
# Thèse de Doctorat

## *Mention Mathématiques*

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# Objets rigides : de la combinatoire des catégories amassées supérieures à l'algèbre homotopique.

**Soutenue le 29 septembre 2017, après avis des rapporteurs, devant le jury d'examen :**

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# Table des matières

Dans cette thèse, nous décrivons une réalisation géométrique des carquois de type Dynkin, et certains carquois euclidiens. Nous traitons le cas $\tilde{D_n}$ en profondeur et démontrons quelques résultats complémentaires aux travaux de Baur, Marsh et Torkildsen sur les réalisations géométriques des catégories amassées supérieures. Pour le cas $\tilde{D_n}$, on trouve la figure qui correspond à l'étude, on démontre la compatibilité entre le flip d'une $(m+2)$-angulation, et la mutation de carquois coloré. On trouve une bijection entre les objets $m$-rigides et chaque arc dit admissible, puis entre les objets amas-basculants et les $(m+2)$-angulations. De plus, on démontre la compatibilité entre la réduction d'Iyama-Yoshino, et le fait de couper le long d'un arc, qu'on définira formellement.

Nous démontrons aussi qu'une catégorie exacte est une catégorie de préfibration au sens de Anderson-Brown-Cisinski, qui vérifie le théorème de Quillen, et une catégorie de Frobenius est munie d'une structure de modèle, compatible avec le passage à la catégorie stable, qui est triangulée.

We show that a subcategory of the $m$-cluster category of type $\tilde{D_n}$ is isomorphic to a category consisting of arcs in an $(n-2)m$-gon with two central $(m-1)$-gons inside of it. We show that the mutation of colored quivers and $m$-cluster-tilting objects is compatible with the flip of an $(m+2)$-angulation.

In this thesis, we study the geometric realizations of $m$-cluster categories of Dynkin types A, D, $\tilde{A}$ and $\tilde{D}$. We show, in those four cases, that there is a bijection between $(m+2)$-angulations and isoclasses of basic $m$-cluster tilting objects. Under these bijections, flips of $(m+2)$-angulations correspond to mutations of $m$-cluster tilting objects. Our strategy consists in showing that certain Iyama-Yoshino reductions of the $m$-cluster categories under consideration can be described in terms of cutting along an arc the corresponding geometric realizations. This allows to infer results from small cases to the general ones.

Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective and projective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective and projective objects, and stable under taking direct sums and summands. In this paper, $\mathcal{E}$ is equipped with the structure of a prefibration category with cofibrant replacements. As a corollary, we show, using the results of Demonet and Liu in [DL13], that the category of finite presentation modules on the costable category $\overline{\mathcal{M}}$ is a localization of $\mathcal{E}$. We also deduce that $\mathcal{E} \to \mathrm{mod}\overline{\mathcal{M}}$ admits a calculus of fractions up to homotopy. These two corollaries are analogues for exact categories of results of Buan and Marsh in [BM13], [BM12] (see also [Bel13]) that hold for triangulated categories.

If $\mathcal{E}$ is a Frobenius exact category, we enhance its structure of prefibration category to the structure of a model category (see the article of Palu in [**?**] for the case of triangulated categories). This last result applies in particular when $\mathcal{E}$ is any of the Hom-finite Frobenius categories appearing in relation to cluster algebras.

Chapitre 1

# Introduction

## 1.1 Vers les catégories amassées supérieures

Le concept d'algèbre amassée est né au début des années 2000, introduit par Fomin et Zelevinsky dans [FZ02], afin entre autres, de donner un cadre combinatoire pour étudier les bases canoniques et la positivité totale, deux sujets de théorie de Lie algébrique.

Il s'est avéré plus tard que les algèbres amassées répondaient à beaucoup d'autres questions, notamment en théorie des représentations des carquois et algèbres de dimensions finie. Une belle introduction des algèbres amassées est donnée par Keller dans [Kel11].

Une algèbre amassée est une $\mathbb{Q}$-algèbre commutative, munie de générateurs appelés variables d'amas, regroupés en amas, possédant toujours le même nombre de variables. Ces amas sont construits à partir de mutations successives d'un amas particulier appelé graine initiale, contenant un carquois à $n$ sommets et $n$ variables. Ainsi, l'ensemble des variables d'amas peut être fini ou infini. Fomin et Zelevinsky ont démontré dans [FZ03] que l'algèbre amassée contient un nombre fini de variables d'amas si et seulement si le carquois de la graine initiale est équivalent par mutation à un carquois de type Dynkin (où il est à noter que l'algèbre amassée est un invariant de la classe de mutation du carquois initial).

Afin de démontrer certains résultats liés aux algèbres amassées, il a été nécessaire de catégoriser cette notion. Dans le cas où $Q$ est un carquois acyclique (et $k$ un corps algébriquement clos), la catégorie amassée associée à $Q$ a été créée en 2006 par Buan, Marsh, Reineke, Reiten et Todorov (voir [BMR$^+$06] pour plus de

détails). Il s'agit de la catégorie d'orbite

$$\mathcal{D}^b(kQ)/\tau^{-1}[1]$$

où $\mathcal{D}^b(kQ)$ désigne la catégorie dérivée associée à l'algèbre $kQ$, $\tau$ est la translation d'Auslander-Reiten et [1] le shift. Les détails seront fournis dans les préliminaires. Le cas où $Q$ est de type $A_n$ fut traité en détail par Caldero, Chapoton et Schiffler dans [CCS06]. Le lien entre catégories amassées et algèbres amassées est le suivant : via l'application de Caldero-Chapoton (voir [CC06]), il y a une bijection explicite entre les amas et les objets amas-basculants (basiques), et entre les variables d'amas et les objets rigides indécomposables. Pour une étude détaillée, voir l'article de Keller, [Kel11]. Peu après la définition de catégories amassées au sens de [BMR$^+$06], Keller a prouvé que ces catégories étaient triangulées dans l'article [Kel05]. Il a en réalité démontré un résultat plus général. Sous certaines conditions, la catégorie d'orbite d'une catégorie dérivée associée à une algèbre héréditaire par une gentille auto-équivalence est une catégorie triangulée. Cet article a notamment attiré l'attention d'autres catégories, les

$$\mathcal{D}^b(kQ)/\tau^{-1}[m]$$

où $m$ est un entier naturel non nul. Elle sont alors baptisées catégories amassées supérieures.

L'étude des catégories amassées supérieures peut commencer. Dans son article [Tho07], Thomas en donne une définition formelle, où les objets $m$-amas basculants correspondent aux $m$-amas (définis par Fomin et Reading dans l'article [FR05]), de la même manière que, dans les catégories amassées, les objets amas basculants correspondent aux amas de l'algèbre amassée. On définit les catégories amassées supérieures à la section 3.2

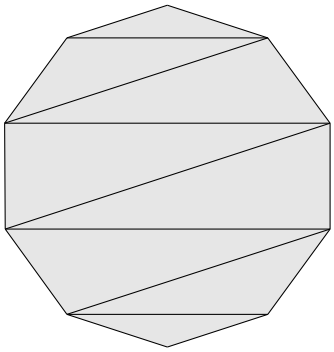
En 2009, Amiot dans [Ami11] a défini les catégories amassées généralisées, pour un carquois quelconque en tant que localisation de Verdier, puis Guo, en 2010, définit les catégories $m$-amassées généralisées associées à une algèbre satisfaisant certaines conditions.

## 1.2 Réalisation géométrique des catégories amassées (supérieures)

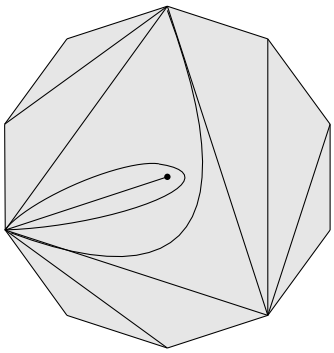
La définition de catégorie amassée de Caldero-Chapoton-Schiffler dans l'article [CCS06] a suscité chez certains l'envie de trouver une description géométrique de

ces catégories dans les cas les plus simples de carquois, d'abord, les carquois de Dynkin, le type $A$ ayant déjà été traité, puis dans le cas $\tilde{A}$. Pour plus de détails, voir la section 3.5.

Dans tous les cas, il s'agit de trouver une surface simple, d'effectuer sur celle-ci des triangulations, puis de démontrer que chaque triangulation est isomorphe à un objet amas basculant. Pour le cas $A_n$, il s'agit d'un polygone à $n+3$ côtés. Voici un exemple de triangulation :



Le cas $D_n$ a été traité par Schiffler dans [Sch08]. Cette fois-ci, un polygone ne suffisait pas à illustrer le carquois, donc il y a rajouté une ponction au centre. Voici un exemple de triangulation :



Fomin, Schapiro et Thurston dans l'article [FST08], généralisèrent des résultats au cas d'une surface marquée épointée (sans ponction). Brüstle et Zhang (voir [BZ11] pour plus de détails) catégorisent ces résultats. Marsh et Palu dans [MP14] démontrent la compatibilité de cette réalisation géométrique avec les mutations de carquois et la réduction d'Iyama-Yoshino. Ils utilisent la définition de catégorie

amassée généralisée d'Amiot, et un résultat de Keller-Yang montrant l'invariance de la catégorie amassée par mutation de carquois.

Baur et Marsh, quant à eux, s'intéressèrent au cas des catégories amassées supérieures, et plutôt que d'utiliser des triangulations, tracèrent des $(m+2)$-angulations, pour une catégorie $m$-amassée (de la même manière que l'on trace des triangles pour une catégorie amassée). Le cas $A_n$ a été traité dans [BM08], et le cas $D_n$ dans [BM07].

Torkildsen traita le cas $\tilde{A}_n$ dans le cas général, en illustrant par un "anneau" formé de deux polygones l'un dans l'autre, le premier ayant $p$ sommets, où $p$ est le nombre de flèches dans le carquois allant dans un sens, et le second $q$, où $q = n + 1 - p$ (qui correspond au nombre de flèches dans l'autre sens). On pourra consulter [Tor12a] pour une étude en détail.

Dans cette thèse, nous traitons le cas $\tilde{D}_n$ en profondeur et démontrons quelques résultats additionnels aux travaux de Baur, Marsh et Torkildsen sur les réalisations géométriques des catégories amassées supérieures. Pour le cas $\tilde{D}_n$, on trouve la figure qui correspond à l'étude, on démontre la compatibilité entre le flip d'une $(m+2)$-angulation, et la mutation de carquois coloré au théorème 2.1.4. On trouve une bijection entre les objets $m$-rigides et chaque arc dit admissible, puis entre les objets amas-basculants et les $(m+2)$-angulations au théorème 2.1.8. De plus, on démontre la compatibilité entre la réduction d'Iyama-Yoshino, et le fait de couper le long d'un arc, qu'on définira formellement. Ceci est fait au théorème 2.1.6.

## 1.3 Catégories triangulées, algèbre homotopique et généralisation aux catégories exactes

Buan, Marsh et Reiten dans [BMR07], souhaitent retrouver les différentes catégories de modules sur les algèbres d'endomorphismes des objets amas basculants à partir de la catégorie amassée. Ils ont démontré que le quotient d'une catégorie amassée par le shift d'un objet amas basculant était une catégorie de module. Buan et Marsh ont généralisé ce résultat aux catégories triangulées dans l'article [BM13], et la localisation d'une catégorie triangulée est une catégorie de module. Puis, Palu étudia le problème d'un point de vue algèbre homotopique, et démontra en 2014 dans [Pal14] qu'on pouvait munir une catégorie triangulée d'une structure de mo-

dèle faible, ce qui permettait de respecter les hypothèses du théorème de Quillen sur l'algèbre homotopique (voir ici le théorème 3.12.3 ou [Qui67]). On rappelle que ce théorème donne le résultat suivant : pour une catégorie de modèle faible $\mathcal{C}$, il existe une équivalence de catégorie entre la sous-catégorie des objets fibrants et cofibrants modulo une relation d'homotopie, et la catégorie homotopique de $\mathcal{C}$.

La réflexion part de ce constat : La catégorie stable d'une catégorie de Frobenius est munie d'une structure triangulée (voir pour une démonstration l'excellent [Hap88] de Happel). Et celles-ci sont (faiblement) des catégories de modèle, et respectent le théorème de Buan et Marsh. Est-il alors possible de "remonter" aux catégories de Frobenius, et, plus encore, aux catégories exactes contenant assez d'injectifs (ou de projectifs) ?

C'est le but de la seconde partie de cette thèse. Nous démontrons qu'une catégorie exacte est une catégorie de préfibration au sens de Anderson-Brown-Cisinski (voir 2.2.2), qui vérifie le théorème de Quillen, et une catégorie de Frobenius est munie d'une structure de modèle (voir 2.2.8), compatible avec le passage à la catégorie stable, qui est triangulée.

Chapter 2

# Summary of results

All throughout this thesis, if $\mathcal{C}$ is a category, we denote by $\mathcal{C}(X, Y)$ the set of morphisms from $X$ to $Y$. All the notions will be introduced properly in chapter 3.

## 2.1 Geometric realization of a higher cluster category of type $A_n$, $D_n$, $\tilde{A}_n$ and $\tilde{D}_n$

In the first part of this thesis, we are interested in geometric realizations of higher cluster categories in different cases. Indeed, though the cases $A_n$, $D_n$, and $\tilde{A}_n$ have been treated respectively by Baur and Marsh in [BM08] and [BM07], and by Torkildsen in [Tor12a], we show on these cases that the Iyama-Yoshino reduction corresponds to cutting along an arc. Moreover, we treat the case $\tilde{D}_n$, by giving an explicit bijection between rigid objects and admissible arcs. We also show that cutting along an arc corresponds to Iyama-Yohsino reduction.

### 2.1.1 Notation

The letter $K$ denotes a field (we prefer to use the small letter $k$ for an integer). For a triangulated category $\mathcal{C}$ endowed with a shift functor $\Sigma$, we consider that the sets $\mathrm{Ext}^1(X, Y)$ and $\mathcal{C}(X, \Sigma Y)$ are the same.

We denote by $Q$ a quiver. In this part, $n$ and $m$ are integers, where $n$ is the number of vertices of $Q$, $n \geq 4$.

If $A$ is an object in a category $\mathcal{C}$, $A^{\perp}$ is the class of all objects $X$ such that $\mathrm{Ext}^i(X, A) = 0$ for all $i \in \{1, \cdots, m\}$. For a quiver $Q$ (respectively for an $(m+2)$-angulation $\Delta$), we note $\mu_k(Q)$ (respectively $\mu_k(\Delta)$ the mutation (respectively the flip) at vertex $k$ (respectively at arc $k$) of $Q$ (respectively $\Delta$).

We denote by $\mathrm{mod}(KQ)$ the category of finitely generated right modules over the path algebra $KQ$. The letter $\tau$ stands for the Auslander-Reiten translation. The application [1] is the shift functor of the (triangulated) higher cluster category.

### 2.1.2 Results

We start with a definition, common to all cases, of an $(m+2)$-angulation.

**Definition 2.1.1.** *Let $n$ be a positive integer, and let $\mathcal{C}_m$ be the higher cluster category associated with a quiver $Q$ of type $A_n$, $D_n$, $\tilde{A}_n$ or $\tilde{D}_n$. Let $P$ be:*

- *A polygon with $nm+2$ sides for case $A_n$.*
- *A polygon with $nm-m+1$ sides for case $D_n$.*
- *An annulus with $p+q$ marked points, where $p$ and $q$ are the number of arrows in one direction and the other for type $\tilde{A}_n$.*
- *An $(n-2)m$-gon with two $(m+1)$-gon inside of it for case $\tilde{D}_n$.*

*An $(m+2)$-angulation of $P$ is a maximal set of noncrossing $m$-diagonals.*

**Remark 2.1.2.** *The notion of $m$-diagonal is defined differently in each case.*

As Buan and Thomas defined a colored quiver from an $m$-cluster-tilting object, with each $(m+2)$-angulation, we define a colored quiver.

**Definition 2.1.3.** *Let $\Delta$ be an $(m+2)$-angulation of the figure $P$ in one of the above cases. We define the colored quiver $Q_\Delta$ associated with $\Delta$ in the following way:*

1. *The vertices of $Q_\Delta$ are in bijection with the $m$-diagonals of $\Delta$.*
2. *If $i$ and $j$ form two sides of the same polygon and only one, then we draw an arrow from $i$ to $j$. The color of the corresponding arrow is the number of edges between both $m$-diagonals, counted clockwise from $i$.*

As Buan and Thomas defined the mutation of colored quiver, we can define the flip of an $(m+2)$-angulation. An important result in this thesis, is that both notions coincide.

**Theorem 2.1.4.** *Let $n$ be a positive integer, and let $\mathcal{C}_m$ be the higher cluster category associated with a quiver $Q$ of type $A_n$, $D_n$, $\tilde{A}_n$ or $\tilde{D}_n$. Let $P$ be as in definition 2.1.1.*

*Let $\Delta$ be any $(m+2)$-angulation of $P$. Let $Q_\Delta$ be the colored quiver associated with $\Delta$. Then, for any $k \in \{1, \cdots, n\}$, we have the equality*

$$\mu_k(Q_\Delta) = Q_{\mu_k(\Delta)}.$$

From the $m$-diagonals we define in this thesis, we can build a new category as the mesh category of $m$-diagonals, with well-chosen morphisms between the arcs. This gives rise to an equivalence of categories between the category of arcs, and a subcategory of the higher cluster category. Moreover, with each $m$-diagonal $\alpha$, we can associate an $m$-rigid object $X_\alpha$. The following theorem helps to build a bijection between $m$-cluster-tilting objects and $(m+2)$-angulations.

**Theorem 2.1.5.** *Let $\alpha$ and $\beta$ be two arcs in the figure $P$. Let $X_\alpha$ and $X_\beta$ be the $m$-rigid objects associated with. Then, $\forall i \in \{1, \cdots, m\}, \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) = 0$ if and only if $\alpha$ and $\beta$ do not cross each other.*

The following theorem shows that cutting along an arc corresponds to Iyama-Yoshino reduction, as in the paper of Marsh and Palu, [MP14, Section 4].

**Theorem 2.1.6.** *Let $\mathcal{C}$ be the $m$-cluster category of type $A_n$, $D_n$, $\tilde{A}_n$ or $\tilde{D}_n$. Let $\alpha$ be an $m$-ear. Let $X_\alpha$ be the $m$-rigid object associated with $\alpha$. Let $\mathcal{U} = \{Y \in \mathcal{C},\ \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, Y) = 0\}$. Let $\mathcal{C}'$ be the Iyama-Yoshino reduction of $\mathcal{C}$, i.e. $\mathcal{C}' = \mathcal{U}/(X_\alpha)$. Then, we have the following equivalence of categories :*

$$\mathcal{C}' \simeq \mathcal{C}^{(m)}_{Q/\alpha}$$

*where $Q/\alpha$ is the quiver obtained from $Q$ by removing $\alpha$ and all its incident arrows.*

Finally, as Buan and Thomas showed that there was a compatibility between mutation of $m$-cluster-tilting objects and colored quivers, we show that the flip of an $(m+2)$-angulation and mutation of $m$-cluster-tilting object is compatible.

This was not obvious. Indeed, a colored quiver is not associated with a unique $m$-cluster-tilting object (or $(m+2)$-angulation, as we will see at theorem 2.1.8). There can be several $m$-cluster-tilting objects associated with the same colored quiver. Then two compatibilities did not imply directly the third.

**Theorem 2.1.7.** *Let $X$ be an $m$-cluster-tilting object in $\mathcal{C}$. Let $\Delta_X$ be its associated $(m+2)$-angulation. Let $\mu_i$ be the flip at the arc $\alpha_i$ in $\Delta_X$ as well as the mutation of the $m$-cluster-tilting object $X$ at summand $i$. Then we have:*

$$\mu_i(\Delta_X) = \Delta_{\mu_i(X)}$$

**Theorem 2.1.8.** *There is a bijection between $(m+2)$-angulations and $m$-cluster-tilting objects.*

Th corollary of the theorem is that the quivers of an $(m+2)$-angulation and of an $m$-cluster-tilting object coincide.

**Theorem 2.1.9.** *Let $\Delta$ be an $(m+2)$-angulation. Let $Q_\Delta$ be the associated coloured quiver. Let $X_\Delta$ be the $m$-cluster-tilting object associated with $\Delta$, and let $Q_{X_\Delta}$ be the quiver associated with $X_\Delta$ in the sense of Buan and Thomas in [BT09]. Then*

$$Q_\Delta = Q_{X_\Delta}$$

We also work on a deeper study of case $\tilde{D}_n$ which had not been treated yet: we give the admissible arcs corresponding to projective modules, injective modules and regular modules. We give an isomorphism between regular modules and some particular $m$-diagonals in the geometric realization.

## 2.2 On the homotopy category of an exact category

The end of the thesis is devoted to the construction of a model structure on exact categories. Indeed, Demonet and Liu in [DL13] showed that the quotient of an exact category $\mathcal{E}$ by a homotopy relation is a module category. The goal of this second part is to show that this quotient is actually the homotopy category of $\mathcal{E}$. This permits to satisfy the hypotheses of the theorem of Quillen, and to provide an explicit homotopy category.

In the same way Palu in [Pal14] showed that a triangulated category was equipped with a weak model structure, we show that a Frobenius category is equipped with a model structure, thus checks the hypotheses of the theorem of Quillen, and then we show that an exact category $\mathcal{E}$ (under some quite nice assumptions) is a prefibration category, which still permits to apply the theorem of Quillen.

### 2.2.1 Notations

We denote by $\mathcal{E}$ an exact category. Let $f : X \to Y$ and $g : A \to B$ be two morphisms in $\mathcal{E}$. We say that $f \square g$ when, for any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

there exists a morphism $B \to X$ such that both triangles commute. For a set $\mathcal{A}$ of morphisms of $\mathcal{E}$, we call by

$$\mathcal{A}^\square = \{g,\ \forall f \in \mathcal{A}, f \square g\} \text{ and } {}^\square\mathcal{A} = \{f,\ \forall g \in \mathcal{A}, f \square g\}.$$

### 2.2.2 Results

This part of the thesis is devoted to the study of exact categories in a homotopical point of view. Indeed, we show the theorem of Quillen on homotopy theory for exact categories. Moreover, we show that there is a prefibration structure on such categories, and that Frobenius categories are equipped with a model structure.

**Lemma 2.2.1.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective and projective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective and projective objects, and stable under taking direct sums and summands. Let*

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

*and*

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

*Then, there exist some $\mathrm{pr}\mathcal{M}$-approximation.*

**Theorem 2.2.2.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands.*

*Then there exist two classes of morphisms $\mathcal{F}ib$ and $\mathcal{W}$ (the same weak equivalences as in the Frobenius case) such that $(\mathcal{E}, \mathcal{F}ib, \mathcal{W})$ has a prefibration sutrcture in the sense of Anderson-Brown-Cisinski (see [RB06] for a deeper study). All the objects are fibrant, and an object is cofibrant if and only if it belongs to $\mathrm{pr}\mathcal{M}$. Moreover, under the assumption that $\mathcal{M}$ contains the projective objects, there exist some cofibrant replacements for any object of $\mathcal{E}$.*

**Corollary 2.2.3.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands. Suppose moreover that $\mathcal{M}$ contains the projective objects, and that there exist some $\mho\mathcal{M}$-approximation. Let*

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

*and*

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

*Let*

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-, X)/\overline{\mathcal{M}} \end{array}$$

*which induces the following equivalence of categories*

$$\mathrm{pr}\mathcal{M}/\mho\mathcal{M} \simeq \mathrm{mod}\ \overline{\mathcal{M}}.$$

*Then the homotopy category is equivalent to* $\mathrm{mod}\ \overline{\mathcal{M}}$.

**Corollary 2.2.4.** *We deduce from this that there is a homotopy calculus of fractions in the sense of Radulescu-Banu in [RB06].*

The theorem 2.2.2 not only shows the previous result. It shows that the objects of $\mathrm{pr}\mathcal{M}$ are exactly the fibrant and cofibrant objects (as all objects are fibrant, we say that the subcatebory $\mathrm{pr}\mathcal{M}$ is the subcategory of cofibrant objects).

**Theorem 2.2.5.** *Each object of an exact category* $\mathcal{E}$ *satisfying the hypotheses of corollary 2.2.3 has a cofibrant replacement. This means that* $\mathrm{pr}\mathcal{M}$*-approximations are weak equivalences (they even are trivial fibrations).*

We also characterize trivial cofibrations (morphisms which satisfy a lift property on fibrations):

**Lemma 2.2.6.** *Suppose that* $\mho M \to Y$ *is a right* $\mho\mathcal{M}$*-approximation. A morphism* $f : X \to Y$ *is in* ${}^{\square}(J^{\square})$ *if and only if it is a retract of the canonical injection* $X \to X \oplus \mho M$.

We find a homotopy relation on morphisms of the category $\mathcal{E}$, compatible with the usual notion of left and right homotopy.

**Lemma 2.2.7.** *For two morphisms* $f$ *and* $g$ *from an object* $X$ *to* $Y$*,*$f$ *and* $g$ *are homotopic if and only if* $f - g$ *factorizes through* $\overline{\mathcal{M}}^{\perp}$.

Next, from now we suppose that $\mathcal{E}$ is a Frobenius category. As it has much more properties, we can equip it with a stronger structure : a model structure.

**Theorem 2.2.8.** *Let* $\mathcal{E}$ *be a Frobenius category. Let* $\mathcal{M}$ *be a full subcategory of* $\mathcal{E}$ *containing the injective objects, and assume that* $\mathcal{M}$ *is contravariantly finite. Then, there exist three classes of morphisms,* $\mathcal{F}ib$*,* $\mathcal{C}of$ *and* $\mathcal{W}$ *(the last one being the weak equivalences, which means the morphisms such that their image under the functor* $G$ *is an isomorphism), forming a model structure on* $\mathcal{E}$*. More precisely, all the objects are fibrant, and an object is cofibrant if and only if it belongs to* $\mathrm{pr}\mathcal{M}$.

We characterize precisely weak equivalences, fibrations and trivial fibrations:

**Lemma 2.2.9.** *Let $X, Y \in \mathcal{E}$. Let $f : X \to Y$. Then $f$ is a weak equivalence if and only if, for*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle g} \\ I_X & \xrightarrow{u} & Z \\ {\scriptstyle \pi_X}\downarrow & & \downarrow \\ \mho X & = & \mho X \end{array}$$

*we have $I_X \oplus Y \xrightarrow{\overline{(u\ g)}} Z \in \overline{\mathcal{M}}^{\perp}$ and, for*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle \tilde{g}} & & \uparrow \\ Z & \longrightarrow & P_Y \\ \uparrow & & \uparrow \\ \mho Y & = & \mho Y \end{array}$$

*we have $Z \longrightarrow X \oplus P_Y \in \overline{\mathcal{M}}^{\perp}$.*

**Lemma 2.2.10.** *Suppose that $\mathcal{E}$ is a weakly idempotent complete category (see the article of Bühler, [Büh10]). Then $f$ is a fibration if and only if $f$ is a deflation, and $g \in \mho\overline{\mathcal{M}}^{\perp}$, where $g$ is a cone of $f$, defined by*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle g} \\ I_X & \xrightarrow{u} & Z \\ {\scriptstyle \pi_M}\downarrow & & \downarrow \\ \mho X & = & \mho X \end{array} .$$

Note that all these results are compatible by studying the stable category, which is triangulated (it means that, stabilizing the Frobenius category $\mathcal{E}$, we find back the results of Palu in [Pal14]).

Chapter 3

# Preliminaries

## 3.1 Triangulated categories

Triangulated categories behave like exact categories. Triangles are the equivalent for short exact sequences in exact categories. For a gentle introduction to triangulated categories, see the book of Holm and Jorgensen [HJ10]. For a deeper study of triangulated categories, see the book of Zimmermann, [Zim14].

### 3.1.1 Axioms of a triangulated category

**Definition 3.1.1.** *Let $\mathcal{C}$ be an additive category. Let $\Sigma$ be an automorphism of $\mathcal{C}$, called the suspension functor. We call by triangles the diagrams of the form $X \to Y \to Z \to \Sigma X$, stables under isomorphisms. Then $\mathcal{C}$, endowed with such triangles is triangulated if it satisfies the following axioms :*

**TR 1** *For any morphism $X \to Y$, there exists a triangle $X \to Y \to Z \to \Sigma X$. For any $X in \mathcal{C}$, the diagram $X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle.*

**TR 2** *The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle is and only if the diagram $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ .*

**TR 3** *For any triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

*and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ , for any commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{f'} & Y' \end{array} ,$$

*there exists $\gamma : Z \to Z'$ such that*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

*is a morphism of triangle.*

**TR 4** *Suppose that there exist three triangles*

$$X \xrightarrow{a} Y \xrightarrow{b} Z' \xrightarrow{c} \Sigma X \ ,$$

$$Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X' \xrightarrow{\gamma} \Sigma Y$$

*and*

$$X \xrightarrow{\mu} Z \xrightarrow{\nu} Y' \xrightarrow{\xi} \Sigma X$$

*such that $\mu = \alpha \circ a$.*
*Then there exists a triangle*

$$Z' \xrightarrow{d} Y' \xrightarrow{e} X' \xrightarrow{f} \Sigma Z'$$

*such that*

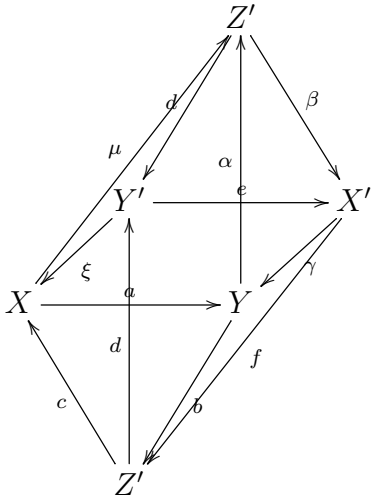
$$\begin{cases} d \circ b = \nu \circ \alpha \\ c = \xi \circ d \\ b[1] \circ \gamma = f \\ a[1] \circ \xi = \gamma \circ e \\ e \circ \nu = \beta \end{cases} \tag{3.1}$$

*We can illustrate this axiom by the following diagram: if we have the full arrows, then we have the dotted arrows.*

$$\begin{array}{ccccccc} & & \Sigma^{-1}X' & = & \Sigma^{-1}X' & & \\ & & \downarrow{\scriptstyle\Sigma^{-1}\gamma} & & \vdots{\scriptstyle\Sigma^{-1}f} & & \\ X & \xrightarrow{d} & Y & \xrightarrow{b} & Z' & \xrightarrow{c} & \Sigma X \\ \| & & \downarrow{\scriptstyle\alpha} & & \vdots{\scriptstyle d} & & \| \\ X & \xrightarrow{\mu} & Z & \xrightarrow{\nu} & Y' & \xrightarrow{\xi} & \Sigma X \\ & & \downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle e} & & \\ & & X' & = & X' & & \end{array}$$

*with all the squares commutative.*

**Remark 3.1.2.** *The last axiom is called the octahedral axiom. It can be formulated differently. The following one illustrates well the name of the axiom:*




*but here the triangles are quite difficult to see.*

**Remark 3.1.3.** *The morphism in TR 3 is unique, up to non-unique isomorphism. This causes many problems we do not have in abelian categories.*

### 3.1.2 Some properties of triangulated categories

Throughout this section, $\mathcal{C}$ is a triangulated category.

**Lemma 3.1.4** (May's lemma)**.** *The axiom TR 3 is a consequence of the axioms TR 1, TR 2 and TR 4.*

It is thought that axiom TR 4 is a consequence of TR1, TR 2 and TR 3, but the question is still without any answer.

**Proposition 3.1.5.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *be a triangle. Then we have*

$$g \circ f = 0 \text{ and } h \circ g = 0.$$

**Lemma 3.1.6** (Triangulated five lemma)**.** *Let*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

*be a morphism of triangle. If two of the vertical arrows are isomorphisms, then the third one is one too.*

We now tell a useful lemma about long exact sequences.

**Lemma 3.1.7.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *be a triangle. Then, we have a long exact sequence:*

$$\cdots \to \mathcal{C}(U, \Sigma^{-1}Z) \to \mathcal{C}(U, X) \to \mathcal{C}(U, Y) \to \mathcal{C}(U, Z) \to \mathcal{C}(U, \Sigma X) \to \cdots$$

*for any* $U \in \mathcal{C}$*. We also have*

$$\cdots \to \mathcal{C}(\Sigma X, U) \to \mathcal{C}(Z, U) \to \mathcal{C}(Y, U) \to \mathcal{C}(X, U) \to \mathcal{C}(\Sigma^{-1}X, U) \to \cdots$$

*for any* $U \in \mathcal{C}$*.*

### 3.1.3 Examples of triangulated categories

Now, the category $\mathcal{C}$ is not necessarily triangulated anymore.

#### Derived categories

Historically, derived categories are the first example of triangulated categories, due to Verdier in his posthumous printed thesis [Ver96]. Here, we define derived categories. For a nive lecture note, see [Kel96]. For a deep study of derived categories, see [Zim14].

**Definition 3.1.8.** *Let* $\mathcal{C}$ *be an abelian category (we often study* mod $A$*, which is the category of finitely generated right modules over a finite dimensional* $k$*-algebra, where* $k$ *is an algebraically closed field). We introduce* $C(\mathcal{C})$ *the new category of complexes, given by:*

- *Objects: Complexes of objects of* $\mathcal{C}$*.*
- *Morphisms: morphisms of degree* $0$*, which commute with the differential.*
- *Natural composition.*

From this category, we define another category, the homotopy category $K(\mathcal{C})$.

**Definition 3.1.9.** *Let* $f : M \to N$ *be a morphism of complexes. The morphisms* $f$ *is said to be null-homotopic if there exists a morphism* $h : M \to N$*, of degree* $-1$*, such that* $f = d \circ h + h \circ d$*, chere* $d$ *is the differential.*

*The homotopy category* $K(\mathcal{C})$ *is defined in the following way:*

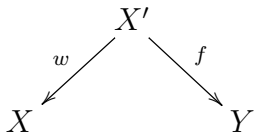
- *Objects: same as the ones of* $\mathcal{C}$*.*
- *Morphisms: classes of morphisms of* $C(\mathcal{C})$ *modulo the null-homotopic relation.*
- *Natural composition.*

Now we can define the derived category of $\mathcal{C}$.

**Definition 3.1.10.** *The morphisms which induce isomorphisms in homology are called quasi-isomorphisms.*

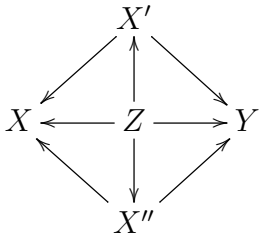
*The derived category $\mathcal{D}(\mathcal{C})$ is the localization of $K(\mathcal{C})$ with respect to the class of quasi-isomorphisms.*

Let us explain the morphisms of $\mathcal{D}(\mathcal{C})$: a morphisms from $X$ to $Y$ in $\mathcal{D}(\mathcal{C})$ is the class of diagrams of the type

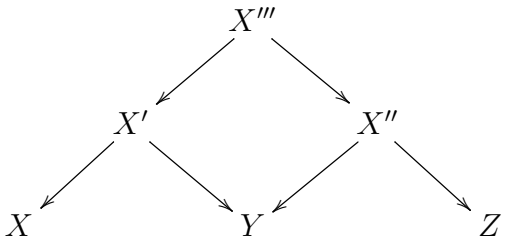



where $f \in K(X', Y)$, and $w$ is a quasi-isomorphism.

Two such diagrams are called equivalent when there exists a commutative diagram:




The composition of two diagrams is given by the following diagram:




**Theorem 3.1.11** (Verdier, [Ver96])**.** *Let $\mathcal{C}$ be an additive category. Then the derived category $\mathcal{D}(\mathcal{C})$ of $\mathcal{C}$ is triangulated.*

For more details, see the excellent book of Alexander Zimmermann, [Zim14].

### The stable category of a Frobenius category

We define at section 3.6 what a Frobenius category is. In his book [Hap88], Happel discovered a new example of triangulated categories.

**Definition 3.1.12.** *Let $\mathcal{E}$ be a Frobenius category. For any two objects $A$ and $B$, we denote by $I(A,B)$ the subgroup of morphisms from $A$ to $B$ which factors through an injective object.*

*The stable category $\underline{\mathcal{E}}$, associated with $\mathcal{E}$, is the category whose objects are the same as the ones of $\mathcal{E}$, and the set of morphisms, is*

$$\underline{\mathcal{E}}(A,B) = \mathcal{E}(A,B)/I(A,B).$$

The following theorem is due to Happel.

**Theorem 3.1.13** (Happel, [Hap88])**.** *The stable category of a Frobenius category is triangulated, with suspension functor the object $\Sigma A$ appearing in the triple*

$$A \rightarrowtail I_A \twoheadrightarrow \Sigma A.$$

**Remark 3.1.14.** *The homotopy category, as above, is the stable category of a Frobenius category. Then it is automatically a triangulated category.*

## 3.2 Higher cluster categories

In this thesis, we are interested in higher cluster categories. Cluster categories have been defined in [BMR$^+$06], in order to categorify the concept of cluster algebras, Keller in [Kel05] studied these and also higher cluster categories. For a survey of the two next section, see the article of Buan, [Bua11].

### 3.2.1 Definition of higher cluster categories

We recall the definition of a cluster category: Consider $H$ a basic hereditary finite dimensional algebra over an algebraically closed field $k$. Let $\mathcal{D}^b(H)$ be the bounded derived category of $H$. For convenience, we denote by [1] the shift functor of $\mathcal{D}^b(H)$, and $\tau$ the Auslander-Reiten translation on the derived category. We refer to [Zim14] for a deeper study of derived categories. The cluster category associated with $H$ is the orbit category $\mathcal{D}^b(H)/\tau^{-1}[1]$.

**Definition 3.2.1.** *Let $H$ be a basic hereditary finite dimensional algebra over an algebraically closed field $k$. Let $\mathcal{D}^b(H)$ be the bounded derived category of $H$. Then the $m$-cluster category associated with $H$ of degree $m$ is defined by*

$$\mathcal{C}_H^{(m)} = \mathcal{D}^b(H)/\tau^{-1}[m]$$

*where $[m]$ is the functor $[1]$ repeated $m$ times.*

**Theorem 3.2.2** (Keller, [Kel05])**.** *The higher cluster category is triangulated.*

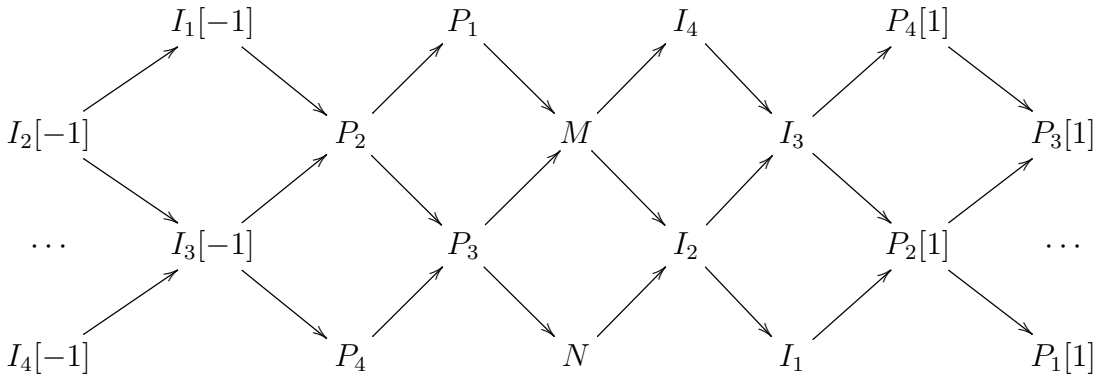
**Theorem 3.2.3** (Buan, Marsh, Reineke, Reiten, Todorov, [BMR$^+$06])**.** *The higher cluster category is Krull-Schmidt.*

### 3.2.2 Example of a 2-cluster category

We take this example from Buan, in [Bua11]. Let $Q$ be the quiver

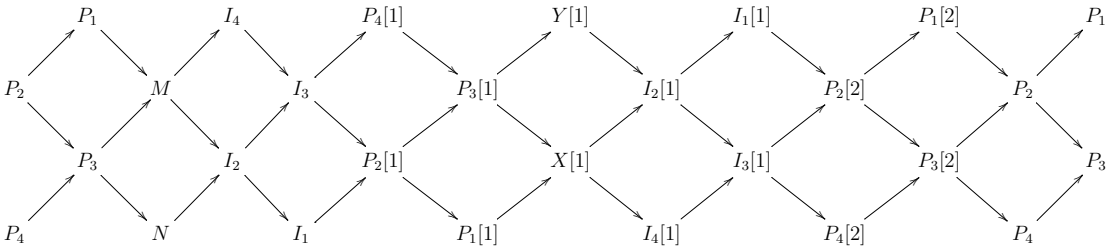
$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \,.$$

Let $A$ be the path algebra $kQ$. We draw the infinite Auslander-Reiten quiver of $Q$:




Then, the 2-cluster category of $Q$ consists in taking each object and the first copy (first shift) of each object. Then, it is composed of 24 indecomposable objects (10 plus 10 shifts plus one additional copy of the projective objects shifted twice).

The Auslander-Reiten quiver of the higher cluster category is thus cyclic:




The left and right most objects are identified.

## 3.3 Cluster-tilting subcategories

In triangulated categories, there are some interesting objects or subcategories, which reveal the category, and helps to understand it: the cluster-tilting objects (or subcategories). In this section, we present this.

### 3.3.1 Definition of cluster-tilting categories

**Definition 3.3.1.** *Let $k$ be a field, and let $\mathcal{C}$ be a triangulated $k$-linear category with split idempotents and a suspension functor* [1]. *We suppose that $\mathcal{C}$ is* Hom*-finite, and admits a Serre functor $S$. Then $\mathcal{C}$ is said to be $(m+1)$-Calabi-Yau if there exists an isomorphism*

$$[m+1] \simeq S.$$

We now define $m$-cluster-tilting subcategory.

**Definition 3.3.2.** *Let $\mathcal{T}$ be a subcategory of $\mathcal{C}$. We say that $\mathcal{T}$ is an $m$-cluster-tilting subcategory if it satisfies the following:*

- *The category $\mathcal{T}$ is a $k$-linear subcategory.*
- *The functors $\mathcal{C}(-,X)|\mathcal{T}$ and $\mathcal{C}(X,-)|\mathcal{T}$ are finitely generated for any $X \in \mathcal{C}$.*
- *For all $i \in \{1,\cdots,m\}$, and any $T, T' \in \mathcal{T}$, we have $\mathrm{Ext}^i(T,T') = 0$.*
- *For $X \in \mathcal{C}$, if $\mathrm{Ext}^i(T,X) = 0$ for all $i \in \{1,\cdots,m\}$ and any $T \in \mathcal{T}$, then $X \in \mathcal{T}$.*

**Definition 3.3.3.** *Let $\mathcal{C}$ be an $m$-cluster category. An object $T$ of $\mathcal{C}$ is called rigid if*

$$\mathrm{Ext}^i_{\mathcal{C}}(T,T) = 0$$

*for all $i \in \{1,\cdots,m\}$. An $m$-cluster-tilting object is a rigid object $T$ satisfying moreover the following property:*

$$\forall X \in \mathcal{C}, \mathrm{Ext}^i_{\mathcal{C}}(T,X) = 0 \ \forall i \in \{1,\cdots,m\} \Longrightarrow X \in \mathrm{add}T.$$

**Remark 3.3.4.** *We note that, if $T \in \mathcal{C}$, then $T$ is rigid if and only if the above implication is an equivalence. Zhu in [Zhu08] and Wraalsen in [Wra09] showed independently that maximal rigid objects and $m$-cluster-tilting objects coincide. If $A$ is a finite dimensional hereditary algebra containing exactly $n$ simple modules up to isomorphism, they also showed that a rigid object $T$ is $m$-cluster-tilting if and only if it has $n$ isomorphism classes of indecomposable direct summands.*

### 3.3.2 Some properties of $m$-cluster-tilting objects

**Theorem 3.3.5** (Wraalsen, Zhou-Zhu, [Wra09],[ZZ09])**.** *Let $T = \oplus T_k$ be an $m$-cluster-tilting object. We denote by $B_k = T/T_k$ an almost $m$-cluster-tilting object. The objects $X$ such that $B_k \oplus X$ is an $m$-cluster-tilting object are called complement to $B_k$. Then $B_k$ has $m+1$ complements.*

**Definition 3.3.6.** *Let $T = \oplus T_k$ be an $m$-cluster-tilting object in a triangulated category $\mathcal{C}$. Then the map $f : T_k \to B'_k$ is a minimal left add $B_k$-approximation of $T_k$ if:*

— $B'_k \in \text{add } B_k$

— *Any map* $T_k \to B$, $B \in \text{add } B_k$ *factors through* $f$.

— *If, for any* $g : B'_k \to B'_k$, $g \circ f = f$, *then the map* $g$ *is an isomorphism.*

*By TR 1 in the category* $\mathcal{C}$, *we have the following triangle*

$$T_k \to B'_k \to T_k^* \to T_k[1].$$

*This triangle is called an exchange triangle.*

**Theorem 3.3.7** (Iyama-Yoshino, [IY08])**.** *The* $m+1$ *complements found in the previous theorem appear in the* $m+1$ *exchange triangles. They are exactly the* $T_k^{(c)}$ *for* $c \in \{0, \cdots, m\}$ *occurring in the exchange triangles*

$$T_k^{(c)} \to B_k^{(c)} \to T_k^{(c+1)} \to T_k^{(c)}[1].$$

Here we only cite a very useful theorem of Keller and Reiten in [KR08], we will use all throughout the first part of the thesis.

