Random Perturbation of Certain Interacting Particle Systems related to Quantum Mechanics

Amirali Hannani

▶ **To cite this version:**

Amirali Hannani. Random Perturbation of Certain Interacting Particle Systems related to Quantum Mechanics. Mathematical Physics [math-ph]. Université Paris sciences et lettres, 2021. English. NNT : 2021UPSLD034 . tel-03663326

HAL Id: tel-03663326

https://theses.hal.science/tel-03663326

Submitted on 10 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PSL
UNIVERSITÉ PARIS

THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à l'Université Paris-Dauphine

Random Perturbation of Certain Interacting Particle Systems related to Quantum Mechanics

Soutenue par
Amirali HANNANI
Le 16 Dec 2021

École doctorale n°543
Ecole Doctorale SDOSE

Spécialité
Sciences

Composition du jury :

M. Mathieu LEWIN DR CNRS, Université Paris-Dauphine	*Président*
M. Herbert SPOHN Prof., Technical University of Munich	*Rapporteur*
M. Cédric BERNARDIN Prof., Université Côte d'Azur	*Rapporteur*
Mme. Serena CENATIEMPO RtdB, Gran Sasso Science Institute	*Examinateur*
M. Wojciech DE ROECK Prof., KU Leuven	*Examinateur*
M. François HUVENEERS MCF, Université Paris-Dauphine	*Examinateur*
Mme. Marielle SIMON CR INRIA, Université de Lille	*Examinateur*
M. Stefano OLLA Prof., Université Paris-Dauphine, IUF	*Directeur de thèse*

Dauphine | PSL
UNIVERSITÉ PARIS

Dauphine | PSL
UNIVERSITÉ PARIS

Université Paris-Dauphine PSL

École doctorale **SDOSE**

Unité de recherche **Centre de Recherche en Mathématiques de la Décision**, **Université Paris-Dauphine PSL**

Thèse présentée par **Amirali HANNANI**

En vue de l'obtention du grade de docteur de l'Université Paris-Dauphine PSL

Discipline **Mathématiques appliquées**

Random perturbation of certain interacting particle systems related to quantum mechanics

Perturbation aléatoire de certains systèmes de particules en interaction, liée à la mécanique quantique

Thèse dirigée par Stefano OLLA directeur

Composition du jury

Rapprteurs	Herbert SPOHN	prof. à Technical University of Munich
	Cédric BERNARDIN	prof. à l'Université Côte d'Azur
Examinateurs	Mathieu LEWIN	DR CNRS à l'Université Paris-Dauphine
	Wojciech DE ROECK	prof. à KU Leuven
	François HUVENEERS	MCF à l'Université Paris-Dauphine
	Serena CENATIEMPO	RtdB à Gran Sasso Science Institute
	Marielle SIMON	CR INRIA à l'Université de Lille
Directeur de thèse	Stefano OLLA	prof. à l'Université Paris-Dauphine, IUF

This dissertation is prepared in

CEREMADE
CEntre de REcherche en MAthémtiques de la Décision UMR 7534
Université Paris-Dauphine PSL Research University

Place du Maréchal De Lattre De Tassigny, 75016 Paris, France
http://ceremade.dauphine.fr

Dedication

To my family, Kimia, Aresou, Tina, Hami.
To my grand parents Mansour.
In memory of Reyhaneh and Mohammad Ebrahim.

Say not all are fighting, what use is my lone call for peace?

You're not one, but thousands; light your beacon

Rumi

Acknowledgment

First and foremost, I would like to express my sincere gratitude to Prof. Stefano Olla, my Ph.D. advisor, for his continuous support, invaluable advice, and endless patience. I have been fortunate to benefit from his guidance since my Master's studies, where he accepted to be my mentor. His wealth of knowledge, kindness, and constant encouragement made the past five years pleasant and fruitful.

I am deeply grateful to Prof. Herbert Spohn and Prof. Cédric Bernardin for accepting to be the reporter for my thesis. Prof. Spohn's comments during my Ph.D. were particularly insightful, and I find discussions with Prof. Bernardin enlightening. I would also like to thank other jury members: Prof. Mathieu Lewin, Prof. Wojciech De Roeck, Prof. François Huveneers, Prof. Marielle Simon, and Prof. Serena Cenatiempo for agreeing to evaluate this dissertation. In particular, François deserves special thanks: his office's door was always open to me, and I genuinely appreciate his help in every stage of the first research project.

During these years, I had the opportunity to participate in several seminars and conferences; this became possible thanks to the financial support of the ANR-15-CE40-0020-01 LSD grant of the French National Research Agency. I thank Stefano again for providing these opportunities.

We benefited from several insightful comments, discussions, and suggestions during these projects; I am particularly grateful to Paolo Antonelli, Sourav Chatterjee, Serena Cenatiempo, Benjamin Doyon, and Nikolay Tzvetkov in that regard.
I want to offer my special thanks to Arya Akhavan for reading this manuscript and pointing out many typos. I am also indebted to Jean Cazalis for his generous assistance regarding the French part of this thesis.

I acknowledge CEREMADE and Dauphine's ecole doctoral's support. I should thank all the CEREMADE's members for cultivating a productive atmosphere, which I benefited from during my thesis. Particularly, I should mention François Simenhaus and Julien Poisat for helping me be a better teacher; Alessandra for her kindness; Cyril Labbé, Paul Gassiat, and Nejla Nouaili for fruitful discussions; Cézare, Isabelle, and Marie Belle for their constant help; it is impossible not to thank all the current CEREMADE's Ph.D. students, postdocs, and ATERs as well as the ones who left during my studies. Especially, I am thankful to Adrien (specially for his help regarding the French part of the thesis), Arnaud, Changqing, Claudia,

Donato (life of the lab), Grégoire (taught me MCMC), Ivailo, Jean (thank you again for numerous discussions), Kathy, Laëtitia (for her help during the TP), Laurent, Quan, Reka, and Samuel.

Being a brown immigrant in France with limited French knowledge can be challenging. I could not have finished my thesis without my friend's support, making me feel less alien and helping me to fit in here. Therefore, I would like to thank them all here: Ali, Baran, Zeinab, Mathieu (for helping me with accommodation); Payam, Niloufar (for helping me survive French administration), Milad; Shiva R. (for helping me with my French; I also owe her a special thanks for her help with the French part of the thesis), Arya, Amirhossein; Marzie, Koohzad, Shiva T., Hamze; Mohammad, Masoud, Mohsen, Pooriya, Mona, Leila. Furthermore, I am grateful to Behnaz, Payam, and Shahab (for his valuable advice) as well. And a special thank goes to François, Charlotte, Esther, and Thibaut for constantly reminding me of beauty of the life.

And my biggest thanks to my family for all the support you have shown me through these years and before, to my mom, Arezou, for her endless kindness, to my father, Hami, for his hearth of gold, to Tina for her constant care. Last but not least, I want to thank Kimia, I can write pages describing her support, but just let me say: these years would have been colorless without you (and I promise, now I clean all the papers all over our studio).

Abstract

Abstract

This thesis is devoted to the study of random perturbation of two models of interacting particle systems related to quantum mechanics. First is a one-dimensional unpinned disordered chain of quantum harmonic oscillators, where a hydrodynamic limit in the hyperbolic scaling of time and space is proven; distribution of the elongation, momentum, and energy converges to the solution of the Euler equation in this scaling. Anderson localization decouples the mechanical and thermal energy, providing the closure of the macroscopic equation out of thermal equilibrium, and indicating that the temperature profile does not evolve in time. Decay of correlation-type phenomena facilitates dealing with the quantum nature of the system. To the best of our knowledge, this is among the first examples where one can prove the hydrodynamic limit for a quantum system rigorously.
In the second model, a mass conserving stochastic perturbation of a certain class of discrete non-linear Schrödinger equations is introduced, modeling the action of a heat bath at a given temperature. The corresponding Gibbs measure is the unique invariant measure of the dynamics, providing ergodicity and time-mixing properties. As an application, it is proved that in the one-dimensional cubic focusing case, the large time, continuum approximation, and the low-temperature limit of the solution converges to the steady wave of the continuous non-perturbed equation that minimizes the energy for a given mass.

Keywords: Hydrodynamic limits, Disordered quantum harmonic chains, Euler equation, Thermal conductivity, Anderson localization, Decay of correlation, Discrete non-linear Schrödinger equation, Non-linear Schrödinger equation, Stochastic processes, Hypoelliptic diffusion, Solitary waves, Large deviations

---

Résumé

Cette thèse est consacrée à l'étude de la perturbation aléatoire de deux modèles de systèmes de particules en interaction, liés à la mécanique quantique. Le premier est une chaîne désordonnée unidimensionnelle d'oscillateurs harmoniques quantiques, où une limite hydrodynamique dans l'échelle hyperbolique du temps et de l'espace

est prouvée ; l'élongation, la quantité de mouvement et l'énergie convergent vers la solution de l'équation d'Euler dans cette échelle.
Dans le deuxième modèle, une perturbation stochastique conservant la masse d'une certaine classe d'équations de Schrödinger non-linéaires discrètes est introduite, modélisant l'action d'un bain de chaleur à une température donnée. La mesure de Gibbs correspondante est la seule mesure invariante de la dynamique, fournissant des propriétés d'ergodicité et de mélange temporel. En guise d'application, on étudie la limite de grand temps, l'approximation du continuum, et la limite de basse température dans le cas cubique unidimensionnelle focalisant.

Mots clés : Limites hydrodynamiques, Chaînes harmoniques quantiques désordonnées, Équation d'Euler, Conductivité thermique, Localisation d'Anderson, Décroissance de la corrélation, Équation de Schrödinger non-linéaire discrète, Équation de Schrödinger non-linéaire, Processus stochastiques, Diffusion hypoelliptique, Ondes solitaires, Grandes déviations

Preface

This dissertation concerns two distinct research projects. They share the same spirit in the following sense: in both of these problems, an interacting particle system related to quantum mechanics is perturbed by some "random" noise/disorder. These perturbations change the nature of the dynamics and lead to novel and interesting phenomena.

The first project concerns a hydrodynamic limit for a disordered chain of quantum harmonic oscillators. The second one is about a stochastic perturbation of a certain class of discrete non-linear Schrödinger equations and its applications. A detailed introduction to these problems, including the obtained results, brief reviews of some related works, descriptions of our contributions, and collections of open problems, is given in Section 1.1, and Section 1.2 of Chapter 1, respectively. Section 1.1 contains a part concerning Anderson localization providing a short introduction to this subject which can be skipped completely by readers familiar with the matter (except Lemma 1.1.2).

Treatment of the disordered chain constitutes the second chapter of this thesis, which is extracted from [125]: Hannani, A. Hydrodynamic Limit for a Disordered Quantum Harmonic Chain. Commun. Math. Phys. 390, 349–423 (2022). Chapter 3 is devoted to the project concerning discrete non-linear Schrödinger equations. This is extracted from arXiv:2109.01389 [math-ph] [127] which is a joint work with Prof. Stefano Olla.
One can read Chapters 2 and 3 independently; all the notations are recalled when needed. The same rule applies to Sections 1.1 and 1.2 of Chapter 1.
A French summary (résumé) of this dissertation is provided before the first Chapter. This "résumé" is a summarized version of Chapter 1.

Contents

Acknowledgment **vi**

Abstract **viii**

Preface **x**

Résumé **xiv**

I Chaîne désordonnée . . . xv
I.I L'hydrodynamique en quelques mots . . . xv
I.II Modèle et résultats . . . xviii
I.III Le phénoménologie et l'idée de la preuve . . . xxiv
I.IV Autres commentaires . . . xxvii
II NLS . . . xxviii
II.I NLS en un coup d'œil . . . xxviii
II.II Approche de la mécanique statistique et mesure de Gibbs . . . xxxiv
II.III Bruit stochastique et résultats . . . xxxvi

1 Introduction **1**

1.1 Disordered Chain . . . 1
1.1.1 Hydrodynamic limits in a nutshell . . . 1
1.1.2 Model and Results . . . 6
1.1.3 Phenomenology and idea of the proof . . . 13
1.1.4 Localization . . . 22
1.1.5 Further comments, and open problems . . . 28
1.2 NLS . . . 30
1.2.1 NLS in a glance . . . 30
1.2.2 Statistical mechanics approach and Gibbs measure . . . 38
1.2.3 Stochastic noise and results . . . 42
1.2.4 Open problems, and future direction . . . 49

2 Hydrodynamics **53**

2.1 Model Description and Results . . . 53
2.2 Preliminary Bounds . . . 59
2.2.1 Hamiltonain diagonalization . . . 59
2.2.2 Density operator diagonalisation . . . 64
2.2.3 Ensemble average . . . 67

2.3 Wave Equation . . . 72
2.4 Energy at Time Zero . . . 83
2.4.1 Strong law of large numbers for energy at time zero . . . 83
2.4.2 Thermal equilibrium . . . 91
2.5 Energy Evolution . . . 92
2.5.1 Mechanical energy . . . 92
2.5.2 Thermal energy and localization . . . 94
2.5.3 Decomposition . . . 104
2.5.4 From C^1 to C^0 . . . 106
2.A Properties of the Limiting Function . . . 108
2.B Alternative Coordinates . . . 117
2.C Average of Bosonic Operators . . . 122

3 Stochastic DNLS **125**
3.1 Stochastic Dynamics . . . 125
3.2 Large Scale Limit and Main Result . . . 129
3.2.1 Preliminaries about periodic cubic non-linear Schrödinger equation . . . 129
3.2.2 Stochastic perturbation of discrete focusing NLS . . . 131
3.3 Discrete "Soliton" . . . 134
3.4 Large Deviation Estimates . . . 139
3.A LD estimates for μ_m^n . . . 143
3.A.1 More Precise LD estimates . . . 146
3.B Hypoellipticity . . . 152
3.C Discrete Gagliardo-Nirenberg . . . 155

Bibliography **159**

Résumé

I Chaîne désordonnée

I.I L'hydrodynamique en quelques mots

Considérons un système macroscopique, un récipient de gaz, par exemple. Supposons que nous soyons intéressés par la description de l'évolution de ce système. Acceptant les lois fondamentales de la physique non-relativiste, qu'il s'agisse de la *mécanique classique* ou *quantique*, on peut modéliser ce système comme une collection de particules évoluant selon ces règles. Sur l'échelle microscopique, si nous faisons un zoom, disons dans une fenêtre de 10^{-8} à 10^{-10} m de long, et que nous suivons un couple de particules pendant un temps très court, leur trajectoire confirme la validité de notre modèle de manière satisfaisante. Cependant, la résolution simultanée de l'équation d'évolution pour toutes les particules ne semble pas utile, ni possible dans de nombreux cas, même numériquement [1].

D'autre part, la dynamique macroscopique de ce système peut être décrite par des équations autonomes phénoménologiques telles que l'équation d'Euler ou les équations de Navier-Stokes. Ces équations ont été comprises depuis des siècles à partir de considérations sur le continuum. Cependant, la dérivation exacte de ces équations à partir de lois fondamentales est extrêmement difficile.

État d'équilibre

Afin de mieux comprendre le lien entre les images microscopiques et macroscopiques, nous suivons l'approche de la physique statistique initiée par Boltzmann et Maxwell (cf. [40], [170]). Pour le moment, nous supposons que la dynamique sous-jacente est donnée par la mécanique newtonienne. Tout d'abord, nous examinons l'état d'équilibre du système. Cet état peut être caractérisé par certains paramètres *macroscopiques* correspondant aux caractéristiques thermodynamiques du système, noté $\{\mathcal{P}_\alpha\}_{\alpha=1}^k$. Par exemple, dans notre cas (récipient de gaz), on peut penser à la température, la densité et la pression, comme les quantités macroscopiques souhaitées. Remarquez que ces paramètres sont conjugués aux quantités localement conservées du système.

[1] En gros, la mémoire nécessaire pour stocker l'espace de phase d'un système classique à 10^{23} particules est cent fois supérieure à la capacité de stockage de données mondiale actuelle.

Équilibre local

Bien que la connaissance des états d'équilibre puisse être intéressante dans de nombreux systèmes, en pratique, les caractéristiques d'équilibre de notre gaz varient lentement à l'échelle macroscopique. Par conséquent, nous introduisons la notion d'équilibre local. Supposons que notre gaz soit confiné dans la boîte $\Lambda = [0, L]^d$; considérons un point macroscopique $u \in \Lambda$, soit $B(u)$ un voisinage de u. $B(u)$ doit être suffisamment petit par rapport au volume macroscopique du système, et suffisamment grand à l'échelle microscopique, c'est-à-dire qu'il doit être suffisamment grand pour être considéré comme un système «macroscopique»(bien que petit). Pour notre système, on pourrait penser que cette échelle est de l'ordre de 10^{-4}m, c'est-à-dire l'échelle typique où varierait une caractéristique macroscopique du système telle que la température. On pourrait s'attendre à ce que dans chaque voisinage $B(u)$, le système atteigne un état d'équilibre, décrit par les caractéristiques thermodynamiques susmentionnées $\{\mathcal{P}_\alpha(u)\}_{\alpha=1}^k$ en fonction du point macroscopique u, où ces fonctions varient de manière régulière dans l'espace.
Nous nous attendons à ce que l'évolution de cet état d'équilibre local soit négligeable à l'échelle de temps microscopique[2], puisque cet état est localement invariant sous la dynamique, grâce à la construction. Cependant, si nous observons notre récipient de gaz dans une fenêtre de temps macroscopique, nous nous attendons à ce que l'énergie, la quantité de mouvement et les particules soient transportées sur des distances macroscopiques. En fait, si l'équilibre local est maintenu, au temps macroscopique t, on peut s'attendre à ce que l'image de l'équilibre local évolue et que le système puisse être décrit par de nouveaux paramètres : $\{\mathcal{P}_\alpha(u, t)\}_{\alpha=1}^k$, où ces champs varient régulièrement dans le temps en suivant le système approprié d'équations différentielles partielles autonomes, c'est-à- dire l'équation hydrodynamique.

Espace-temps mise à l'échelle

Avant de poursuivre, nous soulignons que la relation entre les images microscopiques et macroscopiques implique une mise à l'échelle spatio-temporelle. Prenons un paramètre d'échelle n, qui va à l'infini. L'espace et le temps macroscopiques, y et t, sont alors donnés par $ny = x$, $t = \vartheta(n)\tau$, où x et τ désignent respectivement l'espace et le temps microscopiques. Nous nous attendons à ce que l'image précédente devienne "exacte" dans la limite où $n \to \infty$. Remarquez que le choix de $\vartheta(n) = n$, correspond à l'échelle hyperbolique où nous nous attendons à obtenir l'équation d'Euler. En revanche, certains phénomènes tels que la conductivité thermique sont généralement censés se produire dans une échelle de temps plus lente. Pour observer ces phénomènes, nous devons redimensionner le temps dans une échelle de temps diffusive $\vartheta(n) = n^\alpha$, avec $\alpha = 2$ ou dans une échelle de temps super-diffusive ($\alpha < 2$).

[2]Par exemple, on pourrait considérer l'échelle de temps microscopique comme le temps libre moyen entre les collisions de particules.

Ergodicité

Bien que l'idée d'équilibre local remonte à C.Morrey (cf. [177]) dans les années cinquante, le traitement rigoureux de ce problème pour les systèmes hamiltoniens est encore inaccessible. La principale difficulté consiste à prouver le fait que l'équilibre local est maintenu (*hypothèse d'équilibre local*). Le consensus est que l'ergodicité du système est nécessaire à cette fin, et prouver l'ergodicité pour les systèmes purement hamiltoniens semble hors de portée pour le moment. Soulignons la conviction que l'ergodicité est une partie essentielle de ce tableau avec les citations suivantes :

« *Despite many efforts, this program has not been completely achieved for Hamiltonian systems where particles evolve deterministically according to Newton's equations, mainly due to the lack of good ergodic properties of the system.* »
du *Scaling Limits of Interacting Particle Systems* par C. Kipnis, et C. Landim (cf.[137] Page un).

« *Unfortunately, despite intense efforts, for many degrees of freedom deterministic chaos remains a poorly understood subject. I am convinced that deterministic chaos on the small scale is needed in order to have the kind of large scale dynamics we actually see.* »
du *Large Scale Dynamics of Interacting Particles* par H. Spohn (cf. [212] Page deux).

Afin de clarifier les concepts susmentionnés, nous donnons un exemple concret (chaîne d'oscillateurs (an-)harmoniques classiques) dans la Section 1.1.1. Dans cet exemple, nous définissons rigoureusement l'état d'équilibre. Nous apportons une définition de l'équilibre local dans (1.1.8) et (1.1.9). Remarquez que, mathématiquement, la notion d'équilibre local a plus de sens dans la limite d'échelle. Enfin, nous esquissons la limite hydrodynamique en supposant l'équilibre local dans (1.1.10) et (1.1.12). Nous ne présentons pas cet exemple ici par souci de brièveté, car il n'est pas crucial pour notre propos, et il ne fait qu'expliquer les concepts abstraits que nous avons exposés ci-dessus.

La preuve de l'équilibre local est penseé être liée à l'ergodicité du système. D'ailleurs, Olla, Varadhan et Yau ([186]) ont établi que l'ergodicité du système infini est suffisante pour prouver la limite hydrodynamique. Plus précisément, dans notre cas, le système infini est ergodique, si chaque probabilité "régulière" (densité d'entropie finie), invariante en translation et stationnaire ν est une combinaison convexe de mesures de Gibbs $d\mu_{\tau,\mathfrak{p},\beta}$ (1.1.4), de manière équivalente par équivalence d'ensemble cela signifie que les seules quantités conservées locales, invariantes par translation, pour la dynamique infinie sont données par l'énergie, la quantité de mouvement et la densité.
Bien que l'on s'attende à ce que le fait d'avoir un potentiel non-linéaire dans l'hamiltonien puisse fournir une telle ergodicité, prouver ceci est difficile (cf. [102] de Fritz, Funaki et Lebowitz pour plus de détails).
L'une des principales stratégies pour obtenir l'ergodicité consiste à ajouter du bruit

stochastique (échange aléatoire de vitesses, changement aléatoire du signe de la vitesse, etc. par exemple) pour éliminer toutes les quantités conservées (possibles), sauf celles souhaitées (cf. [186],[49], [144] comme exemples de cette stratégie). Le principal outil mathématique permettant de contrôler l'évolution macroscopique dans ces exemples est la méthode de l'entropie relative introduite dans [226] par Yau (cf. [186] également). Il est intéressant de mentionner que dans certaines limites, lorsque nous avons la conservation de l'énergie, cette méthode ne fonctionne pas et les méthodes basées sur les distributions de Wigner peuvent être exploitées (cf. [146]).

Notes complémentaires et commentaires. Pour prouver les limites hydrodynamiques, au lieu de perturber la dynamique hamiltonienne avec un bruit conservateur, nous pouvons considérer la dynamique microscopique stochastique plutôt que la dynamique hamiltonienne. Dans cette approche, la dynamique sous-jacente est supposée être complètement aléatoire, par exemple cf. [119], [226], [137] et les références y afférentes, [212] Section II et les références y afférentes, [83] et les références y afférentes, notamment [138], [198]. Ces modèles peuvent ne pas sembler aussi "fondamentaux" que les modèles hamiltoniens. Cependant, ils sont capables de décrire une gamme assez vaste de phénomènes. Dans ce cas, nous voudrions mentionner que le travail pionnier de Guo, Papanicolaou et Varadhan [119], inspiré par Fritz ([101]), est particulièrement important. Ils suivent l'évolution de l'entropie relative entre la loi du processus et l'équilibre global. Cette approche a été généralisée par Yau [226] en prenant l'entropie relative par rapport à l'équilibre local. En citant Yau : « On pense que la principale difficulté dans l'étude de l'hydrodynamique est l'absence d'une norme stable contrôlable mathématiquement », nous aimerions mentionner que récemment d'autres normes utiles ont été exploitées à cette fin, telles que la norme H_{-1} et la distance de Wasserstein (cf. [117]).

I.II Modèle et résultats

Cette section est consacrée à l'explication du modèle principal de la première partie de cette thèse, de sa signification et des résultats.

Jusqu'à présent, la situation peut être résumée comme suit : l'obtention de l'évolution macroscopique des quantités conservées et de leurs courants correspondants pour un système "physique" à partir de sa dynamique microscopique, également connue sous le nom de limite hydrodynamique, est un problème mathématique difficile. L'un des cas les plus intéressants de ce programme est celui où la dynamique microscopique sous-jacente est donnée par les théories "fondamentales", c'est-à-dire soit la dynamique newtonienne/hamiltonienne, soit la dynamique de Schrödinger/Heisenberg pour les systèmes classiques et quantiques, respectivement. Dans les systèmes classiques, la preuve mathématique reste difficile à obtenir en raison des difficultés à prouver les propriétés ergodiques du système. De plus, il est généralement admis que l'ergodicité est un élément omniprésent de ce tableau.

La situation n'est pas différente lorsque la dynamique sous-jacente est donnée par la mécanique quantique. En fait, à notre connaissance, il n'existe aucun exemple où l'on prouve rigoureusement une limite hydrodynamique pour un système quantique déterministe, sauf dans [181], où est adaptée aux systèmes quantiques la méthode de l'entropie relative. Cependant, leur travail est basé sur une hypothèse d'ergodicité (cf. hypothèse III de [181]), qui est assez difficile à prouver pour les systèmes physiques.

L'un des *principaux objectifs* et l'une des *principales nouveautés* de ce travail est de prouver la limite hydrodynamique pour un système quantique simple en interaction : une chaîne unidimensionnelle non épinglée d'oscillateurs harmoniques quantiques avec des masses aléatoires. La principale nouveauté vient du fait que ce modèle est parmi les premiers exemples où l'on peut prouver la limite hydrodynamique d'un système quantique de façon rigoureuse. En effet, nous prouvons qu'à partir d'un état d'équilibre local (locally Gibbs state), après une remise à l'échelle hyperbolique du temps et de l'espace, la distribution de l'élongation, de la quantité de mouvement et de l'énergie convergent vers la solution de l'équation d'Euler. De plus, notre résultat indique que le profil de température n'évolue dans aucune échelle spatio-temporelle ; en particulier, le coefficient de diffusion thermique disparaît.
Un résultat similaire a été obtenu pour l'homologue classique de ce système dans [35]. Ces deux modèles partagent une caractéristique surprenante : ils ne sont pas *ergodiques* ; en fait, ils sont complètement intégrables avec un ensemble complet de quantités conservées. Cependant, nous sommes toujours capables de fermer l'équation macroscopique avec trois quantités conservées, à savoir la quantité de mouvement, l'élongation et l'énergie, hors de l'équilibre thermique. Nous tenons à souligner que ce résultat semble contre-intuitif (en raison du manque d'ergodicité), si l'on croit au tableau général bien établie que nous avons décrite dans la Section I.I. Nous soulignerons plus tard les différences entre les systèmes classique et quantique (cf. Section I.III).

Nous obtenons le résultat mentionné ci-dessus grâce à deux phénomènes principaux dans cette chaîne. Le premier est la *localisation d'Anderson*, qui dissocie l'énergie mécanique et l'énergie thermique, ce qui permet de fermer l'équation de l'énergie ; un phénomène similaire se produit également dans la chaîne classique [35]. Le second phénomène découle simplement de la nature quantique du système : il est similaire à une sorte de phénomène de *décroissance de la corrélation*, qui nous permet de contourner les difficultés découlant du fait que les états de Gibbs ne sont pas des produits en raison de la nature quantique du système.

Nos résultats mettent en lumière les propriétés de transport de la chaîne désordonnée non épinglée à l'échelle de l'espace-temps hyperbolique ; le transport de l'énergie mécanique est balistique, tandis que le transport de l'énergie thermique et des autres quantités conservées sera supprimé grâce à la **localisation d'Anderson**. En outre, concernant l'autre échelle, nous observons que le transport de l'énergie thermique sera supprimé à toute échelle de temps plus grande.

Modèle et résultats

Formellement, ce système peut être décrit par l'hamiltonien suivant :

$$H_n = \frac{1}{2}\sum_{x=1}^{n}\Big(\frac{p_x^2}{m_x} + (q_{x+1}-q_x)^2\Big), \tag{i}$$

où q_x est l'opérateur de position de la particule x, p_x est l'opérateur d'impulsion correspondant, avec $[q_x, p_y] = i\delta_{xy}$, où l'on prend $\hbar = 1$. De plus, m_x désigne la masse de la particule x, où elles seront prises comme des variables aléatoires i.i.d., définies sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Nous désignons l'espérance par rapport à $\mathbb{P}$ par $\mathbb{E}$ et $\mathbb{E}(m_x) = \bar{m}$; de plus, nous désignons la loi de distribution des masses par μ. Enfin, nous avons des conditions aux bords libres : $q_0 = q_1$ et $q_n = q_{n+1}$.
Nous aimerions mentionner que ce modèle pourrait être compris comme un modèle des degrés de liberté de vibration dans le réseau cristallin[3] (cf. [141] Chapitre 2, [13] Chapitre 23, [180]). En particulier, cela signifie que les particules sont marquées par des points du réseau et qu'elles sont *distinguables*, même si toutes les masses sont similaires. La distinguabilité devient encore plus évidente pour la chaîne désordonnée, puisque toutes les particules ont des masses différentes, presque sûrement.

L'évolution temporelle est donnée par la dynamique de Heisenberg générée par H_n, c'est à dire :

$$\dot{p}_x = i[H_n, p_x] = (\Delta q)_x, \quad \dot{q}_x = i[H_n, q_x] = \frac{p_x}{m_x}, \tag{ii}$$

où Δ est le laplacien discret défini dans (2.1.8).

Cette dynamique possède n quantités conservées ; cependant, nous ne nous intéressons qu'aux trois principales suivantes : l'énergie totale : $H_n := \sum_{x=1}^{n} e_x$, la quantité de mouvement totale : $\sum_{x=1}^{n} p_x$, et l'élongation totale : $\sum_{x=1}^{n-1} r_x$, où nous avons noté :

$$r_x = q_{x+1} - q_x, \qquad e_x = \frac{1}{2}\left(\frac{p_x^2}{m_x} + r_x^2\right). \tag{iii}$$

Remarquez que l'énergie totale H_n et la quantité de mouvement totale $\sum_{x=1}^{n} p_x$ sont réellement conservées. En revanche, l'élongation totale $\sum_{x=1}^{n-1} r_x$ est conservée localement, et la conservation globale de l'élongation se rompt à la frontière en raison des conditions aux bords $q_0 = q_1$ et $q_n = q_{n+1}$. Nous discutons des autres quantités conservées dans la Remarque I.6.

Tout d'abord, nous laissons la chaîne se trouver dans un état de Gibbs local correspondant aux quantités conservées susmentionnées. Cet état est hors équilibre

[3] On pourrait imaginer ces particules comme des ions/atomes dans un isolant, interagissant avec leurs voisins les plus proches. En prenant les masses aléatoires, nous pouvons imaginer que nous avons un *alliage aléatoire* où nous avons une combinaison de différents ions/atomes dans le réseau. En particulier, en ayant cette image à l'esprit, il devient clair que bien que chacun de ces ions puisse être un boson/fermion en tant que particule quantique, ils sont distinguables.

thermique avec un profil de température régulier $\beta \in C^0([0,1])$ et hors équilibre mécanique avec un profil régulier de quantité de mouvement et d'élongation $\bar{p}, \bar{r} \in C^1([0,1])$, avec $\bar{r}(0) = \bar{r}(1) = 0$. De plus, nous supposons que le centre de masse macroscopique de la chaîne est fixe, c'est-à- dire que $\int_0^1 \bar{p}(y)dy = 0$ (Nous discutons de cette hypothèse dans l'Annexe 2.B et la Remarque 2.B.1). L'opérateur de densité de cet état est noté $\rho^n_{\beta,\bar{p},\bar{r}}$, et est défini dans (2.1.9). Ensuite, nous laissons la chaîne évoluer dans le temps, selon l'équation de Heisenberg et désignons par $r_x(t), p_x(t),$, $e_x(t)$ la solution de cette équation, c'est-à-dire la solution aux lois de conservation locales suivantes correspondant à nos quantités conservées :

$$\begin{aligned} \dot{r}_x(t) &= \frac{p_{x+1}(t)}{m_{x+1}} - \frac{p_x(t)}{m_x}, \quad \dot{p}_x(t) = r_x(t) - r_{x-1}(t), \\ \dot{e}_x(t) &= \frac{p_{x+1}(t)r_x(t) + r_x(t)p_{x+1}(t)}{2m_{x+1}} - \frac{p_x(t)r_{x-1}(t) + r_{x-1}(t)p_x(t)}{2m_x}, \end{aligned} \tag{iv}$$

où dans la deuxième équation, nous avons souligné le fait que $p_{x+1}r_x \neq r_x p_{x+1}$.
Étant donné le profil de température $\beta \in C([0,1])$ et la distribution des masses μ, le profil d'énergie thermique macroscopique est noté par $\mathrm{f}^\mu_\beta : [0,1] \to \mathbb{R}$, défini dans (2.4.5). Nous discutons de cette fonction dans la Section 2.4.1 et l'Annexe 2.A. Soulignons qu'a priori, d'après la définition, il n'est pas clair si cette fonction est "locale". Cependant, dans la Proposition 2.A.1.2 nous prouvons que $\mathrm{f}^\mu_\beta(y) = \mathrm{f}^\mu(\beta(y))$, où $\mathrm{f}^\mu : (0,\infty) \to \mathbb{R}$ est la fonction correspondante en équilibre thermique (cf. (2.4.35) pour la définition, voir également les Remarques I.5, I.1).
En conséquence, en désignant par $\bar{m} = \mathbb{E}(m_x)$, où $\mathbb{E}$ désigne l'espérance par rapport à μ, les profils macroscopiques $\mathrm{r}(y,t)$, $\mathrm{p}(y,t)$ et $\mathrm{e}(y,t)$ évoluent selon les lois de conservation suivantes avec des conditions aux bords appropriées :

$$\begin{aligned} &\partial_t \mathrm{r}(y,t) = \frac{1}{\bar{m}}\partial_y \mathrm{p}(y,t), \quad \partial_t \mathrm{p}(y,t) = \partial_y \mathrm{r}(y,t), \quad \partial_t \mathrm{e}(y,t) = \frac{1}{\bar{m}}\partial_y(\mathrm{r}(y,t)\mathrm{p}(y,t)), \\ &\mathrm{r}(y,0) = \bar{r}(y), \qquad \mathrm{p}(y,0) = \bar{p}(y), \qquad \mathrm{e}(y,0) = \frac{1}{2}\Big(\frac{\bar{p}(y)^2}{\bar{m}} + \bar{r}(y)\Big) + \mathrm{f}^\mu_\beta(y), \\ &\mathrm{r}(0,t) = \mathrm{r}(1,t) = 0. \end{aligned} \tag{v}$$

Remarquez que la quantité de mouvement totale $\int_0^1 \mathrm{p}(y,t)dy$, est conservée par l'équation d'évolution (v), grâce aux conditions aux bords $\mathrm{r}(0,t) = \mathrm{r}(1,t) = 0$. Ce fait justifie par ailleurs l'hypothèse $\int_0^1 \bar{p}(y)dy = 0$.

Le principal résultat de ce manuscrit est que, après une remise à l'échelle hyperbolique du temps et de l'espace, la densité empirique d'élongation, de quantité de mouvement et d'énergie, c'est-à-dire r, p, e, moyennée sous l'état localement Gibbs $\rho^n_{\bar{p},\bar{r},\beta}$, converge vers la solution de l'équation macroscopique (v). L'énoncé précis du résultat est donné dans le Théorème 2.1.1.

Remarque I.1. *Remarquez que dans le cas classique pour $\tilde{\beta} \in (0,\infty)$, $\mathrm{f}^\mu(\tilde{\beta}) = \tilde{\beta}^{-1}$, alors que dans le cas quantique on peut observer que cette fonction dépend de tous les*

moments de la distribution de probabilité des masses, μ. Grâce à cette observation, il est intéressant de mentionner que dans le cas classique, les équations macroscopiques ne reflètent que l'espérance des masses $\mathbb{E}(m_x)$, alors que dans le cas quantique, la distribution entière apparaît dans l'image macroscopique à travers la fonction f^{μ}*. Ce fait pourrait être contre-intuitif, dans le sens où pour deux distributions des masses avec une moyenne similaire, nous pouvons observer une énergie thermique moyenne différente.*

L'histoire du modèle

Les oscillateurs harmoniques ont été utilisés pour modéliser les propriétés thermodynamiques des solides et des fluides. En équilibre, le modèle des oscillateurs harmoniques peut prédire la chaleur spécifique des cristaux à haute température comme dans la loi de Dulong-Petit (cf. [89], [151] Section VI). Ces modèles peuvent être utilisés pour prédire la capacité thermique des solides à plus basse température, comme dans le modèle d'Einstein en considérant la nature quantique des oscillateurs. Par la suite, en prenant en compte les interactions comme dans le modèle de Debye, nous pouvons également prédire la capacité thermique à basse température (cf. [81], [91], Section VI de [151]).

Bien que ces prédictions concordent de manière satisfaisante avec l'expérience, les modèles purement harmoniques[4] tant quantiques que classiques ne sont pas suffisamment complexes pour prédire les phénomènes hors-équilibre tels que la conduction thermique.
Expérimentalement, depuis les travaux de Fourier en 1822 [21], on s'attend à ce que le transfert de chaleur dans le matériau soit diffusif [5], et satisfasse la loi de Fourier :

$$J(T) = -\kappa(T)\nabla T; \qquad c_v(T)\frac{\partial T}{\partial t} = \nabla.(\kappa(T)\nabla T), \tag{vi}$$

où $\kappa(T)$ est le coefficient de diffusion, correspondant à la conductivité du matériau, et $c_v(T)$ est la chaleur spécifique par unité de volume. $T : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ est le profil de température du matériau à l'instant $t \in \mathbb{R}_+$, et J est le flux thermique, c'est-à-dire la quantité de chaleur transportée à travers une unité de surface par unité de temps.
Rider, Lebowitz et Lieb ([201]), ont étudié une chaîne classique unidimensionnelle d'oscillateurs harmoniques de masses identiques. Ils ont préparé la chaîne en contact avec deux bains de chaleur modélisés par des opérateurs de type Fokker-Planck aux frontières avec des températures respectives T_L et T_R. Ils ont observé qu'à l'état stationnaire, le flux thermique est proportionnel à la différence de température $|T_R - T_L|$, plutôt qu'au gradient de température $|T_L - T_R|/n$ [6] comme cela est

[4] Par «modèles purement harmoniques», nous entendons modèles où toutes les masses sont égales et où l'interaction est donnée par des formes quadratiques.

[5] Nous parlons ici du comportement "général" des solides, et nous ne sommes pas concernés par les contre-exemples techniques ou exotiques.

[6] Ici n représente le nombre de particules dans la chaîne.

prescrit par la loi de Fourier. Cela indique que l'énergie se propage de manière balistique plutôt que diffuse, et que la conductivité thermique est proportionnelle à la taille du système. Ce résultat confirme que les modèles (purement) harmoniques ne sont pas assez complexes pour modéliser le transfert de chaleur, ce qui a été compris depuis les travaux de Peierls [189], [190].

Dans les modèles purement harmoniques, les phonons se propagent de manière balistique, et la conductivité thermique diverge en fonction de la taille du système. Afin d'enrichir le modèle pour refléter les observations physiques, nous devons prendre en compte la diffusion des phonons. Ceci peut être fait en ajoutant des caractéristiques supplémentaires au modèle (anharmonicité, bruit stochastique ou impuretés). Bien qu'il existe une littérature considérable concernant les propriétés thermiques des chaînes d'oscillateurs (plus d'études numériques et d'arguments heuristiques et moins de résultats rigoureux), nous n'avons pas l'intention de les passer en revue ici. Nous nous référons à [41], [155] pour des revues générales. Nous apportons également des exemples concernant les mécanismes ci-dessus dans la Section 1.1.2, juste pour donner une image générale.

Ici, nous nous intéressons principalement au cas des harmoniques désordonnées, où les effets de diffusion des phonons peuvent être observés en ajoutant des impuretés. Pour modéliser les impuretés, on peut considérer des chaînes harmoniques désordonnées (classiques). Ce cas a notamment été étudié par Casher et Lebowitz dans [63], et Rubin et Greer dans [203]. Par désordonné, nous entendons que les masses des particules sont des variables aléatoires i.i.d.. En présence d'un potentiel de pinning, cette chaîne sera un isolant parfait, et la conductivité thermique disparaît lorsque $n \to \infty$ (ceci est démontré rigoureusement dans [33]). Cependant, en l'absence de pinning, en conséquence de la conservation de la quantité de mouvement, nous avons un effet surprenant comme il est observé dans [85], [86] : la conductivité thermique dépend des conditions aux bords et des propriétés spectrales du bain de chaleur.
Dans [63], les auteurs ont étudié la chaîne harmonique classique désordonnée en contact avec des réservoirs de chaleur modélisés par des processus d'Ornstein-Uhlenbeck à des températures T_L, T_R. Ils en ont déduit que la conductivité thermique disparaît et que le courant thermique à l'état stationnaire décroît comme $\mathbb{E}(J_n) \sim (T_L - T_R)n^{-\frac{3}{2}}$, où n désigne la taille du système et $\mathbb{E}$ désigne l'espérance par rapport aux masses. [6] contient une preuve rigoureuse de cette dernière affirmation. [203] est examiné un problème similaire, où est modélisé un bain de chaleur avec des chaînes infinies d'oscillateurs. Cette étude a été rendue rigoureuse dans [222].
Enfin, nous pouvons mentionner [2] et ses références, plus particulièrement [180] comme exemples d'étude de chaînes harmoniques quantiques désordonnées. Dans ce corpus de travaux, la localisation dynamique de la chaîne en termes de borne de Lieb-Robinson à vitesse nulle a été étudiée. Il est à noter que ces modèles sont similaires au cas pinné et qu'ils ne présentent pas la divergence de la longueur de localisation dans la partie basse du spectre (nous en discutons dans la section suivante).

I.III Le phénoménologie et l'idée de la preuve

Équilibre thermique

Rappelons l'hamiltonien (i), prenons $m_x = 1$ pour tout x, on appelle cette chaîne une «chaîne propre». Pour une chaîne propre, en équilibre thermique (en prenant $\beta(y) = \beta$ pour $y \in [0,1]$) le résultat du Théorème 2.1.1 est toujours vrai. La distribution de l'élongation, de la quantité de mouvement et de l'énergie, moyennée sous l'état de Gibbs local, convergent vers la solution de (v) dans l'échelle hyperbolique, où l'on doit modifier les conditions aux bords et la constante $\bar{m} = 1$ dans (v), en conséquence. Remarquez qu'ici notre état de Gibbs est en équilibre thermique avec la température inverse $\beta \in (0, \infty)$, et hors de l'équilibre mécanique avec le profil de quantité de mouvement et le profil des élongations $\bar{p}, \bar{r} \in C^1([0,1])$. Nous avons prouvé le théorème susmentionné dans la Section 1.1.3, en particulier voir le Théorème 1.1.1, pour une chaîne avec des conditions aux bords périodiques, où l'on doit modifier l'équation macroscopique comme dans (1.1.33).
Remarquez que dans ce cas la fonction $\mathrm{f}^{\mu}_{\beta} \equiv \mathrm{f}_{\beta}$ est constante dans l'espace et qu'elle est donnée comme suit (nous apportons une expression générale, où la masse de la particule est égale à m et la dépendance à la $\hbar$ est également représentée): $\forall y \in (0,1)$,

$$\mathrm{f}_{\beta}(y) = \mathrm{f}_{\beta} = \beta^{-1} \int_0^1 \frac{\beta\hbar}{\sqrt{m}} \sin(\pi k) \coth\left(\frac{\beta\hbar}{\sqrt{m}} \sin(\pi k)\right) dk. \tag{vii}$$

Remarque I.2. *Le Théorème 1.1.1 implique que notre résultat est vrai en équilibre thermique pour une chaîne propre. La même conclusion est valable dans le cas classique, pour une chaîne propre en équilibre thermique, où dans l'équation macroscopique nous avons* $\mathrm{f}_{\beta}(y) = \beta^{-1}$ *(cf. [35] Section 1.1). Remarquez que la constante dans* (vii)*, dans la limite classique* $\hbar \to 0$*, converge vers la valeur précédente de* $\mathrm{f}_{\beta} = \beta^{-1}$*. De plus, l'asymptotique de* f_{β} *dans* (vii) *pour les grandes masses ou les hautes températures (*$m \gg 1$ *ou* $\beta^{-1} \gg 1$*) est également égale à* β^{-1}*.*

Remarque I.3. *Le Théorème 1.1.1 se base sur l'homogénéité de l'espace, sinon ce théorème échouait même dans l'homologue classique de notre système. Par exemple, si l'on considère que notre système est hors équilibre thermique avec un profil de température inhomogène, on peut observer que le Théorème 1.1.1 n'est plus valide. En fait, on peut facilement adapter l'argument de la Section 1.1 de [35] et observer que pour une chaîne propre, avec un profil de température variable, bien que nous ayons l'équation d'onde, l'évolution de l'énergie n'est pas gouvernée par l'équation d'Euler, aussi bien dans le cas quantique que classique. Dans ce cas, l'évolution de l'énergie thermique n'est pas autonome (cf. [35] pour plus de détails dans le cas classique).*

Chaîne désordonnée : phénoménologie

Dans la section précédente, le comportement hydrodynamique d'une chaîne propre est expliqué. Puisque la chaîne est harmonique, l'évolution microscopique de r, p

est linéaire. Par conséquent, dans l'échelle hyperbolique, la limite macroscopique est donnée par les équations d'onde dans (v). Pour une chaîne propre, en équilibre thermique, l'évolution de l'énergie suit l'équation d'Euler : l'énergie mécanique suit l'équation d'Euler et la fluctuation thermique n'évolue pas dans le temps. Hors de l'équilibre thermique, l'évolution de r, p est toujours gouvernée par l'équation des ondes. Par conséquent, l'évolution de l'énergie mécanique est donnée par la solution de l'équation d'Euler. Cependant, en raison du gradient de température, la fluctuation thermique n'est plus constante dans le temps, et les ondes thermiques transportent l'énergie thermique dans le système.

Le fait de considérer les masses des particules comme des variables aléatoires i.i.d. modifie considérablement cette image : l'évolution de r et de p est toujours gouvernée par l'équation d'onde ; cependant, la preuve de ce fait nécessite une certaine homogénéisation sur les masses. Remarquez que seule la valeur d'espérance des masses, $\mathbb{E}(m_x)$, apparaît à l'échelle macroscopique. L'énergie mécanique est traitée de manière similaire à la chaîne propre. L'effet principal du caractère aléatoire des masses apparaît dans la suppression des ondes thermiques et nous permet de déduire que l'énergie thermique n'évolue pas dans le temps. Dans ce qui suit, nous tentons d'expliquer ce phénomène.

Un résultat similaire a été obtenu pour l'homologe classique de ce système dans [35]. Comme nous l'avons mentionné, la première caractéristique de ces systèmes qui supprime l'onde thermique, nous permet de fermer l'équation de l'énergie, et prouve la limite hydrodynamique est le phénomène de localisation exprimé dans le Lemme 2.5.2, mathématiquement. Comme nous l'avons observé dans la Section I.II, les modèles de chaînes harmoniques désordonnées (classiques et quantiques) ont été beaucoup étudiés dans la littérature (cf. [6], [222], [63], [85], [203] pour les classiques et [180] pour les quantiques). La plupart de ces modèles présentent des propriétés similaires à l'isolant d'Anderson [9], c'est-à-dire que certains modes propres de ces chaînes sont spatialement localisés (nous donnons une brève introduction à la localisation dans la Section 1.1.4). Dans notre cas, la matrice aléatoire apparaissant dans l'analyse de notre système est $M^{-1}\Delta$, où M est la matrice diagonale des masses. Nous désignons l'ensemble des vecteurs propres de $M^{-1}\Delta$ par $\{\psi^k\}_{k=0}^{n-1}$, ordonnés de manière croissante en fonction de leurs valeur propre correspondante. Remarquez que ces vecteurs apparaissent dans la solution de l'équation d'évolution temporelle (2.2.22). Dans le cas d'une chaîne propre, cette solution est donnée par (1.1.31), où dans ce cas ces vecteurs sont simplement des coefficients de Fourier.
Dans le cas désordonné, on peut déduire de la conservation du momentum que l'état fondamental de $M^{-1}\Delta$ est fixe, également connu sous le nom de «mode protégé par symétrie»[124]. Par conséquent, la longueur de localisation diverge lorsqu'on s'approche de l'état fondamental, à savoir pour $k \sim n^\gamma$, où $\gamma \in [0, \frac{1}{2})$ est choisi correctement. En fait, on peut observer que la longueur de localisation ξ_k se comporte asymptotiquement comme $\xi_k \sim \omega_k^2 \sim (\frac{k}{n})^2$ (si nous prenons la limite $\frac{k}{n} \to 0$ correctement, cf. [222] Théorème 1, estimation 3.8, ou (4.2) dans [6]), où ω_k désigne la valeur propre de la chaîne propre avec $m = 1$, c'est-à-dire $\omega_k = |2\sin\left(\pi\frac{k}{n}\right)|$. De plus,

les modes élevés (lorsque k n'est pas proche de zéro, $k \gg \sqrt{n}$) sont complètement localisés.
Au niveau microscopique, les modes avec $k \ll \sqrt{n}$ restent étendus, c'est-à-dire similaires aux modes propres de la chaîne propre (modes de Fourier). D'abord, l'évolution macroscopique de r, p suit ces modes bas, ceci est prouvé via une homogénéisation de masse dans la Section 2.3. Au niveau microscopique et macroscopique, nous décomposons l'énergie en partie thermique et mécanique. Au niveau macroscopique, en résolvant (v), l'évolution de l'énergie est purement mécanique ; au niveau microscopique, les modes bas transportent l'énergie mécanique, ce fait est prouvé dans le Lemme 2.5.1. En fait, la convergence de l'énergie mécanique microscopique vers l'énergie macroscopique peut être déduite des résultats de la Section 2.3 ; en particulier, la convergence de la quantité de mouvement et de l'élongation à l'étape 4 de la Section 2.3. À l'échelle macroscopique, la partie thermique de l'énergie n'évolue pas dans le temps, puisque l'évolution macroscopique de l'énergie est purement mécanique. Au niveau microscopique, le phénomène similaire peut être prouvé par la localisation que nous avons exprimée dans les Lemmes 2.5.2 et 2.5.3, en exploitant les estimations apparues dans [35], [6], [222]. Cette estimation est donnée explicitement dans le Lemme 1.1.2. En effet, puisque les modes élevés, i.e., $k \gg \sqrt{n}$ sont localisés, la partie thermique de l'énergie (fluctuations thermiques) n'évolue pas dans le temps, ceci a été prouvé dans le Lemme 2.5.4. La remarque suivante est une conséquence directe de ce lemme.

Remarque I.4. *Dans le Lemme 2.5.4, la preuve peut être facilement adaptée pour d'autres échelles de temps, où nous rééchelonnons le temps par un facteur n^α avec $\alpha \geq 1$. Par conséquent, le profil de la température n'évolue pas dans le temps à toute échelle de temps $n^\alpha t$ pour $\alpha \geq 1$, y compris à l'échelle de temps de diffusion, ce qui signifie que le coefficient de diffusion thermique disparaît.*

Nature quantique du système : nouveauté technique

La principale différence de notre modèle avec son homologue classique [35] provient du fait que l'état de Gibbs $\rho^n_{\bar{p},\bar{r},\beta}$ (2.1.9) n'est pas un état produit, puisque les énergies des particules les plus proches ne commutent pas entre elles dans le cas quantique. Cela a des conséquences à la fois mathématiques et physiques. Sur le plan mathématique, cette caractéristique entraîne certaines difficultés techniques qui peuvent être traitées par des arguments similaires à ceux du type décroissance de la corrélation. Physiquement, cette différence apparaît dans l'équation d'évolution macroscopique via la fonction $\mathfrak{f}^\mu_\beta(y)$ (cf. (2.4.5)). Soulignons le fait que cette fonction dépend de la distribution des masses (et pas seulement de ses premiers moments tels que la moyenne et la variance), alors que dans le cas classique elle était égale à $\frac{1}{\beta(y)}$. Expliquons un peu plus cette différence.
Tout d'abord, cette question entraîne quelques difficultés techniques pour l'obtention de certaines limites, qui sont traitées en diagonalisant le pseudo-hamiltonien H^n_β (2.1.10), apparaissant dans la définition de $\rho^n_{\bar{p},\bar{r},\beta}$.
Le problème plus fondamental qui se pose ici est que pour une réalisation fixe des masses, $\langle \tilde{e}_x \rangle_{\rho^n}$, c'est-à-dire, la moyenne de l'énergie thermique de la particule x, qui

est calculée dans (2.2.47) et (2.2.46), dépend de toute la configuration des masses et de tout le profil de la température $(\beta(\frac{1}{n}),\dots,\beta(\frac{n}{n}))$. En revanche, dans le cas classique, elle était simplement égale à $\beta_x^{-1} = \beta(\frac{x}{n})^{-1}$. Cette différence se reflète également dans l'équation macroscopique (v) dans la fonction $\mathfrak{f}^{\mu}_{\beta}$, qui est définie dans (2.4.5). Contrairement au cas classique, où elle était égale à $\frac{1}{\beta(y)}$. Remarquez que la nature quantique de notre système ne survit dans la limite macroscopique que grâce à cette fonction. En effet, dans la limite $\hbar \to 0$, cette fonction converge vers son homologue classique $\frac{1}{\beta(y)}$.
Puisque pour une réalisation fixe des masses, l'énergie thermique microscopique $\left\langle \tilde{e}_{[ny]} \right\rangle_{\rho^n}$ dépend de toute la configuration des masses, on doit la considérer comme une variable aléatoire. De plus, a priori, il n'est pas clair que l'objet limite désiré $\lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \langle \tilde{e}_x \rangle_{\rho^n}$, pour une fonction test f, soit déterministe. De plus, il n'est pas évident, d'après la construction, que cet objet soit local au sens où il ne dépendrait que de la température macroscopique $\beta(y)$. Nous consacrons la Section 2.4.1 et l'Annexe 2.A à ces questions. Afin de montrer que cette limite est déterministe, nous prouvons qu'en chaque point x, la dépendance de l'énergie moyenne à la masse d'une particule y, éloignée de x, décroît suffisamment. Nous utilisons ensuite la loi forte des grands nombres pour les variables aléatoires faiblement dépendantes. Afin de prouver ce fait, nous utilisons des arguments similaires à la décroissance de la corrélation. Ici, nous utilisons le fait que $\mathfrak{f}(z) = \sqrt{z}\coth\sqrt{z}$ est analytique dans un certain domaine, alors nous représentons l'énergie thermique comme $\langle \tilde{e}_x \rangle_{\rho^n}$ en termes de $\mathfrak{f}(A_p^{\beta}) + \mathfrak{f}(A_r^{\beta})$, pour certaines matrices A_p^{β}, A_r^{β} (cf. (2.2.52), (2.2.25)). Nous développons $\mathfrak{f}$ en série de Taylor, et nous utilisons le fait que les matrices A_p^{β}, A_r^{β} apparaissant dans le développement sont locales, c'est-à-dire, la masse de la particule x n'apparaît que sur les entrées proches du terme diagonal $(A_p^{\beta})_{xx}$; ainsi, l'espérance des premiers termes $\frac{|x-y|}{2}$ est factorisée, et le reste est petit. Ceci fournit une décomposition suffisante pour prouver la loi des grands nombres pour $\frac{1}{n}\sum_x f(\frac{x}{n}) \langle \tilde{e}_x \rangle_{\rho}$. Par des arguments similaires, on peut montrer que $\mathfrak{f}^{\mu}_{\beta}(y)$ est continu et local.

Remarque I.5. *Remarquez que lorsque n est fini, $\left\langle \tilde{e}_{[ny]} \right\rangle_{\rho}$ dépend de tout le profil de température $(\beta(\frac{1}{n}),\dots,\beta(1))$. Nous prouvons que lorsque $n \to \infty$, pour tout $y \in (0,1)$, $\mathfrak{f}^{\mu}_{\beta}(y)$ ne dépend que de la température macroscopique $\beta(y)$. En fait, dans la Proposition 2.A.1.2, nous prouvons que $\mathfrak{f}^{\mu}_{\beta}(y) = \mathfrak{f}^{\mu}(\beta(y))$ pour tout $y \in (0,1)$. Ici, $\mathfrak{f}^{\mu}(\tilde{\beta})$ est la fonction correspondante en équilibre thermique à température inverse $\tilde{\beta}$, c'est-à-dire le cas où $\beta(y) = \tilde{\beta}$, pour tout $y \in [0,1]$. Nous définissons cette fonction dans* (2.A.1.2) *et nous observons qu'elle est bien définie (ne dépend pas de y) dans le Corollaire 2.A.21.*

I.IV Autres commentaires

Autres quantités conservées

Ces dernières années, l'idée d'obtenir l'évolution macroscopique des quantités conservées et de leurs charges correspondantes (limites hydrodynamiques) a été adaptée

aux systèmes intégrables avec un nombre infini de quantités conservées via la théorie de l'hydrodynamique généralisée et en introduisant l'ensemble de Gibbs généralisé (GGE) : [38],[64], [213].
Puisque la chaîne harmonique désordonnée est intégrable, on peut légitimement se questionner sur les autres quantités conservées, nous abordons cette question dans la remarque suivante.

Remarque I.6. *Puisque d'autres quantités conservées peuvent être écrites comme des gradients supplémentaires de p et r (voir I_n dans* (2.3.6) *comme exemple), en utilisant la même stratégie que nous avons utilisée pour l'énergie, nous pouvons les décomposer en deux parties : l'une impliquant des contributions mécaniques, et l'autre impliquant des contributions thermiques, de manière similaire à* (2.5.49)*. En utilisant l'argument similaire que nous avons utilisé pour l'énergie thermique, grâce à la localisation les termes thermiques sont constants dans le temps. Cependant, puisque les quantités conservées ont été obtenues en prenant le gradient supplémentaire de r et p, les termes mécaniques disparaissent dans l'échelle hyperbolique du temps et de l'espace, grâce à la régularité du profil initial macroscopique. Par conséquent, dans l'échelle hyperbolique, toutes les autres quantités conservées n'évoluent pas dans le temps.*

Nous soulignons que la dernière remarque concerne l'opérateur de densité correspondant à $\beta, \mathtt{p}, \mathtt{r}$. Cela signifie que nous considérons initialement l'état localement Gibbs correspondant à $\beta, \mathtt{p}, \mathtt{r}$. En fait, le rôle du désordre est de fournir la fermeture de l'équation macroscopique correspondant à e, de sorte que nos équations deviennent autonomes.
Nous devons souligner que cette remarque ne concerne pas la situation où initialement nous préparons notre système dans le GGE avec un ensemble complet de quantités conservées. Cependant, grâce à la construction, on peut s'attendre à ce que ces quantités aient un faible poids dans le GGE.

Nous fournissons quelques problèmes ouverts et esquissons les directions futures possibles dans la section 1.1.5.

II NLS

II.I NLS en un coup d'œil

Considérons l'équation de Schrödinger non-linéaire (NLS) en dimension d :

$$\begin{aligned} i\partial_t\psi(x,t) &= -\Delta\psi(x,t) + \kappa|\psi|^{p-1}\psi(x,t); \\ \psi : \Omega^d \times \mathbb{R}_+ &\to \mathbb{C}; \quad \psi(x,0) := \psi_0(x), \end{aligned} \tag{i}$$

où Δ désigne le laplacien sur $\mathbb{R}^d$, $p > 1$ détermine l'intensité de la non-linéarité, et $\kappa \in \{1, -1\}$, avec $\kappa = -1$ correspond au cas focalisant et $\kappa = 1$ au cas défocalisant,

$\Omega = \mathbb{R}$, ou $\Omega = \mathbb{T}^1_L$, le cercle de longueur L, pour le cas des conditions aux bords périodiques. De plus, nous supposons que $\psi_0 \in H^1(\Omega^d)$.
Cette équation et ses variations apparaissent dans de nombreux domaines de la physique, tels que l'optique non-linéaire, la physique de la matière condensée, la biophysique et d'autres domaines (cf. [228] : NLS en mécanique des fluides ; [224] : étude mathématique du minimiseur de DNLS ; [205], [97] : mécanique statistique de DNLS ; BEC et NLS : [39] ; biophysique : [105], [106], [176]).
Mentionnons quelques-uns de ces exemples. Par exemple dans $d = 3$, $p = 3$ avec $\kappa = -1$, (i) décrit, dans certains régimes, l'onde de Langmuir dans un plasma (évolution temporelle de l'enveloppe du champ électrique se propageant), $d = 2$, $p = 3$ correspond à un champ laser se propageant dans un milieu non-linéaire (cf. [154] et ses références). L'évolution de la condensation de Bose-Einstein (BEC) peut également être modélisée par cette équation. Celle-ci peut être dérivée rigoureusement de modèles microscopiques (cf. [93], [94], [140] pour une dérivation rigoureuse de l'équation NLS à partir de la dynamique microscopique ; cf. le Chapitre un de [217] et les références qui s'y trouvent pour une introduction plus générale aux applications physiques de l'équation NLS ; cf. [3] également).

Mathématiquement, cette équation appartient à la famille des *EDP non-linéaires dispersives*, qui est un domaine de recherche très actif avec de nombreuses conjectures et problèmes ouverts (même au niveau de l'heuristique, il y a beaucoup d'inconnues dans ce domaine). De cette famille, on peut citer KdV, l'équation des ondes non-linéaires (NLW), l'équation de Benjamini Ono, etc. Pour une liste assez complète (mais ancienne) de résultats concernant l'existence globale et locale de la solution, ainsi que certaines estimations de décroissance et des résultats de diffusion, on peut consulter la page web suivante de Terry Tao et. al. https://www.math.ucla.edu/ tao/Dispersive/. Ce livre de Tao (cf.[218]) fournit également une très bonne introduction. Concernant l'équation NLS, nous nous référons aux notes de cours de Cazenave (cf. [66]) et au livre de Bourgain (cf. [45]) pour les résultats classiques concernant la théorie de Cauchy et le comportement qualitatif des solutions. Pour une revue des résultats relativement récents sur le blow-up dans le cas focalisant, on peut se reporter aux notes de cours de Raphaël (cf. [194]).

NLS sur $\mathbb{R}^d$

Prenons $\Omega = \mathbb{R}$. Dans ce qui suit, nous rappelons certaines des caractéristiques bien connues de NLS, majoritairement nous suivons [194]. Tout d'abord, observons que l'équation NLS possède deux quantités conservées importantes, l'hamiltonien $\mathcal{H}$ et la masse $\mathcal{M}$:

$$\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(x)|^2 dx + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |\psi(x)|^{p+1} dx, \qquad \mathcal{M}(\psi) = \int_{\mathbb{R}^d} |\psi(x)|^2 dx. \quad \text{(ii)}$$

Le momentum est une autre quantité conservée $\mathcal{I}m\,(\int \nabla u u^*)$. En fait, on peut considérer l'équation NLS comme un flux hamiltonien de dimension infinie avec

l'hamiltonien $\mathcal{H}$.

Le cas défocalisant ($\kappa = 1$) est considéré comme mieux compris. En plus de la théorie générale et de l'existence de la solution (cf. [66], [218]), dans certains cas nous avons également des informations sur son comportement à long terme. En fait, sous des hypothèses assez générales, on s'attend à ce que les solutions de l'équation NLS défocalisant se dispersent et se comportent qualitativement de manière similaire au cas linéaire ($\kappa = 0$) . En d'autres termes, pour une classe générale de conditions initiales ψ_o, pour tout ensemble compact K, nous avons : $\lim_{t\to\infty} \int_K |\psi(x,t)|^2 \to 0$ où $\psi(x,t)$ désigne la solution forte de (i). Ceci peut être obtenu grâce à certaines estimations de décroissance (estimations de Strichartz et Morawetz cf. [218] Section 3.5, [66] Section 7).

Dans ce manuscrit, nous nous concentrons principalement sur le cas focalisant ($\kappa = -1$). Dans ce cas, l'existence locale de la solution peut être déduite des travaux de Ginibre et Velo (cf. [112]) dans un certain régime de paramètres. Un outil important dans l'étude de l'équation NLS est les inégalités d'interpolation de Gagliardo-Nirenberg : pour tout $u \in H^1(\mathbb{R}^d)$ avec $p \leq 1 + \frac{4}{d}$, il existe une constante $C(p,d) > 0$ telle que :

$$\int_{\mathbb{R}^d} |u(x)|^{p+1} dx \leq C(d,p) \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \right)^{\frac{d(p-1)}{4}} \left(\int_{\mathbb{R}^d} |u(x)|^2 dx \right)^{\frac{p+1}{2} - \frac{d(p-1)}{4}} . \quad \text{(iii)}$$

Par (iii), on peut déduire l'existence globale de la solution pour $p < 1 + \frac{4}{d}$: pour tout t, $\psi(t)$ est borné dans H^1, et c'est une solution de (i) avec une donnée initiale dans H^1. Ce régime ($p < 1 + \frac{4}{d}$) est appelé régime sous-critique. À partir de maintenant, nous supposons que $p < 1 + \frac{4}{d}$, sauf indication contraire.

Ondes solitaires sur $\mathbb{R}^d$

Il est généralement admis que le comportement à long terme du cas focalisant est sensiblement différent de celui du cas défocalisant. En particulier, l'absence de dispersion peut être observée grâce à la classe spéciale suivante de solutions : *ondes solitaires*, ou *ondes stationnaires*, que nous appelons également solitons. Ce sont des solutions périodiques dans le temps de la forme : $\psi(x,t) = e^{i\omega t}\phi(x)$ avec $\omega > 0$. $\psi(x,t)$ est une solution H^1 de (i), si ϕ résout l'EDO elliptique non-linéaire suivante :

$$\Delta\phi - \omega\phi + |\phi|^{p-1}\phi = 0, \quad \text{(iv)}$$

avec $\phi \in H^1(\mathbb{R}^d)$. Prenez $\omega = 1$; en général, on peut construire différentes solutions à (iv) (cf. [28]). Cependant, sous certaines hypothèses, (iv) a une solution unique. Par exemple, dans $d = 1$, toutes les solutions de (iv), sont des translations spatiales de $Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{(p-1)x}{2})} \right)^{p-1}$. De plus, pour $d > 1$, si nous supposons que ϕ est une solution de (iv) et, *de plus nous supposons que ϕ est non négatif*, alors ϕ devrait être la translation d'un profil radial $Q(r)$ exponentiellement décroissant. En fait,

$Q(r)$ est la solution unique, non négative et radialement symétrique de (iv), qui est appelée *soliton d'état fondamental*, (rappelons que nous prenons $\omega = 1$) (cf. [194] Proposition 1.2 et Proposition 1.3 ; pour plus d'informations [218] Annexe B ; originellement : [111] [150]).

Le profil Q mentionné ci-dessus a une caractérisation variationnelle (ceci est dû à Cazenave et à Lions cf. [65]), qui s'avère être très utile ; en particulier, pour nos objectifs. Rappelons que nous sommes dans le régime masse-sous critique $p < 1 + \frac{4}{d}$.

Proposition II.0.1. *Prenons $\omega = 1$ et soit Q la solution unique, radialement symétrique et non-négative de* (iv). *Fixez $m > 0$, et considérons le problème de minimisation suivant :*

$$\mathcal{E}_o(m) := \inf_{\psi \in H^1, \mathcal{M}(\psi)=m} \mathcal{H}(\psi). \tag{v}$$

Le minimiseur de (v) *est atteint par la famille*

$$Q_{\lambda(m)}(. - x_o)e^{i\gamma_o}, \quad x_o \in \mathbb{R}^d, \gamma_o \in \mathbb{R}, \tag{vi}$$

où $Q_\lambda(x) := \lambda^{\frac{2}{p-1}} Q(\lambda x)$ et où $\lambda(m)$ est l' unique solution de l'équation $\mathcal{M}(Q_{\lambda(m)}) = m$.
De plus, toute séquence minimisante ϕ_n de (v) *est relativement compacte dans H^1 à une translation spatiale, et multiplication par une phase constante près: $e^{i\gamma_{n_k}}\phi_n(. + x_{n_k}) \to Q_{\lambda(m)}$ dans H^1.*

En fait, la proposition mentionnée ci-dessus est similaire à la Proposition 1.5 de [194] dont nous esquissons la preuve (qui repose sur l'argument de «concentration-compacité») dans la Section 1.2.4.

L'une des conséquences directes de la caractérisation variationnelle ci-dessus est la *stabilité orbitale* des solitons. Nous disons que l'état fondamental Q est stable orbitalement dans l'espace fonctionnel $\mathcal{X}$ si pour tout $\epsilon > 0$ il existe $\delta(\epsilon)$, tel que pour tout $\psi_o \in \mathcal{X}$, avec $||\psi_o - Q||_{\mathcal{X}} < \delta(\epsilon)$, nous avons pour tout $t \in \mathbb{R}_+$:

$$\inf_{x_o \in \mathbb{R}^d, \gamma \in \mathbb{R}} ||\psi(t,x) - Q(x - x_o)e^{i\gamma}||_{\mathcal{X}} \leq \epsilon, \tag{vii}$$

où $\psi(t,x)$ désigne la solution de (i), avec la donnée initiale ψ_o. La stabilité orbitale de Q dans $H^1(\mathbb{R}^d)$ est une conséquence directe de la Proposition II.0.1, et de la conservation de l'énergie et de la masse (ceci est prouvé pour la première fois dans [65]).

Une autre signification des solitons est qu'on croit généralement qu'ils apparaissent dans le comportement à long terme de l'équation NLS focalisant (i). En fait, l'étude du comportement à long terme de l'équation NLS est un domaine de recherche très actif. Nous en donnons un très bref aperçu dans la Section 1.2.1.

En particulier, une des questions dans cette direction concerne le *comportement générique à long terme* de NLS. En effet, de nombreuses évidences numériques et théoriques suggèrent que le comportement asymptotique à long terme de l'équation

NLS, pour des données initiales "génériques", se décompose en une partie dispersive et une autre partie non-dispersive correspondant à un couple de solitons se déplaçant à des vitesses différentes (partie "localisée"). C'est le contenu de ce qu'on appelle la *conjecture de résolution des solitons* (cf. [218] p. 154), qui est un problème ouvert majeur dans le domaine. Remarquez que cet énoncé est plutôt vague, et que l'on doit formuler ces mots vagues en un énoncé rigoureux, afin de résoudre cette conjecture. En particulier, cette description n'est pas vraie pour toutes les données initiales, et rendre ces mots rigoureux implique également la définition de "générique". Plus précisément, nous nous intéressons au travail de [67], où Chatterjee a étudié cette conjecture dans un certain sens probabiliste, en discrétisant le domaine (nous y reviendrons plus tard). En fait, notre travail dans le Chapitre 3 est motivé par cette conjecture et le travail de Chatterjee.

NLS sur $\mathbb{T}$

Rappelons l'équation de Schrödinger non-linéaire (i), dans cette section nous prenons $\Omega = \mathbb{T}$. Les NLS sur $\mathbb{T}^d$ sont considérés comme plus compliqués et moins explorés que ceux sur $\mathbb{R}^d$. En particulier, la compréhension du comportement en temps long semble plus complexe. Naïvement, cette difficulté provient du fait que sur un domaine borné, notre champ ne peut pas "s'échapper" à l'infini (cf. Section 1.2.1).

Dans ce qui suit, nous allons rappeler certains résultats concernant les solutions de type «ondes stationnaires»dans le cadre d'une configuration périodique. À notre connaissance, ces solutions sont surtout étudiées pour $d = 1$. Afin de profiter de la littérature sur les EDPs, nous nous limitons à la configuration suivante. Comme précédemment, nous nous intéressons au cas focalisant, nous nous limitons au cas cubique ($p = 3$), unidimensionnel, avec des conditions aux bords périodiques [7].

Plus précisément, considérons l'équation de Schrödinger cubique non-linéaire suivante :

$$\begin{aligned} &i\partial_t\psi(x,t) = -\partial_{xx}\psi(x,t) - |\psi(x,t)|^2\psi(x,t),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ &\psi(x,0) = \psi_0(x),\ \psi_0 \in H^1_{per}([0,L]), \end{aligned} \tag{viii}$$

où nous supposons les conditions aux bords périodiques par la définition de $H^1_{per}([0,L]) = H^1(\mathbb{T}_L)$ comme : $H^1(\mathbb{T}_L) := H^1_{per}([0,L]) = \{u \in H^1_{loc}(\mathbb{R},\mathbb{C}) | \forall x \in \mathbb{R},\ u(x+L) = u(x)\}$, avec des normes et un produit scalaire approprié.
L'existence de la solution globale de ce problème est établie dans [45], [66], en particulier : $\forall t > 0,\ \psi(x,t) \in H^1_{per}[0,L]$. Comme pour (ii), cette équation a trois importantes quantités conservées [8]. L'énergie ou l'hamiltonien $\mathcal{H}$, la masse $\mathcal{M}$ et le

[7] Bien que cette équation soit intégrable, nous n'utilisons pas l'intégrabilité, et à partir du Chapitre 3 nous brisons l'intégrabilité par discrétisation et ajout d'une perturbation stochastique.

[8] En fait, comme cette équation est complètement intégrable, nous avons une infinité de quantités conservées. Cependant, la plupart des résultats de cette note peuvent être généralisés aux non-linéarités qui ne sont pas intégrables: nous pouvons changer le terme de non-linéarité dans (viii) en $|\psi|^{p-1}\psi$ avec sous-critique p : $1 < p < 5$.

momentum $\mathcal{P}$, qui peuvent être définis comme ($\mathfrak{Im}(z)$ représente la partie imaginaire de z) :

$$\begin{aligned}\mathcal{H}_L(\psi) &= \frac{1}{2}\int_0^L |\partial_x\psi|^2 - \frac{1}{4}\int_0^L |\psi|^4.\\ \mathcal{M}_L(\psi) &= \int_0^L |\psi|^2,\ \ \mathcal{P}(\psi) = \mathfrak{Im}(\int_0^L \partial_x\psi\bar{\psi}).\end{aligned} \tag{ix}$$

De nouveau, comme pour (iv), nous pouvons construire différentes classes de solutions périodiques dans le temps pour (viii). Ces solutions appelées *«ondes stationnaires»* ou *«ondes périodiques»* (puisqu'elles sont spatialement périodiques), et ont la forme suivante :

$$\psi(x,t) = e^{i\omega t}u(x). \tag{x}$$

Si $\psi(x,t) = e^{i\omega t}u(x)$ est une solution de (viii), alors $u(x)$ devrait satisfaire l'EDO suivante :

$$u''(x) - \omega u(x) + |u(x)|^2 u(x) = 0, \tag{xi}$$

où u est spatialement périodique.
Une des différences entre la situation précédente et la situation périodique repose sur le fait que les solutions de (iv) sont bien comprises en général par rapport aux solutions de (xi).

On peut trouver différentes classes de solutions régulières de (xi) (cf.[120], [188], [108]), en utilisant les propriétés des fonctions elliptiques de Jacobi ($\mathfrak{cn}(\cdot), \mathfrak{sn}(\cdot), \mathfrak{dn}(\cdot)$), également connues sous le nom d'ondes dnoïdales, cnoïdales et snoïdales. Nous définissons ces fonctions dans la Section 1.2.1. (cf. Chapitre 2,3 de [152] pour les propriétés de ces fonctions).
Rappelons la définition de la stabilité orbitale (vii). La stabilité orbitale des ondes stationnaires par rapport à différentes perturbations (différents espaces fonctionnels $\mathcal{X}$) n'est pas triviale. Cette question a été étudiée au cours des dernières décennies (cf. [188] pour une introduction générale). Nous rappelons quelques résultats dans cette direction dans la Section 1.2.1. Ces résultats reposent principalement sur des méthodes similaires à celles utilisées par Grillakis-Shatah-Strauss (cf. [115], [116]), où il s'agit de propriétés spectrales de l'opérateur $\mathcal{L} = -\frac{d^2}{dx^2} - c + \mathfrak{dn}_c$, contrairement au cas précédent (NLS sur $\mathbb{R}^d$), où la stabilité orbitale était prouvée par des méthodes variationnelles.

