q^*(\mu_h)$ with $\mu_h \in [0, 1]$:

$$\begin{aligned}F'(Q)L''(Q) + \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)Q} &> F''(Q)L'(Q) \\ \Leftrightarrow QF'''(Q) + 2F''(Q) + nF''(Q) &> F''(Q)L'(Q) - \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)QF'(Q)} \\ \Leftrightarrow \eta''(Q) &> \frac{F''(Q)(QF''(Q) + (n+1-n)F'(Q) + an)}{F'(Q)} - \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)QF'(Q)} \\ \Leftrightarrow \eta''(Q) &> \rho - \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)QF'(Q)}\end{aligned}$$

■

Corollary 3. When $(n-1)(QF''(Q) + F'(Q))^2 - 4QF'(Q)(QF'''(Q) + (n+2)F''(Q)) = (n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) \leq 0$ the surplus is increasing with information precision.

Note

$$a_{1s} = -\frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

and

$$a_{2s} = \frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

When $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) > 0$ the surplus is increasing (resp. decreasing) with information precision if:

$$\left\{ \begin{array}{l} \text{in case where } a_{2s} < 0 \\ \text{in case where } a_{2s} > 0 \text{ and } a_{1s} < 0, \text{ if } a > (<)a_{2s} \\ \text{in case where } a_{1s} > 0, \text{ if } a > a_{2s} \text{ or } a < a_{1s}, (\text{ if } a_{1s} < a < a_{2s}) \end{array} \right.$$

Proof. By the proof of [Proposition 14](#), we know that the surplus is increasing with information precision if:

$$F'(Q)L''(Q) + \frac{L'(Q)(L'(Q) - (n-1)F'(Q))}{(n-1)Q} - F''(Q)L'(Q) > 0$$

$$\Leftrightarrow \left(F'(Q)L''(Q) + \frac{L'(Q)(L'(Q) - (n-1)\eta'(Q))}{(n-1)Q} > 0 \right)$$

Assume $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) \leq 0$. Then,

$$F'(Q)L''(Q) \geq \frac{(n-1)\eta'(Q)^2}{4Q}$$

Thus,

$$\begin{aligned} & F'(Q)L''(Q) + \frac{L'(Q)(L'(Q) - (n-1)\eta'(Q))}{(n-1)Q} \\ & > \frac{(n-1)\eta'(Q)^2}{4Q} + \frac{L'(Q)(L'(Q) - (n-1)\eta'(Q))}{(n-1)Q} \\ & = \frac{(n-1)^2\eta'(Q)^2 + 4L'(Q)^2 - 4L'(Q)(n-1)\eta'(Q)}{4(n-1)Q} \\ & = \frac{((n-1)\eta'(Q) - 2L'(Q))^2}{4(n-1)Q} > 0 \end{aligned}$$

So, if $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) \leq 0$, the surplus is maximized with full information, whatever the value of a .

Now, assume that $(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q) > 0$.

$F'(Q)L''(Q) + \frac{L'(Q)(L'(Q) - (n-1)\eta'(Q))}{(n-1)Q} = 0$ is equivalent to the following polynomial:

$$L'(Q)^2 - (n-1)\eta'(Q)L'(Q) + (n-1)QF'(Q)L''(Q) = 0$$

It is equivalent with respect to a to:

$$\begin{aligned} & (QF''(Q) + (n+1)F'(Q) + an)^2 - (n-1)\eta'(Q)(QF''(Q) + (n+1)F'(Q) + an) \\ & + (n-1)QF'(Q)L''(Q) = 0 \\ \Leftrightarrow & a^2n^2 + ((n+3)F'(Q) - (n-3)QF''(Q))an \\ & + (QF''(Q) + (n+1)F'(Q))(2F'(Q) - (n-2)QF''(Q)) + (n-1)QF'(Q)L''(Q) = 0 \end{aligned}$$

One can check that the solutions to this polynomial with respect to a are:

$$a_{1s} = -\frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

and

$$a_{2s} = \frac{\sqrt{(n-1)}\sqrt{(n-1)\eta'(Q)^2 - 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - \frac{F'(Q) - QF''(Q)}{2}$$

It is direct that this polynomial is convex since the only term of order 2 is $a^2n^2 > 0$. Thus the polynomial is negative between the roots and positive elsewhere. ■

Lemma 30. $\mathbb{E}(\Pi(Q)|\lambda)$ is increasing (resp. decreasing) in λ if the profit is convex in the belief μ_h , i.e. if $\frac{QF'(Q)}{L'(Q)}$ is an increasing (resp. decreasing) function. Then in the first case, the profit maximizer precision is $\lambda_P = 1$, and in the second case $\lambda_P = \frac{1}{2}$. As well, the profit is increasing (resp. decreasing) with information precision if:

$$\eta''(Q) < (>)\rho(Q) + \underbrace{\frac{L'(Q)}{Q}}_{>0} \equiv \bar{\rho}$$

Proof. Note $Q = Q^*(\mu_h)$.

$$\Pi(Q) = QF(Q) - \int_0^Q F(x)dx$$

Then, by differentiating the profit with respect to μ_h :

$$\begin{aligned} \frac{\partial \Pi(Q)}{\partial \mu_h} &= \frac{\partial Q}{\partial \mu_h} [QF(Q) + F(Q)] - \frac{\partial Q}{\partial \mu_h} F(Q) \\ \frac{\partial \Pi(Q)}{\partial \mu_h} &= \frac{\partial Q}{\partial \mu_h} QF(Q) \\ \frac{\partial \Pi(Q)}{\partial \mu_h} &= \frac{n\zeta QF'(Q)}{L'(Q)} \end{aligned}$$

Thus, the expected profit is increasing if $\frac{QF'(Q)}{L'(Q)}$ is increasing in Q . The decreasing profit case works as well.

To conclude:

$$\begin{aligned} &\left(\frac{QF'(Q)}{L'(Q)}\right)' > 0 \\ \Leftrightarrow &\frac{L'(Q)F'(Q)}{L'(Q)^2} + \frac{QL'(Q)F''(Q)}{L'(Q)^2} > \frac{QL''(Q)F'(Q)}{L'(Q)^2} \\ \Leftrightarrow &L'(Q)(F'(Q) + QF''(Q)) > QL''(Q)F'(Q) \\ \Leftrightarrow &Q\eta''(Q) + nQF''(Q) < \frac{L'(Q)(F'(Q) + QF''(Q))}{F'(Q)} \\ \Leftrightarrow &\eta''(Q) + nF''(Q) < \frac{L'(Q)}{Q} + \frac{(\eta'(Q) + an + nF'(Q))F''(Q)}{F'(Q)} \\ \Leftrightarrow &\eta''(Q) + nF''(Q) < \frac{L'(Q)}{Q} + \rho + nF''(Q) \\ \Leftrightarrow &\eta''(Q) < \rho(Q) + \underbrace{\frac{L'(Q)}{Q}}_{>0} \equiv \bar{\rho} \end{aligned}$$

Corollary 4. Note $a_P = \frac{QF'(Q)L''(Q)}{n\eta'(Q)} - \frac{\eta'(Q)}{n} - F'(Q)$. The profit is increasing (resp. decreasing) with information precision if:

$$\begin{cases} \text{In case where } \eta'(Q) < 0, \text{ if } a > (<)a_P \\ \text{In case where } \eta'(Q) < 0, \text{ if } a < (>)a_P \end{cases}$$

When the supply curve F is convex, increasing parameter a increases unambiguously the span (of η'') where the owners prefer information. Otherwise, increasing parameter a decreases the span (of η'') where the owners prefer information if $\eta'(Q) = QF''(Q) + F'(Q) < 0$.

Proof. By [Lemma 30](#), the profit is increasing with information precision if:

$$\eta''(Q) < \rho(Q) + \underbrace{\frac{L'(Q)}{Q}}_{>0} \equiv \bar{\rho}$$

Re-writing the expression and assuming that $\eta'(Q) > 0$, we get:

$$\begin{aligned} an \frac{\eta'(Q)}{QF'(Q)} &> \eta''(Q) - \frac{F''(Q)\eta'(Q)}{F'(Q)} - \frac{\eta'(Q) + nF'(Q)}{Q} \\ \Leftrightarrow a &> \frac{QF'(Q)\eta''(Q)}{n\eta'(Q)} - \frac{QF''(Q)}{n} - \frac{F'(Q)(\eta'(Q) + nF'(Q))}{n\eta'(Q)} \\ \Leftrightarrow a &> \frac{QF'(Q)\eta''(Q)}{n\eta'(Q)} - \frac{\eta'(Q)}{n} - \frac{F'(Q)^2}{\eta'(Q)} \\ \Leftrightarrow a &> \frac{QF'(Q)L''(Q)}{n\eta'(Q)} - \frac{QF'(Q)nF''(Q)}{n\eta'(Q)} - \frac{\eta'(Q)}{n} - \frac{nF'(Q)^2}{n\eta'(Q)} \\ \Leftrightarrow a &> \frac{QF'(Q)L''(Q)}{n\eta'(Q)} - \frac{F'(Q)n(QF''(Q) + F'(Q))}{n\eta'(Q)} - \frac{\eta'(Q)}{n} \\ \Leftrightarrow a &> \frac{QF'(Q)L''(Q)}{n\eta'(Q)} - \frac{\eta'(Q)}{n} - F'(Q) \end{aligned}$$

We solve by analogy the case with $\eta'(Q) < 0$. ■

Proposition 15. Whenever $n \geq 1$, if $Q \left(1 + \frac{F'(Q)}{L'(Q)}\right)$ is increasing (resp. decreasing), then the welfare is increasing (resp. decreasing) with information precision. As well, the welfare is increasing (resp. decreasing) with information precision if:

$$\eta''(Q) < (>)\rho(Q) + \underbrace{\frac{L'(Q)(L'(Q) + F'(Q))}{QF'(Q)}}_{>0} \equiv \bar{\rho}$$

Proof. The economic welfare is the sum of the buyers' surplus and the owners' profit. Then, we can write:

$$W(Q) = \int_0^Q D^{-1}(x) - P(x)dx = \int_0^Q \mathbb{E}(Z|S) - ax - F(x)dx$$

Therefore, the expected economic welfare at equilibrium can be written (with $Q = Q^*(\mu_h)$):

$$W(Q) = Q \left(\mu_h \zeta + z_l - \frac{1}{2}aQ \right) - \int_0^Q F(q)dq$$

We can then deduce the derivative with respect to λ :

$$\frac{\partial W(Q)}{\partial \mu_h} = \frac{\partial Q}{\partial \mu_h} (\mu_h \zeta + z_l - aQ - F(Q)) + Q\zeta$$

From [sFOC](#) we know that $\mu_h \zeta + z_l - aQ - F(Q) = \frac{Q}{n}F'(Q)$. It follows:

$$\begin{aligned} \frac{\partial W(Q)}{\partial \mu_h} &= \frac{n\zeta}{L'(Q)} \frac{QF'(Q)}{n} + Q\zeta \\ &= \zeta \times Q \left(1 + \frac{F'(Q)}{L'(Q)} \right) \end{aligned}$$

Consequently, the economic welfare is increasing in the precision if $x \mapsto x \left(1 + \frac{F'(x)}{L'(x)} \right)$ is an increasing function.

