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Optimization of the asset portfolio of life insurance companies under capital guarantee constraints

Runsheng Gu

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L'UNIVERSITÉ DE ANGERS
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Par

Runsheng GU

**Optimization of the asset portfolio of life insurance companies
under capital guarantee constraints**

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To my dearest motherland and beloved family

Runsheng
December 2021

"All models are wrong, but some are useful."

George E. P. Box (1978)

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GENERAL INTRODUCTION

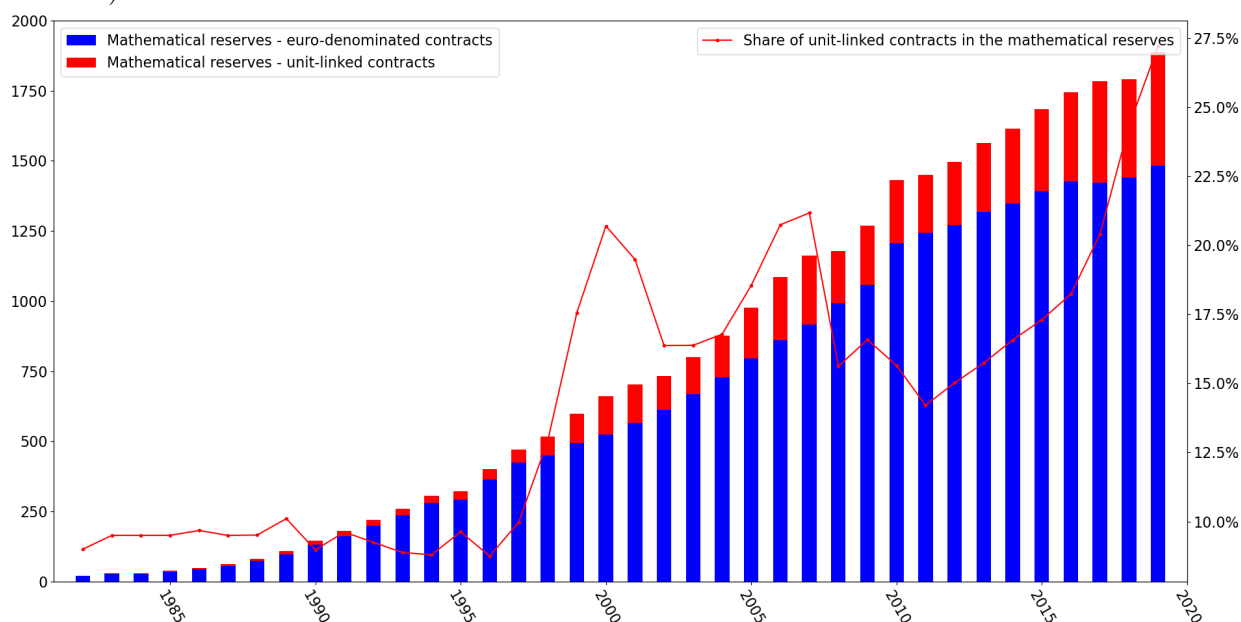
Life insurers are important financial intermediaries on account of their capacity to reallocate risks in the economy and their often long-term investment horizons. In most European countries, life insurance benefits from its characteristics in terms of the capital guarantee, tax incentives, long-term investment horizon, diversification effect, and investment return. In the good years, insurers build up surpluses by offering lower returns to savers than those of their portfolio, enabling them to draw on this reserve during the bad years to offer higher returns. Thereby reducing the shortfall risk of the life insurers while providing for higher guaranteed interest rates.

About 40% of their net worth is invested in life insurance and pension funds by European households in an average year from 2016 to 2018 [ECB, 2019]. According to the Bank of France¹, the value of euro-denominated life insurance contracts held by French households accounts for approximately 32% of their financial wealth by the end of September 2020. At the same time, the corresponding proportion of unit-linked contracts is about 7%.

As shown by the progress in outstandings (see Figure 1), the French have invested massively in euro-denominated life insurance contracts over the past forty years, from 20 billion euros in 1982 to 1500 billion euros in 2019. Nevertheless, a slowdown in euro-denominated contracts has also been witnessed in recent years. The development of unit-linked contracts has been accelerated since the end of the 20th century, their share in life insurance contracts exceeded 27% in 2019.

1. Banque de France, <https://www.banque-france.fr/presentation-trimestrielle-de-lepargne-des-menages>

Figure 1 – Mathematical reserves by type of product in France, 1982 - 2019 (in billions of euros)



Source: ACPR

However, while the overall prospect for life insurers during the period seems to be very positive, some challenges have been forming in both investment activity and the life insurance business. The declining rate of return of their investment portfolios² and the underwriting profitability driven by low premium growth have created concerns for all life insurance companies in the past several years.

The current loose monetary policy conducted by the European Central Bank results in extraordinarily low-interest rates. Life insurers typically allocate a large proportion of sovereign bonds and corporate bonds in their portfolio, therefore a generalized decrease in interest rates directly affects the rate of return of these portfolios. Lower-for-longer yields prompt life insurers to seek riskier and more illiquid investments to earn their targeted return. This increased risk-taking may lead to a further buildup of vulnerabilities among them [IMF, 2019]. Low-interest rates are becoming a threat to the stability of the life insurance industry, especially in countries where products with relatively high guaranteed returns sold in the past still represent a prominent share of the liabilities [Berdin and Gründl, 2015].

2. According to the analysis and summaries in [ACPR, 2020b], the average investment rate of return of euro-denominated funds in the past five years is less than 3%, while the return offered to the insured has also declined below 1.5% in 2019.

According to the European Insurance and Occupational Pension Authority (EIOPA)'s recent Financial Stability Report ([[EIOPA, 2019a](#)]), the current low yield environment represents the prominent risk for life insurers, which are currently struggling to pay guaranteed rates of return and to maintain strong profitability in the long term. In a survey conducted by the EIOPA to assess the risks and vulnerabilities of the European insurance and pension sectors, the obtained results suggest that in addition to the prolonged period of ultra-low rates, credit risk, equity risk, and property risk are also relevant for the life insurance sector [[EIOPA, 2020b](#)]. Moreover, the prolonged period of low yields raises negative prospects on the profitability of life insurers' investment portfolios because of reinvestment risk.

Since the return on life insurer's assets tends to adjust quicker to low interest rates than the growth rate of the liabilities, which includes the return promises to policyholders, a low interest rate environment poses a serious threat to life insurers' solvability [[Niedrig, 2015](#)]. Besides, policyholders used to keep their life insurance contracts for very long periods of time to accumulate wealth and benefit from the fiscal advantage after holding eight years in France. The duration of the liability side might have become higher than the duration of the asset side. Under the Solvency II regulation, the current level of interest rates increases the present value of current liabilities more than the present value of assets. This, in turn, reduces the market value of equity capital with adverse effects on the solvency position of the life insurance company.

In the context of the gradual decrease in the return on euro-denominated contracts related to the decline in long-term interest rates, the excellent performance of the developed stock markets allowed unit-linked products (which accounts for 27% of outstanding in 2019 as shown in [Figure 1](#)) to be shown as an attractive alternative that enabled investors to benefit at least partly from the rise in stock market prices while remaining within the attractive taxation framework. Net inflows into unit-linked life insurance contracts have notably increased during the past two decades. At the same time, net inflows into euro-denominated life insurance contracts have declined not only because of the diminishing rate of returns but also due to the age pyramid changes in France. On the one hand, outflows comprised of claims paid, benefits and redemptions remain increasing significantly. On the other hand, inflows mainly on premiums show a declining trend in past several years.

Life insurance companies have tried to drive the inflows into unit-linked contracts in order to limit the inflows into euro-denominated contracts and reduce their buying of

bonds which offered a very small return. In 2020, there was a negative net inflow of life insurance contracts in France, including a net inflow of 23.9 billion euros in unit-linked contracts and a net outflow of 30.9 billion euros from euro-denominated contracts [Frayssé et al., 2021].

To continue offering favorable returns to their clients while guaranteeing their invested capital, the life insurance companies decided to modify the structure of their assets gradually in line with the emerging economic model of life insurance in euros. However, by turning to more risky assets, which potentially carry a positive risk premium, they are less likely to be able to guarantee the capital (even if they maintain a liquidity "cushion" as a cash reserve). All these factors now pose a significant risk on the life insurance model, in its euro version.

Inspired by the general framework of the classical ruin theory and the stochastic properties of the risk model that considers two major processes comprised of the basic insurance business and the investment activity, we apply such methodologies to model multiple risk exposures of life insurers and search for optimal investment strategies and risk management. The theoretical foundation of ruin theory, known as the classical risk process model (or the Cramér–Lundberg model), was introduced in [Lundberg, 1903]. The initial goal of early researchers of the field was to determine the probability for the surplus to become negative. In the second half of the 20th century, this theory was far developed in different directions, with one of the endeavors to incorporate risky investments. Inspired by ideas from mathematical finance, the model was suggested where capital is allowed to be invested in risky assets (e.g., [Harrison, 1977], [Delbaen and Haezendonck, 1987], [Paulsen, 1993]). Then, the integrated risk process comprises two basic processes, i.e., a basic risk process and a return on investment generating process. It is assumed that the two processes are independent. The basic risk process describes an insurance company that experiences two opposing cash flows: inflows of cash premiums and outflows of claims. In the return on investment generating process, the asset portfolio is capable of allocating risk-free and risky investments.

The objective of this thesis is to continue on the efforts undertaken with recent development in ruin theory and risk models in order to better understand and manage the portfolio optimization problems related to the asset management of life insurers. Many attempts have been made to life insurance portfolio optimization. However, much more remains to be done, especially in such a low-interest-rate environment, because of the variety and complexity of existing and potential risks. The challenges, in this thesis, are

to model the income of a life insurance company and to provide closed-form formulas for their first two moments and then to generate optimal investment strategies for life insurers in such a lower-for-longer interest rate environment. This thesis is organized as follows.

The first chapter is an introductory of euro-denominated life insurance. It presents a literature survey on the consequences of low interest rates and a background analysis of the economic environment of French life insurance companies. It highlights and compares the situations of the life insurance industry in France and Europe. The prudential framework, Solvency II, is discussed and analyzed. The chapter examines the declining returns on euro-denominated life insurance contracts and the paradigm shift in the activity of life insurance companies facing significant potential outflows. It provides practical solutions to the problems faced by life insurance companies.

The second chapter is dedicated to the mathematical modeling of the income of a life insurance company. It provides indispensable tools for grasping expositions of results of the next chapter. It shows the elements of stochastic calculus that are needed in the calculation of our mathematical modeling. It formulates the explicit form of the first two moments of the income of a life insurance company with three case studies where the basic risk process and the return on investment generating process are modeled by Brownian Motion with drift, the sum of a Brownian Motion and a compound Poisson process and Lévy Process, respectively. By calculating the first-order partial derivatives, we obtain the relationships between each variable and the first two moments of the income of a life insurance company. This chapter closes with a discussion on the optimal investment strategy.

The third chapter is devoted to find the optimal asset allocation of the portfolio of a life insurance company. A dataset consisting of one risk-free asset and four categories of risky assets is displayed. We construct the optimal investment strategy based on the obtained formulae in *Chapter 2*. The optimization problem is solved numerically with real data by incorporating certain practical constraints, such as the probability of insolvency, reserves, and the maximum investment ratios by asset class. We investigate how the optimal asset allocation depends on a series of parameters through sensitivity analysis. Some implications for both life insurers and regulations are proposed with numerical results.

The last chapter focuses on the performance evaluation of our proposed asset allocation strategy in comparison with the buy-and-hold (B&H) strategies invested in one asset,

as well as the customized security-based benchmarks that reflected our proposed strategy in the long term. The backtesting design of the empirical analysis is presented, including portfolio rebalancing disciplines, performance appraisal measures, transaction costs, and the backtesting procedures. The effectiveness of the strategies is examined comprehensively. We further analyze the robustness of our asset allocation strategies under different scenarios and compare the performance with real-world data.

Finally, this dissertation ends with a general conclusion, the main limitations of the study, and future research perspectives.

ECONOMIC ANALYSIS OF EURO-DENOMINATED LIFE INSURANCE IN FRANCE

1.1 Introduction

The French life insurance market is one of the largest in Europe. It represents more than EUR 2.1 trillion in technical provisions¹ at the end of 2019 [ACPR, 2020a]. In the life insurance business, euro-denominated funds have been the cornerstone of any life insurance contracts in France since the mid-1980s, and they represent around 80% of the outstanding managed assets in life insurance. The capital guarantee and tax advantages provided by the life insurance product in euro-denominated funds are massively sought by households [Cazenave-lacrouts et al., 2018]. Life insurers used to derive most of their financial income from the investment of premiums received from their policyholders in fixed-rate bonds. Besides these attractive features, the success of euro-denominated funds was due to the relative high-interest rates mainly during 1980-2010, with highly weighted default-risk-free bonds in the portfolios of life insurers. For example, at the end of the 1990s, the public bonds used to represent about one-third of the overall investment of Life

1. Technical provisions are intended to represent the current amount the insurance company would have to pay for an immediate transfer of its obligations to a third party. Technical provisions comprise two components: the best estimate of the liabilities and a risk margin.

insurance companies.

However, the economic model of euro-denominated life insurance has recently experienced a real upheaval because of

- Very low, sometimes negative, interest rates: For example, the French ten-year bond yield decreased to 1.00 % by 2014 and further dropped to -0.40 % by 2019.
- A rise in outflows and a decrease in inflows²: not only because of the declining returns but also due to the age pyramid changes.

To continue offering favorable returns to their clients while guaranteeing their invested capital, the life insurance companies first decided to modify gradually the structure of their assets in line with the emerging economic model of life insurance in euros. Secondly, they have developed unit-linked contracts and euro-growth contracts in an attempt to limit the flows into euro-denominated life insurance contracts. Some companies have included new rules to limit the percentage of inflows in euro-denominated funds in recent years. However, in the first case, by turning to more risky assets, which potentially carry a positive risk premium, they are less likely to be able to guarantee the capital (even if they maintain a liquidity "cushion" as a cash reserve). All these factors now pose a significant risk on the life insurance model, in its euro version.

The new prudential framework Solvency II came into force on January 1, 2016. It posted a solvency capital requirement (SCR) and a minimum capital requirement (MCR) under insurers, providing ample protection against the risks they are exposed to. The continued low-interest-rate environment weighted on life insurers, leading to a reduction in their own funds and an increase in their capital requirements. The positive point is that the French insurance sector has displayed high solvency levels [ACPR, 2020a], which indicates high financial stability, allowing French insurers to face an unfavorably low-interest-rate environment for more years.

The remainder of this chapter proceeds as follows. Section 1.2 provides a synthetic review of the related literature in life insurance and the background analysis of the economic environment for life insurers. Section 1.3 outlines the situations of the life insurance industry in France and Europe. Section 1.4 presents a comprehensive analysis of the declining returns on euro-denominated funds and feasible solutions to life insurers. The final section concludes.

2. The flow of euro-denominated funds is shown and analyzed in Section 1.3.2

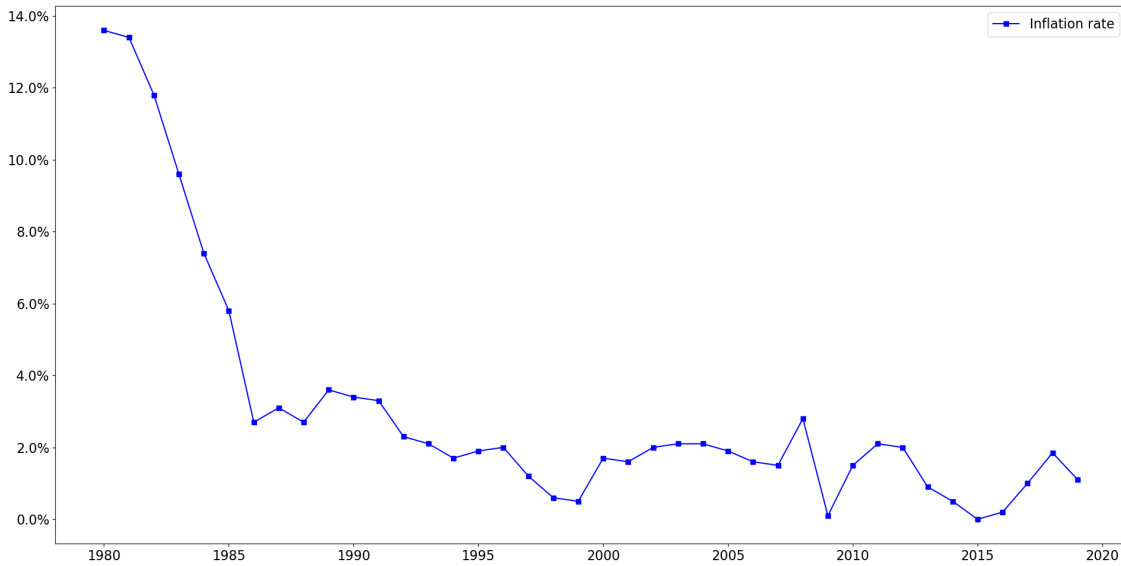
1.2 Literature review and background analysis

1.2.1 Literature review

According to the European Insurance and Occupational Pension Authority (EIOPA)'s recent Financial Stability Report ([EIOPA, 2019a]), the risk of a prolonged low yield environment remains the key challenge for European life insurers, putting pressure on both solvency positions and long-term profitability. The economic model of life insurance has been changed in a very low-interest-rate environment. This phenomenon leads to a reduction in the rate of return on euro-denominated funds since the majority is invested in bonds. If the 10-year French bond remains negative for too long, it will force insurers to invest both collected savings and the proceeds from the sale of maturing bonds into well-rated lower rates bonds. What is more, we notice that the emergence of numerous old contracts will become a completely free reinvestment in other financial products thanks to the tax exemption on the benefits after the holding period of 8 years.

Since the peak of long-term interest rates in Europe in the 1980s, the impact of low-interest-rate on life insurance companies has been studied for decades. Long-term interest rates serve as the valuation basis to determine premiums, policy reserves, guaranteed rates of return, and profit-sharing [Holsboer, 2000]. As capital market rates approach the valuation interest rate, life insurers have a problem: even if their existing portfolios are invested in assets that yield above the valuation rate, they immediately lock in a loss with cash flow from new business reinvestment. [Boubel and Séjourné, 2001] deal with the development of European life insurance markets through diversification of life insurance products and delivery networks. Since the mid-1980s, the decline in rates of return on traditional contracts denominated in the national currency and invested in the bond markets caused by the decline in long-term interest rates was noticeable in countries with a finishing inflationary environment (e.g., France and many countries in southern Europe). In France, the inflation rate was still superior to 10% in the early 1980s (13,4% in 1981), while in 1986, it was not more than 2.7% as shown in Figure 1.1. This led insurers to switch to other assets. This fall in long-term nominal rates reflects a steady decline over more than two decades in the long-term risk-free real interest rate, rather than a fall in expected inflation since the 2000s, which has remained broadly stable in [Bean et al., 2015] until recently.

Figure 1.1 – Average annual inflation rate in France since 1980.



Source: France-Inflation.com

[Berdin and Gründl, 2015] showed that a prolonged period of low-interest rate would markedly affect the solvency situation of life insurers, leading to a relatively high cumulative probability of default, especially for less capitalized companies. This has become even more true since their publication. In a study of the secular determinants of the world’s long-term real interest rates, [Rachel and Smith, 2015] attribute about two-thirds of the decline in real-world rates since the 1980s to secular factors that determine the desired saving and investment rates. [Sobrun and Turner, 2016] argue that recent estimates of unobserved concepts, such as the theoretical policy rate, the natural rate, and the long-term rate premium, suggest that the “new normal” world interest rate is lower than before.

Numerical analysis in [Kling et al., 2007] suggests that allowing to accumulate resources during years of high rate of returns in order to distribute them in the case of poor performance in bad years substantially reduces the shortfall risk of the life insurers while allowing for higher guaranteed interest rates. The sensitivity analysis in [Schmeiser and Wagner, 2015] shows that if the risk-free interest rate approaches the interest rate guarantee (which is currently the case in many European countries), the insurer’s position follows with equity capital drifting to zero. The policyholder guarantees offered become worthless, and that the insurer faces residual costs due to excessive equity capital, which may not be reduced at short notice. [Gründl et al., 2016] investigate the extent to which changes in macroeconomic conditions, market developments, and insurance regulation

may affect the role of insurers in long-term investment financing. They conclude that regulation should neither unduly favor nor hinder long-term investment as such but place a priority on incentivizing prudent asset-and-liability management with mechanisms that allow for a “true and fair view” of insurers’ risk exposures.

Several studies examine the relationship between interest rates and surrenders³ and explore reasons for life insurance surrender (e.g., [Dar and Dodds, 1989], [Kuo et al., 2003], [Haefeli and Ruprecht, 2012], [Gemmo and Götz, 2016], and [Nolte and Schneider, 2017]). [Gollier, 2015] pointed out the possibility of the insurance crisis: should the interest rates start to rise in the euro area, in particular rapidly, insurers would end up with a considerable stock of bonds showing an unrealized loss. Policyholders would be drawn to other products on the market that would be more attractive than today. In [Kubitza et al., 2020], their empirical results show that policyholders’ surrender activity is more sensitive toward interest rate changes when policies are relatively young and have low guaranteed returns. An increase in interest rates leads to withdrawals of life insurance policies. Moreover, the long-standing policyholders should also potentially exercise their surrender option when the tax advantage is over after 8 years. Many life insurance products around the world involve guaranteed rates joined with a profit-sharing policy. France is characterised by relatively short-term, low guaranteed rates in comparison to other European countries [Borel-Mathurin et al., 2018]. These characteristics are extremely country-dependent as they are the result of a combination of legal obligations and competition. The legal, fiscal, accounting and regulatory environments do vary from one country to another: even though the entry into force of the Solvency II is supposed to bring more standardisation, large differences still remain in Europe.

1.2.2 Background analysis

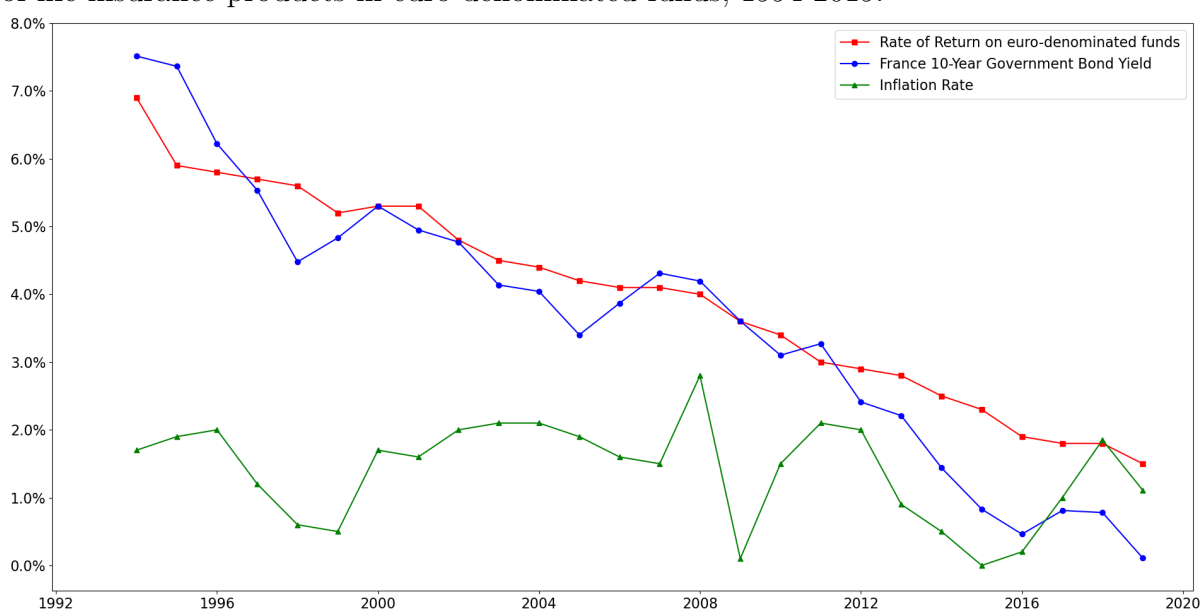
As mentioned in Section 1.1, there have been two general trends in French life insurance in the last two decades: the declining rate of return on euro-denominated funds under sustained low-interest-rate and the net inflow reduction of euro-denominated life insurance contracts, respectively. We present a background analysis on the two general trends with a data set built from reports collected by the French insurance and banking supervisor, namely the Autorité de Contrôle Prudentiel et de Résolution (ACPR), and the French national institute of statistics and economic studies, namely the Institut national de la

3. A surrender is a full cancellation of a life insurance policy.

statistique et des études économiques (Insee).

As shown in Figure 1.2⁴, since the 1990s, there has been a general downward trend in the real return of euro-denominated funds from 7.00% to 5.00% in 2000 and 1.46% in 2019, which means only 1.20% net of social contributions⁵. However, as life insurers invest the majority of insurance savings in long-term government bonds, such a decrease should not come as a surprise in a low inflation environment and an always increasing demand for low-risk assets in Europe.

Figure 1.2 – France 10-year government bonds yield, inflation rate and the performance of life insurance products in euro-denominated funds, 1994-2019.



Source: Bloomberg, ACPR, Insee

Indeed, the European central bank’s monetary policy decisions addressing the consequences of the 2007-2008 financial crises (both quantitative easing and the cut of the reference rates) reinforced a drop in the 10-year bond yields too. This trend had already been observed after introducing the single currency, which stabilized the nominal interest rates at low levels across the Eurozone. As the inflation rate has been stable over the last two decades, around the ECB target of 2.00%, the convergence in nominal interest

4. France 10-year Bond Yield is estimated by the average of France 10-year Monthly Bond Yields.

5. The interests generated by single life insurance contracts are subject to social security contributions every year. The rate of deductions applicable is that in force on the date of acquisition of the interest. That was 17.20% since the beginning of 2019. They are directly deducted by the insurer who transfers them to the tax authorities when the interest is entered into the contract.

rates engendered a drop in real ones [Franks et al., 2018]. Under these circumstances, the return of euro-denominated funds also plunged whenever new assets had to be added into the portfolio for replacing the maturing ones or placing the net inflows. In 2019, 10-year French government bonds yielded between the range of -0.40% to 0.60%. Moreover, when inflation is considered, the real return rates on euro-denominated funds decrease more rapidly in the past three years. If this level of bond yields persist several years more, the returns on euro-denominated funds will approach zero.

In our contention, a good benchmark to use as “secure savings” for making an informed investment choice is the Passbook A (*Livret A* in French), which is the most popular financial product in France. This secure and liquid savings account has a regulated ceiling deposit amount to 22,500 euros. It bears a guaranteed interest rate, which has been decreased from 0.75% in 2016 to 0.50% in 2019 and is tax-exempt. Such features reveal the fierce competition that the life insurers must now face when considering the annual return. There is no limit for the amount invested in the life insurance products, and as far as life insurance has tax advantages under the French inheritance law, the French citizens still perceive euro-denominated funds as good investments. However, the spread between the returns of those two products is diminishing and, according to the trend, is going to approach zero in the foreseeable future. Moreover, the reduction of the return on euro-denominated funds creates a real risk of outflows from the companies since, at the same time, most contracts have become fiscally free of reinvestment. Having massive amounts free of reinvestment (free of withdrawals) after the 8-year holding period of tax-advantage, the management of the life insurance company becomes closer to the one of banks [Bobtcheff et al., 2016]. They face a liquidity problem and are exposed to information asymmetry: they do not know if and when the policyholders will take their money back. It is not only a problem of life expectancy any more.

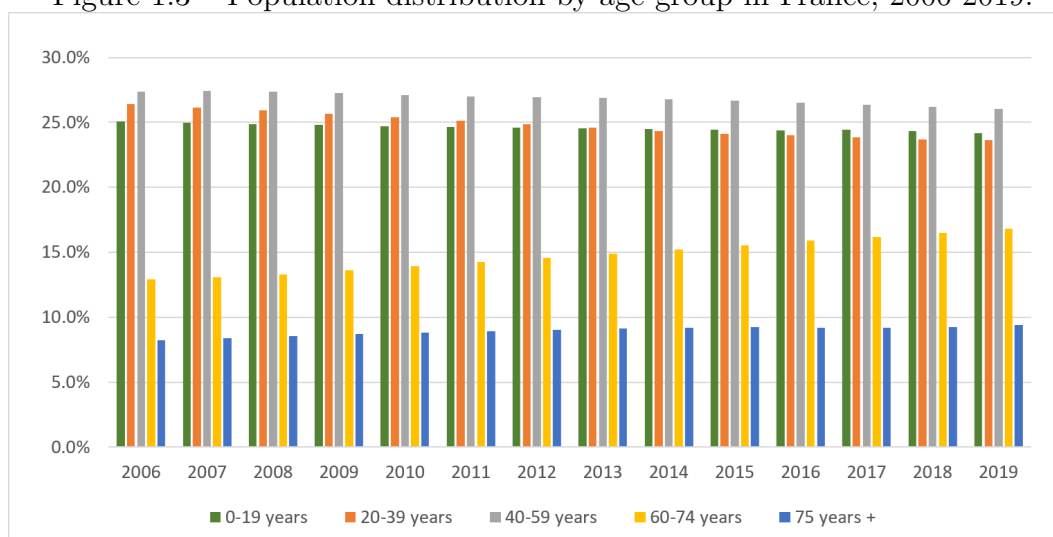
One of the motivations for holding life insurance is to create an additive pension to the pay-as-you-go systems. The recent creation of a new retirement savings product⁶ dedicated to individual or collective retirement could create incentives for people who have long-term investment prospects. A survey of French savings and retirement by Le

6. In French, le Plan d'épargne retraite (PER). It is a new retirement savings product in the application of the retirement savings reform by the French law of May 22, 2019, relating to the growth and transformation of companies (PACTE) and its implementing texts: the order of July 24, 2019, reforming supplementary retirement savings and the decree of July 30, 2019. It has been available since October 1, 2019. The aim is to enhance the attractiveness of long-term savings and direct them more towards corporate financing. <https://www.economie.gouv.fr/>

Cercle des Épargnants⁷ revealed several behavioral changes. Life insurance and Passbook A remain the most popular savings choices, while retirement savings products attract growing interest. Considering retirement savings, life insurance as the best investment for retirement has declined for several years. There is more and more competition from the retirement savings plan. In 2021, the PER is positioned as the leading option for retirement savings from respondents in the survey.

The second trend observed from euro-denominated life insurance is a rise in outflows and a decrease in inflows: not only because of the decline in the rate of returns but also due to changes in the age pyramid. Like in other advanced industrial countries, in France, demographic aging has become a widely debated topic [Béland and Viriot Durandal, 2013].

Figure 1.3 – Population distribution by age group in France, 2006-2019.



Source: Insee

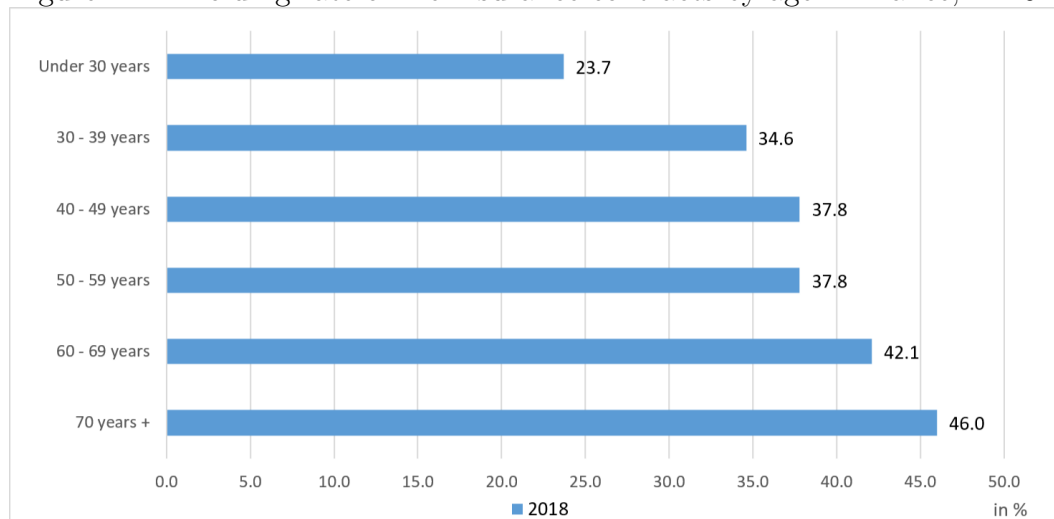
[Boutillier et al., 2001] conclude that the share of 40-59 years old in the total population positively influences the outstanding amount of life insurance contracts. Marked by the baby boom period, the proportion of the population between the ages of 40-59 has increased until 2006. A decline in this proportion is presented in Figure 1.3: the proportion declined from a peak of 27.4% in 2006 to 26% in 2019. A decreasing outstanding amount of life insurance contracts caused by a gradually declining proportion of the 40-59 years old population leads to the reasonably declining inflows to euro-denominated life

7. Sondage 2021: Les Français, l'épargne et la retraite. <https://www.cerledesepargnants.com/barometre-de-lepargne/>

insurance business.

In the survey of household wealth by Insee, the holding rate of life insurance contracts increases with age: the holding rate of life insurance contracts by households aged 60 and more (around 44%) is far larger than those aged under 60 [Cazenave-Lacrouts et al., 2018], as shown in Figure 1.4. This generation has reached retirement age, and those who have accumulated life insurance contracts to increase their pensions have begun to withdraw some of their contracts, either in the form of an annuity or in the form of capital. Both the retirement of many generations and their higher holding rates of life insurance contracts indicate larger outflows from the asset portfolios of life insurance business by the elderly based on their needs, in case of withdrawal for retirement and inheritance transfers.

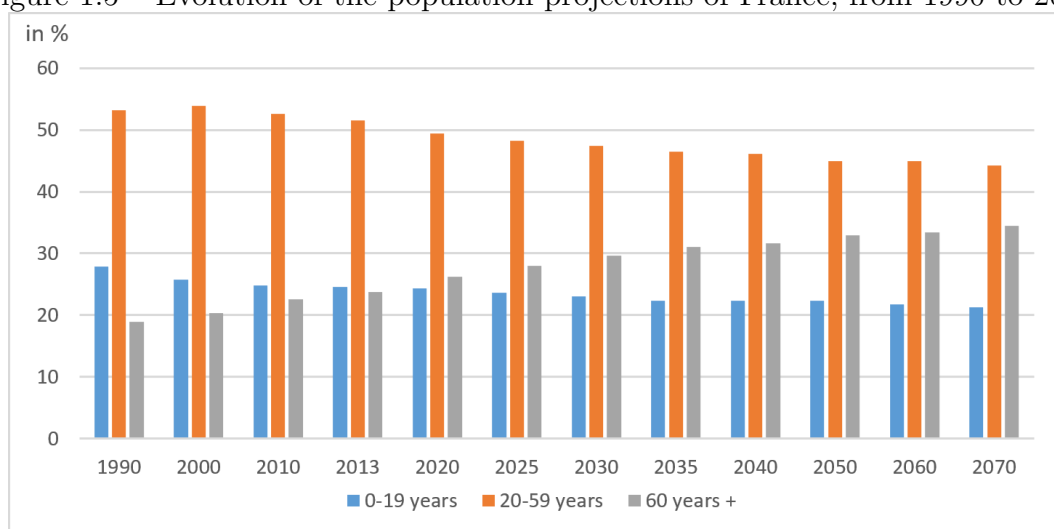
Figure 1.4 – Holding rate of life insurance contracts by age in France, in 2018.



Source: Insee

The proportion of the French population aged 60 and over has been increasing. The population projections in France in Figure 1.5 further shows that the proportion of the French population aged 60 and over will increase gradually by 2070 [Blanpain and Buisson, 2016].

Figure 1.5 – Evolution of the population projections of France, from 1990 to 2070.



Source: Insee

Nevertheless, on the brighter side, life insurance remains the preferred financial investment of households after the passbook ([Coppoletta-Solotareff et al., 2015] and [Cazenave-Lacrouts et al., 2018]): 39% of metropolitan households own at least one contract in 2018, which was 36.5% in 2015 as shown in Table 1.1. Despite the significant drop in the rate of returns of euro-denominated funds, French households still seem to favor the security of this investment rather than its returns, especially since the financial crisis of 2008.

Table 1.1 – Holding rate of life insurance contracts in mainland France between 1998 and 2018.

Year	1998	2004	2010	2015	2018
Holding rate	28.9%	26.2%	34.7%	36.5%	39.0%

Source: Insee

Indeed, this product offers savers an attractive tax framework and the opportunity to secure at least part of their savings. Life insurance thus retains its attractiveness because it allows one to accumulate wealth over the course of life and transmit it in succession in favorable tax conditions.

1.3 Life insurance in a low-interest-rate environment

1.3.1 Taxation, investment supports & constraints

The French life insurance market is mainly segmented into euro-denominated and unit-linked contracts, where the former makes it possible to guarantee the invested capital while the latter carries a risk of losses. From the end of 2014, life insurers offered a third type: euro-growth contracts. The synthesis comparison between the three funds is shown in Table 1.2.

Table 1.2 – Comparison between Euro-denominated funds, Unit-linked products and Euro-growth funds.

Thematic	Euro-denominated funds	Unit-linked products	Euro-growth funds
Capital guarantee	Yes	No	Yes, with a minimum holding period of 8 years
Annual Guaranteed Interest Rate and the profit-sharing clause	Yes	No	Yes, with a minimum holding period of 8 years
Risk-bearer	Insurer	Insured	Insurer
Target return and financial performance	French government bonds yield + margin	Investment support yield	Mathematical provision + diversification provision

Source: [Agbojan et al., 2016]

The common characteristic of the three different funds is related to taxation incentives of life insurance contracts: life insurance maintains a favorable tax regime for both income tax and inheritance tax. Any contract that has been subscribed for at least 8 years will benefit from a reduction in interest when it comes to redemption. While this taxation remains very attractive, it is also very complicated because the successive amendments have introduced rules that vary according to the quality of the beneficiary, the subscription date, the date of payment of funds, and the age of the subscriber.

In the event of partial or total redemption, the interest (capital gains) are subject to tax. For a full redemption, the interest is determined by the difference between the value of the contract at the time of redemption and all of the payments made on the contract. For a partial redemption, the interest is determined by the proportion between the payments and the capital obtained. The government has introduced a new tax system

for redemptions made on life insurance contracts. A distinction must be made between payments (and subscriptions) made on the contract before and after September 27, 2017.

- Payments (and subscriptions) until September 26, 2017

In principle, interest from payments made until September 26, 2017, is taxable at the progressive income tax scale. Policyholders can opt for the application of a tax. The tax rate depends on the seniority of the contract, which is shown in Table 1.3.

Table 1.3 – Tax rates for payments (and subscriptions) until September 26, 2017.

The seniority of the contract	Tax rate
Between 0 and 4 years	35%
Between 4 years and 8 years	15%
Over 8 years	7.5%

Source: FFA

- Payments (and subscriptions) from September 27, 2017

The government has implemented the "flat tax" or single lump-sum tax, which applies in principle to interest linked to payments after 09/27/2017. The flat tax standardizes the taxation of life insurance. The tax rate also depends on the seniority of the contract, which is shown in Table 1.4.

Table 1.4 – Tax rates for payments (and subscriptions) from September 27, 2017.

The seniority of the contract	Tax rate
Between 0 and 8 years	12.8%
Over 8 years	15% (payments less than or equal to €150,000) 7.5% (payments over €150,000)

Source: FFA

In the event of retirement, the life insurance contract offers the possibility of taking out a life annuity and choosing this option during the life of the contract. The annuity resulting from a life insurance contract is subject to taxation according to a fraction depending on the age of the annuitant when the annuity is set up, which is shown in Table 1.5.

Interest generated in life insurance contracts is also subject to social security contributions annually as soon as they are entered into an account at the overall rate of 17.2%

Table 1.5 – Taxable annuity fraction by the age of the annuitant.

	Taxable annuity fraction
Up to 49 years old	70%
50 to 59 years old	50%
60 to 69 years old	40%
From 70 years old	30%

Source: FFA

(since January 1, 2018) directly taken by the life insurance company.

1.3.1.1 Euro-dominated funds (les fonds en euros)

Euro-denominated funds offer a great deal of security for the saver, whatever the economic situation, because the fund on which the savings are invested is made up of low-risk investments, mainly on bonds that are not very sensitive to the markets. More specifically, euro-denominated funds are generally invested in bonds from 60% to 80%, the rest being invested in shares, real estate, and other financial assets. This diversification promotes the return on euro-denominated funds, whose rates of return offered in recent years would not have been achieved if they had been 100% fixed-rate bonds.

In addition to this diversification, life insurers have an obligation to guarantee the sums of the investments, which makes the euro-denominated funds guaranteed capital. Moreover, life insurers also ensure that the investor retains the earnings generated each year.

The main features of euro-denominated funds are as follows:

- Capital guarantee and annual minimum rate guaranteed by the insurer. Historically, life insurance contracts have generally had a guaranteed minimum rate. In order to limit the risks taken by insurers, regulation has come to limit the possibilities in this area. If the proceeds of the investments are not sufficient to pay the principal and interest, the own funds of the life insurers will come as collateral.
- "Ratchet effect" (l'effet cliquet). The interests served are definitively acquired. This mechanism ensures the policyholders the profits of the past year and thus protects against a possible reversal of stock market trends. Then, the profit generated by the life insurance is always acquired, in addition to the amounts initially paid to the contract.

- Life insurers mainly bear risk: various legal (French Insurance Code⁸) and regulatory constraints (Solvency II) govern the management of euro-denominated funds.
- The security of euro-denominated funds remains closely linked to the solidity and competence of the life insurance company.

The advantages of euro-denominated funds are the security through the capital guarantee provided by insurance organizations; the stability: low volatility of financial returns; and liquidity, which allows redemption within a reasonable time. However, there is one limitation: lack of transparency to investigate their compositions and compare with different life insurers, except for the annual returns published and analyzed every year in the financial press.

The compensation of contracts in euro-denominated funds is based on several elements: technical interest rates, profit sharing, and the minimum guaranteed rate [CielEden, a].

- The technical interest rate (le taux d'intérêt technique):

Throughout the period of validity of the contract, insurance companies have the possibility to remunerate the capital invested based on a minimum rate: this is called the technical interest rate. It can neither exceed 75% of the semi-annual average rate of French sovereign bonds at the date of subscription for the first eight years nor 60% of this rate or 3.5% annually after the eighth year (see article A.132-1 of the Code des Assurances [Légifrance, 2020]).

- Profit-sharing (la participation aux bénéfices):

Life insurance companies have an obligation to redistribute part of their profits during the year to the policyholders of euro-denominated contracts. This redistribution cannot be less than:

- 85% of the financial benefits: They result from the gains that insurers have made by investing the capital entrusted to them by the contract holders;
- 90% of the technical benefits: They arise intrinsically from the difference between the premiums collected and the technical interests (see article A.331-4 of the Code des Assurances).

Profit-sharing is theoretically donated every year. The company has the choice to be able to allocate a portion of these profits to a reserve account commonly known as "the provision for participation in profits" (la provision pour participation aux bénéfices, PPB). When the total amount is not redistributed, the benefits in that reserve must be returned to the insured within a maximum of eight years (see article A.132-7 of the Code

8. Code des Assurance in French.

des Assurances). This "reserve" allows the insurers to smooth the financial results of euro-denominated contracts over time.

· The minimum guaranteed rate (le taux minimum garanti):

Some life insurers offer a minimum guaranteed rate. In this case, this minimum rate of return includes the minimum interest paid based on the technical interest rate and profit-sharing (see articles A.132-2 and A.132-3 of the Code des Assurances). Since 2010, this minimum guaranteed rate is not offered for the entire duration of the contract but on an annual basis and are limited to 85% of the average rate of return on assets over the previous two years.

1.3.1.2 Unit-linked products (les unités de compte)

In unit-linked life insurance contracts, savings are invested in financial markets mainly through funds, even if some contracts offer the possibility of a direct holding of financial assets. The main advantage of unit-linked contracts is that they offer various investments, providing different investment strategies to investors searching for diversity and performance. The number and types of funds chosen differ from contracts and companies. The more elaborated contracts offer an open-architecture, with an infinite choice of investments.

While in euro-denominated contracts, the invested amount and capitalized interest are guaranteed by the insurer. In unit-linked contracts, the insurer only commits to the number of units held. The value can fluctuate depending on the performance in financial markets or other investment instruments. Therefore, unit-linked contracts present a directional risk to the saver: in the event of a decline in the markets, the value of the contract may decrease. After numerous cases of failure to provide the right advice at the end of the 1990s, information on the risk taken by the investors has been reinforced during the last years, mainly thanks to MiFID and MiFID II⁹ regulations.

9. The Markets in Financial Instruments Directive (MiFID) is a European regulation that increases the transparency across the European Union's financial markets and standardizes the regulatory disclosures required for firms operating in the European Union. MiFID implemented measures, such as pre and post-trade transparency requirements, and set out the conduct standards to be followed by financial companies. MiFID has a defined scope that primarily focuses on stocks. The directive has been in force across the European Union (EU) since 2007. MiFID II replaced MiFID in 2018, which extends the requirements to more financial instruments, including commodities, debt instruments, futures and options, exchange-traded funds, and currencies. <https://www.esma.europa.eu/>

1.3.1.3 Euro-growth funds (les fonds euro-croissance)

The euro-growth contract was designed by the public authorities and insurers, launched at the end of 2014. Its objective is to generate more efficient returns than those of euro-denominated funds without taking excessive risks. Life insurers see it as a way of transferring the amount invested in euro-denominated funds and serving as a financing source for small and medium-sized enterprises [CielEden, b].

The commitments in euro-growth funds are expressed contractually in two ways: in euros and diversification parts. The commitment expressed in diversification parts includes a minimum value expressed in euros for each part. The valuation of each part will depend on the technical and financial results of the fund.

Table 1.6 – The outstanding amount of euro-growth funds between 2014 and 2019 (direct business, in billions of euros).

Year	2014	2015	2016	2017	2018	2019
Amount	1.0	1.7	1.9	2.2	2.5	3.1

Source: FFA

Unlike euro-denominated funds that offer a capital guarantee and availability of the amounts invested at any time, the capital invested in a euro-growth fund will be guaranteed only after 8 years following the first payment. They are considered for a long-term investment. Until now, this formula seems particularly unsuccessful: at the end of 2019, the overall outstanding was 3.1 billion euros as shown in Table 1.6.

1.3.1.4 Two alternatives to classic euro-denominated funds

In the French life insurance market, there are two alternative products with higher expected average rates of return relative to traditional euro-denominated funds¹⁰: the real estate euro-denominated funds and the dynamic euro-denominated funds. For the savers, the major difference between the classic and the alternatives is that insurers will require the savers to invest part of their savings in unit-linked products (with a minimum of 20, 30, or even 50%) to access the alternative euro-funds.

1. Real estate euro-denominated funds

They are characterized by a higher relative weight of real estate investments in the alloca-

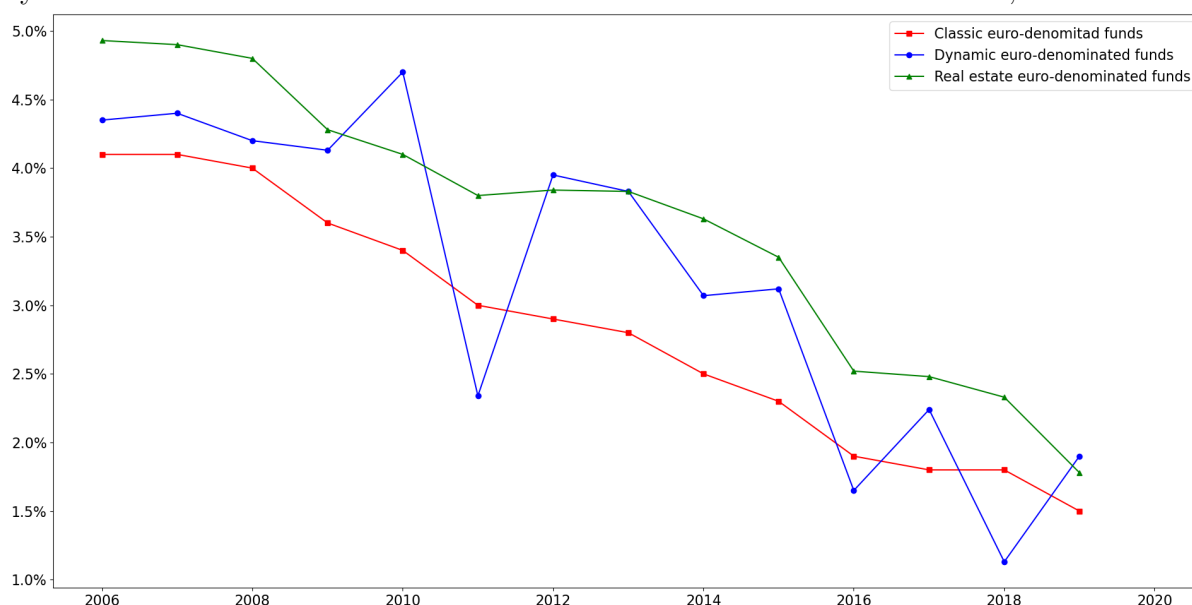
10. Good Value for Money (GVfM) <https://www.goodvalueformoney.eu/espace-documentaire/>

tion. However, faced with the difficulty of investing large amounts of net inflows into the funds at reasonable prices, life insurers have severely limited the investment possibilities for their clients in recent years, making these funds a rare product.

2. Dynamic euro-denominated funds

They are characterized by higher risk exposures of assets to stocks and real estate. Life insurers usually adopt the constant proportion portfolio insurance strategy (CPPI, see [Black and Perold, 1992]) to manage this fund.

Figure 1.6 – The performance of life insurance products in classic euro-denominated funds, dynamic euro-denominated funds and real estate euro-denominated funds, 2006-2019.



Source: ACPR, GVfM

Figure 1.6 presents the relative performance of the three euro-denominated funds. Compared with dynamic euro-denominated funds, the returns of classic and real estate ones are less volatile. Under a downward trend in the rate of returns, real estate euro-denominated funds have relatively higher returns. But all three rate of returns fall below 2% in 2019.

1.3.1.5 Investment constraints on euro-denominated funds

French law (see article R332-3 of the Code des Assurances) dictates the list of assets that may be eligible for insurance liabilities. These assets are classified into broad categories: bonds and other interest rate securities, shares or equivalent securities, real estate

assets, loans, and deposits. The asset portfolio structure representing the commitments must meet certain constraints to limit the impact of the market risk and the liquidity risk.

The main limits of the composition of euro-denominated funds in terms of investment (%) are (maximum investment ratios by asset class):

- 100% for bonds and bond funds;
- 65% for equities and equity funds;
- 40% for real estate;
- 10% for the loans;
- 0.5% for the premiums of financial derivatives.

The assets held in the portfolio must also meet constraints to limit the counterparty risk, requiring sufficient counterparties dispersion. The rules of dispersion are:

- 5% for bonds and loans of the same issuer outside the OECD (Organization for Economic Co-operation and Development), this ratio can be raised to 10%, but the share of issuers exceeding 5% cannot exceed 40%;
- 10% on one building or one real estate fund;
- 1% for risky mutual fund and mutual fund in innovation issued by the same organization.

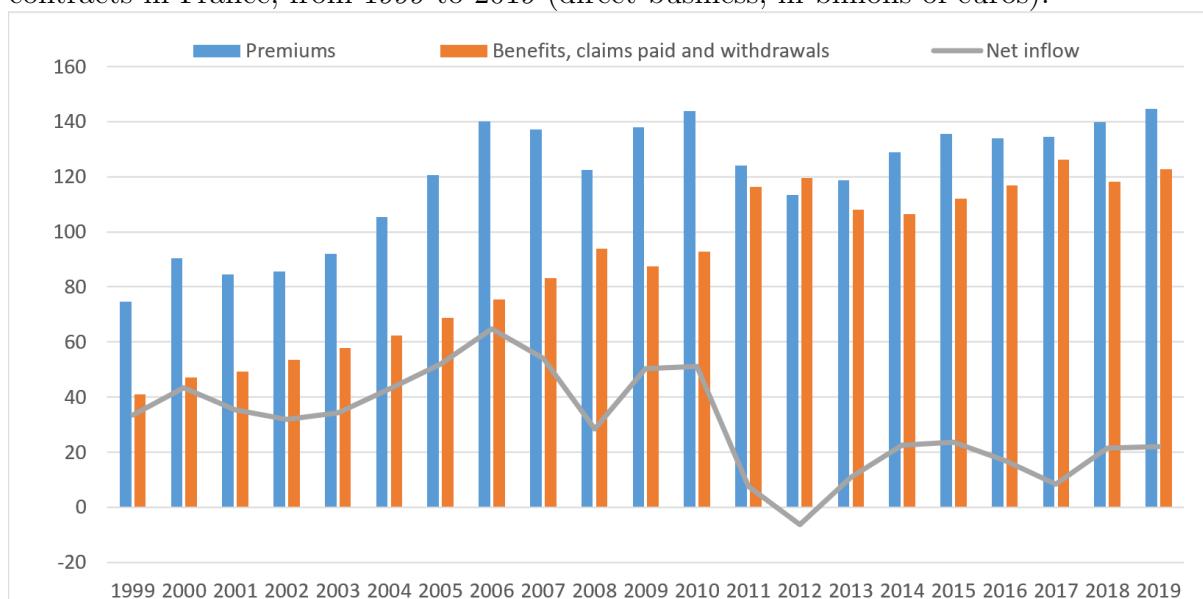
1.3.2 Life insurance business

In this subsection, we analyze the life insurance business in France and the operating conditions of the different life insurance products presented in the previous Section 1.3.1 based on the annual data set (e.g., [FFSA, 2009], [FFA, 2016] and [FFA, 2020b]) obtained from the French Insurance Federation¹¹.

When observing the life insurance business ([FFA, 2020a]) in Figure 1.7, the premiums remain at around 120 - 140 billion euros for the past 15 years. At the same time, the trend for outflows including benefits, claims paid, and withdrawals has been increasing, causing a relative decline in net inflow.

11. The French Insurance Federation (la Fédération Française de l'Assurance, FFA) was created in July 2016. It gathers under a single umbrella organisation the French Federation of Insurance Companies (la Fédération française des sociétés d'assurances, FFSA) and the Association of Mutual Insurance Companies (le Groupement des entreprises mutuelles d'assurance, GEMA). <https://www.ffa-assurance.fr>

Figure 1.7 – Evolution of the premiums, withdrawals and net inflows of life insurance contracts in France, from 1999 to 2019 (direct business, in billions of euros).

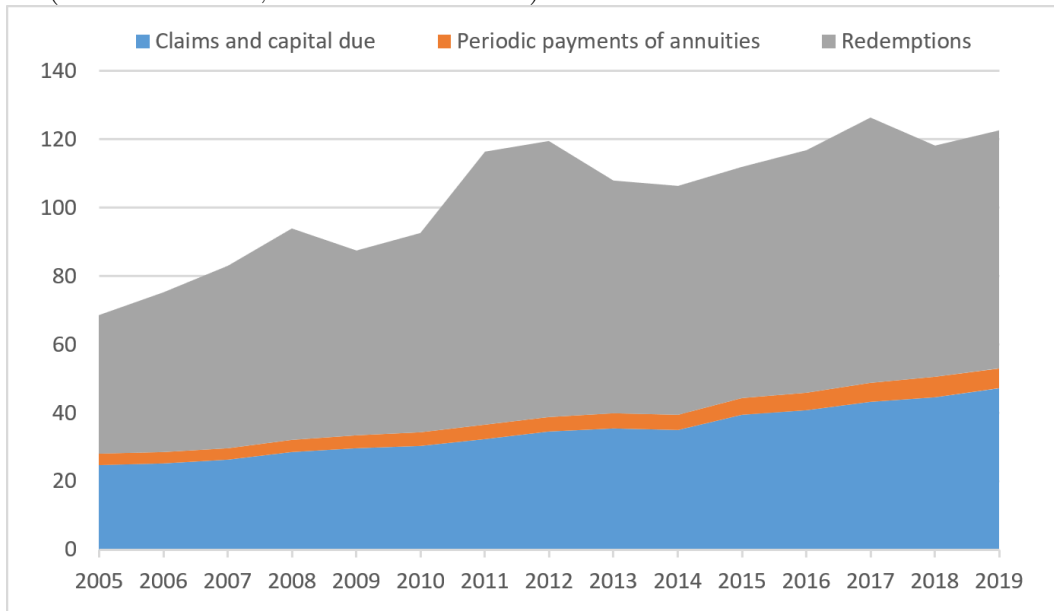


Source: FFA

Firstly, we decompose the outflow of benefits, claims paid, and withdrawals. It shows that the increase in outflows since 2006 is due to the increase in redemptions, mainly to the jumps in redemptions during the 2007-08 financial crisis, 2011-12 public debt crisis in Eurozone, and the promulgation of the Sapin 2 law¹² on December 2016 as shown in Figure 1.8.

12. In the law of Sapin 2, the high council of financial stability is authorized to suspend, delay or limit the withdrawals of money or shifts on the life insurance in the event of a serious or characterized threat to the financial system. Withdrawals will be blocked for 3 months. (Loi Sapin 2, see article L631-2-1 of the Code Monétaire et Financier)

Figure 1.8 – Breakdown of withdrawals of life insurance contracts in France, from 2005 to 2019 (direct business, in billions of euros).



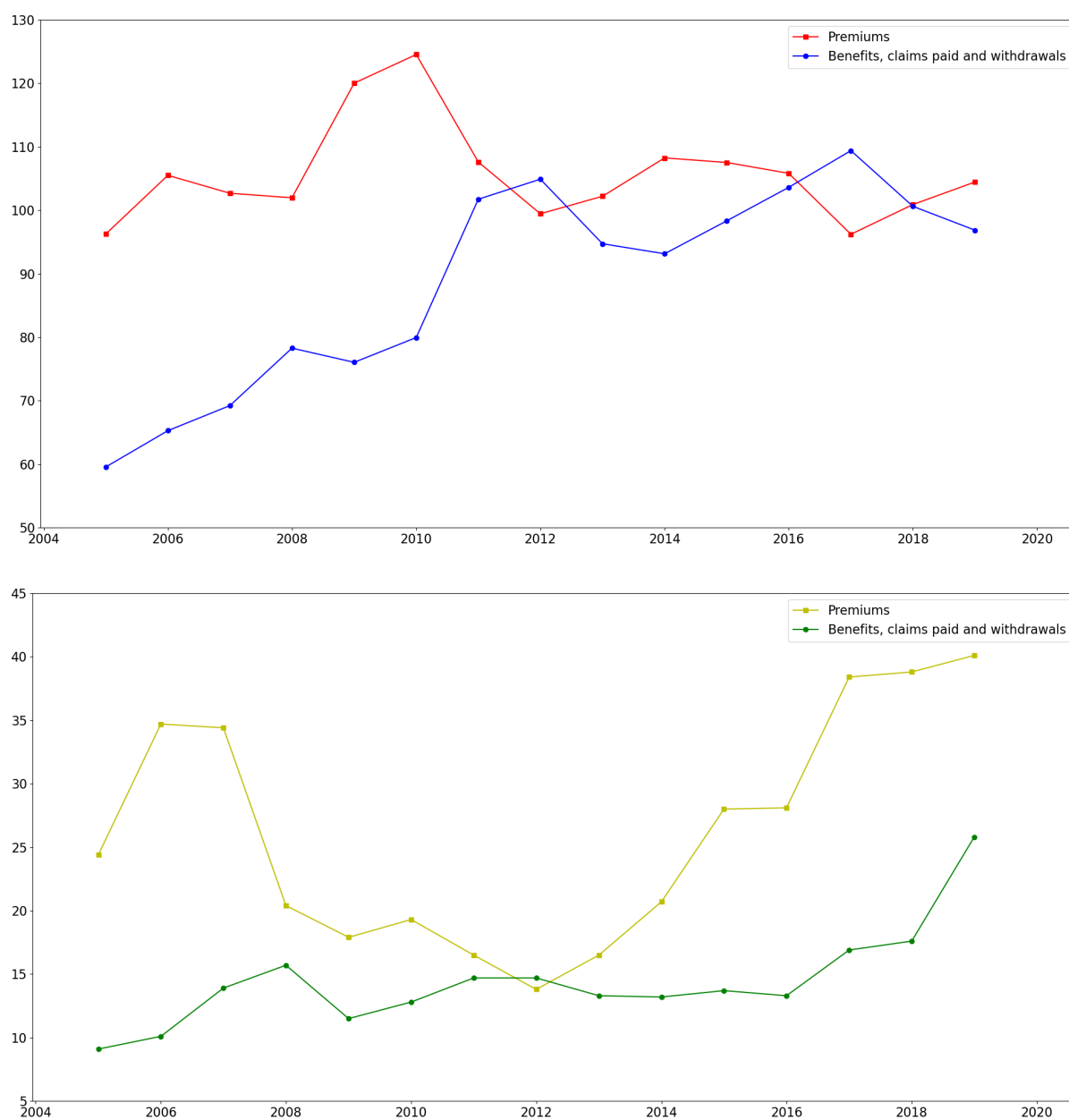
Source: FFA

The outflows of euro-denominated funds have increased from 60 billion euros in 2005 to approximately 100 billion euros in 2019 [FFA, 2020b]. However, the outflows have stabilized at around 90-110 billion euros in the last decade¹³. The maximum outflow amount for unit-linked products in 2019 is around 25 billion euros, far less than that of euro-denominated funds since euro-denominated funds take the largest part in the composition of life insurance contracts, as shown in Figure 1.9. Secondly, the net inflows from the unit-linked contracts have increased before the 2007-08 financial crisis and after the 2011-12 public debt crisis in Eurozone. This phenomenon was noticed for the first time at the end of the 1990s when the inflow of unit-linked products exceeded the inflow of euro-denominated products [Boubel and Séjourné, 2001]. This trend is highly related to the performance of the stock markets, where we show the CAC 40 stock index as an example in Figure 1.10. The gradual decline in the return on euro-denominated funds, related to the fall in long-term interest rates, spreads throughout Europe [EIOPA, 2020a]. The excellent performance of the stock markets allowed unit-linked products to perform as an attractive alternative that enabled investors to benefit at least partly from the rise

13. This is shown in more detail in the monthly data of euro-denominated funds in ACPR. Since we do not have the authority for this monthly data, we quote the chart in the ACPR article [Perdu and Marionnet, 2019] and refer it in Figure 15 in Appendix .1 for review.

in stock market prices while remaining within the attractive taxation framework of life insurance.

Figure 1.9 – Evolution of the premiums, withdrawals and net inflows of euro-denominated funds (upper) and unit-linked products (bottom) in France, from 2005 to 2019 (direct business, in billions of euros).



Source: FFA

Figure 1.10 – The performance of unit-linked contracts and the evolution of the CAC 40 index at the end of the year, from 2000 to 2019.



Source: FFA

The Fourgous amendment¹⁴ can also account for this trend: since the law of July 26th, 2005, there exists a possibility of transforming the mono-support contract into a multi-support contract without having to close the original contract and losing the related tax precedence. This measure aims to encourage investment in shares, which are available on multi-support contracts. In addition to euro-denominated funds, it will offer the opportunity to invest the savings in unit-linked products. It is a symbol of flexibility, but it includes a requirement: the capital invested in the units of account must be greater than 20% of the assets, leading to a net inflow of unit-linked products.