Ce n'est que récemment qu'une approche variationnelle similaire a été adaptée dans [120] pour prouver la stabilité orbitale des ondes stationnaires périodiques, où ils récupèrent les résultats susmentionnés via une caractérisation variationnelle des ondes dnoïdales, cnoïdales et snoïdales, sur des espaces fonctionnels appropriés. Parmi les caractérisations dans [120], le résultat suivant sera utilisé (de manière cruciale) dans notre analyse. Il est donné dans ([120], Proposition 3.2), et caractérise $\mathfrak{dn}$ comme le minimiseur de $\mathcal{H}_L(\psi)$ sous la contrainte que $\mathcal{M}_L(\psi) = m$.

Proposition II.0.2. *Fixez $m, L \in \mathbb{R}_+$, et considérez le problème de minimisation suivant :*

$$E_0(m, L) := \inf\{\mathcal{H}_L(u) | \mathcal{M}_L(u) = m,\ u \in H_{per}([0, L])\}. \tag{xii}$$

alors on a : $-\infty < E_0(m, L) < 0$, et

1. *Si $0 < m \leq \frac{\pi^2}{L}$, alors la fonction constante $Q_{m,L}(x) = (\frac{2m}{L})^{\frac{1}{2}}$ est l'unique minimiseur de* (xii)*. Cette unicité est jusqu'à une multiplication par une phase constante près.*

2. *Si $\frac{\pi^2}{L} < m$, alors $Q_{m,L}(x) := \alpha \mathfrak{dn}(\lambda x, k)$ est l'unique minimiseur de* (xii)*, à une translation et une multiplication par une phase constante près. De plus, $\alpha, \lambda > 0, k \in (0, 1)$ sont déterminés de manière unique par m, L.*

De plus, nous avons la compacité de la séquence minimisante jusqu'à multiplication par une phase constante et translation près dans $H^1_{per}([0, L])$, c'est-à-dire, pour toute séquence u_n dans $H^1_{per}([0, L])$, telle que $\mathcal{H}(u_n) \to E_0(m, L)$, lorsque $n \to \infty$, il existe une suite u_{n_k} et des suites $\gamma_k \in [0, 2\pi)$ et $x_k \in [0, L]$, telles que $e^{i\gamma_k} u_{n_k}(. + x_k) \to Q_{m,L}$ dans $H^1_{per}([0, L])$.

La preuve de la Proposition II.0.2 est plus ou moins similaire à la preuve de la Proposition II.0.1. En fait, parmi les ondes périodiques, $\mathfrak{dn}$ a la caractérisation la plus "évidente", et on peut suivre les étapes similaires, avec quelques modifications (cf. Section 1.2.1). En particulier, nous utilisons ici l'inégalité de Gagliardo-Nirenberg dans le cas périodique :

$$\begin{aligned} |u|^4_{L^4(\mathbb{T})} &= \int_0^1 |u|^4 \leq C(|\partial_x u|_{L^2(\mathbb{T})} |u|^3_{L^2(\mathbb{T})} + |u|^4_{L^2(\mathbb{T})}) \\ &= C\left(\left(\int_0^1 |u|^2\right)^{\frac{3}{2}} \left(\int_0^1 |\partial_x u|^2\right)^{\frac{1}{2}} + \left(\int_0^1 |u|^2\right)^2\right). \end{aligned} \tag{xiii}$$

II.II Approche de la mécanique statistique et mesure de Gibbs

Les mesures de Gibbs sont généralement perçues comme fournissant des informations utiles concernant le comportement "typique" (à long terme) des systèmes hamiltoniens. Éclairons un peu plus cette idée. La "mesure de Lebesgue" est invariante pour le flux hamiltonien, par conséquent, la mesure de Gibbs avec la densité $\exp(-\beta H)$ par rapport à la "mesure de Lebesgue" est également invariante. Si l'hamiltonien fournit des propriétés "ergodiques" suffisantes, on peut s'attendre à ce que cette dernière soit la seule mesure invariante, et que le comportement à long terme de notre système puisse être décrit par cette mesure. En d'autres termes, notre système se thermalise après un temps suffisamment long. En considérant NLS comme un flux hamiltonien, l'argument mentionné ci-dessus donne une forte motivation pour étudier la mesure de Gibbs associée à NLS. Soulignons que prouver l'ergodicité même dans le cas d'une discrétisation est extrêmement difficile, et

nous fournissons l'ergodicité par une perturbation stochastique appropriée. De plus, l'équation NLS cubique dans $d = 1$ est complètement intégrable et ce schéma (sans la perturbation stochastique) échoue.

Mesure de Gibbs : bref examen

L'idée d'étudier la mesure de Gibbs de l'équation NLS remonte aux travaux pionniers de Lebowitz, Rose et Speer [154]. Ils procèdent comme suit. Le premier obstacle à l'étude de cette mesure est que l'équation NLS est un flux de dimension infinie, et qu'une telle mesure de Lebesgue n'existe pas en dimension infinie. Pour NLS sur $\mathbb{T}_L$, cette mesure peut être écrite formellement :

$$d\nu_L^\beta = \frac{1}{Z_L(\beta)} \exp(-\beta \mathcal{H}_L^p(u)) \Pi_{y \in \mathbb{T}_L} du(y), \tag{xiv}$$

où $\mathcal{H}_L^p = \frac{1}{2}\int_0^L |u'|^2 - \frac{1}{p+1}\int_0^L |u|^{p+1}$. Ce problème peut être contourné en absorbant l'énergie cinétique dans la mesure de Lebesgue "formelle" et en obtenant la mesure de Wiener appropriée :

$$d\nu_L^\beta = \frac{1}{Z_L(\beta)} \exp\left(\frac{\beta}{p+1}\int_0^L |u(y)|^{p+1} dy\right) dW_{\beta,L}, \tag{xv}$$

où $dW_{\beta,L}$ est la mesure de Wiener appropriée, induite par le pont brownien. L'autre difficulté provient du fait que $\mathcal{H}_L^p$ est non minorée, ceci peut être résolu en restreignant la mesure aux boules telles que $\mathcal{M}_L \leq m$, pour une constante m. Par conséquent, dans [154], il est prouvé qu'une telle mesure est bien définie dans $d = 1$, pour $1 < p < 5$ et pour $p = 6$ avec de petits m (c'est le même domaine de paramètres où l'équation a des solutions globales).
Avant de poursuivre, mentionnons que la construction de la mesure de Gibbs dans le cas du continuum est un problème difficile. Nous passons brièvement en revue certains des résultats les plus célèbres dans cette direction dans la Section 1.2.2.

Le comportement qualitatif de la mesure de Gibbs est informatif et intéressant en soi (cela devient encore plus intéressant, lorsque nous ajoutons la perturbation stochastique à notre dynamique). Pour autant que nous le sachions, ce problème est moins étudié dans la communauté mathématique. Nous décrivons brièvement certains des résultats intéressants dans cette direction dans la Section 1.2.2 (cf. [139] pour une revue). Remarquez que dans de nombreux cas, la version "discrétisée " de la mesure de Gibbs est étudiée pour éviter les complications concernant l'existence. Nous comparons également la force et la faiblesse de nos résultats par rapport aux premiers dans la Section 1.2.3.

Mesure microcanonique. Du point de vue de la mécanique statistique, en considérant la mesure de Gibbs de l'équation NLS, nous étudions le système dans l'ensemble "canonique"[9]. Si l'on croit en l'efficacité de l'approche de la mécanique

[9]Citation de [154] : NLS is a effective equation describing many degrees of freedom. The idea

statistique, il serait naturel d'explorer également l'ensemble microcanonique de l'équation NLS. Dans la communauté des physiciens, il existe certains travaux concernant cette question pour l'équation de Schrödinger non-linéaire discrète (DNLS) (cf. [205], [206], [114] et leurs références).
Mathématiquement, Chatterjee et ses coauteurs ont une série de travaux dans cette direction (cf. [69], [68], en particulier [67]), remarquez que dans ce cas la discrétisation semble inévitable. En particulier, il prouve une version probabiliste de la conjecture de résolution de soliton dans [67] en analysant l'ensemble microcanonique. Comme notre travail est partiellement motivé par ce travail, nous expliquons son résultat principal dans la Section 1.2.2.

II.III Bruit stochastique et résultats

Jusqu'à présent, la situation peut être résumée comme suit : le comportement à long terme de l'équation NLS focalisant est insaisissable. Les approches de mécanique statistique, comme l'étude de la mesure de Gibbs, se sont avérées utiles. Bien qu'elles fournissent une mesure invariante naturelle pour la dynamique, on ne peut pas garantir la convergence du flux NLS vers cette mesure, principalement en raison de la difficulté à prouver l'ergodicité (dans le cas du NLS cubique dans $d = 1$, nous n'avons même pas d'ergodicité du tout).
L'un des principaux objectifs du Chapitre 3 est d'introduire une perturbation stochastique conservant la masse ($\mathcal{M}$) de NLS, de telle sorte que la mesure de Gibbs devienne l'unique mesure invariante. On pourrait considérer l'action de ce bruit, comme un bain de chaleur à température constante β^{-1}.
Rappelons l'équation (viii) (avec une non-linéarité générale p), avec l'hamiltonien $\mathcal{H}_L$ et la masse $\mathcal{M}_L$ (ix), où nous considérons une non-linéarité générale p. Formellement, une façon de définir une telle dynamique stochastique est de considérer l'EDP stochastique :

$$\begin{aligned} i\partial_t\psi(x,t) = -\Delta\psi(x,t) + \kappa|\psi|^{p-2}\psi(x,t) - \gamma\psi(x,t)\left(i\beta^{-1} - \frac{\delta\mathcal{H}(\psi))}{\delta\theta(x)}\right) \\ +\sqrt{2\gamma\beta^{-1}}\psi(x,t)W(x,t), \end{aligned} \tag{xvi}$$

où $\theta(x)$ est la phase de $\psi(x)$ ($\psi(x) = |\psi(x)|e^{i\theta(x)}$), $W(x,t)$ est le bruit blanc standard (blanc dans l'espace et le temps), et $\gamma > 0$ est un paramètre qui régule l'intensité du contact avec le bain de chaleur. Cela correspond à des rotations aléatoires mais continues de la phase de $\psi(x)$ indépendamment en chaque point x avec une drift qui a en quelque sorte pour but d'aligner les phases de la particule. Remarquez que $\frac{\delta\mathcal{H}(\psi))}{\delta\theta(x)} = \mathcal{I}m[\psi(x)^*\Delta\psi(x)]$, et que (xvi) doit être entendu au sens d'Itô. Par conséquent, la masse $\mathcal{M}(\psi)$ est toujours formellement conservée par cette dynamique. En raison de la singularité dans l'espace du bruit blanc multiplicatif W et des non-

of coarsening the system into an effective equation, then refining it by considering the Canonical ensemble may seem artificial; however, "there are often good reasons for wanting to focus attention on the collective variables of the field rather than on the atomic degree of freedom e.g, in fluid turbulence it is in hydrodynamic modes that interesting action take place."

linéarités présentes dans (xvi), il est très difficile de donner un sens à la solution de cette équation. Il existe une littérature considérable sur l'équation NLS avec un bruit multiplicatif corrélé dans l'espace (cf. [75], [76])[10], mais elle n'inclut pas les non-linéarités comme $\psi(x)\Delta\psi^*(x)$. Des bruits additifs ont également été introduits (cf. [153], [60]), mais ils ne conservent généralement pas la masse et la dynamique correspondante a la mesure de Gibbs grand canonique comme stationnaire. Nous introduisons à la place une discrétisation spatiale de (xvi), voir (3.1.8), dont la solution peut être définie globalement. La version à température infinie de cette évolution stochastique a été introduite dans (cf. [156]).
Alternativement, ce bruit peut être décrit par son générateur. Bien que cette dynamique soit définie dans la Section 3.1[11] , rappelons-la brièvement ici . En général, nous considérons le domaine $h\mathbb{T}_n^d$ comme une discrétisation de la boîte $[0, nh]^d$, où h est la "mesh size". Corrélativement, la discrétisation de $\mathcal{H}_L$ et $\mathcal{M}_L$ à la "mesh size" h est définie comme suit : pour tout $\psi \in \mathbb{C}^{n^d}$ nous avons

$$H_n := s \sum_{|x-y|=1} h^{-2}|\psi(x) - \psi(y)|^2 + \frac{s\kappa}{p+1} \sum_{x\in\mathbb{T}_n^d} |\psi(x)|^{p+1}, \quad M_n(\psi) = s\sum_x |\psi(x)|^2. \tag{xvii}$$

Le choix de $h = \frac{L}{n}$ et $s = \frac{1}{n^d}$ correspond à la discrétisation de (ix), tandis que prendre $s = h^d$, et n, h tels que $h \to 0, n \to \infty, nh \to \infty$ correspond à la discrétisation de (ii).
L'équation de Schrödinger non-linéaire discrète peut être vue comme une dynamique hamiltonienne correspondant à H_n, avec le générateur suivant :

$$A_n = \frac{1}{s} \sum_{x\in\mathbb{T}_n^d} (\partial_{\psi_i(x)} H_n)\partial_{\psi_r(x)} - (\partial_{\psi_r(x)} H_n)\partial_{\psi_i(x)}, \tag{xviii}$$

où ψ_r, ψ_i désignent la partie réelle et imaginaire de ψ: $\psi(x) = \psi_r(x) + i\psi_i(x)$. On désigne la phase de $\psi(x)$ par $\theta(x)$ ($\psi(x) = |\psi(x)|e^{i\theta(x)}$), le générateur du bruit à la température β^{-1} est donné par :

$$S_n := \beta^{-1} \sum_{x\in\mathbb{T}_n^d} e^{\beta H_n}\partial_{\theta(x)} e^{-\beta H_n}\partial_{\theta(x)}. \tag{xix}$$

Par conséquent, pour toute $\gamma > 0$, le générateur de notre dynamique est donné par :

$$L_n = A_n + \gamma S_n. \tag{xx}$$

De la définition du générateur, il est clair que la masse $M_n(\psi)$ est conservée par la dynamique. Par conséquent, en partant d'une configuration avec $M_n(\psi_o) = m$, la dynamique est confinée dans la sphère $S_m^n := \{\psi \in \mathbb{C}^{n^d} | M_n(\psi) = m\}$. Dénotant la mesure uniforme sur S_m^n par μ_m^n, la mesure de Gibbs est donnée par $d\mu_{\beta,m}^n =$

[10]Nous n'avons pas l'intention de passer en revue cette vaste littérature ici, et nous ne mentionnons que le travail pionnier de de Bouard, et Debussche, par la suite il y a beaucoup de travaux dans la littérature concernant l'équation NLS stochastique et ses variations ; cependant, leur point de vue est assez différent du nôtre, où nous nous intéressons à la thermalisation.

[11]Bien que ces définitions soient répétitives, elles rendent notre introduction plus compréhensible.

$\frac{1}{Z_n}\exp(-\beta H_n)d\mu_m^n$, et est l'analogue discret de (xv) (que nous devrions contraindre à la boule de masse inférieure à m cf. (1.2.21)). Cette mesure est bien définie et invariante pour la dynamique engendrée par L_n.

Le premier résultat du Chapitre 3 est que $\mu_{\beta,m}^n$ est la mesure invariante unique pour la dynamique générée par L_n (xx). Le générateur est hypoelliptique et pour tout n fixé, la distribution du processus partant d'une condition initiale arbitraire converge exponentiellement en norme de variation totale vers la mesure stationnaire. L'énoncé de ce résultat est donné dans le Théorème 3.1 et la Proposition 3.1.3.
La nouveauté de notre travail est qu'il s'agit du premier résultat concernant la perturbation *conservatrice de masse* de l'équation DNLS telle que la mesure de Gibbs soit *l'unique mesure invariante*, déterminant le comportement à long terme de la dynamique. Nous devons mentionner que dans la Section 6 de [59], un bruit conservateur de masse est proposé de manière que la mesure canonique de Gibbs reste invariante par la dynamique. Cependant, cette dynamique n'est pas étudiée dans [59], et ce travail est concerné principalement une autre dynamique, qui ne conserve pas la masse et converge vers la mesure grand-canonique. De plus, la dynamique conservant la masse mentionnée ci-dessus est différente de la nôtre ; en fait, une analyse directe suggère que notre dynamique est plus dégénérée (nous avons un processus de Wiener à n-dimensions sur la sphère $2n-1$, alors qu'ils ont une mesure de Wiener à $2n-1$-dimensions sur une sphère de dimension $2n-1$). De plus, la généralisation de leur bruit à des cas plus généraux ne semble pas simple.

Remarque II.1. *Il convient de mentionner que le résultat rapporté ci-dessus est général dans le sens où il ne dépend pas des paramètres h, s, γ, β dans notre définition. Plus important encore, ce résultat est vrai pour toute puissance de non-linéarité et toute valeur de $\kappa \in \{-1, 0, 1\}$. En fait, on peut remplacer la non-linéarité $|\psi|^{p+1}$, par toute non-linéarité de la forme $F(|\psi|)$, avec $F \in C^2$, et obtenir le même résultat. Par conséquent, on peut perturber n'importe lequel des modèles discrets de la Section 1.2.2, ou d'autres DNLS appropriés (cf. [135] et ses références) avec notre bruit, et grâce au théorème susmentionné, le comportement à long terme de la dynamique émergente sera donné par la mesure de Gibbs correspondante. Remarquez que dans beaucoup des modèles mentionnés ci-dessus, le comportement de la mesure de Gibbs est relativement bien compris. Bien que notre résultat soit général dans le sens ci-dessus, notre choix de discrétisation est important (jusqu'à changer les paramètres d'échelle), et le générateur hamiltonien A_n a un rôle important.*

Le reste du Chapitre 3, concerne une application du résultat mentionné ci-dessus. A partir de la Section 3.2 du Chapitre 3, nous nous concentrons sur l'équation NLS cubique focalisant unidimensionnelle sur le tore ($d = 1$, $p = 3$, $\kappa = -1$). Rappelons la solution d'onde stationnaire de l'équation NLS cubique focalisant unidimensionnelle sur le tore, et en particulier, la définition et les propriétés du minimiseur de l'énergie sous contrainte de masse $Q_{m,L}$ à partir de la proposition II.0.2.
Tout d'abord, nous discrétisons cette équation. Cela correspond au choix de $d = 1$, $s = h = \frac{1}{n}$ (prendre le volume macroscopique égal à un), $p = 3$ dans l'hamiltonien (xvii), et par conséquent dans les générateurs A_n et S_n. Soit $\psi_n(j, t)$ la solution de

la dynamique discrète stochastique correspondante. Le but du Chapitre 3 est de montrer que ψ_n est "proche" de $Q_{m,1} \equiv Q_m$ pour de grands t et de grands n, où nous ré-échelonnons β par n de telle sorte que dans cette limite, la température converge vers zéro assez rapidement. Remarquez que pour donner un sens à cette affirmation, nous devrions définir une norme pour comparer ψ_n et Q_m. Pour cela, nous prenons simplement l'interpolation linéaire de ψ_n, (3.2.17) et nous la comparons à Q_m dans une norme H^1 qui ne prend pas en compte la translation et de la multiplication par une phase constante (3.2.18) : $||u-v||_{\tilde{H}^1} = \inf_{\theta,x} ||e^{i\theta}u(\cdot - x) - v||_{H^1}$.
En d'autres termes, si $\mu_t^{\beta_n,n,m}$ est la loi du processus généré par L_n alors notre résultat principal est que pour tout $\epsilon > 0$:

$$\lim_{n\to\infty} \lim_{t\to\infty} \mu_t^{\beta_n,m,n} \left(||\bar{\psi}_n - Q_m||_{\tilde{H}^1} < \epsilon \right) \to 1, \tag{xxi}$$

où $\bar{\psi}_n$ désigne l'interpolation linéaire de ψ_n, et $\beta_n = \vartheta(n)\beta_*$ avec $\vartheta(n) \gg n$. L'énoncé du résultat est donné dans le Théorème (3.2.2).

La dernière limite (xxi) est une façon d'interpréter la *conjecture de résolution des solitons* (SRC) dans le cas périodique, où il n'y a aucune possibilité pour l'énergie de s'échapper à l'infini. Intuitivement, dans le cas périodique, notre dynamique dans la limite de température nulle, agit comme un bain de chaleur et dissipe l'excès d'énergie sans perdre de masse, forçant le système à se rapprocher de l'état fondamental lorsque $t \to \infty$. Ce mécanisme imite en quelque sorte la dynamique de l'équation DNLS dans [67], où l'énergie se disperse à l'infini via une partie "rayonnante" du champ, portant une masse arbitrairement petite. Comme nous l'avons mentionné, notre dynamique est partiellement motivée par [67], où Chatterjee prouve une version "probabiliste" de la SRC (cf. Section 1.2.2). En particulier, le Théorème 3.1 de [67] prouve que *presque toutes* les mesures invariantes ergodiques satisfont le SRC au sens de la moyenne temporelle. Notre dynamique stochastique fournit l'unicité de la mesure ergodique invariante et la propriété de mélange temporel.

Idées de la preuve

La dynamique dans la limite de $t \to \infty$ converge vers la mesure de Gibbs. Ceci suit l'ergodicité de la dynamique qui repose sur deux faits cruciaux. Le premier est l'hypoellipticité du générateur de la dynamique qui est prouvée dans le Lemme 3.B.1. Puisque les parties réelle et complexe de notre champ sont quelque peu symétriques dans le bruit, cela rend la preuve du Lemme 3.B.1 plus compliquée que d'habitude, et le calcul de trois commutateurs imbriqués est nécessaire cf. (3.B.4) (plutôt qu'un ou deux dans la chaîne d'oscillateurs cf. [32] Lemme 5.3). Le deuxième ingrédient est l'unicité de la mesure invariante, qui est prouvée dans le Théorème 3.1, grâce à la séparation du bruit en une partie phase et une partie amplitude.

Après avoir étudier la limite en t, la deuxième étape de la preuve, concerne la compréhension du comportement de la mesure de Gibbs $\mu_{\beta,m}^n$ (3.2.15). En fait, nous devons prouver que la mesure de Gibbs se concentre sur des configurations

discrètes avec une "énergie proche du minimum" pour n suffisamment grand, où β_n est rééchelonné par un facteur $\vartheta(n) \gg n$. Ici, par énergie minimale, nous faisons référence au minimiseur de l'hamiltonien discret sous la contrainte de masse ($E_0^n(m)$ défini dans (3.3.1)). Cette tâche est réalisée dans le Théorème 3.4.1, esquissons ce théorème : notre dynamique est confinée sur la sphère à $2n$ dimensions S_m^n, et la mesure de Gibbs a une densité $\exp(-\beta H_n)$ par rapport à la mesure uniforme sur cette sphère μ_m^n. En divisant l'énergie en parties cinétique G_n et potentielle V_n : $H_n = G_n - V_n$ comme dans (3.2.8), on peut observer qu'une configuration "typique" par rapport à μ_m^n a une énergie cinétique d'ordre n^2. Plus généralement, les estimations des grandes déviations dans la Section 3.A, en particulier la Proposition 3.A.1, revient à dire que pour $0 \leq a \leq 2$, le "facteur d'entropie" se comporte comme $\mu_m^n(G_n \sim n^a) \sim \mu_m^n(H_n \sim n^a) \sim n^{-(2-a)n\ln n}$, où la première estimation est une conséquence de l'inégalité discrète de Gagliardo-Nirenberg (GN) (3.C.8) pour $a > 0$. Par conséquent, en tenant compte du facteur dit de Boltzmann $\exp(-\beta H_n)$, on observe pour $0 < a \leq 2$: $\mu_{\beta,m}^n(H_n \sim n^a) \sim e^{-\beta_n n^a} e^{-(2-a)n\ln n}$. En optimisant cette dernière pour tout $a \in [0,2)$, on obtient pour $\beta_n \sim O(1)$ que la mesure de Gibbs se concentre sur des configurations plutôt non régulières avec $|\psi(j) - \psi(j-1)| \sim \frac{1}{\sqrt{n}}$ et $H_n \sim G_n \sim n$. Cela correspond au fait que la mesure de Wiener est concentrée sur les configurations régulières de $\frac{1}{2}^-$-Hölder. La dernière estimation suggère que pour que $\mu_{\beta_n,m}^n$ se concentre sur des configurations régulières ($|\psi(j) - \psi(j-1)| \sim \frac{1}{n}$) avec $H_n \sim O(1)$, nous devons rééchelonner β_n par un facteur d'au moins n. Remarquez que les configurations d'énergie minimale, que nous appelons Solitons discrets, ont également une énergie (négative) d'ordre un, grâce à l'inégalité de GN. Cependant, cette mise à l'échelle n'est pas suffisante pour que cette mesure se concentre sur un petit voisinage de solitons discrets et nous devons aller plus loin. Enfin, grâce à l'estimation de la grande déviation (3.A.11), nous déduisons dans le Théorème 3.4.1 que la mise à l'échelle de β_n par tout facteur $\vartheta(n) \gg n$ (3.2.19) est suffisante.
Dans la dernière étape de la preuve, nous montrons dans la Proposition 3.3.1 que si ψ_n est une configuration avec une énergie proche de $E_n^0(m)$, alors son interpolation linéaire ($\bar{\psi}_n$) est proche de $Q_{m,L}$ dans la norme $\tilde{H}^1$ pour n suffisamment grand. À cet égard, nous observons d'abord qu'avoir une énergie proche de $E_0^n(m)$ signifie que la configuration est régulière $G_n \sim O(1)$, grâce à l'inégalité de GN discret. Ensuite, nous observons que pour une configuration régulière ψ_n nous avons : $H_n(\psi_n)$ est proche de $\mathcal{H}(\bar{\psi}_n)$ (cf. Corollaire 3.3.3.1, en effet, ce n'est pas vrai pour une configuration "typique" non régulière). Enfin, nous concluons le résultat grâce à l'observation ci-dessus, où nous utilisons la compacité de la séquence minimisante correspondant au problème de minimisation continu qui caractérise les solitons (3.2.6).

Le plan du Chapitre 3 est le suivant : les propriétés d'ergodicité et de mélange temporel de la dynamique sont prouvées dans la Section 3.1. Le fait que la mesure de Gibbs discrète (finie n) se concentre assez rapidement dans un petit voisinage de solitons discrets, repose sur les propriétés de grande déviation de la mesure de Gibbs discrète, qui sont prouvées dans la Section 3.4. Ces estimations de grandes déviations sont basées sur certaines grandes déviations précises de la mesure de probabilité uniforme sur la sphère complexe de dimension m, que nous prouvons dans

l'Annexe 3.A, et la version discrète de l'inégalité de Gagliardo-Nirenberg, prouvée dans l'Appendice 3.C. Dans la Section 3.3, la relation souhaitée entre les *solitons discrets* et continus est prouvée. L'Annexe 3.B contient la preuve de l'hypoellipticité de la dynamique stochastique discrète, nécessaire pour la preuve de l'ergodicité de la Section 3.1.

Background

Nos résultats, peuvent être comparés à deux corpus de travaux différents dans la littérature. Le premier concerne la perturbation stochastique de l'équation NLS, et le second le comportement de la mesure de Gibbs de de l'équation DNLS. Dans la Section 1.2.2, nous fournissons une liste non exhaustive des travaux connexes concernant la mesure de Gibbs. Dans la Section 1.2.3, nous soulignons les similitudes et les différences cruciales entre notre modèle et les résultats concernant la mesure de Gibbs. Nous présentons ici un bref résumé de ces comparaisons, et nous renvoyons à la Section 1.2.3 pour plus de détails.
Avant de poursuivre, notez que nous avons déjà souligné la nouveauté de notre dynamique stochastique par rapport à des travaux existants dans la section précédente, et que pour plus de détails, vous pouvez vous reporter à la Section 1.2.3 (pour faire court : notre bruit conserve la masse, il "thermalise" le système, et fait de la mesure de Gibbs la seule mesure invariante, et ces caractéristiques en font une nouveauté).

La mesure de Gibbs des "DNLS" a été étudiée à la fois dans la communauté des mathématiciens (cf. [69], [67]) et des physiciens cf. [135] et les références qui s'y trouvent, en particulier : [196], [205], [132] ; voir également [114]. Dans la communauté des physiciens, il est habituel de prendre l'énergie cinétique avec un signe négatif et d'étudier la mesure correspondant au hamiltonien (xvii), en prenant $h = 1$. Bien que ce régime soit sensiblement différent du nôtre, et ne corresponde plus à une discrétisation d'un profil continu, des phénomènes intéressants tels que le breather discret sont observés (cf. [97], [224]).
Dans la communauté mathématique, on peut citer notamment [69] et [67] (cf. [139] pour une revue). Dans [69], l'hamiltonien (xvii) est considéré dans le régime $Nh^2 \to 0$, $h \to 0$ et $N \to \infty$, où N désigne le nombre de particules, et h la distance interparticulaire. Ces hypothèses ne semblent naturelles qu pour $d \geq 3$. Dans ce régime, une certaine transition de phase se manifeste : lorsque $\beta m^2 < \theta_c$, la mesure de Gibbs se concentre sur des configurations telles que $\psi_n(j) \sim o(n)$, alors que pour $\beta m^2 > \theta_c$ des structures de type breather apparaissent où un seul site a une masse macroscopique. En [67], le modèle est défini sur la boîte $[0, nh]^d$ telle que $h \to 0$, $n \to \infty$ avec $nh \to \infty$. Dans ce régime, la mesure microcanonique correspondant à l'énergie E et à la masse m se concentre sur les configurations de type soliton dans $\mathbb{R}^d$.
En comparant nos résultats avec ceux de [69] et de [67], nous soulignons le fait qu'une mise à l'échelle différente des paramètres h, n conduit à des phénomènes substantiellement différents : dans [69], la limite $Nh^2 \to 0$ rend le terme de gradient négligeable et la transition de phase est une conséquence de la compétition entre

l'énergie potentielle et la contrainte de masse. Dans le cas de [67], $nh \to \infty$, l'énergie cinétique et l'énergie potentielle deviennent comparables ; cependant, la masse par particule va jusqu'à zéro dans la limite, démontrant le volume infini macroscopique, facilitant la fuite de l'énergie vers l'infini et résultant en un comportement de type soliton. En revanche, dans notre cas, nous prenons $n \to \infty$ et $nh = 1$, représentant le volume macroscopique fini et une masse par particule positive dans la limite macroscopique. Cette mise à l'échelle donne une énergie cinétique dominante pour les configurations typiques sur la sphère de masse constante, et la remise à l'échelle de β_n rend les énergies cinétique et potentielle comparables.
En particulier, ces différentes mises à l'échelle modifient nos estimations de grands deviation (3.A.1) et (3.A.11) par rapport aux estimations de [67] (voir la Section 10 du [67]).

Problèmes ouverts et directions futures

Il existe de nombreux problèmes ouverts et inconnus concernant la dynamique que nous avons introduite. En fait, comme cette dynamique est plutôt nouvelle, beaucoup de ses caractéristiques n'ont pas été étudiées, et de nombreuses recherches peuvent être extraites dans cette direction. Nous présentons divers problèmes ouverts dans la Section 1.2.4. Ces questions peuvent être divisées selon le thème général suivant : question concernant le temp de convergence vers l'équilibre pour n fixe, changement de l'ordre des limites en t, n (ou les prendre ensemble avec une certaine mise à l'échelle), limite hydrodynamique dans l'échelle de temps diffusive, comportement général de notre dynamique lorsque nous changeons les paramètres de mise à l'échelle et les grandes déviations plus précises. Pour une description détaillée de ces questions, nous renvoyons à la Section 1.2.4.

Chapter 1

Introduction

1.1 Disordered Chain

1.1.1 Hydrodynamic limits in a nutshell

Consider a macroscopic system, a container of gas, for instance. Suppose that we are interested in describing the evolution of this system. Then, believing in the fundamental rules of non-relativistic physics, either *classical* or *quantum* mechanics, one can model this system as a collection of particles evolving following these rules. On the microscopic scale, if we zoom in, let's say in a window of length 10^{-8} to 10^{-10}m, and follow a couple of particles for a very short time, their trajectory confirms the validity of our model in a satisfactory manner. However, solving the evolution equation for all the particles simultaneously does not seem useful, or possible in many cases, even numerically [1].

On the other hand, the macroscopic dynamics of this system can be described by phenomenological autonomous equations such as the Euler equation or the Navier-Stokes equations. These equations have been understood for centuries from continuum considerations. However, the exact derivation of these equations from fundamental laws is extremely challenging.

Equilibrium state

In order to better understand the connection between microscopic and macroscopic pictures, we follow the statistical mechanics' approach initiated by Boltzmann and Maxwell (cf. [40], [170]). For the moment, let us assume that the underlying dynamics is given by the Newtonian mechanics. First, we examine the equilibrium state of the system. This state can be characterized by a few *macroscopic* parameters corresponding to the thermodynamic characteristics of the system, and denoted by $\{\mathcal{P}_\alpha\}_{\alpha=1}^k$. For example, in our case (container of gas), one could think of temperature, density, and pressure, as the desired macroscopic quantities. Notice that these

[1] Roughly speaking, memory which is needed to store the phase space of a classical system with 10^{23} particles is hundred times the current global data storage capacity.

parameters are conjugate to the locally conserved quantities of the system.

Local equilibrium

Although knowing the equilibrium states can be interesting in many systems, in practice, the equilibrium characteristics of our gas vary slowly on a macroscopic scale. Hence, we introduce the notion of local equilibrium. Assume that our gas is confined in the box $\Lambda = [0, L]^d$; consider a macroscopic point $u \in \Lambda$, let $B(u)$ be a neighborhood of u. $B(u)$ should be small enough compared to the macroscopic volume of the system, and large enough microscopically, i.e., it should be large enough to be considered a "macroscopic" system (although small). Mathematically, this notion makes more sense in the scaling limit, and we clarify it a bit more later by a concrete example (see (1.1.8)). On the physical ground one could think of this scale to be of order 10^{-4}m in our system, i.e., the typical scale where a macroscopic characteristics of the system such as temperature varies. One could expect that in each neighborhood $B(u)$, the system reaches an equilibrium state, described by the aforementioned thermodynamic characteristics $\{\mathcal{P}_\alpha(u)\}_{\alpha=1}^k$ depending on the macroscopic point u, where these functions vary smoothly in space.
We expect this local equilibrium state's evolution to be negligible in the microscopic time scale,[2] since this state is locally invariant under the dynamics thanks to the construction. However, if we observe our container of gas in a macroscopic time window, we expect the energy, momentum, and particles to be transported over macroscopic distances. In fact, if the local equilibrium is maintained, at macroscopic time t, we could expect the local equilibrium picture to be evolved and the system can be described by new parameters: $\{\mathcal{P}_\alpha(u, t)\}_{\alpha=1}^k$, where these fields vary smoothly over time following the appropriate system of autonomous partial differential equations, i.e., hydrodynamic equation.

Space-time scaling

Before proceeding, we emphasize that the relation between the microscopic and macroscopic pictures involves space-time scaling. Take a scaling parameter n, which eventually goes to infinity. Then the macroscopic space and time, y and t, is given by $ny = x$, $t = \vartheta(n)\tau$, where x and τ denote the microscopic space and time, respectively. We expect the previous picture to become "exact" in the limit as $n \to \infty$. Notice that the choice of $\vartheta(n) = n$, corresponds to the hyperbolic scaling, where we expect to obtain the Euler equation. In contrast, certain phenomena such as thermal conductivity are generally believed to happen in a slower time scale. We need to rescale time in a diffusive time scale $\vartheta(n) = n^\alpha$, with $\alpha = 2$ or in a super-diffusive time scale ($\alpha < 2$) to observe these phenomena.

[2]For instance, one could think of the microscopic time scale as the mean free time between particle's "collision".

Ergodicity

Although the idea of local equilibrium can be traced back to C.Morrey (cf. [177]) in the fifties, the rigorous treatment of this problem for purely Hamiltonian systems is still elusive. The main difficulty lies in proving the fact that the local equilibrium is maintained. The consensus is that the ergodicity of the system is needed for this purpose, and proving ergodicity for purely Hamiltonian systems seems out of reach for the moment. Let us highlight the belief that ergodicity is an essential part of this picture with the following quotes:

"*Despite many efforts, this program has not been completely achieved for Hamiltonian systems where particles evolve deterministically according to Newton's equations, mainly due to the lack of good ergodic properties of the system.*"
from "Scaling Limits of Interacting Particle Systems" by C. Kipnis, and C. Landim (cf.[137] Page one).

"*Unfortunately, despite intense efforts, for many degrees of freedom deterministic chaos remains a poorly understood subject. I am convinced that deterministic chaos on the small scale is needed in order to have the kind of large scale dynamics we actually see.*"
from "Large Scale Dynamics of Interacting Particles" by H. Spohn (cf. [212] Page two).

Example: classical chain of oscillators

In order to further illuminate the above concepts, we bring a concrete, simple example, mostly from [145]. Consider a *classical* chain of one dimensional (an-)harmonic oscillators. The phase space is given by $\Omega_n = (\mathbb{R} \times \mathbb{R})^n$ for the finite system, and $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ for the infinite system. A typical configuration is given by $\{(\mathsf{q}_x, \mathsf{p}_x) | x = 1, \dots n\}$, where we label the particles by indices $x \in \{1, \dots, n\}$; furthermore, q_x, p_x, and m_x denote the position, momentum, and mass of the particle x, respectively. Each particle interacts with its nearest neighbors, and the interaction is described by a potential $V(\mathsf{r}_x)$, where r_x is the inter-particle distance: $\mathsf{r}_x := \mathsf{q}_{x+1} - \mathsf{q}_x$, and $V : \mathbb{R} \to \mathbb{R}$, is a "reasonable" potential. The Hamiltonian of the system reads

$$H_n = \sum_{x=1}^{n} \Big(\frac{\mathsf{p}_x^2}{2m_x} + V(\mathsf{r}_x) \Big) =: \sum_{x=1}^{n} \mathsf{e}_x. \tag{1.1.1}$$

Formally, the evolution of the chain is given by the following Hamiltonian dynamics:

$$\begin{aligned} \partial_t \mathsf{r}_x(t) &= \frac{\mathsf{p}_{x+1}(t)}{m_x} - \frac{\mathsf{p}_x(t)}{m_x}, \\ \partial_t \mathsf{p}_x(t) &= V'(\mathsf{r}_x(t)) - V'(\mathsf{r}_{x-1}(t)), \\ \partial_t \mathsf{e}_x(t) &= \frac{\mathsf{p}_{x+1}(t)}{m_{x+1}} V'(\mathsf{r}_x(t)) - \frac{\mathsf{p}_x(t)}{m_x} V'(\mathsf{r}_{x-1}(t)). \end{aligned} \tag{1.1.2}$$

We take all the masses equal to one. This dynamics has three obvious (locally) conserved quantities:

$$\sum_{x=1}^{n} \mathsf{r}_x, \qquad \sum_{x=1}^{n} \mathsf{p}_x, \qquad \sum_{x=1}^{n} \mathsf{e}_x, \tag{1.1.3}$$

corresponding to the total *volume*, *momentum*, and *energy* of the chain. One could regard these quantities as the macroscopic thermodynamic characteristics of the system. Correspondingly, these conserved quantities assist us to describe the **equilibrium state** by a family of stationary probability measures over the phase space, i.e., (Canonical) Gibbs states. These states are parameterized by tension, velocity, and inverse temperature $\tau, \mathfrak{p}, \beta$, and given by:

$$d\mu^n_{\tau,\mathfrak{p},\beta} = \prod_{x=1}^{n} \exp\Big(-\beta\Big(e_x - \mathfrak{p}\mathsf{p}_x - \tau\mathsf{r}_x\Big) - \mathcal{G}(\tau,\mathfrak{p},\beta)\Big) d\mathsf{r}_x d\mathsf{p}_x, \tag{1.1.4}$$

where $\mathcal{G}(\tau,\mathfrak{p},\beta)$ is the free energy, which is defined such that $d\mu^n_{\tau,\mathfrak{p},\beta}$ be a probability measure.

Denote the average w.r.t (1.1.4), by $\langle . \rangle_{\tau,\mathfrak{p},\beta}$. Thanks to the local equilibrium, we can obtain the internal energy $u(\tau,\beta) = \langle \mathsf{e}_x \rangle_{\tau,\mathfrak{p},\beta} - \frac{\mathfrak{p}^2}{2}$, and length $r(\tau,\beta) = \langle \mathsf{r}_x \rangle_{\tau,\mathfrak{p},\beta}$ as functions of inverse temperature, velocity, and tension:

$$\begin{aligned} &r(\tau,\beta) = \beta^{-1}\partial_\tau \mathcal{G}(\tau,\mathfrak{p},\beta), \\ &u(\tau,\beta) = -\partial_\beta \mathcal{G}(\tau,\mathfrak{p},\beta) + \tau r, \\ &\mathfrak{p} = \beta^{-1}\partial_{\mathfrak{p}} \mathcal{G}(\tau,\mathfrak{p},\beta). \end{aligned} \tag{1.1.5}$$

Accordingly, we can define the thermodynamic entropy $S(u,r)$ by the Legendre transform:

$$S(u,r) = \inf_{\tau,\beta} \Big\{ -\beta\tau r + \beta u + \mathcal{G}(\tau,0,\beta)\Big\}, \tag{1.1.6}$$

which leads us to the following relations:

$$\beta(u,r) = \partial_u S(u,r), \qquad \tau(u,r) = -\beta^{-1}(u,r)\partial_r S(u,r). \tag{1.1.7}$$

Let us take a chain with periodic boundary, and macroscopic volume one. Corresponding to the smooth macroscopic profiles $\tau, \mathfrak{p} : \mathbb{T} \to \mathbb{R}$, $\beta : \mathbb{T} \to \mathbb{R}_+$ a prototypical **local equilibrium state** can be defined as:

$$d\mu_{\tau(.),\mathfrak{p}(.),\beta(.)} = \prod_{x=1}^{n} \exp\Big(-\beta(\frac{x}{n})\Big(e_x - \mathfrak{p}(\frac{x}{n})\mathsf{p}_x - \tau(\frac{x}{n})\mathsf{r}_x\Big) - \mathcal{G}_n(\tau(.),\mathfrak{p}(.),\beta(.))\Big) d\mathsf{r}_x d\mathsf{p}_x. \tag{1.1.8}$$

One can observe that for a suitable potential, and proper boundary conditions, this state has the "nice" properties we outlined above; in particular, in the macroscopic point y we are locally in equilibrium with parameters $\tau(y), \mathfrak{p}(y)$, and $\beta(y)$. This can be expressed in the following weak sense: for any smooth function $f : \mathbb{T} \to \mathbb{R}$, we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} f\left(\frac{x}{n}\right) \langle \mathsf{r}_x \rangle_{\mu_{\tau(.),\mathfrak{p}(.),\beta(.)}} \to \int_{\mathbb{T}} G(y) r(y) dy, \tag{1.1.9}$$

where $r(.)$ is determined from $\tau(.)$, and $\beta(.)$ via (1.1.7). Notice that the same relation is true for the corresponding profile of energy and velocity as well. In fact, many authors define the local equilibrium state as any sequence of states having the property (1.1.9), or slightly modified version of this property (cf. [37] Section 4, [212] 2.40 and 2.39, [137] Section 2, definition 2.2).

Starting from the local equilibrium state, we are interested in the macroscopic evolution of $(\mathsf{r}_x, \mathsf{p}_x, \mathsf{e}_x)$, in the hyperbolic space-time scaling (nt, ny). In this scaling we expect that $(\mathsf{r}_x, \mathsf{p}_x, \mathsf{e}_x)$ converges weakly to the solution of the following hyperbolic system of PDEs equipped with proper boundary conditions and initial datum (a.k.a. compressible Euler system of equations) in the smooth regime:

$$\begin{aligned}
\partial_t \mathtt{r}(t,y) &= \partial_y \mathtt{p}(t,y),\\
\partial_t \mathtt{p}(t,y) &= \partial_y \tau\Big(\mathtt{u}(t,y), \mathtt{r}(t,y)\Big),\\
\partial_t \mathtt{e}(t,y) &= \partial_y \bigg(\tau\Big(\mathtt{u}(t,y), \mathtt{r}(t,y)\Big)\mathtt{p}(t,y)\bigg),
\end{aligned} \tag{1.1.10}$$

where $\mathtt{u} = \mathtt{e} - \frac{\mathtt{p}^2}{2}$ is the internal energy. This means for any continuous test function $f : \mathbb{T} \to \mathbb{R}$, we expect:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} f\big(\frac{x}{n}\big) \begin{pmatrix} \mathsf{r}_x(nt) \\ \mathsf{p}_x(nt) \\ \mathsf{e}_x(nt) \end{pmatrix} \to \int_0^1 f(y) \begin{pmatrix} \mathtt{r}(t,y) \\ \mathtt{p}(t,y) \\ \mathtt{e}(t,y) \end{pmatrix} dy. \tag{1.1.11}$$

Assuming the maintenance of local equilibrium, we have the following heuristics for (1.1.11):

$$\begin{aligned}
&-\frac{d}{dt}\frac{1}{n}\sum_{x=1}^{n} f\big(\frac{x}{n}\big) \begin{pmatrix} \langle \mathsf{r}_x(nt)\rangle_{\mu^n} \\ \langle \mathsf{p}_x(nt)\rangle_{\mu^n} \\ \langle \mathsf{e}_x(nt)\rangle_{\mu^n} \end{pmatrix} = \\
&-\sum_{x=1}^{n} f\big(\frac{x}{n}\big) \begin{pmatrix} \langle \mathsf{p}_{x+1}(nt)\rangle_{\mu^n} - \langle \mathsf{p}_x(nt)\rangle_{\mu^n} \\ \langle V'(\mathsf{r}_x(nt))\rangle_{\mu^n} - \langle V'(\mathsf{r}_{x-1}(nt))\rangle_{\mu^n} \\ \langle \mathsf{p}_{x+1}(nt)V'(\mathsf{r}_x(nt))\rangle_{\mu^n} - \langle \mathsf{p}_x(nt)V'(\mathsf{r}_{x-1}(nt))\rangle_{\mu^n} \end{pmatrix} \sim \\
&\frac{1}{n}\sum_{x=1}^{n} f'\big(\frac{x}{n}\big) \begin{pmatrix} \langle \mathsf{p}_{x+1}(nt)\rangle_{\mu^n} \\ \langle V'(\mathsf{r}_x(nt))\rangle_{\mu^n} \\ \langle \mathsf{p}_{x+1}(nt)V'(\mathsf{r}_x(nt))\rangle_{\mu^n} \end{pmatrix} \sim \int_0^1 f'(y) \begin{pmatrix} \mathtt{p}(t,y) \\ \tau\Big(\mathtt{u}(t,y), \mathtt{r}(t,y)\Big) \\ \mathtt{p}(t,y)\tau\Big(\mathtt{u}(t,y), \mathtt{r}(t,y)\Big) \end{pmatrix} dy,
\end{aligned} \tag{1.1.12}$$

where in the first line we used (1.1.2), then we performed a summation by parts. In the last step, *local equilibrium* is assumed.

It is believed that proving local equilibrium is related to ergodicity of the system. In fact, in [186] Olla, Varadhan, and Yau established that ergodicity of the infinite system is sufficient for proving the hydrodynamic limit. More precisely, in our case,

the infinite system is ergodic, if every "regular" (finite entropy density), translation invariant, and stationary probability ν be a convex combination of Gibbs measures $d\mu_{\tau,\mathfrak{p},\beta}$ (1.1.4), equivalently by equivalence of ensemble this means that the only local translation invariant conserved quantities for the infinite dynamics are given by energy momentum, and density.
Although it is expected that non-linear potential V could provide such ergodicity, proving this is still quite challenging and unreachable, (cf. [102] by Fritz, Funaki, and Lebowitz for more details).
One of the main strategies to obtain ergodicity is adding certain stochastic noise (random exchange of velocities, random velocity flip, etc. for example) to kill all the (possible) conserved quantities except the desired ones (cf. [186],[49], [144] as examples of this strategy). The main mathematical tool for controlling the macroscopic evolution in these examples is the relative entropy method introduced in [226] by Yau (cf. [186] as well). It is worth mentioning that in certain limits when we have energy conservation, this method does not work, and methods based on Wigner distributions can be exploited (cf. [146]).

Further notes and comments. For proving hydrodynamic limits, instead of perturbing the Hamiltonian dynamics with a conservative noise, we can consider stochastic microscopic dynamics rather than the Hamiltonian ones. In this approach, the underlying dynamics are assumed to be completely random, for example, (cf. [119], [226], [137] and references therein, [212] Section II and references therein, [83] and references therein, notably [138], [198]). These models may not seem as "fundamental" as Hamiltonian models. However, they are able to describe a fairly vast range of phenomena. In this case, we would like to mention that the pioneering work of Guo, Papanicolaou, and Varadhan [119], inspired by Fritz (cf. [101]), is particularly important. They follow the evolution of the relative entropy between the law of the process and global equilibrium. This approach has been generalized by Yau [226] by taking the relative entropy w.r.t. local equilibrium. Quoting Yau: "It is believed that the main difficulty in studying hydrodynamics is the lack of a stable mathematically controllable norm", we would like to mention that recently other useful norms have been exploited for this purpose, such as H_{-1} norm and Wasserstein distance (cf. [117]).

1.1.2 Model and Results

This section is devoted to explaining the main model of this thesis' first part, its significance and the results. We also provide a very rough background of the model by mentioning a handful of the most famous results in the literature concerning chain of oscillators. We also bring a few other examples where we find them inspiring.

So far the situation can be summarized as follows: obtaining the macroscopic evolution of conserved quantities and their corresponding currents for a "physical" system from its microscopic dynamics, also known as hydrodynamic limit, is a chal-

lenging mathematical problem. One of the most interesting cases of this program is when the underlying microscopic dynamics is given by the fundamental theories, i.e., either Newtonian/Hamiltonian dynamics or Schrödinger/Heisenberg dynamics for classical and quantum systems, respectively. In classical systems, having a mathematical proof is still elusive due to difficulties in proving the ergodic properties of the system. Moreover, it is believed that ergodicity is an omnipresent part of the picture. The situation is not different when the underlying dynamics is given by quantum mechanics. In fact, to the best of our knowledge, there are no examples where one rigorously proves a hydrodynamic limit for a deterministic quantum system, except in [181], where they adapted the relative entropy method to quantum systems. However, their work is based on an ergodicity assumption (cf. assumption III of [181]), which is quite challenging to prove for physical systems.

One of the *main purposes and novelties* of this work is proving the hydrodynamic limit for a simple interacting quantum system (one-dimensional unpinned chain of quantum harmonic oscillator with random masses). The main novelty comes from the fact that this model is among the first examples where one can prove the hydrodynamic limit for a quantum system rigorously. In fact, we prove that starting from a locally Gibbs state, after hyperbolic rescaling of time and space, the distribution of the elongation, momentum, and energy converges to the solution of the Euler equation. Moreover, our result indicates that the temperature profile does not evolve in any space-time scale; in particular, the thermal diffusion coefficient vanishes.
A similar result has been obtained for the classical counterpart of this system in [35]. These models both share a surprising feature: they are not *ergodic*; actually they are completely integrable with full set of conserved quantities. However, we are still able to close the macroscopic equation with three conserved quantities namely momentum, elongation, and energy, out of thermal equilibrium. We would like to emphasize that this result seems counter-intuitive (due to lack of ergodicity), if one believes in the general well-established picture we outlined in Section 1.1.1. We will outline the differences between the classical and quantum systems later (cf. Section 1.1.3).

We obtain the above-mentioned result thanks to two main phenomena in this chain. First is the *Anderson localization*, which decouples the mechanical and thermal energy, providing the closure of the equation for energy, similar phenomena happens in the classical chain as well [35]. The second phenomenon is purely quantum mechanical: it is similar to some sort of *decay of correlation* phenomena, which lets us circumvent the difficulties arising from the fact that Gibbs states are not product due to the quantum nature of the system.

Our result shed light on the transport properties of disordered unpinned chain in the hyperbolic space-time scale; mechanical energy's transport is ballistic, while thermal energy and other conserved quantities' transport will be suppressed thanks to **Anderson localization**. In addition, concerning the other scaling, we observe

that the transport of the thermal energy will be suppressed at any larger time-scale.

Model and Results

Formally, this system can be described by the following Hamiltonian:

$$H_n = \frac{1}{2}\sum_{x=1}^{n}\left(\frac{p_x^2}{m_x} + (q_{x+1}-q_x)^2\right), \tag{1.1.13}$$

where q_x is the position (multiplication) operator acting on particle x, p_x is the corresponding momentum operator with $[q_x, p_x] = i$, where we take $\hbar = 1$. Moreover, m_x denotes the mass of particle x, where they will be taken as i.i.d random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the the expectation w.r.t $\mathbb{P}$ by $\mathbb{E}$, and $\mathbb{E}(m_x) = \bar{m}$; furthermore, we denote the law of the distribution of the masses by μ. Finally, we have free boundary conditions: $q_0 = q_1$ and $q_n = q_{n+1}$. Notice the difference between (1.1.1), and (1.1.13), where we quantized the former. Moreover, the state space in the later is given by $L^2(\mathbb{R}^n)$ rather than $(\mathbb{R}\times\mathbb{R})^n$.
We would like to mention that this model could be understood as a model of vibration degrees of freedom in crystal lattices [3] (cf. [141] Chapter 2, [13] Chapter 23, [180]). In particular, this means that the particles are labeled by lattice points and they are *distinguishable*, even in case of the clean chain. Distinguishability becomes even more obvious for the disordered chain, since all the particles have different masses, almost surely.

The time evolution is given by the Heisenberg dynamics generated by H_n, i.e.

$$\dot{p}_x = i[H_n, p_x] = (\Delta q)_x, \quad \dot{q}_x = i[H_n, q_x] = \frac{p_x}{m_x}, \tag{1.1.14}$$

where Δ is the discrete Laplacian defined in (2.1.8).

This dynamics has n conserved quantities; however, we are only interested in the following three main ones: total energy: $H_n := \sum_{x=1}^{n} e_x$, total momentum: $\sum_{x=1}^{n} p_x$, and total elongation: $\sum_{x=1}^{n-1} r_x$, where we denoted:

$$r_x = q_{x+1} - q_x, \qquad e_x = \frac{1}{2}\left(\frac{p_x^2}{m_x} + r_x^2\right). \tag{1.1.15}$$

Notice that total energy H_n and total momentum $\sum_{x=1}^{n} p_x$ are truly conserved. In contrast, total elongation $\sum_{x=1}^{n-1} r_x$ is locally conserved, and global conservation of elongation breaks at the boundary due to boundary conditions $q_0 = q_1$, and $q_n = q_{n+1}$. We discuss the other conserved quantities in Remark 1.1.7.

[3] On the physical ground, one could think of these particles as ions/atoms in an insulator crystal interacting with their nearest neighbors. By taking masses random we can imagine that we have a *random alloy* where we have a combination of different ions/atoms in the lattice. In particular, having this picture in mind, it becomes clear that although each of these ions could be bosons/fermions as a quantum particle, they are distinguishable.

First, we let the chain to be in a locally Gibbs state corresponding to the aforementioned conserved quantities. This state is out of thermal equilibrium with a smooth temperature profile $\beta \in C^0([0,1])$, and out of mechanical equilibrium with a smooth profile of momentum and elongation $\bar{p}, \bar{r} \in C^1([0,1])$, with $\bar{r}(0) = \bar{r}(1) = 0$. Furthermore, we assume the chain's macroscopic center of the mass is fixed i.e., $\int_0^1 \bar{p}(y)dy = 0$ (We discuss this assumption in Appendix 2.B and Remark 2.B.1). The density operator of this state is denoted by $\rho^n_{\beta,\bar{p},\bar{r}}$, and is defined in (2.1.9). Then, we let the chain evolve in time, according to the Heisenberg equation of motion and denote $r_x(t), p_x(t), e_x(t)$ to be the solution of this equation, i.e. the solution to the following local conservation laws corresponding to our conserved quantities:

$$\begin{aligned} &\dot{r}_x(t) = \frac{p_{x+1}(t)}{m_{x+1}} - \frac{p_x(t)}{m_x}, \quad \dot{p}_x(t) = r_x(t) - r_{x-1}(t), \\ &\dot{e}_x(t) = \frac{p_{x+1}(t)r_x(t) + r_x(t)p_{x+1}(t)}{2m_{x+1}} - \frac{p_x(t)r_{x-1}(t) + r_{x-1}(t)p_x(t)}{2m_x}, \end{aligned} \tag{1.1.16}$$

where in the second equation, we emphasized the fact that $p_{x+1}r_x \neq r_x p_{x+1}$.
Given the temperature profile $\beta \in C([0,1])$, and the distribution of the masses μ, the macroscopic thermal energy profile is denoted by $\mathrm{f}^\mu_\beta : [0,1] \to \mathbb{R}$, defined in (2.4.5). We discuss this function in Section 2.4.1, and Appendix 2.A. Let us emphasize that a priori from the definition it is not clear if this function is "local". However, in Proposition 2.A.1.2 we prove that $\mathrm{f}^\mu_\beta(y) = \mathrm{f}^\mu(\beta(y))$, where $\mathrm{f}^\mu : (0,\infty) \to \mathbb{R}$ is the corresponding function in thermal equilibrium (cf. (2.4.35) for the definition, see also Remarks 1.1.6, 1.1.1).

Correspondingly, denoting $\bar{m} = \mathbb{E}(m_x)$, where $\mathbb{E}$ denotes the expectation w.r.t μ, the macroscopic profiles $\mathrm{r}(y,t)$, $\mathrm{p}(y,t)$, $\mathrm{e}(y,t)$ evolve according to the following conservation laws with proper boundary conditions:

$$\begin{aligned} &\partial_t \mathrm{r}(y,t) = \frac{1}{\bar{m}}\partial_y \mathrm{p}(y,t), \quad \partial_t \mathrm{p}(y,t) = \partial_y \mathrm{r}(y,t), \quad \partial_t \mathrm{e}(y,t) = \frac{1}{\bar{m}}\partial_y(\mathrm{r}(y,t)\mathrm{p}(y,t)), \\ &\mathrm{r}(y,0) = \bar{r}(y), \qquad \mathrm{p}(y,0) = \bar{p}(y), \qquad \mathrm{e}(y,0) = \frac{1}{2}\Big(\frac{\bar{p}(y)^2}{\bar{m}} + \bar{r}(y)\Big) + \mathrm{f}^\mu_\beta(y), \\ &\mathrm{r}(0,t) = \mathrm{r}(1,t) = 0. \end{aligned} \tag{1.1.17}$$

Notice that total momentum $\int_0^1 \mathrm{p}(y,t)dy$, is conserved by the evolution equation (1.1.17), thanks to the boundary condition $\mathrm{r}(0,t) = \mathrm{r}(1,t) = 0$. This fact further justifies our assumption $\int_0^1 \bar{p}(y)dy = 0$.

The main result of this manuscript is that, after hyperbolic rescaling of time and space, the empirical density of elongation, momentum, and energy, i.e. r, p, e, averaged under the locally Gibbs state $\rho^n_{\bar{p},\bar{r},\beta}$, converge to the solution of the macroscopic equation (1.1.17). The precise statement of the result is given in Theorem 2.1.1.

Remark 1.1.1. *Notice that in the classical case for $\tilde{\beta} \in (0,\infty)$, $\mathrm{f}^\mu(\tilde{\beta}) = \tilde{\beta}^{-1}$, while in the quantum case one can observe that this function depends on all the*

moments of the probability distribution of the masses, μ. Thanks to this observation, it is worth mentioning that in the classical case, the macroscopic equations only reflect the expectation of the masses $\mathbb{E}(m_x)$, whereas in the quantum case the whole distribution appears in the macroscopic picture through the function $\mathfrak{f}^\mu$. This fact could be counterintuitive, in the sense that for two distribution of the masses with similar mean, we can observe different average thermal energy.

Background

Before proceeding, let us briefly highlight some of the well-known results connected to this model in the literature [4]. In this section, we generally talk about one-dimensional models.
Harmonic oscillators have been used to model the thermodynamic properties of solids and fluids. In equilibrium, model of harmonic oscillators can predict the specific heat of crystals at high temperature as in Dulong–Petit law (cf. [89], [151] Section VI). These models can be used to predict the heat capacity of the solids at lower temperatures, as in the Einstein's model by considering the quantum nature of the oscillators. Subsequently, taking into account the interactions as in Debye's model, we can predict the heat capacity at low temperature as well (cf. [81], [91], Section VI of [151]).

Although these predictions agree with the experiment satisfactorily, pure classical and quantum harmonic models [5] are not complex enough to predict out of equilibrium phenomena such as heat conduction.
On the physical ground, since the work of Fourier in 1822 [21], one expects heat transfer in the bulk of material to be diffusive and satisfies the Fourier's law [6]:

$$J(T) = -\kappa(T)\nabla T; \qquad c_v(T)\frac{\partial T}{\partial t} = \nabla.(\kappa(T)\nabla T), \tag{1.1.18}$$

where $\kappa(T)$ is the diffusion coefficient, corresponding to conductivity of the material, and $c_v(T)$ is the specific heat per unit volume. $T : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is the material's temperature profile at time $t \in \mathbb{R}_+$, and J is the heat flux, i.e. the amount of heat transported through a unit surface per unit time.
In [201], Rider, Lebowitz, and Lieb studied a classical one-dimensional chain of harmonic oscillators. The Hamiltonian of this model is given by (1.1.1) with $V(\mathsf{r}_x) = \mathsf{r}_x^2$, and all the masses are equal. They prepared the chain in contact with two heat baths modeled by Fokker-Planck type operators at the boundaries with respective temperatures T_L and T_R. They observed that in the stationary state the heat flux is proportional to the temperature difference $|T_R - T_L|$, rather than temperature gradient $|T_L - T_R|/n$ [7] as it is prescribed by Fourier's law. This indicates that the

[4] This section narrates my personal history toward this problem, rather than a brief literature review.

[5] By pure harmonic models, we refer to models where all the masses are equal, and the interaction is given by quadratic forms.

[6] Here we talk about "general" behavior of solids, and we are not concerned by engineered or exotic counterexamples

[7] Here n represents the number of particles in the chain.

energy propagates ballistically rather than diffusively, and heat conductivity is proportional to the size of the system. This result confirms that pure harmonic models are not complex enough to model the heat transfer, this has been understood since the works of Peierls [189], [190].

In purely harmonic models, phonons propagate ballistically, and thermal conductivity diverges by the system's size. In order to enrich the model to reflect physical observations, we need to take into account the phonon scattering. This can be done by adding extra features to the model (anharmonicity, stochastic noise, impurities). Although there exists an extensive literature concerning thermal properties of chain of oscillators (more numerical studies and heuristic arguments and less fully rigorous results), we do not intend to review them here. We refer to (cf. [41], [155]) for general reviews. We bring here a few examples which are related to our analysis, and a handful of results only to give a general picture (most of these results have been chosen based on personal taste).

Either of the above-mentioned mechanisms can somehow mimic the phonon scattering and produce rich phenomena: considering anharmonic interactions as it has been done in the seminal work of Fermi–Pasta–Ulam–Tsingou (cf. [96]) may seems the most "natural" way; however, its rigorous treatment is extremely cumbersome. In this case, anharmonicity affects the conductivity, and it is possible to observe diffusive behavior $J \sim \frac{1}{n}$ (cf. [11]), or anomalous behavior $J \sim n^{-\frac{3}{5}}$ (cf. [165]). Let us mention that it is believed that when potential only depends on the displacement (unpinned case) the anomalous behavior is expected in $d = 1$ i.e, $J \sim n^{\alpha}$, with $\alpha < 1$, and in $d = 2$, $J \sim n^{\alpha}$, or $J \sim \log n$.

The second idea imitates anharmonicity effect qualitatively by a stochastic noise, conserving the total energy and possibly total momentum. Although this family of models may seem "artificial", their rigorous treatment is more plausible. Actually one can argue that these noises can model the "unknown" effect of the environment, or other degrees of freedom, and they are not completely "hypothetical". Moreover, they can be designed to reproduce the thermal behavior of anharmonic solids suitably at a qualitative level. As an example, the conductivity of a one-dimensional unpinned chain[8] subject to momentum and energy conserving noise diverges in $d = 1, d = 2$, and it is finite and positive in $d = 3$. Furthermore, the conductivity is positive and finite ($J \sim \frac{1}{n}$) in the case of pinned chain in all dimensions in this model. (cf. [23], [36], cf. [22] for a review on rigorous results in this domain).

Lastly, phonon scattering effects can be seen by adding impurities. To model impurities, one can consider disordered harmonic (classical) chains. This case was first investigated by Casher and Lebowitz in [63], and Rubin and Greer in [203]. By disordered we mean that the mass of the particles are i.i.d random variables with nice law in (1.1.13). In the presence of pinning potential, this chain will be a perfect insulator, and thermal conductivity vanishes as $n \to \infty$ (this is shown rigorously in

[8]By "pinned" we refer to a chain, with pinning potential at each site, i.e., in (1.1.1) we have a term of the form: $\sum_{x=1}^{n} \mathcal{V}(\mathsf{q}_x)$.

[33]). However, in the absence of pinning, as a consequence of momentum conservation, we have a surprising effect as it is observed in [85], [86] : thermal conductivity depends on the boundary conditions and spectral properties of the bath.
In [63], authors studied the classical disordered harmonic chain in contact with heat reservoirs modeled by Ornstein–Uhlenbeck processes at temperatures T_L, T_R. They deduced that thermal conductivity vanishes, and the steady state heat current decays as $\mathbb{E}(J_n) \sim (T_L - T_R)n^{-\frac{3}{2}}$, where n denotes the system's size, and $\mathbb{E}$ denotes the expectation with respect to the masses. This has been made rigorous in [6]. In [203], they examined the similar problem, where they model their heat bath with infinite chains of oscillators. Their study made rigorous in [222].

The classical disordered harmonic chains have been studied in higher dimensions as well [70], where they put the chain in contact with Langevin thermostats. They observed heuristically (by kinetic theory type arguments) and numerically, that steady state current's scaling depends on the boundary conditions, as well as the presence of the pinning potential. In particular, they recovered Fourier's law heuristically for three dimensional pinned system.

One can apply any combination of the aforementioned machineries (disorder, noise, anharmonicity) and study the corresponding dynamics. In fact, there are numerous works (rigorous and non rigorous) in that direction, such as [33] [210], [34], [130], [51], [87].

Finally, we can mention [2] and references therein, most notably [180] as instances of investigating disordered quantum harmonic chains. In this body of work, the dynamical localization of the chain in terms of zero-velocity Lieb Robinson bound has been studied. Notice that these models, are similar to the pinned case and they do not have the divergence of the localization length in the lower part of the spectrum (we discuss this in the next Section). Moreover, we should emphasize that although these systems and our model are many body quantum systems, embody localization phenomena, they are not example of "real" Many body localization (cf. [24], [1], [193]), since we can map them to effective one-particle Hamiltonian, and the localization mechanism in these models are more or less similar to one-body Anderson localization (we discuss this further in the next section).

Before proceeding, we should like to recall that studying quantum chains and their thermal properties (quantum spin chains) is a very active and growing field of research. However, reviewing this immense literature is out of our scope. Concerning these systems, their thermalization, and questions such as many body localization, entanglement, and eigenstate thermalization hypothesis we refer to [1], and [74] for a general review (mostly non-rigorous and physics oriented). Regarding open quantum systems, we refer to historical papers (cf. [26], [100], [99], [98]) modeling quantum heat bath by chain of oscillators and obtaining quantum Langevin equation (cf. [14] for a rigorous treatment). For a general theory of open quantum systems, where one can model the heat bath by Lindblad operators, we refer to [50]. (cf.

[7], cf. [15] for more rigorous account). We mention [88] as an example of a clean quantum harmonic chain in contact with a heat bath. Notice that engineering a bath (rigorously) such that it thermalizes a quantum chain by acting from the chain's two ends with the same temperature is a challenging problem.

1.1.3 Phenomenology and idea of the proof

In this section, first we consider a clean[9] quantum harmonic chain in thermal equilibrium, and out of mechanical equilibrium. We show Theorem 2.1.1 holds in this case. Then we argue that out of thermal equilibrium, our result does not remain true. Subsequently, we illustrate the role of the disorder. We explain that disorder changes the nature of the system by modifying the chain's eigenmodes. Then we sketch the proof of Theorem 2.1.1.

Thermal equilibrium

Let us emphasize that for the sake of brevity, our definitions and arguments in this section are not fully rigorous. However, it is possible to fill the gaps easily. In fact, all the hand-waving arguments of this section is made rigorous in Chapter 2.
Denote the space variable by $\xi \in \mathbb{R}^n$, and let the Hilbert space $\mathcal{H}_n = L^2(\mathbb{R}^n, d\xi)$, where $d\xi$ denotes the Lebesgue measure on $\mathbb{R}^n$, be the state space of our model with n distinguishable particles. We denote the elements of $\mathcal{H}_n$ by $|.\rangle$ notation [10]. As usual for $x \in \mathbb{T}_n = \{1, 2, \dots, n\}$, q_x denotes the position operator acting on particle x, i.e,

$$q_x \left|\psi(\xi_1, \dots, \xi_x, \dots, \xi_n)\right\rangle = \xi_x \left|\psi(\xi_1, \dots, \xi_x, \dots, \xi_n)\right\rangle .$$

Moreover, $p_x = -i\frac{\partial}{\partial \xi_x}$ denotes the corresponding momentum operator, where we take $\hbar = 1$, and we have the canonical commutation relation (CCR): $[q_x, p_y] = i\delta_{xy}$, $[q_x, q_y] = 0$, $[p_x, p_y] = 0$, where $[a, b] = ab - ba$. The Hamiltonian of the system is given by:

$$H_n = \frac{1}{2}\sum_{x=1}^{n} \left(p_x^2 + (q_x - q_{x-1})^2\right), \tag{1.1.19}$$

where we assumed periodic boundary conditions with fixed total volume, i.e, there exists a constant operator $\hat{\bar{r}}_o$ such that $q_0 \equiv q_n - \hat{\bar{r}}_o$, and $q_1 \equiv q_{n+1} - \hat{\bar{r}}_o$ [11]. We take mass of all the particles equal to one. Let us define the elongation operator as $r_x := q_x - q_{x-1}$, representing the operator corresponding to the length of the spring between particles x, and $x + 1$. Notice that the (CCR) in terms of this operator reads:

$$[r_x, p_y] = i\Big(\delta_{xy} - \delta_{(x-1)y}\Big), \qquad [r_x, r_y] = [p_x, p_y] = 0. \tag{1.1.20}$$

[9] As usual, by clean we refer to the case where all the masses are equal to one.

[10] Notice that starting from next chapter, $|\psi\rangle$ denotes the element of $\mathbb{R}^n$, and $\boldsymbol{|\psi\rangle}$ denotes the elements of $\mathcal{H}_n$. These notation will be defined accordingly.

[11] This assumption is not necessary; in fact, starting from next chapter we describe our system with fixed boundary condition. These assumptions make the computation easier. One should think of $\hat{\bar{r}}_o$ as a vanishing constant in large n limit.

In terms of the elongation operator, Hamiltonian reads:

$$H_n = \frac{1}{2}\sum_{x=1}^{n}(p_x^2 + r_x^2) =: \sum_{x=1}^{n} e_x, \tag{1.1.21}$$

where we defined $e_x := \frac{1}{2}(p_x^2 + r_x^2)$ as the energy of the x-th oscillator. Notice that p_x, and r_x, and consequently e_x, and H_n are essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ (space of Schwartz function from $\mathbb{R}^n \to \mathbb{C}$). We denote their closure with the same symbols (cf. [197], [180]).
The time evolution of these operators is given by the Heisenberg dynamics generated by H_n. Thanks to spectral theory for self adjoint operators, for all $t \in \mathbb{R}$, e^{itH_n} is well defined. Therefore, by using the Stone's theorem, we are able to define a one parameter group of automorphism τ_t^n on $\mathcal{B}(\mathcal{H}_n)$ (space of bounded operators on $\mathcal{H}_n$):

$$a(t) := \tau_t^n(a) = e^{itH_n} a e^{-itH_n}. \tag{1.1.22}$$

By using Stone's theorem, we can extend the domain of this dynamics to certain unbounded operators. In particular, $r_x(t) := \tau_t^n(r_x)$, $p_x(t) := \tau_t^n(p_x)$, and $e_x(t) := \tau_t^n(e_x)$ are well defined (cf. [197], [180]). Moreover, one can verify that these operators (and all the subsequent operators we will encounter) solve the Heisenberg equation in the strong sense on a proper dense domain:

$$\partial_t a = i[H_n, a]. \tag{1.1.23}$$

Notice that thanks to the canonical commutation relation (1.1.20), denoting $\dot{a} := \partial_t a$, we have :

$$\dot{p}_x = r_{x+1} - r_x, \qquad \dot{r}_x = p_x - p_{x-1}. \tag{1.1.24}$$

The evolution equation (1.1.24), can be solved by diagonalizing the Hamiltonian via a discrete Fourier transform and introducing proper bosonic operators. For $k \in \mathbf{I}_n := \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ define

$$\hat{r}_k = \frac{1}{\sqrt{n}}\sum_{x=1}^{n} e^{-(2\pi i)kx} r_x, \qquad \hat{p}_k = \frac{1}{\sqrt{n}}\sum_{x=1}^{n} e^{-(2\pi i)kx} p_x. \tag{1.1.25}$$

Denote the adjoint of an operator a by a^* and observe $\hat{r}_k^* = \hat{r}_{1-k}$, $\hat{p}_k^* = \hat{p}_{1-k}$. Moreover, define the bosonic operators a_k, and a_k^* for $k \in \mathbf{I}_n^o := \{\frac{1}{n}, \dots, \frac{n-1}{n}\}$ as follows ($\omega_k := 2|\sin(\pi k)|$ denotes the dispersion relation):

$$a_k = \frac{1}{\sqrt{2\omega_k}}\Big(\frac{\omega_k \hat{r}_k}{1 - e^{-2\pi i k}} + i\hat{p}_k\Big), \quad a_k^* = \frac{1}{\sqrt{2\omega_k}}\Big(\frac{\omega_k \hat{r}_k^*}{1 - e^{2\pi i k}} - i\hat{p}_k^*\Big). \tag{1.1.26}$$

From (1.1.20), and (1.1.25), we have for $k, k' \in \mathbf{I}_n$:

$$[\hat{r}_k, \hat{p}_{k'}] = i\delta_{k+k'}(1 - e^{-2\pi i k}), \quad [\hat{r}_k, \hat{r}_{k'}] = 0, \quad [\hat{p}_k, \hat{p}_{k'}] = 0. \tag{1.1.27}$$

Consequently, from (1.1.26), and (1.1.27), we get the following commutation relations for $k, k' \in \mathbf{I}_n^o$:

$$[a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0, \quad [a_k, a_{k'}^*] = \delta_{kk'}. \tag{1.1.28}$$

The Hamiltonian in terms of these coordinates is given by[12]:

$$H_n = \frac{1}{2}\left(\hat{\bar{r}}_o^*\hat{\bar{r}}_o + \hat{p}_0^*\hat{p}_0\right) + \sum_{k\in\mathbf{I}_n^o}\omega_k\Big(a_k^*a_k + \frac{1}{2}\Big) = \frac{1}{2}\sum_{k\in\mathbf{I}_n}\left(\hat{r}_k^*\hat{r}_k + \hat{p}_k^*\hat{p}_k\right). \tag{1.1.29}$$

Thanks to (1.1.29), and commutation relations (1.1.28), we can solve the evolution equation (1.1.24):

$$a_k(t) = e^{-i\omega_k t}a_k(0), \qquad a_k^*(t) = e^{i\omega_k t}a_k^*(0), \qquad \hat{p}_0(t) = \hat{p}_0(0), \quad \forall k \in \mathbf{I}_n^o, \tag{1.1.30}$$

$$\begin{aligned} p_x(t) &= \frac{1}{\sqrt{n}}\left(\hat{p}_0 + \sum_{k\in\mathbf{I}_n^o} e^{2\pi ikn}\sqrt{\frac{\omega_k}{2}}(-i)\Big(e^{-i\omega_k t}a_k(0) - e^{i\omega_k t}a_{1-k}^*(0)\Big)\right), \\ r_x(t) &= \frac{1}{\sqrt{n}}\left(\hat{\bar{r}}_0 + \sum_{k\in\mathbf{I}_n^o} e^{2\pi ikn}\frac{1-e^{-2\pi ik}}{\sqrt{2\omega_k}}\left(e^{-i\omega_k t}a_k(0) + e^{i\omega_k t}a_{1-k}^*(0)\right)\right). \end{aligned} \tag{1.1.31}$$

Take $\beta > 0$, and $\mathtt{p}, \mathtt{r} \in C^1([0,1])$, such that $\mathtt{p}(0) = \mathtt{p}(1)$, $\mathtt{r}(0) = \mathtt{r}(1)$, and $\int_0^1 \mathtt{r}(y)dy = \int_0^1 \mathtt{p}(y)dy = 0$. Initially, assume that our chain is in a mixed state, described by a density operator $\rho_n \in \mathcal{B}(\mathcal{H}_n)$, ($\rho_n$ is a trace class, positive, bounded operator on $\mathcal{H}_n$, with $\mathrm{Tr}(\rho_n) = 1$). For any operator a, such that $a\rho_n$ is trace-class, we denote the average of a w.r.t ρ_n by $\langle a\rangle_{\rho_n} := \mathrm{Tr}(\rho_n a)$. Moreover, we assume ρ_n has the following properties. By abusing notation, for any two operators a, b, denote $\mathrm{Cov}(a,b) := \langle ab\rangle_{\rho_n} - \langle a\rangle_{\rho_n}\langle b\rangle_{\rho_n}$ [13]:

1. $\langle p_x\rangle_{\rho_n} = \mathtt{p}(\frac{x}{n})$, $\langle r_x\rangle_{\rho_n} = \mathtt{r}(\frac{x}{n})$, $1 \le x \le n$ [14],
2. $\mathrm{Cov}_{\rho_n}(r_x, r_y) = \mathrm{Cov}_{\rho_n}(p_x, p_y) = \tilde{C}_n^\beta(x-y) \quad 1 \le x, y \le n$, with

$$\tilde{C}_n^\beta(x-y) = \epsilon_o + \frac{1}{\beta n}\sum_{k\in\mathbf{I}_n^o} e^{2\pi ik(x-y)}\frac{\omega_k\beta}{2}\coth\left(\frac{\omega_k\beta}{2}\right). \tag{1.1.32}$$

3. $\mathrm{Cov}_{\rho_n}(r_x, p_y) = \frac{i}{2}(\delta(x-y) - \delta(x-y-1)) \quad 1 \le x, y \le n$,

where ϵ_o is a vanishing constant, we choose later. Notice that the "locally Gibbs state" corresponding to β, $\mathtt{p}$, and $\mathtt{r}$ satisfies these conditions as well [15]. We define this state at the end of this section.