To conclude:

$$\begin{aligned} &\left(Q \left(1 + \frac{F'(Q)}{L'(Q)} \right) \right)' > 0 \\ \Leftrightarrow &\frac{QL''(Q)F'(Q)}{L'(Q)^2} < \frac{L'(Q)^2 + L'(Q)F'(Q)}{L'(Q)^2} + \frac{QL'(Q)F''(Q)}{L'(Q)^2} \\ \Leftrightarrow &QL''(Q)F'(Q) < L'(Q)(L'(Q) + F'(Q) + QF''(Q)) \\ \Leftrightarrow &Q\eta''(Q) + nQF''(Q) < \frac{L'(Q)(L'(Q) + F'(Q) + QF''(Q))}{F'(Q)} \\ \Leftrightarrow &\eta''(Q) + nF''(Q) < \frac{L'(Q)(L'(Q) + F'(Q))}{Q} + \frac{(\eta'(Q) + an + nF'(Q))F''(Q)}{F'(Q)} \\ \Leftrightarrow &\eta''(Q) + nF''(Q) < \frac{L'(Q)(L'(Q) + F'(Q))}{Q} + \rho + nF''(Q) \\ \Leftrightarrow &\eta''(Q) < (>)\rho(Q) + \underbrace{\frac{L'(Q)(L'(Q) + F'(Q))}{QF'(Q)}}_{>0} \end{aligned}$$

■

Corollary 5. The welfare is increasing with information precision when:

$$(QF''(Q) + F'(Q))^2 + 4QF'(Q)(QF'''(Q) + (n+2)F''(Q)) = \eta'(Q)^2 + 4QF'(Q)L''(Q) \leq 0$$

Note

$$a_{1w} = -\frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

and

$$a_{2w} = \frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

And, the welfare is increasing (resp. decreasing) when $\eta'(Q)^2 + 4QF'(Q)L''(Q) > 0$ if:

$$\begin{cases} \text{in case where } a_{2w} < 0 \\ \text{in case where } a_{2w} > 0 \text{ and } a_{1w} < 0, \text{ if } a > (<)a_{2w} \\ \text{in case where } a_{1w} > 0, \text{ if } a > a_{2w} \text{ or } a < a_{1w}, (\text{ if } a_{1w} < a < a_{2w}) \end{cases}$$

When the supply curve F is convex, increasing parameter a increases unambiguously the span (of η'') where the welfare increases with information. Otherwise, increasing parameter a decreases the span (of η'') where the welfare increases with information if $\eta'(Q) = QF''(Q) + F'(Q) < 0$.

Proof. By the proof of [Proposition 15](#), the welfare is increasing with information precision if:

$$\begin{aligned} QL''(Q)F'(Q) &< L'(Q)(L'(Q) + F'(Q) + QF''(Q)) \\ (\Leftrightarrow QL''(Q)F'(Q) - L'(Q)(L'(Q) + \eta'(Q)) &< 0) \end{aligned}$$

Assume $\eta'(Q)^2 + 4QF'(Q)L''(Q) \leq 0$. Then,

$$QL''(Q)F'(Q) \leq -\frac{\eta'(Q)^2}{4}$$

Thus,

$$\begin{aligned} QF'(Q)L''(Q) - L'(Q)(L'(Q) + \eta'(Q)) &< -\frac{\eta'(Q)^2}{4} - L'(Q)(L'(Q) + \eta'(Q)) \\ &= -\frac{\eta'(Q)^2}{4} - L'(Q)^2 - L'(Q)\eta'(Q) \\ &= -\frac{(\eta'(Q) - 2L'(Q))^2}{4} < 0 \end{aligned}$$

So, if $\eta'(Q)^2 + 4QF'(Q)L''(Q) \leq 0$, the welfare is maximized with full information, whatever the value of a .

Now, assume that $\eta'(Q)^2 + 4QF'(Q)L''(Q) > 0$.

$QL''(Q)F'(Q) - L'(Q)(L'(Q) + \eta'(Q)) = 0$ is equivalent to the following polynomial:

$$(QF''(Q) + (n+1)F'(Q) + an)^2 + \eta'(Q)(QF''(Q) + (n+1)F'(Q) + an) - QL''(Q)F'(Q) = 0$$

It is equivalent with respect to a to:

$$\begin{aligned} & a^2n^2 + ((2n+3)F'(Q) + 3QF''(Q))an \\ & + (QF''(Q) + (n+1)F'(Q))((n+2)F'(Q) + 2QF''(Q)) + QL''(Q)F'(Q) = 0 \end{aligned}$$

One can check that the solutions to this polynomial with respect to a are:

$$a_{1w} = -\frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

and

$$a_{2w} = \frac{\sqrt{\eta'(Q)^2 + 4QF'(Q)L''(Q)}}{2n} - \frac{3\eta'(Q)}{2n} - F'(Q)$$

It is direct that this polynomial is convex since the only term of order 2 is $a^2n^2 > 0$. Thus the polynomial is negative between the roots and positive elsewhere. ■

Lemma 31. When $\mu_0 = \frac{1}{2}$, whenever $n \geq 1$ and (1) $\frac{F'(Q)}{L'(Q)}$ increases, $\lambda_C = 1$. If (2) $\frac{F'(Q)}{L'(Q)}$ decreases, $\lambda_C = 1$ when (2i) $m(Q) > 0$ and $\lambda_C = \frac{1}{2}$ when (2ii) $m(Q) < 0$.

Proof. First note that if (1) $\frac{F'(Q)}{L'(Q)}$ increases, by [Lemma 28](#), the *ex ante* mean price increases with respect to λ . Then, by [Lemma 29](#), all owners prefer the maximal precision (strictly for the ones who sell their capacity unit).

Now, assume $\frac{F'(Q)}{L'(Q)}$ decreases. Owners that always produce prefers $\lambda_C = \frac{1}{2}$ since their profit increase with the expected price, which is maximal when $\lambda = \frac{1}{2}$.

Owners with bids included in $[F(Q(\mu_h = 0)), \hat{b}]$ prefers $\lambda_C = \frac{1}{2}$ as well. Recalling that for them: $F(Q^*(\mu_0)) - b_i > \mu_h(F(Q^*(1)) - b_i)$, one can find that for any $\tilde{\lambda} \in (\frac{1}{2}, 1]$, $\Pi(\tilde{\lambda}) < \Pi(\lambda = \frac{1}{2})$. Indeed, we know that: $\mathbb{E}(F(Q^*|\tilde{\lambda})) < F(Q^*|\lambda = \frac{1}{2})$, then $\mathbb{E}(F(Q^*|\tilde{\lambda})) - b < F(Q^*|\lambda = \frac{1}{2}) - b$. And the owner prefers to always produce since if he does not produce in the low signal case, he gets at most $\Pi(\lambda = 1) (= \frac{1}{2}[F(Q^*(\mu_h)(\lambda = 1)) - b])$ which is less than $\Pi(\lambda = \frac{1}{2})$ by hypothesis.

Thus, if $m(Q) < 0$, a majority prefers the lowest precision and $\lambda_c = \frac{1}{2}$.

To see why $m(Q) > 0$ implies that a majority prefers $\lambda_C = 1$, note that owners that produce only in the high signal case choose $\lambda_C = 1$ to maximize $F(Q_h^*(\lambda))$. Then, we just have to show that each owner with $b > \hat{b}$ wants to produce only with the high signal.

We recall that for them: $\mathbb{E}(F(Q^*(\lambda = \frac{1}{2})) - b(Q)) < \frac{1}{2}(F(Q_h^*(\lambda = 1)) - b(Q))$.

For owners whose the bid is included in $(\hat{b}, F(Q^*(\lambda = \frac{1}{2}))]$, if they want to produce both when s_h and s_l , they should prefer $\lambda_C = \frac{1}{2}$ to maximize their profit. However

$\Pi(\lambda = 1) > \Pi(\lambda = \frac{1}{2})$ for them, since $b > \hat{b}$. Finally they should only produce in high signal case.

For owners whose the bid is included in $(F(Q^*(\lambda = \frac{1}{2})), F(Q_h^*(\lambda = 1))]$, they prefer $\lambda_c = 1$, because they never sell their capacity when $\lambda = \frac{1}{2}$ while they do produce in the high signal case.

Finally, if $m(Q) > 0$, a majority prefers the highest precision and $\lambda_c = 1$. ■

Lemma 32. Total quantities at the high and low equilibria are increasing concave functions with respect to n .