**Theorem 3.3.8.** *[KR08, Theorem 4.2] Let* $\mathcal{C}$ *be a Hom-finite algebraic* $(m+1)$*-Calabi-Yau category. Let* $T$ *be an* $m$*-cluster-tilting object in* $\mathcal{C}$, *such that* $\forall i \in \{1, \cdots, m\}$ $\text{Ext}^{-i}_{\mathcal{C}}(T, T) = 0$ *and* $\text{End}_{\mathcal{C}}(T) \simeq KQ_T$, *for some quiver* $Q_T$.

*Then* $\mathcal{C} \simeq \mathcal{C}^m_{Q_T}$.

**Remark 3.3.9.** *These categories from theorem 3.3.8 are Iyama-Yoshino reductions. They are algebraic (see the article of Buan, Iyama, Reiten and Scott [BIRS09, Theorem I.1.8] for example).*

**Remark 3.3.10.** *In his paper [Kel05], Keller has shown that orbit categories are also algebraic.*

### 3.3.3 Example

We still take the previous example due to Buan. Let $B = I_4 \oplus I_1 \oplus Y[1]$ be an almost $m$-cluster-tilting object. Then it has three complements which are given by the exchange triangles. They are $X, P_2[1]$ and $I_3[1]$.

## 3.4 Quivers and colored quivers

Before defining colored quivers, let us introduce the mutation of a classic Gabriel quiver.

**Definition 3.4.1.** *Let $Q$ be a quiver, it means a set of vertices $Q_0$, a set of arrows $Q_1$ and two maps $s$ and $t$ from $Q_1$ to $Q_0$, which associate respectively with an arrow its source and its target. Then the mutation at vertex $i$ of the quiver $Q$, denoted as $\mu_i(Q)$ is given by:*

— *Replace each $j \to i \to k$ by an arrow $j \to k$.*

— *Reverse all arrow coming from or going to $i$.*

— *Remove all 2-cycles.*

### 3.4.1 Definition of a colored quiver

A deeper study on colored quivers can be found in [BT09] written by Buan and Thomas.

**Definition 3.4.2.** *Let $m, n$ be positive integers. An $m$-colored quiver consists of:*

— *A set $Q_0$ of $n$ vertices.*

— *A set $Q_1$ of arrows.*

— *Two maps $s, t : Q_1 \to Q_0$ which associate with each arrow respectively the source and the target.*

— *A map $Q_1 \to \{1, \cdots, m\}$ which associates with each arrow a color $c \in \{1, \cdots, m\}$.*

*We write the color above the arrow. The term $q_{ij}^{(c)}$ corresponds to the number of arrows from $i$ to $j$ of color $c$.*

For the study, we ask our colored quivers to satisfy the following properties:

— No loops which means that $q_{ii}^{(c)} = 0$ for all $c \in \{1, \cdots, m\}$ and $i \in \{1, \cdots, n\}$.

— Monochromaticity which means that if $q_{ij}^{(c)} \neq 0$, then $q_{ij}^{(c')} = 0$ for all $c' \neq c$.

— Skew-symmetry which means that $q_{ij}^{(c)} = q_{ji}^{(m-c)}$.

We now define the mutation of colored quivers.

**Definition 3.4.3.** *[BT09] Let $Q$ be a colored quiver, and let $k$ be a vertex of $Q$. We define the new quiver $\mu_k(Q)$ with the same vertices, and the new number of arrows $\tilde{q}_{ij}^{(c)}$ given by:*

$$\tilde{q}_{ij}^{(c)} = \begin{cases} q_{ij}^{(c+1)} \text{ if } j = k \\ q_{ij}^{(c-1)} \text{ if } i = k \\ \max\{0, q_{ij}^{(c)} - \sum_{t \neq c} q_{ij}^{(t)} + (q_{ik}^{(c)} - q_{ik}^{(c-1)})q_{kj}^{(0)} + q_{ik}^{(m)}(q_{kj}^{(c)} - q_{kj}^{(c+1)})\} \text{ else.} \end{cases}$$

In section 3.4.3, an example is shown.

### 3.4.2 Main theorem about colored quivers

In this section, if $T$ is an $m$-cluster-tilting object, we define $T'$, the $m$-cluster-tilting object which is obtained by mutation of $T$. Unfortunately, if $Q_T$ is the Gabriel quiver associated with $T$, for $m > 1$, there does not exist a quiver mutation $\mu_j$ such that $Q_{T'} = \mu_j(Q_T)$. Then, to remedy this lack, Buan and Thomas in [BT09] built a new quiver from $T$, which is denoted as the colored quiver associated with $T$.

**Proposition 3.4.4.** *Mutating a colored quiver as in definition 3.4.3 is equivalent to the following procedure:*

1. *For any $i \xrightarrow{(c)} k \xrightarrow{(0)} j$, if $i \neq j$ and $c$ is an integer in $\{0, \cdots, m\}$, then draw an arrow $i \xrightarrow{(c)} j$ and an arrow $j \xrightarrow{(m-c)} i$.*
2. *If condition 2 of monochromaticity in the restriction of colored quivers is not satisfied anymore from one vertex $i$ to one vertex $j$, then remove the same number of arrows of each colour, in order to restore the condition.*
3. *For any arrow $i \xrightarrow{(c)} k$, add 1 to the color $c$, and for any arrow $k \xrightarrow{(c)} j$, subtract 1 to the color $c$.*

We recall from Buan and Thomas in [BT09] that there are exchange triangles

$$T_k^{(c)} \xrightarrow{f_k^{(c)}} B_k^{(c)} \xrightarrow{g_k^{(c+1)}} T_k^{(c+1)} \xrightarrow{h_k^{(c+1)}} T_k^{(c)}[1] \tag{3.2}$$

With any $m$-cluster-tilting object $T$ in the $m$-cluster category, we associate a corresponding colored quiver $Q_T$ such that:

1. The vertices of $Q_T$ are the integers from 1 to $n$ where $n$ is the number of indecomposable summands of $T$.
2. The number $q_{ij}^{(c)}$ is the multiplicity of $T_j$ in $B_i^{(c)}$ in the exchange triangle (2.13).

We now state the main theorem about colored quivers and $m$-cluster-tilting objects a proof of which can be found in [BT09]:

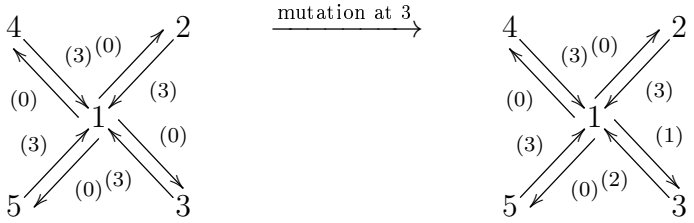
**Theorem 3.4.5.** *[BT09, Theorem 2.1] Let $T = \bigoplus_{i=1}^{n} T_i$ and $T' = T/T_k \bigoplus T_k^{(1)}$ be $m$-cluster-tilting objects, where there is an exchange triangle $T_k \to B_k^{(0)} \to T_k^{(1)} \to T_k[1]$. Then*

$$Q_{T'} = \mu_k(Q_T).$$

**Remark 3.4.6.** *In particular, the coloured quiver $Q_{T'}$ only depends on the colored quiver $Q_T$.*

### 3.4.3 Example of the mutation of colored quivers

We give here a simple example of the mutation of a colored quiver:




## 3.5 Geometric realization of cluster categories of type $A$, $D$, $\tilde{A}$ and marked surfaces

This section is more an explanation of what have been done until now than a real preliminary background, necessary to understand the results of this thesis.

Researchers have tried to give a geometric realization of cluster categories, which could explain some curious results such as how the morphisms in the Auslander-Reiten quiver are set, and give a visual understanding of these categories. We now consider the cases $A$, $D$, and the very general case of marked surfaces.

### 3.5.1 Case $A$

This is the first case studied, by Caldero, Chapoton and Schiffler in [CCS06]. Let $Q$ be a quiver of type $A_n$. Let $P$ be the regular $(n+3)$-gon. Then with each diagonal (we keep the vocabulary of [CCS06]) of a triangulation of $P$, we can associate a vertex of $Q$. We can draw arrows between diagonals (indeed, we draw an arrow from one diagonal to another if they are consecutive clockwise). This makes the triangulation correspond to a quiver.

We introduce the flip of a triangulation in the following way: let $T$ be a triangulation. Let $i$ be a diagonal of $T$. Then, the new triangulation $T' = \mu_i(T)$, flipped at vertex $i$, is defined in the following way: $T'$ contains all the diagonals of $T$ except $i$. The diagonal $i'$ obtained by removing $i$ and taking the other diagonal of the quadrilateral created by removing $i$ completes the triangulation $T'$. In this situation, Caldero, Chapoton and Schiffler in [CCS06] show the following lemma:

**Lemma 3.5.1.** *Let $Q$ be a quiver of type $A_n$. Let $P$ be the regular $(n+3)$-gon. Let $T$ be the triangulation corresponding to $Q$ in $P$. Let $i$ be a vertex of $Q$, corresponding to a diagonal of $T$. Then,*

$$\mu_i(Q) = \mu_i(T).$$

Continuing from this, Caldero Chapoton and Schiffler built a new category where the objects arise from diagonals in P and where the irreducible morphisms arise from the arrows between diagonals, with morphisms arising from compositions of arrows between the objects, modulo mesh relations. They showed the following theorem:

**Theorem 3.5.2** (Caldero-Chapoton-Schiffler, Theorem 2.14, [CCS06])**.** *There is an equivalence of categories between the cluster category of type $A$ and the category geometrically built from diagonals.*

They even showed more than this: particular objects such as injective, projective, corresponded to special diagonals in the polygon $P$.

### 3.5.2 Case $D$

After this, Schiffler in [Sch08] provided a model for type $D_n$, for $n \geq 4$. Now, $P$ is a polygon with $n$ vertices, and one puncture in its center. Schiffler defines what is an admissible diagonal, and elementary moves between two diagonals. In the same way as in case $A_n$, he shows the following lemma:

**Lemma 3.5.3.** *Let $Q$ be a quiver of type $D_n$. Let $P$ be the regular $n$-gon with one puncture. Let $T$ be the triangulation corresponding to $Q$ in $P$. Let $i$ be a vertex of $Q$, corresponding to a diagonal of $T$. Then,*

$$\mu_i(Q) = \mu_i(T).$$

Finally, by building a category whose objects arise from the diagonals, and the morphisms the composition of elementary moves, modulo mesh relations, he shows the following theorem:

**Theorem 3.5.4** (Schiffler, Theorem 3.3, [Sch08])**.** *There is an equivalence of category between the category of diagonals, and the cluster category of type $D$.*

### 3.5.3 Case $\tilde{A}$

This case has been treated by Torkildsen in [Tor12a]. He shows similar results, taking into account the different form of the Auslander-Reiten quiver of $Q$ (presence of tubes, which must correspond to regular modules). For a deeper study of these Euclidean quivers, see the book [SS07].

The first part of his thesis consists in finding these results for case $\tilde{A}_n$, and to show that cutting along an arc corresponds to apply Iyama-Yoshino reduction.

### 3.5.4 General case of the marked surface

Brüstle and Zhang in [BZ11], introduce the cluster category associated with a marked surface. Marsh and Palu in [MP14] generalized the results given above for these generalized cluster categories. Indeed, they use a theorem of Keller and Yang (see [KY11]). Before giving it, we need to see the definition of the generalized cluster category.

**Definition 3.5.5.** *Let $Q$ be a quiver, associated with a potential $W$ (a linear combination of cycles in $Q$, up to cyclic permutation). We define the Ginzburg dg algebra $\Gamma(Q,W)$ as follows: First define a graded quiver $Q$ which arrows are defined as follows:*

- *The arrows of $Q$, of degree $0$*
- *For each arrow $\alpha : i \to j$, add an arrow $\alpha^* : j \to i$ of degree $-1$.*
- *For each vertex $i$ of $Q$, add a loop $e_i^*$ at $i$ of degree $-2$.*

*The Ginzburg dg algebra is the path algebra of the graded quiver.*

We now define the cluster category for an arbitrary quiver Q (without assuming acyclicity).

**Definition 3.5.6** (Amiot)**.** *The generalized cluster category $\mathcal{C}_Q$ associated with $Q$ is the Verdier localization*

$$\mathrm{per}\Gamma(Q,W)/\mathcal{D}^b(\Gamma(Q,W)).$$

Amiot has shown that this category corresponds in the acyclic case to the classical cluster category. One can define a quiver with potential from a triangulation of a marked surface. Then, Keller and Yang have shown this powerful theorem:

**Theorem 3.5.7** (Keller-Yang, [KY11])**.** *Let $\mathcal{T}$ be a triangulation, and $(Q_\mathcal{T}, W_\mathcal{T})$ be the associated quiver with potential. Then, for any arc $i$ of $\mathcal{T}$, if $\mu_i(\mathcal{T})$ is the new triangulation obtained from flipping the arc $i$, there is a triangle equivalence: Then we have the equivalence*

$$\mathcal{C}_{Q_\mathcal{T}} \simeq \mathcal{C}_{Q_{\mu_i(\mathcal{T})}}.$$

This theorem permits to associate with a surface, a unique cluster category. Indeed, Marsh and Palu show the following result:

**Theorem 3.5.8.** *There is a bijection between the arcs of the triangulation of a marked surface, and the cluster-tilting objects of the associated cluster category. Moreover, cutting along an arc corresponds to Iyama-Yoshino reduction. Finally, the quiver mutation is compatible with flip, as in the $A_n$ and $D_n$ cases.*

## 3.6 Exact and Frobenius categories

In this section, we introduce exact and Frobenius categories as in Bühler (see [Büh10]). Exact categories have been invented by Quillen in [Qui10] in order to generalize the notion of short exact sequences in abelian categories.

### 3.6.1 Axioms of exact categories

In this section, we choose as a reference the article of Bühler [Büh10].

**Definition 3.6.1.** *Let $\mathcal{E}$ be an additive category. A short exact sequence $(i,p)$ in $\mathcal{E}$ is a pair of composable morphisms $A \xrightarrow{i} A' \xrightarrow{p} A''$ such that $i$ is a kernel of $p$ and $p$ is a cokernel of $i$. If a class $C$ of short exact sequences on $\mathcal{E}$ is fixed, an inflation is a morphism $i$ for which there exists a morphism $p$ such that $(i,p) \in C$. Deflations are defined dually.*

*An exact structure on $\mathcal{E}$ is a class $C$ of short exact sequences, which is closed under isomorphisms and satisfies the following axioms:*

**E0** *The identity morphism is an inflation.*

**E0'** *The identity morphism is a deflation.*

**E1** *The class of inflations is closed under composition.*

**E1'** *The class of deflations is closed under composition.*

**E2** *The push-out of an inflation along an arbitrary morphism exists and yields an inflation.*

**E2'** *The pull-back of a deflation along an arbitrary morphism exists and yields a deflation.*

**Remark 3.6.2** (Vocabulary)**.** *Short exact sequences are also called kernel-cokernel pairs. Moreover, there exists a further pair of notions for inflation and deflation, which is admissible monic and admissible epic. Here we prefer to use the notations of Keller.*

**Remark 3.6.3.** *In order to be clear, we note inflation with an arrow $\rightarrowtail$ and a deflation with an arrow $\twoheadrightarrow$.*

### 3.6.2 Some properties of exact categories

Let $\mathcal{E}$ be an exact category. To have proofs of these properties, we refer to Bühler ([Büh10])

**Proposition 3.6.4.** *For any objects $A$ and $B$, the diagram*

$$A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(0\ 1)} B$$

*is a short exact sequence.*

**Proposition 3.6.5.** *The direct sum of two short exact sequences is short exact.*

**Proposition 3.6.6.** *Consider the following commutative square:*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow{i'} & B' \end{array} .$$

*Suppose that the horizontal arrows are inflations. Then the following assertions are equivalent:*

*i* *The square is a push-out*

*ii* *The sequence*

$$A \xrightarrow{\binom{i}{-f}} B \oplus A' \xrightarrow{(f'\ i')} B'$$

*is short exact.*

*iii* *The square is both a push-out and a pull-back.*

*iv* *The square makes part of a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \| \\ A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C \end{array}$$

*where the rows are short exact.*

**Proposition 3.6.7.** *The pull-back of an inflation along a deflation yields an inflation.*

### 3.6.3 Examples of exact categories

1. We remark that any abelian category is in particular exact.
2. Any full, closed under extension subcategory of an abelian category is exact.
3. If $\mathcal{C}$ is a category which has finite limits, is cartesian closed, and has a subobject classifier (it is a morphism such that any monomorphism in the topos is the pullback of this morphism along a unique morphism), the category $\mathcal{C}$ is called a topos, then $\mathcal{C}$ is an exact category.

### 3.6.4 The particular case of Frobenius categories

Now we recall the definition of a Frobenius category. For further information, see the book of Happel, [Hap88].

**Definition 3.6.8.** *Let $\mathcal{E}$ be an additive category.*

*An object $P \in \mathcal{E}$ is said to be projective if, for any deflation $\pi : A \twoheadrightarrow B$, and any morphism $f : P \to B$, there exists a lift $g : P \to A$ such that $\pi \circ g = f$.*

*An object $I \in \mathcal{E}$ is said to be injective if, for any inflation $\iota : A \twoheadrightarrow B$, and any morphism $f : A \to I$, there exists a lift $g : B \to I$ such that $\iota \circ g = f$.*

**Definition 3.6.9.** *An additive category $\mathcal{E}$ has enough projective objects if, for any $B \in \mathcal{E}$, there exists a deflation $\pi : P \twoheadrightarrow B$ with $P$ projective.*

*An additive category $\mathcal{E}$ has enough injective objects if, for any $A \in \mathcal{E}$, there exists an inflation $\iota : A \twoheadrightarrow I$ with $I$ injective.*

**Definition 3.6.10.** *An exact category $\mathcal{E}$ is said to be Frobenius if is has enough projective and injective objects, and if the projective coincide with the injective.*

## 3.7 Weakly idempotent complete categories

We now see a particularity of some categories, which we will use for exact categories (See [Büh10, Section 7]). This type of categories permits to build easily short exact sequences.

**Definition 3.7.1.** *In a category $\mathcal{E}$ (not necessarily exact), a morphism $r : A \to B$ is called a retraction if there exists a section $s : B \to A$ such that $r \circ s = 1_B$. Dually, a morphism $c : A \to B$ is a coretraction if there exists a section $s :\to A$ such that $s \circ c = 1$.*

**Proposition 3.7.2.** *If $\mathcal{E}$ is an additive category (not necessarily exact), the following are equivalent:*

*(i) Any coretration has a cokernel*

*(ii) Any retraction has a kernel*

**Definition 3.7.3.** *If one of the conditions of the previous lemma holds, then the category $\mathcal{E}$ is said to be weakly idempotent complete.*

We now have the following useful proposition:

**Proposition 3.7.4.** *Let $\mathcal{E}$ be an exact category. Then the following are equivalent:*

*(i) The category $\mathcal{E}$ is weakly idempotent complete.*

*(ii) For any two morphisms $f : A \to B$ and $g : B \to C$, if $g \circ f : A \twoheadrightarrow C$ is a deflation then $g$ is a deflation.*

## 3.8 Quotient of exact categories by cluster-tilting subcategories as module categories

In 2007, Buan, Marsh and Reiten in [BMR07] showed that we can pass from triangulated categories to abelian categories by factoring out any cluster-tilting subcategory. The aim of this section is to present the analogous result shown by Demonet and Liu in [DL13] for exact categories. In fact, the main theom is more interesting than that, and this thesis uses it a lot in order to prove the theorem of Quillen in some cases (see the dedicated section on preliminaries).

**Theorem 3.8.1** ([DL13]). *Let $\mathcal{E}$ be an exact category. Let $\mathcal{M}$ be a full rigid subcategory which contains the injective objects. Let*

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

*and*

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

*These are short exact sequences. Let $G$ be the following functor:*

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-,X)/\overline{\mathcal{M}} \end{array}$$

*Then, the functor $G$ induces the following equivalence:*

$$\mathrm{pr}\mathcal{M}/\mho\mathcal{M} \simeq \mathrm{mod}\overline{\mathcal{M}}.$$

They show analogously:

**Theorem 3.8.2** ([DL13]). *Let $\mathcal{E}$ be an exact category. Let $\mathcal{M}$ be a full rigid subcategory which contains the projective objects. Let*

$$\mathrm{copr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to X \to M_0 \to M_1 \to 0\}$$

*and*

$$\Omega\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, P \in \mathrm{Proj}, 0 \to X \to P \to X \to M\}.$$

*These are short exact sequences. Let $H$ be the following functor:*

$$\begin{array}{rccc} H & : & \mathcal{E} & \to & \mathrm{Mod}\underline{\mathcal{M}} \\ & & X & \mapsto & \mathrm{Ext}^1_{\mathcal{E}}(-,X)/\mathcal{M} \end{array}$$

*Then, the functor $H$ induces the following equivalence:*

$$\mathrm{copr}\mathcal{M}/\Omega\mathcal{M} \simeq \mathrm{mod}\underline{\mathcal{M}}.$$

This has important consequences:

**Corollary 3.8.3.** *In the first case, if we denote by* $\overline{\mathcal{M}}^{\perp}$ *the set* $\{X \in \mathcal{E}, \overline{\mathcal{E}}(\mathcal{M}, X) = 0\}$*, then we have that*

$$\mho\mathcal{M} = \overline{\mathcal{M}}^{\perp} \cap \mathrm{pr}\mathcal{M}.$$

*In the dual case, if* $\mathcal{M}$ *is rigid and contravariantly finite (it means that there exist some* $\mathcal{M}$*-approximations), then the quotient category* $\mathrm{copr}\mathcal{M}/\Omega\mathcal{M}$ *is abelian.*

## 3.9 Localization of triangulated categories

In their article [BM13], Buan and Marsh built an equivalence of categories between a localization of a triangulated category, and a module category. This is what we generalize to Frobenius categories, and then exact categories.

**Theorem 3.9.1** (Buan-Marsh, [BM13])**.** *Let* $\mathcal{C}$ *be a skeletally small, Hom-finite, Krull-Schmidt triangulated category with a rigid object* $T$*. Let* $\chi_T$ *be the full subcategory of* $\mathcal{C}$ *whose objects are the objects of* $\mathcal{C}$ *having no non-zero maps from* $T$*. Let* $\tilde{S}$ *be the class of maps* $f$ *that can be completed to a triangle* $f, g, h$ *in which* $g, h \in (\chi_T)$*. Then, the induced functor* $G : \mathcal{C}_{\tilde{S}} \to \mathrm{mod}\ \mathrm{End}_C(T)^{\mathrm{op}}$ *is an equivalence.*

## 3.10 Model categories

Model categories have been invented in order to give a reasonable context to do homotopy theory. It has been developed by Quillen in [Qui67]. For a gentle introduction, see [DS95], or for a deeper study, see the book of Hovey, [Hov99].

### 3.10.1 Definition of a model category

We start with a preliminary definition.

**Definition 3.10.1.** *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array}$$

*a lift of the diagram is a morphism* $g : B \to X$ *such that* $g \circ h = a$ *and* $f \circ g = b$ *(we say that both triangles commute).*

**Definition 3.10.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{W}$ be the class of weak equivalences, $\mathcal{F}ib$ be the class of fibrations, and $\mathcal{C}of$ be the class of cofibrations. We call an element of $\mathcal{F}ib \cap \mathcal{W}$ (respectively an element of $\mathcal{C}of \cap \mathcal{W}$) an acyclic fibration (respectively an acyclic cofibration). The category $\mathcal{C}$ is said to have a model structure if the triple $(\mathcal{W}, \mathcal{F}ib, \mathcal{C}of)$ and $\mathcal{C}$ satisfy the following properties:*

**MC 1** *Finite limits and colimits exist in $\mathcal{C}$.*

**MC 2** *$\mathcal{W}$ has the "two-out-of-three" property, it means that, for two composable maps $g$ and $f$, if two of the three maps $f$, $g$ or $gf$ are in $\mathcal{W}$, so is the third.*

**MC 3** *The three sets $\mathcal{F}ib$, $\mathcal{C}of$ and $\mathcal{W}$ are stable under retracts.*

**MC 4** *For a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array}$$

*a lift exists if: either $h$ is a cofibration and $f$ is an acyclic fibration (meaning that $f \in \mathcal{F}ib \cap \mathcal{W}$) or $h$ is an acyclic cofibration and $f$ is a fibration.*

**MC 5** *Any morphism can be factored in two ways: a cofibration followed by an acyclic fibration, and an acyclic cofibration followed by a fibration.*

**Remark 3.10.3.** *Here, we observe that this is in fact the definition of a closed model category. Indeed, all throughout this thesis, we only work with closed model categories.*

**Proposition 3.10.4.** *If $\mathcal{C}$ is a model category, then $\mathcal{C}^{\mathrm{op}}$ is also a model category, with fibrations and cofibrations exchanged.*

**Definition 3.10.5.** *An object $X$ of a model category $\mathcal{C}$ is said to be fibrant if the morphism $X \to *$ is a fibration (where $*$ is the terminal object of $\mathcal{C}$, existing from MC 1). It is said to be cofibrant if the morphism $\emptyset \to X$ is a cofibration (where $\emptyset$ is the initial object of $\mathcal{C}$).*

### 3.10.2 Some properties of model categories

The following lemma is actually a corollary of the retract argument, which we do not need here.

**Lemma 3.10.6.** *If $\mathcal{C}$ is a model category, then cofibrations (and acyclic cofibrations) are closed under pushouts. That is to say, if we have a diagram*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array},$$

*if $h$ is a cofibration (respectively an acyclic cofibration), then $f$ is one too. Dually, fibrations (and acyclic fibrations) are closed under pullbacks.*

In his book [Hov99], Hovey showed a very useful theorem about model categories. He has shown an equivalent definition for categories with all small limits and colimits.

**Theorem 3.10.7** ([Hov99])**.** *Let $\mathcal{C}$ be a category. Let $\mathcal{W}$, $\mathcal{F}ib$, $\mathcal{C}of$ be three classes of morphisms, where $\mathcal{F}ib$ (respectively $\mathcal{C}of$) is obtained from an orthogonal relation $\square$ on a set of morphisms $J$ (respectively $I$). Then there is a model structure on $\mathcal{C}$ if and only if:*

*(i) The set of morphisms $\mathcal{W}$ has the "two-out-of-three" property, and is stable under retracts.*

*(ii) We have $J^{\square} \cap \mathcal{W} = I^{\square}$.*

*(iii) We have ${}^{\square}(J^{\square}) \subseteq \mathcal{W} \cap {}^{\square}(I^{\square})$.*

*(iv) Any morphism can be factored through a weak cofibration followed by a fibration. Any morphism can be factored through a cofibration followed by a weak fibration.*

**Remark 3.10.8.** *In his article, Hovey does not write the theorem in this way. As he uses the small object argument, he does not need the factorizations, but he needs that the domains of $I$ (respectively $J$) are small relative to the class of colimits of elements of $I$ (respectively $J$). It is this way he finds the factorizations.*

*We did not use it like this, because in some examples (for instance, see [BIKR08]), we do not have all the colimits. Then, we find directly the factorization without using the small object argument.*

### 3.10.3 Examples of model categories

The category of topological spaces Top is equipped with a structure of model category with:

— The morphism $f$ is a weak equivalence if it is a weak homotopy equivalence.

— The morphism $f : X \to Y$ is a cofibration if it is a retract of a map $X \to Y'$, where $Y'$ is obtained from $X$ by attaching cells.

— The fibrations are Serre fibrations.

If $\mathcal{C}$ and $\mathcal{D}$ are model categories, we can construct $\mathcal{C} \times \mathcal{D}$, the product category, which is, in the natural way, a model category.

## 3.11 Prefibration and precofibration categories

Some categories are not nice enough in order to build a model structure on them. For example, one can have cofibrations, but no fibrations. Then, to deal with such situations, Anderson, Brown and Cisinski have given axioms for precofibration categories, and prefibration categories. This section is dedicated to this. A quick definition is given by Brown in [Bro73]. For a further study, see the book of Radulescu [RB06].

### 3.11.1 Definition of prefibration and precofibration categories

**Definition 3.11.1.** *An ABC prefibration category (ABC stands for Anderson-Brown-Cisinski) consists of a category $\mathcal{E}$, with two classes of maps, the fibrations $\mathcal{F}ib$ and the weak equivalences $\mathcal{W}$ satisfying the following axioms:*

**F 1** *$\mathcal{E}$ has a final object which is fibrant. Fibrations are stable under composition. All isomorphisms are weak equivalences, and all isomorphisms with fibrant codomain are trivial functions (fibrations which are also weak equivalences).*

**F 2** *$\mathcal{W}$ has the "two-out-of-three" property, meaning that, for two composable maps $g$ and $f$, if two of the three maps $f$, $g$ or $gf$ are in $\mathcal{W}$, so is the third.*

**F 3** *For any diagram*

$$\begin{array}{ccc} A & \overset{a}{\dashrightarrow} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array}$$

*the pullback exists in $\mathcal{E}$. Moreover, if $f$ is a fibration (respectively a trivial fibration), then $h$ is a fibration (respectively a trivial fibration).*

**F 4** *Any morphism $f : A \to B$, with $B$ fibrant, can be factored through a weak equivalence followed by a fibration.*

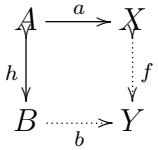
Dually, one can define precofibration categories.

**Definition 3.11.2.** *An ABC precofibration category consists of a category $\mathcal{E}$, with two classes of maps, the cofibrations $\mathcal{C}of$ and the weak equivalences $\mathcal{W}$ satisfying the following axioms:*

**COF 1** *$\mathcal{E}$ has an initial object which is cofibrant. Cofibrations are stable under composition. All isomorphisms are weak equivalences, and all isomorphisms with cofibrant domain are trivial functions (cofibrations which are also weak equivalences).*

**COF 2** *$\mathcal{W}$ has the "two-out-of-three" property, meaning that, for two composable maps $g$ and $f$, if two of the three maps $f$, $g$ or $gf$ are in $\mathcal{W}$, so is the third.*

**COF 3** *For any diagram*



*the pushout exists in $\mathcal{E}$. Moreover, if $f$ is a cofibration (respectively a trivial cofibration), then $h$ is a cofibration (respectively a trivial cofibration).*

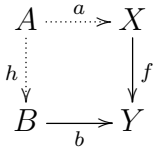
**COF 4** *Any morphism $f : A \to B$, with $A$ cofibrant, can be factored through a cofibration followed by a weak equivalence.*

### 3.11.2 Properties of such categories

In the following theorem, $\mathcal{C}$ does not need to be a prefibration (or precofibration) category.

**Theorem 3.11.3** (Radulescu-Banu,[RB06], Theorem 6.4.2)**.** *If $\mathcal{E}, \mathcal{W}$ is a category pair satisfying the following:*

1. *The two-of-the-three axiom.*
2. *For a diagram*

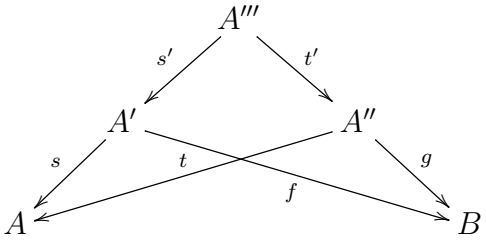


*the pullback exists in $\mathcal{E}$. Moreover, if $f$ is a weak equivalence, then $h$ is a weak equivalence.*

3. *For any maps $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{t} B'$ with $t \in \mathcal{W}$, and $t \circ f = t \circ g$, there exists $t' : A' \to A \in \mathcal{W}$ such that $f \circ t' = g \circ t'$.*

*Then, we have the following results:*

1. *Each map $h : A \to B$ in the homotopy category can be written as a right fraction $f \circ s^{-1}$, with $s \in \mathcal{W}$.*
2. *Two fractions $f \circ s^{-1}$ and $g \circ s^{-1}$ are equal in the homotopy category if and only if there exist weak equivalences $s'$, $t'$, as in the diagram below, and such that $s \circ s' = t \circ t'$ and $f \circ s' = g \circ t'$:*




## 3.12 The homotopy category of a model category

In this section, we explain the main theorem of Quillen on homotopy categories. He actually shows that the homotopy category of a nice category $\mathcal{C}$ is equivalent to the category of fibrant and cofibrant objects quotiented by a homotopy relation. This result needs a litte bit of background on homotopy categories. For the reader who wants to know more, see [DS95] of [Hov99].

### 3.12.1 Definition of the homotopy category

This definition comes from Hovey in [Hov99].

**Definition 3.12.1.** *Let $\mathcal{C}$ be a category which contains a subcategory of weak equivalences, called $\mathcal{W}$. Let us form the free category $\mathcal{F}_{\mathcal{C}}$ defined in the following way: the objects of $\mathcal{F}_{\mathcal{C}}$ are the same as the ones of $\mathcal{C}$. A morphism of $\mathcal{F}_{\mathcal{C}}$ is a finite string of composable elementary morphism where the elementary morphisms are either morphisms of $\mathcal{C}$ or weak inverses of morphisms of $\mathcal{W}$. The empty string is the identity, and the composition is defined naturally by the composition of strings.*

*Then the homotopy category* Ho $\mathcal{C}$ *of $\mathcal{C}$ is the quotient of $\mathcal{F}_{\mathcal{C}}$ by the relations (the notation* dom $f$ *stands for the domain of $f$, and* codom $f$ *stands for the codomain of $f$):*

- $1_A = (1_A)\ \forall A \in \mathcal{C}$
- $(f, g) = (g \circ f)$ *for any composable* $f, g$
- $1_{\text{dom } f} = (f, f^{-1})$ *for all* $f \in \mathcal{W}$
- $1_{\text{codom } f} = (f^{-1}, f)$ *for all* $f \in \mathcal{W}$.

At this moment, we do not know that Ho $\mathcal{C}$ is a category. Indeed, the class of morphisms is not a set. The theorem of Quillen will tell us that , if $\mathcal{C}$ is a model category, then Ho $\mathcal{C}$ is a category. If $\mathcal{C}$ is not a model category, it is necessary to find a higher universe to consider Ho $\mathcal{C}$ as a category. We do not consider universe problems, and then we assume Ho $\mathcal{C}$ is a category.

**Definition 3.12.2.** *Let $\mathcal{C}$ be a model category. We call by $\mathcal{C}_{cf}$ the full subcategory of $\mathcal{C}$ containing the objects which are both fibrant and cofibrant.*

**Theorem 3.12.3.** *There is an equivalence of categories*

$$\text{Ho } \mathcal{C}_{cf} \to \text{Ho } \mathcal{C}.$$

Now we define cylinder and path objects.

**Definition 3.12.4.** *Let $X \in \mathcal{E}$. A cylinder object for $X$ is a factorization of the morphism $\nabla : X \oplus X \to X$ (which is the identity on each copy of $X$).*

*Let $f, g : X \to Y$ be two morphisms. A left homotopy from $f$ to $g$ is a morphism $h : X' \to Y$, where $X'$ is a cylinder object for $X$, such that $h \circ \nabla = (f \ g)$.*

Dually, we can define path objects and right homotopies.

**Definition 3.12.5.** *Let $Y \in \mathcal{E}$. A path object for $Y$ is a factorization of the morphism $\Delta : Y \to Y \oplus Y$ (which is the identity on each copy of $Y$).*

*Let $f, g : X \to Y$ be two morphisms. A right homotopy from $f$ to $g$ is a morphism $k : X \to Y'$, where $Y'$ is a path object for $Y$, such that $\Delta \circ k = \begin{pmatrix} f \\ g \end{pmatrix}$.*

**Definition 3.12.6.** *Two morphisms $f$ and $g$ are said to be homotopic if they are both right and left homotopic. We note $f \sim g$.*

*We say that a morphism $f : A \to B$ is a homotopy equivalence if there exists $g : B \to A$ such that $g \circ f \sim 1$ and $f \circ g \sim 1$.*

### 3.12.2 Theorem of Quillen

Before claiming the theorem, we start with a useful lemma.

**Lemma 3.12.7.** *The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation compatible with composition. Thus, the category $\mathcal{C}_{cf}/\sim$ exists.*

*Moreover, weak equivalences are exactly homotopy equivalences.*

**Theorem 3.12.8** (Quillen, [Qui67])**.** *Let $\mathcal{C}$ be a model category. Then there is an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\sim} \text{Ho } \mathcal{C}\ .$$

*Isomorphisms in the homotopy category are exactly weak equivalences.*

**Remark 3.12.9.** *In his proof in [Hov99], Hovey only needs the existence of the second factorization on cofibrant objects.*

## 3.13 A weak model structure on triangulated categories

In his paper [Pal14], Palu has shown that there was a weak model structure on triangulated categories, which satisfy the theorem of Quillen.

**Theorem 3.13.1** (Palu,[Pal14]). *Let $\mathcal{C}$ be a triangulated category with a covariantly finite, rigid subcategory $\mathcal{T}$. There is a left-weak model category structure on $\mathcal{C}$. It means that $\mathcal{C}$ contains three classes of morphisms $\mathcal{W}$, $\mathcal{F}ib$, $\mathcal{C}of$ such that:*

1. *Pullbacks of trivial fibrations along deflations exist and are trivial fibrations.*
2. *If $A$ is cofibrant, then for any cofibration $A \to B$, $B$ is cofibrant. If $B$ is cofibrant, then the inclusions $A \to A \oplus B$ and $A \to B \oplus A$ are cofibrations.*
3. *The class $\mathcal{W}$ has the "two-out-of-three" property.*
4. *The three classes $\mathcal{W}$, $\mathcal{F}ib$, $\mathcal{C}of$ contain identity, are stable under retracts and composition.*
5. *We have $\mathcal{W} \cap \mathcal{C}of \subseteq^{\square} \mathcal{F}ib$ and $\mathcal{C}of \subseteq^{\square} (\mathcal{W} \cap \mathcal{F}ib)$*
6. *Any morphism can be factored through a weak cofibration followed by a fibration. Any morphism with cofibrant domain can be factored through a cofibration followed by a weak fibration.*

## 3.14 Some examples of such categories

As it has been said in the introduction, there are several examples of applications of this article.

**Definition 3.14.1.** *Let $A$ be a finite-dimensional algebra over an algebraically closed field $K$. Let $Q = (Q_0, Q_1, s, t)$ be a Dynkin quiver. Let $\overline{Q}$ be the quiver obtained from $Q$ by adding, for each arrow $\alpha$ of $Q$ from $i$ to $j$, an arrow $\alpha^*$ from $j$ to $i$. Let $K\overline{Q}$ be the path algebra over $\overline{Q}$. Then the preprojective algebra associated with $Q$ is defined as*

$$\Lambda_Q = K\overline{Q}/\mathcal{I}$$

*where $\mathcal{I}$ is the ideal generated by the element*

$$\sum_{\alpha \in Q_1} (\alpha^* \circ \alpha - \alpha \circ \alpha^*).$$

**Lemma 3.14.2.** *Let $Q$ and $Q'$ be two quivers of the same Dynkin type. Then $\Lambda_Q$ is isomorphic to $\Lambda_{Q'}$.*

**Theorem 3.14.3.** *Let $T$ be a maximal rigid $\Lambda$-module. Then the functor*

$$\mathrm{Hom}_\Lambda(-, T) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\mathrm{End}_\Lambda(T))$$

*is fully faithful, and its image is the category of $\mathrm{End}_\Lambda(T)$-modules of projective dimension at most one.*

*Moreover, if $T_1$ and $T_2$ are two maximal rigid modules, then $\mathrm{End}_\Lambda(T_1)$ and $\mathrm{End}_\Lambda(T_2)$ are derived equivalent.*

They also have shown similar results about more general quivers.

**Definition 3.14.4.** *Let $Q$ be a finite connected quiver without oriented cycles and $n$ vertices, and let $\Lambda$ be its associated preprojective algebra. As $KQ$ is a subalgebra of $\Lambda$, we can introduce $\pi_Q : \mathrm{mod}(\Lambda) \to \mathrm{mod}(KQ)$ as the restriction functor.*

*Let $M$ be a $KQ$-module. Then $M$ is called terminal if the following hold:*

- *$M$ is preinjective*
- *If $X$ is an indecomposable $KQ$-module, with $\mathrm{Hom}_{KQ}(M, X) \neq 0$, then $X \in \mathrm{add}\ M$.*
- *Any indecomposable injective $KQ$-module belongs to $\mathrm{add}\ M$.*

**Theorem 3.14.5.** *Let $\mathcal{C}_M$ be the subcategory of all $\Lambda$-modules which image under $\pi_Q$ belongs to $\mathrm{add}\ M$. Then the following holds:*

- *The category $\mathcal{C}_M$ is Frobenius with $n$ indecomposable projectives.*
- *The stable category of $\mathcal{C}_M$ is 2-Calabi-Yau.*

If we consider the non-Frobenius case, Adachi in [Ada15] has shown the following results:

**Theorem 3.14.6.** *Let $\Lambda$ be a finite-dimensional algebra with radical square zero. Then the following are equivalent:*

1. *The algebra $\Lambda$ is $\tau$-rigid-finite*
2. *Any single subquiver of the separated quiver for $\Lambda$ is a disjoint union of Dynkin quivers.*

**Corollary 3.14.7.** *Under the same assumptions, if any indecomposable $\Lambda$-module is $\tau$-rigid, then $\Lambda$ is representation-finite.*

# Part I

# Geometric realizations of $m$-cluster categories

Chapter 1

# A geometric realization of the $m$-cluster categories of type $\tilde{D}_n$

## 1.1 How to associate a colored quiver with an $(m+2)$-angulation

In this section we introduce the geometric realizations. We define the flip of an $(m+2)$-angulation, and build a colored quiver from an $(m+2)$-angulation. The main theorem of the section, namely theorem 2.1.12, states the compatibility between the flip of an $(m+2)$-angulation and the mutation of the associated colored quiver.

### 1.1.1 Geometric realization and flips

Let $n > 4$ and $m$ be a positive integer. Let $P$ be an $(n-2)m$-gon with two central $(m-1)$-gons $R$ and $S$ inside of it (cf figure 1.1). We replace each vertex of $R$ and $S$ by a disk, which we henceforth call a thick vertex. If $m = 1$, then we consider an $(n-2)$-gon with two disks inside of it.

We now define the notion of a tangent arc.

**Definition 1.1.1.** *Consider one moment an arc starting at an arbitrary vertex of $P$, and ending at a thick vertex of $P$ is called to be left tangent at $R$ if this arc is $\mathcal{C}^\infty$, tangent to a thick vertex, and if there exists a neighborhood of this arc such that the thick vertex of $R$ is situated at the right of the arc. We similarly define right tangency.*

**Definition 1.1.2.** *Let us number the vertices of the polygon $P$ from $P_1$ to $P_{(n-2)m}$ clockwise. Then, an admissible arc between $P_i$ and $P_j$ is defined in the following way:*

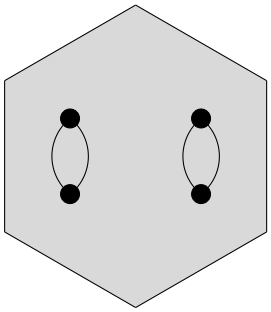


Figure 1.1 – The $(n-2)m$-gon with two digons. Here $m = 3$ and $n = 4$.

1. *If $i \neq j$, then an admissible arc is an oriented path from $P_i$ to $P_j$, lying inside of $P$, which does not cross any of the two inner polygons, satisfying one of the following conditions:*
   - *Either the arc crosses the space between both central polygons and cuts the figure into a $km+1$-gon and a $k'm+1$-gon, for some $k$ ($k'$ is entirely determined by $k$). This arc is of type 1.*
   - *Or, the arc is homotopic to the boundary path, and cuts the figure into a $km$-gon with both central polygons inside of it and a $k'm+2$-gon (where $k'$ is still entirely determined by $k$). This arc is of type 2.*
2. *If $P_i = P_j$, then an admissible arc is a path ending in $P_i$, and the other end of the path is tangent to one of the thick vertices placed around $R$ and $S$. There are two more admissible arcs starting and ending at $i$, going around one of the inner polygons: the left loop and the right loop. They look the same on the picture, but they are labeled differently, we will see later why.*
3. *Any arc being tangent to two disks, one arising from $R$, and one from $S$ is admissible.*

**Remark 1.1.3.** *We will often write $i$ for $P_i$.*

**Definition 1.1.4.** *We denote by $B_{ij}$ the boundary path from $i$ to $j$ going clockwise around the boundary (where $B_{ii}$ denotes the trivial path at $i$). We call a "t-angle", a figure delimited by arcs and/or $B_{ij}$ for some distinct $i$ and $j$ and/or sides of the inner polygons $R$ and $S$, where the total number of sides and arcs is $t$.*

Note that we only consider unoriented arcs, the order of $i$ and $j$ does not matter. For convenience, we will nevertheless use the terminology "from $i$ to $j$".

