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Stochastic Analysis of non-Markovian irregular phenomena

Alan Teixeira Nicácio de Messias

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Stochastic Analysis of non-Markovian irregular phenomena.

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(EDMH)

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THÈSE DE DOCTORAT (PhD)

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Alan Teixeira

Analyse stochastiques de phénomènes irréguliers non-markoviens.

Contents

Résumé	1
Introduction	5
1 ON PATH-DEPENDENT SDEs INVOLVING DISTRIBUTIONAL DRIFTS	17
1.1 Introduction	17
1.2 Notations and Preliminaries	20
1.2.1 General notations	20
1.3 Non-Markovian SDE: the function case.	21
1.3.1 General considerations.	21
1.3.2 Comments about the distributional case	22
1.4 The case when the drift is the derivative of a continuous function	23
1.4.1 The Markovian case	23
1.4.2 The path-dependent framework.	26
1.4.3 Existence	29
1.4.4 Uniqueness.	33
1.4.5 Results on pathwise uniqueness	33
1.5 Appendix: different notions of solutions when b' is a function	37
2 ON SDEs FOR BESSEL PROCESSES IN LOW DIMENSION AND PATH-DEPENDENT EXTENSIONS	39
2.1 Introduction	39
2.2 About path-dependent martingale problems	42
2.2.1 Preliminary notations, definitions and results	42
2.3 Martingale problem for Bessel processes	45
2.3.1 Preliminary considerations	45
2.3.2 The framework for $0 \leq \delta < 1$	46
2.3.3 The martingale problem in the full line case when $0 \leq \delta < 1$	48
2.3.4 The martingale problem in the \mathbb{R}_+ -case	52
2.3.5 On an alternative approach to treat the martingale problem on the full line. . .	53

2.3.6	The framework for $\delta = 1$	54
2.4	Martingale problem related to the path-dependent Bessel process	55
2.4.1	Generalities	55
2.4.2	The martingale problem in the path-dependent case: existence in law.	56
2.4.3	Some preliminary results on a path-dependent SDE	57
2.4.4	A new class of solutions to the martingale problem	58
2.4.5	Equivalence between martingale problem and SDE in the path-dependent case	62
2.4.6	The martingale problem in the path-dependent case: uniqueness in law.	64
2.4.7	Path-dependent Bessel process: results on pathwise uniqueness.	64
3	ROUGH PATHS AND REGULARIZATION	67
3.1	Introduction	67
3.2	Preliminary notions	69
3.2.1	Basic notations	69
3.2.2	Matrix-valued integrals via regularization	71
3.3	Stochastically controlled paths and Gubinelli derivative	72
3.4	The second order process and rough integral via regularization	78
3.5	Rough stochastic integration via regularizations	80

Résumé

Cette thèse se concentre sur certains aspects d'analyse stochastique de modèles non-markoviens irréguliers. Nous formulons existence et unicité pour certains problèmes de martingales impliquant deux types de dérive irrégulière perturbée par des fonctionnelles dépendant de la trajectoire.

Nous nous donnons une fonctionnelle non-causale

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (0.1)$$

où

$$\Lambda := \{(t, \eta) \in [0, T] \times C([0, T]); \eta = \eta^t\},$$

et

$$\eta^t(s) := \begin{cases} \eta(s), & \text{if } s \leq t \\ \eta(t), & \text{if } s > t. \end{cases} \quad (0.2)$$

Par convention, nous prolongeons Γ de Λ à $[0, T] \times C([0, T])$ en définissant (de manière non-anticipative)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]). \quad (0.3)$$

Nous considérons alors la **EDS (équation différentielle stochastique) dépendant de la trajectoire**

$$dX_t = \sigma(X_t)dW_t + [\beta(X_t) + \Gamma(t, X^t)]dt, \quad X_0 = x_0 \in \mathbb{R}, \quad (0.4)$$

où nous soulignons que X^t est défini selon (0.2).

Le chapitre 1 est dédié au cas où σ est une fonction continue strictement positive et β est la dérivée d'une fonction continue b . Une motivation pour étudier cela c'est l'introduction d'une version non-markovienne de la diffusion de Brox. Nous modélisons cette dernière par une EDS dépendant de la trajectoire à drift distributionnel

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + \Gamma(t, X^t)dt + dW_t, \quad (0.5)$$

où B est un mouvement brownien bilatéral indépendant de W . Dans ce cas nous rencontrons deux difficultés combinées: la présence de la distribution de Schwartz β et la fonctionnelle Γ dépendant de la trajectoire.

L'équation (0.4) est interprétée à l'aide d'un problème de martingale associé à l'opérateur

$$\mathcal{L}f := Lf + \Gamma f', \quad (0.6)$$

voir Définition 1.3.3.

Nous disons qu'un couple (X, \mathbb{P}) résout le problème de martingale associé à un domaine \mathcal{D}_L et à l'opérateur \mathcal{L} si

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (0.7)$$

est une \mathbb{P} -martingale locale pour tout $f \in \mathcal{D}_L$. Nous introduisons également une notion de problème de martingale fort associé à \mathcal{L} et à un mouvement brownien W donné, voir Définition 1.3.4, qui correspond à l'analogue de solutions fortes d'équations différentielles stochastiques. L'existence et l'unicité de solution du problème de martingale sont énoncées dans le théorème 1.4.23 et la proposition 1.4.24 respectivement. Nous montrons que la résolution d'un problème de martingale fort équivaut la résolution d'une vraie EDS dépendant de la trajectoire (sans drift distributionnel), à savoir (4.10), voir Proposition 1.4.12. Finalement, sous certaines conditions, nous prouvons que (4.10) admet existence forte et unicité trajectorielle, voir définitions 1.5.2 et 1.5.3 dans la section 1.5.

Lorsque cela se produit, cela entraîne que le problème de martingale fort admet une solution unique, voir le théorème 1.4.30 et le corollaire 1.4.31.

Le chapitre 2 enquête (0.4), lorsque $\sigma = 1$ et β est la dérivée de la fonction b définie en (0.28) qui explose en zéro. L'EDS considérée est formellement

$$dX_t = dW_t + \left[\frac{\delta - 1}{2} X_t^{-1} + \Gamma(t, X^t) \right] dt. \quad (0.8)$$

Nous formulons (0.8) à nouveau au travers d'un problème de martingale comme en (0.7) par rapport à deux domaines spécifiques de fonctions $\mathcal{D}_L = \mathcal{D}_{L^\delta}$ or $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ (qui est le domaine des fonctions appartenant à \mathcal{D}_{L^δ} restreintes à \mathbb{R}_+).

L'existence de solutions du problème de martingale associé au domaine \mathcal{D}_{L^δ} et

$$\mathcal{L}f := L^\delta f + \Gamma f', \quad (0.9)$$

est démontré dans la proposition 2.4.2 à condition que Γ soit borné. Nous montrons qu'en général l'unicité n'est pas vérifiée. Néanmoins elle a lieu si nous remplaçons \mathcal{D}_{L^δ} par $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$, voir Remarque 2.4.3. De façon similaire à ce qui se passe quand β est la dérivée d'une fonction continue b et σ est une fonction continue strictement positive, nous pouvons montrer que le problème de martingale (resp. le problème de martingale fort) est équivalent à la résolution d'une EDS dépendant de la trajectoire, voir Proposition 2.4.9, dont la solution est indiquée par S . Sous certaines conditions appropriées cette dernière EDS admet existence forte et unicité pathwise (comme définie en Section 1.5) ce qui nous permet conclure que le problème de martingale fort associé à $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ et un mouvement brownien W fixé admet existence et unicité, voir théorème 2.4.16 et corollaire 2.4.17.

La thèse explore également des relations et analogies entre la théorie des chemins rugueux (rough paths en anglais) et le calcul stochastique via régularisation.

Au chapitre 3, nous proposons une version stochastique de l'intégrale "rough paths" au sens de Gubinelli. En particulier, nous illustrons le lien fort entre la notion de processus de Dirichlet faible et celle de processus stochastiquement contrôlé, qui est une version stochastique de celle proposée par M. Gubinelli [56]. Selon la définition 3.3.2 tel processus satisfait

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, s < t, \quad (0.10)$$

où

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds = 0, \quad (0.11)$$

en probabilité pour chaque $t \in [0, T]$. Ici, X est le processus constitue le processus intégrateur de référence, Y' est un processus (dont les trajectoires ne sont pas nécessairement γ -Hölder continues). La condition d'orthogonalité (0.51) qui caractérise une composante des processus de Dirichlet faible remplace la condition de régularité 2γ -Hölder réminiscente de [56].

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a maturity $T > 0$. Let $(\mathcal{F}_t)_t$ a filtration satisfying the usual conditions and $(W_t)_t$ be a standard $(\mathcal{F}_t)_t$ -Brownian motion defined on that space. Let $\sigma, \beta : \mathbb{R} \rightarrow \mathbb{R}$ be Borel functions. Classical stochastic differential equations (SDEs) are expressed as

$$dX_t = \sigma(X_t)dW_t + \beta(X_t)dt, \quad X_0 = x_0 \in \mathbb{R}. \quad (0.12)$$

Indeed, even though the framework could be multidimensional we will keep a one-dimensional formalism. When σ and β are Lipschitz, then (0.12) has an unique (strong) solution X , which has moreover the Markov property. Solutions X of previous equation are called **diffusions** or **diffusion processes**. There are several notions of existence and uniqueness for SDEs. In this thesis we will refer to strong and weak solutions and to strong existence, pathwise uniqueness, existence and uniqueness in law (or weak). For those notions the reader can refer to [61, Sections 5.2, 5.3].

Suppose now that σ and β have only linear growth. Consider the so called generator $L : C^2(\mathbb{R}) \rightarrow C(\mathbb{R})$ of the SDE (or the related semigroup) given by

$$Lf = \frac{1}{2}\sigma^2 f'' + \beta f'. \quad (0.13)$$

According to the Stroock-Varadhan theory a solution to the (classical) martingale problem related to L is a pair $(X, \bar{\mathbb{P}})$ such that the process

$$M_t := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \quad (0.14)$$

is a $\bar{\mathbb{P}}$ -local martingale (with respect to the canonical filtration of X) for all $f \in C^2(\mathbb{R})$. In fact, to be more precise, in [93], Ω is the canonical path space $C([0, T])$ of real-valued continuous functions on $[0, T]$ and the authors only refer to $\bar{\mathbb{P}}$ instead to the couple $(X, \bar{\mathbb{P}})$.

The following well-known result establishes an equivalence between (0.12) and (0.14), see Corollary 4.8 in [61, Section 5.4].

Lemma 0.0.1. *Let $\bar{\mathbb{P}}$ be a probability on some measurable space (Ω, \mathcal{F}) and X be a (continuous) stochastic process. The pair $(X, \bar{\mathbb{P}})$ is a solution to the martingale problem related to L if and only if X is a (weak) solution of (0.12) with respect to $\bar{\mathbb{P}}$.*

In [93] several results are available for existence and uniqueness in law for (0.14). For instance (in dimension 1 and even 2, since the coefficients of (0.12) are time-homogeneous) existence and uniqueness in law holds if σ and β are bounded and σ is strictly larger than a positive constant on each compact, see Exercises 7.3.2–7.3.4 of [93], and also Chapter 5, Refinements 4.32.

If the drift β is very irregular (for instance a distribution), it is difficult to express (0.12) as an “equation”. The equivalence stated in Lemma 0.0.1 will motivate us in the sequel to formulate a substitutive notion of SDE in term of martingale problems.

Suppose now that the SDE (0.12) is perturbed by a locally bounded path-dependent functional

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (0.15)$$

where

$$\Lambda := \{(t, \eta) \in [0, T] \times C([0, T]); \eta = \eta^t\},$$

and

$$\eta^t(s) := \begin{cases} \eta(s), & \text{if } s \leq t \\ \eta(t), & \text{if } s > t. \end{cases} \quad (0.16)$$

By convention, we extend Γ from Λ to $[0, T] \times C([0, T])$ by setting (in non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]). \quad (0.17)$$

We consider then the following **path-dependent SDE**

$$dX_t = \sigma(X_t)dW_t + [\beta(X_t) + \Gamma(t, X^t)]dt, \quad X_0 = x_0 \in \mathbb{R}, \quad (0.18)$$

where we emphasize that X^t is defined according to (0.16). In general, solutions of (0.18) are not Markov processes. Path-dependent SDEs were investigated under several aspects. For instance, if σ and β are Lipschitz and Γ is uniformly Lipschitz with respect to the path on $C([0, T])$ equipped with the sup-norm, it is known, (see e.g Theorem 11.2 [80, chapter V]) that strong existence and pathwise uniqueness hold.

We come back now to Markovian SDEs. Suppose now that β is the derivative (in the sense of Schwartz distributions) of a real function b . In this case (0.12) becomes in principle an SDE with distributional drift. When β is a Radon measure, those equations (suitably interpreted) have been intensively studied over the years, starting by [77]. Later on, still in the case β as a measure, many authors considered special cases of SDEs with generalized coefficients, see for instance [12, 34, 81]. In those cases the solutions are semimartingales. The first example of non-semimartingale solution of a stochastic differential equation with generalized drift, was considered by [35].

In [40] and [41], the authors studied systematically time-independent one-dimensional SDEs of the form

$$dX_t = \sigma(X_t)dW_t + \beta(X_t)dt, \quad t \in [0, T], \quad (0.19)$$

whose solutions are possibly non-semimartingale processes, where σ is a strictly positive continuous function and β is the derivative of a continuous real function b . The only supplementary assumption was the existence of the function

$$\Sigma(x) := 2 \int_0^x \frac{b'}{\sigma^2}(y)dy, x \in \mathbb{R}, \tag{0.20}$$

considered as a suitable limit via regularizations. In those papers well-posedness for the corresponding martingale problem was discussed, where the map L was defined on a specific domain \mathcal{D}_L instead of $C^2(\mathbb{R})$. The domain was characterized as

$$\mathcal{D}_L = \{f \in C^1(\mathbb{R}) \text{ for which there exists } \phi \in C^1(\mathbb{R}) \text{ such that } f' = \exp(-\Sigma)\phi\}. \tag{0.21}$$

The domain \mathcal{D}_L is a subset of $C^1(\mathbb{R})$ but its intersection with $C^2(\mathbb{R})$ (even C^∞) is empty. If we denote Σ as in (0.20), then the operator L can be written as

$$Lf = (e^\Sigma f')' \frac{e^{-\Sigma} \sigma^2}{2}. \tag{0.22}$$

One says that a couple $(X, \bar{\mathbb{P}})$ solves the martingale problem related to L if (0.14) is a $\bar{\mathbb{P}}$ -martingale for all $f \in \mathcal{D}_L$. Proposition 3.13 in [40] established existence, uniqueness and non-explosion for the martingale problem related to L .

Now, as anticipated earlier, the SDE (0.19) does not make sense a priori, since it involves the composition of the Schwartz distribution β with the continuous path $X(\omega)$. If β were a function, then Lemma 0.0.1 would state that (0.19) is equivalent to a martingale problem. Under this inspiration, we interpret the ‘‘SDE’’ (0.19) as the martingale problem, which makes perfectly sense replacing $C^2(\mathbb{R})$ with \mathcal{D}_L . Nevertheless in some cases (when $\frac{\sigma^2}{2} + \beta$) is a Radon measure, the authors give in fact a precise SDE interpretation of (0.19) as an equation in terms of occupation measure, see Proposition 3.11 of [40]. [41] showed in Corollary 5.11 that the solution of the martingale problem is a semimartingale if and only if Σ has bounded variation. In particular if $\sigma = 1$ this happens if b has locally bounded variation (i.e. β is a σ -finite measure). Moreover, in the particular case when β is a bounded function, [41] in Theorem 4.4 provided Itô’s formula under weak conditions, expanding $f(X)$, where X is the (this time strong) solution of (0.19) and $f \in W_{loc}^{1,2}$.

The SDE (0.19) was also investigated by [8], where the authors provided a well-stated framework when σ and b are γ -Hölder continuous, $\gamma > \frac{1}{2}$. In [82], the authors introduce the notion of *strong martingale problem* for which they establish existence and uniqueness. That notion corresponds to the one of strong existence and pathwise uniqueness for classical stochastic differential equations. One says that a process X solves the **strong martingale problem** with respect to a Brownian motion W on some probability space and with respect to some domain \mathcal{D}_L if

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \tag{0.23}$$

is an (\mathcal{F}_t) -local martingale where (\mathcal{F}_t) is the canonical filtration of W . In particular, if \mathcal{D}_L is an algebra, by the Brownian martingale representation one can show that

$$M_t^f = \int_0^t f'(X_s)dW_s.$$

When the underlying filtration (\mathcal{F}_t) is clearly identified we will often omit it.

More recently, in the time-dependent framework (but still one-dimensional), a significant contribution was made by [26]. As far as the multidimensional case is concerned, some important steps were done in [39], in [14] when the diffusion matrix is the identity and β is a time-dependent drift in some proper negative Sobolev space. In the case of stable noise we also mention the significant contribution of [16]. We also refer to [9], where the authors have focused on (0.19) in the case of a time independent drift β which is a measure of Kato class.

In the literature there are some significant examples of (Markovian) solutions of SDE of distributional drift in the framework of (0.19). For example setting $\sigma = 1, b = B$, where B is a two-sided real-valued Brownian motion independent of W , (0.19) reads

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + dW_t. \quad (0.24)$$

The solution of (0.24) is the so-called Brox diffusion, see e.g [13, 59, 92] and other references therein. This is a famous *random environment* model.

Another process which is expected to be a solution of an SDE of distributional drift is the so called Bessel process in low dimension. The class of Bessel processes is one of the most important classes of diffusion processes with values in \mathbb{R}_+ . It is a family of (strong) Markov processes parameterized by $\delta \in \mathbb{R}_+$ (called the *dimension*). Bessel processes have been largely investigated in the literature, we refer the reader to e.g. [70, 95, 79] (Section 2.3, Chapter 3 and Chapter XI, respectively), for an overview on Bessel processes.

Let $x_0 \geq 0$. We denote by $\text{BES}^\delta(x_0)$ the *Bessel process* X with initial condition x_0 and dimension $\delta \geq 0$, which is defined as the square root of $\text{BESQ}^\delta(x_0^2)$ the so-called *squared Bessel process*, with initial condition x_0^2 and dimension δ . This latter is characterized as the pathwise unique solution of the SDE

$$dS_t = 2\sqrt{|S_t|}dW_t + \delta t, S_0 = x_0^2.$$

When $\delta > 2$ it is well-known how to characterize X as (pathwise unique non-negative) solution of

$$dX_t = \frac{\delta - 1}{2}X_t^{-1}dt + dW_t, \quad (0.25)$$

in particular X is an Itô process. For $0 \leq \delta \leq 1$, the Lebesgue integral $\int_0^t X_s^{-1}ds$ does not exist and $\text{BES}^\delta(x_0)$ is a non-semimartingale process, except for $\delta = 1$ and $\delta = 0$, see [79, 60], Chapter XI Section 1 and Section 6.1, respectively. If $0 < \delta < 1$, it can be represented as

$$X_t = x_0 + \frac{\delta - 1}{2} \text{p.v.} \int_0^t \frac{1}{X_s} ds + W_t, t \geq 0, \quad (0.26)$$

where p.v. stands for principal value (defined in a suitable way via local time). The drift in decomposition (0.26) is a zero energy additive functional in the language of Markov processes and $\text{BES}^\delta(x_0)$ is a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process. For further details, we refer the reader to the works [95, 35, 70] and other references therein. Existence

and uniqueness of a non-negative strong solution of (0.25) with $\delta \in]1, 2]$ was proved in [18]. Existence and uniqueness of a non-negative solution to some strong martingale problem for with $\delta \in]0, 2]$ is investigated in [3] under the assumption that the process spends zero time at 0. In Chapter 2 we propose an alternative method to study well-posedness of a corresponding martingale problem which does not suppose that assumption, even in the Markovian framework.

Let us go into some details of the ‘‘Markovian’’ contribution of Chapter 2. We show that the Bessel process in dimension $0 \leq \delta \leq 1$ is the unique solution of some specific strong martingale problem. We define \mathcal{D}_{L^δ} as the set of $f \in C^2(\mathbb{R})$ such that $f'(0) = 0$ and, for $\delta \in [0, 1[$, we define

$$L^\delta f(x) = \begin{cases} \frac{f''(x)}{2} + \frac{(\delta - 1)f'(x)}{2x} & : x \neq 0 \\ \frac{f''(0)}{2} & : x = 0. \end{cases} \quad (0.27)$$

We remark that the first line of previous expression is ‘‘natural’’; the value at zero of $L^\delta f$ comes out taking the limit of the first line taking $x \rightarrow 0$ and applying L’Hospital rule. The case of $\delta = 1$ can be discussed similarly. We also set $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ as the set of restrictions of functions in \mathcal{D}_{L^δ} to \mathbb{R}_+ .

In fact, this is compatible to the formalism of L characterized in (0.22) with $\sigma = 1$ and β being the derivative of b , which is given by

$$b(x) = \begin{cases} \frac{\delta-1}{2} \log |x|, x \in \mathbb{R}^*, & | \delta \neq 1 \\ H(x), x \in \mathbb{R}, & | \delta = 1, \end{cases} \quad (0.28)$$

where H is the Heaviside function and $\mathbb{R}^* = \mathbb{R} - \{0\}$. Even though b is no longer a continuous function, (0.20) can still be defined in such a way that $\Sigma \equiv 2b$ and (0.22) holds.

We distinguish two cases: $0 \leq \delta < 1$ and $\delta = 1$.

- $0 \leq \delta < 1$. (0.28) and (0.20) imply

$$\exp(-\Sigma(x)) = |x|^{1-\delta}. \quad (0.29)$$

Representation (0.22) for $L^\delta = L$ yields (0.27).

- $\delta = 1$. In this case, $b(x) = H(x)$. So (0.22) yields

$$L^1 f(x) = \frac{f''(x)}{2} + \delta_0 f'(x), \quad x \neq 0, \quad (0.30)$$

where δ_0 is the Dirac measure at zero.

Adopting the notation (0.29) the first line of (0.27) gives (for $x > 0$),

$$L^\delta f = (x^{\delta-1} f')' x^{1-\delta}. \quad (0.31)$$

Our main contributions to what concerns Markov Bessel process in low dimension are the following.

- We prove that for $0 \leq \delta \leq 1$ and $x_0 \geq 0$, a $\text{BES}^\delta(x_0)$ -process is a solution to the strong martingale problem with respect to \mathcal{D}_{L^δ} starting from any $X_0 = x_0$ in the sense of (0.23), see Proposition 2.3.4. More generally, for $0 \leq \delta \leq 1$, the strong martingale problem with respect to \mathcal{D}_{L^δ} starting from $x_0 \in \mathbb{R}$ (the sign does not matter) admits existence, see Corollary 2.3.6. We also prove that if a process X starting from x_0 (the sign does not matter) is a strong solution to the martingale problem with respect to \mathcal{D}_{L^δ} then X^2 is a $\text{BESQ}^\delta(x_0^2)$ process, see Proposition 2.3.10.
- Concerning uniqueness of the aforementioned strong martingale problem, excepting when $\delta = 0$, we prove that it does not hold, see Propositions 2.3.7 and Remark 2.3.11. However, if we replace \mathcal{D}_{L^δ} with $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ then (strong) martingale problem admits uniqueness, see Proposition 2.3.13.

We come back to (0.18), i.e. with our SDE with distributional drift and path-dependent perturbation. Chapter 1 is devoted to the case where σ is a continuous strictly positive function and β is the derivative of a continuous function b such that Σ , as defined in (0.20), exists. A motivation for this comes from introducing a non-Markovian version of the Brox diffusion. We model it by the path-dependent SDE with distributional drift

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + \Gamma(t, X^t)dt + dW_t, \quad (0.32)$$

where B is still a two-sided Brownian motion independent of W . In this case we have the combined difficulty due to the presence of the Schwartz distribution β and the path-dependent functional Γ . Equations (0.18) and (0.35) will be interpreted as a martingale problem related to the operator

$$\mathcal{L}f := Lf + \Gamma f', \quad (0.33)$$

see Definition 1.3.3. A couple (X, \mathbb{P}) is said to solve the martingale problem related to a domain \mathcal{D}_L and the operator \mathcal{L} if

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (0.34)$$

is a \mathbb{P} -local martingale for all $f \in \mathcal{D}_L$. We also introduce the suitable notion of strong martingale problem related to \mathcal{L} and a given Brownian motion W , see Definition 1.3.4. Existence and uniqueness of solution to the martingale problem are proven in Theorem 1.4.23 and Proposition 1.4.24 respectively. The strong martingale problem is shown to be equivalent to the problem of solving a path-dependent SDE without distributional drift, namely (4.10), see Proposition 1.4.12. Finally we prove that (4.10) admits strong existence and pathwise uniqueness, see Definitions 1.5.2 and 1.5.3 in Section 1.5. This implies that the strong martingale problem admits a unique solution, see Theorem 1.4.30 and Corollary 1.4.31.

Besides the martingale problem related to the Bessel process, Chapter 2 investigates (0.18), when $\sigma = 1$ and β is the derivative of the function b defined in (0.28). The considered SDE appears formally to be

$$dX_t = dW_t + \left[\frac{\delta - 1}{2} X_t^{-1} + \Gamma(t, X^t) \right] dt. \quad (0.35)$$

We formulate (0.35) by means of a martingale problem as in (0.34) with respect to the domain $\mathcal{D}_L = \mathcal{D}_{L^\delta}$ or $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$.

Existence of solutions to the martingale problem related to the domain \mathcal{D}_{L^δ} and

$$\mathcal{L}f := L^\delta f + \Gamma f', \tag{0.36}$$

is proven in Proposition 2.4.2 provided Γ is bounded. As in the Markovian case, uniqueness does not hold in general. Nevertheless it holds if we replace \mathcal{D}_{L^δ} with $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$, see Remark 2.4.3. Similarly to what happens when β is the derivative of a continuous function b and σ is a non-vanishing continuous function, see Chapter 1, the martingale problem (resp. the strong martingale problem) can be shown to be equivalent to a path-dependent SDE, see Proposition 2.4.9 whose solution is indicated with S . Under some suitable conditions the path-dependent SDE admits strong existence and pathwise uniqueness (as defined in Section 1.5) which allows to conclude that the strong martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and some Brownian motion W admits existence and uniqueness, see Theorem 2.4.16 and Corollary 2.4.17.

The first two chapters focused in particular on solutions of SDE with distributional drift. Let us come back to the Markovian case. In the classical stochastic analysis, solutions of classical SDEs with Lipschitz coefficients and generator L (see (0.13)), are useful to represent solutions of semilinear Kolmogorov type PDEs

$$\begin{cases} (\partial_t + L)u + F(\cdot, \cdot, u, \sigma \partial_x u) = 0 \\ u(T) = g, \end{cases} \tag{0.37}$$

where F, g are continuous f has linear growth and is Lipschitz in the fourth argument and $g \in C^2$ has linear growth.

This can be done by Feynman-Kac representations or more elegantly and efficiently via forward-backward SDE, see for instance [76]. For simplicity of formulation we will remain in the one-dimensional framework.

Let $0 \leq s \leq t \leq T$. Consider the flow of diffusions

$$X_t^{s,x} = x + \int_s^t \beta(X_r^{s,x})dr + \int_s^t \sigma(X_r^{s,x})dW_r, \quad t \in [0, T]. \tag{0.38}$$

The first natural observation is that classical solution of (0.37) (i.e. in $C^{1,2}$) provide solution for the BSDEs

$$Y_t^{s,x} = g(X_T^{s,x}) - \int_t^T f(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})dr - \int_t^T Z_r^{s,x}dW_r, \quad t \in [s, T]. \tag{0.39}$$

A solution of (0.39) is a couple of progressively measurable square integrable processes (Y, Z) : here $Y = Y^{s,x}, Z = Z^{s,x}$. For instance, in [76] one shows that solutions of (0.39) provide solutions (in the viscosity sense) to (0.37). In particular $u(s, x) := Y_s^{s,x}$ defines a viscosity solution of (0.37).

Suppose we replace (0.38) with the strong martingale problem related to Bessel processes in low dimension $\delta \in [0, 1]$, with initial time s and initial position x . Let W be a standard Brownian motion

on some probability space and let $X^{s,x}$ be the solution of the strong martingale problem with respect to L^δ , where $0 \leq \delta \leq 1$ on $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$, see (0.27). This constitutes the rigorous meaning of

$$X_t^{s,x} = x + \frac{\delta - 1}{2} \int_s^t \frac{1}{X_r^{x,s}} dr + W_r, \quad t \in [0, T]. \quad (0.40)$$

When $s = 0$, $X^{0,x}$ is a δ -dimensional Bessel process starting from x . Let us consider (0.37) where L is replaced with L^δ . Even when $f = 0$ that PDE involves a singular (distributional) coefficient. The natural question for us are the following.

1. How do we define the notion of ‘‘classical’’ solution for (0.37)?
2. In which sense those ‘‘classical’’ solutions produce solutions of the BSDE (0.39)?
3. How is it possible to define ‘‘generalized solutions’’ of (0.37)?
4. Let $(Y^{s,x}, Z^{s,x})$ be the solutions of the BSDE (0.39) and set $u(s, x) := Y_s^{s,x}$. Does u produce a ‘‘generalized solution’’ of (0.37)?
5. Can we solve the so called identification problem, i.e. can we associate $Z = Z^{s,x}$ explicitly with u ?

In the paper in preparation [73], we answer the first two questions.

We introduce here the subdomain $D := \{\varphi \in C_0^2(\mathbb{R}_+); \partial_x \varphi(0) = 0\}$ of $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$.