Proof. By transforming sFOC, we get $\mathbb{E}(Z|s) - \left(\frac{\partial F(Q^*)}{\partial q} \frac{Q^*}{n} + F(Q^*) + a \times Q^*\right) = 0$. The implicit differentiation with respect to n gives:

$$\begin{aligned} \frac{\partial Q}{\partial n} &= \frac{\frac{F'(Q^*)Q^*}{n^2}}{\left(\frac{F''(Q^*)Q^*}{n} + \frac{F'(Q^*)}{n} + F'(Q^*) + a\right)} = \frac{F'(Q^*)Q^*}{n(F''(Q^*)Q^* + (n+1)F'(Q^*) + an)} > 0 \\ \frac{\partial^2 Q}{\partial n^2} &= -\frac{F'(Q^*)Q^*(F''(Q^*)Q^* + (2n+1)F'(Q^*) + 2an)}{n^2(F''(Q^*)Q^* + (n+1)F'(Q^*) + an)^2} < 0 \end{aligned}$$
■

Lemma 33. For any thrice differentiable supply curve, it exists $N \in \mathbb{N}$ such that $\forall n > N$: If (a) $F'' < 0$, then (a2i) $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$ and $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$. If (b) $F'' > 0$, then (b2i) $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$ and $\frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$.

Proof. When buyers behave in a competitive way (i.e. $n \rightarrow \infty$), previous computations simplify since in equilibrium, the marginal value is equalized with the price. Essentially, $L'(Q^*)$ becomes $L'(Q^*) \equiv F'(Q) + a$, since we have the next competitive equilibrium condition:

$$\mathbb{E}(Z|s) - aQ^* = F(Q^*) \tag{CEC}$$

Again, by implicit differentiation, we can write:

$$\frac{dQ^*}{d\mu_h} = \frac{\zeta}{L'(Q^*)} = \frac{\zeta}{F'(Q^*) + a} > 0 \Rightarrow \frac{\partial^2 Q^*}{\partial \mu_h^2} = -\frac{\zeta F''(Q^*)}{(F'(Q^*) + a)^2} \tag{E.1}$$

The price's second derivative with respect to μ_h is:

$$\frac{\partial P^*(\mu_h)}{\partial \mu_h} = \frac{\partial F(Q^*(\mu_h))}{\partial Q^*(\mu_h)} \frac{\partial Q^*(\mu_h)}{\partial \mu_h} = F'(Q^*) \times \frac{\zeta}{F'(Q^*) + a} \Rightarrow \frac{\partial^2 P^*(\mu_h)}{\partial \mu_h^2} = \frac{a\zeta F''(Q)}{(F'(Q^*) + a)^2} \tag{E.2}$$

The result is direct. ■

Proposition 17. If (a) $F'' > 0$, then $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0 \Rightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$.

If (b) $F'' < 0$, then $\frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0 \Rightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$.

Proof. First recall that L' and F' are both positive by hypothesis. We know that $L''(Q) < 0 \Leftrightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$. As well, we know that $F''(Q)L'(Q) - F'(Q)L''(Q) > 0 \Leftrightarrow \frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$. It is directly seen that if (b), $L''(Q) > 0 \Leftrightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} < 0$ implies that $F''(Q)L'(Q) - F'(Q)L''(Q) < 0 \Leftrightarrow \frac{\partial \mathbb{E}(P^*)}{\partial \lambda} < 0$. Similarly, if (a), $L''(Q) < 0 \Leftrightarrow \frac{\partial \mathbb{E}(Q^*)}{\partial \lambda} > 0$ implies that $F''(Q)L'(Q) - F'(Q)L''(Q) > 0 \Leftrightarrow \frac{\partial \mathbb{E}(P^*)}{\partial \lambda} > 0$. By contrapositive, we can deduce both statements. ■

Proposition 18. When n is large enough, preferences can be characterized with respect to the curvature of the bid function:

F''		$-\frac{F'(F'+a)}{aQ}$	0	$\frac{F'+a}{Q}$	
Buyers' Surplus		$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = 1$	$\lambda_S = \frac{1}{2}$
Owners' Profit		$\lambda_P = \frac{1}{2}$	$\lambda_P = 1$	$\lambda_P = 1$	$\lambda_P = 1$
Majority Winning	if $m(Q) > 0$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_C = 1$
	if $m(Q) < 0$	$\lambda_C = \frac{1}{2}$	$\lambda_C = \frac{1}{2}$	$\lambda_C = 1$	$\lambda_C = 1$
Welfare		$\lambda_W = 1$			

Table E.1: Maximizing precision under perfect competition

Proof. Combining [Lemma 33](#) and [Lemma 31](#), we deduce directly the statements on the Majority Winning precision.

For any total demanded quantity Q , the buyers' surplus is:

$$S(Q) = \int_0^Q D^{-1}(x) - P(Q)dx = \int_0^Q \mathbb{E}(Z|S) - ax - F(Q)dx$$

Therefore, the expected buyers' surplus at equilibrium can be written²:

$$\mathbb{E}(S(Q^*(\mu_h))) = Q^*(\mu_h) \left(\mu_h \zeta + z_l - \frac{1}{2}aQ^*(\mu_h) - F(Q^*(\mu_h)) \right)$$

Then, we can deduce the derivative with respect to μ_h in the competitive case:

$$\begin{aligned} \frac{\partial \mathbb{E}(S(Q^*))}{\partial \mu_h} &= \frac{\partial Q^*}{\partial \mu_h} \underbrace{(\mu_h \zeta + z_l - aQ^* - F(Q^*) - F'(Q^*)Q^*)}_{=-Q^*F'(Q^*) \text{ by CEC}} + Q^* \zeta \\ \Rightarrow \frac{\partial \mathbb{E}(S(Q^*))}{\partial \mu_h} &= -\frac{\zeta Q^* F'(Q^*)}{F'(Q^*) + a} + Q^* \zeta = \frac{a\zeta Q^*}{F'(Q^*) + a} \\ \Rightarrow \frac{\partial^2 \mathbb{E}(S(Q^*))}{\partial \mu_h^2} &= \frac{\partial Q^*}{\partial \mu_h} \frac{a\zeta (F'(Q^*) + a) - a\zeta Q^* F''(Q^*)}{(F'(Q^*) + a)^2} \\ \Rightarrow \frac{\partial^2 \mathbb{E}(S(Q^*))}{\partial \mu_h^2} &= \frac{a\zeta^2}{(F'(Q^*) + a)^3} \times (F'(Q^*) + a - Q^* F''(Q^*)) \end{aligned} \quad (\text{E.3})$$

²Cf. proof of [Proposition 13](#).

The profit is expressed as follow (with Q is the aggregate quantities sold to the buyers):

$$\Pi(Q) = \int_0^Q F(Q) - F(x)dx$$

The derivative of the expected profit can be written³:

$$\begin{aligned} \frac{\partial \mathbb{E}(\Pi(Q^*))}{\partial \mu_h} &= \frac{d Q^*}{d \mu_h} \times Q^* F'(Q^*) = \frac{\zeta Q^* F'(Q^*)}{F'(Q^*) + a} \\ \Rightarrow \frac{\partial^2 \mathbb{E}(\Pi(Q^*))}{\partial \mu_h^2} &= \zeta \frac{d Q^*}{d \mu_h} \times \frac{(Q^* F''(Q^*) + F'(Q^*))(F'(Q^*) + a) - F''(Q^*)Q^* F'(Q^*)}{(F'(Q^*) + a)^2} \\ \Rightarrow \frac{\partial^2 \mathbb{E}(\Pi(Q^*))}{\partial \mu_h^2} &= \frac{\zeta^2}{(F'(Q^*) + a)^3} \times (aQ^* F''(Q^*) + aF'(Q^*) + F'(Q^*)^2) \end{aligned} \quad (\text{E.4})$$

Finally,

$$\begin{aligned} \frac{\partial^2 \mathbb{E}(W(Q^*))}{\partial \mu_h^2} &= \frac{\partial^2 \mathbb{E}(S(Q^*))}{\partial \mu_h^2} + \frac{\partial^2 \mathbb{E}(\Pi(Q^*))}{\partial \mu_h^2} \\ \Rightarrow \frac{\partial^2 \mathbb{E}(W(Q^*))}{\partial \mu_h^2} &= \frac{\zeta^2}{(F'(Q^*) + a)^3} \times (aF'(Q^*) + a^2 - aQ^* F''(Q^*)) + \\ &\quad \frac{\zeta^2}{(F'(Q^*) + a)^3} \times (aQ^* F''(Q^*) + aF'(Q^*) + F'(Q^*)^2) \\ \Rightarrow \frac{\partial^2 \mathbb{E}(W(Q^*))}{\partial \mu_h^2} &= \frac{\zeta^2}{(F'(Q^*) + a)} > 0 \end{aligned} \quad (\text{E.5})$$

The result is direct. ■

³Cf. proof of [Lemma 30](#).