[12]Notice that here $\hat{\bar{r}}_o$ is the constant corresponding to length of the chain, and $\hat{p}_0$ is the total momentum operator.

[13]Note that $\mathrm{Cov}(a,b)$ is not equal $\mathrm{Cov}(b,a)$, unless in case $[a,b] = 0$.

[14]This assumption implicitly means that we should take the constant $\hat{\bar{r}}_o$ corresponding to length of the chain equal to $\hat{\bar{r}}_o := \sum_{x=1}^n \bar{r}(\frac{x}{n})$.

[15]Later, we discuss the construction of our locally Gibbs state, every "reasonable construction", satisfy the above mentioned conditions up to a vanishing error (cf. Appendix 2.B in Chapter 2).

In the macroscopic level, recall the following system of hyperbolic conservation laws, where we modify the initial condition:

$$\begin{aligned}
&\partial_t \mathrm{r}(y,t) = \partial_y \mathrm{p}(y,t), \quad \partial_t \mathrm{p}(y,t) = \partial_y \mathrm{r}(y,t), \quad \partial_t \mathrm{e}(y,t) = \partial_y(\mathrm{r}(y,t)\mathrm{p}(y,t)),\\
&\mathrm{r}(y,0) = \mathtt{r}(y), \qquad \mathrm{p}(y,0) = \mathtt{p}(y), \qquad \mathrm{e}(y,0) = \frac{1}{2}\Big(\mathtt{p}(y)^2 + \mathtt{r}(y)^2\Big) + \mathrm{f}_\beta,\\
&\mathrm{r}(0,t) = \mathrm{r}(1,t), \quad \mathrm{p}(0,t) = \mathrm{p}(1,t),
\end{aligned} \tag{1.1.33}$$

where f_β can be obtained explicitly: we express following general relation, in case all the particles have mass m instead of one:

$$\mathrm{f}_\beta = \beta^{-1} \int_0^1 \frac{\beta\hbar}{\sqrt{m}} \sin(\pi k) \coth\Big(\frac{\beta\hbar}{\sqrt{m}} \sin(\pi k)\Big) dk. \tag{1.1.34}$$

For any $k \in \mathbf{I}_n$, we denote the discrete Fourier transform of $\tilde{C}_n^\beta$ by $\hat{C}_n^\beta(k)$. One can observe that $\hat{C}_n^\beta(k) = \frac{\omega_k}{2} \coth\big(\frac{\beta\omega_k}{2}\big)$. Notice that $\mathrm{f}_\beta = \lim_{n\to\infty} \sum_k \hat{C}_n^\beta(k)$. Assuming ρ_n satisfies assumptions 1, 2, and 3, we have the following theorem as a counterpart of Theorem 2.1.1: let $\bar{r}_x(nt) := \langle r_x(nt)\rangle_{\rho_n}$, $\bar{p}_x(nt) = \langle p_x(nt)\rangle_{\rho_n}$, $\bar{e}_x(nt) := \langle e_x(nt)\rangle_{\rho_n}$ for every $x \in \mathbb{T}_n$, and $t \in [0,T]$ for a fixed $T > 0$.

Theorem 1.1.1. *Denote $\mathfrak{u}_n(x,t) := (\bar{r}_x(nt), \bar{p}_x(nt), \bar{e}_x(nt))$, for $x \in \mathbb{T}_n$, then $\mathfrak{u}_n(x,t)$ converges weakly to $\mathfrak{u}(y,t) := (\mathrm{r}(y,t), \mathrm{p}(y,t), \mathrm{e}(y,t))$ i.e, for every smooth test function $f \in C^0([0,1])$:*

$$\frac{1}{n}\sum_{x=1}^n \mathfrak{u}_n(x,t) f\Big(\frac{x}{n}\Big) \to \int_0^1 \mathfrak{u}(y,t) f(y) dy, \qquad \forall t \in [0,T], \tag{1.1.35}$$

as $n \to \infty$.

In the rest of this section, we sketch the proof of Theorem 1.1.1. As we mentioned before, filling the gaps are rather easy thanks to the arguments in Chapter 2.

Remark 1.1.2. *Theorem 1.1.1 implies that our result is true in thermal equilibrium for a clean chain. The same conclusion holds in the classical case, for a clean chain in thermal equilibrium i.e., the microscopic dynamics converges to the solution of the Euler equation* (1.1.33)*, with $\mathrm{f}_\beta(y) = \beta^{-1}$ (cf. [35] Section 1.1).*

Notice that the the constant in (1.1.34)*, in the classical limit $\hbar \to 0$, converges to the previous value of $\mathrm{f}_\beta = \beta^{-1}$. Moreover, the asymptotic of f_β in* (1.1.34) *for large masses or high temperatures ($m \gg 1$ or $\beta^{-1} \gg 1$) is equal to β^{-1} as well.*

Sketch of proof of Theorem 1.1.1. Recall the evolution equation (1.1.24), since this equation is linear, thanks to linearity of the trace, and Assumption 1, we deduce that $\bar{p}_x(nt)$ and $\bar{r}_x(nt)$ satisfy the following system of ODEs as functions on $\mathbb{R}^n$:

$$\begin{aligned}
&\dot{\bar{p}}_x = \bar{r}_{x+1} - \bar{r}_x, \qquad \dot{\bar{r}}_x = \bar{p}_x - \bar{p}_{x-1}.\\
&\bar{p}_x(0) = \mathtt{p}(\frac{x}{n}), \quad \bar{r}_x(0) = \mathtt{r}(\frac{x}{n}).
\end{aligned} \tag{1.1.36}$$

For any $\mathtt{k} \in \mathbb{Z}$, $\hat{\mathtt{p}}(\mathtt{k})$, $\hat{\mathtt{r}}(\mathtt{k})$ denote the Fourier transform of $\mathtt{r}$, and $\mathtt{p}$ on the circle. Observe that for any fixed $\mathtt{k} \in \mathbb{Z}$, $\frac{1}{\sqrt{n}}\hat{\bar{r}}_{\frac{\mathtt{k}}{n}} \to \hat{\mathtt{r}}(\mathtt{k})$ and $\frac{1}{\sqrt{n}}\hat{\bar{p}}_{\frac{\mathtt{k}}{n}} \to \hat{\mathtt{p}}(\mathtt{k})$, where $\hat{\bar{p}}_{\frac{\mathtt{k}}{n}}$, $\hat{\bar{r}}_{\frac{\mathtt{k}}{n}}$ are discrete Fourier transform of $\bar{r}_x$ and $\bar{p}_x$. Consequently, for $\mathtt{k} \in \mathbb{Z}$, define

$$c\Big(\frac{\mathtt{k}}{n}\Big) := \Big(\frac{\omega_{\frac{\mathtt{k}}{n}}}{1 - e^{-2\pi i \frac{\mathtt{k}}{n}}}\hat{\bar{r}}_{\frac{\mathtt{k}}{n}} + i\hat{\bar{p}}_{\frac{\mathtt{k}}{n}}\Big),$$

and observe: $\frac{1}{\sqrt{n}}c\Big(\frac{\mathtt{k}}{n}\Big) \to i(\hat{\mathtt{p}}(\mathtt{k}) - \hat{\mathtt{r}}(\mathtt{k})) := \varphi_-(\mathtt{k})$, and $\frac{1}{\sqrt{n}}c^*\Big(1 - \frac{\mathtt{k}}{n}\Big) \to i(\hat{\mathtt{p}}(\mathtt{k}) - \hat{\mathtt{r}}^*(\mathtt{k})) := \varphi_+(\mathtt{k})$.
Thanks to (1.1.36), we have for $k := \frac{\mathtt{k}}{n}$: $c(k,t) = e^{-i\omega_k t}c(k)$, and $c^*(k,t) = e^{i\omega_k t}c^*(k)$. Therefore, we get for $\mathtt{k} \in \mathbb{Z}$:

$$\frac{1}{\sqrt{n}}c\Big(\frac{\mathtt{k}}{n}, nt\Big) \to e^{-2\pi i \mathtt{k} t}\varphi_-(\mathtt{k}), \tag{1.1.37}$$

as $n \to \infty$. From (1.1.37), using the definition of c we have:

$$\begin{aligned}
\frac{1}{\sqrt{n}}\hat{\bar{r}}\Big(\frac{\mathtt{k}}{n}\Big) &\to \frac{i}{2}(e^{-2\pi i \mathtt{k} t}\varphi_-(\mathtt{k}) + e^{2\pi \mathtt{k} i t}\varphi_+(\mathtt{k})) = \hat{\mathrm{r}}(\mathtt{k}, t),\\
\frac{1}{\sqrt{n}}\hat{\bar{p}}\Big(\frac{\mathtt{k}}{n}\Big) &\to \frac{-i}{2}(e^{-2\pi i \mathtt{k} t}\varphi_-(\mathtt{k}) - e^{2\pi i \mathtt{k} t}\varphi_+(\mathtt{k})) = \hat{\mathrm{p}}(\mathtt{k}, t).
\end{aligned} \tag{1.1.38}$$

where $\mathrm{r}(y,t)$, $\mathrm{p}(y,t)$ are solutions to the wave equation in (1.1.33), and $\hat{\mathrm{p}}(\mathtt{k},t), \hat{\mathrm{r}}(\mathtt{k},t)$ are their Fourier transforms. Notice that in (1.1.38), we take advantage of the explicit solution of the wave equation.
Finally, (1.1.38) basically finishes the proof of (1.1.35) for r and p. Notice that thanks to regularity of $\mathtt{r}$, and $\mathtt{p}$ one can say more: we can deduce that

$$\bar{r}_{[ny]}(nt) \to \mathrm{r}(y,t), \quad \bar{p}_{[ny]}(nt) \to \mathrm{p}(y,t). \tag{1.1.39}$$

as $n \to \infty$. Notice that one can adapt the argument in Section 2.3 of Chapter 2 for this purpose as well. In fact, this case is much simpler since we do not need any homogenization over the masses and estimates (2.3.7), and (2.3.8), are enough to prove the pointwise convergence. These estimates are direct consequences of regularity of $\mathtt{r}$ and $\mathtt{p}$, as well as conservation laws of the dynamics.

Proof of the limit in (1.1.35) corresponding to e, relies crucially on the following observation (recall the bosonic operator (1.1.26)):

$$\mathrm{Cov}(a_k^*, a_{k'}) = \delta_{k,k'}\left(\frac{1}{\exp(\beta\omega_k) - 1}\right) \implies \mathrm{Cov}(a_k^*(t), a_{k'}(t)) = \delta_{k,k'}\left(\frac{1}{\exp(\beta\omega_k) - 1}\right), \tag{1.1.40}$$

where the first identity follows by a straightforward computation from the definition of a_k, a_k^*, (1.1.26), definition of $\hat{r}_k$, $\hat{p}_k$ (1.1.25), and assumptions (2), and (3). The second equality comes from the time evolution identity in (1.1.30).

Plugging (1.1.40) into the time evolution (1.1.31) we get (up to a vanishing error, thanks to the definition of f_β):

$$\begin{aligned}\Big\langle p_x^2(nt)\Big\rangle_{\rho_n} &= \bar{p}_x^2(nt) + \frac{1}{\beta n}\sum_{k\in\mathbf{I}_n^o}\frac{\beta\omega_k}{2}\coth\left(\frac{\beta\omega_k}{2}\right) + \epsilon_n = \bar{p}_x^2(nt) + \mathrm{f}_\beta + \epsilon_n' \\ \Big\langle r_x^2(nt)\Big\rangle_{\rho_n} &= \bar{r}_x^2(nt) + \frac{1}{\beta n}\sum_{k\in\mathbf{I}_n^o}\frac{\beta\omega_k}{2}\coth\left(\frac{\beta\omega_k}{2}\right) + \epsilon_n = \bar{r}_x^2(nt) + \mathrm{f}_\beta + \epsilon_n'.\end{aligned} \tag{1.1.41}$$

Thanks to the definition of $e_x(nt)$ we deduce:

$$\big\langle e_x(nt)\big\rangle_\rho = \bar{p}_x^2(nt) + \bar{r}_x^2(nt) + \mathrm{f}_\beta + \varepsilon_n. \tag{1.1.42}$$

Taking the limit $n\to\infty$ in the later expression, we can conclude (1.1.35), thanks to the convergence results corresponding to r, p, namely (1.1.39). □

Remark 1.1.3. *Theorem 1.1.1 is still true, if one replace $\tilde{C}_n^\beta$ in Assumption 2, with an appropriate sequence of functions $\tilde{C}_n : \{1,\dots n\}\to\mathbb{R}$. More specifically, it is sufficient that $\tilde{C}_n$ has the following properties: denoting the discrete Fourier transform of $\tilde{C}_n$ by $\hat{C}_n$, $\hat{C}_n$ is positive, and symmetric ($\hat{C}_n(k) = \hat{C}_n(1-k)$ for $k\in\mathbf{I}_n$). Moreover, $\mathrm{f}_C := \lim_{n\to\infty}\sum_k \hat{C}_n(k)$ exists. In this case one should replace f_β in* (1.1.33)*, with f_C.*

Notice that in the proof, proving the propagation of the local equilibrium is not needed.

One can take ρ_n to be the locally Gibbs state of the local equilibrium corresponding to the profile $\mathtt{p}$, and $\mathtt{r}$, and temperature $\beta^{-1}>0$. Inspiring from the classical counterpart of this system one is tempted to define this state as:

$$\begin{aligned}\rho_n &= \frac{1}{Z_n'}\exp\left(-\beta H_n - \beta\sum_{x=1}^n\Big(p_x\mathtt{p}(\frac{x}{n}) + r_x\mathtt{r}(\frac{x}{n})\Big)\right) \\ &= \frac{1}{Z_n}\exp\left(-\frac{\beta}{2}\sum_{x=1}^n\left(\Big(p_x - \mathtt{p}(\frac{x}{n})\Big)^2 + \Big(r_x - \mathtt{r}\Big(\frac{x}{n}\Big)\Big)^2\right)\right).\end{aligned} \tag{1.1.43}$$

Notice that ρ_n can be viewed as $\exp\Big(-\beta\tilde{H}_n\Big)$, where $\tilde{H}_n$ is a "displaced" version of H_n. Defining $\tilde{p}_x = p_x - \mathtt{p}\Big(\frac{x}{n}\Big)$, and $\tilde{r}_x = r_x - \mathtt{r}\Big(\frac{x}{n}\Big)$, one could define the discrete Fourier transform of $\tilde{p}_x$ and $\tilde{r}_x$ as $\hat{\tilde{p}}_k$, $\hat{\tilde{r}}_k$ similar to (1.1.25), and diagonalize $\tilde{H}_n$ in terms of these new operators:

$$\tilde{H}_n = \frac{1}{2}\hat{\tilde{p}}_0^2 + \frac{1}{2}\sum_k\omega_k(\hat{\tilde{p}}_k^*\hat{\tilde{p}}_k + \hat{\tilde{r}}_k^*\hat{\tilde{r}}_k) = \frac{1}{2}\hat{\tilde{p}}_0^2 + \sum_k\omega_k(\tilde{a}_k^*\tilde{a}_k + \frac{1}{2}), \tag{1.1.44}$$

where $\tilde{a}_k$, $\tilde{a}_k^*$ is defined similar to a_k, and a_k^* (1.1.26) in terms of $\hat{\tilde{p}}_k$ and $\hat{\tilde{r}}_k$ rather than $\hat{r}_k$, and $\hat{p}_k$. The later diagonalization suggests that $\tilde{H}_n$ has a continuous spectrum. Moreover, one can observe that ρ_n is not trace-class. This technical difficulty stems from the fact that in our description we do not consider the center of the mass, and

center of mass momentum $\hat{p}_0 = \sum_x p_x$. However, $\hat{p}_0$ is conserved by the dynamics, and one could circumvent this difficulty in different ways: classicaly, we can describe our system by an observer in the center of the mass. Hence, our system is defined on an $n-1$ dimensional Manifold $\mathcal{M}$. Then, we can quantize the corresponding system. In this case ρ_n will be a trace-class operator on $L^2(\mathbb{R}^{n-1})$. We follow this line in Section 2.1 of Chapter 2. Alternatively, we can "fix" the total momentum as follows: we can separate our Hilbert space into two parts: one corresponding to the center of the mass, and the rest (classicaly corresponds to other $n-1$ degrees of freedom). Initially, we assume that the center of the mass is described by a pure state $\phi \in L^2(\mathbb{R})$. Then we can modify ρ_n as tensor product of the pure state $|\phi\rangle\langle\phi|$ and $\tilde{\rho}_n = \exp\Big(-\tilde{H}_n^+\Big)$, where $\tilde{H}_n^+ = H_n - \frac{\hat{p}_0^2}{2}$, we proceed this way in Appendix 2.B of Chapter2 [16].
In either case, ρ_n is a density operator, mimicking the behavior of (1.1.44). In fact, effectively it can be considered as $\exp\Big(-\beta\tilde{H}_n^+\Big)$, where $\tilde{H}_n^+$ is digonalized in (1.1.44) as sum of independent displaced oscillators. Moreover, its eigenvalue and eigenfunctions are completely understood in terms of Hermite polynomials (cf. Section 2.2 in Chapter 2). Therefore, computing the averages of $\tilde{a}_k$, $\tilde{a}_k^*$, and $\tilde{a}_k^*\tilde{a}_k$ w.r.t ρ_n is straightforward, upto a vanishing error which comes from our construction (cf. Proposition 5.2.28 of [48], and Appendix 2.C):

$$\langle \tilde{a}_k \rangle_{\rho_n} = \langle \tilde{a}_k^* \rangle_{\rho_n} = \langle \tilde{a}_k^* \tilde{a}_{k'}^* \rangle_{\rho_n} = \langle \tilde{a}_k \tilde{a}_{k'} \rangle_{\rho_n} = 0, \quad \langle \tilde{a}_k^* \tilde{a}_{k'}^* \rangle_{\rho_n} = \delta_{kk'} \left(\frac{1}{\exp(\beta\omega_k) - 1} \right). \tag{1.1.45}$$

Therefore, thanks to (1.1.45), and definition of $\tilde{a}_k$ we can verify that ρ_n satisfies all the assumptions of Theorem 1.1.1.

Remark 1.1.4. *Theorem 1.1.1 relies on the translation invariance of $\tilde{C}$, this theorem fails even in the classical counterpart of our system, when $\tilde{C}$ is space inhomogeneous. For example, if we take our system to be out of thermal equilibrium with an inhomogenous temperature's profile, one can observe that Theorem 1.1.1 is not valid anymore. In fact, one can easily adapt the argument in Section 1.1 of [35] and observe that for a clean chain, with a varying temperature's profile, although we have the wave equation, the evolution of the energy does not govern by the Euler equation, both in quantum and classical case. In this case, the evolution of the thermal energy is not autonomous (cf. [35] for more details in the classical case).*

Disordered chain: phenomenology

In the previous section, the hydrodynamic behavior of a clean chain is understood. Since the chain is harmonic, the microscopic evolution of r, p is linear and similar to the classical case. Therefore, in the hyperbolic scaling the macroscopic limit is

[16] There are other ways to overcome this difficulty. For example, taking an site potential, which makes $\tilde{H}_n$ having a discrete spectrum, and sending this potential to zero very fast in large n limit.

given by the wave equations in (1.1.33). Notice that this does not depend on the temperature's profile.
In the macroscopic level (1.1.33), the evolution of the energy can be solved explicitly thanks to the regularity of $\mathtt{r}$, and $\mathtt{p}$. This evolution is given by a mechanical part $\frac{1}{2}(\mathrm{p}^2(y,t)+\mathrm{r}^2(y,t))$, and a thermal part f_β which is constant in time. In the microscopic level, we have a similar decomposition thanks to (1.1.45), and (1.1.30). For a clean chain, out of thermal equilibrium, the evolution of r, p are still governed by the wave equation. Consequently, the evolution of the mechanical energy is given by the solution of the Euler equation. However, due to temperature gradient, the thermal fluctuation is not constant in time anymore, and thermal waves transport the thermal energy in the system.

Taking mass of the particles to be i.i.d random variables modifies this picture extensively: the evolution of r, and p is still governed by the wave equation; however proving this fact needs certain homogenization over the masses. Notice that only the expectation value of the masses, $\mathbb{E}(m_x)$, appears in the macroscopic scale. The mechanical energy is treated similar to the clean chain. The main effect of randomness appears in suppressing the thermal waves and permitting us to deduce that thermal energy does not evolve in time. In the following, we try to explain this phenomena.

A similar result has been obtained for the classical counterpart of this system in [35]. As we mentioned, the first feature of these systems which suppresses the thermal wave, permits us to close the equation for the energy, and proves the hydrodynamic limit is the localization phenomena expressed in Lemma 2.5.2, mathematically. As we observed in Section 1.1.2, models of disordered harmonic chains (both classical and quantum mechanical) have been studied extensively in the literature (cf. [6], [222], [63], [85], [203] for classical and [180] for quantum mechanical). Most of these models exhibit properties similar to the Anderson insulator [9], i.e., certain eigenmodes of these chains are spatially localized (we give a brief introduction to localization in Section 1.1.4). In our case, the random matrix appearing in the analysis of our system is $M^{-1}\Delta$, where M is the diagonal matrix of the masses. We denote the set of eigenvectors of $M^{-1}\Delta$ by $\{\psi^k\}_{k=0}^{n-1}$, where they are ordered increasingly according to their corresponding eigenvalues. Notice that these vectors appear in the solution of the time evolution equation (2.2.22). In the case of a clean chain this solution is given by (1.1.31), where in this case these vectors are simply Fourier coefficients.
In the disordered case, one can deduce from the conservation of momentum that the ground state of $M^{-1}\Delta$ is fixed, also known as "symmetry protected mode" [124]. Consequently, the localization length diverges as we approach the ground state, namely for $k \sim n^\gamma$, such that $\gamma \in [0, \frac{1}{2})$ is chosen properly. In fact, one can observe that the localization length ξ_k behaves asymptotically as $\xi_k \sim \omega_k^2 \sim (\frac{k}{n})^2$, as we take the limit $\frac{k}{n} \to 0$ properly (cf. [222] Theorem 1, estimate 3.8, or (4.2) in[6]), where ω_k denotes the eigenvalue of the clean chain with unit mass, i.e. $\omega_k = |2\sin\Big(\pi\frac{k}{n}\Big)|$.

Moreover, high modes (when k is not close to zero, $k \gg \sqrt{n}$) are completely localized.
In the microscopic level, the modes with $k \ll \sqrt{n}$ remain extended, i.e., similar to the eigenmodes of the clean chain (Fourier modes). First, the macroscopic evolution of r, p follows this low modes, this is proven via a mass homogenization in Section 2.3. In the microscopic and macroscopic level, we decompose the energy into the thermal and mechanical part. In the macroscopic level, by solving (1.1.17), the evolution of the energy is purely mechanical; in the microscopic level low modes transport the mechanical energy, this fact is proven in Lemma 2.5.1. In fact, the convergence of the microscopic mechanical energy to the macroscopic one can be deduced from the results of Section 2.3; in particular, the pointwise convergence of the momentum and elongation in Step 4 of Section 2.3. In the macroscopic scale, the thermal part of the energy does not evolve in time, since the macroscopic evolution of the energy is purely mechanical. In the microscopic level, the similar phenomenon can be proven thanks to the localization where we expressed in lemma 2.5.2, 2.5.3, exploiting the estimates appeared in [35], [6], [222]. In fact, since the high modes, i.e. $k \gg \sqrt{n}$ are localized, the thermal part of the energy (Thermal fluctuations) does not evolve in time, this has been proven in Lemma 2.5.4. Notice that the following remark is a direct consequence of this lemma:

Remark 1.1.5. *In Lemma 2.5.4, the proof can be adapted easily for other time scales, where we rescale time by a factor n^α with $\alpha \geq 1$. Therefore, the temperature's profile does not evolve in time at any time scale $n^\alpha t$ for $\alpha \geq 1$, including diffusive timescale, which means that the thermal diffusion coefficient vanishes.*

Quantum nature of the system: technical novelty

The main difference of our model with its classical counterpart [35] stems from the fact that the Gibbs state $\rho^n_{\bar{p},\bar{r},\beta}$ (2.1.9) is not a product state, since the energies of nearest neighbor particles do not commute with each other in the quantum case. This has both mathematical and physical consequences. Mathematically, this feature causes certain technical difficulties which can be treated by arguments similar to decay of correlation type arguments. Physically, this difference appears in the macroscopic evolution equation via the function $\mathfrak{f}^\mu_\beta(y)$ (cf. (2.4.5)). Let us emphasize the fact that this function depends on the distribution of the masses (and not only on its low moments such as mean and variance), whereas in the classical case it was equal to $\frac{1}{\beta(y)}$. Let us explain this difference a bit more.
First, this issue leads to some technical difficulties for obtaining certain bounds, which is treated by diagonalizing the pseudo-Hamiltonian H^n_β (2.1.10), appearing in the definition of $\rho^n_{\bar{p},\bar{r},\beta}$.
The more fundamental issue arising here is that for a fixed realization of the masses, $\langle \tilde{e}_x \rangle_{\rho^n}$, i.e., the average of the thermal energy of the particle x, which is computed in (2.2.47) and (2.2.46), depends on the whole configuration of the masses and the whole profile of the temperature $(\beta(\frac{1}{n}), \dots, \beta(\frac{n}{n}))$. In contrast, in the classical case it was simply equal to $\beta_x^{-1} = \beta(\frac{x}{n})^{-1}$. This difference also reflects in the macroscopic equation (1.1.17) in the function $\mathfrak{f}^\mu_\beta$, which is defined in (2.4.5). In contrast to the

classical case, where it was equal to $\frac{1}{\beta(y)}$. Notice that the quantum nature of our system survives in the macroscopic limit only through this function. In fact, in the limit $\hbar \to 0$, this function converges to its classical counterpart $\frac{1}{\beta(y)}$.
Since for a fixed realization of the masses the microscopic thermal energy $\left\langle \tilde{e}_{[ny]} \right\rangle_{\rho^n}$ depends on the whole configuration of the masses, one should think of it as a random variable. Moreover, a priori, it is not clear that the desired limiting object $\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} f(\frac{x}{n}) \langle \tilde{e}_x \rangle_{\rho^n}$, for a test function f, is deterministic. Furthermore, it is not obvious from the construction, if this object would be local in the sense that it only depends on macroscopic temperature $\beta(y)$. We devote Section 2.4.1 and Appendix 2.A to these issues. In order to show that this limit is deterministic, we prove that at each point x, the average energy's dependence on the mass of a particle y, far away from x, decays sufficiently fast. Then we use the Strong Law of Large Numbers for weakly dependent random variables. In order to prove this fact, we use arguments similar to decay of correlation. Here, we use the fact that $\mathfrak{f}(z) = \sqrt{z} \coth \sqrt{z}$ is analytic in a certain domain, then we represent the thermal energy as $\langle \tilde{e}_x \rangle_{\rho^n}$ in terms of $\mathfrak{f}(A_p^\beta) + \mathfrak{f}(A_r^\beta)$, for certain matrices A_p^β, A_r^β (cf. (2.2.52), (2.2.25)). We expand $\mathfrak{f}$ in terms of its Taylor series, and we use the fact that the matrices A_p^β, A_r^β appearing in the expansion are local, i.e., the mass of particle x only appears on the entries close to the diagonal term $(A_p^\beta)_{xx}$; hence, the expectation of the first $\frac{|x-y|}{2}$ terms is factorized, and the rest is small. This provides a sufficient decay to prove the law of large numbers for $\frac{1}{n} \sum_x f(\frac{x}{n}) \langle \tilde{e}_x \rangle_\rho$. By similar arguments we can show $\mathfrak{f}_\beta^\mu(y)$ is continuous and local.

Remark 1.1.6. *Notice that when n is finite, $\left\langle \tilde{e}_{[ny]} \right\rangle_\rho$ depends on the whole profile of temperature $(\beta(\frac{1}{n}), \dots, \beta(1))$. We prove that as $n \to \infty$, for any $y \in (0,1)$, $\mathfrak{f}_\beta^\mu(y)$ depends only on the macroscopic temperature $\beta(y)$. In fact, in Proposition 2.A.1.2, we prove that $\mathfrak{f}_\beta^\mu(y) = \mathfrak{f}^\mu(\beta(y))$ for all $y \in (0,1)$. Here, $\mathfrak{f}^\mu(\tilde{\beta})$ is the corresponding function in thermal equilibrium at inverse temperature $\tilde{\beta}$, i.e., the case where $\beta(y) = \tilde{\beta}$, for all $y \in [0,1]$. We define this function in* (2.A.1.2)*, and we observe that it is well defined (does not depend on y) in Corollary 2.A.21.*

1.1.4 Localization

In this section, we present a very brief introduction to localization. As we already mentioned, our result in Chapter 2 crucially rests on localization of the eigenmodes of the random matrice $M^{-1}\Delta$. The necessary localization estimates for our proof is stated in Lemma 2.5.2 without any proof, directly from [35], [6], and [222]. Using this estimate we prove 2.5.3 as another necessary estimate. This makes our result rather mysterious for readers without a priori knowledge about localization, since we bring Lemma 2.5.2 without any explanation. In this section, we briefly introduce the notion of localization, review related results concerning localization in the literature, and explain the underlying mechanism, heuristically, without proof. Notice that the concept of localization is studied extensively in the context of disordered

quantum systems, and the underlying mechanism is similar to the one we are using. Therefore, in this section we use notations and definitions from the former context. This part can skipped by readers familiar with localization. For more general introduction one can see (cf. [62], [216]).

We begin with a cartoonish picture. Consider a deterministic large matrice with size n, such as the discrete Laplacian Δ: $(\Delta f)_x = f_{x+1} - 2f_{x+1} + f_{x-1}$. This matrix can be diagonilized and its eigenvectors, $\psi^1, \ldots, \psi^n$, are Fourier modes $\psi^k_x \sim \frac{e^{2\pi i \frac{k}{n} x}}{\sqrt{n}}$. The normalized eigenvectors behave typically as follows: most of their entries are of order $\frac{1}{\sqrt{n}}$, and they seem "extended". Now, take certain entries of the matrix to be i.i.d random variables, for example, the diagonal entries. Then "typical behavior" of this new matrice's eigenvectors changes significantly. Notice that these new vectors are random. However, it turns out that in large n limit, they "typically" (for "most"of realizations of the random variables) share a similar feature, namely, most of their entries are extremely small, except few of them whom carry the mass [17] of the vector.

Definition

Let us make the above-mentioned heuristics more clear via a concrete example: Anderson tight-binding model [9] [18].

Let d be the space dimension, $\Lambda_n = [0, n]^d \cap \mathbb{Z}^d$, $|.|$ denotes the lattice norm, and consider the Hilbert spaces $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, with inner product $\langle ., . \rangle$, and $\mathcal{H}_n = \ell^2(\Lambda_n)$. We use the Dirac notation: denote the elements of $\mathbb{Z}^d$, by $|\psi\rangle$. In particular, for $x \in \mathbb{Z}^d$, we denote $|x\rangle := \delta_x$. Define the "discrete Laplacian" [19] as an operator on $\mathcal{H}$: for $\psi \in \mathcal{H}$ $(H_o\psi)(x) = -\sum_{y \in \mathbb{Z}^d, |y-x|=1} \psi(y)$. Let $(v_x(\omega))_{x \in \mathbb{Z}^d}$ be i.i.d random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with "nice"(compactly supported and smooth) law μ. Define the random potential $V_\omega : \mathbb{Z}^d \to \mathbb{R}$, as $V_\omega(x) = v_x(\omega)$. Correspondingly, for $\lambda > 0$ the Anderson model is defined as the following Hamiltonian operator on $\mathcal{H}$:

$$H_\omega := H_o + \lambda V_\omega. \tag{1.1.46}$$

This model is attributed to Anderson [9] [20], describing behavior of a single electron in a disordered media (more on this latter).
First, observe that H_ω is an operator-valued function on $(\Omega, \mathcal{F}, \mathbb{P})$, and its spectral properties depend on ω. We are mainly interested in "almost sure" properties. For any operator-valued functions $\mathcal{H}_\omega$, denote its spectrum by $\sigma(\mathcal{H}_\omega)$, and its pure point spectrum by $\sigma_p(\mathcal{H}_\omega)$. Notice that $\sigma(H_\omega) = \sigma(H_o) + \text{supp}(\mu) = [-2d, 2d] + \text{supp}(\mu)$.

First notion of localization can be defined as follows: we say $\mathcal{H}_\omega$ exhibit *spectral localization* in an energy interval $[a, b]$, if $\sigma(\mathcal{H}_\omega) \cap [a, b] \subseteq \sigma_p(\mathcal{H}_\omega)$, almost surely

[17] By mass we refer to ℓ^2 norm.
[18] All the notations of this sections are only used here.
[19] We absorb the diagonal terms in the potential
[20] In the Anderson model, the random potential is uniformly distributed on the interval $[-\frac{W}{2}, \frac{W}{2}]$.

w.r.t $\mathbb{P}$ [21].
Moreover, $\mathcal{H}_\omega$ exhibits *exponential spectral localization* in $[a,b]$, if $\mathcal{H}_\omega$ has spectral localization in $[a,b]$, and eigenfunctions corresponding to the eigenvalues in $[a,b]$ decay exponentially: for almost all ω, $\mathcal{H}_\omega$ has a complete set of eigenfunctions $(\psi_{\omega,n})_{n\in\mathbb{N}}$ in the energy interval $[a,b]$, such that:

$$|\psi_{\omega,n}(x)| \leq C_{n,\omega}\exp(-c|x-x_{n,\omega}|), \tag{1.1.47}$$

where $c>0$ depends on $[a,b]$, and it corresponds to the localization length, $C_{n,\omega}$ is uniformly bounded in n and ω, a.s., and $x_{n,\omega}$ is the localization center.

If one thinks of $\mathcal{H}$ as the Hamiltonian of a quantum system ($\mathcal{H}$ is a "nice" self-adjoint operator), localization of $\mathcal{H}$ affects transport properties of this system. By transport properties, one should think of the Schrödinger evolution of a wave function, corresponding to $\mathcal{H}$. More precisely, how does this evolution "spread" a finitely supported wave function in space. For example, one should recall that for a free Hamiltonian we have a ballistic transport. In particular, from spectral localization of $\mathcal{H}$ we can deduce absence of the ballistic transport in the following sense (cf. [208]).
Recall that the Hamiltonian of the system is given by $\mathcal{H}$, let the initial state be given by $\psi_o \in \mathcal{H}$, then at time t the time evolution is given by Schrödinger dynamics: $\psi(t) = e^{-it\mathcal{H}}\psi_o$, where $e^{-it\mathcal{H}}$ is defined thanks to spectral theory. Take an energy interval $I := [a,b]$, let χ_I be the characteristic function of I, and denote: $\psi_I(t) := e^{-i\mathcal{H}t}\chi_I(\mathcal{H})\psi_o$. Denote the position operator on $\mathcal{H}$ by X, i.e, for $\psi\in\mathcal{H}$, $(X\psi)(y) = y\psi(y)$. Then we define the p-th moment of X, projected into I, at time t, starting from ψ_o as $\langle |X|^p(t)\rangle_{I,\psi_o} := \langle \psi_I(t), |X|^p\psi_I(t)\rangle$.
Then the relation among spectral localization, and absence of ballistic transport can be expressed as follows. If $\mathcal{H}$ has spectral localization in I, and ψ_o be compactly supported, then we have(cf. [208]):

$$\lim_{t\to\infty}\frac{\langle X^2(t)\rangle_{I,\psi_o}}{t^2} = 0,$$

almost surely. Having in mind the Anderson model, one would expect stronger form of transport's suppression. However, there exists pathological examples (cf. [82], Appendix 2) of non-random operators $\mathcal{H}$, exhibiting spectral localization with "almost" ballistic transport: $\mathcal{H}$ has exponential spectral localization in the whole spectrum, and for a suitable compactly supported ψ_o, we have: $\lim_{t\to\infty}\frac{\langle X^2(t)\rangle_{I,\psi_o}}{t^{2-\delta}} = \infty$, for all $\delta>0$.
These examples inevitably lead us to a stronger definition for localization. *dynamical localization*. Recall $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, and for $x\in\mathbb{Z}^d$, $|x\rangle = \delta_x$.[22] We say $\mathcal{H}$ exhibits

[21] Notice that localization may depends on the corresponding "eigenvalue", in some part of the spectrum the corresponding eigenvectors are localized, and in other part of the spectrum the eigenvectors are extended.

[22] By this notation we mean $|x\rangle$ is a function from $\mathbb{Z}^d$ to $\mathbb{C}$ such that it returns 1 at point x and zero otherwise.

dynamical localization in an interval $I \subset \mathbb{R}$, if there exists $0 < c, C < \infty$, such that

$$\mathbb{E}\left(\sup_t \left|\langle x, e^{-it\mathcal{H}}\chi_I(\mathcal{H})y\rangle\right|\right) \leq Ce^{-c|x-y|}, \forall x, y \in \mathbb{Z}^d, \tag{1.1.48}$$

where c corresponds to the localization length.
From (1.1.48), one can deduce absence of "any transport" in the following sense. For all $p > 0$, and ψ_o compactly supported, we have:

$$\sup_t || \, |X|^p e^{-it\mathcal{H}}\chi_I(\mathcal{H})\psi_o||^2 < \infty, \tag{1.1.49}$$

almost surely. Notice that combining (1.1.49) with RAGE (Ruelle, Amrein, Georcescu, and Enss) theorem, (cf. [92], [8] [204]) we can deduce that Dynamical localization implies spectral localization.

Brief review

Let us briefly review a couple of the most important results concerning Anderson localization. Subsequently, we observe the connection between some of these results and our model.
Recall the Anderson model (1.1.46). In his seminal paper [9], Anderson predicted that this model exhibits localization in the sense of (1.1.49), for $d = 1, 2$, and for $d = 3$ with sufficiently large noise (λ should be sufficiently large). Recall that Anderson Model, try to describe the behavior of an electron in a disordered media. Therefore, localization in the sense of (1.1.49), means that we do not have any electric transport in the system and *electron diffusion coefficient* vanishes, this means that our system is an electric insulator a.k.a *Anderson insulator*. We stress that our system of disordered chain was a *thermal insulator*. In both of these systems, mechanism of transport's suppression is due to localization.
Localization in the strong sense in $d = 1$ with proper μ has been proven by Kunz and Souillard in 81' (cf. [149]). Subsequently, Carmona et. al. extended this result ($d = 1$) to certain singular potentials in 87' (cf. [61]).

Before proceeding, we should mention that the mechanism of localization in $d = 1$ is fundamentally different from $d > 1$. In particular, mathematical methods which have been used to proof localization in $d = 1$, are substantially different from $d > 1$. In fact, in $d = 1$, transfer matrix approach is proven to be useful and let us to benefit from strong tools from one dimensional dynamical systems, such as asymptotic of product of two by two random matrices. This approach fails in higher dimensions. Let us emphasize that in this manuscript we are mainly interested in one-dimensional systems; particularly, in our system in chapter (2), localization estimates has been obtained based on variation of these methods.

Although we are not interested in higher dimensions, let us review some of the results in $d > 1$ concerning H_ω (Anderson model). There are two main tools in proving localization in $d > 1$. First, is the multiscale analysis exploited by Frölich

and Spencer in 83' (cf. [103]). In case of large disorders or small energies, they obtained estimates for the decay of resolvent: $\langle x,(H_\omega - E)^{-1}y\rangle \leq C_{\omega,x}e^{-c|x-y|}$, for fixed $E \in \sigma(H_\omega)$, and a.a. ω, this also implies vanishing conductivity, using Kubo formula. Subsequently, Martinelli, and Scoppola 85'(cf. [169]) deduced the absence of absolutely continuous spectrum for large disorders or small energies. In 86' Simon and Wolff (cf. [209]) concluded the spectral localization in the sense of $\sigma(H_\omega)\cap[a,b]\subset\sigma_p(H_\omega)$, and exponential decay of eigenfunctions for large disorders in the entire spectrum, or for low energies and small disorders, by using methods developed in [103].
The second method in $d > 1$ is known as Fractional moments and is due to Aizenman and Molchanov 93' (cf.[5]). They proved localization at large disorder and extreme energies. Their method is based on decay of fractional moments of the Green function. Let $G(x,y;z) := \langle x,(H_\omega - z)^{-1}y\rangle$, then they showed that $\mathbb{E}(|G(x,y;E+i\epsilon)|^s) \leq Ce^{-c|x-y|}$, for a fixed $0 < s < 1$, uniformly in $\epsilon \neq 0$, and $E \in (a,b)$. Notice that these estimates are quite powerful, since they give dynamical localization in (a,b), as well as level repulsion (cf. [175]).

Transfer matrix approach: naive scheme

As we mentioned, for proving localization in $d = 1$, one can benefit from rather strong tools from dynamical systems, thanks to transfer matrix approach. In this section, we give a general picture about this scheme, and illustrate its connection to the model of disordered chain.
Recall the Anderson Hamiltonian H_ω (1.1.46), for a fixed energy E, we are looking for a solution $u \in \mathcal{H}$, to the following eigenvalue problem: $(H_\omega - E)u = 0$. By using the definition of H_ω, this means:

$$u(x+1) + u(x-1) + (V_\omega - E)u(x) = 0. \tag{1.1.50}$$

Define the vector $\underline{u} := \begin{pmatrix} u(x+1) \\ u(x) \end{pmatrix}$, and the matrix

$$T_x(\omega) := T_x(E,\omega) := \begin{pmatrix} E - V_x(E) & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1.51}$$

We can observe that $\underline{u}(x+1) = T_{x+1}\underline{u}(x)$.
Let $\Phi_x(E) := \Phi_x(\omega,E) := T_x(E,\omega)T_{x-1}(E,\omega)\dots T_1(E,\omega)$. Assuming the initial condition $\underline{u}(0)$, we define the solution of $(H_\omega - E)u = 0$ "to the right" as

$$\underline{u}(x) = \Phi_x(E)\underline{u}(0). \tag{1.1.52}$$

Similarly, one can define the solution of $(H_\omega - E)u = 0$ "to the left" as $\underline{u}(-x) = \Phi_{-x}(E)\underline{u}(0)$, where $\Phi_{-x}(E) := T^{-1}_{-x+1}(E)\dots T_0^{-1}(E)$, and $T_y^{-1}(E)$ denotes the inverse of $T_y(E)$.
Accordingly, we can define the Lyapunov exponents

$$\xi^{\pm}(\omega,E) := \lim_{N\to\pm\infty}\frac{1}{|N|}\ln(||\Phi_N(\omega,E)||),$$

where $||.||$ denotes the matrix norm. In an intuitive level, one can see that if $\xi^{\pm}$ be well-defined, and positive, almost surely, we can observe the exponential spectral localization, thanks to (1.1.52).
In fact, from the work of Fustenberg and Kesten in 60' (cf. [104])[23] one can deduce that for a fixed E, and almost all ω, both limits in the definition of $\xi^{\pm}$ exists, they are independent of ω, and $\xi(E) := \xi^{+}(E) = \xi^{-}(E)$ is a well defined Lyapunov exponent. Furthermore, by Oseledets multiplicative ergodic theorem (cf. [187], [12]), one can deduce the following. Positive $\xi(E)$ implies exponentially growing or exponentially decaying solution for $H_{\omega}u = Eu$. Finally, by Kotani's result in 84' (cf. [147]) we can deduce $\xi(E) > 0$ for almost all $E \in \mathbb{R}$. Concluding the dynamical localization in this model can be done thanks to the result of Kunz and Souillard (cf. [149]).

Connection to disordered chains

So far, we observed mathematical definition of localization, and outlined the scheme of transfer matrix approach in Anderson model. As we pointed out, in disordered harmonic chains, both quantum and classical, for our purposes, it is essential to understand the eigenmode's behavior of the random matrix $M^{-1}\Delta$, where Δ is the discrete Laplacian, and M is the diagonal matrix of the (random) masses. We apply the previous scheme here, where one should modify the transfer matrices properly. Let ψ be the eigenmode corresponding to the fixed eigenvalue γ. Then we define: (let us neglect the dependence on the event ω).

$$T_x(\gamma) := \begin{pmatrix} 2 - m_x\gamma^2 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.1.53}$$

Similar to (1.1.52), localization of the eigenmodes is closely connected to the norm of the following matrix product: $||\Phi_n(\gamma)|| := ||T_n(\gamma)T_{n-1}(\gamma)\dots T_1(\gamma)||$. Again, we can use similar multiplicative ergodic theorems (cf. [12]) and deduce that $||\Phi_n(\gamma)||$ grows as $e^{n\xi(\gamma)}$, where $\xi(\gamma) \geq 0$ is the Lyapunov exponent. The main difference of our disordered chain (without pining, with momentum conservation) with Anderson model is the Lyapunov exponent's scaling as $\gamma \to 0$. In the Anderson model ($d = 1$), $\xi(\gamma)$ is positive uniformly in γ. However, for the disordered chain $\xi(\gamma)$ scales as γ^2, when $\gamma \to 0$. This means that localization length ($\sim \frac{1}{\xi(\gamma)}$) diverges as $\gamma \to 0$, representing extended states in the lower edge of the spectrum. The later estimate is observed in [63], [184] [6], [222] (cf. [222] Theorem 1, estimate 3.8, or (4.2) in [6]). In fact, since the work of O'Connor (cf. [184]) it is understood that for $\gamma > n^{-\frac{1}{2}+\epsilon}$ with small $\epsilon > 0$, $\Phi_n(\gamma)$ decays exponentially. The behavior of the low modes, $\gamma < n^{-\frac{1}{2}+\epsilon}$, is further studied in [222], [6]. In [6], by a change of variable each transfer matrix is viewed as a Möbius transform. Subsequently, elements of $\Phi_n(\gamma)$ are represented in terms of an amplitude and a phase. In this representation, one can see the phase evolution as a discrete time Markov chain on a circle. This facilitates the essential estimates corresponding to low modes, by enabling to control the phase and amplitude jointly.

[23] Their result concerns a rather general class of stationary stochastic processes, instead of T_x.

Combing the above-mentioned tools, in particular estimates from [6], and [222], one can deduce the following form of localization [24] (cf. [35] Section 5.3). Recall the definition of $M^{-1}\Delta_n$, where Δ_n is the $n \times n$ matrix of discrete Laplacian with *free boundary condition*, M is the diagonal matrix of masses (i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with nice law). Denote the normalized eigenvectores of this matrix by $\{\psi^k\}_{k=0}^{n-1}$, where we ordered them in the increasing order corresponding to their eigenvalues, i.e, if γ_k denotes the eigenvalue corresponding to ψ^k, then $0 = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{n-1}$.

Lemma 1.1.2. *Let $0 < \alpha < \frac{1}{2}$, and let $I(\alpha) := (N^{1-\alpha}, N-1) \cap \mathbb{Z}$, then there exists $0 < c, C < \infty$ uniformly in n such that*

$$\mathbb{E}\left(\sum_{k \in I(\alpha)} |\psi_x^k \psi_y^k|\right) \leq C e^{-c\xi(\alpha)|x-y|}, \tag{1.1.54}$$

where $\xi(\alpha) = n^{-2\alpha}$.

Let us emphasize that Lemma 2.5.2 is a direct consequence of (1.1.54). In fact, Lemma 2.5.2 can be obtained from (1.1.54) by Chebychev's inequality.

1.1.5 Further comments, and open problems

We conclude the first part of this chapter, by a couple of rather scattered remarks, and outline possible future directions.

Other conserved quantities

In recent years, the idea of obtaining the macroscopic evolution of conserved quantities and their corresponding charges (hydrodynamic limits) has been adapted to integrable systems with infinite conserved quantities via the theory of generalized hydrodynamic and introducing the generalized Gibbs ensemble (GGE) [38],[64],[213]. Since the disordered harmonic chain is integrable, one could rightfully ask about other conserved quantities, we address this question in the following remark.

Remark 1.1.7. *Since other conserved quantities can be written as further gradients of p and r (See I_n in* (2.3.6) *as an example), by using the same strategy as we used for the energy, we can decompose them into two parts: one involving mechanical contributions, and the other involving thermal contributions, similar to* (2.5.49)*. The thermal terms is constant in time, using the similar argument we used for the thermal energy, thanks to the localization. However, since the conserved quantities have been obtained by taking further gradient of r and p, the mechanical terms vanish in hyperbolic scaling of time and space, thanks to regularity of the macroscopic initial profile. Therefore, in the hyperbolic scaling, all the other conserved quantities do not evolve in time.*

[24] This estimate is similar to dynamical localization, however, it is more convenient to express it in terms of finite dimensional spaces.

We emphasize that the later remark concerns the density operator corresponding to $\beta, \mathtt{p}, \mathtt{r}$. This means initially we consider the locally Gibbs state corresponding to $\beta, \mathtt{p}, \mathtt{r}$. In fact, the role of disorder is that it provide the closure of the macroscopic equation corresponding to e, such that our equations become autonomous.
We should emphasize that this remark does not address the situation where initially we prepare our system in the GGE with full set of conserved quantities. However, thanks to the construction one would expect that these quantities bears a small weight in the GGE.

Possible open problems

In this section, we briefly mention possible future directions and open problems. These problems have been chosen based on personal interest.
In chapter 2, we study the average of the fields $r_{[ny]}(nt)$, $p_{[ny]}(nt)$, $e_{[ny]}(nt)$ w.r.t the Gibbs state $\rho^n_{\bar{p},\bar{r},\beta}$. One could investigate the fate of the higher moments of these fields, such as their fluctuation.
In classical systems, hydrodynamic limits can be proven in the following sense: let μ^n_t be the law of the microscopic dynamics, this can be understood as the evolved Liouville measure in the Hamiltonian dynamics, or law of the microscopic stochastic process either in the case of lattice systems or stochastic perturbation of the Hamiltonian dynamics. Let $\varrho_{[ny]}$ denotes a microscopic quantity of interest, (in our model it could be either r, p, e), and $\mathrm{r}(y,t)$ denotes its macroscopic counterpart, which is obtained as a solution to the corresponding hydrodynamic equation. Then for a proper initial condition, and a proper test function f we have for any $\epsilon > 0$:

$$\lim_{n\to\infty} \mu^n_{n^\alpha t}\left(|\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\varrho_x - \int f(y)\mathrm{r}(y,t)| > \epsilon\right) \to 0, \tag{1.1.55}$$

or alternatively one can prove $\mu^n_{n^\alpha t}(|...|) = 0$. Notice that heuristically, the difference among (1.1.55) and our result (2.1.1), (1.1.35), is as follows: in (1.1.55), we are basically saying that starting from "most" of the initial configurations, we end up in the hyperbolic scaling near the solution to the macroscopic equation, where this "most" should be understood w.r.t the initial local equilibrium measure. However, (2.1.1) means we end up near the solution to the macroscopic equation in average. Somehow, this difference involves a law of large numbers type argument.
The open question is how can we define such a measure for a quantum system, corresponding to macroscopic profiles $\beta, \mathtt{r}, \mathtt{p}$? Notice that the density operator could be understood as a measure on eigenvectors of the Hamiltonian, where we would like to define a measure on the whole configuration space. If we could define such a measure, does our result hold in the sense of (1.1.55) for this measure? More precisely, can we prove the following statement: in the hyperbolic scaling, for "most" of the initial states (pure state) the average of r, p, e in those states, is given by solution to the Euler equations, where "most" should be understood w.r.t this measure? Notice that for our system, where at each lattice point we have an infinite dimensional Hilbert space, this problem could be more challenging, mathematically. We expect that in spin systems with finite dimensional state space, the formulation

of this question be less difficult mathematically. Moreover, one could expect that this question be closely related to the notion of ETH (eigenstate thermalization hypothesis, (cf. [214], [74])). Finally, we hope that the previous question (higher moments) could be related to this question as well.

The last open question is rather vague. In classical systems, there exists a successful strategy in proving hydrodynamic limits, namely, adding conservative noises. We are wondering if it is possible to adapt such strategies to quantum systems? One should probably first think of an easier problem: finding the quantum counterpart of hydrodynamic results concerning lattice models such as simple exclusion processes (probably these are good starting points in that regard: cf. [30], [31]).

1.2 NLS

In this Section, we provide a brief introduction to Chapter 3. First, we briefly review some basic preliminaries concerning non-linear Schrödinger equation (NLS) on $\mathbb{R}^d$, and $\mathbb{T}^d$. Then we review some results regarding NLS's Gibbs measure, and discrete non-linear Shcrödinger equation. Laying the groundwork, we describe our model as a mass conserving stochastic perturbation of the DNLS that models the action of a heat bath at a given temperature. Afterwards, we present our result, where we prove that the corresponding Gibbs distribution is the unique invariant measure. Moreover, in the one-dimensional cubic focusing case on the torus, we prove that in the limit for large time, continuous approximation and low temperature, the solution converges to the steady wave of the continuous equation that minimizes the energy for a given mass. Then we compare our results with other relevant works. Finally, we finish this section by plenty of open questions arising from the above-mentioned dynamics.

1.2.1 NLS in a glance

NLS on $\mathbb{R}^d$

Consider the following non-linear Schrödinger equation[25] (NLS) in d space dimension.

$$\begin{aligned} i\partial_t\psi(x,t) &= -\Delta\psi(x,t) + \kappa|\psi|^{p-1}\psi(x,t); \\ \psi : \Omega^d \times \mathbb{R}_+ \to \mathbb{C}; &\quad \psi(x,0) := \psi_0(x), \end{aligned} \tag{1.2.1}$$

where Δ denotes the Laplacian on $\mathbb{R}^d$, $p > 1$ determines the strength of the non-linearity, and $\kappa \in \{1, -1\}$, with $\kappa = -1$ corresponds to the focusing, and $\kappa = 1$

[25] It is called semi-linear Schrödinger equation as well.

to the defocusing case, in this section, we take $\Omega = \mathbb{R}$. Moreover, we assume that $\psi_0 \in H^1(\mathbb{R}^d)$.
This equation and its variations appear in many areas of physics, such as non-linear optics, condensed matter physics, biophysics and other areas (cf. [228]: NLS in fluid mechanics; [224]: mathematical study of DNLS's minimizer; [205], [97]: statistical mechanics of DNLS; BEC and NLS: [39]; biophysics: [105], [106], [176]).

Let us mention a couple of these examples from [154]. For instance in $d = 3$, $p = 3$ with $\kappa = -1$, (1.2.1) describes, in certain regimes, the Langmuir wave in a plasma (time evolution of the propagating electric field's envelope), $d = 2$, $p = 3$ corresponds to laser field propagating in non-linear medium (cf. [154] and references therein). The evolution of Bose Einstein condensation (BEC) can also be modeled by this equation. This can be derived rigorously from microscopic models (cf. [93], [94], [140] for rigorous derivation of NLS from microscopic dynamics; cf. Chapter one of [217] and references therein for a more general introduction into the physical applications of NLS; cf. [3] as well).

Mathematically, this equation belongs to the family of *dispersive non-linear PDEs*, which is a very active field of research with numerous conjectures and open problems (even at the level of heuristics, there are many unknowns in this field). From this family, one can mention KdV, non-linear wave equation (NLW), Benjamini Ono equation, etc. For a rather comprehensive (but old) list of results concerning global and local wellposedness of these equations, as well as certain decay estimates and scattering results, one can see the following webpage by Terry Tao et. al. https://www.math.ucla.edu/ tao/ Dispersive/. This book of Tao (cf.[218]) also provides a very good introduction. Concerning NLS, we refer to lecture notes of Cazenve (cf. [66]), and Bourgain's book (cf. [45]), for classical results such as Cauchy theory, and solutions' qualitative behavior. For a review of relatively recent results about blow-up in the focusing case, one can see lecture notes of Raphaël (cf. [194]).

In the following, we recall some of the well-known features of NLS, which we are interested in, mostly from (cf. [194]). First, observe that NLS has two important conserved quantities. Hamiltonian $\mathcal{H}$, and mass $\mathcal{M}$:

$$\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(x)|^2 dx + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |\psi(x)|^{p+1} dx, \qquad \mathcal{M}(\psi) = \int_{\mathbb{R}^d} |\psi(x)|^2 dx. \tag{1.2.2}$$

Momentum is another conserved quantity $\mathcal{I}m\,(\int \nabla u u^*)$. In fact, one can view NLS as an infinite dimensional Hamiltonian flow with Hamiltonian $\mathcal{H}$.
The defocusing case ($\kappa = 1$) is considered to be more understood. Besides the general theory, and wellposedness (cf. [66], [218]), in certain cases we have information about its long time behavior as well. In fact, under rather general assumptions, one expects that solutions to the defocusing NLS disperse, and behave qualitatively similar to the linear case ($\kappa = 0$) . In other words, for a general class of $\psi_o \in H^1$, for any compact set K, we have: $\lim_{t\to\infty} \int_K |\psi(x,t)|^2 \to 0$, where $\psi(x,t)$ denotes the strong solution to (1.2.1). This can be obtained thanks to the proper decay

estimates (Strichartz and Morawetz estimates cf. [218] Section 3.5, [66] Section 7).

In this manuscript, we mainly focus on the focusing case ($\kappa = -1$). The local well-posedness of (1.2.1) can be deduced from (cf. [112]), for $1 < p < q^*(d) - 1$, where q^* is the Sobolev exponent of the injection $\dot{H}^1 \to L^{q^*}$: $q^*(d) = \infty$ for $d = 1, 2$, and $q^*(d) = \frac{2d}{d-2}$ for $d \geq 3$. This means that there exists $0 < T \leq \infty$, such that $\psi(t,x) \in C([0,T), H^1(\mathbb{R}^d))$. Notice that (cf. [112]) also implies that for the maximal T: if $T < \infty$, then $\lim_{t \to T} ||\psi(x,t)||_{H1} = \infty$.
An important tool in studying NLS is Gagliardo-Nirenberg interpolation inequalities: for any $u \in H^1(\mathbb{R}^d)$ with $p \leq 1 + \frac{4}{d}$, there exists a constant $C(p,d) > 0$, such that:

$$\int_{\mathbb{R}^d} |u(x)|^{p+1} dx \leq C(d,p) \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \right)^{\frac{d(p-1)}{4}} \left(\int_{\mathbb{R}^d} |u(x)|^2 dx \right)^{\frac{p+1}{2} - \frac{d(p-1)}{4}} . \quad (1.2.3)$$

Thanks to (1.2.3), we have a priori bound on $||\psi(t)||_{H^1}$ for $p < 1 + \frac{4}{d}$; therefore, we can deduce the global well-posedness for $p < 1 + \frac{4}{d}$. Hence, for any t, $\psi(t)$ is bounded in H^1, and it is a solution to (1.2.1) with initial datum in H^1. This regime ($p < 1 + \frac{4}{d}$) is called the *mass sub-critical* regime. From now on, we assume $p < 1 + \frac{4}{d}$, unless stated otherwise.

Solitons on $\mathbb{R}^d$

It is believed that the long time behavior of the focusing case is substantially different from the defocusing case. In particular, absence of dispersion can be observed thanks to the following special class of solutions: *solitary waves*, or *standing waves*, where we also call them solitons. These are time periodic solutions of the form: $\psi(x,t) = e^{i\omega t}\phi(x)$, with $\omega > 0$. $\psi(x,t)$ is an H^1 solution to (1.2.1), iff ϕ solves the following non-linear elliptic ODE:

$$\Delta\phi - \omega\phi + |\phi|^{p-1}\phi = 0, \quad (1.2.4)$$

with $\phi \in H^1(\mathbb{R}^d)$. Take $\omega = 1$, generally, one can construct different solutions to (1.2.4) (cf. [28]). However, under certain assumptions (1.2.4) has a unique solution. For instance in $d = 1$, all the solutions to (1.2.4), are space translation of $Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{(p-1)x}{2})} \right)^{p-1}$. Moreover, in $d > 1$, if we assume that ϕ is a solution to (1.2.4), and *additionally we suppose ϕ to be nonnegative*, then ϕ should be translation of an exponentially decreasing C^2, radial profile $Q(r)$. In fact, $Q(r)$ is the unique, nonnegative radially symmetric solution to (1.2.4), which is called *ground state soliton* (recall that we take $\omega = 1$) (cf. [194] Proposition 1.2 and Proposition 1.3; for more information [218] Appendix B; originally: [111] [150]). Let us emphasize the fact that the uniqueness of positive solutions will be crucial later.
The above-mentioned profile Q, has a variational characterization (this is due to Cazenave, and Lions cf. [65]), which turns out to be very useful; in particular, for our purposes. Recall that we are in mass-sub critical regime $p < 1 + \frac{4}{d}$:

Proposition 1.2.0.1. *Take $\omega = 1$, and let Q be the unique, radially symmetric, and non-negative solution to (1.2.4). Fix $m > 0$, and consider the minimization problem*

$$\mathcal{E}_o(m) := \inf_{\psi \in H^1, \mathcal{M}(\psi)=m} \mathcal{H}(\psi). \tag{1.2.5}$$

The minimizer of (1.2.5) is attained on the family

$$Q_{\lambda(m)}(. - x_o)e^{i\gamma_o}, \quad x_o \in \mathbb{R}^d, \gamma_o \in \mathbb{R}, \tag{1.2.6}$$

where $Q_\lambda(x) := \lambda^{\frac{2}{p-1}} Q(\lambda x)$, and $\lambda(m)$ is uniquely determined by m such that $\mathcal{M}(Q_{\lambda(m)}) = m$.
Moreover, any minimizing sequence ϕ_n of (1.2.5), is relatively compact in H^1 up to a translation and phase shift: $e^{i\gamma_{n_k}}\phi_n(. + x_{n_k}) \to Q_{\lambda(m)}$ in H^1.

In fact, the above-mentioned proposition is similar to Proposition 1.5 of [194], and one can find the proof there.
First, notice that both $\mathcal{M}$, and $\mathcal{H}$ are invariant under spatial shift, and multiplication by a constant phase, this justifies (1.2.6). The proof of Proposition 1.2.0.1, rests on Lions' *concentration-compactness* technique (cf. [161], [162]), which let us deduce that any minimizing sequence of (1.2.5) has a convergent subsequence in L^{p+1}, and consequently in H^1. Afterwards, by using the fact that $\int |\nabla |u||^2 \leq \int |\nabla u|^2$, we can assume our minimizer u is real and positive. Therefore, by an Euler-Lagrange argument we can deduce that u is a positive solution to (1.2.4), where ω plays the role of Lagrange multiplier (in fact, by Pohazaev integration ω is uniquely determined by m: $\omega(m) = \frac{d+2-p(d-2)}{m(\frac{d(p-1)}{2}-2)}\mathcal{E}_o(m) > 0$). Finally, we observe that $v(x) = \omega^{\frac{1}{p-1}} u(\omega^{\frac{1}{2}} x)$ is positive and satisfies (1.2.4) (with $\omega = 1$); therefore, by using the fact that positive solutions to the later is unique and equal to Q up to a translation, we can conclude the proof of Proposition (1.2.0.1) [26].
We stress that in Proposition 1.2.0.1, the characterization of the minimizer relies on the good understanding of the solution to the ODE (1.2.4).

One of the direct consequences of the above variational characterization, is the *orbital stability* of the ground state solitons. We say the ground state Q is orbitaly stable in the functional space $\mathcal{X}$, if for all $\epsilon > 0$, there exists $\delta(\epsilon)$, such that for all $\psi_o \in \mathcal{X}$, with $||\psi_o - Q||_{\mathcal{X}} < \delta(\epsilon)$, we have for all $t \in \mathbb{R}_+$:

$$\inf_{x_o \in \mathbb{R}^d, \gamma \in \mathbb{R}} ||\psi(t, x) - Q(x - x_o)e^{i\gamma}||_{\mathcal{X}} \leq \epsilon, \tag{1.2.7}$$

where $\psi(t, x)$ denotes the solution to (1.2.1), with initial datum ψ_o. Orbital stability of Q in $H^1(\mathbb{R}^d)$ is a straightforward consequence of Proposition 1.2.0.1, and conservation of the energy and the mass (this is first proven in [65]).

Another significance of the solitons is that they are believed to be involved in the long time behavior of focusing NLS (1.2.1). In fact, studying NLS's long time

[26] Notice that we used the fact that for the minimizer $\int |\nabla u|^2 = \int |\nabla |u||^2$, and therefore, we only can multiply v by a constant phase or translate is spatially, in case of non-real solutions.

behavior is a very active field of research. For example, one would like to know if besides stability, do we have *asymptotic stability*? In the following sense that: do the solutions converge to the ground state in certain norms or semi-norms as $t \to \infty$? (cf. [58], [110], [202], [211], [148], [167] for progress in case of other non-linearities/dispersive equations).
Another closely related question concerns the *generic long time behavior* of NLS. In fact, numerous numerical and theoretical evidences suggest that asymptotic long time behavior of the NLS, for a "generic" initial data, decouples into a dispersing part (vanishing in L^∞/"radiating part"), and another non-dispersive part corresponding to a couple of solitons moving at different speeds ("localized" part). This is the content of so-called *Soliton Resolution Conjecture* (cf. [218] p. 154,), which is a major open problem in the field. Notice that this statement is rather vague, and one needs to formulate these vague words into a rigorous statement, in order to solve this conjecture. Particularly, this description is not true for all initial data, and making these words rigorous involves the definition of "generic", as well. Regarding current progress in this direction, one can mention (cf [202], where a similar result is proven for KdV equation, and [168], where similar theorem is proven for a very specific non-linearity for NLS (not a pure-power)). Notice that in case $p = 3$, $d = 1$, due to integrability, the asymptotic stability is wrong (cf. [227]).
More specifically, we are intrested in [67], where Chatterjee studied this conjecture in certain probabilistic sense, by discretizing the domain (more on this later). In fact, our work in chapter 3 is motivated by this conjecture, and Chatterjee's work.

Let us conclude this section by mentioning that in the super-critical regime $p > 1 + \frac{4}{d}$, solitons can be used to construct blow-up solutions (solutions with diverging H^1 norm in finite time). In case we do not have any blow-up, it is believed that soliton resolution conjecture is still true in this regime.

NLS on $\mathbb{T}$

Recall the non-linear Schrödinger equation (1.2.1), in this section we take $\Omega = \mathbb{T}$. NLS on $\mathbb{T}^d$ is believed to be more complicated and less explored comparing to $\mathbb{R}^d$. In particular, understanding long time behavior seems to be more involved. Naively, this difficulty stems from the fact that on a bounded domain, our field cannot "escape" to infinity. Let us illustrate this complication by an example. The defocusing ($\kappa = 1$), cubic ($p = 3$), NLS on $\mathbb{R}^d$ with $d > 1$ has dispersive behavior under general assumptions. However, if we take the cubic defocusing NLS on $\mathbb{T}^d$ (or any rational tori) with $d > 1$, one can construct solutions with arbitrarily small initial H^s norm (for any $s > 1$), such that after a sufficiently large time, their H^s norm be arbitrarily large. This happens due to energy transfer from low modes to higher modes (energy cascade) [27] (cf. [71], [207]).

In the following, we are going to recall certain results concerning "soliton-like"

[27] Notice that for $d = 1$ the cubic NLS is integrable, and energy cascade is absent, since we can control higher Sobolev norms thanks to higher order conserved quantities.

solutions in the periodic set-up. As far as we are aware, these solutions are mostly studied in $d = 1$. In order to benefit from PDEs' literature, we restrict ourselves to the following set-up. As before, we are interested in the focusing case, we restrict ourselves to the cubic ($p = 3$), one-dimensional case, with periodic boundary condition [28].

More specifically, consider the following non-linear cubic Schrödinger equation:

$$\begin{aligned} &i\partial_t\psi(x,t) = -\partial_{xx}\psi(x,t) - |\psi(x,t)|^2\psi(x,t),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ &\psi(x,0) = \psi_0(x),\ \psi_0 \in H^1_{per}([0,L]), \end{aligned} \tag{1.2.8}$$

where we assume the periodic boundary condition by the definition of $H^1_{per}([0,L]) = H^1(\mathbb{T}_L)$ as: $H^1(\mathbb{T}_L) := H^1_{per}([0,L]) = \{u \in H^1_{loc}(\mathbb{R},\mathbb{C}) | \forall x \in \mathbb{R}, \quad u(x+L) = u(x)\}$, with the following norms and inner product ($\bar{v}$ indicates the complex conjugate):

$$||u||_{L^p} = (\int_0^L |u|^p)^{\frac{1}{p}},\ \ (u,v) = \int_0^L u\bar{v},\ \ \ ||u||_{H^1} = (\int_0^L |\partial_x u|^2 + |u|^2)^{\frac{1}{2}}. \tag{1.2.9}$$

Global wellposedness of this problem is established in [45], [66], in particular, $\forall t > 0, \psi(x,t) \in H^1_{per}[0,L]$. Similar to (1.2.2) this equation has three important conserved quantities [29]. The energy or Hamiltonian $\mathcal{H}$, mass $\mathcal{M}$, and momentum $\mathcal{P}$, which can be defined as ($\mathfrak{Im}(z)$ stands for the imaginary part of z):

$$\begin{aligned} &\mathcal{H}_L(\psi) = \frac{1}{2}\int_0^L |\partial_x\psi|^2 - \frac{1}{4}\int_0^L |\psi|^4. \\ &\mathcal{M}_L(\psi) = \int_0^L |\psi|^2,\ \ \mathcal{P}(\psi) = \mathfrak{Im}(\int_0^L \partial_x\psi\bar{\psi}). \end{aligned} \tag{1.2.10}$$

Again, similar to (1.2.4), we can construct different class of time periodic solutions to (1.2.8). These solutions called the *"standing waves"* or *"periodic waves"* (since they are spatially periodic), and have the following form:

$$\psi(x,t) = e^{i\omega t}u(x). \tag{1.2.11}$$

If $\psi(x,t) = e^{i\omega t}u(x)$ be a solution to (1.2.8), then $u(x)$ should satisfy the following ODE:

$$u''(x) - \omega u(x) + |u(x)|^2 u(x) = 0, \tag{1.2.12}$$

where u is spatially periodic.
One of the differences between the previous set-up and the periodic set-up rests on the fact that solutions to (1.2.4) are well understood in general in comparison to

[28] Although this equation is integrable, we do not use integrability, and from Chapter 3, we break integrability by discretization and adding a stochastic perturbation.

[29] In fact, since this equation is completely integrable, we have infinitely many conserved quantities. However, most of the results in this note can be generalized to the non-linearities that are not integrable, i.e., we can change the non-linearity term in (3.2.1) into $|\psi|^{p-1}\psi$ with sub-critical p: $1 < p < 5$.

solutions to (1.2.12)[30].

One can find different class of smooth solutions to (1.2.12) (cf. [120], [188], [108]), using the properties of Jacobi elliptic functions $(\mathfrak{cn}(\cdot), \mathfrak{sn}(\cdot), \mathfrak{dn}(\cdot))$, also known as dnoidal, cnoidal, and snoidal waves. We define these functions in Section 1.2.1. (cf. Chapter 2,3 of [152] for properties of these functions). Recall the definition of orbital stability (1.2.7). Orbital stability of standing waves w.r.t different perturbations (different functional spaces $\mathcal{X}$) is not trivial. This question has been studied in recent decades (cf. [188] for a general introduction). Let us briefly recall a couple of results in this direction. Adapting the celebrated methods of Grillakis-Shatah-Strauss (cf. [115], [116]) to our setup, Pava proved that dnoidal waves with period L are orbitaly stable in $\mathcal{X} = H^1_{per}([0, L])$ (cf. [10]), the same result is proven for snoidal waves under certain conditions (cf. [107]); moreover, in the same work, the orbital stability w.r.t $\frac{L}{2}$ periodic perturbations of cnoidal waves with small amplitude is established. By methods using integrability (cf. [42], [109]), one can prove the stability of the snoidal waves w.r.t perturbations with any multiple of the principal period: $\mathcal{X} = H^1_{per}(0, nL)$. On the other hand, cnoidal waves are not orbitaly stable w.r.t perturbations with large enough multiple of their period (cf. [120], see also [108]).

The previous results mostly rest on Grillakis-Shatah-Strauss type methods, where it involves spectral properties of the operator $\mathcal{L} = -\frac{d^2}{dx^2} - c + \mathfrak{dn}_c$, in contrast to the previous case (NLS on $\mathbb{R}^d$), where orbital stability was proved by varitional methods. Notice that in the later the proof only relies on the variational characterization of the ground state and conservation laws.

Only recently a similar variational approach has been adapted in [120] to prove the orbital stability of standing waves, where they recover the above-mentioned results via variational characterization of the dnoidal, cnoidal, and snoidal waves, on proper functional spaces (a variational problem similar to (1.2.0.1), with $\psi \in H^1_{per}([0, L])$, for $\mathfrak{dn}$, and ψ belongs to half anti-periodic functions for $\mathfrak{cn}$). Among characterizations in [120], the following result will be used (crucially) in our analysis. It is given in ([120], Proposition 3.2), and characterizes $\mathfrak{dn}$ as the minimizer of $\mathcal{H}_L(\psi)$ under the constraint that $\mathcal{M}_L(\psi) = m$.

Proposition 1.2.0.2. *Fix $m, L \in \mathbb{R}_+$, and consider the following minimization problem:*

$$E_0(m, L) := \inf\{\mathcal{H}_L(u) | \mathcal{M}_L(u) = m,\ u \in H_{per}([0, L])\}. \tag{1.2.13}$$

then we have: $-\infty < E_0(m, L) < 0$, and

1. *If $0 < m \leq \frac{\pi^2}{L}$, then the constant function $Q_{m,L}(x) = (\frac{2m}{L})^{\frac{1}{2}}$ is the unique minimizer of (1.2.13). This uniqueness is up to a multiplication by a constant phase.*

[30]In fact, solution to the periodic ODE for a general p are poorly understood; therefore, we restrict ourselves to the case $p = 3$.

2. *If* $\frac{\pi^2}{L} < m$, *then* $Q_{m,L}(x) := \alpha \mathfrak{dn}(\lambda x, k)$ *is the unique minimizer of* (1.2.13), *up to a translation and multiplication by a constant phase. Moreover,* $\alpha, \lambda > 0, k \in (0,1)$ *are determined uniquely by* m, L.