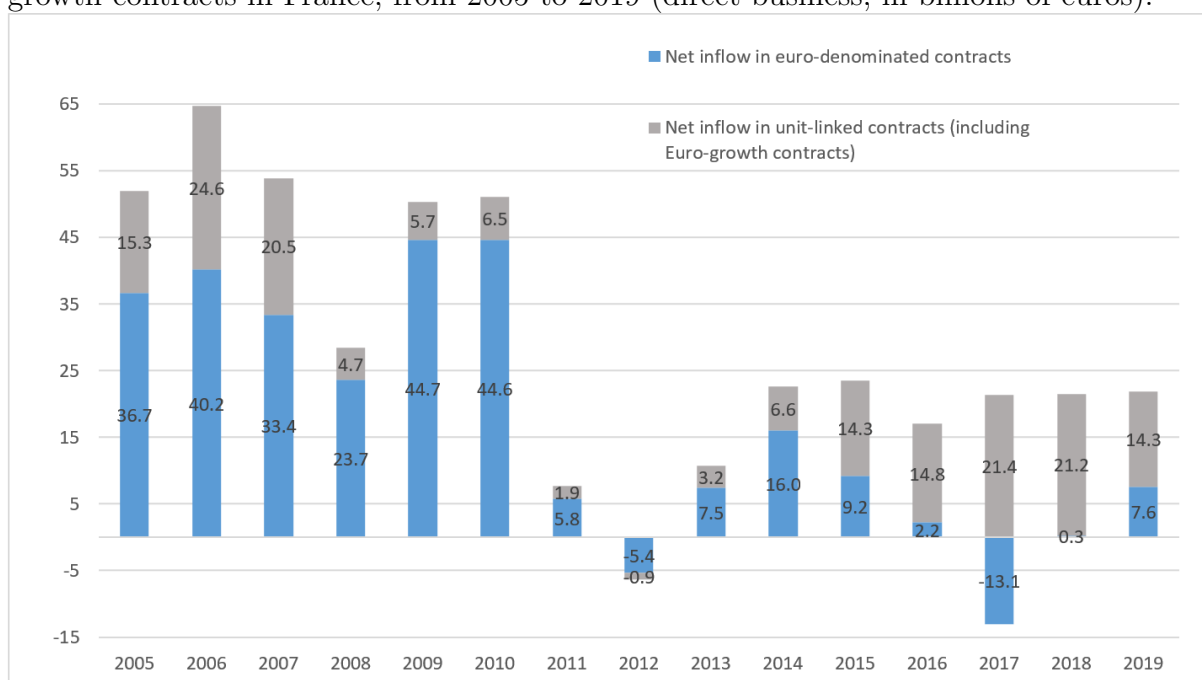
By the act of Pact (La Loi Pacte), savers can benefit from an exceptional tax advantage in the event of a transfer from life insurance to a new retirement savings product (PER). Any redemption carried out on a life insurance contract of holding more than 8 years will be subject to a doubled tax deduction from October 1, 2019, to January 1, 2023, provided that the sums are reinvested in a PER and that the redemption is made for at least 5 years before retirement. During this period, the outflow of euro-denominated contracts will also potentially increase with the transfer of policyholders.

A partial explanation of the decreasing net inflow from euro-denominated contracts

14. L'amendement Fourgous, loi n.2005-842 du 26 juillet 2005 pour la confiance et la modernisation de l'économie

since the early 2010s lies in the issuance of euro-growth contracts in 2014 (see Figure 1.11). Since the objective of euro-growth funds is to generate more efficient returns than those of euro-denominated funds, without taking excessive risk in the long-term (at least eight years), it is reasonable that euro-growth contracts absorb one part of the shares in the net inflows from euro-denominated contracts. What is more, the approval of the Pact law¹⁵ for the transformation and growth of businesses, which modified the terms of investment in life insurance contracts, provides for the possibility of transforming part or total of the existing contracts into a euro-growth contract within the same insurer, while retaining the tax precedence of the policyholders. This law will also potentially increase the net inflows into euro-growth contracts in the future. However, these contracts remain largely unknown, and a bigger part of the inflows is invested in unit-linked contracts.

Figure 1.11 – Life insurance net inflow from euro-denominated, unit-linked, and euro-growth contracts in France, from 2005 to 2019 (direct business, in billions of euros).



Source: FFA

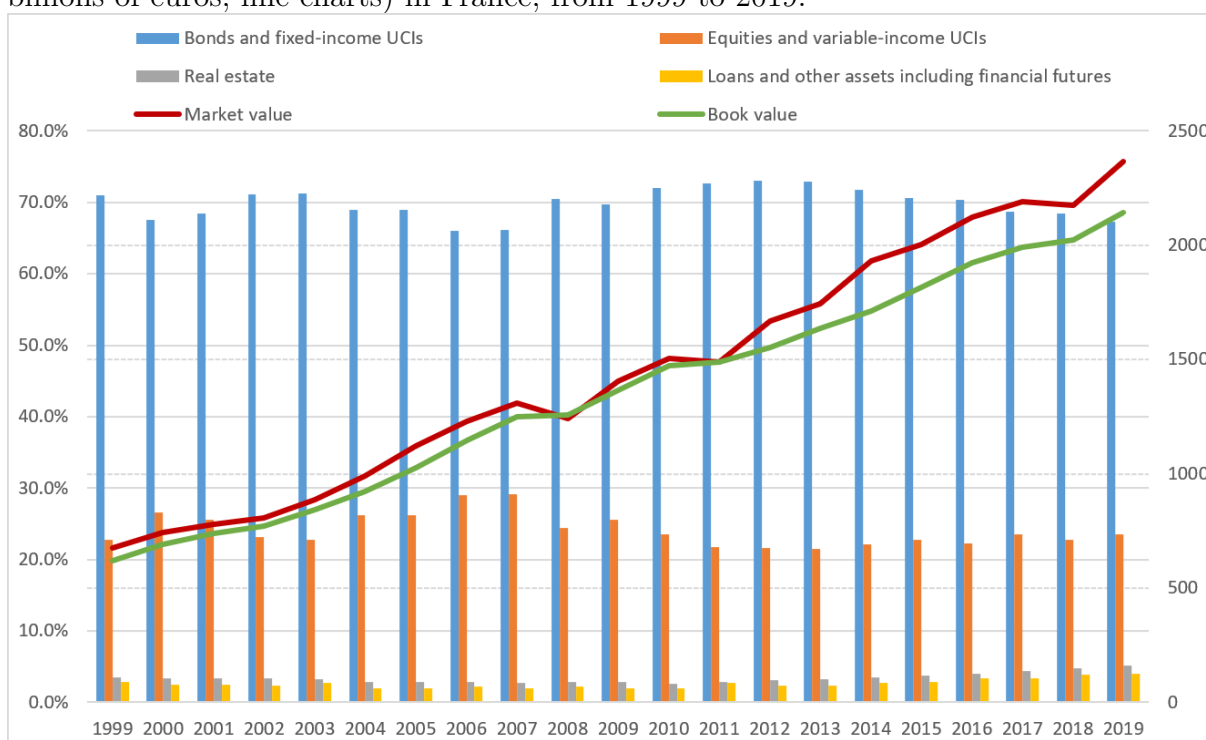
Lastly, [Séjourné, 2004] analyzed the volatility of the stock markets and the behavior of French households. A return to an upward trend in the financial markets could encourage

15. La Loi Pacte: "Loi n. 2019-486 du 22 mai 2019 relative à la croissance et la transformation des entreprises"

savers to regain a reflex of securing capital gains. The incentive to change the portfolio seems to rise with volatility. However, it is noted that this was not the case during the downturn in the financial markets, which perhaps indicates the asymmetric behavior according to the direction of the stock market fluctuations. It is reasonable that the net inflows from unit-linked contracts have been increasing in recent years, and that could remain on an increasing trend when it keeps in a bull market. In the context of the trend that the subscription of unit-linked contracts is preferred over euro-denominated funds, it is necessary to pay attention in a long-term financial stability perspective to the shift of market risk exposures from insurers to policyholders.

1.3.3 Life insurance investment

Figure 1.12 – Structure of the investments of life, capitalization and composite companies (book value, breakdown in %, bar charts) and evolution of outstanding investments of life, capitalization and composite companies (market value and book value, direct business, in billions of euros, line charts) in France, from 1999 to 2019.



Source: FFA

Life insurance plays a key role in financing the economy. The outstanding investments of life insurance companies in France have increased during the past two decades, from 600 billion euros to 1200 billion euros in book value, as shown in the line chart in Figure 1.12. The life insurance industry continued to make an important contribution to financing the economy in France.

In the past two decades, the market value of investments by life insurance companies fell below the book value in 2008 and 2011 due to the financial crisis in 2007-08 and the Eurozone public debt crisis in 2011. In an environment of declining long-term interest rates, the investment ratio of bonds and fixed-income UCIs¹⁶ has declined year by year since the Eurozone debt crisis, from the peak of 73% in 2012 to 67% in 2019. The investment ratios in both equities and real estate have increased since 2012.

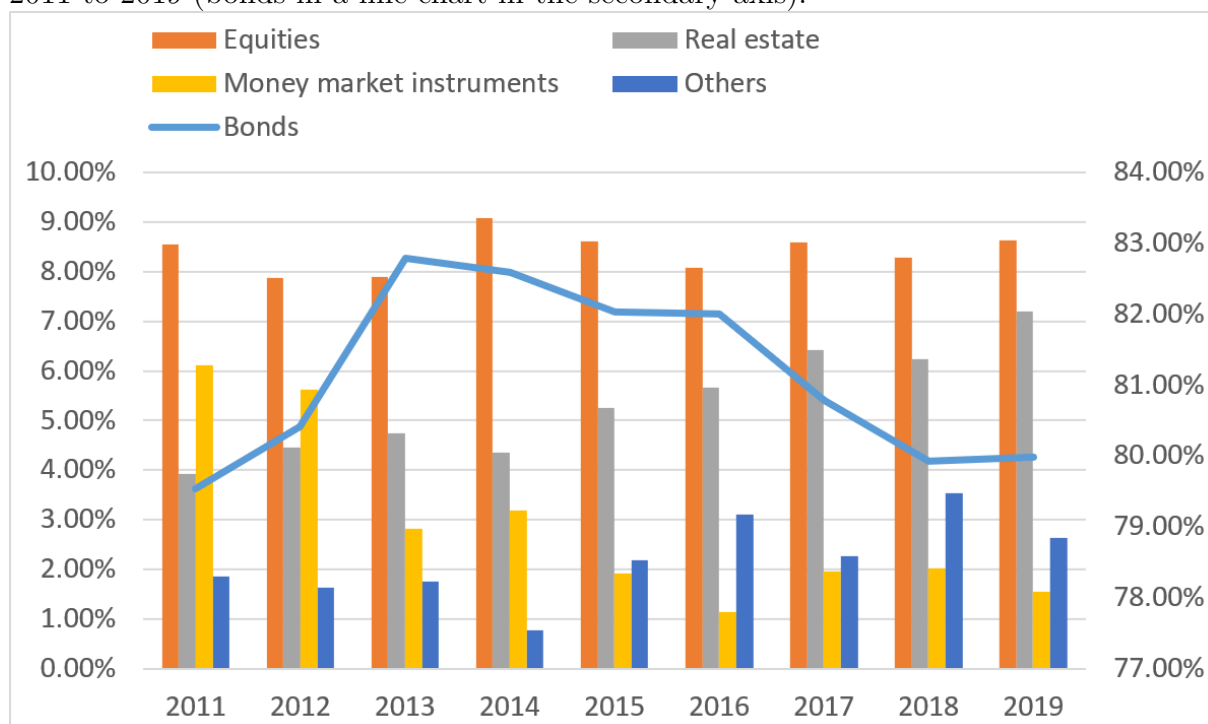
Although the decline in the investment proportion of fixed-income securities is not obvious, in the context of a continued increase in the book value of outstanding investments, which means a higher proportion of assets have shifted to risky investments (equities and real estate). This gradual shift has partly contributed to the increase in the market value of investments, which positively impacts the portfolio of life insurance companies.

Since the allocation structure of the life insurance industry contains the investments of all products in life, capitalization and composite companies, we focus on euro-denominated funds and analyze their investment structure based on the data collated from GVfM.

The proportion allocated to bonds and money market instruments in euro-denominated funds has a downward trend in recent years. In contrast, the proportion of real estate investment shows a gradual rise. The allocation to equities has remained basically stable, as shown in Figure 1.13. It is worth noting that the allocation ratio of money market securities has been declining from 2011 to 2016, and the allocation ratio is below 2% by 2019. This declining trend is in contrast to the analysis in Section 1.3.2. Since euro-denominated funds face an increase in outflows, life insurers will have to consider meeting the rising liquidity needs by increasing the allocation ratio of money markets products.

16. The Undertakings for the Collective Investment in Transferable Securities (UCITS) is a regulatory framework of the European Commission that creates a harmonized regime throughout Europe for the management and sale of mutual funds. The Autorité des Marchés Financiers (AMF) distinguishes six main families of UCITS (les organismes de placement collectif en valeurs mobilières, OPCVM) according to the type of funds managed: money market, bond, equity, alternative, diversified, and formula (e.g., guaranteed and protected) funds. <https://www.amf-france.org/>

Figure 1.13 – Structure of the investments of euro-denominated funds in France, from 2011 to 2019 (bonds in a line chart in the secondary axis).



Source: GVfM

Table 1.7 – Structure of the bond investment in euro-denominated funds from 2015 to 2019.

	2015	2016	2017	2018	2019
Corporate bonds	51.60%	63.30%	57.90%	59.40%	59.40%
Government bonds	36.70%	37.00%	42.10%	40.60%	40.60%

Source: GVfM

Table 1.7 presents the bond investment structure in euro-denominated funds from 2015 to 2019. It reflects the investment trend of euro-denominated funds from government bonds to corporate bonds.

When comparing the asset allocation of euro-denominated funds with that of the life insurance industry, the investment strategy in euro-denominated funds is more conservative with allocating more bonds and fewer stocks. The characteristics of euro-denominated

funds, including capital guarantee, require life insurers to allocate lower-risk assets in their portfolio.

1.3.4 Life insurance companies & Solvency II

1.3.4.1 Solvency framework

In the financial and insurance sector, the French law (Code des Assurances) imposes strict obligations on insurance companies on the one hand on their solvency and the other hand on the representation of their commitments by regulated assets. Solvency is a central issue for the insurance business. The supervisory authorities conduct regular monitoring of this solvency for the insured behalf through a regulatory state. The insurers should consider the balance between the long-term interest of the insured, the fulfillment of the contractual commitments, and sustainable solvency [Chelly and Robert, 2017].

Solvency II, which came into force on January 1, 2016, represents a major development in the insurance world, leading to a shift in risk management practices and the strategic management of insurance companies. The main objective of the Solvency II regime is to protect insurance companies by setting up a harmonized European prudential framework in line with the best professional practices [Hull and Gruson, 2018].

The principal characteristics and the three pillars of Solvency II are as follows [Morin and Thourot, 2017]: The new European Solvency system completely changes the vision of financial policy for insurance companies. Under Solvency II, the insurer will have to settle an amount of its capital used to cover the investment risk. This capital level is estimated by Value-at-Risk models, and it depends on the portfolio risk level. The riskier the portfolio is, the more the insurer must enhance the capital. This market risk covering method modifies objectives according to which the insurers guide their investment policy. The insurers must take account, in addition to the traditional factors, of the settled amount in its own capital.

- Pillar I: Quantitative requirements

Pillar I covers quantitative requirements for calculating technical provisions and Solvency Capital Requirement (SCR) using either a standard formula given by the regulators or an internal model developed by the insurance company. The SCR is the capital required to ensure that the insurance company will meet its obligations over the next 12 months with a probability of at least 99.5%. In addition to the SCR capital, a Minimum capital requirement (MCR) must be calculated, representing the threshold below which the

national supervisor would intervene. The MCR is intended to correspond to an 85% probability of adequacy over a one-year period and is bounded between 25% and 45% of the SCR. For supervisory purposes, the SCR and MCR can be regarded as "soft" and "hard" floors, respectively. That is, a regulatory ladder of intervention applies once the capital holding of the insurance undertaking falls below the SCR, with the intervention becoming progressively more intense as the capital holding approaches the MCR [Guégan and Hassani, 2019].

- Pillar II: Qualitative requirements

It enables supervisory authorities to assess the internal control, risk management and governance of each insurance company. This pillar is based on the control of authorities and the internal control of companies. It requires insurers to identify, measure, monitor, manage and report risks they are exposed to. Insurers must put risk management at the heart of decision-making and are required to conduct an own risk and solvency assessment.

- Pillar III: Information requirements

This pillar aims to redefine the supervisory reporting and public disclosure of financial and other information by insurance companies, especially on the accessibility and transparency of information produced, as well as the comparability at the European level.

1.3.4.2 Analysis of the solvency of insurance companies

Under Solvency II standards, equity capital is classified following its potential to be called up to absorb losses [ACPR, 2017]. Solvency II established a minimum capital requirement (MCR) below which an undertaking is considered to be no longer financially viable. To avoid losing their authorization to operate, insurers must ensure that their MCR coverage ratio is always more than 100% (i.e., the amount of their eligible equity capital must exceed their MCR level). Since 2016, the MCR coverage ratio displays a rising trend in insurance companies, as shown in Table 1.8. The coverage ratio in non-life insurance companies (792%) is higher than life and composite insurance companies (602%), which means more financially viable in non-life insurance companies. In 2019, the MCR coverage ratio remained at a high level of 651% for the population as a whole.

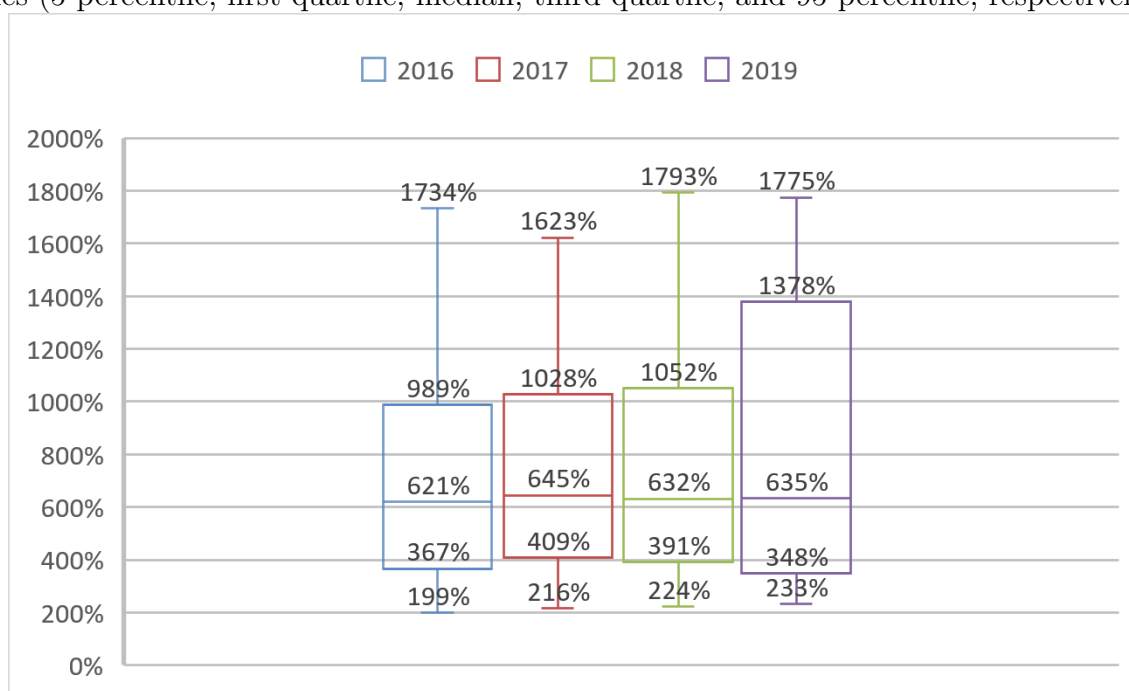
Table 1.8 – The average MCR coverage rate of insurance companies in France, from 2016 to 2019.

	2016	2017	2018	2019
All insurance companies	550%	593%	593%	651%
Life and composite insurance companies	484%	528%	531%	602%
Non-life insurance companies	746%	773%	763%	792%

Source: ACPR

Next, we analyze the distribution of the MCR coverage ratio in life and compound insurance companies and obtain complementary results for the average ratio, as shown in Figure 1.14.

Figure 1.14 – Distribution of MCR coverage ratio in life and composite insurance companies (5 percentile, first quartile, median, third quartile, and 95 percentile, respectively).



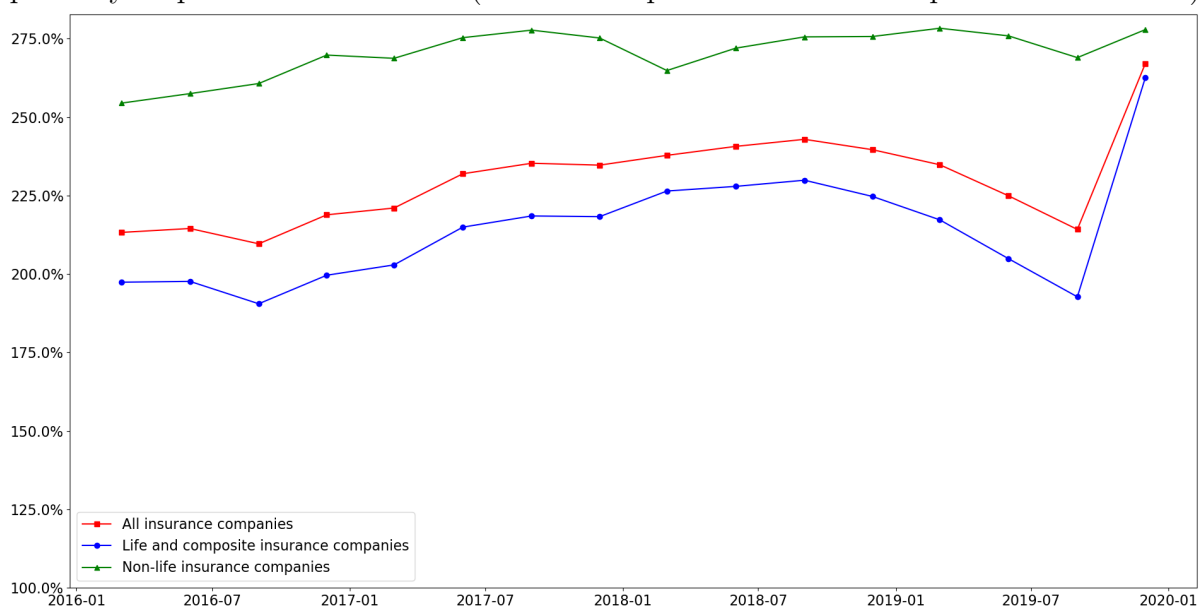
Source: ACPR

The average MCR ratio in life and composite insurance companies from 2016 to 2019 is less than the median, which means that more insurance companies have lower MCR ratios, lying between the first quartile and the median. The third quartile of the MCR

coverage ratio has increased significantly compared with the past three years, from an average of 1000% to 1378%.

Figure 1.15 displays the average SCR coverage ratio in quarterly scope. This ratio in life and composite insurance companies reveals an identical trend with the French 10-year Government bond yield, shown in Figure 1.2. The French 10-year Government bond yield moved into negative for most of the second half of 2019 before returning to a slightly positive level at the end of the year. This situation weighed on the solvency of life insurers, leading to a mechanical reduction in their own funds and an increase in their capital requirements. The negative impact was maximum in the third quarter of 2019 (193%) with a decrease in the SCR coverage rate of 37% compared to the same time in 2018 (230%).

Figure 1.15 – The average SCR coverage rate of insurance companies in France, the quarterly scope from 2016 to 2019 (life and composite insurance companies data labeled).



Source: ACPR

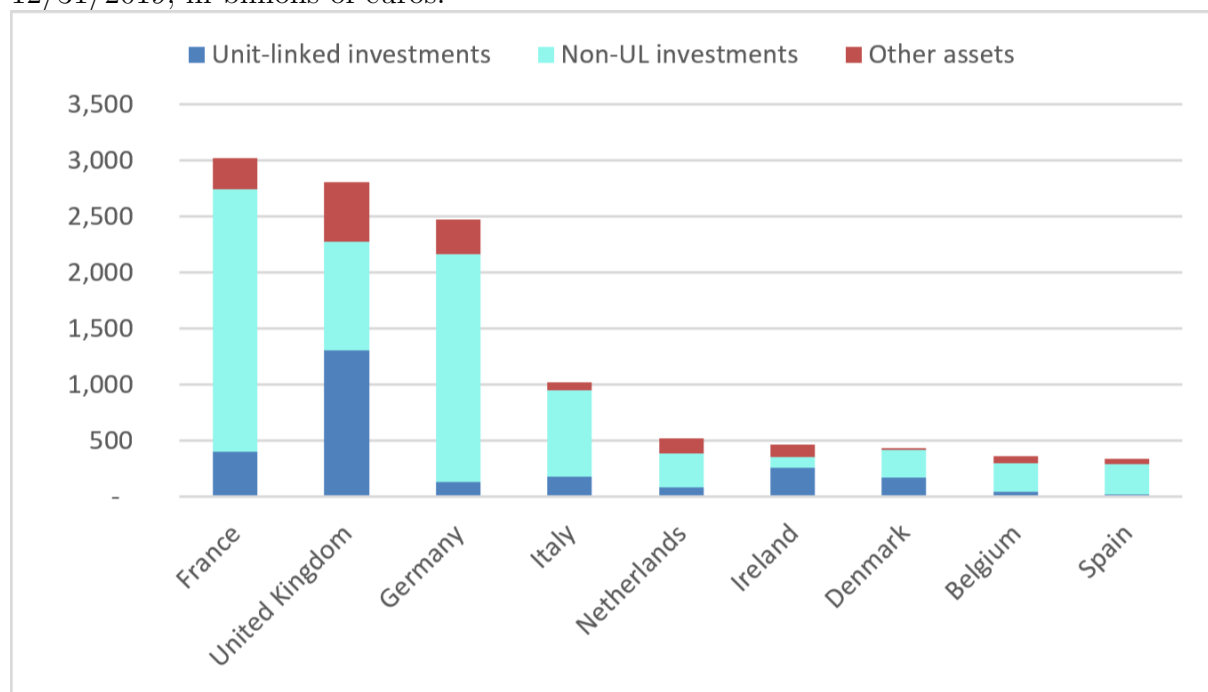
However, the rise in MCR and SCR ratios at the end of 2019 enabled an improvement that was directly reflected in assessing the prudential balance sheet at market value and calculating insurers' capital requirements. The steepening of the EIOPA risk-free rate curve used for discounting future cash flows as part of the assessment of technical provisions of insurers made it possible to ease prudential capital requirements [EIOPA, 2019b].

In addition, several organizations have implemented measures to strengthen their capital base (through capital increases or the issue of subordinated debt) and protect themselves against a further decline in interest rates. Lastly, life insurance companies benefited from the provisions of the decree¹⁷ of December 24, 2019, which amended the rules for recognizing provisions for profit-sharing in shareholders' equity eligible for regulatory capital. The inclusion of these reserves in surplus equity strengthened capital requirements by an average of 30 points [ACPR, 2020a].

1.3.5 Life insurance in EU level

In this subsection, we compare life insurance in Europe based on data from insurance organizations published by the European Insurance and Occupational Pensions Authority (EIOPA). The French insurance market ranks as the leading market in continental Europe in terms of size and the volume of gross premiums (life plus non-life) [EIOPA, 2019b].

Figure 1.16 – Total assets of insurance companies in main European countries as of 12/31/2019, in billions of euros.

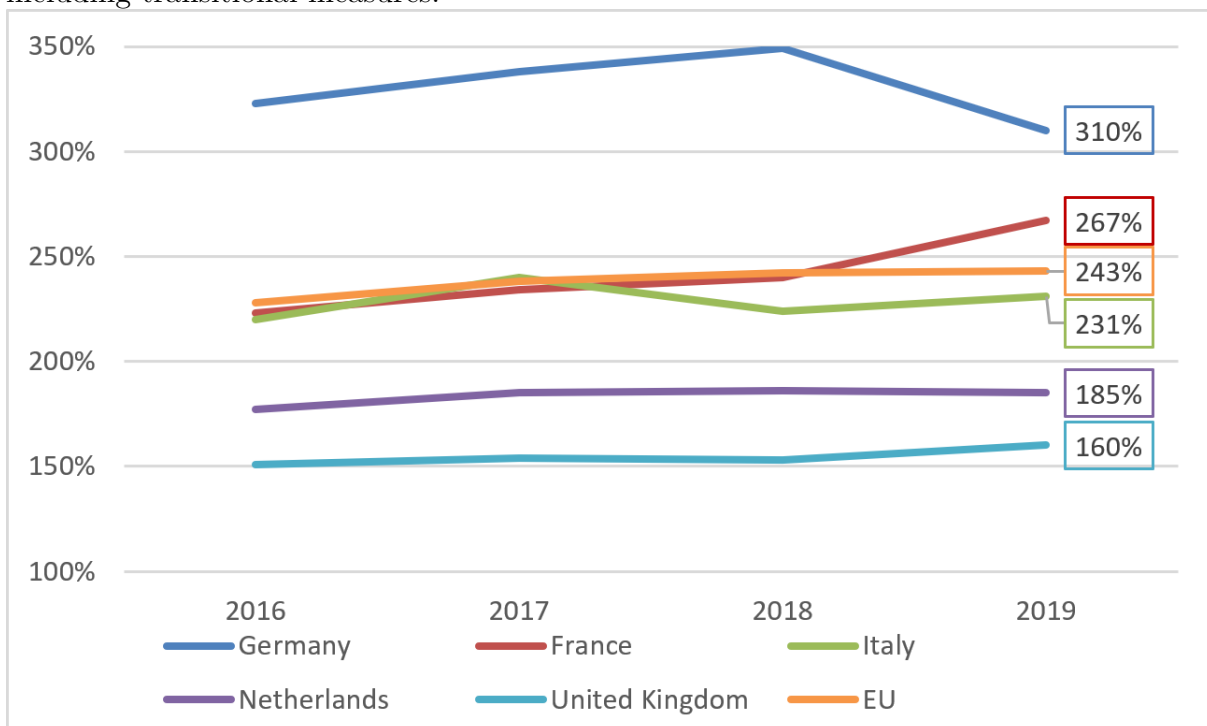


Source: EIOPA

17. Arrêté du 24 décembre 2019 relatif aux fonds excédentaires en assurance vie. <https://www.legifrance.gouv.fr/jorf/id/JORFTEXT000039682972/>

At the end of 2019, the French insurance market (3,108 billion euros) thus ranked first in Europe, just ahead of the United Kingdom (2,805 billion euros) and Germany (2,473 billion euros) in assets as shown in Figure 1.16. In the asset composition of major European countries, a larger part of the assets of insurance companies in continental European countries is made up of non-unit-linked investments (for example, euro-denominated funds in France). In comparison, assets in non-continental countries such as the United Kingdom and Ireland are mainly made up of unit-linked investments.

Figure 1.17 – The SCR coverage rate in leading European countries from 2015 to 2019, including transitional measures.



Source: EIOPA

The solvency condition in leading European countries is presented by the SCR coverage ratios in Figure 1.17. Although the SCR coverage ratio in Germany dropped by around 40% in 2019, its ratio of more than 300% still ranks first among major European countries. The SCR coverage ratio of the French market (267%) rose, at the end of 2019, above the average solvency ratio observed in the European Union (243 % end of 2019).

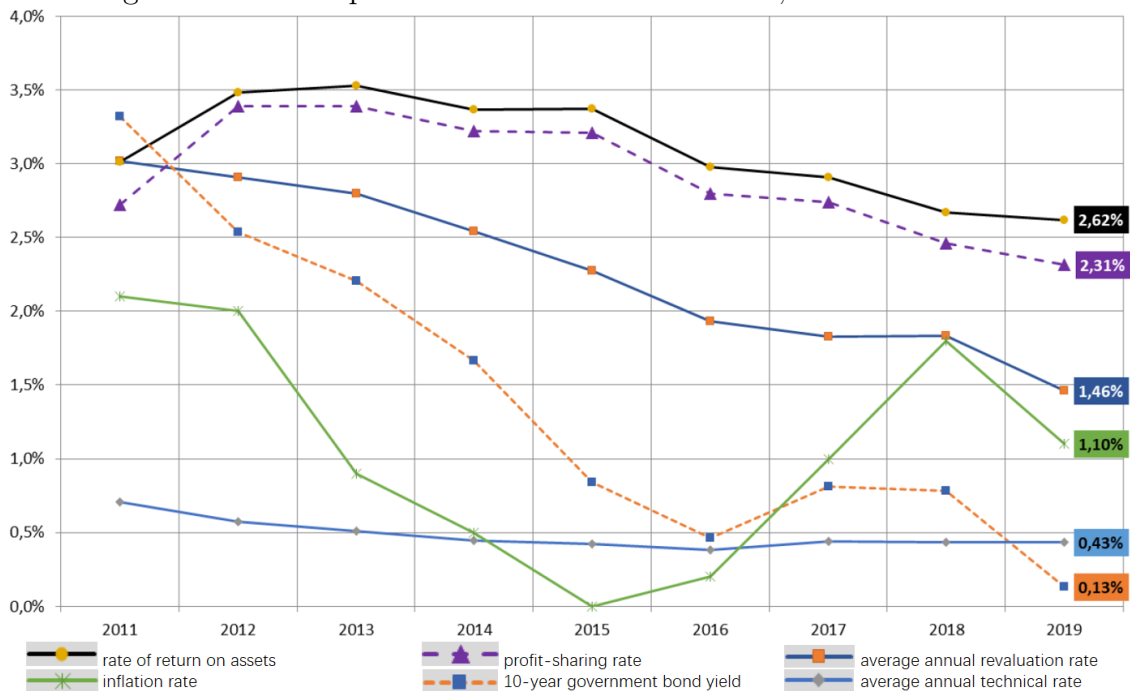
1.4 Diminishing rate of return on euro-denominated funds

1.4.1 Analysis of different rates

The profit-sharing rate¹⁸ of policyholders depends directly on the financial results of the insurers. It is determined by regulatory constraints and the distribution and provisioning strategy specific to each insurer.

Analysis of the impact of the low interest rate environment on life insurance, therefore, requires a study of the rate of return on insurer assets¹⁹. In addition to the profit-sharing rate and the rate of return on assets, the revaluation rate²⁰, technical rate, inflation rate, and 10-year government bond yield must be considered.

Figure 1.18 – Comparative trend of different rates, from 2011 to 2019.



Source: ACPR, [Capitaine and Ahado, 2020]

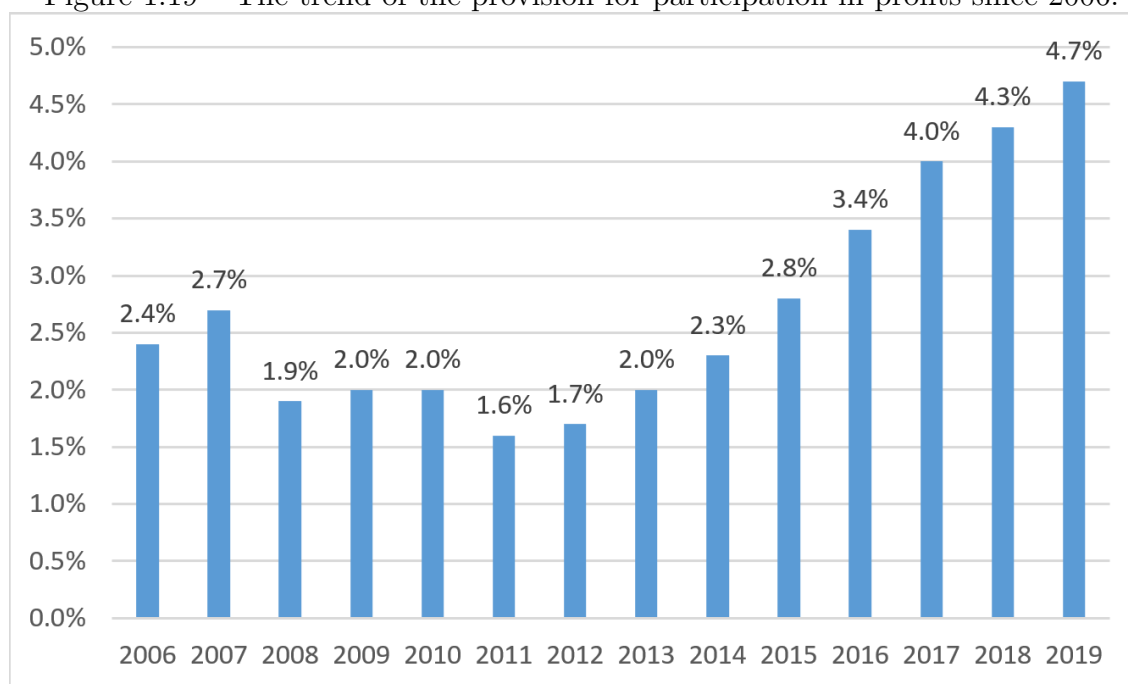
18. Profit-sharing rate = (technical interest + profit-sharing paid + net allocation to provision for profit sharing) / average mathematical provisions

19. Rate of return on assets = net financial income / average net book value

20. Revaluation rate: Interest rate consisting of the guaranteed return and participation in the technical and financial profits of the contract, as defined in Articles L.132-22 and A.132-7 of the Insurance Code. Gross of the technical rate and tax and social security contributions, but net of charges on outstandings, this is the rate actually paid to policyholders in respect of year N.

The sharp drop in revaluation rates is by an acceleration in the decline in interest rates and the appearance of negative rates on benchmark bonds. The 10-year government bond yield hit its lowest in 2019, as shown in Figure 1.18. The revaluation rate follows the decline in the rate of return on assets observed since 2015 with a variation fluctuating over the years [Ahado et al., 2017], which allows the profit-sharing provision (PPB) to be supplied. In 2019, the difference between the revaluation rate and the rate of return on assets was 1.16%, which automatically increases the reserve by insurance organizations of unserved profit sharing. Thus, the provision for participation in profits continued to increase in 2019 at a sustained rate to stand at 4.7% of life insurance provisions, shown in Figure 1.19.

Figure 1.19 – The trend of the provision for participation in profits since 2006.



Source: ACPR

Besides, lower inflation rates (on average around 1%, less than the ECB inflation target ²¹) offset some of the falls in the revaluation rates. The real returns net of inflation paid to policyholders is 0.36% in 2019, where around zero in 2018.

Moreover, as observed for the first time in 2019, the average annual technical rate guaranteed for the entire duration of the contract was higher than that of the 10-year

21. The European Central Bank aims at inflation rates of below, but close to, 2% over the medium term.

government bond yield, i.e., 0.43% and 0.13%, respectively (Figure 1.18).

1.4.2 Changes in investment behaviours

In a low-interest-rate environment, insurers have been changing their investment policy towards higher-paying investments by increasing the proportion of shares (at the cost of raising additional capital), private bond investments, or even real estate in their investments, or by accepting more geographical diversification for example, even if this is accompanied by greater risk-taking. [EIOPA, 2017] investigated both a quantitative and qualitative section focusing on the asset side of the balance sheet at a European level. Thirteen French groups participated in this survey. Several trends are identified: a small decrease in the debt portfolio against a small increase in other investments; a trend towards lower credit rating quality fixed income securities with downgrades of a large number of sovereign and corporate bonds; a trend towards more illiquid investments such as non-listed equity and loans excluding mortgages; the increased average maturity of the bond portfolio; the tendency to invest into new asset classes such as infrastructure, mortgages, loans and real estate. Natixis Investment Managers commissioned a global survey [Natixis, 2019] of 200 Chief Investment Officers (CIOs) at insurers in Europe, North America, and Asia. The survey results reveal three key trends driving investment strategy for insurance CIO teams: Three-quarters of insurers rank interest rates as key portfolio risk. 89 % of insurers globally say regulations deter them from investing in higher-risk assets. Two-thirds of insurers outsource at least some of their portfolio, mainly to gain access to expertise.

1.4.3 Concerns and solutions at different levels

French life insurers have multiple defense lines that should allow them to meet their commitments for many years to come if the current low-interest-rate environment continues [Juilliard, 2020]. If it were to continue, such a situation would gradually weigh on their long-term financial performance and credit quality. The following concerns and solutions are proposed from the perspective of life insurers and insurance industry supervision.

Life insurer

1. Mean reversion and long-term investments

It has been acknowledged that the returns of most financial assets do not follow a random

walk. For example, certain studies ([Fama and French, 1988] and [Bansal and Yaron, 2004]) indicate that equity prices tend to follow the mean reversion. [Campbell and Viceira, 2002] estimate a vector autoregressive model to describe the US yield processes. Their results show that the relative risk of holding shares decreases rapidly for holding periods of between 1 and 20 years, while the risk of holding long-term bonds increases sharply for holding periods of between 1 and 5 years. For time horizons between 20 and 50 years, the volatilities of aggregate returns on shares and long-term bonds are close. [Bec and Gollier, 2009] obtain similar results using French data.

Thanks to the smoothing of shocks on asset returns, and the temporal diversification of equity risk, savers with a long-term investment horizon may benefit from a more favorable risk-return performance than short-term investors [Gollier, 2015]. In euro-denominated funds, both the tax advantage when holding the contract at least 8 years and the relative long liability durations should encourage life insurers to benefit from a long-term risky investment.

2. Optimal asset allocation

The assets of a life insurance company consist of the policyholders' premiums, so each contract represents a specific liability for the life insurer. The main goal of life insurance companies may not be to maximize investment returns but instead to pay more attention to the management of premiums such that the guaranteed returns of policyholders can appropriately meet future benefits. Therefore, the optimal asset allocation strategy to maximize investment returns should be based on certain constraints, such as guaranteed capital and insolvency probability.

Based on this starting point, we will first apply the ruin theory to model the income of a life insurance company, which including the basic life insurance business process and return on investment generating process, and then build the optimal asset allocation strategy based on the obtained formulae of the income of a life insurance company. We introduce and discuss the optimal asset allocation strategy in detail in *Chapter 3*.

Life insurance industry

1. On the minimum guaranteed rate

French life insurers have a large scope to adjust the rates of return served to policyholders in line with changes in the financial markets. For more than 20 years, the minimum guaranteed returns granted to policyholders on their savings contracts have been systematically reduced to the extent that there are no longer any policies with multi-year return

guarantees [Juilliard, 2020]. The low level of minimum guaranteed rates gives them the flexibility to adjust the yields served to policyholders downwards. As shown in Figure 1.18, the average annual technical rate declined to 0.43% in 2019.

This same technique could be applied for several more years if the low-interest-rate environment persists. However, if this were to be the long-term case, insurers could be forced to stop any profit-sharing paid and pay policyholders the minimum guaranteed rate of return.

2. On tax rates and management fees linked to the holding period

Since the tax advantage disappears completely after only eight years, developing incentives on tax rates linked to the holding period after eight years should potentially reduce the withdrawals from the long-standing policyholders if the tax advantage that they would lose in the process is sufficient. Moreover, they potentially benefit from a more favorable risk-return performance with a longer-term investment horizon.

The reduction in managing fees for longer holding periods in life insurance contracts will also have similar influences as increasing incentives on tax rates.

3. On the development of euro-growth funds

In recent years, measures have been taken to limit the inflow of euro-denominated funds and encourage investment in unit-linked funds in the life insurance market. The combined effect of the two should have prompted the share of euro-growth funds to increase notably because of its characteristics, as shown in Section 1.3.1. However, the outstanding amount of euro-growth funds only increased from 1 billion euros to 3.1 billion euros since its issuance, which accounts for around 0.15% in overall life insurance markets in 2019. Incentives that may promote euro-growth funds can be further expanded to potential investors and existing policyholders of traditional life insurance contracts.

1.5 Conclusion

In this chapter, we dealt with the asset portfolio issues of euro-denominated life insurance in a low-interest-rate environment in France.

First of all, we investigated the background focused on the economic environment of the French life insurance industry. Secondly, we gave a comprehensive overview of the situation of the life insurance sector in France based on datasets from several institutions: including investment supports and constraints on euro-denominated funds and other life insurance products; life insurance activities and investments; the solvency (Solvency II)

regulations imposed on life insurance companies and relevant solvency condition of the life insurance industry, as well as a comparison in EU level. Next, we analyzed the declining rates of return on euro-denominated funds through comparison with different rates.

The insurers should consider the balance between the long-term interest of the insured, the fulfillment of the contractual commitments, and sustainable solvency. We proposed solutions from different perspectives to solve multiple problems faced by life insurance companies.

In the following chapters, we will address those risk exposures by applying ruin theory and risk models. In the modelization, the risk process comprises two independent processes, a basic risk process and a return on investment generating process. The basic risk process describes an insurance company that experiences two opposing cash flows: inflows of premiums and outflows of claims and withdrawals. In return on investment generating process, the portfolio is capable of allocating risk-free and risky assets. The risk process covers the multiple risks of the insurance business and the investment process of life insurers. The optimal asset allocation strategy is derived through the risk process. We get implications by examining the sensitivity of the optimal investment strategy to some parameters, including interest rates, probabilities of insolvency, guaranteed capital levels, and premium rates.

ANALYSIS OF LIFE INSURANCE CONTRACTS VIA MATHEMATICAL MODELING

2.1 Introduction

The economic result of a life insurance company depends not only on its life insurance business, but also on how well life insurers invest their capital. Euro-denominated life insurance faces multiple challenges and risks in terms of their life insurance business and investment activities in a low-interest-rate environment, as discussed in *Chapter 1*. In the second chapter, we construct a mathematical model of the income of life insurance companies by applying the ruin theory and risk models.

The ruin theory and risk models with investment income has a long history, going back to [Lundberg, 1903] with the ruin problem. In Lundberg's model, the company did not earn any investment income on its capital. [Cramér, 1938] modeled the value of an insurance company using a compound Poisson process with a drift, and estimated the probability of ruin of insurance companies. In the first half of the 20th century, the theory of stochastic process was far less developed, and the first attempt which incorporate investment incomes was undertaken by [Segerdahl, 1942] with the assumption that capital earns interest at a fixed rate r_f , which can be understood as the risk-free rate. Three decades later, with the inspiration from mathematical finance, the model was suggested that capital is allowed

to be invested in risky assets, with a risky rate of return R .

A path-breaking in the studies of ruin problems with compounding assets was made by [Harrison, 1977], considering a generalization of the classical model of collective risk theory. He obtains a general upper bound for the probability of ruin, a general solution for the case where the cumulative income process of an insurance company has no jumps. The results show that if the income process is well approximated by a Brownian motion with a drift, then the process of the asset is well approximated by a certain diffusion process, which he calls compounding Brownian motion, and the probability of ruin is well approximated by a corresponding first passage probability. [Delbaen and Haezendonck, 1987] give a general description of the classical risk process when macro-economic factors such as interest and inflation are taken into account, and they study the effects of factors on bounds on ruin probabilities. Their numerical results turned out that if the real interest rate tends to zero, the probability of ruin is a function of premiums, of the initial surplus and of the claims size distribution.

The problem of ruin in a risk model when assets earn investment income is treated in [Paulsen, 1993], [Paulsen, 1998], [Paulsen, 2008] and [Paulsen and Gjessing, 1997]. Their studies cover presentations of the relevant integro-differential equations, exact and numerical solutions, asymptotic results, bounds on the probability of ruin and also the possibility of minimizing the ruin probability by investment and possibly reinsurance control. They mainly focus on the continuous time models, but the discrete time models are also considered. The situation where the reserve of an insurance business is currently invested in an asset that may yield negative interest is addressed in [Kalashnikov and Norberg, 2002]. They obtain upper and lower bounds for the probability of ruin which are in general power functions of the initial reserve. [Morales and Schoutens, 2003] present a risk model achieved by incorporating a Lévy process when the aggregate claims and premium fluctuations evolve by jumps. They show how the infinite activity feature of such a family of processes can be used to account for discrete premium fluctuations as well as for semi-heavy tailed claims. [Klüppelberg and Kostadinova, 2008] consider an integrated insurance risk process when the insurance business is modeled by a compound Poisson process and the price of the risky asset is modeled by an exponential Lévy process. Their results show that the model carries a high risk which may originate either from large insurance outflows or from the risky investment. [Vostrikova and Spielmann, 2020] study the ruin problem with investment where the insurance business part X is a Lévy process, and the return on investment R is a semi-martingale. They obtain upper bounds on the

finite and infinite time ruin probabilities that decrease as a power function when the initial capital increases. [Devolder, 2011] investigates the market risk of a life insurer offering a fixed guaranteed rate and investing the premium in a risky fund by comparing various risk measurements, including a static approach, a dynamic approach, and a continuous approach based on the probability of ruin. The results are useful to measure and compare various liquidity and solvency conditions on long-term life insurance products. The ruin theory could also be useful to address the problem of determining appropriate target levels of Solvency Capital Requirement coverage ratio ([Loisel and Gerber, 2012]). In risk management, insurance companies have already started to set risk limits to guarantee that the Solvency Capital Requirement coverage ratio stays above a certain level with a large enough probability because of the new regulation framework Solvency II.

In our research, we analyze the income of life insurance companies through three case studies, where the insurance business and the investment are modeled by different processes to reflect different economic conditions. We study the effects of factors in the insurance business and investment processes on the income of life insurance companies based on the obtained formulae for the first two moments of the integrated risk process, i.e., the result of the insurance business and the net gains of the investment.

The work is arranged as follows. In Section 2.2, we present the elements of stochastic calculus which are needed in the calculation of our mathematical modeling. Then, we introduce the risk models and formulate the explicit form of the first two moments of the income of a life insurance company with three case studies, followed by analysis and discussions in Section 2.3. Section 2.4 shows the optimal asset allocation strategy to apply in Chapter 3. Finally, we give the conclusion in Section 2.5.

2.2 The elements of Stochastic Calculus

The first step of our work is to prepare the elements of stochastic calculus required in deriving the formulae of the moments of the integrated risk processes in Section 2.3. We present the notions of independence, conditional expectations, and properties of conditional expectations in Section 2.2.1, *Information and Conditioning*. Section 2.2.2, *Brownian Motion*, introduces Brownian motion and its properties. Itô integral and Itô's formula are introduced in Section 2.2.3, *Stochastic Calculus*. Section 2.2.4, *Jump Processes*, defines a jump process and presents stochastic integrals and stochastic calculus with respect to jump processes. We refer to [Shreve, 2004, chapter. 2, 3, 4 and 11] and [Gallardo, 2008,

chapter. 4 and 5] for the details.

2.2.1 Information and Conditioning

We first introduce the definitions related to information and conditioning in stochastic calculus.

Definition 2.2.1.1. *Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ -algebra $\mathcal{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ -algebras $\mathcal{F}(t)$, $0 \leq t \leq T$, a filtration.*

Definition 2.2.1.2. *Let X be a random variable defined on a nonempty sample space Ω . The σ -algebra generated by X , denoted $\sigma(X)$, is the collection of all subsets of Ω of the form of $\{\omega \in \Omega | X(\omega) \in B\}$, where B ranges over the Borel subsets of \mathbb{R} .*

Definition 2.2.1.3. *Let M be a random variable defined on a nonempty sample space Ω . Let \mathcal{G} be a σ -algebra of subsets of Ω . If every set in $\sigma(X)$ is also in \mathcal{G} , we say that X is \mathcal{G} -measurable.*

Definition 2.2.1.4. *Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}(t)$, $0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is an adapted stochastic process if, for each t , the random variable $X(t)$ is $\mathcal{F}(t)$ -measurable.*

In the continuous-time models of this thesis, asset prices and risk processes will all be adapted to a filtration that we regard as a model of the flow of public information.

Definition 2.2.1.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let \mathcal{G} and \mathcal{H} be sub- σ -algebras of \mathcal{F} . We say these two σ -algebras are independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \text{ for all } A \in \mathcal{G}, B \in \mathcal{H}.$$

Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We say these two random variables are independent if the σ -algebras they generate, $\sigma(X)$ and $\sigma(Y)$, are independent. We say that the random variable X is independent of the σ -algebra \mathcal{G} if \mathcal{G} and $\sigma(X)$ are independent.

Definition 2.2.1.6. *Let X be a square-integrable random variable. The variance of X , denoted $\text{Var}(X)$, is*

$$\text{Var}(X) = \mathbb{E} \left[(X - \mathbb{E}(X))^2 \right].$$

Because $(X - \mathbb{E}(X))^2$ is nonnegative, $\text{Var}(X)$ is always defined, although it may be infinite. The standard deviation of X is $\sqrt{\text{Var}(X)}$. The linearity of expectations shows that

$$\text{Var}(X) = \mathbb{E} [X^2] - (\mathbb{E}(X))^2 .$$

Let Y be another random variable and assume that $\mathbb{E}(X)$, $\text{Var}(X)$, $\mathbb{E}(Y)$ and $\text{Var}(Y)$ are all finite. The covariance of X and Y is

$$\text{Cov}(X, Y) = \mathbb{E} [(X - \mathbb{E}(X)) \cdot (Y - \mathbb{E}(Y))] .$$

The linearity of expectations shows that

$$\text{Cov}(X, Y) = \mathbb{E} [XY] - \mathbb{E}(X) \cdot \mathbb{E}(Y) .$$

In particular, $\mathbb{E}[XY] = \mathbb{E}(X) \cdot \mathbb{E}(Y)$ if and only if $\text{Cov}(X, Y) = 0$. Assume, in addition to the finiteness of expectations and variances, that $\text{Var}(X) > 0$ and $\text{Var}(Y) > 0$. The correlation coefficient of X and Y is

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} .$$

If $\rho(X, Y) = 0$ (or equivalently, $\text{Cov}(X, Y) = 0$), we say that X and Y are uncorrelated.

Definition 2.2.1.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let \mathcal{G} be a sub- σ -algebra of \mathcal{F} , and let X be a random variable that is either nonnegative or integrable. The conditional expectation of X given \mathcal{G} , denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies

- (i) $\mathbb{E}[X|\mathcal{G}]$ is \mathcal{G} -measurable, and
- (ii)

$$\int_A \mathbb{E}[X|\mathcal{G}(\omega)] d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{G} .$$

Proposition 2.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let \mathcal{G} be a sub- σ -algebra of \mathcal{F} .

- (i) If X and Y are integrable random variables and c_1 and c_2 are constants, then

$$\mathbb{E} [c_1 X + c_2 Y | \mathcal{G}] = c_1 \mathbb{E} [X | \mathcal{G}] + c_2 \mathbb{E} [Y | \mathcal{G}] .$$

- (ii) If X , Y and XY are integrable random variables, and X is \mathcal{G} -measurable, then

$$\mathbb{E} [XY | \mathcal{G}] = X \mathbb{E} [Y | \mathcal{G}] .$$

(iii) If \mathcal{H} is a sub- σ -algebra of \mathcal{G} (\mathcal{H} contains less information than \mathcal{G}) and X is an integrable random variable, then

$$\mathbb{E} \left[\mathbb{E} [X | \mathcal{G}] | \mathcal{H} \right] = \mathbb{E} [X | \mathcal{H}].$$

(iv) If X is integrable and independent of \mathcal{G} , then

$$\mathbb{E} [X | \mathcal{G}] = \mathbb{E} [X].$$

(v) If $\varphi(x)$ is a convex function of a dummy variable x and X is integrable, then

$$\mathbb{E} [\varphi(X) | \mathcal{G}] \geq \varphi (\mathbb{E} [X | \mathcal{G}]).$$

Definition 2.2.1.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration of sub- σ -algebra of \mathcal{F} . Consider an adapted integrable stochastic process $M(t)$, $0 \leq t \leq T$.

(i) If

$$\mathbb{E}[M(t) | \mathcal{F}(s)] = M(s) \text{ for all } 0 \leq s \leq t \leq T,$$

we say this process is a martingale. It has no tendency to rise or fall.

(ii) If

$$\mathbb{E}[M(t) | \mathcal{F}(s)] \geq M(s) \text{ for all } 0 \leq s \leq t \leq T,$$

we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise.

(iii) If

$$\mathbb{E}[M(t) | \mathcal{F}(s)] \leq M(s) \text{ for all } 0 \leq s \leq t \leq T,$$

we say this process is a submartingale. It has no tendency to rise; it may have a tendency to fall.

Definition 2.2.1.9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration of sub- σ -algebra of \mathcal{F} . Consider an adapted stochastic process $X(t)$, $0 \leq t \leq T$. Assume that for all $0 \leq s \leq t \leq T$ and for every nonnegative, Borel-measurable function f , there is another Borel-measurable function g such that

$$\mathbb{E}[f(X(t)) | \mathcal{F}(s)] = g(X(s)).$$

Then we say that the X is a Markov process.

Definition 2.2.1.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let τ be a random variable defined on the probability space taking values in $[0, \infty]$. Then τ is called a stopping time if

$$\{\tau \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0.$$

Remark 2.2.1. A stopping time τ has the property that the decision to stop at time t must be based on information available at time t .

2.2.2 Brownian Motion

In this subsection, we present Brownian motion and develop its basic properties.

Definition 2.2.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω . Then $W(t)$, $t \geq 0$, is a standard Brownian motion if for all $0 = t_0 < t_1 < \dots < t_m$ the increments

$$W(t_0), W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0,$$

$$\text{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

Theorem 2.2.2.1. Brownian motion is a \mathbb{F} -martingale, where \mathbb{F} is natural filtration of this Brownian motion.

Proof. Let $0 \leq s \leq t$ be given. Then

$$\begin{aligned} \mathbb{E}[W(t) | \mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s)) + W(s) | \mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s) | \mathcal{F}(s)] + \mathbb{E}[W(s) | \mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned}$$

□

Theorem 2.2.2.2. *Let W be a standard Brownian motion. Then the quadratic variation of W , $[W, W](T) = T$ for all $T \geq 0$ almost surely.*

Proof. Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$. Define the sampled quadratic variation corresponding to this partition to be

$$Q_\Pi = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2.$$

The sampled quadratic variation is the sum of independent random variables. Therefore, its mean and variance are the sums of the means and variances of these random variables. By Definition 2.2.1.10, we have

$$\mathbb{E} \left[(W(t_{j+1}) - W(t_j))^2 \right] = \text{Var} [W(t_{j+1}) - W(t_j)] = t_{j+1} - t_j,$$

which implies

$$\mathbb{E}(Q_\Pi) = \sum_{j=0}^{n-1} \mathbb{E} \left[(W(t_{j+1}) - W(t_j))^2 \right] = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = T.$$

Moreover,

$$\begin{aligned} & \text{Var} \left[(W(t_{j+1}) - W(t_j))^2 \right] \\ &= \mathbb{E} \left[\left((W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j) \right)^2 \right] \\ &= \mathbb{E} \left[(W(t_{j+1}) - W(t_j))^4 \right] - 2(t_{j+1} - t_j) \mathbb{E} \left[(W(t_{j+1}) - W(t_j))^2 \right] \\ & \quad + (t_{j+1} - t_j)^2. \end{aligned}$$

The fourth moment of a normal random variable with zero mean is three times its variance squared, therefore,

$$\begin{aligned} \mathbb{E} \left[(W(t_{j+1}) - W(t_j))^4 \right] &= 3(t_{j+1} - t_j)^2, \\ \text{Var} \left[(W(t_{j+1}) - W(t_j))^2 \right] &= 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 \\ &= 2(t_{j+1} - t_j)^2, \end{aligned}$$

and

$$\begin{aligned} \text{Var}(Q_\Pi) &= \sum_{j=0}^{n-1} \text{Var} \left[(W(t_{j+1}) - W(t_j))^2 \right] = 2 \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \\ &\leq \sum_{j=0}^{n-1} 2\|\Pi\| (t_{j+1} - t_j) = 2\|\Pi\| T. \end{aligned}$$

In particular, $\lim_{\|\Pi\| \rightarrow 0} \text{Var}(Q_\Pi) = 0$, and we conclude that when $\lim_{\|\Pi\| \rightarrow 0} \|\Pi\| = 0$, the limit in probability, $\lim_{\|\Pi\| \rightarrow 0} [Q_n - \mathbb{E}(Q_n)] = 0$ and since by definition, $[W, W](T) = \lim_{\|\Pi\| \rightarrow 0} Q_n$, we get $[W, W]_T = T$. \square

Remark 2.2.2. Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$. We can compute the cross variation of $W(t)$ with t and the quadratic variation of t with itself, which are

$$\lim_{\|\Pi\| \rightarrow 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j)) (t_{j+1} - t_j) = 0, \quad (2.2.2.1)$$

and

$$\lim_{\|\Pi\| \rightarrow 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0. \quad (2.2.2.2)$$

To see that the limit in (2.2.2.1) is 0, we observe that

$$\left| (W(t_{j+1}) - W(t_j)) (t_{j+1} - t_j) \right| \leq \max_{0 \leq k \leq n-1} \left| W(t_{k+1}) - W(t_k) \right| (t_{j+1} - t_j),$$

and so

$$\left| \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j)) (t_{j+1} - t_j) \right| \leq \max_{0 \leq k \leq n-1} \left| W(t_{k+1}) - W(t_k) \right| T.$$

Since W is continuous, $\max_{0 \leq k \leq n-1} \left| W(t_{k+1}) - W(t_k) \right|$ has limit zero as $\|\Pi\| \rightarrow 0$.

To see that the limit in (2.2.2.2) is 0, we observe that

$$\sum_{j=0}^{n-1} (t_{j+1} - t_j) \leq \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \sum_{j=0}^{n-1} (t_{j+1} - t_j) = \|\Pi\| \cdot T,$$

which obviously has limit zero as $\|\Pi\| \rightarrow 0$. We capture (2.2.2.1) and (2.2.2.2) by writing

$$dW(t) dt = 0, \quad dt dt = 0. \quad (2.2.2.3)$$

Theorem 2.2.2.3. Let $W(t)$, $t \geq 0$, be a Brownian motion with a filtration $\mathcal{F}(t)$, $t \geq 0$, and let σ be a constant. The process

$$Z(t) = \exp \left\{ \sigma W(t) - \frac{1}{2} \sigma^2 t \right\}, \quad (2.2.2.4)$$

$t \geq 0$, is a martingale.

Proof. For $0 \leq s \leq t$, we have

$$\begin{aligned} & \mathbb{E} \left[Z(t) \middle| \mathcal{F}(s) \right] \\ &= \mathbb{E} \left[\exp \left\{ \sigma W(t) - \frac{1}{2} \sigma^2 t \right\} \middle| \mathcal{F}(s) \right] \\ &= \mathbb{E} \left[\exp \{ \sigma (W(t) - W(s)) \} \exp \left\{ \sigma W(s) - \frac{1}{2} \sigma^2 t \right\} \middle| \mathcal{F}(s) \right] \\ &= \exp \left\{ \sigma W(s) - \frac{1}{2} \sigma^2 t \right\} \mathbb{E} \left[\exp \{ \sigma (W(t) - W(s)) \} \middle| \mathcal{F}(s) \right] \\ &= \exp \left\{ \sigma W(s) - \frac{1}{2} \sigma^2 t \right\} \mathbb{E} \left[\exp \{ \sigma (W(t) - W(s)) \} \right], \end{aligned} \quad (2.2.2.5)$$

where we have used Proposition 2.2.1 (ii) and (iv) for the last two steps, respectively. To compute the expected value of $\exp \{ \sigma (W(t) - W(s)) \}$, we apply transfer theorem. Because $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$,

$$\begin{aligned} & \mathbb{E} \left[\exp \{ \sigma (W(t) - W(s)) \} \right] \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} \exp \left\{ x\sigma - \frac{x^2}{2t} \right\} dx \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} \exp \left\{ \frac{(t-s)\sigma^2}{2} - \frac{1}{2(t-s)} [x - (t-s)\sigma]^2 \right\} \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \exp \left\{ \frac{(t-s)\sigma^2}{2} \right\} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{1}{2(t-s)} [x - (t-s)\sigma]^2 \right\} dx \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \exp \left\{ \frac{(t-s)\sigma^2}{2} \right\} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{1}{2(t-s)} u^2 \right\} du \\ &= \exp \left\{ \frac{1}{2} \sigma^2 (t-s) \right\}. \end{aligned}$$

Substituting this expected value into (2.2.2.5), we obtain the martingale property

$$\mathbb{E} [Z(t) | \mathcal{F}(s)] = \exp \left\{ \sigma W(s) - \frac{1}{2} \sigma^2 s \right\} = Z(s).$$

□

2.2.3 Stochastic Calculus

This subsection is devoted to the Itô integrals and develops their properties. These are used to model the value of a portfolio that results from trading assets in continuous time. The fact that the calculus differs from ordinary calculus is because Brownian motion has nonzero quadratic variation and is the source of the volatility term in Black-Scholes-Merton partial differential equation.

We fix a positive number T and an expression such as the following:

$$\int_0^t \Delta(t) dW(t). \tag{2.2.3.1}$$

The basic ingredients here are a standard Brownian motion $W(t)$, $t \geq 0$, together with a filtration $\mathcal{F}(t)$, $t \geq 0$, for this Brownian motion. We let the integrand $\Delta(t)$ be an adapted stochastic process. Requiring $\Delta(t)$ to be adapted means that we require $\Delta(t)$ to be $\mathcal{F}(t)$ -measurable for each $t \geq 0$. In other words, the information available at time t is sufficient to evaluate $\Delta(t)$ at that time.

The problem we face when trying to assign meaning to the Itô integral (2.2.3.1) is that Brownian motion paths cannot be differentiated with respect to time.