Appendix F

Majority-winning with a low probability of having a high state of the world

if $\mu_0 < \frac{1}{2}$, the probability to get a high signal decreases when the precision increases. It means that a part of the owners who prefer 1 to $\frac{1}{2}$ will prefer an other λ . As well for a part of the owners who prefer $\frac{1}{2}$ to 1. We show that there is a set of λ such that each λ maximizes the profit of an owner i . Note:

Precision	Bid
$\lambda = \frac{1-\mu_0}{1-2\mu_0} - \frac{P^*(\mu_0) - b^o}{(1-2\mu_0)^2(P^*(\mu_h) - b^o)}$	$\underline{b}^o = P^*(\mu_0) - \frac{(1-\mu_0)\mu_0 P^{*'}(\mu_h)}{1-2\mu_0}$
$\bar{\lambda} = 1$	$\bar{b}^o = P^*(1) - \frac{(1-\mu_0)P^{*'}(1)}{1-2\mu_0}$

Table F.1: Bounds on the interior precisions and their associated bids

Lemma 34. *If $\mu_0 < \frac{1}{2}$, a non negligible part of owners prefer a precision different from $\frac{1}{2}$ and 1. All the owners bidding in the interval $[\underline{b}^o, \bar{b}^o]$ prefer a different precision between $\underline{\lambda}$ and 1*

Proof. Considering the profit $\Pi_{S_h}(\lambda, b) = \mathbb{P}(s = S_h)(P^*(\mu_h) - b)$ of owners selling their capacity only when the high signal is displayed, the set of the interior precisions maximizing the profit of at least (in fact at most as well) one owner is the set of precisions such that the first order condition is null and such that the profit exceeds the profit with $\lambda = \frac{1}{2}$, i.e.:

$$\lambda^M \in]\frac{1}{2}, 1[\text{ such that } (i) \lambda^M \text{ maximizes } \Pi_{S_h}(\lambda, b)$$

$$\text{and such that } (ii) \mathbb{P}(s = S_h | \lambda^M)(P^*(\mu_h | \lambda^M) - b) \geq P^*(\mu_0) - b$$

For (i), we characterize the set of precisions such that the first order condition of $\Pi_{S_h}(\lambda, b)$ is null. For each bid b , one can find the precision λ^m that maximizes the profit (conditionally on selling only when the signal is high):

$$\begin{aligned} & \arg \max_{\lambda} \mathbb{P}(s = S_h)(P^*(\mu_h) - b) \\ \Leftrightarrow & \frac{\partial \mathbb{P}(s = S_h)}{\partial \lambda} \times (P^*(\mu_h) - b) + \mathbb{P}(s = S_h) \times P^{*'}(\mu_h) = 0 \\ \Leftrightarrow & \lambda^m = \frac{(1 - \mu_0)((1 - 2\mu_0)(P^*(\mu_h) - b) - \mu_0 P^{*'}(\mu_h))}{(1 - 2\mu_0)^2(P^*(\mu_h) - b)} \\ \Leftrightarrow & \lambda^m = \frac{1 - \mu_0}{1 - 2\mu_0} - \frac{(1 - \mu_0)\mu_0 P^{*'}(\mu_h)}{(1 - 2\mu_0)^2(P^*(\mu_h) - b)} \end{aligned}$$

For (ii), we need to define $b^o(\lambda)$ such that the owner who bids b^o is indifferent between $\lambda = \frac{1}{2}$ and a certain $\lambda \in (\frac{1}{2}, 1]$ that induces μ_h when the high signal is sent, i.e. (with $P^*(\mu_h)$ the price when the precision λ and the high signal S_h induce μ_h):

$$\mathbb{P}(s = S_h)(P^*(\mu_h) - b^o) = P^*(\mu_0) - b^o \Leftrightarrow b^o(1 - \mathbb{P}(s = S_h)) = P^*(\mu_0) - \mathbb{P}(s = S_h)P^*(\mu_h)$$

Note that $\forall b > b^o(\lambda)$, $\Pi_{S_h}(\lambda, b) > P^*(\mu_0) - b$, since:

$$\mathbb{P}(s = S_h)(P^*(\mu_h) - b) > P^*(\mu_0) - b \Leftrightarrow b > b^o(\lambda) = \frac{P^*(\mu_0) - \mathbb{P}(s = S_h)P^*(\mu_h)}{1 - \mathbb{P}(s = S_h)}$$

Thus, the set of λ^M is the set:

$$\begin{aligned} \lambda \in]\frac{1}{2}, 1[\text{ such that: } & \exists i \text{ such that } \lambda = \lambda^m(b_i) \\ & \text{and such that } b_i > b^o(\lambda^m) \end{aligned}$$

Now, we establish the bounds of the sets of λ^M and the associated bids b_i .

By implicit differentiation of the first order condition on λ^m , we find:

$$\frac{d\lambda^m}{db} = -\frac{(1 - 2\mu_0) \times \mathbb{P}(s = S_h)^3}{P^{*''}(\mu_h) \times \mu_0^2(1 - \mu_0)^2} > 0$$

We find by implicit differentiation of b^o :

$$\frac{db^o}{d\lambda} = -\frac{\frac{\partial \Pi}{\partial \lambda}}{1 - \mathbb{P}(s = S_h)} \text{ and } \frac{d^2 b^o}{d\lambda^2} = \frac{(1 - 2\mu_0)\frac{\partial \Pi}{\partial \lambda} - (1 - \mathbb{P}(s = S_h))\frac{\partial^2 \Pi}{\partial \lambda^2}}{(1 - \mathbb{P}(s = S_h))^2}$$

$\frac{db^o}{d\lambda}$ is negative if $\frac{\partial \Pi}{\partial \lambda}$ is positive and vice versa.

$\frac{\partial \Pi}{\partial \lambda}$ being decreasing everywhere, b^o is decreasing in λ for $\lambda < \underline{\lambda}$, with $\underline{\lambda}$ being the precision such that $\frac{db^o}{d\lambda} = 0$. Conversely, b^o is increasing in λ for $\lambda > \underline{\lambda}$, making $\underline{b^o} = b^o(\underline{\lambda})$ the minimal bid such that an owner is indifferent between $\frac{1}{2}$ and a certain $\lambda > \frac{1}{2}$. However,

this certain λ , that we noted $\underline{\lambda}$ is not the lowest precision such that an owner is indifferent between $\frac{1}{2}$ and the said precision. Nevertheless, as we will see below, it is the lowest precision (but $\frac{1}{2}$) that can be selected by an owner.

Since λ^m is increasing in b , there is an unique $\lambda^* = \underline{\lambda}$ that maximizes the profit of the owner who bids \underline{b}^o , and there is an unique bid \bar{b}^o such that $\lambda^* = 1$.

For \underline{b}^o , since \underline{b}^o is defined as $\frac{\partial \Pi}{\partial \lambda} = 0$, $\lambda^m = \lambda^o$. For \bar{b}^o , it is sufficient to replace λ^* by 1 in the first order condition.

Then, for each bid $\bar{b} > b > \underline{b}$, a different $\lambda^m(b) > \frac{1}{2}$ maximizes the profit of the owner who bids b . λ^m being strictly increasing, it means that between \underline{b}^o and \bar{b}^o , all owners choose a different λ .

Precision	Bid
$\lambda = \frac{1-\mu_0}{1-2\mu_0} - \frac{P^*(\mu_0) - \underline{b}^o}{(1-2\mu_0)^2(P^*(\mu_h) - \underline{b}^o)}$	$\underline{b}^o = P^*(\mu_0) - \frac{(1-\mu_0)\mu_0 P^{*'}(\mu_h)}{1-2\mu_0}$
$\bar{\lambda} = 1$	$\bar{b}^o = P^*(1) - \frac{(1-\mu_0)P^{*'}(1)}{1-2\mu_0}$

Since $\underline{\mu}_h$ maximizes the profit for the bid \underline{b}^o , and since $\frac{d\lambda^*}{db} > 0$, for all the owners with $b > \underline{b}^o$, they prefer $\lambda^* > \underline{\lambda}$.

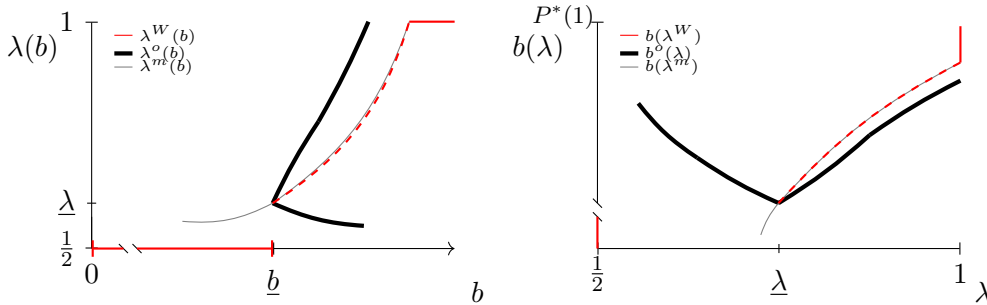


Figure F.1: Graphical example - Profit maximizing and indifference precisions (left-hand side), profit maximized and indifference bids (right-hand side)

On the left-hand side of [Figure F.1](#), we consider for each bid the precision that maximizes the profit (in red), and the precisions such that the owner is indifferent between $\frac{1}{2}$ and the said precisions (in black). Reciprocally, on the right-hand side, we draw the set of bids such that the owners maximize their profit with the given precision (in red) while the bids such that the owners are indifferent between $\frac{1}{2}$ and the said precision are in black. In both graphs, the grey line corresponds to the precisions that maximize the profit of the owner who sells only in the high signal case.

■

Few computations give:

$$\begin{aligned}\frac{\partial \hat{b}}{\partial \mu_0} &= (1 - \mu_0)P^{*'}(\mu_0) - (P^*(1) - P^*(\mu_0)) > 0 \text{ since } P^*(\mu_h) \text{ is concave,} \\ \frac{\partial \underline{b}}{\partial \mu_0} &= (1 - 2\mu_0)^2(P^{*'}(\mu_0) - P^{*'}(\mu_h)) + (1 - \mu_0)\mu_0(2P^{*'}(\mu_h) - (1 - 2\mu_0)(1 - \lambda)\lambda \frac{P^{*''}(\mu_h)}{\mathbb{P}(s=S_h)^2}) > 0, \\ \frac{\partial \bar{b}}{\partial \mu_0} &= -\frac{P^{*'}(1)}{(1-2\mu_0)^2} < 0\end{aligned}$$

Therefore, \hat{b} is increasing in μ_0 . Thus, if $\mu_0 > \frac{1}{2}$, there are more and more owners preferring $\frac{1}{2}$ as μ_0 increases –and less preferring 1.