Furthermore, we have compactness of the minimizing sequence up to a phase shift and translation in $H^1_{per}([0,L])$, *i.e., for any sequence* u_n *in* $H^1_{per}([0,L])$, *such that* $\mathcal{H}(u_n) \to E_0(m,L)$, *as* $n \to \infty$, *there is a subsequence* u_{n_k}, *and sequences* $\gamma_k \in [0, 2\pi)$, *and* $x_k \in [0,L]$, *where* $e^{i\gamma_k} u_{n_k}(. + x_k) \to Q_{m,L}$, *in* $H^1_{per}([0,L])$.

The proof of Proposition 1.2.0.2, is more or less similar to proof of Proposition 1.2.0.1. In fact, among the periodic waves, $\mathfrak{dn}$ has the most "obvious" characterization, and one can follow the similar steps, with following modifications (we only sketch the differences). First, one should recall that the Gagliardo-Nirenberg inequality in the periodic case reads:

$$\begin{aligned} |u|^4_{L^4(\mathbb{T})} &= \int_0^1 |u|^4 \leq C(|\partial_x u|_{L^2(\mathbb{T})} |u|^3_{L^2(\mathbb{T})} + |u|^4_{L^2(\mathbb{T})}) \\ &= C\left(\Big(\int_0^1 |u|^2 \Big)^{\frac{3}{2}} \Big(\int_0^1 |\partial_x u|^2 \Big)^{\frac{1}{2}} + \Big(\int_0^1 |u|^2 \Big)^2 \right). \end{aligned} \tag{1.2.14}$$

Similar to the previous case, we can deduce that any minimizing sequence has a convergent subsequent, converging to u, subsequently, by an Euler-Lagrange argument we can characterize the minimizer as a real solution to (1.2.12). Moreover, one can observe that if we fix the L and $\omega > 0$, and assume u to be real-valued, positive, and *non constant* solution to (1.2.12), then u can be written in terms of Jacobi elliptic functions as $u(x) = \alpha \mathfrak{dn}(\lambda x, k)$, where $k \in (0,1), \alpha$, and $\lambda > 0$ are uniquely determined by $\omega(m)$, and L (cf. Lemma 2.1 [120]).
The last step involves proving the fact that $Q_{m,L}$ is constant iff $\frac{\pi^2}{L} \leq m$.
Let us finish this section by the definition of Jacobi elliptic functions.

Jacobi elliptic functions

Given $k \in (0,1)$, the incomplete elliptic integral of the first kind, for any $\phi \in \mathbb{R}$ is defined as:

$$x = F(\phi; k) := \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.$$

Consequently, one can define $\mathfrak{cn}(\cdot), \mathfrak{sn}(\cdot), \mathfrak{dn}(\cdot)$ via the inverse of $F(\cdot; k)$:

$$\mathfrak{sn}(x,k) := \sin(\phi), \quad \mathfrak{cn}(x,k) := \cos(\phi), \quad \mathfrak{dn}(x,k) := \sqrt{1 - k^2 \sin^2(\phi)}. \tag{1.2.15}$$

From (1.2.15), it is straightforward to see that for all x:

$$\mathfrak{sn}^2(x,k) + \mathfrak{cn}^2(x,k) = k^2 \mathfrak{sn}^2(x,k) + \mathfrak{dn}^2(x,k) = 1. \tag{1.2.16}$$

Moreover, the derivative (w.r.t x) of these functions can be obtained directly from the definition:

$$\begin{aligned} \partial_x \mathfrak{sn}(x,k) = \mathfrak{cn}(x,k)\mathfrak{dn}(x,k), \quad & \partial_x \mathfrak{cn}(x,k) = -\mathfrak{sn}(x,k)\mathfrak{dn}(x,k), \\ & \partial_x \mathfrak{dn}(x,k) = -k^2 \mathfrak{cn}(x,k)\mathfrak{sn}(x,k). \end{aligned} \tag{1.2.17}$$

Furthermore, the period of these functions is given via the following complete elliptic integral:

$$K(k) := F(\frac{\pi}{2}; k), \tag{1.2.18}$$

where $\mathfrak{dn}$ is $2K$ periodic and even, $\mathfrak{sn}$ and $\mathfrak{cn}$ are $4K$ periodic, where $\mathfrak{sn}$ is $2K$ anti-periodic and odd, and $\mathfrak{cn}$ is $2K$ anti-periodic and even.
Notice the limiting cases: $K(k) \to \frac{\pi}{2}$, as $k \to 0$, and $K(k) \to \infty$ as $k \to 1$. Moreover, as for $k = 0$, $\mathfrak{sn}(x, 0) = \sin(x)$, $\mathfrak{cn}(x, 0) = \cos(x)$, $\mathfrak{dn}(x, 0) = 1$. Furthermore, $\mathfrak{sn}(x, 1) = \tanh(x)$, $\mathfrak{cn}(x, 1) = \mathfrak{dn}(x, 1) = \operatorname{sech}(x)$.
Finally, notice that from (1.2.17), one can deduce that $\frac{1}{\alpha}\mathfrak{dn}(\frac{x}{\beta}, k)$, $\frac{1}{\alpha}\mathfrak{cn}(\frac{x}{\beta}, k)$, and $\frac{1}{\alpha}\mathfrak{sn}(\frac{x}{\beta}, k)$ are solutions to (1.2.12) where α, β, k are determined by ω, L in each case, respectively. Notice that as $L \to \infty$, we have: $k \to 1$, and $\mathfrak{dn}$ converges to solitons on real line.

1.2.2 Statistical mechanics approach and Gibbs measure

Gibbs measures are generally perceived to provide useful information regarding the "typical" (long time) behavior of Hamiltonian systems. Let us illuminate this idea a bit more. The "Lebesgue measure" is invariant for the Hamiltonian flow, consequently, the Gibbs measure with density $\exp(-\beta H)$ w.r.t the "Lebesgue measure" is invariant as well. If the Hamiltonian provides sufficient "ergodic" properties, one could expect that the later is the unique invariant measure, and long time behavior of our system can be described by this measure. In other words, our system thermalizes after sufficiently long time. Considering NLS as a Hamiltonian flows the aforementioned argument gives a strong motivation to study the Gibbs measure associated to NLS. Let us emphasize that proving ergodicity even in the "discretize" setup is out of reach, and we provide ergodcity by an appropriate stochastic perturbation. Moreover, the cubic NLS in $d = 1$ is completely integrable and this scheme (without the noise) fails.

Gibbs measure: brief review

The idea of investigating the NLS's Gibbs measure dates back to the pioneer work of Lebowitz, Rose, and Speer [154]. They proceed as follows. First obstacle to study this measure is that NLS is an infinite dimensional flow, and such a Lebesgue measure does not exist in infinite dimension. For NLS on $\mathbb{T}_L$, this measure can be written formally:

$$d\nu_L^\beta = \frac{1}{Z_L(\beta)} \exp(-\beta \mathcal{H}_L^p(u)) \Pi_{y \in \mathbb{T}_L} du(y), \tag{1.2.19}$$

where $\mathcal{H}_L^p = \frac{1}{2}\int_0^L |u'|^2 - \frac{1}{p+1}\int_0^L |u|^{p+1}$. This problem can be circumvented by absorbing the kinetic energy into the "formal" Lebesgue measure and obtaining the proper Wiener measure:

$$d\nu_L^\beta = \frac{1}{Z_L(\beta)} \exp\left(\frac{\beta}{p+1}\int_0^L |u(y)|^{p+1} dy\right) dW_{\beta,L}, \tag{1.2.20}$$

where $dW_{\beta,L}$ is the proper Wiener measure, induced by the Brownian bridge. The other difficulty arises from the fact that $\mathcal{H}_L^p$ is unbounded from below, this can be solved by restricting the measure to the balls such that $\mathcal{M}_L \leq m$, for a constant m. Consequently, in [154], it is proven that such a measure is well defined in $d = 1$, for $1 < p < 5$, and for $p = 6$ with small m (this is the same range of parameters where the equation is globally well-posed).
Before proceeding, let us mention that constructing the Gibbs measure in the continuum setup is a challenging problem, we briefly review some of the most famous results in this direction. Notice that this measure in the defocusing case in the continuum is closely related to the quantum field theory. In the defocusing case, Glimm and Jaffe (cf.[113]) showed that in $d = 1$, this measure exists for all p. In $d = 2$, after proper wick ordering, the existence is proven for $p \leq 5$, and in $d = 3$ for $p \leq 3$. Notice that this problem in $d = 4$ is very difficult. Moreover, Bourgain proved that this measure is actually invariant for the dynamics of NLS for $p = 3$, and $p = 5$ (cf. [45], [43]) (in his construction, he truncates the Fourier modes and takes the limit properly). In the focusing case in $d = 1$, for application and further investigation of this measure, one can see (cf. [171], [174], [173], [172], [200], [199]). In $d = 2$ the construction of Jaffe (see references in [153]), and in $d = 3$ work of Brydge and Slade can be mentioned, where they observed that the "natural construction" (due to Jaffe) does not work in $d = 3$ (cf. [52]).
Although construction of the Gibbs measure is challenging by itself, it has several applications concerning constructing Global solutions with rough initial data (H^s with proper $s < 1$). In this situation, the flow is not globally well-posed; however, combining the Gibbs measure with Bourgain spaces ($X^{s,b}$) one can construct global solutions for "most" (w.r.t this Gibbs measure) initial data. In that regard, one could mention pioneering work of Bourgain (cf. [43], [44], [46]), and further development by Tzvetkov, Oh and coauthors concerning dispersive equations (cf. [54], [55], [56], [220], [219], [221], [72], [182], [57], [27] and references therein, see also [84], [183], [185]) .

Somewhat "inverse" idea has received attention recently. In the previous setup, one extracts information about the dynamics by constructing the Gibbs measure and observing that it is invariant under the dynamics. Conversely, one could construct a dynamics by modern methods such as theory of rough paths, paracontrol distributions, and regularity structure such that the "formal" Gibbs measure be the unique invariant measure, and correspondingly one can "construct" such measure by using properties of the dynamics (In this direction one can see the construction of ϕ_3^4 cf. [118] and references therein).

Gibbs measure's behavior

The qualitative behavior of the Gibbs measure is informative and interesting by itself (this becomes even more interesting in the next section, when we add the noise to our dynamics). As far as we know, this problem is less investigated in the mathematics community. In this subsection, we briefly outline some of the relevant

results in this direction (cf. [139] for a review). We also compare the strength and weakness of our results in comparison to the former in the next section.
Recall that the Gibbs measure in the focusing case in $d = 1$ is well defined thanks to [154]. Fix $p = 3$, this measure can be written as (take $0 < \delta < 1$):

$$d\nu_L^{\beta,m} = \frac{1}{Z_L(\beta, m)} \exp\left(\frac{\beta}{4}\int_0^L |u(y)|^{p+1} dy\right) \mathbb{1}_{\mathcal{M}_L \in [mL(1-\delta), mL]} dW_{\beta,L}, \tag{1.2.21}$$

where we restrict ourselves to the ball with L^2-norm equal to mL. In [154], numerical simulations suggests a possible phase transition: as we move from high temperatures to low temperatures (and from low masses to higher masses), a "typical field" under this Gibbs measure looks more "localized" (typical should be understood as result of a Monte-Carlo Simulation). Another numerical simulation in (cf. [53]) excluded the phase transition's possibility (actually, we believe that our result could shed light on this problem: cf. Section 1.2.3).
In [200], following [172], [173], [174], the Gibbs measure studied in the thermodynamic limit, i.e., as $L \to \infty$, where the mass of the field is equal to mL, with a finite m. In this regime, Rider proved that

$$\lim_{L\to\infty} \frac{1}{L^3} \log Z_L(1, m) = -\mathcal{E}_o(m), \tag{1.2.22}$$

where $\mathcal{E}_o(m)$ is the ground state energy (1.2.5). Moreover, he proved that in the thermodynamic limit, $\nu_L^{\beta,m}$ converges to the trivial measure, where all the mass concentrates on the zero path. Notice that at a heuristic level this result is not surprising as it has been mentioned mentioned in [154]. In the thermodynamic limit, since the Hamiltonian is unbounded, field concentrates the possible mass locally.
Subsequently, Rider in [199] investigated the fluctuation of the same measure with spatial discritization, and assuming the field to be real. He concluded that in the thermodynamic limit for m sufficiently large, the rescaled field $\sqrt{L}u$ converges to the white noise.
In [134], [133] Jordan, Josseran, and Turkington studied the Gibbs measure in the finite (fixed) volume with fixed mass, where they modified the non-linearity to $g(|u|^2)u$ instead of $|u|^2u$, where g should be positive, bounded, increasing, and $xg'(x)$ is bounded as x approaches ∞. In this setup, they considered the spectral truncation of the measure $\nu_n^{\beta_n}$, and they proved that as $n \to \infty$, we have: $\lim_{n\to\infty} \frac{1}{\beta_n} \nu_n^{\beta_n}(u \in S) = -\inf_{u\in S} \mathcal{I}(u)$, where they rescaled β_n by n. Here $S \subset L^2$, and $\mathcal{I}(u) = \mathcal{H}_L^g(u) - E_0^g(m, L)$ is a good rate function. Moreover, $E_0^g(m, L)$ denotes the ground-state energy of $\mathcal{H}_L^g$ with fixed mass, where we replace $|u|^{p+1}$ by the primitive of g in the Hamiltonian. Notice that, this means that this measure concentrates on the ground states in this limit. However, we should mention that their methods is limited to a specific class of non-linearities and one cannot adapt their methods to the cubic NLS.
Finally, in $d \geq 3$, Chatterjee and Kirkpatrick (cf. [69]), studied the spatially discretization of the the measure (1.2.21), in the thermodynamic limit as $n \to \infty$ (number of lattice sites), with $nh^2 \to \infty$, where h is the interparticle distance. Notice that $nh^2 \to \infty$ only seems natural in $d \geq 3$. They proved a phase transition

in this setup: under this Gibbs measure, one can observe the following phenomena with probability one as $n \to \infty$ [31]: if $\beta m^2 < \theta_c$, with θ_c is a critical value, the free energy is equal to $\log(m\pi e)$, and all the sites has macroscopically vanishing mass. If $\beta m^2 > \theta_c$, then the free energy is given by $\log(m\pi e) + g(\beta m^2)$ with a explicit g. In this regime, we have "localization" in the sense that there is a single site with macroscopic non-vanishing mass. Moreover, if we begin with a configuration sampled from this measure (therefore, it is "localized"), the "localization" persists for a exponentially long time, under the flow of DNLS (since we are in a dicretized setup, our flow is the dicretized version of NLS). In both regimes, the discrete H^1 norm of the field diverges.

Microcanonical measure

From a statistical mechanics' point of view, by considering the NLS's Gibbs measure, we study the system in the "canonical" ensemble[32]. If you believe in efficacy of statistical mechanics approach, it would be natural to explore the NLS's microcanonical ensemble as well. In physics community, there are certain works regarding this issue for the discrete non-linear Schrödinger equation (DNLS) (cf. [205], [206], [114] and references therein).
Mathematically, Chatterjee and coauthors have a series of works in this direction (cf. [69], [68], in particular [67]). In particular, he proves a probabilistic version of soliton resolution conjecture in [67] by analyzing the microcanonical ensemble. Since our work is partially motivated by [67], we explain its main result here.

For the NLS, microcanonical ensemble corresponds to the "uniform measure" on set of functions $u \in H^1$, such that for proper values of E and m, we have $\mathcal{H}(u) = E$, and $\mathcal{M}(u) = m$, i.e, $S(E, m) := \{u \in H^1(\mathbb{R}^d) | \mathcal{H}(u) = E, \quad \mathcal{M}(u) = m\}$. Notice that at a heuristic level this "measure" would be invariant for the NLS via Liouville type argument. Constructing such a measure in the continuum is challenging and spatial discretization seems inevitable. In fact, in [67] the "discretized" version of $S(E, m)$ is considered.
Let us consider the focusing NLS in d space dimension, with $1 < p < 1 + \frac{4}{d}$. Let $H_{h,n}$ and $M_{h,n}$ denotes the descretized version of $\mathcal{H}$, and $\mathcal{M}$, with n^d lattice points, and mesh size h (we define $H_{h,n}$, $M_{h,n}$ in (1.2.25), we restrict ourselves to the box $[0, nh]^d$: $s = h^d$).
In this setup, the configuration space is given by discrete functions $u : \mathbb{T}_n^d \to \mathbb{C}$, and their continuum image should be understood as proper step functions $\tilde{u} : [0, nh]^d \to \mathbb{C}$, where for $y \in [0, nh]^d$, $\tilde{u}(y) = u([\frac{y}{h}])$. Accordingly, we can define $S_{\epsilon,n,h}(E, m) := \{u \in \mathbb{C}^{n^d} \Big| \quad |M_{h,n}(u) - m| \leq \epsilon, \ |H_{h,n}(u) - E| \leq \epsilon\}$. This set is supposed to mimic

[31] In fact, these events' probabilities can be bounded from below by $1 - e^{-cn^a}$ with $c > 0$, and $a > 0$.

[32] Quoting from [154]: NLS is a effective equation describing many degrees of freedom. The idea of coarsening the system into an effective equation, then refining it by considering the Canonical ensemble may seem artificial; however, "there are often good reasons for wanting to focus attention on the collective variables of the field rather than on the atomic degree of freedom e.g, in fluid turbulence it is in hydrodynamic modes that interesting action take place."

the set $S(E,m)$ in the limit as $n\to\infty$, $h\to 0$, with $nh\to\infty$. Denote the uniform measure on $S_{\epsilon,n,h}$ by $\mu_{\epsilon,n,h}$. Chatterjee proved (cf. [67] Theorem 2.1) that in the proper limit ($n\to\infty$, $h\to 0$, $nh\to\infty$), this measure concentrates on solitons in the following sense: there exists a sequence (ϵ_k,h_k,n_k), with $\epsilon_k\to 0$, $h_k\to 0$, and $n_k h_k\to\infty$ as $k\to\infty$, such that for any $\delta>0$ and $q\in(2,\infty]$, we have:

$$\lim_{k\to\infty}\mu_{\epsilon_k,n_k,h_k}\Big(||\tilde{u}-Q_{\lambda(m)}||_{\tilde{L}^q}>\delta\Big)=0, \tag{1.2.23}$$

where $Q_{\lambda(m)}$ is the Soliton appeared in Proposition 1.2.0.1, and $\tilde{L}^q$ is a semi-norm similar to L^q-norm, which does not take into account the phase multiplication and translation symmetry: $||u-v||_{\tilde{L}^q}=\inf_{y,\theta}||u-e^{i\theta}v(\cdot-y)||_{L^q}$.
Although (1.2.23) implies that the microcanonical measure concentrates on the solitons, it does not provide information about the long time behavior of the corresponding DNLS flow by itself, since we do not know if the dynamics is ergodic. In [67], author settles this difficulty in the following way: he proves (cf. [67] Theorem 3.1) that in the proper limit of n,h,ϵ nearly all ergodic components of the $\mu_{\epsilon,n,h}$ satisfy the soliton resolution conjecture in the Césaro sence:

$$\nu\Big(u\in\mathbb{C}^{n^d}:\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{||\widetilde{T}_s u-Q||_{\tilde{L}^\infty}>\delta}ds<\delta\Big)=1,$$

where $\widetilde{T}_s u$ denotes the flow of the DNLS acting on u, where the proper continuum image is taken afterwards. Moreover, almost all should be understood w.r.t the measure induced by $\mu_{\epsilon,n,h}$, on the set of all ergodic invariant probability measures of the map T_1, via Choquet representation theorem.
One of our motivations in the next section is to introduce a stochastic noise that provides such ergodicity, which is needed in the later theorem. Connecting the later theorem to soliton resolution conjecture could be a matter of contention. In particular, the relation between the abstract measure in this theorem with the "physical" configurations is not straightforward: we do not know starting from which class of initial conditions, we end up in a "good" ergodic component. For example, is it possible to end up in a "bad" ergodic component starting from a large class of initial conditions? Moreover, behaviors such as multi-solitons moving with different speeds, has not been captured in this theorem.

1.2.3 Stochastic noise and results

So far, the situation can be summarized as follows: long time behavior of the focusing NLS is elusive. Statistical mechanic approaches, such as studying the Gibbs measure is proven to be useful. Although they provide a natural invariant measure for the dynamics, one cannot guarantee the convergence of the NLS flow to this measure, mainly due to difficulty in proving ergodicity (in case of cubic NLS in $d=1$, we even do not have ergodicity at all).
One of the main purposes of Chapter 3 is to introduce a mass ($\mathcal{M}$) conserving stochastic perturbation of NLS, such that the Gibbs measure becomes the unique invariant measure. One could think of action of this noise as a heat bath at constant

temperature β^{-1}.
Recall (1.2.8) (with any non-linearity p), with Hamiltonian $\mathcal{H}_L$, and mass $\mathcal{M}_L$ (1.2.10), where we consider a general non-linearity p. Formally one way to define such stochastic dynamics is to consider the SPDE:

$$i\partial_t\psi(x,t) = -\Delta\psi(x,t) + \kappa|\psi|^{p-1}\psi(x,t) - \gamma\psi(x,t)\left(i\beta^{-1} - \frac{\delta\mathcal{H}(\psi))}{\delta\theta(x)}\right) + \sqrt{2\gamma\beta^{-1}}\psi(x,t)W(x,t), \tag{1.2.24}$$

where $\theta(x)$ is the phase of $\psi(x)$ ($\psi(x) = |\psi(x)|e^{i\theta(x)}$), $W(x,t)$ is the standard space-time white noise, and $\gamma > 0$ is a parameter that regulates the intensity of the contact with the heat bath. This corresponds to random but continuous rotations of the phase of $\psi(x)$ independently at each point x with a drift which somehow intends to align the particle's phases.
Notice that $\frac{\delta\mathcal{H}(\psi))}{\delta\theta(x)} = \mathcal{I}m[\psi(x)^*\Delta\psi(x)]$, and that (1.2.24) should be intended in the Ito's sense. Consequently, the mass $\mathcal{M}(\psi)$ is still formally conserved by this dynamics.
Because of the singularity in space of the multiplicative white noise W and the non-linearities present in (1.2.24), it is very hard to give sense to the solution of this equation. There is an extensive literature on the NLS with space correlated multiplicative noise [33] (cf. [75], [76]), but it does not include non-linearities like $\psi(x)\Delta\psi^*(x)$. Additive noises have also been introduced (cf. [153], [60]), but usually do not conserve the mass, and the corresponding dynamics have the grand canonical Gibbs measure as stationary. We introduce instead a space discretization of (1.2.24), see (3.1.8), whose solution can be defined globally. The infinite temperature version of this stochastic evolution was introduced in (cf. [156]).

Alternatively, this noise can be described via its generator. Although this dynamics is defined in Section 3.1, let us briefly recall it here [34]. Generally, we consider the domain $h\mathbb{T}_n^d$ as a discretization of the box $[0, nh]^d$, where h is the mesh size. Correspondingly, the discretization of $\mathcal{H}_L$, and $\mathcal{M}_L$ at mesh size h is defined as follows: for any $\psi \in \mathbb{C}^{n^d}$ we have:

$$H_n := s\sum_{|x-y|=1} h^{-2}|\psi(x) - \psi(y)|^2 + \frac{s\kappa}{p+1}\sum_{x\in\mathbb{T}_n^d}|\psi(x)|^{p+1}, \quad M_n(\psi) = s\sum_x |\psi(x)|^2. \tag{1.2.25}$$

The choice of $h = \frac{L}{n}$, and $s = \frac{1}{n^d}$ corresponds to the descritization of (1.2.10), whereas taking $s = h^d$, and n, h such that $h \to 0, n \to \infty, nh \to \infty$ corresponds to the discretization of (1.2.2).
The discrete non-linear Schrödinger equation, can be viewed as a Hamiltonian dy-

[33] We do not intend to review this extensive literature here, and we only mention the pioneering work of de Bouard, and Debussche, subsequently there are many works in the literature concerning the stochastic NLS and its variations; however, their point of view is rather different from ours, where we are concerned with thermalization.

[34] Although these definitions are repetitive, they make our introduction more understandable.

namics corresponding to H_n, with the following generator:

$$A_n = \frac{1}{s} \sum_{x \in \mathbb{T}_n^d} (\partial_{\psi_i(x)} H_n) \partial_{\psi_r(x)} - (\partial_{\psi_r(x)} H_n) \partial_{\psi_i(x)}, \tag{1.2.26}$$

where ψ_r, ψ_i denote the real and imaginary part of ψ: $\psi(x) = \psi_r(x) + i\psi_i(x)$. Denoting the phase of $\psi(x)$ by $\theta(x)$ ($\psi(x) = |\psi(x)|e^{i\theta(x)}$), the generator of the noise at temperature β^{-1} is given by:

$$S_n := \beta^{-1} \sum_{x \in \mathbb{T}_n^d} e^{\beta H_n} \partial_{\theta(x)} e^{-\beta H_n} \partial_{\theta(x)}. \tag{1.2.27}$$

Consequently, for any $\gamma > 0$, the generator of our dynamics is given by

$$L_n = A_n + \gamma S_n. \tag{1.2.28}$$

From the definition of the generator it is clear that the mass $M_n(\psi)$ is conserved by the dynamics. Therefore, starting from a configuration with $M_n(\psi_o) = m$, the dynamics is confined in the sphere $S_m^n := \{\psi \in \mathbb{C}^{n^d} | M_n(\psi) = m\}$. Denoting the uniform measure on S_m^n by μ_m^n, the Gibbs measure is given by $d\mu_{\beta,m}^n = \frac{1}{Z_n} \exp(-\beta H_n) d\mu_m^n$, and is the discrete analogous of (1.2.21). This measure is well-defined and invariant for the dynamics generated by L_n.

The first result of Chapter 3 is that $\mu_{\beta,m}^n$ is the unique invariant measure for the dynamics generated by L_n (1.2.28), the generator is hypoelliptic and for any fixed n, the distribution of the process starting from an arbitrary initial condition converges exponentially in total variation norm to the stationary measure. The statement of the result is given in Theorem 3.1, and Proposition 3.1.3.
Let us emphasize that to the best of our knowledge, the novelty of our work is that it is the first result concerning *mass conserving* perturbation of the DNLS (discrete non-linear Schrödinger) such that the Gibbs measure is the *unique invariant measure*, determining dynamics' long-time behavior.We should mention that in Section 6 of [59], a mass conserving noise is proposed such that the canonical Gibbs measure remains invariant by the dynamics. However, this dynamics is not studied, and [59] mainly concerns another dynamics, which does not conserve the mass and converges to the grand canonical measure. Moreover, the above-mentioned mass conserving dynamics is different from ours; in fact, a straightforward analysis suggests that our dynamics is more degenerate (we have a n- dimensional Wiener process on the $2n-1$ sphere, where they have a $2n-1$ dimensional Wiener measure on a $2n-1$ dimensional sphere . Moreover, generalization of their noise to more general cases does not seem straightforward).

Remark 1.2.1. *It is worth mentioning that the above mentioned result is general in the sense that it does not depend on the parameters h, s, γ, β in our definition. More importantly, this result is true for other powers of non-linearity and any value of $\kappa \in \{-1, 0, 1\}$. In fact, one can replace the non-linearity $|\psi|^{p+1}$, with any non-linearity of the form $F(|\psi|^2)$, with $F \in C^2$, and obtain the similar result.*

Therefore, one can perturb any of the discrete models outlined in Section 1.2.2, or other appropriate DNLSs (cf. [135] and references therein) with our noise, and thanks to the above-mentioned theorem the long time behavior of the emerging dynamics is given by the corresponding Gibbs measure. Notice that in many of the aforementioned models the Gibbs measure's behavior is understood relatively well. Although our result is general in the above sense, our choice of discretization is important (up to changing the scaling parameters), and the Hamiltonian generator A_n plays an important role.

The remaining of Chapter 3, concerns an application of the above-mentioned result. From Section 3.2 of Chapter 3, we concentrate on the one-dimensional focusing cubic NLS on the torus ($d = 1$, $p = 3$, $\kappa = -1$). Recall the standing wave solution of one dimensional focusing cubic NLS on the torus. In particular, recall the definition and properties of the minimizer of the energy under mass constraint $Q_{m,L}$, from Proposition 1.2.0.2.
First, we discretize this equation. This corresponds to the choice of $d = 1$, $s = h = \frac{1}{n}$ (take the macroscopic volume one), $p = 3$ in the Hamiltonian (1.2.25), and consequently in the generators A_n, and S_n. Let $\psi_n(j,t)$ denotes the solution to the corresponding stochastic discrete dynamics. The purpose of Chapter 3 is to show that ψ_n is "close" to $Q_{m,1} \equiv Q_m$ for large t, and large n, where we rescale β by n such that in this limit, temperature goes to zero fast enough. Notice that in order to give sense to this statement, we should define a norm to compare ψ_n and Q_m. For this purpose, we simply take the linear interpolation of ψ_n, (3.2.17) and compare it with Q_m in a H^1 norm which does not take into account the translation and phase multiplication (3.2.18): $||u - v||_{\tilde{H}^1} = \inf_{\theta,x} ||e^{i\theta}u(\cdot - x) - v||_{H^1}$.
In the other words, let $\mu_t^{\beta_n,n,m}$ be the law of the process generated by L_n, then our main result states that for any $\epsilon > 0$:

$$\lim_{n\to\infty} \lim_{t\to\infty} \mu_t^{\beta_n,m,n} \left(||\bar{\psi}_n - Q_m||_{\tilde{H}^1} < \epsilon \right) \to 1, \tag{1.2.29}$$

where $\bar{\psi}_n$ denotes the linear interpolation of ψ_n, and $\beta_n = \vartheta(n)\beta_*$ with $\vartheta(n) \gg n$. The statement of the result is given in Theorem (3.2.2).

The later limit (1.2.29) is a way to interpret the *soliton resolution conjecture* (SRC) in the periodic case, where there is no possibility for the energy to escape to infinity. Intuitively, in the periodic case, our dynamics in the zero temperature limit, acts as a heat bath and dissipate the excess of the energy without losing any mass, forcing the system to approach the ground state as $t \to \infty$. This mechanism is somehow mimicking the dynamics of DNLS in [67], where energy disperse to infinity via a "radiating" part of the field carrying arbitrarily small mass. As we mentioned, our dynamics is partially motivated by [67], where Chatterjee proves a "probabilistic" version of the SRC (cf. Section 1.2.2). In particular, in Theorem 3.1 in [67] it is proven that *almost every* ergodic invariant measure satisfies the SRC in the time average sense. Our stochastic dynamics provides the uniqueness of the invariant ergodic measure and the time mixing property.

Idea of the proof

In the following, we give a brief sketch of proof of Theorem 3.2.2. The dynamics in limit of $t \to \infty$ converges to the Gibbs measure. This follows from ergodicity of the dynamic which relies on two crucial facts. First, is the hypoellipticity of the dynamics' generator which is proven in Lemma 3.B.1. Since the real and complex part of our field are somehow symmetric in the noise, this makes the proof of Lemma 3.B.1 more complicated than usual, and computing three nested commutator is necessary cf. (3.B.4) (rather than one or two in chain of oscillators cf. [32] Lemma 5.3). The second ingredient is uniqueness of the invariant measure, which is proven in Theorem 3.1, thanks to separation of the noise into phase and amplitude part.

Knowing the limit in t, the second step of the proof concerns understanding the behavior of the Gibbs measure $\mu^n_{\beta,m}$ (3.2.15). In fact, we need to prove that the Gibbs measure concentrates on discrete configurations with "close to minimal energy" for sufficiently large n, where β_n is rescaled by a factor $\vartheta(n) \gg n$. Here by minimal energy, we refer to the minimizer of the discrete Hamiltonian under the mass constraint ($E^n_0(m)$ defined in (3.3.1)). This task is done in Theorem 3.4.1, let us sketch this theorem. Our dynamics is confined on the $2n$-dimensional sphere S^n_m, and the Gibbs measure has a density $\exp(-\beta H_n)$ with respect to the uniform measure on this sphere μ^n_m. Splitting the energy into the kinetic G_n and potential V_n parts: $H_n = G_n - V_n$ as in (3.2.8), one can observe that a "typical" configuration w.r.t μ^n_m has kinetic energy of order n^2. More generally, large deviations estimates in Section 3.A, in particular Proposition 3.A.1, is basically saying that for $0 \leq a \leq 2$, the "entropy factor" behaves as $\mu^n_m(G_n \sim n^a) \sim \mu^n_m(H_n \sim n^a) \sim n^{-(2-a)n \ln n}$, where the first estimate is a consequence of discrete Gagliardo-Nirenberg (GN) inequality (3.C.8) for $a > 0$. Therefore, taking into account the so-called Boltzmann factor $\exp(-\beta H_n)$, we observer for $0 < a \leq 2$: $\mu^n_{\beta,m}(H_n \sim n^a) \sim e^{-\beta_n n^a} e^{-(2-a)n \ln n}$. Optimizing the later for any $a \in [0, 2)$, yields: for $\beta_n \sim O(1)$, the Gibbs measure concentrates on rather rough configurations with $|\psi(j) - \psi(j-1)| \sim \frac{1}{\sqrt{n}}$, and $H_n \sim G_n \sim n$. This corresponds to the fact that Wiener measure is concentrated on $\frac{1}{2}^-$-Hölder regular configurations. The later estimate suggests that in order for $\mu^n_{\beta_n,m}$ to concentrates on smooth configurations ($|\psi(j) - \psi(j-1)| \sim \frac{1}{n}$) with $H_n \sim O(1)$, we should rescale β_n by a factor of at least n. Notice that minimal energy configurations, where we call them discrete Solitons, has (negative) energy of order one as well, thanks to the GN inequality. However, this scaling is not enough for this measure to concentrate on a small neighborhood of discrete solitons and we need to go further. Finally, thanks to large deviation estimate (3.A.11), we deduce in Theorem 3.4.1 that scaling β_n by any factor $\vartheta(n) \gg n$ (3.2.19) is sufficient.
In the last step of the proof, we show in Proposition 3.3.1 that if ψ_n be a configuration with energy close to $E^0_n(m)$, then its linear interpolation ($\bar{\psi}_n$) is close to $Q_{m,L}$ in $\tilde{H}^1$ norm for n sufficiently large. In that regard, first we observe that having energy close to $E^n_0(m)$ means that the configuration is smooth $G_n \sim O(1)$, thanks to discrete GN inequality. Subsequently, since for smooth configurations $H_n(\psi_n)$ is close to $\mathcal{H}(\bar{\psi}_n)$ (cf. Corollary 3.3.3.1, in fact, this is not true for a "typical" non smooth configuration), one can conclude by compactness of the minimizing sequence

corresponding to the continuous minimization problem which characterizes solitons (3.2.6).

The plan of Chapter 3 is as follows. The ergodic and time mixing properties of the dynamics is proved in Section 3.1. The fact that the discrete Gibbs measure (finite n) concentrate fast enough in a small neighbor of *discrete solitons*, relies on large deviation properties of the discrete Gibbs measure, which are proved in Section 3.4. These large deviations estimate are based on some precise large deviations of the uniform probability measure on the complex n-dimensional sphere, that we prove in Appendix 3.A, and the discrete version of the Gagliardo-Nirenberg inequality, proven in Appendix 3.C. In Section 3.3 the desired relation between discrete and continuous Solitons is proved. Appendix 3.B contains the proof of the hypoellipticity of the discrete stochastic dynamics, necessary for the proof of the ergodicity of section 3.1.

Background

Our results, can be compared to two different body of works in the literature. First is the stochastic perturbation of the NLS, and the second is the behavior of DNLS's Gibbs measure. In Section 1.2.2, we provide a very brief (and not comprehensive by any means) list of related works concerning the Gibbs measure. In the following, we highlight similarities and crucial differences among our model and those results concerning Gibbs measure.
Before addressing these issues, we mention a handful of results regarding stochastic perturbation of NLS.

As we mentioned before, various stochastic perturbation of NLS have been studied in both mathematics and physics literature. In physics literature, one can mention [19], where a stochastic NLS is derived as an effective equation, describing dynamics of excitons in contact with thermal phonons (see also [195]). This equation has a multiplicative noise, colored in space, which conserve the mass. Subsequently, in mathematics literature, this equation and its variation have been considered by many authors. In fact, there is an extensive literature on NLS with space correlated stochastic noise. Most remarkably one can mention (cf. [75], [76], [77] and references therein). Notice that these works are usually concerned with questions such as well-posedness of the dynamics in different regimes, as well as blow up of the solutions (cf. [80], [20]), rather than thermalization and invariant measure.
Besides these multiplicative perturbations, there are numerous works regarding perturbation of NLS by additive white noise (cf. [153], [60] and references therein). In these works, the grand canonical measure is invariant and convergence to this measure is observed. However, they are not mass conserving. In this direction, we can also mention [78], where convergence to the Gibbs measure is proven for a one-dimensional defocsuing NLS with an additive noise. Notice that in this work the mass is not conserved and the Gibbs measure should be understood without confining to configurations with a fixed mass.
Finally, as we mentioned, in Section 6 of (cf. [59]) a strategy to constructing a mass

conserving multiplicative white noise which leaves the Gibbs measure invariant is introduced. However, this noise have not been studied and it is different from ours. Notice that by discretizing the space, and studying the long time behavior of the dynamics through understanding the Gibbs measure's behavior, somehow we followed a rather different path comparing to the above-mentioned works.

Regarding the statistical mechanics and Gibbs measure of DNLS, there exists several studies both in mathematics (cf. [69], [67]) and physics community (cf. [135] and references therein: in particular, [196], [205], [132]; see also [114]). In physics community, one usually takes the kinetic energy with a negative sign and study the measure corresponding to Hamiltonian (3.1.2), by taking $h = s = 1$. Although this regime is substantially different from ours, and does not correspond to discretization of a continuous profile anymore, interesting phenomena such as phase transition and localized structures (discrete breathers (cf. [97], [224])) is observed .
The fate of the dynamics, as well as Gibbs measure typical behavior depends on the parameter h, s, and mass per particle. Consequently, one can consider different regimes: taking $h = 1$, $s = 1$ and finite mass per particle, corresponds to the above-mentioned literature. On the other hand, one can take $h = s = 1$, and define the model on an unbounded domain $\mathbb{Z}^d$, with total mass ν. This corresponds to vanishing mass per particle regime. In this regime, certain localized modes have been observed in mass subcritical regime (cf. [139], [224], See also [215]); moreover, the scaling $\nu \to \infty$ is studied as well, which leads to configurations with single site concentration (cf. [224]; cf.[192], [136], [191], for spectral properties and stability).

Before proceeding, let us mention that discretization of NLS can be subtle: we dicretize NLS by taking the Hamiltonian generator (1.2.26). However, there exists other discretization of this equation. We can mention Ablowitz-Ladik (AL) discretization (cf. [3]), where the term $|\psi(x)|^2\psi(x)$ is replaced by $|\psi(x)|^2(\psi(x-1) + \psi(x+1)$. We observed that NLS with $p = 3, d = 1$ is integrable; however, our discretization is not integrable generally in this case, whereas the AL discretization remains integrable.

In mathematics community, we already mentioned a couple of related results in Section 1.2.2. Notice that, here we are mainly concerned with the regime $h \to 0$, which approximates the continuum. However, there are still different ways of sending h to zero.
In particular, [69], and [67] is related to our work. In fact, we inspired by [67], and some of his ideas has been adapted to our set up. For example, we get the discrete GN on $\mathbb{T}$ thanks to his result on GN inequality on $\mathbb{R}^d$. However, the differences of our work with [67], and [69] is summarized as follows: recall in [69], the Hamiltonian (1.2.25) with $s = \frac{1}{N}$ is considered such that $Nh^2 \to 0$ as $h \to 0$, and $N \to \infty$ where N denotes the number of particles and h is the inter-particle distance. These assumptions only seem natural in $d \geq 3$. In [67], the model is defined on the box $[0, nh]^d$, such that $h \to 0$, $n \to \infty$, with $nh \to \infty$, and the microcanonical measure is considered. Comparing our result with [69], and [67], we highlight the fact that

different scaling among the parameters h, n leads to substantially different phenomena. In [69], $Nh^2 \to 0$ makes the Gradient term negligible, reducing the analysis of the system to Bose-Hubbard model (cf. [67], [131]). Consequently, the phase transition is a consequence of competition among potential energy and mass constraint. In [67], $nh \to \infty$, kinetic and potential energy become comparable; however, mass per particle goes to zero in the limit, demonstrating the macroscopic infinite volume, facilitating the escape of the energy to infinity, and resulting in soliton like behavior. In contrast, in our case we take $n \to \infty$, and $nh = 1$, representing the finite macroscopic volume and positive mass per particle in the macroscopic limit. This scaling yields a dominant kinetic energy for typical configurations on the sphere of constant mass. However, by rescaling β_n, our measure concentrates on smooth configurations, such that kinetic and potential energy become comparable.
In particular, these different scalings change our large deviation estimates (3.A.1), and (3.A.11) comparing to estimates in [67] (cf. Section 10 of [67]).

Let us mention that the result of Rider (cf. [200]), stated in (1.2.22) about the continuum version of our measure in the infinite volume, is different from ours in the following sense. In [200], the concentration of the Gibbs measure on the "zero" path is due to the fact that in the infinite volume, gradient term explodes for non-trivial configurations, and the measure concentrate on the minimal energy at any temperature. However, in our case since the volume is finite, we should send the temperature to zero to see the minimal energy.

Finally, let us mention that comparing our result with other large deviation estimates, it is apparent that choice of scaling plays an important role. In particular, we believe that the phase transition observed in [154] is due to the fact that in their simulation they had only $n = 40$ lattice sites, and consequently the temperature, and mass vary in different scaling regimes in their simulation, causing the "phase transitionish" graph (cf. Fig 1 in [154]).

1.2.4 Open problems, and future direction

We finish our introduction by proposing a couple of open questions and possible future directions. Since our noise is relatively new, we believe that many interesting questions related to this noise can be studied. In this section, we discuss a couple of these questions, in the order of personal preferences.

As we mentioned, our dynamics can be applied to many variations of NLS by changing several parameters (κ, d, n, h, even form of the non-linearity). In each of these cases the dynamics become ergodic and the long time behavior is given by the corresponding Gibbs measure. However, Gibbs measure's behavior is substantially different in each of these cases. Moreover, the dynamics' feature such as convergence rate depends on the choice of these parameters. Therefore, each of these setups should be studied separately. In particular, we find he following questions interesting.

The first obvious question which we are currently investigating, concerns the Gibbs measure's behavior in case where we rescale β_n by n. In this case, we expect the energy to remain finite, and the measure concentrates on a set of configurations with finite energy, far from the ground sate.

The second question concerns the exchange of limits in (1.2.29), we expect that if we first take the limit of $n \to \infty$, such that we rescale $\beta_n \to \infty$ fast enough, we recover the deterministic NLS. In particular, we expect that if take $t, n \to \infty$, where we scale t_n by n such that $t_n \ll \beta_n$, then the noise vanishes and we get the deterministic NLS. This will be investigated in a future work [126] (see also Remark 3.2.1.)

There are two shortcomings in our result (1.2.29), first , is that in this limit, we first take the limit in t, and then the limit in n. A natural question would be to study these limits together. In particular, we would like to find the rate of convergence as a function of n. Presence of the non-linearity, as well as the degeneracy of the dynamics makes this problem hard. Therefore, classical methods such as Poincaré inequality, or Log-Sobolev inequalities do not work here, and one needs to use more advanced tools such as hypocoercivity (cf. [223]), and entropic hypocoercivity (cf. [157]), or probably more "miraculous" ideas similar to [163] is needed. However, a "naive" version of hypocoercivity (where the inner product is constructed by taking only one commutator) is not sufficient and one probably needs to define more complicated inner products due to the fact that in our noise the real and imaginary part of the field are "mixing". Although in the first glimpse this noise is similar to the ones where we perturb the positions in a chain of classical oscillators (cf. [23]), this degeneracy and "mixing" makes the treatment of this problem harder.

One way to make the above-mentioned problem more accessible is to consider the linear case $\kappa = 0$. In this case, our Gibbs measure (with proper scaling) corresponds to the Berlin-Kac model (cf. [29]). Our dynamics in this case is ergodic and this measure is the unique invariant measure. In fact, we know that in $d \geq 3$, this measure exhibits condensation (cf. [164]) similar to Bose-Einstein condensation. As before, one could study the rate of convergence in this setup (even in this case most of the above-mentioned challenges is hard to overcome). Another interesting question is the formation of the condensate. Starting from an arbitrary configuration, how long does it take for the condensate to appear in our dynamics and how does it evolve afterwards?

Another set of problems concerns the *hydrodynamic limit* of the model in the diffusive time-scale. In the linear case $\kappa = 0$, since the grand canonical measure is Gaussian, proving the hydrodynamic limit seems doable by adapting ideas from [143]. Notice that in the linear case, considering the infinite temperature version of our noise, one can prove the convergence of the empirical distribution of $|\psi(x)|$ to the solution of the heat equation in the diffusive time-scale (cf. [156]).
In the finite temperature case, hydrodynamic limit can be connected to the previous questions. Can we understand the evolution of the condensate from the macroscopic

equation appearing in the hydrodynamic limit?
In the non-linear setup, questions regarding the hydrodynamic limit seem relevant; however, they will be much more challenging (even computing the diffusion coefficient, using Kubo formula does not seem straightforward).

Besides the aforementioned questions, where at least we have a clear picture about the way we should proceed, we can mention a couple of other open questions. These questions seem a bit more vague. Since our dynamics is inspired by the soliton resolution conjecture, we are wondering if our dynamics can be connected to the long time behavior of DNLS? More specifically, is there any class of configurations, such that our dynamics and DNLS remain close to each other, starting from these configurations? Having a clear answer for this question, may lead to a better understanding about SRC.

The second shortcoming of our result, comes from the semi-norm $\tilde{H}^1$. This semi-norm, does not capture the evolution of the profile. Using this norm in our result, we can determine the limiting shape of the profile; however, it is not clear how does this shape evolve itself in the macroscopic time-scale due to our dynamics. This shortcoming stems from the fact that our proof relies on the variational characterization of the solitons.

Applying our dynamics to mass super critical dynamics, may lead to new information concerning blow-up solutions, since one expect the Gibbs measure concentrate on "exploding" configurations in this setup.

Finally, questions regarding the definition of the dynamics in the continuum setup, and properties of this noise in the continuum such as well-posedness, blow-up, connection to NLS and rate of convergence in continuous case all seem interesting, but out of scope of our work.

Chapter 2

Hydrodynamics

We study the hydrodynamic limit, in the hyperbolic space-time scaling, for a one-dimensional unpinned chain of quantum harmonic oscillators with random masses. To the best of our knowledge, this is among the first examples where one can prove the hydrodynamic limit for a quantum system rigorously. In fact, we prove that after hyperbolic rescaling of time and space, the distribution of the elongation, momentum, and energy averaged under the proper locally Gibbs state converges to the solution of the Euler equation. Moreover, our result indicates that the temperature profile is frozen in any space-time scale; in particular, the thermal diffusion coefficient vanishes. There are two main phenomena in this chain that enable us to deduce this result. First is the Anderson localization, which decouples the mechanical and thermal energy, providing the closure of the equation for energy. The second phenomenon is similar to some sort of decay of correlation phenomena, which let us circumvent the difficulties arising from the fact that our Gibbs state is not a product state due to the quantum nature of the system.
This chapter is extracted from [125]: Hannani, A. Hydrodynamic Limit for a Disordered Quantum Harmonic Chain. Commun. Math. Phys. 390, 349–423 (2022) [125], where here certain proofs is explained in more detail.

2.1 Model Description and Results

We set the following conventions:

- $\mathbb{I}_n := \{1, \dots, n\}$.
- $\mathbb{I}_n^0 := \{0, 1, \dots, n\}$.
- Denote the set of $n \times n$ real matrices by $\mathcal{M}_n(\mathbb{R})$, for $A \in M_n(\mathbb{R})$ define $A^\dagger$ to be the transpose of A.
- We denote the inner product in $\mathbb{R}^n$ by $\langle , \rangle_n$. Moreover, for $x \in \mathbb{I}_n$, $|x\rangle$ stands for the following member of the canonical basis of $\mathbb{R}^n$: $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears in the xth position. We usually denote vectors of $\mathbb{R}^n$ by Greek letters $|\psi\rangle, |\phi\rangle, |\varphi\rangle, \dots$.

- Let a be an operator (possibly unbounded), defined on a dense subset of the Hilbert space $\mathcal{H}_n = L^2(\mathbb{R}^{n-1})$, we denote the adjoint of a by a^*.

- $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ denotes the diagonal matrix with values $\lambda_i, i \in \mathbb{I}_n$ on the diagonal.

The finite volume system of size n is defined as follows: Let $\mathcal{H}_n$ be the Hilbert space $\mathcal{H}_n = L^2(\mathbb{R}^{n-1}) = \bigotimes_{x=1}^{n-1} L^2(\mathbb{R})$, denote the elements of $\mathcal{H}_n$ by bold ket notation: $|\boldsymbol{\psi}\rangle$, and $\langle \boldsymbol{\psi}|\boldsymbol{\phi}\rangle$ stands for the usual inner product in $\mathcal{H}_n$.

We denote the space variable by $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$. Let $\mathcal{S}(\mathbb{R}^{n-1})$ be the Schwartz space of functions from $\mathbb{R}^{n-1}$ to $\mathbb{C}$, which is dense in $\mathcal{H}_n$. For each $x \in \mathbb{I}_{n-1}$, define the elongation operator r_x on $\mathcal{S}(\mathbb{R}^{n-1})$ as follows: $\forall \boldsymbol{\xi} \in \mathbb{R}^{n-1}$, and $|\boldsymbol{\psi}\rangle \in \mathcal{S}(\mathbb{R}^{n-1})$, we have

$$r_x|\boldsymbol{\psi}(\boldsymbol{\xi_1}, \ldots, \boldsymbol{\xi_x}, \ldots, \boldsymbol{\xi_{n-1}})\rangle = \boldsymbol{\xi_x}|\boldsymbol{\psi}(\boldsymbol{\xi_1}, \ldots, \boldsymbol{\xi_x}, \ldots, \boldsymbol{\xi_{n-1}})\rangle.$$

One could think of r_x as the operator denoting the elongation of the spring between the particle x and $(x+1)$. Moreover, for each $x \in \mathbb{I}_n$, define the momentum operator p_x on $\mathcal{S}(\mathbb{R}^n)$ as: $p_x = -i(\frac{\partial}{\partial \boldsymbol{\xi}_{x-1}} - \frac{\partial}{\partial \boldsymbol{\xi}_x})$ i.e. for $|\boldsymbol{\psi}\rangle \in \mathcal{S}(\mathbb{R}^{n-1})$,

$$p_x|\boldsymbol{\psi}(\boldsymbol{\xi_1}, \ldots, \boldsymbol{\xi_x}, \ldots, \boldsymbol{\xi_{n-1}})\rangle = -i\Big(\frac{\partial}{\partial \boldsymbol{\xi}_{x-1}} - \frac{\partial}{\partial \boldsymbol{\xi}_x}\Big)|\boldsymbol{\psi}(\boldsymbol{\xi_1}, \ldots, \boldsymbol{\xi_x}, \ldots, \boldsymbol{\xi_{n-1}})\rangle,$$

where, we assume the free boundary condition $r_0 = r_n = 0$. This means $p_1 = i\frac{\partial}{\partial \boldsymbol{\xi}_1}$, and $p_n = -i\frac{\partial}{\partial \boldsymbol{\xi}_{n-1}}$, or $\frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial \xi_n} = 0$. This condition means that the center of mass momentum vanishes, i.e. $\hat{p}_o := \sum_{x=1}^n p_x = 0$. Notice that this boundary condition can be understood as $q_0 = q_1$, and $q_n = q_{n+1}$, in the position picture (cf. Appendix 2.B).

Although this system is usually described in terms of the position and momentum operators as in (1.1.13), it is more convenient to work with elongation operators instead of position operators. For the reader's convenience, in Appendix 2.B we illustrate the relation between these two pictures.

The canonical commutation relations (CCR) in this coordinates read:

$$[r_x, r_y] = [p_x, p_y] = 0, \quad [r_x, p_y] = i\Big(\delta_{x,(y-1)} - \delta_{x,y}\Big), \quad \forall x \in \mathbb{I}_{n-1}, y \in \mathbb{I}_n, \tag{2.1.1}$$

where $[a, b] = ab - ba$.

In terms of these operators, Hamiltonian operator is defined on $\mathcal{S}(\mathbb{R}^{n-1})$ by

$$H_n = \frac{1}{2}\sum_{x=1}^{n}\Big(\frac{p_x^2}{m_x} + r_x^2\Big), \tag{2.1.2}$$

where $\{m_x\}_{x=1}^{\infty}$ are i.i.d positive random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the the law of these random variables have a smooth density $\mu(x)$, compactly supported on the set $[m_{min}, m_{max}]$, where $m_{min} > 0$. We denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$, and $\mathbb{E}(m_x) = \bar{m}$. These assumptions

inspired from [35], [6], in order to facilitate the application of the results concerning *Anderson localization*.

Furthermore, for any $x \in \mathbb{I}_n$, define the operator $e_x = \frac{1}{2}(\frac{p_x^2}{m_x} + r_x^2)$ as the energy of the particle x. It is well known that p_x, r_x, hence e_x and H_n are essentially self-adjoint [197]. Consequently we can consider their closure on $\mathcal{H}_n$, that we denote with the same symbols (cf. [197], [180]). The domain of H_n will be denoted by $\mathcal{D}(H_n) \subset L^2(\mathbb{R}^{n-1})$.

The time evolution of the chain is given by the so-called Heisenberg dynamics generated by the Hamiltonian H_n. Since H_n is self-adjoint, by spectral theory e^{-iH_nt} is well defined for any $t \in \mathbb{R}$. Furthermore, using Stone's theorem, we define the one parameter group of authomorphism τ_t^n on $\mathcal{B}(\mathcal{H}_n)$ as follows:

$$a(t) := \tau_t^n(a) = e^{itH_n} a e^{-itH_n}, \quad \forall a \in \mathcal{B}(\mathcal{H}_n), \tag{2.1.3}$$

where $\mathcal{B}(\mathcal{H}_n)$ denotes the set of bounded operators on $\mathcal{H}_n$. Notice that $a(t)$ is the solution of Heisenberg equation:

$$\dot{a} = i[H_n, a], \qquad a(0) = a_0, \tag{2.1.4}$$

where $[a, b] = ab - ba$, and we use the notation $\dot{a}(t) := \partial_t a(t)$. This equation holds in the strong sense on the proper domain.
Again by using Stone's theorem, one can extend the domain of this dynamic to certain unbounded operators. Here we can do this task for operators such as r_x, q_x, and $e_x = \frac{1}{2}(\frac{p_x^2}{m_x} + r_x^2)$ (cf. [180],[197],[47] [48]). We denote $\tau_t^n(r_x)$, $\tau_t^n(p_x)$, and $\tau_t^n(e_x)$ by $r_x(t), p_x(t), e_x(t)$, respectively. In particular, using the canonical commutation relations (CCR), these operators satisfy the following equations:

$$\begin{aligned} \dot{p}_x(t) &= r_x(t) - r_{x-1}(t), \quad \forall x \in \mathbb{I}_n, \\ \dot{r}_x(t) &= \frac{p_{x+1}(t)}{m_{x+1}} - \frac{p_x(t)}{m_x}, \quad \forall x \in \mathbb{I}_{n-1}, \end{aligned} \tag{2.1.5}$$

where one should recall the boundary condition: $r_0 = r_n = 0$, in the first equation.

Let $M = \mathrm{diag}(m_1, \ldots, m_n))$ denotes matrix of masses, $\nabla_- \in \mathbb{R}^{n\times(n-1)}$ be the matrix of discrete gradient with fixed boundary condition, and $\nabla_+ \in \mathbb{R}^{(n-1)\times n}$ be the discrete gradient with free boundary condition. These matrices have the following explicit form:

$$\begin{aligned} M(x,y) &= m_x \delta_{x,y} \qquad 1 \le x, y \le n, \\ \nabla_-(x,y) &= \begin{cases} 1 & x = y, 1 \le x, y \le n-1, \\ -1 & x = y+1, 1 \le y \le n-1, \\ 0 & otherwise, 1 \le x \le n, 1 \le y \le n-1, \end{cases} \\ \nabla_+ &= -(\nabla_-)^\dagger, \end{aligned} \tag{2.1.6}$$

where $(.)^\dagger$ represents the transpose (in case of complex matrices complex conjugate) of a matrix. Formally, we have: for $f \in \mathbb{R}^{n-1}$, $(\nabla_- f)_x = f_x - f_{x-1}$, for $x \in \mathbb{I}_n$ with

$f_n = 0$, and for $f \in \mathbb{R}^n$, $(\nabla_+ f)_x = f_{x+1} - f_x$, for every $x \in \mathbb{I}_{n-1}$.

Taking advantage of these matrices, one can write the equation of motion (2.1.5) in the following form:

$$\dot{p} = \nabla_- r, \qquad \dot{r} = \nabla_+ M^{-1} p, \tag{2.1.7}$$

where p and r denote the vector of momentum and elongation operators, i.e. $p = p(t) = (p_1(t), p_2(t), \ldots, p_n(t))$, and $r(t) = (r_1(t), r_2(t), \ldots, r_{n-1}(t))$. Notice that this equation can be solved explicitly, we do this task in the next section. In fact, given the explicit expression for $p(t)$ and $r(t)$ from (2.2.22), one can check that (2.1.7) holds in strong sense on $\mathcal{D}(H_n)$, and this solution can be extended to the proper domain [1] by linearity.
Finally, define Δ, the discrete Laplacian matrix with free boundary condition, as

$$\Delta = \nabla_- \nabla_+. \tag{2.1.8}$$

Hence, we have $\forall f \in \mathbb{R}^n, (\Delta f)_x = f_{x+1} - 2f_x + f_{x-1}$, for $x \in \mathbb{I}_n$, with free boundary condition $f_{n+1} = f_n, f_0 = f_1$.

Given $\bar{p}$, $\bar{r} \in C^1([0,1])$, such that $\bar{r}(0) = \bar{r}(1) = 0$, $\int_0^1 \bar{p}(y)dy = 0$, and $\beta \in C^0([0,1])$, such that $\forall y \in [0,1]$, $0 < \beta_{min} \leq \beta(y) \leq \beta_{max}$, correspondingly, we define the locally Gibbs state as the density matrix operator acting on $\mathcal{H}_n$ given by

$$\rho^n_{\bar{p},\bar{r},\beta} = \frac{1}{Z_n} \exp\left(-\frac{1}{2} \sum_{x=1}^{n} \left[\frac{\beta(\frac{x}{n})}{m_x} (p_x - \bar{p}(\frac{x}{n}) \frac{m_x}{\bar{m}})^2 + \beta(\frac{x}{n})(r_x - \bar{r}(\frac{x}{n}))^2 \right] \right), \tag{2.1.9}$$

where one should recall $\bar{m} = \mathbb{E}(m_x)$ is the expectation of the masses. This means initially we let the chain to be in the locally Gibbs state such that the elongation and momentum be out of mechanical equilibrium, and their initial "average" coming from a smooth profile. Moreover, we have a smooth profile of temperature. We drop the subscript and superscripts of ρ whenever it makes no confusion. Here Z_n is a normalizing constant such that $\mathrm{Tr}\left(\rho^n_{\bar{p},\bar{r},\beta}\right) = 1$. Note that in this expression $\bar{p}(\frac{x}{n})$, $\bar{r}(\frac{x}{n})$ have been multiplied by the identity operator. It is more convenient to define the density operator in terms of the following pseudo-Hamiltonian:

$$H^n_\beta = \frac{1}{2} \sum_{x=1}^{n} \left(\frac{\beta_x}{m_x} \left(p_x - (\frac{m_x}{\bar{m}}) \bar{p}_x \right)^2 + \beta_x (r_x - \bar{r}_x)^2 \right), \tag{2.1.10}$$

where we wrote the following terms in shorthanded manner: $\beta_x := \beta(\frac{x}{n}), \bar{p}_x := \bar{p}(\frac{x}{n}), \bar{r}_x := \bar{r}(\frac{x}{n})$. Note that it would be more appropriate to write this operator as $H^n_{\beta(.)}$, since it actually depends on the function $\beta(.)$. However, we use the notation H^n_β, whenever it does not make any confusion. Then we have

$$\rho^n_{\bar{p},\bar{r},\beta} = \exp\left(-H^n_\beta \right). \tag{2.1.11}$$

[1] One can take $\cap_{k=1}^{n-1} (\mathcal{D}(\hat{b}_k) \cap \mathcal{D}(\hat{b}^*_k))$ as the proper domain, which is dense in $\mathcal{H}_n$.

First, observe that H^n_β is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^{n-1})$ (cf. [197], [180], [48]), and denote its closure with the same symbol. Furthermore, one can check that H^n_β has a discrete spectrum with non-negative eigenvalues. In fact, we can write H^n_β in terms of the sum of free bosonic operators, and we can obtain the spectrum explicitly. We do this task in details in Section 2.2 . Hence, using spectral theory and properties of $\beta(.)$ in our assumption, one can observe that ρ is well defined and trace-class.
Therefore, for every operator a, if $a\rho$ is a trace class operator, we can define the "average of the observable a in the state ρ", i.e. $\langle a \rangle_\rho$ as:

$$\langle a \rangle_\rho = \mathrm{Tr}(\rho a). \tag{2.1.12}$$

In particular, one can observe that $\langle p_x \rangle_\rho$, $\langle r_x \rangle_\rho$, and $\langle e_x \rangle_\rho$ are well defined.

Fix $T > 0$, and consider the following system of conservation laws:

$$\begin{aligned} \partial_t \mathrm{r}(y,t) &= \frac{1}{\bar{m}} \partial_y \mathrm{p}(y,t), \\ \partial_t \mathrm{p}(y,t) &= \partial_y \mathrm{r}(y,t), \\ \partial_t \mathrm{e}(y,t) &= \frac{1}{\bar{m}} \partial_y (\mathrm{r}(y,t)\mathrm{p}(y,t)), \end{aligned} \tag{2.1.13}$$

where $\mathrm{r}(y,t) \in C^1([0,1]\times[0,T])$, $\mathrm{p}(y,t), \mathrm{e}(y,t) \in C^1([0,1]\times[0,T])$. We impose the following boundary condition

$$\forall t \in [0,T], \quad \mathrm{r}(0,t) = \mathrm{r}(1,t) = 0, \tag{2.1.14}$$

and we add the following initial datum:

$$\mathrm{r}(y,0) = \bar{r}(y), \;\; \mathrm{p}(y,0) = \bar{p}(y), \;\; \mathrm{e}(y,0) = \frac{1}{2}\Big(\frac{\bar{p}(y)^2}{\bar{m}} + \bar{r}(y)^2\Big) + \mathrm{f}^\mu_\beta(y). \tag{2.1.15}$$

Here $\bar{p}$ and $\bar{r}$ are the same functions that appeared in the definition of the Gibbs state (2.1.9), and $\mathrm{f}^\mu_\beta(y) = \mathrm{f}^\mu(\beta(y))$, where $\mathrm{f}^\mu(\beta_{eq})$ for $\beta_{eq} \in (0,\infty)$, is a function which can be determined uniquely by the law of the distribution of the masses μ. (cf. (2.4.5), (2.4.35) for the definition, Section 2.4.1 and Appendix 2.A).
One should recall $\int_0^1 \bar{p}(y)dy = 0$, and $\bar{r}(0) = \bar{r}(1)$. Observe that $\int_0^1 \mathrm{p}(y,t)dy$ is conserved by (2.1.13), thanks to the boundary condition (2.1.14), justifying the assumption $\int_0^1 \bar{p}(y)dy = 0$ (cf. Remark 2.B.2, 2.B.1).

Note that since the initial datum for r and p is regular, this equation has a unique classical solution in r and p, (cf. [95], Section 7).

Now we are prepared to present the main theorem of this manuscript, which states that the empirical distribution of the average of (r,p,e) with respect to initial state ρ, after hyperbolic scaling of time and space converges to the solution of (2.1.13) with (2.1.14) and initial datum (2.1.15). Precisely, we have:

Theorem 2.1.1. *Let $f \in C^0([0,1])$ be an arbitrary test function. Fix $T > 0$ and let $t \in [0,T]$. Recall the definition of the initial state ρ (2.1.9), and let $\bar{p}$, $\bar{r}$, and β satisfy the assumptions stated in the definition of (2.1.9). Then, let $(r(nt), p(nt), e(nt))$ be the evolved operators in the Heisenberg picture with the dynamics generated by the Hamiltonian (2.1.2). Moreover, let the $(\mathrm{r}(y,t), \mathrm{p}(y,t), \mathrm{e}(y,t))$ be the solution to (2.1.13) with boundary condition (2.1.14) and initial datum (2.1.15). Then as $n \to \infty$, we have the following almost sure convergence with respect to distribution of the masses:*

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle r_x(nt) \right\rangle_\rho \to \int_0^1 f(y)\mathrm{r}(y,t)dy, \tag{2.1.16}$$

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle p_x(nt) \right\rangle_\rho \to \int_0^1 f(y)\mathrm{p}(y,t)dy, \tag{2.1.17}$$

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle e_x(nt) \right\rangle_\rho \to \int_0^1 f(y)\mathrm{e}(y,t)dy. \tag{2.1.18}$$

The rest of this paper is devoted to the proof of Theorem 2.1.1. The sketch of the proof and organization of the paper is as follows: One can recall a similar theorem for the classical harmonic chain from ([35], Theorem 1), the main difference here lies in the fact that we cannot write our Gibbs state ρ as a product state, since the energy of each site does not commute with its neighbors. In order to deal with this difficulty, we write H_β^n, which appeared in the definition of $\rho = \exp\left(-H_\beta^n\right)$, in terms of free bosonic operators (using Bogoliubov or quantum canonical transformation), and then compute the average of suitable operators in this new basis. These computations enable us to obtain appropriate bounds in order to prove Theorem 2.1.1.

Moreover, we use the explicit form of the solution to the equation of motion (2.1.7) in our proof; hence, we use another Bogoliubov transformation in order to solve the equation of motion. We devote Section 2.2 to these transformations and corresponding bounds.

Then we deal with the evolution of (r,p) in Section 2.3. This part is similar in spirit to the classical case, since all the operators can be written as the linear combination of bosonic operators. However, our proof will be different from the one in [35].

We devote the last two sections to prove (2.1.18). As we sketched before, to prove (2.1.18) at $t = 0$, we need SLLN (Strong Law of Large Numbers) for $\langle e_x \rangle_\rho$. In order to prove the SLLN, we need to show that the dependence of $\langle e_x \rangle$ on m_z is exponentially decaying for z being far away from x, we devote Section 2.4.1 to prove this fact.

Finally, in Section 2.5, we exploit the localization phenomenon in order to prove (2.1.18) for any $t \in (0,T]$, as we explained in Section 1.1.3.

2.2 Preliminary Bounds

2.2.1 Hamiltonain diagonalization

Recall the Hamiltonian H_n (2.1.2), and the operator $H^n_{\beta(.)}$ acting on $L^2(\mathbb{R}^{n-1})$. These operators have discrete spectrums, and the full set of their eigenvalues and eigenfunctions can be represented rather explicitly, thanks to the quantum canonical transformation also known as Bogoliubov transformation (cf. [47],[79],[180] for a through discussion). We recall here these transformations in details.
Let us mention that the idea of diagonalizing many-body Hamiltonians via Bogoliubov transformations for one-dimensional integrable systems have been used extensively since 1960s (cf. Lieb-Liniger model as an example: [159], [158]; see also [160] Chapter 4,5 and references therein). The new feature of our model comparing to the later is as follows: in the later examples, desired "deterministic" transformation is obtained rather explicitly. Moreover, one needs only one set of transformation in order to "solve" the system. In our case, we have two sets of transformations: one for solving the equations of motion (cf. φ^k in (2.2.3)), and the other for computing certain averages w.r.t ρ (cf. ψ^k (2.2.26)). Solution to the equations of motion is expressed in terms of the former coordinates which are functions of the masses and random; moreover, they have the localization property, enabling us to close the equation and prove the hydrodynamic limit. In thermal equilibrium, ψ^k, and φ^k coincide with each others. However, out of thermal equilibrium, we need to understand the relation among these transformations (cf. (2.2.56)) in order to obtain the crucial bounds (2.2.55). The difference among ψ^k and φ^k leads to complications in proving the fact that $\mathfrak{f}^\mu_\beta$ is local (cf. Appendic 2.A).

Let $\langle , \rangle_n$ denotes the inner product in $\mathbb{R}^n$, we drop the subscript whenever it does not make any confusion. We express the canonical basis of $\mathbb{R}^n$ by $|x\rangle$, for $x \in \mathbb{I}_n$. By abusing the notation, we use the same symbol for a linear combination of operators or for the product of two vector of commuting operators. Using this notation we have:

$$H_n = \frac{1}{2}\Big(\langle p, M^{-1}p\rangle_n + \langle r, r\rangle_{n-1}\Big). \tag{2.2.1}$$

Consider the following matrix:

$$A^0_p := M^{-\frac{1}{2}}(-\Delta)M^{-\frac{1}{2}} \in \mathbb{R}^{n\times n}, \tag{2.2.2}$$

this matrix is symmetric, positive semidefinite and almost surely, it has a non-degenerate spectrum (it is evident by using proposition II.1 of [149]). Let $0 = \omega_0^2 < \omega_1^2 < \cdots < \omega_{n-1}^2$, be the set of eigenvalues of this matrix in the increasing order, and let $\{\varphi^k\}_{k=0}^{n-1}$ be their corresponding eigenvectors, such that they form an orthonormal basis for $\mathbb{R}^n$, where we have $\langle \varphi^k, \varphi^j\rangle = \delta_{k,j}$. Observe that we have $\varphi^0 = (\sum_{x=1}^n m_x)^{-\frac{1}{2}} M^{\frac{1}{2}} |\mathbf{1}\rangle$, where $|\mathbf{1}\rangle$ denotes the vector $(1, 1, \ldots, 1)^\dagger$.