**Notation 1.1.5.** *We call a Dehn twist around $R$ the action of rotating $R$. It means that if we consider an arc $\alpha$ hung to $R$, applying a Dehn twist of $R$ makes $\alpha$ roll around $R$ only.*

**Definition 1.1.6.** *An $m$-diagonal is an equivalence class of admissible arcs where the different classes are:*

1. *If $i \neq j$, type 1: If the arc cuts the figure into a $km+1$-gon and a $k'm+1$-gon, then a class up to homotopy of $m$-diagonals of type 1 is built for each value of $k$.*
2. *If $i \neq j$, type 2: If the arc cuts the figure into a $km$-gon with both central polygons inside of it and a $k'm+2$-gon, then a class up to homotopy of $m$-diagonals of type 2 is built for each value of $k$.*
3. *If $i = j$: a class contains all the images of Dehn twists of admissible arcs tangent to the left of a thick vertex of $R$ plus the left loop described in the remark below. The other classes are given by all the images of Dehn twists of admissible arcs tangent to the right of a thick vertex. Similar classes are built in the same way for $S$.*
4. *Any admissible arc up to homotopy linking both central polygons form a class of $m$-diagonal.*

**Remark 1.1.7.** *Let $i$ be a vertex of $P$, and $\alpha$ be an arc tangent to the left of a thick vertex of $R$. Then the loop belonging to the class of $\alpha$ is the arc up to homotopy going around $R$ without crossing $\alpha$ (and not homotopic to the boundary path). The same description can be done for right-tangency and $S$.*

*Note that there is only one loop for each class containing a tangent $m$-diagonal.*

**Remark 1.1.8.** *If $i = j$, then, there exists an infinity of classes of arcs tangents to the left or $R$. Indeed, the $m$-diagonals can roll around $R$ and $S$, and this leads to two different classes of arcs. See figure 1.2 for an illustration.*

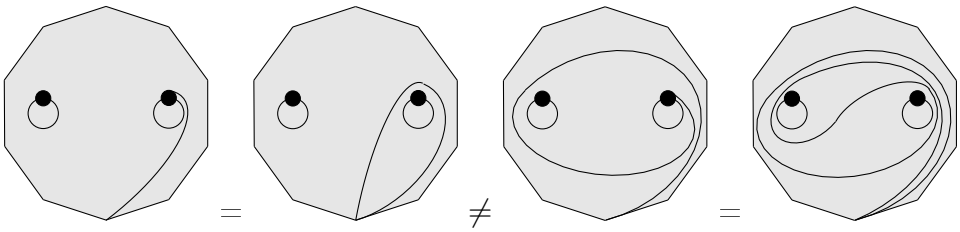



Figure 1.2 – Both figures on the left represent the same $m$-diagonal, which is different from the $m$-diagonal on both figures on the right.

In figure 1.4, we can see an example of two arcs corresponding to the same $m$-diagonal, where we have applied a Dehn twist to one of the polygons.

**Definition 1.1.9.** *1. — If two arcs are not tangent to the same inner polygon, they are said to be noncrossing if their class under homotopy contains representatives which do not cross.*

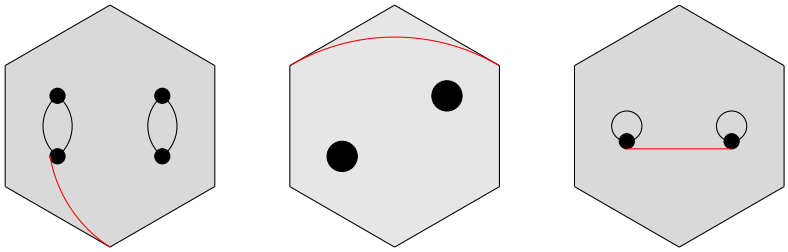


Figure 1.3 – Some examples of arcs in the hexagon. For example, if $m \neq 1$, the second arc is not admissible because it cuts the polygon into a 3-gon and a 5-gon.

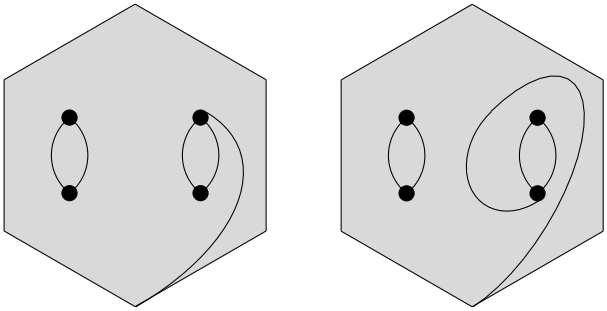


Figure 1.4 – These two arcs represent the same $m$-diagonal.

- *If they both are tangent to the same inner polygon:*
  - *If they have the same tangency (left for example), they are said to be noncrossing if their class under homotopy contain representatives which do not cross.*
  - *If not, replace one of them by a loop, and they are said to be noncrossing if their class under homotopy contain representatives which do not cross.*
- *Two $m$-diagonals linking both inner polygons are said to be noncrossing if their classes under homotopy contain representatives which do not cross.*

2. *An $(m+2)$-angulation is a set of noncrossing $m$-diagonals, such that there exists representative admissible arcs which cut the figure into $(m+2)$-angles. Such a choice of representatives is called a good set of representatives.*

See figure 1.5 for examples of an $(m+2)$-angulation.

**Definition 1.1.10.** *For any $m$ and $n \geq 4$, we define the initial $(m+2)$-angulation in the following way: From the vertex 1 of $P$, we draw:*

1. *An arc tangent to the right of the disk composing a thick vertex $R_i$ of $R$.*

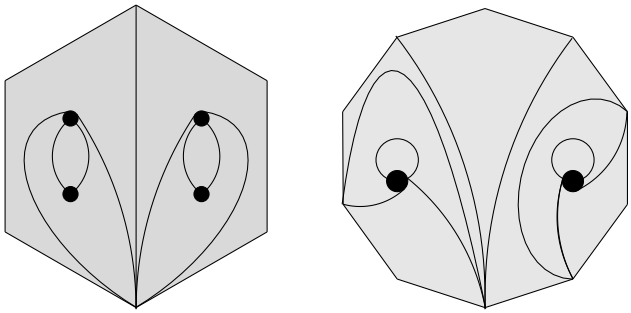


Figure 1.5 – On the left, a collection of arcs that is not a good set of representatives since it does not cut the figure into $(m+2)$-angles. An example of a good set of representatives is shown on the right.

2. *A left loop around $R$.*
3. *An arc tangent to the right of the disk composing a thick vertex $S_i$ of $S$.*
4. *A left loop around $S$.*
5. *The remaining arcs of the initial $(m+2)$-angulations are all of type 1. The first is from 1 to $m+1$, the second from 1 to $2m+1$... and the last from 1 to $(n-3)m+1$.*

*There is an exception if $n=4$. Then $P$ has $2m$ sides, and there is only one arc of type 1, which links vertex 1 to vertex $m$. The arcs hung to $R$ are defined similarly, and the arcs hung to $S$ start at $m$ (and not 1 as previously). See figure 1.6 for the example of $m=3$.*

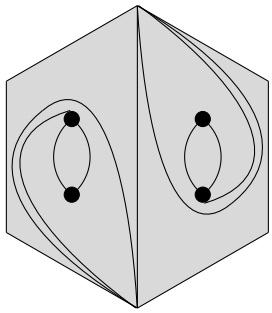


Figure 1.6 – Exception of 1.1.10: the initial 4-angulation of $Q$ for $n=4$ and $m=3$

It is obvious that the arcs of the previous definition form a good set of representatives for the initial $(m+2)$-angulation. See figure 1.7 for an example of initial 4-angulation.

Note that we do not have to label the loops, since their labels are automatically deduced from the other arc ending in the corresponding inner polygon.

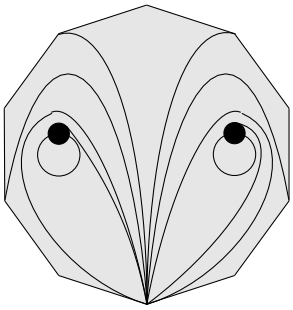


Figure 1.7 – The initial 4-angulation for $m = 2$ and $n = 7$.

**Lemma 1.1.11.** *Let $\Delta$ be an $(m+2)$-angulation. Let $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ be two good sets of representatives for $\Delta$. Then there exists $\theta = (\theta_R, \theta_S)$ a Dehn twist on $R$ and $S$ such that, after exchanging some choices of left or right loops if necessary, for all $i$, $\theta(\beta_i) = \alpha_i$ up to homotopy.*

*Proof.* We may assume that all the loops which appear in the good sets of representatives are left loops. We are only interested in the arcs attached to $R$. The case is similar for $S$.

Let $\alpha_i$ be an arc an end point of which is a thick vertex of $R$. Let $\theta_i$ be a Dehn twist such that $\theta_i(\beta_i) = \alpha_i$. Let $\alpha_j$ be the neighbor of $\alpha_i$ according to $R$. Let $\theta_j$ such that $\theta_j(\beta_j) = \alpha_j$. Then the number of sides of $R$ between $\alpha_i$ and $\alpha_j$ is the same as the number of sides between $\beta_i$ and $\beta_j$. Indeed, the sets $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ are $(m+2)$-angulations, and the other end point of $\alpha_i$ and $\beta_i$ (respectively $\alpha_j$ and $\beta_j$) is a vertex $a_i$ of $P$ (respectively $a_j$). Then to respect the notion of $(m+2)$-angles in an $(m+2)$-angulation, it means that if $k$ is the number of edges of $R$ between $\alpha_i$ and $\alpha_j$, $k$ is also the number of edges of $R$ between $\beta_i$ and $\beta_j$. Then $\theta_i = \theta_j$.

We can reproduce this process until we have treated all the arcs ending on $R$. □

We now introduce the twist which will be used in order to define the flip of an $(m+2)$-angulation at some $m$-diagonal.

**Definition 1.1.12.** *Let $\Delta$ be an $(m+2)$-angulation. Let $\alpha$ be an $m$-diagonal of $\Delta$, linking the vertices $a$ and $b$. If, in $\Delta$, there are two $m$-diagonals which are left and right tangent to the same inner polygon, replace one of them by a loop. When $\alpha$ is tangent to an inner polygon (say $R$) and there are only two arcs tangent to $R$, always replace $\alpha$ by a loop. If $\alpha$ is tangent to $R$ and the other arc is a loop, then replace this loop by the corresponding arc tangent to $R$, and $\alpha$ by a loop. The twist of $\alpha$ in $\Delta$ is defined as follows:*

*Pick any good set of representatives for $\Delta$ containing $\alpha$. Let $\alpha_a$ (respectively $\alpha_b$) be the side of the $(m+2)$-angle ending at $a$ (respectively at $b$) consecutive to $\alpha$. Then the twist of $\alpha$, namely $\kappa_\Delta(\alpha)$ is the path $\alpha_a\alpha\alpha_b$. See figure 1.8 for an illustration of the twist.*

**Remark 1.1.13.** *If necessary, we can always choose a set of representative where $\alpha$ is a loop. Indeed, if $\alpha$ is not a loop (hung to vertex $i$ of $P$, left tangent for instance), the chosen good set of representatives for the $(m+2)$-angulation contains an arc which is tangent to $R$ to the right, and has a common endpoint with $\alpha$ (called $i$). Exchange $\alpha$ for the left loop.*

*After applying the loop, note that the set of representatives is still a good one.*

**Lemma 1.1.14.** *The new equivalence class obtained by twisting the $m$-diagonal does not depend on the choice of the set of good representatives to which $\alpha$ belongs.*

**Remark 1.1.15.** *Here, as in lemma 1.1.11, we note that we choose a set of good representatives in which the arc corresponding to the $m$-diagonal $\alpha$ is a loop. As explained in remark 1.1.13, this is always possible when necessary.*

*Proof.* Let $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ be two good sets of representatives for $\Delta$. There exists $\theta$ such that for all $i$, $\theta(\beta_i) = \alpha_i$, and $\theta$ commutes with the twist $\kappa_\Delta$. □

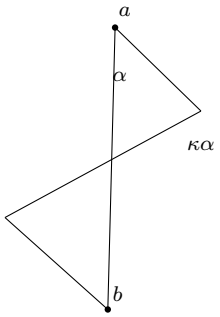



Figure 1.8 – Definition of a twist

In fact, applying the twist to $\alpha$ consists in "slipping" $\alpha$ clockwise. The previous lemma says that "slipping along" a side of $R$ does not depend on the choice of a class under Dehn twist.

**Definition 1.1.16.** *Consider $\Delta$ an $(m+2)$-angulation. Choose a good set $\{\gamma_1, \cdots, \gamma_n\}$ of representatives for $\Delta$. Let $\alpha$ be an arc in it. From remark 1.1.13, we may assume that $\alpha$ is not a loop. The flip of the $(m+2)$-angulation at $\alpha$ is defined by $\mu_\alpha\Delta = \Delta \setminus \{\alpha\} \cup \{\alpha^*\}$ where $\alpha^*$ is given by $\kappa_\Delta(\alpha)$, the twist of $\alpha$.*

**Remark 1.1.17.** *Here again, the flip does not depends on the choice of the set of good representatives before flipping.*

**Remark 1.1.18.** *If we want to flip an m-diagonal $\alpha$ which is tangent to the left to one side of R (when at the same time there is an m-diagonal tangent to the right on the same thick vertices as $\alpha$), then we have to choose the left loop, in order to make sure we have chosen the good representative. The issue is the same for any arcs tangent to the right.*

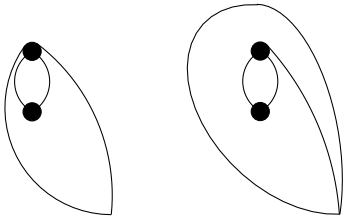


Figure 1.9 – The arc tangent to the left and the loop belong to the same class of $m$-diagonals. The figure on the right is a 5-angle (going twice through the same arc).

**Remark 1.1.19.**
1. *Note that the twist has an inverse, which consists in moving the arc counterclockwise. Then the flip is also invertible.*
2. *We note that the twist is not the same as the Dehn twist.*
3. *A flip does not change the number of m-diagonals in the $(m+2)$-angulation.*

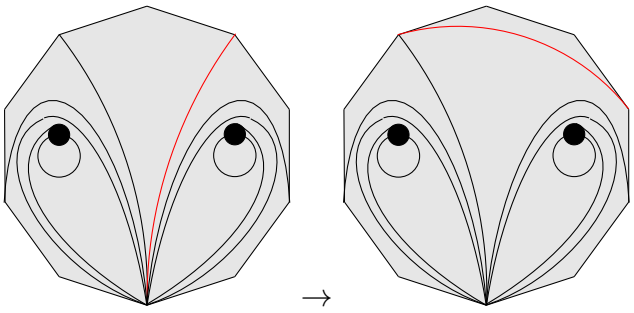


Figure 1.10 – First example of a flip

The twist is defined on the $m$-diagonals, which is a class of equivalence under Dehn twist, and the flip is defined on an $(m+2)$-angulation, which is a set of $m$-diagonal. Nonetheless, we will sometimes apply them on arcs, implicitly assuming that we have chosen a good set of representatives.

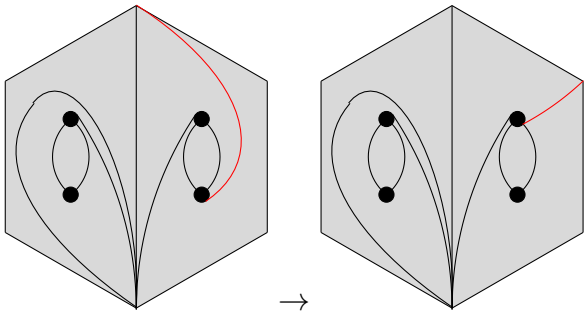


Figure 1.11 – Another example of a flip

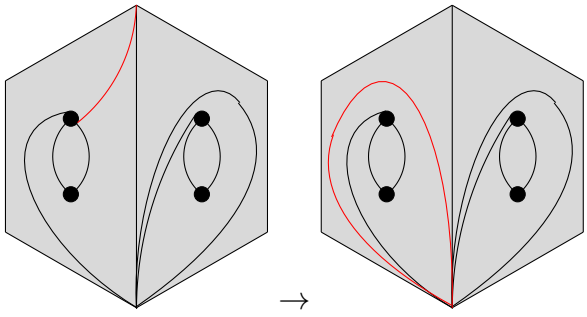


Figure 1.12 – Example of a flip with a loop

**Lemma 1.1.20.** *Any two $(m+2)$-angulations are related by a sequence of flips.*

*Proof.* We show that any $(m+2)$-angulation can be obtained from the initial $(m+2)$-angulation by applying a sequence of flips.

We know that there are at least five $m$-diagonals (as $n \geq 4$). If needed, we flip the $(m+2)$-angulation at some $m$-diagonals ending at $R$, in order to reduce to the case where only two arcs are attached to $R_i$, a thick vertex of $R$. We do the same for $S$. We fix a set of good representatives of the $(m+2)$-angulation.

Indeed, there cannot be less than two arcs ending at $R$, and if there are more than two, then at most two end at the same disk $R_i$, and others end at $R_j$ where $j \neq i$. It suffices to flip several times the arcs which do not end at $R_i$ in a way that they link two different thick vertices of $P$. Then we are reduced to the case of figure 1.13 (which is not completed by the other arcs of the $(m+2)$-angulation).

Then, we notice that this figure is similar to the $A_n$ case. The $A_n$ case has been treated by Baur and Marsh in [BM08]. From here, the remaining $(m+2)$-angulation is related by a sequence of flip to any $(m+2)$-angulation, including the initial $(m+2)$-angulation. □

**Corollary 1.1.21.** *With this lemma and the fact that the flip does not change the*

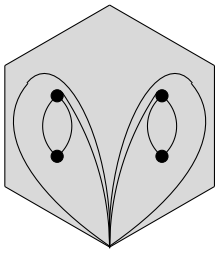


Figure 1.13 –

*number of m-diagonals in an $(m+2)$-angulation, we notice that all the $(m+2)$-angulations contain exactly $n+1$ m-diagonals.*

The following lemma comes from a result of Freedman, Hass, and Scott in [FHS82] on general geodesics, which allows us to consider that the $m$-diagonals do not cross each other.

**Lemma 1.1.22.** *If $\Delta = \{\gamma_1, \cdots, \gamma_n\}$ is an $(m+2)$-angulation, then there exist $\{\alpha_1, \cdots, \alpha_n\}$ representatives of $\{\gamma_1, \cdots, \gamma_n\}$, such that for any $i$ and any $j \neq i$, $\alpha_i$ and $\alpha_j$ do not cross.*

*Proof.* We know from [FHS82] that under homotopy, there exist $\{\alpha_1, \cdots, \alpha_n\}$ representatives which do not cross. It remains the case where some $m$-diagonals are hung to $R$ or $S$, which can be treated in the same way. Now we treat the case where there exist $\gamma_i$ from a vertex of $P$ to a thick vertex of $R$, and $\gamma_j$ a loop around $R$. If an arc crosses $\gamma_i$, then it obviously crosses $\gamma_j$. Else, if it crosses the loop $\gamma_j$, it cannot be hung to a thick vertex of $R$, because that would not cut into $(m+2)$-angles. So it crosses also $\gamma_i$, and this shows the lemma. □

**Remark 1.1.23.** *We can also apply the "flip" on a maximal set $\tilde{\Delta}$ of non-crossing m-diagonals. Indeed, the previous lemma allows us to pick representatives which are pairwise noncrossing. For $\alpha$ an m-diagonal, we can set $\kappa_{\tilde{\Delta}}(\alpha)$ in exactly the same way as in $(m+2)$-angulations.*

Before starting our next result on $(m+2)$-angulations, we recall that a $t$-angle is a figure with $t$ sides, made of:

— Sides of $P$

— Sides of $R$ or $S$

— $m$-diagonals.

**Proposition 1.1.24.** *The definition of an $(m+2)$-angulation is equivalent to the following one: an $(m+2)$-angulation is a maximal set of noncrossing m-diagonals.*

*Proof.* On the one hand, a maximal set of noncrossing $m$-diagonals cuts the polygon into $(m+2)$-angles. Indeed, let us fix $\Delta$ a maximal set of noncrossing $m$-diagonals.

— If $\Delta$ contains two loops. Then, these two loops cut the figure into a $(n-2)m+2$-gon (where $+2$ stands for the loops which become a new side of $P$) without punctures. This reduces to case $A_{n-2}$ (since the polygon in the description of Baur and Marsh in [BM08] in type $A_p$ contains $pm+2$ sides). As the set is maximal, it cuts the figure into $(m+2)$-angles.

— If $\Delta$ only contains one loop (at $S$, for instance). Then, this $m$-diagonal cuts the polygon into a new figure with one inner polygon (which behaves like a puncture) and $(n-2)m+1$ sides. This reduces to case $D_{n-1}$. As the set is maximal, there exist two arcs $i$ and $j$ from two vertices of $P$ to $R$. If they hang the same vertex, then replace one by a loop and this is the previous point. If not, we can apply the twist to one of them (for instance $i$) several times. Any times we apply the twist, it is impossible for $i$ to hang both ends of it to vertices of $P$. Indeed, this contradicts the maximality of the $m$-diagonals (and the definition of $m$-diagonals). But one end of $i$ finishes to "slip along" $j$. The only way is to become a loop. This leads us to the first case.

— If $\Delta$ has no loops, we use the same argument as the previous case to create loops.

— If $\Delta$ contains an arc $i$ from $R$ to $S$, the maximality of $\Delta$ implies that there exist one arc from a vertex of $P$ to $R$ and one arc from $P$ to $S$ (indeed, if this was not the case, we have only $m$-diagonals linking different vertices of $P$, plus an arc from $R$ to $S$. We then can add an $m$-diagonal from a vertex of $P$ to $R$, which does not cross th others and is admissible, and this contradicts the maximality). Then, we apply the twist to $i$ as many times as necessary to hang one of its ends to the polygon $P$. Then, we come to the previous case.

Note that two arcs linking $R$ to $S$ are not compatible (as in [FST08], for the case where $m=1$).

This finally shows that $\Delta$ cuts the figure into $(m+2)$-angles.

On the other hand, let $\Delta$ be a set of noncrossing $m$-diagonals, cutting the picture into $(m+2)$-angles. We first note that the $m$-diagonals do not cross. In addition, the set is maximal. Indeed, if it was not the case: Let us add a noncrossing arc $\alpha$ in an $(m+2)$-gon, then it cannot be an $m$-diagonal:

If $m=1$, then we cannot cut any triangle without being homotopic to one of the edges. Else, the only way to cut an $(m+2)$-angle by forming a $(km+1)$-gon is to form a triangle. This is impossible given the definition of an $m$-diagonal. □

**Remark 1.1.25.** *We notice that the number of $(m+2)$-angles in the figure is fixed whatever the $(m+2)$-angulation, because of the previous proposition and the fact that the flip does not change the number of $(m+2)$-angles.*

**Lemma 1.1.26.** *Any set of noncrossing m-diagonals, can be completed to an $(m+2)$-angulation.*

*Proof.* Suppose $\Lambda = \{\alpha_1, \cdots, \alpha_k\}$ is a set of noncrossing $m$-diagonals. If the set is maximal, then it is already an $(m+2)$-angulation. Else, we can add an new $m$-diagonal $\alpha_{k+1}$. We repeat the operation with $\Lambda \cup \{\alpha_{k+1}\}$ until the set is maximal. The process ends since $P$ has a finite number of sides, and once we have cut into $(m+2)$-angles, we cannot go further. □

With these definitions on the polygon, we are now able to associate a quiver with an $(m+2)$-angulation. We will see that the flip of an $(m+2)$-angulation is compatible with quiver mutation at the same vertex.

### 1.1.2 Colored quivers and $(m+2)$-angulations

First, we indicate how to associate a quiver with an $(m+2)$-angulation and then we will see how to draw a colored quiver from an $(m+2)$-angulation.

**Definition 1.1.27.** *Let $\Delta$ be an $(m+2)$-angulation (where we have chosen good representatives, without any loops). We define the quiver $Q_\Delta$ associated with $\Delta$ in the following way:*

1. *The vertices of $Q_\Delta$ are in bijective correspondence with the m-diagonals of $\Delta$.*
2. *Between two vertices $i$ and $j$ (corresponding to two m-diagonals in $\Delta$), we draw an arrow from $i$ to $j$ when both diagonals share an oriented angle. We require $j$ to be consecutive to $i$ clockwise.*

*If $i$ and $j$ are hung to the same disk of an internal polygon, the arrows incident to $i$ are found by forgetting $j$, and the arrows incident to $j$ are found by forgetting $i$.*

**Remark 1.1.28.** *From there, we call m-diagonals and vertices by the same name. It should be clear from the context whether we talk about polygons or quivers.*

*This implies that $Q_\Delta$ from lemma 1.1.11 is well-defined, because it is independent from the choice of a system of representatives. Indeed, the Dehn twist does not affect the order of the m-diagonals, and this matter to find the arrows of the quiver. More precisely, if we apply $\theta_R$ and $\theta_S$ two Dehn twists to the polygons $R$ and $S$, let us recall $i'$ and $j'$ the images if $i$ and $j$ by $\theta_S \circ \theta_R$. The end of $i'$ (respectively $j'$) hung to $P$ si the same as the one of $i$ (respectively $j$). Then the order is the same, and there is an arrow if $'i$ is consecutive to $j'$ (which is the same as $i$ being consecutive to $j$).*

For example in case $m = 1$, if we take the following triangulation $\Delta$ (figure 1.14):

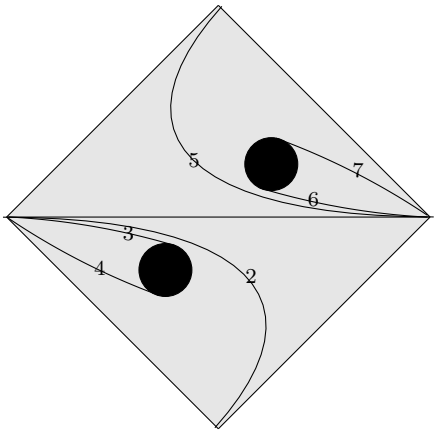
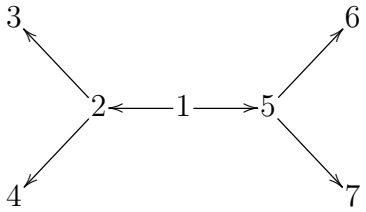
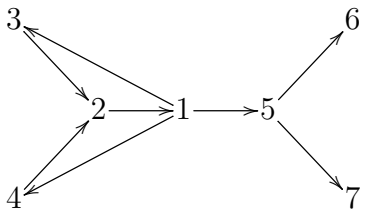



Figure 1.14 – A triangulation $\Delta$

Now, following the rule, we associate the following quiver $Q_\Delta$ with this triangulation:




Let us now mutate this quiver at vertex 2. We obtain this new quiver $Q'$:




We remark that if we flip the previous triangulation at the $m$-diagonal 2, we obtain the new triangulation $\Delta'$ of figure 1.15.

It is noticeable that the quiver $Q'_\Delta$ arising from the new triangulation corresponds to the mutation at 2 of the quiver $Q_\Delta$. This is a result showed by Fomin, Shapiro and Thurston for $m = 1$.

**Proposition 1.1.29** ([FST08], proposition 4.8)**.** *Let $\Delta$ be any triangulation. Let $Q_\Delta$ be the quiver associated with $\Delta$ as before. If $\Delta_i$ is the new triangulation flipped*

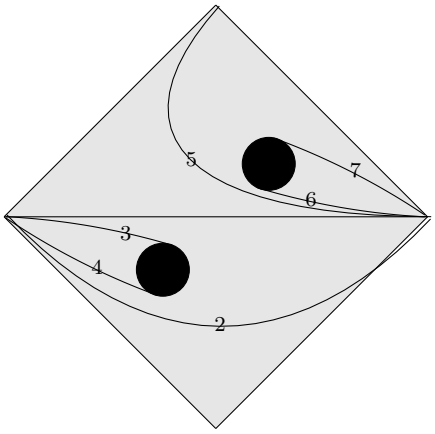



Figure 1.15 – The triangulation associated with $\Delta'$

*at $i$ from $\Delta$, then the quiver $Q_{\Delta_i}$ associated with $\Delta_i$ corresponds to the mutation at vertex $i$ of $Q_\Delta$.*

We will later exhibit a proof for the case of colored quivers of mutation-type $\tilde{D}$. We now describe how to associate a colored quiver with an $(m+2)$-angulation.

**Lemma 1.1.30.** *If $m = 1$, then the $m$-diagonals of $P$ are in bijection with the tagged arcs of Fomin, Schapiro and Thurston in [FST08] for the twice punctured $(n-2)m$-gon and the obvious bijection respects the noncrossing conditions.*

*Proof.* It suffices to read the article of [FST08] to see that the arcs are in bijection. Arcs hung to a thick vertex of $R$ (or $S$) correspond to arcs linking a puncture in the realization of [FST08]. In our point of view, there are two types of arcs, one left tangent, and one right tangent. In Fomin Schapiro and Thurston case, there are plain, or notched arcs, so two types of arcs again. □

**Definition 1.1.31.** *Let $\Delta$ be an $(m+2)$-angulation. We define the colored quiver $Q_\Delta$ associated with $\Delta$ in the following way:*

1. *The vertices of $Q_\Delta$ are in bijection with the $m$-diagonals of $\Delta$.*
2. *If $i$ and $j$ form two sides of the same polygon, then we draw an arrow from $i$ to $j$. The color of the corresponding arrow is the number of edges between both $m$-diagonals, counted clockwise from $i$.*

**Remark 1.1.32.** *We note that if we only draw the arrows of color 0, then we find the classical quiver associated with the $(m+2)$-angulation as in definition 1.1.27.*

**Proposition 1.1.33.** *There is an equivalent definition: the vertices are similarly defined, and for $i$ and $j$ two vertices, and $c$ an integer,*

$$q_{ij}^{(c)} = \begin{cases} 1 & \text{if } \kappa_\Delta^c(i) \text{ and } j \text{ share an oriented angle} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We only have to show that the arrows are the same. If $i$ and $j$ form two sides of the polygon, with a color $c$, it means that if we apply the twist to $i$, then there will be $c-1$ edges between $\kappa_\Delta(i)$ and $j$. Then if we apply the twist $c$ times, there will be no edge between $\kappa_\Delta^c(i)$ and $j$, and they will share an oriented angle.

On the other hand, if $\kappa_\Delta^c(i)$ and $j$ share an oriented angle, it suffices to apply the inverse of the twist $c$ times to make sure that $i$ and $j$ form two sides of a polygon, and that there are $c$ edges between $i$ and $j$. □

**Theorem 1.1.34.** *Let $\Delta$ be any $(m+2)$-angulation. Let $Q_\Delta$ be the colored quiver associated with the $(m+2)$-angulation $\Delta$. If $\Delta_k$ is the new $(m+2)$-angulation flipped at $k$ from $\Delta$, then the colored quiver $Q_{\Delta_k}$ associated with $\Delta_k$ is the mutation at vertex $k$ of the colored quiver $Q_\Delta$ (see figures 1.16 and 1.17 for an illustration).*

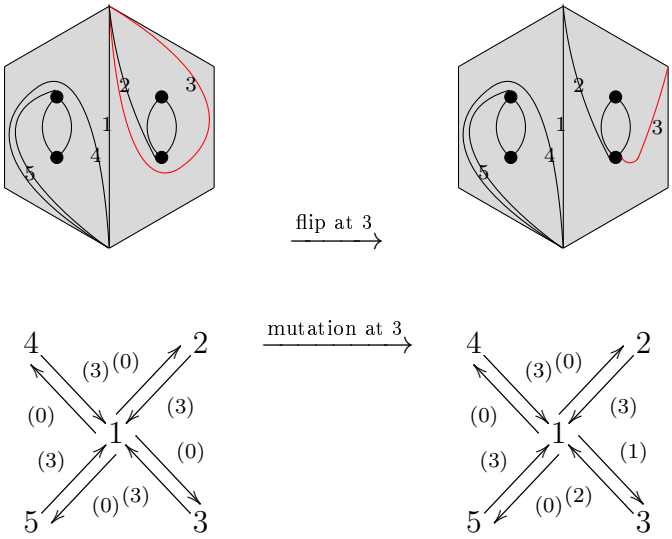
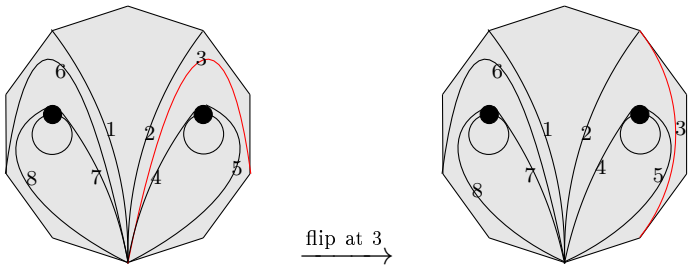



Figure 1.16 – Quiver mutation is compatible with the flip of an $(m+2)$-angulation, for $m = 3$. Note that before flipping the arc 3, we have to choose a good set of representatives. Nonetheless, in order to keep a clear figure, we have not replaced the arcs 8 and 5 by loops.



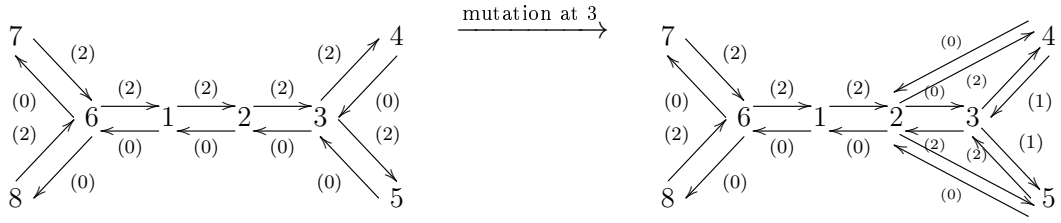



Figure 1.17 – Quiver mutation is compatible with the flip of an $(m+2)$-angulation, for $m = 2$. In order to keep a clear figure, we have not replaced the arcs 8 and 5 by loops.

**Lemma 1.1.35.** *The quiver fulfills the conditions asked for colored quivers in section* **??**. *In particular it is symmetric.*

*Proof of the lemma.* By definition, the quiver contains no loops (i.e. no arrows from $i$ to $i$).

If there is an arrow from $i$ to $j$ of color $c$, it means that $i$ and $j$ share two sides of an $(m+2)$-angle. If we count from $i$ to $j$, there are $c$ edges between them. But if we count from $j$ to $i$, as we deal with $(m+2)$-angles, it means that from $j$ to $i$ there are $m-c$ edges. So there is an arrow from $j$ to $i$ of color $m-c$. Then the symmetry is respected.

In addition, for condition 2, we divide the proof into three cases.

First case: $m = 1$. If we have $i \xrightarrow{(1)} j$ and $i \xrightarrow{(0)} j$ (note that $c$ can be either 1 or 0). It suffices to draw a triangle to ensure this situation is impossible.

Second case: $m > 2$. If we have $i \xrightarrow{(c)} j$ and $i \xrightarrow{(c')} j$ , it means that $i$ and $j$ share two $(m+2)$-gons, one with $c$ edges between $i$ and $j$, and the other one with $c'$ edges. So we are in the type of situation of figure 1.18).

Let us denote by $P_1$ (respectively $P_2$) the number of vertices delimited from $i$ to $j$ (respectively from $j$ to $i$). Let $T_i$ (respectively $U_i$) be the vertices along the dotted edges cutting on the side of $P_1$ (respectively $P_2$) the figure into two parts, one of type $A$ and one of type $D$. On the figure, we can visualize $T_i$ in red and $U_i$ in blue. Let $R_1$ and $R_2$ (respectively $S_1$ and $S_2$) be the number of vertices as shown in figure 1.18 in $R$ (respectively in $S$).

Then, by making a computation on the total number of vertices, we are led to a contradiction.

Indeed, we have the following equations:

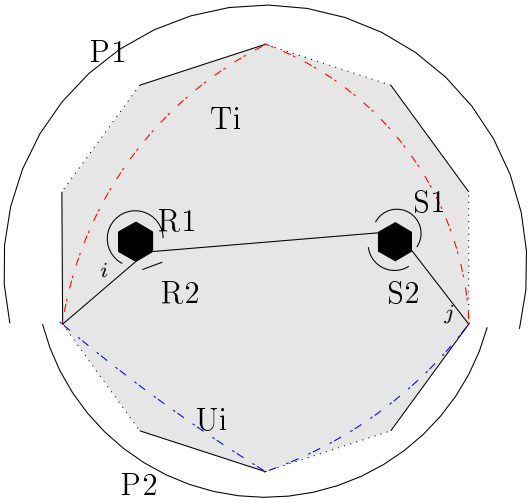



Figure 1.18 – Pathological case

$$\begin{cases} \sum T_i + R_1 + S_1 = m-1 \\ \sum U_i + R_2 + S_2 = m-1 \\ R_1 + R_2 = m-1 \\ S_1 + S_2 = m-1 \\ \sum T_i + \sum U_i = n-2 \\ P_1 + P_2 = (n-2)m \end{cases} \tag{1.1}$$

Then we have that

$$\sum T_i + R_1 + S_1 + \sum U_i + R_2 + S_2 = 2(m-1).$$

So

$$\sum T_i + \sum U_i = 0$$

This leads to $\sum T_i = \sum U_i = 0$ and this is impossible.

Third case: $m = 2$. We are going to number the different possible sub-cases.

First sub-case: if we have $i \xrightarrow{(0)} j$ and $i \xrightarrow{(1)} j$ it means that $i$ and $j$ belong to a triangle, which is impossible since $m = 2$.

Second sub-case: if we have $i \xrightarrow{(0)} j$ and $i \xrightarrow{(2)} j$, it means that we have $i \xrightarrow{(0)} j$ and $j \xrightarrow{(0)} i$ which is also impossible.

Third sub-case: if we have $i \xrightarrow{(1)} j$ and $i \xrightarrow{(2)} j$, this reduces to the first case, since we have $j \xrightarrow{(1)} i$ and $j \xrightarrow{(0)} i$.

We thus have shown that there is no way to have two arrows from $i$ to $j$ of different colors This show the lemma. □

*Proof of the theorem.* Let $\Delta$ be an $(m+2)$-angulation, let $Q_\Delta$ be the associated colored quiver, and let $k$ be a vertex of $Q$.

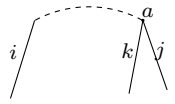
We want to show that $Q_{\mu_k(\Delta)} = \mu_k(Q_\Delta)$. Let us call by $\mathcal{T}$ the new $(m+2)$-angulation obtained from $\Delta$ by flipping the arc $k$. There is an evident bijection between the vertices of $Q_\Delta$ and $Q_T$. Let $i$ and $j$ be two vertices of $Q_{\mu_k(\Delta)}$ (and of $\mu_k(Q_\Delta)$).

Let $\tilde{q}_{ij}^{(c)}$ (respectively $\overline{q}_{ij}^{(c)}$) be the number of arrows of color $c$ from $i$ to $j$ in $Q_{\mu_k(\Delta)}$ (respectively $\mu_k(Q_\Delta)$). Let us show that $\tilde{q}_{ij}^{(c)} = \overline{q}_{ij}^{(c)}$.

First case: If $k = i$, then $\overline{q}_{ij}^{(c)} = q_{ij}^{(c-1)}$ and $\kappa_{\mathcal{T}}^c(i) = \kappa_\Delta^{c+1}(i)$. Consequently $\tilde{q}_{ij}^{(c)} = q_{ij}^{(c-1)}$.

Second case: If $k = j$, then $\overline{q}_{ij}^{(c)} = q_{ij}^{(c+1)}$ and $\kappa_{\mathcal{T}}^c(i) = \kappa_\Delta^{c-1}(i)$. Consequently $\tilde{q}_{ij}^{(c)} = q_{ij}^{(c+1)}$.

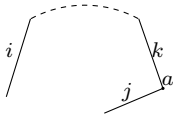
Third case: If, in $Q_\Delta$, we have $i \xrightarrow{(c)} k \xrightarrow{(0)} j$ and $i$ and $j$ are not two sides of the same $(m+2)$-angle. Then $\overline{q}_{ij}^{(c)} = q_{ij}^{(c)} + 1$. We notice that we cannot have two arrows from $k$ to $j$ of color 0. We are in a situation of the following type:



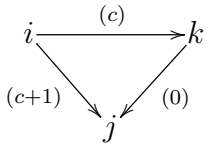

Let then $a$ be the common vertex of $k$ and $j$. Let $b$ be the other vertex of $j$ (where $a$ or $b$ can be central polygons). Then $\kappa_\Delta(k)$ and $j$ have $b$ as a common vertex. As, in $\Delta$, $\kappa_\Delta^c(i)$ and $k$ have one common vertex $a$, then $\kappa_{\mathcal{T}}^c(i)$ and $j$ have one common vertex. As a consequence, $\tilde{q}_{ij}^{(c)} = q_{ij}^{(c)} + 1$. There is no loss because there are not any arrows from $i$ to $j$ of color different from $c$. We note that there can be two arrows of different color appearing from $j$ to $k$, but as this is forbidden, we remove them. This case is symmetric to the case where $i$ and $k$ share a common vertex.

To be clear, we draw an example at the remark just below the proof.

Fourth case: If we are in a situation of the following type:



It means that we have a quiver:




Then $\overline{q}_{ij}^{(c)} = q_{ij}^{(c)} - 1 = 0$, because the arrows get erased. Moreover, under the notations of the third case, $\kappa_\Delta(k)$ and $j$ have $b$ as a common vertex, and $i$ and $j$ do not share a common polygon anymore. Then $\tilde{q}_{ij}^{(c)} = 0$. □

**Remark 1.1.36.** *Let us draw an example in figure 1.19.*

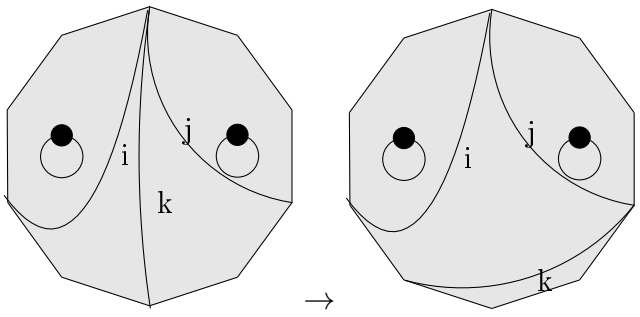
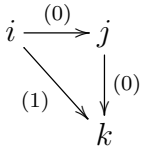


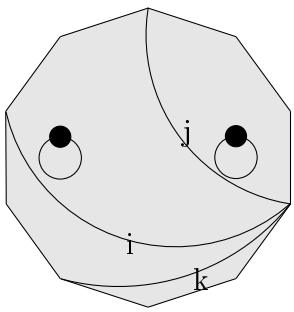

Figure 1.19 – The mutation at $k$ in the third case

*We have $i \xrightarrow{(0)} k \xrightarrow{(0)} j$. When we mutate $k$, we obtain the right figure. The quiver becomes*




*(where we do not note the inverse arrow for sake of clearance).*

*Now let us mutate at vertex $i$. The figure becomes:*



*If we mutate the colored quivers, since two arrows of different color have been built from $j$ to $k$ (one of color (0) and one of color (1)), they get erased and the quiver becomes $j \to i \to k$ of color (0). This exactly corresponds to the quiver of the figure.*

## 1.2 Construction of the category of $m$-diagonals

We are now able to construct a new category, $\mathcal{C}$, equivalent to a subcategory of the higher cluster category of tyê $\tilde{D}$ by using the $m$-diagonals. We remark that the indecomposables of $\mathcal{C}$ correspond to the rigid indecomposables of the higher cluster category.

Before starting with the construction of the category we define what will be the generators the morphisms of $\mathcal{C}$: the elementary moves.

### 1.2.1 Elementary moves

Elementary moves are applications that send an $m$-diagonal to another one. They should not be confused with flips.

Let $Q$ be a quiver of type $\tilde{D}_n$. We want the elementary moves to correspond to the arrows in the Auslander-Reiten quiver of $\mathcal{C}_Q^m$.

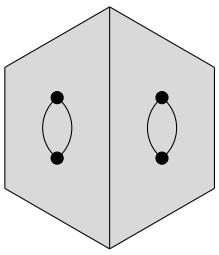


Figure 1.20 – An example of an $m$-diagonal of type 1.

We now define the length of an $m$-diagonal of type 1.

**Definition 1.2.1.** *Let $a < b$ be two integers between 1 and $(n-2)m$. The length of the boundary path from $a$ to $b$, denoted by $l(a,b)$ is the number of edges between $a$ and $b$ counted clockwise.*

**Definition 1.2.2.** *Let $P$ be the polygon considered in section 1.1. Let us number the vertices of $P$ clockwise. Let $\alpha$ be an $m$-diagonal from $i$ to $j$. The translation $\tau\alpha$ of $\alpha$ is defined as follows:*

1. *If $i \neq j$ and $\alpha$ is of type 1, then $\tau\alpha$ is the unique new arc obtained by composing $\alpha$ with the boundary paths $B_{i\ i+m}$ and $B_{j\ j+m}$, linking $i+m$ and $j+m$, and of the same type.*

2. *If $i \neq j$ and $\alpha$ is homotopic to a boundary path, then $\tau\alpha$ is the unique new arc obtained by by composing $\alpha$ with the boundary paths $B_{i\ i+m}$ and $B_{j\ j+m}$, linking $i+m$ and $j+m$, and of the same type.*
3. *If $i = j$, then $\tau\alpha$ is the new arc tangent to the same vertex of the inner polygon, ending at $i+m$. If $m$ is odd, then change the side of the tangency.*
4. *If $\alpha$ links both inner polygons, from $R_k$ to $S_{k'}$ for instance, the new arc $\tau\alpha$ links $R_{k-m}$ and $S_{k'+m}$ modulo $m-1$. The side of the tangency changes only if $m$ is odd.*

For the last point, the new arc is uniquely defined, because it does not depends on the eventual Dehn twist of $R$ or $S$, but only of the relative position before applying $\tau$.

**Remark 1.2.3.** *Applying $\tau$ several times to an arc of type 1 makes the arc "roll around" both central polygons.*

Here in figure 1.22 we can see an example of a translation:

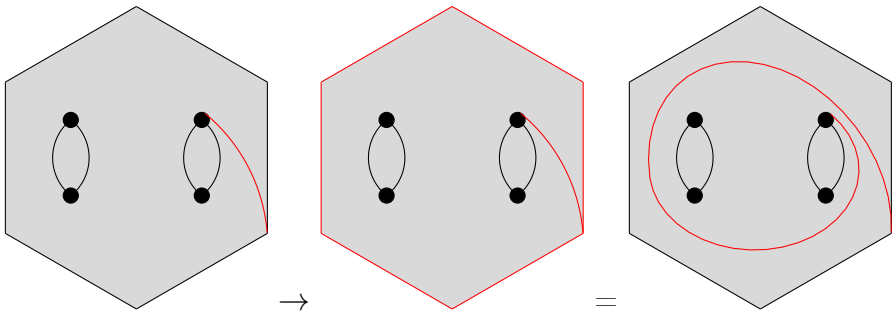


Figure 1.21 – The translation of the arc on the left picture becomes the arc on the middle picture, which is the same as the arc on the right picture.