If $\mu_0 < \frac{1}{2}$, there are also more owners preferring $\frac{1}{2}$ to 1 when μ_0 increases. As well, there are more owners preferring $\frac{1}{2}$ to any other precision when μ_0 increases because $P^*(\mu_0)$ increases. And, when μ_0 increases, more owners prefer 1 to the interior precisions, because the probability of receiving a high signal has increased. In other words, the preferences are more and more polarized, and both $\frac{1}{2}$ and 1 gain voters. At the limit $\mu_0 = \frac{1}{2}$, we have $\underline{b} = \hat{b} = \bar{b}$: $\frac{1}{2}$ and 1 are the only selected precisions.

If the owners are strategic in their choice, they all choose either $\frac{1}{2}$ or 1. Indeed, since $\lambda^m(b)$ is an increasing function, interior precisions can not be supported by more than one owners. Then, results remain unchanged:

Lemma 35. $\forall \mu_0$, whenever $n \geq 1$ and (1) $\frac{F'(Q)}{L'(Q)}$ increases, $\lambda_C = 1$. If (2) $\frac{F'(Q)}{L'(Q)}$ decreases and $\mu_0 \geq \frac{1}{2}$, $\lambda_C = 1$ when (2i) $m(Q) > 0$ and $\lambda_C = \frac{1}{2}$ when (2ii) $m(Q) < 0$.

Appendix G

Figures

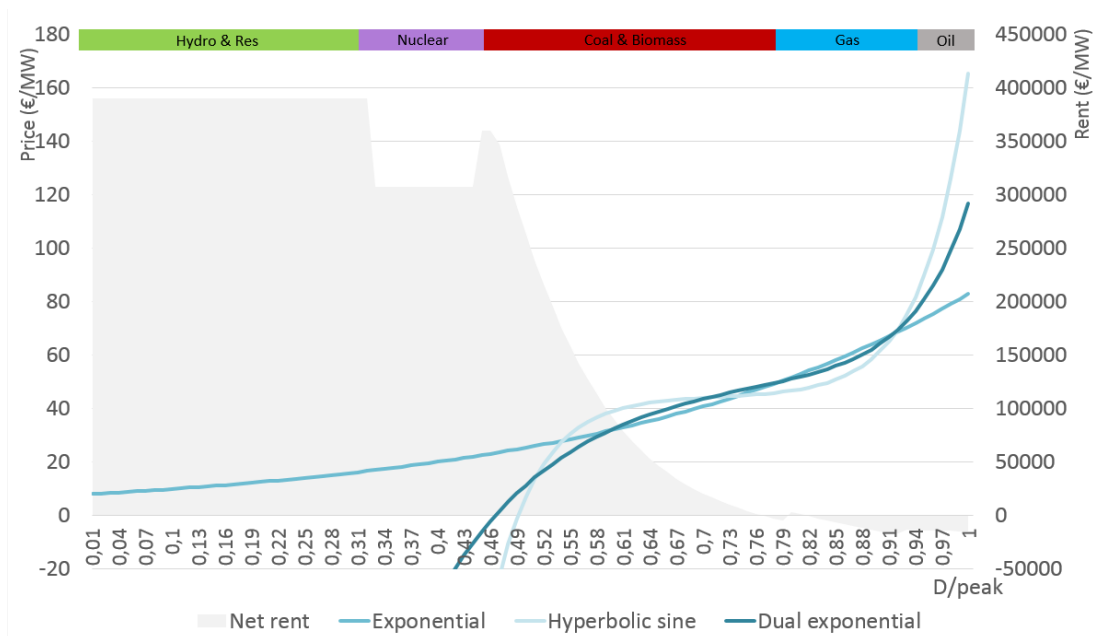


Figure G.1: Merit orders (He et al. [2013]) and net rent estimation (own calculations)

Interval	Interval (MW)	α	β	γ	δ
Quantile 1	[72000; 80000]	-2717529,46	108,35	-0,0014	$6,39 * 10^{-9}$
Quantile 2	(80000; 95000]	1895652,64	-66,68	0,0008	$-2,94 * 10^{-9}$
Quantile 3	(95000; 101000]	-3016576,63	89,09	-0,0008	$2,86E - 09 * 10^{-9}$

Table G.1: Coefficients of the Spline fit for the supply curve of Germany in 2010

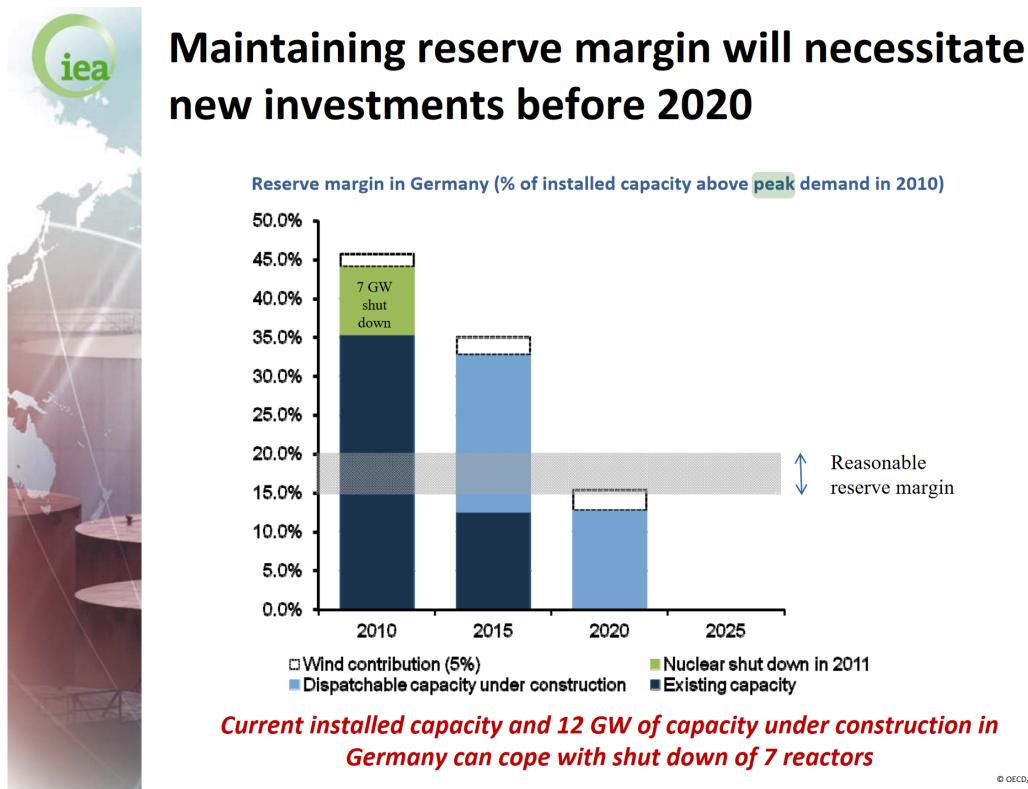


Figure G.2: Expected reserve margin evolution from IEA International Energy Agency [2010]

Forecast total obligation for France under different demand variants (in GW)

Demand variant	2016	2017	2018	2019	2020
High	90.4	91.2	92.0	92.6	93.4
Baseline	89.7	90.0	90.3	90.6	90.8
Stronger DSM	89.2	89.4	89.5	89.6	89.9
Low	88.8	88.6	88.4	88.3	88.0
Security factor	0.93*	0.93*	0.93*	0.93**	0.93**

* Security factor validated for the first two years of the mechanism in the order of 22 January 2015
 ** Security factor assumed to be the same for the following two years

Figure G.3: Forecast total obligation for France under different demand variants (in GW) RTE [2015]

Appendix H

Structural under procurement

$z_h = z_l$		$Q_1^t = 79.9 \text{ GW}$	$Q_2^t = 87.1 \text{ GW}$	$Q_3^t = 100.1 \text{ GW}$	$Q_4^t = 104.5 \text{ GW}$
$n = \infty$	$\forall a$	-	-	-	-
$n = 65$	$a = 50$	-12	-26	-2	-3
	$a = 20$	-29	-62	-6	-8
	$a = 10$	-55	-119	-12	-17
	$a = 5$	-104	-218	-24	-33
$n = 4$	$a = 50$	-178	-414	-40	-53
	$a = 20$	-401	-988	-102	-128
	$a = 10$	-694	-1 819	-213	-244
	$a = 5$	-1 112	-3 081	-467	-444

Table H.1: Differentials in capacity procurement (MW)

In terms of procurement (Table H.1), under procurements structurally increase as the level of competition (n) decreases. However, the size of market power is limited by the slope of the demand curve (implicit penalty). Indeed, it works as a retracting force since deviations from the competitive equilibrium are increasingly costly as a increases. Now, the slope of the supply curve also matters in terms of market power expression: the steeper it is, the more expensive it is to buy additional capacity. For instance, the Q_2^t case is characterized both by a greater F' and a much greater level of under procurement than the three other cases. Interestingly, we can observe that the effect of a on under procurement is not linear, and is modified with respect to the considered target. This effect obviously depends on the supply curve. When F' , is low (as in the third and fourth cases), the second derivative has a role to play. Indeed, when F'' is negative (as in the third case), the market power is decreasing in quantities: as a decreases, less and less quantities are procured. Under procurement is not only a way to decrease prices, but it also allows for more market power expression. On the contrary, F'' is positive at Q_4^t meaning that the marginal effect of reducing procurement on prices is decreasing. This somehow bounds the expression of market power and explains why under procurement is

relatively more important in the Q_3^t case than with Q_4^t (for $a = 5$) although the slope of the supply curve is locally more gentle.