Define the operators $\hat{p}_k, \forall k \in \mathbb{I}^0_{n-1}$ to be the following linear combination of p_x:

$$\hat{p}_k = \langle \varphi^k, M^{-\frac{1}{2}}p\rangle_n = \sum_{x=1}^n \frac{\varphi^k_x}{\sqrt{m_x}} p_x. \tag{2.2.3}$$

Taking into account the definition of φ^0, one can check that $\hat{p}_o := \sum_{x=1}^n p_x = 0 = (\sum_{x=1}^n m_x)^{\frac{1}{2}}\hat{p}_0$, thanks to the imposed boundary condition.
On the other hand let

$$A_r^0 := -\nabla_+ M^{-1}\nabla_-, \tag{2.2.4}$$

due to the positivity assumption on M, one can observe that A_r^0 is a $(n-1)\times(n-1)$ positive symmetric matrix. Moreover, if we let for $k \in \mathbb{I}_{n-1}$, $\phi^k := \frac{1}{\omega_k}\nabla_+ M^{-\frac{1}{2}}\varphi^k$, we have:

$$\begin{aligned}
A_r^0\phi^k &= -\frac{1}{\omega_k}\nabla_+ M^{-1}\nabla_-\Big(\nabla_+ M^{-\frac{1}{2}}\varphi^k\Big) = \frac{1}{\omega_k}\nabla_+ M^{-\frac{1}{2}}\Big(M^{-\frac{1}{2}}(-\Delta)M^{-\frac{1}{2}}\Big)\varphi^k \\
&= \frac{1}{\omega_k}\nabla_+ M^{-\frac{1}{2}}(\omega_k^2)\varphi^k = \omega_k^2\phi^k, \\
\langle\phi^k,\phi^{k'}\rangle_{n-1} &= \frac{1}{\omega_k\omega_{k'}}\langle\nabla_+ M^{-\frac{1}{2}}\varphi^k, \nabla_+ M^{-\frac{1}{2}}\varphi^{k'}\rangle_{n-1} = \frac{1}{\omega_k\omega_{k'}}\langle\varphi^k, M^{-\frac{1}{2}}(-\Delta)M^{-\frac{1}{2}}\varphi^{k'}\rangle_n \\
&= \frac{\omega_{k'}}{\omega_k}\langle\varphi^k,\varphi^{k'}\rangle_n = \delta_{k,k'}.
\end{aligned} \tag{2.2.5}$$

Hence, $\{\phi^k\}_{k=1}^{n-1}$ is the full set of eigenvectors for A_r^0 with the same set of eigenvalues as A_p^0: $\{\omega_1^2,\dots\omega_{n-1}^2\}$. Therefore, they form an orthonormal basis for $\mathbb{R}^{n-1}$. Now define $\hat{r}_k$, for every $k \in \mathbb{I}_{n-1}$ as

$$\hat{r}_k := \langle\phi^k, r\rangle_{n-1}. \tag{2.2.6}$$

Let us denote the operator $s_x := -i\partial/\partial_{\boldsymbol{\xi}_x}$, so we have $p_x = -(s_x - s_{x-1})$, more precisely, thanks to the definition of ∇_- and the boundary condition $r_0 = r_n = 0$, we have:

$$p = -\nabla_- s. \tag{2.2.7}$$

The canonical commutation relation in the s coordinates reads:

$$\forall x, y \in \mathbb{I}_{n-1}, \quad [r_x, s_y] = i\delta_{x,y}. \tag{2.2.8}$$

Therefore, one can compute $[r_k, p_{k'}]$:

$$\begin{aligned}
[r_k, p_k'] &= \left[\left\langle\phi^k, r\right\rangle_{n-1}, \left\langle\varphi^{k'}, M^{-\frac{1}{2}}p\right\rangle_n\right] = \left[\left\langle\phi^k, r\right\rangle_{n-1}, -\left\langle\varphi^{k'}, M^{-\frac{1}{2}}\nabla_- s\right\rangle_n\right] \\
&= \left[\left\langle\phi^k, r\right\rangle_{n-1}, \left\langle\omega_{k'}\nabla_+ M^{-\frac{1}{2}}\varphi^{k'}, s\right\rangle_{n-1}\right] = \omega_{k'}\left[\left\langle\phi^k, r\right\rangle_{n-1}, \left\langle\phi^{k'}, s\right\rangle_{n-1}\right] \\
&= \omega_{k'}\sum_{x,y=1}^{n-1}\phi_x^k\phi_y^{k'}[r_x, s_y] = \omega_{k'}\sum_{x,y=1}^{n-1}\phi_x^k\phi_y^{k'}i\delta_{x,y} = i\omega_{k'}\sum_{x=1}^{n-1}\phi_x^k\phi_x^{k'} = i\omega_k\delta_{k,k'},
\end{aligned} \tag{2.2.9}$$

where we used the identities $(\nabla_-)^\dagger = -\nabla_+$ and $\nabla_+ M^{-\frac{1}{2}}\varphi^k = \omega_k\phi^k$ as well as the fact that $\{\phi^k\}_{k=1}^{(n-1)}$ is an orthonormal basis. Since $[r_x, r_y] = [p_x, p_y] = 0$, we can sum up the commutation relation for our new coordinates as follows:

$$\forall k, k' \in \mathbb{I}_{n-1}^0, \quad [\hat{r}_k, \hat{p}_{k'}] = i\omega_k\delta_{k,k'}, \qquad [\hat{r}_k, \hat{r}_{k'}] = [\hat{p}_k, \hat{p}_{k'}] = 0. \tag{2.2.10}$$

Later, we benefit from the inverse of (2.2.3) and (2.2.6). Let O and $\tilde{O}$ be orthogonal matrices of eigenvectors of A_p^0 and A_r^0, respectively. Hence, $O^\dagger O = OO^\dagger = I_n$, $\tilde{O}^\dagger \tilde{O} = \tilde{O}\tilde{O}^\dagger = I_{n-1}$. In other words: $\sum_{k=0}^{n-1} \varphi_x^k \varphi_y^k = \sum_{k=1}^{n-1} \phi_x^k \phi_y^k = \delta_{x,y}$. Therefore, the inverse expressions read:

$$\begin{aligned} p &= M^{\frac{1}{2}} O\hat{p}, \qquad & p_x &= \sqrt{m_x} \sum_{k=0}^{n-1} \varphi_x^k \hat{p}_k, \\ r &= \tilde{O}\hat{r}, & r_x &= \sum_{k=0}^{n-1} \phi_x^k \hat{r}_k. \end{aligned} \tag{2.2.11}$$

Here, $\hat{p}, \hat{r}$ denote the vector of corresponding operators in the transformed coordinate.
The Hamiltonian H_n can be written in terms of these new coordinates:

$$\begin{aligned} H_n &= \frac{1}{2}(\langle p, M^{-1}p\rangle_n + \langle r, r\rangle_{n-1}) = \frac{1}{2}\Big(\langle M^{\frac{1}{2}} O\hat{p}, M^{-\frac{1}{2}} O\hat{p}\rangle_n + \langle \tilde{O}\hat{r}, \tilde{O}\hat{r}\rangle_{n-1}\Big) \\ &= \frac{1}{2}(\langle \hat{p}, \hat{p}\rangle_n + \langle \hat{r}, \hat{r}\rangle_{n-1}) = \frac{1}{2}\hat{p}_0^2 + \frac{1}{2}\sum_{k=1}^{n-1}(\hat{p}_k^2 + \hat{r}_k^2) = \frac{1}{2}\sum_{k=1}^{n-1}(\hat{p}_k^2 + \hat{r}_k^2), \end{aligned} \tag{2.2.12}$$

where we used the identities $O^\dagger O = OO^\dagger = I_n, \tilde{O}^\dagger \tilde{O} = \tilde{O}\tilde{O}^\dagger = I_{n-1}$ and the fact that $\hat{p}_0 = 0$. We introduce the operators $\hat{b}_k$ with their adjoints $\hat{b}_k^*$ as follows:

$$\hat{b}_k = \frac{1}{\sqrt{2\omega_k}}(\hat{r}_k + i\hat{p}_k), \tag{2.2.13}$$

$$\hat{b}_k^* = \frac{1}{\sqrt{2\omega_k}}(\hat{r}_k - i\hat{p}_k). \tag{2.2.14}$$

Notice that (2.2.14) was deduced from the fact that $\hat{p}_k, \hat{r}_k$ are self-adjoint. These operators are the bosonic creation and annihilation operators. Using (2.2.10), one can verify that they satisfy the annihilation-creation form of the canonical commutation relation:

$$[\hat{b}_k, \hat{b}_{k'}^*] = \delta_{k,k'} \quad [\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^*, \hat{b}_{k'}^*] = 0. \tag{2.2.15}$$

Furthermore, we can express the Hamiltonian as follows, using the identity $\omega_k \hat{b}_k^* \hat{b}_k = \frac{1}{2}(\hat{r}_k^2 + \hat{p}_k^2 - \frac{\omega_k}{2})$, thanks to (2.2.10):

$$H_n = \sum_{k=1}^{n-1} \omega_k (\hat{b}_k^* \hat{b}_k + \frac{1}{2}). \tag{2.2.16}$$

Having (2.2.16), with the commutation relation (2.2.15), full spectrum of H_n (which is discrete) and the corresponding eigenfunctions are quite well known (cf. [123] Section 9 or [180]).
We describe this spectrum and its corresponding eigenfunctions as follows: recall the space variable $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$, similar to (2.2.6), define the new space coordinate

$\hat{\boldsymbol{\xi}}_k := \left\langle \phi^k, \boldsymbol{\xi} \right\rangle_{n-1}$ for $k \in \mathbb{I}^0_{n-1}$. Let $|\boldsymbol{\Phi_0}\rangle \in L^2(\mathbb{R}^{n-1})$ be the unique normalized solution of

$$\forall k \in \mathbb{I}_{n-1}, \quad \hat{b}_k|\boldsymbol{\Phi_0}\rangle = \frac{1}{\sqrt{2}}\Big(\frac{\hat{r}_k}{\sqrt{\omega_k}} + \sqrt{\omega_k}\partial/\partial_{\hat{\boldsymbol{\xi}}_k}\Big)\Big|\boldsymbol{\Phi_0}(\hat{\boldsymbol{\xi}}_{\mathbf{1}}, \ldots, \hat{\boldsymbol{\xi}}_{\boldsymbol{n-1}})\Big\rangle = 0.$$

Then, $|\boldsymbol{\Phi_0}\rangle$ will be the ground state of H_n with the corresponding eigenvalue (energy) $E_0 = \frac{1}{2}\sum_{k=1}^{n-1}\omega_k = \frac{1}{2}\operatorname{Tr}\big(\omega^{\frac{1}{2}}\big) = \frac{1}{2}\operatorname{Tr}\big((-\Delta M^{-1})^{\frac{1}{2}}\big)$, where $\omega = \operatorname{diag}(\omega_1^2, \ldots, \omega_{n-1}^2)$. Let $\mathbb{N}$ be the set of nonnegative integers, for every $\bar{\theta} \in \mathbb{N}^{n-1}$, $\bar{\theta} = (\theta_1, \ldots, \theta_{n-1})$ define $E_{\bar{\theta}}$ as:

$$E_{\bar{\theta}} = E_0 + \sum_{k=1}^{n-1}\omega_k\theta_k. \tag{2.2.17}$$

Then $\mathcal{E} = \{E_{\bar{\theta}} | \bar{\theta} \in \mathbb{N}^{n-1}\}$ is the set of eigenvalues of H_n with the corresponding eigenfunctions $|\boldsymbol{\Phi}_{\bar{\boldsymbol{\theta}}}\rangle$, where they form an orthonormal basis for $\mathcal{H}_n$. One can write these eigenfunctions as follows:

$$|\boldsymbol{\Phi}_{\bar{\boldsymbol{\theta}}}\rangle = \prod_{k=1}^{n-1}\frac{(\hat{b}_k^*)^{\theta_k}}{\sqrt{\theta_k!}}|\boldsymbol{\Phi_0}\rangle. \tag{2.2.18}$$

One can describe $|\boldsymbol{\Phi}_{\bar{\boldsymbol{\theta}}}\rangle$ more precisely as

$$\Big|\boldsymbol{\Phi}_{\bar{\boldsymbol{\theta}}}(\hat{\boldsymbol{\xi}}_{\mathbf{1}}, \ldots, \hat{\boldsymbol{\xi}}_{\boldsymbol{n-1}})\Big\rangle = \prod_k f_{\theta_k}(\hat{\boldsymbol{\xi}}_k), \quad f_{\theta_k}(x) = \frac{1}{\sqrt{\theta_k!}}\Big(\frac{\omega_k}{\pi}\Big)^{\frac{1}{4}}e^{-\frac{\omega_k x^2}{2}}\mathrm{H}_{\theta_k}(\sqrt{2\omega_k}x),$$

where H_j stands or the j-th hermite polynomial. Note that the aforementioned set $\mathcal{E}$ is the full spectrum of H_n, and it's a complete orthonormal basis for $L^2(\mathbb{R}^{n-1})$.

Finally, we address the time evolution of the momentum and elongation operator. First, the time evolution of the bosonic operators $\hat{b}_k$ and $\hat{b}_k^*$, i.e. the action of the Heisenberg dynamic τ_t^n (2.1.3) on these operators is given by:

$$\tau_t^n(\hat{b}_k) =: \hat{b}_k(t) = e^{-i\omega_k t}\hat{b}_k(0), \qquad \tau_t^n(\hat{b}_k^*) =: \hat{b}_k^*(t) = e^{i\omega_k t}b_k^*(0), \tag{2.2.19}$$

where the current form of the Hamiltonian (2.2.16), and the commutation relations (2.2.15), have been used, with $\hat{b}_k(0) = \hat{b}_k$ and $\hat{b}_k^*(0) = \hat{b}_k^*$. Note that $\tau_t^n(\hat{b}_k) =: \hat{b}_k(t) = e^{-i\omega_k t}\hat{b}_k(0)$ is the unique solution of the Heisenberg evolution equation $\dot{\hat{b}}_k = i[H_n, \hat{b}^k] = -i\omega_k\hat{b}_k$, where we used (2.2.15). This equation holds in the strong sense on $\mathcal{D}(H_n)$ and the solution can be extended to $\mathcal{D}(\hat{b}_k)$ by linearity.
Rewriting $\hat{p}_k(0)$, $\hat{r}_k(0)$, for $k \in \mathbb{I}_{n-1}$ in terms of creation and annihilation operators from (2.2.13) and (2.2.14) gives:

$$\hat{p}_k(0) = i\sqrt{\frac{\omega_k}{2}}(\hat{b}_k^*(0) - \hat{b}_k(0)), \qquad \hat{r}_k(t) = \sqrt{\frac{\omega_k}{2}}(\hat{b}_k(0) + \hat{b}_k^*(0)). \tag{2.2.20}$$

By applying (2.2.19), we end up with:

$$\begin{aligned}
\hat{p}_k(t) &= i\sqrt{\frac{\omega_k}{2}}(\hat{b}_k^*(0)e^{-i\omega_k t} - \hat{b}_k(0)e^{i\omega_k t}) = \cos(\omega_k t)\hat{p}_k(0) - \sin(\omega_k t)\hat{r}_k(0) \\
&= \langle M^{-\frac{1}{2}}\varphi^k, p(0)\rangle \cos(\omega_k t) - \langle \phi^k, r(0)\rangle \sin(\omega_k t), \\
\hat{r}_k(t) &= \sqrt{\frac{\omega_k}{2}}(\hat{b}_k(0)e^{-i\omega_k t} + \hat{b}_k^*(0)e^{i\omega_k t}) = \cos(\omega_k t)\hat{r}_k(0) + \sin(\omega_k t)\hat{p}_k(0) \\
&= \langle \phi^k, r(0)\rangle \cos(\omega_k t) + \langle M^{-\frac{1}{2}}\varphi^k, p(0)\rangle \sin(\omega_k t).
\end{aligned} \tag{2.2.21}$$

Lastly, exploiting the relation (2.2.11) the time evolution of p and r i.e. $p(t) := \tau_t^n(p)$ and $r(t) := \tau_t^n(r)$ is given by:

$$\begin{aligned}
p(t) &= \sum_{k=0}^{n-1} M^{\frac{1}{2}}\varphi^k \hat{p}_k(t) = \sum_{k=0}^{n-1} \Big(\cos(\omega_k t)\hat{p}_k(0) - \sin(\omega_k t)\hat{r}_k(0)\Big) M^{\frac{1}{2}}\varphi^k \\
&= \sum_{k=0}^{n-1} \Big(\langle M^{-\frac{1}{2}}\varphi^k, p(0)\rangle \cos(\omega_k t) - \langle \phi^k, r(0)\rangle \sin(\omega_k t)\Big) M^{\frac{1}{2}}\varphi^k, \\
p_x(t) &= \sum_{k=0}^{n-1} \sqrt{m_x}\varphi_x^k \hat{p}_k(t) = \sum_{k=0}^{n-1} \Big(\cos(\omega_k t)\hat{p}_k(0) - \sin(\omega_k t)\hat{r}_k(0)\Big) \sqrt{m_x}\varphi_x^k, \\
r(t) &= \sum_{k=1}^{n-1} \phi^k r_k(t) = \sum_{k=1}^{n-1} \Big(\cos(\omega_k t)\hat{r}_k(0) + \sin(\omega_k t)\hat{p}_k(0)\Big) \phi^k \\
&= \sum_{k=1}^{n-1} \Big(\langle \phi^k, r(0)\rangle \cos(\omega_k t) + \langle M^{-\frac{1}{2}}\varphi^k, p(0)\rangle \sin(\omega_k t)\Big) \phi^k, \\
r_x(t) &= \sum_{k=1}^{n-1} \phi_x^k r_k(t) = \sum_{k=1}^{n-1} \Big(\cos(\omega_k t)\hat{r}_k(0) + \sin(\omega_k t)\hat{p}_k(0)\Big) \phi_x^k.
\end{aligned} \tag{2.2.22}$$

Remark 2.2.1. *It is worth mentioning that following [180], the process of rewriting a quadratic Hamiltonian in terms of free bosons can be done in a more general setting. In fact, this task is doable for any Hamiltonian of the form* $H = \frac{1}{2}\langle (q,p), A(q,p)\rangle_{2n}$, *such that* $A \in \mathcal{M}_{2n}(\mathbb{R})$ *is a positive symmetric matrix. This fact is a direct consequence of Williamson's theorem (cf. [225], and [79] Section 8.3). This theorem states that any positive symmetric matrix* $A \in \mathcal{M}_{2n}(\mathbb{R})$ *can be diagonalized via a sympletic matrix* S. *Recall that* S *is sympletic, if and only if we have* $SJS^\dagger = J$, *where*

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Using this theorem, one can define the new coordinates $(\hat{p},\hat{q})^\dagger = S^{-1}(p,q)^\dagger$. *Here* $v^\dagger$ *denotes the transpose of the vector* v. *Thanks to the properties of* S *(cf. Remark 2.4 of [79]) the canonical commutation relation for these new coordinates is evident i.e.* $\forall x,y \in \mathbb{I}_n, \quad [\hat{p}_x,\hat{p}_y] = [\hat{q}_x,\hat{q}_y] = 0$, *and* $[\hat{q}_x,\hat{p}_y] = i\delta_{x,y}$. *Moreover,* S *diagonalizes* A, *i.e.* $SAS^\dagger = \lambda^2$, *where* $\lambda^2 = \mathrm{diag}(\lambda_1^2,\dots,\lambda_n^2,(\lambda_1')^2,\dots,(\lambda_n')^2)$. *And the Hamiltonian reads:* $H = \frac{1}{2}\sum_{x=1}^n (\lambda_x')^2(\hat{p}_x^2 + \gamma_x^2\hat{q}_x^2)$, *with* $\gamma_x = \frac{\lambda_x}{\lambda_x'}$. *Finally, the bosonic operators are given by* $B_x = \frac{1}{\sqrt{2}}(\sqrt{\gamma_x}\hat{q}_x + \frac{i\hat{p}_x}{\sqrt{\gamma_x}})$, *where* $[B_x,B_y] = [B_x^*,B_y^*] = 0$, *and* $[B_x,B_y^*] = \delta_{x,y}$.

Moreover, the Hamiltonian can be written as $H = \sum_{x=1}^{n} (\lambda'_x)^2 \gamma_x (B_x^* B_x + \frac{1}{2})$*, which is sum of free bosons as we desired.*

In a more physical setup, we have $H = \frac{1}{2}(\langle p, V_p p\rangle_n + \langle q, V_q q\rangle_n)$*, such that* $V_p, V_q \in \mathcal{M}_n(\mathbb{R})$ *are positive and symmetric. In this situation, we can express the desired transformation in the following explicit manner: In this case since* $V_q^{\frac{1}{2}} V_p V_q^{\frac{1}{2}}$ *is positive and symmetric, let* O *be the orthogonal matrix such that* $O^\dagger V_p^{\frac{1}{2}} V_q V_p^{\frac{1}{2}} O = \gamma^2 =: \operatorname{diag}(\gamma_1^2, \dots, \gamma_n^2)$*. Then* S *(the sympletic transformation introduced above) has the following form:*

$$S = \begin{bmatrix} V_p^{\frac{1}{2}} O & 0 \\ 0 & V_p^{-\frac{1}{2}} O \end{bmatrix}.$$

Moreover, the Hamiltonian can be written as:

$$H = \frac{1}{2} \sum_{k=1}^{n} \hat{p}_k^2 + \gamma_k^2 \hat{q}_k^2 = \sum_{k=1}^{n} \gamma_k (B_k^* B_k + \frac{1}{2}),$$

where the bosonic operators are defined as before: $B_k := \frac{1}{\sqrt{2}}(\sqrt{\gamma_k}\hat{q}_k + i\frac{\hat{p}_k}{\sqrt{\gamma_k}})$*. In this remark, we followed the notation in [73]. Notice that in order to diagonalize* H_n *and* H_n^β*, we adapted the same strategy to our setup, where we have the Hamiltonian in terms of* r *coordinates.* [2]

2.2.2 Density operator diagonalisation

Recall the definition (2.1.9) of locally Gibbs state $\rho = \exp\big(-H_\beta^n\big)$, where H_β^n is defined in (2.1.10). In this section, we recall the necessary transformation for rewriting H_β^n in terms of free bosons, following the lines of Remark 2.2.1. This helps us to compute certain averages with respect to ρ.
We begin by defining the following operators:

$$\begin{aligned} \tilde{\mathtt{p}}_x &:= p_x - \frac{m_x}{\bar{m}} \bar{p}_x, \qquad \forall x \in \mathbb{I}_n, \\ \tilde{\mathtt{r}}_x &= r_x - \bar{r}_x, \qquad \forall x \in \mathbb{I}_{n-1}, \end{aligned} \tag{2.2.23}$$

Observe that H_β^n can be written as:

$$H_\beta^n = \frac{1}{2}\Big(\langle \tilde{\mathtt{p}}, M_\beta^{-1} \tilde{\mathtt{p}}\rangle_n + \langle \tilde{\mathtt{r}}, \beta^o \tilde{\mathtt{r}}\rangle_{n-1}\Big), \tag{2.2.24}$$

where $M_\beta = M\tilde{\beta}^{-1}$, with $\tilde{\beta} := \operatorname{diag}(\beta(\frac{1}{n}), \dots, \beta(\frac{n}{n}))$ and $\beta^o := \operatorname{diag}(\beta(\frac{1}{n}), \dots, \beta(\frac{n-1}{n}))$. Define A_p^β and A_r^β, similar to A_r^0 and A_p^0 as:

$$A_p^\beta = M_\beta^{-\frac{1}{2}}(-\nabla_- \beta^0 \nabla_+) M_\beta^{-\frac{1}{2}}, \qquad A_r^\beta = (\beta^o)^{\frac{1}{2}}(-\nabla_+ M_\beta^{-1} \nabla_-)(\beta^o)^{\frac{1}{2}}. \tag{2.2.25}$$

[2] This modification stems from the fact that in our case V_x is not positive definite and has a zero eigenvalue.

Since A_p^β is symmetric positive semidefinite with almost sure non-degenerate spectrum, let $\{\psi^k\}_{k=0}^{n-1}$ be the orthonormal set of eigenvectors for A_p^β, such that they form a basis for $\mathbb{R}^n$. Then the corresponding set of increasing eigenvalues is given by $\{0=\gamma_0^2<\gamma_1^2<\cdots<\gamma_{n-1}^2\}$. Denote the matrix of these eigenvectors by O_β. We have $O_\beta O_\beta^\dagger = O_\beta^\dagger O_\beta = I_n$, $\langle \psi^k, \psi^{k'}\rangle = \sum_{x=1}^n \psi_x^k \psi_x^{k'} = \delta_{k,k'}$, and $\sum_{k=0}^{n-1}\psi_x^k\psi_y^{k'} = \delta_{x,y}$. Note that we have $\psi^0 = (\sum_{x=1}^n \frac{m_x}{\beta_x})^{-\frac{1}{2}} M_\beta^{\frac{1}{2}} |\mathbf{1}\rangle$. Moreover, A_r^β is symmetric positive definite, and one can see if for $k\in\mathbb{I}_{n-1}$,

$$\tilde{\psi}^k := \frac{1}{\gamma_k}(\beta^o)^{\frac{1}{2}}\nabla_+ M_\beta^{-\frac{1}{2}}\psi^k,$$

then $\{\tilde{\psi}_k\}_{k=1}^{n-1}$ is the set of eigenvectors of A_r^β with similar eigenvalues $\gamma_1^2<\cdots<\gamma_{n-1}^2$. Hence, they form an orthonormal basis for $\mathbb{R}^{n-1}$. Denote the matrix of these eigenvectors by $\tilde{O}_\beta$, with $\tilde{O}_\beta\tilde{O}_\beta^\dagger = \tilde{O}_\beta^\dagger\tilde{O}_\beta = I_{n-1}$. This claim follows from the following computation:

$$A_r^\beta\tilde{\psi}_k = \frac{1}{\gamma_k}(\beta^o)^{\frac{1}{2}}(-\nabla_+ M_\beta^{-1}\nabla_-)(\beta^o)\nabla_+ M_\beta^{-\frac{1}{2}}\psi^k = \frac{1}{\gamma_k}(\beta^o)^{\frac{1}{2}}\nabla_+ M_\beta^{-\frac{1}{2}}A_p^\beta\psi^k = \gamma_k^2\tilde{\psi}^k.$$

Define another set of coordinates $\tilde{\mathfrak{p}}_k, \tilde{\mathfrak{r}}_k$, similar to 2.2.3, 2.2.6, for $k\in\mathbb{I}_{n-1}^0$:

$$\begin{aligned}
&\tilde{\mathfrak{p}} = O_\beta^\dagger M_\beta^{-\frac{1}{2}}\tilde{\mathfrak{p}}, \qquad \tilde{\mathfrak{p}}_k = \langle \psi^k, M_\beta^{-\frac{1}{2}}\tilde{\mathfrak{p}}\rangle_n = \sum_{x=1}^n \sqrt{\frac{\beta_x}{m_x}}\psi_x^k\tilde{\mathfrak{p}}_x,\\
&\tilde{\mathfrak{r}} = \tilde{O}_\beta^\dagger(\beta^o)^{\frac{1}{2}}\tilde{\mathfrak{r}}, \qquad \tilde{\mathfrak{r}}_k = \langle(\beta^o)^{\frac{1}{2}}\tilde{\psi}^k, \tilde{\mathfrak{r}}\rangle_{n-1}.
\end{aligned} \tag{2.2.26}$$

Let us define $\mathfrak{p}_k, \mathfrak{r}_k$ as

$$\mathfrak{p}_k = \langle \psi^k, M_\beta^{-\frac{1}{2}}p\rangle_n, \qquad \mathfrak{r}_k = \langle(\beta^o)^{\frac{1}{2}}\tilde{\psi}^k, r\rangle_{n-1}. \tag{2.2.27}$$

Notice that $\mathfrak{p}_k$ and $\mathfrak{r}_k$, differs with $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{r}}_k$ only by a constant, respectively. In particular, thanks to the expression of ψ^0 we have:

$$\tilde{\mathfrak{p}}_0 = \left(\sum_{x=1}^n \frac{m_x}{\beta_x}\right)^{-\frac{1}{2}}\left(\sum_{x=1}^n p_x - \sum_{x=1}^n \bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}\right) =: -\lambda\Pi_0, \tag{2.2.28}$$

where we take advantage of the fact $\sum_{x=1}^n p_x = 0$, and we defined the (random) constants

$$\lambda := \left(\sum_{x=1}^n \frac{m_x}{\beta_x}\right)^{-\frac{1}{2}}, \qquad \Pi_0 := \sum_{x=1}^n \bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}. \tag{2.2.29}$$

Similar to the previous section, we have the inverse transformations:

$$\begin{aligned}
&\tilde{\mathfrak{p}} = M_\beta^{\frac{1}{2}}O_\beta\tilde{\mathfrak{p}}, \qquad \tilde{\mathfrak{p}}_x = \sqrt{\frac{m_x}{\beta_x}}\sum_{k=0}^{n-1}\psi_x^k\tilde{\mathfrak{p}}_k,\\
&\tilde{\mathfrak{r}} = (\beta^o)^{-\frac{1}{2}}\tilde{O}_\beta\tilde{\mathfrak{r}}, \qquad \tilde{\mathfrak{r}}_x = \frac{1}{\sqrt{\beta_x}}\sum_{k=1}^{n-1}\tilde{\psi}_x^k\tilde{\mathfrak{r}}_k.
\end{aligned} \tag{2.2.30}$$

We express H^n_β in terms of these new coordinates, and obtain the desired diagonalization in terms of independent oscillators:

$$\begin{aligned} H^n_\beta &= \frac{1}{2}(\langle \tilde{\mathfrak{p}}, M_\beta^{-1}\tilde{\mathfrak{p}}\rangle_n + \langle \tilde{\mathfrak{r}}, \beta^o \tilde{\mathfrak{r}}\rangle_{n-1}) = \\ &\frac{1}{2}\Big(\langle M_\beta^{\frac{1}{2}} O_\beta \tilde{\mathfrak{p}}, M_\beta^{-\frac{1}{2}} O_\beta \tilde{\mathfrak{p}}\rangle + \left\langle (\beta^o)^{-\frac{1}{2}} \tilde{O}_\beta \tilde{\mathfrak{r}}, (\beta^o)^{\frac{1}{2}} \tilde{O}_\beta \tilde{\mathfrak{r}}\right\rangle_{n-1}\Big) = \frac{1}{2}(\langle \tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}\rangle_n + \langle \tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}\rangle_{n-1}) = \\ &\frac{\tilde{\mathfrak{p}}_0^2}{2} + \frac{1}{2}\sum_{k=1}^{n-1}(\tilde{\mathfrak{p}}_k^2 + \tilde{\mathfrak{r}}_k^2) = \frac{\lambda^2 \Pi_0^2}{2} + \frac{1}{2}\sum_{k=1}^{n-1}(\tilde{\mathfrak{p}}_k^2 + \tilde{\mathfrak{r}}_k^2), \end{aligned} \tag{2.2.31}$$

where we used the identities $O_\beta O_\beta^\dagger = I_n$ and $\tilde{O}_\beta \tilde{O}_\beta^\dagger = I_{n-1}$. Moreover, we take advantage of (2.2.28) to replace $\tilde{\mathfrak{p}}_0^2$ with a constant. The fact that (2.2.31) is sum of free uncoupled oscillators is a direct consequence of the following commutation relations:

$$[\tilde{\mathfrak{p}}_k, \tilde{\mathfrak{p}}_{k'}] = [\tilde{\mathfrak{r}}_k, \tilde{\mathfrak{r}}_{k'}] = 0, \quad [\tilde{\mathfrak{r}}_k, \tilde{\mathfrak{p}}_{k'}] = i\gamma_k \delta_{k,k'}, \quad \forall k, k' \in \mathbb{I}_{n-1}. \tag{2.2.32}$$

The first relation in (2.2.32) is evident from the definition. The second relation can be justified as follows: Recall the operator $s_x = -i\partial/\partial_{\boldsymbol{\xi}_x}$, where we had $[r_x, s_y] = i\delta_{x,y}$. For any $v, w \in \mathbb{R}^{n-1}$ we have (here we drop the subscript $n-1$ in the $\langle\rangle_{n-1}$):

$$[\langle v, r\rangle, \langle w, s\rangle] = \sum_{x,y=1}^{n-1} v_x w_y [r_x, s_y] = \sum_{x,y=1}^{n-1} v_x w_y i\delta_{x,y} = i\sum_{x=1}^{n-1} v_x w_x = i\langle v, w\rangle_{n-1}. \tag{2.2.33}$$

Therefore, by using the relation $p = -\nabla_- s$, and thanks to(2.2.33), the definition of A_p^β and its eigenvectors ψ^k, we compute:

$$\begin{aligned} [\tilde{\mathfrak{r}}_k, \tilde{\mathfrak{p}}_{k'}] &= [\langle (\beta^o)^{\frac{1}{2}} \tilde{\psi}^k, r\rangle_{n-1}, \langle \psi^{k'}, M_\beta^{-\frac{1}{2}} p\rangle_n] = -[\langle (\beta^o)^{\frac{1}{2}} \tilde{\psi}^k, r\rangle_{n-1}, \langle \psi^{k'}, M_\beta^{-\frac{1}{2}} \nabla_- s\rangle_n] \\ &= [\langle (\beta^o)^{\frac{1}{2}} \tilde{\psi}^k, r\rangle_{n-1}, \langle \nabla_+ M_\beta^{-\frac{1}{2}} \psi^{k'}, s\rangle_{n-1}] = i\langle (\beta^o)^{\frac{1}{2}} \tilde{\psi}^k, \nabla_+ M_\beta^{-\frac{1}{2}} \psi^{k'}\rangle_{n-1} \\ &= i\frac{1}{\gamma_k}\langle M_\beta^{-\frac{1}{2}}(-\nabla_- \beta^o \nabla_+) M_\beta^{-\frac{1}{2}} \psi^k, \psi^{k'}\rangle_n = \frac{i}{\gamma_k}\langle A_p^\beta \psi^k, \psi^{k'}\rangle_n = i\gamma_k \delta_{k,k'}. \end{aligned} \tag{2.2.34}$$

In the first equality we substitute $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{p}}$ with r and p, since they only differ in a constant.

Now designate the bosonic operators $\tilde{\mathfrak{b}}_k, \tilde{\mathfrak{b}}_k^*$ for $k \in \mathbb{I}_{n-1}$ similar to (2.2.13). Their commutation relations, which can be deduced from (2.2.32), reads:

$$\begin{aligned} &\tilde{\mathfrak{b}}_k = \frac{1}{\sqrt{2\gamma_k}}(\tilde{\mathfrak{r}}_k + i\tilde{\mathfrak{p}}_k), \qquad \tilde{\mathfrak{b}}_k^* = \frac{1}{\sqrt{2\gamma_k}}(\tilde{\mathfrak{r}}_k - i\tilde{\mathfrak{p}}_k), \\ &[\tilde{\mathfrak{b}}_k, \tilde{\mathfrak{b}}_{k'}^*] = \delta_{k,k'}, \quad [\tilde{\mathfrak{b}}_k, \tilde{\mathfrak{b}}_{k'}] = [\tilde{\mathfrak{b}}_k^*, \tilde{\mathfrak{b}}_{k'}^*] = 0. \end{aligned} \tag{2.2.35}$$

The expression of H^n_β in terms of these operators is as follows:

$$H^n_\beta = \frac{\lambda^2 \Pi_0^2}{2} + \sum_{k=1}^{n-1} \gamma_k (\tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_k + \frac{1}{2}). \tag{2.2.36}$$

Let us denote the constant $\frac{\lambda^2\Pi_0^2}{2}$ by E_0. Owing to (2.2.36), and the commutator relations in (2.2.35), we can treat H_β^n as sum of independent oscillator. Therefore, we can deduce that this operator has a full discrete spectrum, which can be described along with their corresponding eigenfunctions explicitly; Let $|\mathbf{\Psi_0}\rangle \in L^2(\mathbb{R}^{n-1})$ be the the ground state of H_β^n, with corresponding energy (eigenvalue) $\tilde{E}_0 = E_0 + \frac{1}{2}\sum_{k=1}^{n-1}\gamma_k$. Then the set of eigenvalues and eigenfunctions can be labeled by $\bar{\theta} := (\theta_1, \dots, \theta_{n-1}) \in \mathbb{N}_0^{n-1}$:

$$E_{\bar{\theta}} = \tilde{E}_0 + \sum_{k=1}^{n-1}\theta_k\gamma_k, \qquad |\mathbf{\Psi_{\bar{\theta}}}\rangle = \prod_{k=1}^{n-1}\frac{(\tilde{\mathfrak{b}}_k^*)^{\theta_k}}{\sqrt{\theta_k!}}|\mathbf{\Psi_0}\rangle. \tag{2.2.37}$$

By abusing the notation, we write $\left|\bar{\boldsymbol{\theta}}\right\rangle \equiv |(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_{n-1}})\rangle$ instead of $|\mathbf{\Psi_{\bar{\theta}}}\rangle$. Here the eigenfunction $\left|\bar{\boldsymbol{\theta}}\right\rangle$ corresponds to the eigenvalue $E_{\bar{\theta}}$. Moreover, $\{\left|\bar{\boldsymbol{\theta}}\right\rangle|\bar{\theta} \in \mathbb{N}^{n-1}\}$ forms an orthonormal basis for $L^2(\mathbb{R}^{n-1})$. Notice that these functions can be obtained rather explicitly, similar to $|\mathbf{\Phi_{\bar{\theta}}}\rangle$. Here, they are functions of $\tilde{\boldsymbol{\xi}}_1, \dots \tilde{\boldsymbol{\xi}}_{n-1}$, where $\forall k \in \mathbb{I}_{n-1}, \tilde{\boldsymbol{\xi}}_k$ can be defined similar to (2.2.26) as: $\tilde{\boldsymbol{\xi}}_k := \langle(\beta^o)^{\frac{1}{2}}\tilde{\psi}^k, \tilde{\boldsymbol{\xi}}\rangle_{n-1}$, while in the previous case they are written as a function of $\hat{\boldsymbol{\xi}}_1, \dots, \hat{\boldsymbol{\xi}}_{n-1}$.
Let us mention the fact that (2.2.36) is sum of "shifted" harmonic oscillators, rather than normal harmonic oscillators, since all the bosonic operators are shifted by a constant. Therefore, obtaining the ground state $|\mathbf{\Psi_0}\rangle$, and other eigenstates of this operator is slightly different from $|\mathbf{\Phi_0}\rangle$ and other eigenstates of H_n. Notice that, this shift *does not* change the spectrum, and consequently it does not affect our computation.
In fact, if one construct $\mathfrak{b}_k$, and $\mathfrak{b}_k^*$ from (2.2.27), similar to (2.2.35), then H_β^n is shifted version of $\tilde{H}_\beta^n = E_0 + \sum_k \gamma_k(\mathfrak{b}_k^*\mathfrak{b}_k + \frac{1}{2})$. The ground state of the later is well-understood and can be obtained similar to $|\mathbf{\Phi_0}\rangle$. Let us denote the ground state of $\tilde{H}_\beta^n$ by $\left|\tilde{\mathbf{\Psi}}_\mathbf{0}\right\rangle$, then applying the displacement operator we obtain $\left|\tilde{\mathbf{\Psi}}_\mathbf{0}\right\rangle = \mathcal{D}\left|\tilde{\mathbf{\Psi}}_\mathbf{0}\right\rangle$. Here the displacement operator $\mathcal{D}$ is a unitary operator which can be define as follow: let $b_k := \tilde{\mathfrak{b}}_k - \mathfrak{b}_k$ be the displacement constant for k-th oscillator. Then $\mathcal{D}_k = \exp(b_k\mathfrak{b}_k^* - b_k^*\mathfrak{b}_k)$ and $\mathcal{D} = \mathcal{D}_1 \dots \mathcal{D}_{n-1}$.

2.2.3 Ensemble average

Since we established the eigenfunctions and eigenvalues of H_β^n, using spectral theorem one can write $H_\beta^n = \sum_{\bar{\theta}\in\mathbb{N}_0^{n-1}} E_{\bar{\theta}}\left|\bar{\boldsymbol{\theta}}\right\rangle\!\left\langle\bar{\boldsymbol{\theta}}\right|$, where $\left|\bar{\boldsymbol{\theta}}\right\rangle\!\left\langle\bar{\boldsymbol{\theta}}\right|$ is the projection operator on the subspace generated by $\{\left|\bar{\boldsymbol{\theta}}\right\rangle\}$, for $\bar{\theta} \in \mathbb{N}_0^{n-1}$. Moreover, one can observe that $\exp\left(-H_\beta^n\right)$ is trace-class, thanks to the properties of γ_ks, and can be written as $\exp\left(-H_\beta^n\right) = \sum_{\bar{\theta}\in\mathbb{N}^{n-1}} e^{-E_{\bar{\theta}}}\left|\bar{\boldsymbol{\theta}}\right\rangle\!\left\langle\bar{\boldsymbol{\theta}}\right|$, thanks to spectral theorem. Recall the number operator $\tilde{\mathfrak{b}}_k^*\tilde{\mathfrak{b}}_k$, then we can compute the average of the bosonic operators and number operators in the locally Gibbs state ρ (also known as thermal state). This computation is classical (cf. Proposition 5.2.28 of [48]).

Lemma 2.2.2. *Recall the density state ρ (2.1.9), and the operators $\tilde{\mathfrak{b}}_k^*, \tilde{\mathfrak{b}}_k$ defined in (2.2.35). Moreover, recall the definition of $\langle\rangle_\rho$ (2.1.12). Then, we have the followings*

$\forall k \in \mathbb{I}_{n-1}$:

$$\begin{aligned} &\left\langle \tilde{\mathfrak{b}}_k^* \right\rangle_\rho = \left\langle \tilde{\mathfrak{b}}_k \right\rangle_\rho = \left\langle \tilde{\mathfrak{b}}_k \tilde{\mathfrak{b}}_{k'} \right\rangle_\rho = \left\langle \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_{k'}^* \right\rangle_\rho = 0, \\ &\left\langle \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_{k'} \right\rangle_\rho = \frac{\delta_{k,k'}}{e^{\gamma_k}-1}, \quad \left\langle \tilde{\mathfrak{b}}_k \tilde{\mathfrak{b}}_{k'}^* \right\rangle_\rho = \frac{\delta_{k,k'}}{e^{\gamma_k}-1} + 1. \end{aligned} \tag{2.2.38}$$

Proof. One can find the rather straightforward computation of this lemma in Appendix 2.C. □

In the rest of this section, we compute the average of certain observables namely momentum, elongation, and energy at each site at time zero, as an application of (2.2.38). Before proceeding, let us define for $x \in \mathbb{I}_n$:

$$\mathcal{E}_n^x := \frac{m_x}{\beta_x}\lambda^2 \Pi_0 = \frac{m_x}{\beta_x}\left(\sum_{x=1}^n \frac{m_x}{\beta_x}\right)^{-1}\left(\sum_{x=1}^n \bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}\right), \tag{2.2.39}$$

Note that $\frac{m_x}{\beta_x}\left(\sum_{x=1}^n \frac{m_x}{\beta_x}\right)^{-1} \leq \frac{C}{n}$, where $C = \frac{m_{max}\beta_{max}}{\beta_{min}m_{min}}$ is a constant independent of n. Therefore, thanks to the Strong Law of Large numbers and the assumption $\int_0^1 \bar{p}(y)dy = 0$, we have for $y \in [0,1]$, $\mathcal{E}_n^{[ny]} \to 0$ almost surely:

$$|\mathcal{E}_n^{[ny]}| \leq C\Big|\frac{1}{n}\sum_{x=1}^n \bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}\Big| = C\frac{\Pi_0}{n} \to \int_0^1 \bar{p}(y)dy = 0, \tag{2.2.40}$$

almost surely as $n \to \infty$.

Corollary 2.2.2.1. *Recall the coordinates $\tilde{\mathfrak{p}}_x(0)$ and $\tilde{\mathfrak{r}}_x(0)$ from (2.2.23), as a direct consequence of Lemma 2.2.2 we have:*

$$\begin{aligned} \forall x \in \mathbb{I}_n, \quad &\langle \tilde{\mathfrak{p}}_x(0) \rangle_\rho = \langle \tilde{\mathfrak{p}}_x \rangle_\rho = -\mathcal{E}_n^x, \\ &\langle \tilde{\mathfrak{r}}_x(0) \rangle_\rho = \langle \tilde{\mathfrak{r}}_x \rangle_\rho = 0. \end{aligned} \tag{2.2.41}$$

Note that these expressions denote the average of momentum and elongation at time zero. This also implies:

$$\langle p_x \rangle_\rho = \frac{m_x}{\bar{m}}\bar{p}_x - \mathcal{E}_n^x, \quad \langle r_x \rangle_\rho = \bar{r}_x. \tag{2.2.42}$$

Proof. By (2.2.35), and (2.2.30) $\forall x \in \mathbb{I}_n$, we can write $\tilde{\mathfrak{p}}_x$ and $\tilde{\mathfrak{r}}_x$ as linear combination of $\tilde{\mathfrak{b}}_k$s and $\tilde{\mathfrak{b}}_k^*$s; moreover, $\tilde{\mathfrak{p}}_0$ appears in case of $\tilde{\mathfrak{p}}_x$ as well. Then, since $\langle . \rangle_\rho$ is a linear operator, we can deduce (2.2.41).
To be more precise, from (2.2.35) we have:

$$\tilde{\mathfrak{p}}_k = i\sqrt{\frac{\gamma_k}{2}}(\tilde{\mathfrak{b}}_k^* - \tilde{\mathfrak{b}}_k), \qquad \tilde{\mathfrak{r}}_k = \sqrt{\frac{\gamma_k}{2}}(\tilde{\mathfrak{b}}_k^* + \tilde{\mathfrak{b}}_k). \tag{2.2.43}$$

Hence, if we substitute $\tilde{\mathfrak{p}}_k$ and $\tilde{\mathfrak{r}}_k$ with the later in (2.2.30), use the definition $\psi_x^0 = \lambda\sqrt{\frac{m_x}{\beta_x}}$ as well as the expression (2.2.28), we obtain

$$\begin{aligned} \tilde{\mathfrak{p}}_x &= \sqrt{\frac{m_x}{\beta_x}}\left[\psi_x^0\tilde{\mathfrak{p}}_0 + \sum_{k=1}^{n-1}\psi_x^k\Big(i\sqrt{\frac{\gamma_k}{2}}(\tilde{\mathfrak{b}}_k^* - \tilde{\mathfrak{b}}_k)\Big)\right] = \frac{-m_x}{\beta_x}\lambda^2\Pi_0 + \sum_{k=1}^{n-1}\psi_x^k\Big(i\sqrt{\frac{\gamma_k}{2}}(\tilde{\mathfrak{b}}_k^* - \tilde{\mathfrak{b}}_k)\Big), \\ \tilde{\mathfrak{r}}_x &= \frac{1}{\sqrt{\beta_x}}\sum_{k=1}^{n-1}\tilde{\psi}_x^k\Big(\sqrt{\frac{\gamma_k}{2}}(\tilde{\mathfrak{b}}_k^* + \tilde{\mathfrak{b}}_k)\Big). \end{aligned} \tag{2.2.44}$$

Since $\langle . \rangle_\rho$ is linear, thanks to identities $\left\langle \tilde{\mathfrak{b}}_k^* \right\rangle_\rho = \left\langle \tilde{\mathfrak{b}}_k \right\rangle_\rho = 0$ (2.2.38), we get (2.2.41). In order to obtain (2.2.42) from (2.2.41) it's enough to recall these definitions: $\tilde{\mathfrak{p}}_x = p_x - \frac{m_x}{\bar{m}} \bar{p}_x, \quad \tilde{\mathfrak{r}}_x = r_x - \bar{r}_x$.

□

The average of momentum and elongation in our thermal state is understood in (2.2.42). Later we need their fluctuation as well. Hence, we define the following operators, we may refer to them as thermal coordinate, since they correspond to the thermal fluctuation:

$$\begin{aligned} \tilde{p}_x &:= p_x - \langle p_x \rangle_\rho = p_x - \bar{p}_x \frac{m_x}{\bar{m}} + \mathcal{E}_n^x = \tilde{\mathfrak{p}}_x + \mathcal{E}_n^x, \qquad \forall x \in \mathbb{I}_n, \\ \tilde{r}_x &:= r_x - \langle r_x \rangle_\rho = r_x - \bar{r}_x = \tilde{\mathfrak{r}}_x, \qquad \forall x \in \mathbb{I}_{n-1}, \end{aligned} \tag{2.2.45}$$

Notice that these coordinate are similar to $\tilde{\mathfrak{p}}$, and $\tilde{\mathfrak{r}}$ up to a vanishing error.
Let us compute the fluctuation of p_x and r_x in the state ρ:

Corollary 2.2.2.2. *As another straightforward consequence of Lemma 2.2.2 we have:*

$$\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} = \frac{1}{\beta_x} \sum_{k=1}^{n-1} (\psi_x^k)^2 \frac{\gamma_k}{2} \coth\Big(\frac{\gamma_k}{2}\Big), \tag{2.2.46}$$

$$\left\langle \tilde{r}_x^2 \right\rangle_\rho = \frac{1}{\beta_x} \sum_{k=1}^{n-1} (\tilde{\psi}_x^k)^2 \frac{\gamma_k}{2} \coth\Big(\frac{\gamma_k}{2}\Big). \tag{2.2.47}$$

Proof. From (2.2.43) and (2.2.38) one can observe:

$$\begin{aligned} \langle \tilde{\mathfrak{p}}_i \tilde{\mathfrak{p}}_j \rangle_\rho &= \langle \tilde{\mathfrak{r}}_i \tilde{\mathfrak{r}}_j \rangle_\rho = \delta_{i,j} \gamma_i \Big(\frac{1}{e^{\gamma_i} - 1} + \frac{1}{2} \Big) = \delta_{i,j} \frac{\gamma_i}{2} \coth\Big(\frac{\gamma_i}{2}\Big), \quad \forall i,j \in \mathbb{I}_{n-1}, \\ \langle \tilde{\mathfrak{p}}_i \tilde{\mathfrak{p}}_0 \rangle_\rho &= \delta_{i,0} \lambda^2 \Pi_0^2, \qquad \forall i \in \mathbb{I}_n, \end{aligned} \tag{2.2.48}$$

where in the second expression we take advantage of the fact that $\tilde{\mathfrak{p}}_0$ is a constant given in (2.2.28), and $\langle \tilde{\mathfrak{p}}_i \rangle_\rho = 0$ for $i > 0$.
Thanks to (2.2.45), we have $\tilde{p}_x = \tilde{\mathfrak{p}}_x + \mathcal{E}_n^x$, and $\tilde{r}_x = \tilde{\mathfrak{r}}_x$. If one replace $\tilde{\mathfrak{p}}_x$, and $\tilde{\mathfrak{r}}_x$ with corresponding expressions from (2.2.30) and use the definition of $\mathcal{E}_n^x$, and $\tilde{\mathfrak{p}}_0$ it is clear that:

$$\tilde{p}_x = \sqrt{\frac{m_x}{\beta_x}} \sum_{k=1}^{n-1} \psi_x^k \tilde{\mathfrak{p}}_k, \qquad \tilde{r}_x = \frac{1}{\sqrt{\beta_x}} \sum_{k=1}^{n-1} \tilde{\psi}_x^k \tilde{\mathfrak{r}}_k. \tag{2.2.49}$$

We compute $\frac{\langle \tilde{p}_x^2 \rangle}{m_x}_\rho$ and $\langle \tilde{r}_x^2 \rangle_\rho$by squaring these expressions, and then taking the ensemble average $\langle \rangle_\rho$. Therefore, the linearity of the trace and the identities in (2.2.48) give us the result:

$$\begin{aligned} \tilde{p}_x^2 = \frac{m_x}{\beta_x} \sum_{k,k'=1}^{n-1} \psi_x^k \psi_x^{k'} \tilde{\mathfrak{p}}_k \tilde{\mathfrak{p}}_{k'}, \implies \left\langle \tilde{p}_x^2 \right\rangle_\rho &= \frac{m_x}{\beta_x} \sum_{k,k'=1}^{n-1} \psi_x^k \psi_x^{k'} \left\langle \tilde{\mathfrak{p}}_k \tilde{\mathfrak{p}}_{k'} \right\rangle_\rho . \\ \frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} = \frac{1}{\beta_x} \sum_{k,k'=1}^{n-1} \psi_x^k \psi_x^{k'} \delta_{k,k'} \frac{\gamma_k}{2} \coth\Big(\frac{\gamma_k}{2}\Big) &= \frac{1}{\beta_x} \sum_{k=1}^{n-1} (\psi_x^k)^2 \frac{\gamma_k}{2} \coth\Big(\frac{\gamma_k}{2}\Big). \end{aligned} \tag{2.2.50}$$

The exact same computation using (2.2.30) and (2.2.48), gives us corresponding expression in (2.2.47) for $\langle \tilde{r}_x \rangle_\rho$. □

Remark 2.2.3. *Notice that for a clean chain (all masses equal to m), in thermal equilibrium at temperature β_{eq}^{-1}, with periodic boundary conditions, we can obtain ψ^k by discrete Fourier transform. In this case, $\gamma_k = \omega_k = 2|\sin\Big(\pi(\frac{k}{n})\Big)|$. Therefore, since boundary effects disappear in the limit as $n \to \infty$, we get up to a vanishing error:*

$$\left\langle \tilde{r}_x^2 \right\rangle_\rho = \frac{1}{\beta_{eq}} \frac{1}{2n} \sum_{k=1}^{n-1} \frac{\beta_{eq}\omega_k}{2\sqrt{m}} \coth\left(\frac{\omega_k \beta_{eq}}{2\sqrt{m}}\right) + \epsilon_n. \tag{2.2.51}$$

The same expression can be obtained for $\frac{\langle \tilde{p}_x \rangle_\rho}{m}$, for a clean chain in thermal equilibrium. Taking the limit of $n \to \infty$, we obtain the constant f_β in (1.1.34).

Notice the difference of (2.2.47) and (2.2.46) with the classical case, where these averages are simply equal to β_x^{-1}. Moreover observe that clearly for each configuration of the masses the averages $\langle \tilde{p}_x^2 \rangle_\rho / m_x$ and $\langle \tilde{r}_x^2 \rangle_\rho$ depend on the whole configuration of the masses, due to the fact that γ_k and ψ^k ($\tilde{\psi}^k$) are eigenvalues and eigenvectors of $A_p^\beta = M_\beta^{-\frac{1}{2}}(-\nabla_-\beta^0\nabla_+)M_\beta^{-\frac{1}{2}}$ $\Big(A_r^\beta = (\beta^o)^{\frac{1}{2}}(-\nabla_+ M_\beta^{-1}\nabla_-)(\beta^o)^{\frac{1}{2}}\Big)$.

For our purposes, in particular, in Section 2.4.1, it would be useful to rewrite (2.2.46),(2.2.47) in the following form: recall the definition of $|x\rangle_n$[3] for $x \in \mathbb{I}_n$, as the canonical basis of $\mathbb{R}^n$, i.e., $|x\rangle_n = (0, \dots, 0, 1, 0 \dots, 0)^\dagger$, where 1 is at the xth position. So we can write $\psi_x^k = \langle x, \psi^k \rangle_n$. Similarly, let $|x\rangle_{n-1}$ for $x \in \mathbb{I}_{n-1}$, denotes the canonical basis for $\mathbb{R}^{n-1}$. We drop the subscript of $|x\rangle_n$, whenever it does not make any confusion. Then we have:

$$\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} = \frac{1}{\beta_x}\left\langle x, \frac{(A_p^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_p^\beta)^{\frac{1}{2}}}{2}\right)x\right\rangle_n, \left\langle \tilde{r}_x^2 \right\rangle_\rho = \frac{1}{\beta_x}\left\langle x, \frac{(A_r^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_r^\beta)^{\frac{1}{2}}}{2}\right)x\right\rangle_{n-1}. \tag{2.2.52}$$

Here by convention, formally we denote $0\coth(0) = 0$[4].
Since A_p^β (A_r^β) is positive semidefinite (definite) one can define by spectral theorem the following matrices

$$\mathsf{A}_p := \frac{(A_p^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_p^\beta)^{\frac{1}{2}}}{2}\right), \quad \mathsf{A}_r = \frac{(A_r^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_r^\beta)^{\frac{1}{2}}}{2}\right).$$

So if we expand (2.2.52) in the basis of ψ^k, and use the identity $\sum_{k=0}^{n-1} \left|\psi^k\right\rangle\left\langle\psi^k\right| = I_n$, we get the exact same expression as in (2.2.46):

$$\begin{aligned}\langle x, \mathsf{A}_p x\rangle &= \sum_{k=0}^{n-1} \left\langle x, \psi^k \right\rangle \left\langle \psi^k, \mathsf{A}_p x \right\rangle = \sum_{k=0}^{n-1} \frac{\gamma_k}{2}\coth\left(\frac{\gamma_k}{2}\right)\left\langle x, \psi^k \right\rangle \left\langle \psi^k, x \right\rangle \\ &= \sum_{k=1}^{n-1} \frac{\gamma_k}{2}\coth\left(\frac{\gamma_k}{2}\right)(\psi_x^k)^2.\end{aligned} \tag{2.2.53}$$

[3]Notice the difference between the notation $|\mathbf{\Psi}\rangle$ which is used for denoting the member of the Hilbert space, and $|\psi\rangle$ which denotes the finite dimensional vector spaces.

[4]We will modify this convention later.

Similarly, we can justify the expression in (2.2.52) for $\langle \tilde{r}_x^2 \rangle$. Using spectral properties of A_r.

Recall the canonical transformations (2.2.3), (2.2.6), (2.2.11), we define yet another set of operators: let $\hat{\tilde{p}}$ and $\hat{\tilde{r}}$ to be the transformed form of $\tilde{r}$ and $\tilde{p}$, respectively:

$$\begin{aligned} \hat{\tilde{p}} &:= O^\dagger M^{-\frac{1}{2}} \tilde{p}, \qquad & \hat{\tilde{p}}_k &:= \langle M^{-\frac{1}{2}} \varphi^k, \tilde{p} \rangle_n, \\ \hat{\tilde{r}} &:= \tilde{O}^\dagger \tilde{r}, & \hat{\tilde{r}}_k &:= \langle \phi^k, \tilde{r} \rangle_{n-1}. \end{aligned} \tag{2.2.54}$$

Since we use the explicit solution of equations of motion, terms like $\left\langle (\hat{\tilde{p}}_k)^2 \right\rangle_\rho$ and $\left\langle (\hat{\tilde{r}}_k)^2 \right\rangle_\rho$ arise in our calculations, and the following lemma permits us to deal with them. Note that this lemma reflects one of the technical differences of this model with its classical counterpart.

Lemma 2.2.4. *Considering the above definitions, there exists a deterministic constant $C > 0$ independent of n, such that for any realization of the masses and any $k \in \mathbb{I}_n$:*

$$\left\langle (\hat{\tilde{p}}_k)^2 \right\rangle_\rho < C, \quad \left\langle (\hat{\tilde{r}}_k)^2 \right\rangle_\rho < C, \qquad \left\langle \hat{\tilde{p}}_k \right\rangle_\rho = \left\langle \hat{\tilde{r}}_k \right\rangle_\rho = 0. \tag{2.2.55}$$

Proof. Plugging $\tilde{p}_x$ and $\tilde{r}_x$ from (2.2.49) into the definition (2.2.54), we get the following linear relation between these two sets of transformed operators:

$$\begin{aligned} \hat{\tilde{p}}_k &= \sum_{i=1}^{n-1} \langle \varphi^k, \beta^{-\frac{1}{2}} \psi^i \rangle \tilde{\mathfrak{p}}_i, \qquad & \forall k \in \mathbb{I}_{n-1}^0, \\ \hat{\tilde{r}}_k &= \sum_{i=1}^{n-1} \langle \phi^k, \beta^{-\frac{1}{2}} \tilde{\psi}^i \rangle \tilde{\mathfrak{r}}_i, \qquad & \forall k \in \mathbb{I}_{n-1}. \end{aligned} \tag{2.2.56}$$

Since $\tilde{\mathfrak{p}}_i$ and $\tilde{\mathfrak{r}}_i$ are linear combinations of bosonic operators $\tilde{\mathfrak{b}}_i, \tilde{\mathfrak{b}}_i^*$; obviously we have $\langle \tilde{\mathfrak{r}}_i \rangle_\rho = \langle \tilde{\mathfrak{p}}_i \rangle_\rho = 0$. Hence, by linearity of $\langle . \rangle_\rho$, we have $\left\langle \hat{\tilde{p}}_k \right\rangle_\rho = \left\langle \hat{\tilde{r}}_k \right\rangle_\rho = 0$. For the purpose of establishing the bounds in (2.2.55), we square the expression (2.2.56) and by using (2.2.48), we have:

$$\begin{aligned} &\left\langle (\hat{\tilde{p}}_k)^2 \right\rangle_\rho = \sum_{i,j=1}^{n-1} \langle \varphi^k, \beta^{-\frac{1}{2}} \psi^i \rangle \langle \beta^{-\frac{1}{2}} \psi^j, \varphi^k \rangle \left\langle \mathfrak{p}_i \mathfrak{p}_j \right\rangle_\rho = \\ &\sum_{i,j=1}^{n-1} \langle \varphi^k, \beta^{-\frac{1}{2}} \psi^i \rangle \langle \beta^{-\frac{1}{2}} \psi^j, \varphi^k \rangle \delta_{i,j} \frac{\gamma_i}{2} \coth\left(\frac{\gamma_i}{2}\right) = \sum_{i=1}^{n-1} \left| \langle \beta^{-\frac{1}{2}} \varphi^k, \psi^i \rangle \right|^2 \frac{\gamma_i}{2} \coth\left(\frac{\gamma_i}{2}\right). \end{aligned} \tag{2.2.57}$$

Since γ_i are the eigenvalues of $A_p^\beta = M_\beta^{-\frac{1}{2}} (-\nabla_- \beta^0 \nabla_+) M_\beta^{-\frac{1}{2}}$, $\forall i$, $\gamma_i \leq ||A_p^\beta||_2$, where $||.||_2$ denotes the matrix norm induced by the Euclidean norm.[5] However, the

[5] For every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ with corresponding $m \times n$ matrix A, we define $||A||_2 = \sup_{|x|_n = 1} \frac{|Ax|_m}{|x|_n}$, where $|.|$ denotes the Euclidean norm in $\mathbb{R}^n$: $|x|_n = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

following bound is evident from the definition of the matrices appearing in A_p^β: $||A_p^\beta||_2 \leq ||M_\beta^{-\frac{1}{2}}||_2^2||\beta^o||_2||\nabla_-||_2||\nabla_+||_2 \leq \frac{4\beta_{max}^2}{m_{min}}$. This bound holds uniformly in n, for any realization of the masses, since the distribution of the masses is bounded, and β is continuous. Therefore, we deduce that there is a deterministic $c > 0$, independent of n such that for any realization of the masses $||A_p^\beta||_2 \leq c$. Furthermore, since the function $f(x) = x \coth x$ is continuous in $(0, c)$, the expression $\frac{\gamma_i}{2}\coth\left(\frac{\gamma_i}{2}\right)$ is nonnegative and bounded by a constant c', independent of n, so we have for any realization of the masses:

$$\left\langle (\hat{\tilde{p}}_k)^2 \right\rangle \leq c' \sum_{i=1}^{n-1} |\langle \beta^{-\frac{1}{2}}\varphi^k, \psi^i\rangle|^2 = c'|\beta^{-\frac{1}{2}}\varphi^k|^2 \leq c'||\beta^{-\frac{1}{2}}||_2^2|\varphi^k|^2 \leq c'\beta_{min} \leq C. \tag{2.2.58}$$

The sum in this expression is the expansion of the vector $\beta^{-\frac{1}{2}}\varphi^k$ in the basis of ψ^i. We also used the fact that $|\varphi^k|_2 = 1$, and norm of $\beta^{-\frac{1}{2}}$ is bounded by $\beta_{min}^{-\frac{1}{2}}$. For $\left\langle (\hat{\tilde{r}}_k)^2 \right\rangle_\rho$, we proceed similarly and get the following expression:

$$\left\langle (\hat{\tilde{r}}_k)^2 \right\rangle_\rho = \sum_{i=1}^{n-1} |\langle \beta^{-\frac{1}{2}}\phi^k, \tilde{\psi}^i\rangle|^2 \frac{\gamma_i}{2}\coth\left(\frac{\gamma_i}{2}\right), \tag{2.2.59}$$

which can be treated exactly similar to the previous bound. □

Remark 2.2.5. *Recall the averages* $\langle \tilde{r}_x^2 \rangle_\rho$ *and* $\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x}$ *from* (2.2.52)*:*

$$\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} = \frac{1}{\beta_x}\left\langle x, \frac{(A_p^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_p^\beta)^{\frac{1}{2}}}{2}\right)x \right\rangle_n, \left\langle \tilde{r}_x^2 \right\rangle_\rho = \frac{1}{\beta_x}\left\langle x, \frac{(A_r^\beta)^{\frac{1}{2}}}{2}\coth\left(\frac{(A_r^\beta)^{\frac{1}{2}}}{2}\right)x \right\rangle_{n-1}.$$

In the proof of Lemma 2.2.4, we observed that the the norm of the matrices appearing in the above expression are bounded by a constant c'*, uniformly in* n*. Therefore, we can deduce that there exists a constant* C *uniform in* n *such that for any realization of the masses:*

$$\left\langle \tilde{r}_x^2 \right\rangle \leq C, \quad \frac{\langle \tilde{p}_x^2 \rangle}{m_x} \leq C, \tag{2.2.60}$$

where we used the fact that β *is continuous, with* $\beta_{min} \leq \beta(y)$ *for all* $y \in [0, 1]$*, with* β_{min} *strictly positive.*

2.3 Wave Equation

In this section, we are going to show the limits (2.1.16) and (2.1.17). Since our system is harmonic, the dynamic is linear. As we already observed our chain evolves in time according to the Heisenberg dynamics generated by the Hamiltonian (2.2.16). Recall the definition of the dynamic: $\forall a \in \mathcal{B}(\mathcal{H}_n)$ define $a(t)$ as follows:

$$a(t) := e^{iH_nt}ae^{-iH_nt}. \tag{2.3.1}$$

Since H_n is self-adjoint, using Stone's theorem, (2.3.1) is the continuous one parameter group of authomorphisms. Moreover, we can extend the definition of this evolution to certain unbounded operators, such as b_k and b_k^*, where we have:

$$b_k(t) = e^{-i\omega_k t} b_k, \qquad b_k^*(t) = e^{i\omega_k t} b_k^*. \tag{2.3.2}$$

By using linearity, we obtain the explicit time evolution for elongation and momentum operators in (2.2.22). One may use this explicit solutions in order to demonstrate the limits (2.1.17) and (2.1.16). However, we proceed using the equation of motions and certain homogenization lemmas.
Recall the definition of the thermal state ρ^n (2.1.9), and the thermal average: $\langle a \rangle_{\rho^n} := \mathrm{Tr}(\rho^n a)$ for an observable a, such that $a\rho^n$ be trace class. Since $\rho^n = \exp\Big(-H_\beta^n\Big)$, thanks to the spectral theory we observed that $\forall x$, $\rho^n p_x$, $r_x \rho^n$, $\rho^n r_x^2$, and $\rho^n p_x^2$ are trace class. Since the solution in (2.2.22) is linear, we can deduce that $\forall x, \forall t, \in [0,T]$, $p_x(nt)$, $r_x(nt)$, $p_x^2(nt)$, and $r_x^2(nt)$ are trace class. Hence, we can introduce the following notation:

$$\bar{p}_x(nt) := \langle p_x(nt) \rangle_{\rho^n}, \quad \bar{r}_x(t) := \langle r_x(t) \rangle_{\rho^n}. \tag{2.3.3}$$

Recall that according to (2.2.42), we have $\forall x \in \mathbb{I}_n$:

$$\bar{p}_x(0) = \frac{m_x}{\bar{m}} \bar{p}_x - \mathcal{E}_n^x, \quad \bar{r}_x(0) = \bar{r}_x. \tag{2.3.4}$$

Moreover, the time evolution of $\bar{p}_x(nt)$ and $\bar{r}_x(nt)$ can be represented as a system of coupled ODEs. First, observe that the dynamic defined in (2.3.1), gives us the equations of motions as (2.1.7), then by the following simple computation we have:

$$\begin{aligned}
&\frac{d}{dt}\bar{p}_x(t) = \langle \dot{p}_x(t) \rangle_{\rho^n} = \langle r_x(t) - r_{x-1}(t) \rangle_{\rho^n} = (\bar{r}_x(t) - \bar{r}_{x-1}(t)) = (\nabla_- \bar{r}(t))_x, \\
&\frac{d}{dt}\bar{r}_x(t) = \langle \dot{r}_x(t) \rangle_{\rho^n} = \left\langle \frac{p_{x+1}(t)}{m_{x+1}} - \frac{p_x(t)}{m_x} \right\rangle_{\rho^n} = \left(\frac{\bar{p}_{x+1}(t)}{m_{x+1}} - \frac{\bar{p}_x(t)}{m_x} \right) = (\nabla_+ M^{-1} \bar{p}(t))_x,
\end{aligned} \tag{2.3.5}$$

where, ∇_- and ∇_+ were defined in (2.1.6). The justification for this computation is as follows: the operator $\rho^n p_x(nt)$ is bounded, and the time derivative exists in the operator norm (for this bounded operator) and we can change the trace and derivative by a simple argument.

Comparing the functions $\bar{p}(nt) : \mathbb{R}^n \to \mathbb{R}$, and $\bar{r}(nt) : \mathbb{R}^{n-1} \to \mathbb{R}$ for $t \in [0,T]$, with their classical counterpart in [35], it is evident that they satisfiy the same coupled system of linear ODEs, with the same initial conditions up to a vanishing purturbation $\mathcal{E}_n^x$ (for each realization of the masses). Therefore, these functions are very similar, and we can adapt the method of section 3 of [35], and prove (2.1.16) and (2.1.17) in theorem 2.1.1. This conclusion is obtained from the fact that in the harmonic systems, both the classical and quantum evolutions are linear. Although,

the result of [35] is applicable here, their proof is not optimal and has a certain gap[6]. Moreover, we need to take care of $\mathcal{E}_n^x$ separately, hence we state a modified version of that proof here. In fact, we prove this theorem assuming the function β is Lipschitz continuous, since this proof is shorter and better illustrate the idea, then we bring the proof of general case $\beta \in C^0([0,1])$ afterwards.

Proof of (2.1.16)*, and* (2.1.17) *with* β *Lipschitz.* We divide the proof into two steps:
Step1. A priori bound:
Define $\bar{H}_n(t)$ and $\bar{I}_n(t)$ as follows:

$$\begin{aligned}
\bar{H}_n(t) :=& \sum_{x=1}^{n} \frac{\bar{p}_x^2(t)}{2m_x} + \sum_{x=1}^{n-1} \frac{\bar{r}_x^2(t)}{2} = \frac{1}{2}\left\langle \bar{p}(t), M^{-1}\bar{p}(t)\right\rangle_n + \frac{1}{2}\left\langle \bar{r}(t), \bar{r}(t)\right\rangle_{n-1},\\
\bar{I}_n(t) :=& \frac{1}{2}\sum_{x=1}^{n} \frac{\left(\bar{r}_x(t) - \bar{r}_{x-1}(t)\right)^2}{m_x} + \frac{1}{2}\sum_{x=1}^{n-1}\left(\frac{\bar{p}_{x+1}(t)}{m_{x+1}} - \frac{\bar{p}_x(t)}{m_x}\right)^2 \\
=& \frac{1}{2}\left\langle \nabla_-\bar{r}(t), M^{-1}\nabla_-\bar{r}(t)\right\rangle_n + \frac{1}{2}\left\langle \nabla_+ M^{-1}\bar{p}(t), \nabla_+ M^{-1}\bar{p}(t)\right\rangle_{n-1}.
\end{aligned} \tag{2.3.6}$$

From the time evolution (2.3.5), it is evident that $\forall n \in \mathbb{N}$ the quantities in (2.3.6) are conserved. The first quantity $\bar{H}_n(t)$, can be viewed as the mechanical energy. We will see later that the average of the energy $\langle H_n\rangle_{\rho^n}$, can be decomposed into the mechanical and thermal parts, where $\bar{H}_n$ is the mechanical part. The second quantity $\bar{I}_n$, shows us a typical way of constructing the other conserved quantities by taking further gradients (cf. Remark 1.1.7).

The conservation of $\bar{I}_n(t)$ and $\bar{H}_n(t)$ in (2.3.6), the regularity assumptions, where $\bar{r}, \bar{p} \in C^1([0,1])$, and the properties of the masses, give us the following bounds: there exists a deterministic $C > 0$, such that for every $n \in \mathbb{N}$ and $t \in [0,T]$, we have:

$$\sum_{x=1}^{n}\left(\bar{p}_x^2(nt) + \bar{r}_x^2(nt)\right) \le Cn, \tag{2.3.7}$$

$$\sum_{x=1}^{n}\left(\bar{r}_x(nt) - \bar{r}_{x-1}(nt)\right)^2 \le \frac{C}{n}, \quad \sum_{x=1}^{n-1}\left(\frac{\bar{p}_{x+1}(nt)}{m_{x+1}} - \frac{\bar{p}_x(nt)}{m_x}\right)^2 \le \frac{C}{n}. \tag{2.3.8}$$

Notice that these bounds hold for every time scale $n^\alpha t$, for $\alpha > 0$. First, observe that (2.3.7), is bounded by $2(1+m_{max})\bar{H}_n(nt)$, and expressions in (2.3.8) are bounded by and $2(1+m_{max})\bar{I}_n(nt)$. By the conservation of $\bar{I}_n(t)$ and $\bar{H}_n(t)$ in (2.3.6), it is enough to show $\bar{H}_n(nt) \le Cn$, and $\bar{I}_n(nt) \le \frac{C}{n}$ for $t = 0$. But this is obvious: Since $\bar{r}, \bar{p} \in C^1([0,1])$, $\bar{p}_x^2$ and $\bar{r}_x^2$ are bounded for every x, which gives (2.3.7). Moreover, $(\bar{r}_x - \bar{r}_{x-1})^2 = (\bar{r}(\frac{x}{n}) - \bar{r}(\frac{x-1}{n}))^2 \le \frac{c_2}{n^2}$, since $\bar{r} \in C^1([0,1])$ (choose $c_2 = ||\bar{r}'||_{L^\infty}$). Moreover, thanks to (2.3.4) we have

$$\left(\frac{\bar{p}_{x+1}(0)}{m_{x+1}} - \frac{\bar{p}_x(0)}{m_x}\right)^2 \le \frac{2}{\bar{m}^2}\left(\bar{p}\Big(\frac{x+1}{n}\Big) - \bar{p}\Big(\frac{x}{n}\Big)\right)^2 + 2\Big(\frac{\mathcal{E}_n^{x+1}}{m_{x+1}} - \frac{\mathcal{E}_n^x}{m_x}\Big)^2 \le \frac{c_3}{n}. \tag{2.3.9}$$

[6]The gap is as follows: comparing the relations (3.11), (3.12) with (3.14),(3.15) is not sufficient to close the argument, since the derivative of f and g appears in the RHS instead of f and g.

Note that in (2.3.9), first, we apply the property $\bar{p} \in C^1([0,1])$. Then we take advantage of the identity $\frac{\xi_n^x}{m_x} = (\beta(\frac{x}{n}))^{-1}(\sum_{x=1}^n \frac{m_x}{\beta_x})^{-1}(\sum_{x=1}^n \bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}})$ and we bounded $|\frac{1}{\beta_x} - \frac{1}{\beta_{x-1}}| \leq \frac{c_3'}{n}$ by using the assumption that β is Lipschitz and $0 < \beta_{min} < \beta(y) < \beta_{max}$. Finally, we bounded the rest by a constant thanks to the properties of $\bar{p}$ and m_x. (Note that this is the only place where we use the assumption that β is Lipschitz.

From (2.3.7), (2.3.8) by Cauchy-Schwartz inequality we can deduce that $\bar{r}_x(nt)$, $\bar{p}_x(nt)$ are Hölder regular and bounded, in the following sense: there exists a deterministic constant $C > 0$, such that for every n and every $x, x' \in \mathbb{I}_n$ we have:

$$|\bar{r}_x(nt) - \bar{r}_{x'}(nt)| \leq \frac{C|x-x'|^{\frac{1}{2}}}{\sqrt{n}}, \qquad \Big|\frac{\bar{p}_x(nt)}{m_x} - \frac{\bar{p}_{x'}(nt)}{m_{x'}}\Big| \leq \frac{C|x-x'|^{\frac{1}{2}}}{\sqrt{n}}. \tag{2.3.10}$$

Moreover thanks to (2.3.7) and (2.3.10), there exists $C' > 0$, such that $\forall n$ and $\forall x \in \mathbb{I}_n$, we have:

$$|\bar{r}_x(nt)| \leq C', \qquad |\bar{p}_x(nt)| \leq C', \tag{2.3.11}$$

Step2. Mass Homogenization
For every $f \in C^0([0,1])$ and $t \in [0,T]$, as $N \to \infty$ we have:

$$\frac{1}{N}\sum_{x=1}^N f(\frac{x}{N})\frac{\bar{p}_x(Nt)}{m_x}(m_x - \bar{m}) \to 0, \tag{2.3.12}$$

$$\frac{1}{N}\sum_{x=1}^N f(\frac{x}{N})\Big(\frac{\bar{p}_x(Nt)}{m_x}\Big)^2(m_x - \bar{m}) \to 0, \tag{2.3.13}$$

almost surely with respect to the distribution of the masses, where $\bar{m} = \mathbb{E}(m_x)$. This step permits us to deal with the randomness of the masses by homogenizing them. The second limit (2.3.13) will be used in the next section.

Proof. For the proof of this step, one can see Lemma 2 in [35]. Let us emphasize the fact that this lemma's proof only need estimates (2.3.10), and (2.3.7). Therefore, we can use Lemma 2 of [35] here. □

Step3. Weak Convergence to the solution of the wave equation with a C^2 test function.
In this step, we prove the convergences (2.1.16), (2.1.17), for a special class of test functions. In the next step, we complete the proof by using the Hölder bounds in (2.3.10). Notice that here, we follow a different path in comparison to [35]. Let $\forall t \in [0,T]$, and $f, g \in C^2([0,1])$, such that $f(0) = f(1) = 0$, $g(0) = g(1)$. Moreover, f and g are continuously differentiable at 0 and 1, with $g'(0) = g'(1) = 0$ and $f'(0) = f'(1)$ (f, g are periodic, with Dirichlet boundary condition for f, and Neumann boundary condition for g). Then, we have:

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\left\langle r_x(nt)\right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{r}_x(nt) \to \int_0^1 f(y)\mathrm{r}(y,t)dy, \tag{2.3.14}$$

$$\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\left\langle p_x(nt)\right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\bar{p}_x(nt) \to \int_0^1 g(y)\mathrm{p}(y,t)dy, \tag{2.3.15}$$

almost surely, with respect to the distribution of the masses as $n \to \infty$, where $\mathrm{r}(y,t)$ and $\mathrm{p}(y,t)$ are the unique strong solutions to the following system of conservation laws (2.1.13), (2.1.14), and (2.1.15).