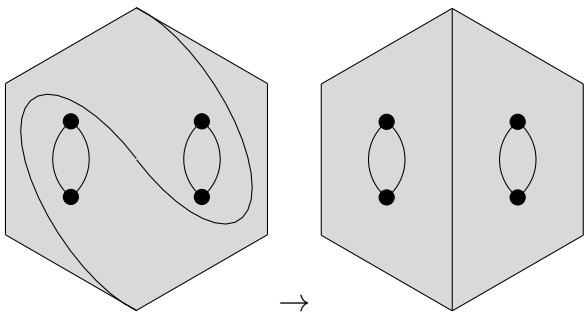


Figure 1.22 – An example of a translation

We now define elementary moves, which will correspond to arrows in the Auslander-Reiten quiver of $\mathcal{C}^m_{\tilde{D}_n}$ as shown in figure 1.23.

Let us recall that the Auslander-Reiten quiver $\Gamma$ of $\mathcal{C}^m_{\tilde{D}_n}$ is defined as follows:

1. The vertices of $\Gamma$ are the isomorphy classes of indecomposable objects in $\mathcal{C}^m_{\tilde{D}_n}$.
2. Between two representants $X$ and $Y$, the arrows $X \to Y$ are in bijective correspondence with the vectors of a basis of $\mathrm{Irr}(X, Y)$, the irreducible morphisms from $X$ to $Y$.

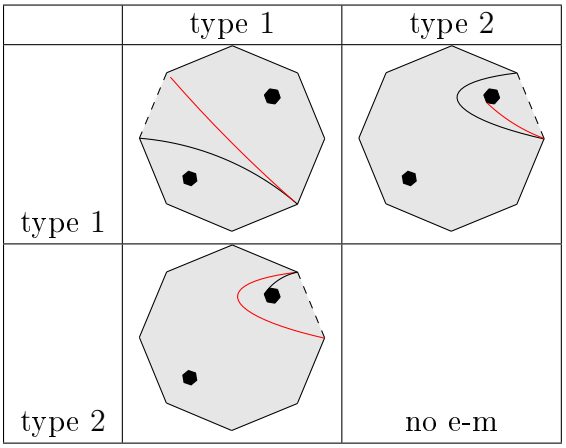
**Definition 1.2.4.** *Let $\alpha$ and $\beta$ be two $m$-diagonals. There is an elementary move from $\alpha$ to $\beta$ when:*

1. *$\alpha$ and $\beta$ are of type 1, share a vertex $a$, and if $l(b, c) = m$ (where $\alpha$ ends at $b$ and $\beta$ ends at $c$).*
2. *$\alpha$ is of type 1 and $\beta$ links a vertex of $P$ to a thick vertex of $R$ (or equivalently $S$), $\alpha$ and $\beta$ share a vertex $a$, and $\beta$ is consecutive to $\alpha$ clockwise.*
3. *$\beta$ is of type 1 and $\alpha$ links a vertex of $P$ to a thick vertex of $R$ (or equivalently $S$), $\beta$ and $\alpha$ share a vertex $a$, and $\beta$ is consecutive to $\alpha$ clockwise.*
4. *$\alpha$ and $\beta$ are homotopic to boundary paths, share a vertex $a$, and if $l(b, c) = m$ (where $\alpha$ ends at $b$ and $\beta$ ends at $c$).*
5. *$\alpha$ and $\beta$ are from a unique thick vertex of $R$ (respectively $S$) to a thick vertex of $S$ (respectively $R$) and $\beta$ is consecutive to $\alpha$ clockwise.*

**Remark 1.2.5.** *The cases 2 and 3 are included in case 1 by using the loops.*

Here is the tabular of elementary moves in the three first cases for a polygon with $(n-2)m$ sides (where e-m stands for "elementary move" and the $m$-diagonal $\alpha$ is black whereas $\beta$ is red). For sake of clarity, we have not drawn the disks on each thick vertex of the central polygons. The dotted lines hide $m$ edges.

The remaining cases in the definition will be treated in section 4.



in the $m$-cluster category

### 1.2.2 The category of $m$-diagonals

In this section, we will make use of the elementary moves defined above in order to generate the morphism of a category of $m$-diagonals. Let us just give a lemma before starting.

**Lemma 1.2.6.** *If $\alpha$ and $\beta$ are two $m$-diagonals, then there exists an elementary move $f : \alpha \to \beta$ if and only if there exists one $\overline{f} : \tau\beta \to \alpha$.*

*Proof.* There are several cases to examine. We can always reduce to the case where $\alpha$ or $\beta$ are not loops. Note that if $\alpha$ is a transjective arc, there exists an elementary move from $\alpha$ to $\beta$ if and only if $\beta$ is also a transjective arc.

1. Let us first study the case of transjective arcs.
   (a) If $\beta$ is of type different from 1.
   Then let $a$ be its only vertex. To be compatible with the definition of an elementary move, it is necessary for $\alpha$ to be of type 1. Let $b$ be the other vertex of $\alpha$ (the first one being $a$, since both arcs have to share an oriented angle).
   By definition, $l(a, b) = m$. If $a'$ is the end point of the $m$-diagonal $\tau\beta$, then $l(a', a) = m$. So $a' = b$ and $\tau\beta$ and $\alpha$ share an oriented angle. Thus there exists $\overline{f} : \tau\beta \to \alpha$
   (b) If $\beta$ is of type 1. There are two sub-cases:
      i. If $\alpha$ is of type different from 1.
      Let $a$ and $b$ be the vertices of $\beta$, where $a$ is also the one of $\alpha$.
      Then $l(a, b) = m$. So, if $c$ and $d$ are the vertices of $\tau\beta$ then $d = a$ (and $l(c, d) = m$). Thus there exists $\overline{f} : \tau\beta \to \alpha$.
      ii. If $\alpha$ is of type 1.
      Let $a$ and $b$ be the vertices of $\beta$ and $a$ and $c$ the ones of $\alpha$. We have $l(c, b) = m$.
      So, if $d$ and $e$ are the vertices of $\tau\beta$, we have $e = c$ and $l(d, a) = m$. Thus there exists $\overline{f} : \tau\beta \to \alpha$.
2. Now, if $\alpha$ and $\beta$ are not in the transjective part of the Auslander-Reiten quiver of $Q$.
   (a) If $\alpha$ is homotopic to the boundary path. Then $\beta$ must be also homotopic to the boundary path. Let $a$ and $b$ be the vertices of $\beta$, where $a$ is also the one of $\alpha$. Then, $l(c, b) = m$. So, if $d$ and $e$ are the vertices of $\tau\beta$ then $d = c$ (and $l(a, e) = m$). Thus there exists $\overline{f} : \tau\beta \to \alpha$.

(b) If $\alpha$ is from a thick vertex of $R$ to a thick vertex of $S$, then $\beta$ is of the same type. Let $a$ and $b$ be the vertices of $\beta$, where $a$ is also the one of $\alpha$. Then, $l(c,b) = m$. So, if $d$ and $e$ are the vertices of $\tau\beta$ then $d = c$ (and $l(a,e) = m$). Thus there exists $\overline{f} : \tau\beta \to \alpha$

The converse is similar. □

**Remark 1.2.7.** *This lemma seems natural since applying $\tau^{-1}$ corresponds to apply an elementary move to each end of the arc.*

**Definition 1.2.8.** *The quiver $\tilde{Q}$ is defined as follows:*

1. *The vertices are the $m$-diagonals.*
2. *There is an arrow between two $m$-diagonals $\alpha$ and $\beta$ when there is an elementary move from $\alpha$ to $\beta$.*

*The category $\mathcal{C}$ is defined to be the additive mesh category of $\tilde{Q}$.*

It means that the category $\mathcal{C}$ is the additive Krull-Schmidt category where:

1. The indecomposable objects of $\mathcal{C}$ are the $m$-diagonals of definition 2.1.21. So the objects of $\mathcal{C}$ are the finite direct sums of $m$-diagonals.
2. The set of morphisms between two indecomposable objects $X$ and $Y$ is given by the $k$-vector space generated by the paths from $X$ to $Y$ in $\tilde{Q}$ by the subvector-space generated (elements in the ideal generated by) mesh relations.
3. The composition is induced by the concatenation of paths in $\tilde{Q}$.

Mesh relations: By the previous lemma, we know that if we have $f : \beta \to \alpha$, then we also have $\overline{f} : \tau\alpha \to \beta$. Then the mesh relations are the relations:

$$R_\alpha = \sum_{\beta, f:\beta\to\alpha} f\overline{f}.$$

We will see that $\mathcal{C}$ is equivalent to a sub-category of the higher cluster category $\mathcal{C}^{(m)}_{\tilde{D}_n}$.

**Proposition 1.2.9.** *The quiver $(\tilde{Q}, \tau)$ is a stable translation quiver.*

*Proof.* Let $x$ be a vertex in $\tilde{Q}$. Given a vertex $y$, we have to show that if there is an arrow from $y$ to $x$, then there is an arrow from $\tau x$ to $y$. This is exactly the lemma 1.2.6.

In addition, no vertex is projective, and as $\tau$ is defined on all vertices, so is $\tau^m$. □

Figure 1.23 illustrates a small part of the Auslander-Reiten quiver of $\mathcal{C}^{(m)}_Q$.

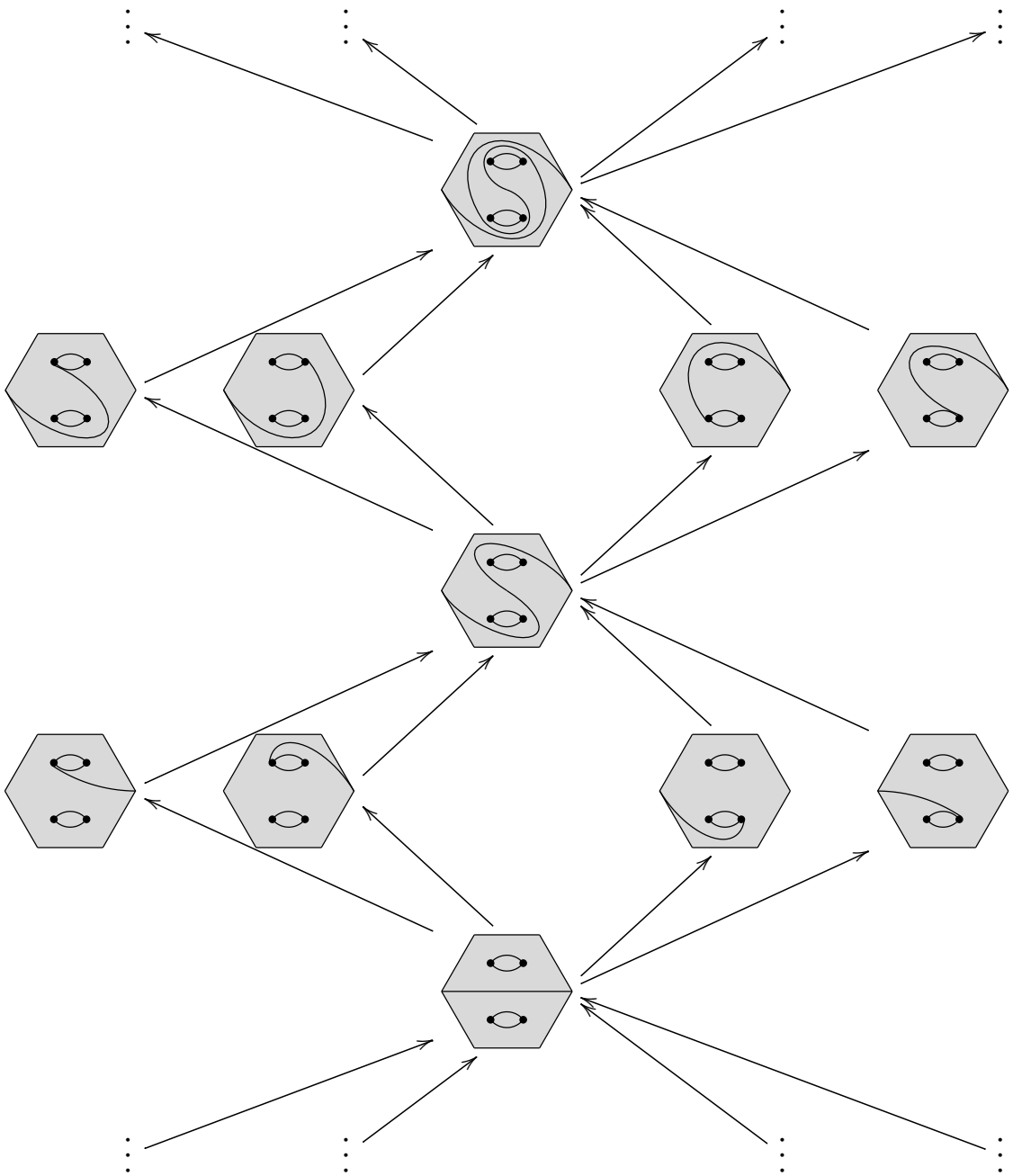


Figure 1.23 – A small part of the Auslander-Reiten quiver of $\mathcal{C}_Q^{(m)}$ with vertices $m$-diagonals, for $m = 3$ and $n = 4$

**Remark 1.2.10.** *We can find on a slice of the Auslander-Reiten quiver of $\mathcal{C}_Q^m$ the following initial $(m+2)$-angulation. In fact, each arc of the initial $(m+2)$-angulation corresponds to the image in the m-cluster category of a preprojective indecomposable representation.*

**Remark 1.2.11.** *We note that, on the previous pictures, we have only drawn the first preprojective component of the Auslander-Reiten quiver of $Q$, but there are $m$ copies of it, where we can find the m-diagonals shifted $1, \cdots, m-1$ times. Indeed, between the tubes, there are $m$ copies of the preinjective and preprojective components. On figure 1.24 we can see how the preprojective, regular and preinjective components of a Auslander-Reiten quiver of type $\tilde{D}_n$ are set.*

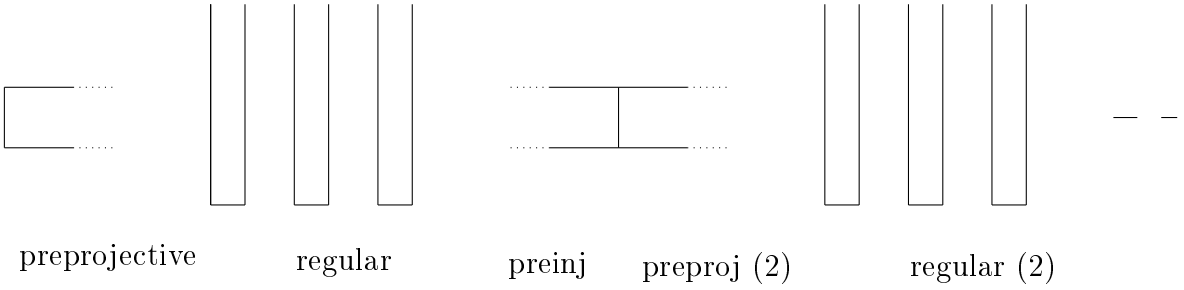
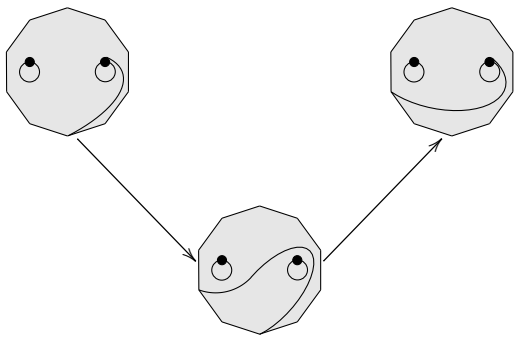



Figure 1.24 –

### 1.2.3 Mesh relations

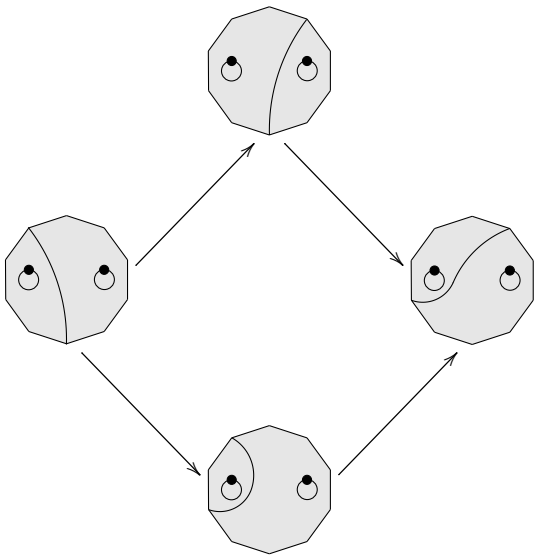
Is is possible to make use of the $m$-diagonals so as to easily picture mesh relations. There are several types of mesh relations that can thus be described: mesh relations containing two or three monomials in the transjective component and mesh relations in the three tubes of rank greater than one.

The first type of relation is the one of the following:



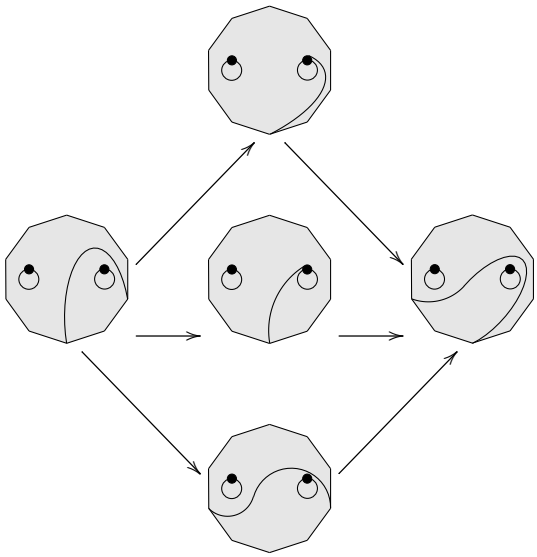
It is easy to see that the end points of both arcs of type different from 1 correspond to the end points of the arc of type 1 linked with the other arcs, and this is a general case.

The second one is of this shape:



As in the first mesh relation, we can see that the end points coincide.

The third one is of this shape:



The pictures on the end points on the left and the right are $m$-diagonals $\alpha$ and the end points of $\beta = \tau\alpha$ are always of length $m$. The upper two pictures in the middle column (or the lower two if we are situated at the bottom of the

Auslander-Reiten quiver) on the figures 1.23 and 1.29 are arcs tangent to one side of a central polygon, and the other end point is at the vertex shared by both $\alpha$ and $\beta$ (as they are of length $m$ and $\beta = \tau\alpha$, they share one vertex). Finally the lowest (or uppermost) picture is an $m$-diagonal of type 1, of length $2m$, where the vertices are the ones of $\alpha$ and $\beta$ which are not shared by each other. The bases of the tube will be described in the next section.

### 1.2.4 Equivalence of categories

In this section, assuming that $Q$ is still a quiver of type $\tilde{D}_n$, we build an additive functor $F$ from the category $\mathcal{C}$ to the $m$-cluster category.

**Definition 1.2.12.** *Let $P$ be the polygon considered in section 1.1. Let us number the vertices of $P$ clockwise. Let $\alpha$ be an $m$-diagonal from $i$ to $j$. The shift $\alpha[1]$ of $\alpha$ is defined as follows:*

1. *If $i \neq j$ and $\alpha$ is of type 1, then $\alpha[1]$ is the unique new arc obtained by composing $\alpha$ with the boundary paths $B_{i\ i+1}$ and $B_{j\ j+1}$, linking $i+1$ and $j+1$, and of the same type.*
2. *If $i \neq j$ and $\alpha$ is homotopic to a boundary path, then $\alpha[1]$ is the unique new arc obtained by by composing $\alpha$ with the boundary paths $B_{i\ i+1}$ and $B_{j\ j+1}$, linking $i+1$ and $j+1$, and of the same type.*
3. *If $i = j$, then $\alpha[1]$ is the new arc tangent to the same vertex of the inner polygon, ending at $i+1$. If $m$ is odd, then change the side of the tangency.*
4. *If $\alpha$ links both inner polygons, from $R_k$ to $S_{k'}$ for instance, the new arc $\alpha[1]$ links $R_{k-1}$ and $S_{k'+1}$. The side of the tangency changes only if $m$ is odd.*

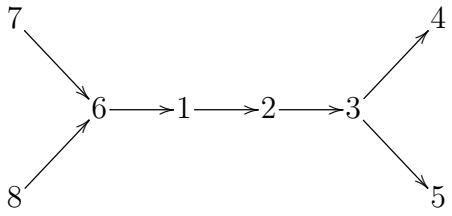
**Definition 1.2.13.** *If $d$ is an integer in $\{1, \cdots, m\}$, then we define $S^d$ as the connected component in the quiver of definition 1.2.8 containing the $d$-th shift of the initial $(m+2)$-angulation, the first component being the one corresponding to the initial $(m+2)$-angulation, and the following one corresponding to the initial $(m+2)$-angulation where $[1], [2], \cdots$ has been applied.*

*Let $\mathcal{S}^d$ be the component in the Auslander-Reiten quiver of $\mathcal{C}^{(m)}_{\tilde{D}_n}$, containing objects of type $\tau^s P[d]$ where $P$ is a projective indecomposable object and $s$ is an integer.*

**Theorem 1.2.14.** *Let $d \in \{1, \cdots, m\}$. Then we have an isomorphism between both components:*

$$S^d \simeq \mathcal{S}^d$$

*Proof.* For $n \geq 6$, we fix an orientation for our quivers. The orientation we choose is the one of the quiver corresponding to the initial $m$-angulation:




Let $\Delta_1^0$ be the initial $(m+2)$-angulation in the first copy of the quiver $\tilde{Q}$. It corresponds to the $m$-cluster-tilting object $T = \bigoplus P_i$ at the initial slice of the Auslander-Reiten quiver of $Q$. With the $m$-diagonal $\alpha_i$ (at vertex $i$), we associate the projective module $P_i = F(\alpha_i)$. Then we associate with the elementary moves of $m$-diagonals, the irreducible morphisms in the cluster category. Indeed, according to the tabular of section 1.2.1, there is an elementary move from $\alpha$ to $\beta$ when both share an oriented angle. As it has been told in definition 1.1.27, in the quiver it means that there is an arrow from $\alpha$ to $\beta$. So there is an irreducible morphism from $F(\alpha)$ to $F(\beta)$.

Now that we have treated the case of the initial slice in the first copy, still calling $\alpha_i$ the $m$-diagonal at vertex $i$ in this slice, we note that every $m$-diagonal in the component $S^d$ in the quiver $\tilde{Q}$ is of the form $\tau^t \alpha_i[d]$, for some $t \in \mathbb{Z}$ and $i \in \{1, \cdots, n+1\}$. So it is natural to define

$$F(\tau^t \alpha_i[d]) = \tau^t(P_i[d]) \forall t.$$

Therefore, there is a bijection between the arcs of $S^1$ and the indecomposables of the transjective components in the Auslander-Reiten quiver of $\mathcal{C}^{(m)}_{\tilde{D}_n}$ containing the image of the indecomposable projective objects.

Indeed, there are two things to show: First, for any arc $\alpha$, there is a unique way to write it $\tau^t \alpha_i[d]$. If we had $\tau^t \alpha_i[d] = \tau^s \alpha_j[d]$ in a component $d$, then $\tau^t \alpha_i = \tau^s \alpha_j$, so $i = j$ because of the type of the diagonal (if it is tangent one or another side of the central polygon, or if it is of type 1, with length $l$, etc...), and if $t \neq s$ it would mean that there is the same $m$-diagonal at two different vertices in the quiver $\tilde{Q}$, which is impossible. Second, if $X$ is an indecomposable in the $d$-th component of $\tilde{Q}$, it can be written $\tau^t P_i[d]$, and this has a unique antecedent by $F$, which is $\tau^t \alpha_i[d]$.

As a matter of conclusion, $F$ is bijective on the objects. Let us show that it is also a bijection on the elementary moves. Suppose that $f : \alpha \to \beta$ is an elementary move. Then there exist $s \in \mathbb{N}, i \in \{1, \cdots, n+1\}$, $d \in \{1, \cdots, m\}$ such that $\tau^s \alpha = \alpha_i[d]$ and $\tau^s \beta = \beta^*[d]$ (i.e $\alpha = \tau^{-s} \alpha_i[d]$ and $\beta = \tau^{-s} \beta^*[d]$, where $\beta^* = \alpha_j$ when there is an elementary move $\alpha_i \to \alpha_j$ or $\beta^* = \tau^{-1} \alpha_j$ where

there is an elementary move $\alpha_j \to \alpha_i$. In the first case, there is an irreducible morphism from $P_i$ to $P_j$ thus from $F(\alpha_i)$ to $F(\beta^*)$ and from $F(\alpha) = \tau^{-s}F(\alpha_i)[d]$ to $F(\beta) = \tau^{-s}F(\beta^*)[d]$. In the second case, there is an irreducible morphism from $P_j$ to $P_i$ thus from $P_i$ to $\tau^{-1}P_j$ and from $F(\tau^s\alpha) = F(\alpha_i[d]) \to F(\beta_i[d]) = F(\tau^s\beta)$, thus an arrow $F(\alpha) \to F(\beta)$.

This proves the statement. □

## 1.3 The case of regular modules

Now that we have seen an equivalence for the transjective components, we are interested in the case of tubes, and their correspondences in terms of arcs. When we introduced the different types of admissible $m$-diagonals, we voluntarily forgot some other types of diagonals. These remaining diagonals have the particularity of being cyclic. Indeed, if we apply $\tau$ several times to them, they come to their first position at a rank $r$. This cannot be the case of the types of $m$-diagonals defined in 2.1.21 (because they wrap around the central polygons).

In figure 1.25, we can see the other types of diagonals. Note that we only have drawn the first ones, but there are also their successive images under $\tau$ (of which there are a finite number because they are cyclic).

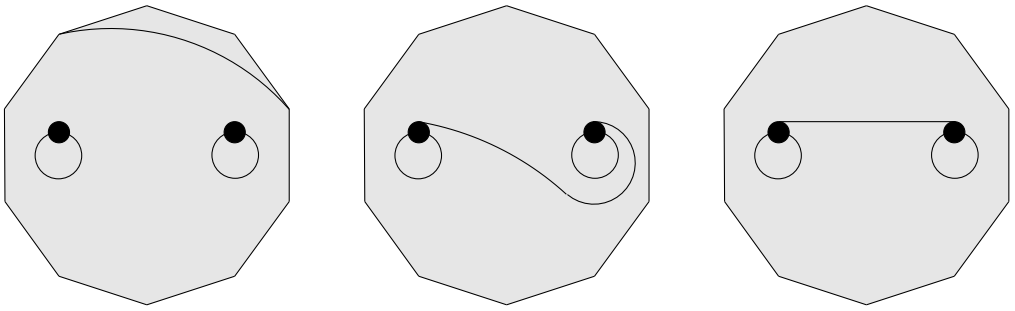


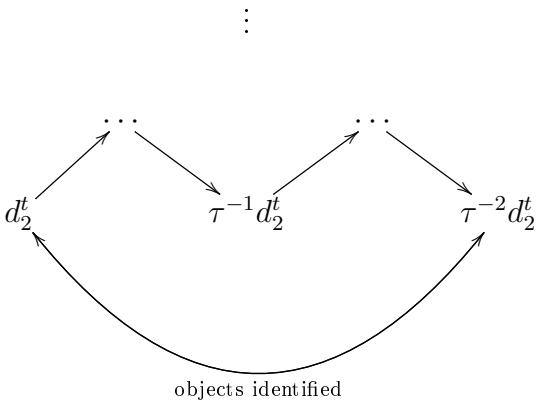
Figure 1.25 – The three different types of arcs in the tube, $d_1^t$, $d_2^t$ and $d_3^t$ for $m = 2$ and $n = 7$

These diagonals have to figure in the Auslander-Reiten quiver of $Q$, and it appears that we can see them in the tubes (it means in the regular part of the quiver).

In case $\tilde{D}_n$, as we can see in the course of Crawley-Boevey [CB], there are three types of tubes, two of period 2, and one of period $n-2$.

As we deal with the higher cluster categories, we have to notice, that we have $m$ copies of each tube, it means we have $3m$ tubes. The first tube contains the first picture of figure 1.25 and all its images under $\tau$. There are $m$ copies of this

tube, which correspond to the successive shifts of the arcs. This is the same proces for the second and third tube.




It is known in [CB] that in a tube of size $r$, only the first $r-1$ layers contain rigid objects. Graphically this corresponds to noncrossing arcs. It means that in $\tilde{Q}$, for the example of the only tube of size $n-2$, the $(n-1)$ lowest lines are made of noncrossing arcs. Then the arc crosses itself, and this does not correspond anymore to an $m$-rigid object in the higher cluster category, and this is the same for the shifted arcs in the successive copies of this first tube. Then with an arc situated on the $r-1$ first lines of the tube, we associate the $m$-rigid object which actually takes place in the Auslander-Reiten quiver of $\mathcal{C}^{(m)}_{\tilde{D}_n}$.

In order to make sure that an arc corresponds to a unique $m$-rigid object in the tube, we have to choose a convention. Let $\Delta$ be the initial $(m+2)$-angulation. Let $\beta$ be the red $m$-diagonal in figure 1.26. Then the vertex of $Q$ associated with $\beta$ is $n-1$ from the isomorphism of theorem 1.2.14. Let $\alpha = \mu_\Delta(\beta)$. We only need to associate an $m$-rigid object in order to find all the objects of the tube of size $n-2$. We choose to associate with $\alpha$ the simple object $\tau^{-1}S_{n-1}$ at vertex $n-1$.

To be precise, we set at the base of the first tube of size $n-2$ the flip of the arc corresponding to the preprojective at the first slice of the Auslander-Reiten quiver. This arc is thus associated with the simple regular module which corresponds to the mutation of the preprojective. This means that we have mutated the object $P_{n-1}$ in the $(m+2)$-angulation containing the arcs corresponding to the sum of the indecomposable projective objects.

For the tubes of size 2, we set the only arcs linking both central polygons without self-crossing (see figure 1.25).

In this way we have a $1-1$-correspondence between the arcs linking two different vertices which are homotopic to the border and the objects of the tube of size $n-2$ in the Aulslander-Reiten quiver of $\mathcal{C}^{(m)}_Q$.

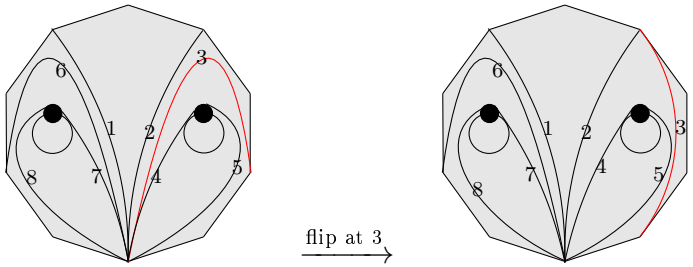



Figure 1.26 – We flip $\beta$ in order to define $\alpha$ in the base of the first tube

## 1.4 About non-self-crossing arcs and $m$-rigid objects

We know by theorem 1.2.14 that an arc in the geometric realization can be interpreted as an object in the $m$-cluster-category. We use the following notation: if $\alpha$ is an arc in the category $\mathcal{C}$ let $X_\alpha$ be the object associated in the $m$-cluster category.

**Lemma 1.4.1.** *The application*

$$\begin{array}{ccc} \{m\text{-}diagonals\} & \to & \{m\text{-}rigid\ indecomposable\ objects\} \\ \alpha & \mapsto & X_\alpha \end{array}$$

*up to isomorphism, is bijective.*

*Proof.* With each arc $\alpha$ which does not cross itself we associate by theorem 1.2.14 an $m$-rigid object $X_\alpha$. Then $\forall i \in \{1, \cdots, m\}$ $\mathrm{Ext}^i(X_\alpha, X_\alpha) = 0$.

Indeed, first, if $\alpha$ belongs to the preprojective part of the Auslander-Reiten quiver, it belongs in particular to a slice of it which forms a quiver of type $\tilde{D}_n$. If we add all the arcs forming this quiver, we obtain an $(m+2)$-angulation. In the Auslander-Reiten quiver of $\mathcal{C}_Q^m$, by theorem 1.2.14, this slice corresponds to an $m$-cluster-tilting object, made of the sum of all $m$-rigid objects at the vertices of the slice. Then, with $\alpha$ is associated an $m$-rigid object $X_\alpha$, and as $\alpha$ is in this part of the Auslander-Reiten quiver of $Q$, it does not cross itself.

Second, if $\alpha$ belongs to a tube, then if it does not cross itself, it is at the base of it, and by the previous section, this corresponds to an $m$-rigid object (because in the case of a tube of size $r$, $X_\alpha$ is situated at one of the $r-1$ first lines). Conversely, if $X_\alpha$ is $m$-rigid, then it is at the base (in the first $r-1$ lines) of the tube of size $r$, it means that it corresponds to an arc of the first lines, which does not cross itself.

We have thus shown that $\alpha$ is an arc which does not cross itself if and only if $\forall i \in \{1, \cdots, m\}, \mathrm{Ext}^i(X_\alpha, X_\alpha) = 0$. □

Then, there is a one-to-one bijection between arcs without any self-crossing and $m$-rigid objects.

**Remark 1.4.2.** *We can extend this bijection to the arcs in the tubes including arcs with an auto-intersection. This can be easily visualized in figure 1.32.*

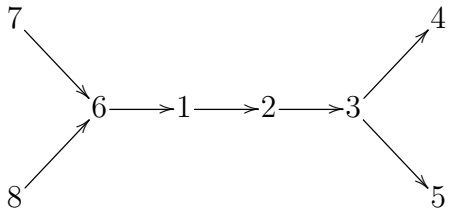
**Remark 1.4.3.** *In a paper which is being drawn up, we show that there is a bijection between $(m+2)$-angulations and $m$-cluster-tilting objects.*

## 1.5 An example

Let us resume this paper with a complete study of an Euclidean quiver of type $\tilde{D}_n$, and its $m$-cluster category, taking $n = 7$ and $m = 2$. So we study the following category

$$\mathcal{C}^2_{\tilde{D}_7} = \mathcal{D}^b(K\tilde{D}_7)/\tau^{-1}[2],$$

where $\tilde{D}_7$ is the following quiver:




### 1.5.1 Quadrangulations of the decagon and colored quiver

The geometric realization shown in figure 1.27 for this quiver is a decagon with two squares inside of it.

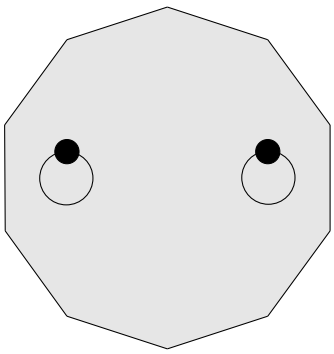


Figure 1.27 – The decagon with two monogons

We are now drawing eight 2-diagonals in order to cut the decagon in quadrangles. This is in figure 1.28 the initial $(m+2)$-angulation from which we will build later the translation quiver isomorphic to a sub-quiver of the Auslander-Reiten quiver of $\mathcal{C}^{(m)}_{\tilde{D}_7}$.

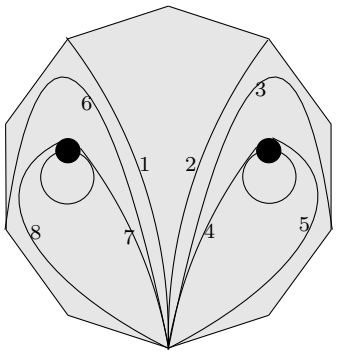
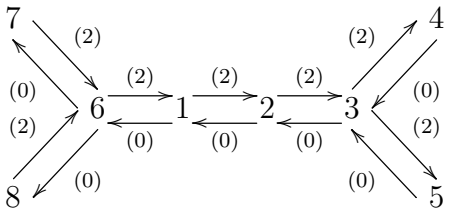



Figure 1.28 – The initial quadrangulation for $n = 7$ and $m = 2$

If we follow the rules of definition 1.1.27, we associate with this quadrangulation the quiver at the beginning of this section.

Moreover, if we associate the colored quiver with this quadrangulation according to definition 2.1.9, we obtain this one.

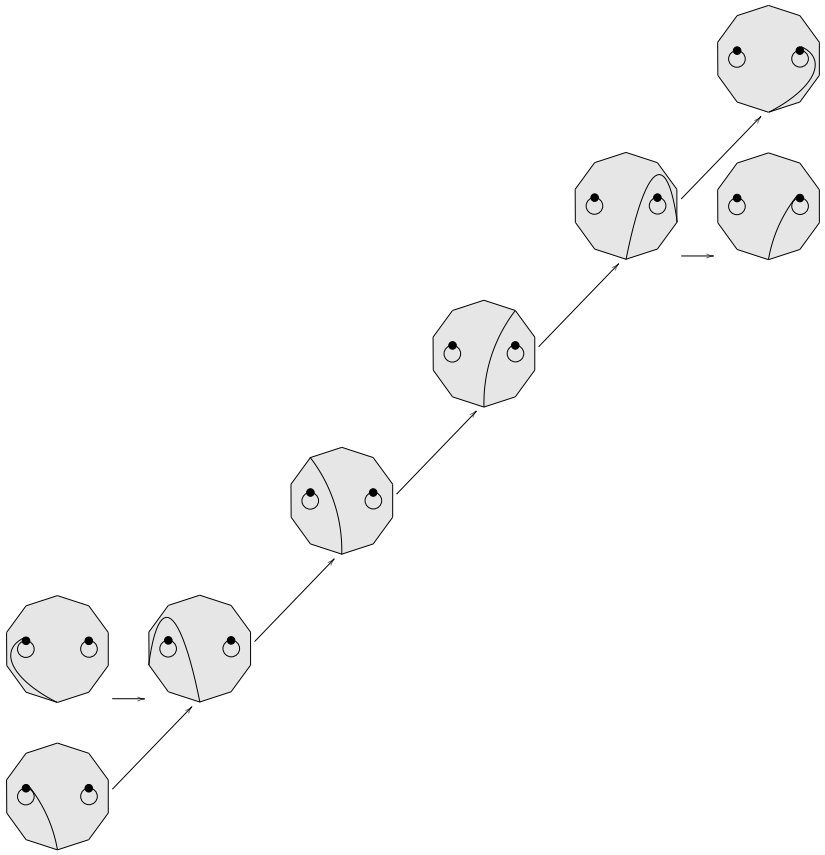



Now we know that mutating a colored quiver corresponds to flipping an arc in an $(m+2)$-angulation, we can flip in any way the quadrangulation, draw the corresponding colored quiver, and note that the colored quivers are related by mutation at the corresponding vertex.

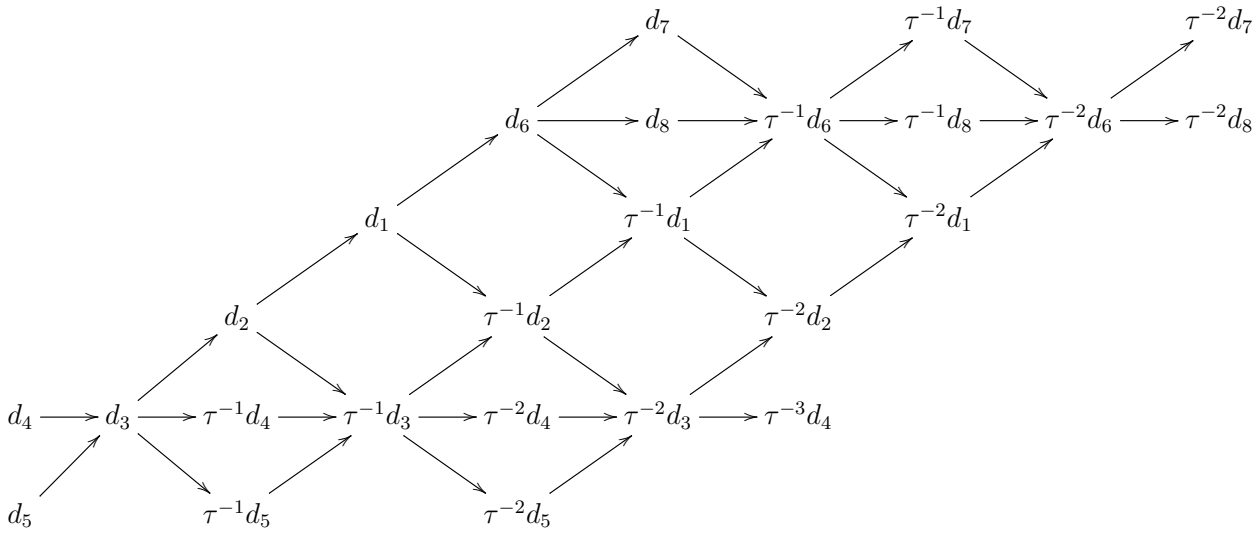
### 1.5.2 2-diagonals and preprojective component of the Auslander-Reiten quiver

From the initial quadrangulation, we extract the arcs and put them in the first component of the Auslander-Reiten quiver of $\tilde{D}_7$.

As the preprojective component of the Auslander-Reiten quiver is of the following form (where we have associated the vertices with the types of the arcs we



are working with), we just have to apply $\tau$ and the shift functor to the arcs to be able to complete the preprojective component and obtain the result in figure 1.29.


$d_7$
$\tau^{-1}d_7$
$\tau^{-2}d_7$
$d_6$
$d_8$
$\tau^{-1}d_6$
$\tau^{-1}d_8$
$\tau^{-2}d_6$
$\tau^{-2}d_8$
$d_1$
$\tau^{-1}d_1$
$\tau^{-2}d_1$
$d_2$
$\tau^{-1}d_2$
$\tau^{-2}d_2$
$d_4$
$d_3$
$\tau^{-1}d_4$
$\tau^{-1}d_3$
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$\tau^{-1}d_5$
$\tau^{-2}d_5$

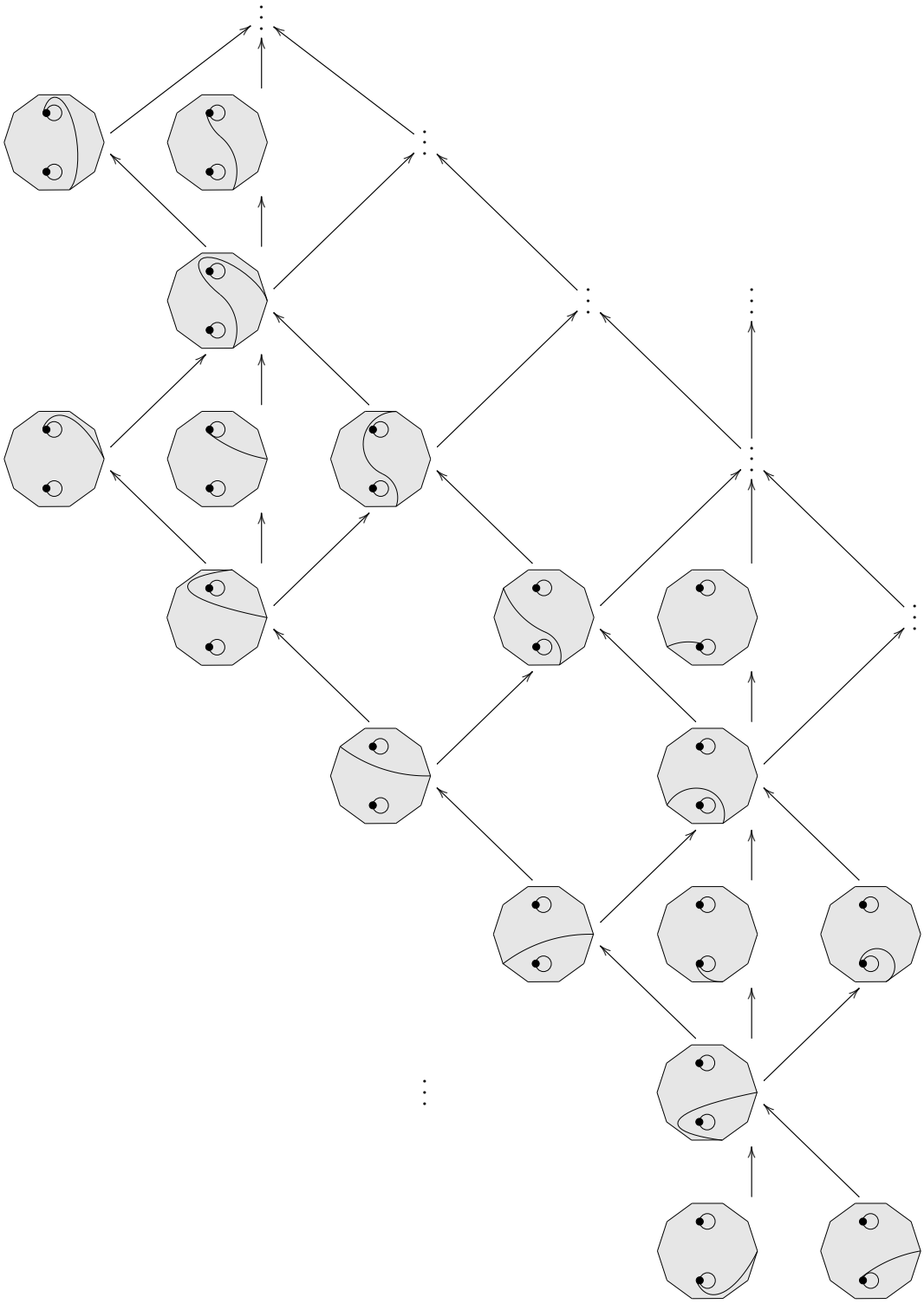


Figure 1.29 – Arcs on the Auslander-Reiten quiver of $\tilde{D}_7$

Note that there are two copies of this component, and the other one is about the same but all the arcs have been shifted once clockwise.