Taking the most extreme case of under procurement, Q_2^t with a low slope, the effect of competition on procured quantities is represented graphically. The market power induced by 65 capacity homogeneous buyers with respect to perfect competition reduces procurement by 0.25% from 87100 MW to 86882 MW. If only 4 LSEs compete in the market, the capacity bought is further decreased to 84019 MW: the expression of market power allows capacity buyers to reduce their actual demand by 3.54% compared to the competitive state. As illustrated by Figure H.1, market power is not linearly decreasing in n . The purple curves represent L : as F' increases, the difference between the light curve ($n = 4$) and the dark curve ($n = \infty$) increases which means that the first one will equal $\mathbb{E}(Z)$ (first order condition) sooner than the other. It is also noticeable from the figure that high competition ($n = 65$) results in an almost linear L contrary to the low competition case ($n = 4$). This is only natural since the smaller the oligopsony, the more self conscious they are of the steepness of the supply curve. A steep supply curve means that additional capacity is increasingly costly, and the less numerous they are, the greater their impact on price. Consequently, they benefit more from their own decision of buying less. All in all, L in the low competition case and the medium one tend to diverge when the supply curve changes shape. It is coherent with what we found in the previous section: when the supply curve shape is modified, the relative increases of both the markdown derivative and the supply curve derivative are modified in different ways. When the markdown has a large weight in sFOC, it may affect the preferences for precision.

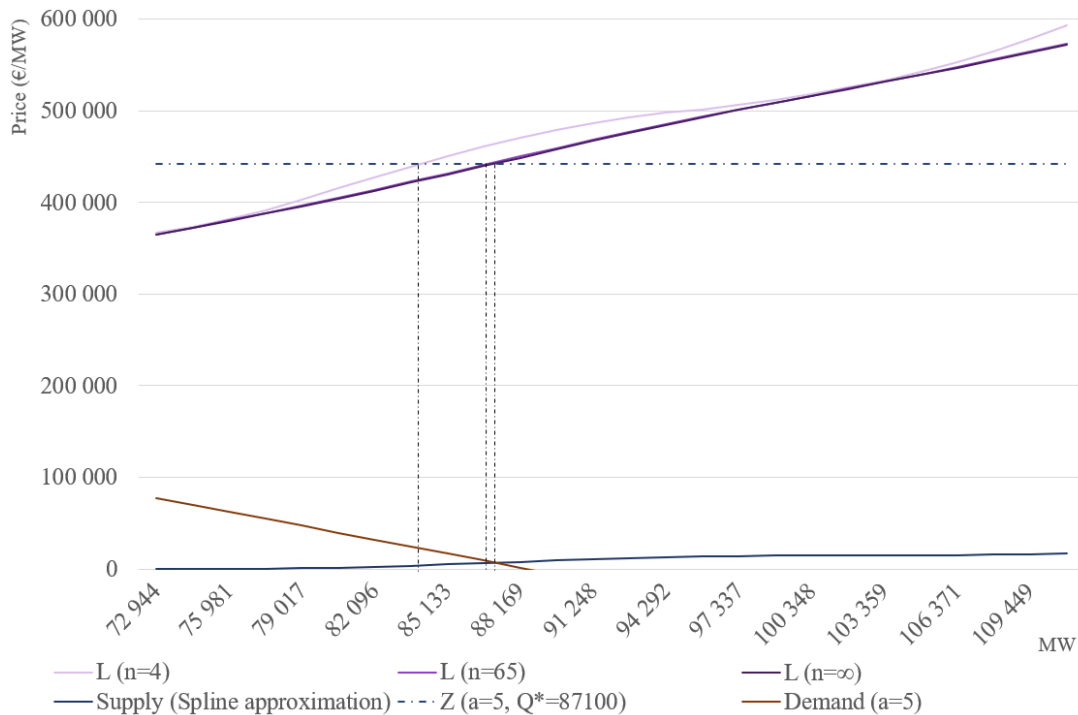


Figure H.1: Effect of competition on procured quantities for $a = 5$, $Z \approx 442382, 41$

Appendix I

Sensitivity on uncertainty

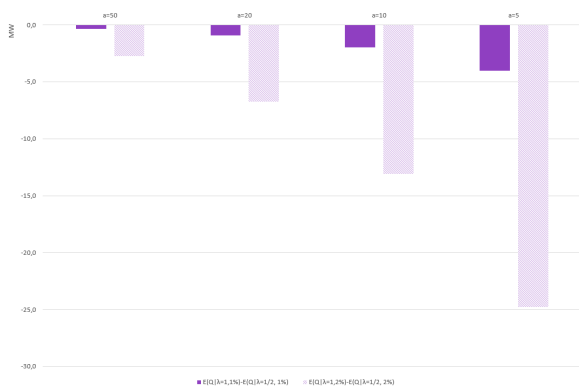


Figure I.1: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_1^t = 79900$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

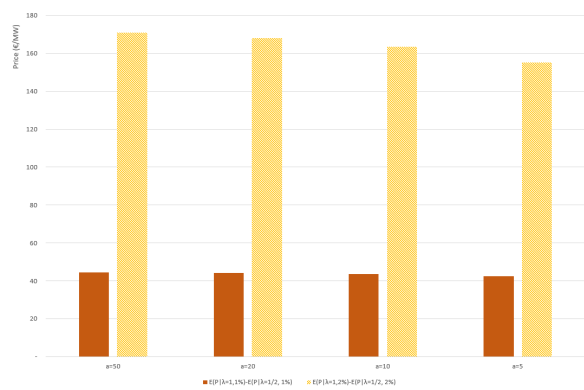


Figure I.2: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_1^t = 79900$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

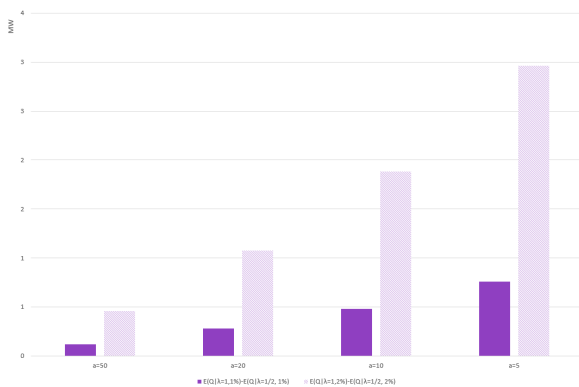


Figure I.3: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_2^t = 87100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

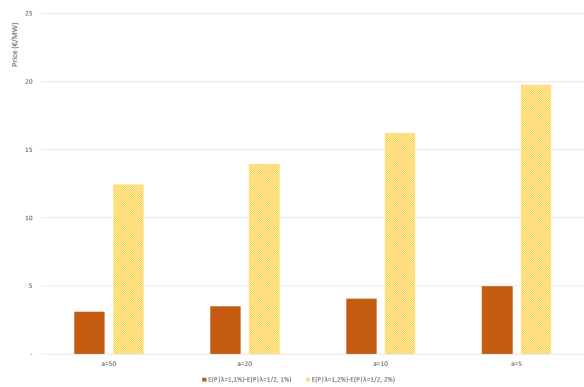


Figure I.4: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_2^t = 87100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

I. Sensitivity on uncertainty

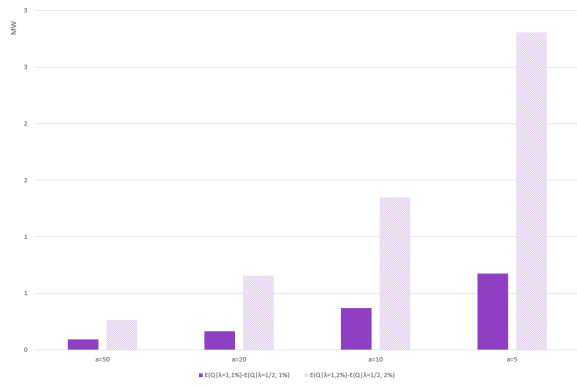


Figure I.5: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_3^t = 100100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

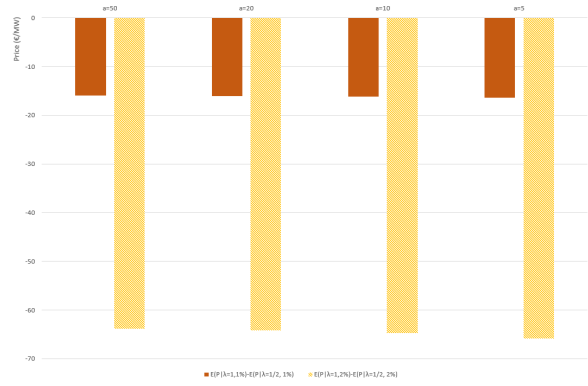


Figure I.6: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_3^t = 100100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)



Figure I.7: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_4^t = 104500$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)

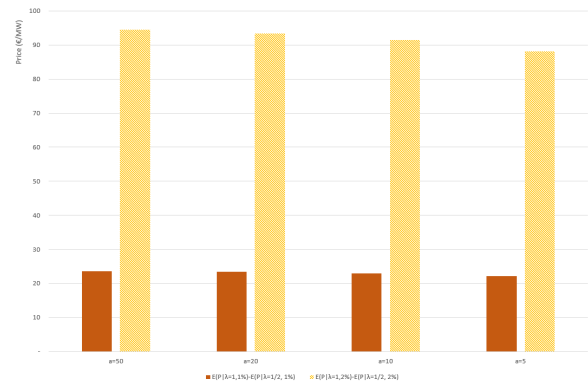


Figure I.8: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_4^t = 104500$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 65$)



Figure I.9: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_1^t = 79900$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

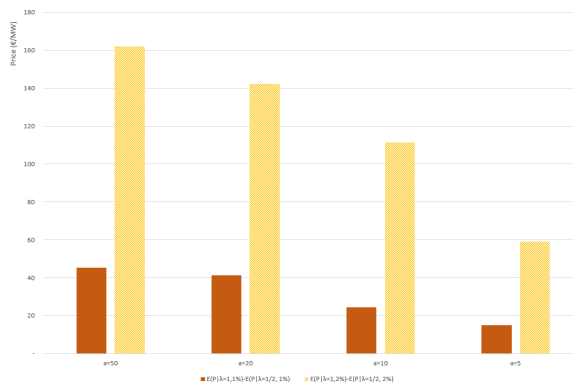


Figure I.10: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_1^t = 79900$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