Proof of Step 3. First, for every $f, g \in C^0([0,1])$, such that $f(0) = f(1) = 0$, $g(0) = g(1)$, define the following kernels:

$$R(f,t) := \int_0^1 \mathrm{r}(y,t)f(y)dy, \quad P(g,t) := \int_0^1 \mathrm{p}(y,t)g(y)dy. \tag{2.3.16}$$

In particular, if we assume $f, g \in C^2([0,1])$ satisfying the assumption of this Step, we can argue as follows: Since the solutions of (2.1.13) are explicit, and the unique weak solution coincide with the strong solution (cf. [95]), by using the explicit form of the solution, we have a uniform bound on the time derivative of r and p. Therefore, by using the dominated convergence theorem we have:

$$\begin{aligned}\partial_s R(f,s) &= \int_0^1 \partial_s \mathrm{r}(y,t)f(y)dy = \frac{1}{\bar{m}}\int_0^1 \partial_y \mathrm{p}(y,t)f(y)dy = -\frac{1}{\bar{m}}\int_0^1 \mathrm{p}(y,t)f'(y)dy \\ &= -\frac{1}{m}P(f',t),\end{aligned} \tag{2.3.17}$$

where we integrated by parts, and used the property $f(0) = f(1) = 0$. Similarly, using the boundary condition $\mathrm{r}(1,s) = \mathrm{r}(0,s)$, implies:

$$\begin{aligned}\partial_s P(g,s) &= \int_0^1 \partial_s \mathrm{p}(y,s)g(y) = \int_0^1 \partial_y \mathrm{r}(y,t)g(y)dy = -\int_0^1 \mathrm{r}(y,t)g'(y)dy \\ &= R(g',t).\end{aligned} \tag{2.3.18}$$

Therefore, by using (2.3.17), (2.3.18), we can characterize $R(f,t)$ and $P(g,t)$, for the aforementioned $f, g \in C^2([0,1])$, with proper boundary conditions, as follows:

$$\begin{aligned}R(f,t) &= R(f,0) - \frac{1}{\bar{m}}\int_0^t P(f',s)ds, \\ P(g,t) &= P(g,0) - \int_0^t R(g',s)ds.\end{aligned} \tag{2.3.19}$$

Notice that we have:

$$R(f,0) = \int_0^1 \bar{r}(y)f(y)dy, \quad P(g,0) = \int_0^1 \bar{p}(y)g(y)dy. \tag{2.3.20}$$

In the microscopic level, for $n \in \mathbb{N}$, we define the kernels $R_n(f,t)$ and $P_n(g,t)$, for $t \in [0,T]$ and $f, g \in C^0([0,1])$:

$$\begin{aligned} R_n(f,t) &= \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle r_x(nt)\right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{r}_x(nt), \\ P_n(g,t) &= \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n}) \left\langle p_x(nt)\right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\bar{p}_x(nt). \end{aligned} \tag{2.3.21}$$

In particular, for $f, g \in C^2([0,1])$, satisfying the assumptions of this step, we can characterize $R_n(f,t), P_n(f,t)$ as follows:

$$\begin{aligned} &R_n(f,t) = R_n(f,0) + \int_0^t \partial_s R_n(f,s)ds = R_n(f,0) + \int_0^t \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\partial_s\Big(\bar{r}_x(ns)\Big)ds = \\ &R_n(f,0) + \int_0^t \sum_{x=1}^{n} f(\frac{x}{n})\Big(\frac{\bar{p}_{x+1}(ns)}{m_{x+1}} - \frac{\bar{p}_x(ns)}{m_x}\Big)ds = \\ &R_n(f,0) - \int_0^t \sum_{x=1}^{n} \Big(f(\frac{x}{n}) - f(\frac{x-1}{n})\Big)\frac{\bar{p}_x(ns)}{m_x}ds, \end{aligned} \tag{2.3.22}$$

where we used the time evolution of $\bar{r}(t)$ from (2.3.5) in the second line, then we performed a summation by parts, using the assumption $f(0) = f(1) = 0$. Since $f \in C^2([0,1])$, we get $f(\frac{x}{n}) - f(\frac{x-1}{n}) = \frac{1}{n}f'(\frac{x}{n}) + \frac{\epsilon_n^x}{n^2}$, where ϵ_n^x is bounded by a constant $C > 0$, uniformly in x and n. Hence, we have:

$$R_n(f,t) = R_n(f,0) - \int_0^t \frac{1}{n}\sum_{x=1}^{n} f'(\frac{x}{n})\frac{\bar{p}_x(ns)}{m_x}ds + \epsilon_n, \tag{2.3.23}$$

where ϵ_n is the remainder term and one can observe $\epsilon_n \to 0$ in a deterministic way, as $n \to \infty$. Moreover, we can use the result of Step 2, namely (2.3.12), in order to replace m_x by $\bar{m}$ and get

$$\frac{1}{n}\sum_{x=1}^{n} f'(\frac{x}{n})\frac{\bar{p}_x(ns)}{m_x} - \frac{1}{n}\sum_{x=1}^{n} f'(\frac{x}{n})\frac{\bar{p}_x(ns)}{\bar{m}} = \epsilon_n'(s),$$

where $\epsilon_n' \to 0$, almost surely as $n \to \infty$. Note that $\epsilon'(s)$ is bounded thanks to (2.3.11), and by dominated convergence theorem, $\bar{\epsilon}_n = \int_0^t \epsilon'(s)ds \to 0$, almost surely. Hence, by using the definition of $P_n(g,s)$, we have:

$$R_n(f,t) = R_n(f,0) - \frac{1}{\bar{m}}\int_0^t P_n(f',s)ds + \bar{\epsilon}_n, \tag{2.3.24}$$

where $\bar{\epsilon}_n \to 0$, as $n \to \infty$, almost surely. We can proceed similarly in order to obtain the counterpart of (2.3.24) for $P_n(f,t)$. Notice that for $P_n(g,t)$, homogenization over

the masses is not necessary.

$$P_n(g,t) = P_n(g,0) + \int_0^t \partial_s P_n(g,s)ds = P_n(g,0) + \int_0^t \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\partial_s \bar{p}_x(ns)\Big)ds =$$
$$P_n(g,0) + \int_0^t \sum_{x=1}^{n} g(\frac{x}{n})\Big(\bar{r}_x(ns) - \bar{r}_{x-1}(ns)\Big)ds =$$
$$P_n(g,0) - \int_0^t \sum_{x=1}^{n} \bar{r}_x(ns)\Big(g\big(\frac{x+1}{n}\big) - g\big(\frac{x}{n}\big)\Big)ds, \tag{2.3.25}$$

where we advanced similar to (2.3.19), using (2.3.5) and the microscopic boundary condition $\bar{r}_0(s) = \bar{r}_n(s) = 0$, for the summation by parts. Again, since $g \in C^2([0,1])$, we shall write $g(\frac{x+1}{n}) - g(\frac{x}{n}) = \frac{1}{n}g'(\frac{x}{n}) + \tilde{\epsilon}^n_x$, where $|\tilde{\epsilon}^n_x| \leq \frac{C'}{n^2}$, for a C', uniform in x and n. Therefore, by using the bound $|\bar{r}_x(ns)| \leq C$, and the definition of $R_n(g',s)$, we have:

$$P_n(g,t) = P_n(g,0) - \int_0^t \frac{1}{n}\sum_{x=1}^{n} \bar{r}_x(ns)g'(\frac{x}{n}) + \tilde{\epsilon}_n = P_n(g,0) - \int_0^t R_n(g',s)ds + \tilde{\epsilon}_n, \tag{2.3.26}$$

such that $\tilde{\epsilon}_n \to 0$ as $n \to \infty$.

For every continuous f, g (therefore, for f, g satisfying our assumptions), we have the following observation: One can obtain the solution to the system of ODEs (2.3.5) explicitly [7] and observe that for any fixed continuous (f,g): $\forall n, R_n(f,t), P_n(g,t)$ are smooth in time (at least C^1). Moreover, by computing their derivatives, we obtain a uniform bound $\bar{C} > 0$ (uniform in t and n):

$$|\partial_t R_n(f,t)| = \Big|\sum_{x=1}^{n-1} f(\frac{x}{n})\Big(\frac{\bar{p}_{x+1}(nt)}{m_{x+1}} - \frac{\bar{p}_x(nt)}{m_x}\Big)\Big| \leq C_1 \sum_{x=1}^{n}\Big|\frac{\bar{p}_{x+1}(nt)}{m_{x+1}} - \frac{\bar{p}_x(nt)}{m_x}\Big| \leq$$
$$C_1(\sum_{x=1}^{n}\Big|\frac{\bar{p}_{x+1}(nt)}{m_{x+1}} - \frac{\bar{p}_x(nt)}{m_x}\Big|^2)^{\frac{1}{2}}(n)^{\frac{1}{2}} \leq C_1 n^{\frac{1}{2}}(\frac{C}{n})^{\frac{1}{2}} \leq \bar{C}, \tag{2.3.27}$$

where we used the Cauchy Schwartz inequality and the bound in (2.3.8). Similarly, using the other inequality in (2.3.8), we have the similar uniform bound $\bar{C}$ for $|\partial_t P_n(g,t)|$.

For proving this step, we show that $R_n(f,t) \to R(f,t)$ and $P_n(g,t) \to P(g,t)$, almost surely as $n \to \infty$, for every f and g, satisfying the smoothness and boundary condition assumption, and every $t \in [0,T]$. First, notice that for $t = 0$, we have $R_n(f,0) = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{r}(\frac{x}{n})$, hence, the convergence to $R(f,0) = \int_0^1 f(y)\bar{r}(y)dy$ is evident.
For $P_n(g,0) = \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\bar{p}_x(0) = \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\frac{m_x}{\bar{m}}(\bar{p}_x - \mathcal{E}^x_n)$, where we used the

[7] These solutions are exactly similar to (2.2.22), where the operators at time zero replaced by the averaged function at time zero.

definition of $\bar{p}_x(0)$ from (2.3.4), we can deduce $\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\frac{m_x}{\bar{m}}(\bar{p}_x) \to P(g,0) = \int_0^1 g(y)\bar{p}(y)$ almost surely, by the Strong Law of Large Numbers. Moreover,

$$\Big|\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\mathcal{E}_n^x\Big| \leq \frac{C}{n}\sum_{x=1}^{n}|\mathcal{E}_n^x| = \frac{C}{n}\Big|\sum_{x=1}^{n}\bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}\Big| \to 0,$$

almost surely, where we use the definition of $\mathcal{E}_n^x$ (2.2.39), and the fact that $\frac{\Pi_0}{n} \to 0$ almost surely (2.2.40). Combining the last two convergences we obtain $P_n(g,0) \to P(g,0)$ almost surely for any continuous g.

We use the characterizations (2.3.19), (2.3.24), and (2.3.26) in order to prove the result for $t \in (0,T]$.

Fix proper f and g, and consider the families of functions $\{(R_n(f,.)\}_n, \{P_n(g,.))\}_n$. These families are equicontinuous, since we established a uniform bound on their derivatives[8]. Hence, by Arzelà-Ascoli theorem, there exist continuous functions $\tilde{\varphi}_r^f(t), \tilde{\varphi}_p^g(t)$ on $[0,T]$, such that a subsequence of $R_n(f,t)$ and $P_n(g,t)$ converges to these functions, respectively.

In particular, we can take $f_k(y) = \sin(k\pi y)$ and $g_k(y) = \cos(k\pi y)$, for every $k \in \mathbb{N}_0$. Denote $R(f_k,t)$ and $P(f_k,t)$ by $\varphi_r^k(t)$ and $\varphi_p^k(t)$, respectively. Since R and P are linear in their first arguments, $f_k' = \pi k g_k$ and $g_k' = -\pi k f_k$, by using the characterization (2.3.19), we have φ_p^k and φ_r^k for every $k \in \mathbb{N}_0$, satisfy the following system of ODEs:

$$\varphi_r^k(t) = \varphi_r^k(0) - \frac{\pi k}{\bar{m}}\int_0^t \varphi_p^k(s)ds, \quad \varphi_p^k(t) = \varphi_p^k(0) + \pi k\int_0^t \varphi_r^k(s)ds. \tag{2.3.28}$$

Recall the continuous functions $\tilde{\varphi}_r^{f_k}$ and $\tilde{\varphi}_p^{g_k}$ as the limit of a subsequence of $R_n(f_k,t)$ and $P_n(f_k,t)$ and denote them by $\tilde{\varphi}_r^k$ and $\tilde{\varphi}_p^k$, respectively. Recall the characterizations (2.3.24) and (2.3.26), observe that P_n and R_n are linear in their first argument and take the limit of these characterizations for the subsequences converging to $\tilde{\varphi}_r^k$ and $\tilde{\varphi}_p^k$. By using the dominated convergence theorem (since these functions are bounded on a compact domain), we deduce that $\tilde{\varphi}_r^k$ and $\tilde{\varphi}_p^k$ satisfy the exact same system of ODEs as (2.3.28), almost surely. Moreover, we observed earlier the convergence at time zero i.e $\tilde{\varphi}_r^k(0) = \varphi_r^k(0)$ and $\tilde{\varphi}_p^k(0) = \varphi_p^k(0)$ almost surely. Therefore, by a uniqueness argument, we have $\tilde{\varphi}_r^k(t) = \varphi_r^k(t)$ and $\tilde{\varphi}_p^k(t) = \varphi_p^k(t)$, for every t in $[0,T]$ almost surely.
Notice that this argument is valid for any limiting point of $R_n(f_k,t)$ and $P_n(f_k,t)$. Hence, $\forall k \in \mathbb{N}_0$, and $\forall t \in [0,T]$, we have $R_n(f_k,t) \to R(f_k,t)$ and $P_n(g_k,t) \to P(g_k,t)$, as $n \to \infty$, almost surely.

Finally, using the fact that the sets $\{f_k\}_{k=0}^{\infty}$ and $\{g_k\}_{k=0}^{\infty}$ are orthonormal (Fourier) basis of $L^2([0,1])$, we can finish the proof with an $\frac{\epsilon}{3}$ argument, thanks to the fact that $R_n(.,t)$ and $P_n(.,t)$ are linear in their first argument. □

[8]For every $\epsilon > 0$, take $\delta = \frac{\epsilon}{C}$, then for every ϕ in this family, and $t, t' \in [0,T]$, if $|t-t'| < \delta$, we have $|\phi(t) - \phi(t')| \leq |t-t'||\phi'(t^*)| \leq \bar{C}|t-t'| \leq \epsilon$, for $t < t^* < t'$, since all the functions in this family are smooth, and their derivative are uniformly bounded.

Step4: The pointwise convergence of $\bar{r}$ and $\bar{p}$:
In this step, we prove the pointwise convergence $\bar{r}_{[ny]}(nt) \to \mathrm{r}(y,t)$ and $\frac{\bar{p}_{[ny]}}{m_{[ny]}} \to \frac{\mathrm{p}(y,t)}{\bar{m}}$, almost surely, exploiting the "Hölder" bounds in (2.3.10). Concretely, $\forall y \in (0,1)$ and $t \in [0,T]$ we have:

$$\bar{r}_{[ny]}(nt) \to \mathrm{r}(y,t), \quad \frac{\bar{p}_{[ny]}(nt)}{m_{[ny]}} \to \frac{\mathrm{p}(y,t)}{\bar{m}}, \tag{2.3.29}$$

almost surely, with respect to the distribution of the masses.

Proof. Fix $y \in (0,1)$, and take $\epsilon_1 > 0$ such that $y \in (2\epsilon_1, 1-2\epsilon_1)$. Let ζ be a positive symmetric mollifier, i.e., $\zeta \in C_c^\infty(\mathbb{R})$ (infinitely differentible, compactly supported, with $supp(\zeta) = [-1,1]$), and

$$\int_{\mathbb{R}} \zeta(y')dy' = 1; \quad \zeta(y) \geq 0; \quad \zeta(y) = \zeta(-y).$$

Let $\zeta_\epsilon := \frac{1}{\epsilon}\zeta(\frac{y}{\epsilon})$, for $0 < \epsilon < \epsilon_1$, be a regularizing family, notice that we have $\zeta_\epsilon \in C_c^\infty(\mathbb{R})$, $supp(\zeta_\epsilon) = [-\epsilon, \epsilon]$, and $\int_{\mathbb{R}} \zeta_\epsilon(y')dy' = 1$, as well as $\zeta_\epsilon(y) \geq 0$.
Since $\int_{\mathbb{R}} \zeta_\epsilon(y')dy' = 1$, multiplying by $\bar{r}_{[ny]}(nt)$, we have $\forall \epsilon_1 > \epsilon > 0$:

$$\begin{aligned} &\bar{r}_{[ny]}(nt) = \bar{r}_{[ny]}(nt)\int_{\mathbb{R}} \zeta_\epsilon(y-y')dy' = \int_{\mathbb{R}} \zeta_\epsilon(y-y')\bar{r}_{[ny]}(nt)dy' = \\ &\int_{\mathbb{R}} \zeta_\epsilon(y-y')\big(\bar{r}_{[ny]}(nt) - \bar{r}_{[ny']}(nt)\big)dy' + \int_{\mathbb{R}} \zeta_\epsilon(y-y')\bar{r}_{[ny']}(nt)dy'. \end{aligned} \tag{2.3.30}$$

Since $\zeta_\epsilon(y-y')$ is supported on $(y-\epsilon, y+\epsilon)$, we have:

$$\begin{aligned} &\Big|\int_{\mathbb{R}} \zeta_\epsilon(y-y')(\bar{r}_{[ny]}(nt) - \bar{r}_{[ny']}(nt))dy'\Big| \leq \sup_{y' \in (y-\epsilon, y+\epsilon)} |\bar{r}_{[ny]}(nt) - \bar{r}_{[ny']}(nt)| \int_{\mathbb{R}} \zeta_\epsilon(y')dy' \\ &= \sup_{y' \in (y-\epsilon, y+\epsilon)} |\bar{r}_{[ny]}(nt) - \bar{r}_{[ny']}(nt)| \leq C\sqrt{\epsilon}, \end{aligned} \tag{2.3.31}$$

where the last bound is deduced from (2.3.10), since $|y-y'| \leq \epsilon$. In order to deal with the second term in (2.3.30), notice that by the choice of ϵ_1, $supp(\zeta_\epsilon) \subset (0,1)$, hence we have

$$\begin{aligned} &\int_{\mathbb{R}} \zeta_\epsilon(y-y')\bar{r}_{[ny']}(nt)dy' = \int_0^1 \zeta_\epsilon(y-y')\bar{r}_{[ny']}(nt)dy' = \sum_{x=0}^{n-1} \int_{\frac{x}{n}}^{\frac{x+1}{n}} \zeta_\epsilon(y-y')\bar{r}_{[ny']}(nt)dy' \\ &= \sum_{x=1}^{n} \bar{r}_x(nt) \int_{\frac{x}{n}}^{\frac{x+1}{n}} \zeta_\epsilon(y-y')dy' = \frac{1}{n}\sum_{x=1}^{n} \bar{r}_x(nt)\zeta_\epsilon(y - \frac{x}{n}) \\ &+ \sum_{x=1}^{n} \bar{r}_x(nt) \int_{\frac{x}{n}}^{\frac{x+1}{n}} \Big(\zeta_\epsilon(y-y') - \zeta_\epsilon(y-\frac{x}{n})\Big)dy', \end{aligned} \tag{2.3.32}$$

where we used the fact that ζ_ϵ is smooth, and $\bar{r}_{[ny']}(nt)$ is a step function. Since ζ_ϵ is smooth (C^∞) and compactly supported, $|\int_{\frac{x}{n}}^{\frac{x+1}{n}} (\zeta_\epsilon(y-y') - \zeta_\epsilon(y-\frac{x}{n}))dy'| \leq \frac{M}{n^2}$,

where M is a constant independent of n (for example, one can take M as $\sup|\zeta'_\epsilon|$). Therefore, thanks to the bound on $\bar{r}_x(nt)$ in (2.3.11), the last term is bounded by $\frac{c'}{n}$, where c' is a constant uniform in y and n [9]. After all, we have $\forall\, 0<\epsilon<\epsilon_1$:

$$\bar{r}_{[ny]}(nt)=\frac{1}{n}\sum_{x=1}^{n}\bar{r}_x(nt)\zeta_\epsilon(y-\frac{x}{n})+\epsilon'(\epsilon)+\epsilon''(n), \tag{2.3.33}$$

where $|\epsilon'(\epsilon)|$ is bounded by $C\sqrt{\epsilon}$, and $|\epsilon''(n)|$ is bounded by $\frac{c'}{n}$.

However, observe that thanks to the choice of ϵ, $\forall\epsilon$, $\zeta_\epsilon(y-.)$ satisfies the properties of the test function f, in the step 3. Therefore, by using (2.3.14), and (2.3.15), for $f(.)=\zeta_\epsilon(y-.)$ and taking the limit $n\to\infty$ in (2.3.33), we get $\forall\epsilon>0$, as $n\to\infty$:

$$\bar{r}_{[ny]}(nt)\to\int_0^1\zeta_\epsilon(y-y')\mathrm{r}(y',t)+\epsilon'(\epsilon), \tag{2.3.34}$$

almost surely. Taking the limit $\epsilon\to 0$, in (2.3.34), since the left hand side is independent of ϵ, thanks to the continuity of $\mathrm{r}(y,t)$, and properties of ζ_ϵ, the first term converges to $\mathrm{r}(y,t)$. The second term converges to zero, thanks to the bound $\epsilon'(\epsilon)\leq C\sqrt{\epsilon}$. This finishes the proof of $\bar{r}_{[ny]}(nt)\to\mathrm{r}(y,t)$.
In order to deal with $\bar{p}_{[ny]}(nt)$, we can proceed similarly. First, for the sake of obtaining the counterpart of (2.3.31), we may use the second bound in (2.3.10), where $|\frac{\bar{p}_x(nt)}{m_x}-\frac{\bar{p}_{x'}(nt)}{m_{x'}}|\leq C\frac{|x-x'|^{\frac{1}{2}}}{\sqrt{n}}$. Then we will get the following expression, similar to (2.3.33) (where we used the bound $\frac{\bar{p}_x}{m_x}\leq C$, as well):

$$\frac{\bar{p}_{[ny]}(nt)}{m_{[ny]}}=\frac{1}{n}\sum_{x=1}^{n}\frac{\bar{p}_x(nt)}{m_x}\zeta_\epsilon(y-\frac{x}{n})+\tilde{\epsilon}'(\epsilon)+\tilde{\epsilon}''(n), \tag{2.3.35}$$

where similar to the previous case, $|\tilde{\epsilon}'(\epsilon)|\leq C\sqrt{\epsilon}$, and $|\tilde{\epsilon}''(n)|\leq\frac{c'}{n}$. Here, since

$$\frac{1}{n}\sum_{x=1}^{n}\zeta_\epsilon(y-\frac{x}{n})\frac{\bar{p}_x(nt)}{m_x}=\frac{1}{n\bar{m}}\sum_{x=1}^{n}\zeta_\epsilon(y-\frac{x}{n})\frac{\bar{p}_x(nt)}{m_x}(\bar{m}-m_x)+\frac{1}{n}\sum_{x=1}^{n}\zeta_\epsilon(y-\frac{x}{n})\frac{p_x(nt)}{\bar{m}},$$

we can replace m_x with $\bar{m}$ in (2.3.35), with the cost of an error term (the first term in the last relation), that goes to zero almost surely, thanks to the result (2.3.12) from Step 2. Since $\zeta_\epsilon(y-.)$ satisfies the criteria of g in Step 3, by using the result of this step and taking $n\to\infty$ in (2.3.35), we have:

$$\frac{\bar{p}_{[ny]}}{m_{[ny]}}\to\int_0^1\frac{\mathrm{p}(y',t)}{\bar{m}}\zeta_\epsilon(y-y')dy'+\tilde{\epsilon}'(\epsilon), \tag{2.3.36}$$

almost surely. Taking $\epsilon\to 0$ similar to the previous case, the continuity of $\mathrm{p}(y,t)$, and the fact that $\tilde{\epsilon}'(\epsilon)\leq C\sqrt{\epsilon}$, finish the proof of this step. □

[9] Notice that we misuse the bound of the sums in the last expression thanks to the support of ζ_ϵ.

Step 5. Finishing the proof
In this step, we finish the proof of the first part of 2.1.1, namely the convergences (2.1.16), (2.1.17). Take $f \in C^0([0,1])$, then as $n \to \infty$ we have:

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle r_x(nt) \right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{r}_x(nt) \to \int_0^1 f(y)\mathrm{r}(y,t)dy, \tag{2.3.37}$$

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \left\langle p_x(nt) \right\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{p}_x(nt) \to \int_0^1 f(y)\mathrm{p}(y,t)dy, \tag{2.3.38}$$

almost surely, with respect to the distribution of the masses.

Proof. In order order to prove (2.3.37), notice that we have:

$$\frac{1}{n}\sum_{x=1}^{n} \bar{r}_x(nt) f\left(\frac{x}{n}\right) = \int_0^1 f\Big(\frac{[ny]}{n}\Big)\bar{r}_{[ny]}(nt)dy.$$

Using the pointwise convergence result (2.3.29), from the previous step, and the continuity of f, we have the pointwise convergence: $f(\frac{[ny]}{n})\bar{r}_{[ny]}(nt) \to f(y)\mathrm{r}(y,t)$, almost surely. Thanks to the bound $|\bar{r}_{[ny]}(nt)| \leq C$, in (2.3.11), we deduce the result i.e. (2.3.37), by dominated convergence theorem. Finally, to prove (2.3.38), we write the right hand side as:

$$\frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\bar{p}_x(nt) = \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n})\frac{\bar{p}_x(nt)}{m_x}(m_x - \bar{m}) + \frac{\bar{m}}{n}\sum_{x=1}^{n} f(\frac{x}{n})\frac{\bar{p}_x(nt)}{m_x}. \tag{2.3.39}$$

The first term in the latter goes to zero almost surely, thanks to (2.3.12). For the second term we can argue similar to the term corresponding to r: it converges to $\int_0^1 f(y)\mathrm{p}(y,t)$, almost surely, thanks to the pointwise convergence: $f(\frac{[ny]}{n})\frac{\bar{p}_{[ny]}(nt)}{m_{[ny]}} \to f(y)\frac{\mathrm{p}(y,t)}{\bar{m}}$ in (2.3.29), the bound $|\bar{p}_x(nt)| \leq C$ in (2.3.11), and the dominated convergence theorem. This finishes the proof. □

□

Remark 2.3.1. *Notice that we only use the fact that $\beta : \mathbb{R} \to [0,1]$ is Lipschitz in the bound* (2.3.9)*. In particular, if we have $\mathcal{E}_n^x = 0$ for all x, the same result* (2.3.37)*,* (2.3.38) *holds with $\beta \in C^0([0,1])$. In fact, if $\bar{p}_x(nt)$ and $\bar{r}_x(nt)$ denote the solution of time evolution* (2.3.5) *with initial condition $\bar{p}_x(0) = \frac{m_x}{\bar{m}}$ and $\bar{r}_x(0) = \bar{r}_x$ then we can deduce* (2.3.37)*,* (2.3.38)*, and* (2.3.29) *with $\beta \in C^0([0,1])$.*

Proof of (2.1.16) *and* (2.1.17) *in general case.* As we observed in Remark 2.3.1, in case $\mathcal{E}_n^x = 0$ for all x, we have (2.3.37), and (2.3.38). Since the evolution of $\bar{p}(nt)$, $\bar{r}(nt)$ (2.3.5) is linear, to prove (2.1.16) and (2.1.17) it is sufficient to prove the following: For any n, let $\pi(nt) \in \mathbb{R}^n$ $\varrho(nt) \in \mathbb{R}^{n-1}$ be the solution of (2.3.5) with initial datum

$$\pi_x(0) = -\mathcal{E}_x^n, \qquad \varrho_x(0) = 0.$$

Then for any test function $f \in C^0([0,1])$, we have:

$$\begin{aligned} &\frac{1}{n}\sum_{x=1}^{n} f\Big(\frac{x}{n}\Big)\pi_x(nt) \to 0, \\ &\frac{1}{n}\sum_{x=1}^{n} f\Big(\frac{x}{n}\Big)\varrho_x(nt) \to 0, \end{aligned} \tag{2.3.40}$$

almost surely as $n \to \infty$. Since $\pi_x(nt)$, $\rho_x(nt)$ solves (2.3.5) we have the conservation of the mechanical energy $\bar{H}_n(t)$:

$$\bar{H}_n(t) = \frac{1}{2}\sum_{x=1}^{n}\left(\frac{(\pi_x(nt))^2}{m_x} + (\varrho_x(nt))^2\right).$$

By a Cauchy-Schwartz inequality we get:

$$\begin{aligned} &\Big|\frac{1}{n}\sum_{x=1}^{n} f\Big(\frac{x}{n}\Big)\pi_x(nt)\Big| \leq \left(\frac{1}{n}\sum_{x=1}^{n} f^2\Big(\frac{x}{n}\Big)\right)^{\frac{1}{2}}\left(\frac{1}{n}\sum_{x=1}^{n}\pi_x^2(nt)\right)^{\frac{1}{2}} \leq \\ &\left|\frac{C}{n}\bar{H}_n(t)\right|^{\frac{1}{2}} = \left|\frac{C}{n}\bar{H}_n(0)\right|^{\frac{1}{2}} = \left|\frac{C}{n}\sum_{x=1}^{n}(\mathcal{E}_n^x)^2\right|^{\frac{1}{2}} \to 0, \end{aligned} \tag{2.3.41}$$

almost surely, where we used the initial datum, as well as the fact that $\mathcal{E}_n^{[ny]} \to 0$ almost surely thanks to its definition (2.2.39), and (2.2.40). We can proceed similarly for $\varrho(nt)$, and this finishes the proof. □

2.4 Energy at Time Zero

2.4.1 Strong law of large numbers for energy at time zero

In order to prove the convergence of the distribution of the energy to the solution of the Euler equation (2.1.18), first, we need to show this convergence at time zero, for any test function $g \in C^0([0,1])$, almost surely i.e.

$$\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\left\langle e_x\right\rangle_\rho \to \int_0^1 g(y)\Big(\frac{\bar{p}(y)^2}{2\bar{m}} + \frac{\bar{r}(y)^2}{2} + \mathfrak{f}_\beta^\mu(y)\Big)dy. \tag{2.4.1}$$

First, we decompose the energy into the mechanical and thermal (fluctuation) part: Recall the definition of $\tilde{p}_x$ and $\tilde{r}_x$ as $\tilde{p}_x = p_x - \left\langle p_x\right\rangle_\rho = p_x - \frac{m_x}{\bar{m}}\bar{p}_x + \mathcal{E}_n^x$, and $\tilde{r}_x = r_x - \left\langle r_x\right\rangle_\rho = r_x - \bar{r}_x$, respectively. Then we have:

$$\left\langle e_x\right\rangle_\rho = \left\langle \frac{p_x^2}{2m_x} + \frac{r_x^2}{2}\right\rangle_\rho = \frac{1}{2}\Big(\frac{\left\langle p_x\right\rangle_\rho^2}{m_x} + \left\langle r_x\right\rangle_\rho^2 + \frac{\langle \tilde{p}_x^2\rangle_\rho}{m_x} + \left\langle \tilde{r}_x^2\right\rangle_\rho\Big). \tag{2.4.2}$$

Moreover, observe that

$$\frac{1}{n}\sum_{x=1}^{n}\frac{1}{2}g(\frac{x}{n})\Big(\frac{\left\langle p_x\right\rangle_\rho^2}{m_x} + \left\langle r_x\right\rangle_\rho^2\Big) \to \int_0^1 g(y)\Big(\frac{\bar{p}(y)^2}{2\bar{m}} + \frac{\bar{r}(y)^2}{2}\Big)dy, \tag{2.4.3}$$

almost surely, with respect to the distribution of the masses. Notice that (2.4.3) is a direct consequence of Corollary (2.2.2.1), where we have $\langle r_x \rangle_\rho = \bar{r}_x = \bar{r}(\frac{x}{n})$ and $\langle p_x \rangle_\rho = \frac{m_x}{\bar{m}} \bar{p}_x - \mathcal{E}^n_x$. Then we applied the Strong Law of Large Numbers for $\{m_x\}$ and take advantage of the fact that p, r, g are continuous. Moreover, we use the fact that terms corresponding to $\mathcal{E}^x_n$ are vanishing (2.2.39), (2.2.40). Therefore, in order to deduce (2.4.1), we shall show the following convergence:

$$\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\frac{1}{2}\Big(\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} + \big\langle \tilde{r}_x^2 \big\rangle_\rho\Big) \to \int_0^1 g(y)\mathrm{f}^\mu_\beta(y), \tag{2.4.4}$$

almost surely. Before proceeding, we define the function $\mathrm{f}^\mu_\beta(y)$ as follows: denote $\tilde{e}_{[ny]} := \frac{1}{2}(\frac{\tilde{p}^2_{[ny]}}{m_x} + \tilde{r}^2_{[ny]})$ and define:

$$\mathrm{f}^\mu_\beta(y) := \lim_{n\to\infty} \mathbb{E}\Big(\big\langle \tilde{e}_{[ny]} \big\rangle_\rho\Big). \tag{2.4.5}$$

We prove the existence of this limit in the Appendix 2.A. Moreover, we show that f^μ_β is continuous.

In pursuance of establishing the limit (2.4.1), it is sufficient to prove a sufficient decay of the following covariance: $\mathbb{E}(\langle \tilde{e}_x \rangle_\rho \langle \tilde{e}_{x'} \rangle_\rho) - \mathbb{E}(\langle \tilde{e}_x \rangle_\rho)\mathbb{E}(\langle \tilde{e}_{x'} \rangle_\rho)$. The rest will be the proof of SLLN for weakly correlated random variables, where we will follow the line of [166]. Precisely, we express this decay in the following lemma: First, for every random variables X, Y, define $\mathrm{Cov}(X, Y) := \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$,

Lemma 2.4.1. *There exists $0 < c, C, \mathcal{C} < \infty$, independent of n, such that for every n we have $\forall\, x, x' \in \mathbb{I}_n$:*

$$|\mathrm{Cov}(\langle \tilde{e}_x \rangle_\rho, \langle \tilde{e}_y \rangle_\rho)| = \Big|\mathbb{E}\Big(\langle \tilde{e}_x \rangle_\rho \langle \tilde{e}_{x'} \rangle_\rho\Big) - \mathbb{E}\Big(\langle \tilde{e}_x \rangle_\rho\Big)\mathbb{E}\Big(\langle \tilde{e}_{x'} \rangle_\rho\Big)\Big| < C\exp(-c|x-x'|) + \frac{\mathcal{C}}{n}. \tag{2.4.6}$$

In order to proof (2.4.1), first, we rewrite $\langle \tilde{e}_x \rangle_\rho$ as:

$$\begin{aligned}\langle \tilde{e}_x \rangle_\rho =& \frac{1}{2}\Big(\big\langle \tilde{r}_x^2 \big\rangle_\rho + \frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x}\Big) \\ =& \frac{1}{2\beta_x}\left(\left\langle x, \frac{\sqrt{A^\beta_r}}{2}\coth\left(\frac{\sqrt{A^\beta_r}}{2}\right)x\right\rangle_{n-1} + \left\langle x, \frac{(A^\beta_p)^{\frac{1}{2}}}{2}\coth\left(\frac{(A^\beta_p)^{\frac{1}{2}}}{2}\right)x\right\rangle_n\right),\end{aligned} \tag{2.4.7}$$

thanks to (2.2.46). Then, we use the analyticity of the function $f(y) = \sqrt{y}\coth\Big(\sqrt{y}\Big)$, and expand the matrix $\frac{\sqrt{A^\beta_p}}{2}\coth\Big(\frac{\sqrt{A^\beta_p}}{2}\Big)$, in terms of its Taylor series around an appropriate point. By using the fact that A^β_p is tridiagonal, and mass terms appear locally in this matrix, we observe that first $|x-x'|$ terms in $\mathbb{E}(\langle e_x \rangle_\rho \langle e'_x \rangle_\rho)$, can be factorized, and the rest of the expansion is exponentially small. This proves (2.4.6). In the rest of this section, first, we make this argument rigorous, and then, we prove the SLLN, by using the results form [166].

In the succeeding lemma we observe that $((A^\beta_p)^k)_{xx} = \langle x, (A^\beta_p)^k x\rangle$, only depends on the masses in the interval $[x - [\frac{k}{2}], x + [\frac{k}{2}]] \cap \mathbb{N}$.

Lemma 2.4.2. *Recall the definition of the matrix* $A_p^\beta = M_\beta^{-\frac{1}{2}}(-\nabla_-\beta^0\nabla_+)M_\beta^{-\frac{1}{2}}$, *where* $M_\beta = M\beta^{-1}$, $M = \mathrm{diag}(m_1,\dots,m_n)$, *and* $\{m_x\}$ *are i.i.d random variables with smooth (w.r.t Lebesgue) density,* $d\mu = \mu(x)dx$. *Moreover, recall the notation of the canonical basis of* $\mathbb{R}^n$ *i.e.* $|x\rangle$ *for* $x \in \mathbb{I}_n$. *Take* $x \in \mathbb{I}_n$, *then for any* $k \in \mathbb{I}_n$, *denote the* x*th diagonal element of* $(A_p^\beta)^k$ *by* $\vartheta^k(x) := (A_\beta^p)^k_{xx} = \langle x, (A_p^\beta)^k x\rangle$, *as a function of masses* $m_1,\dots m_n$. *Then,* $\vartheta^k(x)$ *only depends on* m_i, *for* $i \in I(x,k)$, *where* $I(x,k)$ *is defined as follows:*

$$I(x,k) = \Big[\min\{x - [\frac{k}{2}], 1\}, \max\{x + [\frac{k}{2}], n\}\Big] \cap \mathbb{N}. \tag{2.4.8}$$

In other words, for a fixed realization of the masses, the function $\tilde{\vartheta}(y) := \frac{\partial\vartheta^k(x)}{\partial m_y}$ *is supported on* $I(x,k)$.

Before proving lemma 2.4.2, we deduce the following corollary:

Corollary 2.4.2.1. *For* $x, y \in \mathbb{I}_n$, *and* $k + k' < 2|x-y|$, *we have:*

$$\mathbb{E}\Big(\langle x, (A_p^\beta)^k x\rangle\langle y, (A_p^\beta)^{k'} y\rangle\Big) = \mathbb{E}\Big(\langle x, (A_p^\beta)^k x\rangle\Big)\mathbb{E}\Big(\langle y, (A_p^\beta)^{k'} y\rangle\Big). \tag{2.4.9}$$

Proof. We deduce this corollary directly from (2.4.8), since the assumption $k+k' < 2|x-y|$, implies that $I(x,k)\cap I(y,k') = \emptyset$. Therefore, $\langle x, (A_p^\beta)^k x\rangle$, and $\langle y, (A_p^\beta)^{k'} y\rangle$, are functions of two disjoint set of random variables, and we get the result (2.4.9). □

Before proceeding, we state the proof of Lemma 2.4.2:

Proof of Lemma 2.4.2. Recall the definition of $A_p^\beta = M_\beta^{-\frac{1}{2}}(-\nabla_-\beta^0\nabla_+)M_\beta^{-\frac{1}{2}}$, and observe that it's a symmetric matrix, which can be expressed in the following explicit way (Here $\beta_{n+1} = 0$):

$$\begin{aligned}(A_p^\beta)_{11} = \frac{\beta_1^2}{m_1}, \qquad (A_p^\beta)_{xx} = \frac{\beta_x(\beta_x+\beta_{x+1})}{m_x}, \quad 1 < x \le n,\\ (A_p^\beta)_{x(x+1)} = -\frac{\beta_x\sqrt{\beta_x\beta_{x+1}}}{\sqrt{m_x m_{x+1}}}, \qquad 1 \le x \le n-1.\end{aligned} \tag{2.4.10}$$

Now, consider the expression $\langle x, (A_p^\beta)^k x\rangle$, we rewrite this expression by multiplying the identity matrix for $k-1$ times. Denote these $k-1$ matrices by $\sum_{x_j=1}^n |x_j\rangle\langle x_j|$, for $j = 1,\dots,k-1$:

$$\langle x, (A_p^\beta)^k x\rangle = \sum_{x_1,\dots x_{k-1}=1}^{n} \langle x, A_p^\beta x_1\rangle\langle x_1, A_p^\beta x_2\rangle \dots \langle x_{k-1}, A_p^\beta x\rangle. \tag{2.4.11}$$

Since A_p^β is tridiagonal, each term of the form $\langle x_j, A_p^\beta x_{j+1}\rangle$ is zero, unless $x_j = x_{j+1}$, or $x_j = x_{j+1} \pm 1$. Hence, each non-zero term in the sum (2.4.11), is corresponding to a sequence $(x, x_1, \dots, x_{k-1}, x)$. This sequence can be interpreted as a discrete random walk path from point x, at time zero, to the same point x, at time k, where at each time-step, one can choose to go right i.e., $z \to z+1$, or left, i.e., $z \to z-1$,

or stay at the same position, i.e., $z \to z$. However, since the path should return to point x, it is supported on the interval $I(x,k)$. Furthermore, using the explicit form of A_p^β from (2.4.10), each term $\langle x_j, A_p^\beta x_{j'}\rangle$, depends on $\frac{1}{\sqrt{m_j m_{j'}}}$, for $j = j'$, or $j = j' \pm 1$. Hence, for each path, the corresponding contribution to (2.4.11) only depends on the masses of the points where the path is crossing. Therefore, $\langle x, (A_p^\beta)^k x\rangle$ only depends on the following set of masses: $\{m_i | i \in I(x,k)\}$. □

Note that the same line of reasoning can be done for A_r^β. Since A_r^β is symmetric, and can be expressed as follows:

$$\begin{aligned}(A_r^\beta)_{xx} &= \beta_x\Big(\frac{\beta_x}{m_x} + \frac{\beta x+1}{m_{x+1}}\Big),\\ (A_r^\beta)_{xx+1} &= \frac{\beta_{x+1}}{m_{x+1}}\sqrt{\beta_x\beta_{x+1}}.\end{aligned} \tag{2.4.12}$$

By using the similar argument as in the previous lemma, we deduce that $\forall x \in \mathbb{I}_{n-1}$, and $k < n-1$, $\langle x, (A_r^\beta)^k x\rangle$, only depends on the masses m_i, for $i \in \tilde{I}(x,k)$, where we have:

$$\tilde{I}(x,k) = \Big[\min\{1, x - [\frac{k}{2}]\}, \max\{n, x + [\frac{k}{2}] + 1\}\Big]. \tag{2.4.13}$$

Similar to the corollary 2.4.2.1, with the exact same argument we deduce the following:

Corollary 2.4.2.2. *For $x, y \in \mathbb{I}_n$, and $k + k' < 2|x-y| - 2$, we have:*

$$\begin{aligned}\mathrm{Cov}\Big(\langle x, (A_r^\beta)^k x\rangle, \langle y, (A_r^\beta)^{k'} y\rangle\Big) = 0,\\ \mathrm{Cov}\Big(\langle x, (A_p^\beta)^k x\rangle, \langle y, (A_r^\beta)^{k'} y\rangle\Big) = 0.\end{aligned} \tag{2.4.14}$$

Remark 2.4.3. *One can observe that in* (2.4.14)*, and* (2.4.9)*, we can substitute A_p^β and A_r^β, respectively with $A_p^\beta - cI_n$ and $A_r^\beta - cI_{n-1}$, for some constant c. This is straightforward, since adding a constant to the diagonal elements does not change the support of $\vartheta(y)$ and $\tilde{\vartheta}(y)$. Therefore, we can repeat the same argument with these new set of matrices, and obtain for any $x, y \in \mathbb{I}_n$, and $k + k' < |x-y| + 1$:*

$$\begin{aligned}\mathrm{Cov}\Big(\langle x, (A_p^\beta - cI_n)^k x\rangle, \langle y, (A_r^\beta - cI_{n-1})^{k'} y\rangle\Big) = 0,\\ \mathrm{Cov}\Big(\langle x, (A_p^\beta - cI_n)^k x\rangle, \langle y, (A_p^\beta - cI_n)^{k'} y\rangle\Big) = 0,\\ \mathrm{Cov}\Big(\langle x, (A_r^\beta - cI_{n-1})^k x\rangle, \langle y, (A_r^\beta - cI_{n-1})^{k'} y\rangle\Big) = 0.\end{aligned} \tag{2.4.15}$$

In order to proof lemma 2.4.1, define the function $\mathfrak{f} : \mathbb{C} \to \mathbb{C}$ as follows:

$$\mathfrak{f}(z) = \begin{cases} z^{\frac{1}{2}} \coth\Big(z^{\frac{1}{2}}\Big), & z \neq 0,\\ 1, & z = 0.\end{cases} \tag{2.4.16}$$

One can easily observe that the poles of the function $z^{\frac{1}{2}} \coth\Big(z^{\frac{1}{2}}\Big)$ is the following set: $\{z \in \mathbb{C} | z = -k^2\pi^2, k \in \mathbb{Z}\}$, and this function is analytic on the rest of the

complex plane. However, the point zero is a removable pole, and by redefining the function at zero, we can remove this pole: It is well known that the function $\coth(z)$ has the following Taylor series expression for $0 < |z| < \pi$: $\coth(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^{2n-1}$, where $a_n = \frac{2^{2n} B_{2n}}{(2n)!}$, and B_{2n} are Bernoulli numbers. Hence, we have $z\coth(z) = 1 + \sum_{n=1}^{\infty} a_n z^{2n}$, and $z^{\frac{1}{2}}\coth\Big(z^{\frac{1}{2}}\Big)$ is given by the following Taylor series: $1 + \sum_{n=1}^{\infty} a_n z^n$ for $0 < |z| < \pi^2$. Hence, the pole of $\mathfrak{f}$ is given by the set $\{z \in \mathbb{C} | z = -k^2\pi^2, k \in \mathbb{N}, k > 0\}$.

Finally, we can state the proof of Lemma 2.4.1:

Proof of Lemma 2.4.1. First, from Section 2.2.3, recall that there is a constant $c_0 > 0$, uniform in n[10], such that for any configuration of the masses, we have $||A_p^\beta||_2, ||A_r^\beta||_2 \leq c_0$. Define $\alpha := \frac{1}{2}(c_0 + 1)$, let $\mathcal{R} := \alpha + \pi^2$, by the above argument $\mathfrak{f}(z)$ is analytic in the open disk $|z - \alpha| < \mathcal{R}$, and $\mathcal{R}$ is the radius of convergence for the Taylor expansion of $\mathfrak{f}$, $\mathfrak{f}(z) = \sum_{k=0}^{\infty} a_k (z - \alpha)^k$. Moreover, by the choice of α and c_0, one can easily observe that all the eigenvalues of A_p^β and A_r^β lies in the disk $|z - \alpha| < \mathcal{R}$. Explicitly, $\forall k \in \mathbb{I}_{n-1}$, we have $|\gamma_k^2 - \alpha| < \mathcal{R}$. Hence, we can write the following Taylor expansions for $\mathfrak{f}(A_p^\beta)$ and $\mathfrak{f}(A_r^\beta)$ (For the proof of this fact one can see Theorem 4.7 of [128]).

$$\mathfrak{f}(A_p^\beta) = \sum_{k=0}^{\infty} a_k (A_p^\beta - \alpha I_n)^k, \qquad \mathfrak{f}(A_r^\beta) = \sum_{k=0}^{\infty} a_k (A_r^\beta - \alpha I_{n-1})^k. \tag{2.4.17}$$

Comparing the definition of $\mathfrak{f}$ in (2.4.16), where $\mathfrak{f}(0) = 1$, with the expression (2.2.52), where we had $0\coth(0) = 0$ by convention, we deduce the following expression for $\langle \tilde{e}_x \rangle_\rho$:

$$\begin{aligned} \langle \tilde{e}_x \rangle_\rho &= \frac{1}{\beta_x}\Big(\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1} + \langle x, \mathfrak{f}(A_p^\beta)x\rangle_n - (\psi_x^0)^2\Big) \\ &= \frac{1}{\beta_x}\Big(\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1} + \langle x, \mathfrak{f}(A_p^\beta)x\rangle_n - \frac{\frac{m_x}{\beta_x}}{\sum_{x=1}^n \frac{m_x}{\beta_x}}\Big), \end{aligned} \tag{2.4.18}$$

where we used the equality $\psi^0 = (\sum_{x=1}^n \frac{m_x}{\beta_x})^{-\frac{1}{2}} M_\beta^{\frac{1}{2}} |\mathbf{1}\rangle$. We denote $\frac{\frac{m_x}{\beta_x}}{\sum_{x=1}^n \frac{m_x}{\beta_x}}$ by ϵ_n^x. Notice that $|\epsilon_n^x|$ is bounded by $\frac{\mathcal{C}_0}{n}$, where $\mathcal{C}_0$ is independent from n [11], thanks to the assumptions on the distribution of the masses, and temperature profile $\beta(y)$. Moreover, since $||A_p^\beta||_2$ and $||A_r^\beta||_2$ are bounded by c_0, uniformly in n for any realization of the masses. By using the fact that $\mathfrak{f}$ is continuous and increasing in the interval $[0, c_0]$, $||\mathfrak{f}(A_r^\beta)||_2$ and $||\mathfrak{f}(A_p^\beta)||_2$ are bounded by a constant c_1 [12], independent of n, for any realization of the masses. Therefore, $|\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1} + \langle x, \mathfrak{f}(A_p^\beta)x\rangle_n|$ is bounded by $2c_1$. Taking advantage of the aforementioned bounds, we can deduce

[10] This constant can be taken equal to $4\frac{\beta_{max}^2}{m_{min}}$.

[11] Precisely, one can choose $\mathcal{C}_0$ to be equal to $\frac{m_{max}\beta_{max}}{m_{min}\beta_{min}}$.

[12] One can choose $c_1 = \mathfrak{f}(c_0)$.

the following inequality:

$$\begin{aligned}
&\Big|\mathrm{Cov}(\langle\tilde{e}_x\rangle_\rho\,,\langle\tilde{e}_{x'}\rangle_\rho)\Big| = \\
&\frac{1}{\beta_x\beta_{x'}}\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n-\epsilon_x^n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n-\epsilon_{x'}^n)\Big)\Big| \\
&\leq\frac{1}{\beta_{min}^2}\Big[\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big|+ \\
&\Big|\mathrm{Cov}\Big(\epsilon_x^n,(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n\Big)\Big|+ \\
&\Big|\mathrm{Cov}\Big(\epsilon_{x'}^n,(\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n\Big)\Big|+\Big|\epsilon_{x,x'}^n\Big|\Big],
\end{aligned}\tag{2.4.19}$$

where we denote $\epsilon_{x,x'}^n := \mathrm{Cov}(\epsilon_x^n,\epsilon_{x'}^n)$. Since $|\epsilon_x^n|,|\epsilon_{x'}^n|\leq\frac{\mathcal{C}_0}{n}$, and $|\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n|$, $|\langle x,\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n|\leq 2c_1$, $|\epsilon_{x,x'}^n|$ is bounded by $\frac{1}{\beta_{min}^2}8c_1\frac{\mathcal{C}_0}{n}+o(\frac{1}{n^2})$. Therefore, there exists $\mathcal{C}$ independent of n, such that

$$\begin{aligned}
&\Big|\mathrm{Cov}(\langle\tilde{e}_x\rangle_\rho\,,\langle\tilde{e}_{x'}\rangle_\rho)\Big|\leq \\
&\frac{1}{\beta_{min}^2}\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big|+\frac{\mathcal{C}}{n}.
\end{aligned}\tag{2.4.20}$$

First term can be written as the sum of the following terms:

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big| = \\
&\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)+\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}),(\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)+ \\
&\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)+\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big|\leq \\
&\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)\Big|+\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_r^\beta)x\rangle_{n-1}),(\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big| \\
&+\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)\Big|+\Big|\mathrm{Cov}\Big((\langle x,\mathfrak{f}(A_p^\beta)x\rangle_n),(\langle x',\mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big|.
\end{aligned}\tag{2.4.21}$$

In order to complete the proof, we observe that each of the terms in (2.4.21) are exponentially small. We show this fact for one of these terms, and the rest can be treated exactly in the same way. We do this task, using the Taylor series (2.4.17). We divide the series into two parts: The first $|x-x'|$ terms, and the rest, which is exponentially small. Let us define $\mathfrak{f}_\prec$ and $\mathfrak{f}_\succ$, as follows [13]:

$$\begin{aligned}
\mathfrak{f}_\prec(A_r^\beta) &:= \sum_{k=0}^{|x-x'|-1} a_k(A_r^\beta-\alpha I_{n-1})^k, \qquad \mathfrak{f}_\prec(A_p^\beta) := \sum_{k=0}^{|x-x'|-1} a_k(A_p^\beta-\alpha I_n)^k, \\
\mathfrak{f}_\succ(A_r^\beta) &:= \sum_{k>|x-x'|-1} a_k(A_r^\beta-\alpha I_{n-1})^k, \qquad \mathfrak{f}_\succ(A_p^\beta) := \sum_{k>|x-x'|-1} a_k(A_p^\beta-\alpha I_n)^k.
\end{aligned}\tag{2.4.22}$$

[13] Note that these definition depends on the n,x,x'.

Notice that $\mathfrak{f}(A_r^\beta) = \mathfrak{f}_\prec(A_r^\beta)+\mathfrak{f}_\succ(A_r^\beta)$, and $\mathfrak{f}(A_p^\beta) = \mathfrak{f}_\prec(A_p^\beta)+\mathfrak{f}_\succ(A_p^\beta)$, we substitute $\mathfrak{f}(A_r^\beta)$ in the first term of (2.4.21) with this expression (all the inner products should be understood in $\mathbb{R}^{n-1}$ in this expression, and we drop the subscript $n-1$ here):

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)\Big| = \\
&\Big|\mathrm{Cov}\Big((\langle x, (\mathfrak{f}_\prec+\mathfrak{f}_\succ)(A_r^\beta)x\rangle_{n-1}), (\langle x', (\mathfrak{f}_\prec+\mathfrak{f}_\succ)(A_r^\beta)x'\rangle_{n-1})\Big)\Big| = \\
&\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}_\prec(A_r^\beta)x\rangle), (\langle x', \mathfrak{f}_\prec(A_r^\beta)x'\rangle)\Big) + \mathrm{Cov}\Big((\langle x, \mathfrak{f}_\succ(A_r^\beta)x\rangle), (\langle x', \mathfrak{f}_\prec(A_r^\beta)x'\rangle)\Big)+ \\
&\mathrm{Cov}\Big((\langle x, \mathfrak{f}_\prec(A_r^\beta)x\rangle), (\langle x', \mathfrak{f}_\succ(A_r^\beta)x'\rangle)\Big) + \mathrm{Cov}\Big((\langle x, \mathfrak{f}_\prec(A_r^\beta)x\rangle), (\langle x', \mathfrak{f}_\prec(A_r^\beta)x'\rangle)\Big)\Big|.
\end{aligned} \tag{2.4.23}$$

The first term in (2.4.23) is equal to zero, thanks to the third equality in (2.4.15) in Remark 2.4.3:

$$\begin{aligned}
&\mathrm{Cov}\Big((\langle x, \mathfrak{f}_\prec(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}_\prec(A_r^\beta)x'\rangle_{n-1})\Big) = \\
&\sum_{k,k'<|x-x'|-1} a_k a_{k'} \mathrm{Cov}\Big((\langle x, (A_r^\beta - \alpha I_{n-1})^k x\rangle), (\langle x', (A_r^\beta - \alpha I_{n-1})^{k'} x'\rangle)\Big) = 0,
\end{aligned} \tag{2.4.24}$$

where each term in the sum is equal to zero, thanks to Remark 2.4.3.

We take care of the remaining terms, by using properties of Taylor series. First, observe that by the choice of α, we have $||A_r^\beta - \alpha I_{n-1}||_2 \leq \alpha$. Moreover, $\alpha + 1 < \mathcal{R}$ so the series $\sum_{k=0}^\infty a_k(\alpha+1)^k$ is convergent, and there exist a constant $M > 0$ (independent of x, x' and n), such that $\forall k$, $|a_k(\alpha+1)^k| \leq M$, and by denoting $\varrho = \frac{\alpha}{\alpha+1}$, we get:

$$\begin{aligned}
&||\mathfrak{f}_\succ(A_r^\beta)||_2 = ||\sum_{k>|x-x'|-1} a_k (A_r^\beta - \alpha I_{n-1})^k||_2 \leq \sum_{k>|x-x'|-1} |a_k| ||A_r^\beta - \alpha I_{n-1}||_2^k \\
&\leq \sum_{k>|x-x'|-1} |a_k|\alpha^k = \sum_{k>|x-x'|-1} |a_k|(\alpha+1)^k \varrho^k \leq M \sum_{k>|x-x'|-1} \varrho^k \leq \mathcal{C}_1 \varrho^{(|x-x'|)}.
\end{aligned} \tag{2.4.25}$$

Considering the fact that $||\mathfrak{f}(A_r^\beta))||_2 \leq c_1$, and $\mathfrak{f}(A_r^\beta) = \mathfrak{f}_\prec(A_r^\beta) + \mathfrak{f}_\succ(A_r^\beta)$, as a direct consequence of (2.4.25), we have: $||\mathfrak{f}_\prec(A_r^\beta)||_2 \leq c_2$, where c_2 is a constant uniform in n. Therefore, we have:

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}_\succ(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}_\prec(A_r^\beta)x'\rangle_{n-1}))+ \\
&\mathrm{Cov}((\langle x, \mathfrak{f}_\prec(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}_\succ(A_r^\beta)x'\rangle_{n-1})\Big)\Big| \leq 4||\mathfrak{f}_\succ(A_r^\beta)||_2||\mathfrak{f}_\prec(A_r^\beta)||_2 \leq 4c_2\mathcal{C}_1\varrho^{|x-x'|}, \\
&\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}_\succ(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}_\succ(A_r^\beta)x'\rangle_{n-1})\Big)\Big| \leq 2||\mathfrak{f}_\succ(A_r^\beta)||_2||\mathfrak{f}_\succ(A_r^\beta)||_2 \leq 2\mathcal{C}_1^2\varrho^{2|x-x'|}.
\end{aligned} \tag{2.4.26}$$

Notice that all the bounds here are deterministic and independent of the realization of the masses. Combining (2.4.23), (2.4.24) and (2.4.26), there exists a deterministic constant independent of n and realization of the masses, $\mathcal{C}_2$, such that we have:

$$\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1}), (\langle x', \mathfrak{f}(A_r^\beta)x'\rangle_{n-1})\Big)\Big| \leq \mathcal{C}_2\varrho^{|x-x'|}. \tag{2.4.27}$$

Recall (2.4.21)-the other terms can be treated exactly similar to (2.4.27)- $\mathfrak{f}$ can be divided as in (2.4.22). Then, we obtain the same expression as in (2.4.23), where the first term is equal to zero, thanks to the second and third equality in (2.4.15), and the remainder can be bounded with the exact same bound. Hence, there exist a constant $C > 0$ uniform in n, such that:

$$\frac{1}{\beta_{min}^2}\Big|\mathrm{Cov}\Big((\langle x, \mathfrak{f}(A_r^\beta)x\rangle_{n-1}+\langle x, \mathfrak{f}(A_p^\beta)x\rangle_n), (\langle x', \mathfrak{f}(A_r^\beta)x'\rangle_{n-1}+\langle x', \mathfrak{f}(A_p^\beta)x'\rangle_n)\Big)\Big| \leq C\varrho^{|x-x'|}. \tag{2.4.28}$$

Comparing (2.4.28) and (2.4.20), and recalling the fact that $0 < \varrho = \frac{\alpha}{\alpha+1} < 1$, where α is independent of n, we get the constants $0 < c, \mathcal{C}, C < \infty$, independent of n, such that:

$$|\mathrm{Cov}(\langle \tilde{e}_x\rangle_\rho, \langle \tilde{e}_{x'}\rangle_\rho)| \leq C\exp(-c|x-x'|) + \frac{\mathcal{C}}{n}. \tag{2.4.29}$$

□

Thanks to the the exponential decay of covariances (2.4.6), we have the SLLN (2.4.1). For proving the SLLN from this decay, we follow the lines of [166]:

Theorem 2.4.4. *Recall the definition of density state ρ^n from (2.1.9), and the definition of the ensemble average with respect to this state by $\langle . \rangle_{\rho^n}$ from (2.1.12). Let $g \in C^0([0,1])$, be a test function. Then we have the following convergence, almost surely with respect to the distribution of the masses.*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\langle \tilde{e}_x\rangle_{\rho^n} \to \int_0^1 g(y)\mathrm{f}_\mu^\beta(y)dy, \tag{2.4.30}$$

where $\mathrm{f}_\mu^\beta(y)$ is defined in (2.4.5).

Proof. Define the random variable $Y_x^n := g(\frac{x}{n})\Big(\langle \tilde{e}_x\rangle_{\rho^n} - \mathbb{E}(\langle \tilde{e}_x\rangle_{\rho^n})\Big)$. First, notice that $\mathbb{E}(Y_x^n) = 0$, and by the definition of g, and the fact that $\langle e_x\rangle_{\rho^n}$ is bounded (we established this fact in Lemma 2.2.3), $|Y_x^n|$ is uniformly bounded by a constant C_0. Moreover, thanks to the Lemma 2.4.1, and inequality (2.4.6), and the fact that g is bounded we have:

$$\mathbb{E}(Y_x^n Y_y^n)| \leq C_1\exp(-c|x-y|) + \frac{\mathcal{C}}{n}, \tag{2.4.31}$$

for some constant $C_1, c, \mathcal{C}$, uniform in n. Let $S_n = \frac{1}{n}\sum_{x=1}^n Y_x^n$, by using (2.4.31), we have $\mathbb{E}(S_n^2) \leq \frac{\mathcal{C}_2}{n}$. Hence, $\forall \epsilon > 0$, $\sum_{n=1}^\infty \frac{1}{n}\frac{\mathbb{E}(S_n^2)}{\epsilon^2} < \infty$. On the other hand, by Cauchy condensation lemma, we know if $\sum_{n=1}^\infty \frac{b_n}{n} < \infty$, with $b_n \geq 0$, then there exists a sequence n_k of integers, such that $\sum_{k=1}^\infty b_{n_k} < \infty$, and $\lim_{k\to\infty}\frac{n_{k+1}}{n_k} = 1$, (for the proof of this fact one can see Lemma 3 in [90]). Therefore, there exists a subsequence S_{n_k}, such that $\lim_{k\to\infty}\frac{n_{k+1}}{n_k} \to 1$, and

$$\forall \epsilon > 0, \quad \sum_{k=1}^\infty \frac{\mathbb{E}(S_{n_k}^2)}{\epsilon^2} < \infty.$$

Hence, by Borel-Contelli lemma, since $\forall \epsilon > 0$, $\sum_{k=1}^{\infty} \mathbb{P}(|S_{n_k}| > \epsilon) \leq \sum_{k=1}^{\infty} \frac{\mathbb{E}(S_{n_k}^2)}{\epsilon^2}$, we have $\lim_{k\to\infty} S_{n_k} \to 0$, almost surely. Now take n such that $n_k \leq n < n_{k+1}$; then, by using the fact that $\forall n$, $\forall x \in \mathbb{I}_n$, $|Y_x| \leq C_0$, we have:

$$|S_n - S_{n_k}| \leq \frac{1}{n_k} \sum_{x=1}^{n_k} |Y_x^n - Y_x^{n_k}| + \frac{1}{n_k} \sum_{x>n_k}^{n} |Y_x^n| \leq \frac{1}{n_k} \sum_{x=1}^{n_k} |Y_x^n - Y_x^{n_k}| + C_0 \frac{n - n_k}{n_k}. \tag{2.4.32}$$

Since $\frac{n_{k+1}}{n_k} \to 1$, for any $\varepsilon > 0$, there exits N_*, such that if $n_k > N_*$, then for the second term in (2.4.32), we have: $C_0 \frac{n-n_k}{n_k} < \frac{\varepsilon}{4}$. Moreover, by the Lemma 2.A.2 in Appendix 2.A, for n_k sufficiently large, we have $\forall x \in \mathbb{I}_{n_k}$, with $n_* < x < n - n_*$: $|Y_x^n - Y_x^{n_k}| \leq \frac{\varepsilon}{2}$, where n_* is independent of n_k. Moreover, the terms corresponding to $1 \leq x \leq n_*$ can be bounded by $\frac{\varepsilon}{4}$, since n_* does not depend on n_k, and Y_x^n is uniformly bounded. Hence, the first term in (2.4.32) is bounded by $\frac{3\varepsilon}{4}$, for n_k sufficiently large. Therefore, for every $\varepsilon > 0$, there exist N, such that for $n_k > N$, $|S_n - S_{n_k}| < \varepsilon$ [14]. Since $\lim_{k\to\infty} S_{n_k} \to 0$, almost surely, we deduce that $S_n \to 0$ almost surely. Hence,

$$\frac{1}{n} \sum_{x=1}^{n} g(\frac{x}{n}) \Big(\langle \tilde{e}_x \rangle_{\rho^n} - \mathbb{E}(\langle \tilde{e}_x \rangle_{\rho^n}) \Big) \to 0, \tag{2.4.33}$$

almost surely. This gives us the result (2.4.30), thanks to the definition of $\mathfrak{f}_\mu^\beta$ in (2.4.5), Corollary 2.A.1.1, and dominated convergence theorem.

□

2.4.2 Thermal equilibrium

Recall the definition (2.1.9) of density operator $\rho^n_{\beta,\bar{p},\bar{r}}$ corresponding to our locally Gibbs state. We denote the density operator corresponding to thermal equilibrium at inverse temperature $\beta_{eq} \in (0, \infty)$, by $\rho^{n,\beta_{eq}}_{\bar{p},\bar{r}}$. Recall that by thermal equilibrium, we refer to the case where the temperature profile $\beta(.)$ is constant, i.e., $\forall y \in [0,1]$, $\beta(y) = \beta_{eq}$. In this case, the matrices A_p^β, and A_r^β have the following form:

$$A_p^{\beta_{eq}} = (\beta_{eq})^2 M^{-\frac{1}{2}}(-\Delta) M^{-\frac{1}{2}} = \beta_{eq}^2 A_p^0, \qquad A_r^{\beta_{eq}} = (\beta_{eq})^2 \nabla_+ M^{-1} \nabla_- = \beta_{eq}^2 A_r^0.$$

Therefore, in thermal equilibrium, the average of the fluctuation part of the kinetic and potential energy ($\langle \tilde{r}_x^2 \rangle_\rho$ and $\frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x}$) is given as follows:

$$\Big\langle \tilde{r}_x^2 \Big\rangle_\rho = \left\langle x, \frac{(A_r^0)^{\frac{1}{2}}}{2} \coth\left(\frac{\beta_{eq}(A_r^0)^{\frac{1}{2}}}{2} \right) x \right\rangle_{n-1}, \quad \frac{\langle \tilde{p}_x^2 \rangle_\rho}{m_x} = \left\langle x, \frac{(A_p^0)^{\frac{1}{2}}}{2} \coth\left(\frac{\beta_{eq}(A_p^0)^{\frac{1}{2}}}{2} \right) x \right\rangle_n. \tag{2.4.34}$$

In the finite system, one can see from these expressions (by analyzing their Taylor expansions) that $\mathbb{E}(\Big\langle \tilde{e}_{[ny]} \Big\rangle) \neq \mathbb{E}(\Big\langle \tilde{e}_{[ny']} \Big\rangle)$, for $y, y' \in (0,1)$. However, if we take

[14] This bound is true for every configuration of masses.

the limit as $n \to \infty$, thanks to Corollary 2.A.1.2 in Appendix 2.A, we recover the space homogeneity in the bulk, i.e., denoting the $\lim_{n\to\infty} \mathbb{E}(\langle \tilde{e}_{[ny]} \rangle_{\rho^{n,\beta_{eq}}})$ in thermal equilibrium by $\mathrm{f}^{\mu}_{\beta_{eq}}$ we have:

$$\forall y, y' \in (0,1), \quad \mathrm{f}^{\mu}_{\beta_{eq}}(y) = \mathrm{f}^{\mu}_{\beta_{eq}}(y').$$

Take $y \in (0,1)$, since $\mathrm{f}^{\mu}_{\beta_{eq}}(y)$ is independent of y, we define the function $\mathrm{f}^{\mu}(\beta_{eq})$ to be the thermal equilibrium average at inverse temperature $\beta_{eq} \in (0,\infty)$:

$$\mathrm{f}^{\mu}(\beta_{eq}) := \mathrm{f}^{\mu}_{\beta_{eq}}(y) = \lim_{n\to\infty} \mathbb{E}\Big(\langle \tilde{e}_{[ny]} \rangle_{\rho^{n,\beta_{eq}}} \Big). \tag{2.4.35}$$

Thanks to Proposition 2.A.1.2, out of thermal equilibrium with a proper $\beta \in C^0([0,1])$ satisfying the assumption of definition (2.1.9), we can express the function $\mathrm{f}^{\mu}_{\beta(.)}$ (2.4.5), in terms of $\mathrm{f}^{\mu}(\beta)$ as follows:

$$\forall y \in (0,1), \quad \mathrm{f}^{\mu}_{\beta(.)}(y) = \mathrm{f}^{\mu}(\beta(y)), \tag{2.4.36}$$

where in the second expression, the equilibrium average is computed at inverse temperature $\beta_{eq} = \beta(y)$.

2.5 Energy Evolution

In this section, we finish the proof of Theorem 2.1.1 by proving (2.1.18). The idea is as follows: We can decompose the energy into the mechanical and thermal parts in both microscopic ($\langle e_x(nt) \rangle_{\rho}$) and macroscopic ($\mathrm{e}(y,t)$) scale. The contribution of the mechanical part in the left hand side (LHS) of (2.1.18) converges to the mechanical part in the right hand side, thanks to (2.3.13) and (2.3.29). The contribution of the thermal energy in the LHS of (2.1.18) converges to the thermal part of the RHS, thanks to (2.4.1), at time zero. Finally, the thermal energy in the LHS of (2.1.18) remains constant in the limit as $n \to \infty$, thanks to the localization phenomena, similar to the RHS of (2.1.18), where the contribution of the thermal part is given by a function constant in time. This constant function can be obtained by solving (2.1.13), and finding the explicit solution for $\mathrm{e}(y,t)$. We make this heuristic rigorous in this section.

2.5.1 Mechanical energy

Before proceeding, we state the following lemma, in order to deal with the mechanical part.

Lemma 2.5.1. *For any test function $g \in C^0([0,1])$, by recalling the notation $\bar{p}_x(nt) = \langle p_x(nt) \rangle_{\rho^n}$, and $\bar{r}_x(nt) = \langle r_x(nt) \rangle_{\rho^n}$, we have:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} g(\frac{x}{n}) \Big(\frac{\langle p_x(nt) \rangle^2_{\rho^n}}{2m_x} + \frac{\langle r_x(nt) \rangle^2_{\rho^n}}{2} \Big) \to \int_0^1 g(y) \Big(\frac{\mathrm{p}(y,t)^2}{2\bar{m}} + \frac{\mathrm{r(y,t)}^2}{2} \Big) dy, \tag{2.5.1}$$

almost surely, w.r.t the distribution of the masses.

Proof. Denote the solution of the evolution equation (2.3.5), with initial datum $\bar{p}_x(0) = \bar{p}_x \frac{m_x}{\bar{m}}$, and $\bar{r}_x(0) = \bar{r}_x$ with $\bar{\pi}_x(nt)$ and $\bar{\varrho}_x(nt)$. Moreover, we denote the solution of (2.3.5) with initial datum $\pi_x(0) = -\mathcal{E}_n^x$, and $\varrho_x(0) = 0$ by $\pi_x(nt)$, and $\varrho_x(nt)$, respectively. Thanks to linearity of the evolution equation we have for any x:

$$\langle p_x(nt)\rangle_\rho = \bar{\pi}_x(nt) + \pi_x(nt), \qquad \langle r_x(nt)\rangle_\rho = \bar{\varrho}_x(nt) + \varrho_x(nt). \tag{2.5.2}$$

First, since $\bar{\varrho}_{[ny]}(nt) \to \mathrm{r}(y,t)$, a.s. by (2.3.29), we have $(\bar{\varrho}_{[ny]}(nt))^2 \to \mathrm{r}^2(y,t)$. By using the fact that $g(\frac{[ny]}{n}) \to g(y)$, we get:

$$\frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})(\bar{\varrho}_x(nt))^2 = \frac{1}{2}\int_0^1 g(\frac{[ny]}{n})(\bar{r}_{[ny]}(nt))^2 dy \to \int_0^1 g(y)\mathrm{r}^2(y,t)dy, \tag{2.5.3}$$

almost surely, by dominated convergence theorem. The fact that $\bar{\varrho}_{[ny]}(nt)^2$ is bounded and hence integrable is obvious from the conservation of the energy bounds in (2.3.7). The momentum part can be treated by (2.3.13) as follows:

$$\begin{aligned}
&\frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})\frac{(\bar{\pi}_x(nt))^2}{m_x} = \frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})\frac{(\bar{\pi}_x(nt))^2}{m_x^2}m_x = \frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})(\frac{\bar{\pi}_x(nt)}{m_x})^2\bar{m} + \\
&\frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\bar{\pi}_x(nt)}{m_x}\Big)^2(m_x - \bar{m}) = \bar{m}\int_0^1 g(\frac{[ny]}{n})\Big(\frac{\bar{\pi}_{[ny]}}{m_{[ny]}}\Big)^2 dy + \\
&\frac{1}{2n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\bar{\pi}_x(nt)}{m_x}\Big)^2(m_x - \bar{m}) \to \int_0^1 g(y)\frac{\mathrm{p}(y,t)^2}{2\bar{m}},
\end{aligned} \tag{2.5.4}$$

almost surely as $n \to \infty$, where the last sum converges to 0, almost surely, thanks to (2.3.13). The last integral in the second line converges to $\int_0^1 g(y)\frac{\mathrm{p}(y,t)^2}{2\bar{m}}$, using the convergence $\frac{\bar{\pi}_{[ny]}}{m_{[ny]}} \to \frac{\mathrm{p}(y,t)}{\bar{m}}$ a.s, in (2.3.29), and dominated convergence theorem. On the other hand, for $\pi_x(nt)$ we have:

$$\Big|\frac{1}{n}\sum_{x=1}^{n} g\Big(\frac{x}{n}\Big)\frac{\pi_x^2(nt)}{m_x}\Big| \le \frac{C}{n}\sum_{x=1}^{n}\frac{\pi_x^2(nt)}{m_x} \le \frac{C'}{n}\sum_{x=1}^{n}(\mathcal{E}_n^x)^2 \to 0, \tag{2.5.5}$$

where we used the conservation of the energy as well as the properties of $\mathcal{E}_n^x$ (2.2.40). We can argue similarly and obtain

$$\Big|\frac{1}{n}\sum_{x=1}^{n} g\Big(\frac{x}{n}\Big)\frac{\varrho^2(nt)}{m_x}\Big| \to 0. \tag{2.5.6}$$

almost surely, as $n \to \infty$. Using the decomposition (2.5.2), and combining (2.5.3) and (2.5.4), with (2.5.5) and (2.5.6), and taking advantage of a Cauchy-Schwartz inequality yield the result (2.5.1). □

One can think of $\frac{\langle p_x(nt)\rangle_{\rho^n}^2}{2m_x} + \frac{\langle r_x(nt)\rangle_{\rho^n}^2}{2}$ as the mechanical energy of the particle $x \in \mathbb{I}_n$, and $\frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2}$ as the mechanical energy of the macroscopic material coordinate $y \in \mathbb{I}$. This lemma is basically saying that the microscopic Mechanical energy converges to the macroscopic one, almost surely, in the sense of (2.5.1).

2.5.2 Thermal energy and localization

In this section, we provide the necessary tool in order to deal with the thermal energy, i.e., localization of the "high modes" of the chain, which enables us to close the equation in (2.1.18). We state the localization in the sense of the following lemma. This lemma is a consequence of the well known locaization phenomena in the disordered chain of harmonic ocillators (cf. [149], [6], [4],[222],[35]), and we bring it here directly from ([35] Lemma 3, Section 5) without a proof.

Lemma 2.5.2. *Recall the definition of the random matrix $A_p^0 = M^{-\frac{1}{2}}(-\Delta)M^{-\frac{1}{2}}$, from Section 2.2, where M is the diagonal matrix of the masses, and Δ is the matrix of discrete Laplacian* (2.1.8)*. Moreover, recall the ordered eigenvalues of A_p^0 : $0 = \omega_0 < \omega_1 \cdots < \omega_{n-1}$, and their corresponding eigenvectors: $\{\varphi^k\}_{k=0}^{n-1}$. Denote $\tilde{\varphi}^k := M^{-\frac{1}{2}}\varphi^k$.*
Fix $\alpha, \eta > 0$, such that $0 < 2\alpha < \eta < 1$. Recall the distribution of the masses $\mathbb{P}$, then there exists almost surely, $n_0 \in \mathbb{N}$ such that $\forall\, n > n_0$, and $\forall\, k \in I(\alpha) := (n^{(1-\alpha)}, n-1] \cap \mathbb{Z}$, there exists an interval $J(k) \subset [0, n]$ with $|J(k)| \leq 2n^\eta$, such that:

$$\forall x \notin J(k), \quad |\tilde{\varphi}_x^k| \leq n^{-\frac{1}{\eta}}. \tag{2.5.7}$$

Equivalently, $|\varphi_x^k| \leq n^{-\frac{1}{\eta}}\sqrt{m_{max}}$.

Exploiting this lemma, one can deal with the contribution of the momentum to the thermal energy. In order to deal with the contribution of the elongation, one needs to establish the localization of the eigenvectors of A_r^0. However, since the eigenvectors of A_r^0 i.e. ϕ^k are related to φ^k by the following identity $\phi^k = \frac{1}{\omega_k}\nabla_+ M^{-\frac{1}{2}}\varphi^k$, we do not establish the localization directly. Instead, we control the contribution of the elongation to the thermal energy, by using lemma 2.5.2 as well as the following lemma (Notice that the contribution of the elongation to the thermal energy has not been discussed throughly in [35]):

Lemma 2.5.3. *Recall the set-up of Lemma 2.5.2, for every $\tilde{\varphi}^k$ satisfying* (2.5.7)*, i.e. there exists an interval $J(k)$ with $|J(k)| \leq 2n^\eta$, such that $\forall\, x \notin J(k)$, $|\tilde{\varphi}_x^k| \leq n^{-\frac{1}{\eta}}$, there exists a constant c independent of n, with $\frac{1}{\omega_k} \leq cn^{\frac{3\eta}{2}}$. In particular, we have:*

$$\forall x \in \tilde{J}(k), \quad |\phi_x^k| = \left|\frac{1}{\omega_k}\nabla_+\tilde{\varphi}_x^k\right| \leq 2cn^{-\frac{1}{\eta}+\frac{3\eta}{2}}, \tag{2.5.8}$$

where $\tilde{J}(k)$ is the interval: $[\min\{J(k)\}+1, \max\{J(k)\}-1]$.