As the preprojective component is infinite, we can along it, watch the arcs wrap around the squares.

### 1.5.3 2-diagonals in the tubes of the Auslander-Reiten quiver of $\tilde{D}_7$

Concerning the regular modules, there are three types of tubes: The first type, containing $m$ tubes of size 5 and the two other types of tubes (each one containing $m$ copies of it) of size 2. The one of size 5 corresponds to the following cyclic arc (in figure 1.30) and its images under the shift.

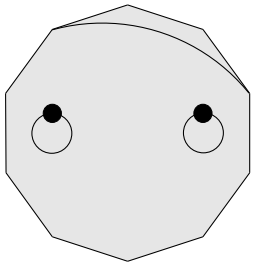


Figure 1.30 – Arc of order 5

Indeed, if we apply $\tau$ five times, the arc returns to the origin, so it is cyclic of order 5. The two remaining arcs corresponding to the tubes of order 2 are the ones of figure 1.31, plus their successive images under the shift.

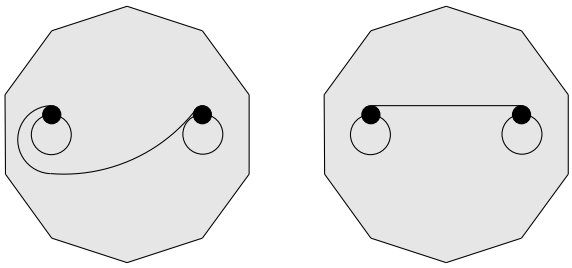
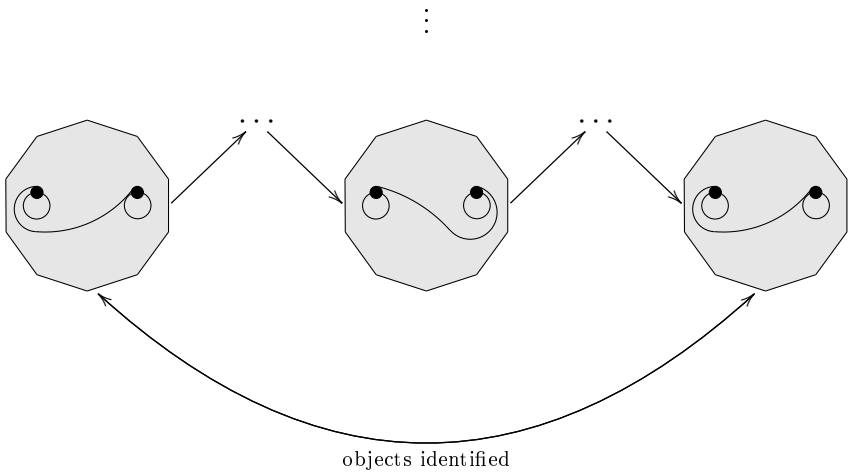


Figure 1.31 – Arcs of order 2

Here we draw the tube of size 2 corresponding to the first picture of figure 1.31.




Note that in this tube, only the first line corresponds to rigid objects. The upper arcs cross themselves and thus no not correspond to rigid objects anymore.

The following picture gives the tube of size 5, with the corresponding arcs. We can notice that in the first four ranks, the arcs do not cross since they match to a rigid object, and this is not the case anymore from rank 5. In order to have clear pictures, we have replaced the monogon inside of $P$ with a simple disk.

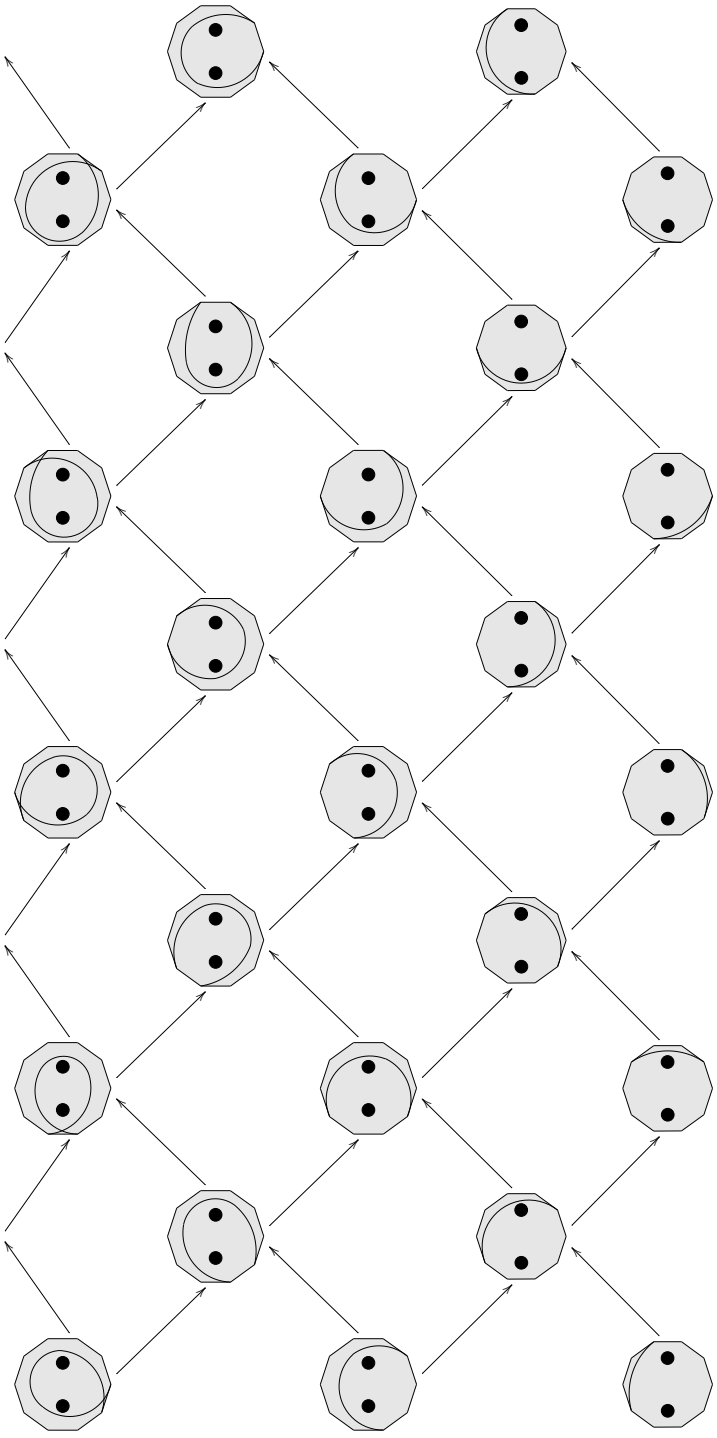


Figure 1.32 – A tube of size 5. We can see that from the 5th layer, the arcs cross themselves and do not correspond to an $m$-rigid object anymore

Chapter 2

# A bijection between $m$-cluster-tilting objects and $(m+2)$-angulations in $m$-cluster categories

## 2.1 Geometric realizations

### 2.1.1 Case $A$

[BM08]

In this section, we recall the geometric realization of the $m$-cluster category of a quiver of type $A_n$, for an integer $n$, after Baur and Marsh [BM08].

Let $Q$ be a quiver of type $A_n$, with $n$ vertices, and let $\mathcal{C}_Q^m$ be the $m$-cluster category associated with $Q$. Let $P$ be a polygon with $nm+2$ sides, numbered clockwise.

**Definition 2.1.1.** *An $m$-diagonal $\alpha$ from the vertex $i$ to $j \neq i$ in $P$ is a diagonal of $P$ linking $i$ and $j$ such that $\alpha$ cuts the figure into two polygons, one with $km+2$ sides, for some $k \in \mathbb{N}$ and one with $lm+2$ sides, for some $l \in \mathbb{N}$.*

In figure 2.1 we draw an example of an $m$-diagonal.

**Definition 2.1.2.** *An $(m+2)$-angulation of $P$ is a maximal set of noncrossing $m$-diagonals.*

**Remark 2.1.3.** *We note that this definition is equivalent to the following one : an $(m+2)$-angulation is a set of $m$-diagonals cutting the polygon into $(m+2)$-angles.*

We can introduce the initial $(m+2)$-angulation, that we will use later. All its $m$-diagonals have one end at the same vertex 1 (see figure 2.3).

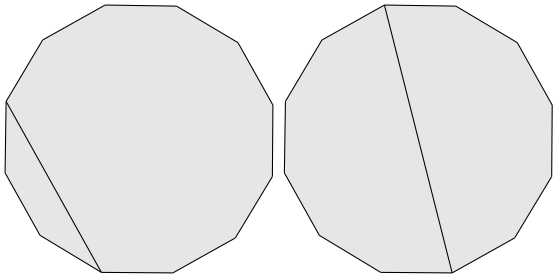


Figure 2.1 – The first is a 2-diagonal, the second is not

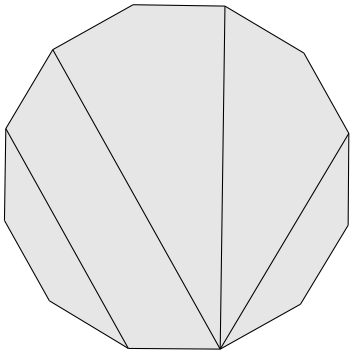


Figure 2.2 – This is an example of a 4-angulation

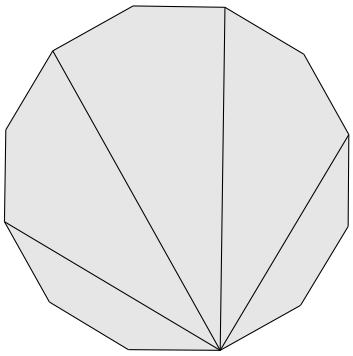


Figure 2.3 – The initial $(m+2)$-angulation, for $n = 5$ and $m = 2$. The vertices can be numbered from 1 to 12.

We can define the twist and the flip of an $(m + 2)$-angulation, as Buan and Thomas did in [BT09].

**Definition 2.1.4.** *Let $\Delta$ be an $(m+2)$-angulation. Let $\alpha$ be an $m$-diagonal of $\Delta$, linking the vertices $a$ and $b$. The twist of $\alpha$ in $\Delta$ is defined as follows:*

*Let $(a', a)$ (respectively $(b, b')$) be the side of the $(m + 2)$-angle ending at $a$ (respectively at $b'$) consecutive to $a$ clockwise (respectively preceding $b$). Then the*

*twist of $\alpha$, namely $\kappa_\Delta(\alpha)$ is the $m$-diagonal $(a', b')$.*

**Definition 2.1.5.** *Consider $\Delta$ an $(m+2)$-angulation. Let $\alpha$ be an arc in $\Delta$. The flip of the $(m+2)$-angulation at $\alpha$ is defined by $\mu_\alpha\Delta = \Delta \setminus \{\alpha\} \cup \{\alpha^*\}$ where $\alpha^*$ is given by $\kappa_\Delta(\alpha)$, the twist of $\alpha$.*

**Remark 2.1.6.**
1. *Note that the twist has an inverse, which consists in moving the arc counterclockwise. Then the flip is also invertible.*
2. *A flip does not change the number of $m$-diagonals in the $(m+2)$-angulation.*

In figure 2.4, we can see an example of a flip.

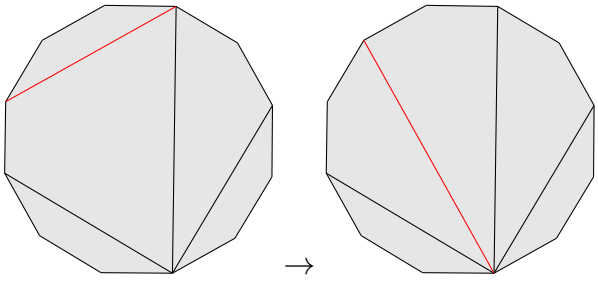


Figure 2.4 – Example of a flip

**Lemma 2.1.7.** *Any two $(m+2)$-angulations are related by a sequence of flips.*

*Proof.* Let $\Delta$ be an $(m+2)$-angulation. We show that we can reach the initial $(m+2)$-angulation by applying a sequence of flips.

If one of the arc has one end at vertex 1, then is suffices to flip several times the neighbour arc in order to hang one end of it to vertex 1. We repeat the operation until all arcs are hung to vertex 1. This leads exactly to the initial $(m+2)$-angulation. If no arc has one end hung to vertex 1, then consider the $(m+2)$-gon containing the vertex 1. Flip one arc of this polygon as many times as necessary in order to hang it at vertex 1. □

**Corollary 2.1.8.** *[Tza06] With this lemma and the fact that the flip does not change the number of $m$-diagonals in an $(m+2)$-angulation, we notice that all the $(m+2)$-angulations contain exactly $n$ $m$-diagonals.*

Note that if $\Lambda$ is a set of noncrossing $m$-diagonals, it can be completed in order to form an $(m+2)$-angulation.

We now associate a colored quiver with an $(m+2)$-angulation.

**Definition 2.1.9.** *Let $\Delta$ be an $(m+2)$-angulation. We define the colored quiver $Q_\Delta$ associated with $\Delta$ in the following way:*

1. *The vertices of $Q_\Delta$ are in bijection with the $m$-diagonals of $\Delta$.*
2. *If $i$ and $j$ form two sides of some $(m+2)$-gon in $\Delta$, then we draw an arrow from $i$ to $j$ and an arrow from $j$ to $i$. The color of the corresponding arrow is the number of edges between both $m$-diagonals, counted clockwise from $i$ (respectively from $j$).*

**Proposition 2.1.10.** *There is an equivalent definition: the vertices are similarly defined, and for $i$ and $j$ two vertices, and $c$ an integer,*

$$q_{ij}^{(c)} = \begin{cases} 1 & \text{if } \kappa_\Delta^c(i) \text{ and } j \text{ share a counterclockwise oriented angle} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We only have to show that the arrows are the same. If $i$ and $j$ form two sides of the polygon, with a color $c$, it means that if we apply the twist to $i$, then there will be $c-1$ edges between $\kappa_\Delta(i)$ and $j$. Then if we apply the twist $c$ times, there will be no edge between $\kappa_\Delta^c(i)$ and $j$, and they will share an oriented angle.

On the other hand, if $\kappa_\Delta^c(i)$ and $j$ share an oriented angle, it suffices to apply the inverse of the twist $c$ times to make sure that $i$ and $j$ form two sides of a polygon, and that there are $c$ edges between $i$ and $j$. This only works if $c \leq m$. If $c > m$, then apply this method to the other end of $i$. □

**Lemma 2.1.11.** *The quiver fulfils the conditions asked for colored quivers in the article of Buan and Thomas [BT09]. In particular it is symmetric.*

*Proof.* By definition, the quiver contains no loops (it means, no arrows from $i$ to $i$).

The condition of monochromaticity is respected since two arcs can only share one polygon.

If there is an arrow from $i$ to $j$ of color $c$, it means that $i$ and $j$ share two sides of a triangle. If we count from $i$ to $j$, there are $c$ edges between them. But if we count from $j$ to $i$, as we deal with $(m+2)$-angles, it means that from $j$ to $i$ there are $m-c$ edges. So there is an arrow from $j$ to $i$ of color $m-c$. Then the symmetry is respected. □

We remark that we have the compatibility between the mutation of a colored quiver in the sense of Buan and Thomas, and the flip of an $(m+2)$-angulation.

**Theorem 2.1.12** (Buan-Thomas, [BT09])**.** *Let $\Delta$ be any $(m+2)$-angulation. Let $Q_\Delta$ be the colored quiver associated with the $(m+2)$-angulation $\Delta$. If $\Delta_k$ is the new $(m+2)$-angulation flipped at $k$ from $\Delta$, then the colored quiver $Q_{\Delta_k}$ associated with $\Delta_k$ is the mutation at vertex $k$ of the colored quiver $Q_\Delta$.*

*Proof.* This is immediate given the previous proposition. □

Baur and Marsh in [BM08, Theorem 5.6] have shown that a category geometrically built from these $m$-diagonals is equivalent to the $m$-cluster category. In the next section, we will go further and show that cutting along an arc corresponds to the Iyama-Yoshino reduction, as Marsh and Palu showed it for the general case of Riemann surfaces for $m = 1$.

We also have the following theorem:

**Theorem 2.1.13** ([BM08], Proposition 5.4)**.** *There is an explicit bijection between the $m$-diagonals without any self-crossing and the $m$-rigid objects of the $m$-cluster category.*

This bijection is found in the following way: Buan and Marsh build a quiver from $m$-diagonals, which is aimed to be isomorphic to the Auslander-Reiten quiver of $Q$.

**Remark 2.1.14.** *We notice that if we only draw the arrows of color 0, then we find the Gabriel quiver of the endomorphism algebra of the $m$-cluster-tilting object associated with the $(m+2)$-angulation.*

### 2.1.2 Case $D$

[BM07]

In this case, treated by Baur and Marsh in [BM07], we use a slightly different geometric realization, in order to simplify the notion of flip of an $(m+2)$-angulation. Baur and Marsh use a polygon with $nm - m + 1$ sides with a punction inside of it. We replace the puntion by an $(m-1)$-gon with, on each vertex of it, a disc. The realization is the same as the one of $\tilde{D}$ (see [JM]), but with only one polygon inside ot it. The arcs are defined in the same way, and this respects the article of Baur and Marsh, since we have an evident bijection between the arcs of Baur and Marsh, and the ones we defined as in case $\tilde{D}$ in [JM].

We also have the following theorem:

**Theorem 2.1.15** ([BM07], Theorem 3.6)**.** *There is an explicit bijection between the $m$-diagonals without any self-crossing and the $m$-rigid objects of the $m$-cluster category.*

Here again, Buan and Marsh build a quiver from $m$-diagonals, which is aimed to be isomorphic to the Auslander-Reiten quiver of $Q$.

### 2.1.3 Case $\tilde{A}$

[Tor12b] The geometric description of case $\tilde{A}$ has been completely treated by Torkildsen in [Tor12b]. We recal part of his description.

Let $m$ be an integer. Let $Q$ be a quiver of type $\tilde{A}_n$, with $p$ arrows going one direction, and $q$ arrows going the other. Let $P$ be a regular $mp$-gon, with a regular $mq$-gon inside of it. In the following, we give the example with $p = 4$ and $q = 3$, for $m = 2$. We number the vertices of the outer polygon $O_1, \cdots, O_{mp-1}$ and the vertices of the inner polygon $I_1, \cdots, I_{mq-1}$.

There are three types of $m$-diagonals:

— A path from a vertex of the outer polygon to a vertex of the inner polygon

— A path from $O_i$ to $O_{i+km+2}$, where $i + km + 2 \geq 1$, is counted modulo $pm + 1$ and $i \in \{1, \cdots, pm + 1\}$ homotopic to the boundary path of the outer polygon.

— A path from $I_i$ to $I_{i+m-1}$ for some $i$ and some $k \geq 3$ homotopic to the boundary path of the inner polygon.

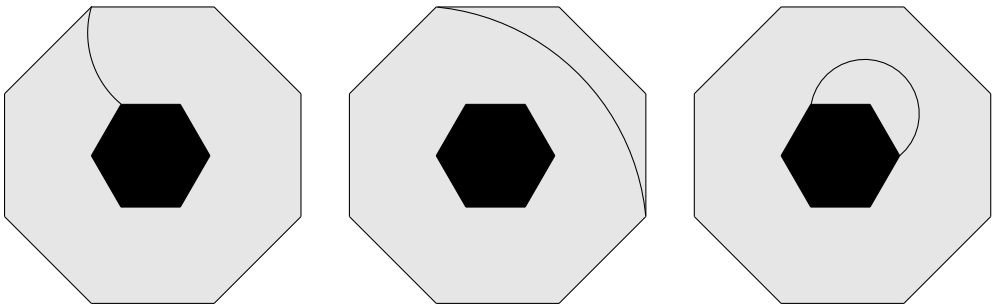


Figure 2.5 – Examples of $m$-diagonals

**Definition 2.1.16.** *We call "$(m + 2)$-angulation" a set of $p + q$ noncrossing $m$-diagonals cutting the figure into $(m + 2)$-angles.*

With an $(m + 2)$-angulation, we can associate a colored quiver defined in this way:

**Definition 2.1.17.** *Let $\Delta$ be an $(m+2)$-angulation. Let $Q_\Delta$ be the quiver defined as follows:*

— *The vertices are the $m$-diagonals*

— *There is an arrow from $i$ to $j$ if $i$ and $j$ bound a common $(m + 2)$-angle.*

— *The color of the arrow is the number of edges between $i$ and $j$ counted from $i$ counterclockwise.*

Then, Torkildsen in [Tor12b] defined the mutation of an $(m + 2)$-angulation:

**Definition 2.1.18.** *Let $\Delta$ be an $(m+2)$-angulation. Let $\alpha$ be an $m$-diagonal of $\Delta$. Then, remove $\alpha$. There exist $m+1$ $m$-diagonals which can complete the almost $(m+2)$-angulation. Let $\alpha'$ be the neighbor of $\alpha$ clockwise.*

*The mutation of $\Delta$ at $\alpha$ is defined to be the $(m+2)$-angulation $\Delta \setminus \{\alpha\} \cup \{\alpha'\}$*

**Theorem 2.1.19.** *[Tor12b, Proposition 5.1] Let $\Delta$ be any $(m+2)$-angulation. Let $Q_\Delta$ be the colored quiver associated with the $(m+2)$-angulation $\Delta$. If $\Delta_k$ is the new $(m+2)$-angulation flipped at $k$ from $\Delta$, then the colored quiver $Q_{\Delta_k}$ associated with $\Delta_k$ is the mutation at vertex $k$ of the colored quiver $Q_\Delta$.*

In the following article, Torkildsen builds an equivalence of categories and shows the following result:

**Theorem 2.1.20** ([Tor12b], Theorem 7.3)**.** *There is an explicit bijection between the $m$-diagonals (called "diagonals" in his article) without any self-crossing and the $m$-rigid indecomposable objects of the $m$-cluster category.*

### 2.1.4 Case $\tilde{D}$

This case has been treated in [JM]. We give some quick explanations here:

Let $P$ be an $(n-2)m$-gon with two central $(m-1)$-gons $R$ and $S$ inside of it (cf figure 2.6). We replace each vertex of $R$ and $S$ by a disk, which we henceforth call a thick vertex. If $m = 1$, then we consider an $(n-2)$-gon with two disks inside of it.

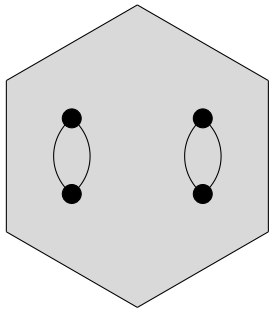


Figure 2.6 – The $(n-2)m$-gon with two digons. Here $m = 3$ and $n = 4$.

We consider the arcs up to rotation of one of the central polygons. It means that, once the arc is set, we can rotate the central polygons, this does not change the nature of the arc. See figure 2.7 for an example:

**Definition 2.1.21.** *An $m$-diagonal is an equivalence class of admissible arcs where the different classes are:*

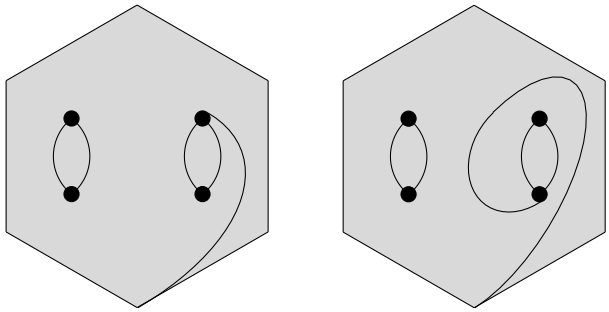


Figure 2.7 – These two arcs make part of the same class of $m$-diagonals.

1. *If $i \neq j$: the homotopy classes of arcs crossing the space between both central polygons and cutting the figure into a $km+1$-gon and a $k'm+1$-gon, for some $k$ and $k'$. In this case, it is said that the class is of type* 1.
2. *If $i \neq j$: the homotopy classes of arcs cutting the figure into a $km$-gon with both central polygons inside of it and a $k'm+2$-gon.*
3. *If $i = j$: all the admissible arcs tangent to the left of the disks of the inner polygon $R$, plus the left loop, form a first class. The second class is given by all the arcs tangent to the right of the disks, plus the right loop. There are two more classes built in the same way for $S$. For more details, see [JM].*
4. *Any admissible arc linking both central polygons form a class of $m$-diagonal. To be clear, if $\alpha$ is an arc whose first end is tangent to a side of $R$, and second end is tangent to a side of $S$, then it is an $m$-diagonal.*

**Definition 2.1.22.** *An $(m+2)$-angulation is a set of noncrossing $m$-diagonals cutting $P$ into $(m+2)$-angles.*

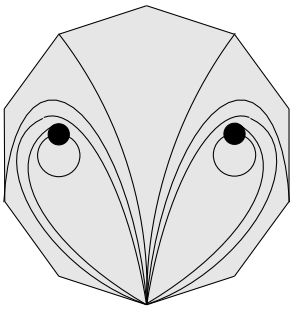


Figure 2.8 – A 4-angulation for $m = 2$ and $n = 7$.

In a similar way to that of Torkildsen, we define the flip of an $(m+2)$-angulation, and build a colored quiver associated with an $(m+2)$-angulation. We show the following result:

**Theorem 2.1.23.** *Let $\Delta$ be any $(m+2)$-angulation. Let $Q_\Delta$ be the colored quiver associated with the $(m+2)$-angulation $\Delta$. If $\Delta_k$ is the new $(m+2)$-angulation flipped at $k$ from $\Delta$, then the colored quiver $Q_{\Delta_k}$ associated with $\Delta_k$ is the mutation at vertex $k$ of the colored quiver $Q_\Delta$.*

Then we show that there are some common points between the higher cluster category and a geometrically built category. More precisely, we build a category from these $m$-diagonals which is equivalent to a subcategory of the higher cluster category. We explicit all $m$-diagonals in the Auslander-Reiten quiver of $\mathcal{C}_Q^m$, where $Q$ is a quiver of type $\tilde{D}_n$.

We also show the following theorem:

**Theorem 2.1.24** ([JM]). *There is an explicit bijection between the $m$-diagonals without any self-crossing and the $m$-rigid indecomposable objects of the $m$-cluster category.*

From now and all throughout the paper, we fix such a bijection.

## 2.2 Noncrossing arcs and extensions

In this section, we are going to show in types $A$, $D$, $\tilde{A}$ and $\tilde{D}$ the following theorem :

**Theorem 2.2.1.** *Let $\alpha$ and $\beta$ be two arcs in the polygon $P$. Let $X_\alpha$ and $X_\beta$ be the associated $m$-rigid objects. If $\forall i \in \{1, \cdots, m\}, \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) = 0$, then $\alpha$ and $\beta$ do not cross each other.*

**Remark 2.2.2.** *The result in cases $A$ and $D$ has already been shown by Thomas in [Tho07] and by Baur and Marsh in [BM08] for case $A$ and [BM07] for case $D$.*

*We nontheless include a proof as it illustrates the method that will be applied in types $\tilde{A}$ and $\tilde{D}$.*

Our strategy to prove this consists in showing that cutting along an arc corresponds to applying the Iyama-Yoshino reduction. But first, let us show a useful lemma:

**Lemma 2.2.3.** *Let $\mathcal{C}$ be a Hom-finite triangulated category with a Serre functor. Let $X \in \mathcal{C}$ an $m$-rigid object. Let $Y$ be an object of $\mathcal{C}$ which belongs to $X^\perp$. Suppose that $\mathcal{C}(Y, X) = 0$ and for all $i \in \{1, \cdots, k\}$, where $k \leq m$, we have $\mathrm{Ext}^{-i}_{\mathcal{C}}(Y, X) = 0$, then*

$$\forall i \in \{1, \cdots, k\} Y\langle -i\rangle \simeq Y[-i]$$

*where $\langle 1\rangle$ denotes the shift in $\mathcal{C}$ and in the Iyama-Yoshino reduction $X^\perp/(X)$.*

*Proof.* We show by induction on $i$ that $Y\langle -i\rangle = Y[-i]$, where $[1]$ denotes the shift in $\mathcal{C}$ and $\langle 1\rangle$ is the shift in the Iyama-Yoshino reduction. It is defined on objects as follows: Let

$$T_k^{(c)} \xrightarrow{f_k^{(c)}} B_k^{(c)} \xrightarrow{g_k^{(c+1)}} T_k^{(c+1)} \xrightarrow{h_k^{(c+1)}} T_k^{(c)}[1]$$

be the exchange triangles as seen in the preliminaries. If we use the notations of the theorem, we have that $Y = T_k^{(c)}$. Then $Y\langle 1\rangle$ is in fact the object $T_k^{(c+1)}$ in the exchange triangle.

First, $Y\langle -1\rangle \simeq Y[-1]$. Indeed, $\mathcal{C}(X, Y) = 0$. Let us take an $\mathrm{add}_X$-approximation of $Y$. Then we have the following triangle:

$$Y[-1] \longrightarrow Y\langle -1\rangle \longrightarrow 0 \xrightarrow{0} Y.$$

As the right morphism is zero, we have that $Y\langle -1\rangle \simeq Y[-1]$.
Suppose $Y\langle -i+1\rangle \simeq Y[-i+1]$. Then

$$\mathcal{C}(X, Y\langle -i+1\rangle) = \mathcal{C}(X, Y[-i+1]) = 0$$

Let us once again take an $\mathrm{add}_X$-approximation of $Y\langle -i+1\rangle$. Then we have

$$Y[-i] \longrightarrow Y\langle -i\rangle \longrightarrow 0 \longrightarrow Y[-i+1].$$

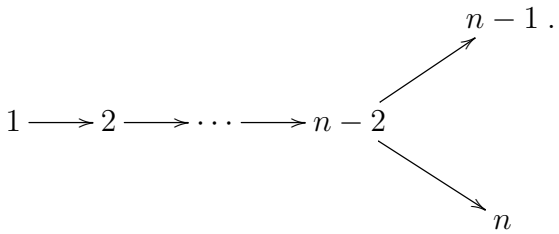
Then $Y\langle -i\rangle \simeq Y[-i]$ □

### 2.2.1 Cases $A$ and $D$

We start by a useful lemma.

**Definition 2.2.4.** *We call an $m$-diagonal $\alpha$ an $m$-ear, when $\alpha$ divides $P$ into an $m+2$-gon and an $(n-1)m+2$-gon for the $A$ case (respectively $(n-1)m-m+1$-gon containing the interior polygon for case $D$).*

Let $Q$ be the quiver $1 \to 2 \to \cdots \to n$ or



**Lemma 2.2.5.** *Let $\mathcal{C}$ be the $m$-cluster category of type $A_n$ (respectively $D_n$). Let $\alpha$ be an $m$-ear. Let $X_\alpha$ be the $m$-rigid object associated with $\alpha$. Let $\mathcal{U} = \{Y \in \mathcal{C},\ \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, Y) = 0\}$. Let $\mathcal{C}'$ be the Iyama-Yoshino reduction of $\mathcal{C}$: $\mathcal{C}' = \mathcal{U}/(X_\alpha)$. Then, we have an equivalence of categories :*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}$$

*where $Q/\alpha$ is the quiver obtained from $Q$ by removing $\alpha$ and all incident arrows.*

*Proof.* This lemma is a consequence of theorem 3.3.8 of Keller and Reiten.

Let us find an $m$-cluster-tilting object $T$ in $\mathcal{C}'$ satisfying the assumptions of theorem 3.3.8, such that $\mathrm{End}(T) \simeq KA_{n-1}$ (respectively $\mathrm{End}(T) \simeq KD_{n-1}$). We recall that we choose the clockwise convention, it means that we draw the arrows of the quiver of an $(m+2)$-angulation clockwise. Moreover, we name by 1, the vertex of $P$ which corresponds to the common vertex of the arcs of the first slice of the Auslander-Reiten quiver of $Q$ (this ensures that 1 is a source in $Q$).

We know from the bijection between $m$-rigid objects and $m$-diagonals that the $m$-ear $\alpha$ corresponds to a translation of the first projective module $P_1$ up to some shift $[j]$, for some $j \leq m$. We may thus assume that $X_\alpha = P_1$. Let $T$ be the sum of all $i$ in $Q_0$ for $i \neq 1$, viewed both as an object in $\mathcal{C}^m_Q$ and of $\mathcal{C}'$. We have that $\mathrm{End}_{\mathcal{C}'}(T) \simeq KA_{n-1}$. Indeed, first, we have that $\mathrm{End}_{\mathrm{mod}}(T) = KA_{n-1}$ (respectively $KD_{n-1}$) because $X_\alpha = P_1$ and 1 is a source in $Q$. Then we show that this remains the same in the higher cluster category by drawing the Auslander-Reiten quiver of $Q$. Applying the Iyama-Yoshino reduction does not change anything since no morphism is incident to $\alpha$. Moreover, from Keller and Reiten in [KR08, Section 4], we have that $\forall i \in \{1, \cdots, m\}\ \mathrm{Ext}^{-i}_{\mathcal{C}}(T, T) = 0$. From lemma 2.2.3 and the fact that 1 is a source in $Q$, we then have $\forall i \in \{1, \cdots, m\}\ \mathrm{Ext}^{-i}_{\mathcal{C}'}(T, T) = 0$ Then we have shown the lemma. □

Before showing the next lemma, we are going to explicit the bijection

$$\{\beta \text{ does not cross } \alpha \leftrightarrow \{\beta \in S/\alpha\}$$

in cases $A$ and $D$. For case $A$ (respectively for case $D$), if $\beta$ is an arc which does not cross $\alpha$, then we can "cut along $\alpha$" in order to have two new figures of type $A$ (respectively one of type $D$ and one of type $A$) and one of these contains the same arc as $\beta$, which we still call as $\beta$.

**Lemma 2.2.6.** *Let $P$ be an $nm+2$-gon (respectively an $mn-m+1$-gon) associated with a quiver $Q$ of type $A_n$ (respectively $D_n$). Let $\alpha$ be an $m$-ear from $i$ to $j$. Then cutting along $\alpha$ corresponds to applying the Iyama-Yoshino reduction of $\mathcal{C}^m_{A_n}$ (respectively $\mathcal{C}^m_{D_n}$) applied on $X_\alpha$. More precisely, let $\mathcal{C}$ be the $m$-cluster category associated with a quiver of type $A_n$ (respectively $D_n$), and let $\mathcal{C}' = \mathcal{U}/X_\alpha$, where*

*$X_\alpha$ is the $m$-rigid object associated with $\alpha$, and $\mathcal{U} = \{Y, \mathrm{Ext}^l_{\mathcal{C}}(\bigoplus X_\alpha, Y) = 0 \ \forall l \in \{1, \cdots, m\}\}$. Let $Q/\alpha$ be the quiver $Q$ where the vertex corresponding to $\alpha$ and all the incident arrows have been removed. The previous lemma tells us that we have the following result:*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}.$$

*Moreover, the following diagram is commutative.*

$$\begin{array}{ccc} \{\beta \text{ does not cross } \alpha, \beta \neq \alpha\} & \longrightarrow & \{X \in \mathcal{U}; X \ncong X_\alpha\}/\simeq \\ \updownarrow & & \updownarrow \\ \{\beta \in S/\alpha\} & \longrightarrow & \{X \in \mathcal{C}'\}/\simeq \end{array}$$

*where the second is given by the Iyama-Yoshino reduction. The horizontal arrows are maps sending $\beta$ to $X_\beta$. The symbol $S/\alpha$ means the surface obtained from $S$ by cutting along $\alpha$. Up to homeomorphism, this does not depends on the choice of a representative of $\alpha$.*

*Proof.* It suffices to show that the arcs that cross $\alpha$ exactly correspond to the $m$-rigid which do not lie in $\mathcal{U}$. Let us then take an $m$-diagonal $\beta$ which cuts $\alpha$ (see figure 2.9). Let $X_\beta$ be the associated $m$-rigid object. Let us show that there exists $k \in \{1, \cdots, m\}$ such that $\mathrm{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$.

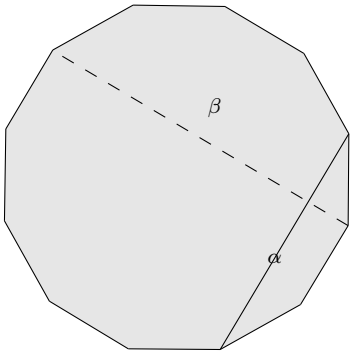



Figure 2.9 – Example of an $m$-ear in case $A$

Indeed, as there are only $m-1$ vertices between $\alpha$ and the border, we can shift $\beta$ $k < m$ times in order to have $\alpha$ and $\beta$ sharing an end point. Let us show that there is a morphism from $X_\alpha$ and $X_\beta[k]$.

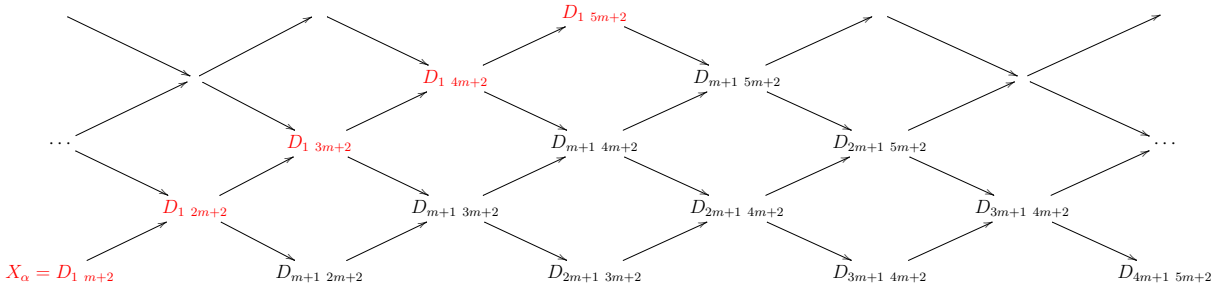
If we take $\beta$, an arc which does not cross $\alpha$. As there is a bijection between the Auslander-Reiten quiver of $Q$ and the translation quiver built in [BM08] for case $A$ and [BM07] for case $D$, the arc $\beta$ corresponds to a unique object $X_\beta$ situated on the Auslander-Reiten quiver of $Q$. Cutting the Auslander-Reiten quiver first,

or the translation quiver built by Baur and Marsh first (and then associating $X_\beta$ with $\beta$ on the new cut quiver) is the same. Thus, the diagram is commutative.

**Remark 2.2.7.** *We need to note that the cases are symmetric. Indeed, to shift $\beta$ $k$ times in order to find a morphism from $\alpha$ to $\beta[k]$ is the same as to shift $\alpha$ $k$ times, which gives a morphism from $\beta[k-(m+1)]$ to $\alpha$. Thanks to the $(m+1)$-Calabi-Yau property, we know that $\mathcal{C}(\beta[k-(m+1)],\alpha) \simeq D(\alpha,\beta[k])$. This means no matter which vertex we shift.*

1. In case $A$: we assume that $\alpha$ is the arc from 1 to $m+2$ with no loss of generality.

   In the Auslander-Reiten quiver of $\mathcal{C}$, the $m$-rigid $X_\alpha$ is situated at the bottom as we can see in the next picture where we identify an arc with the associated object in the higher cluster category. We give the name of the arcs by $D_{1j}$, where the arcs links 1 to $j$. Moreover, we draw the Hom-hammock in red.

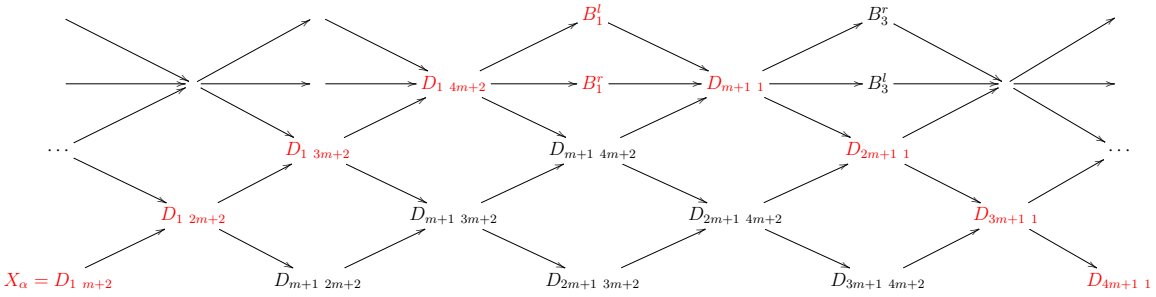



   If we draw the corresponding arcs on the Auslander-Reiten quiver, we realize that the ones on the slice arising from $X_\alpha$ (on the figure, $P_2$, $P_3$, $P_4$, $P_5$) have an end point equal to 1. We note moreover that those are all arcs having 1 as an end. Then $\beta[k]$ belongs to one of them.

   It is also known that these modules exactly correspond to the ones which have a nonzero morphism from $X_\alpha$. Then $\mathrm{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$.

2. In case $D$:

   In the Auslander-Reiten quiver of $\mathcal{C}$, the $m$-rigid $X_\alpha$ is situated at the bottom as we can see in the next picture. We name the diagonals by $D_{ij}$ in the same way as in $A_n$ case. Both particular diagonals are called $B_1^l$ or $B_1^r$. We draw the figure for the case $i=1$ for the sake of simplicity.




The Hom-hammock starting at $X_\alpha$ contains precisely those $X_\gamma$'s for which $\gamma$ contains vertex 1. Then $\beta[k]$ belongs to one of them.

It is also known that these modules exactly correspond to the ones which have a nonzero morphism from $X_\alpha$. Then $\mathrm{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$.

□

We have shown that cutting along an arc corresponds to the Iyama Yoshino reduction. Let us now prove theorem 2.2.1.

*Proof of theorem 2.2.1.* Let us suppose that $\alpha$ and $\beta$ cross each other. If $\alpha$ is an $m$-ear, then the result is already shown.

Else, if we can shift $\beta$ $k < m$ times so that they have one common end point, we prove in a similar way to that of previously, that $\beta[k]$ is situated on the Hom-hammock of $\alpha$, and then there is a nonzero extension from $X_\alpha$ to $X_\beta$. If we cannot shift $\beta$ $k < m$ times as needed, it means that we can draw an $m$-ear $\gamma$, which does not cross $\alpha$ neither $\beta$, and from the previous theorem, we cut along it. By induction, there is some $k \in \{1, \cdots, m\}$ such that $\mathrm{Ext}^k_{\mathcal{C}'}(X_\alpha, X_\beta) \neq 0$. From Iyama-Yoshino, we have that $\mathcal{C}(X, Y[i])$ and $\mathcal{C}'(X, Y\langle i\rangle)$ are isomorphic and this finishes the proof of the theorem. □

## 2.2.2 Case $\tilde{A}$

In this subsection, we will use the same sketch of proof. Let us now define the notion of an $m$-ear:

**Definition 2.2.8.** *Let $\alpha$ be an $m$-diagonal. Then $\alpha$ is an $m$-ear if it lies in the outer or inner polygon, and links a vertex $i$ to $i+m+1$, and is homotopic to the boundary path (see figure 2.10 for an example of $m$-ear).*

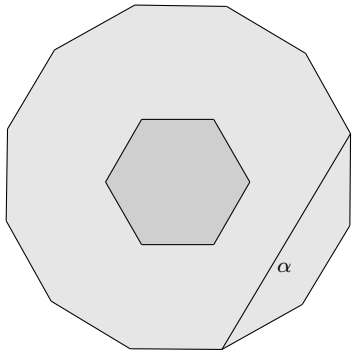



Figure 2.10 – Example of an $m$-ear in case $\tilde{A}$

**Lemma 2.2.9.** *Let $\mathcal{C}$ be the $m$-cluster category of type $\tilde{A}_n$. Let $\alpha$ be an $m$-diagonal which is either an $m$-ear or in the transjective component of the Auslander-Reiten quiver of $\tilde{A}_n$. Let $X_\alpha$ be the $m$-rigid associated with $\alpha$. Let $\mathcal{U} = \{Y \in \mathcal{C},\ \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, Y) = 0\}$. Let $\mathcal{C}'$ be the Iyama-Yoshino reduction of $\mathcal{C}$: $\mathcal{C}' = \mathcal{U}/(X_\alpha)$. Then, we have an equivalence of categories :*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}$$

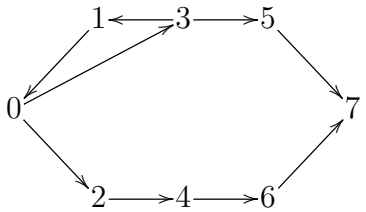
*where $Q/\alpha$ is the quiver obtained from $Q$ by removing $\alpha$ and all incident arrows.*

**Remark 2.2.10.** *We can show exactly the same results if $\alpha$ links two sides of the internal polygon, and is homotopic the the boundary of it.*

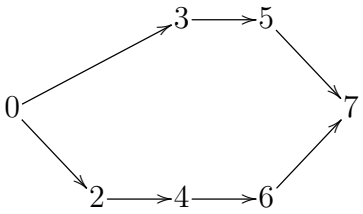
*Proof.* There are two different cases.

First, if $\alpha$ is an $m$-ear:

Let us begin by illustrating this fact with the Gabriel quivers. The mutation at vertex 1 leads to the following quiver:




Using the Iyama-Yoshino reduction at vertex 1 corresponds to forgetting this vertex and all incident arrows. By doing this, we are ensured to be reduced to a quiver of type $\tilde{A}_{n-1}$:




Here again, we use the theorem of Keller and Reiten in [KR08], as in type $A$ and $D$. We have to find an $m$-cluster-tilting object $T$ such that $\mathrm{End}_{\mathcal{C}'}(T) \simeq K\tilde{A}_{n-1}$, and $\forall i \in \{1, \cdots, m\} \mathrm{Ext}^{-i}_{\mathcal{C}'}(T, T) = 0$.