Definition 0.0.2. We say that $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a **classical solution** of

$$\partial_t u - L^\delta u = -F(\cdot, \cdot, u, \partial_x u), \quad u(T) = g, \quad (0.41)$$

if it is of class $C^{1,2}([0, T] \times \mathbb{R}_+)$ with $\partial_x u(t, 0) = 0$ (in particular $u(t) \in D$ for every $t \in [0, T]$) and it solves (0.41).

For every $s \in [0, T]$, $x \geq 0$, let $X^{s,x}$ be the solution of the strong martingale problem with respect to L^δ and an underlying Brownian motion W and u a classical solution of (0.41). By means of that definition, using the notion of strong martingale problem, we can show that (Y, Z) where

$$Y_t = Y_t^{s,x} := u(t, X_t^{s,x}), \quad Z_t = Z_t^{s,x} := \partial_x u(t, X_t^{s,x}), \quad t \in [0, T],$$

is a solution of the BSDE (0.39). This answers questions 1. and 2. In [73] we only partially answer to question 3. in the sense that generalized solutions of (0.41) will appear below in the form of mild and weak solutions, which are properly defined in Definitions 0.0.3 and 0.0.4. The link to more general type of solutions like *viscosity* or *decoupled mild*, see e.g. [7] will be object of future research.

Let us shortly describe some basic elements contained in [73] about the PDE (0.41) as far as question 3. is concerned. In view of introducing the notion of mild solution for (0.41), We consider the Bessel semigroup (P_t^δ) defined, for every bounded Borel function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, by

$$P_t^\delta f(x) = \int_0^\infty f(y) p_t^\delta(x, y) dy, \quad t \geq 0,$$

where $p_t^\delta(x, \cdot)$ is the marginal density law at t of a δ -Bessel process starting at time 0 from x .

We define on \mathbb{R}_+ the Borel σ -finite positive measure

$$\mu(dx) = x^{\delta-1} dx.$$

By using the formulation L^δ appearing in (0.31) one can see that μ is invariant in the sense that it fulfills

$$\int_0^\infty L^\delta \varphi d\mu = 0,$$

for every smooth function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ belonging to D .

By the explicit expressions of densities in Appendix A.1.2 of [60] we immediately see that

$$p_t^\delta(x, y) y^{1-\delta} = p_t^\delta(y, x) x^{1-\delta}; x, y > 0. \tag{0.42}$$

This allows us to show that $P_t^\delta : L^2(\mu) \rightarrow L^2(\mu)$ has the contraction property, for all $t > 0$. For that notion, the reader can consult Section 1.3. of [48]. In particular the space $L^2(\mu)$ is preserved by P^δ . So $L^2(\mu)$ is a natural choice for the state space of the mild solution if f would not depend on $\partial_x u$. Since it indeed does depend on $\partial_x u$, we set \mathcal{H} to be the sub-space of $f \in L^2(\mu)$ for which there exists an unique (in $L^2(\mu)$ sense) $g \in L^2(\mu)$ such that for all $x, y > 0$ $f(x) - f(y) = \int_y^x g(z) dz$. On \mathcal{H} we set the norm

$$\|f\|_{\mathcal{H}} := \|f\|_{L^2(\mu)} + \|g\|_{L^2(\mu)}. \tag{0.43}$$

Notice that for $f \in D$ $f(x) - f(y) = \int_y^x f'(z) dz$, so $g = f'$ and $\|f\|_{\mathcal{H}} := \|f\|_{L^2(\mu)} + \|f'\|_{L^2(\mu)}$.

We denote by B the space of functions $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $u : [0, T] \rightarrow \mathcal{H}$ and

$$\|u\|_B := \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{H}} < \infty. \tag{0.44}$$

From now on, suppose that the final condition g of the PDE (0.41) belongs to $L^2(\mu)$. Let $u : [0, T] \rightarrow \mathcal{H}$ such that $\int_0^T (\partial_x u)^2(t, x) d\mu(x) dt < \infty$.

Definition 0.0.3. We say that $u \in B$ is a *mild solution* of (0.41) if it satisfies

$$u(t, x) = P_{T-t}^\delta[g](x) + \int_t^T P_{s-t}^\delta[F(s, \cdot, u(s, \cdot), \partial_x u(s, \cdot))](x) ds, \quad t \in [0, T]. \tag{0.45}$$

Definition 0.0.4. We say that $u \in B$ is a *weak solution* of (0.41) if $\forall \phi \in D$

$$\langle u(t), \phi \rangle_{L^2(\mu)} = \langle u(T), \phi \rangle_{L^2(\mu)} - \int_t^T \langle u(s), L^\delta \phi \rangle_{L^2(\mu)} ds - \int_t^T \langle F(s, u(s), \partial_x u(s)), \phi \rangle_{L^2(\mu)} ds, \tag{0.46}$$

where $u(t)$ (resp. $\partial_x u(t)$) denotes $u(t, \cdot)$ (resp. $\partial_x u(t, \cdot)$.)

Remark 0.0.5. In [73], by classical arguments, we prove uniqueness of weak solution of (0.46) when $f \equiv 0$. This allows us to show that weak and mild solution are equivalent.

Proposition 0.0.6. *The PDE (0.41) admits an unique (mild) solution.*

The proof of Proposition 0.0.6 is based on a fixed point argument on $C([0, T]; \mathcal{H})$. A crucial estimate for the proof is

$$\|\partial_x P_t^\delta[f]\|_{L^2(\mu)} \leq \frac{\|f\|_{L^2(\mu)}}{\sqrt{t}}, \quad t \geq 0.$$

That estimate can be obtained by extending L^δ to its Von Neumann extension and then applying Lemma 1.3.3 in Chapter 1 of [48].

In the last part of this thesis, we study non-Markovian systems from the point of view of Rough Paths [69, 56] and in the context of stochastic calculus via regularization, see e.g [83, 84, 87, 88, 89] for the basic notions. More precisely, we provide a fundamental connection between a stochastic version of the rough path integral (in the sense of Gubinelli [56]) and the forward and symmetric-Stratonovich integrals, in the sense of regularization, when the driving noise process is given by a multi-dimensional continuous semimartingale.

The connection between Rough Paths theory with semimartingales has been investigated by some authors. [22] shows pathwise Wong-Zakai-type theorems for Stratonovich SDEs driven by continuous semimartingales. In particular, the integral defined by rough paths theory agrees with Stratonovich integrals for real-valued functions $f(X)$ of the driving noise X , see also Proposition 17.1 in [47]. Recently, [44] introduces a concept of rough semimartingales and develops the corresponding stochastic integration having a deterministic rough path in the background and mixing with p -variation regularity. Beyond semimartingale driving noises, we draw attention to the recent work of [67]. The authors have established the connection between rough integrals and trapezoidal Riemann sum approximations for controlled processes integrands (in the pathwise sense of [56]) and a general class of Gaussian driving noises.

The origin of controlled Rough Paths theory (inspired by the seminal article of T. Lyons [69, 68]) developed by Gubinelli [56] is purely deterministic because the role of the probability measure is totally restricted in constructing a lifting \mathbb{X} of a driving noise X with suitable algebraic and analytic constraints in two-parameters. It is fundamentally based on Taylor-like local expansions of the form

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, \quad (0.47)$$

where (Y, Y') and R^Y are dominated by a control compatible in some sense with the path regularity of X . Once one is able to lift X into \mathbb{X} through the use of a probability measure, then path by path one can make fundamental use of the so-called Sewing Lemma (see [56]) to insure almost sure convergence of the rough path integral

$$\int_0^t Y_s d\mathbf{X}_s = \lim_{|\Pi| \rightarrow 0} \sum_{t_i \in \Pi} \left(Y_{t_i} X_{t_i, t_{i+1}} + Y'_{t_i} \mathbb{X}_{t_i, t_{i+1}} \right), \quad (0.48)$$

as the mesh $|\Pi|$ of refined partitions Π goes to zero. For further details, see e.g [44]. Calculus via

regularization is based on integral-type approximations of the form

$$\int_0^t Y d^* X_s = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t Y_s X(\epsilon, s) ds; \quad * = +, -, \circ, \quad (0.49)$$

where $\frac{1}{\epsilon} X(\epsilon, s)$ encodes a sort of “derivative approximation” of X and convergence (0.49) should be interpreted in probability. This gives rise to three different types of stochastic integrals called backward $* = +$, forward $* = -$ and symmetric $* = \circ$ integrals. Stochastic calculus via regularization was started first by F. Russo and P. Vallois in [83]. The calculus was later continued in [84, 87, 88] in the framework of continuous integrators, essentially with finite quadratic variation integrators. The case of processes with higher variation was first introduced in [36, 37] and continued in [24, 55, 54, 53, 90, 10], especially in relation with fractional Brownian motion and related processes, whose stochastic calculus was first studied in [25]. A not very recent survey paper in the framework of finite dimensional processes is [89]. Stochastic calculus via regularization for processes taking values in Banach spaces, with applications to the path-dependent case, was realized in [31, 29] and in [21]. The case of real-valued jump integrators was first introduced in [85, 86] and then deeply investigated in [4] and later by [5]. Applications to mathematical finance (resp. to fluidodynamics modeling) were published in [23] (resp. [38]). Stochastic differential equations driven by rough noises, i.e. processes which are not of finite quadratic variation were investigated in the framework of rough paths, see e.g. [68], but also (under a more probabilistic point of view) in the context of calculus via regularization, see e.g. [24].

The connection of stochastic calculus via regularization with semimartingale theory, Young and Skorohod integrals has been studied over the last years by many authors, see e.g [89], [1], [2], [17]. An important notion which emerged in calculus via regularization is the notion of weak Dirichlet processes, started in [36, 52] in the framework of continuous processes and continued in [5] in the framework of jump processes. That notion was applied in several contexts of stochastic analysis, see e.g. [51] for a verification theorem in stochastic control theory and for solving the identification problem in solutions of BSDEs, see e.g. [6].

We recall that a process is weak Dirichlet if it can be decomposed as a sum of a local martingale M and an orthogonal process A such that $[A, N] = 0$ for any continuous martingale N . This constitutes a natural generalization of the notion of semimartingale and of Dirichlet process (in the sense of Föllmer), see [42]. We recall that a Dirichlet process is the sum of a local martingale M and a zero quadratic variation process A (i.e. $[A, A] = 0$). Typical examples of (weak) Dirichlet processes emerge from SDEs with distributional drifts as studied in Chapters 1 and 2 and, more recently, from stochastic rough differential equations as described in the recent preprint [46]. We emphasize that the notion of Dirichlet process (which constitutes a first level of generalization of semimartingale) is not the suitable one for treating jump processes. Indeed a zero quadratic variation process A is necessarily continuous.

In Chapter 3, we propose a stochastic version of the Rough Path integral in the sense of Gubinelli. In particular, we emphasize the strong link between the notion of weak Dirichlet process and one of

stochastically controlled process, which is a stochastic version of the one proposed by Gubinelli [56]. According to Definition 3.3.2 such a process fulfills

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, s < t, \quad (0.50)$$

where

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds = 0, \quad (0.51)$$

in probability for each $t \in [0, T]$. Here, X is the reference driving noise process, Y' is a process (not necessarily admitting γ -Hölder continuous paths). The orthogonality condition (0.51) resembles the 2γ -Hölder-regularity condition reminiscent from [56].

The main results of Chapter 3 are the following. Propositions 3.3.7 and 3.3.9 present the connection between weak Dirichlet processes and stochastically controlled processes. In particular, when the reference driving noise is a martingale, then both concepts coincide. As a side effect, Theorem 3.5.6 shows Stratonovich integration as a stochastic rough-type integration for weak Dirichlet integrands and continuous semimartingale integrators. In Chapter 3, we take full advantage of the probability measure and the stochastic controllability (0.50) to establish consistency between stochastic rough-type and Stratonovich integrals for more general integrands.

The results of Chapter 3 are the starting point for the study on stochastic rough-type integrals driven by Gaussian rough paths and their connection with Stratonovich and Skorohod integrals. This analysis will be explored in a forthcoming paper.

Chapter 1

ON PATH-DEPENDENT SDEs INVOLVING DISTRIBUTIONAL DRIFTS

This chapter is object of the paper [74].

1.1 Introduction

This paper discusses in detail a framework of one-dimensional stochastic differential equations (henceforth abbreviated by SDEs) with distributional drift and possible path-dependency. To our best knowledge, this is the first paper which approaches a class of non-Markovian SDEs with distributional drifts.

The main objective of this paper is to analyze the solution (existence and uniqueness) of the martingale problem associated with SDEs of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \quad (1.1)$$

where $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $\sigma > 0$, $x_0 \in \mathbb{R}$ and W is a standard Brownian motion. The assumptions on b , which will be formulated later, imply that b' is a Schwartz distribution. Concerning the path-dependent component of the drift, we consider a locally bounded functional

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (1.2)$$

where

$$\Lambda := \{(s, \eta) \in [0, T] \times C([0, T]); \eta = \eta^s\}$$

and

$$\eta^s(t) := \begin{cases} \eta(t), & \text{if } t \leq s \\ \eta(s), & \text{if } t > s, \end{cases}$$

and $C([0, T])$ is the space of real-valued continuous functions on $[0, T]$, for a given terminal time $0 < T < \infty$. By convention, we extend Γ from Λ to $[0, T] \times C([0, T])$ by setting (in non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]). \quad (1.3)$$

Setting $\sigma = 1, b = B$, where B is a two-sided real-valued Brownian motion which is independent from W ; then (1.1) reads

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + \Gamma(t, X^t)dt + dW_t. \quad (1.4)$$

When $\Gamma = 0$, (1.4) constitutes the so-called Brox diffusion, see e.g [13, 59, 92] and other references therein. This is a celebrated *random environment* model. This paper includes the study of (1.4), where Γ is a bounded path-dependent functional, which appears to be a non-Markovian variant of Brox diffusion.

Path-dependent SDEs were investigated under several aspects. Under standard Lipschitz regularity conditions on the coefficients, it is known, (see e.g Theorem 11.2 [80, chapter V]) that strong existence and uniqueness hold. In the case the path-dependence takes the form of delayed stochastic equations, one-sided Lipschitz condition ensures strong existence and uniqueness, see e.g [91, 71]. Beyond Lipschitz regularity on the coefficients of the SDE, [43] shows uniqueness in law under structural conditions on an underlying approximating Markov process, where local-time and running maximum dependence are considered. The existence in law for infinite-dimensional SDEs with additive noise on the configuration space with path-dependent drift functionals with sublinear growth is studied by [27]. In all those contributions the drift is a non-anticipative functional. Beyond Brownian motion based driving noises, [15] establishes existence of solutions for some path-dependent differential equation driven by an Hölder process.

The Markovian case ($\Gamma = 0$) with distributional drift has been intensively studied over the years. Diffusions in the generalized sense were first considered in the case when the solution is still a semimartingale, beginning with [77]. Later on, many authors considered special cases of SDEs with generalized coefficients. It is difficult to quote them all; in the case when the drift b' is a measure and the solutions are semimartingales we refer the reader to [12, 34, 81]. We also recall that [35] considered even special cases of non-semimartingales solving stochastic differential equations with generalized drift.

In [40] and [41], the authors studied time-independent one-dimensional SDEs of the form

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \quad t \in [0, T], \quad (1.5)$$

whose solutions are possibly non-semimartingale processes, where σ is a strictly positive continuous function and b' is the derivative of a real-valued continuous function. They presented well-posedness of the martingale problem, Itô's formula under weak conditions, semimartingale characterization and Lyons-Zheng decomposition. The only supplementary assumption was the existence of the function

$$\Sigma(x) := 2 \int_0^x \frac{b'}{\sigma^2}(y)dy, x \in \mathbb{R}, \quad (1.6)$$

considered as a suitable limit via regularizations. Those authors considered solutions in law. The SDE (1.5) was also investigated by [8], where the authors provided a well-stated framework when σ and b are γ -Hölder continuous, $\gamma > \frac{1}{2}$. In [82], the authors have also shown that in some cases strong solutions exist and pathwise uniqueness holds. More recently, in the time-dependent framework (but still one-dimensional), a significant contribution was done by [26]. As far as the multidimensional case is concerned, some important steps were done in [39] and more recently in [14], when the diffusion matrix is the identity and b is a time-dependent drift in some proper negative Sobolev space. We also refer to [9], where the authors have focused on (1.1) in the case of a time independent drift b which is a measure of Kato class.

Let us come back to the objective of the present paper in which the path-dependent drift contains the derivative in the sense of distributions of a continuous function b . We remark that in case b' is a bounded measurable function and Γ is a bounded path-dependent functional, then the problem can be easily treated by applying Girsanov's theorem, see e.g. [61] Proposition 3.6 and 3.10, chapter 5. Here, the combination of a Schwartz distribution b' with a path-dependent functional Γ requires a new set of ideas. Equation (1.1) will be interpreted as a martingale problem with respect to an operator

$$\mathcal{L}f := Lf + \Gamma f',$$

see (3.3), where L is the Markovian generator

$$Lf = \frac{\sigma^2}{2} f'' + b' f', \quad (1.7)$$

where we stress that b' is the derivative of some continuous function b . If we denote Σ as in (1.6), then the operator L can be written as

$$Lf = (e^\Sigma f')' \frac{e^{-\Sigma} \sigma^2}{2}, \quad (1.8)$$

see [40]. We define a notion of martingale problem related to \mathcal{L} (see Definition 1.3.3) and a notion of strong martingale problem related to $\mathcal{D}_{\mathcal{L}}$ and a given Brownian motion W , see Definition 1.3.4. That definition has to be compared with the notion of strong existence and pathwise uniqueness of an SDE. In this article, the notion of martingale problem extends the usual one by replacing the space C^2 of twice continuously differentiable real-valued functions with a more suitable set $\mathcal{D}_{\mathcal{L}}$. In the Markovian case, the notion of *strong martingale problem* was introduced in [82]. As anticipated, we will concentrate on the case when b is continuous, the case of special discontinuous functions is investigated in Chapter 2.

The strategy of this paper consists in eliminating the distributional drift of the so-called Zvonkin's transform, see [96]. In this direction, we transform the equation via an L -harmonic function h which exists under the assumption that the function (1.6) is well-defined. The case with $\Gamma = 0$ was already implemented in [40] and [41], where the drift in the transformed SDE was null. In our non-Markovian context, the transformed equation is essentially a path-dependent SDE with measurable coefficients.

Under some linear growth conditions (see Assumption 1.4.16), Theorem 1.4.23 illustrates the existence of the martingale problem related to (1.1). Proposition 1.4.24 states uniqueness under more restrictive conditions. Consider indeed the example when $\sigma = 1$ and b' is the derivative (in the sense of distributions) of a bounded continuous function b and Γ is a bounded measurable functional. In this case, $h = \int_0^\cdot e^{-2b(y)} dy$. Then the study of the well-posedness of the martingale problem is equivalent to the well-posedness of the path-dependent SDE

$$Y = Y_0 + \int_0^\cdot \exp(-2b(h^{-1}(Y_s))) dW_s + \int_0^\cdot \Gamma(s, h^{-1}(Y_s)) \exp(-2b(h^{-1}(Y_s))) ds. \quad (1.9)$$

Existence and uniqueness for (1.9) can be established via Girsanov's theorem.

Moreover, Corollary 1.4.31 establishes well-posedness for the strong martingale problem associated to (1.1). This holds under suitable Lipschitz regularity conditions on the functional $\tilde{\Gamma}$, which is related to Γ via (4.14), and a specific assumption on the function σ_0 defined in Remark 4.9. We suppose that σ_0 is bounded, uniformly elliptic and it fulfills a Yamada-Watanabe type condition. One typical example is given when σ_0 is γ -Hölder continuous function for $\frac{1}{2} \leq \gamma \leq 1$.

Several results of the present paper can be partially extended to the multidimensional case, by using the techniques developed in the Markovian case by [39]. In this direction, the harmonic function h should be replaced by the "mild" solution ϕ of the parabolic Kolmogorov equation

$$\begin{cases} \partial_t \phi + \frac{1}{2} \Delta \phi + b' \nabla \phi & = \lambda(\phi - \text{id}), \\ \phi(T, \cdot) & = \text{id}, \end{cases} \quad (1.10)$$

see Section 3.2 of [39].

1.2 Notations and Preliminaries

1.2.1 General notations

Let I be an interval of \mathbb{R} . $C^k(I)$ is the space of real functions defined on I having continuous derivatives till order k . Such space is endowed with the uniform convergence topology on compact sets for the functions and all derivatives. Generally, $I = \mathbb{R}$ or $[0, T]$ for some fixed positive real T . The space of continuous functions on I will be denoted by $C(I)$. Often, if there is no ambiguity $C^k(\mathbb{R})$ will be simply indicated by C^k . Given an a.e. bounded real function f , $|f|_\infty$ will denote the essential supremum.

We recall some notions from [40]. For us, all filtrations \mathfrak{F} fulfill the usual conditions. When no filtration is specified, we mean the canonical filtration of the underlying process. Otherwise the canonical filtration associated with a process X is denoted by \mathfrak{F}^X . An \mathfrak{F} -Dirichlet process X is the sum of an \mathfrak{F} -local martingale M^X with an \mathfrak{F} -adapted zero quadratic variation process A^X . We will fix by convention that $A_0^X = 0$ so that the decomposition is unique. A sequence (X^n) of continuous

processes indexed by $[0, T]$ is said to converge u.c.p. to some process X whenever $\sup_{t \in [0, T]} |X_t^n - X_t|$ converges to zero in probability. Finally the notion of covariation between two general càdlàg processes (whenever it exists) is denoted by $[X, Y]$ and we set $[X] = [X, X]$, see e.g. [86]. If $[X]$ exists, X is called finite quadratic variation process.

Remark 1.2.1.

1. An \mathfrak{F} -continuous semimartingale Y is always an \mathfrak{F} -Dirichlet process. The A^Y process coincides with the continuous bounded variation component. Moreover the quadratic variation $[Y]$ is the usual quadratic variation for semimartingales.
2. Any \mathfrak{F} -Dirichlet process is a finite quadratic variation process and its quadratic variation gives $[X] = [M^X]$.
3. If $f \in C^1(\mathbb{R})$ and $X = M^X + A^X$ is an \mathfrak{F} -Dirichlet process, then $Y = f(X)$ is again an \mathfrak{F} -Dirichlet process and $[Y] = \int_0^\cdot f'(X)^2 d[M^X]$.

1.3 Non-Markovian SDE: the function case.

1.3.1 General considerations.

As in the case of Markovian SDEs, it is possible to formulate the notions of strong existence, pathwise uniqueness, existence and uniqueness in law for path-dependent SDEs of the type (1.1), see e.g. Section 1.5. Let us suppose for the moment that $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions. We will consider solutions X of

$$\begin{cases} dX_t &= \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt \\ X_0 &= \xi, \end{cases} \quad (3.1)$$

for some initial condition ξ .

The previous equation will be denoted by $E(\sigma, b', \Gamma; \nu)$ (where ν is the law of ξ), or simply by $E(\sigma, b', \Gamma)$ if we omit the initial condition. For simplicity, the initial conditions will always be considered as deterministic. The functional Γ is non-anticipative in the sense of (1.3).

Definition 1.3.1. Let ν be the Dirac probability measure on \mathbb{R} such that $\nu = \delta_{x_0}, x_0 \in \mathbb{R}$. A stochastic process X is called **solution** of $E(\sigma, b', \Gamma; \nu)$ with respect to a probability \mathbb{P} if there is a Brownian motion W on some filtered probability space, such that X solves (3.1) and $X_0 = x_0$. We also say that the couple (X, \mathbb{P}) solves $E(\sigma, b', \Gamma)$ with initial condition distributed according to ν .

Suppose $\Gamma \equiv 0$. A very well-known result in [93], Corollary 8.1.7, concerns the equivalence between martingale problems and solution in law of SDEs. Suppose for a moment that b' is a continuous function. According to [61, chapter 5], a process X and probability \mathbb{P} solve the classical martingale

problem, if and only if, X is a solution of (1.1). The proof of that result can be easily adapted to the path-dependent case, i.e. when $\Gamma \neq 0$. This provides the statement below.

Proposition 1.3.2. *A couple (X, \mathbb{P}) is a solution of $E(\sigma, b', \Gamma)$, if and only if, under \mathbb{P} ,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds \quad (3.2)$$

is a local martingale, where $Lf = \frac{1}{2}\sigma^2 f'' + b' f'$, for every $f \in C^2$.

1.3.2 Comments about the distributional case

When b' is a distribution, it is not obvious to introduce the notion of SDE, except in the case when L is close to the divergence form, i.e. when $Lf = (\sigma^2 f')' + \beta f'$ and β is a Radon measure, see e.g. Proposition 3.1 of [40]. For this reason, we replace the notion of solution in law with the notion of martingale problem. Suppose for a moment that L is a second order PDE operator with possible generalized coefficients. In general, as it is shown in [40], C^2 is not included in the natural domain of operator L and, similarly to [40], we will replace C^2 with some set \mathcal{D}_L . Suppose that $L : \mathcal{D}_L \subset C^1(\mathbb{R}) \rightarrow C(\mathbb{R})$. Nevertheless \mathcal{D}_L is not the domain of L in the sense of the generator of a semigroup.

Definition 1.3.3. 1. *We say that a continuous stochastic process X solves (with respect to a probability \mathbb{P} on some measurable space (Ω, \mathcal{F})) the martingale problem related to*

$$\mathcal{L}f := Lf + \Gamma f', \quad (3.3)$$

with initial condition $\nu = \delta_{x_0}$, $x_0 \in \mathbb{R}$, with respect to a domain \mathcal{D}_L if

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (3.4)$$

is a \mathbb{P} -local martingale for all $f \in \mathcal{D}_L$.

We will also say that the couple (X, \mathbb{P}) is a solution of (or (X, \mathbb{P}) solves) the martingale problem with respect to \mathcal{D}_L .

2. *If a solution exists, we say that existence holds for the martingale problem above.*
3. *We say that uniqueness holds for the martingale problem above, if any two solutions (X^i, \mathbb{P}^i) , $i = 1, 2$ (on some measurable space (Ω, \mathcal{F})) have the same law.*

We remark that in the classical literature of martingale problems, see [93], a solution is a probability on the path space $C([0, T])$ and X is the canonical process. If (X, \mathbb{P}) is a solution according to our notations, a solution in the classical framework would be the probability law of X with respect to \mathbb{P} . We have preferred to conserve our notations in conformity with [40, 41].

In the sequel, when the measurable space (Ω, \mathcal{F}) is self-explanatory it will be often omitted. As already observed in Proposition 1.3.2, the notion of martingale problem is (since the works of Stroock and Varadhan [93]) a concept related to solutions of SDEs in law. In the case when b' and σ are continuous functions (see [93]), \mathcal{D}_L corresponds to the space $C^2(\mathbb{R})$, in agreement with Remark 1.4.6 below.

Below we introduce the analogous notion of strong existence and pathwise uniqueness for our martingale problem.

Definition 1.3.4.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathfrak{F} = (\mathcal{F}_t)$ be the canonical filtration associated with a fixed Brownian motion W . Let $x_0 \in \mathbb{R}$ be a constant. We say that a continuous \mathfrak{F} -adapted real-valued process X such that $X_0 = x_0$ is a **solution to the strong martingale problem** (related to (3.3)), with respect to \mathcal{D}_L and W (with related filtered probability space), if for all $f \in \mathcal{D}_L$ (3.4) is a \mathfrak{F} -local martingale given by

$$\int_0^t f'(X_s)\sigma(X_s)dW_s. \quad (3.5)$$

2. We say that **strong existence** for the martingale problem related to (3.3) with respect to \mathcal{D}_L holds, if for every $x_0 \in \mathbb{R}$, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{F})$, where $\mathfrak{F} = (\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W , there is a process X solving the strong martingale problem (related to (3.3)) with respect to \mathcal{D}_L and W with $X_0 = x_0$.
3. We say that **pathwise uniqueness** holds for the martingale problem (related to (3.3)) with respect to \mathcal{D}_L , if given $(\Omega, \mathcal{F}, \mathbb{P})$, a Brownian motion W on it, two solutions $X^i, i = 1, 2$, to the strong martingale with respect to \mathcal{D}_L and W and $\mathbb{P}[X_0^1 = X_0^2] = 1$, then X^1 and X^2 are indistinguishable.

1.4 The case when the drift is the derivative of a continuous function

1.4.1 The Markovian case

In this section we recall some basic notations and results from [40] but in a way that simplifies the presentation of our framework. We will also add some useful new elements. Let σ and b be functions in $C(\mathbb{R})$ with $\sigma > 0$. In [40], in view of defining \mathcal{D}_L and L in the spirit of (1.7), the authors define the function

$$\Sigma(x) = 2 \lim_{n \rightarrow \infty} \int_0^x \frac{b'_n(y)}{\sigma_n^2} dy, \forall x \in \mathbb{R}, \quad (4.1)$$

where the limit is intended to be in $C(\mathbb{R})$, i.e. uniformly on each compact. Here,

$$\sigma_n^2 := \sigma^2 * \Phi_n, b_n := b * \Phi_n, \quad (4.2)$$

where $\Phi_n(x) := n\Phi(nx)$, $n \geq 1$, and $\Phi \in \mathcal{S}(\mathbb{R})$ (the Schwartz space), with $\int \Phi(x) dx = 1$. For concrete examples, one can take either σ^2 or b are of locally bounded variation, see [40] for other examples and other details. Proposition 2.3, Lemma 2.6 and 2.9 of [40] allow us (equivalently) to define a subspace \mathcal{D}_L of $C^1(\mathbb{R})$ on which the definition of Lf by (1.8) makes sense.