Proof. First, recall $\sum_{x=1}^n |\varphi_x^k|^2 = 1$, therefore $\sum_{x=1}^n |\tilde{\varphi}_x^k|^2 \geq \frac{1}{m_{max}}$. On the other hand, using the assumption (2.5.7), since $|J(k)^c| < n$, we have $\sum_{x\notin J(k)} |\tilde{\varphi}_x^k|^2 \leq n^{-\frac{2}{\eta}+1}$. Combining last two inequalities, we have

$$\sum_{x\in J(k)} |\tilde{\varphi}_x^k|^2 \geq \frac{1}{m_{max}} - \frac{1}{n^{\frac{2}{\eta}-1}} \geq \frac{c_0}{m_{max}},$$

where c_0 is a constant independent of n, and the last inequality is deduced from the fact that $0 < \eta < 1$, and m_{max} is bounded. Consequently, since $|J(k)| = 2n^\eta$, we can choose $x_0 \in J(k)$ such that:

$$\frac{c_1}{n^{\frac{\eta}{2}}} \leq |\tilde{\varphi}^k_{x_0}|, \tag{2.5.9}$$

where c_1 is a constant independent of n. Now, choose $x_1 \notin J(k)$ to be the closest member of $J(k)^c$ to x_0. Using the assumption $|\tilde{\varphi}^k_x| \leq n^{-\frac{1}{\eta}}$, and the inequality (2.5.9), we have(assume $x_0 > x_1$, the other situation will be exactly similar):

$$\begin{aligned} c_2 n^{-\frac{\eta}{2}} \leq c_1 n^{-\frac{\eta}{2}} - n^{-\frac{1}{\eta}} \leq ||\tilde{\varphi}^k_{x_0}| - |\tilde{\varphi}^k_{x_1}|| \leq |\tilde{\varphi}^k_{x_0} - \tilde{\varphi}^k_{x_1}| = |\sum_{j=x_1}^{x_0-1} \nabla_+ \tilde{\varphi}^k_j| \leq \sum_{j=x_1}^{x_0-1} |\nabla_+ \tilde{\varphi}^k_j| \\ \leq |x_1 - x_0| \max_{j\in[x_0,x_1]} |\nabla_+ \tilde{\varphi}^k_j| \leq 2n^\eta \max_{j\in[x_0,x_1]} |\nabla_+ \tilde{\varphi}^k_j|, \end{aligned} \tag{2.5.10}$$

where $\nabla_+ \tilde{\varphi}^k_j = \tilde{\varphi}^k_{j+1} - \tilde{\varphi}^k_j$. Here, c_2 is a constant independent of n, and we used the choice of x_1: since $x_0 \in J(k)$, $|x_0 - x_1| \leq |J(k)| = 2n^\eta$. Therefore, there exists $j_0 \in \mathbb{I}_{n-1}$, and a constant c_3, such that:

$$|\nabla_+ \tilde{\varphi}^k_{j_0}| \geq c_3 n^{-\frac{3\eta}{2}}. \tag{2.5.11}$$

Finally, we use the fact $\sum_{x=1}^{n-1} |\phi^k_x|^2 = \sum_{x=1}^{n-1} \frac{|\nabla_+ \tilde{\varphi}^k_x|^2}{\omega_k^2} = 1$, and thanks to (2.5.11), we obtain:

$$\omega_k^2 \geq |\nabla_+ \tilde{\varphi}^k_{j_0}|^2 \geq cn^{-3\eta}, \tag{2.5.12}$$

where, this finishes the proof of the bound $\frac{1}{\omega_k} \leq cn^{\frac{3\eta}{2}}$. Since we assumed $\forall x \notin J(k)$, $|\tilde{\varphi}^k_x| \leq n^{-\frac{1}{\eta}}$, hence, $\forall x \notin \tilde{J}(k)$, $|\tilde{\varphi}^k_{x+1} - \tilde{\varphi}^k_x| \leq 2n^{-\frac{1}{\eta}}$. Now, using the bound $\frac{1}{\omega_k^2} \leq cn^{3\eta}$, and the definition $\phi^k_x = \frac{1}{\omega_k}(\tilde{\varphi}^k_{x+1} - \tilde{\varphi}^k_x)$, give us the estimate (2.5.8) as well. □

We finish this section, by expressing the following lemma: Recall the "thermal" operators $\tilde{p}_x$ and $\tilde{r}_x$, defined in (2.2.25). These operators can be defined at any time t:

$$\tilde{p}_x(nt) = p_x(nt) - \langle p_x(nt)\rangle_\rho, \quad \tilde{r}_x(nt) = r_x(nt) - \langle r_x(nt)\rangle_\rho. \tag{2.5.13}$$

Notice that here $\langle p_x(nt)\rangle_\rho$, should be understood as the constant times the identity operator. Then we have:

Lemma 2.5.4. *For any test function $g \in C^0([0,1])$, define $T^g_N(t)$ as follows:*

$$T^g_n(t) := \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\langle \tilde{p}^2_x(nt)\rangle_\rho}{2m_x} + \frac{\langle \tilde{r}^2_x(nt)\rangle_\rho}{2}\Big). \tag{2.5.14}$$

Then, $\forall g \in C^1([0,1])$:

$$\lim_{n\to\infty} T^g_n(t) - T^g_n(0) \to 0, \tag{2.5.15}$$

almost surely.

One can see $T_n^g(t)$ as the contribution of the thermal energy in the LHS of (2.1.18). Observe that $\lim_{n\to\infty} T_n^g(0) = \int_0^1 g(y)\mathfrak{f}_\beta^\mu(y)dy$, $\forall g \in C^0([0,1])$, by (2.4.1).

Proof. We denoted the average with respect to ρ with $\langle . \rangle_\rho$, and the inner product in $\mathbb{R}^n$ with $\langle ., . \rangle_n$. Only in this proof, for the convenience and in order to prevent any confusion, we will denote the average with respect to ρ with $\boldsymbol{\langle . \rangle_\rho}$, whenever these two appear in the same expression.
We use the explicit solution of the evolution equation, since $(p_x(nt), r_x(nt))$, and $(\langle p_x(nt)\rangle_\rho, \langle r_x(nt)\rangle_\rho)$ are respectively solutions to the similar linear equations (2.1.7), and (2.3.5), by linearity, $\tilde{p}(nt)$ and $\tilde{r}(nt)$ can be obtained directly from (2.2.22):

$$\begin{aligned}
\tilde{p}(nt) &= \sum_{k=0}^{n-1} \Big(\cos(\omega_k nt)\hat{\tilde{p}}_k(0) - \sin(\omega_k nt)\hat{\tilde{r}}_k(0))M^{\frac{1}{2}}\varphi^k =: \sum_{k=0}^{n-1} (\hat{\tilde{p}}_k(nt)\Big) M^{\frac{1}{2}}\varphi^k, \\
\tilde{r}(nt) &= \sum_{k=1}^{n-1} \Big(\cos(\omega_k nt)\hat{\tilde{r}}_k(0) + \sin(\omega_k nt)\hat{\tilde{p}}_k(0)\Big)\phi^k =: \sum_{k=1}^{n-1} \hat{\tilde{r}}_k(nt)\phi^k,
\end{aligned} \tag{2.5.16}$$

where

$$\begin{aligned}
\hat{\tilde{r}}_k(0) &= \left\langle \phi^k, \tilde{r}(0) \right\rangle_{n-1} = \left\langle \phi^k, r(0) \right\rangle_{n-1} - \left\langle \phi^k, \bar{r}(0) \right\rangle_{n-1}, \\
\hat{\tilde{p}}_k(0) &= \left\langle M^{\frac{1}{2}}\varphi^k, \tilde{p}(0) \right\rangle_{n} = \left\langle M^{\frac{1}{2}}\varphi^k, p(0) \right\rangle_{n} - \left\langle M^{\frac{1}{2}}\varphi^k, \bar{p}(0) \right\rangle_{n},
\end{aligned} \tag{2.5.17}$$

were defined in (2.2.54). Moreover, the definition of $\hat{\tilde{r}}_k(nt)$, and $\hat{\tilde{p}}_k(nt)$ are implicit in this expression.

We prove this lemma in the following steps:
Step1. Contribution of the low modes tends to zero.
Define $\tilde{p}^o$ and $\tilde{r}^o$ to be the low mode portion of $\tilde{p}$ and $\tilde{r}$, respectively, for proper $0 < \alpha < 1$. The choice of α will become clear later:

$$\tilde{p}_x^o(nt) := \sum_{k\in\mathbb{Z}\cap[0,n^{(1-\alpha)}]} \hat{\tilde{p}}_k(nt)\sqrt{m_x}\varphi_x^k, \qquad \tilde{r}_x^o(nt) := \sum_{k\in\mathbb{Z}\cap[1,n^{(1-\alpha)}]} \hat{\tilde{r}}_k(nt)\phi_x^k. \tag{2.5.18}$$

Then we have for any fixed $t \in [0,T]$:

$$\mathcal{L}_n^g(t) := \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\langle (\tilde{p}_x^o(nt))^2\rangle_\rho}{2m_x} + \frac{\langle (\tilde{r}_x^o(nt))^2\rangle_\rho}{2} \Big) \to 0, \tag{2.5.19}$$

as $n \to \infty$.

First, observe that $\frac{\langle (\tilde{p}_x^o)^2\rangle_\rho}{2m_x}$, and $\frac{\langle (\tilde{r}_x^o)^2\rangle_\rho}{2}$ are positive. This is elementary since $\tilde{p}_x^o$ and $\tilde{r}_x^o$ are self-adjoint. Therefore, since g is bounded, we proceed as follows (Notice

that $\tilde{r}_n^o$ is zero by boundary condition):

$$
\begin{aligned}
&|\mathcal{L}_n^g(t)| \leq \frac{||g||_\infty}{2n} \sum_{x=1}^{n} \left\langle \frac{(\tilde{p}_x^o)^2}{2m_x} + \frac{(\tilde{r}_x^o)^2}{2} \right\rangle_\rho \leq \\
&\frac{C}{2n} \left(\left\langle \sum_{k,k'=0}^{[n^{1-\alpha}]+1} \hat{\tilde{p}}_k(nt) \hat{\tilde{p}}_{k'}(nt) \sum_{x=1}^{n} \varphi_x^k \varphi_x^{k'} \right\rangle_\rho + \left\langle \sum_{k,k'=1}^{[n^{1-\alpha}]+1} \hat{\tilde{r}}_k(nt) \hat{\tilde{r}}_{k'}(nt) \sum_{x=1}^{n-1} \phi_x^k \phi_x^{k'} \right\rangle_\rho \right) = \\
&\frac{C}{2n} \left\langle \sum_{k,k'=0}^{[n^{1-\alpha}]+1} (\hat{\tilde{p}}_k(nt) \hat{\tilde{p}}_{k'}(nt) + \hat{\tilde{r}}_k(nt) \hat{\tilde{r}}_{k'}(nt))(\delta_{k,k'}) \right\rangle_\rho = \\
&\frac{C}{2n} \left\langle \sum_{k=0}^{[n^{1-\alpha}]+1} (\hat{\tilde{p}}_k(nt))^2 + (\hat{\tilde{r}}_k(nt)^2) \right\rangle_\rho ,
\end{aligned}
\tag{2.5.20}
$$

where we substitute $\tilde{p}_x^o$ and $\tilde{r}_x^o$ by their definitions in (2.5.19), and obtain the double sum, then we benefited from the linearity of tr: $\langle . \rangle_\rho$, and the fact that $\{\varphi^k\}$ and $\{\phi^k\}$ are orthonormal basis for $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively, hence $\sum_{x=1}^n \varphi_x^k \varphi_x^{k'} = \sum_{x=1}^n \phi_x^k \phi_x^{k'} = \delta_{k,k'}$. Notice that by abusing the notation, we start the last two sums from $k = 0$, in spite of the fact that $\hat{\tilde{r}}_0$ has not been defined and by convention one can take $\hat{\tilde{r}}_0 \equiv 0$, at any time.
On the other hand, one can see that $\hat{\tilde{p}}_k^2 + \hat{\tilde{r}}_k^2$ is conserved in time, $\forall k \in \mathbb{I}_n$, by the direct computation from the definition (2.5.16):

$$
\begin{aligned}
&(\hat{\tilde{p}}_k(nt))^2 + (\hat{\tilde{r}}_k(nt))^2 = (\hat{\tilde{p}}_k(0))^2 + (\hat{\tilde{r}}_k(0))^2 + \left(\hat{\tilde{p}}_k \hat{\tilde{r}}_k + \hat{\tilde{r}}_k \hat{\tilde{p}}_k \right) \times \\
&\left(\sin(\omega_k nt) \cos(\omega_k nt) - \cos(\omega_k nt) \sin(\omega_k nt) \right) = (\hat{\tilde{p}}_k(0))^2 + (\hat{\tilde{r}}_k(0))^2.
\end{aligned}
\tag{2.5.21}
$$

Hence, using the bounds in (2.2.55), i.e., $\left\langle (\hat{\tilde{p}}_k(0))^2 \right\rangle_\rho < C$, $\left\langle (\hat{\tilde{r}}_k(0))^2 \right\rangle_\rho < C$, from Lemma 2.2.4, we obtain:

$$
\begin{aligned}
&|\mathcal{L}_n^g(t)| \leq \frac{C}{2n} \sum_{k=0}^{[n^{1-\alpha}]+1} \left\langle (\hat{\tilde{p}}_k(nt))^2 + (\hat{\tilde{r}}_k(nt))^2 \right\rangle_\rho = \frac{C}{2n} \sum_{k=0}^{[n^{1-\alpha}]+1} \left\langle (\hat{\tilde{p}}_k(0))^2 + (\hat{\tilde{r}}_k(0))^2 \right\rangle_\rho \leq \\
&C' \frac{n^{(1-\alpha)}}{n},
\end{aligned}
\tag{2.5.22}
$$

which clearly goes to zero as $n \to \infty$, by the choice of $0 < \alpha < 1$. Hence, we get (2.5.19).
Step2. Localization and freezing of the high modes.
In this step, we prove that the part of thermal energy coming from high modes is frozen in time, thanks to the localization Lemmas 2.5.2 and 2.5.3. In the same spirit of the previous step (2.5.18), recall $I(\alpha) = (n^{(1-\alpha)}, n-1] \cap \mathbb{Z}$, and define $\tilde{p}_x^\bullet(nt)$ and

$\tilde{r}_x^\bullet(nt)$ as:

$$\begin{aligned}\tilde{p}_x^\bullet(nt) &:= \tilde{p}_x(nt) - \tilde{p}_x^o(nt) = \sum_{k\in I(\alpha)} \hat{\tilde{p}}_k(nt)\sqrt{m_x}\varphi_x^k,\\ \tilde{r}_x^\bullet(nt) &:= \tilde{r}_x(nt) - \tilde{r}_x^o(nt) = \sum_{k\in I(\alpha)} \hat{\tilde{r}}_k(nt)\phi_x^k.\end{aligned} \tag{2.5.23}$$

Moreover, define $\mathcal{U}_n^g(t)$ as:

$$\mathcal{U}_n^g(t) := \frac{1}{n}\sum_{x=1}^n g(\frac{x}{n})\Big(\frac{\langle(\tilde{p}_x^\bullet(nt))^2\rangle_\rho}{2m_x} + \frac{\langle(\tilde{r}_x^\bullet(nt))^2\rangle_\rho}{2}\Big). \tag{2.5.24}$$

Then we have for any $t\in[0,T]$, and $g\in C^1[0,1]$:

$$\mathcal{U}_n^g(t) - \mathcal{U}_n^g(0) \to 0, \tag{2.5.25}$$

almost surely.

In order to prove (2.5.25), we decompose $\mathcal{U}_n^g(t)$ into two parts, one which is constant in time, and the other which is small.
For a fixed n, a function $g:[0,1]\to\mathbb{R}$, and a vector of n operators p, let $g.p$ denotes the following vector of operators: $g.p(x) = g(\frac{x}{n})p_x, \forall x\in\mathbb{I}_n$. Moreover, for a vector of $n-1$ operators r, $g.r$ is the following vector of operators: $g.r(x) = g(\frac{x}{n})r_x, \forall x\in\mathbb{I}_{n-1}$. Since n is fixed in our computation, this notation does not cause any confusion. Using this notation and linearity of the trace one can rewrite $\mathcal{U}_n^g(t)$ as follows:

$$\begin{aligned}\mathcal{U}_n^g(t) &= \frac{1}{2n}\Big\langle \sum_{x=1}^n g(\frac{x}{n})(\frac{\tilde{p}_x^\bullet(nt)^2}{m_x} + \tilde{r}_x^\bullet(nt)^2)\Big\rangle_{\boldsymbol{\rho}} =\\ &\frac{1}{2n}\Big\langle \Big\langle g.\tilde{p}^\bullet(nt), M^{-1}\tilde{p}^\bullet(nt)\Big\rangle_n + \langle g.\tilde{r}^\bullet(nt), \tilde{r}^\bullet(nt)\rangle_{n-1}\Big\rangle_{\boldsymbol{\rho}}.\end{aligned} \tag{2.5.26}$$

By the resolution of the identity i.e. $I_n = \sum_{k=0}^{n-1}\left|\varphi^k\right\rangle\left\langle\varphi^k\right|$, and $I_{n-1} = \sum_{k=1}^{n-1}\left|\phi^k\right\rangle\left\langle\phi^k\right|$ in $\mathbb{R}^n$, and $\mathbb{R}^{n-1}$, we expand the later in the basis of ϕ^k and φ^k. We also split M^{-1} and recall the definition $\tilde{\varphi}^k = M^{-\frac{1}{2}}\varphi^k$:

$$\begin{aligned}\mathcal{U}_n^g(t) &= \frac{1}{2n}\Big\langle \sum_{k=1}^n \Big\langle g.\tilde{p}^\bullet(nt), \tilde{\varphi}^k\Big\rangle_n \Big\langle \tilde{\varphi}^k, \tilde{p}^\bullet(nt)\Big\rangle_n + \sum_{k=1}^{n-1}\Big\langle g.\tilde{r}^\bullet, \phi^k\Big\rangle_{n-1}\Big\langle \phi^k, \tilde{r}^\bullet(nt)\Big\rangle_{n-1}\Big\rangle_{\boldsymbol{\rho}}\\ &= \frac{1}{2n}\Big\langle \sum_{k\in I(\alpha)}\Big(\Big\langle g.\tilde{p}^\bullet(nt), \tilde{\varphi}^k\Big\rangle_n \hat{\tilde{p}}_k(nt) + \Big\langle g.\tilde{r}^\bullet(nt), \phi^k\Big\rangle_{n-1}\hat{\tilde{r}}_k(nt)\Big)\Big\rangle_{\boldsymbol{\rho}}.\end{aligned} \tag{2.5.27}$$

Notice that in order to obtain the second line, we used the following identities:

$$\forall k\in I(\alpha):\ \Big\langle\tilde{\varphi}^k, \tilde{p}^\bullet(nt)\Big\rangle_n = \hat{\tilde{p}}_k(nt), \quad \forall k\notin I(\alpha):\ \Big\langle\tilde{\varphi}^k, \tilde{p}^\bullet(nt)\Big\rangle_n = 0,$$

and their counterparts for $\tilde{r}^\bullet(nt)$, thanks to the definition of $\tilde{p}^\bullet(nt)$ and $\tilde{r}^\bullet(nt)$ in (2.5.23).
Now, let us split g for each k. For each $k\in I(\alpha)$, recall the interval $J(k)$ given by

the Lemma 2.5.2, and let x_k be the center of this interval. Then, let $g_k(x) := g(\frac{x_k}{n})$, $\forall x \in \mathbb{I}_n$, be the constant vector for each $k \in I(\alpha)$, and define $\tilde{g}_k(x) = g(x) - g_k(x)$, $\forall x \in \mathbb{I}_n$. We simply have: $\forall k \in I(\alpha)$, $g = \tilde{g}_k + g_k$. By linearity of $g.p$, and $\langle . \rangle_n$, we rewrite (2.5.27) as follows:

$$\mathcal{U}_n^g(t) = \frac{1}{2n}\Big\langle \sum_{k\in I(\alpha)} \Big(\Big\langle g_k.\tilde{p}^\bullet(nt), \tilde{\varphi}^k \Big\rangle_n \hat{\tilde{p}}_k(nt) + \Big\langle g_k.\tilde{r}^\bullet(nt), \phi^k \Big\rangle_{n-1} \hat{\tilde{r}}_k(nt) \Big) \Big\rangle_{\boldsymbol{\rho}} +$$
$$\frac{1}{2n}\Big\langle \sum_{k\in I(\alpha)} \Big(\Big\langle \tilde{g}_k.\tilde{p}^\bullet(nt), \tilde{\varphi}^k \Big\rangle_n \hat{\tilde{p}}_k(nt) + \Big\langle \tilde{g}_k.\tilde{r}^\bullet(nt), \phi^k \Big\rangle_{n-1} \hat{\tilde{r}}_k(nt) \Big) \Big\rangle_{\boldsymbol{\rho}} . \tag{2.5.28}$$

In the later decomposition the first line is constant in time, and the second line vanishes as $n \to \infty$. Let $\mathcal{U}_n^g(t) = \bar{\mathcal{U}}_n^g(t) + \tilde{\mathcal{U}}_n^g(t)$, where $\bar{\mathcal{U}}_n^g(t)$ is the first line in (2.5.28), and $\tilde{\mathcal{U}}_n^g(t)$ is the second line. Since for each k, g_k is constant in x, we can factor it and observe:

$$\begin{aligned}\bar{\mathcal{U}}_n^g(t) &:= \frac{1}{2n}\Big\langle \sum_{k\in I(\alpha)} g(\frac{x_k}{n}) \Big(\Big\langle \tilde{p}^\bullet(nt), \tilde{\varphi}^k \Big\rangle_n \hat{\tilde{p}}_k(nt) + \Big\langle \tilde{r}^\bullet(nt), \phi^k \Big\rangle_{n-1} \hat{\tilde{r}}_k(nt) \Big) \Big\rangle_{\boldsymbol{\rho}} \\ &= \frac{1}{2n}\Big\langle \sum_{k\in I(\alpha)} g(\frac{x_k}{n}) \Big(\hat{\tilde{p}}_k(nt)^2 + \hat{\tilde{r}}_k(nt)^2 \Big) \Big\rangle_{\boldsymbol{\rho}} = \bar{\mathcal{U}}_n^g(0),\end{aligned} \tag{2.5.29}$$

where we take advantage of the identity $\Big\langle \tilde{\varphi}^k, \tilde{p}^\bullet(nt) \Big\rangle_n = \hat{\tilde{p}}_k(nt)$, for $k \in I(\alpha)$, and the similar identity for $\tilde{r}^\bullet(nt)$, as in (2.5.27). Moreover, we already observed in (2.5.22), that the expression $\hat{\tilde{p}}_k(nt)^2 + \hat{\tilde{r}}_k(nt)^2$ for each k -which represents the thermal energy of the kth mode- is conserved by the dynamics. Hence, $\bar{\mathcal{U}}_n^g(t) = \bar{\mathcal{U}}_n^g(0)$ is constant in time.

In the rest of this step, we prove that $\tilde{\mathcal{U}}_n^g \to 0$, almost surely as $n \to \infty$. In preparation of this proof, we need to use the following form of the Cauchy Schwartz inequality. For certain operators a and b, we have:

$$|\langle ab^* \rangle_\rho|^2 \leq \langle aa^* \rangle_\rho \langle bb^* \rangle_\rho . \tag{2.5.30}$$

In order to deal with $\tilde{\mathcal{U}}_n^g(t)$, first, we bound $\Big\langle \Big\langle \tilde{g}_k.\tilde{p}^\bullet(nt), \tilde{\varphi}^k \Big\rangle_n^2 \Big\rangle_{\boldsymbol{\rho}}$ as follows: Recall Lemma 2.5.2, since this lemma is valid for any choice of $0 < 2\alpha < \eta < 1$, we take $\eta < \frac{2}{3}$. Then, there exists a constant $c > 0$, independent of n, such that for any $k \in I(\alpha)$, and $n > n_0$, where n_0 is given by 2.5.2, we have:

$$\Big\langle \Big\langle \tilde{g}_k.\tilde{p}^\bullet(nt), \tilde{\varphi}^k \Big\rangle_n^2 \Big\rangle_{\boldsymbol{\rho}} \leq \frac{c}{n^{1-3\eta}}, \qquad \Big\langle \Big\langle \tilde{g}_k.\tilde{r}^\bullet(nt), \phi^k \Big\rangle_{n-1}^2 \Big\rangle_{\boldsymbol{\rho}} \leq \frac{c}{n^{1-3\eta}}, \tag{2.5.31}$$

almost surely. These bounds can be achieved by expanding the inner product, and

performing the following computation:

$$\begin{aligned}
&\Big\langle \big\langle \tilde{g}_k.\tilde{p}^{\bullet}(nt), \tilde{\varphi}^k \big\rangle_n^2 \Big\rangle_{\boldsymbol{\rho}} = \sum_{x,y=1}^{n} \tilde{g}_k(\frac{x}{n})\tilde{g}(\frac{y}{n})\tilde{\varphi}_x^k\tilde{\varphi}_y^k \langle \tilde{p}_x^{\bullet}(nt)\tilde{p}_y^{\bullet}(nt)\rangle_{\boldsymbol{\rho}} \leq \\
&\sum_{x,y=1}^{n} \Big|\tilde{g}_k(\frac{x}{n})\tilde{g}(\frac{y}{n})\tilde{\varphi}_x^k\tilde{\varphi}_y^k\Big| \Big|\langle \tilde{p}_x^{\bullet}(nt)\tilde{p}_y^{\bullet}(nt)\rangle_{\boldsymbol{\rho}}\Big| \leq \\
&\sum_{x,y=1}^{n} \Big|\tilde{g}_k(\frac{x}{n})\tilde{g}(\frac{y}{n})\tilde{\varphi}_x^k\tilde{\varphi}_y^k\Big| \Big(\langle (\tilde{p}_x^{\bullet}(nt))^2\rangle_{\boldsymbol{\rho}}\Big)^{\frac{1}{2}} \Big(\langle (\tilde{p}_y^{\bullet}(nt))^2\rangle_{\boldsymbol{\rho}}\Big)^{\frac{1}{2}} = \\
&\Big(\sum_{x=1}^{n} \Big|\tilde{g}_k(\frac{x}{n})\tilde{\varphi}_x^k\Big|\Big(\langle (\tilde{p}_x^{\bullet}(nt))^2\rangle_{\boldsymbol{\rho}}\Big)^{\frac{1}{2}}\Big)^2 \leq \Big(\sum_{x=1}^{n} \Big|\tilde{g}_k(\frac{x}{n})\tilde{\varphi}_x^k\Big|^2\Big)\Big(\sum_{x=1}^{n} \langle (\tilde{p}_x^{\bullet}(nt))^2\rangle_{\boldsymbol{\rho}}\Big).
\end{aligned} \tag{2.5.32}$$

The second inequality obtained using the aforementioned form of Cauchy-Schwartz inequality in (2.5.30), and the last inequality is evident, using the Cauchy- Schwartz inequality for finite dimensional vectors.
In the following computation, using the definition of $\tilde{p}_x^{\bullet}(nt)$ (2.5.23), and getting a double sum, we bound the second term in (2.5.32) by $c_0 n$, where $c_0 > 0$ is a constant independent of n:

$$\begin{aligned}
&\sum_{x=1}^{n} \langle \tilde{p}_x^{\bullet}(nt)^2\rangle_{\boldsymbol{\rho}} \leq \sum_{x=1}^{n} \frac{m_{max}}{m_x} \sum_{k,k'\in I(\alpha)} m_x \varphi_x^k \varphi_x^{k'} \langle \hat{\tilde{p}}_k(nt)\hat{\tilde{p}}_{k'}(nt)\rangle_{\boldsymbol{\rho}} \leq \\
&c_1 \sum_{k,k'\in I(\alpha)} \langle \hat{\tilde{p}}_k(nt)\hat{\tilde{p}}_{k'}(nt)\rangle_{\boldsymbol{\rho}} \sum_{x=1}^{n} \varphi_x^k\varphi_x^{k'} = \sum_{k\in I(\alpha)} \langle (\hat{\tilde{p}}_k(nt))^2\rangle_{\boldsymbol{\rho}} \leq c_1 4C|I(\alpha)| \leq c_0 n.
\end{aligned} \tag{2.5.33}$$

In the last line, we bounded $\langle (\hat{\tilde{p}}_k(nt))^2\rangle_{\boldsymbol{\rho}} \leq 4C$ (2C works as well, with conservation argument), by using the explicit form of $\hat{\tilde{p}}_k(nt) = \cos(\omega_k nt)\hat{\tilde{p}}_k - \sin(\omega_k nt)\hat{\tilde{r}}_k$:

$$\begin{aligned}
&\langle (\hat{\tilde{p}}_k(nt))^2\rangle_{\boldsymbol{\rho}} = \langle \hat{\tilde{p}}_k(0)^2\cos^2(\omega_k nt) + \hat{\tilde{r}}_k(0)^2\sin^2(\omega_k(nt)) - \\
&\sin(\omega_k nt)\cos(\omega_k nt)(\hat{\tilde{r}}_k(0)\hat{\tilde{p}}_k(0) + \hat{\tilde{p}}_k(0)\hat{\tilde{r}}_k(0))\rangle_{\boldsymbol{\rho}} \leq \\
&\langle (\hat{\tilde{p}}_k(0))^2\rangle_{\boldsymbol{\rho}} + \langle (\hat{\tilde{r}}_k(0))^2\rangle_{\boldsymbol{\rho}} + |\langle \hat{\tilde{p}}_k(0)\hat{\tilde{r}}_k(0)\rangle_{\boldsymbol{\rho}}| + |\langle \hat{\tilde{r}}_k(0)\hat{\tilde{p}}_k(0)\rangle_{\boldsymbol{\rho}}| \leq \\
&\langle \hat{\tilde{p}}_k(0))^2\rangle_{\boldsymbol{\rho}} + \langle (\hat{\tilde{r}}_k(0))^2\rangle_{\boldsymbol{\rho}} + 2\langle (\hat{\tilde{p}}_k(0))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}} \langle \hat{\tilde{r}}_k(0))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}} \leq 4C,
\end{aligned} \tag{2.5.34}$$

where we bounded $|\sin(.)|$ and $|\cos(.)|$ by one, used Cauchy Schwartz (2.5.30) in the second line, and used the appropriate bound from Lemma 2.2.4 in the last line. Moreover, we can bound the first term in the RHS of (2.5.31) by $\frac{c_1}{n^{2-3\eta}}$, almost surely, thanks to (2.5.7), and regularity of g. Recall the definition of $J(k)$ from lemma 2.5.2, then we can rewrite the later as:

$$\begin{aligned}
&\sum_{x=1}^{n} |\tilde{g}_k(\frac{x}{n})\tilde{\varphi}_x^k|^2 = \sum_{x\in J(k)} |\tilde{g}_k(\frac{x}{n})\tilde{\varphi}_x^k|^2 + \sum_{x\notin J(k)} |\tilde{g}_k(\frac{x}{n})\tilde{\varphi}_x^k|^2 \leq \\
&c_1' \sum_{x\in J(k)} |\tilde{g}_k(\frac{x}{n})|^2 + c_2' \sum_{x\notin J(k)} |\tilde{\varphi}_x^k|^2 \leq c_1' C_1 \sum_{x\in J(k)} \frac{|x-x_k|^2}{n^2} + c_2' n^{-\frac{2}{\eta}} |J(k)^c| \\
&\leq 2c_1' C_1 \sum_{j=0}^{n^\eta} j^2 + c_2' n^{-\frac{2}{\eta}+1} \leq C_1' \frac{n^{3\eta}}{n^2} + c_2' n^{-\frac{2}{\eta}+1} \leq \frac{c_1}{n^{2-3\eta}},
\end{aligned} \tag{2.5.35}$$

where in the second inequality, we bound $|\tilde{\varphi}^k_x|$, in the second term by $n^{-\frac{1}{\eta}}$, almost surely, for $x \notin J(k)$, thanks to the bound (2.5.7) in Lemma (2.5.2). In the second inequality, for the first term, we exploited the definition of $\tilde{g}_k(\frac{x}{n}) = g(\frac{x}{n}) - g(\frac{x_k}{n})$, as well as the fact that $g \in C^1([0,1])$, and therefore, there exists a constant $C_1 > 0$ [15], such that $|g(\frac{x}{n}) - g(\frac{x_k}{n})| \leq C_1|\frac{x}{n} - \frac{x_k}{n}|$. In the third inequality, we applied the definition of x_k as the center of the interval $J(k)$, we also used the fact that $|J(k)| = 2n^\eta$, and $|J(k)^c| = n - 2n^\eta < n$.

By inserting the bounds

$$\sum_{x=1}^{n} \langle \tilde{p}^\bullet_x(nt)^2 \rangle_{\boldsymbol{\rho}} \leq c_0 n, \quad \sum_{x=1}^{n} |\tilde{g}_k(\frac{x}{n})\tilde{\varphi}^k_x|^2 \leq \frac{c_1}{n^{2-3\eta}},$$

from (2.5.33), and (2.5.35), in the expression (2.5.32), we obtain the first bound in (2.5.31), which is the product of these two bounds, namely:

$$\Big\langle \big\langle \tilde{g}_k . \tilde{p}^\bullet(nt), \tilde{\varphi}^k \big\rangle^2_n \Big\rangle_{\boldsymbol{\rho}} \leq \Big(\sum_{x=1}^{n} \langle \tilde{p}^\bullet_x(nt)^2 \rangle_{\boldsymbol{\rho}} \Big) \Big(\sum_{x=1}^{n} |\tilde{g}_k(\frac{x}{n})\tilde{\varphi}^k_x|^2 \Big) \leq c_0 n \times \frac{c_1}{n^{2-3\eta}} \leq \frac{c}{n^{1-3\eta}}.$$

The second bound in (2.5.31), corresponding to the elongation operator r, can be treated similarly, with a small modification (using Lemma 2.5.3 instead of Lemma 2.5.2).
Performing the exact same computation as in (2.5.32), one can obtain the following bound using the Cauchy Shwartz inequality twice:

$$\Big\langle \big\langle \tilde{g}_k . \tilde{r}^\bullet(nt), \phi^k \big\rangle^2_{n-1} \Big\rangle_{\boldsymbol{\rho}} \leq \Big(\sum_{x=1}^{n-1} |\tilde{g}_k(\frac{x}{n})\phi^k_x|^2 \Big) \Big(\sum_{x=1}^{n-1} \langle (\tilde{r}^\bullet_x(nt))^2 \rangle_{\boldsymbol{\rho}} \Big), \tag{2.5.36}$$

Again, similar to (2.5.33), we have:

$$\begin{aligned} &\sum_{x=1}^{n} \langle (\tilde{r}^\bullet_x(nt))^2 \rangle_{\boldsymbol{\rho}} = \sum_{x=1}^{n-1} \sum_{k,k' \in I(\alpha)} \phi^k_x \phi^{k'}_x \langle \hat{\tilde{r}}_k(nt) \hat{\tilde{r}}_{k'}(nt) \rangle_{\boldsymbol{\rho}} = \sum_{k,k' \in I(\alpha)} \langle \hat{\tilde{r}}_k(nt) \hat{\tilde{r}}_{k'}(nt) \rangle_{\boldsymbol{\rho}} \sum_{x=1}^{n-1} \phi^k_x \phi^{k'}_x \\ &= \sum_{k \in I(\alpha)} \langle \hat{\tilde{r}}_k(nt)^2 \rangle_{\boldsymbol{\rho}} \leq 4C|I(\alpha)| \leq c_0' n, \end{aligned} \tag{2.5.37}$$

where we bounded $\langle \hat{\tilde{r}}_k(nt)^2 \rangle_{\boldsymbol{\rho}} \leq 4C$, exactly similar to (2.5.34), using the explicit form of $\hat{\tilde{r}}_k(nt) = \cos(\omega_k nt)\hat{\tilde{r}}_k(0) + \sin(\omega_k nt)\hat{\tilde{r}}_k(0)$, and the bounds in Lemma 2.2.4. Dealing with the first term in the RHS of (2.5.36) requires more attention (this problem have not been addressed in [35]). We can proceed similar to (2.5.35),

[15] One can simply take $C_1 = ||g'||_\infty$.

recalling $\tilde{J}(k)$ from Lemma 2.5.3:

$$\sum_{x=1}^{n-1}|\tilde{g}_k(\frac{x}{n})\phi_x^k|^2=\sum_{x\in\tilde{J}(k)}|\tilde{g}_k(\frac{x}{n})\phi_x^k|^2+\sum_{x\notin\tilde{J}(k)}|\tilde{g}_k(\frac{x}{n})\phi_x^k|^2\leq$$
$$c_1''\sum_{x\in\tilde{J}(k)}|\tilde{g}_k(\frac{x}{n})|^2+c_2''\sum_{x\notin\tilde{J}(k)}|\phi_x^k|^2\leq(c_1''C_1\sum_{x\in\tilde{J}(k)}\frac{|x-x_k|^2}{n^2})+(c_3n^{-\frac{2}{\eta}+3\eta}|\tilde{J}(k)^c|)\leq$$
$$(2c_1''C_1\sum_{j=0}^{n^\eta}j^2)+(c_3n^{-\frac{2}{\eta}+3\eta+1})\leq C_1'\frac{n^{3\eta}}{n^2}+c_3n^{-\frac{2}{\eta}+3\eta+1}\leq\frac{c_1}{n^{2-3\eta}},\tag{2.5.38}$$

where these computation can be justified similar to (2.5.35), except from the fact that on the second inequality we exploited the bound (2.5.8): $|\phi_x^k|\leq 2cn^{-\frac{1}{\eta}+\frac{3\eta}{2}}$ from Lemma 2.5.3. Moreover, in the last inequality, by using the assumption $\eta\in(0,\frac{2}{3})$, we deduced that $3\eta-2>-\frac{2}{\eta}+3\eta+1$, and $n^{3\eta-2}>n^{-\frac{2}{\eta}+3\eta+1}$. Hence, we have the last bound by choosing the proper constant. Finally, notice that here we can replace $J(k)$ by $\tilde{J}(k)$, since $|J(k)|\geq|\tilde{J}(k)|$.
By inserting (2.5.37) and (2.5.38) into (2.5.36), we obtain the second bound in (2.5.31):

$$\Big\langle\big\langle\tilde{g}_k.\tilde{r}^\bullet(nt),\phi^k\big\rangle_{n-1}^2\Big\rangle_{\boldsymbol{\rho}}\leq\Big(\sum_{x=1}^{n-1}|\tilde{g}_k(\frac{x}{n})\phi_x^k|^2\Big)\Big(\sum_{x=1}^{n-1}\langle(\tilde{r}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}}\Big)\leq\frac{c_1}{n^{2-3\eta}}\times c_0'n\leq\frac{c}{n^{1-3\eta}}.\tag{2.5.39}$$

Recall the definition of $\tilde{\mathcal{U}}_n^g(t)$

$$\tilde{\mathcal{U}}_n^g(t):=\frac{1}{2n}\sum_{k\in I(\alpha)}\Big\langle\big\langle\tilde{g}_k.\tilde{p}^\bullet(nt),\tilde{\varphi}^k\big\rangle_n\hat{\tilde{p}}_k(nt)+\big\langle\tilde{g}_k.\tilde{r}^\bullet(nt),\phi^k\big\rangle_{n-1}\hat{\tilde{r}}_k(nt)\Big\rangle_{\boldsymbol{\rho}},\tag{2.5.40}$$

by using the Cauchy Schwartz inequality (2.5.30), we have[16]:

$$|\tilde{\mathcal{U}}_n^g(t)|\leq\frac{1}{2n}\sum_{k\in I(\alpha)}\Big|\Big\langle\big\langle\tilde{g}_k.\tilde{p}^\bullet(nt),\tilde{\varphi}^k\big\rangle_n\hat{\tilde{p}}_k(nt)\Big\rangle_{\boldsymbol{\rho}}\Big|+\Big|\Big\langle\big\langle\tilde{g}_k.\tilde{r}^\bullet(nt),\phi^k\big\rangle_{n-1}\hat{\tilde{r}}_k(nt)\Big\rangle_{\boldsymbol{\rho}}\Big|$$
$$\leq\frac{1}{2n}\sum_{k\in I(\alpha)}\Big\langle\big\langle\tilde{g}_k.\tilde{p}^\bullet(nt),\tilde{\varphi}^k\big\rangle_n^2\Big\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}\langle\hat{\tilde{p}}_k(nt)^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}+\Big\langle\big\langle\tilde{g}_k.\tilde{r}^\bullet(nt),\phi^k\big\rangle_{n-1}^2\Big\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}\langle\hat{\tilde{r}}_k(nt)^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}$$
$$\leq\frac{\sqrt{C}}{n}\sum_{k\in I(\alpha)}\Big\langle\big\langle\tilde{g}_k.\tilde{p}^\bullet(nt),\tilde{\varphi}^k\big\rangle_n^2\Big\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}+\Big\langle\big\langle\tilde{g}_k.\tilde{r}^\bullet(nt),\phi^k\big\rangle_{n-1}^2\Big\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}\leq\frac{\sqrt{C}}{n}\frac{2\sqrt{c}}{n^{\frac{1}{2}-\frac{3\eta}{2}}}|I(\alpha)|$$
$$\leq\frac{C_0}{n^{\frac{1}{2}-\frac{3\eta}{2}}}\to 0,\tag{2.5.41}$$

where in the first inequality, we used Cauchy-Schwartz (2.5.30). In the second inequality, we exploited the bounds $\langle\hat{\tilde{p}}_k(nt)^2\rangle_{\boldsymbol{\rho}}\leq 4C,\langle\hat{\tilde{r}}_k(nt)^2\rangle_{\boldsymbol{\rho}}\leq 4C$, as we already

[16]We already discussed the positivity of the terms under the square root in the second line, which can be obtained by the fact that these operators are linear combination of bosonic operators $\tilde{\mathfrak{b}}_k$, and their adjoints, and they are self adjoint, hence, by using Lemma 2.2.38, we obtain the positivity (more abstract proof is of course possible).

did in (2.5.33) and (2.5.37). In the third inequality, we benefited from the bounds in (2.5.31). Hence, if one takes $\eta < \frac{1}{3}$, one can deduce $|\tilde{\mathcal{U}}_n^g(t)| \to 0$, almost surely. Lastly, recall that we expressed $\mathcal{U}_n^g(t) = \bar{\mathcal{U}}_n^g(t) + \tilde{\mathcal{U}}_n^g(t)$, we sum up the result of this step as follows:

$$\begin{aligned} &|\mathcal{U}_n^g(t) - \mathcal{U}_n^g(0)| = |\bar{\mathcal{U}}_n^g(t) + \tilde{\mathcal{U}}_n^g(t) - (\bar{\mathcal{U}}_n^g(t) + \tilde{\mathcal{U}}_n^g(t))| = |\tilde{\mathcal{U}}_n^g(t) - \tilde{\mathcal{U}}_n^g(0)| \leq \\ &|\tilde{\mathcal{U}}_n^g(t))| + |\tilde{\mathcal{U}}_n^g(0))| \to 0, \end{aligned} \tag{2.5.42}$$

almost surely, thanks to $\bar{\mathcal{U}}_n^g(t) = \bar{\mathcal{U}}_n^g(0)$ from (2.5.29), and (2.5.41). This finishes the proof of this step (2.5.25).

Step3. Summing up.
In this step, we finish the proof of Lemma 2.5.4,(the limit (2.5.15)), by combining the results from previous steps (2.5.19), (2.5.25). In fact, the later expressions let us conclude that $|\mathcal{U}_n^g(t) + \mathcal{L}_n^g - \mathcal{U}_n^g(0) - \mathcal{L}_n^g(0)| \to 0$, almost surely. By comparing $T_n^g(t)$, and $\mathcal{U}_n^g(t) + \mathcal{L}_n^g$, one can see it is sufficient to control the following term in order to obtain (2.5.15) [17].

$$\mathcal{E}_n^g(t) := \frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\langle \tilde{p}_x^o(nt)\tilde{p}_x^\bullet(nt)\rangle_{\boldsymbol{\rho}}}{m_x} + \langle \tilde{r}_x^o(nt)\tilde{r}_x^\bullet(nt)\rangle_{\boldsymbol{\rho}}\Big). \tag{2.5.43}$$

However, this term can be treated as usual, using the Cauchy Schwartz inequality (2.5.30):

$$\begin{aligned} &|\mathcal{E}_n^g(t)| \leq \Big|\frac{1}{n}\sum_{x=1}^{n}\Big|\frac{g(\frac{x}{n})}{m_x}\Big||\langle \tilde{p}_x^o(nt)\tilde{p}_x^\bullet(nt)\rangle_{\boldsymbol{\rho}}|\Big| + \Big|\frac{1}{n}\sum_{x=1}^{n-1}|g(\frac{x}{n})||\langle \tilde{r}_x^o(nt)\tilde{r}_x^\bullet(nt)\rangle_{\boldsymbol{\rho}}|\Big| \leq \\ &\Big|\frac{1}{n}\sum_{x=1}^{n}\Big|\frac{g(\frac{x}{n})}{m_x}\Big||\langle (\tilde{p}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}\langle (\tilde{p}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}|\Big| + \Big|\frac{1}{n}\sum_{x=1}^{n-1}|g(\frac{x}{n})||\langle (\tilde{r}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}\langle (\tilde{r}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}}^{\frac{1}{2}}|\Big| \\ &\leq \Big|\frac{1}{n}\sum_{x=1}^{n}\Big|\frac{g(\frac{x}{n})}{m_x}\Big|\langle (\tilde{p}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}\Big|^{\frac{1}{2}}\Big|\frac{1}{n}\sum_{x=1}^{n}|\frac{g(\frac{x}{n})}{m_x}|\langle (\tilde{p}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}}\Big|^{\frac{1}{2}} \\ &+ \Big|\frac{1}{n}\sum_{x=1}^{n-1}|g(\frac{x}{n})|\langle (\tilde{r}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}\Big|^{\frac{1}{2}}\Big|\frac{1}{n}\sum_{x=1}^{n-1}|g(\frac{x}{n})|\langle (\tilde{r}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}}\Big|^{\frac{1}{2}} \end{aligned}$$

$$\leq \sqrt{2\tilde{c}_0 C_1}\left(\frac{1}{n}\sum_{x=1}^{n}|g(\frac{x}{n})|\Big(\frac{\langle (\tilde{p}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}}{m_x} + \langle (\tilde{r}_x^o(nt))^2\rangle_{\boldsymbol{\rho}}\Big)\right)^{\frac{1}{2}} \leq \Big(\frac{\tilde{c}_0 C_1 C' n^{(1-\alpha)}}{n}\Big)^{\frac{1}{2}} \to 0, \tag{2.5.44}$$

The first inequality is obtained by (2.5.30). In the second inequality, the Cauchy-Schwartz has been used. Then by taking $\tilde{c}_0 = \max\{c_0, c_0'\}$, we take advantage of the following bounds:

$$\frac{1}{n}\sum_{x=1}^{n-1}\langle (\tilde{r}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}} \leq c_0', \quad \frac{1}{n}\sum_{x=1}^{n}\langle (\tilde{p}_x^\bullet(nt))^2\rangle_{\boldsymbol{\rho}} \leq c_0$$

[17] Recall that all the expressions corresponding to the nth component of the r operator is zero, and we bring them in the same sum just to lighten the notation.

from (2.5.37) and (2.5.33). Then, we used the inequality $\sqrt{a}+\sqrt{b} \leq \sqrt{2(a+b)}$. The last inequality can be obtained exactly similar to (2.5.20) and (2.5.22), where we bounded $|\mathcal{L}_n^g(t)| \leq C' \frac{n^{(1-\alpha)}}{n}$.

Finally, we prove $|T_n^g(t) - T_n^g(0)| \to 0$, almost surely, by using (2.5.19), (2.5.25), and (2.5.44). First, recall $\tilde{p}_x(nt) = \tilde{p}_x^o(nt) + \tilde{p}_x^\bullet(nt)$, and $\tilde{r}_x(nt) = \tilde{r}_x^o(nt) + \tilde{r}_x^\bullet(nt)$, for all $x \in \mathbb{I}_n$. By using the later, and comparing the definition of $T_n^g(t)$ (2.5.14), $\mathcal{L}_n^g(t)$ (2.5.19), $\mathcal{U}_n^g(t)$ (2.5.24), and $\mathcal{E}_n^g(t)$ (2.5.43), it is evident that:

$$T_n^g(t) = \mathcal{L}_n^g(t) + \mathcal{U}_n^g(t) + \mathcal{E}_n^g(t). \tag{2.5.45}$$

Therefore, we can conclude:

$$\begin{aligned} |T_n^g(t) - T_n^g(0)| =& |\mathcal{L}_n^g(t) + \mathcal{U}_n^g(t) + \mathcal{E}_n^g(t) - (\mathcal{L}_n^g(0) + \mathcal{U}_n^g(0) + \mathcal{E}_n^g(0))| \\ \leq& |\mathcal{U}_n^g(t) - \mathcal{U}_n^g(0)| + |\mathcal{L}_n^g(0)| + |\mathcal{L}_n^g(t)| + |\mathcal{E}_n^g(0)| + |\mathcal{E}_n^g(t)| \to 0, \end{aligned} \tag{2.5.46}$$

almost surely, where $|\mathcal{U}_n^g(t) - \mathcal{U}_n^g(0)| \to 0$ almost surely, by (2.5.25). Moreover, $|\mathcal{L}_n^g(0)| + |\mathcal{L}_n^g(t)| \to 0$, since (2.5.19) is valid for any t, and $|\mathcal{E}_n^g(0)| + |\mathcal{E}_n^g(t)| \to 0$ as a consequence of (2.5.44), which holds for all $t \in [0, T]$. This finishes the proof of (2.5.15) and Lemma 2.5.4. □

Remark 2.5.5. *For a clean chain in thermal equilibrium, we have:* $\left\langle \tilde{e}_{[ny]}(nt) \right\rangle_\rho - \left\langle \tilde{e}_{[ny]}(0) \right\rangle_\rho$ *vanishes as* $n \to \infty$. *One can observe this by using the fact that* A_r^β *and* A_r^0 *can be diagonalized in the same basis for this chain, and hence this difference can be computed explicitly. Therefore, a simple calculation shows that in this case, we can obtain* (2.5.15). *Notice that the rest of the proof for a clean chain in thermal equilibrium is exactly similar to the disordered case; therefore, we can obtain the* (2.1.1) *for this chain, with corresponding* f_β *from* (1.1.34).

2.5.3 Decomposition

In this section, we finish proof of (2.1.18) for any test function $g \in C^1([0,1])$. We will extend this result to continuous test functions in the next section.

Proof of (2.1.18) *with* C^1 *test function.* First, we can solve the macroscopic equation explicitly. In fact, thanks to the regularity assumption: $\bar{r}(y), \bar{p}(y) \in C^1([0,1])$, the wave equation (two first equation in (2.1.13), with boundary conditions (2.1.14), (2.1.15)) for r and p can be solved explicitly, by expanding in the Fourier basis and, we have a smooth in time, strong solutions, such that $\mathrm{p}(y,t), \mathrm{r}(y,t) \in C^1([0,1])$. Moreover, one can observe that the solution to the equation (2.1.13), (2.1.14), and (2.1.15) for e is given by:

$$\mathrm{e}(y,t) = \frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2} + \mathrm{f}_\beta^\mu(y). \tag{2.5.47}$$

One can justify this solution by simply taking the derivative and plug-in the solution to the wave equation. Notice that this formal argument is legitimate, thanks to the

regularity of r and p.
Fix $g \in C^1([0,1])$, then the LHS of (2.1.18) is given by:

$$\int_0^1 g(y)\mathrm{e}(y,t)dy = \underbrace{\int_0^1 g(y)\Big(\frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2}\Big)dy}_{K_g\text{:Macroscopic Mechanical energy}} + \underbrace{\int_0^1 g(y)\mathrm{f}^{\mu}_{\beta}(y)dy}_{T_g\text{:Macroscopic Thermal energy}}, \tag{2.5.48}$$

where we defined $K_g(t)$ and $T_g(t) \equiv T_g$, to be the Macroscopic Mechanical energy, and macroscopic Thermal energy, respectively. At the macroscopic level, we already observed the following decomposition [18]:

$$\begin{aligned}\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\left\langle e_x(nt)\right\rangle_{\rho} = &\underbrace{\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\Big(\frac{\left\langle (p_x(nt))^2\right\rangle_{\rho}}{m_x} + \left\langle (r_x(nt))^2\right\rangle_{\rho}\Big)}_{K_n^g(t)\text{:Microscopic Mechanical Energy}} \\ &+ \underbrace{\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\left(\frac{\left\langle (p_x(nt)-\left\langle p_x(nt)\right\rangle_{\rho})^2\right\rangle_{\rho}}{m_x} + \left\langle (r_x(nt)-\left\langle r_x(nt)\right\rangle_{\rho})^2\right\rangle_{\rho}\right)}_{T_n^g(t)\text{:Miscroscopic Thermal Energy}},\end{aligned} \tag{2.5.49}$$

where we defined $K_n^g(t)$ to be the microscopic Mechanical energy. Recalling the definition $\tilde{p}_x(nt) = p_x(nt) - \left\langle p_x(nt)\right\rangle_{\rho}$, and $\tilde{r}_x(nt) - \left\langle r_x(nt)\right\rangle_{\rho}$, the second term $T_n^g(t)$ is the microscopic Thermal energy, which is defined in (2.5.14).
Comparing these two expressions, (2.5.48) and (2.5.49), the rest of the proof becomes clear. In step one, we proved that the microscopic Mechanical energy converges to the macroscopic counterpart: $K_n^g(t) \to K_g(t)$ in (2.5.1). In the previous section, we proved that the microscopic Thermal energy is frozen in time i.e. $T_n^g(t) - T_n^g(0) \to 0$, in (2.5.15). Finally, in Section 2.4.1, we proved that $T_n^g(0) \to T_g$, in (2.4.30). All these limit are almost surely w.r.t $\mathbb{P}$. Combining these three argument finishes the proof:

$$\begin{aligned}&\frac{1}{n}\sum_{x=1}^{n} g(\frac{x}{n})\left\langle e_x(nt)\right\rangle_{\rho} = K_n^g(t) + T_n^g(t) = K_n^g(t) + T_n^g(0) + (T_n^g(t) - T_n^g(0)) \\ &\to K_g(t) + T_g + 0 = \int_0^1 g(y)\Big(\Big(\frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2}\Big) + \mathrm{f}^{\mu}_{\beta}(y)\Big)dy = \int_0^1 g(y)\mathrm{e}(y,t)dy,\end{aligned} \tag{2.5.50}$$

almost surely, where these three limits have been deduced from (2.5.1), (2.4.30) and (2.5.15) respectively. This finishes the proof of (2.1.18) for $g \in C^1([0,1])$. □

Notice that the only limit among those, where we needed the stronger assumption $g \in C^1([0,1])$, rather than $g \in C^0([0,1])$, was the last one: $T_n^g(t) - T_n^g(0) \to 0$. We will circumvent this obstacle in the next section, using the energy estimate: $\frac{1}{n}\sum_{x=1}^{n}\left\langle e_x(nt)\right\rangle_{\rho} \leq C$.

[18] Notice that we come back to the notation where we denote the thermal average by $\langle . \rangle_{\rho}$ instead of $\langle . \rangle_{\boldsymbol{\rho}}$, since there is no confusion here.

2.5.4 From C^1 to C^0

This section extends the previous result and omits the additional regularity assumption on f, and finishes the proof of Theorem 2.1.1. As we already mentioned, the essential tool for this purpose is the following energy estimate: there exists a constant $C > 0$ independent of n, such that for all $t \in [0, T]$:

$$\frac{1}{n}\sum_{x=1}^{n} \langle e_x(nt)\rangle_\rho \leq C. \tag{2.5.51}$$

The later is rather straightforward due to our previous calculations. First, notice that since the total energy is conserved we have:

$$\frac{1}{n}\sum_{x=1}^{n} \langle e_x(nt)\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n} \langle e_x\rangle_\rho .$$

On the other hand we have:

$$\frac{1}{n}\sum_{x=1}^{n} \langle e_x\rangle_\rho = \frac{1}{n}\sum_{x=1}^{n}\Big(\frac{\langle p_x\rangle_\rho^2}{2m_x} + \frac{\langle r_x\rangle_\rho^2}{2}\Big) + \frac{1}{n}\sum_{x=1}^{n}\Big(\frac{\langle \tilde{p}_x^2\rangle_\rho}{2m_x} + \frac{\langle \tilde{r}_x^2\rangle_\rho}{2}\Big), \tag{2.5.52}$$

where $\tilde{p}_x = p_x - \langle p_x\rangle_\rho$, $\tilde{r}_x = r_x - \langle r_x\rangle_\rho$. The first sum in (2.5.52), is bounded by a constant $C_1 > 0$, since we have $\langle p_x\rangle_\rho = \frac{m_x}{\bar{m}}\bar{p}_x - \mathcal{E}_x^n$, $\langle r_x\rangle_\rho = \bar{r}_x$ from (2.2.42), where $\bar{p}_x = \bar{p}(\frac{x}{n})$, $\bar{r}_x = \bar{r}(\frac{x}{n})$, and $\mathcal{E}_x^n$ is given in (2.2.39). Notice that we used the fact that $|\mathcal{E}_n^x|$ is bounded by a deterministic constant, which is evident from its definition and properties of m_x, β and $\bar{p}$.

The second sum in (2.5.52), is bounded by $C_2 > 0$ independent of n, which is given in (2.2.60) in Remark 2.2.5. Therefore $\frac{1}{n}\sum_{x=1}^{n}\langle e_x\rangle_\rho$ is bounded by another deterministic constant C, uniform in n.

Proof of (2.1.18) *for* $f \in C^0$. After all, we can state the proof of (2.1.18) for a test function $f \in C^0([0,1])$. Fix $f \in C^0([0,1])$ and recall the definition of the regularizing family ζ_ϵ, for any $\epsilon > 0$ as follows.
Let $\zeta \in C_c^\infty(\mathbb{R})$, with

$$\text{supp}(\zeta) = [-1,1]; \quad \int_{\mathbb{R}} \zeta(y')dy' = 1; \quad \zeta(y) \geq 0; \quad \zeta(y) = \zeta(-y).$$

Let $\zeta_\epsilon := \frac{1}{\epsilon}\zeta(\frac{y}{\epsilon})$, for $0 < \epsilon < 1$. Notice that we have $\zeta_\epsilon \in C_c^\infty(\mathbb{R})$, $supp(\zeta_\epsilon) = [-\epsilon, \epsilon]$, $\int_{\mathbb{R}} \zeta_\epsilon(y')dy' = 1$, and $\zeta_\epsilon(y) \geq 0$ for any $0 < \epsilon < 1$.
For any $\delta > 0$, let $f_\delta : [0,1] \to \mathbb{R}$ be defined as

$$f_\delta(y) := (f * \zeta_\delta)(y) = \int f(y')\zeta(y - y')dy'.$$

By properties of convolution, we have $f_\delta \in C^1([0,1]), \forall \delta > 0$.
Fix $t \in [0,T]$, and $\epsilon > 0$, we are going to introduce a proper δ: recall the solution

to the macroscopic equation $\mathrm{e}(y,t) = \frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2} + \mathrm{f}^{\mu}_{\beta}(y)$, as we already observed $\frac{\mathrm{p}^2(y,t)}{2\bar{m}} + \frac{\mathrm{r}^2(y,t)}{2}$ is continuous, thanks to the regularity assumption on $\bar{r}$ and $\bar{p}$. Moreover, we proved the continuity of $\mathrm{f}^{\mu}_{\beta}(y)$ in Proposition 2.A.1.1, hence $\mathrm{e}(y,t)$ is positive and continuous on $[0,1]$ (positivity of $\mathrm{f}^{\mu}_{\beta}(y)$ is evident from the construction), and $\int_0^1 \mathrm{e}(y,t)$ is bounded by a constant $C_0 > 0$. Let $\tilde{C} = \max\{C_0, C\}$, where C is the uniform bound on $\frac{1}{n}\sum_{x=1}^{n} \langle e_x \rangle_\rho$. Since f is continuous on $[0,1]$, it will be uniformly continuous, so there exists $\delta(\epsilon) > 0$, such that if $y, y' \in [0,1]$, $|y - y'| \leq 2\delta$, then $|f(y) - f(y')| \leq \frac{\epsilon}{4\tilde{C}}$. This choice of δ, besides the properties of ζ_δ, in particular, the fact that $supp(\zeta_\delta) = [-\delta, \delta]$, and $\int \zeta_\delta(y)dy = 1$, lead to the following estimate: $\sup_y |f_\delta(y) - f(y)| \leq \frac{\epsilon}{4\tilde{C}}$, this can be observed from the definition of f_δ:

$$
\begin{aligned}
&|(f - f_\delta)(y)| = \Big| f(y) - \int f(y')\zeta_\delta(y - y')dy' \Big| = \Big| \int f(y)\zeta(y - y')dy' - \int f(y')\zeta_\delta(y - y')dy' \Big| \\
&\leq \int |f(y) - f(y')|\zeta_\delta(y - y')dy' = \int_{y-\delta}^{y+\delta} |f(y) - f(y')|\zeta_\delta(y - y')dy' \\
&\leq \sup_{|y-y'|\leq 2\delta} (|f(y) - f(y')|) \leq \frac{\epsilon}{4\tilde{C}}.
\end{aligned}
\tag{2.5.53}
$$

Notice that this bound is valid for any y, so we can take the sup over y. By using the later, again, thanks to the choice of δ, definition of ζ_δ, and $\tilde{C}$ we obtain

$$
\begin{aligned}
&\Big| \frac{1}{n}\sum_{x=1}^{n} (f - f_\delta)(\frac{x}{n}) \langle e_x(nt) \rangle_\rho \Big| \leq \sup_y |(f - f_\delta)(y)| \frac{1}{n}\sum_{x=1}^{n} \langle e_x(nt) \rangle_\rho \leq \frac{\epsilon}{4\tilde{C}} C \leq \frac{\epsilon}{4}, \\
&\Big| \int_0^1 (f - f_\delta)(y) \Big| \mathrm{e}(y,t)dy \leq \sup_y |(f - f_\delta)(y)| \int_0^1 \mathrm{e}(y,t)dy \leq \frac{\epsilon}{4\tilde{C}} C_0 \leq \frac{\epsilon}{4},
\end{aligned}
\tag{2.5.54}
$$

uniformly in n. Now let us bound:

$$
\begin{aligned}
&\Big| \frac{1}{n}\sum_{x=1}^{n} f(\frac{x}{n}) \langle e_x(nt) \rangle_\rho - \int_0^1 f_\delta(y)\mathrm{e}(y,t)dy \Big| = \Big| \Big(\frac{1}{n}\sum_{x=1}^{n} \Big(f(\frac{x}{n}) \langle e_x(nt) \rangle_\rho + \\
&\langle e_x(nt) \rangle_\rho (f - f_\delta)(\frac{x}{n}) \Big) \Big) - \Big(\int_0^1 f_\delta(y)\mathrm{e}(y,t) + \int_0^1 (f - f_\delta)(y)\mathrm{e}(y,t)dy \Big) \Big| \leq \\
&\Big| \frac{1}{n}\sum_{x=1}^{n} f_\delta(\frac{x}{n}) \langle e_x(nt) \rangle_\rho - \int_0^1 f_\delta(y)\mathrm{e}(y,t)dy \Big| + \Big| \frac{1}{n}\sum_{x=1}^{n} (f - f_\delta)(\frac{x}{n}) \langle e_x(nt) \rangle_\rho \Big| + \\
&\Big| \int_0^1 (f - f_\delta)(y)\mathrm{e}(y,t)dy \Big| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4}.
\end{aligned}
\tag{2.5.55}
$$

The last two term were bounded by (2.5.54). In order to treat the first term, note that as we observed $f_\delta \in C^1([0,1])$, so we can use the result of the previous section i.e. (2.5.50) for f_δ:

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^{n} f_\delta(\frac{x}{n}) \langle e_x(nt) \rangle_\rho \to \int_0^1 f_\delta(y)\mathrm{e}(y,t)dy, \tag{2.5.56}
$$

almost surely.
Hence, there exists N, such that for $n > N$, we have $|\frac{1}{n}\sum_{x=1}^{n} f_\delta(\frac{x}{n}) \langle e_x(nt) \rangle_\rho - \int_0^1 f_\delta(y)\mathrm{e}(y,t)dy| \leq \frac{\epsilon}{2}$, almost surely, and this finishes the proof of Theorem 2.1.1. □

2.A Properties of the Limiting Function

In this section, we study the properties of $\mathfrak{f}^{\mu}_{\beta}$, which is defined in (2.4.5). We prove a couple of lemmas to facilitate the proof of Theorem 2.4.4. In particular, we prove the existence of the limit (2.4.5), this means that $\mathfrak{f}^{\mu}_{\beta}$ is well-defined. Moreover, we show that this function is continuous. Furthermore, we treat the equilibrium case and demonstrate that the function $\mathfrak{f}^{\mu}(\beta_{eq})$ is well defined, i.e. $\mathfrak{f}^{\mu}_{\beta_{eq}}(y)$ does not depend on y. Then we demonstrate the relation (2.4.36). Finally, we prove a lemma which has been needed in 2.4.4. All the proofs in this section share the same spirit: We represent $\langle \tilde{e}_x \rangle_{\rho}$ in terms of the Taylor series, then cut the series similar to (2.4.22), and control the expressions depending on the first part of the Taylor series, using the fact that the number of these terms is finite, β is continuous and bounded, and the distribution of the masses is compactly supported. Finally, we use the fact that the terms depending on the remainder of the series is small.

Notice that since n is not fixed here, we denote the ensemble average with $\langle \rangle_{\rho^n}$, in order to emphasize the dependence on n. Before proceeding, recall the definition of A^{β}_{p}, and A^{β}_{r} from (2.2.25), since here n is not fixed and we study matrices with different sizes, we change our notations only in this section and denote these matrices by A^{p}_{n}, and A^{r}_{n}, respectively i.e.

$$A^{p}_{n} := M_{\beta}^{-\frac{1}{2}}(-\nabla_{-}\beta^{0}\nabla_{+})M_{\beta}^{-\frac{1}{2}}, \quad A^{r}_{n} := (\beta^{o})^{\frac{1}{2}}(-\nabla_{+}M_{\beta}^{-1}\nabla_{-})(\beta^{o})^{\frac{1}{2}},$$

where $M_{\beta} = M\beta^{-1}$, with $M = \mathrm{diag}(m_1, \dots, m_n)$, $\beta = \mathrm{diag}(\beta(\frac{1}{n}), \dots, \beta(\frac{n}{n}))$, and $\beta^{o} = \mathrm{diag}(\beta(\frac{1}{n}), \dots, \beta(\frac{n-1}{n}))$.
Since all these proofs share the same spirit, we set a handful of notation here:

Recall the average expression $\langle \tilde{e}_x \rangle_{\rho^n}$ from (2.4.18):

$$\langle \tilde{e}_x \rangle_{\rho} = \frac{1}{\beta_x}\Big(\langle x, \mathfrak{f}(A^{r}_{n})x\rangle_{n-1} + \langle x, \mathfrak{f}(A^{p}_{n})x\rangle_{n} + \epsilon^{x}_{n}\Big). \tag{2.A.1}$$

Thanks to (2.4.17), the part corresponding to p can be written as follows:

$$\langle x, \mathfrak{f}(A^{p}_{n})x\rangle_{n} = \sum_{k=0}^{\infty} a_k \langle x, (A^{p}_{n} - \alpha I_n)^{k}x\rangle_{n}. \tag{2.A.2}$$

As we argued in Section 2.4.1, there exists a constant $c_0 > 0$, such that for every realization of the masses, and $\forall n$, $||A^{n}_{p}||_2 \leq c_0$. Therefore, using the properties of Taylor series as in (2.4.25), we observe that $\forall \epsilon > 0$, there exists $K_*(\epsilon) \in \mathbb{N}$, such that $\forall n$, $\forall x \in \mathbb{I}_n$, and any realization of the masses we have:

$$\Big| \sum_{k > K_*(\epsilon)} a_k \langle x, (A^{p}_{n} - \alpha I_n)^{k}x\rangle_{n} \Big| \leq \epsilon. \tag{2.A.3}$$

Notice that we can choose $K_*(\epsilon)$, such that the same bound (2.A.3) holds when we substitute A^{p}_{n} with A^{r}_{n}. Hence, given $\epsilon > 0$ one can define $\mathfrak{f}^{\epsilon}_{\prec}(.)$ and $\mathfrak{f}^{\epsilon}_{\succ}(.)$ as follows:

$$\mathfrak{f}^{\epsilon}_{\prec}(A^{p}_{n}) := \sum_{k=0}^{K_*(\epsilon)} a_k (A^{p}_{n} - \alpha I_n)^{k}, \qquad \mathfrak{f}^{\epsilon}_{\succ}(A^{p}_{n}) := \sum_{k > K_*(\epsilon)}^{\infty} a_k (A^{p}_{n} - \alpha I_n)^{k}. \tag{2.A.4}$$

In particular, we can rewrite (2.A.3) in the following way: For any $\epsilon > 0$ there exits $K_*(\epsilon)$ such that for any n and $x \in \mathbb{I}_n$ we have:

$$\begin{aligned} &|\langle x, \mathfrak{f}^{\epsilon}_{\succ}(A^p_n)x\rangle_n| \leq \epsilon, \quad |\mathbb{E}(\langle x, \mathfrak{f}^{\epsilon}_{\succ}(A^p_n)x\rangle_n)| \leq \epsilon, \\ &|\langle x, \mathfrak{f}^{\epsilon}_{\succ}(A^r_n)x\rangle_{n-1}| \leq \epsilon, \quad |\mathbb{E}(\langle x, \mathfrak{f}^{\epsilon}_{\succ}(A^r_n)x\rangle_{n-1})| \leq \epsilon. \end{aligned} \tag{2.A.5}$$

Fix $(k, n) \in \mathbb{N}^2$, denote $A^p_n - \alpha I_n$ by $\tilde{A}^p_n$, take $x \in \mathbb{I}_n$, and consider the following term: $\langle x, (\tilde{A}^p_n)^k x\rangle$. Here we represent this term in a more appropriate manner, introducing following notations. First, recall the random walk representation of $\langle x, (\tilde{A}^p_n)^k x\rangle$:

$$\langle x, (\tilde{A}^p_n)^k x\rangle = \sum_{x_1,\dots x_{k-1}=1}^{n} \langle x, \tilde{A}^{\beta}_p x_1\rangle\langle x_1, \tilde{A}^{\beta}_p x_2\rangle \dots \langle x_{k-1}, \tilde{A}^{\beta}_p x\rangle. \tag{2.A.6}$$

Denote the set of indices with non-zero contribution, in RHS of (2.A.6) by $\mathcal{I}^{x,p}_{n,k}$:

$$\mathcal{I}^{x,p}_{n,k} := \{(x_1, \dots x_{k-1}) \in \mathbb{I}^{k-1}_n | \langle x, \tilde{A}^p_n x_1\rangle\langle x_1, \tilde{A}^p_n x_2\rangle \dots \langle x_{k-1}, \tilde{A}^p_n x\rangle \neq 0\}, \tag{2.A.7}$$

where $\tilde{A}^p_n = A^p_n - \alpha I_n$. We denote the elements of $\mathcal{I}^{x,p}_{n,k}$ by $\underline{x} := (x_1, \dots, x_{k-1})$. Notice that there is a bijection between $\mathcal{I}^{x,p}_{n,k}$ and the set of paths in $[0, k]\times[0, n]\cap\mathbb{Z}^2$ from the point $(x, 0)$ to (x, k), consisting of the following vectors: $(-1, 1), (0, 1), (1, 1) \in \mathbb{Z}^2$. In fact, this bijection is given as follows: $\forall \underline{x} \in \mathcal{I}^{x,p}_{n,k}$, assign to $\underline{x}$ the path which is given by the following points: $(0, x)$, $(1, x_1)$,..., (j, x_j), ...,$(k-1, x_{k-1}), (k, x)$.

Let $\tilde{\mathcal{I}}^{x,p}_{n,k}$ be the set where every element of $\mathcal{I}^{x,p}_{n,k}$ shifted by the vector $(x, \dots, x) \in (\mathbb{I}_n)^{k-1}$, precisely define:

$$\tilde{\mathcal{I}}^{x,p}_{n,k} := \{(x_1 - x, \dots, x_{k-1} - x) | \underline{x} \in \mathcal{I}^{x,p}_{n,k}\}. \tag{2.A.8}$$

We denote each element of $\tilde{\mathcal{I}}^{x,p}_{n,k}$ by $\underline{\eta} = (\eta_1, \dots, \eta_{k-1})$. Notice that $\underline{\eta}$ corresponds to a path from $(0, 0)$ to $(0, k)$ in $[0, k]\times[-x, n-x]\cap\mathbb{Z}^2$ consisting of the aforementioned vectors. Finally, define $\tilde{\mathcal{I}}_k$ as follows:

$$\tilde{\mathcal{I}}_k := \{\underline{\eta} \in \mathbb{Z}^{k-1} | \forall i \in \mathbb{I}_k,\ |\eta_i - \eta_{i-1}| \leq 1, \text{ with } \eta_0 = \eta_k = 0\}. \tag{2.A.9}$$

Observe that each $\underline{\eta} \in \mathcal{I}_k$, corresponds to a path from $(0, 0)$ to $(0, k)$ in $\mathbb{Z}^2$ consisting of $(-1, 1), (0, 1), (1, 1) \in \mathbb{Z}^2$.
Having in mind the geometric interpretation of $\tilde{\mathcal{I}}^{x,p}_{n,k}$ and $\tilde{\mathcal{I}}_k$, one can observe[19]:

$$\begin{aligned} &\tilde{\mathcal{I}}^{x,p}_{n,k} \subset \tilde{\mathcal{I}}_k, \quad |\tilde{\mathcal{I}}^{x,p}_{n,k}| \subset |\tilde{\mathcal{I}}_k| \leq 3^k, \\ &\tilde{\mathcal{I}}^{x,p}_{n,k} = \tilde{\mathcal{I}}_k \quad \text{iff} \quad [\frac{k}{2}] \leq x \leq n - [\frac{k}{2}]. \end{aligned} \tag{2.A.10}$$

By using (2.A.8), we can rewrite the sum in (2.A.6), as a sum over the set $\tilde{\mathcal{I}}^{x,p}_{n,k}$. Here we introduce a set of notations in order to rewrite each term in (2.A.6) in a more

[19]This bound is obvious, since $|\tilde{\mathcal{I}}_k|$ is the solution to the problem of the number of path from $(0, 0)$ to $(k, 0)$ consisting of the vectors $(1, 1), (1, -1), (1, 0)$, this is equivalent to the number of solution of $s_1 + \dots + s_k = 0$, for $s_i \in \{-1, 0, 1\}$, which is obviously bounded by 3^k. This bound is not sharp and a better asymptotic will be $\frac{c3^k}{\sqrt{k}}$ for c around $\frac{1}{2}$, but 3^{n_*} is sufficient for our purposes.

suitable way.
Fix $k \in \mathbb{N}$, and take $\underline{\eta} \in \tilde{\mathcal{I}}_k$, then consider the set of indices $\underline{j} = (j_{-[\frac{k}{2}]}, \dots, j_0, \dots, j_{[\frac{k}{2}]+1}) \in \mathbb{Z}^{2[\frac{k}{2}]+2}$, where $j_i \neq j_{i'}$ for $i \neq i'$. Correspondingly, let $\underline{m}^k$ denotes a vector of $2[\frac{k}{2}]+2$ masses indexed by $\underline{j}$, i.e. $\underline{m}^k = (m_{j_{-[\frac{k}{2}]}}, \dots, m_{j_{[\frac{k}{2}]+1}})$, notice that we extended the set of i.i.d random variables $\{m_x\}_{x=1}^n$ to the set of i.i.d random variables $\{m_x\}_{x\in\mathbb{Z}}$[20]. Moreover, let $\underline{b}^k \in [\beta_{min}, \beta_{max}]^{2[\frac{k}{2}]+2}$ denotes the following vector: $\underline{b}^k = (b_{-[\frac{k}{2}]}, \dots, b_{[\frac{k}{2}]+1})$. We define $\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, \underline{b}^k)$ as follows:

$$\begin{aligned}
&\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, \underline{b}^k) = \theta_{\eta_0\eta_1} \dots \theta_{\eta_i\eta_{i+1}} \dots \theta_{\eta_{k-1}\eta_k}, \\
&\theta_{\eta_i\eta_{i+1}} = \begin{cases} b_{\eta_i}\left(\frac{b_{\eta_i}}{m_{j_{\eta_i}}} + \frac{b_{(\eta_i+1)}}{m_{j_{(\eta_i+1)}}}\right) - \alpha \quad \text{if} \quad \eta_i = \eta_{i+1}, \\ -\frac{b_{\hat{\eta}_i}}{m_{j_{\hat{\eta}_i}}}\sqrt{b_{\hat{\eta}_i} b_{(\hat{\eta}_i+1)}} \quad \text{if} \quad \eta_i \neq \eta_{i+1}, \end{cases}
\end{aligned} \tag{2.A.11}$$

where we denoted $\hat{\eta}_i = \min\{\eta_i, \eta_{i+1}\}$, and $\eta_0 = \eta_k = 0$. The following properties of $\mathcal{F}_{k,\underline{\eta}}$ are straightforward, since the distribution of the masses is compactly supported and $\mathcal{F}_{k,\underline{\eta}}$ is continuous on a compact set:

- $\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, \underline{b}^k)$ is uniformly continuous in the second component, uniformly in the distribution of the masses. More precisely, $\forall \epsilon > 0$, there exists $\delta > 0$ such that if $|\underline{b}_1^k - \underline{b}_2^k| < \delta$,[21] then for any realization of the masses, $|\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, \underline{b}_1^k) - \mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, \underline{b}_2^k)| \leq \epsilon$.