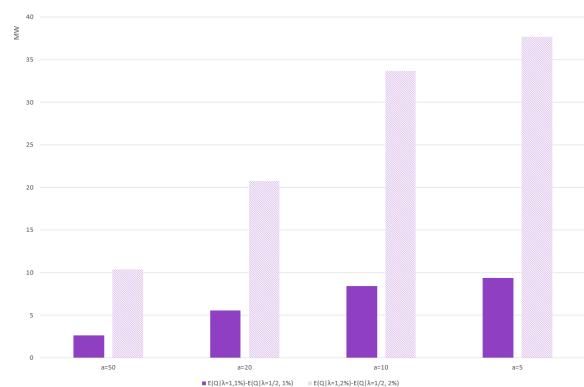


Figure I.11: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_2^t = 87100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

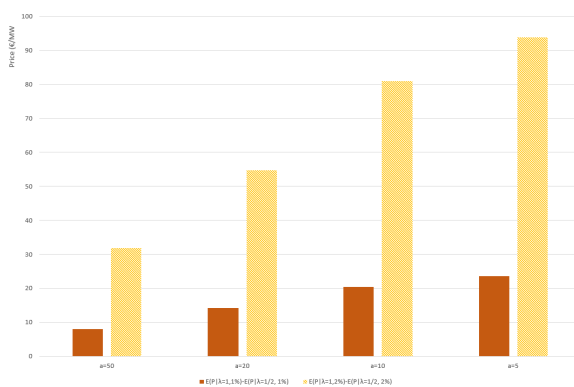


Figure I.12: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_2^t = 87100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

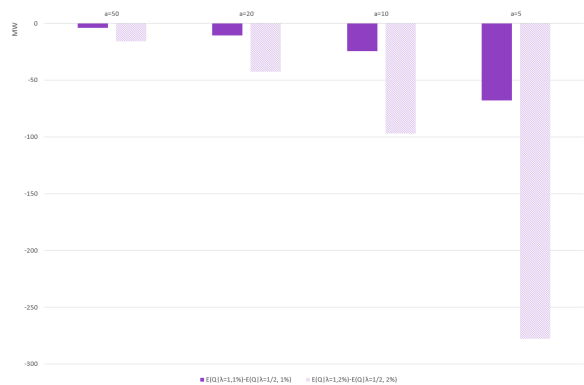


Figure I.13: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_3^t = 100100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

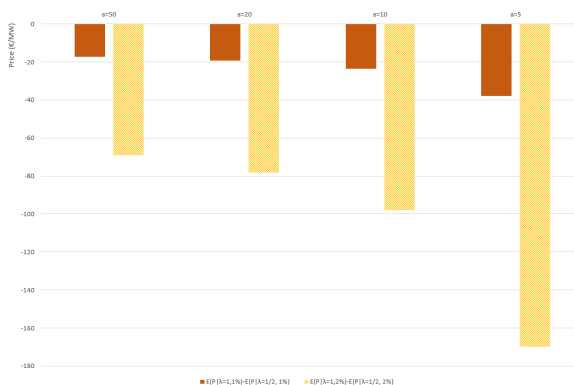


Figure I.14: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_3^t = 100100$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

I. Sensitivity on uncertainty

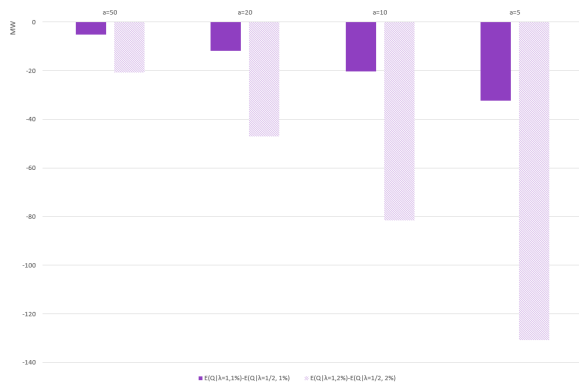


Figure I.15: Difference in mean quantities between the information ($\lambda = 1$) equilibrium ($Q_4^t = 104500$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

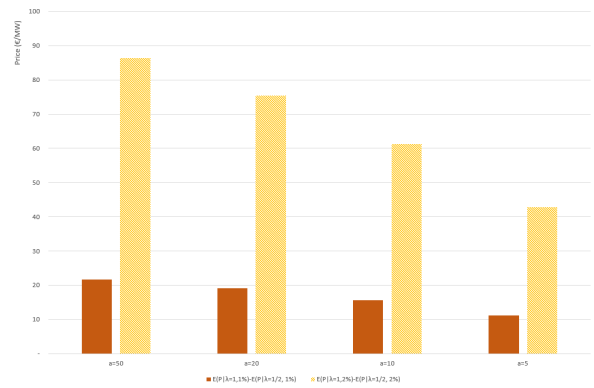


Figure I.16: Difference in mean prices between the information ($\lambda = 1$) equilibrium ($Q_4^t = 104500$ MW) and the no information ($\lambda = \frac{1}{2}$) equilibrium at ($n = 4$)

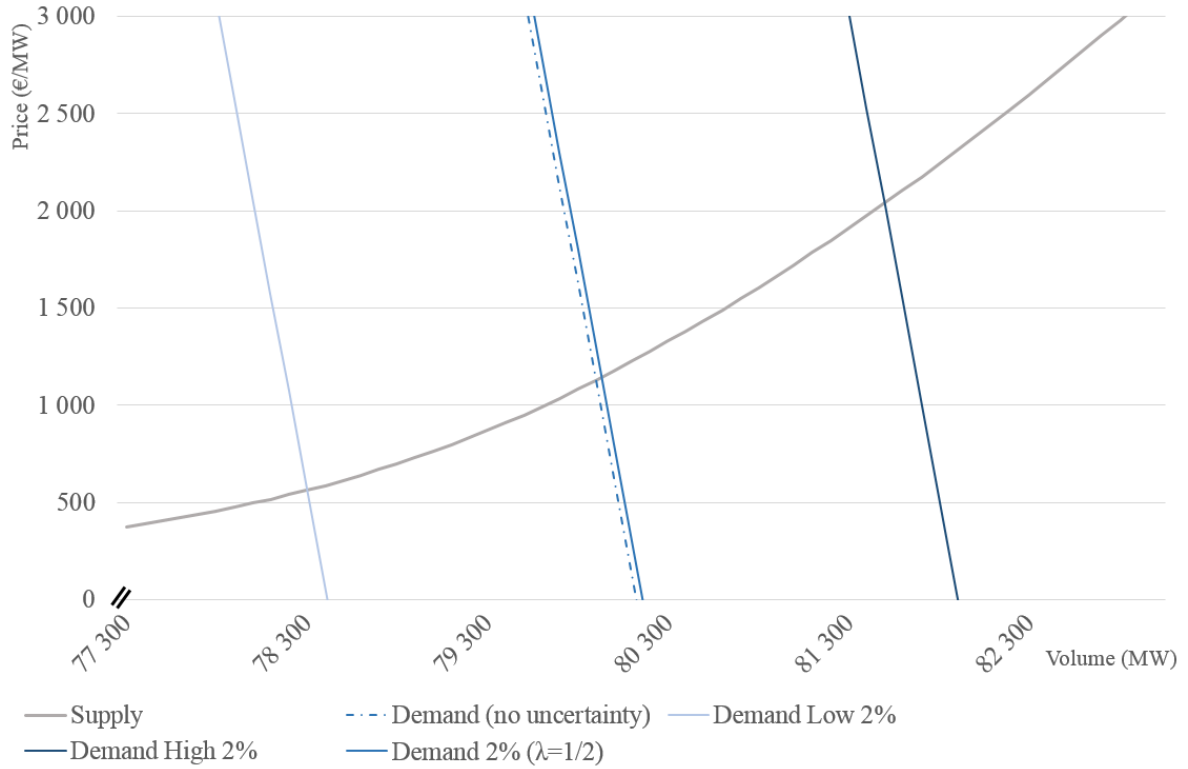


Figure I.17: Representation of the Q_1 case with both no uncertainty and 2% uncertainty with and without information ($a = 5$)

Appendix J

Demand-side design in capacity markets

J.1 Design related Information Precision in Capacity Markets

In a framework where the SoS depends on the ability of LSEs to collectively cover the aggregated peak demand to avoid rationing: the uncertainty is endemic and the forecast error can be costly. This is one of the difficulties in designing capacity markets: what is the fair risk that retailers/consumers should bear with respect forecasting and securing their future demand? With price insensitive end consumers - for technical reasons -, it is a complex trade off. As forward periods of 3 to 4 years are increasingly implemented, the risks related to forecast errors need to be tackled through the capacity market design itself. In addition to the well discussed degree of centralization, two approaches are observed in existing CRMs: (a) *ex ante* requirements hedge market actors against the forecast error while (b) the *ex post* requirements makes them accountable for it (see [Table 4.1](#)). As a consequence, the official peak demand forecast is unevenly informative depending on the capacity market design.

Indeed, in a mechanism that defines the capacity requirements *ex ante* like in Great Britain, any forecast error would only affect the end-users through the cost of capacity (over procurement) or eventual black-outs (under procurements). In case of discontentment, they might turn against the central authority, but the actual electricity market actors are not accountable for reliability: capacity owners receive what they deserve and LSEs pay what they are told to. This solution is broadly implemented in the US as well. The amount of capacity to be procured is fully known *ex ante* either for each LSE (former MISO) or for the whole system (ISO-NE, PJM) which is the central planner case¹. Full regulation equates perfect information: the forecast requirement of the central authority is fully instructive for them and this is true irrespective of actual scarcity in the market

¹Capacity market reality can also lie in between with a mix of bilateral trade, two sided auctions and/or one sided auction with an administrative demand curve (NYISO).

at a given delivery year.