To define the integral (2.2.3.1), Itô devised the following way around the nondifferentiability of the the Brownian paths. We first define the Itô integral for simple integrands $\Delta(t)$ and then extend it to nonsimple integrands as a limit of the integral of simple integrands. Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$; i.e.,

$$0 = t_0 \leq t_1 \leq \dots \leq t_n = T.$$

Assume that $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$. Such a process $\Delta(t)$ is a simple process. We have

$$\begin{aligned} I(t) &= \Delta(t_0)[W(t) - W(t_0)] = \Delta(0)W(t), \quad 0 \leq t \leq t_1, \\ I(t) &= \Delta(t_0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \leq t \leq t_2, \end{aligned}$$

$I(t) = \Delta(t_0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t_2) - W(t_1)]$, $t_2 \leq t \leq t_3$,
and so on. In general, if $t_k \leq t \leq t_{k+1}$, then

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]. \quad (2.2.3.2)$$

The process $I(t)$ in (2.2.3.2) is the Itô integral of the simple process $\Delta(t)$, we write as

$$I(t) = \int_0^t \Delta(u) dW(u).$$

Next, we assume that $\Delta(t)$, $t \geq 0$, is a general stochastic process adapted to the filtration $\mathcal{F}(t)$, $t \geq 0$. We also assume the square-integrability condition

$$\mathbb{E} \int_0^t \Delta^2(t) dt < \infty. \quad (2.2.3.3)$$

We define the Itô integral for $\Delta(t)$ by the formula

$$\int_0^t \Delta(t) dW(u) = \lim_{n \rightarrow \infty} \int_0^t \Delta_n(u) dW(u), \quad 0 \leq t \leq T, \quad (2.2.3.4)$$

where $\Delta_n(t)$, $0 \leq t \leq T$, are simple functions such that

$$\mathbb{E} \int_0^t (\Delta_n(s) - \Delta(s))^2 ds \xrightarrow[n \rightarrow +\infty]{} 0.$$

Theorem 2.2.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies (2.2.3.3). Then $I(t) = \int_0^t \Delta(u) dW(u)$ defined by (2.2.3.4) has the following properties.*

- (i) **(Continuity)** *As a function of the upper limit of integrand t , the paths of $I(t)$ are continuous.*
- (ii) **(Adaptivity)** *For each t , $I(t)$ is $\mathcal{F}(t)$ -measurable.*
- (iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u) dW(u)$ and $J(t) = \int_0^t \Gamma(u) dW(u)$, then*

$$I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u)) dW(u);$$

furthermore, for every constant c , $cI(t) = \int_0^t c\Delta(u) dW(u)$.

- (iv) **(Martingale)** *$I(t)$ is a martingale.*
- (v) **(Itô isometry)** $\mathbb{E} I^2(t) = \mathbb{E} \int_0^t \Delta^2(u) du$.
- (vi) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u) du$.

The processes for which we develop stochastic calculus are the Itô processes defined below.

Definition 2.2.3.1. *Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u) dW(u) + \int_0^t \Theta(u) du, \quad (2.2.3.5)$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes.¹

In order to understand the volatility associated with Itô processes, we must determine the rate at which they accumulate quadratic variations.

Lemma 2.2.1. *The quadratic variation of the Itô process (2.2.3.5) is*

$$[X, X](t) = \int_0^t \Delta^2(u) du.$$

Proof. We write (2.2.3.5) in the differential notation,

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt$$

and then we use the differential multiplication table, $dW(t) dW(t) = dt$, $dt dW(t) = dW(t) dt = 0$, $dt dt = 0$, to compute

$$\begin{aligned} dX(t)dX(t) &= \Delta^2(t) dW(t) dW(t) + 2\Delta(t) \Theta(t) dW(t) dt + \Theta^2(t) dt dt \\ &= \Delta^2(t) dt. \end{aligned}$$

□

This says that, at each time t , the process X is accumulating quadratic variation at rate $\Delta^2(t)$ per unit time, and hence the total quadratic variation accumulated on the time interval $[0, t]$ is $[X, X](t) = \int_0^t \Delta^2(u) du$. This quadratic variation is solely due to the quadratic variation of the Itô integral $\int_0^t \Delta(u)dW(u)$. The ordinary integral $\int_0^t \Theta(u)du$ has zero quadratic variation and thus contributes nothing to the quadratic variation of X .

1. We assume that $\mathbb{E} \int_0^t \Delta^2(u)du$ and $\int_0^t |\Theta(u)|du$ are finite for every $t > 0$ so that the integrals on the right-hand side of (2.2.3.5) are defined and the Itô integral $\int_0^t \Delta(u)dW(u)$ is a martingale.

Definition 2.2.3.2. Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 2.2.3.1, and let $\Gamma(t)$, $t \geq 0$, be an adapted process. We define the integral with respect to an Itô process

$$\int_0^t \Gamma(u) dX(u) = \int_0^t \Gamma(u) \Delta(u) dW(u) + \int_0^t \Gamma(u) \Theta(u) du. \quad (2.2.3.6)$$

Theorem 2.2.3.2. Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 2.2.3.1, and let $f(t, x)$ be a continuous function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$, and $f_{xx}(t, x)$ are defined, continuous and bounded. Then, for every $T \geq 0$, the Itô formula for an Itô process is

$$\begin{aligned} & f(T, X(T)) \\ &= f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) dX(t) \\ & \quad + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) d[X^c, X^c](t) \\ &= f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) \Delta(t) dW(t) \\ & \quad + \int_0^T f_x(t, X(t)) \Theta(t) dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) \Delta^2(t) dt, \end{aligned} \quad (2.2.3.7)$$

where X^c stands for the continuous martingale part of X .

Remark 2.2.3. Every term on the right-hand side has a solid definition, and in the end the right-hand side reduces to a sum of a nonrandom quantity, three ordinary (Lebesgue) integrals with respect to time, and an Itô integral. However, it is easier if this theorem is in differential notation. We may write (2.2.3.7) as

$$\begin{aligned} & df(t, X(t)) \\ &= f_t(t, X(t)) dt + f_x(t, X(t)) dX(t) + \frac{1}{2} f_{xx}(t, X(t)) d[X^c, X^c](t) \\ &= f_t(t, X(t)) dt + f_x(t, X(t)) \Delta(t) dW(t) \\ & \quad + f_x(t, X(t)) \Theta(t) dt + \frac{1}{2} f_{xx}(t, X(t)) \Delta^2(t) dt. \end{aligned} \quad (2.2.3.8)$$

Theorem 2.2.3.3. Let $W(s)$, $s \geq 0$, be a standard Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s) dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s) ds$.

Proof. $(I(t))_{t \geq 0}$ is a martingale and $I(t) = 0$ for each $t \geq 0$, we must have $\mathbb{E}(I(t)) = I(0) = 0$. Itô's isometry (Theorem 2.2.3.1(v)) implies that

$$\text{Var}(I(t)) = \mathbb{E}(I^2(t)) = \int_0^t \Delta^2(s) ds.$$

The next is to show that $I(t)$ is normally distributed. We shall do this by establishing that $I(t)$ has the moment-generating function of a normal random variable with mean zero and variance $\int_0^t \Delta^2(s) ds$, i.e.,

$$\mathbb{E} e^{uI(t)} = \exp \left\{ \frac{1}{2} u^2 \int_0^t \Delta^2(s) ds \right\} \text{ for all } u \in \mathbb{R}. \quad (2.2.3.9)$$

Because $\Delta(s)$ is not random, (2.2.3.9) is equivalent to

$$\begin{aligned} & \mathbb{E} \exp \left\{ uI(t) - \frac{1}{2} u^2 \int_0^t \Delta^2(s) ds \right\} \\ &= \mathbb{E} \exp \left\{ \int_0^t u \Delta(s) dW(s) - \frac{1}{2} \int_0^t (u \Delta(s))^2 ds \right\}. \end{aligned} \quad (2.2.3.10)$$

The last expectation is equal to 1 since the process

$$\exp \left\{ \int_0^t u \Delta(s) dW(s) - \frac{1}{2} \int_0^t (u \Delta(s))^2 ds \right\}$$

is a martingale, and it is a generalized geometric Brownian motion with mean rate of return zero, and volatility $u \Delta(s)$, which takes the value 1 at $t = 0$. \square

2.2.4 Jump Processes

In this subsection, we present the processes with jumps and independent increments, including the Poisson process and compound Poisson process, as well as stochastic integrals and stochastic calculus for jump processes. We will consider the jump process in *Case 2* and *Case 2 bis* in Section 2.3.3 when the basic insurance business process is modeled by the sum of a Brownian Motion with a drift and a compound Poisson process. The withdrawals in life insurance business will be modeled by a compound Poisson process.

The most elementary and useful jump process is the standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ which is a counting process, i.e., $(N_t)_{t \in \mathbb{R}_+}$ has jumps of size +1 only, and its paths are constant between two jumps.

Definition 2.2.4.1. Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < \dots < t_n$ be given. Then the increments

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$$

are stationary and independent, and

$$\mathbb{P}(\{N(t_{j+1}) - N(t_j) = k\}) = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}, \quad k = 0, 1, \dots$$

Then, we calculate the mean and variance of Poisson increments. With the exponential power series, we have:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=2}^{\infty} \frac{x^{k-2}}{(k-2)!},$$

and according to Definition 2.2.4.1, we have

$$\sum_{k=0}^{\infty} \mathbb{P}(\{N(t) - N(s) = k\}) = e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^k}{k!} = e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)} = 1.$$

The expected increment is

$$\begin{aligned} \mathbb{E}[N(t) - N(s)] &= \sum_{k=0}^{\infty} k \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} \\ &= \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1} (t-s)^{k-1}}{(k-1)!} \\ &= \lambda(t-s) \cdot e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)} \\ &= \lambda(t-s). \end{aligned}$$

The second moment of the increment is

$$\begin{aligned} \mathbb{E}[(N(t) - N(s))^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} \\ &= e^{-\lambda(t-s)} \sum_{k=1}^{\infty} (k-1+1) \frac{\lambda^k (t-s)^k}{(k-1)!} \\ &= e^{-\lambda(t-s)} \sum_{k=2}^{\infty} \frac{\lambda^k (t-s)^k}{(k-2)!} + e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^k (t-s)^k}{(k-1)!} \end{aligned}$$

$$\begin{aligned}
 &= \lambda^2(t-s)^2 e^{-\lambda(t-s)} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}(t-s)^{k-2}}{(k-2)!} \\
 &\quad + \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!} \\
 &= \lambda^2(t-s)^2 + \lambda(t-s).
 \end{aligned}$$

And

$$\begin{aligned}
 \text{Var}[N(t) - N(s)] &= \mathbb{E} \left[(N(t) - N(s))^2 \right] - (\mathbb{E}[N(t) - N(s)])^2 \\
 &= \lambda(t-s).
 \end{aligned}$$

Theorem 2.2.4.1. *Let $N(t)$ be a Poisson process with intensity λ . Define the compensated Poisson process*

$$M(t) = N(t) - \lambda t.$$

Then $M(t)$ is a martingale.

Proof. Let $0 \leq s < t$ be given. Because $N(t) - N(s)$ is independent of $\mathcal{F}(s)$ and has expected value $\lambda(t-s)$, we have

$$\begin{aligned}
 \mathbb{E}[M(t) | \mathcal{F}(s)] &= \mathbb{E}[M(t) - M(s) | \mathcal{F}(s)] + \mathbb{E}[M(s) | \mathcal{F}(s)] \\
 &= \mathbb{E}[N(t) - N(s) - \lambda(t-s) | \mathcal{F}(s)] + M(s) \\
 &= \mathbb{E}[N(t) - N(s)] - \lambda(t-s) + M(s) \\
 &= M(s).
 \end{aligned}$$

□

Since the jumps of the Poisson process are of constant size, it is not realistic to develop financial markets based on it. We need jump processes that can have random jump sizes. Let $N(t), t > 0$ be a Poisson process with intensity λ , and let Z_1, Z_2, \dots be a sequence of identically distributed random variables with mean $\beta = \mathbb{E}(Z_i)$. Assume the random variables Z_1, Z_2, \dots are mutually independent and also independent of the Poisson process $N(t), t > 0$.

Definition 2.2.4.2. *The process $Q(t)_{t \in \mathbb{R}_+}$ given by the random sum*

$$Q_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N(t)} Z_k, \quad t \in \mathbb{R}_+,$$

is called a compound Poisson process.

Like the simple Poisson process $N(t)$, the increments of the compound Poisson process $Q(t)$ are independent. For $0 \leq s < t$,

$$Q(s) = \sum_{i=1}^{N(s)} Z_i,$$

which sums up the first $N(s)$ jumps, and

$$Q(t) - Q(s) = \sum_{i=N(s)+1}^{N(t)} Z_i,$$

which sums up jumps $N(s) + 1$ to $N(t)$, are independent. Moreover, $Q(t) - Q(s)$ has the same distribution as $Q(t - s)$ because $N(t) - N(s)$ has the same distribution as $N(t - s)$, and $(Z_k)_{k \in \mathbb{N}^*}$ are i.i.d.

The mean of the compound Poisson process is

$$\begin{aligned} \mathbb{E} Q(t) &= \sum_{k=0}^{\infty} \mathbb{E} \left[\sum_{i=1}^k Z_i \middle| N(t) = k \right] \mathbb{P} \{N(t) = k\} \\ &= \sum_{k=0}^{\infty} \beta k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \beta \lambda t e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{(k-1)}}{(k-1)!} = \beta \lambda t. \end{aligned} \tag{2.2.4.1}$$

On average, there are λt jumps in the time interval $[0, t]$, the average jump size is β , and the number of jumps is independent of the size of the jumps. Before to present the variance of the compound Poisson process, we prove the following theorem.

Theorem 2.2.4.2. *Suppose that X is a random variable with finite second moment $\mathbb{E}[X^2]$. As a consequence $\mathbb{E}[|X|] < \infty$, and for any random variable Y , $\mathbb{E}[X^2|Y]$ and $\mathbb{E}[X|Y]$ are well defined with $\mathbb{E}[\mathbb{E}[X^2|Y]] = \mathbb{E}[X^2] < \infty$ and $\mathbb{E}[\mathbb{E}[|X||Y]] = \mathbb{E}[|X|] < \infty$. Then*

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y]),$$

where $Var(X|Y)$ is defined as

$$Var(X|Y) = \mathbb{E} \left[(X - \mathbb{E}(X|Y))^2 | Y \right] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2.$$

Proof. The two terms on the right are

$$\mathbb{E}[Var(X|Y)] = \mathbb{E} \left[\mathbb{E}[X^2|Y] \right] - \mathbb{E} \left[(\mathbb{E}[X|Y])^2 \right] = \mathbb{E}[X^2] - \mathbb{E} \left[(\mathbb{E}[X|Y])^2 \right],$$

and

$$Var(\mathbb{E}[X|Y]) = \mathbb{E} \left[(\mathbb{E}[X|Y])^2 \right] - (\mathbb{E}[\mathbb{E}[X|Y]])^2 = \mathbb{E} \left[(\mathbb{E}[X|Y])^2 \right] - (\mathbb{E}[X])^2.$$

Combining last two expressions gives the result. □

Theorem 2.2.4.3. *Let N be a nonnegative integer-valued Poisson process with intensity λ . Let $Z_n : n \geq 1$ be a sequence of independent and identically distributed random variables with mean $\beta = \mathbb{E}[Z_1]$ and finite variance $\sigma_J^2 = Var(Z_1)$. Suppose that N is independent of the sequence $Z_n : n \geq 1$. Then*

$$Var \left(\sum_{k=1}^{N_t} Z_k \right) = \sigma_J^2 \mathbb{E}[N_t] + \beta^2 Var(N_t) = \lambda t (\sigma_J^2 + \beta^2).$$

Proof. Let $S = \sum_{i=1}^{N_t} Z_i$ with fixed t , then

$$Var(S|N_t = n) = n\sigma_J^2 \text{ and } \mathbb{E}[S|N_t = n] = n\beta,$$

therefore

$$Var(S|N_t) = N_t\sigma_J^2 \text{ and } \mathbb{E}[S|N_t] = N_t\beta.$$

Then the conditional variance formula gives

$$Var(S) = \mathbb{E}[Var(S|N_t)] + Var(\mathbb{E}[S|N_t]) = \lambda t (\sigma_J^2 + \beta^2).$$

□

Theorem 2.2.4.4. *Let $Q(t)$ be the compound Poisson process as described in Definition*

2.2.4.1. The compensated compound Poisson process

$$Q(t) - \beta\lambda t$$

is a martingale.

Proof. Let $0 \leq s < t$ be given. Because the increments $Q(t) - Q(s)$ is independent of $\mathcal{F}(s)$ and has the mean $\beta\lambda(t - s)$, we have

$$\begin{aligned} \mathbb{E} [Q(t) - \beta\lambda t | \mathcal{F}(s)] &= \mathbb{E} [Q(t) - Q(s) | \mathcal{F}(s)] + Q(s) - \beta\lambda t \\ &= \beta\lambda(t - s) + Q(s) - \beta\lambda t \\ &= Q(s) - \beta\lambda s. \end{aligned}$$

□

Next, we will present the stochastic integral when the integrator is a process with jumps.

Definition 2.2.4.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration on this space. We say that a standard Brownian motion W is a Brownian motion relative to the filtration $\mathcal{F}(t)$, $t \geq 0$ if $W(t)$ is $\mathcal{F}(t)$ -measurable for every t and for every $u > t$ the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. Similarly for a Poisson process $N(t)$ and a compound Poisson process $Q(t)$.

We define the stochastic integral

$$\int_0^t \Phi(s) dX(s),$$

where the integrator X is cad-log process which can have jumps. More precisely,

$$X(t) = X(0) + I(t) + R(t) + J(t), \tag{2.2.4.2}$$

where $X(0)$ is a nonrandom initial condition, the process

$$I(t) = \int_0^t \Gamma(s) dW(s)$$

is an Itô integral of an adapted process $\Gamma(s)$ with respect to a Brownian motion, the process

$$R(t) = \int_0^t \Theta(s) ds$$

is a Riemann integral for some adapted process $\Theta(t)$, and $J(t)$ is an adapted, right-continuous pure jump process with $J(0) = 0$. By right-continuous, it means that $J(t) = \lim_{s \downarrow t} J(s)$ for all $t \geq 0$. The left-continuous version of such a process will be denoted $J(t-)$. We define the continuous part of $X(t)$ to be

$$X^c(t) = X(0) + I(t) + R(t) = X(0) + \int_0^t \Gamma(s) dW(s) + \int_0^t \Theta(s) ds.$$

Definition 2.2.4.4. *Let $X(t)$ be a process of the form (2.2.4.2) and let Φ be an adapted process. The stochastic integral of Φ with respect to X is defined to be*

$$\int_0^t \Phi(s) dX(s) = \int_0^t \Phi(s) \Gamma(s) dW(s) + \int_0^t \Phi(s) \Theta(s) ds + \sum_{0 < s \leq t} \Phi(s) \Delta J(s).$$

In differential notation

$$\begin{aligned} \Phi(t) dX(t) &= \Phi(t) \Gamma(t) dW(t) + \Phi(t) \Theta(t) dt + \Phi(t) dJ(t) \\ &= \Phi(t) dI(t) + \Phi(t) dR(t) + \Phi(t) dJ(t) \\ &= \Phi(t) dX^c(t) + \Phi(t) dJ(t). \end{aligned}$$

To write the Itô formula for processes with jumps, we need to calculate the quadratic variation.

Let $X(t)$ be a jump process, we choose $0 = t_1 < t_2 < \dots < t_n = T$, denote the set of these times by $\Pi = \{t_0, t_1, \dots, t_n\}$, and denote the length of the longest subinterval by $\|\Pi\| = \max_j (t_{j+1} - t_j)$, and define

$$Q_\Pi(X) = \sum_{j=0}^{n-1} (X(t_{j+1}) - X(t_j))^2.$$

The quadratic variation of X on $[0, T]$ is defined to be

$$[X, X](T) = \lim_{\|\Pi\| \rightarrow 0} Q_\Pi(X),$$

as $\|\Pi\| \rightarrow 0$ and as the number of points in Π approach infinity. Let $X(1)$ and $X(2)$ be jump processes, we define the concept of cross variation.

$$C_\Pi(X_1, X_2) = \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j)) (X_2(t_{j+1}) - X_2(t_j)),$$

and

$$[X_1, X_2](T) = \lim_{\|\Pi\| \rightarrow 0} C_\Pi(X_1, X_2).$$

Theorem 2.2.4.5. *Let $X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$ be a jump process, where $I_1(t) = \int_0^t \Gamma_1(s) dW(s)$, $R_1(t) = \int_0^t \Theta_1(s) ds$, and $J_1(t)$ is a right-continuous pure jump process. Then $X_1^c(t) = X_1(0) + I_1(t) + R_1(t)$ and*

$$\begin{aligned} [X_1, X_1](T) &= [X_1^c, X_1^c](T) + [J_1, J_1](T) \\ &= \int_0^T \Gamma_1^2(s) ds + \sum_{0 < s \leq T} (\Delta J_1(s))^2. \end{aligned}$$

Let $X_2(t) = X_2(0) + I_2(t) + R_2(t) + J_2(t)$ be a jump process, where $I_2(t) = \int_0^t \Gamma_2(s) dW(s)$, $R_2(t) = \int_0^t \Theta_2(s) ds$, and $J_2(t)$ is a right-continuous pure jump process. Then $X_2^c(t) = X_2(0) + I_2(t) + R_2(t)$, and

$$\begin{aligned} [X_1, X_2](T) &= [X_1^c, X_2^c](T) + [J_1, J_2](T) \\ &= \int_0^T \Gamma_1(s) \Gamma_2(s) ds + \sum_{0 < s \leq T} \Delta J_1(s) \Delta J_2(s). \end{aligned} \tag{2.2.4.3}$$

Proof. We have

$$\begin{aligned} C_\Pi(X_1, X_2) &= \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j)) (X_2(t_{j+1}) - X_2(t_j)) \\ &= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j) + J_1(t_{j+1}) - J_1(t_j)) \\ &\quad \times (X_2^c(t_{j+1}) - X_2^c(t_j) + J_2(t_{j+1}) - J_2(t_j)) \\ &= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j)) (X_2^c(t_{j+1}) - X_2^c(t_j)) \\ &\quad + \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j)) (J_2(t_{j+1}) - J_2(t_j)) \\ &\quad + \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j)) (X_2^c(t_{j+1}) - X_2^c(t_j)) \\ &\quad + \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j)) (J_2(t_{j+1}) - J_2(t_j)). \end{aligned} \tag{2.2.4.4}$$

For the first term on the right-hand side of (2.2.4.3), by the theory of continuous processes,

we have

$$\begin{aligned} \lim_{\|\Pi\| \rightarrow 0} \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j)) (X_2^c(t_{j+1}) - X_2^c(t_j)) &= [X_1^c, X_2^c](T) \\ &= \int_0^T \Gamma_1(s) \Gamma_2(s) ds. \end{aligned}$$

For the second term on the right-hand side of (2.2.4.4), we fix $\omega \in \Omega$. Then

$$\begin{aligned} & \left| \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j)) (J_2(t_{j+1}) - J_2(t_j)) \right| \\ & \leq \max_{0 \leq j \leq n-1} \left| (X_1^c(t_{j+1}) - X_1^c(t_j)) \right| \cdot \sum_{j=0}^{n-1} \left| (J_2(t_{j+1}) - J_2(t_j)) \right| \\ & \leq \max_{0 \leq j \leq n-1} \left| (X_1^c(t_{j+1}) - X_1^c(t_j)) \right| \cdot \sum_{0 < s \leq T} |\Delta J_2(s)|. \end{aligned}$$

As $\|\Pi\| \rightarrow 0$, the factor $\max_{0 \leq j \leq n-1} \left| (X_1^c(t_{j+1}) - X_1^c(t_j)) \right|$ has limit zero, whereas $\sum_{0 < s \leq T} |\Delta J_2(s)|$ is a finite number not depending on Π . Hence, the second term on the right hand side of (2.2.4.4) has limit zero as $\|\Pi\| \rightarrow 0$. Similarly, the third term on the right hand side of (2.2.4.4) also has limit zero. For the fourth term on the right hand side of (2.2.4.4), we fix an arbitrary $\omega \in \Omega$, which sets the paths of these processes, and choose the points in Π so close together that there is at most one jump of J_1 in each interval $(t_j, t_{j+1}]$, at most one jump of J_2 in each interval $(t_j, t_{j+1}]$, and if J_1 and J_2 have a jump in the same interval, then these jumps are simultaneous. Let A_1 denote the set of indices j for which $(t_j, t_{j+1}]$ contains a jump of J_1 , and let A_2 denote the set of indices j for which $(t_j, t_{j+1}]$ contains a jump of J_2 . Then we have

$$\begin{aligned} & \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j)) (J_2(t_{j+1}) - J_2(t_j)) \\ & = \sum_{j \in A_1 \cap A_2} (J_1(t_{j+1}) - J_1(t_j)) (J_2(t_{j+1}) - J_2(t_j)) \\ & = \sum_{0 < s \leq T} \Delta J_1(s) \Delta J_2(s). \end{aligned}$$

□

Remark 2.2.4. Equality (2.2.4.3) in differential notation says that if

$$X_1(t) = X_1(0) + X_1^c(t) + J_1(t), \quad X_2(t) = X_2(0) + X_2^c(t) + J_2(t),$$

then

$$dX_1(t)dX_2(t) = dX_1^c(t)dX_2^c(t) + dJ_1(t)dJ_2(t).$$

In particular,

$$dX_1^c(t)dJ_2(t) = dX_2^c(t)dJ_1(t) = 0.$$

It shows that the cross variation between two processes is zero if one of them is continuous and the other is pure jump process. In order to get a nonzero cross variation, both processes must have a dW term or the processes must have simultaneous jumps. It means that the cross variation between a Brownian motion and a compensated Poisson process is also zero.

Theorem 2.2.4.6. *Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined, continuous and bounded. Then*

$$\begin{aligned} & f(X(t)) \\ &= f(X(0)) + \int_0^t f'(X(s)) dX^c(s) + \frac{1}{2} \int_0^t f''(X(s)) dX^c(s)dX^c(s) \quad (2.2.4.5) \\ &+ \sum_{0 < s \leq t} [f(X(s)) - f(X(s-))]. \end{aligned}$$

Proof. Fix $\omega \in \Omega$, which fixes the path of X , and let $\tau_0 = 0 < \tau_1 < \tau_2 < \dots < \tau_{n-1} < \tau_n = t$ be the jump times in $[0, t)$ of this path of the process X . We set $\tau_0 = 0$, which is not a jump time, and $\tau_n = t$, which may or may not be a jump time. Whenever $u < v$ are both in the same interval (τ_j, τ_{j+1}) , there is no jump between times u and v , and from the Itô formula for continuous processes, we have

$$f(X(v)) - f(X(u)) = \int_u^v f'(X(s)) dX^c(s) + \frac{1}{2} \int_u^v f''(X(s)) dX^c(s)dX^c(s).$$

Letting $u \downarrow \tau_j$ and $v \uparrow \tau_{j+1}$ and using the right-continuity of X , we conclude that

$$\begin{aligned} & f(X(\tau_{j+1}-)) - f(X(\tau_j)) \\ &= \int_{\tau_j}^{\tau_{j+1}} f'(X(s)) dX^c(s) + \frac{1}{2} \int_{\tau_j}^{\tau_{j+1}} f''(X(s)) dX^c(s)dX^c(s). \end{aligned}$$

And adding the jump in $f(X)$ at time τ_{j+1} , we have

$$\begin{aligned} & f(X(\tau_{j+1})) - f(X(\tau_j)) \\ &= \int_{\tau_j}^{\tau_{j+1}} f'(X(s)) dX^c(s) + \frac{1}{2} \int_{\tau_j}^{\tau_{j+1}} f''(X(s)) dX^c(s)dX^c(s) \\ & \quad + f(X(\tau_{j+1})) - f(X(\tau_{j+1}-)). \end{aligned}$$

And summing over $j = 0, \dots, n-1$, we obtain

$$\begin{aligned} & f(X(t)) - f(X(0)) \\ &= \sum_{j=0}^{n-1} [f(X(\tau_{j+1})) - f(X(\tau_j))] \\ &= \int_0^t f'(X(s)) dX^c(s) + \frac{1}{2} \int_0^t f''(X(s)) dX^c(s)dX^c(s) \\ & \quad + \sum_{j=0}^{n-1} [f(X(\tau_{j+1})) - f(X(\tau_{j+1}-))]. \end{aligned}$$

□

Corollary 2.2.4.1. *Let $W(t)$ be a Brownian motion and let $N(t)$ be a Poisson process with intensity $\lambda > 0$, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration $\mathcal{F}(t)$, $t \geq 0$. Let u_1 and u_2 be fixed real numbers and define*

$$Y(t) = \exp \left\{ u_1 W(t) + u_2 N(t) - \frac{1}{2} u_1^2 t - \lambda (e^{u_2} - 1) t \right\}.$$

Then, if the process $W(t)$ and $N(t)$ are independent, $(Y(t))_{t \geq 0}$ is a martingale and $\mathbb{E}(Y(t)) = 1$ for all $t \geq 0$.

Proof. We use the Itô formula to show that Y is a martingale.

First, we define

$$X(s) = u_1 W(s) + u_2 N(s) - \frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s,$$

and $f(x) = e^x$, so that $Y(s) = f(X(s))$. The process $X(s)$ has Itô integral part $I(s) = u_1 W(s)$, Riemann integral part $R(s) = -\frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s$, and pure jump part $J(s) = u_2 N(s)$. And

$$dX^c(s) = u_1 dW(s), \quad dX^c(s)dX^c(s) = u_1^2 ds.$$

Next, we observe that if Y has a jump at time s , then

$$Y(s) = \exp \left\{ u_1 W(s) + u_2 (N(s-) - \frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s) \right\} = Y(s-) e^{u_2}.$$

Then

$$Y(s) - Y(s-) = (e^{u_2} - 1) Y(s-) \Delta N(s).$$

By the Itô formula (2.2.4.5) for jump processes,

$$\begin{aligned} & Y(t) \\ &= f(X(t)) \\ &= f(X(0)) + \int_0^t f'(X(s)) dX^c(s) + \frac{1}{2} \int_0^t f''(X(s)) dX^c(s) dX^c(s) \\ &\quad + \sum_{0 < s \leq t} [f(X(s)) - f(X(s-))] \\ &= 1 + u_1 \int_0^t Y(s) dW(s) - \frac{1}{2} u_1^2 \int_0^t Y(s) ds - \lambda (e^{u_2} - 1) \int_0^t Y(s) ds \\ &\quad + \frac{1}{2} u_1^2 \int_0^t Y(s) ds + \sum_{0 < s \leq t} [Y(s) - Y(s-)] \\ &= 1 + u_1 \int_0^t Y(s) dW(s) - \lambda (e^{u_2} - 1) \int_0^t Y(s) ds \\ &\quad + (e^{u_2} - 1) \int_0^t Y(s-) \Delta N(s) \\ &= 1 + u_1 \int_0^t Y(s) dW(s) - (e^{u_2} - 1) \int_0^t Y(s-) dM(s), \end{aligned} \tag{2.2.4.6}$$

where $M(s) = N(s) - \lambda s$ is a compensated Poisson process. Because the fact that Y has only finitely many jumps, $\int_0^t Y(s) ds = \int_0^t Y(s-) ds$. In the last line of (2.2.4.6), the Itô integral $\int_0^t Y(s) dW(s)$ is a martingale, and the integral of the left-continuous process $Y(s-)$ with respect to the martingale $M(s)$ is also. Therefore, Y is a martingale. Because $Y(0) = 1$, and Y is a martingale, we have $\mathbb{E} Y(t) = 1$ for all t , and

$$\mathbb{E} \exp \left\{ u_1 W(t) + u_2 N(t) - \frac{1}{2} u_1^2 t - \lambda (e^{u_2} - 1) t \right\} = 1 \text{ for all } t \geq 0.$$

□

Remark 2.2.5. From the last line, we can obtain the joint moment-generating function formula

$$\mathbb{E} \exp \{ u_1 W(t) + u_2 N(t) \} = \exp \left\{ \frac{1}{2} u_1^2 t \right\} \cdot \exp \{ \lambda t (e^{u_2} - 1) \}.$$

This is the product of moment-generating function $\mathbb{E} e^{u_1 W(t)} = \exp\left\{\frac{1}{2}u_1^2 t\right\}$ for $W(t)$ and the moment-generating function $\mathbb{E} e^{u_2 N(t)} = \exp\{\lambda t(e^{u_2} - 1)\}$ for $N(t)$. Since the joint moment-generating function factors into the product of moment-generating functions, the random variables $W(t)$ and $N(t)$ are independent.

Theorem 2.2.4.7. *Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t, x_1, x_2)$ be a continuous function whose first and second partial derivatives appearing in the following formula are defined, continuous and bounded. Then*

$$\begin{aligned}
 & f(t, X_1(t), X_2(t)) \\
 = & f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s)) ds \\
 & + \int_0^t f_{x_1}(s, X_1(s), X_2(s)) dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s)) dX_2^c(s) \\
 & + \frac{1}{2} \int_0^t f_{x_1, x_1}(s, X_1(s), X_2(s)) dX_1^c(s) dX_1^c(s) \\
 & + \int_0^t f_{x_1, x_2}(s, X_1(s), X_2(s)) dX_1^c(s) dX_2^c(s) \\
 & + \frac{1}{2} \int_0^t f_{x_2, x_2}(s, X_1(s), X_2(s)) dX_2^c(s) dX_2^c(s) \\
 & + \sum_{0 < s \leq t} [f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))].
 \end{aligned}$$

Corollary 2.2.4.2. *Let $X_1(t)$ and $X_2(t)$ be jump processes. Then*

$$\begin{aligned}
 X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^t X_2(s) dX_1^c(s) + \int_0^t X_1(s) dX_2^c(s) \\
 &\quad + [X_1^c, X_2^c](t) + \sum_{0 < s \leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\
 &= X_1(0)X_2(0) + \int_0^t X_2(s-) dX_1(s) + \int_0^t X_1(s-) dX_2(s) \\
 &\quad + [X_1, X_2](t). \tag{2.2.4.7}
 \end{aligned}$$

Proof. Take $f(x_1, x_2) = x_1 x_2$ so that

$$f_{x_1} = x_2, \quad f_{x_2} = x_1, \quad f_{x_1 x_1} = 0, \quad f_{x_1 x_2} = 1, \quad f_{x_2 x_2} = 0.$$

The two-dimensional Itô formula implies

$$\begin{aligned} X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) \\ &\quad + \int_0^t 1dX_1^c(s)dX_2^c(s) + \sum_{0 < s \leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)]. \end{aligned}$$

This establishes the first equality in (2.2.4.7). To obtain the second equality, we denote the pure jump parts of $X_1(t)$ and $X_2(t)$ by $J_1(t) = X_1(t) - X_1^c(t)$ and $J_2(t) = X_2(t) - X_2^c(t)$, respectively, and begin with the last line of (2.2.4.7), using (2.2.4.3)

$$\begin{aligned} &X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) \\ &\quad + [X_1, X_2](t) \\ &= X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1^c(s) + \int_0^t X_2(s-)dJ_1(s) \\ &\quad + \int_0^t X_1(s-)dX_2^c(s) + \int_0^t X_1(s-)dJ_2(s) \tag{2.2.4.8} \\ &\quad + [X_1^c, X_2^c](t) + \sum_{0 < s \leq t} [J_1(s)J_2(s)] \\ &= X_1(0)X_2(0) + \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dJ_2^c(s) + [X_1^c, X_2^c](t) \\ &\quad + \sum_{0 < s \leq t} [\Delta X_1(s)X_2(s-) + X_1(s-)\Delta X_2(s) + \Delta X_1(s)\Delta X_2(s)]. \end{aligned}$$

Now, it remains to show that the last sum is the same as

$$[X_1^c, X_2^c](t) + \sum_{0 < s \leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)]$$

in the second line of (2.2.4.7). We expand the typical term in the sum in the second line of (2.2.4.7):

$$\begin{aligned} &[X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\ &= (X_1(s-) + \Delta X_1(s))(X_2(s-) + \Delta X_2(s)) - X_1(s-)X_2(s-) \\ &= X_1(s-)X_2(s-) + X_1(s-)\Delta X_2(s) + \Delta X_1(s)X_2(s-) + \Delta X_1(s)\Delta X_2(s) \\ &\quad - X_1(s-)X_2(s-) \\ &= X_1(s-)\Delta X_2(s) + \Delta X_1(s)X_2(s-) + \Delta X_1(s)\Delta X_2(s). \end{aligned}$$

This is the typical term in the sum appearing at the end of (2.2.4.8). □

For stochastic calculus without jumps, we know from Girsanov's Theorem how to change the measure using the Radon-Nikodym derivative process

$$Z(t) = \exp \left\{ - \int_0^t \Gamma(s) dW(s) - \frac{1}{2} \int_0^t \Gamma^2(s) ds \right\}.$$

This process satisfies the stochastic differential equation

$$dZ(t) = -Z(t)\Gamma(t)dW(t) = Z(t)dX^c(t),$$

where $X^c(t) = - \int_0^t \Gamma(s)dW(s)$ and $[X^c, X^c](t) = \int_0^t \Gamma^2(s)ds$. We may rewrite $Z(t)$ as

$$dZ(t) = \exp \left\{ X^c(t) - \frac{1}{2}[X^c, X^c](t) \right\}. \quad (2.2.4.9)$$

For stochastic calculus with jumps, the analogous stochastic differential equation is

$$dZ^X(t) = Z^X(t-)dX(t), \quad (2.2.4.10)$$

where the integrator X is now allowed to have jumps. The solution to (2.2.4.10) is like (2.2.4.9), except now, whenever there is a jump in X , (2.2.4.10) says there is a jump in Z^X of size

$$\Delta Z^X(s) = Z^X(s-)\Delta X(s).$$

Therefore,

$$Z^X(s) = Z^X(s-) + \Delta Z^X(s) = Z^X(s-) (1 + \Delta X(s)).$$

And, we have the following corollary for this result.

Corollary 2.2.4.3. *Let $X(t)$ be a jump process. The Doleans-Dade exponential of X is defined to be the process*

$$Z^X(t) = \exp \left\{ X^c(t) - \frac{1}{2}[X^c, X^c](t) \right\} \prod_{0 < s \leq t} (1 + \Delta X(s)).$$

This process is the solution to the stochastic differential equation (2.2.4.10) with initial condition $Z^X(0) = 1$, which in integral form is

$$Z^X(t) = 1 + \int_0^t Z^X(s-)dX(s). \quad (2.2.4.11)$$

Proof. We write $X(t)$ as $X(t) = X^c(t) + J(t) + \int_0^t \Theta(s)ds$, where

$$X^c(t) = \int_0^t \Gamma(s)dW(s)$$

is the continuous part of X and $J(t)$ is the pure jump part. We define

$$\begin{aligned} Y(t) &= \exp \left\{ \int_0^t \Gamma(s)dW(s) + \int_0^t \Theta(s)ds - \frac{1}{2} \int_0^t \Gamma^2(s)ds \right\}. \\ &= \exp \left\{ X^c(t) - \frac{1}{2} [X^c, X^c](t) \right\}. \end{aligned}$$

From the Itô formula for continuous processes, we know that

$$dY(t) = Y(t)dX^c(t) = Y(t-)dX^c(t).$$

Next, we define $K(t) = 1$ for t between 0 and the time of the first jump of X , and we set

$$K(t) = \prod_{0 < s \leq t} (1 + \Delta X(s))$$

for t greater than or equal to the first jump time of X . The process $K(t)$ is a pure jump process, and $Z^X(t) = Y(t)K(t)$. If X has a jump at time t , then $K(t) = K(t-) (1 + \Delta X(s))$. Then,

$$\Delta K(t) = K(t) - K(t-) = K(t)\Delta X(t).$$

Because $Y(t)$ is continuous and $K(t)$ is a pure jump process, $[Y, K](t) = 0$. We use Itô product rule for jump processes to obtain

$$\begin{aligned} Z^X(t) &= Y(t)K(t) \\ &= Y(0) + \int_0^t K(s-)dY(s) + \int_0^t Y(s-)dK(s) \\ &= 1 + \int_0^t Y(s-)K(s-)dX^c(s) + \sum_{0 < s \leq t} Y(s-)K(s-)\Delta X(s) \\ &= 1 + \int_0^t Y(s-)K(s-)dX(s) \\ &= 1 + \int_0^t Z^X(s-)dX(s). \end{aligned}$$

This is (2.4.11). □

2.3 Mathematical modeling of euro-denominated life insurance

To make the ideas of the life insurance business transparent, we introduce the risk process employing two basic processes following the survey of [Paulsen, 2008], i.e.,

- a basic risk process X with $X_0 = 0$,
- a return on investment generating process R with $R_0 = 0$.

Suppose that the basic risk process of an insurance company can be described by the process $X = (X_t)_{t \geq 0}$ such that

$$X_t = a_X t + \sigma_X W_t + \sum_{k=1}^{N_t} Z_k. \quad (2.3.0.1)$$

with $a_X \neq 0$, $\sigma_X \neq 0$, and $t \geq 0$, where $W = (W_t)_{t \geq 0}$ is a standard Brownian Motion, $N = (N_t)_{t \geq 0}$ is a nonnegative integer-valued Poisson process with intensity λ , and $Z_k = (Z_k)_{k \in \mathbb{N}}$, a sequence of independent and identically distributed random variables with mean $\mathbb{E}[Z_k] = \beta_Z$ and finite variance $Var(Z_k) = \sigma_Z^2$. We suppose that W , N and $(Z_k)_{k \in \mathbb{N}}$ are independent.

In this modelization, a_X is the premium rate, i.e., the amount paid by the policyholders to the life insurers over a period of time, λ is the intensity of the Poisson process, β_Z is the average size of withdrawals by the policyholders, σ_Z^2 is the variance of withdrawals, while σ_X represents fluctuations in premium income and maybe also small withdrawals.

The life insurance business in France has been developing prosperously and smoothly for most of its history. Households have invested massively in euro-denominated funds. And the withdrawals were not so numerous before the 2010s. In such a situation, the basic risk process does not incorporate the compound Poisson process, and $(X_t)_{t \geq 0}$ is simply given by

$$X_t = a_X t + \sigma_X W_t.$$

Suppose that the log price of the risky asset $R = (R_t)_{t \geq 0}$ corresponds to a Black-Scholes model: for $t > 0$,

$$dR_t = a_R dt + \sigma_R dB_t,$$

with $R_0 = 0$, where a_R is a constant, describing the drift; $\sigma_R > 0$ is a constant, describing the volatility; $B = (B_t)_{t \geq 0}$ is a standard Brownian Motion independent of $(X_t)_{t \geq 0}$. Here,

the log is understood as the stochastic logarithm, i.e., the process such that the price is the Doléans-Dade's exponential of $(R_t)_{t \geq 0}$.

The life insurance company invests the proportion $(1 - \gamma)$ in a risk-free asset with an interest rate of $r \geq 0$ and the proportion $(0 < \gamma < 1)$ in a risky asset with a log price of this asset of R_t , so that the aggregated risk process $(Y_t)_{t \geq 0}$ of this company verifies: for $t > 0$:

$$dY_t = dX_t + (1 - \gamma)Y_t r dt + \gamma Y_t dR_t, \quad (2.3.0.2)$$

with $Y_0 = y$, corresponds to the initial capital of an insurance company. $0 < \gamma < 1$ means short selling is not allowed, and the portfolio is self-financing.

Then, the return on investment generating process is

$$R_t^{(\gamma)} = [\gamma a_R + (1 - \gamma)r]t + \gamma \sigma_R B_t.$$

To give the expressions of the aggregated risk process, we introduce

$$\mu_\gamma = \gamma a_R + (1 - \gamma)r, \quad (2.3.0.3a)$$

$$\sigma_\gamma = \gamma \sigma_R, \quad (2.3.0.3b)$$

Then, we have

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t. \quad (2.3.0.3c)$$

Therefore, (2.3.0.2) is equivalent to:

$$dY_t = dX_t + Y_t dR_t^{(\gamma)}. \quad (2.3.0.4)$$

As known, the solution of (2.3.0.4) is given by [Paulsen, 1996] (Theorem 11.3):

$$Y_t^{(\gamma)} = \mathcal{E}(R^{(\gamma)})_t \left[y + \int_0^t \frac{dX_s}{\mathcal{E}(R^{(\gamma)})_s} \right],$$

where $\mathcal{E}(R^{(\gamma)})$ is Doléans-Dade's exponential,

$$\mathcal{E}(R^{(\gamma)})_t = \exp \left(\mu_\gamma t + \sigma_\gamma B_t - \frac{1}{2} \sigma_\gamma^2 t \right).$$

In the case when $\gamma = 0$, which means a completely risk-free investment,

$$Y_t^{(0)} = \exp(rt) \left[y + \int_0^t \frac{dX_s}{\exp(rs)} \right].$$

The rest of this section is to give the expressions for the expectation, \mathbb{E} , and the variance, Var , of the income of the insurance company with investment in three cases.

In *Case 1* and *Case 1bis*, we consider the situation of one risky asset and multiple risky assets, respectively, when the basic risk process is modeled by Brownian Motion with a drift; in *Case 2* and *Case 2bis*, we consider the situation of one risky asset and multiple risky assets, respectively, when the basic risk process is modeled by the sum of a Brownian Motion and a compound Poisson process; in *General Case*, we consider the situation of one risky asset when $X = (X_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$ are both Lévy Processes, respectively.

2.3.1 Case 1: One Risky Asset Modeled by Brownian Motion

In this case, we suppose the insurance business process, X and the investment generating process, R are modeled by two independent Brownian Motions.

$$X_t = a_X t + \sigma_X W_t,$$

and

$$R_t = a_R t + \sigma_R B_t.$$

We remind that this case corresponds to large investments in euro-denominated funds and small and regular withdrawals (without significant impact on the insurance business) under the development of the life insurance market, as it has been observed during a very long period in France. We assume the possibility to invest in one risk-free and one risky asset². Then, we derive the formulas for the expectation and the variance of the income of a life insurance company.

2.3.1.1 Expectation of the income of a life insurance company

In this subsection, we show that the expectation of the income of the insurance company is a function of μ_γ , t , y and a_X , given below.

Proposition 2.3.1. *For $0 < \gamma < 1$, $t \geq 0$ and $\mu_\gamma \neq 0$:*

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \right] = e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X}{\mu_\gamma}, \quad (2.3.1.1)$$

where μ_γ is defined in (2.3.0.3a). If $\mu_\gamma = 0$, then

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \right] = y + a_X t.$$

2. *Case 1bis* and *Case 2bis* discuss multiple risky assets in investment.

Proof. First, we calculate conditional expectation of Y_t conditionally to $\mathcal{E}(R^{(\gamma)})_t = h_t$. Namely, we show that:

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \middle| \mathcal{E} \left(R^{(\gamma)} \right)_{0 \leq s \leq t} = (h_s)_{0 \leq s \leq t} \right] = h_t \left(y + a_X \int_0^t \frac{ds}{h_s} \right). \quad (2.3.1.2)$$

Since W and B are independent Brownian Motions, and hence, $R^{(\gamma)}$ and X are independent,

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \middle| \mathcal{E} \left(R^{(\gamma)} \right)_{0 \leq s \leq t} = (h_s)_{0 \leq s \leq t} \right] = h_t \left(y + \mathbb{E} \int_0^t \frac{dX_s}{h_s} \right). \quad (2.3.1.3)$$

Taking into account the expression for X_t in *Case 1*:

$$\int_0^t \frac{dX_s}{h_s} = a_X \int_0^t \frac{ds}{h_s} + \sigma_X \int_0^t \frac{dW_s}{h_s}. \quad (2.3.1.4)$$

We note that $(\int_0^t \frac{dW_s}{h_s})$ is a local martingale, and by localisation procedure, we show that:

$$\mathbb{E} \left(\int_0^t \frac{dW_s}{h_s} \right) = 0. \quad (2.3.1.5)$$

From (2.3.1.3), (2.3.1.4) and (2.3.1.5) we get (2.3.1.2).

We know from the properties of conditional expectation that:

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \mathbb{E} \left\{ \mathbb{E} \left[Y_t^{(\gamma)}(y) \middle| \mathcal{E} \left(R^{(\gamma)} \right)_{0 \leq s \leq t} = (h_s)_{0 \leq s \leq t} \right] \right\}.$$

So, we have:

$$\begin{aligned} \mathbb{E} [Y_t^{(\gamma)}(y)] &= \mathbb{E} \left[\mathcal{E}(R^{(\gamma)})_t \left(y + a_X \int_0^t \frac{ds}{\mathcal{E}(R^{(\gamma)})_s} \right) \right] \\ &= y \mathbb{E} [\mathcal{E}(R^{(\gamma)})_t] + a_X \mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right]. \end{aligned} \quad (2.3.1.6)$$

Now, for the first part of (2.3.1.6)

$$\mathbb{E} [\mathcal{E}(R^{(\gamma)})_t] = \mathbb{E} \left[\exp \left(\mu_\gamma t + \sigma_\gamma B_t - \frac{1}{2} \sigma_\gamma^2 t \right) \right] = \exp(\mu_\gamma t), \quad (2.3.1.7)$$

since $\mathbb{E} \left[\exp \left(\sigma_\gamma B_t - \frac{1}{2} \sigma_\gamma^2 t \right) \right] = 1$.

For the second part of (2.3.1.6) we get:

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right] \\ &= \mathbb{E} \left\{ \int_0^t \exp \left[(\mu_\gamma(t-s) + \sigma_\gamma(B_t - B_s) - \frac{1}{2}\sigma_\gamma^2(t-s)) \right] ds \right\}. \end{aligned}$$

It is known that there is equality in law $(B_t - B_s)_{0 \leq s \leq t} \stackrel{\mathcal{L}}{=} (B_{t-s})_{0 \leq s \leq t}$, so that doing time change $u = t - s$ we have:

$$\mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right] = \mathbb{E} \left[\int_0^t \exp \left(\mu_\gamma u + \sigma_\gamma B_u - \frac{1}{2}\sigma_\gamma^2 u \right) du \right].$$

Using Fubini Theorem, we exchange the expectation and integration over $[0, t]$ and since

$$\mathbb{E} \left[\exp \left(\mu_\gamma u + \sigma_\gamma B_u - \frac{1}{2}\sigma_\gamma^2 u \right) \right] = \exp(\mu_\gamma u),$$

we get:

$$\mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right] = \int_0^t \exp(\mu_\gamma u) du.$$

During financial crisis or even more extreme market conditions, for instance, when $a_R \leq 0 \leq r$, there will be one allocation that $\mu_\gamma = \gamma a_R + (1 - \gamma)r = 0$. Then, there should be two expressions for $\mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right]$:

$$\mathbb{E} \left[\int_0^t \frac{\mathcal{E}(R^{(\gamma)})_t ds}{\mathcal{E}(R^{(\gamma)})_s} \right] = \begin{cases} \frac{e^{\mu_\gamma t} - 1}{\mu_\gamma} & \text{if } \mu_\gamma \neq 0, \\ t & \text{if } \mu_\gamma = 0. \end{cases} \quad (2.3.1.8)$$

Finally, from (2.3.1.6), (2.3.1.7) and (2.3.1.8):

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0, \\ y + a_X t, & \text{if } \mu_\gamma = 0. \end{cases} \quad (2.3.1.9)$$

□

Discussion 1:

As we can see in (2.3.1.9), the basic variables that influence the expectation of the income of the insurance company are: initial capital y , premium rate a_X and investment return μ_γ .

The question is now how the expectation of the income depends on μ_γ , y and a_X .

The impact of each variable on $\mathbb{E}[Y_t^{(\gamma)}(y)]$, keeping others unchanged:

- initial capital y ,

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial y} = \begin{cases} e^{\mu_\gamma t}, & \text{if } \mu_\gamma \neq 0 \\ 1, & \text{if } \mu_\gamma = 0 \end{cases}$$

The impact of y on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is positive. The expected income of the life insurance company will increase following a rise in the initial capital.

- premium rate a_X ,

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial a_X} = \begin{cases} \frac{e^{\mu_\gamma t} - 1}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0 \\ t, & \text{if } \mu_\gamma = 0 \end{cases}$$

The impact of a_X on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is positive if $\mu_\gamma > 0$. When the premiums in the insurance business increase, life insurers can generate more income when the investment rate of return is positive.

- investment return μ_γ ,

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \mu_\gamma} = \begin{cases} ye^{\mu_\gamma t} + a_X \frac{(\mu_\gamma t - 1)e^{\mu_\gamma t} + 1}{\mu_\gamma^2}, & \text{if } \mu_\gamma \neq 0 \\ 0, & \text{if } \mu_\gamma = 0 \end{cases}$$

We know that $ye^{\mu_\gamma t} > 0$. Moreover, for $a_X \frac{(\mu_\gamma t - 1)e^{\mu_\gamma t} + 1}{\mu_\gamma^2}$, we obtain as follows.

For $\mu_\gamma \neq 0$, we have:

- when $\mu_\gamma > 0$,

if $\mu_\gamma t - 1 \geq 0$, then $t \geq \frac{1}{\mu_\gamma}$ and $a_X \frac{(\mu_\gamma t - 1)e^{\mu_\gamma t} + 1}{\mu_\gamma^2} > 0$;

if $\mu_\gamma t - 1 < 0$, then $0 < \mu_\gamma t < 1$, $0 < t < \frac{1}{\mu_\gamma}$, and $1 < e^{\mu_\gamma t} < e$, we have $-1 < (\mu_\gamma t - 1)e^{\mu_\gamma t} < 0$, $0 < (\mu_\gamma t - 1)e^{\mu_\gamma t} + 1 < 1$, so $a_X \frac{(\mu_\gamma t - 1)e^{\mu_\gamma t} + 1}{\mu_\gamma^2} > 0$;

- when $\mu_\gamma < 0$, (which means investment loss), then $\mu_\gamma t < 0$, $(\mu_\gamma t - 1) < -1$, and $0 < e^{\mu_\gamma t} < 1$, we have $-1 < (\mu_\gamma t - 1)e^{\mu_\gamma t} < 0$, $0 < (\mu_\gamma t - 1)e^{\mu_\gamma t} + 1 < 1$, so $a_X \frac{(\mu_\gamma t - 1)e^{\mu_\gamma t} + 1}{\mu_\gamma^2} > 0$;

Hence, the impact of μ_γ on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is positive, if the investment rate of return, μ_γ is non-zero.

When $\mu_\gamma = 0$, there is no relation between μ_γ and $\mathbb{E}[Y_t^{(\gamma)}(y)]$, which means the expectation of the income is the sum of initial capital and premiums, without any investment profit.

2.3.1.2 Variance of the income of a life insurance company

In this subsection, we give the expression for the variation of the risk process, as a function of y , a_X , σ_X , μ_γ and σ_γ and hence, as a function of γ .

Proposition 2.3.2. *For $0 < \gamma < 1$ and $t \geq 0$, when $\mu_\gamma \neq 0$, $\mu_\gamma + \sigma_\gamma^2 \neq 0$ and $2\mu_\gamma + \sigma_\gamma^2 \neq 0$:*

$$\begin{aligned}
 & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\
 &= y^2 \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] \\
 &+ 2a_X y \frac{\mu_\gamma \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] - \sigma_\gamma^2 (e^{2\mu_\gamma t} - e^{\mu_\gamma t})}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} \\
 &+ 2a_X^2 \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right] \\
 &+ \sigma_X^2 \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2};
 \end{aligned} \tag{2.3.1.10}$$

when $\mu_\gamma = 0$:

$$\begin{aligned}
 \text{Var} \left[Y_t^{(\gamma)}(y) \right] &= y^2 \left(e^{\sigma_\gamma^2 t} - 1 \right) + 2a_X y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} \\
 &+ a_X^2 \frac{2e^{\sigma_\gamma^2 t} - 1 - (\sigma_\gamma^2 t + 1)^2}{\sigma_\gamma^4} + \sigma_X^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2},
 \end{aligned} \tag{2.3.1.11}$$

where μ_γ and σ_γ are defined in (2.3.0.3a) and (2.3.0.3b).

Before proving Proposition 2.3.2, we propose and prove Lemma 2.3.1.

Lemma 2.3.1. *Suppose that $(g_t)_{t \geq 0}$ is a deterministic continuous function verifying the integral equation*

$$g_t = g_0 + f_t + k \int_0^t g_s ds, \tag{2.3.1.12}$$

with differentiable deterministic function $(f_t)_{t \geq 0}$. Then

$$g_t = e^{kt} \left(g_0 + \int_0^t e^{-ks} f'_s ds \right). \tag{2.3.1.13}$$

Proof. The relation in (2.3.1.12) is equivalent to:

$$g'_t = f'_t + kg_t, \quad g_0 = 0. \quad (2.3.1.14)$$

Differentiating (2.3.1.13) we get (2.3.1.14). And since the solution of the equation is unique, the result claimed follows. \square

Proof. We know that:

$$\text{Var} [Y_t^{(\gamma)}(y)] = \mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\} - \left\{ \mathbb{E} [Y_t^{(\gamma)}(y)] \right\}^2.$$

From Proposition 2.3.1, it follows that it remains to calculate $\mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\}$. We do this calculus using the Itô Formula (Theorem 2.2.3.2). To avoid the complicated notations, we omit for a moment the index (γ) . From (2.3.0.2), (2.3.0.3c), (2.3.0.4) and *Case 1* it follows that:

$$dY_t = a_X dt + \sigma_X dW_t + Y_t \mu_\gamma dt + Y_t \sigma_\gamma dB_t. \quad (2.3.1.15)$$

The Ito Formula with the function $f(x) = x^2$ gives:

$$Y_t^2 = Y_0^2 + 2 \int_0^t Y_s dY_s + \frac{1}{2} \int_0^t 2d\langle Y^c \rangle_s,$$

where Y^c is the continuous martingale part of the process Y , and $\langle Y^c \rangle$ is the predictable quadratic variation of Y^c . Since $Y_0^2 = y^2$ and (2.3.1.14) we get:

$$\begin{aligned} Y_t^2 = & y^2 + 2a_X \int_0^t Y_s ds + 2\sigma_X \int_0^t Y_s dW_s + 2\mu_\gamma \int_0^t Y_s^2 ds \\ & + 2\sigma_\gamma \int_0^t Y_s^2 dB_s + \langle Y^c \rangle_t. \end{aligned} \quad (2.3.1.16)$$

Since $\langle Y^c \rangle_t = \langle \sigma_X \int_0^t dW_s + \sigma_\gamma \int_0^t Y_s dB_s \rangle$, with independent W and B , we get

$$\langle Y^c \rangle_t = \sigma_X^2 t + \sigma_\gamma^2 \int_0^t Y_s^2 ds. \quad (2.3.1.17)$$

Let $\tau_n = \inf \{t \geq 0 : |Y_t| > n\}$ with $\inf \{\emptyset\} = +\infty$. Then $\left(\int_0^{t \wedge \tau_n} Y_s dW_s \right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau_n} Y_s^2 dB_s \right)_{t \geq 0}$ are local martingales. It implies that there exists a sequence of stopping times $(s_n)_{n \geq 1}$ going to $+\infty$ such that $\left(\int_0^{t \wedge \tau_n \wedge s_n} Y_s dW_s \right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau_n \wedge s_n} Y_s^2 dB_s \right)_{t \geq 0}$ are martingales.

We put $\tau'_n = \tau_n \wedge s_n$, then from (2.3.1.16) and (2.3.1.17):

$$\begin{aligned} Y_{t \wedge \tau'_n}^2 = & y^2 + 2a_X \int_0^{t \wedge \tau'_n} Y_s ds + 2\sigma_X \int_0^{t \wedge \tau'_n} Y_s dW_s + 2\mu_\gamma \int_0^{t \wedge \tau'_n} Y_s^2 ds \\ & + 2\sigma_\gamma \int_0^{t \wedge \tau'_n} Y_s^2 dB_s + \sigma_X^2 (t \wedge \tau'_n) + \sigma_\gamma^2 \int_0^{t \wedge \tau'_n} Y_s^2 ds. \end{aligned} \quad (2.3.1.18)$$

We take the mathematical expectation in (2.3.1.18), and since $\left(\int_0^{t \wedge \tau'_n} Y_s dW_s\right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau'_n} Y_s^2 dB_s\right)_{t \geq 0}$ are martingales, we get that:

$$\begin{aligned} \mathbb{E} \left[Y_{t \wedge \tau'_n}^2 \right] = & y^2 + 2a_X \int_0^{t \wedge \tau'_n} \mathbb{E} (Y_s) ds + 2\mu_\gamma \int_0^{t \wedge \tau'_n} \mathbb{E} (Y_s^2) ds \\ & + \sigma_X^2 (t \wedge \tau'_n) + \sigma_\gamma^2 \int_0^{t \wedge \tau'_n} \mathbb{E} (Y_s^2) ds. \end{aligned} \quad (2.3.1.19)$$

Since $Y_s \geq 0$, we can do limit passage, $\lim_{n \rightarrow +\infty}$, in each term in the right-hand side by Lebesgue's Monotone Convergence Theorem. We prove that $\left[Y_{t \wedge \tau'_n}^2 \right]_{n \in \mathbb{N}}$ is uniformly integrable and we pass to the limit in the left-hand side.

This gives using Fubini Theorem that:

$$\mathbb{E} (Y_t^2) = y^2 + 2a_X \int_0^t \mathbb{E} (Y_s) ds + 2\mu_\gamma \int_0^t \mathbb{E} (Y_s^2) ds + \sigma_X^2 t + \sigma_\gamma^2 \int_0^t \mathbb{E} (Y_s^2) ds.$$

We get with

$$g_t = \mathbb{E} (Y_t^2),$$

$$f_t = 2a_X \int_0^t \mathbb{E} (Y_s) ds + \sigma_X^2 t,$$

and

$$k = 2\mu_\gamma + \sigma_\gamma^2,$$

from Proposition 2.3.1 and Lemma 2.3.1 that:

$$\mathbb{E} (Y_t^2) = e^{(2\mu_\gamma + \sigma_\gamma^2)t} \left\{ y^2 + \int_0^t e^{-(2\mu_\gamma + \sigma_\gamma^2)s} \left[2a_X \mathbb{E} (Y_s) + \sigma_X^2 \right] ds \right\}. \quad (2.3.1.20)$$

When $\mu_\gamma \neq 0$, the integral part in the right-hand side of (2.3.1.20) is:

$$\begin{aligned}
 & \int_0^t e^{-(2\mu_\gamma + \sigma_\gamma^2)s} [2a_X \mathbb{E}(Y_s) + \sigma_X^2] ds \\
 &= \int_0^t e^{-(2\mu_\gamma + \sigma_\gamma^2)s} \left[2a_X \left(ye^{\mu_\gamma s} + a_X \frac{e^{\mu_\gamma s} - 1}{\mu_\gamma} \right) + \sigma_X^2 \right] ds \\
 &= \int_0^t \left[2a_X ye^{-(\mu_\gamma + \sigma_\gamma^2)s} + \frac{2a_X^2}{\mu_\gamma} e^{-(\mu_\gamma + \sigma_\gamma^2)s} \right. \\
 &\quad \left. + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) e^{-(2\mu_\gamma + \sigma_\gamma^2)s} \right] ds \tag{2.3.1.21} \\
 &= \left[-2a_X y \frac{e^{-(\mu_\gamma + \sigma_\gamma^2)t} \Big|_0^t}{\mu_\gamma + \sigma_\gamma^2} \right] + \left[-\frac{2a_X^2}{\mu_\gamma} \frac{e^{-(\mu_\gamma + \sigma_\gamma^2)t} \Big|_0^t}{\mu_\gamma + \sigma_\gamma^2} \right] \\
 &\quad + \left[-\left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) \frac{e^{-(2\mu_\gamma + \sigma_\gamma^2)t} \Big|_0^t}{2\mu_\gamma + \sigma_\gamma^2} \right] \\
 &= 2a_X \left(y + \frac{a_X}{\mu_\gamma} \right) \frac{1 - e^{-(\mu_\gamma + \sigma_\gamma^2)t}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) \frac{1 - e^{-(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma + \sigma_\gamma^2}.
 \end{aligned}$$

This gives, when $\mu_\gamma \neq 0$ the following relation

$$\begin{aligned}
 & \mathbb{E}(Y_t^2) \\
 &= e^{(2\mu_\gamma + \sigma_\gamma^2)t} \left[y^2 + 2a_X \left(y + \frac{a_X}{\mu_\gamma} \right) \frac{1 - e^{-(\mu_\gamma + \sigma_\gamma^2)t}}{\mu_\gamma + \sigma_\gamma^2} \right. \\
 &\quad \left. + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) \frac{1 - e^{-(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma + \sigma_\gamma^2} \right] \tag{2.3.1.22} \\
 &= y^2 e^{(2\mu_\gamma + \sigma_\gamma^2)t} + 2a_X \left(y + \frac{a_X}{\mu_\gamma} \right) \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} \\
 &\quad + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}.
 \end{aligned}$$

When $\mu_\gamma = 0$, the integral part in the right-hand side of (2.3.1.20) is:

$$\begin{aligned}
 & \int_0^t e^{-(2\mu_\gamma + \sigma_\gamma^2)s} [2a_X \mathbb{E}(Y_s) + \sigma_X^2] ds \\
 &= \int_0^t e^{-\sigma_\gamma^2 s} [2a_X (y + a_X s) + \sigma_X^2] ds \\
 &= \int_0^t (2a_X y e^{-\sigma_\gamma^2 s} + 2a_X^2 s e^{-\sigma_\gamma^2 s} + \sigma_X^2 e^{-\sigma_\gamma^2 s}) ds \\
 &= -2a_X y \frac{e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} \Big|_0^t - 2a_X^2 \frac{e^{-\sigma_\gamma^2 t} (\sigma_\gamma^2 t + 1)}{\sigma_\gamma^4} \Big|_0^t - \sigma_X^2 \frac{e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} \Big|_0^t \\
 &= 2a_X y \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} + 2a_X^2 \frac{1 - e^{-\sigma_\gamma^2 t} - \sigma_\gamma^2 t e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^4} + \sigma_X^2 \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}.
 \end{aligned} \tag{2.3.1.23}$$

This gives, when $\mu_\gamma = 0$:

$$\begin{aligned}
 & \mathbb{E}(Y_t^2) \\
 &= e^{\sigma_\gamma^2 t} \left(y^2 + 2a_X y \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} + 2a_X^2 \frac{1 - e^{-\sigma_\gamma^2 t} - \sigma_\gamma^2 t e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^4} \right. \\
 & \quad \left. + \sigma_X^2 \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} \right) \\
 &= y^2 e^{\sigma_\gamma^2 t} + 2a_X y \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} + 2a_X^2 \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^4} \\
 & \quad + \sigma_X^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2}.
 \end{aligned} \tag{2.3.1.24}$$

For the calculation of $[\mathbb{E}(Y_t)]^2$ by Proposition 2.3.1, when $\mu_\gamma \neq 0$:

$$\begin{aligned}
 [\mathbb{E}(Y_t)]^2 &= \left[e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X}{\mu_\gamma} \right]^2 \\
 &= e^{2\mu_\gamma t} \left(y^2 + 2y \frac{a_X}{\mu_\gamma} + \frac{a_X^2}{\mu_\gamma^2} \right) - 2e^{\mu_\gamma t} \frac{a_X}{\mu_\gamma} \left(y + \frac{a_X}{\mu_\gamma} \right) + \frac{a_X^2}{\mu_\gamma^2} \\
 &= y^2 e^{2\mu_\gamma t} + 2a_X y \frac{e^{2\mu_\gamma t} - e^{\mu_\gamma t}}{\mu_\gamma} + a_X^2 \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{\mu_\gamma^2};
 \end{aligned} \tag{2.3.1.25}$$

when $\mu_\gamma = 0$:

$$\begin{aligned}
 [\mathbb{E}(Y_t)]^2 &= (y + a_X t)^2 \\
 &= y^2 + 2a_X y t + a_X^2 t^2.
 \end{aligned} \tag{2.3.1.26}$$

We take back the notations of the index (γ).