Notation 1.4.1. 1. \mathcal{D}_L is the subset of all $f \in C^1(\mathbb{R})$ for which there exists $\phi \in C^1$ such that $f' = \exp(-\Sigma)\phi$.

2. If $f \in \mathcal{D}_L$, then we set

$$Lf = \phi' \exp(-\Sigma) \frac{\sigma^2}{2}, \quad (4.3)$$

where ϕ is the function given in item (1) above.

3. We denote by $h : \mathbb{R} \rightarrow \mathbb{R}$ the function the scale function, see also [80], page 178, characterized by

$$h(0) = 0, \quad h' = e^{-\Sigma}. \quad (4.4)$$

In particular h is an L -harmonic function in the sense that $Lh = 0$, as we will see in Proposition 1.4.3.

Remark 1.4.2. (4.3) in Notation 1.4.1 corresponds to (1.8).

Proposition 1.4.3.

1. If $f \in \mathcal{D}_L$, then $f^2 \in \mathcal{D}_L$ and $Lf^2 = \sigma^2 f'^2 + 2fLf$.

2.

$$Lh = 0, \quad Lh^2 = \sigma^2 h'^2.$$

Proof.

1. We observe $f^2 \in \mathcal{D}_L$ because $(f^2)' = 2ff' = (2f\phi) \exp(-\Sigma)$. From Notation 1.4.1 (1) and the fact that $\phi_2 := 2f\phi \in C^1$, we conclude $f^2 \in \mathcal{D}_L$. By (4.3),

$$\begin{aligned} Lf^2 &= \phi_2' \exp(-\Sigma) \frac{\sigma^2}{2} = (f\phi)' \exp(-\Sigma) \sigma^2 \\ &= f' \sigma^2 \exp(-\Sigma) \phi + f \phi' \exp(-\Sigma) \sigma^2 = f'^2 \sigma^2 + 2fLf. \end{aligned}$$

2. The proof follows by setting $\phi = 1$, using item (1) above and Notation 1.4.1 (1).

□

We now formulate a standing assumption.

Assumption 1.4.4.

- Σ given by (4.1) is a well-defined function.

- We suppose the non-explosion condition

$$\int_{-\infty}^0 e^{-\Sigma(x)} dx = \int_0^{+\infty} e^{-\Sigma(x)} dx = \infty. \quad (4.5)$$

Remark 1.4.5.

1. Under Assumption 1.4.4, the L -harmonic function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined in (4.4) is a C^1 -diffeomorphism. In particular h is surjective.
2. It is easy to verify that Assumption 1.4.4 implies the non-explosion condition (3.16) in Proposition 3.13 in [40], which corresponds to the Feller test when the drift is a function. In [40], one could allow h not to be surjective.

Remark 1.4.6. When σ and b' are continuous functions, then $\mathcal{D}_L = C^2$. Indeed, in this case, $\Sigma \in C^1$ and then $f' = \exp(-\Sigma)\phi \in C^1$. In particular, Lf corresponds to the classical definition.

Remark 1.4.7. We suppose that Assumption 1.4.4 is satisfied. Let $g \in \mathcal{D}_L$ be a fixed diffeomorphism of class C^1 such that $g' > 0$. We set

$$\sigma_0^g := (\sigma g') \circ g^{-1}, \quad b^g = ((Lg) \circ g^{-1}) \quad (4.6)$$

and consider

$$L^g v := \frac{1}{2}(\sigma_0^g)^2 v'' + b^g v', \quad v \in \mathcal{D}_{L^g}$$

and where we define \mathcal{D}_{L^g} according to Notation 1.4.1 replacing L with L^g .

By Remark 1.4.6, since L^g has continuous coefficients, then $\mathcal{D}_{L^g} = C^2$.

Proposition 1.4.8. We suppose that Assumption 1.4.4 is satisfied. Let $g \in \mathcal{D}_L$ be a fixed diffeomorphism of class C^1 such that $g' > 0$. Then, $f \in \mathcal{D}_L$ if and only if $f \circ g^{-1}$ belongs to \mathcal{D}_{L^g} and

$$Lf \circ g^{-1} = L^g(f \circ g^{-1}).$$

Proof (of Proposition 1.4.8). By Notation 1.4.1, there exists $\phi^g \in C^1$ such that

$$g' = \exp(-\Sigma)\phi^g. \quad (4.7)$$

Concerning the direct implication, if $f \in \mathcal{D}_L$, first we prove that $f \circ g^{-1} \in \mathcal{D}_{L^g}$. Again by Notation 1.4.1 there exists $\phi^f \in C^1$ such that $f' = \exp(-\Sigma)\phi^f$. So, $(f \circ g^{-1})' = \frac{f'}{g'} \circ g^{-1} = \frac{\phi^f}{\phi^g} \circ g^{-1} \in C^1$ because $g^{-1} \in C^1$ and $\phi^g > 0$; then, $f \circ g^{-1} \in C^2$. Note that, by Remark 1.4.7, $\mathcal{D}_{L^g} = C^2$ and hence $f \circ g^{-1} \in \mathcal{D}_{L^g}$. Moreover, according to Notation 1.4.1 (2),

$$(\phi^f)' = \frac{2Lf}{\sigma^2} \exp(\Sigma), \quad (\phi^g)' = \frac{2Lg}{\sigma^2} \exp(\Sigma).$$

A direct computation gives

$$(f \circ g^{-1})'' = \left(\frac{\phi^f}{\phi^g} \circ g^{-1} \right)' = \left[\frac{2Lf}{g'^2 \sigma^2} - \frac{2Lg f'}{\sigma^2 g'^3} \right] \circ g^{-1}.$$

Consequently,

$$\frac{(\sigma_0^g)^2}{2}(f \circ g^{-1})'' = \left[Lf - \frac{Lg f'}{g'} \right] \circ g^{-1}. \quad (4.8)$$

By (4.8), $Lf \circ g^{-1} = \frac{(\sigma_0^g)^2}{2}(f \circ g^{-1})'' + (Lg) \circ g^{-1}(f \circ g^{-1})' = L^g(f \circ g^{-1})$.

Let us discuss the converse implication. Suppose that $f \circ g^{-1}$ belongs to $\mathcal{D}_{L^g} = C^2$. Again, according to Notation 1.4.1 (1), we need to show that $f' \exp(\Sigma) \in C^1$ which is equivalent to showing that $(f' \exp(\Sigma)) \circ g^{-1}$ belongs to C^1 . If $\phi^g \in C^1$ is such that $g' = \exp(-\Sigma)\phi^g$ (see (4.7)) we have

$$(f' \exp(\Sigma)) \circ g^{-1} = \left(f' \frac{\phi^g}{g'} \right) \circ g^{-1} = (f \circ g^{-1})'(\phi^g \circ g^{-1}),$$

which obviously belongs to C^1 . Therefore $f \in \mathcal{D}_L$. \square

By setting $h = g$ in Proposition 1.4.8 we obtain the following.

Corollary 1.4.9. *Let h be the function defined by (4.4). Then, $f \in \mathcal{D}_L$ if, and only if, $f \circ h^{-1} \in C^2$. Moreover, by setting $\varphi = f \circ h^{-1}$ for $f \in \mathcal{D}_L$, we have*

$$L(\varphi \circ h) \circ h^{-1} = L^h(\varphi) = \frac{1}{2}\sigma_0^2\varphi'',$$

where

$$\sigma_0 := \sigma_0^h = (\sigma h') \circ h^{-1}. \quad (4.9)$$

In [40], the authors also show that the existence and uniqueness of the solution to the martingale problem are conditioned to a non-explosion feature. The proposition below is an easy consequence of Proposition 3.13 in [40], which concerns the well-posedness of the martingale problem with respect to L in the case $\Gamma = 0$.

Proposition 1.4.10. *Let $\nu = \delta_{x_0}, x_0 \in \mathbb{R}$ and suppose that Assumption 1.4.4 holds true. Then, the existence and uniqueness holds for the martingale problem related to L (i.e. with $\Gamma = 0$) with respect to \mathcal{D}_L with initial condition ν .*

Remark 1.4.11. *By Proposition 3.2 of [40], if $\Gamma = 0$ and (X, \mathbb{P}) is a solution of the martingale problem given in Proposition 1.4.10, then there exists a \mathbb{P} -Brownian motion W such that (3.4) equals*

$$\int_0^t (f' \sigma)(X_s) dW_s, t \in [0, T].$$

1.4.2 The path-dependent framework.

Let σ and b be functions in $C(\mathbb{R})$ with $\sigma > 0$ and Γ as defined in (1.2). Let us suppose again Assumption 1.4.4 and let h be the function defined in (4.4). We recall that σ_0 was defined in (4.9).

The first result explains how to reduce our path-dependent martingale problem to a path-dependent SDE.

Proposition 1.4.12. *Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

1. (X, \mathbb{P}) solves the martingale problem related to (3.3) with respect to \mathcal{D}_L if and only if the process $Y = h(X)$ is a solution (with respect to \mathbb{P}) of

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y_s)) ds, \quad (4.10)$$

for some \mathbb{P} -Brownian motion W .

2. Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, X is a solution to the strong martingale problem with respect to \mathcal{D}_L and W if and only if (4.10) holds.

Proof.

1. We start proving the direct implication. According to (3.4) and the notations introduced therein

$$M_t^h = h(X_t) - h(X_0) - \int_0^t Lh(X_s) ds - \int_0^t h'(X_s) \Gamma(s, X_s) ds, \quad (4.11)$$

is a \mathbb{P} -local martingale on some measurable space (Ω, \mathcal{F}) .

In particular, by Proposition 1.4.3, Y satisfies

$$Y_t = Y_0 + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y_s)) ds + M_t^h,$$

where M^h is a local martingale and hence Y is a semimartingale. We need now to evaluate $[M^h]_t = [Y]_t$. We apply (3.4) to $f = h^2$ and again by Proposition 1.4.3 we get

$$Y_t^2 = Y_0^2 + \int_0^t \sigma_0^2(Y_s) ds + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y_s)) ds + M_t^{h^2}, \quad (4.12)$$

where M^{h^2} is a local martingale and we recall that σ_0 was defined in (4.6). By integration by parts,

$$\begin{aligned} [Y]_t &= Y_t^2 - Y_0^2 - 2 \int_0^t Y_s dY_s \\ &= Y_t^2 - Y_0^2 + M_t - 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y_s)) ds, \end{aligned}$$

where $M_t = -2 \int_0^t Y_s dM_s^h$. Therefore

$$Y_t^2 = Y_0^2 - M_t + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y_s)) ds + [Y]_t. \quad (4.13)$$

The semimartingale Y^2 admits the two decompositions (4.12) and (4.13). By uniqueness, $-M = M^{h^2}$ and $\int_0^t \sigma_0^2(Y_s) ds = [Y]_t$. By (4.11)

$$[M^h]_t = [Y]_t = \int_0^t \sigma_0^2(Y_s) ds.$$

Setting

$$W_t := \int_0^t \frac{dM_s^h}{\sigma_0(Y_s)}, t \geq 0,$$

we have

$$[W]_t \equiv t.$$

Therefore, by Lévy's characterization of Brownian motion, W is a standard Brownian motion. Since

$$M^h = \int_0^\cdot \sigma_0(Y_s) dW_s,$$

(4.11) shows that Y solves (4.10).

Next, we prove the converse implication. Suppose that $Y = h(X)$ satisfies (4.10), for some \mathbb{P} -Brownian motion W . We take $f \in \mathcal{D}_L$. By Corollary 1.4.9 we have $\varphi \equiv f \circ h^{-1} \in C^2$. Using Itô's formula and again Corollary 1.4.9, we get

$$\begin{aligned} \varphi(Y_t) &= \varphi(Y_0) + \int_0^t \varphi'(Y_s) dY_s + \frac{1}{2} \int_0^t \varphi''(Y_s) d[Y]_s \\ &= \varphi(Y_0) + \int_0^t \varphi'(Y_s) \sigma_0(Y_s) dW_s + \frac{1}{2} \int_0^t \varphi''(Y_s) \sigma_0^2(Y_s) ds \\ &= \varphi(Y_0) + \int_0^t \varphi'(Y_s) \sigma_0(Y_s) dW_s + \frac{1}{2} \int_0^t L^h \varphi(Y_s) \sigma_0^2(Y_s) ds \\ &= f(X_0) + \int_0^t Lf(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s + \int_0^t f'(X_s) \Gamma(s, X^s) ds. \end{aligned}$$

Therefore

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds = \int_0^t f'(X_s) \sigma(X_s) dW_s$$

is a local martingale, which concludes the proof.

2. The converse implication follows in the same way as for item (1). The proof of the direct implication follows directly by Itô's formula.

□

Corollary 1.4.13. *Let (X, \mathbb{P}) be a solution of the martingale problem related to (3.3) with respect to \mathcal{D}_L . Then X is a Dirichlet process (with respect to its canonical filtration) and $[X]_t = \int_0^t \sigma^2(X_s) ds, t \in [0, T]$.*

Proof.

By Remark 1.4.5, h is a diffeomorphism. By Proposition 1.4.12, $X = h^{-1}(Y)$, where Y is obviously a semimartingale such that $[Y]_t = \int_0^t \sigma_0^2(Y_s) ds, t \in [0, T]$. Consequently, by Remark 1.2.1, X is indeed a Dirichlet process and

$$[X]_t = \int_0^t ((h^{-1})')^2 \sigma_0^2(Y_s) ds = \int_0^t \frac{\sigma_0^2}{(h' \circ h^{-1})^2}(Y_s) ds = \int_0^t \sigma^2(X_s) ds, t \in [0, T].$$

□

Remark 1.4.14. *If X is a solution of the strong martingale problem with respect to \mathcal{D}_L and some Brownian motion W , then X is a Dirichlet process with respect to the canonical filtration of the Brownian motion.*

An immediate consequence of Proposition 1.4.12 is the following.

Corollary 1.4.15. *Suppose that $\Gamma = 0$ and let (X, \mathbb{P}) be a solution to the martingale problem related to L with respect to \mathcal{D}_L . Then, $Y = h(X)$ is an \mathfrak{F} -local martingale where \mathfrak{F} is the canonical filtration of X with quadratic variation $[Y] = \int_0^\cdot \sigma_0^2(Y_s) ds$.*

1.4.3 Existence

We fix here the same conventions as in Section 1.4.2. In the sequel, we introduce the map $\tilde{\Gamma} : \Lambda \rightarrow \mathbb{R}$ defined by

$$\tilde{\Gamma}(s, \eta) = \frac{\Gamma(s, \eta)}{\sigma(\eta(s))}, (s, \eta) \in \Lambda. \quad (4.14)$$

At this point, we introduce the following technical assumption, which is in particular verified if Γ is bounded and $\sigma = 1$.

Assumption 1.4.16. *There is $K > 0$ such that*

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, h^{-1} \circ \eta^s)| \leq K \left(1 + \sup_{s \in [0, T]} |\eta(s)| \right), \forall \eta \in C([0, T]).$$

Remark 1.4.17. *Let X be a stochastic process and set $Y = h(X)$. If Assumption 1.4.16 is fulfilled, then*

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, X^s)| \leq K \left(1 + \sup_{s \in [0, T]} |Y_s| \right). \quad (4.15)$$

In particular $\int_0^T \tilde{\Gamma}^2(s, X^s) ds < \infty$ a.s.

Proposition 1.4.18 below is a well-known extension of Novikov's criterion. It is an easy consequence of Corollary 5.14 in [61, Chapter 3].

Proposition 1.4.18. *Suppose Assumption 1.4.16 holds true. Let W be a Brownian motion and let X be a continuous and adapted process for which there exists a subdivision $0 = t_0 < t_1 < \dots < t_n = T$ such that*

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \right) \right] < \infty,$$

for every $i \in \{1, \dots, n\}$. Then, the process

$$N_t = \exp \left(\int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right),$$

is a martingale.

Next, we need a slight adaptation of the Dambis-Dubins-Schwarz theorem to the case of a finite interval. For the sake of completeness, we give the details here.

Proposition 1.4.19. *Let M be a local martingale vanishing at zero such that $[M]_t = \int_0^t A_s ds$, $t \in [0, T]$. Then, on a possibly enlarged probability space, there exists a copy of M (still denoted by the same letter M) with the same law and a Brownian motion β such that*

$$M_t = \beta \int_0^t A_s ds, t \in [0, T].$$

Proof.

Let us define

$$\tilde{M}_t = \begin{cases} M_t, & t \in [0, T] \\ M_T + B_t - B_T, & t > T, \end{cases}$$

where B is a Brownian motion independent of M . If the initial probability space is not rich enough, one considers an enlarged probability space containing a copy of M (still denoted by the same letter) with the same law and the independent Brownian motion B . Note that \tilde{M} is a local martingale with quadratic variation given by

$$[\tilde{M}]_t = \begin{cases} [M]_t, & t \in [0, T] \\ t - T + [M]_T, & t > T. \end{cases}$$

Observe that $\lim_{t \rightarrow \infty} [\tilde{M}]_t = \infty$. By the classical Dambis, Dubins-Schwarz theorem there exists a standard Brownian motion β such that a.s. $\tilde{M}_t = \beta \int_0^t A_s ds$, $t \geq 0$. In particular

$$M_t = \beta \int_0^t A_s ds, 0 \leq t \leq T.$$

□

The proposition below is an adaptation of a well-known argument for Markov diffusions.

Proposition 1.4.20. *Suppose that Assumption 1.4.16 holds and that σ_0 is bounded. Let (X, \mathbb{P}) be a solution to the martingale problem related to (3.3) with respect to \mathcal{D}_L with $\Gamma = 0$. Let M^X be the local martingale component of X . We set*

$$W_t := \int_0^t \frac{1}{\sigma(X_s)} dM^X, t \in [0, T].$$

Then

$$\exp \left(\int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right), t \in [0, T], \quad (4.16)$$

is a martingale.

Remark 1.4.21. *Let (X, \mathbb{P}) be a solution to the martingale problem related to (3.3) with respect to \mathcal{D}_L with $\Gamma = 0$. We recall that, by Corollary 1.4.13, X is an \mathfrak{F} -Dirichlet process (\mathfrak{F} being the canonical filtration) and $[X] = [M^X] = \int_0^\cdot \sigma^2(X_s) ds$ so that, by Lévy's characterization theorem, W is an \mathfrak{F} -Brownian motion.*

Proof (of Proposition 1.4.20).

Let $Y = h(X)$. By Proposition 1.4.12, we know that $[Y] = \int_0^\cdot \sigma_0^2(Y_s) ds$. Let us choose $k \geq |\sigma_0|_\infty^2 T$ and a subdivision $\{t_0 = 0, \dots, t_n = T\}$ of $[0, T]$ in such way that

$$c_i := \frac{3}{2}(t_i - t_{i-1})K^2k < \frac{1}{2}, \quad (4.17)$$

for $i \in \{1, \dots, n\}$. Here, K comes from Assumption 1.4.16. By (4.15), we know that

$$\int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \leq (t_i - t_{i-1})K^2 \left(1 + \sup_{s \in [0, T]} |Y_s|\right)^2, \quad (4.18)$$

for every $i \in \{1, \dots, n\}$. We set $M_t = Y_t - Y_0, t \in [0, T]$ and we note that

$$\left(1 + \sup_{s \in [0, T]} |Y_s|\right)^2 \leq 3 \sup_{s \in [0, T]} |M_s|^2 + 3(1 + Y_0^2). \quad (4.19)$$

We recall that Y_0 is deterministic. By applying Proposition 1.4.18, taking into account (4.18) and (4.19), we get

$$\begin{aligned} & \mathbb{E} \left(\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right) \leq \\ & \mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |M_s|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right). \end{aligned} \quad (4.20)$$

Since M is a local martingale vanishing at zero, Proposition 1.4.19 states that there is a copy (with the same distribution) of M (still denoted by the same letter) on another probability space, a Brownian motion β such that previous expression gives

$$\begin{aligned} & \mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |\beta_{[M]_s}|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right) \\ & \leq \mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{\tau \in [0, k]} |\beta_\tau|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right), \end{aligned} \quad (4.21)$$

the latter inequality being valid because $[M]_t = \int_0^t \sigma_0^2(Y_s) ds$. By (4.17) we get

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right] & \leq \mathbb{E} \left[\exp \left(\frac{c_i}{k} \sup_{\tau \in [0, k]} |B_\tau|^2 \right) \right] \exp \left(\frac{c_i}{k} (1 + Y_0^2) \right) \\ & \leq \mathbb{E} \left[\sup_{\tau \in [0, k]} \exp \left(\frac{c_i}{k} |B_\tau|^2 \right) \right] \exp \left(\frac{c_i}{k} (1 + Y_0^2) \right), \end{aligned} \quad (4.22)$$

for every $i \in \{1, \dots, n\}$. By Remark 1.4.22 below, for each $i \in \{1, \dots, n\}$, we have

$$\mathbb{E} \left[\exp \left(\frac{c_i}{k} |B_\tau|^2 \right) \right] \leq \mathbb{E} [\exp(c_i G^2)] < \infty,$$

where G is a standard Gaussian random variable. Since $x \mapsto \exp(\frac{c_i}{2k}x)$ is increasing and convex, and $(|B_\tau|^2)_{\tau \geq 0}$ is a non-negative square integrable submartingale, then $(\exp(\frac{c_i}{2k}|B_\tau|^2))$ is also a non-negative submartingale. Consequently, by Doob's inequality (with $p = 2$) the expectation on the right-hand side of (4.22) is finite. Finally by Proposition 1.4.18, (4.16) is a martingale. \square

Remark 1.4.22. Let G be a standard Gaussian random variable. If $c < \frac{1}{2}$ then

$$\mathbb{E}[\exp(cG^2)] < \infty.$$

Proposition 1.4.20 opens the way to the following existence result for the path-dependent martingale problem.

Theorem 1.4.23. Suppose that Assumption 1.4.4 holds and that one of the two conditions below are fulfilled.

1. $\tilde{\Gamma}$ is bounded.
2. $\tilde{\Gamma}$ fulfills Assumption 1.4.16 and σ_0 is bounded.

Then existence holds for the martingale problem related to (3.3) with respect to \mathcal{D}_L .

Proof.

By Proposition 1.4.10, there is a solution (X, \mathbb{P}) to the above-mentioned martingale problem with $\Gamma = 0$. By Remark 1.4.11, there is a Brownian motion W such that

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds = \int_0^t (f'\sigma)(X_s)dW_s, \quad (4.23)$$

for every $f \in \mathcal{D}_L$. We define the process

$$V_t := \exp \left(\int_0^t \tilde{\Gamma}(s, X^s)dW_s - \frac{1}{2} \int_0^t \tilde{\Gamma}^2(s, X^s)ds \right).$$

Under item (1), V is a martingale by the Novikov's condition. Under item (2), Proposition 1.4.20 says that V is a martingale. We define

$$\tilde{W}_t := W_t - \int_0^t \tilde{\Gamma}(s, X^s)ds. \quad (4.24)$$

By Girsanov's theorem, (4.24) is a Brownian motion under the probability \mathbb{Q} such that

$$d\mathbb{Q} := \exp \left(\int_0^T \tilde{\Gamma}(s, X^s)dW_s - \frac{1}{2} \int_0^T \tilde{\Gamma}^2(s, X^s)ds \right) d\mathbb{P}.$$

Applying (4.24) in (4.23), we obtain

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\tilde{\Gamma}(s, X^s)ds = \int_0^t (f'\sigma)(X_s)d\tilde{W}_s,$$

for every $f \in \mathcal{D}_L$. Since $\int_0^t (f'\sigma)(X_s)d\tilde{W}_s$ is a local martingale under \mathbb{Q} , (X, \mathbb{Q}) is proved to be a solution to the martingale problem in the statement. \square

1.4.4 Uniqueness.

We use here again the notation $\tilde{\Gamma}$ introduced in (4.14).

Proposition 1.4.24. *Suppose that Assumption 1.4.4 is satisfied. Then, uniqueness holds for the martingale problem related to (3.3) with respect to \mathcal{D}_L .*

Proof.

Let (X^i, \mathbb{P}^i) , $i = 1, 2$, be two solutions of the martingale problem related to (3.3) with respect to \mathcal{D}_L . Let us fix $i = 1, 2$. By Corollary 1.4.13, X^i is an \mathfrak{F}^{X^i} -Dirichlet process with respect to \mathbb{P}^i , such that $[X^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$. Let M^i be the martingale component of X^i . Since $[M^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$, by Lévy's characterization theorem, the process

$$W_t^i = \int_0^t \frac{dM_s^i}{\sigma(X_s^i)}, t \in [0, T], \quad (4.25)$$

is an \mathfrak{F}^{X^i} -Brownian motion. In particular, W^i is a Borel functional of X^i .

By means of localization (similarly to Proposition 5.3.10 in [61]), without loss of generality we can suppose $\tilde{\Gamma}$ to be bounded. We define the process (whose random variables are also Borel functionals of X^i)

$$V_t^i = \exp \left(- \int_0^t \tilde{\Gamma}(s, X^{i,s}) dW_s^i - \frac{1}{2} \int_0^t \left(\tilde{\Gamma}(s, X^{i,s}) \right)^2 ds \right),$$

which, by Novikov's condition, it is a \mathbb{P}^i -martingale. This allows us to define the probability \mathbb{Q}^i such that $d\mathbb{Q}^i = V_T^i d\mathbb{P}^i$.

By Girsanov's theorem, under \mathbb{Q}^i , $B_t^i := W_t^i + \int_0^t \tilde{\Gamma}(s, X^{i,s}) ds$ is a Brownian motion. Therefore, (X^i, \mathbb{Q}^i) solves the martingale problem related to L ($\Gamma = 0$) with respect to \mathcal{D}_L . By uniqueness of the martingale problem with respect to \mathcal{D}_L and $\Gamma = 0$ (see Proposition 1.4.10), X^i (under \mathbb{Q}^i), $i = 1, 2$ have the same law. Hence, for every Borel set $B \in \mathfrak{B}(C[0, T])$, we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_{\Omega} \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} d\mathbb{Q}^1 = \int_{\Omega} \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} d\mathbb{Q}^2 = \mathbb{P}^2\{X^2 \in B\}.$$

Therefore, X^1 under \mathbb{P}^1 has the same law as X^2 under \mathbb{P}^2 . Finally, uniqueness holds for the martingale problem related to (3.3) with respect to \mathcal{D}_L . \square

1.4.5 Results on pathwise uniqueness

Before exploring conditions for strong existence and uniqueness for the martingale problem, we state and prove Proposition 1.4.27, which constitutes a crucial preliminary step.

Let $\bar{\Gamma} : \Lambda \rightarrow \mathbb{R}$ be a generic Borel functional. Related to it, we formulate the following technical assumption.

Assumption 1.4.25.

1. There exists a function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\int_0^\epsilon l^{-2}(u)du = \infty$ for all $\epsilon > 0$ and

$$|\sigma_0(x) - \sigma_0(y)| \leq l(|x - y|).$$

2. σ_0 has at most linear growth.

3. there exists $K > 0$ such that

$$|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K \left(|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)|dr \right),$$

for all $(s, \eta^1), (s, \eta^2) \in \Lambda$.

4. $\bar{\Gamma}_\infty := \sup_{s \in [0, T]} |\bar{\Gamma}(s, 0)| < \infty$.

Remark 1.4.26. One typical example of non-anticipative functional which satisfies (3) is given by

$$\bar{\Gamma}(s, \eta) = \Phi \left(s, \eta(s), \int_0^s \eta(r)dr \right),$$

and $\Phi : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz real function in the second and third variables.

Proposition 1.4.27. Suppose that Assumption 1.4.25 is satisfied and fix $y_0 \in \mathbb{R}$. Then, pathwise uniqueness holds for the SDE with dynamics

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s)dW_s + \int_0^t \bar{\Gamma}(s, Y^s)ds, \quad (4.26)$$

i.e. $E(\sigma_0, 0, \bar{\Gamma})$.

The proof of the Proposition 1.4.27 generalizes the techniques of Yamada-Watanabe pathwise uniqueness theorem for Markovian SDEs (see e.g. Theorem 5.2.19 in [61]). Before proceeding with that proof, we state a lemma which is an easy consequence of Problem 5.3.15 in [61].

Lemma 1.4.28. Suppose the assumptions in Proposition 1.4.27 are in force. Let Y be a solution of (4.26) and let $m \geq 2$ be an integer. Then, there exists a constant $C > 0$, depending on the linear growth constant of σ_0 , Y_0, T, m , and the quantities $(K, \bar{\Gamma}_\infty)$ given in Assumptions 1.4.25 (3)-(4), such that

$$\mathbb{E} \left(\sup_{t \leq T} |Y_t|^m \right) \leq C.$$

Proof (of Proposition 1.4.27).