Let $T = \oplus P_i$. We know from Torkildsen (see figure 8 in [Tor12b]), that $T$ corresponds summand by summand, to the $(m+2)$-angulation made of all $m$-diagonals linking the external polygon to the internal one (see figure 2.11)

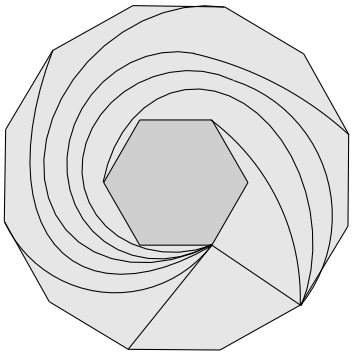


Figure 2.11 – The "initial" $(m+2)$-angulation of type $\tilde{A}$, for $m = 4$, $p = 12$, $q = 6$.

Let $T'$ be the mutation of $T$ at $P_1$ the first preprojective module. Then $T' = \oplus_{i \neq 1} P_i \oplus X$ is also an $m$-cluster-tilting object. Let us show that $\tau X$ corresponds to the simple module at the base of the first tube (see figure 2.12 to visualize the mutation in terms of arcs). However, we do not know yet that the mutation of $m$-cluster tilting objects corresponds to the flip of $(m+2)$-angulations.

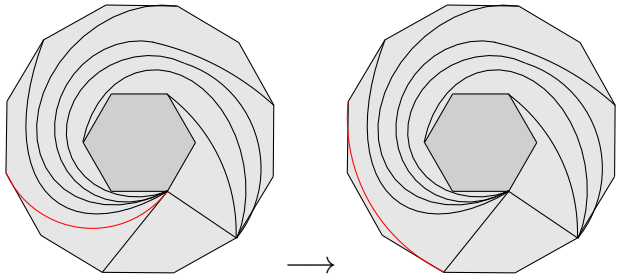


Figure 2.12 – Geometric visualization of $T'$

We have to show that $\tau X = S_k$, the simple module in $k$, which is situated at the bottom of the tube of size $n-2$.

Let us find $\mathcal{C}(X[-1], T)$. For all $i \neq k$, $\mathcal{C}(X[-1], P_i) \simeq \mathcal{C}(X, P_i[1]) = 0$ since $T'$ is an $m$-cluster-tilting object.

Now we focus on $\mathcal{C}(X[-1], P_k)$. We have that $\mathcal{C}(X[-1], P_k) \simeq \mathcal{C}(X, P_k[1]) \simeq K$ from Iyama and Yoshino in [IY08].

From the $(m+1)$-Calabi-Yau property, we have

$$\mathcal{C}(X[-1], T) \simeq \mathcal{C}(X, T[1]) \simeq D\mathcal{C}(T, \tau X)$$

Then $\tau X \simeq S_k$ and $X = X_\alpha$ corresponds to the red arc, named $\alpha$.

Moreover, from the paper of Baur and Torkildsen, we can easily visualize the morphisms in the module category of type $\tilde{A}$.

We have $\mathrm{End}_{\mathcal{C}'}(T') \simeq K\tilde{A}_{n-1}$. Indeed, in the module category, $\mathrm{End}_{\mathrm{mod}}(T') \simeq K\tilde{A}_{n-1}$, because, the objects of $T'$ (apart from $X_\alpha$) are on the projective slice of the Auslander-Reiten quiver of $Q$. Thus there is no relation. If any morphism $f : T' \to T'$ factorizes through $P_k$, then $f = uv$ where $T' \xrightarrow{u} P_k \xrightarrow{v} T'$ and this is impossible given that there is no morphism from a regular module to a preprojective one.

Now we show this for the higher cluster category. We have the following decomposition of morphisms ($G$ is the functor $\tau^{-1}[m]$):

$$\mathcal{C}(M, N) \simeq \bigoplus_{i \in \mathbb{Z}} \mathcal{D}^b(G^i M, N).$$

If $m = 1$, the result is already known, because if $X$ is a preprojective object and $Y$ a regular one, then $\mathrm{Ext}_{\mathcal{C}}(X, Y) = 0$.

If not, we use the decomposition just above. For $i \geq 1$, we have that

$$\mathcal{D}^b(\tau^{-1}[m]T, T) = D\mathcal{D}^b(T, T[m+1])$$

thanks to the duality. From the book [ASS06], the algebra of a quiver of type $\tilde{A}$ is hereditary and then the extension $\mathrm{Ext}^2_{\mathcal{D}^b}(T, T) = 0$. Then, for $i \geq 1$, all the terms of the sum are zero. Then

$$\mathrm{End}_{\mathcal{C}}(T') \simeq \mathrm{End}_{\mathcal{D}^b}(T').$$

Applying the Iyama-Yoshino reduction does not change anything since in the higher cluster category, there is no morphism incident to $\alpha$.

It finally remains to prove that $\mathrm{Ext}^{-i}_{\mathcal{C}'}(T', T') = 0$ for all $i \in \{1, m-1\}$.

Let us first show that $\mathrm{Ext}^{-i}_{\mathcal{C}}(T', T') = 0$, using the shift in $\mathcal{C}$. Then we will use lemma 2.2.3 in order to conclude.

We claim that $\mathcal{C}(T', T'[-i]) = 0$. Indeed, from Keller and Reiten in [KR08], we know that $\mathcal{C}(\oplus_{j\neq k} P_j, T'[-i]) = 0$. Moreover, as $X_\alpha$ and $X_\alpha[-i]$ are not in the same tube, then $\mathcal{C}(X_\alpha, X_\alpha[-i]) = 0$ in the module category (which from Wraalsen [Wra09] or Zhou-Zhu [ZZ09], immediately translates to the higher cluster category). In addition, there cannot be any morphism from a tubular component to a preprojective one. Then $\mathcal{C}(X_\alpha[-i], P_j) = 0$ in the module category (which from Wraalsen [Wra09] or Zhou-Zhu [ZZ09], immediately translates to the higher cluster category). It remains to show that $\mathcal{C}(P_j, X_\alpha[-i]) = 0$ for any $j \neq k$ and $i \in \{1, \cdots, m-1\}$. By definition of the mutation, we have the exchange triangle

$$P_k[-i] \longrightarrow U[-i], U \in \operatorname{add} \oplus_{l\neq k} P_l \longrightarrow X_\alpha[-i] \longrightarrow P_k[-i+1].$$

$$\begin{array}{ccccc} & & \overset{\exists\ g}{\swarrow} & \uparrow{\scriptstyle f} & \overset{0}{\searrow} \\ & & & P_j & \end{array}$$

For $i \neq 1$, there is no morphism from $P_j$ to $P_k[-i+1]$.

Then, there exists $g : P_j \to U[-i]$. As $U$ is only composed with projectives which are not $P_k$, this shows that $g = 0$. Then $f = 0$.

For $i = 1$, the composition $P_j \to X_\alpha[-1] \to P_k$ is zero because there is no morphism from tubular objects to preprojective objects. Then there exists $g$ such as previously, but the composition with $U[-i] \to X_\alpha[-i]$ is zero for the same reason. Then $f = 0$.

This shows that $\operatorname{Ext}^{-i}_{\mathcal{C}}(T', T') = 0$.

From lemma 2.2.3, we have that $T'_l < -i > \simeq T'_l[-i]$. Finally,

$$\operatorname{Ext}^{-i}_{\mathcal{C}'}(T', T') = 0.$$

We now have gathered all the information in order to apply the theorem of Keller and Reiten, and we have that

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}.$$

Else, if $\alpha$ corresponds to a transjective module, we proceed in the same way, we have that $\operatorname{Ext}^{-i}_{\mathcal{C}}(T', T') = 0$ from Keller and Reiten [KR08, Lemma 4.1], and we can apply Keller-Reiten theorem. In details, let $T$ be the $m$-cluster-tilting object corresponding to a slice of the Auslander-Reiten quiver of $Q$ (see the article of Baur and Torkildsen [BT15] for details). Then as there is an isomorphism between the Auslander-Reiten quiver of $Q$ (except the homogeneous tubes) and the translation quiver $\Gamma$ built by Baur and Torkildsen in [BT15, Proposition 3.7], the morphisms in the module category from $T$ to $T$ (where $T$ is a slice of the Auslander-Reiten quiver) correspond to a quiver of type $A_{n-1}$. Then we have $\operatorname{End}_{\mathrm{mod}}(T) \simeq KA_{n-1}$. □

**Lemma 2.2.11.** *Let $\alpha$ be an $m$-ear. Then cutting along $\alpha$ corresponds to applying the Iyama-Yoshino reduction. More precisely, the application which, with an indecomposable rigid object of the higher cluster category, associates an $m$-diagonal, induces a map*

$$\{\textit{rigid indecomposable modules of } \mathcal{U}\} \to \{m\textit{-diagonals which do not cross } \alpha\}.$$

**Remark 2.2.12.** *We need to note that the cases are symmetric. Indeed, to shift $\beta$ $k$ times in order to find a morphism from $\alpha$ to $\beta[k]$ is the same as to shift $\alpha$ $k$ times, which gives a morphism from $\alpha$ to $\beta[m+1-k]$ thanks to the $(m+1)$-Calabi-Yau property. This means, we do not care about which vertex we shift.*

*Proof.* Here, as in cases $A$ and $D$, it suffices to show that, if $\beta$ is an $m$-diagonal crossing $\alpha$, then we can find a morphism from $X_\alpha$ to $X_\beta[k]$, for some $k \in \{1, \cdots, m\}$.

By the geometric realization of Torkildsen in [Tor12b], the $m$-diagonal $\alpha$ is situated at the bottom of the first tube. As $\beta$ crosses $\alpha$, we can shift it $k < m$ times so that one end point of $\beta$ is in common with one of $\alpha$. There are two cases:

First case: $\beta$ corresponds to an $m$-rigid in a tube. Then, by the proof of Proposition 7.2 in [Tor12b], there exists a nonzero morphism from $X_\alpha$ to $X_\beta[k]$ (see figure 13 of the article for a clear picture of this map).

Second case: $\beta$ corresponds to a preinjective arc. Then, by the paragraph 4.1 of the article written by Baur and Torkildsen [BT15], as $\alpha$ and $\beta[k]$ share an oriented angle, there is a so-called "long move", hence a nonzero morphism in the module category from $X_\alpha$ to $X_\beta[k]$.

In any case, we have found a nonzero morphism in the higher cluster category from $X_\alpha$ to $X_\beta[k]$. Then the arcs which cross $\alpha$ exactly correspond to the rigid which do not lie in $\mathcal{U}$. Then the Iyama-Yoshino reduction corresponds to cutting along an arc. □

We are now able to prove theorem 2.2.1:

*Proof of theorem 2.2.1.* If $\alpha$ and $\beta$ are two crossing $m$-diagonals in the geometric realization of a quiver of type $\tilde{A}$ (an external polygon $P$ with $p$ sides together with an internal polygon $R$ with $q$ sides). There are two cases:

1. First case: The $m$-diagonal $\alpha$ links two vertices of $P$, and is homotopic to the boundary path (or in the same way, $\alpha$ lies in the inner polygon). If $\alpha$ is an $m$-ear, then the result is shown. Else:

   - If it is impossible to draw an $m$-ear between both end points of $\alpha$ without crossing $\beta$, then it means that $\alpha$ is an $m$-ear, and the result is shown. It suffices to count the vertices to make sure of it: if $\alpha$ is not an $m$-ear, it cuts the polygon into a $u$-gon, with $u \geq 2m+2$. Then, at one side of $\beta$, there is at least $m+1$ free vertices, where we can draw an $m$-ear.

- Else we can draw an $m$-ear $\gamma$ between an end point of $\alpha$ and an end point of $\beta$, then it suffices to cut along $\gamma$ and repeat the operation as many times as necessary, in order to reduce to the previous case.

See figure 2.13 for an illustration.

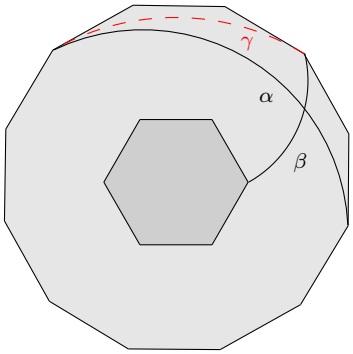



Figure 2.13 – Illustration of the first case for $p = 12$, $q = 6$ and $m = 2$

2. Second case: The $m$-diagonal $\alpha$ is a transjective arc (corresponding to the transjective part of the Auslander-Reiten quiver of $\mathcal{C}^m_{\tilde{A}_n}$).

   (a) If $\beta$ is homotopic to the boundary path of one of the polygons (let us say for instance that $\beta$ is homotopic to the boundary path of the external polygon). Then, we use the same type of argument.

   - If we cannot draw an $m$-ear lying inside the end points of $\beta$ which does not cross $\alpha$, then it suffices to shift $\beta$ $k < m$ times in order to hang one end point of $\beta$ to one end point of $\alpha$. This corresponds to a long move, then to a morphism in the module category in the sense of Baur and Torkildsen in [BT15].

   - Else, we cut along this $m$-ear, and repeat the operation as many times as necessary to reduce to the first case.

   (b) If $\beta$ is a transjective arc.

   - If we can shift $\beta$ $k < m$ times in order to hang one end point of $\beta$ to one end point of $\alpha$, then both arcs $\alpha$ and $\beta[k]$ belong to the same slice in the Auslander-Reiten quiver of $Q$. We can show that there exists a morphism in the module category from $X_\alpha$ to $X_\beta[k]$ with the article of Baur and Torkildsen [BT15, Paragraphs 3.3 and 3.4].

   - Else, there exists an $m$-ear $\gamma$ which does not cross $\alpha$ nor $\beta$. It suffices to cut along $\gamma$ and repeat as many times as necessary in order to reduce to the previous case.

□

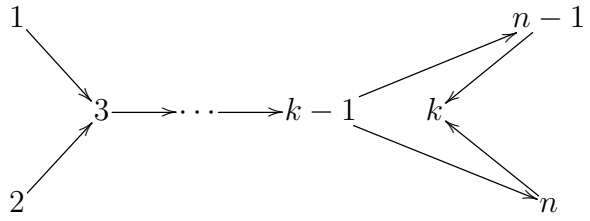
### 2.2.3 Case $\tilde{D}$

**Definition 2.2.13.** *Let $P$ be a polygon with $(n-2)m$ sides with two $m-1$-gons inside of it, associated with a quiver of type $\tilde{D_n}$. Then, an $m$-ear is an $m$-diagonal linking a vertex $i$ to the vertex $i+m+1$ homotopic to the boundary of $P$.*

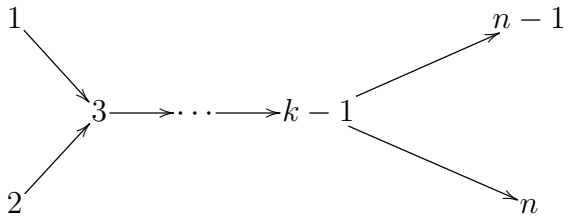
**Lemma 2.2.14.** *Let $P$ be a polygon with $(n-2)m$ sides associated with a quiver $Q$ of type $\tilde{D_n}$ and let $\alpha$ be an $m$-ear. Then the Iyama-Yoshino reduction of $\mathcal{C}^m_{\tilde{D_n}}$ applied on $X_\alpha$ corresponds to cutting along $\alpha$. More precisely, let $\mathcal{C}$ be the $m$-cluster category associated with a quiver of type $\tilde{D_n}$, and let $\mathcal{C}' = \mathcal{U}/X_\alpha$, where $X_\alpha$ is the $m$-rigid object associated with $\alpha$, and $\mathcal{U} = \{Y, \mathrm{Ext}^l_{\mathcal{C}}(\bigoplus X_\alpha, Y) = 0 \ \forall l \in \{1, \cdots, m\}\}$. Let $Q/\alpha$ be the quiver $Q$ where the vertex corresponding to $\alpha$ and all the incident arrows have been removed. Then we have the following result:*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}.$$

Let us begin by illustrating this fact with the Gabriel quivers. The mutation at vertex $k$ leads to the following quiver:




Using the Iyama-Yoshino reduction at vertex $k$ corresponds to forget this vertex and all incident arrows. By doing this, we are ensured to be reduced to a quiver of type $\tilde{D}_{n-1}$:

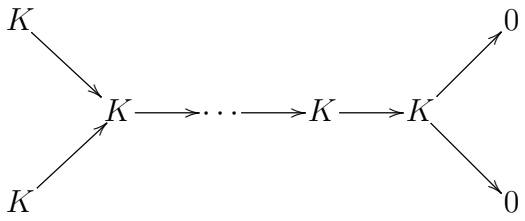



*Proof.* Here again, we use theorem 3.3.8 of Keller and Reiten.

We are going to use this theorem, by building a new $m$-cluster-tilting object respecting the hypotheses.

Let $T = \bigoplus P_i$ be the sum of all projective modules. We know that $T$ is an $m$-cluster-tilting object. This object is naturally associated with the initial $(m+2)$-angulation.

Let $P_k$ be the following projective module:




From Iyama and Yoshino in [IY08], we have an exchange triangle:

$$P_k \to Y \to X \to P_k[1] \tag{2.1}$$

where $Y \in \operatorname{add} \bigoplus_{j \neq k} P_j$.

We mutate the object $T$ as Wraalsen and Zhou, Zhu in [Wra09] and [ZZ09] at $P_k$ in order to study the new $m$-cluster-tilting object

$$T' = \bigoplus_{j \neq k} P_j \oplus X.$$

Let us first show that $X$ corresponds to the arc $\alpha$, which is the arc obtained by flipping the arc of type 1 corresponding to the vertex $k$ of the Auslander-Reiten quiver (see figure 2.14).

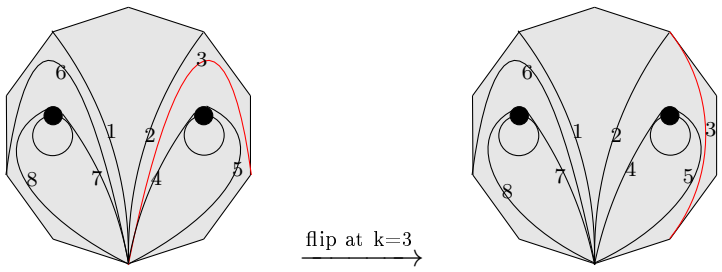



Figure 2.14 – We flip the arc corresponding to $P_k$. The new arc is called by $\alpha$.

We have to show that $\tau X = S_k$, the simple module in $k$, which is situated at the bottom of the tube of size $n-2$ as we set in the previous section.

Let us find $\mathcal{C}(X[-1], T)$. For all $i \neq k$, $\mathcal{C}(X[-1], P_i) \simeq \mathcal{C}(X, P_i[1]) = 0$ since $T'$ is an $m$-cluster-tilting object.

Now we focus on $\mathcal{C}(X[-1], P_k)$. Wa have that $\mathcal{C}(X[-1], P_k) \simeq \mathcal{C}(X, P_k[1]) \simeq K$ from Iyama and Yoshino in [IY08].

From the $(m+1)$-Calabi-Yau property, we have

$$\mathcal{C}(X[-1], T) \simeq \mathcal{C}(X, T[1]) \simeq D\mathcal{C}(T, \tau X)$$

Then $\tau X \simeq S_k$ and $X = X_\alpha$ corresponds to the arc $\alpha$.

It now remains to check the hypotheses of Keller-Reiten's theorem. First, $\mathcal{C}'$ is a Hom-finite algebraic $(m+1)$-Calabi-Yau category. The object $T'$ is our candidate. It is still an $m$-cluster-tilting object. First of all, from [KR08, Lemma 4.1], the morphisms $\mathcal{C}(P_j, P_l[-i])$ are zero for any $j$ and $l$. Moreover, as $X$ and $X[-i]$ are not in the sale tube, we have $\mathcal{C}(X, X[-i]) = 0$. In addition, $\mathcal{C}(X[-i], P_j) = 0$ as there is no morphism from a regular object to a preprojective object. It now remains to show that $\mathcal{C}(P_j, X[-i]) = 0$ for any $j \neq k$ and any $i \in \{1, \cdots, m\}$. We then have the following diagram:

$$P_k[-i] \longrightarrow U[-i], U \in \operatorname{add} \oplus_{l \neq k} P_l \longrightarrow X_\alpha[-i] \longrightarrow P_k[-i+1].$$

$$\exists\ g \qquad f \qquad 0$$

$$P_j$$

and we conclude in the same way as in type $\tilde{A}$.

If $i = 0$, we remark that there can be no relations or factorizations in the slice taken, which means , we have that $\mathcal{C}(T, T) \simeq K\tilde{D}_{n-1}$ (see illustration at the beginning of the proof), and this permits to apply Keller-Reiten's Theorem and finishes the proof. □

**Lemma 2.2.15.** *Let $\alpha$ be an $m$-ear. Then cutting along $\alpha$ corresponds to applying the Iyama-Yoshino reduction at $X_\alpha$. More precisely, the application which, with an indecomposable rigid object of the higher cluster category, associates an $m$-diagonal, induces a map*

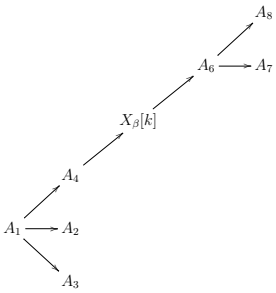
$$\{m\text{-diagonals which do not cross } \alpha\} \to \{\text{rigid indecomposable modules of } \mathcal{U}\}.$$

*Proof.* Here again, it suffices to show that, if $\beta$ is an arc cutting $\alpha$, then there exists some $k \in \{1, \cdots, m\}$ such that $\operatorname{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$.

Let $\beta$ be an arc crossing $\alpha$. Then we can shift $\beta$ $k < m$ times in order for both of them to share a common ending vertex $a$.

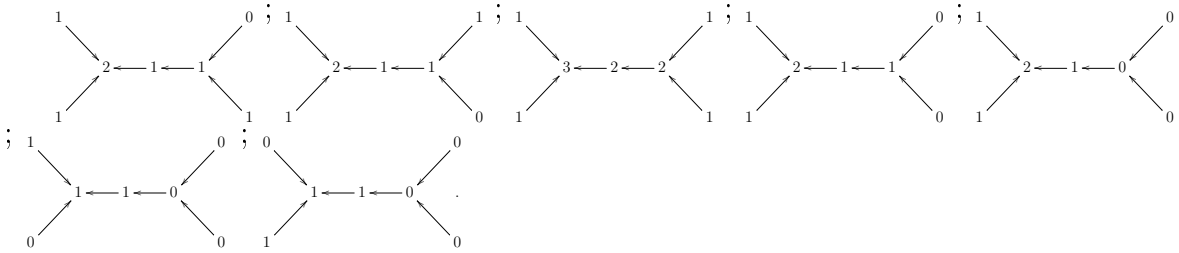
If $\beta[k]$ is in a tube of size $n-2$, then it is situated in the same tube as $\alpha$, higher than it. Then there exists a morphism from $X_\alpha$ to $X_\beta[k]$.

Else, $\beta[k]$ is situated in the preinjective part of the Auslander-Reiten quiver. Whatever the type of $\beta$, is it in a slice of the preinjective part. Then it suffices to show that there is a morphism from $X_\alpha$ to the head of this slice, it means in our orientation, to $A_1$ in the following quiver.

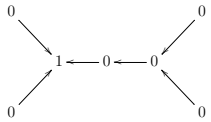



We note that $\beta[k-1]$ is exactly the arc corresponding to $\tau^{-1}P_l$ for an $l \in \{1, \cdots, n+1\}$. We can prove the existence of a morphism in the module category from the simple regular $X_\alpha$ to $\tau^{-1}P_l$ for any $l$.

To draw an example, in case $n = 7$, let us give the dimension vectors of $\tau^{-1}P_l$, for each $l$: They are



In any case there is a morphism from the simple



to any of the $\tau^{-1}P_l$. We can have a deeper analyse in the article of Dlab and Ringel in [DR74]. □

We now generalize this result to all arcs excepted the one in the tubes of size 2.

**Lemma 2.2.16.** *Let $P$ be a polygon with $(n-2)m$ sides associated with a quiver $Q$ of type $\tilde{D}_n$ and let $\alpha$ be an arc which corresponds to a regular module in a tube of size $n-2$. Then $\alpha$ cuts the figure into a polygon $T$ on the one hand and another figure of type $\tilde{D'_n}$ for some $n' < n$ on the other hand. Let $\alpha_1, \cdots, \alpha_k$ be arcs lying in $T$, such that, if we cut along $\alpha_i$, then $\alpha_{i+1}$ becomes an $m$-ear. Then the successive Iyama-Yoshino reduction of $\mathcal{C}^m_{\tilde{D}_n}$ applied on the $X_{\alpha_i}$ corresponds to cutting successively along the $\alpha_i$. More precisely, let $\mathcal{C}$ be the $m$-cluster category associated with a quiver of type $\tilde{D}_n$, and let $\mathcal{C}' = \mathcal{U}/\bigoplus X_{\alpha_i}$, where $X_{\alpha_i}$ is the $m$-rigid object associated with $\alpha_i$, and $\mathcal{U} = \{Y, \mathrm{Ext}^l_{\mathcal{C}}(\bigoplus X_{\alpha_i}, Y) = 0 \; \forall l \in \{1, \cdots, m\}\}$.*

*Let $Q/\alpha_1, \cdots, \alpha_n$ be the quiver $Q$ where the vertex corresponding to $\alpha_1, \cdots, \alpha_n$ and all the incident arrows have been removed. Then we have the following result:*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha_1,\cdots,\alpha_n}.$$

*Proof.* We have that $\alpha$ is an arc linking two different vertices $i$ and $j$ and $\alpha$ is homotopic to the boundary path.

If $\alpha$ is an $m$-ear, this is exactly the previous lemma. Else, it means that $j > i+m+1$. Then there exists an $m$-ear from $i$ to $i+m+1$ which does not cut $\alpha$. We use Iyama-Yoshino reduction in order to cut along this $m$-ear. We do this operation again as many times as necessary, to reduce $n$ until $\alpha$ becomes an $m$-ear. We are ensured that the process stops since $\alpha$ cuts the polygon into a $km$-gon with both $m-1$ gons inside of it on the first side, and into a $km+2$-gon of type $A$ on the other side. This shows the result if $\alpha$ is in a tube of size $n-2$. □

**Lemma 2.2.17.** *Let $P$ be a polygon with $(n-2)m$ sides associated with a quiver $Q$ of type $\tilde{D}_n$ and let $\alpha$ be an arc which is associated to an $m$-rigid object lying in the transjective component of the Auslander-Reiten quiver of $\mathcal{C}^m_Q$. Then the Iyama-Yoshino reduction of $\mathcal{C}^m_{\tilde{D}_n}$ applied on $X_\alpha$ corresponds to cutting along $\alpha$. More precisely, let $\mathcal{C}$ be the $m$-cluster category associated with a quiver of type $\tilde{D}_n$, and let $\mathcal{C}' = \mathcal{U}/X_\alpha$, where $X_\alpha$ is the $m$-rigid object associated with $\alpha$, and $\mathcal{U} = \{Y, \mathrm{Ext}^l_{\mathcal{C}}(\bigoplus X_\alpha, Y) = 0 \ \forall l \in \{1, \cdots, m\}\}$. Let $Q/\alpha$ be the quiver $Q$ where the vertex corresponding to $\alpha$ and all the incident arrows have been removed. Then we have the following result:*

$$\mathcal{C}' \simeq \mathcal{C}^m_{Q/\alpha}.$$

*Moreover, the following diagram is commutative.*

$$\begin{array}{ccc} \{\beta \text{ does not cross } \alpha, \beta \neq \alpha\} & \longrightarrow & \{X \in \mathcal{U}; X \ncong X_\alpha\}/\simeq \\ \updownarrow & & \updownarrow \\ \{\beta \in S/\alpha\} & \longrightarrow & \{X \in \mathcal{C}'\}/\simeq \end{array}$$

*where the first vertical bijection is given in the way of Marsh and Palu in [MP14], and the second is given by the Iyama-Yoshino reduction. The horizontal arrows are maps sending $\beta$ to $X_\beta$.*

*Proof.* We have that $\alpha$ is an arc situated in the preprojective (or preinjective) part of the Auslander-Reiten quiver of $Q$.

Then, we complete $\alpha$ into an $(m+2)$-angulation $\oplus\alpha_i$ composed by the slice containing $\alpha$ in the Auslander-Reiten quiver of $Q$. Then all the arcs are preprojective (or preinjective), and we can use the theorem of Keller and Reiten in [KR08, Theorem 4.2]. Indeed, let $T = \oplus X_{\alpha_i}$, where $X_{\alpha_i}$ corresponds to the arcs $\alpha_i$. We

know that $T$ is an $m$-cluster-tilting object (because, by applying $\tau$ several times if necessary, we go back to the first slice, which contains the projective modules, and is actually an $m$-cluster-tilting object). Moreover, $\mathrm{Ext}^{-i}_{\mathcal{C}'}(T', T') = 0$. Moreover, we can check that $\mathrm{End}_{\mathcal{C}'}(T') = KQ_{T'}$ as in the previous lemma. Then we can apply the theorem of Keller and Reiten, and this shows the result.

Moreover, the diagram is commutative since the application $\beta \to X_\beta$ is the same, no matter if we consider the surface $S$ or the surface cut $S/\alpha$ (this applications does not depends on the type of the surface). Then the upper arrow is the same as the lower arrow. □

We now state a technical lemma which helps us to find morphisms between two $m$-rigid objects.

**Lemma 2.2.18.** *Let $\alpha$ and $\beta$ be two $m$-diagonals. Suppose that there exists an $(m+2)$-angulation $\Delta$ which contains $\alpha$ and not $\beta$ such that $\mu_\alpha(\Delta)$ contains $\beta$ (it means that there exists $\Delta$ such that $\beta = \kappa^i_\Delta(\alpha)$, for an $i \in \{1, \cdots, m-1\}$). Then $\mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$ where $X_\alpha$ (respectively $X_\beta$) is the $m$-rigid object associated with $\alpha$ (respectively $\beta$) thanks to the bijection between $m$-rigid objects and $m$-diagonals.*

*Proof.* We number the arcs in $\Delta$ and consider that $\alpha$ corresponds to $k$. We use Calabi-Yau reduction in order to prove the statement. Let us introduce $\mathcal{C}' = \mathcal{U}/\bigoplus_{j\neq k} X_j$, where

$$\mathcal{U} = \{Y, \mathrm{Ext}^l_{\mathcal{C}}(\oplus X_j, Y) = 0 \ \forall l \in \{1, \cdots, m\}\}.$$

By Iyama and Yoshino in [IY08], we know that $\mathcal{C}'$ is triangulated and $(m+1)$-Calabi-Yau. If $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{C}$, where $X \to Y$ is a $\oplus X_j$-left approximation, then $Z$ is isomorphic to the shift of $X$ in $\mathcal{C}'$.

From the previous lemma, we know that the Iyama-Yoshino reduction corresponds to cutting along an arc. We can suppose that $\Delta$ does not contain any arc lying in a tube of size 2. In this way we have cut along all the arcs of the $(m+2)$-angulation excepted $\alpha$. Then, as $\beta$ is the $i$-th twist of $\alpha$, it becomes the $i$-th shift in the reduced category. From Iyama and Yoshino in [IY08], $X_\beta = \Sigma^i_{\mathcal{C}'} X_\alpha$. Then $\mathrm{Ext}^i_{\mathcal{C}'}(X_\alpha, X_\beta) \simeq \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$ (the first equivalence is due to the fact that $1 \leq i \leq m$). □

Before showing the main lemma of this section, we show that we can reduce to the case $\tilde{D}_4$. The following lemma show that we can reduce to cases where $n \leq 6$, and the next remark treats cases $n = 5$ and $n = 6$.

**Lemma 2.2.19.** *Suppose that $n > 6$. Let $\alpha$ and $\beta$ be two crossing arcs in the $(n-2)$m-gon realizing $\tilde{D}_n$. Then there exists at least $(n-4)$ $m$-ears which do not cut $\alpha$ neither $\beta$.*

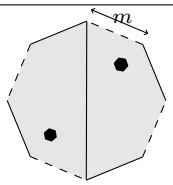
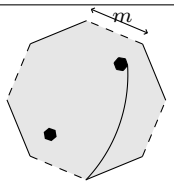
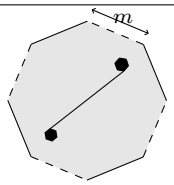
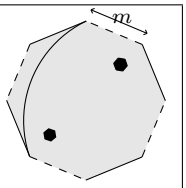
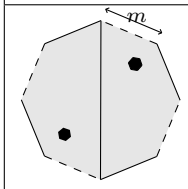
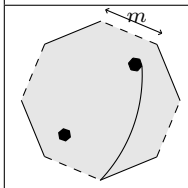
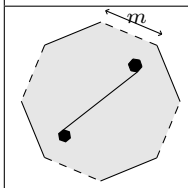
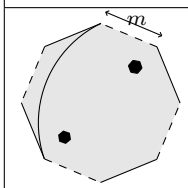
*Proof.* The case where $\alpha$ and $\beta$ are of type one is the most difficult. The arcs cut the polygon $P$ into 4 parts. If we cannot draw an $m$-ear between one of the parts, it means that the number of vertices strictly contained in a part is at most $m-1$ in each part. Then the total number of vertices is at most $4(m-1)+4$. Then $(n-2)m \leq 4m$ this means $n \leq 6$. □

**Remark 2.2.20.** *If $n=5$ or $n=6$, then the only case where we cannot reduce to $\tilde{D}_4$ is when $\alpha$ and $\beta$ are of type 1. But at this moment there exists $k<m$ such that $\alpha=\beta[k]$, then there exists a nonzero extension between $\alpha$ and $\beta$.*

**Lemma 2.2.21.** *Let $\alpha$ and $\beta$ be two arcs in an $(m+2)$-angulation $\Delta$. Let $X_\alpha$ and $X_\beta$ be their associated $m$-rigid object. If $\forall i \in \{1,\cdots,m\}, \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta)=0$, then $\alpha$ and $\beta$ do not cross.*

*Proof.* We recall that we are in the case where $n=4$, it means that we study a $2m$-gon We show that if $\alpha$ crosses $\beta$, then $\mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$, for some $i \in \{1,\cdots,n\}$.

Suppose that $\alpha$ and $\beta$ cross. Then both arcs can be of different type. Let us sum up all the cases to treat in the following tabular:

| $\alpha$, $\beta$ |  |  |  |  |
|---|---|---|---|---|
|  | Case 1 | Case 2 | Case 3 | Case 4 |
|  | Case 2 | Case 5 | Case 6 | Case 7 |
|  | Case 3 | Case 6 | Case 8 | Impossible |
|  | Case 4 | Case 7 | Impossible | Case 9 |

As we have seen previously, we only have to show the result for $\tilde{D}_4$. Then $P$ has $2m$ vertices.

**Remark 2.2.22.** *We need to note that the cases are symmetric. Indeed, to shift $\beta$ $k$ times in order to find a morphism from $\alpha$ to $\beta[k]$ is the same as to shift $\alpha$ $k$ times, which gives a morphism from $\alpha$ to $\beta[m+1-k]$ thanks to the $(m+1)$-Calabi-Yau property.*

First, we have to notice that cases 4,7,9 are already treated from lemmas 2.2.15 and 2.2.16.

Case 1: $\alpha$ and $\beta$ are of type 1 (cf figure 2.15).

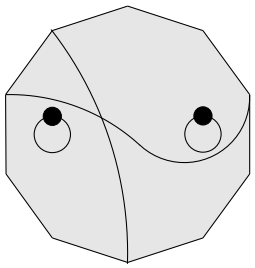


Figure 2.15 – First case, $\alpha$ and $\beta$ are of type 1

Let $i$ and $j$ be the closest vertices of the $2m$-gon, where $i$ is an end of $\alpha$ and $j$ is an end of $\beta$.

We have either $j \leq i+m+1$ or $i \leq j+m+1$, then $\beta$ can be shifted $k < m$ times in order to hang to one end point of $\alpha$. Then $\alpha$ and $\beta[k]$ share an oriented angle. As they are of type 1, and share an oriented angle, they are on the same slice of the Auslander-Reiten quiver. Then, this is a nonzero composition of arrows. In this way, we have found a nonzero morphism from $\Sigma^k\alpha$ to $\beta$. Then $\mathrm{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$.

Case 2: $\alpha$ is of type 1 and $\beta$ is of other type (cf figure 2.16).

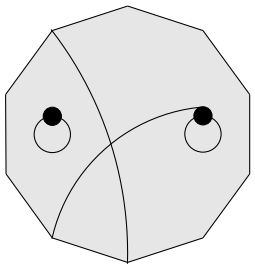


Figure 2.16 – Case where $\alpha$ and $\beta$ are not of the same type

This case is similar to that of the first one. It suffices to shift $\alpha$ $k < m$ times in order to hang both arcs to the same vertex. Consequently, they do not cross a

mesh in the Auslander-Reiten quiver. Then, there is a Hom-hammock from one to another. Then there is a nonzero extension from $\alpha$ to $\beta$.

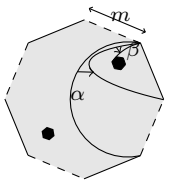



Figure 2.17 – The arrows are elementary move. The nonzero extension corresponds to the composition of the arrows

Case 3: We are in the situation of figure 2.18

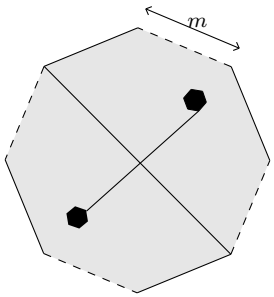



Figure 2.18 – Case where $\alpha$ is of type 1 and $\beta$ is in the tube

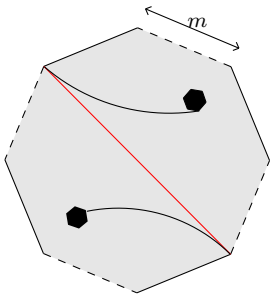



Figure 2.19 –

In this case, it is more difficult to see morphisms in the Auslander-Reiten quiver of $Q$ because one arc is in the transjective component and the other is in a tube. Nonetheless, if we can find an $(m+2)$-angulation where $\beta$ is the $i$-th twist of $\alpha$, then from lemma 2.2.18, there is an extension which is nonzero. We have to complete $\alpha$ to an $(m+2)$-angulation containing this arc (see figure 2.19):

As $\beta$ is the $i$-twist of $\alpha$, then there exists $i \in \{1, \cdots, m\}$ such that

$$\mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0.$$

The case where $\alpha$ is in the tube and $\beta$ is of type 1 is similar.

Case 5: If both $\alpha$ and $\beta$ are of other type (cf figure 2.20).

We can move $\beta$ $k < m$ times in order to hang its end to $\alpha$. Then the composition of elementary moves in figure 2.21 is not zero since it follows a slice of the Auslander-Reiten quiver (so do not cross a mesh).

Then there is a nonzero extension between $\alpha$ and $\beta$.

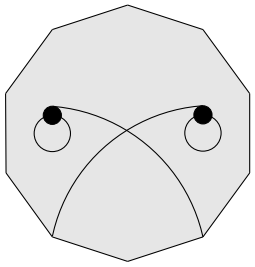


Figure 2.20 – Case where both $\alpha$ and $\beta$ are of other type

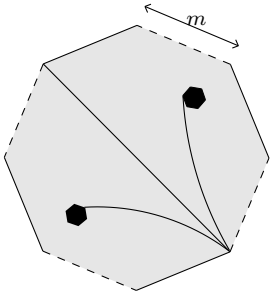



Figure 2.21 –

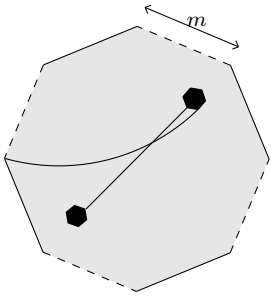



Figure 2.22 – Case where $\alpha$ is of other type and $\beta$ is in a tube of size 2

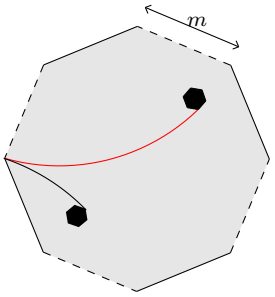



Figure 2.23 –

Case 6: If we are in the situation of figure 2.22:

The same arguments as in case 3 lead to find an $(m+2)$-angulation containing these arcs in figure 2.23.

Here again, there exists a nonzero extension between $X_\alpha$ and $X_\beta$. The inverse case is similar.

Case 8: If we are in a tube of size 2 (cf figure 2.24)

If they cross each other, it means that they are in the same tube. Then one is situated higher than the other and there exists a Hom-hammock between them.

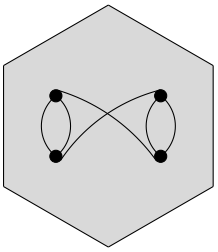


Figure 2.24 – Case of a tube of size 2, for example for $m = 3$

In any case, we have shown that if $\alpha$ crosses $\beta$, then there exists $k$ such that $\mathrm{Ext}^k_{\mathcal{C}}(X_\alpha, X_\beta) \neq 0$. □

## 2.3 Compatibility flip/bijection between $m$-cluster-tilting objects and $(m+2)$-angulations

With theorem 2.2.1, we are able to define an $(m+2)$-angulation from an $m$-cluster-tilting object.

**Definition 2.3.1.** *Let $T = \oplus T_i$ be an $m$-cluster-tilting object, and $T_i$ its $m$-rigid components. With each $T_i$ we associate $\alpha_i$ the corresponding $m$-diagonal. We know that the $\alpha_i$ do not cross each other from the previous section. Then the set $\{\alpha_i, i \in \{1, \cdots, m\}\}$ form a maximal set of noncrossing $m$-diagonals, then an $(m+2)$-angulation, called $\Delta_T$, the $(m+2)$-angulation associated with $T$.*

We first show the theorem of compatibility between the flip of an $(m+2)$-angulation, and the mutation of an $m$-cluster-tilting object.

**Theorem 2.3.2.** *Let $\Delta$ be an $(m+2)$-angulation. Let $X$ be its associated object. Let $\mu_i$ be the flip at the arc $\alpha_i$ in $\Delta$ as well as the mutation of the $m$-cluster-tilting object $X$ at summand $i$. Then we have:*

$$\mu_i(\Delta) = \Delta_{\mu_i(X)}$$

*Proof.* The flip at vertex $\alpha_i$ affects only $\alpha_i$ itself. In the same way, the mutation of $X$ affects only the $i$-th component of the $m$-cluster-tilting object.

By Buan and Thomas in [BT09], we know that there is a triangle $X_i \to B_i^{(0)} \to X_i^{(1)} \to$, where $X_i$ is the $m$-rigid object corresponding to $\alpha_i$ and $B_i^{(0)} \in \mathrm{add}T$. The aim is to show that $X_i^{(1)} \simeq X_{\tilde{i}}$, where $X_{\tilde{i}}$ is the $m$-rigid corresponding to the arc $\tilde{\alpha_i}$ which is the twist of $\alpha_i$.

Let $\overline{X} = X/X_i$ be the almost $m$-cluster-tilting object. Then from Wraalsen ([Wra09]) and Zhou, Zhu ([ZZ09]), $\overline{X}$ has $m+1$ complements. Let $\overline{\Delta}$ be the

"almost" $(m+2)$-angulation, containing all arcs of $\Delta$ except $\alpha_i$. Then by 1.4.1, $\overline{\Delta}$ corresponds to $\overline{X}$.