When $\mu_\gamma \neq 0$, from (2.3.1.22) and (2.3.1.25), we finally have:

$$\begin{aligned}
 & \text{Var} [Y_t^{(\gamma)}(y)] \\
 &= \mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\} - \left\{ \mathbb{E} [Y_t^{(\gamma)}(y)] \right\}^2 \\
 &= \left[y^2 e^{(2\mu_\gamma + \sigma_\gamma^2)t} + 2a_X \left(y + \frac{a_X}{\mu_\gamma} \right) \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} \right. \\
 &\quad \left. + \left(\sigma_X^2 - \frac{2a_X^2}{\mu_\gamma} \right) \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2} \right] \\
 &\quad - \left[y^2 e^{2\mu_\gamma t} + 2a_X y \frac{e^{2\mu_\gamma t} - e^{\mu_\gamma t}}{\mu_\gamma} + a_X^2 \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{\mu_\gamma^2} \right] \tag{2.3.1.27} \\
 &= y^2 \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] \\
 &\quad + 2a_X y \frac{\mu_\gamma \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] - \sigma_\gamma^2 (e^{2\mu_\gamma t} - e^{\mu_\gamma t})}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} \\
 &\quad + 2a_X^2 \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right] \\
 &\quad + \sigma_X^2 \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2};
 \end{aligned}$$

When $\mu_\gamma = 0$, from (2.3.1.24) and (2.3.1.26), we have:

$$\begin{aligned}
 & \text{Var} [Y_t^{(\gamma)}(y)] \\
 &= \mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\} - \left\{ \mathbb{E} [Y_t^{(\gamma)}(y)] \right\}^2 \\
 &= \left(y^2 e^{\sigma_\gamma^2 t} + 2a_X y \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} + 2a_X^2 \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^4} + \sigma_X^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} \right) \\
 &\quad - \left(y^2 + 2a_X y t + a_X^2 t^2 \right) \\
 &= y^2 \left(e^{\sigma_\gamma^2 t} - 1 \right) + 2a_X y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} \\
 &\quad + a_X^2 \frac{2e^{\sigma_\gamma^2 t} - 1 - (\sigma_\gamma^2 t + 1)^2}{\sigma_\gamma^4} + \sigma_X^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2}.
 \end{aligned} \tag{2.3.1.28}$$

□

Discussion 2:

As we can see in (2.3.1.27) and (2.3.1.28), there are more parameters that influence the variance of the income of the insurance company compared with those in the expectation of the income of the insurance company: initial capital y , risky remuneration a_R , riskless interest rate r , volatility of risky investment σ_R , premium rate a_X , fluctuations in premium income σ_X and time t .

The impact of each variable on $Var [Y_t^{(\gamma)}(y)]$, keeping others unchanged:

- initial capital y ,

when $\mu_\gamma \neq 0$,

$$\begin{aligned} \frac{\partial Var [Y_t^{(\gamma)}(y)]}{\partial y} = & 2y \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] \\ & + 2a_X \frac{\mu_\gamma \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t} \right] - \sigma_\gamma^2 (e^{2\mu_\gamma t} - e^{\mu_\gamma t})}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2}, \end{aligned}$$

and it is an increasing linear function of y . So, for sufficiently big y , the relation between $Var [Y_t^{(\gamma)}(y)]$ and y will be positive;

when $\mu_\gamma = 0$,

$$\frac{\partial Var [Y_t^{(\gamma)}(y)]}{\partial y} = 2y (e^{\sigma_\gamma^2 t} - 1) + 2a_X \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} > 0,$$

the impact of y on $Var [Y_t^{(\gamma)}(y)]$ is positive, because $\sigma_\gamma > 0$ by our assumption.

In general, the greater the initial capital investment, the higher the corresponding variance in the income of a life insurance company, regardless of its investment result.

- premium rate a_X and fluctuations in premium income σ_X in the risk process X_t ,

when $\mu_\gamma \neq 0$,

$$\begin{aligned} & \frac{\partial Var [Y_t^{(\gamma)}(y)]}{\partial a_X} \\ = & 2y \frac{\mu_\gamma \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t} \right] - \sigma_\gamma^2 (e^{2\mu_\gamma t} - e^{\mu_\gamma t})}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} \end{aligned}$$

$$+ 4a_X \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right],$$

$$\frac{\partial \text{Var} [Y_t^{(\gamma)}(y)]}{\partial \sigma_X} = 2\sigma_X \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2} > 0,$$

when $\mu_\gamma = 0$,

$$\frac{\partial \text{Var} [Y_t^{(\gamma)}(y)]}{\partial a_X} = 2y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} + 2a_X \frac{2e^{\sigma_\gamma^2 t} - 1 - (\sigma_\gamma^2 t + 1)^2}{\sigma_\gamma^4} > 0,$$

$$\frac{\partial \text{Var} [Y_t^{(\gamma)}(y)]}{\partial \sigma_X} = 2\sigma_X \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} > 0.$$

When $\mu_\gamma \neq 0$, the relation between a_X and $\text{Var} [Y_t^{(\gamma)}(y)]$ is not clear; when $\mu_\gamma = 0$, the impact of a_X on $\text{Var} [Y_t^{(\gamma)}(y)]$ is positive, and the impact of σ_X on $\text{Var} [Y_t^{(\gamma)}(y)]$ is positive.

- investment return μ_γ and the volatility of the risky investment σ_R , since the two derivatives are complicated, we do not list the formulas. Both relations are not clear.

Remark 2.3.1. *As we can see, the impacts of y and a_X on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ are both positive, while the impacts of y and a_X on $\text{Var} [Y_t^{(\gamma)}(y)]$ are not clear when $\mu_\gamma \neq 0$.*

The reason for this situation can be explained as when the initial investment or the premium rate increases, the expectation of the income of the insurance company increases, the uncertainty of the variance of the income also increases, however, it is not clear whether the variance increases, decreases, or remains unchanged.

In other words, increasing the initial investment or the premium rate does not necessarily lead to a higher Sharpe ratio [Sharpe, 1994].

2.3.2 Case 1bis: Multiple Risky Assets Modeled by Brownian Motion

In this modelization, we suppose that there are multiple risky assets in risky investment.

Let $n \in \mathbb{N}^*$ be the number of risky assets. We model the log prices of the risky assets by dependent Brownian Motions $(B_t^{(i)})_{t \geq 0}$ with drifts:

$$R_t^{(i)} = a_R^{(i)} t + \sigma_R^{(i)} B_t^{(i)}, \quad 1 \leq i \leq n.$$

We denote by \mathbb{C} the covariance matrix of n-dimensional Brownian Motion $(\vec{B}_t)_{t \geq 0}$, $\vec{B}_t = (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(n)})^\top$,

$$\mathbb{C} = (c_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} = \begin{bmatrix} c_{11}, & \cdots, & c_{1j}, & \cdots, & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{i1}, & \ddots, & c_{ij}, & \ddots, & c_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1}, & \cdots, & c_{nj}, & \cdots, & c_{nn} \end{bmatrix},$$

with $c_{i,i} = Var(B_1^{(i)}) = (\sigma^{(i)})^2$, $c_{i,j} = cov(B_1^{(i)}, B_1^{(j)})$.

In this case $Y_t^{(\gamma)}$ satisfies the equation:

$$dY_t^{(\gamma)} = dX_t + Y_t^{(\gamma)}(1 - \gamma)r dt + Y_t^{(\gamma)} \sum_{i=1}^n \gamma_i dR_t^{(i)}, \quad (2.3.2.1)$$

where $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$ and $(\gamma_i)_{1 \leq i \leq n}$ are the proportions of the investment in i -th risky asset.

From the properties of Brownian Motion, $(\sum_{i=1}^n \gamma_i R_t^{(i)})_{t \geq 0}$ is Brownian Motion with the drift

$$a_R^{(\gamma)} = \sum_{i=1}^n \gamma_i a_R^{(i)},$$

and the variance is

$$(\sigma_R^{(\gamma)})^2 = \langle \mathbb{C} \vec{\gamma}, \vec{\gamma} \rangle = \sum_{i=1}^n \sum_{j=1}^n \gamma_i \gamma_j \sigma_R^{(i)} \sigma_R^{(j)} c_{i,j}, \quad (2.3.2.2)$$

where $\vec{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$, and $\langle \cdot, \cdot \rangle$ is the scalar product.

Then (2.3.2.1) is equivalent to:

$$dY_t^{(\gamma)} = dX_t + Y_t^{(\gamma)} dR_t^{(\gamma)}, \quad (2.3.2.3)$$

where

$$R_t^{(\gamma)} = [(1 - \gamma)r + a_R^{(\gamma)}] t + \sigma_R^{(\gamma)} B_t,$$

and $(B_t)_{t \geq 0}$ is a *new Brownian Motion* obtained by linear combination of the previous

Brownian Motions $B^{(i)}$, $1 \leq i \leq n$.

More precisely, there is equality in law,

$$\left(\sum_{i=1}^n \gamma_i \sigma_R^{(i)} B_t^{(i)} \right)_{t \geq 0} \stackrel{\mathcal{L}}{=} \left(\sigma^{(\gamma)} B_t \right)_{t \geq 0},$$

which provides the equality of all characteristics of the processes, in particular, the equality of two moments at fixed $t = T$.

To simplify the notations, let

$$\mu_\gamma = (1 - \gamma)r + a_R^{(\gamma)}, \quad (2.3.2.4a)$$

$$\sigma_\gamma = \sigma_R^{(\gamma)}, \quad (2.3.2.4b)$$

Then:

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t. \quad (2.3.2.4c)$$

Finally, by the same reasoning as in the proofs of Proposition 2.3.1 and Proposition 2.3.2 in *Case 1*, we obtain the expressions for the expectation and variance of the income of a life insurance company when there are multiple risky assets in risky investment.

Proposition 2.3.3. For $0 < \gamma < 1$ and $t \geq 0$,

when $\mu_\gamma \neq 0$:

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X}{\mu_\gamma}, \quad (2.3.2.5)$$

if in addition $\mu_\gamma + \sigma_\gamma^2 \neq 0$, $2\mu_\gamma + \sigma_\gamma^2 \neq 0$, then

$$\begin{aligned} & \text{Var} [Y_t^{(\gamma)}(y)] \\ &= y^2 \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] \\ &+ 2a_X y \frac{\mu_\gamma \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] - \sigma_\gamma^2 (e^{2\mu_\gamma t} - e^{\mu_\gamma t})}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} \\ &+ 2a_X^2 \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right] \\ &+ \sigma_X^2 \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}; \end{aligned} \quad (2.3.2.6)$$

when $\mu_\gamma = 0$, we get:

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \right] = y + a_X t, \quad (2.3.2.7)$$

$$\begin{aligned} \text{Var} \left[Y_t^{(\gamma)}(y) \right] &= y^2 \left(e^{\sigma_\gamma^2 t} - 1 \right) + 2a_X y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} \\ &+ a_X^2 \frac{2e^{\sigma_\gamma^2 t} - 1 - \left(\sigma_\gamma^2 t + 1 \right)^2}{\sigma_\gamma^4} + \sigma_X^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2}; \end{aligned} \quad (2.3.2.8)$$

where μ_γ and σ_γ is defined in (2.3.2.4a) and (2.3.2.4b).

2.3.3 Case 2: One Risky Asset, when $(X_t)_{t \geq 0}$ is Modeled by the sum of a Brownian Motion and a compound Poisson process

Here we suppose that the basic risk process is described in (2.3.0.1):

$$X_t = a_X t + \sigma_X W_t + Q_t, \quad (2.3.3.1)$$

where $(Q_t)_{t \in \mathbb{R}_+}$ is the compound Poisson process,

$$Q_t = \sum_{k=1}^{N_t} Z_k,$$

with N_t , a nonnegative integer-valued Poisson process with intensity λ , and $(Z_k)_{k \in \mathbb{N}}$, a sequence of independent and identically distributed random variables with mean $\mathbb{E}[Z_k] = \beta_Z$ and finite variance $\text{Var}(Z_k) = \sigma_Z^2$. Suppose that N_t is independent of the sequence $(Z_k)_{k \in \mathbb{N}}$. Then, from Theorem 2.2.4.4 in Section 2.2.4, we have

$$\mathbb{E}(Q_t) = \mathbb{E} \left(\sum_{n=1}^{N_t} Z_k \right) = \lambda \beta_Z t, \quad (2.3.3.2)$$

and

$$\text{Var}(Q_t) = \text{Var} \left(\sum_{n=1}^{N_t} Z_k \right) = \lambda \beta_Z^2 t + \lambda \sigma_Z^2 t. \quad (2.3.3.3)$$

For the compound Poisson process (Q_t) , λ , β_Z , and σ_Z^2 represent the intensity of the Poisson process, the mean of withdrawal sizes, and the variance of withdrawals in the basic insurance business, respectively.

2.3.3.1 Expectation of the income of a life insurance company

In this subsection, we show that the expectation of the income of the insurance company is a function of μ_γ , t , y , a_X , λ and β_Z , given below.

Proposition 2.3.4. For $0 < \gamma < 1$ and $t \geq 0$:

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \right] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X + \lambda \beta_Z}{\mu_\gamma} \right) - \frac{a_X + \lambda \beta_Z}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0, \\ y + (a_X + \lambda \beta_Z) t, & \text{if } \mu_\gamma = 0, \end{cases} \quad (2.3.3.4)$$

where μ_γ is defined in (2.3.0.3a), and \mathbb{E} is mathematical expectation.

Proof. First, we calculate conditional expectation of Y_t conditionally to $\mathcal{E}(R^{(\gamma)})_t = h_t$. Namely, we show that:

$$\begin{aligned} & \mathbb{E} \left[Y_t^{(\gamma)}(y) \middle| \mathcal{E} \left(R^{(\gamma)} \right)_{0 \leq s \leq t} = (h_s)_{0 \leq s \leq t} \right] \\ &= h_t \left[y + (a_X + \lambda \beta_Z) \int_0^t \frac{ds}{h_s} \right]. \end{aligned} \quad (2.3.3.5)$$

Since W and B are independent Brownian Motions, W , N and $(Z_k)_{k \in N}$ are independent, and hence, $R^{(\gamma)}$ and X are independent,

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \middle| \mathcal{E} \left(R^{(\gamma)} \right)_{0 \leq s \leq t} = (h_s)_{0 \leq s \leq t} \right] = h_t \left(y + \mathbb{E} \int_0^t \frac{dX_s}{h_s} \right). \quad (2.3.3.6)$$

Taking into account (2.3.3.1), and from (2.3.3.2) we have:

$$\mathbb{E} \left(\int_0^t \frac{dX_s}{h_s} \right) = \mathbb{E} \left(a_X \int_0^t \frac{ds}{h_s} \right) + \mathbb{E} \left(\sigma_X \int_0^t \frac{dW_s}{h_s} \right) + \mathbb{E} \left(\int_0^t \frac{dQ_s}{h_s} \right). \quad (2.3.3.7)$$

We note that $\int_0^t \frac{dW_s}{h_s}$ is a local martingale, and by localisation procedure, we show that:

$$\mathbb{E} \left(\sigma_X \int_0^t \frac{dW_s}{h_s} \right) = 0. \quad (2.3.3.8)$$

The property of Q implies that

$$\mathbb{E} \left(\int_0^t \frac{dQ_s}{h_s} \right) = \lambda \beta_Z \int_0^t \frac{ds}{h_s}. \quad (2.3.3.9)$$

From the *Case 1* and the relation in (2.3.3.5), we conclude substituting a_X by $a_x + \lambda\beta_Z$, then the expectation in this case is:

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X + \lambda\beta_Z}{\mu_\gamma} \right) - \frac{a_X + \lambda\beta_Z}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0, \\ y + (a_X + \lambda\beta_Z) t, & \text{if } \mu_\gamma = 0. \end{cases} \quad (2.3.3.10)$$

□

Discussion 3:

As we compared the two expressions of the expectation of the income of the insurance companies between (2.3.3.10) in *Case 2* and (2.3.1.9) in *Case 1*, the difference is that the expectation of the income also depends on the intensity of the Poisson process λ , and the mean of withdrawal sizes β_Z . The impact of each variable on $\mathbb{E} [Y_t^{(\gamma)}(y)]$, keeping others unchanged:

- the intensity of the Poisson process λ ,

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \lambda} = \begin{cases} \frac{\beta_Z (e^{\mu_\gamma t} - 1)}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0 \\ t, & \text{if } \mu_\gamma = 0 \end{cases}$$

when $\beta_Z < 0$ and $\mu_\gamma > 0$, the impact of λ on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is negative,

when $\beta_Z = 0$, there is no relation between λ and $\mathbb{E}[Y_t^{(\gamma)}(y)]$,

when $\beta_Z > 0$ and $\mu_\gamma > 0$, the impact of λ on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is positive;

- the mean of withdrawal sizes β_Z ,

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \beta_Z} = \begin{cases} \frac{\lambda (e^{\mu_\gamma t} - 1)}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0 \\ t, & \text{if } \mu_\gamma = 0 \end{cases}$$

the impact of β_Z on $\mathbb{E}[Y_t^{(\gamma)}(y)]$ is positive when $\mu_\gamma > 0$, because $\lambda > 0$ by the assumption.

2.3.3.2 Variance of the income of a life insurance company

In this subsection, we give the expression for the variance of the risk process, as a function of y , a_X , σ_X , λ , β_Z , σ_Z , μ_γ and σ_γ and hence, as a function of γ . We denote $a_{\lambda,\beta} = a_X + \lambda\beta_Z$, $\sigma_{\lambda,\beta}^2 = \sigma_X^2 + \lambda(\beta_Z^2 + \sigma_Z^2)$ and as before μ_γ and σ_γ are defined in (2.3.0.3a) and (2.3.0.3b).

Proposition 2.3.5. For $0 < \gamma < 1$ and $t \geq 0$,
when $\mu_\gamma \neq 0$, $\mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2$ and $\mu_\gamma \neq -\sigma_\gamma^2$:

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] + 2a_{\lambda, \beta} y \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - e^{\mu_\gamma t}}{\mu_\gamma} \right] \\ & + 2a_{\lambda, \beta}^2 \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right] \\ & + \sigma_{\lambda, \beta}^2 \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}; \end{aligned}$$

when $\mu_\gamma = -\frac{1}{2}\sigma_\gamma^2$:

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left(1 - e^{-\sigma_\gamma^2 t} \right) + 4a_{\lambda, \beta} y \frac{e^{-\sigma_\gamma^2 t} - 2e^{-\frac{1}{2}\sigma_\gamma^2 t} + 1}{\sigma_\gamma^2} \\ & + 4a_{\lambda, \beta}^2 \frac{\sigma_\gamma^2 t - 3 - e^{-\sigma_\gamma^2 t} + 4e^{-\frac{1}{2}\sigma_\gamma^2 t}}{\sigma_\gamma^4} + \sigma_{\lambda, \beta}^2 t; \end{aligned}$$

when $\mu_\gamma = -\sigma_\gamma^2$:

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left(e^{-\sigma_\gamma^2 t} - e^{-2\sigma_\gamma^2 t} \right) + 2a_{\lambda, \beta} y \frac{(\sigma_\gamma^2 t - 1) e^{-\sigma_\gamma^2 t} + e^{-2\sigma_\gamma^2 t}}{\sigma_\gamma^2} \\ & + a_{\lambda, \beta}^2 \frac{1 - 2\sigma_\gamma^2 t e^{-\sigma_\gamma^2 t} - e^{-2\sigma_\gamma^2 t}}{\sigma_\gamma^4} + \sigma_{\lambda, \beta}^2 \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} \end{aligned}$$

when $\mu_\gamma = 0$:

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left(e^{\sigma_\gamma^2 t} - 1 \right) + 2a_{\lambda, \beta} y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} + a_{\lambda, \beta}^2 \frac{2e^{\sigma_\gamma^2 t} - (\sigma_\gamma^2 t + 1)^2 - 1}{\sigma_\gamma^4} \\ & + \sigma_{\lambda, \beta}^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2}. \end{aligned}$$

Proof. We know that:

$$\text{Var} [Y_t^{(\gamma)}(y)] = \mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\} - \left\{ \mathbb{E} [Y_t^{(\gamma)}(y)] \right\}^2.$$

From Proposition 2.3.4, it follows that it remains to calculate $\mathbb{E} \left\{ [Y_t^{(\gamma)}(y)]^2 \right\}$.

We omit (γ) for the simplicity of the notation.

We do the calculus using the Itô Formula from Theorem 2.2.4.7 in Section 2.2.4:

$$\begin{aligned} f(Y(t)) &= f(Y(0)) + \int_0^t f'(Y(s)) dY^c(s) + \frac{1}{2} \int_0^t f''(Y(s)) d\langle Y^c, Y^c \rangle_s \\ &+ \sum_{0 < s \leq t} [f(Y(s)) - f(Y(s-))], \end{aligned} \quad (2.3.3.11)$$

where Y^c is continuous martingale part of Y . In our case,

$$dY_t^c = \sigma_X dW_t + \sigma_R^\gamma Y_{t-} dB_t.$$

Equation (2.3.3.11) in differential form is:

$$\begin{aligned} dY_t &= dX_t + Y_{t-} dR_t \\ &= a_X dt + \sigma_X dW_t + dQ_t + \mu_\gamma Y_{t-} dt + \sigma_\gamma Y_{t-} dB_t, \end{aligned} \quad (2.3.3.12)$$

The Ito formula with the function $f(x) = x^2$ in integral form gives:

$$\begin{aligned} Y_t^2 &= y^2 + 2a_X \int_0^t Y_s ds + 2\sigma_X \int_0^t Y_s dW_s + 2\mu_\gamma \int_0^t Y_s^2 ds \\ &+ 2\sigma_\gamma \int_0^t Y_s^2 dB_s + \langle Y^c \rangle_t + [Q, Q]_t + 2 \int_0^t Y_{s-} dQ_s, \end{aligned} \quad (2.3.3.13)$$

because the last term in the right-side hand of (2.3.3.11) in our case is:

$$\begin{aligned} &\sum_{0 < s \leq t} [Y^2(s) - Y^2(s-)] \\ &= \sum_{0 < s \leq t} [(Y(s-) + \Delta Y_s)^2 - (Y(s-))^2] \\ &= \sum_{0 < s \leq t} [(2Y(s-) + \Delta Q_s) \cdot (\Delta Q_s)] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{0 < s \leq t} \left[2Y(s-) \cdot \Delta Q_s + (\Delta Q_s)^2 \right] \\
 &= [Q, Q]_t + 2 \int_0^t Y_{s-} dQ_s,
 \end{aligned}$$

and $\langle Y^c \rangle_t = \langle \sigma_X \int_0^t dW_s + \sigma_\gamma \int_0^t Y_s dB_s \rangle$, with independent W and B , gives

$$\langle Y^c \rangle_t = \sigma_X^2 t + \sigma_\gamma^2 \int_0^t Y_s^2 ds. \quad (2.3.3.14)$$

Let $\tau_n = \inf \{t \geq 0 : |Y_t| > n\}$ with $\inf \{\emptyset\} = +\infty$. Then $\left(\int_0^{t \wedge \tau_n} Y_s dW_s\right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau_n} Y_s^2 dB_s\right)_{t \geq 0}$ are local martingales. It implies that there exists a sequence of stopping times $(s_n)_{n \geq 1}$ going to $+\infty$ such that $\left(\int_0^{t \wedge \tau_n \wedge s_n} Y_s dW_s\right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau_n \wedge s_n} Y_s^2 dB_s\right)_{t \geq 0}$ are martingales.

We put $\tau'_n = \tau_n \wedge s_n$, then from (2.3.3.13) and (2.3.3.14):

$$\begin{aligned}
 &Y_{t \wedge \tau'_n}^2 \\
 &= y^2 + 2a_X \int_0^{t \wedge \tau'_n} Y_s ds + 2\sigma_X \int_0^{t \wedge \tau'_n} Y_s dW_s + 2\mu_\gamma \int_0^{t \wedge \tau'_n} Y_s^2 ds \\
 &\quad + 2\sigma_\gamma \int_0^{t \wedge \tau'_n} Y_s^2 dB_s + \sigma_X^2 (t \wedge \tau'_n) + \sigma_\gamma^2 \int_0^{t \wedge \tau'_n} Y_s^2 ds \\
 &\quad + [Q, Q]_{t \wedge \tau'_n} + 2 \int_0^{t \wedge \tau'_n} Y_{s-} dQ_s.
 \end{aligned} \quad (2.3.3.15)$$

We take mathematical expectation in (2.3.3.15), and since $\left(\int_0^{t \wedge \tau'_n} Y_s dW_s\right)_{t \geq 0}$ and $\left(\int_0^{t \wedge \tau'_n} Y_s^2 dB_s\right)_{t \geq 0}$ are martingales, we get that:

$$\begin{aligned}
 &\mathbb{E} \left[Y_{t \wedge \tau'_n}^2 \right] \\
 &= y^2 + 2a_X \mathbb{E} \int_0^{t \wedge \tau'_n} Y_s ds + 2\mu_\gamma \mathbb{E} \int_0^{t \wedge \tau'_n} Y_s^2 ds + \sigma_X^2 (t \wedge \tau'_n) \\
 &\quad + \sigma_\gamma^2 \mathbb{E} \int_0^{t \wedge \tau'_n} Y_s^2 ds + \mathbb{E} [Q, Q]_{t \wedge \tau'_n} + 2 \mathbb{E} \int_0^{t \wedge \tau'_n} Y_{s-} dQ_s.
 \end{aligned} \quad (2.3.3.16)$$

From Theorems 2.2.4.3 and 2.2.4.4 in Section 2.2.4, we calculate

$$\begin{aligned}
 \mathbb{E} [Q, Q]_t &= \mathbb{E} \int_0^t \int_{\mathbb{R}} x^2 d\mu_Q(x) = \mathbb{E} \int_0^t \int_{\mathbb{R}} x^2 d\nu_Q(x, t) \\
 &= \mathbb{E} \int_0^t \int_{\mathbb{R}} x^2 dF_Z(x) \lambda ds = \text{Var}(Q_t) \\
 &= \lambda t (\beta_Z^2 + \sigma_Z^2).
 \end{aligned}$$

and, the integral of a continuous stochastic process with respect to the compound Poisson process,

$$\begin{aligned} 2 \mathbb{E} \int_0^t Y_{s-} dQ_s &= 2 \mathbb{E} \int_0^t Y_{s-} \int_{\mathbb{R}} x F_Z(dx) \lambda ds \\ &= 2\lambda\beta_Z \int_0^t \mathbb{E}(Y_s) ds. \end{aligned}$$

Since $Y_s \geq 0$, we can do limit passage, $\lim_{n \rightarrow +\infty}$, in each term in the right-hand side by Lebesgue's Monotone Convergence Theorem. We prove that $\left[Y_{t \wedge \tau'_n}^2 \right]_{n \in \mathbb{N}}$ is uniformly integrable and we pass to the limit in the left-hand side.

This gives using Fubini Theorem that:

$$\begin{aligned} \mathbb{E}(Y_t^2) &= y^2 + 2a_X \int_0^t \mathbb{E}(Y_s) ds + 2\mu_\gamma \int_0^t \mathbb{E}(Y_s^2) ds + \sigma_X^2 t + \sigma_\gamma^2 \int_0^t \mathbb{E}(Y_s^2) ds \\ &\quad + \lambda t (\beta_Z^2 + \sigma_Z^2) + 2\lambda\beta_Z \int_0^t \mathbb{E}(Y_s) ds. \end{aligned}$$

We get with

$$g_t = \mathbb{E}(Y_t^2),$$

$$k = 2\mu_\gamma + \sigma_\gamma^2 + 2\lambda\beta_Z, \text{ and}$$

$$f_t = 2a_X \int_0^t \mathbb{E}(Y_s) ds + \sigma_X^2 t + \lambda t (\beta_Z^2 + \sigma_Z^2) + 2\lambda\beta_Z \int_0^t \mathbb{E}(Y_s),$$

then, the equations for $Var \left[Y_t^{(\gamma)}(y) \right]$ can be obtained from *Case 1*, replacing a_X by $a_X + \lambda\beta_Z$ and σ_X^2 by $\sigma_X^2 + \lambda(\beta_Z^2 + \sigma_Z^2)$. \square

Discussion 4:

As we compared the expressions of the variances of the income of the insurance companies between *Case 1* and *Case 2*, the compound Poisson process (Q_t) in the basic risk process has additional impacts on the variance through the intensity of the Poisson process, λ , the mean of withdrawal sizes, β_Z , and the variance of withdrawals, σ_Z^2 . The impact of the each variable on $Var \left[Y_t^{(\gamma)}(y) \right]$, keeping others unchanged:

- the intensity of the Poisson process, λ , and the mean of withdrawal sizes, β_Z , since their derivatives are complicated, we do not list the formulas. The relations are not clear.
- the standard deviation of withdrawal sizes, $\sigma_Z = \sqrt{\sigma_Z^2}$,

(1) when $\mu_\gamma \neq 0$, $\mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2$ and $\mu_\gamma \neq -\sigma_\gamma^2$:

$$\frac{\partial \text{Var} [Y_t^{(\gamma)}(y)]}{\partial \sigma_Z} = 2\lambda\sigma_Z \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2} > 0;$$

(2) when $\mu_\gamma = -\frac{1}{2}\sigma_\gamma^2$:

$$\frac{d\text{Var} [Y_t^{(\gamma)}(y)]}{d\sigma_Z} = 2\lambda\sigma_Z t > 0;$$

(3) when $\mu_\gamma = -\sigma_\gamma^2$:

$$\frac{d\text{Var} [Y_t^{(\gamma)}(y)]}{d\sigma_Z} = 2\lambda\sigma_Z \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2} > 0;$$

(4) when $\mu_\gamma = 0$:

$$\frac{d\text{Var} [Y_t^{(\gamma)}(y)]}{d\sigma_Z} = 2\lambda\sigma_Z \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2} > 0.$$

The impact of σ_Z on $\text{Var} [Y_t^{(\gamma)}(y)]$ is positive.

Remark 2.3.2. *In this case, the relationships between each variable and the variance of the income of the insurance company are the same as the relationships in Case 1, in addition, the variance of withdrawals, σ_Z^2 , in the compound Poisson process in the insurance business, has a positive impact on $\text{Var} [Y_t^{(\gamma)}(y)]$. It is intuitive that the increase of the fluctuation in withdrawal sizes in the insurance business will contribute to the increase of the variance of the income of the insurance company.*

2.3.4 Case 2bis: Multiple Risky Assets, when $(X_t)_{t \geq 0}$ is Modeled by the sum of a Brownian Motion and a compound Poisson process

By the same reasoning for the proofs of *Case 1bis*, Proposition 2.3.4 and Proposition 2.3.5 in *Case 2*, we give the expressions for the expectation and variance of the income of the insurance company when there are multiple risky assets in risky investment. As in *Case 2*, $a_{\lambda,\beta} = a_X + \lambda\beta_Z$, $\sigma_{\lambda,\beta}^2 = \sigma_X^2 + \lambda(\beta_Z^2 + \sigma_Z^2)$, μ_γ and σ_γ are defined in (2.3.2.4a)

and (2.3.2.4b).

Proposition 2.3.6. For $0 < \gamma < 1$ and $t \geq 0$:

$$\mathbb{E} \left[Y_t^{(\gamma)}(y) \right] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X + \lambda \beta_Z}{\mu_\gamma} \right) - \frac{a_X + \lambda \beta_Z}{\mu_\gamma}, & \text{if } \mu_\gamma \neq 0, \mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2, \mu_\gamma \neq -\sigma_\gamma^2 \\ y + (a_X + \lambda \beta_Z) t, & \text{if } \mu_\gamma = 0 \\ e^{-\frac{1}{2}\sigma_\gamma^2 t} \left(y - \frac{a_X + \lambda \beta_Z}{\frac{1}{2}\sigma_\gamma^2} \right) + \frac{a_X + \lambda \beta_Z}{\frac{1}{2}\sigma_\gamma^2}, & \text{if } \mu_\gamma = -\frac{1}{2}\sigma_\gamma^2 \\ e^{-\sigma_\gamma^2 t} \left(y - \frac{a_X + \lambda \beta_Z}{\sigma_\gamma^2} \right) + \frac{a_X + \lambda \beta_Z}{\sigma_\gamma^2}, & \text{if } \mu_\gamma = -\sigma_\gamma^2 \end{cases}$$

When $\mu_\gamma \neq 0$, $\mu_\gamma \neq -\frac{1}{2}\sigma_\gamma^2$ and $\mu_\gamma \neq -\sigma_\gamma^2$, we get

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left[e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{2\mu_\gamma t} \right] + 2a_{\lambda, \beta} y \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma + \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - e^{\mu_\gamma t}}{\mu_\gamma} \right] \\ & + 2a_{\lambda, \beta}^2 \left[\frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - e^{\mu_\gamma t}}{\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} + \frac{1 - e^{(2\mu_\gamma + \sigma_\gamma^2)t}}{2\mu_\gamma^2 + \mu_\gamma \sigma_\gamma^2} - \frac{e^{2\mu_\gamma t} - 2e^{\mu_\gamma t} + 1}{2\mu_\gamma^2} \right] \\ & + \sigma_{\lambda, \beta}^2 \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)t} - 1}{2\mu_\gamma + \sigma_\gamma^2}. \end{aligned}$$

When $\mu_\gamma = -\frac{1}{2}\sigma_\gamma^2$, we have

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left(1 - e^{-\sigma_\gamma^2 t} \right) + 4a_{\lambda, \beta} y \frac{e^{-\sigma_\gamma^2 t} - 2e^{-\frac{1}{2}\sigma_\gamma^2 t} + 1}{\sigma_\gamma^2} \\ & + 4a_{\lambda, \beta}^2 \frac{\sigma_\gamma^2 t - 3 - e^{-\sigma_\gamma^2 t} + 4e^{-\frac{1}{2}\sigma_\gamma^2 t}}{\sigma_\gamma^4} + \sigma_{\lambda, \beta}^2 t. \end{aligned}$$

When $\mu_\gamma = -\sigma_\gamma^2$, we obtain

$$\begin{aligned} & \text{Var} \left[Y_t^{(\gamma)}(y) \right] \\ = & y^2 \left(e^{-\sigma_\gamma^2 t} - e^{-2\sigma_\gamma^2 t} \right) + 2a_{\lambda, \beta} y \frac{(\sigma_\gamma^2 t - 1) e^{-\sigma_\gamma^2 t} + e^{-2\sigma_\gamma^2 t}}{\sigma_\gamma^2} \\ & + a_{\lambda, \beta}^2 \frac{1 - 2\sigma_\gamma^2 t e^{-\sigma_\gamma^2 t} - e^{-2\sigma_\gamma^2 t}}{\sigma_\gamma^4} + \sigma_{\lambda, \beta}^2 \frac{1 - e^{-\sigma_\gamma^2 t}}{\sigma_\gamma^2}. \end{aligned}$$

Finally, when $\mu_\gamma = 0$,

$$\begin{aligned} & \text{Var} [Y_t^{(\gamma)}(y)] \\ &= y^2 (e^{\sigma_\gamma^2 t} - 1) + 2a_{\lambda, \beta} y \frac{e^{\sigma_\gamma^2 t} - 1 - \sigma_\gamma^2 t}{\sigma_\gamma^2} + a_{\lambda, \beta}^2 \frac{2e^{\sigma_\gamma^2 t} - (\sigma_\gamma^2 t + 1)^2 - 1}{\sigma_\gamma^4} \\ & \quad + \sigma_{\lambda, \beta}^2 \frac{e^{\sigma_\gamma^2 t} - 1}{\sigma_\gamma^2}. \end{aligned}$$

2.3.5 General case: when: $X = (X_t)_{t \geq 0}$ and $R^{(\gamma)} = (R_t^{(\gamma)})_{t \geq 0}$ are both Lévy processes

Let $X = (X_t)_{t \geq 0}$ be a Lévy process with the parameters (a_X, σ_X^2, ν_X) , where a_X is drift parameter, σ_X^2 is the variance coefficient of the Brownian part, and ν_X is the Lévy measure, $\int_{\mathbb{R}} (x^2 \wedge 1) \nu_X(dx) < \infty$. We assume in addition that X is square-integrable, i.e., $\forall t \geq 0, \mathbb{E}(|X_t^2|) < \infty$. Moreover,

$$X_t = X_0 + \delta_X \cdot t + \sigma_X W_t + \int_0^t \int_{\mathbb{R}} x(\mu_X - \nu_X)(ds, dx),$$

where μ_X is the measure of the jumps of X ,

$$\delta_X = a_X + \int_{|x| > 1} x \nu_X(dx). \quad (2.3.5.1)$$

Let $R = (R_t)_{t \geq 0}$ be also a Lévy process with the parameters (a_R, σ_R^2, ν_R) , where a_R is drift parameter, σ_R^2 is the variance coefficient of the Brownian part, and ν_R is the Lévy measure, $\int_{\mathbb{R}} (x^2 \wedge 1) \nu_R(dx) < \infty$.

We suppose that the jumps of the process R , $\Delta R_t = R_t - R_{t-} > -1$, for all $t > 0$, that $\mathbb{E}(e^{2R_t}) < \infty$, and that X and R are independent processes.

We consider a risk process, $Y = (Y_t)_{t \geq 0}$, of the insurance company such that, for $0 \leq t \leq T$,

$$Y_t = y + X_t + \int_0^t Y_{s-} dR_s, \quad (2.3.5.2)$$

where $X = (X_t)_{t \geq 0}$ is a basic process and $R = (R_t)_{t \geq 0}$ is risky remuneration.

If the insurance company invests the proportion of $(1 - \gamma)$ in a risk-free asset with interest

rate ($r > 0$) and the proportion of ($0 < \gamma < 1$) in a risky asset with the return R_t , then

$$\begin{aligned} Y_t &= y + X_t + (1 - \gamma)r \int_0^t Y_{s-} ds + \gamma \int_0^t Y_{s-} dR_s \\ &= y + X_t + \int_0^t Y_{s-} dR_s^{(\gamma)}, \end{aligned} \quad (2.3.5.3)$$

where $R_s^{(\gamma)}$ is also a Lévy process with the characteristics of $(\gamma a_R + (1 - \gamma)r, \gamma^2 \sigma_R^2, \nu_R(s, x/\gamma))$.

For simplicity, we consider the model in (2.3.5.2), since to apply it to (2.3.5.3) it is sufficient to replace the parameter a_R by $\gamma a_R + (1 - \gamma)r$, σ_R^2 by $\gamma^2 \sigma_R^2$ and $\nu_R(s, x)$ by $\nu_R(s, x/\gamma)$. Then, as well-known by [Paulsen, 1996], the equation (2.3.5.2) has a unique strong solution: for $t > 0$,

$$Y_t = \mathcal{E}(R^{(\gamma)})_t \left(y + \int_0^t \frac{dX_s}{\mathcal{E}(R^{(\gamma)})_{s-}} \right), \quad (2.3.5.4)$$

where $\mathcal{E}(R^{(\gamma)})$ is Doléans-Dade's exponential,

$$\mathcal{E}(R^{(\gamma)})_t = \exp \left\{ R_t - \frac{1}{2} \langle R^c, R^c \rangle_t \right\} \prod_{0 < s \leq t} e^{-\Delta R_s} (1 + \Delta R_s).$$

We can show that $\forall t \geq 0$,

$$\begin{aligned} e^{\hat{R}_t} &= \mathcal{E}(R)_t \\ &= \exp \left\{ R_t - \frac{1}{2} \langle R^c, R^c \rangle_t \right\} \prod_{0 < s \leq t} e^{-\Delta R_s} (1 + \Delta R_s), \end{aligned} \quad (2.3.5.5)$$

where $(\hat{R}_t)_{t \geq 0}$ is also a Lévy process.

Then, equation (2.3.5.4) is equivalent to

$$Y_t = e^{\hat{R}_t} \left(y + \int_0^t e^{-\hat{R}_s} dX_s \right).$$

We denote by

$$\Lambda_t(\epsilon) = \mathbb{E} \left(e^{-\epsilon \hat{R}_t} \right),$$

the Laplace transform of \hat{R}_t of the parameter ϵ , for ϵ belonging to the domain of the existence of this Laplace transform.

It is known that

$$\mathbb{E} \left(e^{-\epsilon \hat{R}_t} \right) = e^{-t\Phi(\epsilon)},$$

where $\Phi(\epsilon)$ is the Laplace exponent of the process \hat{R} .

If we denote by $(\hat{a}_R, \hat{\sigma}_R^2, \hat{\nu}_R)$ the characteristics of \hat{R} , then

$$\Phi(\epsilon) = \epsilon \hat{a}_R - \frac{1}{2} \epsilon^2 \hat{\sigma}_R^2 - \int_{\mathbb{R}} (e^{-\epsilon x} - 1) \hat{\nu}_R(dx). \quad (2.3.5.6)$$

Next, we express the characteristics $(\hat{a}_R, \hat{\sigma}_R^2, \hat{\nu}_R)$ via (a_R, σ_R^2, ν_R) , we obtain the expression of $\Phi(\epsilon)$ containing the parameters of the process R ,

$$\Phi(\epsilon) = \epsilon a_R - \frac{1}{2} \sigma_R^2 (\epsilon - \epsilon^2) + \epsilon \int_{\mathbb{R}} \ln(1+x) \nu_R(dx). \quad (2.3.5.7)$$

Proof. From (2.3.5.5) we know $\forall t \geq 0$,

$$R_t - \frac{1}{2} \langle R^c, R^c \rangle_t - \sum_{0 < s \leq t} [\ln(1 + \Delta R_s) - \Delta R_s] = \hat{R}_t. \quad (2.3.5.8)$$

We look on the continuous martingale part of the left-hand side and right-hand side in (2.3.5.8), $\forall t \geq 0$, since $R_t^c = \hat{R}_t^c$, we have $\sigma_R^2 = \hat{\sigma}_R^2$.

From (2.3.5.8), since $R_t^c = a_R t + M_t^c + M_t^d$,

$$\begin{aligned} a_R t + M_t^c + M_t^d - \frac{1}{2} \langle R^c, R^c \rangle_t + \int_0^t \int_{\mathbb{R}} [\ln(1+x) - x] (\mu_R - \nu_R)(dx) \\ + t \int_{\mathbb{R}} [\ln(1+x) - x] \nu_R(dx) = \hat{a}_R t + M_t^c + \hat{M}_t^d, \end{aligned}$$

then $a_R t - \frac{1}{2} \sigma_R^2 t + t \int_{\mathbb{R}} [\ln(1+x) - x] \nu_R(dx) = \hat{a}_R t$,

$$\hat{a}_R = a_R - \frac{1}{2} \sigma_R^2 + \int_{\mathbb{R}} [\ln(1+x) - x] \nu_R(dx).$$

From (2.3.5.8), $\Delta \hat{R}_t = \hat{R}_t - \hat{R}_{t-} = \Delta R_t + \ln(1 + \Delta R_s) - \Delta R_s = \ln(1 + \Delta R_s)$.

$$\begin{aligned} t \int_{\mathbb{R}} (e^{-\epsilon x} - 1) \hat{\nu}_R(dx) &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} (e^{-\epsilon x} - 1) \hat{\nu}_R(dx) \right] = \mathbb{E} \left[\sum_{0 < s \leq t} (e^{\Delta \hat{R}_s} - 1) \right] \\ &= \mathbb{E} \left[\sum_{0 < s \leq t} (e^{\ln(1 + \Delta R_s)} - 1) \right] = \mathbb{E} \left[\sum_{0 < s \leq t} \Delta R_s \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} -\epsilon x \nu_R(dx) \right] = t \int_{\mathbb{R}} -\epsilon x \nu_R(dx). \end{aligned}$$

Hence,

$$\begin{aligned}
 \Phi(\epsilon) &= \epsilon \left\{ a_R - \frac{1}{2} \sigma_R^2 + \int_{\mathbb{R}} [\ln(1+x) - x] \nu_R(dx) \right\} \\
 &\quad - \frac{1}{2} \epsilon^2 \sigma_R^2 + \epsilon \int_{\mathbb{R}} x \nu_R(dx) \\
 &= \epsilon a_R - \frac{1}{2} \sigma_R^2 (\epsilon - \epsilon^2) + \epsilon \int_{\mathbb{R}} \ln(1+x) \nu_R(dx).
 \end{aligned} \tag{2.3.5.9}$$

□

We denote also $\forall t \geq 0$

$$I_t = \int_0^t e^{\hat{R}s} ds, \quad J_t = \int_0^t e^{2\hat{R}s} ds.$$

Let us calculate $\mathbb{E}(Y_t)$.

2.3.5.1 Expectation of the income of a life insurance company

Proposition 2.3.7. *For $t \geq 0$,*

$$\mathbb{E}(Y_t) = \begin{cases} y + \delta_X t, & \text{if } \Phi(-1) = 0, \\ y e^{-t\Phi(-1)} + \delta_X \frac{1 - e^{-t\Phi(-1)}}{\Phi(-1)}, & \text{if } \Phi(-1) \neq 0, \end{cases}$$

where δ_X is defined by (2.3.5.1), $\Phi_\gamma(\cdot)$ is given in (2.3.5.11).

Proof. We denote for fixed $t > 0$, $\mathcal{E}(R) = (\mathcal{E}(R)_s)_{0 \leq s \leq t}$ and $h = (h_s)_{0 \leq s \leq t}$ the trajectory of $\mathcal{E}(R)$. Then

$$\mathbb{E} \left[Y_t | \mathcal{E}(R) = h \right] = \mathbb{E} \left[h_t \left(y + \int_0^t \frac{dX_s}{h_s} \right) \right],$$

since X and R are independent processes. Moreover,

$$X_s = X_0 + \delta_X \cdot s + M_s,$$

where $M = M(s)_{0 \leq s \leq t}$ is a martingale, such that

$$M_s = \delta_X W_s + \int_0^t \int_{\mathbb{R}} x (\mu_X - \nu_X)(ds, dx).$$

So that,

$$\begin{aligned}\mathbb{E} [Y_t | \mathcal{E}(R) = h] &= \mathbb{E} \left[h_t \left(y + \delta_X \int_0^t \frac{ds}{h_s} + \int_0^t \frac{dM_s}{h_s} \right) \right] \\ &= h_t \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right),\end{aligned}$$

since $\mathbb{E} \left(\int_0^t \frac{dM_s}{h_s} \right) = 0$ by the martingale property. Then, replacing h_s by \hat{R}_s and h_t by \hat{R}_t , we get

$$\begin{aligned}\mathbb{E}(Y_t) &= \mathbb{E} \left[e^{\hat{R}_t} \left(y + \delta_X \int_0^t e^{-\hat{R}_s} ds \right) \right] \\ &= \mathbb{E} \left(y e^{\hat{R}_t} \right) + \mathbb{E} \left[\delta_X \int_0^t e^{+(\hat{R}_t - \hat{R}_s)} ds \right],\end{aligned}$$

by the change of the variable $s = t - s$ and the fact $(\hat{R}_t - \hat{R}_s)_{0 \leq s \leq t} \stackrel{\mathcal{L}}{=} (\hat{R}_{t-s})_{0 \leq s \leq t}$,

$$\begin{aligned}\mathbb{E}(Y_t) &= y e^{-t\Phi(-1)} + \delta_X \mathbb{E} \left(\int_0^t e^{\hat{R}_{t-s}} ds \right) \\ &= y e^{-t\Phi(-1)} + \delta_X \mathbb{E} \left(\int_0^t e^{\hat{R}_s} ds \right),\end{aligned}$$

Hence,

$$\mathbb{E}(Y_t) = y e^{-t\Phi(-1)} + \delta_X \mathbb{E}(I_t).$$

For $\mathbb{E}(I_t)$ by Fubini theorem:

$$\begin{aligned}\mathbb{E}(I_t) &= \int_0^t \mathbb{E}(e^{\hat{R}_s}) ds = \int_0^t e^{-s\Phi(-1)} ds = \\ &= \begin{cases} t, & \text{if } \Phi(-1) = 0, \\ \frac{1 - e^{-t\Phi(-1)}}{\Phi(-1)}, & \text{if } \Phi(-1) \neq 0. \end{cases} \end{aligned} \tag{2.3.5.10}$$

From (2.3.5.9), we can calculate

$$\Phi_\gamma(-1) = -[\gamma a_R + (1 - \gamma)r] - \int_{\mathbb{R}} \ln(1 + \gamma x) \nu_R(dx). \tag{2.3.5.11}$$

Finally,

$$\mathbb{E}(Y_t) = y e^{-t\Phi(-1)} + \delta_X \mathbb{E}(I_t)$$

$$= \begin{cases} y + \delta_X t, & \text{if } \Phi(-1) = 0, \\ ye^{-t\Phi(-1)} + \delta_X \frac{1-e^{-t\Phi(-1)}}{\Phi(-1)}, & \text{if } \Phi(-1) \neq 0. \end{cases}$$

□

2.3.5.2 Variance of the income of a life insurance company

Proposition 2.3.8. For $t \geq 0$, when $\Phi(1) \neq 0$, $\Phi(2) \neq 0$ and $\Phi(1) \neq \Phi(2)$,

$$\begin{aligned} \text{Var}(Y_t) = & y^2 \left[e^{-t\Phi(-2)} - e^{-2t\Phi(-1)} \right] \\ & + 2y\delta_X \left[\frac{1}{2} \frac{d}{dt} \mathbb{E}(I_t^2) - e^{-t\Phi(-1)} \mathbb{E}(I_t) \right] \\ & + \delta_X^2 \left\{ \mathbb{E}(I_t^2) - [\mathbb{E}(I_t)]^2 \right\} + \left[\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right] \mathbb{E}(J_t), \end{aligned}$$

where $\Phi(\cdot)$ is defined in (2.3.5.9), $\mathbb{E}(I_t)$ is given in (2.3.5.10), $\mathbb{E}(I_t^2)$ is given in (2.3.5.17), and $\mathbb{E}(J_t)$ is given in (2.3.5.14).

Proof. First, we calculate $\mathbb{E}(Y_t^2)$:

$$\begin{aligned} & \mathbb{E} \left[Y_t^2 \middle| \mathcal{E}(R) = h \right] \\ = & \mathbb{E} \left[h_t^2 \left(y + \int_0^t \frac{dX_s}{h_s} \right)^2 \right] \\ = & \mathbb{E} \left[h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} + \int_0^t \frac{dM_s}{h_s} \right)^2 \right] \\ = & \mathbb{E} \left[h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right)^2 + 2h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right) \cdot \int_0^t \frac{dM_s}{h_s} + h_t^2 \left(\int_0^t \frac{dM_s}{h_s} \right)^2 \right] \end{aligned}$$

Due to the properties of the stochastic integral with respect to a martingale,

$$\begin{aligned} \mathbb{E} \left[Y_t^2 \middle| \mathcal{E}(R) = h \right] &= h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right)^2 + 0 + h_t^2 \mathbb{E} \left[\left(\int_0^t \frac{dM_s}{h_s} \right)^2 \right] \\ &= h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right)^2 + h_t^2 \mathbb{E} \left(\int_0^t \frac{d[M, M]_s}{h_s^2} \right) \\ &= h_t^2 \left(y + \delta_X \int_0^t \frac{ds}{h_s} \right)^2 + h_t^2 \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right) \int_0^t \frac{ds}{h_s^2} \end{aligned}$$

Then, replacing h_s by \hat{R}_s and h_t by \hat{R}_t , we get

$$\begin{aligned} \mathbb{E}(Y_t^2) &= \mathbb{E} \left[e^{2\hat{R}_t} \left(y + \delta_X \int_0^t e^{-\hat{R}_s} ds \right)^2 \right] \\ &\quad + \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right) \cdot \mathbb{E} \left(e^{2\hat{R}_t} \int_0^t e^{-2\hat{R}_s} ds \right) \end{aligned} \quad (2.3.5.12)$$

We calculate two mathematical expectations expressed in the right-hand side. For the second one, by change of the variable we get

$$\begin{aligned} \mathbb{E} \left(e^{2\hat{R}_t} \int_0^t e^{-2\hat{R}_s} ds \right) &= \mathbb{E} \int_0^t e^{2(\hat{R}_t - \hat{R}_s)} ds = \mathbb{E} \int_0^t e^{2\hat{R}_t - s} ds \\ &= \mathbb{E} \int_0^t e^{2\hat{R}_s} ds = \mathbb{E}(J_t). \end{aligned} \quad (2.3.5.13)$$

By Fubini theorem,

$$\begin{aligned} \mathbb{E}(J_t) &= \mathbb{E} \int_0^t e^{2\hat{R}_s} ds = \int_0^t \mathbb{E} e^{2\hat{R}_s} ds = \int_0^t e^{-s\Phi(-2)} ds \\ &= \begin{cases} t, & \text{if } \Phi(-2) = 0, \\ \frac{1 - e^{-t\Phi(-2)}}{\Phi(-2)}, & \text{if } \Phi(-2) \neq 0. \end{cases} \end{aligned} \quad (2.3.5.14)$$

For the first term in (2.3.5.12) we obtain:

$$\begin{aligned} &\mathbb{E} \left[e^{2\hat{R}_t} \left(y + \delta_X \int_0^t e^{-\hat{R}_s} ds \right)^2 \right] \\ &= \mathbb{E} \left[y^2 e^{2\hat{R}_t} + 2y\delta_X e^{2\hat{R}_t} \int_0^t e^{-\hat{R}_s} ds + \delta_X^2 e^{2\hat{R}_t} \left(\int_0^t e^{-\hat{R}_s} ds \right)^2 \right] \\ &= \mathbb{E} \left[y^2 e^{2\hat{R}_t} + 2y\delta_X e^{\hat{R}_t} \int_0^t e^{\hat{R}_t - \hat{R}_s} ds + \delta_X^2 \left(\int_0^t e^{\hat{R}_t - \hat{R}_s} ds \right)^2 \right] \\ &= \mathbb{E} \left[y^2 e^{2\hat{R}_t} + 2y\delta_X e^{\hat{R}_t} \int_0^t e^{\hat{R}_t - s} ds + \delta_X^2 \left(\int_0^t e^{\hat{R}_t - s} ds \right)^2 \right] \\ &= y^2 e^{-t\Phi(-2)} + 2y\delta_X \mathbb{E} \left(e^{\hat{R}_t} \int_0^t e^{\hat{R}_s} ds \right) + \delta_X^2 \mathbb{E} \left[\left(\int_0^t e^{\hat{R}_s} ds \right)^2 \right] \\ &= y^2 e^{-t\Phi(-2)} + 2y\delta_X \cdot \frac{1}{2} \frac{d}{dt} \mathbb{E}(I_t^2) + \delta_X^2 \cdot \mathbb{E}(I_t^2). \end{aligned} \quad (2.3.5.15)$$

Then, from (2.3.5.12), (2.3.5.13) and (2.3.5.15)

$$\mathbb{E}(Y_t^2) = y^2 e^{-t\Phi(-2)} + 2y\delta_X \cdot \frac{1}{2} \frac{d}{dt} \mathbb{E}(I_t^2) + \delta_X^2 \cdot \mathbb{E}(I_t^2)$$

$$+ \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right) \mathbb{E}(J_t),$$

Finally, from Proposition 2.3.7,

$$\begin{aligned} \text{Var}(Y_t) &= \mathbb{E}(Y_t^2) - \left(\mathbb{E}(Y_t) \right)^2 \\ &= y^2 e^{-t\Phi(-2)} + 2y\delta_X \cdot \frac{1}{2} \frac{d}{dt} \mathbb{E}(I_t^2) + \delta_X^2 \cdot \mathbb{E}(I_t^2) \\ &\quad + \left(\sigma_X^2 + \int_{|x| \leq 1} x^2 \nu_X(dx) \right) \mathbb{E}(J_t) - \left[y e^{-t\Phi(-1)} + \delta_X \mathbb{E}(I_t) \right]^2 \\ &= y^2 \left[e^{-t\Phi(-2)} - e^{-2t\Phi(-1)} \right] \\ &\quad + 2y\delta_X \left[\frac{1}{2} \frac{d}{dt} \mathbb{E}(I_t^2) - e^{-t\Phi(-1)} \mathbb{E}(I_t) \right] \\ &\quad + \delta_X^2 \left\{ \mathbb{E}(I_t^2) - [\mathbb{E}(I_t)]^2 \right\} + \left[\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right] \mathbb{E}(J_t). \end{aligned} \tag{2.3.5.16}$$

For the calculation of $\mathbb{E}(I_t^2)$ in (2.3.5.16), we know from Corollary 1 in [Salminen and Vostrikova, 2018], that if Φ is bijective on $[0, n]$ then

$$\begin{aligned} \mathbb{E}(I_t^2) &= 2! \frac{e^{-\Phi(0)t} - e^{-\Phi(2)t}}{\prod_{1 \leq i \leq 2} [\Phi(i) - \Phi(0)]} + 2! \frac{e^{-\Phi(1)t} - e^{-\Phi(2)t}}{\prod_{0 \leq i \leq 2, i \neq 1} [\Phi(i) - \Phi(0)]} \\ &= 2 \frac{e^{-\Phi(0)t} - e^{-\Phi(2)t}}{[\Phi(1) - \Phi(0)][\Phi(2) - \Phi(0)]} + 2 \frac{e^{-\Phi(1)t} - e^{-\Phi(2)t}}{[\Phi(0) - \Phi(1)][\Phi(2) - \Phi(1)]} \\ &= 2 \frac{1}{\Phi(1) - \Phi(0)} \left[\frac{e^{-\Phi(0)t} - e^{-\Phi(2)t}}{\Phi(2) - \Phi(0)} + \frac{e^{-\Phi(2)t} - e^{-\Phi(1)t}}{\Phi(2) - \Phi(1)} \right], \end{aligned}$$

and from (2.3.5.9), we get with $\Phi(0) = 0$. Then, if $\Phi(1) \neq 0$, $\Phi(2) \neq 0$ and $\Phi(1) \neq \Phi(2)$, we get

$$\mathbb{E}(I_t^2) = \frac{2}{\Phi(1)} \left[\frac{1 - e^{-\Phi(2)t}}{\Phi(2)} + \frac{1 - e^{-\Phi(1)t}}{\Phi(2) - \Phi(1)} \right]. \tag{2.3.5.17}$$

□

2.3.6 Analysis and discussions

We tabulated the contents of discussions and remarks in *Case 1* and *Case 2*. By calculating the first-order partial derivatives, we obtain the relationships between each variable and the first two moments of the income of the life insurance company in both

cases. The impact of each variable on the expectation of the income of a life insurance company is filled in Table 2.1. It shows that the expected income of a life insurance company is positively related to the initial capital, the premium rate, and the investment return when the investment return is non-negative. In *Case 2*, the basic risk process negatively impacts the expected income of a life insurance company through the intensity of the Poisson process and the average size in withdrawal events. The expectation of the income of a life insurance company $\mathbb{E} [Y_t^{(\gamma)}(y)]$ is an increasing function of the proportion invested in a risky asset γ , under the conditions for no-arbitrage and positive cash flows from the insurance process.

Table 2.1 – The impact of each variable on the expected income of a life insurance company

Variables	Case 1		Case 2	
	$\mu_\gamma \neq 0$	$\mu_\gamma = 0$	$\mu_\gamma \neq 0$	$\mu_\gamma = 0$
initial capital, y	positive	positive	positive	positive
premium rate, a_X	positive	positive	positive	positive
fluctuations in premium income, σ_X	none	none	none	none
investment rate of return, μ_γ	positive	none	positive	none
volatility, σ_γ	none	none	none	none
proportion invested in a risky asset, γ	positive ^a	none	positive ^b	none
the intensity of the Poisson process, λ	none	none	negative	negative
average size of withdrawals, β_Z	none	none	negative	negative
variance of withdrawal sizes, σ_Z^2	none	none	none	none

- a. if the no-arbitrage condition $a_R > r$ holds and $a_X > 0$.
- b. if the no-arbitrage condition $a_R > r$ holds and $a_X > \lambda\beta_Z$.

Table 2.2 – The impact of each variable on the second moment of the income of a life insurance company

Variables	Case 1		Case 2	
	$\mu_\gamma > 0$	$\mu_\gamma = 0$	$\mu_\gamma > 0$	$\mu_\gamma = 0$
the initial capital, y	positive	positive	positive	positive
the premium rate, a_X	complex	positive	complex	positive
fluctuations in premium income, σ_X	positive	positive	positive	positive
the investment return, μ_γ	complex	none	complex	none
the investment volatility, σ_γ	complex	positive	complex	positive
the intensity of the Poisson process, λ	none	none	complex	positive
average size of withdrawals, β_Z	none	none	complex	positive
the variance of withdrawal sizes, σ_Z^2	none	none	positive	positive

When the investment return is positive, the impact on the second moment of the income of the life insurance company is determined by a mixture of the premium rate,

the investment return, the investment volatility, the intensity of the Poisson process, and the average size in withdrawal events, as shown in Table 2.2. The complex impact of the premium rate on the second moment can be explained as when the premium rate increases, the expectation of the income of the insurance company rises, and the uncertainty of the second moment of the income of the insurance company also increases. However, it is not clear whether the second moment increases, decreases, or remains unchanged. It indicates that a higher premium rate does not necessarily lead to a higher rate of return on life insurance contracts when considering the second moment of the income of a life insurance company.

Compared with *Case 1*, the difference in *Case 2* is that the compound Poisson process also impacts the income of the life insurance company through the intensity of the Poisson process, the mean of withdrawal sizes, and the variance of withdrawals.

2.4 Optimal investment strategy of the life insurance company

Life insurance companies usually evaluate their completion at predetermined times, e.g., every year or every quarter of a year. It seems reasonable to choose an investment strategy that maximizes the wealth at the end of their planning period. Moreover, there are certain regulatory or financial bounds on the amount of risk which a life insurance company may take on. The following portfolio optimization problem is based on these considerations.

The aim is to maximize the expectation of the income of a life insurance company subject to the condition that the probability that the result of the exercise at the end of the period T is less than the saving capital must be less or equal to α , i.e.,

$$\max_{0 < \gamma < 1} \mathbb{E} \left(Y_T^{(\gamma)} \right),$$

subject to the condition

$$P \left(Y_T^{(\gamma)} \leq -c \right) \leq \alpha,$$

where T is the investment period, c is a saving capital and α is the probability of insolvency. Such problems are typical for the financial industry, see e.g., [Korn, 1997], [Jorion, 2000], and [Kostadinova, 2007].

We need to verify if $\mathbb{E} [Y_t^{(\gamma)}(y)]$ is an increasing function of γ .