Let Y^1, Y^2 be two solutions on the same probability space with respect to the same Brownian motion W of (4.26) such that $Y_0^1 = Y_0^2 = y_0$. In the sequel, we set $\Delta_t = Y_t^1 - Y_t^2, t \in [0, T]$. By Lemma 1.4.28, we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} |Y_t^i|^2 \right) < \infty, \quad (4.27)$$

for $i = 1, 2$. By the assumption on σ_0 , this obviously gives

$$\mathbb{E} \left(\int_0^T |\sigma_0(Y_t^i)|^2 dt \right) < \infty, \quad (4.28)$$

for $i = 1, 2$. We observe

$$\Delta_t = \int_0^t (\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})) ds + \int_0^t (\sigma_0(Y_s^1) - \sigma_0(Y_s^2)) dW_s, t \in [0, T]. \quad (4.29)$$

We recall from the proof of Proposition 2.13 in [61, Chapter 5], the existence of the functions

$$\Psi_n(x) = \int_0^{|x|} \int_0^y \rho_n(u) du dy,$$

such that for every $x \in \mathbb{R}$

$$0 \leq \rho_n(x) \leq \frac{2}{nl^2(x)}, \quad |\Psi'_n(x)| \leq 1, \quad |\Psi_n(x)| \leq |x|, \quad \lim_{n \rightarrow \infty} \Psi_n(x) = |x|. \quad (4.30)$$

By applying Itô's formula and using (4.29), we get

$$\begin{aligned} \Psi_n(\Delta_t) &= \int_0^t \Psi'_n(\Delta_s) [\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})] ds + \frac{1}{2} \int_0^t \Psi''_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)]^2 ds \\ &+ \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s. \end{aligned}$$

By using Assumption 1.4.25 and (4.30), we get

$$\Psi_n(\Delta_t) \leq \int_0^t K \left(|Y_s^1 - Y_s^2| + \int_0^s |Y_r^1 - Y_r^2| dr \right) ds + \frac{t}{n} + M_t, \quad (4.31)$$

where $M_t = \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s$ is a local martingale. Since Ψ'_n is bounded, the estimate (4.28), ensures that M is a (even square integrable) martingale.

We take the expectation, applying Fubini's theorem in (4.31), to obtain

$$\mathbb{E} \Psi_n(\Delta_t) \leq K \int_0^t \mathbb{E} |Y_s^1 - Y_s^2| ds + KT \int_0^t \mathbb{E} |Y_s^1 - Y_s^2| ds + \frac{t}{n}, \quad (4.32)$$

since $\mathbb{E} M_t = 0$. Passing to the limit when $n \rightarrow \infty$, by Lebesgue's dominated convergence theorem, we get

$$\mathbb{E} |\Delta_t| \leq (K + TK) \int_0^t \mathbb{E} |\Delta_s| ds. \quad (4.33)$$

By the Gronwall's inequality, we obtain $\mathbb{E} |\Delta_t| = 0$. By the continuity of the sample paths of Y^1, Y^2 we conclude that Y^1, Y^2 are indistinguishable. \square

We come back to the framework of the beginning of Section 1.4.1. We suppose again the validity of Assumption 1.4.4. We recall the definition of the harmonic function h defined by $h(0) = 0, h'(x) = e^{-\Sigma}$, see (4.4). We recall the notation $\sigma_0 = (\sigma h') \circ h^{-1}$. We define

$$\bar{\Gamma}(s, \eta) := h'(h^{-1}(\eta(s))) \Gamma(s, h^{-1}(\eta^s)), \quad s \in [0, T], \eta \in C([0, T]). \quad (4.34)$$

Corollary 1.4.29. *Suppose that Assumptions 1.4.4 and 1.4.25 (related to $\bar{\Gamma}$ introduced in (4.34)) are fulfilled. Then, uniqueness holds for the martingale problem related to (3.3) with respect to \mathcal{D}_L .*

Proof. It follows by using Assumptions 1.4.4 and 1.4.25 on $\bar{\Gamma}$ and applying Proposition 1.4.27 and Proposition 1.4.12. \square

Theorem 1.4.30. *Suppose that Assumptions 1.4.4 and 1.4.25 (related to $\bar{\Gamma}$ introduced in (4.34)) and one of two hypotheses below are in force.*

1. $\tilde{\Gamma}$ defined in (4.14) is bounded.
2. σ_0 is bounded and uniformly elliptic.

Then, strong existence and pathwise uniqueness hold for the SDE with dynamics (4.10).

Proof.

By Proposition 1.4.27, pathwise uniqueness holds. Indeed, taking into account the expression of (4.34), the equation (4.10) is a particular case of (4.26).

In order to prove existence, we will apply Theorem 1.4.23. For this purpose, we need to verify that either Hypothesis (1) or (2) of that theorem hold. Hypothesis (1) in the above statement coincides with Hypothesis (1) in Theorem 1.4.23. Suppose the validity of (2) in the above statement and we check that Assumption 1.4.16 holds true. By (4.14) and the definition of $\bar{\Gamma}$ in (4.34), we obtain

$$\sigma_0(\eta(s))\tilde{\Gamma}(s, (h^{-1} \circ \eta)) = \bar{\Gamma}(s, \eta). \quad (4.35)$$

The fact that $\frac{1}{\sigma_0}$ is bounded jointly with (3) in Assumption 1.4.25 yield the existence of a constant $K_1 > 0$ such that

$$|\tilde{\Gamma}(s, (h^{-1} \circ \eta))| \leq K_1 \left(|\eta(s)| + \int_0^s |\eta(r)| dr \right) + |\bar{\Gamma}(s, 0)|.$$

By (4) in Assumption 1.4.25 and the simple estimate

$$\int_0^s |\gamma(r)| dr \leq s \sup_{r \in [0, s]} |\gamma(r)|,$$

allow us to conclude Assumption 1.4.16 holds.

By Theorem 1.4.23, existence holds for the martingale problem related to (3.3) with respect to \mathcal{D}_L and by Proposition 1.4.12 (1), we have that (4.10) has a solution. At this point, we can apply Yamada-Watanabe theorem to guarantee that the solution is actually strong. We remark that the Yamada-Watanabe theorem (in the path-dependent case) proof is the same as the one in the Markovian case, which is for instance stated in Proposition 3.20 [61, Chapter 5]. \square

As a consequence of Proposition 1.4.12 and Theorem 1.4.30, we obtain the following result.

Corollary 1.4.31. *Under the same assumptions as in Theorem 1.4.30, strong existence and pathwise uniqueness hold for the martingale problem related to (3.3), with respect to \mathcal{D}_L .*

1.5 Appendix: different notions of solutions when b' is a function

Let us suppose below that $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$ are locally bounded Borel functions and Γ as given in (1.2). As already mentioned, for simplicity we will only consider deterministic initial conditions x_0 .

Definition 1.5.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \geq 0}$ a Brownian motion and $x_0 \in \mathbb{R}$. A solution X of $E(\sigma, b', \Gamma; x_0)$ (depending on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Brownian motion W) is an \mathfrak{F}^W -progressively measurable process fulfilling (3.1), with $\xi = x_0$ a.s.

Definition 1.5.2. (Strong existence). We say that **strong existence** holds for equation $E(\sigma, b', \Gamma)$ if for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a Brownian motion $(W_t)_{t \geq 0}$ and $x_0 \in \mathbb{R}$, there exists a process $(X_t)_{t \geq 0}$ which is a solution to $E(\sigma, b', \Gamma; x_0)$.

Definition 1.5.3. (Pathwise uniqueness). We say that **pathwise uniqueness** holds for equation $E(\sigma, b', \Gamma)$ if the following property is fulfilled for every $x_0 \in \mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a Brownian motion $(W_t)_{t \geq 0}$. If two processes X, \tilde{X} are two solutions to $E(\sigma, b', \Gamma; x_0)$, then X and \tilde{X} are indistinguishable.

Definition 1.5.4. (Existence in law). Let $x_0 \in \mathbb{R}$. We say that **existence in law** holds for $E(\sigma, b', \Gamma; x_0)$ if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a Brownian motion $(W_t)_{t \geq 0}$ and a process $(X_t)_{t \geq 0}$ such that (X, \mathbb{P}) is a solution of $E(\sigma, b', \Gamma; x_0)$, see Definition 1.5.1.

We say that **existence in law** holds for $E(\sigma, b', \Gamma)$ if existence in law holds for $E(\sigma, b', \Gamma; x_0)$, for every $x_0 \in \mathbb{R}$.

Definition 1.5.5. (Uniqueness in law). Let $x_0 \in \mathbb{R}$. We say that **uniqueness in law** holds for $E(\sigma, b', \Gamma; x_0)$ if we have the following. Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (respectively $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$) carrying a Brownian motion $(W_t)_{t \geq 0}$ (respectively $(\tilde{W}_t)_{t \geq 0}$). We suppose that a process $(X_t)_{t \geq 0}$ (resp. a process $(\tilde{X}_t)_{t \geq 0}$) is a solution of $E(\sigma, b', \Gamma; x_0)$, such that both $X_0 = x_0, \mathbb{P}$ a.s. and $\tilde{X}_0 = x_0, \tilde{\mathbb{P}}$ a.s. Uniqueness in law means that X and \tilde{X} must have the same law as random elements taking values in $C([0, T])$ or $C(\mathbb{R}_+)$.

We say that **uniqueness in law** holds for $E(\sigma, b', \Gamma)$ if uniqueness in law holds for $E(\sigma, b', \Gamma; x_0)$ for every $x_0 \in \mathbb{R}$.

Chapter 2

ON SDEs FOR BESSEL PROCESSES IN LOW DIMENSION AND PATH-DEPENDENT EXTENSIONS

This chapter is object of the paper [75].

2.1 Introduction

The class of Bessel processes is one of the most important classes of diffusion processes with values in \mathbb{R}_+ . It is a family of strong Markov processes parameterized by $\delta \in \mathbb{R}_+$ (called the *dimension*), which has deep connections with the radial behavior of the Brownian motion, square-root diffusions, conformally invariant processes, etc. Bessel processes have been largely investigated in the literature, we refer the reader to e.g [70, 95, 79] (Section 2.3, Chapter 3 and Chapter XI, respectively) for an overview on Bessel processes.

Let $x_0 \geq 0$. We recall that a *Bessel process* X (with initial condition x_0 , dimension $\delta \geq 0$ and denoted $\text{BES}^\delta(x_0)$) is defined as the square root of the so-called *squared Bessel process* (with initial condition $s_0 = x_0^2$, dimension $\delta \geq 0$ and denoted $\text{BESQ}^\delta(x_0^2)$), which is characterized as the pathwise unique solution of the SDE

$$dS_t = 2\sqrt{|S_t|}dW_t + \delta t, S_0 = x_0^2.$$

When $\delta > 1$ it is possible to characterize X as (pathwise unique non-negative) solution of

$$dX_t = \frac{\delta - 1}{2}X_t^{-1}dt + dW_t, \quad (1.1)$$

in particular X is an Itô process. For $0 \leq \delta \leq 1$, the integral $\int_0^t X_s^{-1}ds$ does not converge and $\text{BES}^\delta(x_0)$ is a non-semimartingale process, except for $\delta = 1$ and $\delta = 0$, see [79] and [60] Chapter XI Section 1 and Section 6.1, respectively. If $0 < \delta < 1$, it can be represented as

$$X_t = x_0 + \frac{\delta - 1}{2} \text{p.v.} \int_0^t \frac{ds}{X_s} + W_t, t \geq 0, \quad (1.2)$$

where p.v. stands for principal value (defined in a suitable way via local time). The drift in decomposition (1.2) is a zero energy additive functional in the language of Markov processes and $BES^\delta(x_0)$ is a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process. For further details, we refer the reader to the works [95, 35, 70] and other references therein.

In this work, we characterize $BES^\delta(x_0)$, for $0 \leq \delta \leq 1$, as the unique solution of an SDE with distributional drift, interpreted as a suitable martingale problem. A non-Markovian extension will also be considered for SDEs with singular drifts of the form

$$\frac{\delta - 1}{2} \text{p.v.} \frac{1}{X_t} + \Gamma(t, X^t),$$

where Γ is a non-anticipative functional satisfying some technical conditions. Our point of view consists however in adopting the definition of p.v. $\frac{1}{x}$ analytically, i.e. as the derivative in the sense of Schwartz distributions of the function $x \mapsto \log|x|$.

Our analysis is inspired by the series of works [40, 41, 82] which treat Markovian SDEs of the form

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.3}$$

where σ and b are continuous functions on \mathbb{R} . Moreover σ is strictly positive and the existence of the function

$$\Sigma(x) := 2 \int_0^x \frac{b'}{\sigma^2}(y)dy, \quad x \in \mathbb{R}, \tag{1.4}$$

considered as a suitable limit via regularization, is supposed. We stress that b' is the derivative of some function b in the sense of distributions. Thereby, the authors define a Markov generator L of the form

$$Lf = \frac{\sigma^2}{2} f'' + b' f'. \tag{1.5}$$

Taking into account (1.4), the operator L can be written as

$$Lf = (e^\Sigma f')' \frac{e^{-\Sigma} \sigma^2}{2}. \tag{1.6}$$

In Chapter 1, we have studied the class of SDEs

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.7}$$

for some classes of functionals Γ .

In this paper, we will investigate existence and uniqueness of an SDE of the type (1.1), where $\sigma = 1$, but b is no more a continuous function. More precisely, we focus on the SDE

$$dX_t = dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \tag{1.8}$$

where b is given by

$$b(x) = \begin{cases} \frac{\delta-1}{2} \log|x|, & x \in \mathbb{R}^*, & | & \delta \neq 1 \\ H(x), & x \in \mathbb{R}, & | & \delta = 1, \end{cases} \tag{1.9}$$

and H is the Heaviside function and $\mathbb{R}^* = \mathbb{R} - \{0\}$. Then, (1.2) is considered as a particular case of the SDE (1.8) with distributional drift b' and $\Gamma = 0$. Even though b is no longer a continuous function, (1.4) can still be defined in such a way that $\Sigma \equiv 2b$ and (1.6) holds. We distinguish the two cases: $0 \leq \delta < 1$ and $\delta = 1$.

- $0 \leq \delta < 1$. If b is given by (1.9), then (1.4) implies

$$\exp(-\Sigma(x)) = |x|^{1-\delta}. \quad (1.10)$$

At this point, representation (1.6) for $L^\delta = L$ yields

$$L^\delta f(x) = \frac{f''(x)}{2} + \frac{(\delta - 1)f'(x)}{2x}, \quad x \neq 0. \quad (1.11)$$

- $\delta = 1$. In this case, $b(x) = H(x)$. So (1.6) yields

$$L^1 f(x) = \frac{f''(x)}{2} + \delta_0 f'(x), \quad x \neq 0, \quad (1.12)$$

where δ_0 is Dirac measure at zero.

Existence and uniqueness of a non-negative strong solution to the (1.1) with $\delta \in]1, 2]$ was proved in [18]. After we finished our work we have found existence and uniqueness of a non-negative solution to the strong martingale problem for with $\delta \in]0, 2]$, see [3]. The domain of the martingale problem there was smaller than ours. However there is additional assumption in [3] that the process spends zero time at 0.

We then study the (possibly non-Markovian) martingale problem associated with the operator

$$\mathcal{L}^\delta f = L^\delta + \Gamma f',$$

in a suitable domain. The notion of martingale problem related to \mathcal{L}^δ is given by Definition 2.2.2. The notion of strong martingale problem related to the domain of L^δ and an underlying Brownian motion W is given by Definition 2.2.3, which borrows the one in [82]. It has to be compared with the notion of strong existence and pathwise uniqueness of an SDE. In particular, it represents the corresponding notion to strong solution of SDEs in the framework of martingale problems.

As mentioned earlier, we divide our analysis into two parts: the Markovian case and a non-Markovian extension. In the Markovian case ($\Gamma = 0$), a series of results of existence and uniqueness for the strong martingale problem are provided in Sections 2.3.2, 2.3.3, 2.3.4. For instance Proposition 2.3.2 characterizes the natural domain of L^δ . Concerning $\delta < 1$, Proposition 2.3.4 shows that the Bessel process solves the strong martingale problem, Proposition 2.3.13 is devoted to pathwise uniqueness. Section 2.3.6 states analogous results in the case $\delta = 1$. In the second part of this article, we devote our attention to the SDE (1.8), when $\Gamma \neq 0$, i.e. the non-Markovian case, for a large class of non-anticipative functionals Γ . (1.8) allows to define a class of path-dependent Bessel process, which does not fall into the framework of Chapter 1, because b defined in (1.9) is not a continuous function

on \mathbb{R} . In Section 2.4, we establish existence and uniqueness of the martingale problem associated with the SDE (1.8) under the condition that Γ is bounded; see Propositions 2.4.2 and 2.4.11. Proposition 2.4.8 and Theorem 2.4.16 illustrate sufficient conditions on Γ to have well-posedness of the strong martingale problem.

Motivations for studying path-dependent Bessel processes are provided below. Similarly to the classical Markovian case with integer dimension $\delta \geq 2$, the path-dependent SDE (1.8) should be interpreted as the radial dynamics of a δ -dimensional Brownian motion β with drift having a radial intensity proportional to a non-anticipative functional Γ . In other words, if Y is a weak solution to

$$dY_t = d\beta_t + \Gamma(t, \|Y_t\|_{\mathbb{R}^\delta}) \frac{Y_t}{\|Y_t\|_{\mathbb{R}^\delta}} dt, \tag{1.13}$$

then $X_t = \|Y_t\|_{\mathbb{R}^\delta}$ solves (1.8) formally for an integer $\delta \geq 2$. Indeed, if Y is a solution of (1.13), then a formal application of Itô's formula to $\rho_t := \|Y_t\|_{\mathbb{R}^\delta}^2$ and Lévy's characterization theorem for local martingales show that

$$d\rho_t = 2\sqrt{\rho_t}dW_t + 2\sqrt{\rho_t}\Gamma(t, \sqrt{\rho_t})dt + \delta dt. \tag{1.14}$$

A subsequent formal application of Itô's formula shows that $X_t = \sqrt{\rho_t}$ solves (1.8). On the other hand, Bessel-type processes with dimension $\delta \in \mathbb{R}$ play an important role in the theory of Schramm-Loewner evolution, see e.g. [64]. In particular, the two-parameter family of Schramm-Loewner evolution $SLE(\kappa, \kappa - 4)$ defined in [63] provides a source of examples of BES^δ flows with very singular behavior when $\delta = 1 - \frac{4}{\kappa}, \kappa > 4$, which are covered by the SDE (1.1). In fact, the final right-boundary of $SLE(\kappa, \kappa - 4)$ processes is described by the excursions of δ -dimensional Bessel processes. We refer the reader to [33] for more details. In this case, equation (1.8) describes a non-Markovian version of those phenomena.

The paper is organized as follows. After this Introduction we recall the notations and some important results from Chapter 1. Then we introduce specific preliminary considerations. Section 2.3 is devoted to the case of Bessel processes in low dimension, under the perspective of strong martingale problems. Section 2.4 discusses the case of non-Markovian perturbations of Bessel processes.

2.2 About path-dependent martingale problems

2.2.1 Preliminary notations, definitions and results

In this section we recall the general notation and some necessary results from Chapter 1.

Let I be an interval of \mathbb{R} . For $k \in \mathbb{N}$, $C^k(I)$ will denote the space of real functions defined on I having continuous derivatives till order k . Such space is endowed with the uniform convergence topology on compact sets for the functions and all derivatives. Generally $I = \mathbb{R}, \mathbb{R}_+ := [0, +\infty[, \mathbb{R}_- :=]-\infty, 0], [0, T]$, for some fixed positive real T . If there is no ambiguity $C^k(\mathbb{R})$ will be simply indicated by C^k . The space of continuous functions on I will be denoted by $C(I)$. Given an a.e. bounded real function f , $|f|_\infty$ will denote the essential supremum.

We recall some notions from [40]. For us all filtrations \mathfrak{F} fulfill the usual conditions. When no filtration is specified, we mean the canonical filtration of an underlying process. Otherwise, the canonical filtration associated with a process X is denoted by \mathfrak{F}^X .

A sequence (X^n) of continuous processes indexed by $[0, T]$ is said to converge u.c.p. to some process X whenever $\sup_{t \in [0, T]} |X_t^n - X_t|$ converges to zero in probability.

We consider a locally bounded functional

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (2.1)$$

where

$$\Lambda := \{(s, \eta) \in [0, T] \times C([0, T]); \eta = \eta^s\}$$

and

$$\eta^s(t) = \begin{cases} \eta(t), & \text{if } t \leq s \\ \eta(s), & \text{if } t > s. \end{cases}$$

By convention, we extend Γ from Λ to $[0, T] \times C([0, T])$ by setting (in a non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]).$$

All along the paper E will denote \mathbb{R} or \mathbb{R}_+ .

Let us consider some locally bounded Borel functions $\sigma, b' : E \rightarrow \mathbb{R}$. In this case the path-dependent SDE

$$\begin{cases} dX_t &= \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt \\ X_0 &= \xi, \end{cases} \quad (2.2)$$

for some deterministic initial condition ξ taking values in E , makes perfectly sense, see Section 5 of Chapter 1, in particular one can speak about strong existence, pathwise uniqueness, existence and uniqueness in law. (2.2) is denominated by $E(\sigma, b', \Gamma)$. Proposition 1.3.2 implies the following.

Proposition 2.2.1. *Let $b' : E \rightarrow \mathbb{R}$ be a locally bounded function. We set $Lf = \frac{\sigma^2}{2}f'' + b'f'$, $f \in C^2(E)$. A couple (X, \mathbb{P}) is a solution of $E(\sigma, b', \Gamma)$, if and only if, under \mathbb{P} ,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds \quad (2.3)$$

is a local martingale, where $Lf = \frac{1}{2}f'' + b'f'$, for every $f \in C^2(E)$.

In this paper, we will be interested in a formal $E(\sigma, b', \Gamma)$ where $\sigma = 1$ but b' is the derivative of some specific Borel discontinuous function. The formulation is inspired by Proposition 2.2.1 which states that the SDE is equivalent to a specific martingale problem. We will consider formal PDE operators of the type $L : \mathcal{D}_L(E) \subset C^1(E) \rightarrow C(E)$, where Lf gives formally $\frac{\sigma^2}{2}f'' + b'f'$. When b', σ are locally bounded functions then $\mathcal{D}_L(E) = C^2(E)$.

Definition 2.2.2. 1. We say that a continuous stochastic process X solves (with respect to a probability \mathbb{P} on some measurable space (Ω, \mathcal{F})) the martingale problem related to

$$\mathcal{L}f := Lf + \Gamma f', \quad (2.4)$$

with initial condition $\nu = \delta_{x_0}$, $x_0 \in E$, with respect to a domain $\mathcal{D}_L(E)$ if

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (2.5)$$

is a \mathbb{P} -local martingale for all $f \in \mathcal{D}_L(E)$.

We will also say that the couple (X, \mathbb{P}) is a solution of (or (X, \mathbb{P}) solves) the martingale problem with respect to $\mathcal{D}_L(E)$.

2. If a solution exists we say that the martingale problem above admits existence.
3. We say that the martingale problem above admits uniqueness if any two solutions (X^i, \mathbb{P}^i) , $i = 1, 2$ (on some measurable space (Ω, \mathcal{F})) have the same law.

In the sequel, when the measurable space (Ω, \mathcal{F}) is self-explanatory it will be often omitted.

The notion of martingale problem is (since the works of Stroock and Varadhan [93]) a concept related to solutions of SDEs in law. The case when b' and σ are continuous functions (see [93]), $\mathcal{D}_L(\mathbb{R})$ corresponds to $C^2(\mathbb{R})$.

Below we introduce the analogous notion of strong existence and pathwise uniqueness for our martingale problem, see also 1 for the case when b' is the derivative of a continuous function and [82] for the case $\Gamma = 0$. In both cases we had $E = \mathbb{R}$.

Definition 2.2.3.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathfrak{F} = (\mathcal{F}_t)$ be the canonical filtration associated with a fixed Brownian motion W . Let $x_0 \in E$. We say that a continuous \mathfrak{F} -adapted E -valued process X such that $X_0 = x_0$ is a **solution to the strong martingale problem** (related to (2.4)) with respect to $\mathcal{D}_L(E)$ and W (with related filtered probability space), if for all $f \in \mathcal{D}_L(E)$ (2.5) is a \mathfrak{F} -local martingale given by

$$\int_0^t f'(X_s)\sigma(X_s)dW_s. \quad (2.6)$$

2. We say that the martingale problem related to (2.4) with respect to $\mathcal{D}_L(E)$ admits **strong existence** if for every $x_0 \in E$, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{F})$, where $\mathfrak{F} = (\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W , there is a process X solving the strong martingale problem (related to (2.4)) with respect to $\mathcal{D}_L(E)$ and W with $X_0 = x_0$.
3. We say that the martingale problem (related to (2.4)) with respect to $\mathcal{D}_L(E)$ admits **pathwise uniqueness** if given $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion W and X^i , $i = 1, 2$ are solutions to the strong martingale problem with respect to $\mathcal{D}_L(E)$ and W with $\mathbb{P}[X_0^1 = X_0^2] = 1$ then X^1 and X^2 are indistinguishable.

The mention E will be often omitted when $E = \mathbb{R}$. For instance $C^1(E), C^2(E), \mathcal{D}_L(E)$, will be simply denoted by C^1, C^2, \mathcal{D}_L .

2.3 Martingale problem for Bessel processes

2.3.1 Preliminary considerations

In this section, we are going to introduce and investigate well-posedness for a martingale problem related to a Bessel process. We recall that the rigorous definition of the Bessel process is the following. A non-negative process X is said to be a Bessel process starting at x_0 with dimension $\delta \geq 0$ (notation $BES^\delta(x_0)$) if $S = X^2$ is a squared Bessel process starting at $s_0 = x_0^2$ of dimension δ . S is denoted by $BESQ^\delta(s_0)$.

In particular S is the pathwise unique solution of

$$S_t = s_0 + 2 \int_0^t \sqrt{|S_s|} dW_s + \delta t, t \geq 0, \quad (3.1)$$

where W is a standard Brownian motion.

As is shown in Proposition 2.13 in chapter 5 of [61] (see also [95, chapter 3]) (3.1) admits pathwise uniqueness. Since $x \mapsto \sqrt{|x|}$ has linear growth (i.e. $\sqrt{|x|} \leq c(1 + |x|)$ for some constant c), it has weak existence and so by Yamada-Watanabe theorem it also admits strong existence.

Remark 2.3.1. For $\delta > 1$, we know that the Bessel process X fulfills

$$X_t = x_0 + \frac{\delta - 1}{2} \int_0^t X_s^{-1} ds + W_t. \quad (3.2)$$

We observe that for $\delta > 2$, X is even transient and it never touches zero, see [79, Chapter XI]. As anticipated, when $\delta = 1$ or $\delta = 0$ X is still a semimartingale. Unfortunately if $0 < \delta < 1$ it is not the case, see Chapter 10 of [70], it is just a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process.

Our point of view consists in rewriting (3.2) under the form

$$X_t = x_0 + \int_0^t b'(X_s) ds + W_t, \quad (3.3)$$

where W is a Brownian motion and b' is the derivative of the function $b(x) = \frac{\delta-1}{2} \log |x|$, when $\delta \neq 1$. Of course, the derivative b' restricted on $]0, +\infty[$ is given by $\frac{\delta-1}{2x}$, which explains (3.2). Unfortunately, for small values of δ , X is recurrent, so it touches zero very often. In this case, the derivative of b has to be considered on $[0, \infty[$ or \mathbb{R} . In those sets it is only a Schwartz distribution. In the case $\delta = 1$, we will have b being a Heaviside function so that b' is the δ -Dirac measure at zero.

We are going to construct two settings: one for $0 \leq \delta < 1$ and another one for $\delta = 1$. In what follows, we should recall $\mathbb{R}^* = \mathbb{R} - \{0\}$.

2.3.2 The framework for $0 \leq \delta < 1$

According to the considerations in Section 2.3.1, the natural form of the operator $L^\delta := L$ (outside zero) is expected to be of the form

$$L^\delta f(x) = \frac{f''(x)}{2} + \frac{(\delta - 1)f'(x)}{2x}, x \neq 0, \quad (3.4)$$

for $f \in C^2(\mathbb{R}^*)$. This can also be expressed as

$$L^\delta f(x) = \frac{|x|^{1-\delta}}{2} (|x|^{\delta-1} f')', x \neq 0. \quad (3.5)$$

The problem is to provide a natural extension for $x = 0$, which constitutes the critical point.

As anticipated, we fix $b : \mathbb{R} \rightarrow \mathbb{R}$, $b(x) = \frac{\delta-1}{2} \log|x|$ and $\sigma \equiv 1$. Formally speaking, Σ as in (1.4) gives $\Sigma(x) = 2b(x)$, so

$$\exp(-\Sigma(x)) = |x|^{1-\delta}, x \in \mathbb{R}. \quad (3.6)$$

We have now to specify the domain \mathcal{D}_{L^δ} which is compatible with (3.5). For this, we naturally define it as the set of $f \in C(\mathbb{R}) \cap C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-)$ such that the following holds.

- (a) There is a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ extending $x \mapsto f'(x)|x|^{\delta-1}$, $x \neq 0$.
- (b) There is a continuous function $G : \mathbb{R} \rightarrow \mathbb{R}$, extending $x \mapsto g'(x)|x|^{1-\delta}$, $x \neq 0$, (i.e. $2L^\delta f(x)$, according to (3.5)) to \mathbb{R} .