Let

$$\mathcal{U} = \{Y \in \mathcal{C}_Q^m, \forall k \in \{1, \cdots, m\}, \mathrm{Ext}_{\mathcal{C}}^k(\overline{X}, Y) = 0\}.$$

Then by lemma 2.2.21, an object in $\mathcal{U}$ corresponds to an arc which does not cross $\overline{\Delta}$, it means that there are $m+1$ possibilities of remaining arcs in order to have an $(m+2)$-angulation. In a way similar to that of Marsh and Palu in [MP14], studying $\mathcal{C}' = \mathcal{U}/\overline{X}$ corresponds to cut along the arcs of $\overline{\Delta}$

Then by Keller, and Iyama and Yoshino in [Kel05] and [IY08], $\mathcal{C}'$ is a triangulated, hom-finite, algebraic and $(m+1)$-Calabi-Yau category. Moreover, each arc which does not cross $\overline{\Delta}$ is an $m$-cluster-tilting object in $\mathcal{C}'$. In addition, $\mathrm{Ext}_{\mathcal{C}}^k(\alpha_i, \alpha_i) = \mathrm{Ext}_{\mathcal{C}}^{-k}(\alpha_i, \alpha_i) = 0$ for all $k \in \{1, \cdots, m\}$ since $\alpha_i$ does not cross itself. The algebra $\mathrm{End}(\alpha_i) = K$ is hereditary since it is of global dimension 0. Then, by [KR08], theorem 4.2, we have an equivalence

$$\mathcal{C}' \simeq \mathcal{C}_{A_1}^m.$$

Therefore we have a distinguished triangle

$$\alpha_i \to E_i \to \Sigma_{\mathcal{C}'}\alpha_i \to \Sigma\alpha_i,$$

where $E_i$ is the set of arcs which follow $\alpha_i$ in the sense of its quiver and where $\Sigma_{\mathcal{C}'}\alpha_i$ is the shift in the category $\mathcal{C}_{A_1}^m$, which means the shift in the remaining $(2m+2)$-gon. Then it follows that $\Sigma_{\mathcal{C}'}\alpha_i = \tilde{\alpha}_i$. Then we have two distinguished triangles:

$$\begin{array}{ccccccc} X_i & \longrightarrow & B_i^{(0)} & \longrightarrow & X_i^{(1)} & \longrightarrow & \Sigma X_i \\ \wr\downarrow & & \wr\downarrow & & & & \wr\downarrow \\ \alpha_i & \longrightarrow & E_i & \longrightarrow & \tilde{\alpha}_i & \longrightarrow & \Sigma\alpha_i \end{array}$$

By TR3, the third axiom of triangulated categories, we have a morphism $X_i^{(1)} \to \tilde{\alpha}_i$

$$\begin{array}{ccccccc} X_i & \longrightarrow & B_i^{(0)} & \longrightarrow & X_i^{(1)} & \longrightarrow & \Sigma X_i \\ \wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow \\ \alpha_i & \longrightarrow & E_i & \longrightarrow & \tilde{\alpha}_i & \longrightarrow & \Sigma\alpha_i \end{array}$$

By the five lemma applied to triangulated categories, we have an isomorphism

$$\begin{array}{ccccccc} X_i & \longrightarrow & B_i^{(0)} & \longrightarrow & X_i^{(1)} & \longrightarrow & \Sigma X_i \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \alpha_i & \longrightarrow & E_i & \longrightarrow & \tilde{\alpha}_i & \longrightarrow & \Sigma\alpha_i \end{array}$$

Then we have shown that

$$X_i^{(1)} \simeq X_{\tilde{i}}.$$

□

**Lemma 2.3.3.** *Let $\alpha$ and $\beta$ be any $m$-diagonals in the the category $\mathcal{C}$. Then*

$$\forall i \in \{1, \cdots, m\}, \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) = 0 \Leftrightarrow \alpha \text{ and } \beta \text{ do not cross.}$$

*Proof.* Associating an $(m+2)$-angulation with an $m$-cluster-tilting object in the natural way above is the same thing as associating an $(m+2)$-angulation with an $m$-cluster-tilting object in the following way:

Take $X$ an $m$-cluster-tilting object. From Buan and Thomas in [BT09, Proposition 7.1], we can introduce $f$ as a sequence of flip necessary to bring $X$ to the sum of the projective modules $\bigoplus P_i$. We associate with $\bigoplus P_i$ the initial $(m+2)$-angulation, and flip back to $\Delta_X$ via the inverse sequence of flips.

From theorem 2.3.2, these two ways of defining an $(m+2)$-angulation from an $m$-cluster-tilting object are the same. Then we can show the result:

If $\alpha$ and $\beta$ do not cross each other, we can complete them into an $(m+2)$-angulation $\Delta$. We associate with $\Delta$ an $m$-cluster-tilting object by applying flips on $\Delta$ as told at the beginning of this proof, and this shows that the associated object is $m$-cluster-tilting. Then $\forall i \in \{1, \cdots, m\}, \mathrm{Ext}^i_{\mathcal{C}}(X_\alpha, X_\beta) = 0$. □

Finally, we show the bijection between $(m+2)$-angulations and $m$-cluster-tilting objects.

**Theorem 2.3.4.** *The natural application from $(m+2)$-angulations to $m$-cluster-tilting objects induces a bijection between these two notions.*

*Proof.* By lemma 2.3.3, with an $(m+2)$-angulation, we associate a unique $m$-cluster-tilting object. Therefore the application is well-defined. Let us call by $\Phi$ this function.

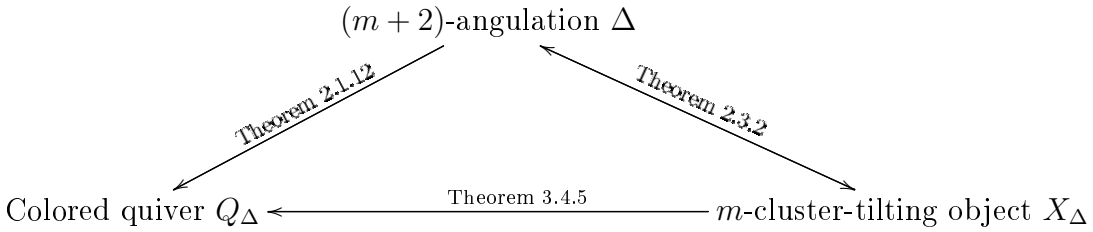
First, $\Phi$ is injective since, if we take $X$ an $m$-cluster-tilting object, we can associate a unique $(m+2)$-angulation. Indeed, $X = \bigoplus X_i$, where the $X_i$ are $m$-rigid. With each summand $X_i$, we associate the corresponding arc $\alpha_i$. By theorem 2.2.1, as the $X_i$ are $m$-rigid, we know that the $\alpha_i$ do not cross, and they are $n+1$, so they form a maximal set of noncrossing arcs, thus an $(m+2)$-angulation. It is uniquely defined. So $\Phi$ is injective.

Finally, we show that $\Phi$ is surjective. If $X$ is an $m$-cluster-tilting object, then it exists $f$, a sequence of flips, such that $f(X) = \bigoplus P_i$. Moreover, $\bigoplus P_i = \Phi(\Delta_{\text{init}})$, then $f(X) = \Phi(\Delta_{\text{init}})$. Let $g$ be the inverse sequence of flips of $f$. Then, $X = g(\Phi(\Delta_{\text{init}}))$ and by theorem 2.3.2,

$$X = \Phi(g(\Delta_{\text{init}})).$$

This finishes to show the bijection between $m$-cluster-tilting objects and $(m+2)$-angulations. □

We can summarize all the important properties between $m$-cluster-tilting objects, colored quivers, and $(m+2)$-angulations in the following diagram:




We now finish this section with a direct consequence of this diagram.

**Theorem 2.3.5.** *Let $\Delta$ be an $(m+2)$-angulation. Let $Q_\Delta$ be the associated colored quiver. Let $X_\Delta$ be the $m$-cluster-tilting object associated with $\Delta$, and let $Q_{X_\Delta}$ be the quiver associated with $X_\Delta$ in the sense of Buan and Thomas in [BT09]. Then*

$$Q_\Delta = Q_{X_\Delta}$$

Note here that theorem 2.1.12 is a direct consequence of theorems 2.3.2 and 3.4.5.

# Part II

# Model categories structures from rigid objects in exact categories

Chapter 1

# The case of exact categories

## 1.1 Study of the properties of $\mathrm{pr}\mathcal{M}$

We show some preliminary lemmas which will be used in section 1.5 in order to associate a prefibration structure with a given rigid subcategory.

We note that Lemma 1.1, which is an analogue of lemma 3.3 shown by Buan and Marsh in [BM13] for exact categories, might be of independent interest.

Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective and projective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands. Let

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

and

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

Let

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-,X)/\overline{\mathcal{M}} \end{array}$$

which induces the following equivalence of categories

$$\mathrm{pr}\mathcal{M}/\mho\mathcal{M} \simeq \mathrm{mod}\ \overline{\mathcal{M}}.$$

For more details, see the article of Demonet and Liu, [DL13].

In the following lemma, we prove that if M is contravariantly finite, then so is $\mathrm{pr}\mathcal{M}$, provided that $\mathcal{M}$ also contains all projective objects.

This lemma tells us that we can replace each object by a cofibrant replacement.

**Lemma 1.1.1.** *Assume that the rigid subcategory $\mathcal{M}$, contains all injectives and all projectives. Then, for any $X$ in $E$, there exist $A$ in* $\mathrm{pr}\mathcal{M}$ *and a right* $\mathrm{pr}\mathcal{M}$ *approximation $A \to X$.*

*Proof.* Let $X \in \mathcal{E}$. Let $M_0 \to X$ be an $\mathcal{M}$-approximation of $X$. Since $\mathcal{E}$ is weakly idempotent complete, with enough projectives and $\mathcal{M}$ contains the projective objects, the morphism $M_0 \to X$ is a deflation. Let

$$0 \to K_0 \to M_0 \to X \to 0$$

be the associated short exact sequence. Similarly, let

$$0 \to K_1 \to M_1 \to K_0 \to 0$$

be a short exact sequence coming from an $\mathcal{M}$-approximation of $K_0$. Then we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & K_1 & & & & & & \\
 & & \downarrow{\scriptstyle \beta} & & & & & & \\
 & & M_1 & = & M_1 & & & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & & & \\
0 & \longrightarrow & K_0 & \xrightarrow{b} & M_0 & \xrightarrow{a} & X & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

We have $a \circ b \circ \alpha = 0$. Let $A$ be the push-out of the square:

$$\begin{array}{ccc}
M_1 & \xrightarrow{b\alpha} & M_0 \\
\downarrow & & \downarrow{\scriptstyle r} \\
I_{M_1} & \longrightarrow & A \\
 & \searrow^{0} & \downarrow{\scriptstyle \exists\varphi} \\
 & & X
\end{array}
\qquad
\begin{array}{c}
\\
\\
\\
\\
\nearrow_{a}
\end{array}$$

Then there exists a morphism $\varphi : A \to X$ such that $\varphi \circ r = a$ and the other triangle commutes.

We have $A \in \mathrm{pr}\mathcal{M}$. Indeed, we have the following short exact sequence:

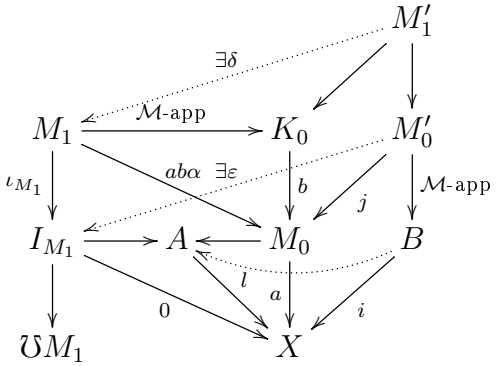
$$0 \to M_1 \to I_{M_1} \oplus M_0 \to A \to 0.$$

Moreover, $A \to X$ is an approximation. Indeed, let $i : B \to X$ be a morphism, with $B \in \mathrm{pr}\mathcal{M}$. Let us show that there exists $B \to A$ which makes the triangle commute.

Let

$$0 \to M_1' \to M_0' \to B \to 0$$

be a short exact sequence with $M_0, M_1 \in \mathcal{M}$. We have the following diagram:




As $a$ is an $\mathcal{M}$-approximation, then there exists a morphism $j : M_0' \to M_0$ which makes the lower-right square commute. Then, there exists a morphism $M_1' \to K_0$ which makes it a morphism of short exact sequences.

As $M_1 \to K_0$ is an $\mathcal{M}$-approximation, then there exists $\delta : M_1' \to M_1$ which makes the upper triangle commute. Since $I_{M_1}$ is injective and $M_1' \to M_0'$ is an inflation, there exists a morphism $\varepsilon : M_0' \to I_{M_1}$ which leads to a morphism of short exact sequences.

All the conditions are required to build a morphism $k : B \to A$ such that $l \circ k = i + 0$ since $I_{M_1} \to A \to X = 0$. Then we have shown the result. □

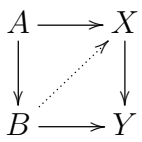
## 1.2 Weak equivalences and fibrations

**Definition 1.2.1.** *We recall that $G$ is the functor*

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-, X)|\overline{\mathcal{M}} \end{array}$$

*We call by $\mathcal{W}$, the weak equivalences, the class of morphisms $f$ for which $Gf$ is an isomorphism.*

**Definition 1.2.2.** *Let $f : X \to Y$ and $g : A \to B$ be two morphisms. We say that $f \square g$ when, for any commutative square*



*there exists a morphism $B \to X$ such that both triangles commute. For a class of morphisms $\mathcal{A}$, we call by*

$$A^{\square} = \{g,\ \forall f \in \mathcal{A}, f \square g\} \text{ and } {}^{\square}A = \{f,\ \forall g \in \mathcal{A}, f \square g\}.$$

Let

$$J = \{f : 0 \to \mho M, M \in \mathcal{M}\}.$$

The morphisms of $J^{\square}$ are called fibrations and compose the class $\mathcal{F}ib$.

The next lemma shows that the $\mathrm{pr}\mathcal{M}$-approximation constructed in lemma 1.1.1 is actually a weak equivalence. This permits to take cofibrant replacements as we will see later.

**Lemma 1.2.3.** *Let $h : A \to X$ be the $\mathrm{pr}\mathcal{M}$-approximation constructed in lemma 1.1.1. Then $h \in J^{\square} \cap \mathcal{W}$.*

*Proof.* We first show that $h \in J^{\square}$. Let $\mho M \to X$ be a morphism. Since $\mho\mathcal{M} \in \mathrm{pr}\mathcal{M}$, and $A \to X$ is a $\mathrm{pr}\mathcal{M}$-approximation, there automatically exists a lift as wanted.

Next, we have to show that the morphism $\overline{\mathcal{E}}(-, A)|\overline{\mathcal{M}} \to \overline{\mathcal{E}}(-, X)|\overline{\mathcal{M}}$ is an isomorphism. It is surjective, since if we take $M \to X$ a morphism, since $M \in \mathrm{pr}\mathcal{M}$, there exists a lift as we have seen in order to show that $h$ is a fibration.

Then, if $a : M \to A$ is a morphism such that $h \circ a = 0$ (we will see later the case where this morphism factorizes through an injective module). Using the same notations as in lemma 1.1.1, as $M$ is rigid, there exists $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : M \to I_1 \oplus M_0$ such that

$$\pi \circ b_1 + r \circ b_2 = a$$

where $\pi : I_1 \to A$. As $\alpha : M_1 \to K_0$ is an $\mathcal{M}$-approximation, there exists $c : M \to M_1$ such that $b \circ \alpha \circ c = b_2$. Then we have

$$a = \pi \circ b_1 + r \circ b_2$$

so

$$a = \pi \circ b_1 + r \circ b \circ \alpha \circ c.$$

By the pushout of lemma 1.1.1, $r \circ b \circ \alpha = \pi \circ \iota_1$ where $\iota_1 : M_1 \to I_1$. Then

$$a = \pi \circ b_1 + \pi \circ \iota_1 \circ c.$$

This shows that $a$ factorizes through an injective module.

Finally, if we suppose that $h \circ a$ factorizes through an injective $J$, for example $h \circ a = \mu \circ \nu$, as $J \in \mathrm{pr}\mathcal{M}$, there exists $\tilde{a}$ such that $h \circ \tilde{a} = \mu$. We then proceed as above with the morphism $a - \tilde{a} \circ \nu$. This finishes to show the result. □

We recall that trivial fibrations are those morphisms that are both fibrations and weak equivalences.

**Lemma 1.2.4.** *Let $f : X \to Y$ be a trivial fibration. If $\alpha$ is a morphism from an element $M$ of $\mathcal{M}$ to $Y$, then there exists $\beta : M \to X$ such that $f \circ \beta = \alpha$.*

*Proof.* As $f$ is a weak equivalence, there exists $\tilde{\beta} : M \to X$, $\iota_M : M \to I_M$ and $\gamma : I_M \to Y$ such that $\alpha + \gamma \circ \iota_M = f \circ \tilde{\beta}$. As $f \in J^{\square}$ and $I_M \in \mho\mathcal{M}$, there exists $\delta : I_M \to X$ such that $f \circ \delta = \gamma$. Then we have

$$\alpha = f \circ (\tilde{\beta} - \delta \circ \iota_M).$$

□

**Lemma 1.2.5.** *Let $f : X \to Y$ be a trivial fibration. Then it is automatically a deflation.*

*Proof.* Let $P_Y$ be a projective cover of $Y$. As $P_Y \in \mathcal{M}$, and $f$ is a trivial fibration, from the previous lemma, there exists a lift from $P_Y \to X$. From Bühler in [Büh10, Proposition 7.6, (ii)], as $P_Y \to Y$ is a deflation, then $f$ is a deflation. □

## 1.3 Factorization

Let us now show a characterization of the morphisms of ${}^{\square}(J^{\square})$.

**Lemma 1.3.1.** *Suppose that $\mho M \to Y$ is a right $\mho\mathcal{M}$-approximation. A morphism $f : X \to Y$ is in ${}^{\square}(J^{\square})$ if and only if it is a retract of the canonical injection $X \to X \oplus \mho M$.*

*Proof.* Let $f : X \to Y \in^{\square} (J^{\square})$. Let $\alpha : \mho M \to Y$ be a $\mho\mathcal{M}$-approximation of $Y$. Then, we have the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\binom{1}{0}} & X \oplus \mho M \\ {\scriptstyle f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle (f\ \alpha)} \\ Y & = & Y \end{array}$$

The morphism $(f\ \alpha) \in J^{\square}$. Indeed, if $\mho M' \to Y$ is a morphism, as $\alpha$ is a $\mho\mathcal{M}$-approximation, there exist a lift as wanted (which is zero on $X$). As $f \in^{\square} (J^{\square})$, there exist $s : Y \to X \oplus \mho M$ which makes both triangles commute. Then, $f$ is a retract of the canonical injection $X \to X \oplus \mho M$.

Conversely, it is well-known that the lifting property $\square$ is stable under retract. □

**Lemma 1.3.2.** *Under the assumption that there exist some $\mho\mathcal{M}$-approximations, any morphism can be factorized through a morphism in ${}^{\square}(J^{\square})$ followed by a morphism in $J^{\square}$.*

*Proof.* Let $f : X \to Y$ be a morphism. It factorizes through $X \to X \oplus \mho M \to Y$ by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} f & \alpha \end{pmatrix}$, where $\alpha$ is a $\mho\mathcal{M}$-approximation (this morphism is a retract of itself). The first morphism is in ${}^{\square}(J^{\square})$. As $\begin{pmatrix} f & \alpha \end{pmatrix}$ is a $\mho\mathcal{M}$-approximation, it satisfies the lifting property of $J^{\square}$. Then $\begin{pmatrix} f & \alpha \end{pmatrix} \in J^{\square}$. □

# 1.4 Cofibrant objects and homotopy

## 1.4.1 Cofibrant objects

Cofibrant objects are objects which have a lift along trivial fibrations.
In this subsection, we characterize fibrant and cofibrant objects.

**Lemma 1.4.1.** *Any object is fibrant.*

*Proof.* For any $X \in \mathcal{E}$, the map $X \to 0$ is a fibration. □

**Lemma 1.4.2.** *Suppose that the subcategory $\mathcal{M}$ contains the projective objects. Let $C \in \mathcal{E}$. Then $C$ is cofibrant if and only if $C \in \mathrm{pr}\mathcal{M}$.*

*Proof.* Let $C \in \mathrm{pr}\mathcal{M}$. We introduce $0 \longrightarrow M_1 \xrightarrow{h'} M_0 \xrightarrow{h} C \longrightarrow 0$. Let $f : X \to Y$ be a trivial fibration and $b : C \to Y$. As $f \in \mathcal{W}$, $Gf$ is an isomorphism, and there exists from lemma 1.2.4 a morphism $a : M_0 \to X$ such that

$$f \circ a = b \circ h.$$

Since $\mathcal{M}$ contains all the projective objects, the lemma 1.2.5 shows that $f$ is a deflation. Let $k : K \to X$ be the kernel of $f$. We then have a morphism of short exact sequences:

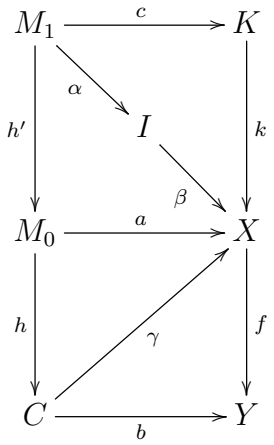
$$\begin{array}{ccc} M_1 & \xrightarrow{c} & K \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle k} \\ M_0 & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[b]{} & Y \end{array}$$

As $k \in \overline{\mathcal{M}}^{\perp}$, there exists $I$ an injective object, $\alpha : M_1 \to I$ and $\beta : I \to X$ such that

$$k \circ c = \beta \circ \alpha.$$

As $h'$ is an inflation, there exists $\beta' : M_0 \to I$ such that

$$\beta' \circ h' = \alpha.$$




So,

$$h' \circ a = \beta \circ \beta' \circ h'$$

and there exists $\gamma : C \to X$ such that

$$\gamma \circ h = \beta \circ \beta' + a.$$

Then,

$$f \circ \gamma \circ h = f \circ a + f \circ \beta \circ \beta'$$

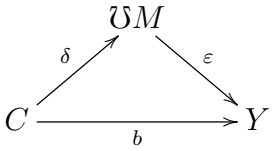
and

$$(f \circ \gamma - b) \circ h = f \circ \beta \circ \beta'.$$

Then, in $\mathrm{mod}\overline{\mathcal{M}}$, we have the good lifting. By Demonet and Liu in [DL13], we have that

$$\mathrm{mod}\overline{\mathcal{M}} \simeq \mathrm{pr}\mathcal{M}/\mho\mathcal{M}.$$

As $C \in \mathrm{pr}\mathcal{M}$, there exists $M \in \mathcal{M}$ such that the morphism $b$ factorizes through $\mho M$, let us say




such that $\varepsilon \circ \delta = b$. As $f \in J^{\square}$, there exists $\iota : \mho M \to X$ such that $f \circ \iota = \varepsilon$. Then $\iota \circ \delta : C \to X$ is the good candidate to lift $b$, it means that

$$f \circ \iota \circ \delta = \varepsilon \circ \delta = b.$$

On the other hand, let $X$ be a cofibrant object. We can say that the pr$\mathcal{M}$-approximation is a trivial fibration from lemma 1.2.3, then a retraction, and $X$ is a direct summand of $A$.

Let

$$0 \to Y \to M_0 \to X \to 0$$

be an $\mathcal{M}$-approximation of $X$. We have the following diagram:

□

## 1.4.2 Homotopy

We define here cylinder objects and left homotopies.

**Definition 1.4.3.** *Let $X \in \mathcal{E}$. A cylinder object for $X$ is a factorization of the morphism $\nabla : X \oplus X \to X$ (which is the identity on each copy of $X$) through $X'$, where $X' \to X$ is a weak equivalence.*

*Let $f, g : X \to Y$ be two morphisms. A left homotopy from $f$ to $g$ is a morphism $h : X' \to Y$, where $X'$ is a cylinder object for $X$, such that $h \circ (\nabla_1 \ \nabla_2) = (f \ g)$ where $(\nabla_1 \ \nabla_2)$ is the morphism $X \oplus X \to X'$ in the factorization of $\nabla$.*

Dually, we can define path objects and right homotopies.

**Definition 1.4.4.** *Let $Y \in \mathcal{E}$. A path object for $Y$ is a factorization of the morphism $\Delta : Y \to Y \oplus Y$ (which is the identity on each copy of $Y$) through $Y'$, where $Y \to Y'$ is a weak equivalence.*

*Let $f, g : X \to Y$ be two morphisms. A right homotopy from $f$ to $g$ is a morphism $k : X \to Y'$, where $Y'$ is a path object for $Y$, such that $\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \circ k = \begin{pmatrix} f \\ g \end{pmatrix}$ where $\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}$ is the morphism $Y' \to Y$ in the factorization of $\Delta$.*

**Lemma 1.4.5.** *For two morphisms $f$ and $g$ from an object $X$ to $Y$, $f$ and $g$ are homotopic if and only if $f - g$ factorizes through $\overline{\mathcal{M}}^{\perp}$.*

*Proof.* We begin by noting this fact: in the next diagram, a factorization of $\Delta$ is a path object if and only if it is isomorphic to $Y \xrightarrow{\binom{1}{0}} Y \oplus V \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ c & d \end{smallmatrix}\right)} Y \oplus Y$ for a $V \in \overline{\mathcal{M}}^{\perp}$.

Indeed, let $Y \xrightarrow{r} Y' \longrightarrow Y \oplus Y$ be a factorization of $\Delta$. As $r$ is a section, it is isomorphic to some $Y \to Y \oplus V$. Then $r \in \mathcal{W}$ if and only if $V \in \overline{\mathcal{M}}^{\perp}$.

Now, let us suppose that $f-g$ factorizes in the following way:

$$\begin{array}{ccc} X & \xrightarrow{f-g} & Y \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & A \end{array}.$$

The object $Y \oplus A$ is a path object. Then we have

$$\begin{array}{ccccc} X & \xrightarrow{(g\ \alpha)} & Y \oplus A = Y' & \xleftarrow{(1\ 0)} & Y \\ & \searrow^{(f\ g)} & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 1 \\ \beta & 0 \end{smallmatrix}\right)} & \swarrow_{\Delta} & \\ & & Y \oplus Y & & \end{array}$$

and the morphism $\begin{pmatrix} 1 & 1 \\ \beta & 0 \end{pmatrix}$ gives a homotopy between $f$ and $g$. Moreover, $s \circ a = \mathrm{Id}$, thus $s \simeq Y \oplus A \to Y$ and $s \in \mathcal{W}$, so $A \in \overline{\mathcal{M}}^{\perp}$. Then $f$ is homotopic to $g$.

Conversely, if $f$ is homotopic to $g$, then there exists $k : X \to Y'$ such that $\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \circ k = \begin{pmatrix} \\ g \end{pmatrix}$. Then, $f - g = (\Delta_1 - \Delta_2) \circ k$. As $Y' = Y \oplus A$, and $A \in \overline{\mathcal{M}}^{\perp}$, this finishes to show the result. □

**Remark 1.4.6.** *The notions of left and right homotopy are the same for the cofibrant objects.*

The following lemma is a corollary of the theorem of Demonet and Liu in [DL13]. However, we give a direct proof here.

**Lemma 1.4.7.** *With the above notations, we have*

$$\mathrm{pr}\mathcal{M} \cap \overline{\mathcal{M}}^{\perp} \simeq \mho\mathcal{M}.$$

*Proof.* The indirect inclusion is obvious since $\mathcal{M}$ is rigid. Now, let $X \in \mathrm{pr}\mathcal{M} \cap \overline{\mathcal{M}}^{\perp}$. As $X \in \mathrm{pr}\mathcal{M}$, we have a short exact sequence

$$0 \to M_1 \to M_0 \to X \to 0,$$

with $M_1, M_0 \in \mathcal{M}$. We have the following diagram:

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_0 & \xrightarrow{c} & X & & \\ \| & & \downarrow{\scriptstyle \alpha} & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle k} & & \\ M_1 & \longrightarrow & I & \xrightarrow{h} & \mho M_1 & & \\ & & \downarrow{\scriptstyle \psi} & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle \xi} & & \\ & & \mho M_0 & = & \mho M_0 & & \end{array}$$

In order to show that $X \in \mho\mathcal{M}$, we will show that the short exact sequence

$$0 \to X \to \mho M_1 \to \mho M_0 \to 0$$

splits. As $X \in \overline{\mathcal{M}}^{\perp}$, the morphism $c : M_0 \to X$ factorizes through $I$. We call by $\alpha : M_0 \to I$ and $\beta : I \to X$ such that $\beta \circ \alpha = c$. Let $k : X \to \mho M_1$ and $h : I \to \mho M_1$. Then,

$$k \circ \beta \circ \alpha = k \circ c = h \circ \alpha.$$

Then, $h - k \circ \beta$ factorizes through $\mho M_0$. Let us call $\varphi : \mho M_0 \to \mho M_1$ and $\psi : I \to \mho M_0$. Then, we have

$$h = k \circ \beta + \varphi \circ \psi.$$

Let $\xi : \mho M_1 \to \mho M_0$ in the short exact sequence (see diagram for sake of clearance). Then we have

$$\xi \circ \varphi \circ \psi = \xi \circ (h - k \circ \beta) = \xi \circ h - \xi \circ k \circ \beta = \psi - 0.$$

As $\psi$ is a deflation, we can conclude that $\xi \circ \varphi = 1$. Then, there is a section to the short exact sequence and $X \in \mho\mathcal{M}$ (which is stable under direct summands since $\mathcal{M}$ is stable under direct summands. □

## 1.5 Prefibration structures from rigid subcategories

In this section, we show that an exact category $\mathcal{E}$ is nearly equipped with a structure of prefibration category in the sense of Anderson-Brown-Cisinski (for more details, see the book of Radulescu-Banu, [RB06]).

We recall that

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-, X)|\overline{\mathcal{M}} \end{array}$$

induces the following equivalence of categories from Demonet and Liu in [DL13]

$$\mathrm{pr}\mathcal{M}/\mho\mathcal{M} \simeq \mathrm{mod}\ \overline{\mathcal{M}}.$$

**Theorem 1.5.1.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands. Suppose moreover that $\mho\mathcal{M}$ is contravariantly finite. Let*

$$J = \{f : 0 \to \mho M, M \in \mathcal{M}\}$$

*and $J^{\square}$ be the class of fibrations. Let $\mathcal{W}$ be the class of morphisms whose image under functor $G$ are isomorphisms.*

*Then $\mathcal{E}$ is almost equipped with a structure of prefibration category, it means that there exist two spaces of morphisms, $\mathcal{W}$, the weak equivalences, and $\mathcal{F}ib$ the fibrations, such that:*

*(i) The space $\mathcal{W}$ is stable under retracts and satisfies the two out of three axiom.*

*(ii) The space $\mathcal{F}ib$ is stable under composition, and all the isomorphisms are fibrations.*

*(iii) Pullbacks exist along fibrations, and the pull-back of a fibration is a fibration. Moreover, if $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} Y \longrightarrow 0$ is a short exact sequence, if $Gi$ is a monomorphism and $f : X \to Y$ is a trivial fibration ($f \in \mathcal{F}ib \cap \mathcal{W}$), then $h$ defined by the following pull-back is a trivial fibration.*

$$\begin{array}{ccc} E & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ A \xrightarrow[i]{} & B \xrightarrow[p]{} & Y \end{array}$$

*(iv) There exist path objects such that for any object $X$, the diagonal map $X \to X \oplus X$ can be factorized through $X'$, where the first morphism is a weak equivalence.*

*(v) For any object $B$, the morphism $0 \to B$ is a fibration.*

Let us first show the following lemma:

**Lemma 1.5.2.** *Let*

$$\begin{array}{ccc} E & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ A \xrightarrow[i]{} & B \xrightarrow[p]{} & Y \end{array}$$

*where $(i, p)$ is a short exact sequence, $Gi$ monomorphism and $f \in J^{\square} \cap \mathcal{W}$. Then $h \in J^{\square} \cap \mathcal{W}$.*

*Proof.* First, we have, without using the fact that $Gi$ is a monomorphism, that $h \in J^{\square}$. Indeed, let

$$\begin{array}{ccc} E & \xrightarrow{a} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & Y \end{array}$$

be a commutative square, with $f \in J^{\square}$. Let us show that $h \in J^{\square}$. Let $M \in \mathcal{M}$ and $b : \mho M \to B$. As $f \in J^{\square}$, there exists $\alpha : \mho M \to X$ such that $f \circ \alpha = p \circ b$. We use the pullback property to claim that there exists $\varphi : \mho M \to E$ such that

$h \circ \varphi = b$. Then $h \in J^{\square}$. Second, let us show that $Gh$ is a monomorphism. As $h$ is a deflation (since it belongs to $J^{\square}$, and from lemma 1.2.5), the morphisms $f$ and $h$ have the same kernel. Then we have the following diagram:

$$\begin{array}{ccccc} & & K & = & K \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{k} & E & \xrightarrow{a} & X \\ \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & B & \xrightarrow[p]{} & Y \end{array}$$

We have a short exact sequence $0 \to A \to E \to X \to 0$. Let us show that $Gh$ is a monomorphism. Let $\beta : D \to GE$ such that $Gh \circ \beta = 0$. We show that $\beta = 0$. We have

$$Gp \circ Gh \circ \beta = 0.$$

Then

$$G(p \circ h) \circ \beta = 0$$

and

$$G(f \circ a) \circ \beta = 0.$$

Thus, we have

$$Gf \circ Ga \circ \beta = 0.$$

As $Gf$ is a monomorphism since $f \in \mathcal{W}$, we have

$$Ga \circ \beta = 0.$$

The fact that $G$ is left exact shows that there exists $c : D \to GA$ such that

$$\beta = Gk \circ c. \tag{1.1}$$

Moreover, by hypothesis

$$Gi \circ c = Gh \circ \beta = 0.$$

As $Gi$ is a monomorphism, $c = 0$ and from 1.1 we have $\beta = 0$. This shows that $Gh$ is a monomorphism.

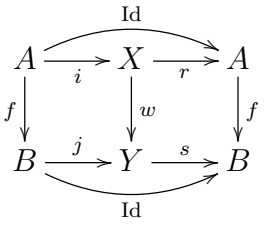
Now we show that $Gh$ is an epimorphism. Let $b : M \to B$ be a morphism, where $M \in \mathcal{M}$. Then we have $p \circ b : M \to Y$. From lemma 1.2.4, there exists $\alpha : M \to X$ such that

$$f \circ \alpha = p \circ h.$$

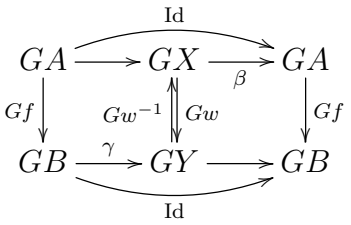
Then, from the pullback property, there exists $\varphi : M \to E$ such that $h \circ \varphi = b$ and this shows that $Gh$ is an epimorphism. $\square$

*Proof of the theorem.* Some of the properties are trivial because of the results on Frobenius categories which did not use the Frobenius condition.

(i) Let $f$ be a retract of $w \in \mathcal{W}$. Let us show that $Gf$ is an isomorphism. We have:




As $Gf$ is a retract of an isomorphism, it is itself en isomorphism. Indeed,




The morphism $\beta \circ Gw^{-1} \circ \gamma$ is an inverse of $Gf$.

Then $Gf$ is an isomorphism.

Now let us show that $\mathcal{W}$ has the 2-out-of-3 property.

— If $f, g \in \mathcal{W}$. Then $G(f \circ g) = G(f) \circ G(g)$, and we have that $f \circ g \in \mathcal{W}$.

— If $f, f \circ g \in \mathcal{W}$. Then $(G(f \circ g))^{-1} \circ G(f \circ g) = \mathrm{Id}$. So $(G(f \circ g))^{-1} \circ G(f) \circ G(g) = \mathrm{Id}$. Let $G(g)^{-1}$ be the morphism $(G(f \circ g))^{-1} \circ G(f)$. It is a left inverse. Let us show that it is also a right inverse:
$G(f \circ g) \circ (G(f \circ g))^{-1} = \mathrm{Id}$. Then $G(f) \circ G(g) \circ (G(f \circ g))^{-1} = \mathrm{Id}$, and $G(g) \circ (G(f \circ g))^{-1} \circ G(f) = \mathrm{Id}$ and it is indeed a right inverse.

— If $g, f \circ g \in \mathcal{W}$. Then we use the same hint to show that $Gf$ is an isomorphism.

(ii) This result is easy, because fibrations are defined by a lifting property.

(iii) Fibrations are deflations (this is because $\mathcal{E}$ is weakly idempotent complete, see lemma 1.2.5), then pullbacks exist along fibrations. The rest of the item is the previous lemma.

(iv) We have the factorization from lemma 1.3.2. It means that any morphism can be factorised through a morphism of ${}^{\square}(J^{\square})$ followed by a morphism of $J^{\square}$. Indeed, it only uses the fact that there exist some $\mathfrak{V}\mathcal{M}$-approximations, which we suppose in the hypotheses of this theorem. Then for any $X$, the diagonal $X \to X \oplus X$ can be factorised $X \to X' \to X \oplus X$, where the first

morphism is a weak equivalence (we have seen that the morphisms of ${}^{\square}(J^{\square})$ are weak equivalences from lemma 1.3.1).

(v) By definition of $J$, any object is fibrant, then $0 \to B$ is a fibration for any object $B$ of $\mathcal{E}$.

□

## 1.6 Theorem of Quillen

We recall that the objects of $\mathrm{pr}\mathcal{M}$ are exactly the same as cofibrant objects (defined here by satisfying the lifting property along trivial fibrations).

Now, we show that the theorem of Quillen is satisfied. We note that we still have the same notion of homotopy.

**Theorem 1.6.1.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands. Suppose moreover that $\mho\mathcal{M}$ is contravariantly finite. Let*

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

*and*

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

*Let* Ho $\mathcal{E}$ *be the localization of $\mathcal{E}$ at the class $\mathcal{W}$ of weak equivalences. Let* mod $\overline{\mathcal{M}}$ *be the category of finitely presented $\overline{\mathcal{M}}$ modules. There is an equivalence of categories*

$$\mathrm{Ho}\ \mathcal{E} \simeq \mathrm{mod}\ \overline{\mathcal{M}}.$$

**Remark 1.6.2.** *In order to prove this theorem, we need the following lemmas. The proof of these is well-known, but we give here some details in order to show that the restriction of (iii) in theorem does not affect the results. Then, some parts of the proofs simplify due to the particular shape of the relation of homotopy.*

**Lemma 1.6.3.** *Let $A, B, X$ be three cofibrant objects. Let $F$ be the functor defined by*

$$F : A \mapsto \mathcal{E}(A, X)/\sim$$

*on objects, and, if $f : A \to B$,*

$$Ff : \mathcal{E}(B, X)/\sim \mapsto \mathcal{E}(A, X)/\sim .$$

*Suppose moreover that $f$ is a weak equivalence. Then $F$ induces an isomorphism*

$$Ff : \mathcal{E}(B, X)/\sim \mapsto \mathcal{E}(A, X)/\sim$$

*where $\sim$ is the right homotopy relation (which is also left taken into account the definition of the homotopy).*

*Proof.* This functor is well-defined because the relation $\sim$ behaves well with the right composition. Let now $f : A \to B$. We factor $f$ through a morphism $g : A \to C \in^{\square} (J^{\square})$ followed by a morphism $p : C \to B \in J^{\square}$. As $g \in \mathcal{W}$ and $f \in \mathcal{W}$, then by the two out of three property, $p$ is also a weak equivalence. In addition, $B$ is cofibrant, then there exists $w : B \to C$ such that $p \circ w = 1$.

Then, we have

$$1 = F(1) = F(p \circ w) = F(p) \circ F(w).$$

Then $F(p) \in \mathcal{W}$. This shows by the two out of three property, that $F(f) \in \mathcal{W}$, and $Ff$ is surjective.

Let us now show that it is injective. If $\alpha : B \to X$ is such that $F\alpha = 0$, then $\alpha \circ f \sim 0$. As $A$ is cofibrant, we have $\alpha \circ f \in (\mho\mathcal{M})$ (from the equivalence of Demonet and Liu in [DL13]). This shows that $G(\alpha \circ f) = 0$ since $\mathcal{M}$ is rigid. But $f \in \mathcal{W}$, then $Gf$ is an isomorphism, then $G\alpha = 0$. As $B$ is cofibrant, $\alpha \in (\mho\mathcal{M})$. Then $\alpha \sim 0$. Then $Ff$ is injective and this shows the lemma. □

Now we can show the following result:

**Lemma 1.6.4.** *If $A$ and $B$ are two cofibrant objects, then*

$$\mathrm{Ho}\ \mathcal{E}(A, B) = \mathcal{E}(A, B)/\sim .$$

*Proof.* Let $A$ and $B$ be two cofibrant objects.

Step 1: We show that, $f : A \to B$ is a weak equivalence if and only if it is a homotopy equivalence.

Suppose $f$ is a weak equivalence. We use the previous lemma with $X = A$. From the surjectivity of $Ff$, there exists $g : B \to A$ such that $g \circ f \sim 1$. Then, $f \circ g \circ f \sim f$. Now we apply the result to $X = B$ and have that

$$Ff(f \circ g) = Ff(1).$$

However, $Ff$ is injective, then $f \circ g \sim 1$ and $f$ is a homotopy equivalence.

Suppose now that $f$ is a homotopy equivalence. Let $f'$ be a homotopy inverse for $f$. We have $f \circ f' \sim 1$ and $f' \circ f \sim 1$. Then $f \circ f' - 1 \in \mho\mathcal{M}$ and $f' \circ f - 1 \in \mho\mathcal{M}$. Then $G(f \circ f') = 1$ and $G(f' \circ f) = 1$. Then $Gf$ is an isomorphism and $f \in \mathcal{W}$.

This shows that $f = g$ in $\mathrm{Ho}\mathcal{E}(A, B)$ if and only if $f \sim g$.

Step 2: We now show the surjectivity of $\mathcal{E}(A, B) \to \mathrm{Ho}(A, B)$. We are going to use the book of Radulescu (see [RB06, Theorem 6.4.2]). Let us check the hypothesis with the pair of categories $(\mathcal{E}/\{f, Gf = 0\}, \underline{\mathcal{W}})$, where $\underline{\mathcal{W}}$ is the image of $\mathcal{W}$ in the quotient of $\mathcal{E}$ by $\{f, Gf = 0\}$.

— The two out of three property is automatically checked.

— If we have a pair of morphisms

$$\begin{array}{ccc} & & A' \\ & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[p]{} & A \end{array}$$

such that $a \in \mathcal{W}$, then there exists $B', h : B' \to B$ and $k : B' \to A'$ such that the following square commutes:

$$\begin{array}{ccc} B' & \xrightarrow{k} & A' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[p]{} & A \end{array}$$

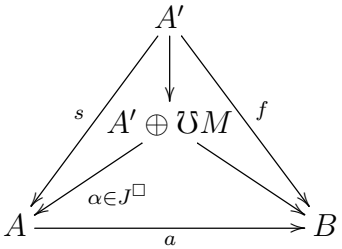
Indeed, we introduce the factorization of $A' \to A$ by $b \in^{\square} (J^{\square})$ followed by $c \in J^{\square}$. As $a, b \in \underline{W}$, we also have $c \in \underline{W}$. Then $c$ is a trivial fibration. Let $B'$ be a pr$\mathcal{M}$-approximation of $B$. We lift the morphism $p \circ h$ to $c$, let us say $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : B' \to A' \oplus \mho M$ (which is the shape of the factor in the factorization we have). Then $h_1$ is a lift from $B'$ to $A'$. We can check that the square commutes and $h \in \underline{W}$.

— Suppose that we have

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{t} B'$$

with $t \in \underline{W}$ and $t \circ f = t \circ g$. Then $Gf = Gg$, so, if $t' : A' \to A \in \underline{W}$, we have $G(f \circ t') = G(g \circ t')$, then $f \circ t' - g \circ t' \in \overline{\mathcal{M}}^{\perp}$. Then $f \circ t' = g \circ t'$ in $\mathcal{E}/\{f, Gf = 0\}$.

We can now apply the theorem of Radulescu 6.4.2 and any morphism in $\mathrm{Ho}(\mathcal{E}(A, B))$ can be written as $f \circ s^{-1}$ with $s \in \underline{\mathcal{W}}$. As $A$ and $B$ are cofibrant, we factor through $A' \to A$ in this way:




As $s \in \underline{\mathcal{W}}$, then $\alpha \in J^{\square} \cap \mathcal{W}$. Then we can lift $A' \oplus \mho M$ to $B$ by $(f\ 0)$.

From the theorem of Radulescu, if $\alpha \in \mathrm{Ho}(A, B)$, then there exists $s \in \mathcal{W}$ and $f \in \mathcal{E}(A, B)$ such that

$$\alpha = \overline{f} \circ \overline{s}^{-1}.$$

Then $\alpha = \overline{a}$ (since $\overline{a} \circ \overline{s} = \overline{f}$).

Then we have shown the surjectivity and then the lemma. □

*Proof of the theorem of Quillen.* First of all, the functor is well-defined, since

$$0 \to \mho M \in \mathcal{W}$$

for any $M \in \mathcal{M}$ (because $\mathrm{Ext}^1(-, M)|\mathcal{M} = 0$ implies $\overline{\mathcal{E}}(-, \mho M)|\mathcal{M} = 0$).

Then, the functor is essentially surjective, since there exist some pr$\mathcal{M}$-approximations, which are weak equivalences.

Next, from lemma 1.6.4, we have that $f = g$ in Ho$\mathcal{E}(A, B)$ if and only if $f \sim g$ which immediately shows that the functor is faithful.

Finally, the functor is full, from the surjectivity of lemma 1.6.4.

The theorem of Demonet and Liu in [DL13] finishes to show the result. □

Chapter 2

# The particular case of Frobenius categories

We recall the notations:

Let $\mathcal{E}$ be a Frobenius category. Let $\mathcal{M}$ be a full rigid subcategory which contains the injective objects. Let

$$\mathrm{pr}\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

and

$$\mho\mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \mathrm{Inj}, 0 \to M \to I \to X \to 0\}.$$

Let

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-,X)/\overline{\mathcal{M}} \end{array}$$

which induces the following equivalence of categories

$$\mathrm{pr}\mathcal{M}/\mho\mathcal{M} \simeq \mathrm{mod}\ \overline{\mathcal{M}}.$$

For more details, see the article of Demonet and Liu, [DL13].

Let $\mathcal{M}$ is a subcategory of $\mathcal{E}$ which is contravariantly finite. As the injective objects are also projective, note that $\mathcal{M}$ contains the projective objects.