In *Case 1*: we know from Proposition 2.3.1,

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X}{\mu_\gamma}, & \mu_\gamma \neq 0, \\ y + a_X t, & \mu_\gamma = 0. \end{cases}$$

For $0 < \gamma < 1$ and $t \geq 0$,

- when $\mu_\gamma \neq 0$:

$$\begin{aligned} & \frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} \\ &= (a_R - r) t e^{\mu_\gamma t} \left(y + \frac{a_X}{\mu_\gamma} \right) - \frac{a_X (a_R - r) e^{\mu_\gamma t}}{\mu_\gamma^2} + \frac{a_X (a_R - r)}{\mu_\gamma^2} \\ &= y (a_R - r) t e^{\mu_\gamma t} + \frac{a_X (a_R - r) [e^{\mu_\gamma t} (\mu_\gamma t - 1) + 1]}{\mu_\gamma^2}, \end{aligned}$$

since the function $f(x) = e^x(x - 1) + 1 \geq 0$ for $x \geq 0$ in the second part of the right-hand side, with the no-arbitrage condition, $a_R > r \geq 0$ and $a_X > 0$ we know that

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} > 0;$$

- when $\mu_\gamma = 0$:

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} = 0.$$

In *Case 2*: we know from Proposition 2.3.4,

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \begin{cases} e^{\mu_\gamma t} \left(y + \frac{a_X + \lambda \beta_Z}{\mu_\gamma} \right) - \frac{a_X + \lambda \beta_Z}{\mu_\gamma}, & \mu_\gamma \neq 0 \\ y + (a_X + \lambda \beta_Z) t, & \mu_\gamma = 0. \end{cases}$$

For $0 < \gamma < 1$ and $t \geq 0$,

- when $\mu_\gamma \neq 0$:

$$\begin{aligned} & \frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} \\ &= y (a_R - r) t e^{\mu_\gamma t} + \frac{(a_X + \lambda \beta_Z) (a_R - r) [e^{\mu_\gamma t} (\mu_\gamma t - 1) + 1]}{\mu_\gamma^2}, \end{aligned}$$

then, with the no-arbitrage condition, $a_R > r \geq 0$ and the assumption that $a_X + \lambda\beta_Z > 0$, we calculate the limit of the partial derivative function

$$\lim_{t \rightarrow 0^+} \frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} = 0,$$

$$\lim_{t \rightarrow +\infty} \frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} = +\infty;$$

- when $\mu_\gamma = 0$:

$$\frac{\partial \mathbb{E} [Y_t^{(\gamma)}(y)]}{\partial \gamma} = 0.$$

In the *General Case*, we know from Proposition 2.3.7 that for $t \geq 0$,

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = \begin{cases} y + \delta_X t, & \text{if } \Phi(-1) = 0, \\ ye^{-t\Phi(-1)} + \delta_X \frac{1 - e^{-t\Phi(-1)}}{\Phi(-1)}, & \text{if } \Phi(-1) \neq 0, \end{cases}$$

where $\delta_X = a_X + \int_{|x|>1} x\nu_X(dx)$, $\Phi_\gamma(-1) = -[\gamma a_R + (1 - \gamma)r] - \int_{\mathbb{R}} \ln(1 + \gamma x)\nu_R(dx)$. If we invest γ part in a risky asset,

$$\mathbb{E} [Y_t^{(\gamma)}(y)] = e^{-t\Phi(-1)} \left[y - \frac{\delta_X}{\Phi_\gamma(-1)} \right] + \frac{\delta_X}{\Phi_\gamma(-1)},$$

where $\Phi_\gamma(-1) = -[\gamma a_R + (1 - \gamma)r] - \int_{\mathbb{R}} \ln(1 + \gamma x)\nu_R(dx)$. We know that $\Phi_\gamma(-1)$ is an analog of $(-\mu_\gamma)$ for the *Case 1* and *Case 2*.

Therefore, we conclude that with a non-zero investment rate of return, the expectation of the income of a life insurance company, $\mathbb{E} [Y_t^{(\gamma)}(y)]$ is an increasing function of the proportion invested in a risky asset, γ , given the no-arbitrage condition and positive cash flows from its insurance business.

2.5 Conclusion

In this chapter, we studied a portfolio optimization problem that is related to the portfolio management of life insurance business – euro-denominated funds.

In a persistently very low-interest-rates environment, euro-denominated life insurance business faces multiple risks: diminishing returns on euro-denominated funds, increasing risk exposures in asset allocations, and unpredicted dramatic redemptions of the contracts.

We apply a risk model based on this background.

Then, we obtain the expressions on the expectations and the variances of the income of a life insurance company with investments in three cases:

1. when both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company are modeled by two independent Brownian Motions;
2. when the basic risk process X_t is modeled by the sum of a Brownian Motion and a compound Poisson process, and the return on investment generating process $R_t^{(\gamma)}$ of the insurance company is modeled by a Brownian Motion;
3. when the two processes are both modeled by Lévy processes.

By calculating the first-order partial derivatives, we show the relationships between each variable and the expectation (and the variance) of the income of life insurance company in the first two cases. We conclude that the optimal expectation-variance of the income of the life insurance company depends on both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$.

Under certain conditions for the insurance business and for the investment process, the expectation of the income of a life insurance company is an increasing function of the investment proportion in a risky asset. We solve the optimization problem based on real-world constraints and data in the next chapter.

OPTIMAL ASSET ALLOCATION STRATEGIES OF EURO-DENOMINATED LIFE INSURANCE FUNDS

3.1 Introduction

In recent years, several investigations on simulations and empirical analysis have been developed which try to find the optimal asset allocation for an insurance company. [Wang et al., 2007] study the optimal investment problem for an insurer through the martingale approach. When the insurer's risk process is modeled by a Lévy process and the security market is described by the standard Black-Scholes model, closed-form solutions to the problems of mean-variance efficient investment and constant absolute risk aversion (CARA) utility maximization are obtained. They analyze the effect of the claim process on the mean-variance efficient investment using their explicit solutions. They find that the mean-variance efficient strategies do depend on the claim process. While in [Kostadinova, 2007], the price of the risky asset follows a general exponential Lévy process and the total claim amount in the risk process is modeled by a compound Poisson process. An approximation of the optimal investment strategy which maximizes the expected wealth of the insurance company under a risk constraint on the Value-at-Risk is provided. The difference between the optimal investment strategy in the two regimes is discussed, namely the dangerous claims regime and the dangerous investment regime. [Brokate et al., 2008] focus

on asymptotic tail estimations and apply numerical methods to find the distribution tail. They also determine the optimal investment by maximizing the expected wealth subject to a risk bound given in terms of a Value-at-Risk. Their method shows higher accuracy when the risky investment is not too small. [Liu et al., 2012] consider the problem of minimizing the ruin probability when a dynamic Value-at-Risk constraint is imposed. They show that the exposure to risky assets could be increased because of the risk diversification effect for the negatively correlated or uncorrelated risky assets and the stochastic of the insurance business.

[Huang and Lee, 2010] use a multi-asset model to investigate the optimal asset allocation of life insurance reserves and obtain formulae for the first two moments of the accumulated asset value. They provide a new perspective for solving both single-period and multi-period asset allocation problems regarding life insurance policies. [Yu et al., 2010] apply the simulation optimization approach to the multi-period asset allocation problem of non-life insurers. They construct a simulation model to simulate operations of a property-casualty insurer and develop multi-phase evolution strategies (MPES) to be used with the simulation model to search for promising asset allocations for the insurer. They find that the re-allocation strategy resulting from MPES outperforms re-balancing strategies. Their approach to the asset allocation decisions for better investment performance is also applicable to other financial institutions, such as life insurance companies. [Fidan Neslihan et al., 2016] do in-sample and out-of-sample simulations for portfolios of stocks from the Dow Jones, S&P 100, and DAX indices to compare portfolio optimization with the Second-Order Stochastic Dominance (SSD) constraints with mean-variance and minimum variance portfolio optimization, with the superior performance of portfolios with SSD constraints.

In the previous chapter, we have obtained the results for the first two moments of the income of a life insurance company in three cases. These solutions will be applied to search for the optimal asset allocation strategies. At the regulatory level, the Prudential Solvency II framework applied since 2016 impacts the asset management of insurance institutions. In order to meet the requirements, life insurance companies must maintain a certain level of capital to prove their financial ability to absorb shocks. To this end, they must optimize their asset allocation strategies while managing potential risks. Our research includes such parameters as the saving capital and the probability of insolvency of the life insurer into the constraints when obtaining the optimal investment strategies and the corresponding asset allocations.

The work is arranged as follows. We select and analyze a dataset consisting of one risk-free asset and the four main categories of risky assets available to life insurance companies in Section 3.2. In Section 3.3, based on the obtained formulae for the first two moments of the income of a life insurance company with investments in *Chapter 2*, we solve the optimization problem numerically with an investment strategy under certain constraints. Then, we conduct a sensitivity analysis to show how the optimal asset allocation depends on selecting a series of parameters in Section 3.4. Finally, we give a conclusion in Section 4.6.

3.2 Dataset statistics

We analyzed and presented the investment structure of French life insurance companies in Section 1.3.3 *Chapter 1*. In general, five main categories of assets describe the asset allocation structure, namely bonds (the sum of government bonds and corporate bonds), equities, real estate, loans and money market instruments and others, as plotted in Figure 1.12 and Figure 1.13. Besides, the life insurance company is assumed to invest in a risk-free asset and a risky asset (multiple risky assets in *Case 1bis* and *Case 2bis*) in the mathematical modeling in *Chapter 2*. Based on these considerations, we choose one risk-free asset and four main risky assets in our dataset in this section.

This section contains two subsections. Section 3.2.1 presents the risk-free asset and its interest rate. Section 3.2.2 describes four main risky assets, consisted of government bonds, corporate bonds, stocks and real estates.

3.2.1 Risk-free asset

The risk-free interest rate, also referred to as the risk-free rate of return, is a theoretical interest rate of an investment that carries zero risk. This rate represents the minimum interest an investor would expect from a risk-free investment over a specific period of time. Technically, the risk-free interest rate is purely theoretical, as all investments have some type of risk attached to them. The analysis in [Tobin and Golub, 1997] provides support to the concept that the risk-free rate may not be directly observable. [Kemp, 2009] states that the risk-free rate means different things to different people, and there is no consensus on how to measure and choose it.

There are different views on what exactly the risk-free rate means [Kemp, 2009, chap-

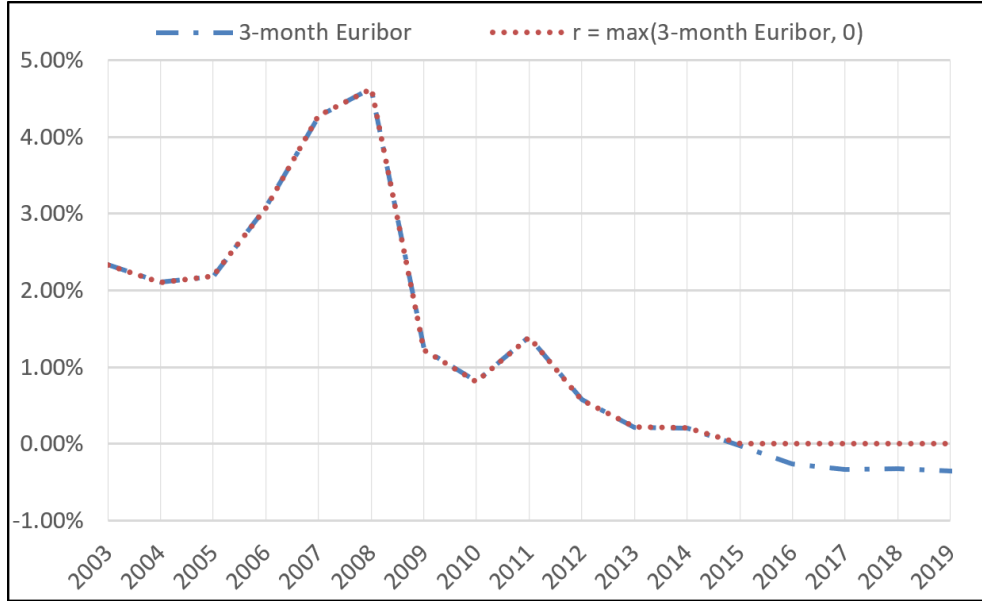
ter. 5]. On the one hand, the ‘risk-free rate’ might merely be the conventional name given to an assumption fed into a pricing algorithm, e.g., the short-term interest rate in the Black-Scholes-Merton model ([Black and Scholes, 1973] and [Merton, 1973]). On the other hand, the ‘risk-free rate’ might be shorthand for what might more accurately be described as a ‘reference rate’, i.e., a convenient interest rate that can be used for a wide range of purposes, but which is not strictly free of all risks. Inter-bank lending rates (e.g., Euribor and Libor) and short-dated government bonds are examples of potential proxies for risk-free assets.

Euribor stands for Euro Interbank Offered Rate. It is a daily reference rate based on the averaged interest rates at which Eurozone banks offer to lend unsecured funds to other banks in the euro wholesale money market (or interbank market). It is considered the most liquid and active interest rate¹. We remind that one of the main categories of assets in the investment structure of life insurance companies is loans and money market instruments. Moreover, long-term government bonds are considered risky assets, which will be described in the next subsection. Based on the above points, the 3-month Euribor is the proxy of the risk-free asset in our research. Recently, this interest rate became negative, life insurance companies can have access to an alternative asset.

We collect the monthly data on the 3-month Euribor from the European Central Bank. The monthly time series covers a period from January 31, 2003, to December 31, 2019. The annualized average interest rate of the 3-month Euribor is calculated based on the collected monthly data. We plot the annualized average interest rate of the 3-month Euribor in Figure 3.1.

1. Euribor is not free of credit risk. Investors depositing money in such a way in the interbank market would become exposed to the risk of default of the bank with which they have deposited their money (because Euribor relates to unsecured funds).

Figure 3.1 – Time series of the 3-month Euribor interest rate and the risk-free rate.



The 3-monthly Euribor interest rate is negative since 2015 without considering the inflation rate. As noticed in *Chapter 1*, the return on investment of life insurance companies suffers from the decline in nominal interest rates, both long-term and short-term, because of their heavy asset allocation in fixed-income securities. Besides, the first time the net outflow of euro-denominated funds in 2011, the liquidity needs of life insurance companies have risen to a certain degree, increasing the demand for liquid assets to meet the insureds' withdrawal and redemption requirements. Consequently, it is reasonable and profitable for life insurance companies to hold cash than to invest in negative interest rates.

We assumed the non-negative risk-free rate in the mathematical modeling in *Chapter 2*. Then we can define the risk-free interest rate:

$$r = \max(\text{Euribor}_{3M}, 0), \tag{3.2.1.1}$$

where Euribor_{3M} represents the 3-month Euribor. Figure 3.1 also displays the time series of non-negative risk-free rates.

3.2.2 Risky assets

In this subsection, four risky indices are presented and analyzed. We employ both government bonds and corporate bonds in our dataset to represent the asset class of fixed income securities because the revaluation rate of euro-denominated funds is evidently

correlated with 10-year government bond yield.

Government bonds are relatively risk-free for domestic investors because, by definition, there is no risk of default – a bond is a form of government obligation which is being discharged by paying another form of government obligation (i.e., its currency). From another point of view, however, the issue is that with coupon-bearing bonds, investors do not know in advance what their income will be on the reinvested coupons [Tobin and Golub, 1997]. Therefore, the return of government bonds cannot be considered risk-free because of the reinvestment risk, especially in a declining interest rate environment. This explains why we use government bonds as risky assets in this study.

3.2.2.1 Data description - period from 2002 to 2019

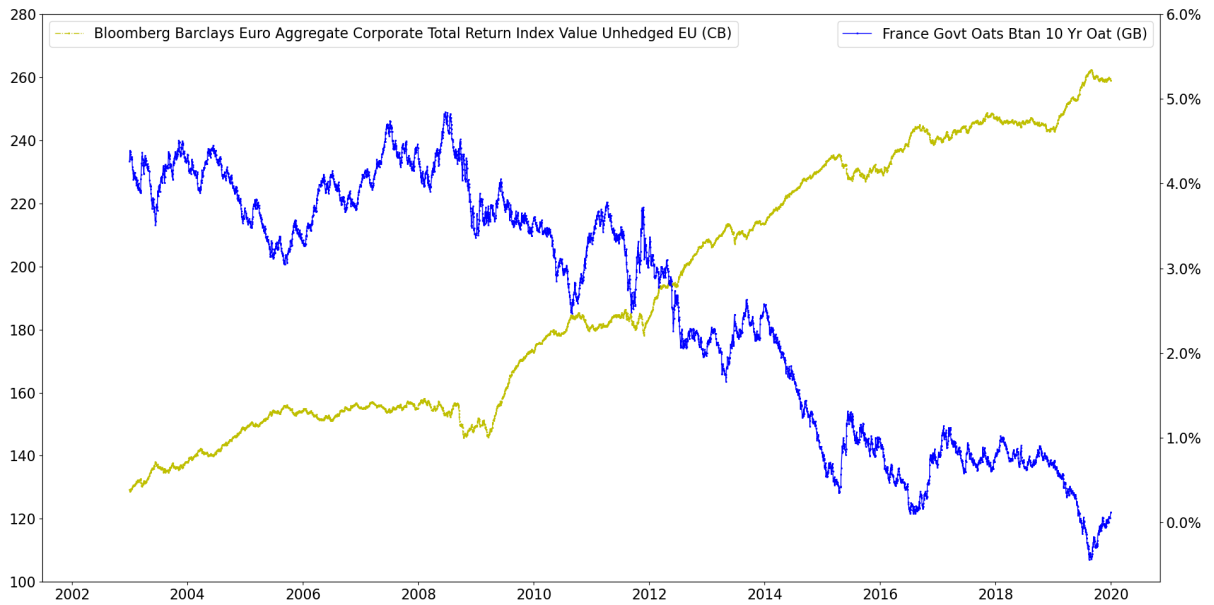
The dataset used for the analysis of the return on investment generating process include the France 10-year Government bond (asset 1), the Bloomberg Barclays Euro Aggregate Corporate Total Return Index Value Unhedged EU² (asset 2), the EURO STOXX 50 Index³ (asset 3) and the Euronext IEIF REIT Europe Index⁴ (asset 4) from December 31, 2002, to December 31, 2019, obtained from the Bloomberg terminal. After data cleaning, each asset covers 4289 daily data, 4288 daily returns. These four assets (and indices) are used as risky assets to represent government bonds, corporate bonds, stocks, and real estate. We plot the time series of four risky assets (and indices) in Figure 3.2 and Figure 3.3.

2. The Bloomberg Barclays Euro Aggregate Corporate Total Return Index is a broad-based benchmark that measures the investment grade, euro-denominated, fixed-rate corporate bond market. Published returns for Bloomberg Barclays benchmark indices measure the total return of a fixed income instrument, which includes capital appreciation and security price movements, interest payments and accruals, and principal repayments (scheduled or unscheduled) in the case of amortizing or sinkable bonds. Source: Bloomberg Barclays Methodology.

3. The EURO STOXX 50 Index, Europe's leading blue-chip index for the Eurozone, provides a blue-chip representation of super-sector leaders in the Eurozone in terms of free-float market capitalization. The index covers 50 stocks from 8 Eurozone countries: Belgium, Finland, France, Germany, Ireland, Italy, the Netherlands and Spain. The EURO STOXX 50 Index is licensed to financial institutions to serve as an underlying for a wide range of investment products such as exchange-traded funds (ETFs), futures, options and structured products worldwide. Source: STOXX Index Methodology Guide

4. The Euronext IEIF Real Estate Investment Trust (REIT) Europe Index selects companies whose market capitalization is more than 0.4% of the Universe (Property companies listed on European regulated markets that have opted for a tax transparency regime), with a minimum free float of 20% and with a minimum daily average turnover representing 0.2% The index serves as an underlying for structured products, funds and exchange traded funds. Source: Euronext Index Rule Book.

Figure 3.2 – Time series of the France 10-year Government bond yield (expressed as a percentage) and the Bloomberg Barclays Euro Aggregate Corporate Total Return Index.



Source: Bloomberg

Figure 3.3 – Time series of the EURO STOXX 50 Index, Stock and the Euronext IEIF REIT Europe Index, REIT.



Source: Bloomberg

The daily return of the risky asset is defined as

$$x_i = \ln(S_i/S_{i-1}), i = 1, 2, \dots, n,$$

where x_i is the return at date i , and S_i is the risky asset price at date i .

We take the arithmetic average rate of return over n time periods of equal length as the mean return:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} (x_1 + \dots + x_n).$$

The volatility is measured by the sample standard deviation of the logarithmic daily returns:

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}}.$$

The expected return and volatility are approximated by the annualized average return and standard deviation of the daily returns in the period, respectively. We assume that $T = 252$ trading days in any given year. Then, the annualized average return:

$$a_R = (1 + \bar{x})^{252} - 1,$$

and the annualized volatility:

$$\sigma_R = s \cdot \sqrt{252}.$$

Consequently, different estimates are considered depending upon investment periods.

Figure 3.4 – Daily rates of return in the Government bond and the Corporate bond.

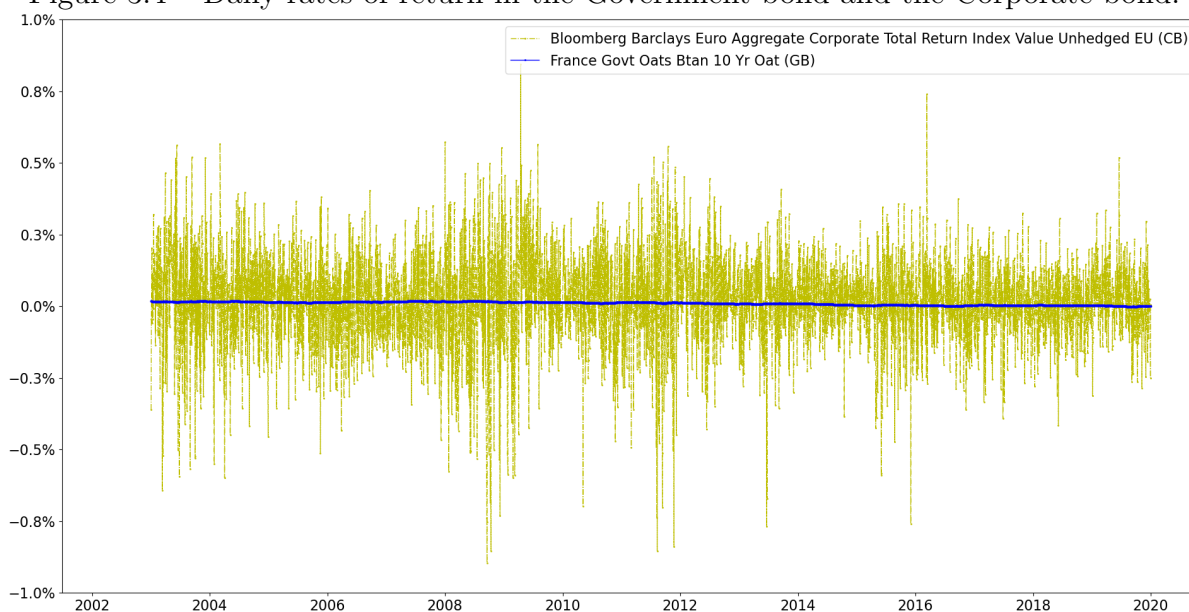


Figure 3.5 – Daily rates of return in the Stock and the REIT.

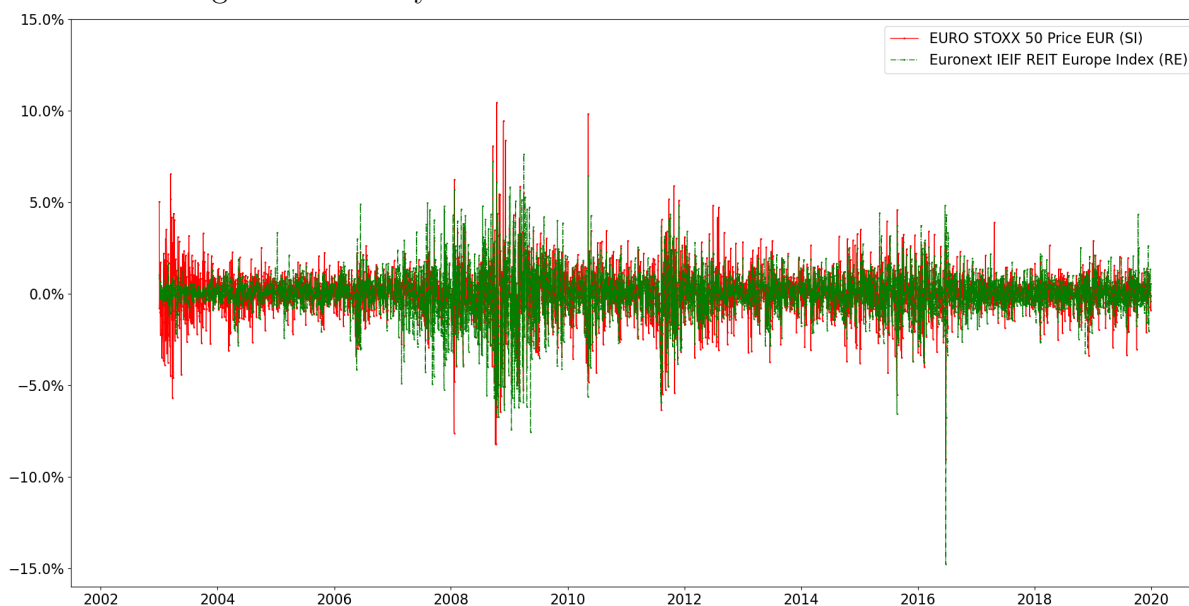


Figure 3.4 and Figure 3.5 present the daily rates of return for the four series of indices. Daily returns fluctuate around zero and are characterized by volatility clustering. Three assets (corporate bonds, stocks and real estate) demonstrate higher volatility in 2007-2009 and 2015-2016. However, the corporate bond index appears volatile during the European sovereign debt crisis. Since for each of these three assets, there is no homogeneity during the entire period from 2002 to 2019, we will use two normality tests to obtain the statistics

year by year in Section 3.2.2.

Then, we present the variance-covariance matrix. Covariance is a measure of the extent to which corresponding elements from two sets of ordered data moved in the same direction. Let $X = (x_i)_{1 \leq i \leq N}$ and $Y = (y_i)_{1 \leq i \leq N}$ be two sets of ordered assets of the size N . We use the following standard formula to compute covariance,

$$Cov(x_i, y_i) = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})(y_i - \bar{y}),$$

where,

\bar{x} is the mean of the N returns in the first data set,

\bar{y} is the mean of the N returns in the second data set.

Table 3.1 – Summary statistics on rates of returns on four series of indices.

	Government bond	Corporate bond	Stock	REIT
Count	4288	4288	4288	4288
Mean	0.000100	0.000162	0.000104	0.000139
Std. Deviation	0.000057	0.001624	0.013449	0.012960
Skewness	-0.352976	-0.532390	-0.075084	-0.644866
Kurtosis	-1.353210	2.675116	6.175035	8.557335
Min	-0.000017	-0.008956	-0.090110	-0.147347
25%	0.000036	-0.000684	-0.006083	-0.005394
50%	0.000118	0.000211	0.000392	0.000561
75%	0.000149	0.001102	0.006548	0.006125
Max	0.000188	0.008443	0.104376	0.076377

Table 3.1 reports the summary of statistics for the return series. Four patterns are evident from the table. Firstly, these four indices exhibit a positive average daily rate of returns in our data period. Secondly, the volatility of the stock index and the REIT are higher than that of the government bond and the corporate bond, as reflected in the standard deviations. The stock index has the largest standard deviation. The government bond is the least volatile series. Thirdly, all series have negative skewness. A negative skewness implies that large negative changes in returns occur more often than positive changes. Besides, the France 10-year government bond yield has negative kurtosis, which means that its frequency distribution of daily returns has thinner tails than a normal curve with the same mean and standard deviation. As shown in Figure 3.6(a), the government bond has a flatter distribution than that of the other three assets and displays a non-

normal distribution that fails to resemble a bell curve. The other three assets display positive kurtosis, which implies that large changes occur more often than would be the case if the series had a normal distribution, as shown in Figure 3.6 (a, b, and c). This is also confirmed by the one-sample Kolmogorov–Smirnov(K-S) tests in Table 3.2, which reject the null hypothesis of the normal distribution for all series.

Figure 3.6 – Histogram of the frequency distribution of daily returns of four risky assets.

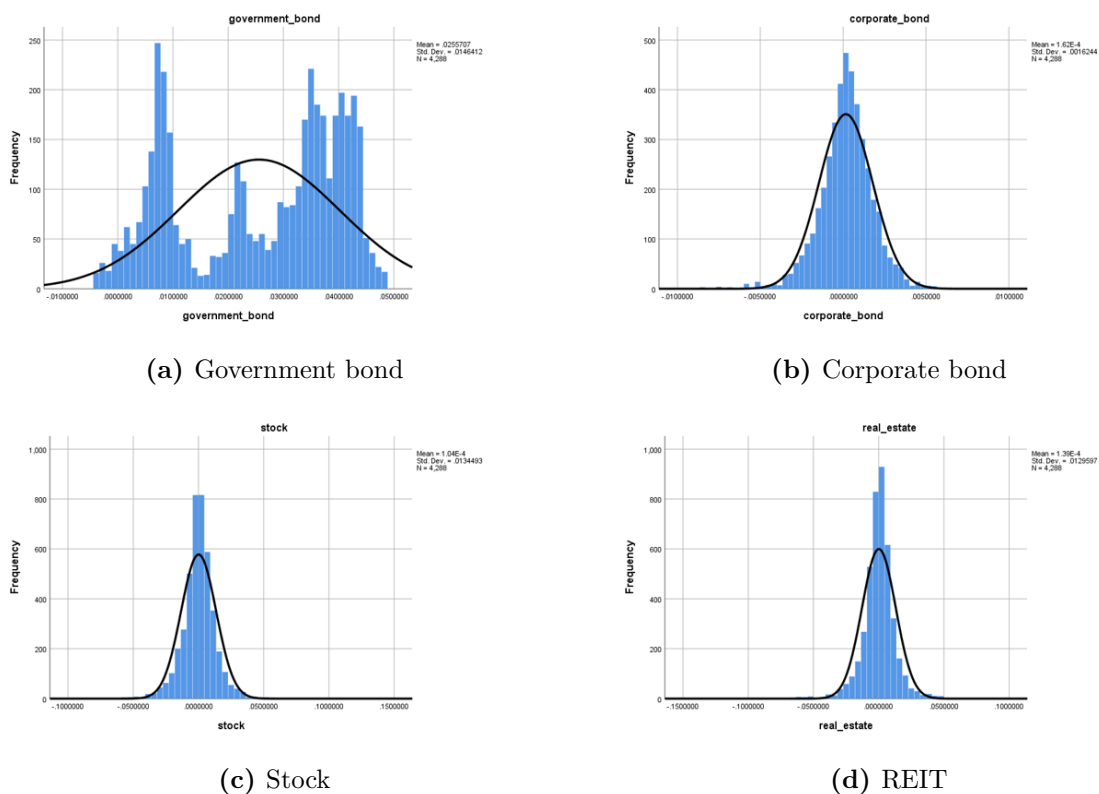


Table 3.2 – One-sample Kolmogorov–Smirnov tests on rates of returns on four series of indices.

		Government bond	Corporate bond	Stock	REIT
Count		4288	4288	4288	4288
Normal Parameters ^{a,b}	Mean	0.000100	0.000162	0.000104	0.000139
	Std. Deviation	0.000057	0.001624	0.013449	0.012960
Most Extreme Differences	Absolute	0.146	0.055	0.075	0.093
	Positive	0.132	0.033	0.069	0.081
	Negative	-0.146	-0.055	-0.075	-0.093
Test statistic		0.146	0.055	0.075	0.093
Asymp.Sig. (2-tailed)		0.000 ^c	0.000 ^c	0.000 ^c	0.000 ^c

a. Test distribution is Normal; b. Calculated from data; c. Lillefors Significance Correction.

We present the variance-covariance matrix of the four assets for the whole data set period, 2003 - 2019.

$$\mathbb{C}_{2003-2019} = \begin{bmatrix} 0.000000813, & -0.000000031, & -0.000001088, & -0.000001497 \\ -0.000000031, & 0.000664797, & -0.001171425, & -0.000327940 \\ -0.000001088, & -0.001171425, & 0.045572271, & 0.028983914 \\ -0.000001497, & -0.000327940, & 0.028983914, & 0.042314258 \end{bmatrix}.$$

Covariance measures the directional relationship between the returns on two assets. As we can see, the covariances between the government bond and the other three assets are all negative, which means that the government bond yield moves inversely with these three more risky assets. This is consistent with the long-term downward trend of government bond yields. Among the other three risky assets, the stock index and the REIT tend to move together. As mentioned, the REIT includes all publicly traded property companies listed on European regulated markets, and such a composition makes it behave like a stock index. This explains why the stock index and the REIT tend to move together. The corporate bond index moves inversely with the stock index and the REIT. The negative covariance between bonds and stocks as well as real estate highlights the diversification benefits of the asset portfolio.

3.2.2.2 Normality test

- One-sample Kolmogorov-Smirnov Test

In statistics, the one-sample K-S test is a nonparametric test of the equality of continuous, one-dimensional probability distributions used to compare a sample with a reference probability distribution. The K-S test is named after Andrey Nikolaevich Kolmogorov and Nikolai Vasilyevich Smirnov. The K-S statistic quantifies a distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution. The null distribution of this statistic is calculated under the null hypothesis that the sample is drawn from the reference distribution. In the one-sample case, the distribution considered under the null hypothesis may be continuous, purely discrete, or mixed. The K-S test can be modified to serve as a goodness of fit test. In the special case of testing for normality of the distribution, samples are standardized and compared with a standard normal distribution. This is equivalent to setting the mean and variance of the reference distribution equal to the sample estimates, and it is known that using these to define the specific reference distribution changes the null distribution of the test statistic.

Given the cumulative distribution function $F(x)$, [Conover, 1999] defined the test statistic proposed by [Kolmogorov, 1933] as,

$$D_n = \sup_x |F_n(x) - F(x)|,$$

where \sup_x is the supremum taken over \mathbb{R} . The empirical distribution function F_n for n independent and identically distributed (i.i.d.) ordered random variables x_i is defined as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I_{[-\infty, x]}(x_i),$$

where $I_{[-\infty, x]}(x_i)$ is the indicator function, equal to 1 if $x_i \leq x$ and equal to 0 otherwise.

- χ^2 Test for Normality

The χ^2 goodness-of-fit test can be used to test the hypothesis that data comes from a normal hypothesis [Cochran, 1952]. An attractive feature of the χ^2 goodness-of-fit test is that it can be applied to any univariate distribution for which we can calculate the cumulative distribution function. The χ^2 goodness-of-fit test is applied to binned data. The χ^2 test is an alternative to the K-S goodness-of-fit test.

The χ^2 test is defined for the hypothesis:

H_0 : The data comes from a normal distribution.

H_a : The data does not come from a normal distribution.

To calculate the χ^2 goodness of fit, the data is divided into k bins. Each bin has approximately 10 data, and the test statistic is defined as

$$\chi^2 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i(1 - E_i)},$$

where O_i is the observed frequency for bin i , and E_i is the expected frequency for bin i . The expected frequency is calculated by

$$E_i = [F(Y_u) - F(Y_l)] \cdot N,$$

where F is the cumulative distribution function for the distribution being tested, Y_u is the upper limit for class i , Y_l is the lower limit for class i , and N is the sample size.

3.2.2.3 Data description - year by year from 2002 to 2019

We denote by $(S_i)_{1 \leq i \leq n}$ the last prices of an asset, n is the total number of the data for the period of one year.

For the four risky assets (government bonds, corporate bonds, stocks and real estate), we calculate the following parameters, respectively:

$$\begin{aligned} x_i &= \log(S_i) - \log(S_{i-1}), \quad 2 \leq i \leq n, \\ \bar{x} &= \frac{1}{n-1} \sum_{i=2}^n x_i, \quad S^2 = \frac{1}{n-2} \sum_{i=2}^n (x_i - \bar{x})^2. \\ u_i &= \frac{x_i - \bar{x}}{S}, \quad 2 \leq i \leq n. \end{aligned}$$

The following four figures show the histograms of the frequency distribution of (x_i) with a normal distribution curve, with Figure 3.7 for the government bond, Figure 3.8 for the corporate bond index, Figure 3.9 for the stock index, and Figure 3.10 for the REIT, respectively. By comparing the frequency and the normal distribution curve, we can have a basic understanding of whether it comes from a normal distribution. For some samples, we cannot determine whether it comes from a normal distribution through the histogram. Next, we perform the K-S test and the χ^2 test.

Figure 3.7 – Histogram of the frequency distribution of (x_i) of the 10-year Government Bond (GB) Yield year by year.

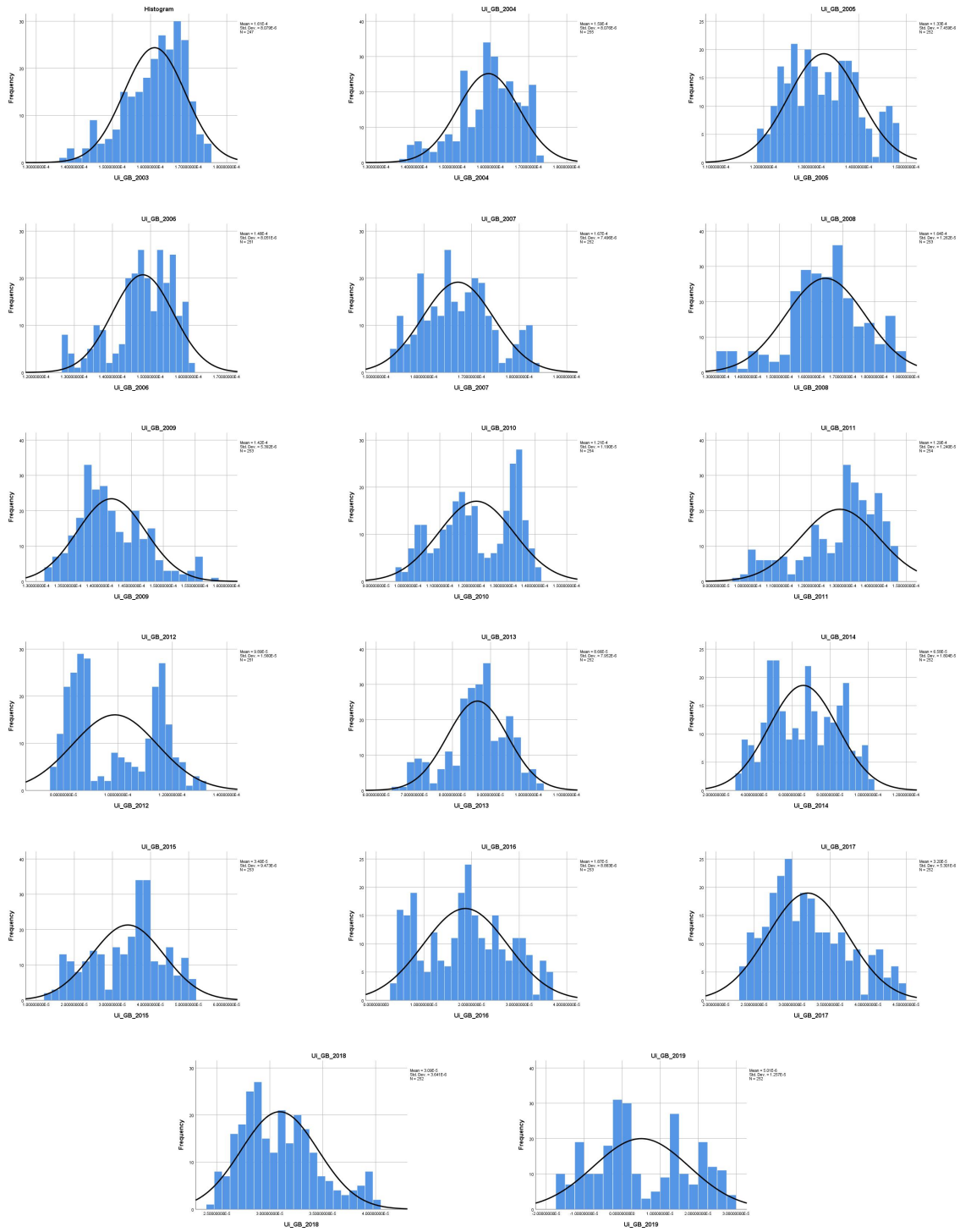


Figure 3.8 – Histogram of the frequency distribution of (x_i) of the Corporate Bond (CB) Index year by year.

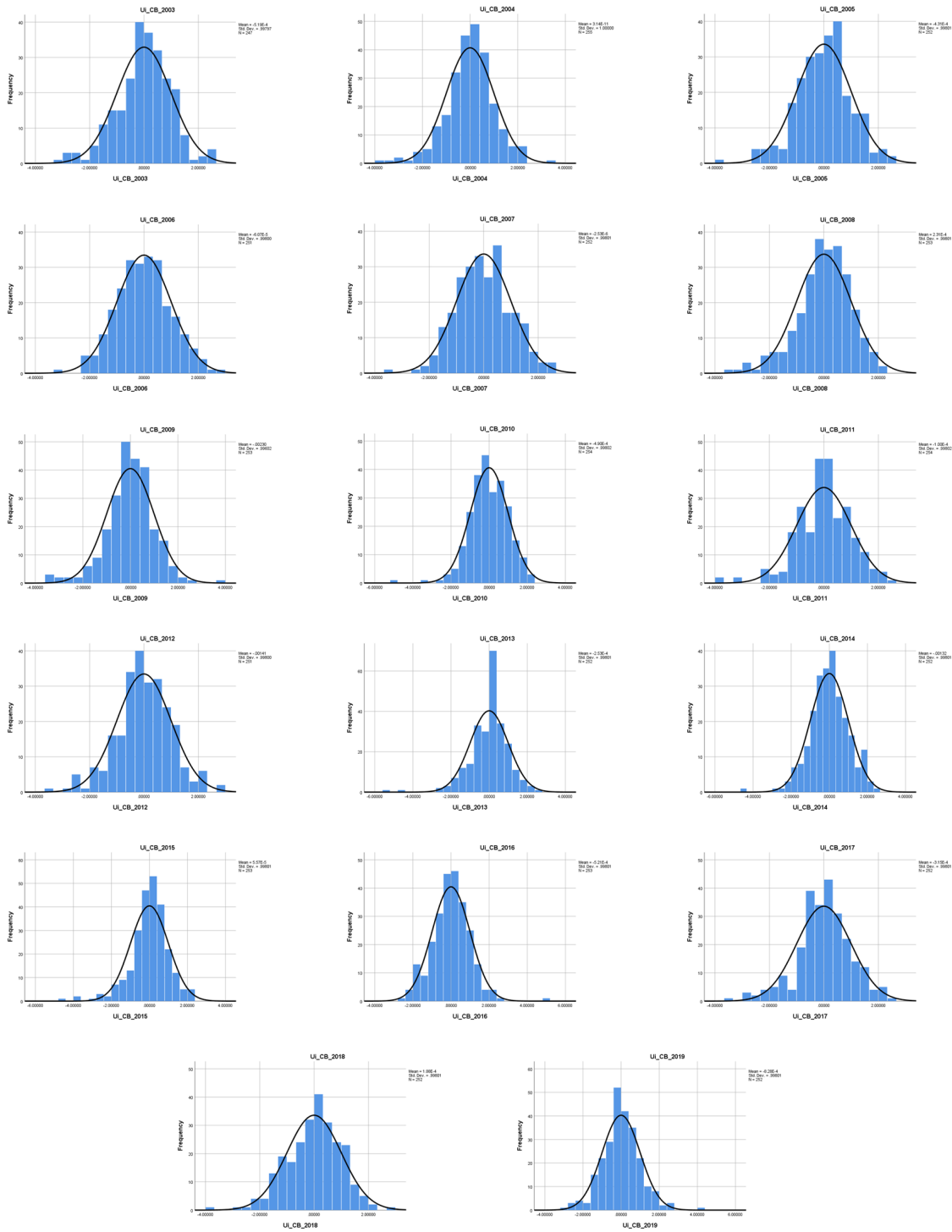


Figure 3.9 – Histogram of the frequency distribution of (x_i) of the Stock Index (SI) year by year.

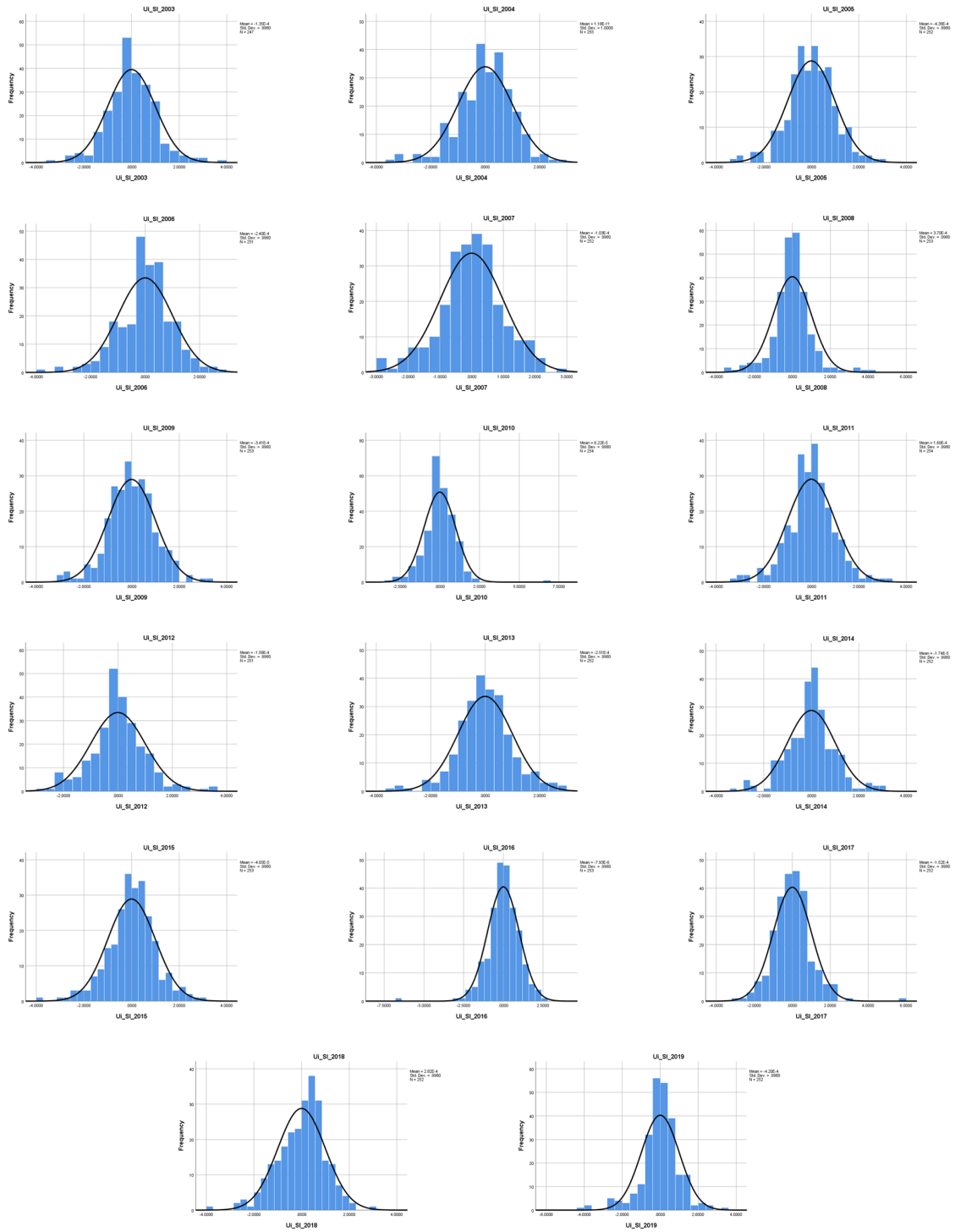
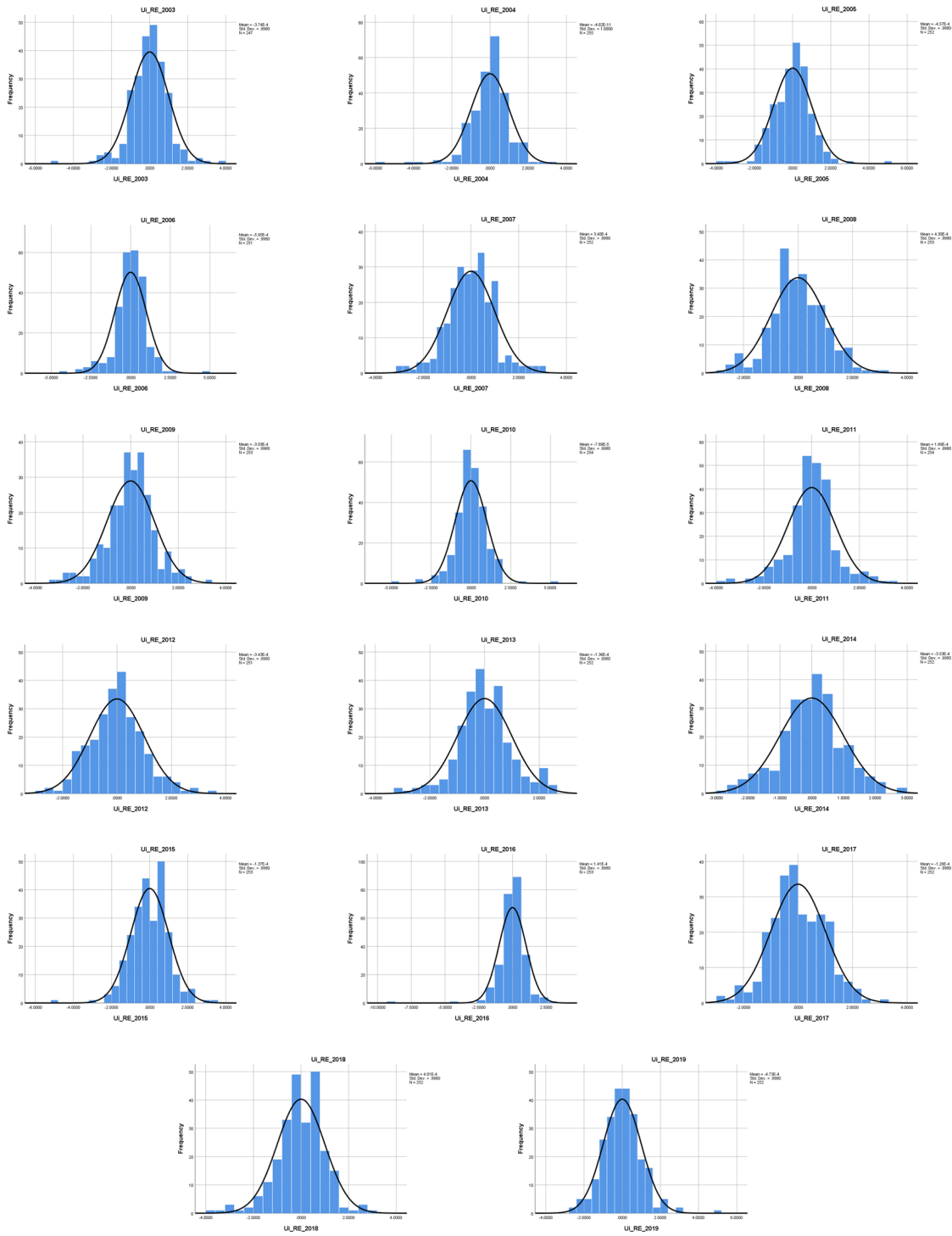


Figure 3.10 – Histogram of the frequency distribution of (x_i) of the REIT (RE) year by year.



As discussed in Section 3.2.2, the government bond displays a non-normal distribution for the whole dataset period, confirmed by the one-sample K-S test that rejects the null hypothesis of the normal distribution for the government bond. In Figure 3.7, we can observe that the distribution of each year is similar to the distribution of the whole dataset period.

For the normality test with respect to the yearly dataset, the null hypothesis, H_0 : the sample $(x_i)_{2 \leq i \leq n}$ comes from a normal distribution. To further verify whether it comes from a normal distribution, we implement the K-S test and display the results in Table 3.3. There is only one p-value (Sig.) larger than 0.05 in the year 2007 for the government bond. Thus, we can conclude that government bonds scarcely come from a normal distribution. For the other three more risky assets (corporate bonds, stocks, and real estate), it is difficult to distinguish in the frequency distribution histograms. We further employ the K-S test and χ^2 test. We present the results of the K-S test in Table 3.3 and the results of χ^2 test in Table 3.4, respectively.

Table 3.3 – One-sample K-S tests on (x_i) on four series of risky assets (indices).

Year	df	statistic-GB	Sig.-GB	statistic-CB	Sig.-CB	statistic-SI	Sig.-SI	statistic-RE	Sig.-RE
2003	247	0.085	0.000	0.074	0.003	0.055	0.067	0.056	0.058
2004	255	0.090	0.000	0.077	0.001	0.060	0.026	0.092	0.000
2005	252	0.063	0.019	0.043	.200*	0.045	.200*	0.092	0.008
2006	251	0.084	0.000	0.023	.200*	0.093	0.000	0.088	0.000
2007	252	0.052	0.100	0.024	.200*	0.050	.200*	0.056	0.054
2008	253	0.072	0.003	0.068	0.007	0.100	0.000	0.046	.200*
2009	253	0.097	0.000	0.061	0.023	0.042	.200*	0.059	0.032
2010	254	0.132	0.000	0.031	.200*	0.089	0.000	0.068	0.006
2011	254	0.148	0.000	0.070	0.005	0.063	0.017	0.077	0.001
2012	251	0.204	0.000	0.049	.200*	0.073	0.003	0.053	0.089
2013	252	0.074	0.002	0.093	0.000	0.059	0.033	0.066	0.010
2014	252	0.087	0.000	0.041	.200*	0.074	0.002	0.045	.200*
2015	253	0.096	0.000	0.088	0.000	0.060	0.028	0.047	.200*
2016	253	0.088	0.000	0.051	.200*	0.077	0.001	0.100	0.000
2017	252	0.087	0.000	0.071	0.003	0.065	0.012	0.046	.200*
2018	252	0.086	0.000	0.047	.200*	0.069	0.005	0.063	0.017
2019	252	0.117	0.000	0.055	0.058	0.107	0.000	0.047	.200*

df: degree of freedom

*: This is a lower bound of the true significance. The SPSS does not give a specific number of the p-value but tells that it is greater than 0.2, which is statistically significant.

As we can see from Table 3.3, for the corporate bond index, the p-value (Sig.) is larger than 0.05 in the following periods: 2005-2007, 2010, 2012, 2014, 2016, and 2018-2019, indicating that these nine yearly periods of samples follow the normal distribution;

for the stock index, the p-value is larger than 0.05 in the following periods: 2003, 2005, 2007 and 2009, indicating that these four yearly periods of samples follow the normal distribution; for the REIT, the p-value is larger than 0.05 in the following periods: 2003, 2007, 2008, 2012, 2014-2015, 2017 and 2019, indicating that these eight yearly periods of samples follow the normal distribution.

In Table 3.4, the χ^2 test shows the same result as in the K-S test for the corporate bond index. For the stock index, there are four more years that show a normal distribution compared with the results of the K-S test, which are 2004, 2011, 2013, and 2017. Concerning the REIT, when we compare with the results of the K-S test, it comes from a normal distribution in the year 2018 but does not in the year 2015 under the χ^2 test.

Table 3.4 – χ^2 tests on (x_i) on three series of risky indices.

Year	χ^2 statistics-CB	χ^2 statistics-SI	χ^2 statistics-RE
2003	16.25	11.59	16.91
2004	19.27	11.81	29.08
2005	10.28	8.14	21.13
2006	2.47	21.64	49.39
2007	2.97	12.59	10.48
2008	19.91	57.03	12.99
2009	21.23	5.56	21.45
2010	4.04	31.50	20.53
2011	29.59	12.00	28.37
2012	8.66	17.85	10.76
2013	17.18	13.10	25.01
2014	8.37	25.17	14.14
2015	27.74	18.94	20.72
2016	7.56	15.70	38.98
2017	16.33	11.04	13.79
2018	7.74	17.67	12.63
2019	12.09	44.35	9.28

In the χ^2 test, the degree of freedom = 8, the probability associated with the χ^2 distribution = 0.05, the critical value (the inverse of the one-tailed probability of the χ^2 distribution) = 15.51.

If the χ^2 test statistic is less than the critical value, there is a good fit. We do not reject the null hypothesis that the data comes from a normal distribution.

As shown in Table 3.3 and Table 3.4, for the stock index and the REIT, their data for 2006, 2010, and 2016 do not come from a normal distribution under both the K-S test and the χ^2 test. Risky assets such as stocks and REIT were very volatile in the 2006 bull market. The fear of the European sovereign debt crisis in 2010 has also increased volatil-

ity in risky assets markets. The Brexit referendum outcome on June 24, 2016, exceeded market expectations, which also caused turmoil in the international financial market. The normality test results of these three years confirm that the normality assumption of the Black-Scholes model does not capture certain extreme movements. Based on the results of the two normality tests, we can conclude that the K-S test is more strict than the χ^2 test since four annual samples of the stock index do not come from a normal distribution under the K-S test.

Table 3.5 – The annualized rate of return and volatility of five series of indices.

	Risk-free	Government Bond		Corporate Bond		Stock Index	Real Estate		
Year	r	$a_R^{(1)}$	$\sigma_R^{(1)}$	$a_R^{(2)}$	$\sigma_R^{(2)}$	$a_R^{(3)}$	$\sigma_R^{(3)}$	$a_R^{(4)}$	$\sigma_R^{(4)}$
2003	0.0233	0.0414	0.000128	0.0692	0.0328	0.1542	0.2710	0.1099	0.0712
2004	0.0211	0.0410	0.000128	0.0746	0.0260	0.0681	0.1402	0.3088	0.0824
2005	0.0218	0.0340	0.000118	0.0396	0.0225	0.2127	0.1101	0.2082	0.1034
2006	0.0308	0.0380	0.000128	0.0054	0.0223	0.1518	0.1479	0.4467	0.1559
2007	0.0428	0.0430	0.000119	0.0002	0.0219	0.0679	0.1587	-0.3020	0.2643
2008	0.0463	0.0423	0.000200	-0.0375	0.0411	-0.4428	0.3927	-0.4765	0.3740
2009	0.0123	0.0364	0.000085	0.1564	0.0315	0.2104	0.2796	0.2577	0.3775
2010	0.0081	0.0311	0.000189	0.0471	0.0233	-0.0576	0.2364	0.0648	0.1973
2011	0.0139	0.0330	0.000196	0.0147	0.0362	-0.1694	0.2881	-0.1506	0.2391
2012	0.0057	0.0252	0.000247	0.1365	0.0228	0.1384	0.2080	0.2348	0.1546
2013	0.0022	0.0221	0.000126	0.0237	0.0232	0.1794	0.1646	0.0774	0.1372
2014	0.0021	0.0167	0.000286	0.0840	0.0152	0.0120	0.1721	0.2104	0.1353
2015	0	0.0086	0.000150	-0.0056	0.0251	0.0383	0.2314	0.1212	0.2083
2016	0	0.0047	0.000141	0.0471	0.0220	0.0070	0.2194	-0.1367	0.2601
2017	0	0.0081	0.000084	0.0241	0.0189	0.0649	0.1032	0.0547	0.1054
2018	0	0.0078	0.000058	-0.0125	0.0168	-0.1435	0.1374	-0.1879	0.1298
2019	0	0.0013	0.000199	0.0624	0.0183	0.2477	0.1316	0.2906	0.1350

r , risk-free rate, $a_R^{(i)}$, annualized mean, and $\sigma_R^{(i)}$, annualized volatility.

Table 3.5 presents the interest rate of one risk-free asset and the annualized return and volatility of the four risky assets, year by year. Regarding bonds, the government bond shows a decreasing return from about 4% to around 0 with minimum volatility, while the corporate bond index is more volatile than the government bond, but comparatively stable volatility than the stock index and the REIT. Both the stock index and the REIT have higher annualized volatility, within 10% to 40%. Most of the time, their annualized rates of return are close. However, in some years, the return varies greatly, such as 2004, 2006-2007, and 2014.

Next, we compute the variance-covariance matrix of the four assets year by year.

$$\mathbb{C}_{2003} = \begin{bmatrix} 0.000000016 & -0.000000427 & 0.000003486 & 0.000000515 \\ -0.000000427 & 0.001083207 & -0.004274875 & -0.000020306 \\ 0.000003486 & -0.004274875 & 0.073734000 & 0.003216463 \\ 0.000000515 & -0.000020306 & 0.003216463 & 0.005083592 \end{bmatrix};$$

$$\mathbb{C}_{2004} = \begin{bmatrix} 0.000000016 & -0.000000353 & -0.000000064 & -0.000000750 \\ -0.000000353 & 0.000679430 & -0.000793603 & -0.000039701 \\ -0.000000064 & -0.000793603 & 0.019724900 & 0.003751312 \\ -0.000000750 & -0.000039701 & 0.003751312 & 0.006815330 \end{bmatrix};$$

$$\mathbb{C}_{2005} = \begin{bmatrix} 0.000000014 & -0.000000187 & -0.000000465 & -0.000000456 \\ -0.000000187 & 0.000508638 & -0.000018206 & 0.000376649 \\ -0.000000465 & -0.000018206 & 0.012174140 & 0.003615727 \\ -0.000000456 & 0.000376649 & 0.003615727 & 0.010740760 \end{bmatrix};$$

$$\mathbb{C}_{2006} = \begin{bmatrix} 0.000000016 & -0.000000046 & -0.000000844 & -0.000002028 \\ -0.000000046 & 0.000498743 & -0.000327960 & -0.000422819 \\ -0.000000844 & -0.000327960 & 0.021962380 & 0.014470500 \\ -0.000002028 & -0.000422819 & 0.014470500 & 0.024399000 \end{bmatrix};$$

$$\mathbb{C}_{2007} = \begin{bmatrix} 0.000000014 & -0.000000140 & 0.000000035 & -0.000001090 \\ -0.000000140 & 0.000480262 & -0.001653039 & -0.002096064 \\ 0.000000035 & -0.001653039 & 0.025287980 & 0.030132280 \\ -0.000001090 & -0.002096064 & 0.030132280 & 0.070122000 \end{bmatrix};$$

$$\mathbb{C}_{2008} = \begin{bmatrix} 0.000000040 & -0.000000782 & 0.000003196 & 0.000003916 \\ -0.000000782 & 0.001695793 & -0.005684324 & -0.004260003 \\ 0.000003196 & -0.005684324 & 0.154827000 & 0.106716000 \\ 0.000003916 & -0.004260003 & 0.106716000 & 0.140458000 \end{bmatrix};$$

$$\mathbb{C}_{2009} = \begin{bmatrix} 0.000000007 & -0.000000100 & 0.000002849 & 0.000003663 \\ -0.000000100 & 0.000996525 & -0.002887472 & -0.002464118 \\ 0.000002849 & -0.002887472 & 0.078462000 & 0.076236000 \\ 0.000003663 & -0.002464118 & 0.076236000 & 0.143105000 \end{bmatrix};$$

$$\mathbb{C}_{2010} = \begin{bmatrix} 0.000000036 & 0.000000179 & -0.000000091 & -0.000001354 \\ 0.000000179 & 0.000544730 & -0.000939434 & -0.000201894 \\ -0.000000091 & -0.000939434 & 0.056126500 & 0.037377480 \\ -0.000001354 & -0.000201894 & 0.037377480 & 0.039090000 \end{bmatrix};$$

$$\mathbb{C}_{2011} = \begin{bmatrix} 0.000000039 & -0.000000024 & 0.000003059 & 0.000003338 \\ -0.000000024 & 0.001319038 & -0.003126853 & -0.000904821 \\ 0.000003059 & -0.003126853 & 0.083322000 & 0.054868000 \\ 0.000003338 & -0.000904821 & 0.054868000 & 0.057409000 \end{bmatrix};$$

$$\mathbb{C}_{2012} = \begin{bmatrix} 0.000000061 & 0.000000140 & -0.000002350 & 0.000000159 \\ 0.000000140 & 0.000519866 & -0.001228831 & -0.000562984 \\ -0.000002350 & -0.001228831 & 0.043457000 & 0.023769140 \\ 0.000000159 & -0.000562984 & 0.023769140 & 0.024004730 \end{bmatrix};$$

$$\mathbb{C}_{2013} = \begin{bmatrix} 0.000000016 & -0.000000340 & 0.000000338 & -0.000000279 \\ -0.000000340 & 0.000539640 & 0.000295744 & 0.000886312 \\ 0.000000338 & 0.000295744 & 0.027189720 & 0.015040650 \\ -0.000000279 & 0.000886312 & 0.015040650 & 0.018896870 \end{bmatrix};$$

$$\mathbb{C}_{2014} = \begin{bmatrix} 0.000000082 & 0.000000183 & 0.000000790 & 0.000000613 \\ 0.000000183 & 0.000232245 & -0.000291637 & 0.000036809 \\ 0.000000790 & -0.000291637 & 0.029752240 & 0.016873390 \\ 0.000000613 & 0.000036809 & 0.016873390 & 0.018403310 \end{bmatrix};$$

$$\mathbb{C}_{2015} = \begin{bmatrix} 0.000000023 & -0.000000483 & -0.000000946 & -0.000001046 \\ -0.000000483 & 0.000633570 & 0.000044250 & 0.001142921 \\ -0.000000946 & 0.000044250 & 0.053770950 & 0.037562580 \\ -0.000001046 & 0.001142921 & 0.037562580 & 0.043561000 \end{bmatrix};$$

$$\mathbb{C}_{2016} = \begin{bmatrix} 0.000000020 & -0.000000329 & -0.000002015 & -0.000000916 \\ -0.000000329 & 0.000486220 & 0.000965777 & 0.001911025 \\ -0.000002015 & 0.000965777 & 0.048318000 & 0.044994450 \\ -0.000000916 & 0.001911025 & 0.044994450 & 0.067919400 \end{bmatrix};$$

$$\mathbb{C}_{2017} = \begin{bmatrix} 0.000000007 & -0.000000013 & 0.000000406 & 0.000000206 \\ -0.000000013 & 0.000358384 & -0.000374121 & 0.000276005 \\ 0.000000406 & -0.000374121 & 0.010688760 & 0.003814368 \\ 0.000000206 & 0.000276005 & 0.003814368 & 0.011159130 \end{bmatrix};$$

$$\mathbb{C}_{2018} = \begin{bmatrix} 0.000000003 & -0.000000127 & -0.000000247 & -0.000000087 \\ -0.000000127 & 0.000283137 & -0.000071295 & 0.000195953 \\ -0.000000247 & -0.000071295 & 0.018944960 & 0.010098740 \\ -0.000000087 & 0.000195953 & 0.010098740 & 0.016920450 \end{bmatrix};$$

$$\mathbb{C}_{2019} = \begin{bmatrix} 0.000000040 & 0.000000341 & 0.000001230 & 0.000000295 \\ 0.000000341 & 0.000335139 & -0.000013636 & 0.000062253 \\ 0.000001230 & -0.000013636 & 0.017398000 & 0.008328682 \\ 0.000000295 & 0.000062253 & 0.008328682 & 0.018285370 \end{bmatrix}.$$

The covariance coefficients between the government bond and the other three more risky assets are mostly negative, and the absolute value is close to 0, which shows that as a risky asset in our study, it does not move together with other risky assets. As we described in the analysis of euro-denominated funds in Section 1.3.1, there are several advantages in euro-denominated life insurance, e.g., capital guarantees and an annual minimum rate of return. Life insurance companies invest a large proportion of their savings in bonds, mostly long-term government bonds. The stable yield of government bonds has been one

of the main points which allowed for the long-term development and prosperity of euro-denominated life insurance in the past forty years. It is noticed that the government bond yields have declined to less than 1% or even negative (below the minimum rate of return) in recent years. Life insurers have to increase investments in more risky assets (including alternative investments) to sustain capital guarantees in euro-denominated contracts.