We define then

$$L^\delta f := \frac{G}{2}. \quad (3.7)$$

Proposition 2.3.2. 1. Suppose $\delta > 0$. Then $\mathcal{D}_{L^\delta} = \mathcal{D}_\delta := \{f \in C^2(\mathbb{R}) | f'(0) = 0\}$ and

$$L^\delta f(x) = \begin{cases} \frac{f''(x)}{2} + \frac{(\delta-1)f'(x)}{2x} & : x \neq 0 \\ \delta f''(0) & : x = 0. \end{cases} \quad (3.8)$$

2. Suppose $\delta = 0$. Then $\mathcal{D}_{L^0} = \mathcal{D}_0$, where

$$\mathcal{D}_0 := \{f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-) | f'(0) = 0\} \quad (3.9)$$

and

$$L^0 f(x) = \begin{cases} \frac{f''(x)}{2} - \frac{f'(x)}{2x} & : x \neq 0 \\ 0 & : x = 0. \end{cases} \quad (3.10)$$

Proof. We first show the inclusion $\mathcal{D}_{L^\delta} \subset \mathcal{D}_\delta$. Suppose $f \in \mathcal{D}_{L^\delta}$. We have

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} |x|^{1-\delta} g(x) = 0. \quad (3.11)$$

This obviously implies that $f \in C^1(\mathbb{R})$ and $f'(0) = 0$. Taking into account (3.4), we have

$$\begin{aligned} f''(0+) &:= \lim_{x \rightarrow 0+} f''(x) = \lim_{x \rightarrow 0+} \left(G(x) - (\delta - 1) \frac{f'(x)}{x} \right) \\ &= G(0) - (\delta - 1) \lim_{x \rightarrow 0+} \frac{f'(x)}{x} = G(0) - (\delta - 1) f''(0+), \\ f''(0-) &:= \lim_{x \rightarrow 0-} f''(x) = \lim_{x \rightarrow 0-} \left(G(x) - (\delta - 1) \frac{f'(x)}{x} \right) \\ &= G(0) - (\delta - 1) \lim_{x \rightarrow 0-} \frac{f'(x)}{x} = G(0) - (\delta - 1) f''(0-), \end{aligned}$$

by L'Hospital rule. This implies that

$$\delta f''(0+) = G(0) = \delta f''(0-). \quad (3.12)$$

To show that $f \in \mathcal{D}_\delta$, it remains to show that $f''(0+) = f''(0-)$ when $\delta \neq 0$ since when $\delta = 0$ there is nothing more to prove. This obviously follows from (3.12). This shows the inclusion $\mathcal{D}_{L^\delta} \subset \mathcal{D}_\delta$. Now, (3.12), (3.4) and (3.7) show in particular (3.8) and (3.10).

We prove now the opposite inclusion $\mathcal{D}_\delta \subset \mathcal{D}_{L^\delta}$. Let $f \in \mathcal{D}_\delta$, in particular such that $f'(0) = 0$. We need to prove that it fulfills the properties (a) and (b) characterizing \mathcal{D}_{L^δ} . We set $g(x) := f'(x)|x|^{\delta-1}$, $x \neq 0$ and $g(0) := 0$. By l'Hospital rule we can show that $\lim_{x \rightarrow 0} g(x) = 0$, so that g is continuous at zero. This proves property (a) characterizing \mathcal{D}_{L^δ} . Taking the derivative of g on \mathbb{R}^* we get

$$g'(x) = f''(x)|x|^{\delta-1} + (\delta - 1)f'(x)\text{sign}(x)|x|^{\delta-2}. \quad (3.13)$$

Concerning property (b), as $x \mapsto G(x) := g'(x)|x|^{1-\delta}$ is continuous on \mathbb{R}^* it is enough to show that $\lim_{x \rightarrow 0} G(x)$ exists. By (3.13) we obtain

$$G(x) = f''(x) + (\delta - 1)\text{sign}(x) \frac{1}{|x|} f'(x) = f''(x) + (\delta - 1) \frac{f'(x)}{x}, \quad x \neq 0.$$

We recall that $f''(0+)$ and $f''(0-)$ exist. Taking the limit when x goes to zero from the right and from the left, by L'Hospital rule, we get

$$\begin{aligned} G(0+) &= f''(0+) + (\delta - 1)f''(0+) = \delta f''(0+), \\ G(0-) &= f''(0-) + (\delta - 1)f''(0-) = \delta f''(0-), \end{aligned}$$

Distinguishing the cases $\delta > 0$ (in this case $f''(0+) = f''(0-)$) and $\delta = 0$, show that $G(0+) = G(0-)$ and finally G extends continuously to 0. This concludes the proof of the two properties (a) and (b) and so the inclusion $\mathcal{D}_\delta \subset \mathcal{D}_{L^\delta}$. \square

In the sequel we will denote by $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ the set of functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ which are restrictions of functions \hat{f} belonging to \mathcal{D}_{L^δ} . Sometimes, we will also denote $\mathcal{D}_{L^\delta}(\mathbb{R}) := \mathcal{D}_{L^\delta}$. We will also denote $L^\delta f$ as the restriction to \mathbb{R}_+ of $L^\delta \hat{f}$. (3.8) shows that this notation is coherent. This convention will be made also for $\delta = 1$ in Section 2.3.6.

Starting from Section 2.3.3, we will make use of convergence properties for functions and processes according to the remark below.

Remark 2.3.3.

1. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous (therefore uniformly continuous on compacts) then $f_n(x) = f(x + \frac{1}{n})$ converges to f uniformly on compacts.
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. If $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is a sequence of functions that converges uniformly on compacts of \mathbb{R} to a function f then $f_n(X)$ converges to $f(X)$ u.c.p.

2.3.3 The martingale problem in the full line case when $0 \leq \delta < 1$.

Proposition 2.3.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W . Let $x_0 \geq 0, 0 \leq \delta < 1$. Let S be the solution of (3.1) (necessarily non-negative by comparison theorem) with $s_0 = x_0^2$, so that $X = \sqrt{S}$ is a $BES^\delta(x_0)$ process.

Then X solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and W . Moreover, for every $f \in \mathcal{D}_{L^\delta}$

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s. \tag{3.14}$$

Remark 2.3.5. 1. Suppose that S is a non-negative solution of an SDE of the type (3.1), where the Brownian motion W is replaced by a continuous semimartingale whose martingale component is a Brownian motion. Then (3.14) still holds for every $f \in \mathcal{D}_{L^\delta}$.

2. For $\delta = 0$ and $x_0 = 0$, $BESQ^0(0)$ is the null process. By Proposition 2.3.2 $L^0 f(0) = 0$ for all $f \in \mathcal{D}_{L^0}$, obviously $f(0) - f(0) - \int_0^t L^\delta f(0) ds \equiv 0$ and (3.14) holds.

Proof (of Proposition 2.3.4).

We consider immediately the case of Remark 2.3.5 (1) and suppose W to be a semimartingale such that $[W]_t \equiv t$. Let $X = \sqrt{S}$, where S is a $BESQ^\delta(s_0)$, let $f \in \mathcal{D}_{L^\delta}$ and define $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ as $f_n(y) = f(\sqrt{y + \frac{1}{n}})$. Clearly $f_n \in C^2(\mathbb{R}_+)$. Applying Itô's formula we have

$$\begin{aligned} f_n(S_t) = f_n(S_0) &+ \int_0^t \frac{f'(\sqrt{S_s + \frac{1}{n}})}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \delta \frac{f'(\sqrt{S_s + \frac{1}{n}})}{2\sqrt{S_s + \frac{1}{n}}} ds \\ &+ \int_0^t \left[\frac{1}{2} f''\left(\sqrt{S_s + \frac{1}{n}}\right) - \frac{1}{2} \frac{f'(\sqrt{S_s + \frac{1}{n}})}{\sqrt{S_s + \frac{1}{n}}} \right] \left[\frac{S_s}{S_s + \frac{1}{n}} \right] ds, \end{aligned} \tag{3.15}$$

which can be rewritten as

$$\begin{aligned} f_n(S_t) &= f_n(S_0) + \int_0^t \frac{f' \left(\sqrt{S_s + \frac{1}{n}} \right)}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \frac{1}{2} f'' \left(\sqrt{S_s + \frac{1}{n}} \right) \left[\frac{S_s}{S_s + \frac{1}{n}} \right] ds \\ &\quad + \frac{1}{2} \int_0^t \frac{f' \left(\sqrt{S_s + \frac{1}{n}} \right)}{\sqrt{S_s + \frac{1}{n}}} \left[\delta - \frac{S_s}{S_s + \frac{1}{n}} \right] ds. \end{aligned} \quad (3.16)$$

The first integral converges to

$$\int_0^t f' \left(\sqrt{S_s} \right) dW_s, \quad (3.17)$$

u.c.p. by Remark 2.3.3, since $f' \in C(\mathbb{R}_+)$.

As $y \mapsto f'' \left(\sqrt{y + \frac{1}{n}} \right)$ is continuous, by Remark 2.3.3 and Lebesgue's dominated convergence theorem the second integral in (3.16) converges u.c.p. to

$$\frac{1}{2} \int_0^t f'' \left(\sqrt{S_s} \right) ds. \quad (3.18)$$

We set $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}$, the continuous function defined by

$$\ell(x) = \begin{cases} \frac{f'(x)}{x} & : x \neq 0. \\ f''(0+) & : x = 0. \end{cases}$$

The third integral can be rewritten as

$$\frac{1}{2} \int_0^t \ell \left(\sqrt{S_s + \frac{1}{n}} \right) \left[\delta - \frac{S_s}{S_s + \frac{1}{n}} \right] ds.$$

By Remark 2.3.3 and Lebesgue's dominated convergence the previous expression converges u.c.p. to

$$\int_0^t \ell \left(\sqrt{S_s} \right) \left(\frac{\delta - 1}{2} \right) ds. \quad (3.19)$$

Finally (3.17), (3.18) and (3.19) allow to conclude the proof of (3.14). \square

Corollary 2.3.6. *Let $x_0 \in \mathbb{R}, 0 \leq \delta < 1$. The martingale problem with respect to \mathcal{D}_{L^δ} , with initial condition $X_0 = x_0$ admits strong existence. More precisely we have the following. If $x \geq 0$, we denote by X^x the $BES^\delta(x)$ process, being the square root of a solution of (3.1) with $s_0 = x^2$.*

1. If $x_0 \geq 0$, X^{x_0} solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and W .
2. If $x_0 \leq 0$, $-X^{-x_0}$ solves the same strong martingale problem with respect to \mathcal{D}_{L^δ} and $-W$.

Proof.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W . We set $s_0 = x_0^2$. We know that (3.1) admits a strong solution S . Then, by Proposition 2.3.4 $X = \sqrt{S}$ is a solution for the strong martingale problem with respect to \mathcal{D}_{L^δ} and W with initial condition $|x_0|$.

So, if $x_0 \geq 0$ then strong existence is established. If $x_0 < 0$ then we show below that $-X$ also solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and $-W$.

Let $f \in \mathcal{D}_{L^\delta}$. Then obviously $f_-(x) := f(-x) \in \mathcal{D}_{L^\delta}$ and

$$L^\delta f_-(x) = L^\delta f(-x).$$

Therefore, since X solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and W , for all $f \in \mathcal{D}_{L^\delta}$ we have

$$f_-(X_t) - f_-(x_0) - \int_0^t L f_-(X_s) ds = \int_0^t f'_-(X_s) dW_s,$$

which implies

$$f(-X_t) - f(-x_0) - \int_0^t L^\delta f(-X_s) ds = \int_0^t f'(-X_s) d(-W)_s.$$

Thus $-X$ also solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and $-W$. □

Proposition 2.3.7. *Let us suppose $0 < \delta < 1$. The martingale problem with respect to \mathcal{D}_{L^δ} does not admit (in general) uniqueness in law.*

Proof.

Let S be the $BESQ^\delta(0)$. By Corollary 2.3.6, we know that $X^+ = \sqrt{S}$ and $X^- = -\sqrt{S}$ solve the martingale problem with respect to an underlying probability \mathbb{P} .

Obviously X does not have the same law as $-X$ since X is positive and $-X$ is negative. □

Remark 2.3.8. *If the initial condition x_0 is different from zero, for instance positive, then uniqueness also fails since we can exhibit two solutions. The first one is still the classical Bessel process, the second one behaving as the first one until it reaches zero and then it behaves like minus a Bessel. Such a stopping time always exists since the Bessel process hits zero, see the considerations after Corollary (1.4) in Chapter XI in [79].*

For proving indeed results for uniqueness, we will need the following.

Proposition 2.3.9. *Let $0 \leq \delta < 1$. Let (X, \mathbb{P}) be a solution (not necessarily positive) of the martingale problem with respect to \mathcal{D}_{L^δ} . Then $S = X^2$ is a squared Bessel process.*

Proof.

We first show that

$$M_t^1 := X_t^2 - x_0^2 - \delta t \tag{3.20}$$

is a local martingale and

$$X_t^4 = x_0^4 + 2(2 + \delta) \int_0^t X_s^2 ds + M_t^2, \quad (3.21)$$

where M^2 is a local martingale. Clearly, $f_1(x) := x^2 \in \mathcal{D}_{L^\delta}$ because $f \in C^2(\mathbb{R})$ and $f'_1(0) = 0$. By Proposition 2.3.2 $L^\delta f_1(x) \equiv \delta$, which shows (3.20). On the other hand, obviously $f_2(x) := x^4 \in \mathcal{D}_{L^\delta}$ and then, by Proposition 2.3.2, $L^\delta f_2(x) = 2(2 + \delta)x^2$, so (3.21) follows. Now, setting $S := X^2$, by integration by parts and using (3.20) we have

$$[M^1]_t = [S]_t = S_t^2 - s_0^2 - 2 \int_0^t S_s dS_s = X_t^4 - x_0^4 - 2\delta \int_0^t X_s^2 ds + M_t, \quad (3.22)$$

where M is a local martingale. This implies

$$X_t^4 = x_0^4 + 2\delta \int_0^t X_s^2 ds + [M^1]_t - M_t. \quad (3.23)$$

We remark that (3.23) and (3.21) provide two decompositions of the semimartingale X^4 . By uniqueness of the semimartingale decomposition we can identify the bounded variation component, which implies

$$[M^1]_t = 4 \int_0^t X_s^2 ds, t \in [0, T]. \quad (3.24)$$

Consequently the process

$$W_t := \int_0^t \frac{dM_s^1}{2|X_s|},$$

is a Brownian motion taking into account the fact that $[W]_t \equiv t$ together with Lévy's characterization of Brownian motion. Hence, the process S is a (weak) solution of the SDE

$$dS_s = \delta ds + 2\sqrt{|S_s|} dW_s, \quad (3.25)$$

which shows that S is a $BESQ^\delta(s_0)$, $s_0 = x_0^2$. \square

With very similar arguments to those in the proof of the previous proposition, we can prove the following.

Proposition 2.3.10. *Let $0 \leq \delta < 1$. Let X be a solution (not necessarily positive) to the strong martingale problem with respect to \mathcal{D}_{L^δ} and a Brownian motion W . Then $S = X^2$ is a solution to (3.1).*

Proof.

Let us suppose that X is a solution of the strong martingale problem with respect to \mathcal{D}_{L^δ} and a Brownian motion W with related canonical filtration \mathfrak{F} . The same arguments as in the first part of the proof of Proposition 2.3.9 until (3.24), allow us to establish that M^1 defined in (3.20) is an \mathfrak{F} -local martingale with quadratic variation given by (3.24). By the Brownian martingale representation theorem, we can prove that

$$M_t^1 = 2 \int_0^t |X_s| dW_s, t \in [0, T].$$

This shows that $S = X^2$ is a solution of

$$S_t = s_0 + 2 \int_0^t \sqrt{|S_s|} dW_s + \delta t, t \in [0, T], s_0 = x_0^2.$$

□

Proposition 2.3.7 shows that no uniqueness on the real line holds when $\delta > 0$. Surprisingly, if $\delta = 0$ then uniqueness holds.

Remark 2.3.11. *Suppose $\delta = 0$. Then pathwise uniqueness holds for the strong martingale problem.*

1. *Assume $x_0 = 0$. By Proposition 2.3.9 if X is a solution of the strong martingale problem, then X^2 is a $BESQ^0(0)$ which is the null process; this fact shows uniqueness.*
2. *Suppose x_0 different from zero (for instance strictly positive). If X is a solution to the strong martingale problem, then, by Proposition 2.3.9 $S := X^2$ is a $BESQ^0(x_0^2)$.*

More precisely, by Proposition 2.3.10, S is a solution to (3.1). By the fact that (3.1) admits pathwise uniqueness, the strong Markov property shows that, whenever S reaches zero it is forced to remain there.

2.3.4 The martingale problem in the \mathbb{R}_+ -case

We remain still with the case $0 \leq \delta < 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a Brownian motion W with canonical filtration \mathfrak{F} . We will be interested in non-negative solutions X for the martingale problem in the strong sense, with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W , which means that

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s, \tag{3.26}$$

for all $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$. Proposition 2.3.12 below states the existence result. It follows directly from the \mathbb{R} -case, see Proposition 2.3.4.

Proposition 2.3.12. *Let $0 \leq \delta < 1$. The process $BES^\delta(x_0)$ as stated in Proposition 2.3.4 solves the strong martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W . In particular, the martingale problem related to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits strong existence.*

Proposition 2.3.13. *The martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W admits pathwise uniqueness.*

Proof.

Let us suppose that X is solution of the strong martingale problem with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W . This implies the same with respect to \mathcal{D}_{L^δ} and W . By Proposition 2.3.10 $S = X^2$ is a solution of (3.1). The result follows by the pathwise uniqueness of the SDE (3.1) and the positivity of X .

□

2.3.5 On an alternative approach to treat the martingale problem on the full line.

A priori we could have approached the martingale problem related to Bessel processes by the technique of [40].

1. Thereby, the authors handled martingale problems related to operators $L : \mathcal{D}_L \subset C^1(\mathbb{R}) \rightarrow \mathbb{R}$ of the form $Lf = \frac{\sigma^2}{2} f'' + b' f'$, where b is the derivative of a continuous function, σ is strictly positive continuous and Σ is defined as (1.4). The idea was to consider an L -harmonic function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(0) = 0$ and $h' = e^{-\Sigma}$. In [40], L was also expressed in the form (1.6). The proof of well-posedness of the martingale problem thereby was based on a non-explosion condition (3.16) in Proposition 3.13 in [40] and the fact that $\sigma_0 := (\sigma e^{-\Sigma}) \circ \sigma^{-1}$ is strictly positive and so the SDE (for every fixed initial condition)

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s) dW_s, \quad (3.27)$$

is well-posed.

2. Consider $\delta \in [0, 1[$. As far as the martingale problem (for the Bessel process) on the full line is concerned, we could have tried to adapt similar methods. We observe that $L := L^\delta$ is also expressed in the form (1.6), which in our case gives (3.5). Taking into account (3.6), we have

$$h(x) = \text{sign}(x) \frac{|x|^{2-\delta}}{2-\delta}, x \in \mathbb{R}.$$

Since h is bijective, one can show that (3.16) in Proposition 3.13 in [40] is automatically satisfied. Moreover

$$\sigma_0(y) = \text{sign}(y) (2-\delta)^{\frac{1-\delta}{2-\delta}} |y|^{\frac{1-\delta}{2-\delta}}. \quad (3.28)$$

Following the same idea as in Proposition 3.2 of [40], one can show that the well-posedness of the Bessel martingale problem is equivalent to the well-posedness (in law) of (3.27). Here $\sigma_0(0) = 0$, but (3.27) is still well-posed even if

$$\int_0^\varepsilon \frac{1}{\sigma_0^2(y)} dy = +\infty, \forall \varepsilon > 0. \quad (3.29)$$

In fact in that case (3.29) corresponds to the Engelbert-Schmidt criterion (see Theorem 5.7 in [61, Chapter 5]).

3. The criterion (3.29) can be reformulated here saying that the quantity

$$\frac{1}{(2-\delta)^{\frac{2-2\delta}{2-\delta}}} \int_0^\varepsilon y^{\frac{2\delta-2}{2-\delta}} dy, \forall \varepsilon > 0, \quad (3.30)$$

is infinite. Now, (3.30) is always finite for any $\delta > 0$. This confirms that (3.27) has no uniqueness in law on \mathbb{R} , with σ_0 defined in (3.28), when $\delta \in]0, 1[$. So, the non-uniqueness observed in Proposition 2.3.7 is not astonishing.

4. On the other hand, when $\delta = 0$, then (3.30) is infinite, which implies uniqueness in law. This agrees with Remark 2.3.11 which even states pathwise uniqueness.

2.3.6 The framework for $\delta = 1$

Let W be a standard Brownian motion on some underlying probability space. By definition, a Bessel process of dimension $\delta = 1$ starting at $x_0 \geq 0$ is a non-negative process X such that $S := X^2$ is a $BESQ^1(x_0^2)$. On the other hand, in the literature such a Bessel process X is also characterized as a non-negative strong solution of

$$X_t = x_0 + W_t + L_t, t \in [0, T], \tag{3.31}$$

where L is a non-decreasing process only increasing when $X = 0$, i.e.

$$\int_{[0, T]} X_s dL_s = \int_{[0, T]} X_s 1_{\{X_s=0\}} dL_s.$$

In particular, X is a semimartingale. Indeed, let X be a non-negative solution of (3.31), then by an easy application of Itô's formula for semimartingales, setting $S := X^2$, we have

$$\begin{aligned} S_t &= x_0^2 + 2 \int_0^t X_s dW_s + \int_0^t X_s dL_s + \frac{1}{2} 2t \\ &= x_0^2 + 2 \int_0^t \sqrt{S_s} dW_s + \int_0^t X_s 1_{\{X_s=0\}} dL_s + t \\ &= x_0^2 + 2 \int_0^t \sqrt{S_s} dW_s + t, \end{aligned}$$

which implies that S is a $BESQ^1(x_0^2)$ and so X is a $BES^1(x_0)$. This shows in particular that (3.31) admits pathwise uniqueness. Existence and uniqueness of (3.31) can be seen via the Skorohod problem, see [58].

In this section, we represent alternatively X as a non-negative solution of a (strong) martingale problem. As we mentioned at the beginning of Section 2.3, we have fixed

$$b(x) = H(x) = \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0. \end{cases}$$

Formally speaking we get

$$\Sigma(x) = 2 \int_0^x \delta_0(y) dy = 2H(x),$$

where H is the Heaviside function. Coming back to the expression (1.6), it is natural to set

$$L^1 f = (\exp(2H) f')' \frac{\exp(-2H)}{2}, f \in C^2(\mathbb{R}^*). \tag{3.32}$$

This gives of course

$$L^1 f = \frac{f''}{2}, f \in C^2(\mathbb{R}^*). \tag{3.33}$$

Analogously to the case $\delta \in]0, 1[$ and applying the same principle as for the domain characterization in the case $\delta \in [0, 1[$, we naturally arrive to

$$\mathcal{D}_{L^1} = \{f \in C^2 \mid f'(0) = 0\}.$$

Since $L^1 f$ has to be continuous, (3.33) gives

$$L^1 f = \frac{f''}{2}. \quad (3.34)$$

The PDE operator L^1 appearing at (3.34) coincides with the generator of Brownian motion. However, the domain of that generator is larger since it is $C^2(\mathbb{R})$.

Remark 2.3.14. *The same preliminary analysis of Section 2.3.3 about the martingale problem related to $0 \leq \delta < 1$ in the \mathbb{R} -case extends to the case $\delta = 1$. More precisely, Proposition 2.3.4, Corollary 2.3.6, Proposition 2.3.7 and Remark 2.3.8 hold. This is stated below.*

Proposition 2.3.15.

1. *There is a process $BES^1(x_0)$ solving the strong martingale problem with respect to \mathcal{D}_{L^1} and W .*
2. *The martingale problem related to L^1 with respect to \mathcal{D}_{L^1} admits (in general) no uniqueness.*

Similarly to Corollary 2.3.6, the processes $BES^1(x_0)$ and $-BES^1(-x_0)$ are solutions to the strong martingale problem with respect to \mathcal{D}_{L^1} and an underlying Brownian motion W . Other solutions on the real line are the so-called *skew Brownian motions* which will be investigated more in detail in a future work. For this last one, we can mention the works of Harrison and Shepp ([58]) and Le Gall ([65]).

Concerning the \mathbb{R}_+ -case, let again $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with the canonical filtration \mathfrak{F}^W of a Brownian motion W .

By using the same arguments as for Propositions 2.3.12 and 2.3.13, we get the following result.

Proposition 2.3.16. *There is a process $BES^1(x_0)$ solving the strong martingale problem with respect to $\mathcal{D}_{L^1}(\mathbb{R}_+)$ and W . Moreover, the martingale problem admits pathwise uniqueness with respect to $\mathcal{D}_{L^1}(\mathbb{R}_+)$.*

2.4 Martingale problem related to the path-dependent Bessel process

2.4.1 Generalities

Now we are going to treat a non-Markovian martingale problem which is a perturbation of the Bessel process $BES^\delta(x_0)$, $0 \leq \delta \leq 1, x_0 \geq 0$. More precisely, we want to analyze existence and uniqueness of solutions to the martingale problem related to the SDE

$$X_t = x_0 + W_t + \int_0^t b'(X_s) ds + \int_0^t \Gamma(s, X^s) ds, \quad (4.1)$$

where Γ is the same path-dependent functional as in (2.1), and b is as in (1.9).

Proposition 2.4.1. *Suppose $\delta = 0, x_0 = 0$. Let W be a standard Brownian motion. The null process is a solution to the strong martingale problem (in the sense of Definition 2.2.3) with respect to \mathcal{D}_{L^δ} and W .*

In presence of a path-dependent drift Γ , under suitable conditions, Corollary 2.4.17 allows to show that the null process is still the unique solution of the corresponding strong martingale problem.

2.4.2 The martingale problem in the path-dependent case: existence in law.

We recall that a pair (X, \mathbb{P}) is a solution for the martingale problem related to \mathcal{L} in the sense of Definition 2.2.2 with $L = L^\delta$ with respect to \mathcal{D}_{L^δ} (resp. $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$), $0 \leq \delta \leq 1$, if for all $f \in \mathcal{D}_{L^\delta}$ (resp. $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$),

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds, \quad (4.2)$$

is a \mathbb{P} -local martingale.

A first criterion of existence can be stated if Γ is measurable and bounded.

Proposition 2.4.2. *Suppose that Γ is bounded. Then the martingale problem related to \mathcal{L} (defined in (2.4)) admits existence with respect to \mathcal{D}_{L^δ} . Moreover we have the following.*

1. *If the initial condition is $x_0 \geq 0$, then the solution can be constructed to be non-negative.*
2. *If the initial condition is $x_0 \leq 0$, then the solution can be constructed to be non-positive.*

Proof. Let $x_0 \geq 0$. Given a Brownian motion W , by Propositions 2.3.12 and 2.3.15, there exists a solution X to the (even strong) martingale problem related to (2.4) (with $\Gamma = 0$) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W . That solution is in fact a $BES^\delta(x_0)$. In particular, for all $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$,

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s. \quad (4.3)$$

Since the Bessel process is non-negative, (4.3) also holds for $f \in \mathcal{D}_{L^\delta}$. As Γ is bounded then, by Novikov's condition

$$N_t = \exp \left(\int_0^t \Gamma(s, X^s) dW_s - \frac{1}{2} \int_0^t \Gamma^2(s, X^s) ds \right),$$

is a martingale. By Girsanov's Theorem

$$B_t := W_t - \int_0^t \Gamma(s, X^s) ds,$$

is a Brownian motion under the probability measure \mathbb{Q} such that $d\mathbb{Q} = N_T d\mathbb{P}$. Then, we can rewrite (4.3) as

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds - \int_0^t f'(X_s) dB_s - \int_0^t f'(X_s) \Gamma(s, X^s) ds = 0.$$

Since $\int_0^t f'(X_s) dB_s$ is a \mathbb{Q} -local martingale, (X, \mathbb{Q}) happens to be a solution to the martingale problem in the sense of Definition 2.2.2 with respect to \mathcal{D}_{L^δ} .

Suppose now that $x_0 \leq 0$. The process X defined as $-BES^\delta(-x_0)$ is a solution of (4.3). Then the same procedure as for the case $x_0 \geq 0$ works. This shows existence for the martingale problem on \mathcal{D}_{L^δ} .

Let us discuss the sign of the solution. Suppose that $x_0 \geq 0$ (resp. $x_0 \leq 0$). Then, our construction starts with $BES^\delta(x_0)$ (resp. $-BES^\delta(-x_0)$) which is clearly non-negative (resp. non-positive). The constructed solution is again non-negative (resp. non-positive) since it is supported by an equivalent probability measure. \square

Remark 2.4.3. *As we have mentioned in Proposition 2.3.7 and its extension to $\delta = 1$, the martingale problem in the sense of Definition 2.2.2 admits no uniqueness in general, at least with respect to \mathcal{D}_{L^δ} , i.e. on the whole line.*

2.4.3 Some preliminary results on a path-dependent SDE

Before studying a new class of path-dependent martingale problems we recall some results stated in Section 4 of Chapter 1.

Let $\sigma_0 : \mathbb{R} \rightarrow \mathbb{R}$. Let $\bar{\Gamma} : \Lambda \rightarrow \mathbb{R}$ be a generic Borel functional. Related to it we formulate the following.

Assumption 2.4.4.

1. *There exists a function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\int_0^\epsilon l^{-2}(u) du = \infty$ for all $\epsilon > 0$ and*

$$|\sigma_0(x) - \sigma_0(y)| \leq l(|x - y|).$$

2. *σ_0 has at most linear growth.*

3. *There exists $K > 0$ such that*

$$|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K \left(|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)| dr \right),$$

for all $s \in [0, T]$, $\eta^1, \eta^2 \in C([0, T])$.

4. $\sup_{s \in [0, T]} |\bar{\Gamma}(s, 0)| < \infty$.

The proposition below was the object of Proposition 1.4.27 in Chapter 1.

Proposition 2.4.5. *Let $y_0 \in \mathbb{R}$. Suppose the validity of Assumption 2.4.4. Then $E(\sigma_0, 0, \bar{\Gamma})$, i.e.*

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t \bar{\Gamma}(s, Y^s) ds, \quad (4.4)$$

admits pathwise uniqueness.

The lemma below was the object of Lemma 1.4.28 in Chapter 1.