## 2.1 A deeper study of $\mathrm{pr}\mathcal{M}$

We have this first lemma:

**Lemma 2.1.1.** *Let $X \in \mathcal{E}$. If there is an $\mathcal{M}$-approximation of $\Omega X$, then there exists a $\mho\mathcal{M}$-approximation of $X$. In particular, if $\mathcal{M}$ is contravariantly finite, then so is $\mho\mathcal{M}$.*

*Proof.* Let $a : M \to \Omega X$ be an $\mathcal{M}$-approximation of $\Omega X$. We can introduce $Y$ as the push-out of the pair of morphisms:

$$\begin{array}{ccc} M & \xrightarrow{a} & \Omega X \\ {\scriptstyle \iota_M}\downarrow & & \downarrow{\scriptstyle c} \\ I_M & \xrightarrow[b]{} & Y \end{array}$$

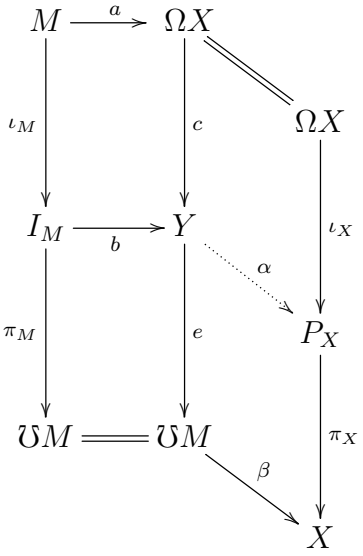
in particular

$$c \circ a = b \circ \iota_M. \tag{2.1}$$

Let

$$0 \longrightarrow \Omega X \xrightarrow{\iota_X} P_X \xrightarrow{\pi_X} X \longrightarrow 0$$

be a short exact sequence expressing the fact that $\Omega X$ is a sizygy of $X$. We then have the following diagram:




Note that we have

$$e \circ b = \pi_M. \tag{2.2}$$

As $\iota_X \circ a$ factorizes through the module $I_M$, there exists $\alpha : Y \to P_X$ such that

$$\iota_X = \alpha \circ c \text{ and } \alpha \circ b = d. \tag{2.3}$$

Then there exists $\beta$, which makes a morphism of short exact sequences:

$$\pi_X \circ \alpha = \beta \circ e \tag{2.4}$$

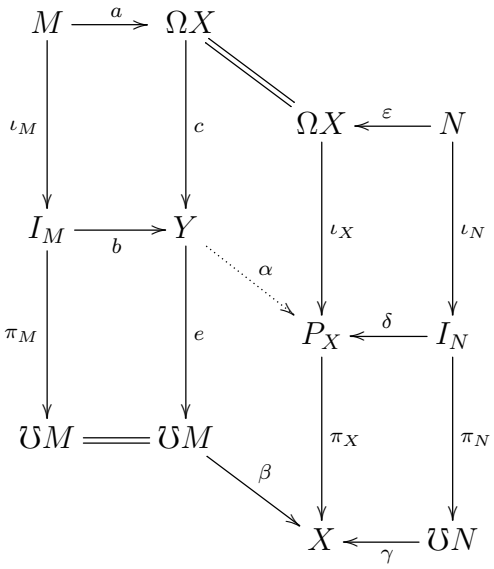
Then we have a short exact sequence

$$0 \to Y \to P_X \oplus \Upsilon M \to X \to 0.$$

We now show that $(\pi_X\ \beta)$ is a $\mho\mathcal{M}$-approximation (we note that, $\mathcal{E}$ being Frobenius, $\mho\mathcal{M}$ contains the projective modules).

Let $\gamma : \mho N \to X$ be a morphism. As $I_N$ is also projective, there exists $\delta : I_N \to P_X$ such that

$$\pi_X \circ \delta = \gamma \circ \pi_N \tag{2.5}$$




We complete into a morphism of short exact sequences by $\varepsilon$, it means that

$$\iota_X \circ \varepsilon = \delta \circ \iota_N. \tag{2.6}$$

As $a : M \to \Omega X$ is an $\mathcal{M}$-approximation, there exists $\zeta : N \to M$ such that

$$a \circ \zeta = \varepsilon. \tag{2.7}$$

As $I_M$ is injective, and $\iota_N$ is an inflation, there exists $\eta : I_N \to I_M$ such that

$$\eta \circ \iota_N = \iota_M \circ \zeta. \tag{2.8}$$

Then there exists $\theta : \mho N \to \mho M$ which makes a morphism of short exact sequences:

$$\theta \circ \pi_N = \pi_M \circ \eta. \tag{2.9}$$

Then we have:

$$\begin{aligned} \delta \circ \iota_N &= \iota_X \circ \varepsilon \text{ by 2.6} \\ &= \iota_X \circ a \circ \zeta \text{ by 2.7} \\ &= \alpha \circ c \circ a \circ \zeta \text{ by 2.3} \\ &= \alpha \circ b \circ \iota_M \circ \zeta \text{ by 2.1} \\ &= \alpha \circ b \circ \eta \circ \iota_N \text{ by 2.8.} \end{aligned} \tag{2.10}$$

Then, the morphism $\alpha \circ b \circ \eta - \delta$ factorizes through $\pi_N$.
Thus, there exists $\kappa : \mho N \to P_X$ such that

$$\kappa \circ \pi_N = \delta - \alpha \circ b \circ \eta. \tag{2.11}$$

Let us finally show that the morphism $\begin{pmatrix} \kappa \\ \theta \end{pmatrix}$ satisfies $(\pi_X \ \beta) \begin{pmatrix} \kappa \\ \theta \end{pmatrix} = \gamma$.
As $\pi_N$ is a deflation, is suffices to show that

$$\pi_X \circ \kappa \circ \pi_N + \beta \circ \theta \circ \pi_N = \gamma \circ \pi_N.$$

We have:

$$\begin{aligned} \pi_X \circ \kappa \circ \pi_N + \beta \circ \theta \circ \pi_N &= \pi_X(\delta - \alpha \circ b \circ \eta) + \beta \circ \theta \circ \pi_N \text{ by 2.11} \\ &= \pi_X(\delta - \alpha \circ b \circ \eta) + \beta \circ \pi_M \circ \eta \text{ by 2.9} \\ &= \gamma \circ \pi_N - \pi_X \circ \alpha \circ b \circ \eta + \beta \circ \pi_M \circ \eta \text{ by 2.5} \\ &= \gamma \circ \pi_N - \beta \circ e \circ b \circ \eta + \beta \circ \pi_M \circ \eta \text{ by 2.4} \\ &= \gamma \circ \pi_N - \beta \circ \pi_M \circ \eta + \beta \circ \pi_M \circ \eta \text{ by 2.2} \\ &= \gamma \circ \pi_N. \end{aligned} \tag{2.12}$$

□

**Definition 2.1.2.** *We call by* copr $\mho\mathcal{M}$ *the class of objects* $X \in \mathcal{E}$ *such that there exist* $M, M' \in \mathcal{M}$ *such that* $0 \to X \to \mho M \to \mho M' \to 0$ *is a short exact sequence.*

$$\text{copr } \mho\mathcal{M} = \{X \in \mathcal{E}, 0 \to X \to \mho M \to \mho M' \to 0\}.$$

**Lemma 2.1.3.** *We have the following equality:*

$$\text{copr } \mho\mathcal{M} = \text{pr}\mathcal{M}.$$

*Proof.* Let $C \in \text{copr } \mho\mathcal{M}$. Let us show that we then have a short exact sequence $0 \to M_1 \to M' \to C \to 0$. We have the diagram:

$$\begin{array}{ccccc} M_1 & \longrightarrow & \Omega\mho M_0 & \longrightarrow & C \\ \| & & \downarrow & & \downarrow \\ M_1 & \longrightarrow & I & \longrightarrow & \mho M_1 \\ & & \downarrow & & \downarrow \\ & & \mho M_0 & = & \mho M_0 \end{array}$$

We have $\Omega\mho M \simeq M'$. Indeed, we have a two-ways short exact sequence

$$\begin{array}{ccccc} M & \longrightarrow & I & \longrightarrow & \mho M \\ \uparrow\downarrow & & \uparrow\downarrow & & \| \\ \Omega\mho M & \longrightarrow & P & \longrightarrow & \mho M \end{array}$$

Then $\Omega\mho M \simeq M \oplus I \in \mathcal{M}$. The other inclusion is proved in part 1, lemma 1.4.7. □

**Lemma 2.1.4.** *The class $\mho\mathcal{M}$ is rigid.*

*Proof.* Let $M, M' \in \mathcal{M}$. We show that a morphism $f : \mho M \to \mho^2 M'$ is zero. Let us introduce the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{h} & \mho M' \\ {\scriptstyle \iota_M}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle \iota_{\mho M'}} \\ I_M & \xrightarrow{g} & I_{\mho M'} \\ {\scriptstyle \pi_M}\downarrow & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle \pi_{\mho M'}} \\ \mho M & \xrightarrow[f]{} & \mho^2 M' \end{array}$$

As $I_M$ is a lso projective and $\pi_{\mho M'}$ is a deflation, there exists $g$ which makes the lower square commute. Then, there exists $h$ such that the upper square commutes. Now, $h$ factorizes through an injective module since $\mathcal{M}$ is rigid. Let us say that there exists $\alpha : I_M \to \mho M'$ such that $\alpha \circ \iota_M = h$. Then, there exists $\beta : \mho M \to I_{\mho M'}$ such that

$$g = \iota_{\mho M'} \circ \alpha + \beta \circ \pi_M.$$

Then,

$$f \circ \pi_M = \pi_{\mho M'} \circ g = \pi_{\mho M'} \circ (\iota_{\mho M'} \circ \alpha + \beta \circ \pi_M) = \pi_{\mho M'} \circ \beta \circ \pi_M$$

As $\pi_M$ is a deflation, this shows that $f = \pi_{\mho M'} \circ \beta$ and thus factorizes through an injective. Then $\mho\mathcal{M}$ is rigid. □

**Lemma 2.1.5.** *Let $A, C$ be objects of $\mathcal{E}$. Let $M \in \mathcal{M}$. Suppose that we have a short exact sequence*

$$0 \to A \to C \to \mho M \to 0.$$

*Then $C \in \mathrm{pr}\mathcal{M}$ if and only if $A \in \mathrm{pr}\mathcal{M}$.*

*Proof.* First, suppose that $A \in \mathrm{pr}\mathcal{M}$. Then, we have the following diagram:

$$\begin{array}{ccccc} A & \longrightarrow & \mho M_1 & \longrightarrow & \mho M_0 \\ \downarrow & & \downarrow & & \| \\ C & \longrightarrow & E & \longrightarrow & \mho M_0 \\ \downarrow & & \downarrow & & \\ \mho M_2 & = & \mho M_2 & & \end{array}$$

Where $E$ is the pullback of the morphisms $A \to \mho M_1$ and $A \to C$. Then there is a section to the short exact sequence

$$0 \to \mho M_1 \to E \to \mho M_2 \to 0,$$

and $E \simeq \mho M_1 \oplus \mho M_2 \simeq \mho M$. Then we have a short exact sequence $0 \to C \to \mho M \to \mho M_0 \to 0$. Then the previous lemma concludes this implication.

Conversely, if $C \in \mathrm{pr}\mathcal{M}$. Then, by the previous lemma, $C \in \mathrm{copr}\mho\mathcal{M}$, so there exists a short exact sequence

$$0 \to C \to \mho M^0 \to \mho M^1 \to 0.$$

Let us take the push-out:

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & \mho M \\ \| & & \downarrow & & \downarrow \\ A & \longrightarrow & \mho M^0 & \longrightarrow & E \\ & & \downarrow & & \downarrow \\ & & \mho M^1 & = & \mho M^1 \end{array}$$

As $\mho M$ is rigid from the previous lemma, then $E \in \mho\mathcal{M}$. Then, $A \in \mathrm{copr}\mho\mathcal{M}$ then $A \in \mathrm{pr}\mathcal{M}$. □

**Remark 2.1.6.** *We have the dual lemma: If there is a short exact sequence* $0 \to M \to C \to A \to 0$*, then* $A \in \mathrm{pr}\mathcal{M}$ *if and only if* $C \in \mathrm{pr}\mathcal{M}$*.*

## 2.2 Weak equivalences

We start with a definition.

**Definition 2.2.1.** *Let* $\mathcal{A}$ *be a subcategory of* $\mathcal{E}$*. Let* $g : Y \to Z$ *be a morphism. We say that* $g \in \overline{\mathcal{A}}^{\perp}$ *if, for any morphism* $b : \mho A \to Y$*, with* $A \in \mathcal{A}$*, the morphism* $g \circ b$ *factorizes through an injective module.*

**Remark 2.2.2.** *We note that* $\overline{\mathcal{A}}^{\perp}$ *is not an ideal.*

This lemma characterizes weak equivalences.

**Lemma 2.2.3.** *Let $X, Y \in \mathcal{E}$. Let $f : X \to Y$. Then $Gf$ is an isomorphism if and only if, for*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle g} \\ I_X & \xrightarrow{u} & Z \\ {\scriptstyle \pi_X}\downarrow & & \downarrow \\ \mho X & = & \mho X \end{array}$$

*we have $I_X \oplus Y \xrightarrow{\overline{(u\ g)}} Z \in \overline{\mathcal{M}}^{\perp}$ and, for*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle \tilde{g}} & & \uparrow \\ Z & \longrightarrow & P_Y \\ \uparrow & & \uparrow \\ \mho Y & = & \mho Y \end{array}$$

*we have $Z \longrightarrow X \oplus P_Y \in \overline{\mathcal{M}}^{\perp}$ .*

*Proof.* If $Gf$ is an isomorphism. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle g} \\ I_X & \xrightarrow{u} & Z \\ {\scriptstyle \pi_X}\downarrow & & \downarrow \\ \mho X & = & \mho X \end{array}$$

be the diagram which induces the short exact sequence

$$0 \longrightarrow X \longrightarrow I_X \oplus Y \xrightarrow{(u\ g)} Z \longrightarrow 0 \, .$$

Let us show that $(\overline{u\ g}) \in \overline{\mathcal{M}}^{\perp}$. Since $u : I_X \to Z$, with $I_X$ injective we already have that $\overline{\mathcal{E}}(\mathcal{M}, u) = 0$. Let

$$\begin{array}{rccc} \mathcal{E}(\mathcal{M}, g) \ : & \mathcal{E}(\mathcal{M}, Y) & \to & \mathcal{E}(\mathcal{M}, Z) \\ & h : M \in \mathcal{M} \to Y & \mapsto & g \circ h \end{array} \, .$$

As $Gf$ is an isomorphism, there exists $h' \in \mathcal{E}(\mathcal{M}, X)$ such that $f \circ \overline{h'} = \overline{h}$. Then

$$\overline{h'} \to f \circ \overline{h'} = \overline{h} \to g \circ f \circ \overline{h'} = g \circ h = 0,$$

and $\overline{\mathcal{E}}(\mathcal{M}, g) = 0$.

In the same way, let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ Z & \longrightarrow & P_Y \\ \uparrow & & \uparrow \\ \mho Y & = & \mho Y \end{array} ,$$

be the diagram which induces the short exact sequence

$$0 \longrightarrow Z \longrightarrow X \oplus P_Y \longrightarrow Y \longrightarrow 0 .$$

Let us show that $\begin{pmatrix} \overline{\tilde{g}} \\ \overline{\tilde{u}} \end{pmatrix} \in \overline{\mathcal{M}}$. We already have that $\overline{\mathcal{E}}(\mathcal{M}, \tilde{u}) = 0$. Let

$$\begin{array}{rccc} \mathcal{E}(\mathcal{M}, \tilde{g}) \quad : & \mathcal{E}(\mathcal{M}, Z) & \to & \mathcal{E}(\mathcal{M}, X) \\ & h : M \in \mathcal{M} \to Z & \mapsto & \tilde{g} \circ h \end{array} .$$

The map $\tilde{g} \circ h$ is sent on 0 by $Gf$. As $Gf$ is an isomorphism, we have that $\tilde{g} \circ h = 0$.

Conversely, let us suppose that

$$I_X \oplus Y \xrightarrow{\overline{(\tilde{u}\ \tilde{g})}} Z \in \overline{\mathcal{M}}^{\perp}$$

and

$$Z \longrightarrow X \oplus P_Y \in \overline{\mathcal{M}}^{\perp} .$$

Let us introduce the short exact sequences

$$0 \longrightarrow X \longrightarrow I_X \oplus Y \xrightarrow{(\tilde{u}\ \tilde{g})} Z \longrightarrow 0 \tag{2.13}$$

and

$$0 \longrightarrow Z \longrightarrow X \oplus P_Y \longrightarrow Y \longrightarrow 0 \tag{2.14}$$

We take the image of equation 2.13 by the functor $G = \mathcal{E}(A, -)$ for an $A \in \mathcal{M}$. As $G$ is left exact, we have the following long exact sequence:

$$0 \to \mathcal{E}(A, X) \to \mathcal{E}(A, Y \oplus I_A) \to \mathcal{E}(A, Z) \to \mathrm{Ext}^1(A, X) \to \cdots$$

If we factorize through the injective modules, then we get:

$$\cdots \to \overline{\mathrm{Ext}}^{-1}(A,Z) \to \overline{\mathcal{E}}(A,X) \to \overline{\mathcal{E}}(A,Y) \to \overline{\mathcal{E}}(A,Z) \to \overline{\mathrm{Ext}}^{1}(A,X) \to \cdots$$

where $\overline{\mathrm{Ext}}$ corresponds to the extension morphisms in the category $\overline{\mathcal{E}}$. Moreover, we have $\overline{\mathcal{E}}(A,g) = 0$ because $A \in \mathcal{M}$ and $g \in \overline{\mathcal{M}}^{\perp}$. Then $Gf$ is an epimorphism.

Now let us apply the functor $G$ to equation 2.14 and factorize through projective modules. We then have:

$$\cdots \to \underline{\mathrm{Ext}}^{-1}(A,Y) \to \underline{\mathcal{E}}(A,Z) \to \underline{\mathcal{E}}(A,X) \to \underline{\mathcal{E}}(A,Y) \to \underline{\mathrm{Ext}}^{1}(A,Z) \to \cdots$$

We have $\underline{\mathcal{E}}(A,\tilde{g}) = 0$ because $A \in \mathcal{M}$ and $\tilde{g} \in \overline{\mathcal{M}}^{\perp} = \underline{\mathcal{M}}^{\perp}$. Then $Gf$ is a monomorphism, thus an isomorphism. □

## 2.3 Fibrations and cofibrations

Let us recall the notations:

Let $\mathcal{W}$ be the class of morphisms $f$ such that $Gf$ is an isomorphism. Such morphisms are called weak equivalences. Let

$$J = \{f : 0 \to \mho M, M \in \mathcal{M}\}$$

and

$$I = \{f : M_0 \to X \oplus I_0, X \in \mathrm{pr}\mathcal{M}\} \cup \{0 \to M, M \in \mathcal{M}\}$$

where $0 \to M_1 \to M_0 \to X \to 0$ is a short exact sequence and $I_0$ appears in the short exact sequence $0 \to M_0 \to I_0 \to \mho M_0$.

The morphisms of $J^{\square}$ are called fibrations and compose the class $\mathcal{F}ib$. The following lemma characterizes fibrations. We recall that $g \in \mho\overline{\mathcal{M}}$ if, for any morphism $b : \mho M \to Y$, with $M \in \mathcal{M}$, the morphism $g \circ b$ factorizes through an injective module.

**Definition 2.3.1.** *For a morphism $f : X \to Y$, we call a cone of $f$, any morphism $g$, obtained from a push-out of an inflation $X \to I$ for some injective $I$ along $f$. Then, we notice that $g$ is unique up to isomorphism.*

**Lemma 2.3.2.** *Suppose that $\mathcal{E}$ is a weakly idempotent complete category (see the article of Bühler, [Büh10]). Then $f$ is a fibration if and only if $f$ is a deflation,*

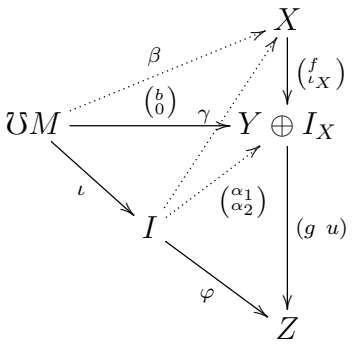
*and* $g \in \mho\overline{\mathcal{M}}^{\perp}$*, where* $g$ *is a cone of* $f$*, defined by*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle g} \\ I_X & \xrightarrow{u} & Z \\ {\scriptstyle \pi_M}\downarrow & & \downarrow \\ \mho X & = & \mho X \end{array} .$$

*Proof.* Let $f : X \to Y$ be a deflation. Let $g$ be a cone of $f$. Suppose that $g \in \mho\overline{\mathcal{M}}^{\perp}$. Let us show that $f \in J^{\square}$. Let $b : \mho M \to Y$, where $\mho M \in \mho\mathcal{M}$. As $(g\ u) \in \mho\overline{\mathcal{M}}^{\perp}$, then the morphism $(g\ u) \circ \begin{pmatrix} b \\ 0 \end{pmatrix}$ factorizes through an injective. We have

$$\begin{pmatrix} g & u \end{pmatrix} \begin{pmatrix} b - \alpha_1 \circ \iota \\ 0 - \alpha_2 \circ \iota \end{pmatrix} = 0.$$

We have the following diagram:




Then, there exists $\beta : \mho M \to X$ such that

$$\begin{pmatrix} b - \alpha_1 \circ \iota \\ \alpha_2 \circ \iota \end{pmatrix} = \beta \circ \begin{pmatrix} f \\ \iota_X \end{pmatrix}.$$

As $I$ is also projective since $\mathcal{E}$ is a Frobenius category, and $f$ is a deflation, there exists $\tilde{\alpha_1} : I \to X$ such that

$$\alpha_1 = f \circ \tilde{\alpha_1}.$$

Then, we have

$$f \circ (\beta - \tilde{\alpha_1} \circ \iota) = b$$

and this shows the first implication.

On the other hand, let $g$ be a cone of $f$. If $f \in J^{\square} : X \to Y$, then it is a deflation. Indeed, let $\pi_Y : P_Y \to Y$ be a projective pre-cover of $Y$ (we can equivalently introduce a $\mho\mathcal{M}$-approximation of $Y$). There exists a morphism $\varphi : P_Y \to X$ which make the diagram commutative. As $f \circ \varphi = \pi_Y$ is a deflation and $\mathcal{E}$ is a weakly idempotent complete category, from the article of Bühler, [Büh10, Theorem 7.6], then $f$ is also a deflation. Let us now show that $g \in \mho\overline{\mathcal{M}}^{\perp}$. Let $h : B \to Y$, where $B \in \mho\mathcal{M}$. We have to show that $g \circ h = 0$. As $f \in J^{\square}$, there exists $\varphi : B \to X$ such that $f \circ \varphi = h$. Then $g \circ h = g \circ f \circ \varphi$. As we know $g \circ f$ factorize through an injective, and so $g \circ h$. Then $g \in \mho\overline{\mathcal{M}}^{\perp}$. □

## 2.4 Factorizations

In addition to the first factorization, we also have a second factorization:

**Lemma 2.4.1.** *Any morphism $f : X \to Y$, where $X \in \mathrm{pr}\mathcal{M}$ can be factorized through a morphism in ${}^{\square}(I^{\square})$ followed by a morphism in $I^{\square}$.*

*Proof.* Let us show that any morphism $f : X \to Y$, where $X \in \mathrm{pr}\mathcal{M}$ can be factorized through a morphism in ${}^{\square}(I^{\square})$ followed by a morphism in $I^{\square}$.

We are going to use item (iv) in the next theorem (whose proof does not need this lemma) We have made a proof of point (iv) lower for sake of clarity (in order to have a one-piece-proof for the main theorem).

Let $f : X \to Y$ be a morphism, such that $X \in \mathrm{pr}\mathcal{M}$. Let

$$0 \to M_1 \to M_0 \to X \to 0$$

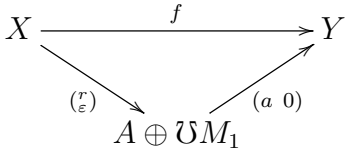
be a short exact sequence, and let $c : M_0 \to X$. From lemma 1.1.1, there exist $a : A \to Y$ a $\mathrm{pr}\mathcal{M}$-approximation, with $A \in \mathrm{pr}\mathcal{M}$. Let $\varepsilon : X \to \mho M_1$ be the induced morphism in the short exact sequence

$$0 \to M_0 \to X \oplus I_{M_0} \to \mho M_1 \to 0.$$

As $A \to Y$ is a $\mathrm{pr}\mathcal{M}$-approximation, and $X \in \mathrm{pr}\mathcal{M}$, there exists $r : X \to A$ such that

$$f = a \circ r.$$

We are going to show that



is the good factorization.

It is immediate that $(a\ 0)$ is a trivial fibration. So it is an element of $I^{\square}$. Let us show that $\begin{pmatrix} r \\ \varepsilon \end{pmatrix} \in^{\square} (I^{\square})$. We introduce the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ {\scriptstyle \binom{r}{\varepsilon}}\downarrow & & \downarrow{\scriptstyle h} \\ A \oplus \mho M_1 & \xrightarrow[(\beta_1\ \beta_2)]{} & V \end{array}$$

with $h \in I^{\square}$.

The object $A \oplus \mho M_1$ is in $\mathrm{pr}\mathcal{M}$, then from lemma 1.4.2, part 1, it is cofibrant. Then, there exists a morphism $(\varphi_1\ \varphi_2) : A \oplus \mho M_1 \to U$ such that

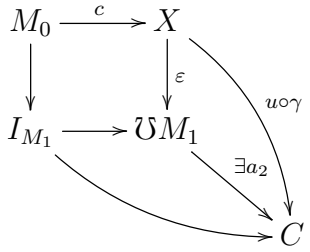
$$h \circ \varphi_1 = \beta_1 \text{ and } h \circ \varphi_2 = \beta_2. \tag{2.15}$$

Unfortunately, this morphism is not the good candidate, because it does not make the upper triangle commute. We have to modify it.

As $h \in J^{\square} \cap \mathcal{W}$, then it is a deflation, we introduce its kernel $u : K \to X$. There exists $\gamma : X \to K$ such that

$$\alpha = u \circ \gamma + \varphi_1 \circ r + \varphi_2 \circ \varepsilon. \tag{2.16}$$

Moreover, we know that $u \in \overline{\mathcal{M}}^{\perp}$. So the morphism $u \circ \gamma \circ c$ factorizes through an injective module, let us say $I_{M_1}$. Then we have the following diagram:




From the push-out property, there exists a morphism $a_2 : \mho M_1 \to C$ such that

$$u \circ \gamma = a_2 \circ \varepsilon. \tag{2.17}$$

Then, from equations 2.16 and 2.17, we have $(\varphi_1\ \varphi_1 + a_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} = \alpha$.

Unfortunately again, this morphism is not the good candidate. Now, the upper triangle commutes, but we have lost the commutativity of the lower one. We have to modify it one last time.

Moreover, the morphism $\begin{pmatrix} r \\ \varepsilon \end{pmatrix}$ is an inflation. We can introduce its cokernel $C$. Let $(c_1\ c_2) : A \oplus \mho M_1 \to C$. We have

$$h \circ (\varphi_1\ \varphi_2 + a_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} = h \circ \alpha = (\beta_1\ \beta_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix}.$$

Then the morphism $(\varphi_1\ \varphi_2 + a_2) - (\beta_1\ \beta_2)$ factorizes through the cokernel $C$ of $\begin{pmatrix} r \\ \varepsilon \end{pmatrix}$. Then there exists $b : C \to V$ such that

$$(\beta_1\ \beta_2) = h \circ (\varphi_1\ \varphi_2 + a_2) - b \circ (c_1\ c_2). \tag{2.18}$$

Moreover, $C \in \mathrm{pr}\mathcal{M}$ from lemma 2.1.5. Then it is cofibrant, as $h \in J^{\square} \cap \mathcal{W}$, there exists $d : C \to U$ such that

$$h \circ d = b. \tag{2.19}$$

We now have enough information in order to choose the good candidate for the morphism $A \oplus \mho M_1 \to U$ which makes both triangles commute. It is:

$$(\varphi_1 - d \circ c_1\ \varphi_2 + a_2 - d \circ c_2).$$

Indeed, for the upper triangle:

$$\begin{aligned}
(\varphi_1 - d \circ c_1\ \varphi_2 + a_2 - d \circ c_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} &= \varphi_1 \circ r - d \circ c_1 \circ r + \varphi_2 \circ \varepsilon + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon \\
&= \alpha - u \circ \gamma - \varphi_2 \circ \varepsilon - d \circ c_1 \circ r + \varphi_2 \circ \varepsilon + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon \text{ from 2.16} \\
&= \alpha - a_2 \circ \varepsilon - d \circ c_1 \circ r + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon \text{ from 2.17} \\
&= \alpha - d \circ (c_1\ c_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} \\
&= \alpha.
\end{aligned} \tag{2.20}$$

The upper triangle commutes.

Now let us show the commutativity of the lower triangle:

$$\begin{aligned}
h \circ (\varphi_1 - d \circ c_1\ \varphi_2 + a_2 - d \circ c_2) &= (h \circ \varphi_1 - h \circ d \circ c_1\ h \circ \varphi_2 + h \circ a_2 - h \circ d \circ c_2) \\
&= (h \circ \varphi_1 - b \circ c_1\ h \circ \varphi_2 + h \circ a_2 - b \circ c_2) \text{ from 2.19} \\
&= (h \circ \varphi_1 + \beta_1 - h \circ \varphi_1\ \beta_2 + b \circ c_2 - c \circ c_2) \text{ from 2.18} \\
&= (\beta_1\ \beta_2)
\end{aligned} \tag{2.21}$$

Then, the lower triangle commutes, so $\begin{pmatrix} r \\ \varepsilon \end{pmatrix} \in^{\square} (I^{\square})$, and we have the factorization when the domain of the morphism is cofibrant. $\square$

## 2.5 The almost model structure on Frobenius categories

We are going to show the following theorem:

**Theorem 2.5.1.** *Let $\mathcal{E}$ be a weakly idempotent complete Frobenius category. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands.*

*Let*

$$J = \{f : 0 \to \mho M, M \in \mathcal{M}\}$$

*and*

$$I = \{f : M_0 \to X \oplus I_0, X \in \mathrm{pr}\mathcal{M}\} \cup \{0 \to M, M \in \mathcal{M}\}.$$

*Let $G$ be the functor*

$$\begin{array}{rccc} G & : & \mathcal{E} & \to & \mathrm{Mod}\overline{\mathcal{M}} \\ & & X & \mapsto & \overline{\mathcal{E}}(-,X)/\overline{\mathcal{M}} \end{array}$$

*Let*

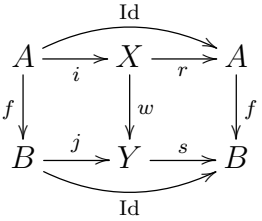
$$\mathcal{W} = \{f, Gf \text{ is an isomorphism}\}.$$

*Let $J^{\square}$ be the class of fibrations. The cofibrations are given by the left-lifting property from trivial fibrations. Then, $(Fib, Cof, \mathcal{W})$ nearly form a model structure for the category $\mathcal{E}$. Indeed, the second factorization is found only when the domain is cofibrant.*

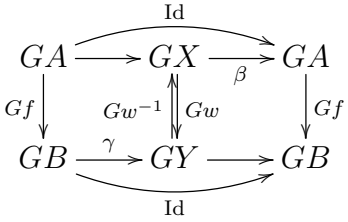
*Proof.* We show the conditions required in order to apply theorem **??** of Hovey in [Hov99, Theorem 2.1.19].

(i) $\mathcal{W}$ is stable under retracts.

Let $f$ be a retract of $w \in \mathcal{W}$. Let us show that $Gf$ is an isomorphism. We have:




As $Gf$ is a retract of an isomorphism, it is itself en isomorphism. Indeed,



The morphism $\beta \circ Gw^{-1} \circ \gamma$ is an inverse of $Gf$.

Then $Gf$ is an isomorphism.

(ii) "2 out of 3" property.

— If $f, g \in \mathcal{W}$. Then $G(f \circ g) = G(f) \circ G(g)$, and we have that $f \circ g \in \mathcal{W}$.

— If $f, f \circ g \in \mathcal{W}$. Then $(G(f \circ g))^{-1} \circ G(f \circ g) = \mathrm{Id}$. So $(G(f \circ g))^{-1} \circ G(f) \circ G(g) = \mathrm{Id}$. Let $G(g)^{-1}$ be the morphism $(G(f \circ g))^{-1} \circ G(f)$. It is a left inverse. Let us show that it is also a right inverse:
$G(f \circ g) \circ (G(f \circ g))^{-1} = \mathrm{Id}$. Then $G(f) \circ G(g) \circ (G(f \circ g))^{-1} = \mathrm{Id}$, and $G(g) \circ (G(f \circ g))^{-1} \circ G(f) = \mathrm{Id}$ and it is indeed a right inverse.

— If $g, f \circ g \in \mathcal{W}$. Then we use the same hint to show that $Gf$ is an isomorphism.

(iii) ${}^{\square}(J^{\square}) \subseteq \mathcal{W} \cap {}^{\square}(I^{\square})$.

As $J \subseteq I$, we automatically have that ${}^{\square}(J^{\square}) \subseteq {}^{\square}(I^{\square})$. From lemma 1.3.1 it is immediate that any morphism of ${}^{\square}(J^{\square})$ is a weak equivalence.

(iv) $I^{\square} = J^{\square} \cap \mathcal{W}$.

First, let us show that $J^{\square} \cap \mathcal{W} \subseteq I^{\square}$. Let $f \in J^{\square} \cap \mathcal{W}$. We introduce the following commutative square:

$$\begin{array}{ccc} M_0 & \xrightarrow{a} & X \\ {\scriptstyle \binom{h}{\iota_0}}\downarrow & & \downarrow{\scriptstyle f} \\ A \oplus I_0 & \xrightarrow[(b_1\ b_2)]{} & Y \end{array}$$

Then we have

$$b_1 \circ h + b_2 \circ \iota_0 = f \circ a. \tag{2.22}$$

As $f$ is a deflation, there exists $K$ such that $0 \to K \to X \to Y \to 0$ is a short exact sequence. If we consider the diagram:

$$\begin{array}{ccc} K & = & K \\ \downarrow & & \downarrow \\ K \oplus I_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ I_0 & \longrightarrow & Y \end{array}$$

as $f$ is a deflation, the morphism $K \oplus I_0 \to I_0$ is a deflation. They moreover have the same kernel (as $K \oplus I_0$ is a pullback), thus the short exact sequence

$$0 \to K \oplus I_0 \to K \oplus I_0 \to Y \to 0$$

exists. As $\begin{pmatrix} h \\ \iota_0 \end{pmatrix} \in I$, then there exist $M_1, M_0 \in \mathcal{M}$ such that

$$0 \to M_1 \to M_0 \to A \to 0$$

is a short exact sequence. We build the following morphism of short exact sequences:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
M_1 & \xrightarrow{\binom{c}{\mu}} & K \oplus I_0 & & \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} \tilde{g} & \alpha \\ 0 & 1 \end{smallmatrix}\right)} & & \\
M_0 & \xrightarrow{\binom{a}{\iota_0}} & X \oplus I_0 & & \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (f \ -b_2)} & & \\
A & \xrightarrow{b_1} & Y & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

From equation 2.22, we have that

$$b_1 \circ h = (f \ -b_2) \circ \begin{pmatrix} a \\ \iota_0 \end{pmatrix}.$$

Then, there exists $\begin{pmatrix} c \\ \mu \end{pmatrix}$ such that

$$\begin{pmatrix} a \circ k \\ \iota_0 \circ k \end{pmatrix} = \begin{pmatrix} \tilde{g} \circ c + \alpha \circ \mu \\ mu \end{pmatrix}.$$

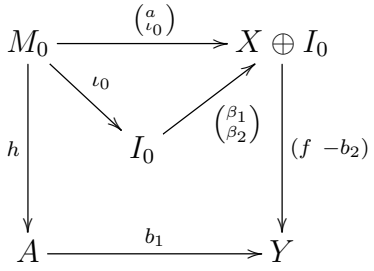
As $\tilde{g} \in \mathcal{M}^{\perp}$, the morphism $M_1 \to K \to X$ factorizes through an injective. We can suppose without loss of generality, that it factorizes through $I_0$. Then we have:

$$\begin{array}{ccccc}
M_1 & & \xrightarrow{\binom{c}{\mu}} & & K \oplus I_0 \\
& \searrow^{\alpha_A} & & & \\
{\scriptstyle k}\downarrow & & I_0 & \xrightarrow{\binom{\beta_1}{\beta_2}} & \downarrow{\scriptstyle \left(\begin{smallmatrix} \tilde{g} & \alpha \\ 0 & 1 \end{smallmatrix}\right)} \\
M_0 & & \xrightarrow{\binom{a}{\iota_0}} & & X \oplus I_0
\end{array}$$

with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \circ \alpha_A = \begin{pmatrix} a \circ k \\ \iota_0 \circ k \end{pmatrix}.$$

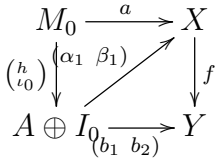
As $I_0$ is injective, and $k$ is an inflation, we can lift $\alpha_A$ to $\iota_0 : M_0 \to I_0$ such that $\iota_0 \circ k = \alpha_A$. Then,




Then, there exists $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} : A \to X \oplus I_0$ such that

$$\begin{pmatrix}a\\ \iota_0\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \circ \iota_0 + \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \circ h. \tag{2.23}$$

Then, we have a morphism $A \oplus I_0 \to X$ which makes the upper triangle commute:

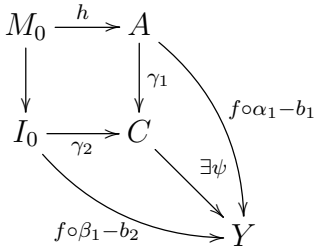



We have

$$(\alpha_1\ \beta_1) \circ \begin{pmatrix}h\\ \iota_0\end{pmatrix} = \alpha_1 \circ h + \beta_1 \circ \iota_0 = a$$

from equation 2.23.

The morphism $M_0 \to A \oplus I_0$ is an inflation. Let us introduce its cokernel $C$. Then we have the push-out:




Then there exists a unique $\psi : C \to Y$ such that:

$$\psi \circ \gamma_2 = f \circ \beta_1 - b_2 \tag{2.24}$$

and

$$\psi \circ \gamma_1 = f \circ \alpha_1 - b_1 \tag{2.25}$$

The push-out $C$ is exactly the cokernel of the morphism $M_1 \to M_0 \to I_0$. Then $C \simeq \mho M_1$. As $f \in J^{\square}$, if we still denote by $\psi$ the morphism from $\mho M_1$ to $Y$, then there exists $\zeta : \mho M_1 \to X$ such that $f \circ \zeta = \psi$. Then the equations 2.24 and 2.25 give respectively

$$f \circ \zeta \circ \gamma_1 = f \circ \alpha_1 - b_1$$

so

$$f \circ (\alpha_1 - \zeta \circ \gamma_1) = b_1$$

and

$$f \circ \zeta \circ \gamma_2 = f \circ \beta_1 - b_2$$

then

$$f \circ (\beta_1 - \zeta \circ \gamma_2) = b_2$$

and, the morphism

$$(\alpha_1 - \zeta \circ \gamma_1 \ \beta_1 - \zeta \circ \gamma_2) : A \oplus I_0 \to X$$

make both triangles commute.

Now, let us show the inverse inclusion. We have immediately $I^{\square} \subseteq J^{\square}$ since $J \subseteq I$. Let $f \in I^{\square}$. Let us show that, following the natural notations, $g \in \overline{\mathcal{M}}^{\perp}$ and $\tilde{g} \in \overline{\mathcal{M}}^{\perp}$. First, let $b : M \to Y$ be a morphism, with $M \in \mathcal{M}$. As $f \in I^{\square}$, there exists $\varphi : M \to X$ such that $f \circ \varphi = b$ then $g \circ b = g \circ f \circ \varphi$ and this morphism factorizes through an injective. This shows that $g \in \overline{\mathcal{M}}^{\perp}$. Second, Let $K$ be the kernel of $f$. Let $b : M \to K$. Let $N$ be an $\mathcal{M}$-approximation of $X$. We denote it by $a : N \to X$. As $a$ is an $\mathcal{M}$-approximation, there exists $h : M \to N$ such that $\tilde{g} \circ b = a \circ h$. Let $\iota_M : M \to I_M$ be the canonical injection. As $\begin{pmatrix} h \\ \iota_M \end{pmatrix}$ is an inflation, we introduce $C$ its cokernel. Let $(k \ u) : N \oplus I_M \to C$. If we put $(a \ 0) : N \oplus I_M \to X$, then there exists $r : C \to Y$ such that there is a morphism of short exact

sequences:

$$\begin{array}{ccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
M & & \xrightarrow{b} & & K \\
{\scriptstyle \binom{h}{\iota_M}}\downarrow & & & & \downarrow{\scriptstyle \tilde{g}} \\
N \oplus I_M & & \xrightarrow{(a\ 0)} & & X \\
{\scriptstyle (k\ u)}\downarrow & & & & \downarrow{\scriptstyle f} \\
C & & \xrightarrow{r} & & Y \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}$$

We are going to build an element of $I$ from $N \oplus I_M \to C$. We add $J$ the injective envelope of $N \oplus I_M$ ($J = I_M \oplus I_N$). Then the morphism

$$\begin{pmatrix} k & u \\ \iota_1 & \iota_2 \end{pmatrix} : N \oplus I_M \to C \oplus J$$

is in $I$. Then the following square is commutative

$$\begin{array}{ccc}
N \oplus I_M & \longrightarrow & X \\
\downarrow & & \downarrow \\
C \oplus J & \longrightarrow & Y
\end{array}$$

and as $f \in I^{\square}$, there exist $(\varphi_1\ \varphi_2) : C \oplus J \to X$ such that $f \circ \varphi_1 = r$ and $f \circ \varphi_2 = 0$ and

$$(\varphi_1 \circ k + \varphi_2 \circ \iota_1\ \varphi_1 \circ u + \varphi_2 \circ \iota_2) = (a\ 0).$$

Then we have

$$\begin{aligned}
\tilde{g} \circ b &= a \circ h \\
&= (a\ 0) \circ \begin{pmatrix} h \\ \iota_M \end{pmatrix} \\
&= (\varphi_1\ \varphi_2) \circ \begin{pmatrix} k & u \\ \iota_1 & \iota_2 \end{pmatrix} \circ \begin{pmatrix} h & \iota_M \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1 \circ k + \varphi_2 \circ \iota_1\ \varphi_1 \circ u + \varphi_2 \circ \iota_2 \end{pmatrix} \circ \begin{pmatrix} h \\ \iota_M \end{pmatrix} \\
&= (\varphi_1\ \varphi_2) \circ \begin{pmatrix} 0 \\ \iota_1 \circ h + \iota_2 \circ \iota_M. \end{pmatrix}
\end{aligned} \tag{2.26}$$

This shows that $\tilde{g} \circ b$ factorizes through an injective, thus $\tilde{g} \in \overline{\mathcal{M}}^{\perp}$.
Then we have shown that $I^{\square} = J^{\square} \cap \mathcal{W}$.

(v) The first required factorization is exactly corollary 1.3.2, part 1.

(vi) The second required factorization is exactly lemma 2.4.1.

□

As a consequence, we have directly the theorem of Quillen for Frobenius categories.

**Theorem 2.5.2.** *Let* Ho $\mathcal{E}$ *be the localization of the quasi-isomorphisms of* $\mathcal{E}$ *at the class* $\mathcal{W}$. *Let* mod $\overline{\mathcal{M}}$ *be the class of* $\overline{\mathcal{M}}$*-modules. There is an equivalence of categories*

$$\text{Ho } \mathcal{E} \simeq \text{mod } \overline{\mathcal{M}}.$$

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Dans cette thèse, nous traitons le cas $\tilde{D}_n$ en profondeur et démontrons quelques résultats additionnels aux travaux de Baur, Marsh et Torkildsen sur les réalisations géométriques des catégories amassées supérieures. Pour le cas $\tilde{D}_n$, on trouve la figure qui correspond à l'étude, on démontre la compatibilité entre le flip d'une $(m+2)$-angulation, et la mutation de carquois coloré. On trouve une bijection entre les objets $m$-rigides et chaque arc dit admissible, puis entre les objets amas-basculants et les $(m+2)$-angulations au théorème. De plus, on démontre la compatibilité entre la réduction d'Iyama-Yoshino, et le fait de couper le long d'un arc, qu'on définira formellement.

Nous démontrons ensuite qu'une catégorie exacte est une catégorie de préfibration au sens de Anderson-Brown-Cisinski, qui vérifie le théorème de Quillen, et une catégorie de Frobenius est munie d'une structure de modèle, compatible avec le passage à la catégorie stable, qui est triangulée.

We show that a subcategory of the $m$-cluster category of type $\tilde{D}_n$ is isomorphic to a category consisting of arcs in an $(n-2)m$-gon with two central $(m-1)$-gons inside of it. We show that the mutation of colored quivers and $m$-cluster-tilting objects is compatible with the flip of an $(m+2)$-angulation.

In this thesis, we study the geometric realizations of $m$-cluster categories of Dynkin types A, D, $\tilde{A}$ and $\tilde{D}$. We show, in those four cases, that there is a bijection between $(m+2)$-angulations and isoclasses of basic $m$-cluster tilting objects. Under these bijections, flips of $(m+2)$-angulations correspond to mutations of $m$-cluster tilting objects. Our strategy consists in showing that certain Iyama-Yoshino reductions of the $m$-cluster categories under consideration can be described in terms of cutting along an arc the corresponding geometric realizations. This allows to infer results from small cases to the general ones.

Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective and projective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective and projective objects, and stable under taking direct sums and summands. In this paper, $\mathcal{E}$ is equipped with the structure of a prefibration category with cofibrant replacements. As a corollary, we show, using the results of Demonet and Liu in [DL13], that the category of finite presentation modules on the costable category $\overline{\mathcal{M}}$ is a localization of $\mathcal{E}$. We also deduce that $\mathcal{E} \to \mathrm{mod}\overline{\mathcal{M}}$ admits a calculus of fractions up to homotopy. These two corollaries are analogues for exact categories of results of Buan and Marsh in [BM13], [BM12] (see also [Bel13]) that hold for triangulated categories.

If $\mathcal{E}$ is a Frobenius exact category, we enhance its structure of prefibration category to the structure of a model category (see the article of Palu in [**?**] for the case of triangulated categories). This last result applies in particular when $\mathcal{E}$ is any of the Hom-finite Frobenius categories appearing in relation to cluster algebras.