Since 2012, the covariance coefficients between the corporate bond index and the stock index and the REIT are no longer mostly negative as in 2003-2012, indicating that the diversification benefits will decline. The covariance coefficients between the stock index and the REIT are larger if the markets are in bull, bear, and bull-bear transitions, such as 2008-2009, 2011, 2015-16, 2018. They tend to move together in a more volatile market. In the rest of the period, the comparatively low correlation with the stock index also makes the REIT an excellent portfolio diversifier that reduces overall portfolio risk and improves returns. These reflect the characteristics of REIT-based alternative investment.

3.3 Optimal asset allocation

In this section, we propose and obtain the optimal investment strategy that maximizes the income of a life insurance company under some practical constraints related to the ruin probability, based on the derived formulae in the previous chapter. Then, we perform a numerical illustration of our optimal investment strategy to find the optimal asset allocation with real data from the French life insurance sector and financial markets in 2019.

3.3.1 Optimal Investment Strategy

In order to illustrate the portfolio optimization problem, we consider the risk-free asset and the four risky assets as described in Section 3.2. The goal of the optimal investment strategy is to maximize the expected income of a life insurance company, $\mathbb{E}(Y_T^{(\gamma)})$ subjected to the condition that the probability that the result of the exercise at each time in the period $[0, T]$ is less than a saving capital must be less or equal to α , i.e.,

$$P\left(Y_T^{(\gamma)} \leq -c\right) \leq \alpha,$$

where $Y_T^{(\gamma)}$ is defined as the income of a life insurance company, c is a saving capital and α is the probability of insolvency, $\alpha = 1\%$ or 0.5% . In other words, it is reasonable to choose an investment strategy that can maximize the income of a life insurance company at the

end of a predetermined investment horizon while controls the risk at specific regulatory or financial bounds.

From (2.3.2.4a), we have

$$\mu_\gamma = a_R^{(1)} \gamma_1 + a_R^{(2)} \gamma_2 + a_R^{(3)} \gamma_3 + a_R^{(4)} \gamma_4 + r\gamma_r,$$

and $a_R^{(1)}$, $a_R^{(2)}$, $a_R^{(3)}$, and $a_R^{(4)}$ are the rates of return of the four risky assets. γ_1 , γ_2 , γ_3 , and γ_4 are the proportions of each risky asset, $\gamma_i \geq 0$. Here, r and γ_r are the risk-free rate and the proportion of the risk-free asset, respectively, and

$$\gamma_r = 1 - \sum_{i=1}^4 \gamma_i = 1 - (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \geq 0.$$

From (2.3.2.2), we also have

$$\sigma_\gamma^2 = \sum_{i=1}^4 \sum_{j=1}^4 \gamma_i \gamma_j \sigma_R^{(i)} \sigma_R^{(j)} \rho_{i,j},$$

where $\sigma_R^{(i)}$ is the standard deviation of the risky asset i and $\rho_{i,j}$ is the correlation coefficient between asset i and j , $i, j \in \{1, 2, 3, 4\}$.

As it was mentioned, the objective is to find

$$\max_{\gamma \in J} \mathbb{E} \left(Y_T^{(\gamma)} \right),$$

subject to the condition

$$P \left(-Y_T^{(\gamma)} \geq c \right) \leq \alpha,$$

where $J = \left\{ (\gamma_r, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \left| \sum_{i=1}^4 \gamma_i \leq 1, 0 \leq \gamma_i \leq \gamma_{\max}^{(i)} \right. \right\}$, $\gamma_{\max}^{(i)}$ is the maximum investment ratio by asset class authorized by French insurance regulations.

Since the explicit formula for the probability $P \left(-Y_T^{(\gamma)} \geq c \right)$ is unknown, we will use the upper estimation of this probability, namely

$$P \left(-Y_T^{(\gamma)} \geq c \right) \leq \frac{\mathbb{E} \left[\left(Y_T^{(\gamma)} \right)^2 \right]}{c^2}. \quad (3.3.1.1)$$

As we can see, as $c \rightarrow +\infty$, the right-hand side of the above inequality tends to zero, and there exists $c_0 > 0$ such that for $c \geq c_0$

$$\frac{\mathbb{E} \left[\left(Y_T^{(\gamma)} \right)^2 \right]}{c^2} \leq \alpha, \quad (3.3.1.2)$$

c_0 is the minimum saving capital required under a given probability of insolvency.

The use of the upper bound is better than the numerical calculation of the mentioned probability because such numerical calculations can lower the probability of ruin, which is a dangerous situation.

Of course, the maximization of $\mathbb{E}\left(Y_T^{(\gamma)}\right)$ under (3.3.1.2) will not give the same result as the maximization of $\mathbb{E}\left(Y_T^{(\gamma)}\right)$ under (3.3.1.1). The maximum of $\mathbb{E}\left(Y_T^{(\gamma)}\right)$ under (3.3.1.2) will be smaller than the maximum of $\mathbb{E}\left(Y_T^{(\gamma)}\right)$ under (3.3.1.1), since (3.3.1.2) is stronger than (3.3.1.1).

Again, for security reasons, it is better to underestimate the maximum of $\mathbb{E}\left(Y_T^{(\gamma)}\right)$ than overestimate it.

It is more common for life insurance companies to get positive investment rates of return $\mu_\gamma > 0$, so that up to now, we consider this situation. For simplicity in the numerical illustration in section 3.3.2, we assume $y = 0$.

3.3.1.1 Case 1 and Case 2

For *Case 2* (and Case 2bis), we get

$$\mathbb{E}\left(Y_T^{(\gamma)}\right) = \frac{a_{\lambda,\beta}}{\mu_\gamma} \left(e^{\mu_\gamma T} - 1\right),$$

$$\mathbb{E}\left[\left(Y_T^{(\gamma)}\right)^2\right] = \frac{2a_{\lambda,\beta}^2}{\mu_\gamma} \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)T} - e^{\mu_\gamma T}}{\mu_\gamma + \sigma_\gamma^2} + \left(\sigma_{\lambda,\beta}^2 - \frac{2a_{\lambda,\beta}^2}{\mu_\gamma}\right) \frac{e^{(2\mu_\gamma + \sigma_\gamma^2)T} - 1}{2\mu_\gamma + \sigma_\gamma^2},$$

where

$$a_{\lambda,\beta} = a_X + \lambda\beta_Z,$$

$$\sigma_{\lambda,\beta}^2 = \sigma_X^2 + \lambda\left(\beta_Z^2 + \sigma_Z^2\right).$$

If we consider *Case 1* (and Case 1bis), $a_{\lambda,\beta}$ and $\sigma_{\lambda,\beta}^2$ should be simply replaced by a_X and σ_X^2 . So, for the maximization problem, we need to consider only the second case for the modeling of X_t because other cases can be obtained by changing the parameters.

We introduce the functions

$$f(x) = \frac{e^x - 1}{x},$$

and

$$g(x) = \frac{e^x}{x}.$$

Then, given c , α , T , a_X , σ_X^2 , $\gamma_{\max}^{(1)}$, $\gamma_{\max}^{(2)}$, $\gamma_{\max}^{(3)}$, $\gamma_{\max}^{(4)}$ and r , we solve numerically the maximization problem to find

$$\max_{\gamma \in J} [a_{\lambda, \beta} T \cdot f(\mu_\gamma T)], \quad (3.3.1.3)$$

subject to the constraint:

$$2a_{\lambda, \beta}^2 T^2 \cdot g(\mu_\gamma T) \cdot f[(\mu_\gamma + \sigma_\gamma^2)T] + (\sigma_{\lambda, \beta}^2 - \frac{2a_{\lambda, \beta}^2}{\mu_\gamma})T \cdot f[(2\mu_\gamma + \sigma_\gamma^2)T] \leq \alpha \cdot c^2. \quad (3.3.1.4)$$

When we find $\vec{\gamma}^* = (\gamma_r^*, \gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)$ which gives the optimal investment proportions, we calculate μ_* and σ_* , the annualized rate of return and volatility of this optimal investment strategy in the return on investment generating process,

$$\mu_* = a_R^{(1)} \gamma_1^* + a_R^{(2)} \gamma_2^* + a_R^{(3)} \gamma_3^* + a_R^{(4)} \gamma_4^* + r \gamma_r^*,$$

$$\sigma_* = \sqrt{\sigma_*^2} = \sqrt{\sum_{i=1}^4 \sum_{j=1}^4 \gamma_i^* \gamma_j^* \sigma_R^{(i)} \sigma_R^{(j)} \rho_{i,j}}.$$

The maximized expected income of a life insurance company will be equal to

$$\frac{a_{\lambda, \beta}}{\mu_*} (e^{\mu_* T} - 1),$$

and the interest rate of the contract for the holding period (HPY) will be equal to

$$\ln \left[\frac{a_{\lambda, \beta}}{\mu_*} (e^{\mu_* T} - 1) \Big/ (a_{\lambda, \beta} T) \right] = \ln \left(\frac{e^{\mu_* T} - 1}{\mu_* T} \right).$$

3.3.1.2 The General Case

Concerning the general case with $y = 0$, we know:

$$\mathbb{E} \left(Y_T^{(\gamma)} \right) = \delta_X \frac{1 - e^{-t\Phi(-1)}}{\Phi(-1)},$$

$$\mathbb{E} \left[\left(Y_T^{(\gamma)} \right)^2 \right] = \delta_X^2 \cdot \mathbb{E}(I_t^2) + \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right) \mathbb{E}(J_t),$$

where

$$\delta_X = a_X + \int_{|x|>1} x \nu_X(dx),$$

$$\Phi(\epsilon) = \epsilon a_R - \frac{1}{2} \sigma_R^2 (\epsilon - \epsilon^2) + \epsilon \int_{\mathbb{R}} \ln(1+x) \nu_R(dx),$$

$$\mathbb{E}(I_t^2) = \frac{2}{\Phi(1)} \left[\frac{1 - e^{-\Phi(2)t}}{\Phi(2)} + \frac{1 - e^{-\Phi(1)t}}{\Phi(2) - \Phi(1)} \right],$$

$$\mathbb{E}(J_t) = \frac{1 - e^{-t\Phi(-2)}}{\Phi(-2)},$$

$\Phi(-1) \neq 0$, $\Phi(1) \neq 0$, $\Phi(-2) \neq 0$, $\Phi(2) \neq 0$ and $\Phi(1) \neq \Phi(2)$.

With the function $f(x) = \frac{e^x - 1}{x}$, we obtain

$$\mathbb{E}(Y_T^{(\gamma)}) = \delta_X t \cdot f[-t \cdot \Phi(-1)],$$

$$\begin{aligned} & \mathbb{E} \left[\left(Y_T^{(\gamma)} \right)^2 \right] \\ &= \delta_X^2 \cdot \mathbb{E}(I_t^2) + \left(\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right) \mathbb{E}(J_t) \\ &= \left[a_X + \int_{|x|>1} x \nu_X(dx) \right]^2 \cdot \left\{ \frac{2t}{\Phi(1)} \cdot f[-t \cdot \Phi(2)] + \frac{2t}{\Phi(2) - \Phi(1)} \cdot f[-t \cdot \Phi(1)] \right\} \\ & \quad + \left[\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right] \cdot t \cdot f[-t \cdot \Phi(-2)]. \end{aligned}$$

Then, we can also solve numerically the maximization problem to find

$$\max_{\gamma \in J} \{ \delta_X t \cdot f[-t \cdot \Phi(-1)] \},$$

subject to the constraint:

$$\begin{aligned} & \left[a_X + \int_{|x|>1} x \nu_X(dx) \right]^2 \cdot \left\{ \frac{2t}{\Phi(1)} \cdot f[-t \cdot \Phi(2)] + \frac{2t}{\Phi(2) - \Phi(1)} \cdot f[-t \cdot \Phi(1)] \right\} \\ & \quad + \left[\sigma_X^2 + \int_{\mathbb{R}} x^2 \nu_X(dx) \right] \cdot t \cdot f[-t \cdot \Phi(-2)] \leq \alpha \cdot c^2. \end{aligned}$$

3.3.2 Numerical illustration

In this subsection, we will perform a numerical illustration of *Case 2bis* on the obtained optimal investment strategy in Section 3.3.1.

As already mentioned, the dataset used for the risky assets include the France 10-year Government Bond (1, government bond), the Bloomberg Barclays Euro Aggregate Cor-

porate Total Return Index Value Unhedged EU (2, corporate bond), the EURO STOXX 50 Index (3, stock) and the Euronext IEIF REIT Europe Index (4, REIT) from December 31, 2018, to December 31, 2019. To illustrate the type of calculation, we use the data for the exercise of the French life insurance companies in the same year as the financial market, i.e., 2019 (at the time of calculation). The expected annualized return and volatility are approximated by the average and standard deviation of the daily rate of returns in the period, see Section 3.2.2.

Let's remind that in a low interest rate environment, short-term interest rates in the Eurozone have been negative for several years, as shown in Figure 3.1. For example, the 3-month Euribor rate is around -0.35% in 2019. For that reason, life insurance companies can also hold cash rather than invest in negative interest rates. From (3.2.1.1), we know the risk-free rate is assumed to be zero in 2019.

To limit the impact of market risk and the liquidity risk, the French Insurance Code specifies certain constraints on the structure of the asset portfolio of life insurance companies. From Section 1.3.1, the main limits of the composition of euro-denominated funds in terms of investment (%) are (maximum investment ratios by asset class):

- 100% for bonds and bond funds;
- 65% for equities and equity funds;
- 40% for real estate.

Then, we have $\gamma_{\max}^{(1)} = \gamma_{\max}^{(2)} = 100\%$, $\gamma_{\max}^{(3)} = 65\%$ and $\gamma_{\max}^{(4)} = 40\%$.

We obtain the premium of 144.6 billion euros, the claims, benefits paid and redemptions of 122.7 billion euros and a corresponding net inflow of 21.9 billion euros in 2019 from the French Insurance Federation. The variance in the basic risk process is estimated based on monthly data from January 2012 to December 2019, when the average annual net inflow of life insurance business has declined to a new level of less than 20 billion euros, as shown in Figure 1.7 of Chapter 1.

The following set of parameters has been used in the numerical calculation of the maximization problem.

In the basic risk process $(X_t)_{t \geq 0}$, the parameters in billion of euros are given by:

$$a_{\lambda, \beta} = 21.9, \text{ and } \sigma_{\lambda, \beta} = 4.97.$$

In the return on investment generating process $(R_t)_{t \geq 0}$:

$$r = 0, a_R^{(1)} = 0.1263\%, a_R^{(2)} = 6.2396\%, a_R^{(3)} = 24.7671\%, \text{ and } a_R^{(4)} = 29.0565\%,$$

$$\sigma_R^{(1)} = 0.0199\%, \sigma_R^{(2)} = 1.8270\%, \sigma_R^{(3)} = 13.1638\% \text{ and } \sigma_R^{(4)} = 13.4955\%,$$

$$(\rho_{i,j})_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq 4}} = \begin{bmatrix} 1 & 0.0935 & 0.0467 & 0.0109 \\ 0.0935 & 1 & -0.0056 & 0.0251 \\ 0.0467 & -0.0056 & 1 & 0.4669 \\ 0.0109 & 0.0251 & 0.4669 & 1 \end{bmatrix}.$$

We set the one-year investment period, $T = 1$ year, the probability of insolvency, $\alpha = 0.5\%$ and assume the saving capital, $c = 330$ billion euros.

Given all the parameters above, we find the optimal allocation of the maximization,

$$\vec{\gamma}^* = (\gamma_r^*, \gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*) = (0.0046, 0.0166, 0.8875, 0.0424, 0.0489),$$

the corresponding optimal annualized rate of return (μ_*) and volatility (σ_*) in the return on investment generating process are 7.8909% and 1.9401%, respectively. The maximized expected income of a life insurance company in one year is 22.7531 billion euros, and the holding period yield on life insurance contracts is 3.8216%.

Using these ex-post data, we observe that a substantial proportion of the savings of the life insurance company should have been invested in corporate bonds in order to offer the optimal holding period yield of 3.82% if not endowed in a smoothing reserve, while the corresponding average rate in the life insurance industry was 2.62% from the ACPR in 2019. The comparison with real data and the analysis of these differences will be presented in Section 3.4.4..

3.4 Sensitivity analysis

The optimal asset allocations depend on the selection of a range of parameters. In this section, we examine the sensitivity of the optimal investment strategy to certain parameters in the integrated risk process, Y_t . The parameters used to find the optimal asset allocation in Section 3.3.2 serves as the benchmark for calculating the sensitivity analysis in the following subsections.

3.4.1 Sensitivity of the optimal asset allocation to different levels of interest rates

Firstly, we examine the sensitivity of the optimal asset allocation to different levels of interest rates. The gradual decrease in the return on euro-denominated contracts was highly related to the decline in the France 10-year government bond yield, as shown in

Figure 1.2. This encouraged the life insurers to allocate more corporate bonds in their portfolio. We propose two rates in this sensitivity analysis: one negative rate (-1%) and one average rate (the average France 10-year government bond yield in the past two decades is about 4%), while keeping other parameters unchanged.

Table 3.6 – Optimal asset allocation of the portfolio under different yields

$a_R^{(1)}$	Cash	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	Holding period yield
-1%	0.0209	0.0000	0.8886	0.0423	0.0488	7.8907%	1.9416%	3.8215%
0.1263%	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%
4%	0.0008	0.3575	0.5158	0.0530	0.0729	7.8934%	1.7378%	3.8227%

In Table 3.6, the results show that as the France 10-year government bond yield increases, the holding period yield rises. μ_* increases while σ_* decreases in the return on investment generating process, which indicates a diversification benefit. When the long-term government bond yield remains low or negative, the investment strategy becomes conservative. Life insurance companies should hold more cash and liquid assets to meet the liquidity needs in case of increasing outflows from the life insurance business. If the rate is at a high level ⁵, the investment strategy will become aggressive, such that most of the proportion will be shifted from corporate bonds to government bonds. At the same time, the proportion of stocks and REITs has also improved.

3.4.2 Sensitivity of the optimal asset allocation to different constraints, α and c

In Section 3.3.2, we assume that the saving capital and the probability of insolvency in the inequality constraint are 330 and 0.5%, respectively. Table 3.7 shows the optimal asset allocation based on two insolvency probabilities α (0.499% and 0.501%). We also

5. In this sensitivity analysis, we suppose the change in the 10-year government bond yields while keeping other returns unchanged. In reality, yield spreads on corporate bonds fall when government bond yields rise [Duffee, 1998]. Our analysis employs the total return corporate bond index, and the total return can be decomposed approximately into several return components, for instance, yield income, rolldown return, expected change in price based on investor's views of yields and yield spreads, expected credit losses and expected currency gains or losses. Such return components may offset each other, and it is not clear how the total returns vary when government bond yields change. We are conscious that other returns may also be affected by this change.

explore the optimal asset allocation under two saving capitals c (329 and 331) in Table 3.8.

Both α and c can be viewed as the risk tolerance measures of the life insurance company, such that as α or c increases, the insurer's risk-averse attitude decreases, and the investment strategy becomes more aggressive, generating a higher holding period yield.

Table 3.7 – Optimal asset allocation of the portfolio under different probabilities of insolvency

α	Cash	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
0.499%	0.0108	0.0341	0.8669	0.0407	0.0475	7.6788%	1.8894%	3.7219%
0.500%	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%
0.501%	0.0064	0.0070	0.8872	0.0451	0.0543	8.1031%	1.9925%	3.9211%

Table 3.8 – Optimal asset allocation of the portfolio under different saving capitals

c	Cash	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
329	0.0329	0.0679	0.8146	0.0391	0.0455	7.2485%	1.7861%	3.5193%
330	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%
331	0.0018	0.0021	0.8781	0.0526	0.0654	8.5341%	2.1119%	4.1226%

These two parameters can be tailored according to the financial situation of life insurers to reflect their attitudes on investment risks. For instance, when the higher returns accumulated in previous years provide a cushion for taking risks, life insurers can increase the value of such parameters on the premise of meeting regulatory requirements. As a result, the investment strategy becomes more aggressive to accumulate investment returns, increase profit-sharing to the policyholders, and enhance the provision for participation in profits.

3.4.3 Sensitivity of the optimal asset allocation to the parameters in the basic risk process, X_t

The sensitivity of the optimal asset allocation to two net inflows (21.84 and 21.96) is investigated, as shown in Table 3.9.

Table 3.9 – Optimal asset allocation of the portfolio under different net inflows

$a_{\lambda,\beta}$	Cash	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
21.84	0.0000	0.0048	0.8814	0.0509	0.0629	8.4450%	2.0857%	4.0811%
21.9	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%
21.96	0.0406	0.0469	0.8275	0.0393	0.0457	7.3400%	1.8083%	3.5624%

As the net inflow of the life insurance business increases, the investment strategy becomes more conservative, resulting in a decline in holding period yield. This explains why life insurers have limited the subscription and capital inflows in euro-denominated life insurance in a low-interest-rate environment in recent years. Keeping other variables unchanged, the increase in the investable funds following the increase in the net inflow will lead to a more defensive investment strategy. In other words, the investment will be more conservative to meet the constraints in (3.3.1.4) when increasing the net inflow without enhancing the saving capital.

We examine the sensitivity of the optimal asset allocation to the variance in the basic insurance process for two values (24.26 and 25.23). We display the corresponding results in Table 3.10. Both the increased capital outflows and the raised volatility in the life insurance business will prompt life insurers to invest in a more conservative strategy. Life insurance companies should hold more cash and cash equivalents to provide sufficient liquidity to deal with unpredictable redemptions.

Table 3.10 – Optimal asset allocation of the portfolio under different levels of variance in the basic risk process

$\sigma_{\lambda,\beta}$	Cash	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
4.93	0.0030	0.0093	0.8941	0.0430	0.0506	7.9950%	1.9647%	3.8704%
4.97	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%
5.02	0.0051	0.0275	0.8776	0.0416	0.0482	7.7889%	1.9157%	3.7737%

3.4.4 Sensitivity of the optimal asset allocation to different financial market conditions in the return on investment generating process, R_t

In Section 3.3.2, we illustrated the optimal investment strategy numerically with the dataset in 2019. The performance of the optimal investment strategy under the market

ups and downs still remain to be analyzed. In this section, we analyze the sensitivity of the optimal asset allocation to different financial market conditions.

Next, we will test the impact of the yearly financial market conditions on the optimal investment strategy from 2003 to 2018. In order to make the yearly optimal asset allocations comparable, we keep the following parameters at their level of calculation in 2019, including the saving capital c , the probability of insolvency α , the investment period T , the premium rate $a_{\lambda,\beta}$, the variance in the basic risk process $\sigma_{\lambda,\beta}^2$, and the maximum investment ratio by asset class $\gamma_{max}^{(i)}$. Table 3.11 presents the optimal asset allocation of each calendar year under the maximization objective in (3.3.1.3) and the constraint in (3.3.1.4).

Table 3.11 – Optimal asset allocation of the portfolio based on different financial market conditions, from 2003 to 2019.

Year	Risk-free asset	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
2003	0.0007	0.1731	0.5967	0.0505	0.1790	7.5898%	2.3082%	3.8189%
2004	0.0008	0.7388	0.1478	0.0002	0.1124	7.6043%	0.9978%	3.8262%
2005	0.0011	0.7351	0.0267	0.1118	0.1253	7.5933%	2.0609%	3.8207%
2006	0.0013	0.9054	0.0001	0.0002	0.0930	7.6009%	1.4534%	3.8245%
2007	0.0031	0.5031	0.0000	0.4938	0.0000	5.5307%	7.8541%	2.7781%
2008	0.5215	0.4785	0.0000	0.0000	0.0000	4.4394%	0.0096%	2.2279%
2009	0.0008	0.6761	0.3070	0.0152	0.0009	7.6052%	0.9272%	3.8267%
2010	0.0000	0.0000	0.6000	0.0000	0.4000	5.4161%	7.9710%	2.7203%
2011	0.0000	1.0000	0.0000	0.0000	0.0000	3.3027%	0.0197%	1.6559%
2012	0.0074	0.5513	0.4157	0.0067	0.0189	7.6047%	0.9527%	3.8265%
2013	0.0000	0.6475	0.0160	0.3364	0.0001	7.5017%	5.5510%	3.7743%
2014	0.0039	0.1640	0.8037	0.0008	0.0276	7.6023%	1.2888%	3.8252%
2015	0.0000	0.6000	0.0000	0.0000	0.4000	5.3632%	8.3482%	2.6936%
2016	0.0007	0.1744	0.8244	0.0005	0.0000	3.9638%	1.8198%	1.9885%
2017	0.0001	0.0002	0.5993	0.2893	0.1111	3.9302%	3.6239%	1.9715%
2018	0.0000	1.0000	0.0000	0.0000	0.0000	0.7825%	0.0058%	0.3915%
2019	0.0046	0.0166	0.8875	0.0424	0.0489	7.8909%	1.9401%	3.8216%

Firstly, the optimal investment proportions among five assets vary notably year after year. The proportion allocated to stocks is less than 10% on average in the yearly optimal asset allocation statistics, except for the following three years, 2007, 2013, and 2017. The average statistics on these yearly optimal asset allocations are presented in Table 3.12. The same finding is that a less than 10% allocation to real estate on average, except for the maximum investment ratio of 40% in 2010 and 2015. In addition to the above five years, bonds accounted for the vast majority of the proportion, of which government

bonds accounted for nearly half of the total investment, and corporate bonds accounted for nearly one-third. The allocation to the risk-free asset is minimal except for the year of the financial crisis in 2008.

The obtained investment proportions over the years are more volatile than in reality, while the yearly average investment proportions resemble the structure of the investments of euro-denominated funds as observed in Figure 1.13. Several reasons can explain this observation. Our dataset includes five assets to represent asset classes in the investment process of life insurers. In reality, there are thousands of assets available for investment. We consider the maximum investment ratios by asset class ($\gamma_{max}^{(i)}$) in our constraints, but not the dispersion rules that limit the counterparty risk (see Section 1.3.1), e.g., the maximum investment ratio of 10% on one building or one real estate fund. In addition, our model looks for the asset allocation that maximizes the return on investment subject to the inequality constraint (3.3.1.4). Therefore, there is a higher probability that the results will be affected by the performance of one specific asset. There are neither transaction costs nor re-balancing costs in our model since we do not consider the initial portfolio. Moreover, it is now in an environment of ever-decreasing interest rates. The historical yields of the government bonds are higher, which offers higher returns for older issued bonds still held by the French life insurance companies.

Table 3.12 – Average statistics on the optimal asset allocation from 2003 to 2019.

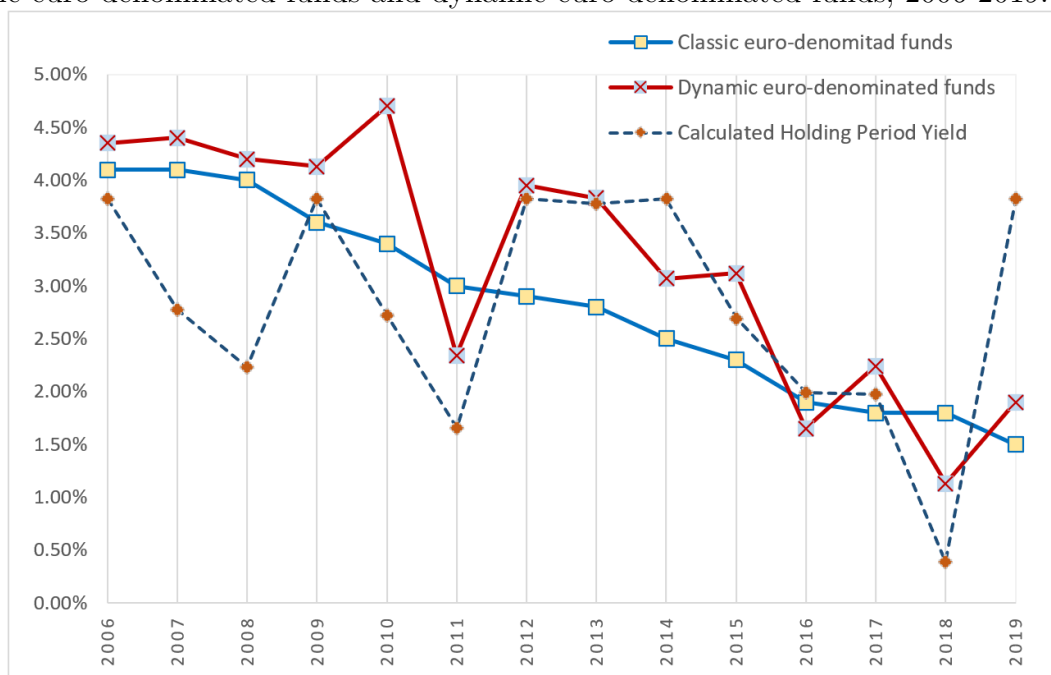
	Risk-free asset	Government bonds	Corporate bonds	Stocks	REITs	μ_*	σ_*	HPY
Mean	0.0321	0.4920	0.3073	0.0793	0.0892	5.9601%	2.7725%	2.9878%

Secondly, the optimal annualized investment return is between 3% and 8%. The investment rate of return is less than 1% due to the unilateral decline in the three main risky assets (corporate bond, stock, and real estate) and a low yield on government bonds in 2018. Thirdly, the annualized volatility is mostly below 5%. There are three years when the volatility is more than 7% in 2007, 2010, and 2015. The large fluctuations in these three years are related to higher allocations to stocks or real estate. This trend of the annualized investment rate of return is similar to the performance of life insurance products in dynamic euro-denominated funds with higher risk exposures of assets to stocks and real estate, as plotted in Figure 1.6.

Lastly, the holding period yield of life insurance contracts is mostly around 2-4%. The results show that this optimal investment strategy can generate a holding period

yield of close to 3% in most financial market conditions given the level of calculation in 2019. The calculated holding period yields in our model and the rate of returns on classic euro-denominated funds and dynamic euro-denominated funds are presented in Figure 3.11.

Figure 3.11 – The performance of the calculated holding period yield and the returns on classic euro-denominated funds and dynamic euro-denominated funds, 2006-2019.



Source: ACPR, GVfM

Compared with the other two returns observed in the real data, the calculated holding period returns are more volatile. As discussed in Section 1.3.1, the return of contracts in euro-denominated funds is based on several elements: management fees, technical interest rates, profit sharing (and the provision for participation in profits), and may also include a minimum guaranteed rate. Euro-denominated contracts designed in this way enable life insurers to smooth the financial results of euro-denominated funds and stabilize the fluctuations of holding such life insurance contracts.

Moreover, the parameters used in the calculations are based on the level in 2019, the optimal rate of return (μ_*) in return on investment generating process (R_t) and the holding period yield could be underestimated. The reason is as follows.

As observed in Figure 1.7, the outflow from life insurance contracts ranged between

40 billion to 80 billion euros from 1999 to 2007. In 2011, it started to exceed 100 billion euros, resulting in a decline in the net inflow of life insurance contracts. If we calculate the optimal asset allocation based on the data of the life insurance business in 2019, certain parameters in the basic insurance business process (X_t) will be overestimated for the past years (i.e., $\sigma_{\lambda,\beta}^2$)⁶, including both the overestimated outflow and the overestimated volatility of the insurance business. Then, keeping other parameters remain unchanged, this overestimation will make the investment strategy more conservative and underestimate the optimal investment rate of return (μ_*) and the holding period yield.

3.5 Conclusion

In this chapter, we selected one government bond, one corporate bond index, one stock index, and one real estate investment trust to represent the four risky assets, and we assumed a non-negative risk-free rate based on the 3-month Euribor interest rate and cash. We performed data description, normality test, and data analysis.

Then, based on the results in the previous chapter of mathematical modeling, we obtained the strategy of maximizing investment return subject to certain constraints to find the optimal asset allocation of the life insurance company. Next, we applied one case analysis with the numerical illustration of this optimal investment strategy through the dataset in 2019. The sensitivity analysis examined the impact of each parameter on the optimal asset allocation and brought certain implications for life insurance companies. We also conducted a sensitivity analysis of the optimal asset allocation under different financial market conditions, including both extreme and less volatile ones from 2003 to 2018. The results provided a holding period yield of around 3% in most financial market conditions, given the unchanged constraints and insurance business in 2019.

The numerical results verified the assumptions and analysis in *Chapter 2*. We concluded that the optimal asset allocation of the investment of the life insurance company depends on both the basic risk process X_t and the return on investment generating process $R_t^{(\gamma)}$.

A small change in the value of benchmark parameters will lead to evident shifts in the optimal asset allocations. By adjusting the value of the parameters α (probability of insolvency) and c (saving capital) in the inequality constraint, life insurers can find an optimal

6. The sensitivity of the optimal asset allocation of the portfolio under different levels of variance in the basic risk process is examined in Section 3.4.3 and presented in Table 3.10.

investment strategy that achieves an authority's requirement. When the investable funds rise following an increase in net flows, the life insurance company should improve the capital level for an optimal asset allocation. Life insurers should hold more liquid and risk-free assets in the contexts of low interest rates and unpredictable redemptions to reduce its liquidity risk and insolvency risk.

PERFORMANCE EVALUATION OF PORTFOLIO OPTIMIZATION STRATEGIES FOR THE LIFE INSURANCE COMPANY

4.1 Introduction

We proposed and analyzed our optimal asset allocation strategy for the life insurance company and implemented the strategy in one-year period in the numerical illustration in the previous chapter. Based on the integrated risk process of the value of a life insurance company which models the basic insurance process and the investment process, the asset allocation strategy is solved to maximize the expected value of a life insurance company subject to some practical constraints in the investment period. As well known, life insurance companies are long-term institutional investors in the financial market. As for the French life insurance products, such as euro-denominated funds, are mostly preferred by their long-term investment performance, tax incentives, and capital guarantees. The long-term performance of our proposed asset allocation strategy is the focus of our study in this chapter.

The objective of this chapter is to analyze the performance of our strategy comprehensively. It is essential because it is the primary concern of life insurers and policyholders. If the final value at the end of each year is lower than the guaranteed capital, this will bring significant challenges to life insurance companies. Performance evaluation is one of the

most important steps of portfolio management. Performance results can be used to assess the quality of the investment strategy and propose changes that might improve it. First, we will turn to the subject of benchmarks. It is necessary to choose benchmarks that reflect the financial assets available to our proposed investment strategies. In the absence of an official benchmark, we will construct a benchmark suitable for our strategy. Next, we will choose tools to evaluate the effectiveness of our strategies and the benchmarks. We will also review different ratios used in performance appraisal.

Our contribution in this chapter is twofold. First, our strategy allows us to analyze how to construct and dynamically rebalance the investment portfolio to control the risks faced by capital guarantees. By parameterizing the probability of insolvency and the saving capital, life insurance companies can select different parameters in each investment period reflecting their financial situation and expectations of the basic insurance business and financial markets; therefore, it is computationally simple and can be easily implemented in practice.

Second, we empirically examine the effectiveness of the proposed strategy for a life insurance company by comparing our strategy with a simple buy-and-hold strategy (B&H) and the benchmarks constructed for our strategy under various scenarios. Some performance measures reflecting the ability to sustain the pre-specified insured value and limit the downward risks while retaining upward returns are considered to demonstrate the return-risk profile of each strategy. In addition, the Omega functions of the strategies are plotted to further illustrate their relative performance. The underlying portfolio of each strategy is rebalanced to reflect changes in market conditions. The rebalancing frequency will have a significant impact on the performance of the strategy under consideration. Because the rebalancing frequency determines the extent to which the strategy participates, and rebalancing entails transaction costs. The parameters that life insurers choose to reflect their risk attitude may also affect the effectiveness of hedging. To demonstrate these effects, different parameter combinations and rebalancing frequencies are considered in this chapter.

We find that both our strategy and the constructed benchmarks are able to limit the negative returns while retaining certain higher returns compared with the B&H strategy except for the B&H strategy that invests only in the corporate bond index. The benchmark is not as good as our less frequent rebalancing strategies in obtaining upside returns. Compared with most of the B&H strategies that invests in one asset, both can limit negative returns while retaining certain high returns, and the income distribution is highly

right-skewed. As the probability of insolvency or the saving capital rises, the upward return of our strategy increases, despite the phenomenon of diminishing marginal increase in the holding period interest rate. In the presence of transaction costs, the maximum drawdown of our strategy with lower rebalancing frequencies (semi-annually and annually rebalancing) is zero during the eight-year investment period. In the presence of transaction costs, the performance of our strategy with lower rebalancing frequencies is better than the real-world results of the life insurance industry in the same dataset period.

This chapter is organized as follows. Section 4.2 briefly reviews our asset allocation strategy and proposes the benchmark choices. Section 4.3 discusses the design of the back-testing. Section 4.4 reports and analyzes the empirical results, and Section 4.5 conducts further research to check the robustness of the obtained results and the validity of the constructed benchmarks and compare with the real-world data. Section 4.6 concludes this chapter.

4.2 Asset allocation strategies and benchmark choices

In this section, the asset allocation strategies that we proposed in *Chapter 3* will be first recalled. Then, we will discuss and present the process of designing investment benchmarks that reflect the investment process and the constraints that govern the construction of the portfolio for the life insurance companies.

4.2.1 Recall: the optimal investment strategies

We select the investment strategy constructed based on *Case 2bis* of Section 3.3.1 in *Chapter 3* for performance evaluation. We recall that the basic risk process and the return on investment generating process of a life insurance company are described in (2.3.0.1) and (2.3.0.3c), where

$$X_t = a_X t + \sigma_X W_t + \sum_{k=1}^{N_t} Z_k, \quad (4.2.1.1)$$

and

$$R_t^{(\gamma)} = \mu_\gamma t + \sigma_\gamma B_t. \quad (4.2.1.2)$$

The related optimal asset allocation strategy is then to solve numerically the maximization problem

$$\max_{\gamma \in J} [a_{\lambda, \beta} T \cdot f(\mu_\gamma T)], \quad (4.2.1.3)$$

subject to the constraint:

$$2a_{\lambda,\beta}^2 T^2 \cdot g(\mu_\gamma T) \cdot f[(\mu_\gamma + \sigma_\gamma^2)T] + (\sigma_{\lambda,\beta}^2 - \frac{2a_{\lambda,\beta}^2}{\mu_\gamma})T \cdot f[(2\mu_\gamma + \sigma_\gamma^2)T] \leq \alpha \cdot c^2, \quad (4.2.1.4)$$

where $J = \left\{ (\gamma_r, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \left| \sum_{i=1}^4 \gamma_i \leq 1, 0 \leq \gamma_i \leq \gamma_{\max}^{(i)} \right. \right\}$, $\gamma_{\max}^{(i)}$ is the maximum investment ratio by asset class authorized by French insurance regulations.

The obtained $\vec{\gamma}^* = (\gamma_r^*, \gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)$ will give the optimal investment proportions. We calculate μ_* and σ_* , the investment rate of return and volatility of this optimal investment strategy in the return on investment generating process,

$$\mu_* = a_R^{(1)} \gamma_1^* + a_R^{(2)} \gamma_2^* + a_R^{(3)} \gamma_3^* + a_R^{(4)} \gamma_4^* + r\gamma_r^*, \quad (4.2.1.5)$$

$$\sigma_* = \sqrt{\sigma_*^2} = \sqrt{\sum_{i=1}^4 \sum_{j=1}^4 \gamma_i^* \gamma_j^* \sigma_R^{(i)} \sigma_R^{(j)} \rho_{i,j}}. \quad (4.2.1.6)$$

Then, the interest rate of the contract for the holding period (HPY) will be equal to

$$HPY = \ln \left[\frac{a_{\lambda,\beta}}{\mu_*} (e^{\mu_* T} - 1) \right] / (a_{\lambda,\beta} T) = \ln \left(\frac{e^{\mu_* T} - 1}{\mu_* T} \right). \quad (4.2.1.7)$$

4.2.2 Benchmark choices

Benchmarks are an important part of the investment process for performance evaluation. An asset-based benchmark is a set of securities that includes assets available to the portfolio. A benchmark should reflect the investment process and the constraints for the construction of the portfolio. If the benchmark does not reflect the investment process, then the evaluation and analysis from the comparison with the benchmark would be biased. The following performance measurements would be incorrect when benchmarks are misspecified, and the evaluation analysis shall be useless.

As far as we know, there are no benchmarks dedicated to euro-denominated funds of the French or European life insurance companies. Only the supervisory authorities and the professional press summarize and compare the returns on the various marketed life insurance contracts. The asset allocation of the French life insurers mainly consists of fixed-income investments. Therefore, practitioners commonly approximate the risk and return profile for life insurance investments with bond investments. As shown in Figure

1.2 of *Chapter 1*, the participation rate of euro-denominated funds is usually compared with the French 10-year government bonds yield. Due to the profit-sharing mechanism and smoothing reserves of euro-denominated funds, since 2011, the difference between the participation rate of euro-denominated funds and the French 10-year government bond yield has enlarged. However, as long-term interest rates continue to decline and life insurers increase allocations to other asset classes, applying the 10-year government bond yield as the benchmark of euro-denominated funds is no longer appropriate.

The choice of benchmark often has a significant effect on the evaluation of portfolio performance. Portfolios should be compared only with benchmarks that reflect the universe of securities available to them. From the definitive list in [Bailey and Tierney, 1998], valid benchmarks should be unambiguous, investable, measurable, appropriate, reflective of current investment opinions, specified in advance, and accountable. We introduce the seven types of benchmarks based on the discussion in [Bailey et al., 2007]: absolute (including target) return benchmarks, broad market indexes, style indexes, factor-model-based benchmarks, returns-based (Sharpe style analysis) benchmarks, manager universes (peer groups), and custom security-based (strategy) benchmarks. Based on the properties of a valid benchmark and the advantages and disadvantages of different types of benchmarks, the most suitable benchmark for our strategy should be the last type, custom security-based (strategy) benchmarks.

Custom security-based benchmarks are also referred to as strategy benchmarks because they should reflect the strategies of a portfolio. After identifying the investment process, the benchmark is constructed by selecting securities and weightings consistent with that process and investment constraints. Custom security-based benchmarks are particularly appropriate when the strategy cannot be closely matched to a broad market index or style index. The disadvantages of such benchmarks are costly to calculate and maintain.

In *Chapter 3*, we have chosen five assets to reflect the investment structure of life insurance companies in France. Therefore, we build custom security-based (strategy) benchmarks that include these five assets to compare with our proposed strategies in Section 4.2.1. The next step is to assign weightings to each asset in the portfolio of the benchmark. Since the daily weighting of each asset in the life insurance company's portfolio is not available, and the annual change of the holdings are relatively small, we propose to calculate the annual average investment ratio of each asset based on the two structures

of the investments¹, which have been plotted in Figure 1.12 and Figure 1.13. Then, we assign the weightings to the five assets in the benchmark.

Table 4.1 – Annual average investment ratios calculated based on the FFA (1999 - 2019) and GVfM (2011 - 2019) datasets.

Data Source	Risk-free rate (RF)	Government Bond (GB)	Corporate Bond (CB)	Stock Index (SI)	REITs (RE)
FFA	2.5%	35%	35%	24%	3.5%
GVfM	3%	32%	50%	9%	6%

From the datasets of the two institutes, FFA and GVfM, we obtain two series of weightings, as shown in Table 4.1. The FFA data (1999-2019) that includes the investment structure of life, capitalization and composite insurance companies has a longer period than the GVfM data (2011-2019) composed of euro-denominated funds in France. The main difference between the two series is the weightings of investment in corporate bonds and stocks. The average weighting of corporate bonds is much higher than the weighting of stocks in the GVfM structure, reflecting the relative conservative investment style of euro-denominated funds because of their capital guarantee characteristics, which is basically consistent with the study from the Bank of France ([Capitaine et al., 2020]). The average annual weightings from these two institutions are assumed to be invested in the two benchmarks when employing the performance evaluation of our strategies.

4.3 Backtesting design

In this section, we present the backtesting design of the empirical analysis. Sections 4.3.1 - 4.3.6 introduce the investigated strategies, portfolio rebalancing disciplines, performance appraisal measures, transaction costs, the value of the parameters, and the backtesting procedures.

4.3.1 The strategies

Three different strategies are investigated herein: our asset allocation strategy, the buy-and-hold (B&H) strategy invested in one asset, and two benchmarks based on the B&H strategy in constant weightings (GVfM and FFA).

1. In Section 4.5.2, we compared another two methods for obtaining the daily hypothetical weightings of the benchmarks, including linear interpolation and equal-to-year-end.

Our asset allocation strategy and two benchmarks are first compared with the B&H strategy invested in one asset to gauge their effectiveness as asset allocation. Following [Barberis, 2000] and [Kandel and Stambaugh, 1996], the B&H strategy invested in one asset is defined as 100% of the investment in each of the five assets until the end of the investment period. Whereas for the two benchmarks, the constant weightings do not change until the investment period is over. In contrast to the B&H strategies. While our strategies frequently adjust the amount invested in four risky assets based on the market movements. It is important to examine the advantages of our strategies over the simple BH strategies in terms of hedging the downside risk and the return performance.

Second, our asset allocation strategy is compared with the two benchmarks to see their relative hedging effectiveness. Our asset allocation strategy has different theoretical foundations than does the strategy used by life insurers [Agbojan et al., 2016]. Consequently, a comparison of the performance of our asset allocation strategy with that of the proposed benchmarks is practical in the French life insurance industry context.

4.3.2 Portfolio rebalancing disciplines

In general, the portfolios can be rebalanced based on three different disciplines: the time discipline, the market move discipline, and the portfolio mix discipline. The first method requires a portfolio rebalancing at predetermined time intervals, such as daily, weekly, monthly, etc. The second method takes a rebalancing when the market has a pre-specified percentage move since the last rebalancing, and the third one suggests adjusting the portfolio whenever the difference between the fixed and the current portfolio mix exceeds a pre-specified range.

Our asset allocation strategy induces a dynamic rebalancing of the underlying portfolio. The weightings based on the market movements and constraints can be calculated at predetermined time intervals. Typically, the strategy with a higher frequency of portfolio rebalancing will provide a better hedge against downside risk than that with lower rebalancing frequency, but it will entail higher transaction costs. To examine how the rebalancing frequency may affect the hedging effectiveness of our asset allocation strategy, portfolios are rebalanced at five predetermined time intervals: weekly, monthly, quarterly, semi-annually, and annually.²

2. In the thesis, we define 252 trading days as a year, 126 trading days as half a year, 63 trading days as a quarter, 21 trading days as a month and 5 trading days as a week.

4.3.3 Performance appraisal measures

Our strategy is to limit downside losses while pursuing upside gains for the life insurance company. The distribution of returns should theoretically be asymmetrical with a short left tail and a long right tail. To provide a profound understanding of the hedging effectiveness, the ability to sustain a guaranteed value and to reshape the return distribution of each strategy is assessed.

We conduct the empirical analysis in the following way. We first compute the 5% ventile (V5) and the average value for the poorest-performed 5% portfolios (AV5), the 75% quartile (Q75) and the average value for the best-performed 25% portfolios (AQ75). Then we calculate the protection ratios of each strategy. In addition, we also analyze the whole return-distributions. Finally, we compute the Sortino ratio, the Omega ratio, and the maximum drawdown for each strategy.

4.3.3.1 V5 and AV5

When we sort the realized portfolio values in ascending order and divide them into 20 equal groups, AV5 is the average of the values in the first group, and V5 is the highest value in this group. Both measures focus on the left tail of the frequency distribution, indicating the ability of the strategy to limit the downside risk. Higher V5 and AV5 values reveal that the strategy is more effective in limiting downside risks.

4.3.3.2 Q75 and AQ75

Q75 and AQ75 are used to measure the ability of the strategy to capture upside returns. Both measures focus on the right tail of the frequency distribution, indicating the ability to capture the upside returns. Higher Q75 and AQ75 implicate a higher ability to retain higher returns in the upward markets.

4.3.3.3 Protection ratio

The protection ratio is widely applied in evaluating the performance of portfolio insurance strategies (we refer to [Black and Perold, 1992] and [Bertrand and Prigent, 2005] for details). The protection ratio is defined as the percentage of the realized portfolios that meet the insured value under a particular strategy (e.g., [Do, 2002] and [Jiang et al., 2009]). This measure shows the ability of the strategy to guarantee an insured value. The

disadvantage of this ratio is that it cannot present the asymmetric characteristics of the return distribution created by the strategy.

4.3.3.4 Sortino ratio

The Sortino ratio was developed by [Sortino and Van der Meer, 1991] to differentiate between good and bad volatilities in the Sharpe ratio. [Chaudhry and Johnson, 2008] found that the Sortino Ratio is the superior performance measure exhibiting more power and less bias than the Sharpe ratio when the distribution of excess returns is skewed. It is calculated by the difference between the portfolio return and the chosen minimum acceptable return (MAR, usually 0%), and then divided by the downside deviation of the returns. The downside deviation uses only the standard deviation of negative returns. The Sortino ratio is goal-oriented in that it measures performance relative to the goal the investor is trying to achieve rather than measuring performance relative to the market. The advantage of this ratio is that it eliminates the volatility of positive returns from its calculation.

4.3.3.5 Maximum drawdown

Maximum Drawdown (MDD) is measured as the maximum cumulative peak-to-trough loss during a continuous period ([Magdon-Ismail and Atiya, 2004]). Maximum drawdown is an indicator used to assess the relative riskiness of one portfolio strategy versus another, as it focuses on capital preservation, which is a main concern of institutional investors such as life insurers. A low maximum drawdown is preferred as it reveals that losses from the portfolio were small. If an investment never lost during the investment period, then the maximum drawdown would be zero.

4.3.3.6 Omega ratio

Omega ratio [Keating and Shadwick, 2002]. It is the ratio of the averages of the gains above a threshold to the averages of the losses below the same threshold. Omega involves all the moments of the whole return distribution, including skewness and kurtosis, so it is an appropriate indicator of the effectiveness of portfolio strategies [Bacmann and Scholz, 2003]. The portfolio with a higher Omega ratio fares better than the one with a lower Omega.

4.3.4 Transaction costs

Transaction costs will affect the performance of investment strategies [Fabozzi et al., 2006, chapter. 3]. To study the effects, we consider the cases with and without transaction costs. According to [Perold, 1988] and [Wagner, 1989], the implementation shortfall (IS) is the most important ex-post trade cost measurement used in finance. The IS provides the total cost³ associated with implementing the investment decision and is calculated as the difference between the return for a paper portfolio, where all transactions are assumed to take place at the decision price, and the portfolio’s actual return, which reflects realized transactions, including all fees and costs.

Transaction costs are assumed to be 0.1% for bonds and 0.3%⁴ for the stock index and the REITs⁵. The cost of trading in the risk-free asset is assumed to be zero, as shown in Table 4.2.

Table 4.2 – Transaction costs (TC) of five assets.

	Risk-free rate (RF)	Government Bond (GB)	Corporate Bond (CB)	Stock Index (SI)	REITs (RE)
Transaction cost	None	0.1%	0.1%	0.3%	0.3%

Transaction costs are deduced from the portfolio value at the first trading date and each rebalancing date in the investment period.

4.3.5 The value of the parameters

In order to measure the performance of our strategy, we also need the value of some parameters in the inequality constraint. On the one side, since the daily data of euro-denominated life insurance business is not available, on the other hand, to evaluate the

3. The IS decomposes the total cost of the trade into four categories: delay cost, trading cost, opportunity cost, and fixed fees. The fixed fees component includes all explicit fees, such as commissions, exchange fees, and taxes. In our backtesting implementation, we only consider the fixed fees as transaction costs.

4. According to the General Directorate of Public Finances in France, the tax on the acquisition of equity securities or similar is 0.3%. Usually, the transaction cost of bonds is lower than that of stocks, and we assume that the transaction cost of bonds is 0.1%. For the 3-month Euribor rate that is assumed as a non-negative risk-free rate, we assume that it has no transaction costs. <https://bofip.impots.gouv.fr/>.

5. See Appendix .2 for a comparison of the performance of our proposed strategy under different combinations of transaction costs of four risky assets.

performance of our strategy under different financial market movements, we need to assume that the following parameters have the same value as used in *Chapter 3*:

The parameters in the inequality constraint are given by:

$$\alpha = 0.5\%, \text{ and } c = 330 \text{ billion euros.}$$

The parameters in billion of euros in the basic risk process X are given by:

$$a_{\lambda,\beta} = 21.9, \text{ and } \sigma_{\lambda,\beta}^2 = 24.745.$$

The maximum investment ratios for bonds, stocks and real estate are known as:

$$\gamma_{max}^{(GB)} = \gamma_{max}^{(CB)} = 100\%, \gamma_{max}^{(SI)} = 65\% \text{ and } \gamma_{max}^{(RE)} = 40\%.$$

4.3.6 Backtesting procedures

Assume that the investment period is from time $t = 0$ to $t = T$. The implementation procedure for our asset allocation strategy can be described as follows:

(1) At time $t = 0$, calculate the proportions $\vec{\gamma}^*$ according to (4.2.1.3) and (4.2.1.4), and then construct the portfolio by investing in the five assets. In the presence of transaction costs, the optimal weightings are calculated based on each investment rate of return after the deduction of the transaction cost.

(2) On the next rebalancing date, record the portfolio value (the value of the life insurance company) and calculate the corresponding interest rate of the contract for the holding period (HPY). The HPY is calculated according to (4.2.1.7). Then, calculate the new proportions $\vec{\gamma}^*$ following the procedure (1) and reconstruct the portfolio based on the information available on this date. This step continues until the end of the investment period, $t = T$. At the end of the eight-year investment period, calculate and record the cumulated HPY.

(3) The above procedure tracks the portfolio values for one realization of the strategy considered. To perform an empirical analysis, the procedure is re-applied on every trading day in the sample period. Consequently, we obtain a series of realizations with overlapped periods in each scenario for each strategy.

4.4 Empirical results

In this analysis, the initial investment of a life insurance company is the net inflow from the life insurance business in the previous year. We assumed it to be a constant as described in Section 4.3.5.

The insured values are 100%, and 104%⁶ of the initial investment.

The investment period is eight years, $T = 8$ years. The reasons are as follows. Firstly, life insurance companies are long-term investors. Secondly, one of the most attractive characteristics of life insurance contracts to policyholders (or investors) is the highest tax advantage when holding the contract for at least eight years. Additionally, as a long-term investment, the capital in euro-growth funds is guaranteed only with a minimum holding period of 8 years. In this way, we can also indirectly determine whether our strategy is suitable for the optimal asset allocation of the euro-growth fund.

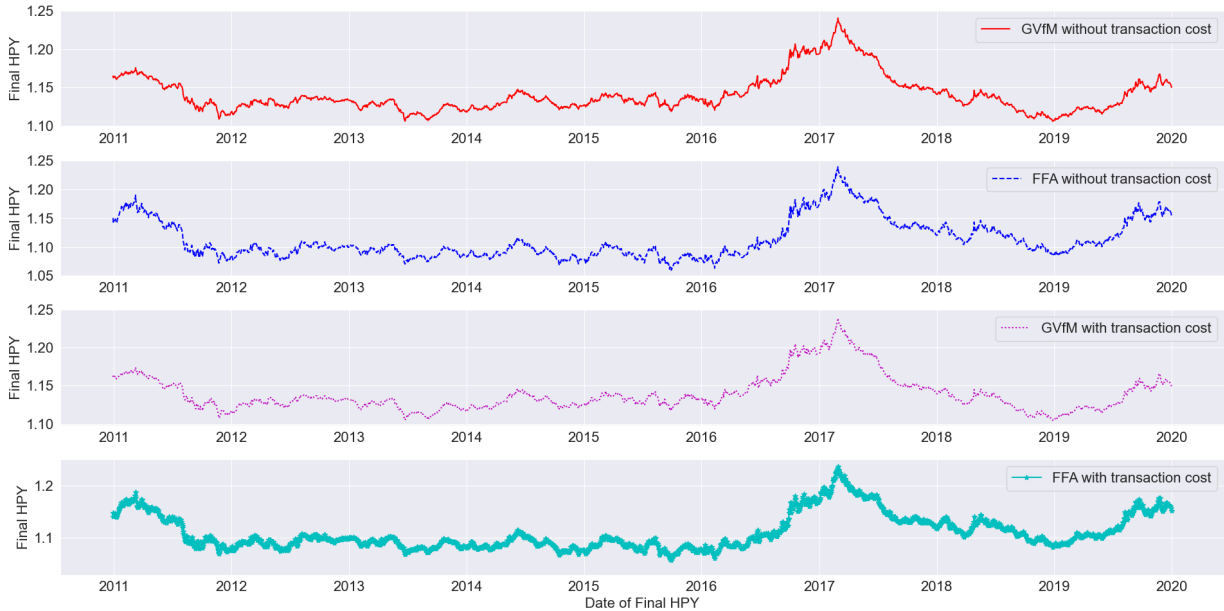
The datasets we used for this analysis are the same as in *Chapter 3*. There are 4289 observations in each of the five data series. The expected return and volatility are approximated by the annualized average return and annualized standard deviation of the daily returns within two rebalancing dates in the investment period, respectively. Consequently, different estimates are used for different investment periods. As the procedure in this chapter is re-employed on each trading day in the sample period, we obtained a series of different realizations for the strategies.

We have obtained 2273 realizations in each scenario for the benchmarks and the B&H strategies, as shown in Figure 4.1 and Figure 4.2, respectively. For our asset allocation strategies, we have obtained 2279, 2231, 2147, 2021, and 1769⁷ different realizations for weekly, monthly, quarterly, semi-annually, and annually rebalancing disciplines, respectively. We plot the realizations of our strategies in Figure 4.3.

6. We know that the average annual technical rate can be regarded as equivalent to the minimum guaranteed rate. From Figure 1.18 in *Chapter 1*, the mean of the average annual technical rate from 2011 to 2019 is less than 0.5%. Then the corresponding eight-year minimum guaranteed interest rate is less than 4%. Consequently, we assume the second insured value to be 104%.

7. For instance, the number of different annually rebalancing realizations is obtained by:
 $4289 - 2 * 252 - 2016 = 1769$.

Figure 4.1 – Realizations of the two benchmarks, GVfM and FFA, without and with transaction costs.

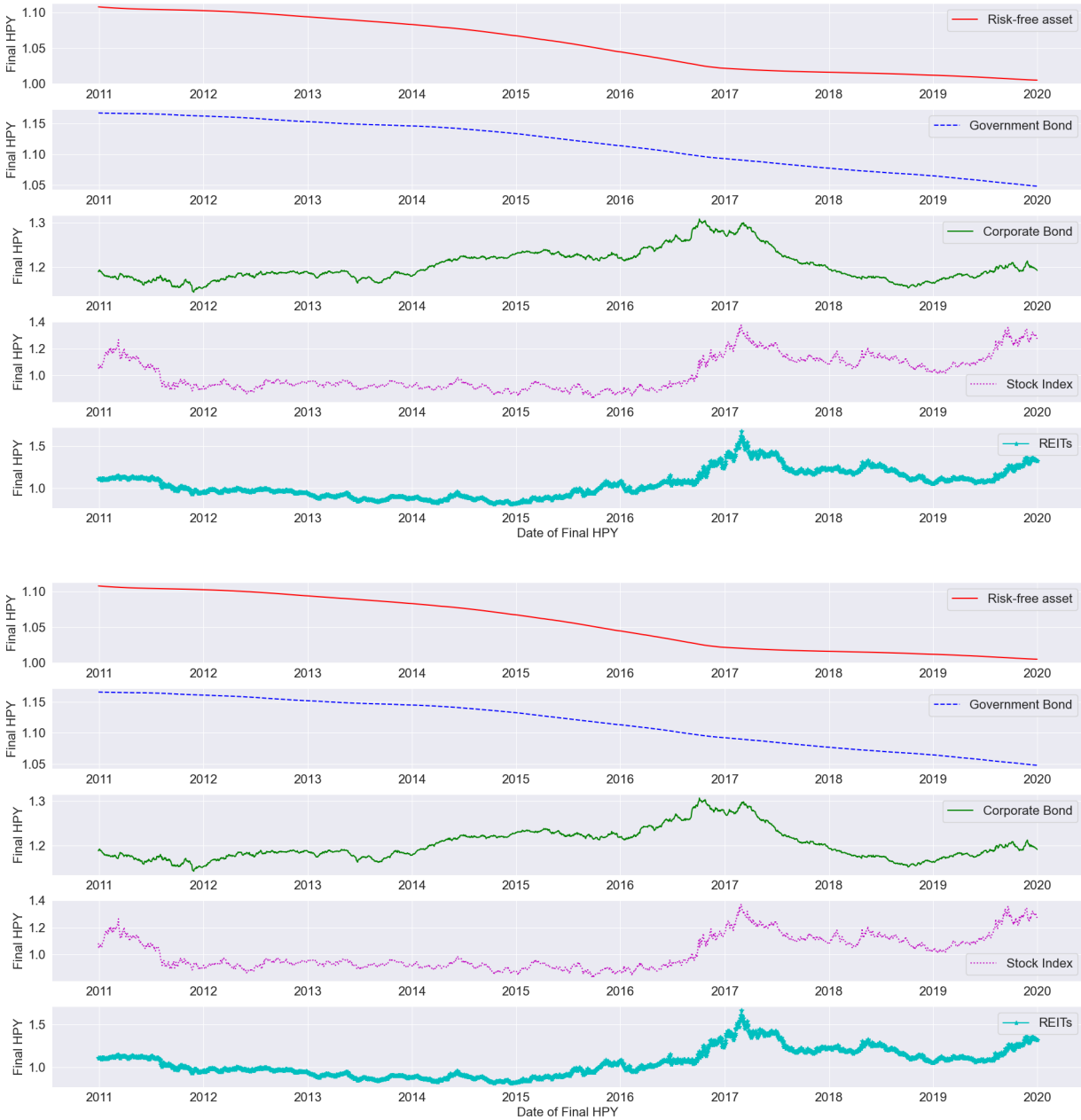


Note: This figure plots the realizations of the final HPYs under the two benchmarks based on the weighings of GVfM and FFA for the cases without and with transaction costs.

By observing Figures 4.1 and 4.2, we can find that the final HPY under the B&H strategy that only invests in the stock index and the REITs and the benchmarks based on two weightings is relatively lower from 2012 to 2016 and higher around 2017. The realizations from 2012 to 2016 (the corresponding starting date of the backtesting, 2004 - 2008) consist of the bear market during the 2007-08 financial crisis, causing the relatively lower final HPYs. The results achieved around 2017 (the corresponding starting date of the backtesting, 2009) do not include the extreme decline of the 2007-08 financial crisis, therefore the final HPYs are relatively higher.