Lemma 2.4.6. *Suppose the validity of the assumptions of Proposition 2.4.5. Let Y be a solution of (4.4) and $m \geq 2$ an integer. Then there exists a constant $C > 0$, depending on the linear growth constant of σ_0 , Y_0 , K, T, m and the quantity (4) in Assumption 2.4.4 such that*

$$\mathbb{E} \left(\sup_{t \leq T} |Y_s|^m \right) \leq C.$$

2.4.4 A new class of solutions to the martingale problem

Besides Proposition 2.4.2, Proposition 2.4.8 below and Proposition 2.4.9 provide a new class of solutions to the martingale problem related to \mathcal{L} with respect to $\mathcal{D}_{L\delta}$. We consider now a particular case of $\bar{\Gamma}$, which is associated with Γ :

$$\bar{\Gamma}(s, \eta) := 2\sqrt{|\eta(s)|}\Gamma(s, \sqrt{|\eta^s|}) + \delta, \quad s \in [0, T] \quad \eta \in C([0, T]). \quad (4.5)$$

Next, we introduce a growth assumption on Γ .

Assumption 2.4.7. Γ is continuous and there exists a constant K such that, for every $(s, \eta) \in \Lambda$ we have

$$|\Gamma(s, \eta)| \leq K \left(1 + \sup_{r \in [0, T]} \sqrt{|\eta(r)|} \right).$$

Proposition 2.4.8. . Let $\delta \in [0, 1]$. Suppose that Γ fulfills Assumption 2.4.7. Then, we have the following.

1. Let $s_0 \geq 0$. The path-dependent SDE

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|}dW_s + \int_0^t 2\sqrt{|S_s|}\Gamma \left(s, \sqrt{|S^s|} \right) ds, \quad \delta \geq 0, \quad (4.6)$$

admits existence in law, see Definition 1.5.4 of Appendix in Chapter 1.

2. The constructed solution of (4.6) in item (1) is non-negative.

3. Let $x_0 \geq 0$. The martingale problem related to $\mathcal{L}f = L^\delta f + \Gamma f'$ (see Definition 2.2.2, (2.4)) admits existence with respect to $\mathcal{D}_{L\delta}(\mathbb{R}_+)$.

Proof.

We remark that the hypothesis on Γ implies that $\bar{\Gamma}$ has linear growth, i.e. there is a constant K such that

$$\bar{\Gamma}(t, \eta^t) \leq K(1 + \sup_{s \in [0, t]} |\eta(s)|), \quad \forall (t, \eta) \in [0, T] \times C([0, T]). \quad (4.7)$$

For item (1), we start truncating Γ . Let $N > 0$. Let us define, for $s \in [0, T], \eta \in C([0, T])$,

$$\begin{aligned} \Gamma^N(s, \eta) &:= (\Gamma(s, \eta^s) \vee (-N)) \wedge N, \\ \bar{\Gamma}^N(s, \eta) &:= 2\sqrt{|\eta(s)|}\Gamma^N(s, \sqrt{|\eta|}) + \delta. \end{aligned}$$

We consider the SDE

$$\begin{cases} dS_t = 2\sqrt{|S_t|}dW_t + \bar{\Gamma}^N(t, S) dt, \\ S_0 = s_0. \end{cases} \quad (4.8)$$

We set $x_0 := \sqrt{s_0}$. Since Γ^N is bounded, by Proposition 2.4.2, the martingale problem related to \mathcal{L} with respect to \mathcal{D}_{L^δ} , admits a solution (X, \mathbb{P}) which is non-negative. By Proposition 2.4.9 the SDE (4.8) admits existence in law and in particular there exists a solution S^N (which is necessarily non-negative) on some probability space $(\Omega, \mathcal{F}, \bar{\mathbb{P}}^N)$. By Itô's formula, this implies that (on the mentioned space),

$$M_t^N := f(S_t^N) - f(S_0^N) - \int_0^t f'(S_s^N)\bar{\Gamma}^N(s, S_s^N) ds - 2 \int_0^t f''(S_s^N)|S_s^N| ds, \quad (4.9)$$

is a martingale for all $f \in C^2$ with compact support. This will be used later.

We want first to show that the family of laws $(\bar{\mathbb{Q}}^N)$ of (S^N) is tight. For this we are going to use the Kolmogorov-Centsov Theorem. We denote by $\bar{\mathbb{E}}^N$ the expectation related to $\bar{\mathbb{P}}^N$. According to Problem 4.11 in Section 2.4 of [61], it is enough to find constants $\alpha, \beta > 0$ realizing

$$\sup_N \bar{\mathbb{E}}^N(|S_t^N - S_s^N|^\alpha) \leq c|t - s|^{1+\beta}; s, t \in [0, T], \quad (4.10)$$

for some constant $c > 0$. Indeed, we will show (4.10) for $\alpha = 6$ and $\beta = 1$. By (4.8) and Burkholder-Davis-Gundy inequality there exists a constant c_6 such that, for $0 \leq s \leq t \leq T$,

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \leq c_6 \left(\bar{\mathbb{E}}^N \left(\int_s^t (|S_r^N|) dr \right)^3 + \bar{\mathbb{E}}^N \left(\int_s^t \bar{\Gamma}^N(r, |S_r^N|) dr \right)^6 \right). \quad (4.11)$$

By (4.7), there exists a constant \mathcal{C}_1 where

$$|\bar{\Gamma}^N(s, \eta)| \leq 2\sqrt{|\eta(s)|}|\Gamma(s, \sqrt{|\eta|})| + \delta = |\bar{\Gamma}(s, \eta)| \leq \mathcal{C}_1 \left(1 + \sup_{r \leq s} |\eta(s)| \right), \quad (4.12)$$

for every $(s, \eta) \in \Lambda$, uniformly in N . By Jensen's inequality and (4.12), there exists a constant $\mathcal{C}_2 > 0$, only depending on T and on $\bar{\Gamma}$, but not on N , such that

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \leq \mathcal{C}_2 \left((t - s)^2 \bar{\mathbb{E}}^N \left(\sup_{s \leq t} |S_s^N|^3 \right) + (t - s)^5 \bar{\mathbb{E}}^N \left(\sup_{s \leq t} |S_s^N|^6 \right) \right).$$

By Lemma 2.4.6, the quantity

$$\bar{\mathbb{E}}^N \left(\sup_{s \leq T} |S_s^N|^3 + \sup_{s \leq T} |S_s^N|^6 \right),$$

is bounded uniformly in N and therefore (4.10) holds. Consequently, the family of laws $(\bar{\mathbb{Q}}^N)$ of (S^N) under $(\bar{\mathbb{P}}^N)$ is tight. We can therefore extract a subsequence which, for simplicity, we will still call $\bar{\mathbb{Q}}^N$ that converges weakly to a probability measure $\bar{\mathbb{Q}}$ on $(C[0, T], \mathfrak{B}(C[0, T]))$.

We denote by \mathbb{E}^N the expectation with respect to $\bar{\mathbb{Q}}^N$. Let $0 \leq s \leq t \leq T$ and let $F : C([0, s]) \rightarrow \mathbb{R}$ be a bounded and continuous function. By (4.9), if S is the canonical process we have

$$\mathbb{E}^N((\tilde{M}_t^N - \tilde{M}_s^N)F(S_r, 0 \leq r \leq s)) = 0, \quad (4.13)$$

where

$$\tilde{M}_t^N := f(S_t) - f(S_0) - \int_0^t f'(S_s) \bar{\Gamma}^N(s, S) ds - 2 \int_0^t f''(S_s) |S_s| ds. \quad (4.14)$$

By Skorokhod's convergence theorem, there exists a sequence of processes (Y^N) and a process Y both on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, converging u.c.p. to Y as $N \rightarrow +\infty$. Indeed (Y^N) and Y can be seen as random elements taking values in the state space $(C[0, T], \mathfrak{B}(C[0, T]))$.

Moreover, the law of Y^N is $\bar{\mathbb{Q}}^N$, so that

$$\mathbb{E}^{\mathbb{Q}}((\bar{M}_t^N - \bar{M}_s^N)F(Y_r^N, 0 \leq r \leq s)) = 0, \quad (4.15)$$

where

$$\bar{M}_t^N := f(Y_t^N) - f(S_0) - \int_0^t f'(Y_r^N) \bar{\Gamma}^N(s, Y^N) ds - 2 \int_0^t f''(Y_s^N) |Y_r^N| dr. \quad (4.16)$$

We wish to pass to the limit when $N \rightarrow \infty$ using Lebesgue dominated convergence theorem and obtain

$$\mathbb{E}^{\mathbb{Q}}((\bar{M}_t - \bar{M}_s)F(Y_r, 0 \leq r \leq s)) = 0, \quad (4.17)$$

with

$$\bar{M}_t := f(Y_t) - f(S_0) - \int_0^t f'(Y_s) \bar{\Gamma}(s, Y) ds - 2 \int_0^t f''(Y_r) |Y_r| dr. \quad (4.18)$$

For this it remains to prove that, when $N \rightarrow \infty$

$$\mathbb{E}^{\mathbb{Q}} \left(\int_s^t f'(Y_r^N) \bar{\Gamma}^N(r, Y^N) dr \right) \rightarrow \mathbb{E}^{\mathbb{Q}} \left(\int_s^t f'(Y) \bar{\Gamma}(r, Y) dr \right) \quad (4.19)$$

and

$$\mathbb{E}^{\mathbb{Q}} \left(\int_s^t f''(Y_r^N) |Y_r^N| dr \right) \rightarrow \mathbb{E}^{\mathbb{Q}} \left(\int_s^t f''(Y_r) |Y_r| dr \right), \quad (4.20)$$

as $N \rightarrow \infty$. Below, we only prove (4.19) since (4.20) follows similarly.

Note that (4.19) is true, if and only if,

$$\lim_{N \rightarrow \infty} I_1(N) = 0, \quad \lim_{N \rightarrow \infty} I_2(N) = 0,$$

where

$$I_1(N) := \mathbb{E}^{\mathbb{Q}} \left[\int_s^t f'(Y_r^N) (\bar{\Gamma}^N(r, Y^N) - \bar{\Gamma}(r, Y^N)) dr \right],$$

$$I_2(N) := \mathbb{E}^{\mathbb{Q}} \left[\int_s^t f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y) dr \right].$$

By (4.7) and (4.12), we have

$$I_1(N) \leq \|f'\|_{\infty} \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{\sup_{r \in [0, T]} |\Gamma(r, Y^N, r)| > N\}} \int_s^t |\bar{\Gamma}^N(r, Y^N) - \bar{\Gamma}(r, Y^N)| dr \right] \leq$$

$$\leq 2KT \|f'\|_{\infty} \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{\sup_{r \in [0, T]} |\Gamma(r, Y^N)| > N\}} \left(1 + \sup_{r \in [0, T]} |Y_r^N| \right) \right].$$

By Cauchy-Schwarz's inequality, there exists a non-negative constant $C(f, T, K)$ such that

$$I_1(N)^2 \leq C(f, T, K)I_{11}(N)I_{12}(N), \quad (4.21)$$

where

$$I_{11}(N) := \mathbb{Q} \left(\sup_{r \in [0, T]} |\Gamma(r, Y^N)| > N \right),$$

$$I_{12}(N) := \mathbb{E}^{\mathbb{Q}} \left[1 + \sup_{r \in [0, T]} |Y_r^N|^2 \right].$$

By Chebyshev's inequality we have

$$I_{11}(N) \leq \frac{1}{N^2} \mathbb{E}^{\mathbb{Q}} \left[\sup_{r \in [0, T]} |\Gamma(r, Y^N)|^2 \right] \leq \frac{2K}{N^2} \mathbb{E}^{\mathbb{Q}} \left[1 + \sup_{r \in [0, T]} |Y_r^N|^2 \right].$$

Consequently, $\lim_{N \rightarrow \infty} I_{11}(N) = 0$ because of Lemma 2.4.6. On the other hand, again by Lemma 2.4.6, $I_{12}(N)$ is bounded in N and so by (4.21), we get $\lim_{N \rightarrow \infty} I_1(N) = 0$.

Concerning $I_2(N)$, we have

$$I_2(N)^2 \leq T \int_s^t \mathbb{E}^{\mathbb{Q}} [|f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^2] dr. \quad (4.22)$$

By Lemma 2.4.6, there exists a constant C not depending on N such that

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{r \in [0, T]} |Y_r^N|^4 \right] \leq C,$$

and, consequently, by Fatou's Lemma

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{r \in [0, T]} |Y_r|^4 \right] \leq C.$$

Let $r \in [0, T]$. We have

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} [|f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^4] \\ & \leq 8 \|f'\|_{\infty}^4 K^4 \left(2 + \mathbb{E}^{\mathbb{Q}} \left[\sup_{r \in [0, T]} |Y_r^N|^4 + \sup_{r \in [0, T]} |Y_r|^4 \right] \right) \\ & \leq 16 \|f'\|_{\infty}^4 K^4 (1 + C). \end{aligned} \quad (4.23)$$

So the sequence

$$|f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^2$$

is uniformly integrable. We fix again $r \in [0, T]$. Since f' and $\bar{\Gamma}$ are continuous it follows that

$$\mathbb{E}^{\mathbb{Q}} [|f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^2] \longrightarrow 0, \quad (4.24)$$

as $N \rightarrow \infty$. Now (4.23) and Cauchy-Schwarz implies that

$$\mathbb{E}^{\mathbb{Q}}[|f'(Y_r^N)\bar{\Gamma}(r, Y^N) - f'(Y_r)\bar{\Gamma}(r, Y)|^2] \leq 4\|f'\|_{\infty}^2 K^2 \sqrt{1+C}. \quad (4.25)$$

This time (4.24), (4.25) and Lebesgue's dominated theorem show that the entire Lebesgue integral of (4.24) on $[s, t]$ converges to 0. Finally, $\lim_{N \rightarrow \infty} I_2(N) = 0$ so that we conclude to (4.19) and, consequently, (4.17). Therefore, (Y, \mathbb{Q}) solve the martingale problem of the type (2.3) as in Proposition 2.2.1 with

$$Lf(x) = 2|x|f''(x) + \delta f'(x)$$

and $\bar{\Gamma}$ replacing Γ . By Proposition 2.2.1, this concludes the proof of item (1).

Concerning item (2), the previously constructed Y is a (weak) solution to (4.6) under the probability \mathbb{Q} . Since it is a limit of non-negative solutions, it will also be non-negative.

Item (3) follows from Proposition 2.4.9 below. □

2.4.5 Equivalence between martingale problem and SDE in the path-dependent case

We state here an important result establishing the equivalence between the martingale problem and a path-dependent SDE of squared Bessel type. Let $0 \leq \delta \leq 1$.

Proposition 2.4.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a stochastic process and we denote $S = X^2$.*

1. *$(|X|, \mathbb{P})$ is a solution to the martingale problem related to (2.4) with respect to \mathcal{D}_{L^δ} , if and only if, the process S is a solution of (4.6) for some \mathfrak{F}^X -Brownian motion W .*
2. *Let W be a standard Brownian motion (with respect to \mathbb{P}). Then $|X|$ is a solution to the strong martingale problem with respect to \mathcal{D}_{L^δ} and W , if and only if, S is a solution of (4.6).*

Remark 2.4.10. *In the statement of Proposition 2.4.9, \mathcal{D}_{L^δ} can be replaced with $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$, provided that $|X|$ is replaced by X .*

Proof (of Proposition 2.4.9). We discuss item (1).

Concerning the direct implication, by choosing $f_1(x) = x^2, f_2(x) = x^4$ we have $L^\delta f_1(x) = \delta, L^\delta f_2(x) = 2(2 + \delta)x^2$. By definition of the martingale problem, the two processes ($t \in [0, T]$)

$$M_t := X_t^2 - X_0^2 - \delta t - \int_0^t 2|X_s|\Gamma(s, |X^s|)ds \quad (4.26)$$

and

$$N_t := X_t^4 - X_0^4 - 2(2 + \delta) \int_0^t X_s^2 ds - 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds, \quad (4.27)$$

are \mathfrak{F}^X -local martingales.

Since $S = X^2$, by (4.26) we have $[S] = [M]$. By integration by parts and (4.26), we have

$$[M]_t = [X^2]_t = X_t^4 - X_0^4 - 2 \int_0^t X_s^2 dX_s^2 = X_t^4 - X_0^4 - 2\delta \int_0^t X_s^2 ds - 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds + M^1,$$

where M^1 is a local martingale. Therefore

$$X_t^4 - X_0^4 = M^1 + 2\delta \int_0^t X_s^2 ds + 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds + [M]_t, \quad t \in [0, T]. \quad (4.28)$$

(4.28) and (4.27) give us two decompositions of the semimartingale X^4 ; by the uniqueness of the semimartingale decomposition, $[M]_t = 4 \int_0^t X_s^2 ds$. We set

$$W_t = \int_0^t \frac{dM_s}{2|X_s|}, \quad t \in [0, T]. \quad (4.29)$$

By Lévy's characterization theorem, W is an \mathfrak{F}^X -Brownian motion and by (4.26), we conclude that $S_t = s_0 + \delta t + \int_0^t 2\sqrt{S_s} dW_s + \int_0^t 2\sqrt{S_s} \Gamma(s, \sqrt{S^s}) ds, t \in [0, T]$.

Concerning the converse implication, suppose that $S = X^2$ solves (4.6) for some Brownian motion W . Then S solves

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|} d\widetilde{W}_s, \quad t \in [0, T], \quad (4.30)$$

where

$$\widetilde{W}_t := W_t + \int_0^t \Gamma(s, \sqrt{|S^s|}) ds, \quad t \in [0, T].$$

Let $f \in \mathcal{D}_{L^\delta}$; by Proposition 2.3.4 and Remark 2.3.5 we have

$$f(|X_t|) - f(|X_0|) - \int_0^t L^\delta f(|X_s|) ds = \int_0^t f'(|X_s|) d\widetilde{W}_s. \quad (4.31)$$

Consequently

$$\begin{aligned} M_t^f &:= f(|X_t|) - f(|x_0|) - \int_0^t L^\delta f(|X_s|) ds - \int_0^t f'(|X_s|) \Gamma(s, |X^s|) ds \\ &= \int_0^t f'(|X_s|) dW_s, \end{aligned}$$

is an \mathfrak{F}^X -local martingale. Then, $(|X|, \mathbb{P})$ solve the martingale problem related to (2.4) with respect to \mathcal{D}_{L^δ} in the sense of Definition 2.2.2. On the other hand, $|X|$ also solves the strong martingale problem with respect to \mathcal{D}_{L^δ} and W . This concludes the proof of item (1).

As far as item (2) is concerned, the converse implication argument has been given above. Concerning the direct implication, we define f_1 as in the proof of item (1). By (2.6), (4.26) and the fact that

$$M = 2 \int_0^\cdot f_1'(|X_s|) dW_s = 2 \int_0^\cdot \sqrt{S_s} dW_s,$$

we obtain (4.6). This concludes the proof. \square

2.4.6 The martingale problem in the path-dependent case: uniqueness in law.

A consequence of Girsanov's theorem gives us the following.

Proposition 2.4.11. *Let $0 \leq \delta \leq 1$. Suppose that Γ is bounded. The martingale problem related to (2.4) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits uniqueness.*

Remark 2.4.12. *Let $x_0 \geq 0$ (resp. $x_0 \leq 0$). By Proposition 2.4.2, every solution of the aforementioned martingale problem is non-negative (resp. non-positive).*

Proof (of Proposition 2.4.11).

Let (X^i, \mathbb{P}^i) , $i = 1, 2$ be two solutions to the martingale problem related to $\mathcal{L}f = Lf + \Gamma f'$ with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$. By Proposition 2.4.9, $S^i = (X^i)^2$ is a solution of (4.6), for some Brownian motion W^i and \mathbb{P}^i . We define the random variable (which is also a Borel functional of X^i)

$$V_t^i = \exp\left(-\int_0^t \Gamma(s, X^i) dW_s^i - \frac{1}{2} \int_0^t (\Gamma(s, X^i))^2 ds\right).$$

By the Novikov condition, it is a \mathbb{P}^i -martingale. This allows us to define the probability $d\mathbb{Q}^i = V_T^i d\mathbb{P}^i$. By Girsanov's theorem, for $i = 1, 2$, under \mathbb{Q}^i , $B_t^i := W_t^i + \int_0^t \Gamma(s, X^{i,s}) ds$ is a Brownian motion. Therefore, S^i is a solution of (4.6) with $\Gamma = 0$, under \mathbb{Q}^i . Now (4.6) (with $\Gamma = 0$) admits pathwise uniqueness and therefore uniqueness in law, by Yamada-Watanabe theorem. Consequently S^i (under \mathbb{Q}^i), $i = 1, 2$ have the same law and the same holds of course for X^i , $i = 1, 2$. Hence, for every Borel set $B \in \mathfrak{B}(C[0, T])$ we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_{\Omega} \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} d\mathbb{Q}^1 = \int_{\Omega} \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} d\mathbb{Q}^2 = \mathbb{P}^2\{X^2 \in B\}.$$

So, X^1 under \mathbb{P}^1 has the same law as X^2 under \mathbb{P}^2 . Finally the martingale problem related to (2.4) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ admits uniqueness. \square

2.4.7 Path-dependent Bessel process: results on pathwise uniqueness.

In this section, $\bar{\Gamma}$ is the same as the one defined in (4.5), i.e.

$$\bar{\Gamma}(s, \eta) := 2\sqrt{|\eta(s)|}\Gamma(s, \sqrt{|\eta^s|}) + \delta, s \in [0, T] \eta \in C([0, T]).$$

At this point, we can state a pathwise uniqueness theorem. For this purpose, we state the following assumption.

Assumption 2.4.13.

1. *There exists a constant $K > 0$ such that, for every $s \in [0, T]$, $\eta^1, \eta^2 \in C([0, T])$, we have $|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K (|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)| dr)$.*

$$2. \sup_{t \in [0, T]} |\bar{\Gamma}(t, 0)| < \infty.$$

Remark 2.4.14.

1. $\sigma_0(y) = 2\sqrt{|y|}$ has linear growth.
2. Defining $l(x) = 2\sqrt{x}$, $x \geq 0$, we have $\int_0^\epsilon l^{-2}(u) du = \infty$ for every $\epsilon > 0$ and $|l(x) - l(y)| \leq l(|x - y|)$, $x, y \in \mathbb{R}_+$.

Remark 2.4.15. Note that, by Remark 2.4.14, Assumption 2.4.13 implies Assumption 2.4.4.

We start the analysis by considering equation (4.6). For the definitions of strong existence and pathwise uniqueness for path-dependent SDEs, see Definitions 1.5.2 and 1.5.3 in Chapter 1.

Theorem 2.4.16. Suppose Assumptions 2.4.13 and 2.4.7.

1. (4.6) admits pathwise uniqueness.
2. (4.6) admits strong existence.
3. Suppose $x_0 \geq 0$. Every solution of (4.6) with $s_0 = x_0^2$ is non-negative.

Proof

1. We remark that (4.6) is of the form (4.4). The result follows from Proposition 2.4.5 and Remark 2.4.14.
2. By Proposition 2.4.8, we have existence in law. By an extension of Yamada-Watanabe theorem to the path-dependent case, strong existence holds for (4.6).
3. Suppose $x_0 \geq 0$. By Proposition 2.4.8 (2), (4.6) admits even existence in law of a non-negative solution. By Yamada-Watanabe theorem extended to the path-dependent case, pathwise uniqueness implies uniqueness in law, so that the above-mentioned solution has to be non-negative.

□

We are now able to state the following.

Corollary 2.4.17. Suppose that $\bar{\Gamma}$ (defined in (4.5)) fulfills Assumptions 2.4.13 and 2.4.7. Then the strong martingale problem related to (2.4) (see Definition 2.2.3) with respect to $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ and W admits strong existence and pathwise uniqueness.

Proof. By Theorem 2.4.16, the equation (4.6) admits a unique strong solution which is non-negative. Proposition 2.4.9 and Remark 2.4.10 allow us to conclude the proof.

□

Chapter 3

ROUGH PATHS AND REGULARIZATION

This chapter is object of the paper [50].

3.1 Introduction

This paper focuses on two variants of stochastic calculus of pathwise type: calculus via regularization and rough paths. The recent literature on rough paths is very rich and it is impossible to list it here completely. It was started in [69] continued by the monograph [68] which focused on rough differential equations. The corresponding integral was introduced later by M. Gubinelli, see [56]. Later, a great variety of contributions on the subject appeared and it is not possible to list all of them. We refer however to the monograph [45] to a fairly rich list of references and for a complete development of the subject. In spite of some recent work mixing probability and deterministic theory, see e.g. [66, 19, 44], the theory of rough paths is essentially deterministic.

Stochastic calculus via regularization was started first by F. Russo and P. Vallois in [83]. The calculus was later continued in [84, 87, 88] in the framework of continuous integrators, essentially with finite quadratic variation. The case of processes with higher variation was first introduced in [36, 37] and continued in [24, 55, 54, 53, 90, 10], especially in relation with fractional Brownian motion and related processes. A not very recent survey paper in the framework of finite dimensional processes is [89]. Stochastic calculus via regularization for processes taking values in Banach spaces, with applications to the path-dependent case, was realized in [28, 29] and in [21]. The case of real-valued jump integrators was first introduced in [85] and then deeply investigated in [4] and later by [5]. Applications to mathematical finance (resp. to fluidodynamics modeling) were published in [23] (resp. [38]).

An important notion which emerged in calculus via regularization is the notion of weak Dirichlet processes, started in [36, 52]. Such a process X is the sum of a local martingale M and an orthogonal process A such that $[A, N] = 0$ for any continuous martingale N . This constitutes a natural gener-

alization of the notion of semimartingale and of Dirichlet process (in the sense of Föllmer), see [42]. In particular, [52] allowed to establish chain rule type decomposition extending Itô formulae with applications to control theory, see [51]. That concept was extended to the jump case by [20] and its related calculus was performed by [5] with applications to BSDEs, see [6]. In [31, 30] one has performed weak Dirichlet decomposition of real functional of Banach space-valued processes. In [21, 32] one has investigated strict solutions of path-dependent PDEs.

In this paper we wish first to give a key to revisit the theory of rough paths under the perspective of stochastic calculus via regularizations. The idea here is not to summarize the theory of rough paths integrals, but to propose a variant version which is directly probabilistic. In particular, we emphasize the strong link between the notion of weak Dirichlet process and one of stochastically controlled process, which is a stochastic version of the one proposed by Gubinelli [56]. According to Definition 3.3.2 such a process fulfills

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, s < t, \quad (1.1)$$

where

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds = 0, \quad (1.2)$$

in probability for each $t \in [0, T]$.

Here, X is the reference driving noise, Y' is a process (not necessarily admitting γ -Hölder continuous paths). The orthogonality condition (1.2) resembles the 2γ -Hölder-regularity condition reminiscent from [56].

Propositions 3.3.7 and 3.3.9 present the connection between weak Dirichlet processes and stochastically controlled processes. In particular, when the reference driving noise is a martingale, then both concepts coincide. As a side effect, Theorem 3.5.6 shows Stratonovich integration as a stochastic rough-type integration for weak Dirichlet integrands and continuous semimartingale integrators. The connection between rough paths theory with semimartingales has been investigated by some authors. [22] shows pathwise Wong-Zakai-type theorems for Stratonovich SDEs driven by continuous semimartingales. In particular, the integral defined by rough paths theory agrees with Stratonovich integrals for real-valued functions $f(X)$ of the driving noise X , see also Proposition 17.1 in [47]. Recently, [44] introduces a concept of rough semimartingales and develops the corresponding stochastic integration having a deterministic rough path in the background and mixing with p -variation regularity. Beyond semimartingale driving noises, we draw attention to the recent work of [67]. The authors have established the connection between rough integrals and trapezoidal Riemann sum approximations for controlled processes integrands (in the pathwise sense of [56]) and a general class of Gaussian driving noises.

In this article, we take full advantage of the probability measure and the stochastic controllability (1.1) to establish consistency between stochastic rough-type and Stratonovich integrals for more general integrands. In the companion paper in preparation [72], a detailed analysis on stochastic

rough-type integrals driven by Gaussian rough paths and their connection with Stratonovich and Skorohod integrals is presented.

The paper is organized as follows. After this introduction, in Section 3.2.2 we introduce some notations about matrix-valued calculus via regularization. In Section 3.3 we introduce the notion of stochastically controlled paths and the one of stochastic Gubinelli derivative, under the inspiration of the classical rough paths theory. We link this with the notion of Dirichlet process. In Section 3.4 we introduce the **second order process** (connected with the Lévy area) and finally in Section 3.5 discuss the notion of rough stochastic integrals via regularization, examining carefully the case when the integrator is a semimartingale.

3.2 Preliminary notions

3.2.1 Basic notations

We introduce here some basic notations intervening in the paper. $T > 0$ will be a finite fixed horizon. Regarding linear algebra, vectors or elements of \mathbb{R}^d will be assimilated to column vectors, so that if x is a vector in \mathbb{R}^d , then x^\top is a row vector.