On the contrary, the French mechanism makes the LSEs directly accountable for their reliability. They have to forecast their own requirements (based on peak demand) and procure its capacity equivalent. Regulatory parameters are set to hedge against temperature related forecast errors and normalize requirements to the SoS objective, but the actual requirements still depend on realized demand. This naturally shifts part of the cost of forecast error towards the LSEs: they have to procure capacity under uncertainty on the future realization of demand levels. The central authority's forecast gives a clue on the expected aggregated level of capacity, but this information is only partially instructive because of the myopic nature of the central authority. Because forecast errors decrease with aggregation, the central authority's forecast is likely to be more accurate than the LSE's one all things equal [Chen, 2017], but it remains imprecise. Accounting for the informational value of the forecast published as well as the uncertainty around their own realization, the LSEs procure the capacity they expect to need.

All in all, three main points make the French mechanism different from the most contemporary CRMs such as the British capacity market: (i) the procurement is decentralized and (ii) the risk related to demand unpredictability is left to the capacity buyers. The first point creates Cournot competition between LSEs because each of them receives the obligation to procure its share of capacity. The second point introduces uncertainty on the future level of capacity demand. Indeed, where the forecast is binding in Great Britain, it is rather the demand realization that is in France. In other words, *ex ante* requirement definition makes the aggregated requirement forecast fully informative to the market participants in Great Britain. It will only be partly informative in the French design because actual requirements (*ex post*) are based on peak demand realization rather than a forecast. Thus, the cost allocation of the mechanism (based on current or future peak share or even peak realization) does affect the uncertainty faced by the LSEs.

J.2 Designs of capacity markets: The British capacity market

The British capacity market was implemented as part of the Electricity Market Reform (EMR) which deals with sustainability, reliability and affordability in the system. The former seeks to comply with the 20-20-20 from the climate and energy package of the European commission² while the second one seeks to meet a reliability standard of 3 hours of expected loss of load per capacity year³. The latter states that price inflation should be limited to ensure an access as universal as possible. In the Electricity Market Reform (EMR), the two first targets are set by the public authority and procured separately under the assumption that their public good attribute prevents efficient market outcomes. These two targets have been thought as complements in order to also achieve

²20% cut in greenhouse gas emissions (from 1990 levels), 20% of EU energy from renewables, 20% improvement in energy efficiency.

³Loss of Load is defined, in the UK, as the time during which an instruction of brown out is issued or emergency options are activated in order to prevent a brown out.

the affordability goal. Otherwise subsidized capacity, as tendered renewables, cannot be eligible for capacity payments. Yet, its inner reliability at peak is considered as secured by the central authority. However, this only represents a small share of the capacity required to cover the peak demand with a loss of load probability (LOLP) lower than 3 hours per year. The rest still needs to be procured through the capacity market. National Grid, the system operator, is mandated by the ministry on behalf of the end consumers to secure the capacity required to meet the reliability target four years in advance. In order to avoid the emergence of pivotal buyer, a slope is built through the definition of a "minimum" capacity where the price cap is reached and a "maximum" one where the price reaches zero. The resulting demand curve is built as follow:

- The capacity target: 46.3 GW for DY 2022-2023
- The net Cost Of New Entry (CONE)⁴ as estimated by the ministry: £49/kW/year for DY 2022-2023
- The price cap at 1.5 Net CONE at: £75/kW/year for DY 2022-2023
- The minimum and maximum capacity at: ± 1.5 GW

On the supply-side, a price taker threshold is defined at 0.5 Net CONE to force price competition between the existing capacity. The existing capacity cannot exit the auction⁵ before the price drops under £25/kW/year. From those characteristics, it results not only a steep demand curve, but also numerous low bids. The choice of the parameters reveals several elements that are key to understanding capacity markets. Firstly, the capacity target is not firm. What is called "target tolerances" reflects the myopic view of the public authority. Indeed, there is no technical way for the end consumer to express its precise preferences in terms of SoS, and when it comes to set a target several years ahead of peak demand realization, the level of procurement can only be an educated guess. For the defenders of capacity markets, its outcomes in terms of SoS are considered more desirable than the energy-only market outcomes. This is because the cost of a black-out is often higher than the cost of additional SoS. From the end consumers' perspective, whatever the reliability target set and reached, the captive consumers still pay for their capacity requirements: a "CM operational levy" and a "CM obligation levy" are charged to LSEs in proportion of their market share at peak⁶. End consumers also still face the cost of any black-out that would come to happen. In this framework, obtaining the least cost solution remains a recurrent discussion among public authorities.

J.3 Designs of capacity markets: The French capacity market

Hoping to minimize the deviation of the procurement from actual needs, France chose a

⁴Cost of a new entrant (CCGT in 2014 and 2015 auctions) after accounting for wholesale and ancillary market revenues.

⁵Descending clock auction.

⁶4pm to 7pm, Monday to Friday during November to February.

decentralized design where the realized peak demand is binding instead of the forecasted peak demand. This design echoes the central authority's myopia with respect to future demand.

As in the UK, the reliability target is centrally defined by the public authority to cope with the non-excludability of SoS. However, instead of procuring the capacity itself, the mandate is given to the LSEs that are then in charge of both the electricity and the SoS provided to the end consumers. Three years in advance, LSEs and network operators have to cover their future peak demand either through the auctions organized by EPEX Spot, or over-the-counter. When the delivery year is over, and peak demand has been quantified, the settlement process requires the agents either to adjust their positions, or to pay a non-compliance penalty if they remain short. This is a two-edged sword since LSEs have a finer knowledge of their own demand and might even be able to reduce their peak demand if they have to pay for it. Nevertheless, demand forecasts are known to be more accurate when markets are centralized [Chen \[2017\]](#). Indeed, demand tends to average out with the size of the market. That being said, the reliability target is translated into regulation through requirement parameters very different from those of the British design. The two following are published 4 years ahead, but individual targets remain uncertain:

- Thermo sensitivity coefficients: the weather normalization forces the LSEs to procure capacity as if extreme temperatures were reached (between -1.6°C and -4°C depending on the time of the day).
- The security factor scales the weather-normalized peak with capacity needs: 0.93.

The weather correction aims both at SoS objectives since demand is highly correlated with temperatures, but also at hedging LSEs against the effect of weather uncertainty on peak demand. As a matter of fact, the LSEs are free to build their expectations individually based on predefined scaling parameters for their future peak demand:

$$\textit{Obligation} = \textit{security coefficient}(\textit{realized consumption} + \textit{weather correction})$$

On the supply-side, capacity is normalized into certificates with compulsory offer on the market. Bids are only constrained by a price cap and the regulator closely monitors the main integrated actors' trades. A capacity register is held by RTE to keep track of every transaction. Quantities and prices are publicly available for transparency reasons, but not the identity of the agents involved.

RÉSUMÉ

L'économie modélise des agents rationnels. Cette hypothèse de rationalité est à la fois un cadre qui permet de faire des prédictions, et un carcan qui peut parfois sembler trop éloigné de la réalité. Cette thèse explore différents pans de la rationalité. D'abord, nous nous intéressons à la cohérence de deux des concepts de théorie des jeux les plus connus, c'est-à-dire l'élimination itérée des stratégies faiblement dominées (IEWDS), et l'élimination itérée des stratégies strictement dominées (IESDS). Nous introduisons deux nouvelles relations de dominance entre stratégies et deux nouvelles procédures d'élimination de stratégies qui apparaissent plus cohérentes que les procédures existantes. En effet, les solutions des procédures que nous proposons sont indépendantes de l'ordre d'élimination, donnant une prédiction unique contrairement aux solutions de IEWDS et IESDS. Par ailleurs, nos concepts correspondent à des relaxations de certaines hypothèses de rationalité que nous identifions. Ensuite, nous relâchons l'hypothèse que les agents économiques sont capables d'utiliser toute l'information disponible. Nous nous intéressons à un jeu d'investissement où les actions sont complémentaires et où les agents paient un coût pour obtenir de l'information. Nous caractérisons les équilibres et montrons quelle est l'influence réciproque de l'acquisition de l'information par les différents agents qui composent l'économie. Finalement, nous adaptons la théorie de la persuasion Bayésienne aux marchés de capacités en électricité. La persuasion Bayésienne permet de définir simplement la quantité d'information optimale qu'un agent peut transmettre à d'autres agents. Nous montrons qu'un régulateur de marchés d'électricité peut avoir intérêt à garder l'information dont il dispose sous des conditions que nous déterminons. Nous appliquons notre modèle à des données sur le marché d'électricité allemand.

MOTS CLÉS

Rationalité, Information, Procédures d'Élimination de Stratégies, Inattention Rationnelle, Complémentarités, Persuasion

ABSTRACT

Economics models rational agents. This assumption of rationality is both a framework for making predictions and a strait-jacket that can sometimes seem too far removed from reality. This thesis explores different aspects of rationality. First, we focus on the consistency of two of the best-known game theory concepts, i.e., the iterated elimination of weakly dominated strategies (IEWDS), and the elimination of strictly dominated strategies (IESDS). We introduce two new dominance relations between strategies and two new strategy elimination procedures that appear more consistent than the existing procedures. Indeed, the solutions of our proposed procedures are independent of the order of elimination, giving a unique prediction contrary to the solutions of IEWDS (in finite games) and IESDS (in infinite games). Moreover, our concepts correspond to relaxations of some rationality assumptions that we identify. That is, we assume that players may explore their available strategies, inducing perturbed beliefs. Second, we relax the assumption that economic agents are able to use all available information. We focus on an investment game where actions are complementary and agents pay a cost to obtain information. We characterize the equilibria and show the reciprocal influence of the acquisition of information by the different agents that compose the economy. Finally, we adapt the theory of Bayesian persuasion to electricity capacity markets. Bayesian persuasion allows us to simply define the optimal amount of information that an agent can transmit to other agents. We show that a regulator of electricity markets may have an incentive to keep the information at his disposal under conditions that we determine. We apply our model to data on the German electricity market.

KEYWORDS

Rationality, Information, Strategies Elimination Procedures, Rational Inattention, Complementarities, Persuasion