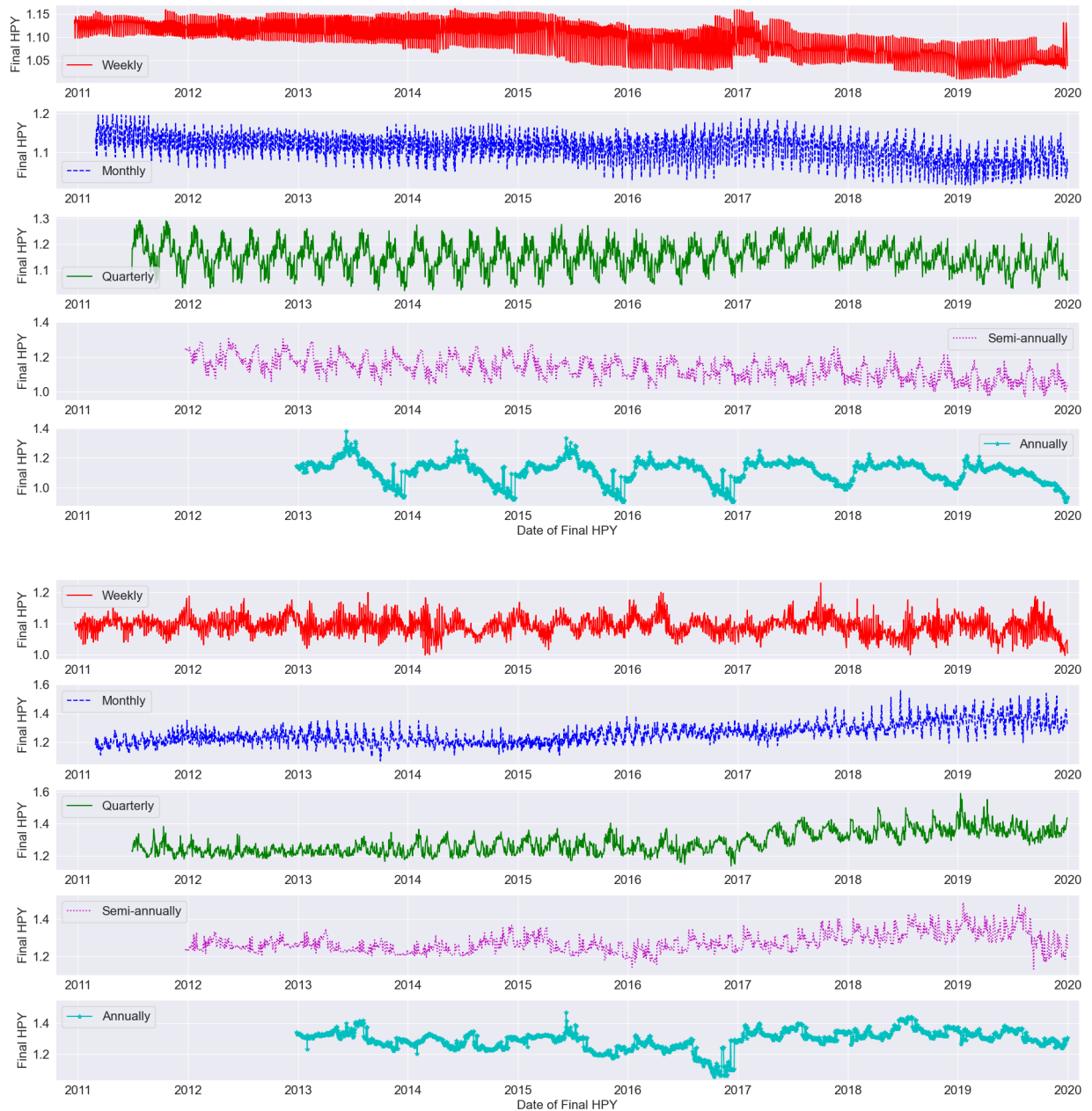
In Figure 4.3, the realizations of our strategy show a certain periodicity. During the backtesting, the final HPYs are close at the interval of the rebalancing frequency (for example, the final HPY is similar every three months under the quarterly rebalancing strategy). Since the investment period is as long as eight years by our backtesting design, the periodicity in the backtesting is generated by the partial overlap of the cumulative HPYs.

Figure 4.2 – Realizations of the B&H strategy invested in one asset, without and with transaction costs.



Note: This figure plots the realizations of the final HPYs under the B&H strategy invested in each of the five assets for the cases without (upper) and with (bottom) transaction costs.

Figure 4.3 – Realizations of our asset allocation strategies under different rebalancing frequencies, without and with transaction costs.



Note: This figure plots the realizations of the final HPYs under our asset allocation strategies by different rebalancing frequencies for the cases without (upper) and with (bottom) transaction costs.

We analyze and compare the realizations of the strategies in the rest of this section.

4.4.1 The effectiveness of asset allocation strategies

Table 4.3 – 5% Ventile (V5) and the average values for the poorest-performed 5% ventile (AV5) realizations for various strategies, without and with transaction costs (TC).

V5									
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE				
Without TC	1.0169	1.0814	1.1770	0.9141	0.9177				
With TC	1.0169	1.0808	1.1764	0.9128	0.9163				
	Benchmark GVfM	Benchmark FFA							
Without TC	1.1258	1.0884							
With TC	1.1243	1.0867							
	Our proposed strategy								
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing				
Without TC	1.0694	1.0902	1.1200	1.0870	1.0673				
With TC	1.0730	1.2048	1.2263	1.2366	1.2603				
AV5									
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE				
Without TC	1.0118	1.0657	1.1671	0.8902	0.8759				
With TC	1.0118	1.0651	1.1665	0.8889	0.8747				
	Benchmark GVfM	Benchmark FFA							
Without TC	1.1189	1.0811							
With TC	1.1175	1.0795							
	Our proposed strategy								
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing				
Without TC	1.0477	1.0662	1.0883	1.0566	1.0054				
With TC	1.0550	1.1793	1.2057	1.2174	1.2122				

Table 4.3 presents the results of the V5 and AV5 for various strategies. As stated in Section 4.3.3, V5 and AV5 measure the ability of hedging against the downside risk. If only one asset is invested under the B&H strategy, the stock index and the REITs investment still have downside risks, which confirms that the diversification among asset classes is necessary even in the long-term (for example, the investment period is 8 years). While investing only in the corporate bond index, it has an outstanding performance in the long-term of hedging the downside risks, which explains why our asset allocation strategy gives the corporate bond index the highest weight in *Chapter 3*.

Both our strategies and the benchmarks are able to limit the negative returns, as they have positive returns in comparison with the B&H strategy invested in only one

asset. The ability to hedge against the downside risk is higher when investing in the GVfM weightings than the FFA weightings. Because the GVfM weightings have lower allocations of the stock index and the REITs and higher allocation of the corporate bond index. In general, the ability of our strategy to hedge against the downside risks increases as the frequency of rebalancing decreases, and this is true regardless of whether or not there are transaction costs.

The impact of transaction costs on B&H strategies and the benchmarks is insignificant because they do not require frequent rebalancing in our backtesting design. Contrary to intuition, the performance of our strategy improves as the transaction costs are present. This is mainly due to the following reason. After the deduction of the transaction cost, the investment rate of return as inputs to the inequality constraints declines at each rebalancing date, which leads to a change in the weightings. Given the four assets with different transaction costs, the government bond and the corporate bond index will be assigned higher weightings than the stock index and the REITs. Therefore, our strategy performed better in the presence of transaction costs.

Table 4.4 reports Q75 and AQ75 to measure the ability to obtain the upside returns for various strategies. Our asset allocation strategies perform better in capturing upside returns when the portfolios are rebalanced quarterly and annually, which means that the transaction costs are important factors when rebalancing the portfolios.

The effects of portfolio rebalancing frequency on the Q75 and AQ75 are not the same as those on the V5 and AV5. As the rebalancing frequencies decrease, the Q75 and AQ75 do not increase too much; on the contrary, the results decrease when the rebalancing frequency shifts from quarterly to annually for our asset allocation strategies.

The results show that the Q75 and AQ75 under the two benchmarks are lower than those under the B&H strategy invested in risky assets and our asset allocation strategies. The differences reflect the implicit costs inherent in the benchmarks with constant weightings in the long term. Because there is a trade-off between the ability to capture the upside return and hedge the downside risk when diversifying without rebalancing, in general, the protection against the market downside risk can only be hedged by sacrificing some upside returns.

Table 4.4 – 75% quartile (Q75) and the average values for the best-performed 25% quartile (AQ75) of realizations for various strategies, without and with transaction costs (TC).

Q75							
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE		
Without TC	1.0915	1.1507	1.2235	1.1163	1.1621		
With TC	1.0915	1.1501	1.2228	1.1146	1.1603		
	Benchmark GVfM	Benchmark FFA					
Without TC	1.1500	1.1330					
With TC	1.1485	1.1312					
	Our proposed strategy						
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing		
Without TC	1.1265	1.1396	1.1862	1.1738	1.1619		
With TC	1.1123	1.2985	1.3305	1.3009	1.3315		
AQ75							
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE		
Without TC	1.1010	1.1604	1.2522	1.1925	1.2838		
With TC	1.1010	1.1598	1.2515	1.1906	1.2818		
	Benchmark GVfM	Benchmark FFA					
Without TC	1.1743	1.1629					
With TC	1.1728	1.1610					
	Our proposed strategy						
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing		
Without TC	1.1413	1.1539	1.2173	1.2132	1.1931		
With TC	1.1307	1.3537	1.3778	1.3455	1.3638		

To see the extent to which the strategies protect against downward risk, we compute the protection ratio for the BH strategy in the same way as calculating the protection ratios for the benchmarks and for our strategies. Table 4.5 presents the protection ratio for different strategies under two insured values.

The results show that both the benchmarks and our strategies generate obviously higher protection ratios than the simple BH strategy invested in the stock index and the REITs. As the insured value increases, the BH strategy with only investing in risk-free assets will not provide policyholders with a minimum guaranteed interest rate for eight years. The BH strategy with only investing in the 10-year government bond could provide the minimum guaranteed interest rate. However, if the yields on government bonds remain low in the long term, life insurers investing in government bonds alone will not be fully

insured.

Table 4.5 – Insured value protection ratios for various strategies, without and with transaction costs (TC).

Insured value = 100%							
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE		
Without TC	100%	100%	100%	43.07%	54.42%		
With TC	100%	100%	100%	42.98%	54.16%		
	Benchmark GVfM	Benchmark FFA					
Without TC	100%	100%					
With TC	100%	100%					
Our proposed strategy							
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing		
Without TC	100%	100%	100%	99.41%	89.60%		
With TC	99.82%	100%	100%	100%	100%		
Insured value = 104%							
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE		
Without TC	57.50%	100%	100%	40.61%	49.09%		
With TC	57.50%	100%	100%	40.48%	48.83%		
	Benchmark GVfM	Benchmark FFA					
Without TC	100%	100%					
With TC	100%	100%					
Our proposed strategy							
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing		
Without TC	93.02%	96.37%	98.56%	93.86%	81.68%		
With TC	95.52%	100%	100%	100%	100%		

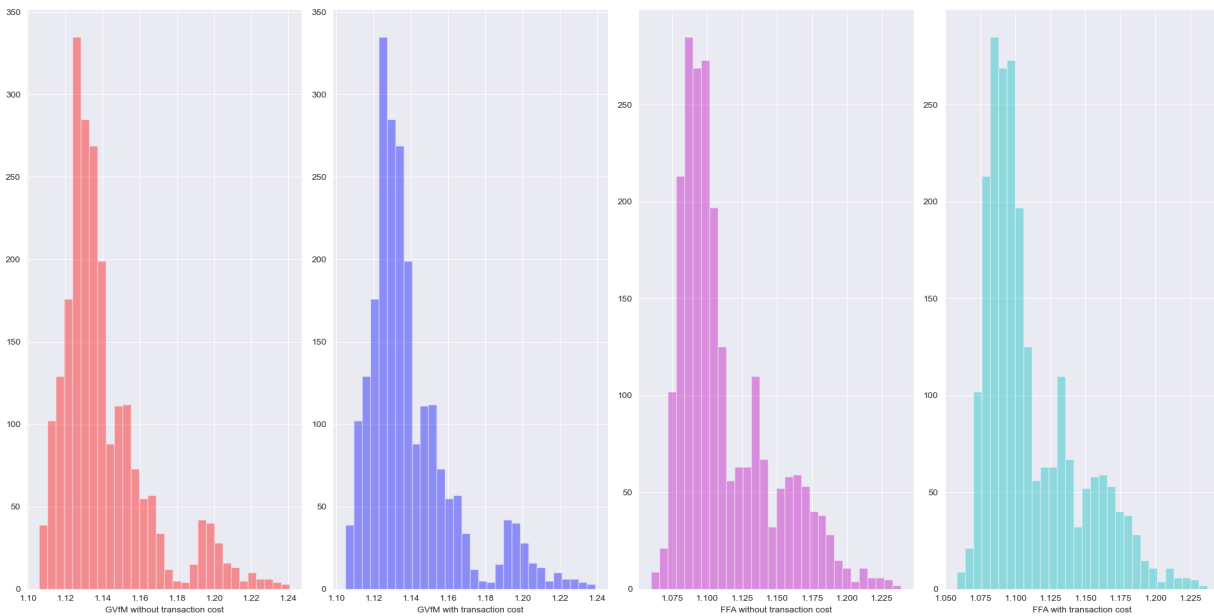
Two benchmarks with different weightings fully protect the insured values regardless of the transaction costs, demonstrating the role of asset allocation in diversifying the portfolio risk again.

In the absence of transaction costs, the performance of our strategies improves as the rebalancing frequency increases in case the insured value is lower; however, when the insured value is higher, the results show that quarterly rebalancing performs best among the rebalancing frequencies. Apart from the weekly rebalancing, other rebalancing frequencies can fully protect the insured values with transaction costs. When the insured value increases, the protection ratios of our strategy decline more in the absence of transaction costs than with transaction costs which are most evident with the weekly rebalancing.

To clearly illustrate the capability of our asset allocation strategies in reshaping the return distributions, the sample distributions for the final HPY values under the two benchmarks, the B&H strategy, and our strategies are plotted. Figure 4.4 is the plot for the two benchmarks, Figure 4.5 is the plot for the B&H strategy, and Figure 4.6 is for our strategies.

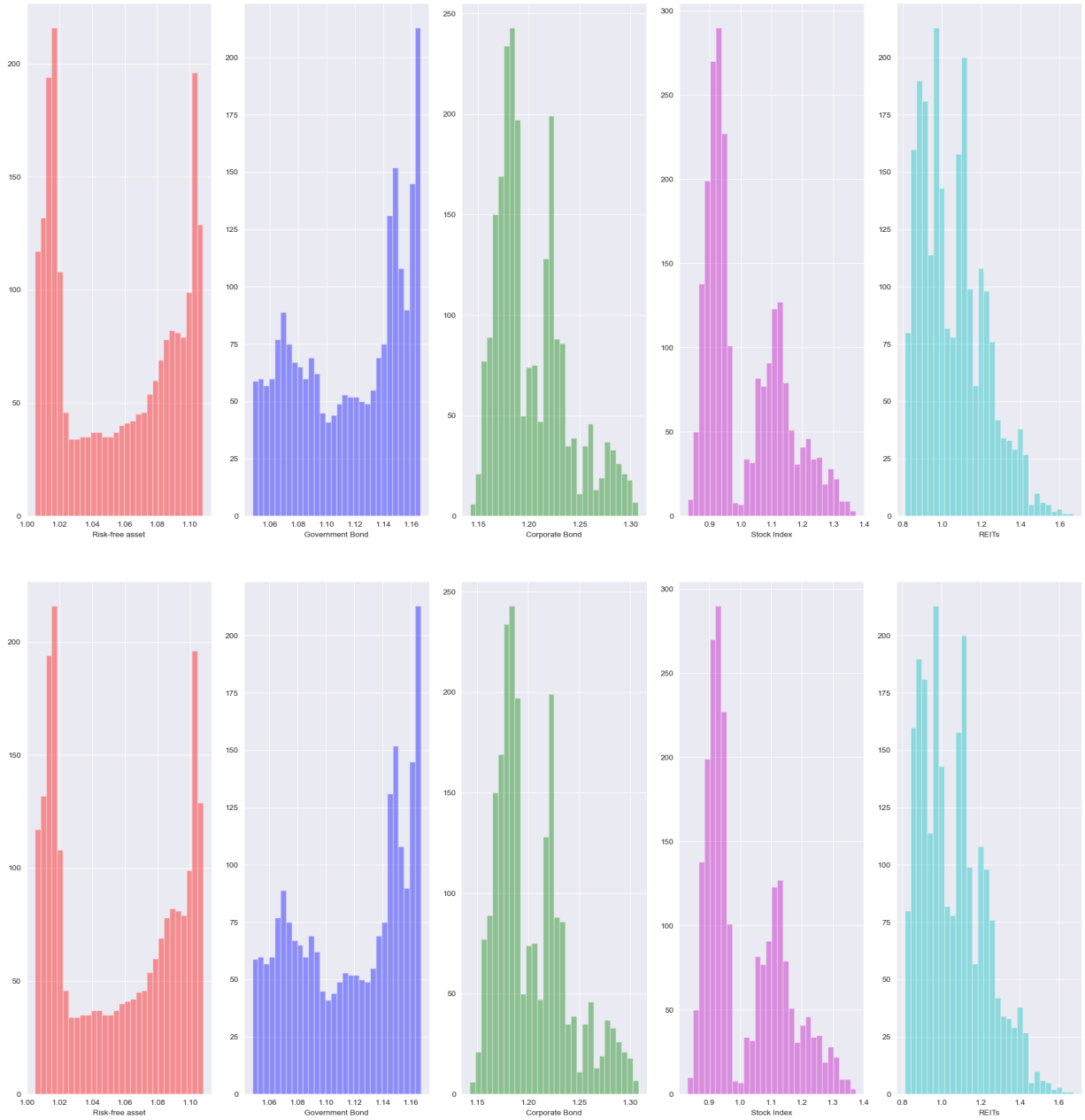
Obviously, both the benchmarks and our strategies have a desirable risk profile with a short left tail and a short right tail in their distributions compared with the BH strategy invested in the stock index and the REITs. In long-term investment, life insurance companies can effectively hedge tail risks through asset allocation, but they also need to pay some extreme returns as the cost. While under our optimal asset allocation strategies, life insurers can hedge tail risks more effectively and retain higher overall returns.

Figure 4.4 – Frequency distributions under the two benchmarks with the B&H strategies, GVfM and FFA, without and with transaction costs.



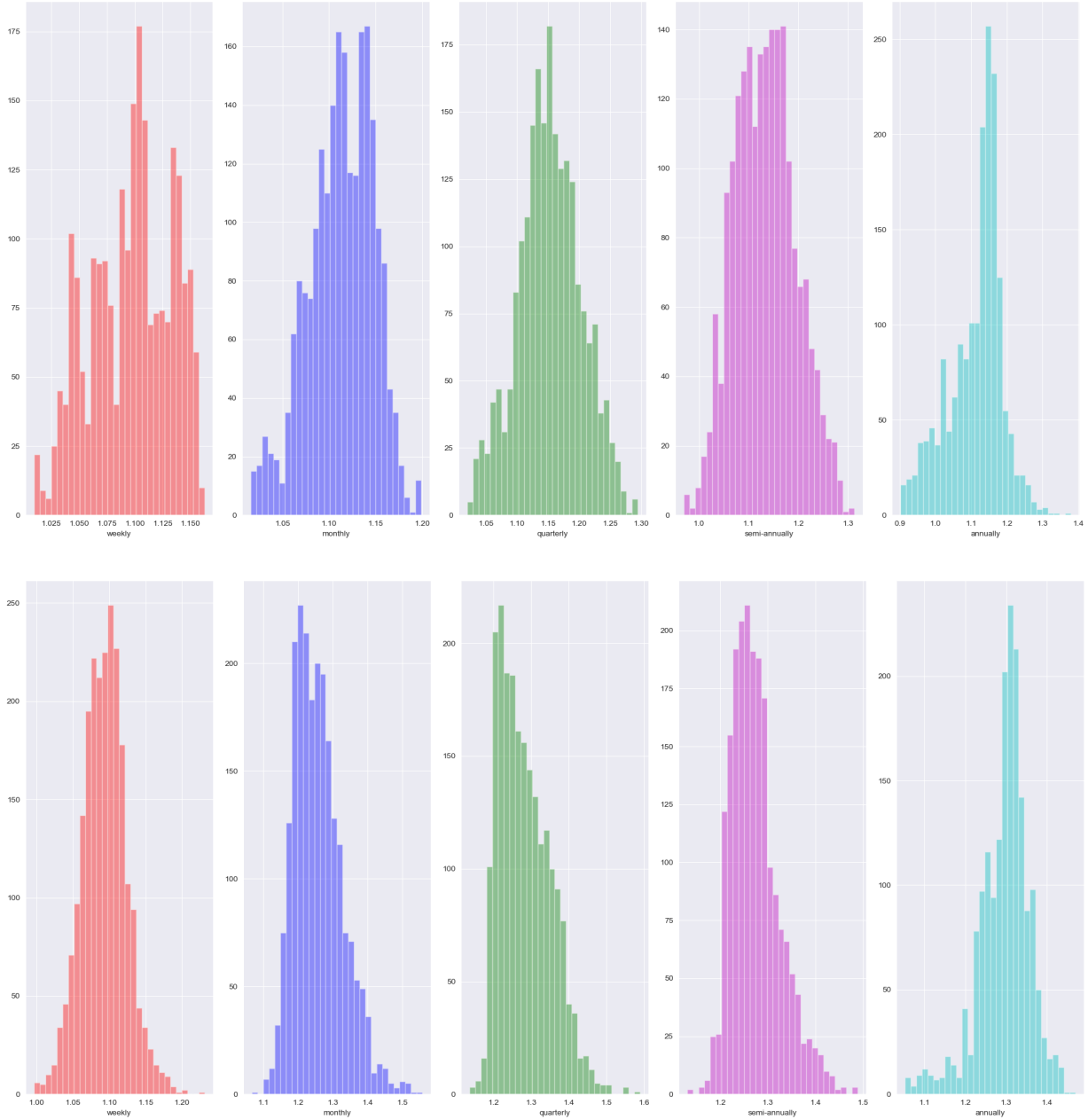
Note: This figure plots the sample distributions of the final HPYs under the two benchmarks based on the weighings of GVfM and FFA for the cases without and with transaction costs.

Figure 4.5 – Frequency distributions under the B&H strategy invested in one asset, without and with transaction costs.



Note: This figure plots the sample distributions for the final HPYs under the B&H strategy for the cases without (upper) and with (bottom) transaction costs.

Figure 4.6 – Frequency distributions under our asset allocation strategies with different rebalancing frequencies, without and with transaction costs.



Note: This figure plots the realizations of the final HPYs under our asset allocation strategies by different rebalancing frequencies for the cases without (upper) and with (bottom) transaction costs.

Table 4.6 presents some statistics of the sample distributions for various strategies. Without transaction costs, the highest average HPY was recorded by the B&H strategy

that only invests in the corporate bond index. However, the B&H strategy that only invests in the stock index or the REITs does not obtain the highest average HPY. The time period of our data (2002-2019) includes the 2007-08 financial crisis, and part of the final HPYs is affected by that bear market. This is why the HPY remains low during the period of 2012-16 (with a corresponding starting period of 2004-08).

Table 4.6 – Statistics of sample distributions for various strategies, without and with transaction costs.

		B&H RF	B&H GB	B&H CB	B&H SI	B&H RE
Without TC	Mean	1.0554	1.1170	1.2033	1.0148	1.0585
	St.dev	0.0363	0.0376	0.0350	0.1250	0.1618
	Skewness	0.0280	-0.2724	0.8810	0.7138	0.6805
With TC	Mean	1.0554	1.1164	1.2026	1.0133	1.0569
	St.dev	0.0363	0.0375	0.0350	0.1248	0.1615
	Skewness	0.0280	-0.2724	0.8810	0.7137	0.6804
		Benchmark GVfM	Benchmark FFA			
Without TC	Mean	1.1408	1.1128			
	St.dev	0.0239	0.0336			
	Skewness	1.5536	1.1383			
With TC	Mean	1.1393	1.1111			
	St.dev	0.0239	0.0335			
	Skewness	1.5536	1.1382			
		Our proposed strategy				
		Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing
Without TC	Mean	1.0964	1.1128	1.1526	1.1333	1.1132
	St.dev	0.0358	0.0346	0.0505	0.0612	0.0756
	Skewness	-0.2531	-0.3922	-0.0261	0.1487	-0.5988
With TC	Mean	1.0929	1.2579	1.2826	1.2741	1.2949
	St.dev	0.0303	0.0700	0.0683	0.0516	0.0625
	Skewness	0.112	0.7424	0.6744	0.8381	-0.8898

Note: This table presents the mean, standard deviation, and skewness of the sample return distributions for various strategies.

As with the previous findings, compared to the benchmark performance based on the FFA weightings, the average HPY under the benchmark of the GVfM weightings is higher but with a lower standard deviation. In the presence of transaction costs, the standard deviation of our strategy is higher than the benchmarks, but it obtains a higher average final HPY.

Since both the benchmarks and our strategies effectively hedge the tail risk, we use

the absolute value of skewness for analysis. More extreme values appear on the right tail of the two benchmarks because of their highest skewness. Our strategy has more extreme values on the left without transaction costs and more extreme values on the right with transaction costs. The above results show that when there are transaction costs, our strategy is likely to obtain a higher HPY, consistent with Q75 and AQ75 analysis.

4.4.2 Our asset allocation strategies versus B&H benchmarks

Now we analyze the relative performance of our asset allocation strategies and the benchmarks under different scenarios.

First, the results of V5 and AV5 presented in Table 4.3 as well as the results of Q75 and AQ75 presented in Table 4.4 show that our strategy, except for the weekly rebalancing, is more capable of limiting the downside losses than the benchmarks in the presence of the transaction costs. Moreover, our strategy can better capture upward profits than the benchmarks in most cases with and without transaction costs.

Second, by comparing the protection ratios for two insured values in Table 4.5, we can see that the protection ratios of our strategy are close to the benchmarks.

Third, as we stated previously, the Sortino ratio measures return adjusted by the downside risk. Our analysis chooses the mean of the average technical interest rate as the minimum acceptable return, consistent with the insured value. Since there are cases in our strategies and the benchmarks with no negative HPY, there is no downside deviation of the HPYs. To make it comparable, we choose to calculate the Sharpe ratio as an alternative indicator, and its risk-free rate is assumed to be equal to the MAR.

The B&H strategy that invests in the corporate bond index achieves the highest Sharpe ratio, followed by our strategy under the semi-annually rebalancing in the presence of transaction costs, and then the benchmark based on GVfM investment weightings, as shown in Table 4.7. In the absence of transaction costs, the Sortino ratio (1.3150) of our quarterly rebalancing strategy is higher than its Sharpe ratio (0.7826). This indicates that the standard deviation of the HPYs, which are less than the MAR of our strategy, is lower than the overall standard deviation. As the Sortino ratio is goal-oriented, life insurers can measure relative performance to obtain the desired results.

Table 4.7 – Sharpe ratios and Sortino ratios for various strategies, without and with transaction costs (TC).

Sharpe Ratio, rf = MAR = 4%								
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE			
Without TC	0.1434	0.7180	1.6397	-0.0732	0.0389			
With TC	0.1434	0.7130	1.6344	-0.0777	0.0354			
	Benchmark GVfM	Benchmark FFA						
Without TC	1.4780	0.7598						
With TC	1.4582	0.7429						
	<u>Our proposed strategy</u>							
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing			
Without TC	0.5503	0.7367	0.7826	0.5346	0.3390			
With TC	0.6087	1.0967	1.2522	1.6002	1.4372			
Sortino Ratio, MAR = 4%								
	B&H RF	B&H GB	B&H CB	B&H SI	B&H RE			
Without TC	0.1103	/	/	-0.0455	0.0195			
With TC	0.1103	/	/	-0.0478	0.0176			
	Benchmark GVfM	Benchmark FFA						
Without TC	/	/						
With TC	/	/						
	<u>Our proposed strategy</u>							
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing			
Without TC	0.3783	0.6681	1.3150	0.3595	0.1258			
With TC	0.2980	/	/	/	/			

"/": all the realizations of the Final HPY are non-negative.

Table 4.8 – Maximum drawdown ratios for various strategies, without and with transaction costs (TC).

	Benchmark GVfM	Benchmark FFA						
Without TC	-7.63%	-10.95%						
With TC	-7.87%	-11.22%						
	<u>Our proposed strategy</u>							
	Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing			
Without TC	-4.09%	-6.01%	-7.21%	-13.62%	-14.49%			
With TC	-7.76%	-8.12%	-0.35%	0	0			

Next, we evaluate the relative riskiness of our asset allocation strategies and the bench-

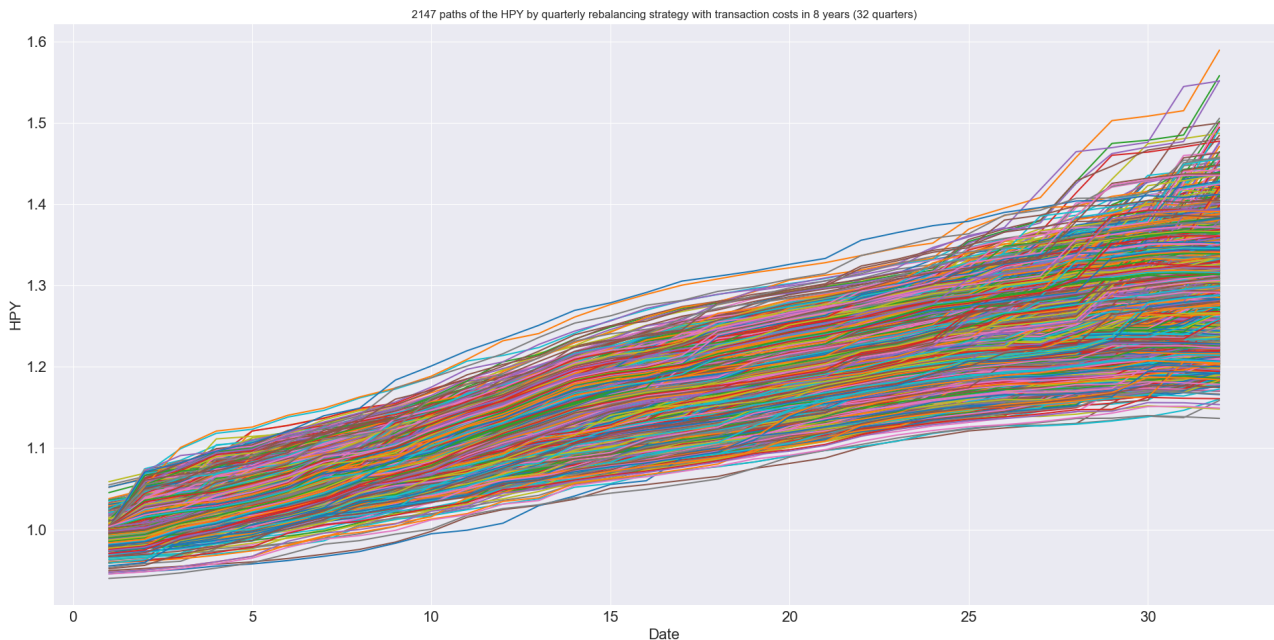
marks by calculating the maximum drawdown, enumerated in Table 4.8. Each ratio represents the maximum drawdown among all the realizations under this strategy.

In the absence of transaction costs, the maximum drawdown of the benchmark is -7.63%. Our strategy performs better when the frequency of rebalancing is relatively high, and the maximum drawdown of weekly, monthly and quarterly rebalancing strategies are -4.09%, -6.01% and -7.21%, respectively.

Unexpectedly, if our strategy is not rebalanced frequently, it performs very well when there are transaction costs. The maximum drawdown of annually, semi-annually and quarterly rebalancing are very few, -0.35%, 0 and 0, respectively. This result is significant because the result of the portfolio is considered for each of the rebalancing dates.

For the annual rebalancing strategy, its 1,769 different realizations were achieved without any drawdown during the eight-year investment period, which means the portfolio value does not decline. In other words, based on the results obtained previously, we know that the portfolio value of the annually rebalancing strategy has been retaining the upward returns without any downside risks.

Figure 4.7 – 2147 paths of the HPY by quarterly rebalancing strategy with transaction costs in 8 years (32 quarters).



We have the same findings for the semi-annually rebalancing strategy with 2021 realizations. Among the 2147 paths of the quarterly rebalancing strategy, the maximum

drawdown of the portfolio value was less than 0.5%, as plotted in Figure 4.7. For illustration purposes, only paths for the quarterly rebalancing strategy with transaction costs are displayed⁸.

Finally, we will compare each strategy through the Omega ratio. As discussed in Section 4.3.3, the Omega ratio is the ratio of the averages of the gains above a threshold to the averages of the losses below the same threshold. To discover the relative performance of different strategies in terms of the Omega measure, we plot their Omega ratio as a function of the threshold, L , in different cases in Figure 4.8 and Figure 4.9.

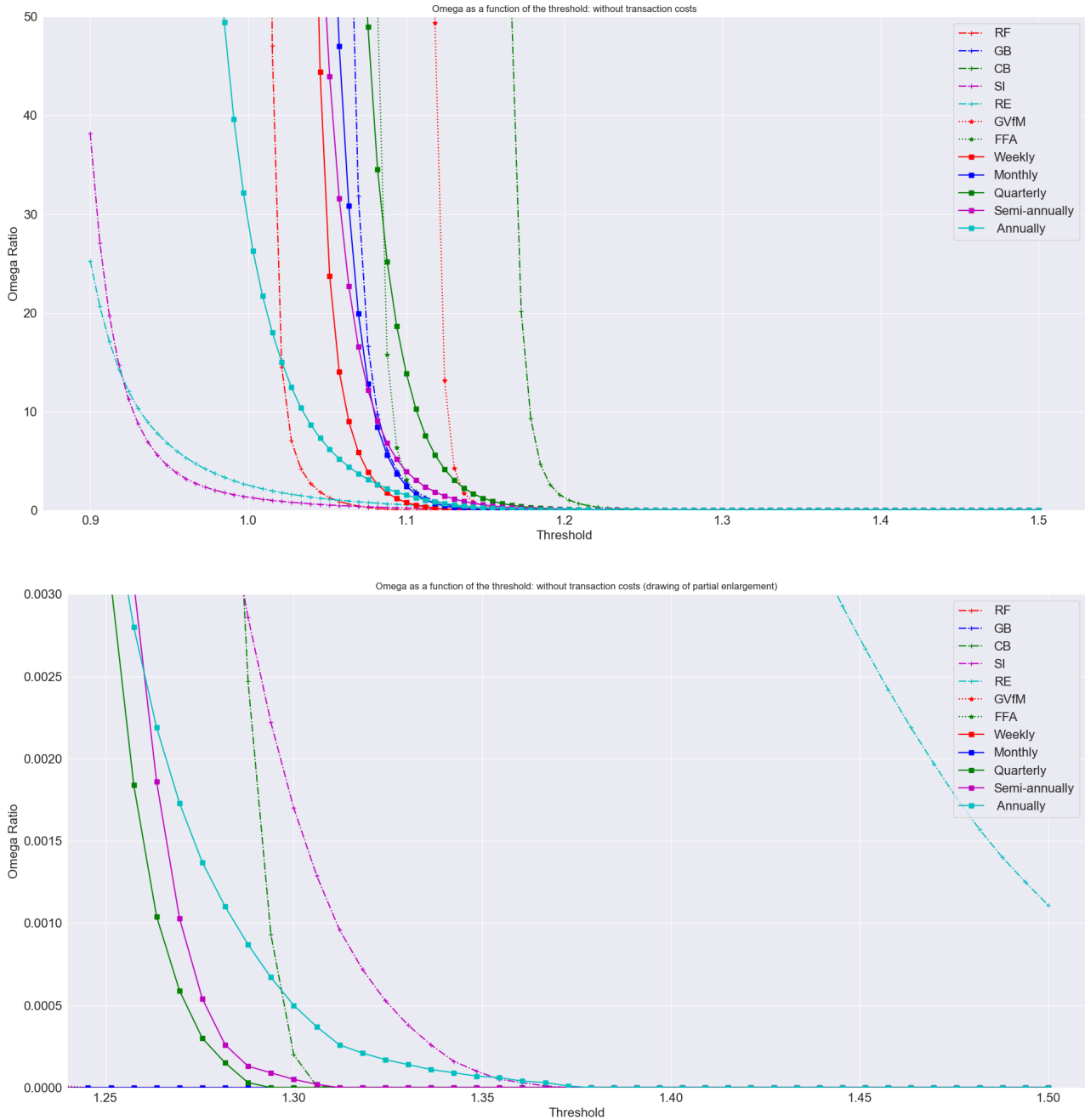
When we focus on the case without transaction costs, the Omega ratio of the B&H strategy that only invests in one asset is distributed on both sides of the graph. When the threshold is lower than 1, the B&H strategy that only invests in the stock index and the REITs performs the worst. When the threshold is higher than 1.1, the B&H strategy that only invests in the corporate bond index is the best. The rest, including the benchmarks and our strategies, are mainly distributed in the middle. As demonstrated in [Bertrand and Prigent, 2011], that a lower threshold level implies that the investors are more concerned with risk control, and a higher level means that they worry more about the return performance of the portfolio.

When we compare the performance of our strategies and the benchmarks, we find that the benchmark seems to have a higher Omega ratio. However, the Omega curve of the benchmark does not seem to be more convex than the Omega curve of our strategy. It means that the benchmark's Omega ratio will decline faster as the threshold increases. At a higher threshold, the benchmark's Omega ratio could be lower than some of our strategies.

Indeed, when we focus on the drawing of partial enlargement, with the threshold between 1.25 and 1.5, the best three performances are the B&H strategy invested in the REITs, the B&H strategy invested in the stock index, and our annually rebalancing strategy. The common point of these three strategies is that their Omega curves are more convex. As the threshold increases, the Omega ratio decreases slower than those with less convex Omega curves. The semi-annual and quarterly rebalancing strategies also rely on greater convexity to surpass other strategies after increasing the threshold. While considering other strategies, their Omega ratios are close to 0 because of poorer performance in obtaining upside returns.

8. See Figure 18 and 19 in Appendix .3 and Figure 20, 21, 22 and 23 in Appendix .4 for a comparison of the paths of the HPY under the benchmarks and other rebalancing frequencies of our strategies.

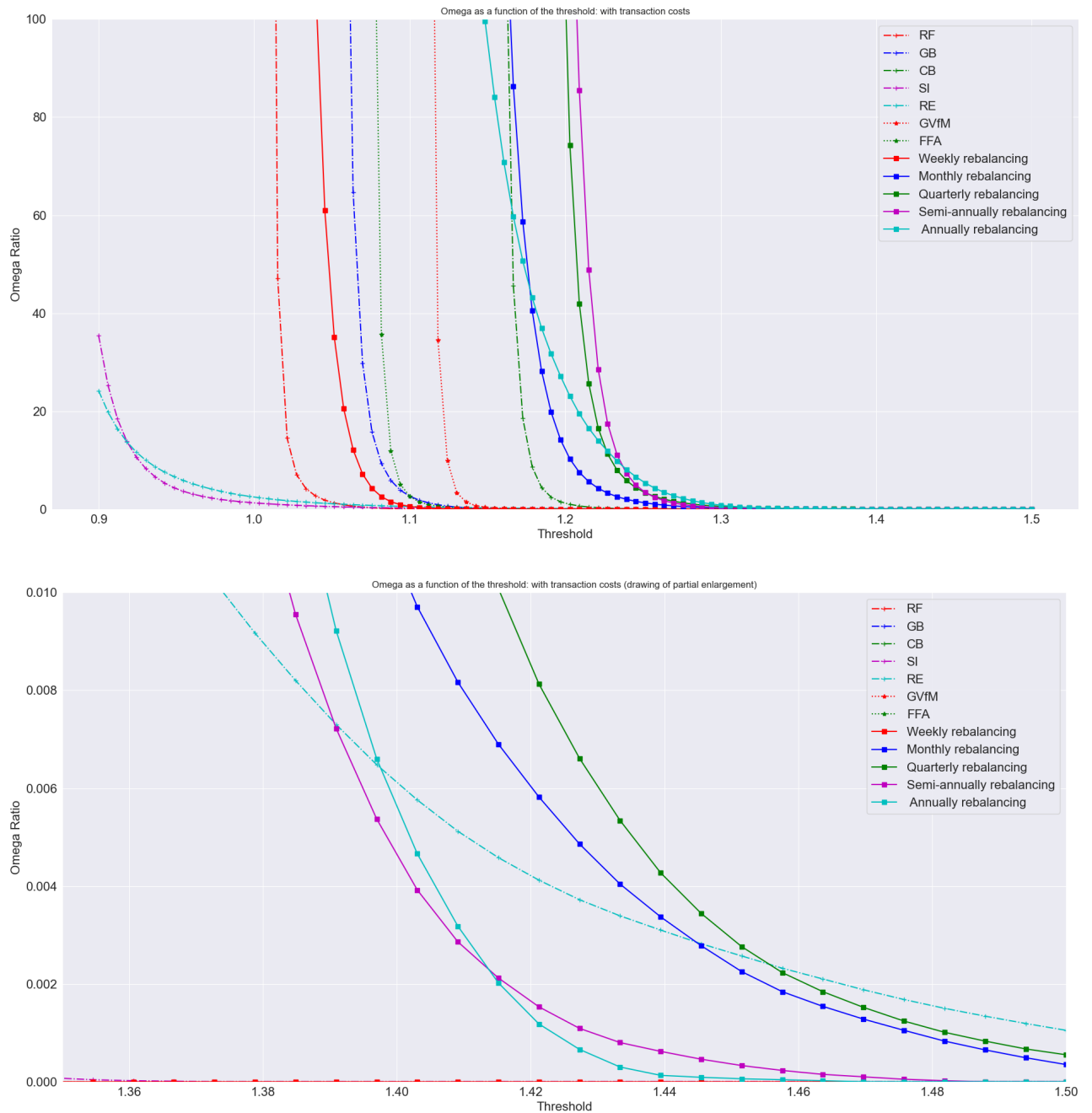
Figure 4.8 – Omega ratio as a function of the threshold: without transaction costs, drawing of complete (upper) and partial enlargement (bottom).



Next, we shift our perspective to the case with transaction costs. In addition to the weekly rebalancing strategy, the other four strategies performed at the top and surpassed the B&H strategy that only invests in the corporate bond index and the two benchmarks.

When the threshold is lower than 1.3, the annual rebalancing strategy performs best. The B&H strategy that only invests in the stock index and the REITs performs the worst in hedging downside risks, but its Omega curve appears to be the most convex.

Figure 4.9 – Omega ratio as a function of the threshold: with transaction costs, drawing of complete (upper) and partial enlargement (bottom).



Similarly, we turn our attention to the drawing of partial enlargement to examine the ability of each strategy to achieve higher upside returns. Not surprisingly, the Omega ratio of the B&H strategy that only invests in the REITs exceeds our semi-annual rebalancing, annual rebalancing, monthly rebalancing, and quarterly rebalancing strategies when the threshold is in the range of 1.39 to 1.46, explaining why life insurance companies have been increasing their weighting in real estate, in recent years [EIOPA, 2017]. By increasing the REITs and real estate investment, life insurers have a higher probability of achieving upside returns. The Omega ratio based on the GVfM weightings is close to zero when the threshold is approximately 1.37. Our asset allocation strategies and the benchmarks can outperform the BH strategy in hedging the downside risk while under-performing the BH strategy to capture the upside return. By observing the changes in the Omega ratio, our strategies' ability to realize higher returns has been further verified.

4.5 Further discussions

In Section 4.4, we provide a full perspective of the effectiveness of the benchmarks, our asset allocation strategies, as well as the B&H strategies. However, the results obtained above rely on the choice of parameter values, such as α and c , and the constraints, for instance, the upper bounds on the weightings of each asset in the investment. We propose to check their robustness along these dimensions. In Section 4.2.2, we assumed that the daily weightings of the benchmark are equal to the annual average investment ratio of each asset during the dataset period. It is also worth discussing whether there will be better results if the benchmark reflects the changes in the weightings each year. In addition, the participation rate provided to policyholders is also known in the market, and the comparison with the real-world data is also of practical significance.

4.5.1 On the constraints, α and c

As we know from the results of the numerical illustration in Section 3.4.2 of *Chapter 3*. Both α and c can be viewed as the risk tolerance measures of the life insurance company, such that as the α or c increases, the insurer's risk-averse attitude decreases, and the investment strategy will be more aggressive. These two parameters can be tailored according to the financial situation of life insurers to reflect their attitudes on investment risks.

According to the results of the previous section, the quarterly rebalancing strategy performed relatively well, and the frequency of quarterly rebalancing was also moderate. We consider the quarterly rebalancing strategy under the cases with and without transaction costs in the following analysis. The values we assigned to α and c were 0.5% and 330, respectively. We choose two values for α , 0.499% and 0.501%, and then two values for c , 329.5 and 330.5. Then four combinations of α and c , each including one basis and one new assigned value, are assumed, as shown in the table below.

Table 4.9 – Five combinations of the parameter α and c under the quarterly rebalancing strategy.

	1: Low c	2: Low α	3: Benchmark c and α	4: High α	5: High c
α	0.5%	0.499 %	0.5%	0.501%	0.5%
c	329.5	330	330	330	330.5

The results of various ratios are presented in Table 4.10. In the absence of transaction costs, as α or c decreases, the performance of our strategies in hedging downside risks increases (with higher V5, AV5, and protection ratios), and the investment style is more conservative, with diminishing maximum drawdown and the increasing of Sharpe ratio and Sortino ratio. The ability to obtain upside returns increases with α and c (with higher Q5 and AQ75).

When considering transaction costs, we find that with the increase of α or c , the strategy continuously enhances the ability to pursue upside returns and does not weaken its protection and hedging against downside risks.

In addition, we also find that based on the comparison with Benchmark c and α , with the increase of α or c , the strategy's performance is reflected with the phenomenon of diminishing marginal growth rates on the ratios regardless of whether transaction costs are considered. For instance, such phenomenon is shown when comparing Low α , Benchmark, and High α , or Low c , Benchmark, and High c .

With the increase of α or c (equivalent to the increase of the right-hand side of the inequality constraint of our strategy⁹), the second moment of the value of the life insurance company on the left-hand side of the inequality can also grow correspondingly, which

9. Recall the constraint:

$$2a_{\lambda,\beta}^2 T^2 \cdot g(\mu_\gamma T) \cdot f[(\mu_\gamma + \sigma_\gamma^2)T] + (\sigma_{\lambda,\beta}^2 - \frac{2a_{\lambda,\beta}^2}{\mu_\gamma})T \cdot f[(2\mu_\gamma + \sigma_\gamma^2)T] \leq \alpha \cdot c^2,$$

Table 4.10 – Various portfolio evaluation ratios for our proposed strategy with quarterly rebalancing under different combinations of α and c .

Without transaction costs	Trans-	Low c	Low α	Benchmark c and α	High α	High c
V5		1.1201	1.1202	1.1200	1.1205	1.1200
AV5		1.0895	1.0890	1.0883	1.0875	1.0866
Q75		1.1837	1.1844	1.1862	1.1880	1.1888
AQ75		1.2130	1.2143	1.2173	1.2197	1.2211
protection ratio (Insured value: 104%)		99.35%	99.07%	98.65%	98.28%	98.04%
Sharpe ratio		0.8054	0.7980	0.7826	0.7681	0.7576
Sortino ratio		2.2902	1.5018	1.3150	1.0592	0.9777
Maximum draw-down		-6.79%	-6.93%	-7.21%	-7.59%	-7.89%
With transaction costs	Transac-	Low c	Low α	Benchmark c and α	High α	High c
V5		1.2018	1.2222	1.2263	1.2299	1.2322
AV5		1.1839	1.2023	1.2057	1.2090	1.2106
Q75		1.2977	1.3237	1.3305	1.3370	1.3392
AQ75		1.3434	1.3708	1.3778	1.3847	1.3880
protection ratio (Insured value: 104%)		100%	100%	100%	100%	100%
Sharpe ratio		1.1910	1.2513	1.2522	1.2523	1.2526
Sortino ratio		/	/	/	/	/
Maximum draw-down		-0.44%	-0.34%	-0.35%	-0.35%	-0.35%

"/": all the realizations of the Final HPY are non-negative.

means that the value of the life insurance company can assume more fluctuations in the investment. However, there are constraints on the investment weightings; namely, the sum of investment proportions cannot be higher than 100%, as well as the lower and upper bounds on the weightings of each asset. Therefore, the HPY will eventually be affected by the constraints on the weightings and cannot increase with α or c .

Following the methodology of the Omega function, we calculate the Omega ratios for each combination of α and c of our quarterly rebalancing strategy under two cases with and without transaction costs, as plotted in Figure 4.10 and Figure 4.11.

We observe that regardless of whether the transaction cost is considered, as α or c increases, the Omega ratio of each combination also rises under each level of the threshold. The same result is also obtained when looking at the drawing of partial enlargement. The robustness of our optimal asset allocation strategy under different constraints of α and c is verified.

Figure 4.10 – Omega ratio as a function of the threshold: the quarterly rebalancing strategy under different combinations of α and c , without transaction costs, drawing of complete (upper) and partial enlargement (bottom).

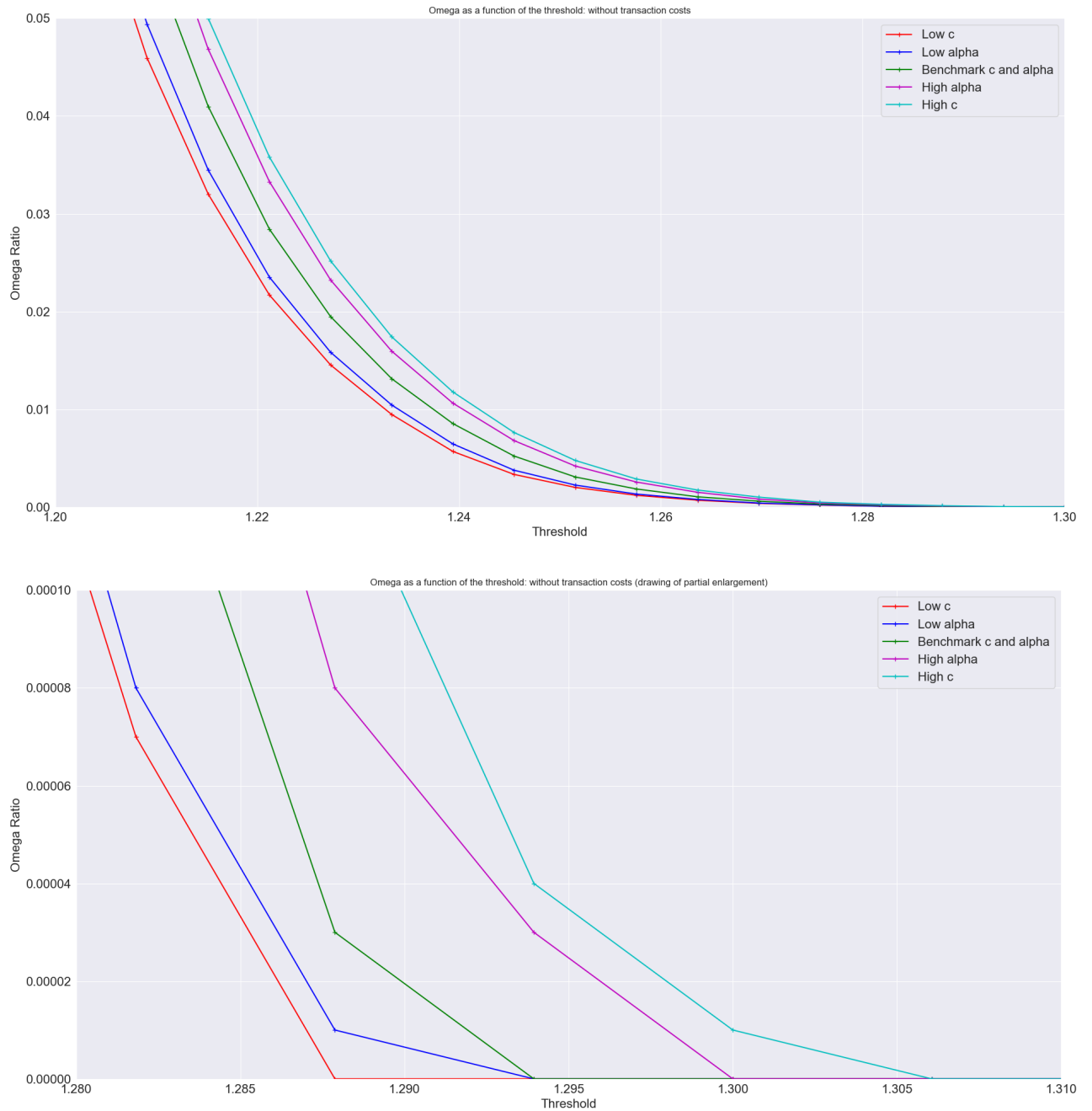
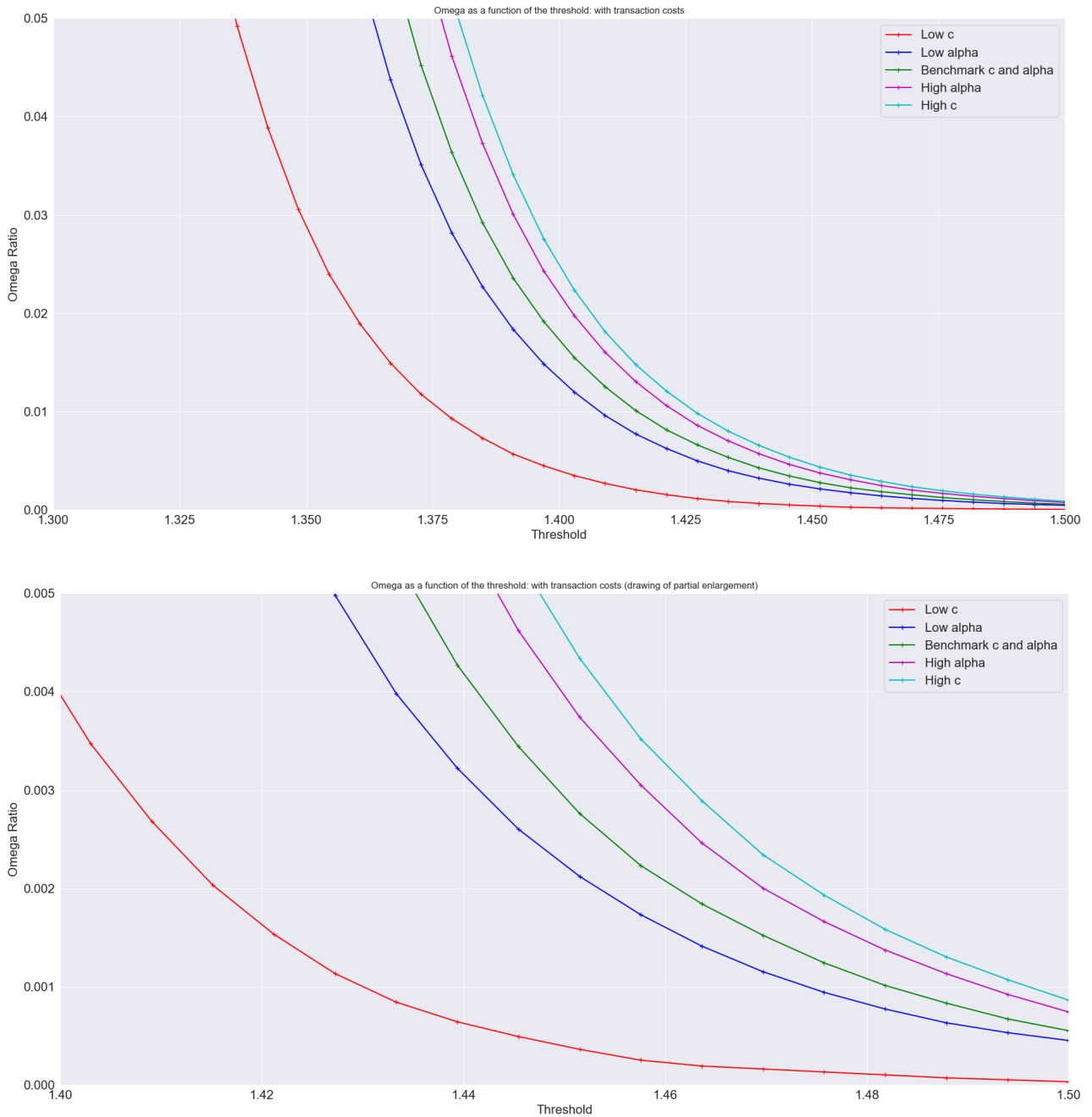


Figure 4.11 – Omega ratio as a function of the threshold: the quarterly rebalancing strategy under different combinations of α and c , with transaction costs, drawing of complete (upper) and partial enlargement (bottom).



4.5.2 On the weightings of the benchmarks

In this section, we will obtain the hypothetical daily investment weightings of the benchmark by applying two methods, linear interpolation and equal-to-year-end.

We know that the annual GVfM dataset is from 2011 to 2019, and each stands for the weighting of the year-end. For the linear interpolation method, first, we assume that the weightings at the end of 2010 are equal to the weightings of 2011. Then, through linear interpolation, we will obtain the daily weightings of the five assets from 2011 to 2019. In the equal-to-year-end method, the weighing of each day in the year has the same weighting as the year-end.

Moreover, in the analysis of performance evaluation in Section 4.3.1, it is assumed that there is no rebalancing during the eight-year investment period. In this section, we will analyze the benchmarks' performance according to different rebalancing disciplines, namely weekly, monthly, quarterly, semi-annually and annually. We will also use the constant weightings with no rebalancing in the 8-year investment period as a reference for the weightings obtained by the two methods.

Since each performance indicator among the cases is close, we only show the Omega ratios for comparison. The Omega functions of the benchmarks without and with transaction costs are plotted in Figures 4.12 and Figure 4.13.

First of all, in the absence of transaction costs, all benchmarks can effectively hedge against downside risks, but they are less capable of retaining upside returns. Among them, the benchmark without rebalancing during the 8-year investment period performed best. The performance of the weightings calculated based on the equal-to-year-end method is better than that of the linear interpolation method.

Next, in the presence of transaction costs, the benchmark without rebalancing during the 8-year investment period performed best as before. The performance of the strategy increases as the rebalancing frequency decreases. Based on the same rebalancing frequency, the performance of the weightings calculated based on the equal-to-year-end method is better than that of the linear interpolation method. The weekly rebalancing strategy can neither effectively hedge against downside risks nor seek upward returns.

In summary, transaction costs have a greater impact on the B&H strategy. The less frequently the portfolio is rebalanced, the B&H strategy can obtain a better risk-return profile.

Figure 4.12 – Omega ratio as a function of the threshold: benchmarks with different weightings under different rebalancing frequencies, without transaction costs, drawing of complete (upper) and partial enlargement (bottom).

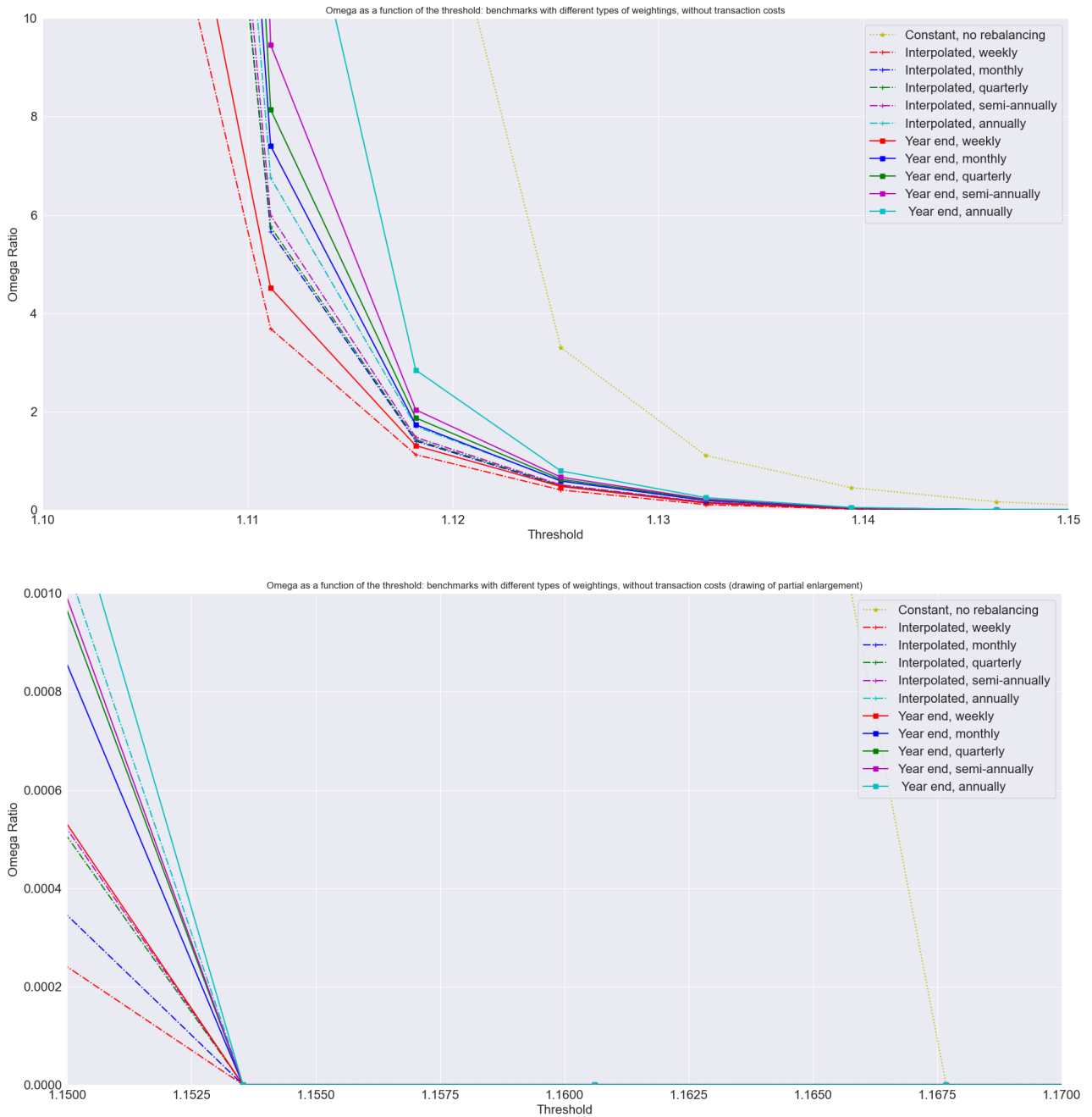
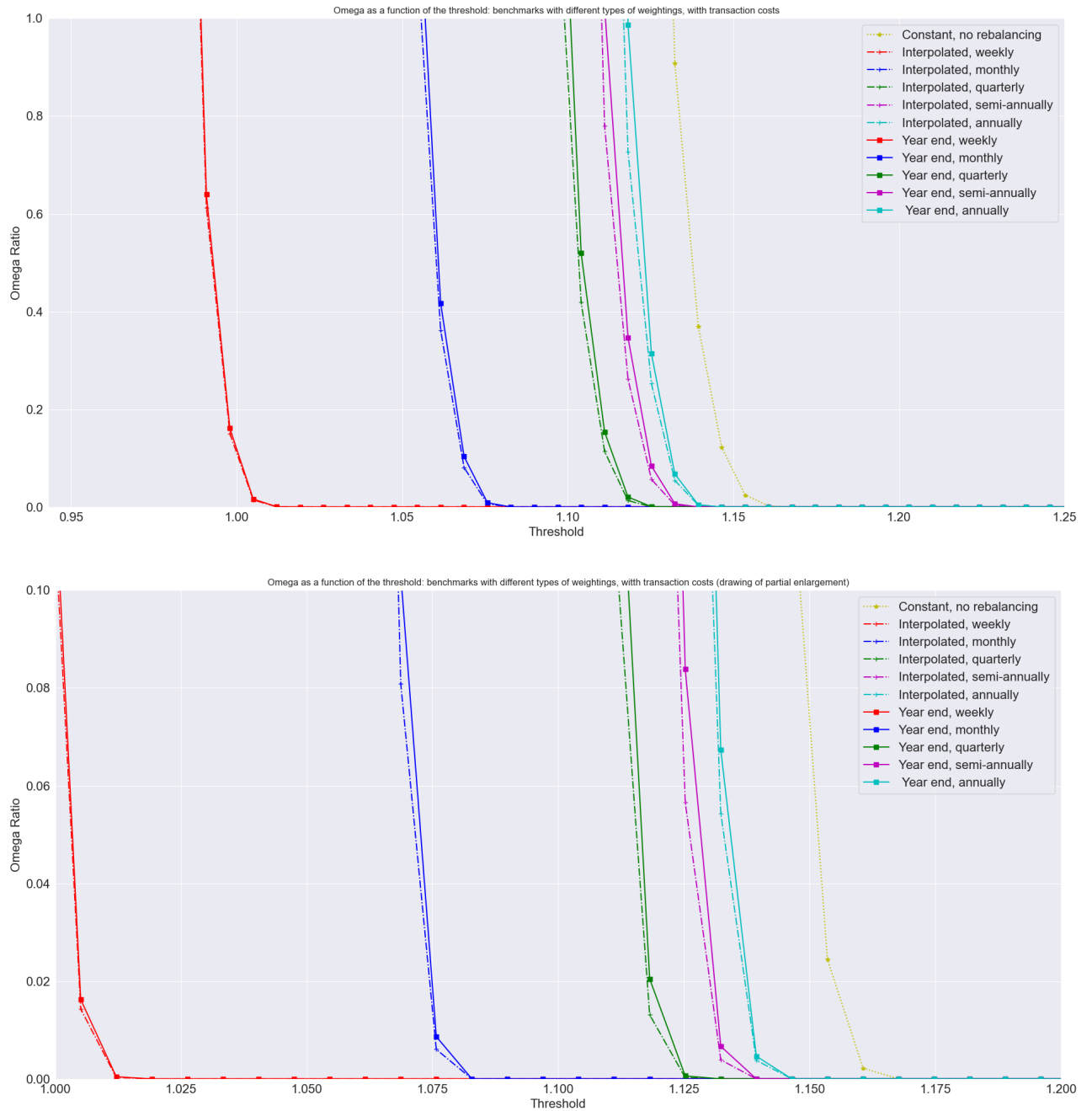


Figure 4.13 – Omega ratio as a function of the threshold: benchmarks with different weightings under different rebalancing frequencies, with transaction costs, drawing of complete (upper) and partial enlargement (bottom).