We continue fixing some notations. In the sequel, finite-dimensional Banach spaces E will be equipped with a norm $|\cdot|$, typically $E = \mathbb{R}^d$. Let $T > 0$ be a fixed maturity. For $\alpha \in]0, 1]$, the notation $C^{[\alpha]}([0, T]; E)$ is reserved for E -valued paths defined on $[0, T]$, Hölder continuous of index $\alpha \in]0, 1]$. For $X \in C^{[\alpha]}([0, T]; E)$, the usual seminorm is given by

$$\|X\|_\alpha := \sup_{s, t \in [0, T], s \neq t} \frac{|X_{s,t}|}{|t - s|^\alpha},$$

where we set

$$X_{s,t} := X_t - X_s, \quad 0 \leq s, t \leq T. \quad (2.1)$$

When $E = \mathbb{R}$ we simply write $C^{[\alpha]}([0, T])$

For a two-parameter function $R : [0, T]^2 \rightarrow \mathbb{R}$, vanishing on the diagonal $\{(s, t) | 0 \leq s = t \leq T\}$, we write $R(s, t) := R_{s,t}$. We say that $R \in C^{[\alpha]}([0, T]^2)$ if

$$\|R\|_\alpha := \sup_{s, t \in [0, T]^2} \frac{|R_{s,t}|}{|t - s|^\alpha} < \infty. \quad (2.2)$$

By convention the quotient $\frac{0}{0}$ will set to zero. In the sequel, if $n \in \mathbb{N}^*$, we will extend a function $R \in C([0, T]^n)$ to $C(\mathbb{R}_+^n)$ by continuity, setting

$$R_{t_1, \dots, t_n} := R_{(t_1 \wedge T), \dots, (t_n \wedge T)}. \quad (2.3)$$

(Ω, \mathcal{F}, P) will be a fixed probability space. Let X^1, X^2 be two stochastic processes, continuous for simplicity.

We introduce

$$C(\varepsilon, X^1, X^2)(t) = \int_0^t \frac{(X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2)}{\varepsilon} ds, \quad t \geq 0. \quad (2.4)$$

In the sequel (\mathcal{F}_t) will be a filtration fulfilling the usual condition.

Definition 3.2.1. 1. The **covariation** of X^1 and X^2 is the continuous process (whenever it exists) $[X^1, X^2]$ such that, for $t \geq 0$,

$$C(\varepsilon, X^1, X^2)(t) \text{ converges in probability to } [X^1, X^2]_t.$$

We say that the covariation $[X^1, X^2]$ exists in the strong sense if moreover

$$\sup_{0 < \varepsilon \leq 1} \int_0^T \frac{|(X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2)|}{\varepsilon} ds < \infty. \quad (2.5)$$

2. A vector of processes $(X^1, \dots, X^d)^\top$ is said to have all its **mutual covariations** if $[X^i, X^j]$ exists for every $1 \leq i, j \leq d$.

3. A real process X is said to be **strong finite cubic variation process**, see [37], if there is a process ξ such that, for every $t \in [0, T]$,

$$\int_0^t \frac{|X_{s+\varepsilon}^1 - X_s^1|^3}{\varepsilon} ds \rightarrow \xi,$$

in probability. If $\xi = 0$ then X is said to have zero cubic variation.

4. A real-valued (continuous) (\mathcal{F}_t) -**martingale orthogonal process** A is a continuous adapted process such that $[A, N] = 0$ for every (\mathcal{F}_t) -local martingale N . A real-valued (continuous) (\mathcal{F}) -**weak Dirichlet process** is the sum of a continuous (\mathcal{F}_t) -local martingale M and an (\mathcal{F}_t) -martingale orthogonal process.

Remark 3.2.2. 1. If X^1, X^2 are two semimartingales then $(X^1, X^2)^\top$ has all its mutual covariations, see Proposition 1.1 of [86] and $[X^1, X^2]$ is the classical covariation of semimartingales.

2. It may happen that $[X^1, X^2]$ exists but $(X^1, X^2)^\top$ does not have all its mutual covariations, see Remark 22 of [89].

3. If X^1 (resp. X^2) has α -Hölder (resp. β -Hölder) paths with $\alpha + \beta > 1$, then $[X^1, X^2] = 0$, see Propositions and 1 of [89].

Suppose that $M = (M^1, \dots, M^d)$, and M^1, \dots, M^d are real-valued local martingales. In particular M^\top is an \mathbb{R}^d -valued local martingale. We denote by $\mathcal{L}^2(d[M, M])$ the space of processes $H = (H^1, \dots, H^d)$ where H^1, \dots, H^d are real progressively measurable processes and

$$\sum_{i,j} \int_0^T H_s^i H_s^j d[M^i, M^j]_s < \infty \quad \text{a.s.} \quad (2.6)$$

$\mathcal{L}^2(d[M, M])$ is an F -space with respect to the metrizable topology d_2 defined as follows: (H^n) converges to H when $n \rightarrow \infty$ if

$$\sum_{i,j} \int_0^T ((H^n)_s^i - H_s^i)((H^n)_s^j - H_s^j) d[M^i, M^j]_s \rightarrow 0,$$

in probability, when $n \rightarrow \infty$.

Similarly as in (27), in Section 4.1 of [89], one can prove the following.

Proposition 3.2.3. *Let X^1, X^2 be two processes such that $(X^1, X^2)^\top$ has all its mutual covariations, and H be a continuous (excepted eventually on a countable number of points) real-valued process, then*

$$\frac{1}{\varepsilon} \int_0^\cdot H_s (X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2) ds \rightarrow \int_0^\cdot H_s d[X^1, X^2]_s$$

in the ucp sense, when $\varepsilon \rightarrow 0$.

3.2.2 Matrix-valued integrals via regularization

Here we will shortly discuss about matrix-valued stochastic integrals via regularizations. Let $\mathbb{M}^{n \times d}$ be the linear space of the real $n \times d$ matrices, which in the rough paths literature are often associated with tensors.

For every $(s, t) \in \Delta := \{(s, t) | 0 \leq s \leq t \leq T\}$, we introduce two $\mathbb{M}^{n \times d}$ -valued stochastic integrals via regularizations. Let X (resp. Y) be an \mathbb{R}^d -valued (resp. \mathbb{R}^n -valued) continuous process (resp. locally integrable process) indexed by $[0, T]$.

So $X = (X^1, \dots, X^d)^\top$ (resp. $Y = (Y^1, \dots, Y^n)^\top$).

$$\int_s^t Y \otimes d^- X := \lim_{\varepsilon \rightarrow 0^+} \int_s^t Y_r \frac{(X_{r+\varepsilon} - X_r)^\top}{\varepsilon} dr, \quad (2.7)$$

$$\left(\text{resp. } \int_s^t Y \otimes d^\circ X := \lim_{\varepsilon \rightarrow 0^+} \int_s^t \frac{Y_r + Y_{r+\varepsilon}}{2} \frac{(X_{r+\varepsilon} - X_r)^\top}{\varepsilon} dr \right), \quad (2.8)$$

provided that previous limit holds in probability and the random function $t \mapsto \int_0^t Y \otimes d^- X$, (resp. $t \mapsto \int_0^t Y \otimes d^\circ X$), admits a continuous version. In particular

$$\left(\int_s^t Y \otimes d^- X \right) (i, j) = \int_s^t Y^i \otimes d^- X^j.$$

We remark that $\int_s^t Y \otimes d^- X$ exists if and only if $\int_s^t Y^i \otimes d^- X^j$ exist for every $1 \leq i \leq n, 1 \leq j \leq d$.

Suppose now that Y is continuous. We denote by $[X, Y]$ the matrix

$$[X, Y](i, j) = [X^i, Y^j], 1 \leq i \leq d, 1 \leq j \leq n,$$

provided those covariations exist. If $n = d$ and $X = Y$, previous matrix exists for instance if and only if X has all its mutual covariations.

We will denote by $[X, X]^{\mathbb{R}}$ the **scalar quadratic variation** defined as the real continuous process (if it exists) such that

$$[X, X]_t^{\mathbb{R}} := [X^{\top}, X] = \lim_{\varepsilon \rightarrow 0} \int_0^t \frac{|X_{s+\varepsilon} - X_s|^2}{\varepsilon} ds, \quad t \in [0, T],$$

when the limit holds in probability. $[X, X]^{\mathbb{R}}$, when it exists, is an increasing process. When X^i are finite quadratic variation processes for every $1 \leq i \leq d$, then

$$[X, X]^{\mathbb{R}} = \sum_{i=1}^d [X^i, X^i].$$

We recall that \mathbb{R}^d -valued continuous process is called semimartingale with respect to a filtration (\mathcal{F}_t) , if all its components are semimartingales.

3.3 Stochastically controlled paths and Gubinelli derivative

In [56], the author introduced a class of controlled paths Y by a reference function.

Definition 3.3.1. (*Gubinelli*). Let X be a function belonging to $C^{[\gamma]}([0, T]; E)$ with $\frac{1}{3} < \gamma < \frac{1}{2}$. An element Y of $C^{[\gamma]}([0, T])$ is called **weakly controlled** (by X) if there exists a function $Y' \in C^{[\gamma]}([0, T]; E)$ (here by convention, Y' will be a row vector), so that the remainder term R defined by the relation

$$Y_{s,t} = Y'_s(X_t - X_s) + R_{s,t}, \quad s, t \in [0, T],$$

belongs to $C^{[2\gamma]}([0, T]^2)$.

From now on X will stand for a fixed \mathbb{R}^d -valued reference continuous process. The definition below is inspired by previous one.

Definition 3.3.2. 1. We say that an \mathbb{R} -valued stochastic process Y is **stochastically controlled** by X if there exists an \mathbb{R}^d -valued stochastic process Y' (here again indicated by a row vector) so that the remainder term R^Y defined by the relation

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, \quad s < t, \tag{3.1}$$

satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds = 0, \tag{3.2}$$

in probability for each $t \in [0, T]$. Y' is called **stochastic Gubinelli derivative**.

2. \mathcal{D}_X will denote the couples of processes (Y, Y') satisfying (3.1) and (3.2).

3. If Y is an \mathbb{R}^n -valued process whose components are Y^1, \dots, Y^n , then Y is said to be stochastically controlled by X if every component Y^i is stochastically controlled by X . The matrix Y' whose rows are stochastic Gubinelli derivatives $(Y^i)'$ of Y^i is called (matrix) stochastic Gubinelli derivative of Y . The relations (3.1) and (3.2) also make sense in the vector setting. $\mathcal{D}_X(\mathbb{R}^n)$ will denote the couples (Y, Y') , where Y is a \mathbb{R}^n -valued process, being stochastically controlled by X and Y' is a Gubinelli derivative. We remark that R^Y also depends on the process X .

Similarly to the theory of (deterministic) controlled rough paths, in general, Y can admit different stochastic Gubinelli derivatives. However Proposition 3.3.7 states sufficient conditions for uniqueness.

Let us now provide some examples of stochastically controlled processes.

Example 3.3.3. Let X be an \mathbb{R}^d -valued continuous process having all its mutual covariations. Let Y be an \mathbb{R} -valued process such that, for every $1 \leq i \leq d$, $[Y, X^i]$ exists in the strong sense and $[Y, X^i] = 0$. Consider for instance the three following particular cases.

- (Y, X^1, \dots, X^d) has all its mutual covariations and $[Y, X] = 0$. In this case, for every $1 \leq i \leq d$, $[Y, X^i]$ exists in the strong sense.
- Let Y (resp. X) be a γ' -continuous (resp. γ -continuous) process with $\gamma + \gamma' > 1$. Again $[Y, X^i]$ admits its mutual covariations in the strong sense and $[Y, X^i] = 0$, for every $1 \leq i \leq d$ since

$$\int_0^T |Y_{s+\varepsilon} - Y_s| |X_{s+\varepsilon}^i - X_s^i| \frac{ds}{\varepsilon} \leq \text{const } \varepsilon^{\gamma+\gamma'-1} \rightarrow 0,$$

when $\varepsilon \rightarrow 0_+$, for every $1 \leq i \leq d$.

We recall that, under those conditions, the Young integral

$\int_0^t Y d^{(y)} X, t \in [0, T]$ exists, see [94, 11].

- If $X^i, 1 \leq i \leq d$, are continuous bounded variation processes and Y is a.s. locally bounded.

1. We claim that Y is stochastically controlled by X with $Y' \equiv 0$.
2. If moreover $[X, X]^{\mathbb{R}} \equiv 0$, then Y' can be any locally bounded process: therefore the stochastic Gubinelli derivative is not unique.

Indeed, for $0 \leq s \leq t \leq T$, write

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y.$$

1. If $Y' \equiv 0$ we have

$$\frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}(X_{s+\varepsilon} - X_s) ds \rightarrow 0,$$

when $\varepsilon \rightarrow 0_+$, since

$$\frac{1}{\varepsilon} \int_0^t (Y_{s+\varepsilon} - Y_s)(X_{s+\varepsilon} - X_s) ds \rightarrow [Y, X] = 0, \quad (3.3)$$

when $\varepsilon \rightarrow 0$.

2. If $[X, X]^{\mathbb{R}} \equiv 0$ and Y' is a locally bounded process, then we also have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t Y'_s (X_{s+\varepsilon} - X_s)(X_{s+\varepsilon} - X_s) ds = 0, \quad t \in [0, T].$$

This follows by

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t |Y'_s| |X_{s+\varepsilon} - X_s|^2 ds = 0, \quad t \in [0, T],$$

$[X, X]^{\mathbb{R}} = 0$ and Kunita-Watanabe inequality, see e.g. Proposition 1.4 of [89]. We leave the detailed proof to the reader. The result follows by (3.3).

In the second example we show that a weakly controlled process in the sense of Gubinelli is a stochastically controlled process.

Example 3.3.4. Let X be an \mathbb{R}^d -valued γ -Hölder continuous process, with $\frac{1}{3} < \gamma < \frac{1}{2}$. Let Y be a γ -Hölder continuous real-valued process such that there exists an \mathbb{R}^d -valued process Y' , so that the remainder term R^Y , given through the relation

$$Y_{s,t} = Y'_s(X_t - X_s) + R_{s,t}^Y,$$

belongs to $C^{[2\gamma]}([0, T]^2)$. In particular ω -a.s., Y is weakly controlled by X . Then, Y is stochastically controlled by X . Indeed a.s.

$$\sup_{0 \leq t \leq T} \left| \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds \right| \leq T \|R\|_{2\gamma} \|X\|_{\gamma} \varepsilon^{3\gamma-1} \rightarrow 0, \quad (3.4)$$

as $\varepsilon \rightarrow 0^+$. In particular the result follows because $\gamma > \frac{1}{3}$.

Example 3.3.5. Let X be an d -dimensional continuous semimartingale. Let $Z = (Z^1, \dots, Z^d)$ where the components Z^1, \dots, Z^d are càglàd progressively measurable processes. We set

$$Y_t = \int_0^t Z_s \cdot dX_s := \sum_{i=1}^d \int_0^t Z_s^i dX_s^i, \quad t \in [0, T].$$

Then, the real-valued process Y is stochastically controlled by X and Z is a Gubinelli stochastic derivative.

Indeed, for $s, t \in [0, T]$ such that $s \leq t$, we define R^Y implicitly by the relation

$$Y_t - Y_s = Z_s^\top (X_t - X_s) + R_{s,t}^Y.$$

We have

$$\frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds = I_1(t, \varepsilon) - I_2(t, \varepsilon),$$

with

$$\begin{aligned} I_1(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t (Y_{s+\varepsilon} - Y_s)(X_{s+\varepsilon} - X_s) ds \\ I_2(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t Z_s^\top (X_{s+\varepsilon} - X_s)(X_{s+\varepsilon} - X_s) ds. \end{aligned} \quad (3.5)$$

$I_1(t, \varepsilon)$ converges in probability to

$$[Y, X]_t = \int_0^t Z_s^\top d[X, X]_s, \quad t \in [0, T], \quad (3.6)$$

by Proposition 9 of [89]. We emphasize that the k -component of the integral on the right-hand side of (3.5) is

$$\frac{1}{\varepsilon} \sum_{j=1}^d \int_0^t Z_s^j (X_{s+\varepsilon}^j - X_s^j)(X_{s+\varepsilon}^k - X_s^k) ds.$$

Reasoning component by component, it can be also shown by Proposition 3.2.3. that $I_2(t, \varepsilon)$ also converges in probability to the right-hand side of (3.6).

Example 3.3.6. Let X be an d -dimensional process whose components are finite strong cubic variation processes and at least one component has a zero cubic variation. Let $f \in C^2(\mathbb{R}^d)$. Then $Y = f(X)$ is a stochastically controlled process by X with stochastic Gubinelli derivative $Y' = (\nabla f)^\top(X)$.

We prove the result for $d = 1$, leaving to the reader the general case. Let $\omega \in \Omega$ be fixed, but underlying. Let $0 \leq s \leq t \leq T$. Then, Taylor's formula yields

$$f(X_t) - f(X_s) = f'(X_s)(X_t - X_s) + R_{s,t}^Y,$$

where

$$R_{s,t}^Y = (X_t - X_s)^2 \int_0^1 f''(X_s + a(X_t - X_s))(1-a) da.$$

$$\left| \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds \right| \leq \sup_{\xi \in I(\omega)} |f''(\xi)| \int_0^t |X_{s+\varepsilon} - X_s|^3 \frac{ds}{\varepsilon},$$

where

$$I(\omega) = [-\min_{t \in [0, T]} X_t(\omega), \max_{t \in [0, T]} X_t(\omega)].$$

Since the integral on the right-hand side converges in probability (even ucp) to zero, R^Y fulfills (3.2).

When X is an (\mathcal{F}_t) -local martingale, Proposition 3.3.7 below shows that somehow a process Y is stochastically controlled if and only if Y is an (\mathcal{F}_t) -weak Dirichlet process.

Proposition 3.3.7. Let $X = M$ be an \mathbb{R}^d -valued continuous (\mathcal{F}_t) -local martingale. Let Y be an \mathbb{R} -valued continuous adapted process.

1. Suppose that Y is a weak Dirichlet process. Then Y is stochastically controlled by M .
2. Suppose that Y is stochastically controlled by M and the stochastic Gubinelli derivative Y' is progressively measurable and càglàd. Then Y is a weak Dirichlet process with decomposition $Y = M^Y + A^Y$ where

$$M_t^Y = \int_0^t Y'_s dM_s, \quad t \in [0, T]$$

and A^Y is an (\mathcal{F}_t) -martingale orthogonal process.

3. (Uniqueness). There is at most one stochastic Gubinelli's derivative Y' in the class of càglàd progressively measurable processes, w.r.t to the Doléans measure $\mu_{[X]}(d\omega, dt) := d[X, X]_t^{\mathbb{R}}(\omega) \otimes dP(\omega)$.

Proof. For simplicity we suppose that $d = 1$.

1. Suppose that Y is a weak Dirichlet process with canonical decomposition

$$Y = M^Y + A^Y,$$

where M^Y is the local martingale and A^Y such that $A_0^Y = 0$, is a predictable process such that $[A^Y, N] = 0$ for every continuous local martingale N . By Galtchouk-Kunita-Watanabe decomposition, see [62, 49], there exist Z and O such that

$$M_t^Y = Y_0 + \int_0^t Z_s dM_s + O_t, \quad t \in [0, T].$$

Moreover O is a continuous local martingale such that $[O, M] = 0$. Then,

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t + A_t^Y, \quad t \in [0, T].$$

We set $Y' := Z$. Hence,

$$Y_t - Y_s = Y'_s(M_t - M_s) + R_{s,t}^Y,$$

where we set

$$R_{s,t}^Y = \int_s^t (Z_r - Z_s)^\top dM_r + O_t - O_s + A_t^Y - A_s^Y. \quad (3.7)$$

Condition (3.2) follows by Remark 3.3.8 and the fact that

$$[O, M] = [A^Y, M] = 0.$$

2. Suppose now that Y is stochastically controlled by M with càglàd stochastic Gubinelli derivative Y' . Then, there is R^Y such that (3.1) and (3.2) hold. Setting $t = s + \varepsilon$, we have

$$Y_{s+\varepsilon} - Y_s = \int_s^{s+\varepsilon} Y'_r dM_r + \int_s^{s+\varepsilon} (Y'_s - Y'_r) dM_r + R_{s,s+\varepsilon}^Y. \quad (3.8)$$

where R^Y fulfills (3.2). We have

$$Y_{s+\varepsilon} - Y_s = \int_s^{s+\varepsilon} Y'_r dM_r + \tilde{R}_{s,s+\varepsilon}^Y, \quad (3.9)$$

where

$$\tilde{R}_{s,s+\varepsilon}^Y = \int_s^{s+\varepsilon} (Y'_s - Y'_r) dM_r + R_{s,s+\varepsilon}^Y,$$

fulfills (3.2) by Remark 3.3.8.

Let N be a continuous local martingale. Multiplying (3.9) by $N_{s+\varepsilon} - N_s$, integrating from 0 to t , dividing by ε , using (3.2) and by Proposition 9 of [89], going to the limit, gives

$$[Y, N]_t = \int_0^t Y'_r d[M, N]_r, \quad t \in [0, T].$$

This obviously implies that Y is a weak Dirichlet process with martingale component $M^Y = Y_0 + \int_0^\cdot Y'_r dM_r$.

3. We discuss now the uniqueness of the stochastic Gubinelli derivative. Given two decompositions of Y , taking the difference, we reduce the problem to the following. Let Y' be a càglàd process and R^Y , such that (3.2) holds for $Y = 0$, i.e. for every $0 \leq s < t \leq T$

$$0 = Y'_s(M_t - M_s) + R^Y_{s,t}, \quad (3.10)$$

satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R^Y_{s,s+\varepsilon}(M_{s+\varepsilon} - M_s) ds = 0, \quad (3.11)$$

in probability for each $t \in [0, T]$. We need to show that Y' vanishes. Setting $t = s + \varepsilon$ in (3.10), multiplying both sides by $M_{s+\varepsilon} - M_s$ integrating, for every $t \in [0, T]$, taking into account (3.11) we get

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^t Y'_s(M_{s+\varepsilon} - M_s)^2 ds = 0,$$

in probability. According to Proposition 3.2.3, the left-hand side of previous expression equals (the limit even holds ucp)

$$\int_0^\cdot Y'_s d[M, M]_s \equiv 0.$$

This concludes the uniqueness result. □

Remark 3.3.8. *It is not difficult to prove the following. Let X be an \mathbb{R}^d -valued continuous semimartingale with canonical decomposition $X = M + V$. Let Z be a process in $\mathcal{L}^2(d[M, M])$. Then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot ds \left(\int_s^{s+\varepsilon} (Z_r - Z_s) dX_r \right) (X_{s+\varepsilon} - X_s) = 0$$

ucp.

The result below partially extends Proposition 3.3.7.

Proposition 3.3.9. *Let $X = M + V$ be an \mathbb{R}^d -valued (\mathcal{F}_t) -continuous semimartingale, where M is a continuous local martingale and V is a bounded variation process vanishing at zero. Let Y be a real-valued weak Dirichlet process*

$$Y = M^Y + A^Y,$$

where M^Y is the continuous local martingale component and A^Y is an (\mathcal{F}_t) -martingale orthogonal process vanishing at zero. Then the following holds.

1. Y is stochastically controlled by X .
2. If Y' is a càglàd stochastic Gubinelli's derivative then

$$[Y, X]_t = \int_0^t Y'_s d[X, X]_s \quad (3.12)$$

Proof. By Galtchouk-Kunita-Watanabe decomposition, there exist Z and O such that

$$M_t^Y = Y_0 + \int_0^t Z_s dM_s + O_t, t \in [0, T],$$

where $Z \in \mathcal{L}^2(d[M, M])$, O is a continuous local martingale such that $[O, M] = 0$. We recall that the space $\mathcal{L}^2(d[M, M])$ was defined at (2.6). Then,

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t + A_t^Y, t \in [0, T].$$

Hence,

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, \quad (3.13)$$

where we set $Y' = Z$, $R_{s,t}^Y = \int_s^t (Z_r - Z_s) dM_r + O_{s,t} + A_{s,t}^Y$.

Now we recall

$$[O, M] = [A^Y, M] = 0. \quad (3.14)$$

Taking into account Remark 3.3.8, (3.13) and (3.14) show condition (3.11), which implies (1).

Then, by Proposition 3.2.3 we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^t (Y_{s+\varepsilon} - Y_s) \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds = \int_0^t Y'_s d[X, X]_s,$$

so that (2) is established. \square

An interesting consequence of Proposition 3.3.9 is given below.

Corollary 3.3.10. *Every continuous (\mathcal{F}_t) -weak Dirichlet process is stochastically controlled by any (\mathcal{F}_t) -continuous semimartingale.*

3.4 The second order process and rough integral via regularization

In the rough paths theory, given a driving integrator function X , in order to perform integration, one needs a supplementary ingredient, often called **second order integral** or improperly called **Lévy area**, generally denoted by \mathbb{X} . The couple $\mathbf{X} = (X, \mathbb{X})$ is often called **enhanced rough path**.

In our setup, we are given, an \mathbb{R}^d -valued continuous stochastic process X , which is our reference. We introduce a stochastic analogue of the second order integral in the form of an $\mathbb{M}^{d \times d}$ -valued random field $\mathbb{X} = (\mathbb{X}_{s,t})$, indexed by $[0, T]^2$, vanishing on the diagonal. \mathbb{X} will be called **second-order process**. For $s \leq t$, $\mathbb{X}_{s,t}$ represents formally a double (stochastic) integral $\int_s^t (X_r - X_s) \otimes dX_r$, which has to be properly defined. By symmetry, \mathbb{X} can be extended to $[0, T]^2$, setting, for $s \geq t$,

$$\mathbb{X}_{s,t} := \mathbb{X}_{t,s}.$$

The pair $\mathbf{X} = (X, \mathbb{X})$ is called **stochastically enhanced process**.

Remark 3.4.1. 1. In the classical rough paths framework, if X is a deterministic γ -Hölder continuous path with $\frac{1}{3} < \gamma < \frac{1}{2}$, \mathbb{X} is supposed to belong to $C^{[2\gamma]}([0, T]^2)$ and to fulfill the so called **Chen's relation** below.

$$-\mathbb{X}_{u,t} + \mathbb{X}_{s,t} - \mathbb{X}_{s,u} = (X_u - X_s)(X_t - X_u)^\top, u, s, t \in [0, T]. \quad (4.1)$$

2. In the literature one often introduces a decomposition of \mathbb{X} into a symmetric and an antisymmetric component, i.e.

$$\begin{aligned} \text{sym}(\mathbb{X}_{s,t})(i, j) &:= \frac{1}{2}(\mathbb{X}_{s,t}(i, j) + \mathbb{X}_{s,t}(j, i)) \\ \text{anti}(\mathbb{X}_{s,t})(i, j) &:= \frac{1}{2}(\mathbb{X}_{s,t}(i, j) - \mathbb{X}_{s,t}(j, i)), \end{aligned}$$

$1 \leq i, j \leq d$, so that

$$\mathbb{X}_{s,t} = \text{sym}(\mathbb{X}_{s,t}) + \text{anti}(\mathbb{X}_{s,t}). \quad (4.2)$$

3. We say that the pair $\mathbf{X} = (X, \mathbb{X})$ is **geometric** if

$$\text{sym}(\mathbb{X}_{st}) = \frac{1}{2}(X_t - X_s)(X_t - X_s)^\top, \quad s, t \in [0, T].$$

A typical second-order process \mathbb{X} is defined setting

$$\mathbb{X}_{s,t} := \int_s^t (X_r - X_s) \otimes d^\circ X_r, \quad (4.3)$$

provided that previous definite symmetric integral exists, for every $0 \leq s \leq t \leq T$, see (2.8).

We can also consider another \mathbb{X} , replacing the symmetric integral with the forward integral, i.e.

$$\mathbb{X}_{s,t} := \int_s^t (X_r - X_s) \otimes d^- X_r, \quad (4.4)$$

provided that previous definite forward integrals exist, exists, for every $(s, t) \in [0, T]^2, 0 \leq s \leq t \leq T, 0$ see (2.7).

Example 3.4.2. Let X be an \mathbb{R}^d -valued continuous semimartingale. Then, for $1 \leq i, j \leq d$, one often considers

$$\mathbb{X}_{s,t}^{\text{stra}}(i, j) := \left(\int_s^t (X_r - X_s) \otimes d^\circ X_r \right) (i, j) = \int_s^t (X_r^i - X_s^i) \circ dX_r^j$$

and

$$\mathbb{X}_{s,t}^{\text{ito}}(i, j) := \left(\int_s^t (X_r - X_s) \otimes d^- X_r \right) (i, j) = \int_s^t (X_r^i - X_s^i) dX_r^j,$$

where the integrals in the right-hand side are respectively intended in the Stratonovich and Itô sense.

3.5 Rough stochastic integration via regularizations

In this section we still consider our \mathbb{R}^d -valued reference process X , equipped with its second-order process \mathbb{X} . Inspired by [56], we start with the definition of the integral.

Definition 3.5.1. *A couple $(Y, Y') \in \mathcal{D}_X$ is rough stochastically integrable if*

$$\int_0^t Y_s d\mathbf{X}_s := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \left(Y_s X_{s, s+\varepsilon}^\top + Y'_s \mathbb{X}_{s, s+\varepsilon} \right) ds \quad (5.1)$$

*exists in probability for each $t \in [0, T]$. Previous integral is called **rough stochastic integral** and it is a row vector.*

We remark that if $Y' = 0$ the rough stochastic integral coincides with the forward integral $\int_0^t Y d^- X$, $t \in [0, T]$. In previous definition, we make an abuse of notation: we omit the dependence of the integral on Y' which in general affects the limit but it is usually clear from the context.

We introduce now a backward version of $\int_0^t Y d\mathbf{X}$, i.e. the **backward rough integral**

$$\int_0^t Y_s \overleftarrow{d}\mathbf{X}_s := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t \left(Y_{s+\varepsilon} X_{s, s+\varepsilon}^\top + Y'_{s+\varepsilon} \mathbb{X}_{s, s+\varepsilon} \right) ds,$$

in probability for $(Y, Y') \in \mathcal{D}_X$. Previous expression is again a row vector.