4.5.3 On the upper bounds of the investment weightings, $\gamma_{max}^{(i)}$

As introduced in Section 4.3.5, the upper bounds on the weightings of each asset in the investment are following the French Insurance Code, which has practical meaning when analyzing the effectiveness of our strategies. However, from the perspective of modelization, different upper bounds on the weightings will cause our optimal asset allocation to be biased.

We assume that the upper bounds on the weightings of each asset are set to 100%. The difference from the regulatory requirements is that the upper bounds on the weightings invested in the stock index and the REITs are 65% and 40%, respectively. We assume that other parameters remain unchanged, as in Section 4. The performance of our strategies is examined based on each of the following two different upper bounds on the weightings (as shown in Table 4.11) by quarterly rebalancing discipline with and without transaction costs.

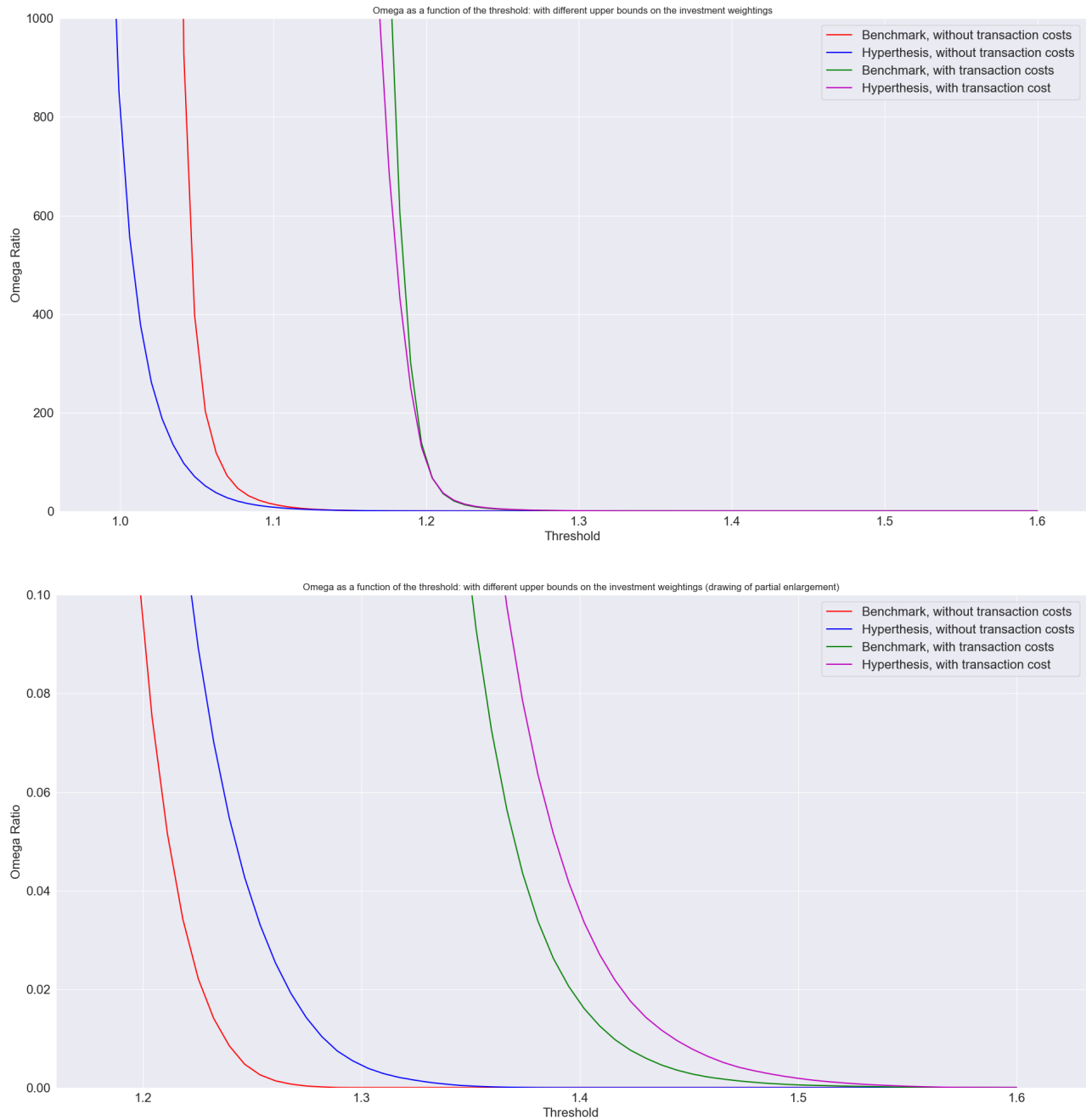
Table 4.11 – Upper bounds on the investment weightings, by the French Insurance Code and hypothesis.

	RF	GB	CB	SI	RE
By the French Insurance Code (Benchmark)	100%	100 %	100%	65%	40%
Hypothesis	100%	100 %	100%	100%	100%

For illustration purposes, we only display the Omega ratios of our strategies based the two upper bounds on the investment weightings for the performance measurement, as plotted in Figure 4.14.

When the threshold is low (as shown in the upper drawing), the strategy under the French insurance law’s upper bounds performs better than the strategy based on our hypothetical upper bounds. If there are no restrictions on the upper bounds on the investment weightings of the stock index and the REITs, our strategy will be less effective in hedging against downside risks. When the threshold is high (as shown in the bottom drawing), the strategy under the hypothetical upper bounds can obtain upside returns.

Figure 4.14 – Omega ratio as a function of the threshold: the quarterly rebalancing strategy under two upper bounds on the investment weightings, with transaction costs, drawing of complete (upper) and partial enlargement (bottom).



For more fluctuated risky assets with relatively low upper bounds on the investment weightings, the hedge comes at the cost of low average upward returns. Our results confirm

that it makes sense for the law to restrict the upper bounds on the investment weightings of risky assets for relatively risk-averse institutional investors such as life insurance companies.

4.5.4 On the performance of our strategies versus the real industry

Through the empirical results in Section 4.4 and the further discussions in this section, we have a comprehensive understanding of our strategy. From a practical point of view, it is worth further comparing our strategy with the real-world results of euro-denominated funds. We know that the comparisons will be based on the fact that there are only 5 investable financial assets under our strategy. These five investable assets are included in the asset pools of life insurance companies. In addition, our strategy cannot consider the historical holdings in long-term bonds prior to our dataset (2003), when long-term bond yields were higher. Thus if we compare the performance of our strategy with real-world results, it will underestimate the performance of our strategy. Nevertheless, we still find that our strategy can outperform the industry's results in the following aspects.

As analyzed before, the participation rate¹⁰ of euro-denominated funds is determined by life insurers at the end of each year and is under the capital guarantee. When deciding this interest rate, life insurance companies perfectly know the annual financial performance and the average 10-year government bond yield over the year. This interest rate should be lower than their investment rate of return of the current year. Since at least one part of the return is endowed in the smoothing reserve (PPB, see Section 1.3.1), distributed to the policyholders within 8 years.

The cumulated HPY of our strategies with quarterly and annually rebalancing (28.26% and 29.46%) can still exceed the cumulative interest rate provided by the life insurers at the industry level (28.10%) as shown in Table 4.12, even if the participation rate of euro-denominated funds includes some of the profits in the PPB for the past eight years that are not considered in our strategy. The average yield of the French 10-year government bonds during that period (1995 - 2002) is around 5.4%¹¹, as shown in Figure 1.2 in *Chapter 1*.

10. It is the interest rate that policyholders receive each year, also known as the revaluation rate

11. An equivalent 8-year cumulative return of investing in 10-year government bonds will be around 52%

Table 4.12 – Average cumulated HPY of an 8-year investment period of the B&H strategy, our benchmarks, our strategies, and the real industry data of euro-denominated funds, with transaction costs, from 2003 to 2019.

B&H RF	B&H GB	B&H CB	B&H SI	B&H RE
5.54%	11.64%	20.26%	1.33%	5.69%
Benchmark GVfM	Benchmark FFA			
13.93%	11.11%			
		<u>Our proposed strategy</u>		
Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing
9.29%	25.79%	28.26%	27.41%	29.49%
Industry				
28.10%				

The 10-year government bond yields decreased to below 3% since 2011. When we only compare the performance after 2011, as seen in Table 4.13, in addition to our weekly rebalancing strategy, the average 8-year cumulative interest rate is higher than that between 2003 and 2019. The industry’s average 8-year cumulative interest rate dropped from 28.10% to 19.77%, mainly due to the decline in 10-year government bond yields.

Table 4.13 – Average cumulated HPY of an 8-year investment period of the B&H strategy, our benchmarks, our strategies, and the real industry data of euro-denominated funds, with transaction costs, from 2011 to 2019.

B&H RF	B&H GB	B&H CB	B&H SI	B&H RE
0.84%	5.62%	18.70%	15.91%	16.02%
Benchmark GVfM	Benchmark FFA			
11.98%	11.14%			
		<u>Our proposed strategy</u>		
Weekly rebalancing	Monthly rebalancing	Quarterly rebalancing	Semi-annually rebalancing	Annually rebalancing
8.35%	35.98%	36.72%	32.18%	31.20%
Industry				
19.77%				
Industry’s 8-year cumulated investment rate of return				
27.17%				

Moreover, as presented in Figure 1.18 in *Chapter 1*, the average annual investment rate of return of euro-denominated funds is about 3.05% between 2011 and 2019; that is, the 8-year cumulative investment rate of return is 27.17%. During this period, the 8-year

cumulative investment rate of return of our strategy, except for the weekly rebalancing one, is more than 50% higher than that of the industry.

This shows that in the period of 2011 to 2019, our strategy is not only more secure in protecting the annual distribution rate to policyholders but also is more capable of obtaining upward returns than the industry.

4.6 Conclusion

In this chapter, we empirically compared our asset allocation strategies with the constructed benchmarks and the B&H strategy to investigate its effectiveness in terms of portfolio protection and performance in an eight-year investment period.

First of all, we chose the type of custom security-based (strategy) benchmarks to build the benchmarks of our asset allocation strategies.

Based on the empirical results, we firstly found that both our strategies and the benchmarks are capable of hedging against the downside risk in the sense that they truncate the left tail of the return distributions compared with the B&H strategy.

Next, based on the analysis of the B&H strategy, asset allocation is essential in long-term investment. The corporate bond index can provide excellent returns and lower risks in the long term, with the highest Sharpe ratio. The B&H strategy that only invested in the stock index and the REITs could neither provide downside protection nor achieve upside returns during the eight-year holding period in our datasets.

Thirdly, compared with the B&H strategy, our strategies and the benchmarks could hedge against downside risks because they truncated the left tail of the return distribution. However, in the absence of transaction costs, the ability to obtain upward returns was not as good as the B&H strategy that only invested in the corporate bond index. This is because some of the upward returns are sacrificed in order to diversify risks through asset allocation. The two benchmarks we assumed had different investment weightings because of their distinct constituents. The performance of the B&H strategy based on the GVM weightings was better than that based on the weightings obtained from the FFA. The main reason was due to the relative risk-averse and the pursuit of safety by euro-denominated life insurance. We also found that the presence of transaction costs has an insignificant effect on the performance of the B&H strategies.

We examined the impact of different rebalancing disciplines on the performance of our strategies in comprehensive scenarios. When comparing our asset allocation strategies,

we found that the performance of the quarterly rebalancing strategy ranked leading in the absence of transaction costs, followed by the monthly and semi-annual rebalancing strategies, weekly and annual rebalancing strategies were relatively poorer. In the presence of transaction costs, our strategies with relatively lower rebalancing frequencies (including quarterly, semi-annual, and annual rebalancing) performed better than those with higher rebalancing frequencies.

Unexpectedly, by calculating the maximum drawdown of different strategies, we found that when there are transaction costs, the semi-annual or annual rebalancing strategy can maintain and increase the value of the life insurance company with no risk among each of the 8-year investment periods in our datasets. The maximum drawdown of the quarterly rebalancing strategy was only -0.35%. From this perspective, our strategy is suitable for euro-denominated funds and euro-growth funds whose capital is guaranteed with a minimum holding period of eight years.

In further discussions, we verified the robustness of our strategy by assigning different values to the parameters in the inequality constraints, α and c , and evaluating the performance of the quarterly rebalancing strategy. We also found that when other variables remain unchanged, increasing the value of alpha or c will result in a diminishing marginal increase in HPY. Two methods to calculate the benchmark weightings also verified the validity of the assumption of the constant benchmark weightings. The lower the rebalancing frequency of the B&H strategy, the better the performance. In addition, by comparing the performance of the quarterly rebalancing strategies based on different upper bounds on the investment weightings, we confirmed that for euro-denominated life insurance, which is built up for the more risk-averse investors, a relatively low upper bounds on the weightings invested in risky assets could provide a higher downside hedge, as the cost of some upward returns. Moreover, We can also conclude that when there are transaction costs, the performance of our strategy at lower rebalancing frequencies is better than the real-world results of the life insurance industry in the same period.

GENERAL CONCLUSION

In Europe, a milestone was reached when the long-term yield for the ten-year benchmark government bond dipped to negative for the first time in 2019 since the peak in the 1980s. Euro-denominated life insurance business is vulnerable to changes in long-term interest rates because those interest rates act as the technical interest rate to determine insurance premiums, reserves, guaranteed rates, and profit-sharing [Holsboer, 2000]. Moreover, life insurers used to assume that the entire portfolio could be invested in long-term bonds, that the returns were high enough to guarantee the policyholder a certain return on their contracts in a perfectly secure environment. However, the asset returns of life insurers could fall below the average guaranteed interest rate in prolonged low interest rates, thus putting policyholders' investments at risk.

When the yield of the long-term bonds is approaching the guaranteed rate, it becomes more difficult to serve the guaranteed returns made long ago when interest rates were high. It creates pressures on the profitability of life insurance investment activities. The net inflow into euro-denominated funds declines with the continuous increase in outflows (e.g., claims, benefits paid, and redemptions) and the gradual decrease in inflows (i.e., premiums), which from another viewpoint, also poses challenges to the life insurance business model.

As we have seen in *Chapter 1*, in a persistent low-interest-rate environment, the conditions under which the life insurance business operates are modified. To continue to offer a favorable return to the insured, life insurers should allocate more risky assets to their portfolio. But doing so, they would be exposed to not being able to guarantee the capital. This thesis has investigated portfolio optimization problems of life insurance companies in the context of the gradual decline in long-term interest rates. To solve this double

problem that obviously challenges the life insurance business model, we applied the ruin theory and risk models. Important developments in both financial economics and mathematics were made for the management of life insurance. However, many remain to do. The works undertaken in this thesis made a modest contribution toward understanding and managing the portfolio optimizations of life insurance companies. In the mathematical modeling of the income of a life insurance company, the explicit expressions of the first two moments have been derived. The obtained formulae enable the analysis of portfolio problems and the optimal investment strategies for life insurance companies. To construct the optimal asset allocation strategy, we incorporated certain practical constraints, such as the probability of insolvency, reserves, and the maximum investment ratios by asset class. In the empirical analysis and back-testing, by applying our proposed strategy to euro-denominated funds and euro-growth funds of the life insurer's portfolio, we found that our proposed strategy has achieved outstanding performance in the long term under multiple conditions, for instance, capital guarantee, risk control, and profitability. Our main results are presented and summarized as follows.

First, we presented an economic analysis of French life insurance based on obtained dataset. We included a literature survey on a low-interest-rate environment and life insurance. A comprehensive overview of the situation of the life insurance sector in France was implemented, including investment supports and constraints on euro-denominated funds and other life insurance products; life insurance activities and investments; the solvency regulations imposed on life insurance companies and solvency conditions of the life insurance industry, as well as a comparison at the EU level. The diminishing rates of return on euro-denominated funds were analyzed by comparing with the technical interest rate, the profit-sharing rate, the rate of return on insurer assets, the inflation rate, and the provision for participation in profits. The conditions in the life insurance business and investment activities indicate that the traditional business model of life insurers is changing. Euro-denominated contracts are partly abandoned and are no longer intensively promoted by most financial intermediaries. Nevertheless, the large stock that is still invested in euro-denominated funds exposes life insurance companies to not being able to keep on offering attractive returns. Dissatisfied investors could then choose to withdraw their capital massively and invest in attractive alternatives. Unit-linked contracts receive more attention in terms of volume and growth rate, and euro-growth funds could take place for a major role as a long-term investment choice. However, other long-term savings products, particularly those dedicated to retirement (e.g., in French, *le Plan d'épargne*

retraite), could also attract household savings.

Those risks analyzed in the first chapter were the starting points of *Chapter 2*. We addressed the mathematical modeling of the income of a life insurance company by applying the ruin theory and risk models following [Paulsen, 2008]. Then we obtained the expressions on the first two moments of the income of a life insurance company in three case studies: 1) when both the basic risk process and the return on investment generating process are modeled by two independent Brownian Motions, 2) when the basic risk process is modeled by the sum of a Brownian Motion and a compound Poisson process, the return on investment generating process remaining the same as in *Case 1*, 3) when two processes are both modeled by Lévy processes. Similar formulae were derived when there are multiple risky assets in the first two cases. The relationships between each variable and the first two moments of the income of a life insurance company were displayed. We concluded that with a non-zero investment rate of return, the expected income of a life insurance company is an increasing function of the investment in risky assets, given the no-arbitrage condition and positive cash flows from its life insurance business. Moreover, the results showed that both the basic risk process and the return on investment generating process have complex impacts on the second moment of the income of a life insurance company.

Third, we derived the optimal asset allocation strategies under some practical constraints for the asset portfolio of life insurance companies. The dataset that we used comprises one risk-free asset and four risky asset classes that life insurance companies can choose to build up their portfolio, namely, one government bond, one corporate bond index, one stock index, and one real estate investment trust. Those four categories of risky assets show large differences in return and risk, which allows the portfolio to be diversified. We conducted data description, normality test and data analysis. Based on the formulas obtained in *Chapter 2*, we solved the maximization problem of the expected income of a life insurance company at the end of the predetermined investment period subject to the ruin probability. We obtained the optimal investment strategies using the upper bound of this probability and some practical constraints, including the maximum investment ratios by asset class, the probability of insolvency, and the saving capital. We numerically illustrated the optimal asset allocation strategy of *Case 2bis* with real data in 2019, in which the compound Poisson process models the outflows in the basic insurance business process to analyze the economic problems observed in *Chapter 1*. The numerical results showed that the current interest rate environment forced life insurers to change

their asset allocation from government bonds to more risky corporate bonds.

The sensitivity analysis investigated how the optimal asset allocations depend on the parameters in X_t and R_t , and the constraints. By varying life insurers' risk tolerance measures in the inequality constraint, such as the saving capital level or the probability of insolvency, we found that the holding period yield of life insurance contracts increases with insurers' stock and REIT investments. The investment strategy would be more conservative when the net inflows in the life insurance business increased without enhancing the saving capital or when the insurance business became more volatile. If the government bond yield becomes negative, the optimal investment strategy would allocate more cash and cash equivalents. Under different market conditions in the period of 2003 to 2018, given the level of calculation in 2019, the results showed a holding period yield of around 3% on average. Our investment strategy appeared to be more volatile than the traditional euro-denominated funds. Because the rate of return served to the policyholders was smoothed by life insurers through the revaluation rate, including the technical interest rates, profit-sharing, and the provision for participation in profits. The performance of our optimal investment strategy resembled one of the alternatives to the traditional euro-denominated funds, namely the dynamic euro-denominated funds.

Fourth, based on the five-asset dataset in *Chapter 3*, we backtested the performance of the proposed asset allocation strategy compared with the buy-and-hold (B&H) strategies and the constructed benchmarks in an eight-year investment period, without and with transaction costs. The empirical results show that in the presence of transaction costs, our strategy with lower rebalancing frequencies performs better in hedging against downside risks, capturing upside returns, reshaping return distributions, and realizing capital guarantees for both euro-denominated funds and euro-growth funds.

In the further discussions section in *Chapter 4*, we tested the robustness of the asset allocation strategy and the constructed benchmarks. When other parameters remain unchanged, with the increase of α or c , the interest rate of the contract for the corresponding holding period (HPY) will increase accordingly. However, due to the bounds on the investment weightings, the HPY showed diminishing marginal growth rates. The empirical results verified the sensitivity analysis in *Chapter 3*. By parameterizing α and c in our modelization, life insurance companies could select different values for this pair of parameters according to their own financial situation, which is easy to compute and implement in practice. Under different time frames, the performance of our strategy was superior to the real results of euro-denominated funds in the industry. Especially in the last decade,

when the government bond yield was lowering, our strategy has not only outperformed the revaluation rate of euro-denominated funds but also surpassed the industry's cumulative investment rate of return.

One of the main limitations in our results is the low data granularity and data availability in the life insurance business, which biased the value of parameters in the insurance business process and the performance of the optimal investment strategy. In addition, portfolio optimization depends on the number of assets that life insurers can invest in. A more comprehensive asset structure could improve the analysis and increase robustness to the results by adding the constraint of the dispersion rules. Moreover, as investment strategies rely on assumptions, the problem with our results is the model risk arising from the differences between the real world and the one assumed in our model. We focus on an analytically tractable case, where the returns are normal and the processes are continuous. Besides, as the implementation of the optimal investment strategy needs estimation for parameters in both the life insurance business and financial markets, such as the variance in the insurance business and the expected return and volatility of financial assets, it is also worth examining how the accuracy of these estimates may affect its effectiveness. These issues are left for future research.

Nevertheless, the results have profound economic implications. Given the importance of the life insurance sector among institutional investors, we show that the stochastic properties of both the life insurance business and the investment activity could greatly impact the economic results of life insurers. In terms of investment, by adjusting the level of saving capital and the probability of insolvency, parameters which reflect their attitude of risk tolerance, life insurers can find the optimal investment strategy that meets an authority's requirement. In terms of the life insurance business, on the one hand, life insurers should continue to limit the inflow into euro-denominated funds and encourage policyholders and potential investors to transfer and invest in unit-linked and euro-growth funds in the long term. On the other hand, they should hold sufficient cash or liquid assets to reduce their liquidity risk and insolvency risk in the context of low-interest rates, rising outflows, and unpredictable redemptions.

The ongoing recession is challenging the profitability of the corporate sector resulting in rating downgrades, increased defaults, and unemployment, especially in the Covid-19 pandemic. In addition, the risk of deterioration of corporates' ratings could affect the market value of insurers' corporate bond holdings [EIOPA, 2020b]. It is worth noting that the levels of government and public debts are increasing dramatically with the activation

of the Public Sector Purchase Programme (PSPP) by the European Central Bank, which indicates there is no need to increase interest rates in the short term. However, it would be essential to consider the case in the medium-long term when interest rates in the euro area start to rise, which would bring new challenges and pressures to the life insurance business.

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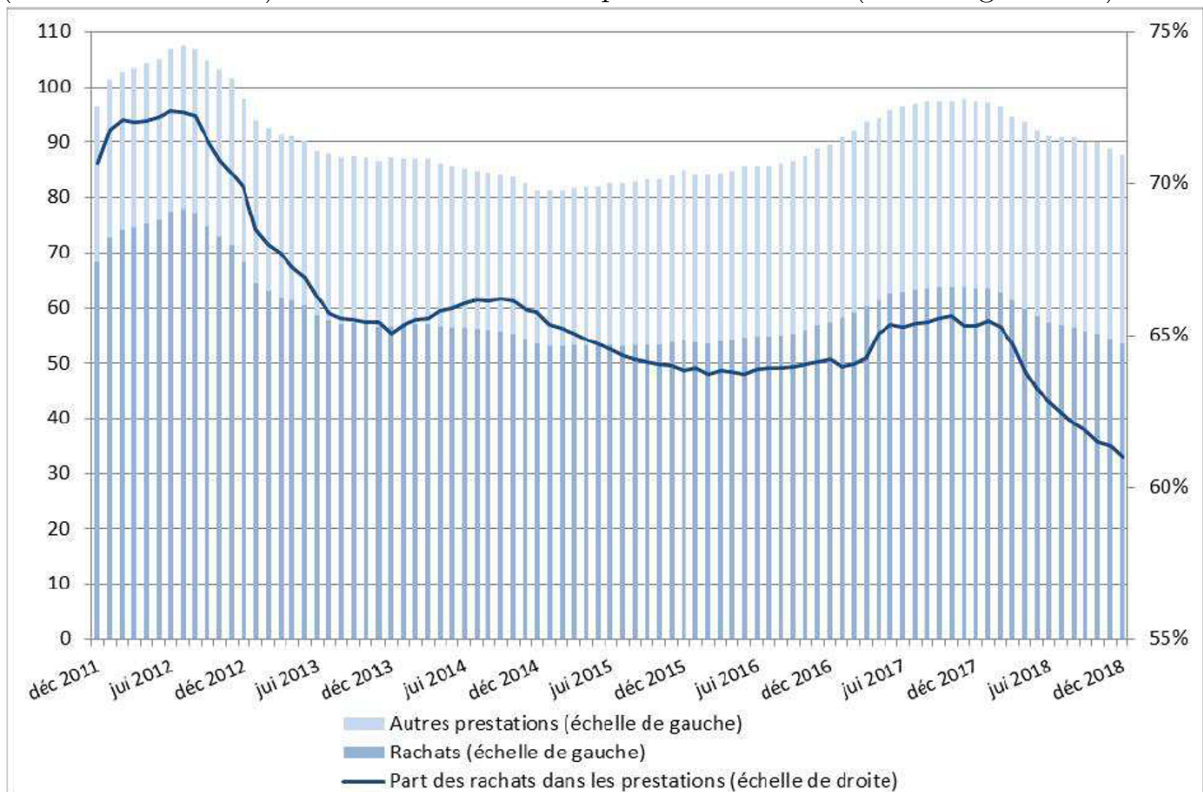
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APPENDICES

.1 Appendix A

Figure 15 – Evolution of redemptions and other outflows from euro-denominated contracts (in billions of euros) and the share of redemptions in outflows (12 rolling months).



Source: [Perdu and Marionnet, 2019], ACPR

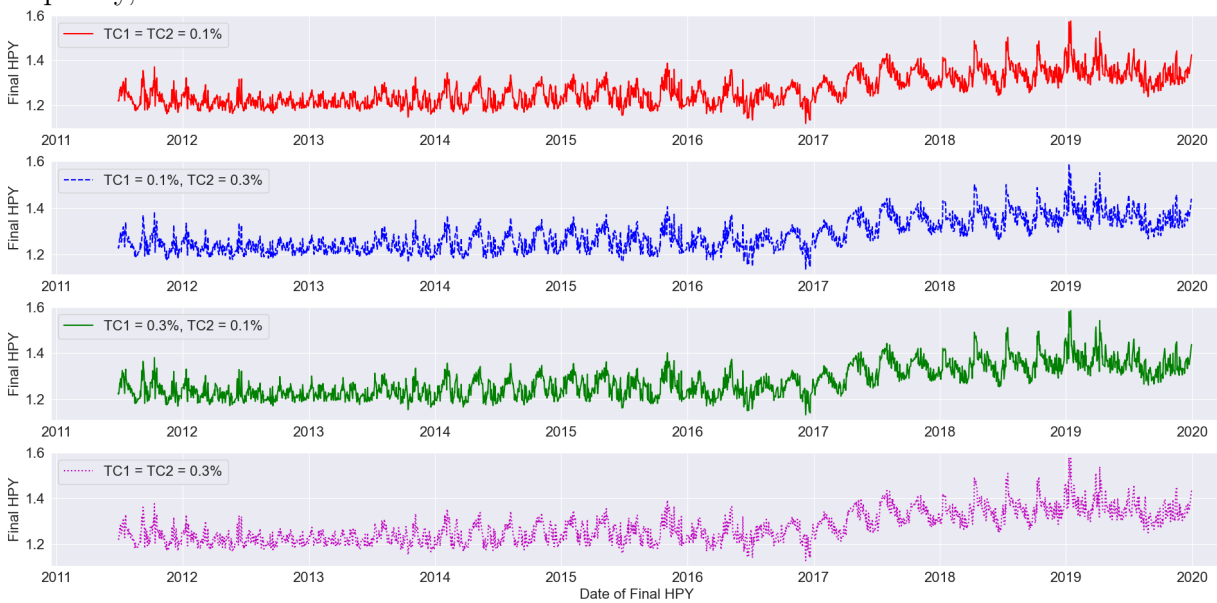
.2 Appendix B

In Table 14, we propose four combinations of transaction costs for the government bond, corporate bond index, stock index, and REITs to compare the impact of transaction costs on the performance of our strategy.

Table 14 – Four different combinations of transaction costs (TC) of four risky assets.

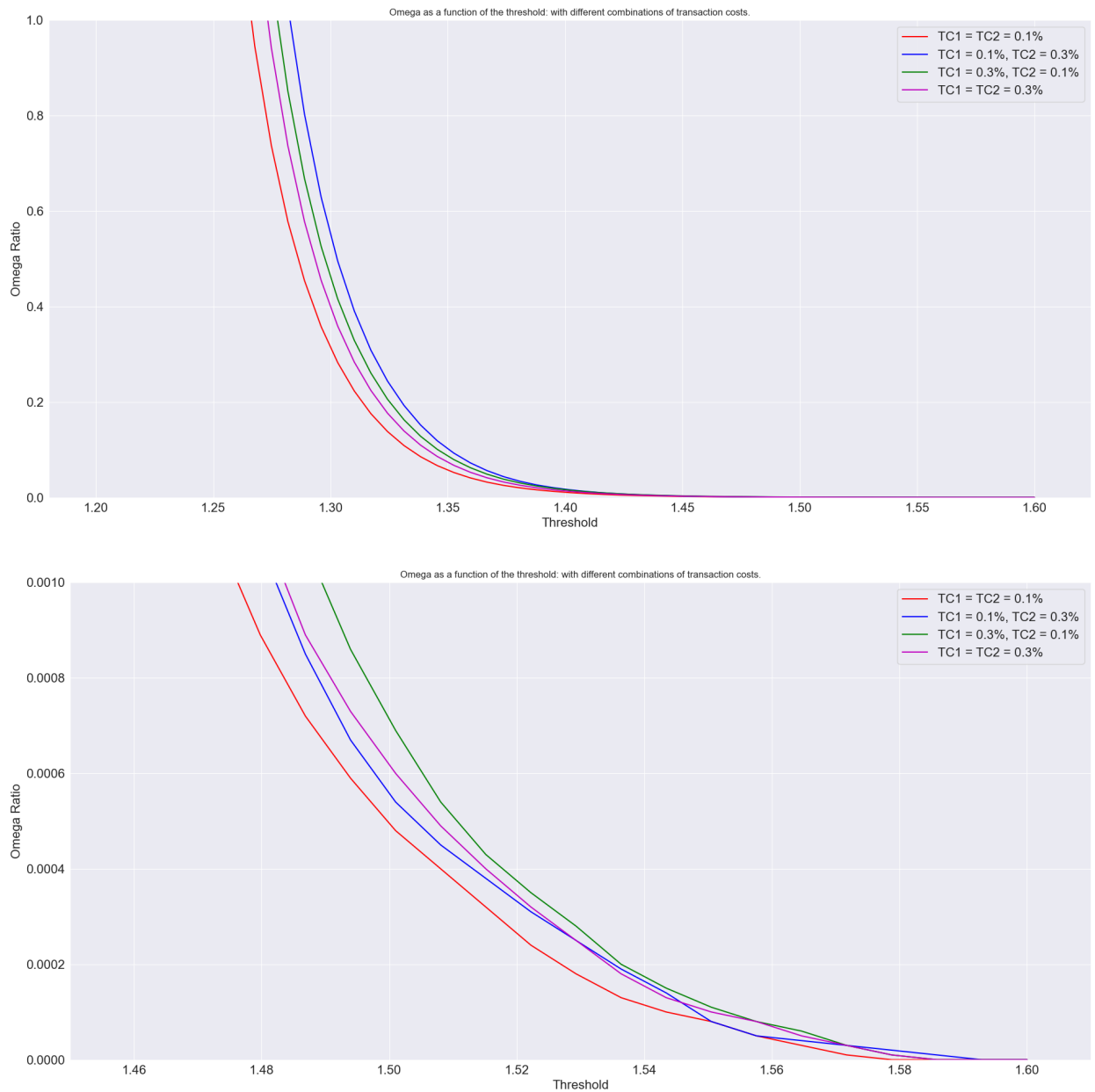
TC Combinations	Government Bond (TC1)	Corporate Bond (TC1)	Stock Index (TC2)	REITs (TC2)
TC1=TC2=0.1%	0.1%	0.1%	0.1%	0.1%
TC1=0.1%, TC2=0.3%	0.1%	0.1%	0.3%	0.3%
TC1=0.3%, TC2=0.1%	0.3%	0.3%	0.1%	0.1%
TC1=TC2=0.3%	0.3%	0.3%	0.3%	0.3%

Figure 16 – Realizations of our asset allocation strategy under quarterly rebalancing frequency, with different combinations of transaction costs.



Note: This figure plots the realizations of the final HPYs under our asset allocation strategy by quarterly rebalancing frequency for the cases with four combinations of transaction costs.

Figure 17 – Omega ratio as a function of the threshold: the quarterly rebalancing strategy under different combinations of transaction costs, drawing of complete (upper) and partial enlargement (bottom).



As shown in Figure 16 and Figure 17, the impact of different transaction costs on the performance of our proposed strategy is not evident in the presence of transaction costs. For illustration purposes, only the quarterly rebalancing strategy is displayed.

.3 Appendix C

Figure 18 – 2273 paths of the HPY by the B&H strategy with the GVfM weightings with transaction costs in 8 years.

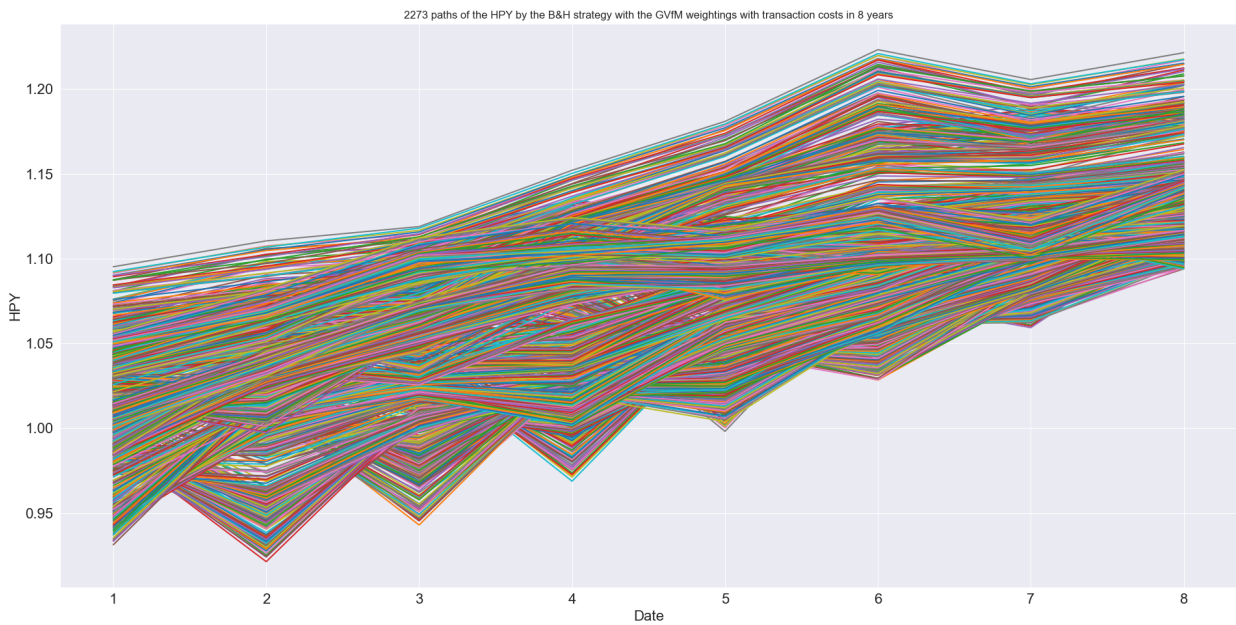
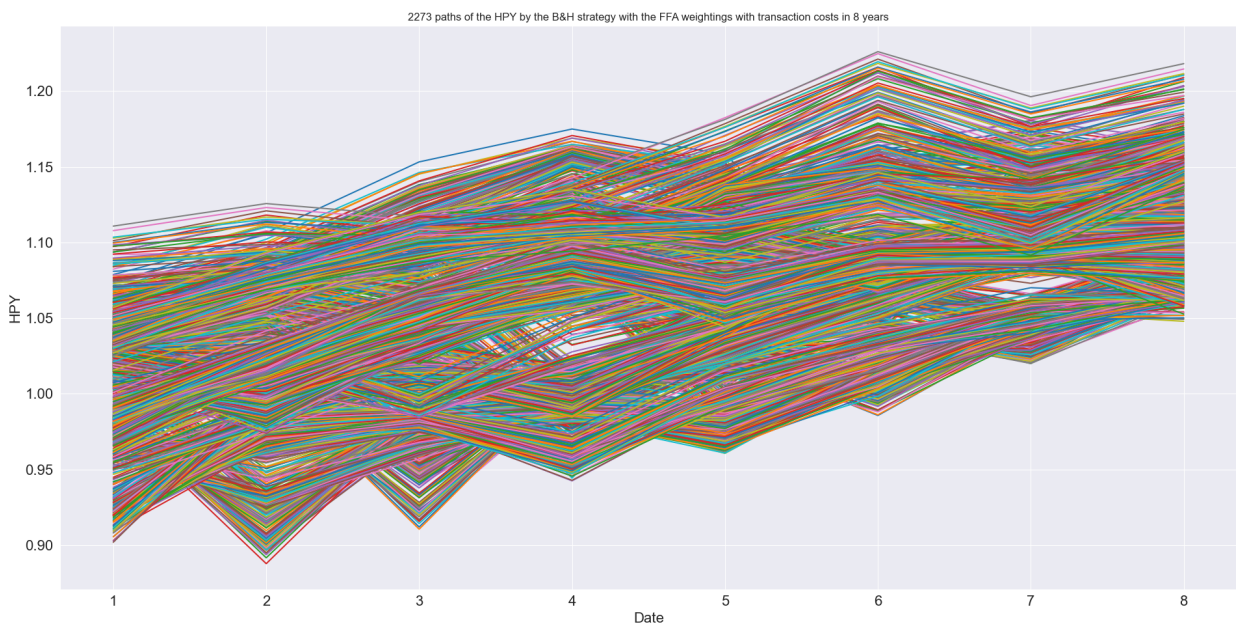


Figure 19 – 2273 paths of the HPY by the B&H strategy with the FFA weightings with transaction costs in 8 years.



.4 Appendix D

Figure 20 – 2279 paths of the HPY by weekly rebalancing strategy with transaction costs in 8 years (400 weeks).

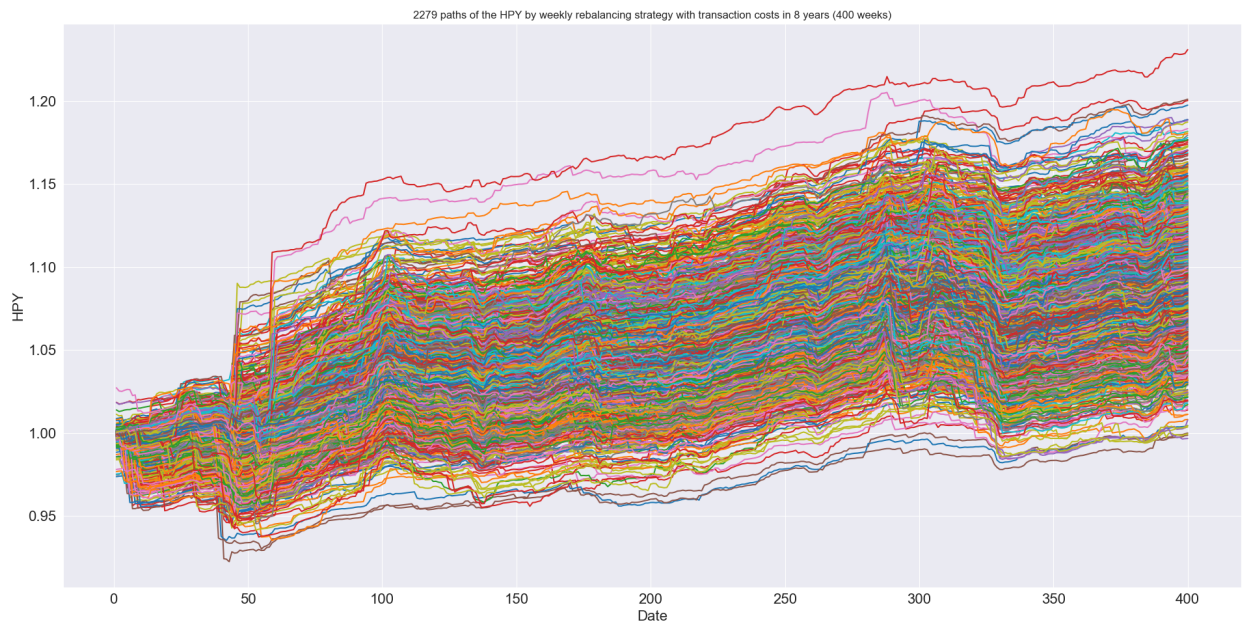


Figure 21 – 2231 paths of the HPY by monthly rebalancing strategy with transaction costs in 8 years (96 months).

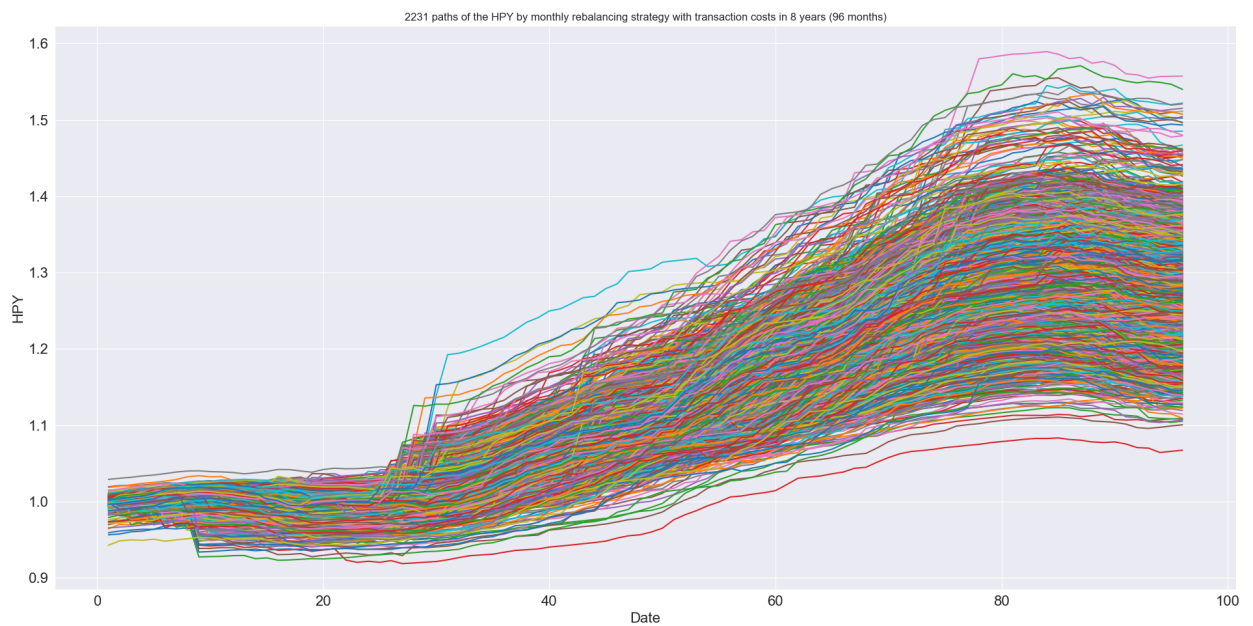


Figure 22 – 2021 paths of the HPY by semi-annually rebalancing strategy with transaction costs in 8 years (16 semi-years).

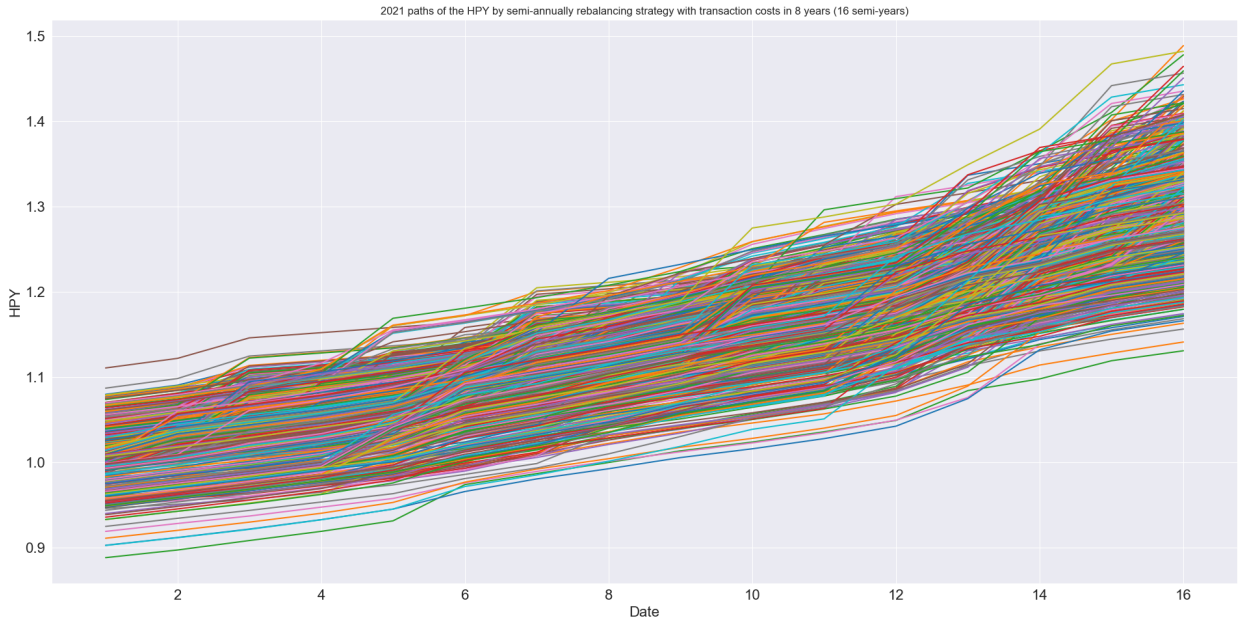
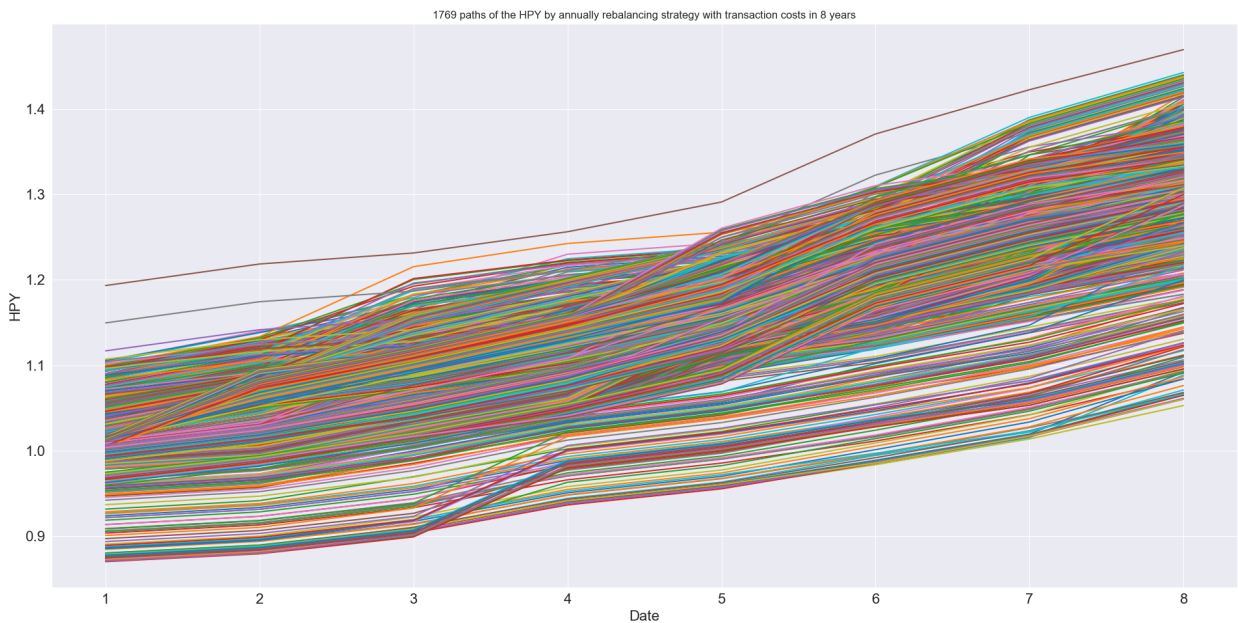


Figure 23 – 1769 paths of the HPY by annually rebalancing strategy with transaction costs in 8 years.



RÉSUMÉ

Les assureurs-vie sont des intermédiaires financiers importants en raison de leur capacité à réaffecter les risques dans l'économie et de leurs horizons d'investissement souvent à long terme. Dans la plupart des pays européens, l'assurance-vie bénéficie de ses caractéristiques en termes de garantie de capital, d'incitations fiscales, d'horizon d'investissement à long terme, d'effet de diversification et de retour sur investissement. Dans les bonnes années, les assureurs accumulent des excédents en offrant aux épargnants des rendements inférieurs à ceux de leur portefeuille, leur permettant de puiser dans cette réserve pendant les mauvaises années pour offrir des rendements plus élevés. Réduisant ainsi le risque de manque à gagner des assureurs-vie tout en prévoyant des taux d'intérêt garantis plus élevés.

Environ 40 % de leur valeur nette est investie dans l'assurance-vie et les fonds de pension par les ménages européens au cours d'une année moyenne de 2016 à 2018 [ECB, 2019]. Selon la Banque de France, la valeur des contrats d'assurance-vie en euros détenus par les ménages français représente environ 32 % de leur patrimoine financier à fin septembre 2020. Parallèlement, la part correspondante des contrats en unités de compte est d'environ 7 %.

La politique monétaire accommodante actuellement menée par la Banque centrale européenne se traduit par des taux d'intérêt extraordinairement bas. Les assureurs-vie allouent généralement une large proportion d'obligations souveraines et d'obligations d'entreprises dans leur portefeuille; par conséquent, une baisse généralisée des taux d'intérêt affecte directement le taux de rendement de ces portefeuilles. Des rendements inférieurs à long terme incitent les assureurs-vie à rechercher des investissements plus risqués et plus illiquides pour obtenir le rendement visé. Cette prise de risque accrue peut conduire à une nouvelle accumulation de vulnérabilités parmi eux [IMF, 2019]. Les faibles taux d'intérêt deviennent une menace pour la stabilité du secteur de l'assurance-vie, en particulier dans les pays où les produits à rendements garantis relativement élevés vendus dans le passé représentent encore une part importante du passif [Berdin and Gründl, 2015].

Dans un contexte de baisse progressive du rendement des contrats en euros liée à la baisse des taux d'intérêt à long terme, l'excellente tenue des marchés boursiers développés

a permis aux produits en unités de compte (qui représentent 27 % de l'encours en 2019) se présenter comme une alternative attractive permettant aux investisseurs de bénéficier au moins en partie de la hausse des cours boursiers tout en restant dans le cadre d'une fiscalité attractive. La collecte nette sur les contrats d'assurance-vie en unités de compte a notamment augmenté au cours des deux dernières décennies. Dans le même temps, la collecte nette sur les contrats d'assurance-vie en euros a diminué non seulement en raison de la baisse des taux de rendement mais aussi en raison de l'évolution de la pyramide des âges en France. D'une part, les sorties constituées de sinistres payés, de prestations et de rachats continuent d'augmenter de manière significative. En revanche, les encaissements principalement sur primes affichent une tendance à la baisse depuis plusieurs années. Les compagnies d'assurance-vie ont tenté de tirer la collecte sur les contrats en unités de compte afin de limiter la collecte sur les contrats en euros et de réduire leurs achats d'obligations qui offraient un rendement très faible.

Pour continuer à offrir des rendements favorables à leurs clients tout en garantissant leur capital investi, les compagnies d'assurance-vie ont décidé de modifier progressivement la structure de leurs actifs en fonction du modèle économique émergent de l'assurance-vie en euros. Cependant, en se tournant vers des actifs plus risqués, potentiellement porteurs d'une prime de risque positive, ils sont moins susceptibles de garantir le capital (même s'ils conservent un « coussin » de liquidité comme réserve de trésorerie). Tous ces facteurs font désormais peser un risque important sur le modèle d'assurance-vie dans sa version euro.

Cette thèse étudie les problèmes d'optimisation de portefeuille des compagnies d'assurance-vie dans le contexte de la baisse progressive des taux d'intérêt à long terme. Pour résoudre ce double problème qui remet évidemment en cause le modèle économique de l'assurance-vie, nous appliquons la théorie de la ruine et les modèles de risque. Des développements importants à la fois en économie financière et en mathématiques ont été réalisés pour la gestion de l'assurance-vie. Cependant, beaucoup restent à faire. Les travaux menés dans cette thèse apportent une modeste contribution à la compréhension et à la gestion des optimisations de portefeuille des compagnies d'assurance-vie. Pour la modélisation mathématique du revenu d'une compagnie d'assurance-vie, les formules des deux premiers moments sont dérivées. Les formules obtenues permettent d'analyser les problèmes de portefeuille et les stratégies d'investissement optimales pour les compagnies d'assurance-vie. Nous intégrons certaines contraintes pratiques, telles que la probabilité d'insolvabilité, les réserves, et les ratios d'investissement maximum par classe d'actifs pour trouver l'allocation d'actifs op-

timale.

Le premier chapitre est une introduction à l'assurance-vie libellée en euros. Il présente une étude bibliographique sur les conséquences de taux d'intérêt bas et une analyse contextuelle de l'environnement économique des sociétés d'assurance-vie françaises. Il met en lumière et compare les situations du secteur de l'assurance-vie en France et en Europe. Le cadre prudentiel Solvabilité II est discuté et analysé. Le chapitre examine la baisse des rendements des contrats d'assurance-vie libellés en euros et le changement de paradigme dans l'activité des compagnies d'assurance-vie confrontées à des sorties potentielles importantes. Les conditions dans les affaires d'assurance-vie et les activités d'investissement indiquent que le modèle d'affaires traditionnel des assureurs-vie est en train de changer. Les contrats libellés en euros sont en partie abandonnés et ne sont plus intensément promus par la plupart des intermédiaires financiers. Néanmoins, le stock important qui est encore investi dans des fonds en euros expose les compagnies d'assurance-vie à ne pas pouvoir continuer à offrir des rendements attractifs. Les investisseurs insatisfaits pourraient alors choisir de retirer massivement leur capital et d'investir dans des alternatives intéressantes. Les contrats en unités de compte reçoivent plus d'attention en termes de volume et de taux de croissance, et les fonds euro-croissance pourraient jouer un rôle majeur en tant que choix d'investissement à long terme. Cependant, d'autres produits d'épargne longue, notamment ceux dédiés à la retraite (le Plan d'Épargne Retraite), pourraient également attirer l'épargne des ménages.

Le deuxième chapitre est consacré à la modélisation mathématique des revenus d'une compagnie d'assurance-vie. Il fournit des outils indispensables pour saisir les expositions des résultats du chapitre suivant. Il montre les éléments du calcul stochastique qui sont nécessaires dans le calcul de la modélisation mathématique. Il formule la forme explicite des deux premiers moments du revenu d'une compagnie d'assurance-vie avec trois études de cas où le processus de risque de base et le processus générateur de retour sur investissement sont modélisés par le Brownian Motion avec la dérive, somme d'un Brownian Motion et d'un processus de Poisson composé et processus de Lévy, respectivement. En calculant les dérivées partielles du premier ordre, on obtient les relations entre chaque variable et les deux premiers moments du revenu d'une compagnie d'assurance-vie. Les relations entre chaque variable et les deux premiers moments du revenu d'une compagnie d'assurance-vie sont affichées. Nous concluons qu'avec un taux de rendement d'investissement non nul, le revenu attendu d'une compagnie d'assurance-vie est une fonction croissante de l'investissement en actifs risqués, compte tenu de la condition de non-arbitrage et des flux

de trésorerie positifs de son activité d'assurance-vie. De plus, les résultats montrent qu'à la fois le processus de risque de base et le processus de génération de retour sur investissement ont des impacts complexes sur le deuxième moment du revenu d'une compagnie d'assurance-vie.

Le troisième chapitre est consacré à la recherche de l'allocation d'actifs optimale du portefeuille d'une compagnie d'assurance-vie. Un ensemble de données composé d'un actif sans risque et de quatre catégories d'actifs risqués est affiché. Nous construisons la stratégie d'investissement optimale sur la base des formules obtenues au *Chapitre 2*. Le problème d'optimisation est résolu numériquement avec des données réelles en incorporant certaines contraintes pratiques, telles que la probabilité d'insolvabilité, les réserves et les ratios d'investissement maximum par classe d'actifs. Nous étudions comment l'allocation d'actifs optimale dépend d'une série de paramètres à travers une analyse de sensibilité. Certaines implications tant pour les assureurs-vie que pour la réglementation sont proposées avec des résultats chiffrés. L'analyse de sensibilité étudie comment les allocations d'actifs optimales dépendent des paramètres de X_t et R_t , et des contraintes. En faisant varier les mesures de tolérance au risque des assureurs-vie dans la contrainte d'inégalité, telles que le niveau de capital d'épargne ou la probabilité d'insolvabilité, nous constatons que le rendement de la période de détention des contrats d'assurance-vie augmente avec le stock des assureurs et les investissements en REIT. La stratégie d'investissement serait plus prudente lorsque la collecte nette dans les affaires d'assurance vie augmentait sans augmenter le capital d'épargne ou lorsque les affaires d'assurance devenaient plus volatiles. Si le rendement des obligations d'État devient négatif, la stratégie d'investissement optimale consisterait à allouer plus de liquidités et d'équivalents de liquidités.

Le dernier chapitre se concentre sur l'évaluation de la performance de notre stratégie d'allocation d'actifs proposée par rapport aux stratégies d'achat et de conservation (B&H) ainsi qu'aux benchmarks dans une période d'investissement de huit ans.. Il montre comment sont construits les repères qui reflètent nos stratégies. La conception de backtesting de l'analyse empirique est présentée, y compris les disciplines de rééquilibrage du portefeuille, les mesures d'évaluation des performances, les coûts de transaction et les procédures de backtesting. L'efficacité des stratégies est examinée en détail. Les résultats empiriques montrent qu'en présence de coûts de transaction, notre stratégie avec des fréquences de rééquilibrage plus faibles est plus performante pour se couvrir contre les risques de baisse, capturer des rendements à la hausse, remodeler les distributions de rendement et réaliser des garanties de capital pour les fonds libellés en euros et les fonds de

croissance en euros. Dans la section des discussions plus approfondies du *Chapitre 4*, nous testons la robustesse de la stratégie d'allocation d'actifs et des indices de référence construits. Lorsque les autres paramètres restent inchangés, avec l'augmentation de α ou c , le HPY augmentera en conséquence. Cependant, en raison des limites sur les pondérations d'investissement, le HPY montre des taux de croissance marginaux décroissants. Les résultats empiriques vérifient l'analyse de sensibilité du *Chapitre 3*. En paramétrant α et c dans notre modélisation, les compagnies d'assurance-vie pourraient sélectionner différentes valeurs pour ce couple de paramètres en fonction de leur propre situation financière, ce qui est facile à calculer et mettre en œuvre dans la pratique. Sous différentes échéances, la performance de notre stratégie est meilleure que les résultats réels des fonds libellés en euros de l'industrie. Surtout au cours de la dernière décennie, lorsque le rendement des obligations d'État baissait, notre stratégie a non seulement surperformé le taux de réévaluation des fonds libellés en euros, mais a également dépassé le taux de rendement cumulé des investissements du secteur.

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Titre : Optimisation du portefeuille d'actifs des compagnies d'assurance vie sous contrainte de garantie en capital

Mot cl s : Assurance-vie, optimisation du portefeuille, fonds en euros, th orie de la ruine

R sum  : La prosp rit  d'une compagnie d'assurance-vie est due non seulement aux b n fices de son activit  principale, mais aussi aux revenus d'investissement du capital   sa disposition. L'environnement prolong  de taux bas a remis en cause ces deux principales activit s de l'assurance-vie. Dans cette th se, nous  tudions les probl mes d'optimisation de portefeuille li s aux fonds en euros des assureurs-vie. Pour continuer   offrir une r mun ration attractive aux assur s, les assureurs vie devraient introduire davantage d'actifs risqu s dans leurs portefeuilles. Mais, ce faisant, ils s'exposeraient   ne pas pouvoir garantir le capital. Par ailleurs, la maturit  du march  fran ais de l'assurance-vie cr e des conditions potentielles de retraits massifs. Nous appliquons la

th orie de la ruine pour mod liser les propri t s stochastiques de ces deux processus dans trois cas et d rivons les expressions explicites des deux premiers moments du revenu d'une compagnie d'assurance-vie. La maximisation du revenu attendu d'une compagnie d'assurance-vie est r solue sous les contraintes de la probabilit  de ruine. La strat gie d'investissement optimale est obtenue en utilisant l'estimation sup rieure de cette probabilit . Nous avons r alis  l'illustration num rique, l'analyse de sensibilit  et le backtesting avec des donn es r elles du secteur de l'assurance-vie et des march s financiers. Nos r sultats indiquent des implications  conomiques pour les compagnies d'assurance-vie dans leurs activit s d'assurance et d'investissement.

Title: Optimization of the asset portfolio of life insurance companies under capital guarantee constraints

Keywords: Life insurance, portfolio optimization, euro-denominated funds, ruin theory

Abstract: The prosperity of a life insurance company is due not only to earnings in its principal business but also to investment profits of the capital at its disposal. The prolonged low-interest-rate environment has been challenging these two main activities of life insurance. In this dissertation, we study portfolio optimization problems related to euro-denominated funds of life insurers. To continue to offer a favorable return to the policyholders, life insurers should allocate more risky assets to their portfolios. But, doing so, they would be exposed to not being able to guarantee the capital. Besides, the maturity of the French life insurance market creates potential conditions for massive withdrawals. We apply the ruin theory to model the

stochastic properties of these two processes in three case studies and derive the corresponding explicit expressions of the first two moments of the income of a life insurance company. The maximization of the expected income of a life insurance company is solved under the constraints of ruin probability. The optimal investment strategy is obtained using the upper estimation of this probability. We conducted the numerical illustration, the sensitivity analysis, and the backtesting with real data from the life insurance sector and financial markets. Our results indicate economic implications for life insurance companies in their insurance business and investment activities.