Remark 3.5.2. *Given an \mathbb{R}^n -valued process $(Y_{t \in [0, T]})$, we denote $\hat{Y}_t := Y_{T-t}$, $t \in [0, T]$.*

1. *The introduction of the backward rough integral is justified by the following observation. By an easy change of variables $s \mapsto T - s$ we easily show that, for every $t \in [0, T]$,*

$$\int_0^t Y_s \overleftarrow{d}\mathbf{X}_s = - \int_{T-t}^T \hat{Y}_s d\hat{\mathbf{X}}_s. \quad (5.2)$$

This holds of course with the convention that \hat{Y} is equipped with \hat{Y}' as Gubinelli derivative.

2. *(5.2) is reminiscent of a well-known property which states that*

$$\int_0^t Y d^+ X = - \int_{T-t}^T \hat{Y} d^- \hat{X},$$

*where the left-hand side is the **backward integral** $\int_0^t Y d^+ X$, see Proposition 1.3, see [89].*

Let us give a simple example which connects deterministic regularization approach with rough paths.

Proposition 3.5.3. *Let $\mathbf{X} = (X, \mathbb{X})$ be an a.s. enhanced rough path, where a.s. $X \in C^{[\gamma]}([0, T])$ with $\frac{1}{3} < \gamma < \frac{1}{2}$. We suppose that a.s. $\mathbb{X} \in C^{[2\gamma]}([0, T]^2)$ and it fulfills the Chen's relation. Let Y be a process such that a.s. its paths are weakly controlled in the sense of Definition 3.3.1 with Gubinelli derivative Y' . The following properties hold.*

1. *The limit*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot \left(Y_s X_{s,s+\varepsilon}^\top + Y'_s \mathbb{X}_{s,s+\varepsilon} \right) ds,$$

exists uniformly on $[0, T]$ and it coincides a.s. with the **Gubinelli integral**. In particular, (5.1) exists.

2. *The limit*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot \left(Y_{s+\varepsilon} X_{s,s+\varepsilon}^\top + Y'_{s+\varepsilon} \mathbb{X}_{s,s+\varepsilon} \right) ds, \quad (5.3)$$

exists uniformly on $[0, T]$ a.s. and it coincides a.s. with the rough Gubinelli integral as described in [56].

3. *The rough stochastic integrals $\int_0^\cdot Y_s d\mathbf{X}_s$ and $\int_0^\cdot Y_s \overleftarrow{d}\mathbf{X}_s$ exist and they are equal a.s. to the Gubinelli integral.*

Remark 3.5.4. When Y is γ' -Hölder continuous and X is γ -Hölder continuous, with $\gamma + \gamma' > 1$, Proposition 3. in Section 2.2 of [89] stated that the Young integral $\int_0^t Y d^{(y)}X$, equals both the forward and backward integrals $\int_0^t Y d^\mathbb{F}X$. Proposition 3.5.3 states an analogous theorem for the Gubinelli integral, which equals both $\int_0^\cdot Y_s d\mathbf{X}_s$ and $\int_0^\cdot Y_s \overleftarrow{d}\mathbf{X}_s$.

We introduce now the notion of multi-increments. Let $k \in \{1, 2, 3\}$. We denote by \mathcal{C}_k the space of continuous functions $g : [0, T]^k \rightarrow \mathbb{R}$, denoted by $(t_1, \dots, t_k) \mapsto g_{t_1, \dots, t_k}$ such that $g_{t_1, \dots, t_k} = 0$ whenever $t_i = t_{i+1}$ for some $1 \leq i \leq k-1$.

For $g \in \mathcal{C}_2$, we have defined $\|g\|_\alpha$ at (2.2). For $g \in \mathcal{C}_3$, we set

$$\|g\|_{\alpha, \beta} := \sup_{s, u, t \in [0, T]} \frac{|g_{tus}|}{|u-s|^\alpha |t-s|^\beta},$$

$$\|g\|_\mu := \inf \left\{ \sum_i \|g^i\|_{\rho_i, \mu - \rho_i}; g = \sum_i g^i, 0 < \rho_i < \mu \right\},$$

where the latter infimum is taken over all sequences $\{g^i \in \mathcal{C}_3\}$ such that $g = \sum_i g^i$ and for all choices of $\rho_i \in]0, \mu[$. We say that $g \in C^\mu([0, T]^3)$ if $\|g\|_\mu < \infty$.

We introduce the maps

1. $\delta_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ defined by $(\delta_1 f)_{s,t} = f(t) - f(s)$.

2. $\delta_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_3$ defined by

$$\delta_2 f_{t_1, t_2, t_3} = -f_{t_2, t_3} + f_{t_1, t_3} - f_{t_1, t_2}.$$

If $k = 1, 2$ and $f \in \mathcal{C}_k$, $\delta_k f$ is called **k -increment** of the function f .

In the proof of Proposition 3.5.3, as in [57], it is crucial to make use of the so called **Sewing Lemma**. The lemma below follows directly from Proposition 2.3 in [57].

Lemma 3.5.5. *Let $g \in \mathcal{C}_2$ such that $\delta_2 g \in C^{[\mu]}([0, T]^3)$, for some $\mu > 1$. Then, there exists a unique (up to a constant) $I \in \mathcal{C}_1$ and $\mathcal{R} \in C^{[\mu]}([0, T]^2)$ such that*

$$g = \delta_1 I + \mathcal{R}.$$

Proof (of Proposition 3.5.3).

1. We set

$$A_{s,t} = Y_s(X_t - X_s)^\top + Y'_s \mathbb{X}_{s,t}, \quad (s, t) \in [0, T]^2. \quad (5.4)$$

Then the 2-increment of A is given by

$$\begin{aligned} (\delta_2 A)_{t_1, t_2, t_3} &= Y_{t_1}(X_{t_3} - X_{t_1})^\top + Y'_{t_1} \mathbb{X}_{t_1, t_3} \\ &\quad - Y_{t_2}(X_{t_3} - X_{t_2})^\top - Y'_{t_2} \mathbb{X}_{t_2, t_3} - Y_{t_1}(X_{t_2} - X_{t_1})^\top - Y'_{t_1} \mathbb{X}_{t_1, t_2} \\ &= (Y_{t_2} - Y_{t_1})(X_{t_2} - X_{t_3})^\top + Y'_{t_1}(\mathbb{X}_{t_1, t_3} - \mathbb{X}_{t_2, t_3} - \mathbb{X}_{t_1, t_2}) \\ &\quad - (Y'_{t_2} - Y'_{t_1}) \mathbb{X}_{t_2, t_3} \\ &= \left\{ Y_{t_2} - Y_{t_1} - Y'_{t_1}(X_{t_2} - X_{t_1}) \right\} (X_{t_2} - X_{t_3})^\top + (\delta_1 Y')_{t_1 t_2} \mathbb{X}_{t_2, t_3} \\ &= R_{t_1, t_2}^Y (\delta_1 X)_{t_2, t_3}^\top + (\delta_1 Y')_{t_1, t_2} \mathbb{X}_{t_2, t_3}, \end{aligned} \quad (5.5)$$

where the third equality follows by Chen's relation. By Definition 3.3.1 we have a.s. $Y' \in C^{[\gamma]}([0, T])$, $R^Y \in C^{[2\gamma]}([0, T]^2)$ and we also have $\mathbb{X} \in C^{[2\gamma]}([0, T]^2)$. Consequently $\delta_2 A \in C^{[3\gamma]}([0, T]^3)$.

Then, setting $\mu = 3\gamma$, outside a null set, Lemma 3.5.5 applied to $g = A$, provides an unique (up to a constant) a continuous process I such that

$$A_{s, s+\varepsilon} = I_{s+\varepsilon} - I_s + \mathcal{R}_{s, s+\varepsilon},$$

where $\mathcal{R} \in C^{[3\gamma]}([0, T]^2)$. For a given $\varepsilon > 0$ and $t \in [0, T]$, we then have

$$\frac{1}{\varepsilon} \int_0^t A_{s, s+\varepsilon} ds = \frac{1}{\varepsilon} \int_0^t I_{s, s+\varepsilon} ds + \frac{1}{\varepsilon} \int_0^t \mathcal{R}_{s, s+\varepsilon} ds$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t I_{s, s+\varepsilon} ds = I_t - I_0, \quad (5.6)$$

uniformly in $[0, T]$. By using the fact that $\mathcal{R} \in C^{[3\gamma]}([0, T]^2)$, we have

$$\frac{1}{\varepsilon} \sup_{s \in [0, T]} |\mathcal{R}_{s, s+\varepsilon}| \leq \frac{\varepsilon^{3\gamma}}{\varepsilon} \|\mathcal{R}\|_{3\gamma} \rightarrow 0,$$

as $\varepsilon \downarrow 0$. This completes the proof.

2. We fix ω . The quantity (5.3) converges to I_t where I is again the (unique) function appearing in the Sewing Lemma 3.5.5. The arguments are similar to those of item 1.
3. This is a direct consequence of previous points and the fact that a.s. I also coincides with the Gubinelli integral.

□

Theorem 3.5.6. *Let $X = (X_t)_{t \in [0, T]}$ be a given continuous (\mathcal{F}_t) -semimartingale with values in \mathbb{R}^d and Y be an (\mathcal{F}_t) -weak Dirichlet process. We set $\mathbb{X} := \mathbb{X}^{\text{stra}}$, see Example 3.4.2.*

Then the rough stochastic integral of Y (with càglàd progressively measurable, stochastic Gubinelli derivative Y') with respect to $\mathbf{X} = (X, \mathbb{X})$ coincides with the Stratonovich integral i.e.

$$\int_0^\cdot Y_s d\mathbf{X}_s = \int_0^\cdot Y_s \circ dX_s. \quad (5.7)$$

Remark 3.5.7. 1. *In Proposition 3.3.9 we have shown the existence of a progressively measurable process Y' such that (Y, Y') belongs to \mathcal{D}_X .*

2. *(5.7) implies that the value of the rough stochastic integral does not depend on Y' .*

Proof (of Theorem 3.5.6).

The rough stochastic integral $\int_0^\cdot Y d\mathbf{X}_s$ defined in (5.1) exists if we prove in particular that the two limits below

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t Y_s X_{s, s+\varepsilon}^\top ds \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s, s+\varepsilon} ds, \quad (5.8)$$

exist in probability. We will even prove the ucp convergence of (5.8). Let us fix $i \in \{1, \dots, d\}$. By Proposition 6. in [89] we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^t Y_s \frac{X_{s+\varepsilon}^i - X_s^i}{\varepsilon} ds = \int_0^t Y_s d^- X_s^i = \int_0^t Y_s dX_s^i, \quad (5.9)$$

ucp, where the second integral in the equality is the usual Itô's stochastic integral.

We show now that

$$\frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s, s+\varepsilon} ds \rightarrow \frac{1}{2} [Y, X]_t, t \in [0, T], \quad (5.10)$$

holds ucp as $\varepsilon \rightarrow 0$.

Let $i \in \{1, \dots, d\}$. We write, for every $t \in [0, T]$, an element of the vector $\frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s, s+\varepsilon} ds$ as

$$\frac{1}{\varepsilon} \left(\int_0^t Y'_s \mathbb{X}_{s, s+\varepsilon} ds \right)_i = \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) \circ dX_r^i \right) ds.$$

The definition of Stratonovich integral yields

$$\begin{aligned} & \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) \circ dX_r^i \right) ds \\ &= \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds \\ &+ \frac{1}{2\varepsilon} \sum_{k=1}^d \int_0^t (Y'_s)^k [X^k - X_s^k, X^i]_{s, s+\varepsilon} ds. \end{aligned}$$

Obviously $[X^k - X_s^k, X^i] = [X^k, X^i]$. Since the covariations $[X^k, X^i]$ are bounded variation processes, item 7. of Proposition 1. in [89] shows that the second term in the right-hand side of the latter identity converges in ucp as $\varepsilon \rightarrow 0$ to

$$\frac{1}{2} \sum_{k=1}^d \int_0^t (Y'_r)^k d[X^k, X^i]_r = \frac{1}{2} \left(\int_0^t Y'_r d[X]_r \right)_i = \frac{1}{2} [Y, X^i]_t,$$

where the latter equality follows by (3.12) in Proposition 3.3.9.

We complete the proof if we show that for every $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, d\}$ the ucp limit

$$\int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (5.11)$$

holds. Let $M^i + V^i$ be the canonical decomposition of the semimartingale X^i . By usual localization arguments we can reduce to the case when

$[M^i], \|V^i\|(T), X^i, (Y')$ are bounded processes. Using the stochastic Fubini's Theorem (see Theorem 64, Chapter 6 in [78]), we can write

$$\begin{aligned} & \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds \\ &= \int_0^{t+\varepsilon} \left(\frac{1}{\varepsilon} \int_{(r-\varepsilon)^+}^{r \wedge t} (Y'_s)^k (X_r^k - X_s^k) ds \right) dX_r^i. \end{aligned}$$

For $\varepsilon > 0$, and $k \in \{1, \dots, d\}$, let us define the auxiliary process

$$\xi^\varepsilon(t) := \frac{1}{\varepsilon} \int_{(r-\varepsilon)^+}^{r \wedge t} (Y'_s)^k (X_r^k - X_s^k) ds.$$

Controlling the border terms as usual, by Problem 5.25 Chapter 1. of [61] (5.11), it remains to show that the limit in probability

$$\int_0^T |\xi^\varepsilon(r)|^2 d[X^i]_r \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{holds.} \quad (5.12)$$

Denoting by $\delta(X, \cdot)$ the continuity modulus of X on $[0, T]$,

$$\int_0^T |\xi^\varepsilon(r)|^2 d[X^i]_r \leq \delta(X, \varepsilon)^2 \sup_{s \in [0, T]} |(Y'_s)^k|^2 [X^i]_T,$$

which obviously converges a.s. to zero. This concludes the proof of (5.10).

Combining (5.9) and (5.10) we finish the proof of (5.7). \square

Through a similar but simpler proof (left to the reader) than the one of Theorem 3.5.6 we have the following.

Theorem 3.5.8. *Let $X = (X_t)_{t \in [0, T]}$ be a given continuous (\mathcal{F}_t) -semimartingale with values in \mathbb{R}^d and let Y be a.s. bounded and progressively measurable. Suppose moreover that Y has a càglàd progressively measurable*

Gubinelli derivative Y' . We set $\mathbb{X} := \mathbb{X}^{\text{ito}}$, see Example 3.4.2. Then the rough stochastic integral of Y with respect to $\mathbf{X} = (X, \mathbb{X})$ coincides with the Itô integral of Y with respect to X , i.e.

$$\int_0^\cdot Y_s d\mathbf{X}_s = \int_0^\cdot Y_s dX_s. \quad (5.13)$$

Theorems 3.5.6 and 3.5.8 somehow extend Proposition 5.1 in [47] and Corollary 5.2 in [45]. In this paper, (Y, Y') does not necessarily have Hölder continuous paths with the classical regularity in the sense of rough paths.

Bibliography

- [1] E. Alos, J. A. León, and D. Nualart. Stochastic Stratonovich calculus fbm for fractional Brownian motion with Hurst parameter less than $1/2$. *Taiwanese Journal of Mathematics*, 5(3):609–632, 2001.
- [2] E. Alòs, O. Mazet, and D. Nualart. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $1/2$. *Stochastic processes and their applications*, 86(1):121–139, 2000.
- [3] O. V. Aryasova and A. Yu. Pilipenko. On the strong uniqueness of a solution to singular stochastic differential equations. *Theory Stoch. Process.*, 17(2):1–15, 2011.
- [4] D. R. Baños, F. Cordoni, G. Di Nunno, L. Di Persio, and E. E. Røse. Stochastic systems with memory and jumps. *ArXiv e-prints*, March 2016.
- [5] E. Bandini and F. Russo. Weak Dirichlet processes with jumps. *Stochastic Process. Appl.*, 127(12):4139–4189, 2017.
- [6] E. Bandini and F. Russo. Special weak Dirichlet processes and BSDEs driven by a random measure. *Bernoulli*, 24(4A):2569–2609, 2018.
- [7] A. Barrasso and F. Russo. Martingale driven BSDEs, PDEs and other related deterministic problems. *Stochastic Processes and their Applications*, 133:193–228, 2021.
- [8] R. F. Bass and Z.-Q. Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields*, 121(3):422–446, 2001.
- [9] R. F. Bass and Z.-Q. Chen. Brownian motion with singular drift. *Ann. Probab.*, 31(2):791–817, 2003.
- [10] B. Bérard-Bergery and P. Vallois. Convergence at first and second order of some approximations of stochastic integrals. In *Séminaire de Probabilités XLIII*, volume 2006 of *Lecture Notes in Math.*, pages 241–268. Springer, Berlin, 2011.
- [11] J. Bertoin. Sur une intégrale pour les processus à α -variation bornée. *Ann. Probab.*, 17(4):1521–1535, 1989.

- [12] S. Blei and H. J. Engelbert. One-dimensional stochastic differential equations with generalized and singular drift. *Stochastic Process. Appl.*, 123(12):4337–4372, 2013.
- [13] Th. Brox. A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14(4):1206–1218, 1986.
- [14] G. Cannizzaro and K. Chouk. Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.*, 46(3):1710–1763, 05 2018.
- [15] R. A. Castrequini and F. Russo. Path dependent equations driven by Hölder processes. *Stoch. Anal. Appl.*, 37(3):480–498, 2019.
- [16] P.-E. Chaudru de Raynal and S. Menozzi. On multidimensional stable-driven stochastic differential equations with Besov drift. *Preprint Arxiv 1907.12263*, 2020.
- [17] P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 41(6):1049–1081, 2005.
- [18] A. S. Cherny. On the strong and weak solutions of stochastic differential equations governing Bessel processes. *Stochastics Stochastics Rep.*, 70(3-4):213–219, 2000.
- [19] I. Chevyrev and P. K. Friz. Canonical RDEs and general semimartingales as rough paths. *Annals of Probability*, 47(1):420–463, 2019.
- [20] F. Coquet, A. Jakubowski, J. Mémin, and L. Śłomiński. Natural decomposition of processes and weak Dirichlet processes. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 81–116. Springer, Berlin, 2006.
- [21] A. Cosso and F. Russo. Functional and Banach space stochastic calculi: path-dependent Kolmogorov equations associated with the frame of a Brownian motion. In *Stochastics of environmental and financial economics—Centre of Advanced Study, Oslo, Norway, 2014–2015*, volume 138 of *Springer Proc. Math. Stat.*, pages 27–80. Springer, Cham, 2016.
- [22] L. Coutin and A. Lejay. Semi-martingales and rough paths theory. *Electronic Journal of Probability*, 10:761–785, 2005.
- [23] R. Coviello, C. Di Girolami, and F. Russo. On stochastic calculus related to financial assets without semimartingales. *Bull. Sci. Math.*, 135(6-7):733–774, 2011.
- [24] R. Coviello and F. Russo. Nonsemimartingales: stochastic differential equations and weak Dirichlet processes. *Ann. Probab.*, 35(1):255–308, 2007.

- [25] L. Decreasefond and A. S. Üstünel. Stochastic analysis of the fractional Brownian motion. *Potential Anal.*, 10(2):177–214, 1999.
- [26] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. volume 165, pages 1–63, 2016.
- [27] D. Dereudre and S. Roelly. Path-dependent infinite-dimensional SDE with non-regular drift: An existence result. *Ann. Inst. H. Poincaré Probab. Statist.*, 53(2):641–657, 2017.
- [28] C. Di Girolami and F. Russo. Infinite dimensional stochastic calculus via regularization and applications. *Preprint HAL-INRIA, inria-00473947 version 1*, (Unpublished), 2010.
- [29] C. Di Girolami and F. Russo. Clark-Ocone type formula for non-semimartingales with finite quadratic variation. *Comptes Rendus Mathématique*, 349(3-4):209 – 214, 2011.
- [30] C. Di Girolami and F. Russo. Generalized covariation and extended Fukushima decompositions for Banach space valued processes. application to windows of Dirichlet processes. *Infinite Dimensional Analysis, Quantum Probability and Related Topics (IDA-QP)*., 15(2), 6 2012.
- [31] C. Di Girolami and F. Russo. Generalized covariation for Banach space valued processes and Itô formula. *Osaka Journal of Mathematics*, 51(3), 6 2014.
- [32] C. Di Girolami and F. Russo. About classical solutions of the path-dependent heat equation. *Random Oper. Stoch. Equ.*, 28(1):35–62, 2020.
- [33] J. Dubédat. Excursion decompositions for SLE and Watts crossing formula. *Probab. Theory Relat. Fields*, 134(3):453–488, 2006.
- [34] H.-J. Engelbert and W. Schmidt. On one-dimensional stochastic differential equations with generalized drift. In *Stochastic differential systems (Marseille-Luminy, 1984)*, volume 69 of *Lecture Notes in Control and Inform. Sci.*, pages 143–155. Springer, Berlin, 1985.
- [35] H.-J. Engelbert and J. Wolf. Strong Markov local Dirichlet processes and stochastic differential equations. *Teor. Veroyatnost. i Primenen.*, 43(2):331–348, 1998.
- [36] M. Errami and F. Russo. Covariation de convolution de martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(5):601–606, 1998.
- [37] M. Errami and F. Russo. n -covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.
- [38] F. Flandoli, M. Gubinelli, and F. Russo. On the regularity of stochastic currents, fractional Brownian motion and applications to a turbulence model. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(2):545–576, 2009.

- [39] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional SDEs with distributional coefficients. *Trans. Am. Math. Soc.*, 369:1665–1688, 2017.
- [40] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.
- [41] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô’s formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.
- [42] H. Föllmer. Dirichlet processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 476–478. Springer, Berlin, 1981.
- [43] N. Frikha and L. Li. Weak uniqueness and density estimates for SDEs with coefficients depending on some path-functionals. *ArXiv preprint: 1707.01295*, 2019.
- [44] P. Friz and P. Zorin-Kranich. Rough semimartingales and p -variation estimates for martingale transforms. *ArXiv preprint arXiv:2008.08897*, 2020.
- [45] P. K. Friz and M. Hairer. *A course on rough paths*. Universitext. Springer, Cham, 2014. With an introduction to regularity structures.
- [46] P. K. Friz, A. Hocquet, and K. Lê. Rough stochastic differential equations. *ArXiv preprint arXiv:2106.10340*, 2021.
- [47] P. K. Friz and N. B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. Theory and applications.
- [48] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes.*, volume 19. Berlin: Walter de Gruyter, 2011.
- [49] L. I. Gal’čuk. A representation of certain martingales. *Teor. Veroyatnost. i Primenen.*, 21(3):613–620, 1976.
- [50] A.O. Gomes, A. Ohashi, F. Russo, and A. Teixeira. Rough paths and regularization. *Journal of Stochastic Analysis (JOSA).*, 2 (4):1–21, 12 2021.
- [51] F. Gozzi and F. Russo. Verification theorems for stochastic optimal control problems via a time dependent Fukushima-Dirichlet decomposition. *Stochastic Process. Appl.*, 116(11):1530–1562, 2006.
- [52] F. Gozzi and F. Russo. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Process. Appl.*, 116(11):1563–1583, 2006.

- [53] M. Gradinaru and I. Nourdin. Approximation at first and second order of m -order integrals of the fractional Brownian motion and of certain semimartingales. *Electron. J. Probab.*, 8:no. 18, 26 pp. (electronic), 2003.
- [54] M. Gradinaru, Ivan Nourdin, F. Russo, and P. Vallois. m -order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(4):781–806, 2005.
- [55] M. Gradinaru, F. Russo, and P. Vallois. Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$. *Ann. Probab.*, 31(4):1772–1820, 2003.
- [56] M. Gubinelli. Controlling rough paths. *Journal of Functional Analysis*, 216(1):86–140, 2004.
- [57] M. Gubinelli and S. Tindel. Rough evolution equations. *The Annals of Probability*, 38(1):1–75, 2010.
- [58] J. M. Harrison and L. A. Shepp. On skew brownian motion. *Ann. Probab.*, 9, 1981.
- [59] Y. Hu, Z. Shi, and M. Yor. Rates of convergences of diffusions with drifted Brownian potentials. *Trans. Amer. Math. Soc.*, 351(10):3915–3934, 1999.
- [60] M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical Methods for Financial Markets*. Springer Finance. Springer London, 2009.
- [61] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [62] H. Kunita and S. Watanabe. On square integrable martingales. *Nagoya Math. J.*, 30:209–245, 1967.
- [63] F. G. Lawler, O. Schramm, and W. Werner. Conformal restriction: The chordal case. *J. Am. Math. Soc.*, 16(4):917–955, 2003.
- [64] G. F. Lawler. *Conformally invariant processes in the plane*, volume 114. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005., 2005.
- [65] K Lê. A stochastic sewing lemma and applications. *ArXiv preprint: 1810.10500*, 2019.
- [66] Khoa Lê. A stochastic sewing lemma and applications. *Electronic Journal of Probability*, 25, 2020.
- [67] Y. Liu, Z. Selk, and S. Tindel. Convergence of trapezoid rule to rough integrals. *ArXiv preprint:2005.06500*, 2020.
- [68] T. Lyons and Zh. Qian. *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002. Oxford Science Publications.

- [69] T. J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.
- [70] R. Mansuy and M. Yor. *Aspects of Brownian motion*. Universitext. Springer-Verlag, Berlin, 2008.
- [71] S-E. A. Mohammed. *Stochastic functional differential equations*, volume 99. Pitman Advanced Publishing Program, 1984.
- [72] A. Ohashi and F. Russo. Rough paths, Skorohod integrals driven by covariance singular Gaussian processes. *In preparation*, 2022.
- [73] A. Ohashi, F. Russo, and A. Teixeira. About a semilinear PDE driven by a Bessel generator. *In preparation*.
- [74] A. Ohashi, F. Russo, and A. Teixeira. On path-dependent SDEs involving distributional drifts. 2020. *Modern Stochastics: Theory and Applications*, to appear. Preprint HAL-02465590.
- [75] A. Ohashi, F. Russo, and A. Teixeira. SDEs for Bessel processes in low dimension and path-dependent extensions. 2020. Preprint.
- [76] E. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic pdes of second order. In *in Stochastic Analysis and Related Topics VI: The Geilo Workshop*, pages 79–127, 1996.
- [77] N. I. Portenko. *Generalized diffusion processes*, volume 83 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990. Translated from the Russian by H. H. McFaden.
- [78] Ph. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.
- [79] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, 1999.
- [80] L.C.G Rogers and D. Williams. *Diffusions, Markov processes, and Martingales*. Cambridge University Press, 2000.
- [81] F. Russo and G. Trutnau. About a construction and some analysis of time inhomogeneous diffusions on monotonely moving domains. *J. Funct. Anal.*, 221(1):37–82, 2005.
- [82] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.*, 35(6):2213–2262, 2007.
- [83] F. Russo and P. Vallois. Intégrales progressive, rétrograde et symétrique de processus non adaptés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):615–618, 1991.

- [84] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993.
- [85] F. Russo and P. Vallois. Noncausal stochastic integration for l ad l ag processes. In *Stochastic analysis and related topics (Oslo, 1992)*, volume 8 of *Stochastics Monogr.*, pages 227–263. Gordon and Breach, Montreux, 1993.
- [86] F. Russo and P. Vallois. The generalized covariation process and It  formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.
- [87] F. Russo and P. Vallois. It  formula for C^1 -functions of semimartingales. *Probab. Theory Related Fields*, 104(1):27–41, 1996.
- [88] F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.
- [89] F. Russo and P. Vallois. Elements of stochastic calculus via regularization. In *S minaire de Probabilit s XL*, pages 147–185. Springer, 2007.
- [90] F. Russo and F. Viens. Gaussian and non-Gaussian processes of zero power variation. *ESAIM Probab. Stat.*, 19:414–439, 2015.
- [91] M. Scheutzow. Stochastic delay equations. *Lecture Notes, CIMPA School*, 2018.
- [92] P. Seignourel. Discrete schemes for processes in random media. *Probab. Theory Related Fields*, 118(3):293–322, 2000.
- [93] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.
- [94] L. C. Young. An inequality of H lder type, connected with Stieltjes integration. *Acta Math.*, 67:251–282, 1936.
- [95] L. Zambotti. *Random obstacle problems*, volume 2181 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. Lecture notes from the 45th Probability Summer School held in Saint-Flour, 2015.
- [96] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.

Title. Stochastic Analysis of non-Markovian irregular phenomena.

Keywords. Stochastic analysis; path-dependent models; distributional drift; rough paths; Bessel processes.

Abstract. This thesis focuses on some particular stochastic analysis aspects of non-Markovian irregular phenomena. It formulates existence and uniqueness for some martingale problems involving two types of irregular drifts perturbed by path-dependent functionals: the first one is related to the case which is the derivative of continuous function and it models irregular path-dependent media; the second one concerns the case when the drift is of Bessel type in low dimension. Finally the thesis also focuses on rough paths techniques and its relation with the stochastic calculus via regularization.

Titre. Analyse stochastiques de phénomènes irréguliers non-markoviens.

Mots clés. Analyse stochastique; dépendance de la trajectoire; dérive distributionnelle; chemin rugueux; processus de Bessel.

Résumé. Cette thèse se concentre sur certains aspects d'analyse stochastique de modèles non-markoviens irréguliers. On formule existence et unicité pour certains problèmes de martingales impliquant deux types de dérive irrégulière perturbée par des fonctionnelles dépendant de la trajectoire. Dans le premier cas on considère le cas où la dérive est la dérivée d'une fonction continue: le modèle correspondant est celui de milieux aléatoires irréguliers dépendant de la trajectoire. Le second concerne le cas où la dérive est celui d'un processus de Bessel en basse dimension: dans ce cas il est bien connu qu'en général les processus ne sont pas des semimartingales. Enfin la thèse explore également des relations et des analogies entre la théorie des chemins rugueux et le calcul stochastique via régularisation.