- Consider two different set of masses $\underline{m}_1^k$ and $\underline{m}_2^k$, since masses are i.i.d, we have: $\mathbb{E}(\mathcal{F}_{k,\underline{\eta}}(\underline{m}_1^k, \underline{b}^k)) = \mathbb{E}(\mathcal{F}_{k,\underline{\eta}}(\underline{m}_2^k, \underline{b}^k))$. Notice that we used the assumption $j_i \neq j_{i'}$ for $i \neq i'$.

- From the above properties, it is clear that taking two set of masses $\underline{m}_1^k$ and $\underline{m}_2^k$ we have: $\forall \epsilon > 0$, there exists $\delta > 0$ such that if $|\underline{b}_1^k - \underline{b}_2^k| < \delta$, then $|\mathbb{E}(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k, {}_1\, \underline{b}_1^k)) - \mathbb{E}(\mathcal{F}_{k,\underline{\eta}}(\underline{m}_2^k, \underline{b}_2^k))| < \epsilon$.

Recall that we fixed k and $\underline{\eta} \in \tilde{\mathcal{I}}_k$. Let us take $n \in \mathbb{N}$ and $x \in \mathbb{I}_n$, then we define the vectors $\underline{m}^k(x,n)$, $\underline{b}^k(x,n)$ as follows:

$$\begin{aligned}
&\underline{m}^k(x,n)_{j_i} = m_{x+i}, \quad -[\frac{k}{2}] \leq i \leq [\frac{k}{2}] + 1, \\
&\underline{b}^k(x,n)_i = \begin{cases} \beta(\frac{x+i}{n}) \quad \text{if } 0 \leq x+i \leq n, \\ \beta(1) \quad \text{if } n < x+i, \\ \beta(0) \quad \text{if } 0 < x+i, \end{cases} \qquad -[\frac{k}{2}] \leq i \leq [\frac{k}{2}] + 1.
\end{aligned} \tag{2.A.12}$$

Finally, by combining the above notations and definitions, in particular (2.A.8), (2.A.11), and (2.A.12), we end-up with the following identity for $[\frac{k}{2}] + 1 \leq x \leq$

[20] This extension is not necessary, it is done to make our notation coherent; however we do not use of this extension.

[21] Here $|.|$ denotes the Euclidean distance in $\mathbb{R}^{2[\frac{k}{2}]+2}$.

$n - [\frac{k}{2}] + 1$:

$$\langle x, (\tilde{A}_n^p)^k x\rangle_n = \sum_{\underline{\eta}\in\tilde{\mathcal{I}}_{n,k}^{x,p}} \mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k(x,n), \underline{b}^k(x,n)\Big) = \sum_{\underline{\eta}\in\tilde{\mathcal{I}}_k} \mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k(x,n), \underline{b}^k(x,n)\Big), \tag{2.A.13}$$

where we have the second equality thanks to (2.A.10) and the choice of x. Notice that to check this identity one should compare the definition of $\mathcal{F}_{k,\eta}$ with the definition of the matrix $\tilde{A}_n^p$ in (2.4.10). Moreover, this identity holds for $x < [\frac{k}{2}] + 1$ and $x > n - [\frac{k}{2}] + 1$ if one slightly modifies the definition of $\mathcal{F}_{k,\underline{\eta}}$. However, the current form of this identity is sufficient for our purposes. It is worth mentioning that $\langle x, (\tilde{A}_n^r)^k x\rangle_{n-1}$ can be written in the similar fashion, where one should define $\tilde{\mathcal{F}}_{k,\underline{\eta}}$ similar to $\mathcal{F}_{k,\underline{\eta}}$. One can check that $\tilde{\mathcal{F}}_{k,\underline{\eta}}$ has the three aforementioned properties. Since this task is rather straightforward, we only treat the terms corresponding to p and the terms corresponding to r can be treated similarly.
Thanks to (2.A.4) and (2.A.13), we can establish the existence of the following limit: $\lim_{n\to\infty} \mathbb{E}(\big\langle \tilde{e}_{[ny]}\big\rangle_{\rho^n})$, for every $y \in [0,1]$. We prove this fact by showing that the sequence $\mathbb{E}(\big\langle \tilde{e}_{[ny]}\big\rangle_{\rho^n})$ is a Cauchy sequence.

Lemma 2.A.1. *Recall the assumption on the distribution of the masses, where $\mu(x)$ is smooth and supported on $[m_{min}, m_{max}]$, $0 < m_{min} < m_{max} < \infty$. We have $\forall y \in [0,1]$, and $\forall \epsilon > 0$, there exists N_0, such that $\forall n, l > N_0$, we have $|\mathbb{E}(\big\langle \tilde{e}_{[ny]}\big\rangle_{\rho^n}) - \mathbb{E}(\big\langle \tilde{e}_{[ly]}\big\rangle_{\rho^l})| \le \epsilon$.*

Proof. Take $\epsilon > 0$, and recall (2.A.1) then we have:

$$\Big|\mathbb{E}\Big(\big\langle \tilde{e}_{[ny]}\big\rangle\Big) - \mathbb{E}\Big(\big\langle \tilde{e}_{[ly]}\big\rangle\Big)\Big| = \Big|\mathbb{E}\Big(\frac{1}{\beta_{[ny]}}\Big(\langle [ny], \mathfrak{f}(A_n^r)[ny]\rangle_{n-1} + \langle [ny], \mathfrak{f}(A_n^p)[ny]\rangle_n + \epsilon_n^{[ny]}\Big)\Big)$$
$$- \mathbb{E}\Big(\frac{1}{\beta_{[ly]}}\Big(\langle [ly], \mathfrak{f}(A_n^r)[ly]\rangle_{l-1} + \langle [ly], \mathfrak{f}(A_l^p)[ly]\rangle_l + \epsilon_l^{[ly]}\Big)\Big)\Big|. \tag{2.A.14}$$

Since β is continuous, and $0 < \beta_{min} \le \beta(y) \le \beta_{max}$, we have $\beta_{[ny]} = \beta(\frac{[ny]}{n})$ and $\beta(\frac{[ly]}{l})$ are sufficiently close. In addition, $\mathfrak{f}(A_n^r)$,$\mathfrak{f}(A_n^p)$ are uniformly bounded in n. Hence, it is sufficient to show

$$\begin{aligned}&\Big|\mathbb{E}\Big(\langle [ny], \mathfrak{f}(A_n^r)[ny]\rangle_{n-1} + \langle [ny], \mathfrak{f}(A_n^p)[ny]\rangle_n + \epsilon_n^{[ny]}\Big) - \\ &\mathbb{E}\Big(\langle [ly], \mathfrak{f}(A_l^r)[ly]\rangle_{l-1} + \langle [ly], \mathfrak{f}(A_l^p)[ly]\rangle_l + \epsilon_l^{[ly]}\Big)\Big| < \epsilon,\end{aligned} \tag{2.A.15}$$

for $n, l > N_0$ with N_0 large enough. But the terms $|\epsilon_n^{[ny]}|$ and $|\epsilon^{[ly]}|$ are bounded by $\frac{\mathcal{C}}{n}$ and $\frac{\mathcal{C}}{l}$, respectively. Therefore, for N_0 large enough, they will be small, and it is enough to show:

$$\Big|\mathbb{E}\Big(\langle [ny], \mathfrak{f}(A_n^r)[ny]\rangle_{n-1} + \langle [ny], \mathfrak{f}(A_n^p)[ny]\rangle_n\Big) - \mathbb{E}\Big(\langle [ly], \mathfrak{f}(A_l^r)[ly]\rangle_{l-1} + \langle [ly], \mathfrak{f}(A_l^p)[ly]\rangle_l\Big)\Big| < \epsilon,$$

for proper N_0. Actually, we prove that there exists N_0, such that for $n, l > N_0$,

$$\Big|\mathbb{E}\Big(\langle [ny], \mathfrak{f}(A_n^p)[ny]\rangle_n\Big) - \mathbb{E}\Big(\langle [ly], \mathfrak{f}(A_n^p)[ly]\rangle_l\Big)| < \epsilon. \tag{2.A.16}$$

The term $|\mathbb{E}(\langle [ny], \mathfrak{f}(A_n^r)[ny]\rangle_{n-1} - \mathbb{E}(\langle [ly], \mathfrak{f}(A_l^r)[ly]\rangle_{l-1}|$ can be treated exactly the same way. In order to demonstrate (2.A.16), recall the definition of $\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^p)$ and $\mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(A_n^r)$ from the expression (2.A.4), and recall $K_*(\frac{\epsilon}{4})$, which is given in this definition. Taking advantage of (2.A.5), we get

$$\Big|\mathbb{E}\Big(\langle [ny], \mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(A_n^p)[ny]\rangle_n\Big) - \mathbb{E}\Big(\langle [ly], \mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(A_l^p)[ly]\rangle_l\Big)\Big| \leq \frac{\epsilon}{2}.$$

By using the fact that $\mathfrak{f}(A_n^p) = \mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^p) + \mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(A_n^p)$, and $\mathfrak{f}(A_l^p) = \mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_l^p) + \mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(A_l^p)$, it is sufficient to prove that for $n, l > N_0$:

$$\Big|\mathbb{E}\Big(\langle [ny], \mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^p)[ny]\rangle\Big) - \mathbb{E}\Big(\langle [ly], \mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_l^p)[ly]\rangle\Big)\Big| \leq \frac{\epsilon}{2}. \tag{2.A.17}$$

Since $K_*(\frac{\epsilon}{4})$ is independent of n and l, it is enough to prove that $\forall \tilde{\epsilon} > 0$, there exist N_0 such that $\forall k \in \{0, \dots, K_*(\frac{\epsilon}{4})\}$ and $\forall n, l > N_0$:

$$\Big|\mathbb{E}\Big(\langle [ny], (A_n^p - \alpha I_n)^k[ny]\rangle\Big) - \mathbb{E}\Big(\langle [ly], (A_l^p - \alpha I_l)^k[ly]\rangle\Big)\Big| < \tilde{\epsilon}. \tag{2.A.18}$$

Then, taking $\tilde{\epsilon} = \frac{\epsilon}{2cK_*(\frac{\epsilon}{4})}$, where c is the bound on $|a_0|, \dots, |a_{K_*(\frac{\epsilon}{4})}|$, completes the proof. We can obtain (2.A.18) by using (2.A.13) and properties of $\mathcal{F}_{k,\underline{\eta}}$ as follows: Let us assume $y \in (0,1)$[22], fix $k \in \{1, \dots K_*(\frac{\epsilon}{4})\}$, and take N' such that for $n, l > N'$ we have: $K_*(\frac{\epsilon}{4}) < [ny] < n - K_*\frac{\epsilon}{4}$ and $K_*(\frac{\epsilon}{4}) < [ly] < l - K_*(\frac{\epsilon}{4})$. By this choice we can use (2.A.13) and observe:

$$\begin{aligned}&\Big|\mathbb{E}\Big(\langle [ny], (A_n^p - \alpha I_n)^k[ny]\rangle_n\Big) - \mathbb{E}\Big(\langle [ly], (A_l^p - \alpha I_l)^k[ly]\rangle_l\Big)\Big| = \\ &\Big|\sum_{\underline{\eta} \in \tilde{\mathcal{I}}_k} \Big(\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k([ny], n), \underline{b}^k([ny], n)\Big)\Big) - \mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k([ly], l), \underline{b}^k([ly], l)\Big)\Big)\Big)\Big|.\end{aligned} \tag{2.A.19}$$

Thanks to the third property of $\mathcal{F}_{k,\underline{\eta}}$, $\forall\ \epsilon' > 0$, there exists $\delta_{\underline{\eta}}(\epsilon')$ such that for $|\underline{b}^k([ly], l) - \underline{b}^k([ny], n)| \leq \delta_{\underline{\eta}}(\epsilon')$, we have

$$\Big|\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k([ny], n), \underline{b}^k([ny], n)\Big)\Big) - \mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}\Big(\underline{m}^k([ly], l), \underline{b}^k([ly], l)\Big)\Big)\Big| \leq \epsilon'.$$

On the other hand, since $\beta(.)$ is continuous, there exist $N_{\underline{\eta}}^k$ such that for $n, l > N_{\underline{\eta}}^k$ and for all $-[\frac{k}{2}] - 1 < i < [\frac{k}{2}] + 1$, we have

$$|\beta(\frac{[ny] + i}{n}) - \beta(\frac{[ly] + i}{l})| \leq \frac{\delta_{\underline{\eta}}(\frac{\tilde{\epsilon}}{3^{K_*(\frac{\epsilon}{4})}})}{\sqrt{k+3}}.$$

[22]The case $y = 0$ or $y = 1$, corresponds to the paths which are constructed by vectors of the form $(1,0), (1,-1)$, for $y = 0$, and $(1,0), (1,1)$ for $y = 1$. In either case, our argument is similar, where we can modify the set $\tilde{\mathcal{I}}_k$ and function $\mathcal{F}$ accordingly.

Hence, thanks to the definition of $\underline{b}^k$ (2.A.12), for $n, l > N^k_{\underline{\eta}}$ we get $|\underline{b}^k([ly], l) - \underline{b}^k([ny], n)| \leq \delta_{\underline{\eta}}(\frac{\tilde{\epsilon}}{3^{K_*(\frac{\epsilon}{4})}})$. Consequently, if we take $N_k = \max_{\{\underline{\eta}\in\tilde{\mathcal{I}}_k\}}\{N^k_{\underline{\eta}}, N'\}$, $\forall n, l > N_k$, we have $\forall \underline{\eta} \in \tilde{\mathcal{I}}_k$:

$$\Big|\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny], n), \underline{b}^k([ny], n))\Big) - \mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ly], l), \underline{b}^k([ly], l))\Big)\Big| \leq \frac{\tilde{\epsilon}}{3^{K_*(\frac{\epsilon}{4})}}.$$

Combining the later with the estimate $|\tilde{\mathcal{I}}_k| \leq 3^k \leq 3^{K_*(\frac{\epsilon}{4})}$, we get (2.A.18). Finally, taking $N_0 = \max_{\{k\in\mathbb{I}_{K_*(\frac{\epsilon}{4})}\}}\{N_k, N'\}$ finishes the proof.
Notice that in order to deal with the term $|\mathbb{E}(\langle[ny], \mathfrak{f}(A^r_n)[ny]\rangle_{n-1} - \mathbb{E}(\langle[ly], \mathfrak{f}(A^r_l)[ly]\rangle_{l-1}|$ one should properly modify the definition of $\mathcal{F}_{\underline{\eta},k}$ and $\tilde{\mathcal{I}}^{x,p}_{n,k}$. In particular, in this case $\mathcal{F}_{\underline{\eta},k}$ is given by:

$$\begin{aligned}
&\tilde{\mathcal{F}}_{n,\underline{\eta}}(\underline{m}^k, \underline{b}^k) = \theta_{\eta_0\eta_1}\dots\theta_{\eta_i\eta_{i+1}}\dots\theta_{\eta_{k-1}\eta_k},\\
&\theta_{\eta_i,\eta_{i+1}} = \begin{cases} \underline{b}^k_{\eta_i}(\frac{\underline{b}^k_{\eta_i}}{m_{\eta_i+k}} + \frac{\underline{b}_{\eta_i+1}}{(m_{\eta_i+1})} - \alpha & \text{if} \quad \eta_i = \eta_{i+1},\\ -\frac{\underline{b}^k_{hat\eta_i+1}}{m_{\hat{\eta}_i}}\sqrt{\underline{b}^k_{\hat{\eta}_i}\underline{b}^k_{\hat{\eta}_i+1}} & \text{if} \quad \eta_i \neq \eta_{i+1}. \end{cases}
\end{aligned} \tag{2.A.20}$$

Since this function satisfies the same properties, the rest of the proof is exactly similar to the previous case.

□

As an obvious consequence of Lemma 2.A.1 we have:

Corollary 2.A.1.1. *$\forall y \in [0, 1]$, the limit $\lim_{n\to\infty}\mathbb{E}(\langle\tilde{e}_{[ny]}\rangle_{\rho^n})$ exists, and the function f^μ_β is well-defined.*

Moreover, following the proof of Lemma 2.A.1, we can deduce the following corollary as well:

Corollary 2.A.1.2. *In thermal equilibrium i.e. when for $\beta_{eq} \in (0, \infty)$, $\beta(y) = \beta_{eq}$ is constant in y, we have:*

$$\forall y, y' \in (0, 1), \qquad \mathrm{f}^\mu_\beta(y) = \mathrm{f}^\mu_\beta(y'). \tag{2.A.21}$$

In particular, the function $\mathrm{f}^\mu(.)$ in (2.4.35) is well defined.

Proof. In order to proof (2.A.21) it is enough to show that

$$\lim_{n\to\infty}\mathbb{E}(\langle\tilde{e}_{[ny]}\rangle_{\rho^{n,\beta_{eq}}}) - \mathbb{E}(\langle\tilde{e}_{[ny']}\rangle_{\rho^{n,\beta_{eq}}}) = 0.$$

We omit the subscript of ρ since it is clear that we are in thermal equilibrium. Take $\epsilon > 0$, first recall the expression of $\langle\tilde{e}_{[ny]}\rangle_\rho$ (2.A.1), then rewrite $\mathfrak{f}(.) = \mathfrak{f}^{\frac{\epsilon}{4}}_\prec(.) + \mathfrak{f}^{\frac{\epsilon}{4}}_\succ(.)$ as it has been defined in (2.A.4) and observe:

$$\begin{aligned}
&\Big|\mathbb{E}\Big(\langle\tilde{e}_{[ny]}\rangle_{\rho^{n,\beta_{eq}}}\Big) - \mathbb{E}\Big(\langle\tilde{e}_{[ny']}\rangle_{\rho^{n,\beta_{eq}}}\Big)\Big| \leq\\
&\frac{1}{\beta_{eq}}\Big|\mathbb{E}\Big(\langle[ny], \mathfrak{f}^{\frac{\epsilon}{4}}_\prec(A^p_n)[ny]\rangle) + \langle[ny], \mathfrak{f}^{\frac{\epsilon}{4}}_\prec(A^r_n)[ny]\rangle\Big) -\\
&\mathbb{E}\Big(\langle[ny], \mathfrak{f}^{\frac{\epsilon}{4}}_\prec(A^p_n)[ny]\rangle + \langle[ny'], \mathfrak{f}^{\frac{\epsilon}{4}}_\prec(A^r_n)[ny']\rangle\Big)\Big| + \epsilon
\end{aligned} \tag{2.A.22}$$

where we bounded the terms involving $\mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}$ by $\frac{\epsilon}{2}$, thanks to (2.A.5). Moreover, $|\epsilon_n^{[ny]}|+|\epsilon_n^{[ny']}|$ is bounded by $\frac{C}{n}$; therefore, it has been bounded by $\frac{\epsilon}{2}$, by taking $n > N_1$, for proper N_1. Lastly, recall $K_*(\frac{\epsilon}{4})$ from (2.A.4), and choose N_2 such that for $n > N_2$,

$$[\frac{K_*(\frac{\epsilon}{4})}{2}]+1 < [ny] < n-[\frac{K_*(\frac{\epsilon}{4})}{2}]-1, \quad [\frac{K_*(\frac{\epsilon}{4})}{2}]+1 < [ny'] < n-[\frac{K_*(\frac{\epsilon}{4})}{2}]-1.$$

Thanks to this choice, and by using (2.A.13), for any $k \in \{1,\dots,K_*(\frac{\epsilon}{4})\}$ we have:

$$|\mathbb{E}(\langle [ny],(A_n^p-\alpha I_n)^k[ny]\rangle_n)-\mathbb{E}(\langle [ny'],(A_n^p-\alpha I_n)^k[ny']\rangle_n)| = \Big|\sum_{\underline{\eta}\in\tilde{\mathcal{I}}_k}\Big(\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny],n),\underline{b}^k([ny],n))\Big)-\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny'],n),\underline{b}^k([ny'],n))\Big)\Big)\Big| = 0, \tag{2.A.23}$$

where, first, we used the fact that in thermal equilibrium we have $\underline{b}^k([ny],n) = \underline{b}^k([ny'],n)$, then we took advantage of the second property of $\mathcal{F}_{k,\underline{\eta}}$. Therefore, by using the definition of $\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}$, the term in second line of (2.A.22) is zero, for $n > N_2$ (the part corresponding to A_n^r is completely analogous). Hence taking $n > \max\{N_1, N_2\}$ gives us the desirable result. □

Proposition 2.A.1.1. *The function* $\mathrm{f}_\beta^\mu(y) : (0,1) \to \mathbb{R}$ *is continuous.*

Proof. Fix $\epsilon > 0$, and observe

$$|\mathrm{f}_\beta^\mu(y)-\mathrm{f}_\beta^\mu(y')| \le |\mathrm{f}_\beta^\mu(y)-\mathrm{f}_{\beta,n}^\mu(y)|+|\mathrm{f}_{\beta,n}^\mu(y)-\mathrm{f}_{\beta,n}^\mu(y')|+|\mathrm{f}_{\beta,n}^\mu(y')-\mathrm{f}_\beta^\mu(y')|, \tag{2.A.24}$$

where $\mathrm{f}_{\beta,n}^\mu(y) := \mathbb{E}(\langle \tilde{e}_{[ny]}\rangle_{\rho^n})$. Since $\lim_{n\to\infty}\mathrm{f}_{\beta,n}^\mu(y) = \mathrm{f}_\beta^\mu(y)$, if we take $n > N_1(\epsilon)$, then

$$|\mathrm{f}_\beta^\mu(y)-\mathrm{f}_{\beta,n}^\mu(y)|+|\mathrm{f}_{\beta,n}^\mu(y')-\mathrm{f}_\beta^\mu(y')| \le \frac{\epsilon}{2}.$$

Moreover, we claim that there exist $N_2(\epsilon)$, such that for $n > N_2(\epsilon)$, there exists δ such that for $|y-y'| < \delta$, $|\mathrm{f}_{\beta,n}^\mu(y)-\mathrm{f}_{\beta,n}^\mu(y')| < \frac{\epsilon}{2}$. Proving this claim completes the proof, since we can take $n > \max\{N_1(\epsilon), N_2(\epsilon)\}$ and observe that for $|y-y'| < \delta$, we have $|\mathrm{f}_\beta^\mu(y)-\mathrm{f}_\beta^\mu(y')| < \epsilon$.

However, the proof of this statement follows the same lines of Lemma 2.A.1 and Corollary (2.A.1.2). Similar to (2.A.22), we divide $\mathfrak{f}(.) = \mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(.)+\mathfrak{f}_{\succ}^{\frac{\epsilon}{4}}(.)$, and take $N_3(\epsilon)$ such that for $n > N_2(\epsilon)$ we have

$$[\frac{K_*(\frac{\epsilon}{4})}{2}]+1 < [ny] < n-[\frac{K_*(\frac{\epsilon}{4})}{2}]-1, \quad [\frac{K_*(\frac{\epsilon}{4})}{2}]+1 < [ny'] < n-[\frac{K_*(\frac{\epsilon}{4})}{2}]-1.$$

Therefore, we have:

$$\Big|\mathbb{E}(\Big\langle\tilde{e}_{[ny]}\Big\rangle_{\rho^n})-\mathbb{E}(\Big\langle\tilde{e}_{[ny']}\Big\rangle_{\rho^n})\Big| \le \frac{1}{\beta(\frac{[ny]}{n})}\Big|\mathbb{E}\Big(\langle [ny],\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^p)[ny]\rangle)+\langle [ny],\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^r)[ny]\rangle\Big)- \mathbb{E}\Big(\langle [ny],\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^p)[ny]\rangle+\langle [ny'],\mathfrak{f}_{\prec}^{\frac{\epsilon}{4}}(A_n^r)[ny']\rangle\Big)\Big|+ \frac{\epsilon}{2}+|\epsilon_n^{[ny]}|+|\epsilon_n^{[ny']}|+C\Big(\beta^{-1}(\frac{[ny']}{n})-\beta^{-1}(\frac{[ny']}{n})\Big), \tag{2.A.25}$$

where we can find N_4 and δ_0 such that the last line will be bounded by $\frac{3\epsilon}{4}$, for $n > N_4$ and $|y-y'| < \delta_0$. Let us take $N_2(\epsilon) = \max\{N_3(\epsilon), N_4\}$. Recall the definition of $\mathfrak{f}^{\frac{\epsilon}{4}}_{\prec}(.)$ (2.A.4) as a Taylor sum up to $K_*(\frac{\epsilon}{4})$ terms. By using the choice of $N_2(\epsilon)$ rewrite each term of this sum as a sum over the paths $\underline{\eta} \in \tilde{\mathcal{I}}_k$ as in (2.A.13). The rest of the proof boils down to demonstrating the fact that for $n > N_2(\epsilon)$, for all $1 \leq k \leq K_*(\frac{\epsilon}{4})$, and for all $\underline{\eta} \in \tilde{\mathcal{I}}_k$, we have: $\forall \hat{\epsilon} > 0$ there exist $\delta_{k,\underline{\eta}}(\hat{\epsilon}) > 0$, such that if $|y - y'| < \delta_{k,\underline{\eta}}(\hat{\epsilon})$ then

$$\Big|\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny],n),\underline{b}^k([ny],n))\Big) - \mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny'],n),\underline{b}^k([ny'],n))\Big)\Big| < \hat{\epsilon}. \tag{2.A.26}$$

However, recalling the definition of $\underline{b}^k([ny],n)$ from (2.A.12), since $\beta(.)$ is continuous, one can observe that if $|y-y'| < \tilde{\delta}$, $|\underline{b}^k([ny],n) - \underline{b}^k([ny'],n)|$ is sufficiently small for a proper choice of $\tilde{\delta}$. Therefore, by using the third property of $\mathcal{F}_{k,\eta}$, we obtain the desired $\delta_{k,\underline{\eta}}(\hat{\epsilon})$. Finally, taking

$$\hat{\epsilon} = \frac{\epsilon}{3^{K_*(\frac{\epsilon}{4})}K_*(\frac{\epsilon}{4})4c}, \quad \delta = \min_{0<k\leq K_*(\frac{\epsilon}{4})}\{\delta_0, \min_{\underline{\eta}\in\tilde{\mathcal{I}}_k}\{\delta_{k,\underline{\eta}}\}\}$$

finishes the proof. Notice that here c is the uniform bound on Taylor coefficients $|a_0|, \dots |a_{K_*(\frac{\epsilon}{4})}|$. Moreover, the choice of $\hat{\epsilon}$ is justified by the bound $|\tilde{\mathcal{I}}_k| \leq 3^k$. Furthermore, as usual the part corresponding to r can be treated exactly in the same way. □

The next proposition proves (2.4.36), and illustrates the fact that $\mathrm{f}^{\mu}_{\beta}(y)$ is in fact a function of inverse temperature at point y i.e. $\beta(y)$. Since the proof of this proposition is similar to the previous lemma and proposition, we just sketch the proof and only highlight the differences:

Proposition 2.A.1.2. *Let $\beta \in C^0([0,1])$ satisfying the assumptions stated in the definition* (2.1.9)*. Recall the definition of $\mathrm{f}^{\mu}_{\beta} : [0,1] \to \mathbb{R}$ from* (2.4.5)*, and $\mathrm{f}^{\mu} : (0,\infty) \to \mathbb{R}$ from* (2.4.35)*, then we have $\forall y \in (0,1)$:*

$$\mathrm{f}^{\mu}_{\beta}(y) = \mathrm{f}^{\mu}(\beta(y)). \tag{2.A.27}$$

Proof. Fix $y \in (0,1)$ and recall that we denote the average in Gibbs state in thermal equilibrium at inverse temperature β_{eq}, with $\langle . \rangle_{\rho^{n,\beta_{eq}}}$. Since in thermal equilibrium, we have translation invariance in the limit thanks to (2.A.21) in Corollary 2.A.1.2, it is enough to prove that $\forall \epsilon > 0$, there exist $N(\epsilon)$, such that for $n > N(\epsilon)$:

$$\Big|\mathbb{E}\Big(\Big\langle \tilde{e}_{[ny]}\Big\rangle_{\rho^n}\Big) - \mathbb{E}\Big(\Big\langle \tilde{e}_{[ny]}\Big\rangle_{\rho^{n,\beta(y)}}\Big)\Big| \leq 2\epsilon. \tag{2.A.28}$$

Let us denote the matrices corresponding to thermal averages at temperature profile $\beta(.)$ (2.2.25) by $A_n^{p,\beta(.)}$ and $A_n^{r,\beta(.)}$, only for the sake of this proposition. Similarly, denote the same matrices in thermal equilibrium at temperature $\beta(y)$ by $A_n^{p,\beta(y)}$, $A_n^{r,\beta(y)}$. The proof goes as follows: we rewrite (2.A.28) in terms of $\mathfrak{f}(A_n^{p,\beta(.)})$, $\mathfrak{f}(A_n^{r,\beta(.)})$, $\mathfrak{f}(A_n^{p,\beta(y)})$, and $\mathfrak{f}(A_n^{r,\beta(y)})$ thanks to (2.A.1), up to a vanishing error. Then we decompose $\mathfrak{f}(.) = \mathfrak{f}^{\frac{\epsilon}{4}}_{\prec}(.) + \mathfrak{f}^{\frac{\epsilon}{4}}_{\succ}(.)$ and we bound all the terms corresponding to $\mathfrak{f}^{\frac{\epsilon}{4}}_{\succ}(.)$ by $\frac{\epsilon}{2}$,

thanks to (2.A.5). Notice that the definition of $\mathfrak{f}$, $\mathfrak{f}^{\epsilon}_{\succ}$ and $\mathfrak{f}^{\epsilon}_{\prec}$ depends on the matrices through the constants α and $K_*(\epsilon)$. Here, we take the definition which is given by matrices $A_n^{p,\beta(.)}$ and $A_n^{r,\beta(.)}$. Then it is straightforward to check that $\mathfrak{f}(A_n^{r,\beta(y)})$ and $\mathfrak{f}(A_n^{p,\beta(y)})$, are well defined and they satisfy the same bounds as in (2.A.5), since $\beta_{\min} \leq \beta(y) \leq \beta_{max}$, and the same uniform bound c_0 in Lemma 2.4.1 holds for $||A_n^{p,\beta(y)}||$ and $||A_n^{r,\beta(y)}||$. The terms involving $\mathfrak{f}^{\frac{\epsilon}{4}}_{\prec}$ can be treated similar to Proposition 2.A.1.1, we sketch the terms related to p, the other ones is similar. First, we choose $N_0(\epsilon)$ such that for $n > N_0(\epsilon)$, $K_*(\frac{\epsilon}{4}) \leq [ny] \leq n - K_*(\frac{\epsilon}{4})$, then we expand each term of the sum appearing in $\mathfrak{f}^{\frac{\epsilon}{4}}_{\prec}(.)$ by using the random walk representation in (2.A.13), similar to (2.A.26), it is enough to show that for any $\hat{\epsilon} > 0$, $1 \leq k \leq K_*(\frac{\epsilon}{4})$ and $\underline{\eta} \in \tilde{\mathcal{I}}_k$ there exists N such that for $n > N$, we have:

$$\Big|\mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny],n),\underline{b}^k([ny],n))\Big) - \mathbb{E}\Big(\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k([ny],n),\tilde{\underline{b}}^k([ny],n))\Big)\Big| \leq \hat{\epsilon}, \quad (2.A.29)$$

where $\tilde{\underline{b}}^k([ny],n)$ is defined analogous to $\underline{b}^k([ny],n)$ (2.A.12) for a constant profile of temperature at inverse temperature $\beta(y)$ i.e. $\tilde{\underline{b}}^k([ny],n)_i = \beta(y)$ for $-[\frac{k}{2}] \leq i \leq [\frac{k}{2}]+1$. However, existence of N such that for $n > N$ (2.A.29) holds is evident from the third property of $\mathcal{F}_{k,\underline{\eta}}$ and the fact that $\beta(.)$ is continuous. □

Recall the definition of $Y_x^n = g(\frac{x}{n})(\langle \tilde{e}_x \rangle_{\rho^n} - \mathbb{E}(\langle \tilde{e}_x \rangle_{\rho^n}))$, and the sequence n_k in the proof of 2.4.4, where $n_k \to \infty$ with $\frac{n_{k+1}}{n_k} \to 1$. The following lemma was an essential part of the proof:

Lemma 2.A.2. *Fix $\epsilon > 0$, then for every realization of the masses, there exists N_* such that, for every $n_k > N_*$ and every n with $n_k \leq n < n_{k+1}$, we have: $\forall x \in \mathbb{I}_{n_k}$, with $n_* < x < n_k - n_*$, $|Y_x^n - Y_x^{n_k}| \leq \epsilon$. Here, n_* is a constant only depending on ϵ.*

Proof. Fix $\epsilon > 0$, and recall the definition of Y_x^n, we have

$$\begin{aligned}\Big|Y_n^x - Y_{n_k}^x\Big| &= \Big|g(\frac{x}{n})\langle \tilde{e}_x \rangle_{\rho^n} - g(\frac{x}{n})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^n}\Big) - g(\frac{x}{n_k})\langle \tilde{e}_x \rangle_{\rho^{n_k}} + g(\frac{x}{n_k})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^{n_k}}\Big)\Big| \leq \\ &\Big|g(\frac{x}{n})\langle \tilde{e}_x \rangle_{\rho^n} - g(\frac{x}{n_k})\langle \tilde{e}_x \rangle_{\rho^{n_k}}\Big| + \Big|g(\frac{x}{n})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^n}\Big) - g(\frac{x}{n_k})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^{n_k}}\Big)\Big|. \end{aligned} \quad (2.A.30)$$

The second term in (2.A.30) can be treated by using Lemma 2.A.1, and continuity of g (note that we used the choice of n and n_k, where $\frac{n}{n_k} \to 1$, as well). Hence, there exits N_1, such that for $n_k > N_1$, we have:

$$\begin{aligned}&\Big|g(\frac{x}{n})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^n}\Big) - g(\frac{x}{n_k})\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^{n_k}}\Big)\Big| \leq \\ &c_1\Big|g(\frac{x}{n}) - g(\frac{x}{n_k})\Big| + c_2\Big|\mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^n}\Big) - \mathbb{E}\Big(\langle \tilde{e}_x \rangle_{\rho^{n_k}}\Big)\Big| \leq \frac{\epsilon}{2}, \end{aligned} \quad (2.A.31)$$

where c_1 is the bound on $|g|$, and c_2 is the bound on $|\mathbb{E}(\langle \tilde{e}_x \rangle_{\rho^{n_k}})|$.

Similarly, for the first term in (2.A.30) $|g(\frac{x}{n})\langle\tilde{e}_x\rangle_{\rho^n} - g(\frac{x}{n_k})\langle\tilde{e}_x\rangle_{\rho^{n_k}}|$, it is enough to deal with $|\langle\tilde{e}_x\rangle_{\rho^n} - \langle\tilde{e}_x\rangle_{\rho^{n_k}}|$. Thanks to the expression of $\langle\tilde{e}_x\rangle_{\rho^n}$ in (2.A.1), for proper N_2, we have for $n > N_2$:

$$\begin{aligned} c_1\Big|\langle\tilde{e}_x\rangle_{\rho^n} - \langle\tilde{e}_x\rangle_{\rho^{n_k}}\Big| \leq c_1C_0\Big(\Big|\langle x, \mathfrak{f}(A_n^p)x\rangle_n - \langle x, \mathfrak{f}(A_{n_k}^p)x\rangle_{n_k}\Big| + \\ \Big|\langle x, \mathfrak{f}(A_n^r)x\rangle_{n-1} - \langle x, \mathfrak{f}(A_{n_k}^r)x\rangle_{n_k-1}\Big| + \frac{\epsilon}{6}\Big), \end{aligned} \tag{2.A.32}$$

where we chose N_2 such that $|\epsilon_n^x| + |\epsilon_{n_k}^x| \leq \frac{\epsilon}{6}$. From now on, let us show $\frac{\epsilon}{c_1C_0}$ by ϵ. Now it is enough to find N_3, such that for $n_k > N_3$,

$$\Big|\langle x, \mathfrak{f}(A_n^p)x\rangle_n - \langle x, \mathfrak{f}(A_{n_k}^p)x\rangle_{n_k}\Big| \leq \frac{\epsilon}{6}.$$

Let us decompose $\mathfrak{f}(.) = \mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}(.) + \mathfrak{f}_{\succ}^{\frac{\epsilon}{12}}(.)$ as in (2.A.4), recall $K_*(\frac{\epsilon}{12})$ from (2.A.4) and let $n_* = K_*(\frac{\epsilon}{12})$, notice that n_* only depends on ϵ. Therefore, thanks to (2.A.5) we have:

$$\Big|\langle x, \mathfrak{f}(A_n^p)x\rangle_n - \langle x, \mathfrak{f}(A_{n_k}^p)x\rangle_{n_k}\Big| \leq \Big|\langle x, \mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}(A_n^p)x\rangle_n - \langle x, \mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}(A_{n_k}^p)x\rangle_{n_k}\Big| + \frac{\epsilon}{12}. \tag{2.A.33}$$

Now, it is sufficient to show that

$$\Big|\langle x, \mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}(A_n^p)x\rangle_n - \langle x, \mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}(A_{n_k}^p)x\rangle_{n_k}\Big| \leq \frac{\epsilon}{12}, \tag{2.A.34}$$

for n_k sufficiently large. In order to control this term, recall the definition of $\underline{b}^k$ from (2.A.12) and notice that for any $\delta > 0$, there exists $N_4(\delta)$ such that for $n_k > N_4(\delta)$, $|\underline{b}^k(x,n) - \underline{b}^k(x,n_k)| < \delta$, since β is continuous, $n_k \leq n < n_{k+1}$, and $\frac{n_{k+1}}{n_k} \to 1$. By using the first property of $\mathcal{F}_{k,\underline{\eta}}$, we choose $\delta > 0$ such that

$$|\mathcal{F}_{k,\underline{\eta}}(\underline{m}^k(x,n), \underline{b}^k(x,n)) - \mathcal{F}_{k,\underline{\eta}}(\underline{m}^k(x,n_k), \underline{b}^k(x,n_k))| \leq \frac{\epsilon}{12c_1n_*3^{n_*}},$$

where c_1 is the uniform bound on coefficients $a_0, \ldots, a_{n_*}$ in (2.A.4).
Taking $n > \max\{N_1, N_2, N_3, N_4(\delta), n_*\}$, and combining this last estimate with the expression of $\mathfrak{f}_{\prec}^{\frac{\epsilon}{12}}$ in (2.A.4) as well as the random walk representation relation (2.A.13), gives us (2.A.34) for any $n_* < x < n - n_*$. The term corresponding to r can be bounded by $\frac{\epsilon}{6}$ similarly. □

2.B Alternative Coordinates

Since it is more fashionable to treat quantum harmonic chains in terms of p and q coordinates, in this section, we introduce our model in terms of these coordinates and rewrite the main transformations in terms of q coordinates, in order to illustrate

the link between our setup and the conventional one. Define on the Hilbert space $L^2(\mathbb{R}^n)$, the following Hamiltonian operator:

$$H_n = \frac{1}{2}\sum_{x=1}^{n}(\frac{p_x^2}{m_x} + (q_{x+1} - q_x)^2), \tag{2.B.1}$$

where for each $x \in \mathbb{I}_n$, q_x denotes the position operator of the particle x, i.e. denoting the space variable by $\boldsymbol{\zeta} \in \mathbb{R}^n$, for any $|\boldsymbol{\psi}\rangle \in L^2(\mathbb{R}^n)$, $q_x|\boldsymbol{\psi(\zeta_1, \ldots, \zeta_x, \ldots, \zeta_n)}\rangle = \boldsymbol{\zeta_x}|\boldsymbol{\psi(\zeta_1, \ldots, \zeta_x, \ldots, \zeta_n)}\rangle$, and p_x denotes the corresponding momentum operator $p_x = -i\partial \setminus \partial_{\boldsymbol{\zeta}_x}$. Moreover, we assume the free boundary condition: $q_0 = q_1$ and $q_n = q_{n+1}$. Here m_x are positive i.i.d random variables.
The Hamiltonian (2.B.1) can be diagonalized with a linear transformation and written as sum of free bosons:

$$H_n = \hat{p}_0^2 + \sum_{k=1}^{n-1}\omega_k(\hat{b}_k^*\hat{b}_k + \frac{1}{2}), \tag{2.B.2}$$

where b_k and b_k^* are bosonic annihilation and creation operators with commutation relations $[\hat{b}_k^*, \hat{b}_{k'}] = \delta_{k,k'}, [\hat{b}_k^*, \hat{b}_{k'}^*] = [\hat{b}_k, \hat{b}_{k'}] = 0$. Here $\hat{p}_o = \sum_{x=1}^n p_x$ is the total momentum operator. Since $[\hat{p}_0, H_n] = 0$, and the Hamiltonian H_n is translation invariant, after a straightforward analysis, the expression (2.B.2) indicates that H_n has a purely continuous spectrum (cf. Remark 3.3 of [179], and Remark 3.6 of [178] for more details). However, one can observe that the Heisenberg dynamics generated by (2.B.2) on $L^2(\mathbb{R}^n)$ is similar to (2.2.22). Up to a $\hat{p}_0 = (\sum_{x=1}^n m_x)^{-1}p_x$ operator which is constant in time.
Another technical problem arising in this description, concerns the density operator $\rho^n_{\bar{p},\bar{r},\beta} = \exp\Big(-\tilde{H}^n_\beta\Big)$ (2.1.9). The pseudo-Hamiltonian $\tilde{H}^n_\beta$ can be written as

$$\tilde{H}^n_\beta = \frac{\tilde{\mathfrak{p}}_0^2}{2} + \sum_{k=1}^{n-1}\gamma_k(\tilde{\mathfrak{b}}_k^*\tilde{\mathfrak{b}}_k + \frac{1}{2}), \tag{2.B.3}$$

where $\tilde{\mathfrak{b}}_k^*$ and $\tilde{\mathfrak{b}}_k$ are another set of bosonic operators (2.2.35), (2.B.14), and

$$\tilde{\mathfrak{p}}_0 = (\sum_{x=1}^{n}\frac{m_x}{\beta_x})^{-\frac{1}{2}}\Big(\sum_{x=1}^{n}p_x - \Pi_0\Big) = (\sum_{x=1}^{n}\frac{m_x}{\beta_x})^{-\frac{1}{2}}(\hat{p}_o - \Pi_0), \tag{2.B.4}$$

Thanks to the definition of $\tilde{\mathfrak{p}}_0$ one can observe that similar to H_n, $\tilde{H}^n_\beta$ has a continuous spectrum as an operator on $L^2(\mathbb{R}^n)$ in this coordinates. Moreover, $\rho^n_{\bar{p},\bar{r},\beta}$ is not trace-class *in this description*, anymore. Although, $\rho^n_{\bar{p},\bar{r},\beta}$ seems to be a natural choice for our locally Gibbs state corresponding to $\bar{r}, \bar{p}, \beta$, we need to slightly modify it, in order to circumvent the above mentioned technicalities. Recall that total momentum $\hat{p}_o = \sum_{x=1}^n p_x$ is conserved by the dynamics. Ideally, one would be tempted to fix the total momentum apriori to the value prescribed by the macroscopic profile i.e.,

$$\Pi_0 = \sum_{x=1}^{n}\bar{p}(\frac{x}{n})\frac{m_x}{\bar{m}}. \tag{2.B.5}$$

However, this is not convenient for technical reasons, instead we modify our initial locally Gibbs state as follows: Let us denote $L^2(\mathbb{R}^n)$ by $\mathcal{H}_n$, only in this section. Inspiring from diagonalization (2.2.31), we decompose $\mathcal{H}_n$ as follows: Let $\tilde{\boldsymbol{\zeta}}_0 := (\sum_{x=1}^n \frac{m_x}{\beta_x})^{-\frac{1}{2}} \sum_{x=1}^n \frac{m_x}{\beta_x}\boldsymbol{\zeta}_x$, and $V_{\tilde{\boldsymbol{\zeta}}_0}^{\perp} \subset \mathbb{R}^n$, be the orthogonal complement of $\mathrm{Span}(\tilde{\boldsymbol{\zeta}}_0)$. Denote the Lebesgue measure on $V_{\tilde{\boldsymbol{\zeta}}_0}^{\perp} \subset \mathbb{R}^n$, by $d\nu_{n-1}$, then we have: $\mathcal{H}_n = L^2(\mathbb{R}, d\tilde{\boldsymbol{\zeta}}_0) \otimes L^2(V_{\tilde{\boldsymbol{\zeta}}_0}^{\perp}, d\nu_{n-1}) \equiv \mathcal{H}_o \otimes \mathcal{H}_{n-1}^-$. Again appealing to (2.2.31), observe that $\tilde{H}_\beta^n = \frac{1}{2}\tilde{\mathfrak{p}}_0^2 + H_\beta^{n,-}$, where

$$\tilde{\mathfrak{p}}_0 = (\sum_{x=1}^n \frac{m_x}{\beta_x})^{-\frac{1}{2}}\Big(\sum_{x=1}^n p_x - \Pi_0\Big) = (\sum_{x=1}^n \frac{m_x}{\beta_x})^{-\frac{1}{2}}(\hat{p}_o - \Pi_0), \tag{2.B.6}$$

only acts on $\mathcal{H}_o$ and

$$H_\beta^{n,-} := \tilde{H}_\beta^n - \frac{1}{2}\tilde{\mathfrak{p}}_0^2, \tag{2.B.7}$$

only acts on $\mathcal{H}_{n-1}^-$.
Take $|\boldsymbol{\phi}\rangle \in L^2(\mathbb{R}, d\tilde{\boldsymbol{\zeta}}_0)$ such that $\langle\boldsymbol{\phi}|\boldsymbol{\phi}\rangle = 1$, $\langle\boldsymbol{\phi}|\tilde{\mathfrak{p}}_0|\boldsymbol{\phi}\rangle = 0$, $\langle\boldsymbol{\phi}|\tilde{\mathfrak{p}}_0^2|\boldsymbol{\phi}\rangle = 1$, this means the total momentum has the following average and uncertainty: $\langle\boldsymbol{\phi}|\hat{p}_o|\boldsymbol{\phi}\rangle = \Pi_0$ and $\langle\boldsymbol{\phi}|\hat{p}_o^2|\boldsymbol{\phi}\rangle - \Pi_0^2 = (\sum_{x=1}^n \frac{m_x}{\beta_x})$, (cf. Remark 2.B.2), then we define the locally Gibbs state with "fixed total momentum" as:

$$\rho_{\bar{p},\bar{r},\beta}^n = \frac{1}{Z_n}\Big(|\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}| \otimes \exp\Big(-H_\beta^{n,-}\Big)\Big), \tag{2.B.8}$$

where $|\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}|$ denotes the projection operator into the subspace spanned by the pure state $|\boldsymbol{\phi}\rangle$. Observe that $H_\beta^{n,-} = \tilde{H}_\beta^n - \frac{\tilde{\mathfrak{p}}_0^2}{2}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^{n-1})$ (cf. [197], [180], [48]), and denote its closure with the same symbol. Notice that $H_\beta^{n,-}$ can be mapped into H_β^n (2.1.10) by a unitary transformation. Consequently, one can check that $H_\beta^{n,-}$ has a discrete spectrum with non-negative eigenvalues (with a process similar to Section 2.2). We can express $H_\beta^{n,-}$ in terms of the sum of free bosonic operators, and obtained the spectrum explicitly similar to what we did in Section 2.2. Hence, using spectral theory, one can observe that $\rho_{\bar{p},\bar{r},\beta}^n$ is well defined and trace-class.
Therefore, for every operator a, if $a\rho$ is a trace class operator, we can define the "average of the observable a in the state ρ", i.e. $\langle a\rangle_\rho$ as:

$$\langle a\rangle_\rho = \mathrm{Tr}(\rho a). \tag{2.B.9}$$

As before, one can observe that $\langle p_x\rangle_\rho$, $\langle r_x\rangle_\rho$, and $\langle e_x\rangle_\rho$ are well defined, this suggests that (2.B.8) is an appropriate modification of (2.1.9) in this coordinate.
In order to avoid the aforementioned difficulties, we describe our model in terms of elongation operators. One can argue that elongation operators are "physically" relevant, since in the classical counterpart of our system the elongation is the "real" physical variable, rather than the position of the particles. Let us highlight the relation between these models by a couple of remarks:

Remark 2.B.1. *Notice that same result of Theorem 2.1.1 holds for this system, as well. In fact, initially the previous description can be mapped into this new description via a unitary transformation (cf. Remark 3.3 in [179]).*
The proof in this new coordinate is basically identical, except for some considerations concerning center of mass, which makes the proof even simpler. For example $\mathcal{E}_x^n$ *in* (2.2.41) *does not appear anymore thanks to the proper choice of* $|\boldsymbol{\phi}\rangle$.
In the previous description (2.1.2)*, by definition we have* $\sum_{x=1}^n p_x = 0$. *This is because we begin the description of the system by quantizing the classical description corresponding to the observer in center of the mass. Consequently, we should take* $\int_0^1 \bar{p}(y)dy = 0$. *In this new coordinates, since we describe the center of mass separately by* $|\boldsymbol{\phi}\rangle$*, we can take any* $\bar{p} \in C^1([0,1])$ *with non zero average.*

Remark 2.B.2. *Physically the initial state* (2.B.8)*, means that initially we prepare our system in the lab such that our system's center of the mass is known and is given by the wave function* $|\boldsymbol{\phi}\rangle$*, such that the above mentioned averages (momentum and kinetic energy contribution) is prescribed by the macroscopic profile of momentum and temperature, and other degrees of freedom are subjected to the thermal and mechanical fluctuations. Mathematically, this state is more convenient. We should emphasize that the assumption,* $\langle\boldsymbol{\phi}|\hat{p}_o^2|\boldsymbol{\phi}\rangle - \Pi_o^2 = (\sum_{x=1}^n \frac{m_x}{\beta_x})$ *is not crucial. Our result holds as long as we replace* $(\sum_{x=1}^n \frac{m_x}{\beta_x})$ *with any constant of order* n*. One can replace the pure state* $|\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}|$ *with any mixed state* ρ_o *acting on* $\mathcal{H}_o$*, with* $\mathrm{Tr}(\rho_o) = 1$*, such that*

$$\mathrm{Tr}(\rho_o \hat{p}_o) = \Pi_0, \qquad \mathrm{Tr}\Big(\rho_o \hat{p}_o^2\Big) - \Pi_o^2 \sim \mathcal{O}(n),$$

and obtain the similar result in Theorem 2.1.1.
Notice that the aforementioned constants are random and they have been defined for a realization of the masses. However, in the thermodynamic limit $\frac{\Pi_0}{n} \to \int_0^1 \bar{p}(y)dy$, *and* $\frac{1}{n}\sum_{x=1}^n \frac{m_x}{\beta_x} \to \int_0^1 \frac{\bar{m}}{\beta(y)}dy$*, almost surely, thanks to law of large numbers, which further justifies our choice.*
Finally, one can construct such $|\boldsymbol{\phi}\rangle$ *easily, via an inverse Fourier transform (in* $\tilde{\boldsymbol{\zeta}}_{\mathbf{0}}$ *variable) of a Gaussian function.*

We show our main transformations in terms of q coordinates: Rewrite the Hamiltonian in q coordinate as:

$$H_n = \frac{1}{2}(\langle p, M^{-1}p\rangle_n + \langle \nabla_+ q, \nabla_+ q\rangle_{n-1})) = \frac{1}{2}\langle p, M^{-1}p\rangle_n + \langle q, -\Delta q\rangle_n).$$

The proper transformation of q in order to diagonalize the Hamiltonian is

$$\hat{q}_k = \langle \varphi^k, M^{\frac{1}{2}} q\rangle_n = \sum_{x=1}^n \sqrt{m_x}\varphi_x^k q_x,$$

where $\hat{p}_k$ is defined as before. We can find the following relation between $\hat{q}_k$ and $\hat{r}_k$:

$$\hat{r}_k = \omega_k \hat{q}_k. \tag{2.B.10}$$

Canonical commutation relation (CCR) in terms of q reads:

$$[\hat{q}_k, \hat{p}_{k'}] = i\delta_{k,k'}, \qquad [\hat{q}_k, \hat{q}_{k'}] = [\hat{p}_k, \hat{p}_{k'}] = 0. \tag{2.B.11}$$

The inverse is given by

$$q = M^{-\frac{1}{2}} O\hat{q}, \qquad q_x = \frac{1}{\sqrt{m_x}} \sum_{k=0}^{n-1} \varphi_x^k \hat{q}_k,$$

and the Hamiltonian reads

$$H_n = \frac{\hat{p}_0}{2}\frac{1}{2}\sum_{k=1}^{n-1}(\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2), \tag{2.B.12}$$

The bosonic operators have the following form in terms of $\hat{q}$ coordinates:

$$\hat{b}_k = \frac{1}{\sqrt{2}}(\sqrt{\omega_k}\hat{q}_k + i\frac{1}{\sqrt{\omega_k}}\hat{p}_k), \quad \hat{b}_k^* = \frac{1}{\sqrt{2}}(\sqrt{\omega_k}\hat{q}_k - i\frac{1}{\sqrt{\omega_k}}\hat{p}_k).$$

In order to expose the aforementioned link further, we introduce the coordinate $\tilde{\mathfrak{q}}_k$ similar to $\hat{q}_k$:

Define $\bar{q}_x$ as follows: First, construct $\bar{q}_x$, for $x \in \mathbb{I}_n$, from $\bar{r}_x$, by defining $\bar{q}_1 = \bar{q}_0 = c$ (c is an arbitrary constant, corresponding to the macroscopic position of the first particle) and let $\bar{q}_x = \sum_{y=1}^{x-1} \bar{r}_y + \bar{q}_1$. Then, we have $\tilde{q}_x = q_x - \bar{q}_x$, which gives us:

$$\tilde{H}_\beta^n = \frac{1}{2}(\langle \tilde{p}, M_\beta^{-1}\tilde{p}\rangle_n + \langle \nabla_+\tilde{q}, \beta^o \nabla_+ \tilde{q}\rangle_{n-1}) = \frac{1}{2}(\langle \tilde{p}, M_\beta^{-1}\tilde{p}\rangle_n + \langle \tilde{q}, -\nabla_-\beta^o\nabla_+\tilde{q}\rangle_n).$$

Therefore, $\tilde{\mathfrak{q}}_k$ is defined as:

$$\tilde{\mathfrak{q}} = O_\beta^\dagger M_\beta^{\frac{1}{2}} \tilde{q}, \qquad \tilde{\mathfrak{q}}_k = \langle \psi^k, M_\beta^{\frac{1}{2}}\tilde{q}\rangle_n = \sum_{x=1}^{n} \sqrt{\frac{m_x}{\beta_x}} \psi_x^k \tilde{q}_x.$$

Moreover, it is illuminating to know the relation between $\tilde{\mathfrak{r}}_k$ and $\tilde{\mathfrak{q}}_k$:

$$\tilde{\mathfrak{r}}_k = \gamma_k \tilde{\mathfrak{q}}_k.$$

The inverse relation for $\tilde{\mathfrak{q}}_k$ is given by:

$$\tilde{q} = M_\beta^{-\frac{1}{2}} O_\beta \tilde{\mathfrak{q}}, \qquad \tilde{q}_x = \sqrt{\frac{\beta_x}{m_x}} \sum_{k=0}^{n-1} \psi_x^k \tilde{\mathfrak{q}}_k.$$

Finally, the pseudo-Hamiltonian is diagonalized as follows:

$$\tilde{H}_\beta^n = \frac{\tilde{\mathfrak{p}}}{2} + \frac{1}{2}\sum_{k=1}^{n-1}(\tilde{\mathfrak{p}}_k^2 + \gamma_k^2 \tilde{\mathfrak{q}}_k^2), \tag{2.B.13}$$

Commutation relation is given by

$$[\tilde{\mathfrak{q}}_k, \tilde{\mathfrak{q}}_{k'}] = [\tilde{\mathfrak{p}}_k, \tilde{\mathfrak{p}}_{k'}] = 0, \qquad [\tilde{\mathfrak{q}}_k, \tilde{\mathfrak{p}}_{k'}] = i\delta_{k,k'}, \quad \forall k \in \mathbb{I}_{n-1}^0,$$

and the bosonic operators are given by:

$$\tilde{\mathfrak{b}}_k = \frac{1}{\sqrt{2}}(\sqrt{\gamma_k}\tilde{\mathfrak{q}}_k + i\frac{1}{\sqrt{\gamma_k}}\tilde{\mathfrak{p}}_k), \tag{2.B.14}$$

$$\tilde{\mathfrak{b}}_k^* = \frac{1}{\sqrt{2}}(\sqrt{\gamma_k}\tilde{\mathfrak{q}}_k - i\frac{1}{\sqrt{\gamma_k}}\tilde{\mathfrak{p}}_k). \tag{2.B.15}$$

2.C Average of Bosonic Operators

Recall the average of bosonic operators (2.2.38) from Lemma 2.2.2. One can compute these averages and proof the lemma as follows (Notice that since E_0 is a constant, we can simply omit it from our computation and it does not change the desired averages) :

Proof. First we compute $Z_n := \mathrm{Tr}\Big(\exp\Big(-H_\beta^n\Big)\Big)$, using spectral theory, we expand the trace in the basis of eigenvalues of H_β^n i.e. $\{\Big|\bar{\boldsymbol{\theta}}\Big\rangle | \bar{\theta} \in \mathbb{N}^{n-1}\}$. Then, the following computation gives us the result: (we abbreviate $\sum_{k=1}^{n-1} \theta_k \gamma_k$ by $\bar{\theta}.\bar{\gamma}$, where $\bar{\gamma}$ stands for the vector $\bar{\gamma} := (\gamma_1, \dots, \gamma_{n-1})$) [23]

$$\begin{aligned} Z_n = \mathrm{Tr}\Big(\exp\Big(-H_\beta^n\Big)\Big) &= \sum_{\bar{\theta}\in\mathbb{N}^{n-1}} \Big\langle\bar{\boldsymbol{\theta}}\Big| \exp\Big(-H_\beta^n\Big)\Big|\bar{\boldsymbol{\theta}}\Big\rangle = \sum_{\bar{\theta}\in\mathbb{N}^{n-1}} \Big\langle\bar{\boldsymbol{\theta}}\Big| \exp\Big(-\bar{\theta}.\bar{\gamma}\Big)\Big|\bar{\boldsymbol{\theta}}\Big\rangle\Big\langle\bar{\boldsymbol{\theta}}\Big|\bar{\boldsymbol{\theta}}\Big\rangle \\ &= \sum_{\bar{\theta}\in\mathbb{N}^{n-1}} \exp\Big(-\bar{\theta}.\bar{\gamma}\Big) = \prod_{k=1}^{n-1}\Big(\sum_{\theta_k=0}^{\infty} e^{-\theta_k\gamma_k}\Big) = \prod_{k=1}^{n-1} \frac{1}{1-e^{-\gamma_k}}. \end{aligned} \tag{2.C.1}$$

Now from (2.2.32), (2.2.37), observe that

$$\tilde{\mathfrak{b}}_k|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k}, \dots, \boldsymbol{\theta_{n-1}})\rangle = \sqrt{\theta_k}|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k - 1}, \dots, \boldsymbol{\theta_{n-1}})\rangle,$$

$$\tilde{\mathfrak{b}}_k^*|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k}, \dots, \boldsymbol{\theta_{n-1}})\rangle = \sqrt{\theta_k + 1}|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k + 1}, \dots, \boldsymbol{\theta_{n-1}})\rangle.$$

Therefore, since $\Big|\bar{\boldsymbol{\theta}}\Big\rangle$ are orthonormal, we have

$$\forall\Big|\bar{\boldsymbol{\theta}}\Big\rangle, \quad \forall k, \quad \Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_k\Big|\bar{\boldsymbol{\theta}}\Big\rangle = \Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_k^*\Big|\bar{\boldsymbol{\theta}}\Big\rangle = \Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_k\tilde{\mathfrak{b}}_{k'}\Big|\bar{\boldsymbol{\theta}}\Big\rangle = \Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_k^*\tilde{\mathfrak{b}}_{k'}^*\Big|\bar{\boldsymbol{\theta}}\Big\rangle = 0.$$

Moreover, if $k \neq k'$, we have

$$\Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_k\tilde{\mathfrak{b}}_{k'}^*\Big|\bar{\boldsymbol{\theta}}\Big\rangle = \Big\langle\bar{\boldsymbol{\theta}}\Big|\tilde{\mathfrak{b}}_{k'}^*\tilde{\mathfrak{b}}_k\Big|\bar{\boldsymbol{\theta}}\Big\rangle = 0.$$

Since $\Big|\bar{\boldsymbol{\theta}}\Big\rangle$ are orthonormal, if we expand $\mathrm{Tr}\Big(\exp\Big(-H_\beta^n\Big)\Big)$ in the basis of H_β^n, we deduce the first set of equalities in (2.2.38). Furthermore, for $k \neq k'$, we deduce$\Big\langle\tilde{\mathfrak{b}}_k\tilde{\mathfrak{b}}_{k'}^*\Big\rangle_\rho = \Big\langle\tilde{\mathfrak{b}}_{k'}^*\tilde{\mathfrak{b}}_k\Big\rangle_\rho = 0$ as well.
On the other hand, if $k = k'$, the same relations (2.4.21), (2.2.32)imply:

$$\tilde{\mathfrak{b}}_k^*\tilde{\mathfrak{b}}_k|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k}, \dots, \boldsymbol{\theta_{n-1}})\rangle = \theta_k|(\boldsymbol{\theta_1}, \dots, \boldsymbol{\theta_k}, \dots, \boldsymbol{\theta_{n-1}})\rangle,$$

[23] We do not explain the proof in details since this is a classical result one can find [48].

thus, we have $\left\langle \bar{\boldsymbol{\theta}} \middle| \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_k \middle| \bar{\boldsymbol{\theta}} \right\rangle = \theta_k$. Consequently, we can compute $\left\langle \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_k \right\rangle_\rho$:

$$\begin{aligned}\left\langle \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_k \right\rangle_\rho &= \mathrm{Tr}\big(\rho \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_{k'}\big) = \frac{1}{Z_n} \mathrm{Tr}\big(e^{-H_\beta^n} \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_{k'}\big) = \frac{1}{Z_n} \sum_{\bar{\theta} \in \mathbb{N}^{n-1}} \left\langle \bar{\boldsymbol{\theta}} \middle| \exp\big(-H_\beta^n\big) \tilde{\mathfrak{b}}_k^* \tilde{\mathfrak{b}}_{k'} \middle| \bar{\boldsymbol{\theta}} \right\rangle \\ &= \frac{1}{Z_n} \sum_{\bar{\theta} \in \mathbb{N}^{n-1}} \theta_k \left\langle \bar{\boldsymbol{\theta}} \middle| \exp\big(-H_\beta^n\big) \middle| \bar{\boldsymbol{\theta}} \right\rangle = \frac{1}{Z_n} \sum_{\bar{\theta} \in \mathbb{N}^{n-1}} \theta_k \exp\left(- \sum_{j=1}^{n-1} \theta_j \gamma_j \right) \\ &= \frac{1}{Z_n} \frac{-\partial Z_n}{\partial \gamma_k} = \frac{1}{e^{\gamma_k} - 1}.\end{aligned} \tag{2.C.2}$$

Since we have the commutator relation $[\tilde{\mathfrak{b}}_k, \tilde{\mathfrak{b}}_k^*] = 1$, we obtain the last equality: $\left\langle \tilde{\mathfrak{b}}_k \tilde{\mathfrak{b}}_k^* \right\rangle_\rho = \frac{1}{e^{\gamma_k} - 1} + 1$. □

Chapter 3

Stochastic DNLS

We introduce a mass conserving stochastic perturbation of the discrete non-linear Schrödinger equation that models the action of a heat bath at a given temperature. We prove that the corresponding canonical Gibbs distribution is the unique invariant measure. In the one-dimensional cubic focusing case on the torus, we prove that in the limit for large time, continuous approximation, and low temperature, the solution converges to the steady wave of the continuous equation that minimizes the energy for a given mass.
This chapter is extracted from arXiv:2109.01389 [math-ph][127] (submitted article), with certain add-ins; in particular, Section 3.A.1 inside Appendix 3.A, containing more precise large deviation estimates, is added.

3.1 Stochastic Dynamics

Fix $n \in \mathbb{N}$, let $\chi = \mathbb{C}^{n^d}$ be the configuration space, and denote a typical element of χ by $\{\psi(x)\}_{x\in\widetilde{\mathbb{T}}_n^d}$, where $\widetilde{\mathbb{T}}_n = \mathbb{Z}/n\mathbb{Z}$ is the discrete Torus of size n. Equivalently, one can see a function on $\widetilde{\mathbb{T}}_n^d$, $\psi : \widetilde{\mathbb{T}}_n^d \to \mathbb{C}$, as the discretization of a function u on the d-dimensional torus of length size nh, $u : \mathbb{T}_{nh}^d \to \mathbb{C}$, with mesh size $h > 0$, i.e., $\psi(x) = u(hx)$, for $x \in \widetilde{\mathbb{T}}_n^d$. Then the discrete non-linear Schrödinger equation (DNLS) is the following system of ODEs:

$$i\frac{d\psi(x,t)}{dt} = -\Delta_d\psi(x,t) + \kappa|\psi(x,t)|^{p-1}\psi(x,t), \qquad x \in \widetilde{\mathbb{T}}_n^d \tag{3.1.1}$$

where Δ_d is the d-dimensional discrete Laplacian:

$$\Delta_d\psi(x) = h^{-2}\sum_{|y-x|=1}\Big(\psi(y) - \psi(x)\Big).$$

These equations conserve the energy, given by the Hamiltonian

$$\mathcal{H}_n(\psi) = s\sum_{\substack{x,y\in\widetilde{\mathbb{T}}_n^d,\\ |x-y|=1}}\frac{h^{-2}}{2}|\psi(x)-\psi(y)|^2 + \frac{s\kappa}{p+1}\sum_{x\in\widetilde{\mathbb{T}}_n^d}|\psi(x)|^{p+1}, \tag{3.1.2}$$

and the mass, given by the ℓ^2 norm:

$$\mathcal{M}_n(\psi) = s \sum_{x \in \widetilde{\mathbb{T}}_n^d} |\psi(x)|^2. \tag{3.1.3}$$

Here $s > 0$ is a scaling parameter that we will choose opportunely later.

Denote $\psi(x) = \psi_r(x) + i\psi_i(x) = |\psi(x)|e^{i\theta(x)}$, the deterministic evolution equation (3.1.1) can be regarded as a Hamiltonian dynamics with the following generator:

$$\mathcal{A} = s^{-1} \sum_{x \in \widetilde{\mathbb{T}}_n^d} (\partial_{\psi_i(x)} \mathcal{H}_n) \partial_{\psi_r(x)} - (\partial_{\psi_r(x)} \mathcal{H}_n) \partial_{\psi_i(x)}. \tag{3.1.4}$$

Moreover, define the operator $\partial_{\theta(x)}$ acting on a suitable function $F : \chi \to \mathbb{C}$ as

$$\partial_{\theta(x)} F(\psi) = (\psi_r(x)\partial_{\psi_i(x)} - \psi_i(x)\partial_{\psi_r(x)})F(\psi). \tag{3.1.5}$$

Corresponding to a positive temperature $\beta^{-1} > 0$, define:

$$\mathcal{S} = \beta^{-1} \sum_{x \in \widetilde{\mathbb{T}}_n^d} e^{\beta \mathcal{H}_n} \partial_{\theta(x)} e^{-\beta \mathcal{H}_n} \partial_{\theta(x)}. \tag{3.1.6}$$

Fix $\beta > 0, \gamma > 0$, and consider the Markov process with values in χ, generated by

$$L = \mathcal{A} + \gamma \mathcal{S}, \tag{3.1.7}$$

where $\mathcal{S}$ and $\mathcal{A}$ are defined in (3.1.6) and (3.1.4). Since $\partial_{\theta(x)}\psi(x) = i\psi(x)$, we have

$$\mathcal{S}\psi(x) = -\psi(x)\left(\beta^{-1} + i\partial_{\theta(x)}\mathcal{H}_n(\psi)\right) = -\psi(x)\left(\beta^{-1} + is\ \mathrm{Im}[\psi^*(x)\Delta_d\psi(x)]\right),$$

hence system of stochastic differential equations generated by (3.1.7) read:

$$\begin{aligned} d\psi(x,t) =& i[\Delta_d\psi(x,t) - \kappa|\psi(x,t)|^{p-1}\psi(x,t)]dt - \gamma\psi(x,t)(\beta^{-1} + i\partial_{\theta(x)}\mathcal{H}_n(\psi))dt \\ & - i\sqrt{2\gamma\beta^{-1}}\psi(x,t)dw(x,t), \qquad x \in \widetilde{\mathbb{T}}_n^d, \end{aligned} \tag{3.1.8}$$

where $\{w(x,t), x \in \widetilde{\mathbb{T}}_n\}$ are real independent Wiener processes.

We observed that $\mathcal{A}\mathcal{M}_n(\psi) = 0$, one can check that $\mathcal{S}\mathcal{M}_n(\psi) = 0$. Therefore, mass is a conserved quantity for the dynamics (3.1.7). Hence, if we assume the initial condition $\psi(0,t) = \psi_0 \in \mathbb{C}^{n^d}$, such that $\mathcal{M}_n(\psi_0) = m$, then our dynamics is confined in the compact manifold with $\mathcal{M}_n(\psi) = m$, which is a $(2n^d - 1)$-sphere. We denote this sphere by $S_{m,s}^n$:

$$S_{m,s}^n = \{\psi \in \mathbb{C}^{n^d} | \mathcal{M}_n(\psi) = m\}. \tag{3.1.9}$$

Proposition 3.1.1. *The generator L is hypoelliptic*[1].

[1]This should be understood up to the above mentioned kernel.

The proof follows from Hörmander characterization, i.e., that the Lie algebra generated by $\{\mathcal{A}, \partial_{\theta(x)}, x \in \widetilde{\mathbb{T}}_n^d\}$ generates the tangent space of $S_{m,s}^n$. This is proven in Appendix 3.B.

Let $d\mu_{m,s}^n$ be the uniform probability measure on $S_{m,s}^n$, one can define this measure as the projection of the Lebesgue measure on $S_{m,s}^n$, properly normalized. Define the canonical Gibbs measure with inverse temperature β on $S_{m,s}^n$ as

$$d\mu_{\beta,m,s}^n = \frac{1}{Z_n(\beta, m, s)} e^{-\beta \mathcal{H}_n(\psi)} d\mu_{m,s}^n, \tag{3.1.10}$$

Here $Z_n(\beta, m, s)$ is the partition function:

$$Z_n(\beta, m, s) = \int_{S_{m,s}^n} e^{-\beta \mathcal{H}_n(\psi)} d\mu_{m,s}^n. \tag{3.1.11}$$

Note that, since $\mathcal{H}_n$ is a smooth function on a compact set and therefore, bounded from below, $Z_n(\beta, m, s)$ is finite, and consequently, the existence of $d\mu_{\beta,m,s}^n$ is evident.

The observation that $\forall f \in C_b(S_{m,s}^n)$, $\int_{\mathbb{C}^n} Lf d\mu_{\beta,m,s}^n = 0$[2], implies that $d\mu_{\beta,m,s}^n$ is an invariant measure for the dynamics (3.1.7)((3.1.8)). In fact, if we fix $m, \gamma, \beta > 0$ this measure is the unique invariant probability measure:

Theorem 3.1.1. *Fix real positive parameters $h, s, \gamma > 0$, the mass of the field $m > 0$, inverse temperature $\beta > 0$, and $\kappa \in \mathbb{R}$, the measure $d\mu_{\beta,m,s}^n$ is the unique invariant measure for the dynamics generated by* (3.1.7).

Proof. Without losing generality we can fix $h = s = 1$. Since the generator L is **hypoelliptic**, the stationary measure must have density w.r.t $d\mu_{m,s}^n$, and then also w.r.t $d\mu_{\beta,m,s}^n$. Denoting $f(\psi)$ the density w.r.t $d\mu_{\beta,m,s}^n$, it must satisfy the equation

$$0 = L^* f = (-\mathcal{A} + \gamma \mathcal{S}) f, \tag{3.1.12}$$

where L^* denotes the adjoint of L in $L^2(d\mu_{\beta,m,s}^n)$. Since L is hypoelliptic, f is smooth and (3.1.12) is valid pointwise. Multiplying by f and integrating w.r.t $d\mu_{\beta,m,s}^n$, we have

$$0 = \gamma < f(-\mathcal{S}) f >= \gamma \sum_{x \in \widetilde{\mathbb{T}}_n^d} < (\partial_{\theta(x)} f)^2 >, \tag{3.1.13}$$

where $< \cdot >$ denotes integration w.r.t $d\mu_{\beta,m,s}^n$. This means that $\partial_{\theta(x)} f = 0$ $d\mu_{\beta,m,s}^n$-a.e., and $\mathcal{A} f = 0$. We want to conclude that $f = 1$, $d\mu_{\beta,m,s}^n$-a.e.. Since $\partial_{\theta(x)} f = 0$ for any x, then $f = \tilde{f}(|\psi(x)|^2, x \in \widetilde{\mathbb{T}}_n^d)$. The operator $\mathcal{A}$ can be written as $\mathcal{A}^0 + \mathcal{A}^p$, with

$$\mathcal{A}^0 = \sum_{x \in \widetilde{\mathbb{T}}_n^d} \left\{ (\Delta_d \psi_i(x)) \, \partial_{\psi_r(x)} - (\Delta_d \psi_r(x)) \, \partial_{\psi_i(x)} \right\}, \tag{3.1.14}$$

and

$$\mathcal{A}^p = \kappa \sum_{x \in \widetilde{\mathbb{T}}_n^d} |\psi(x)|^{p-1} \left\{ \psi_i(x) \partial_{\psi_r(x)} - \psi_r(x) \partial_{\psi_i(x)} \right\} = \kappa \sum_{x \in \widetilde{\mathbb{T}}_n^d} |\psi(x)|^{p-1} \partial_{\theta(x)}. \tag{3.1.15}$$

[2]Here $C_b(S_{m,s}^n)$ denotes the set of continuous bounded functions on $S_{m,s}^n$.

It is immediate that $\mathcal{A}^p f = 0$, hence, $\mathcal{A}^0 f = 0$, pointwise. Let us denote $a(x) := |\psi(x)|^2$, and the canonical basis of $\mathbb{R}^d$ by $\{e_j\}_{j=1}^d$, then we have:

$$
\begin{aligned}
0 = \mathcal{A}^0 f &= 2 \sum_{x \in \widetilde{\mathbb{T}}_n^d} \{(\Delta_d \psi_i(x))\, \psi_r(x) - (\Delta_d \psi_r(x))\, \psi_i(x)\} \left[\partial_{a(x)} \tilde{f}\right] (|\psi(y)|^2, y \in \widetilde{\mathbb{T}}_n^d) \\
&= 2 \sum_{x \in \widetilde{\mathbb{T}}_n^d} \sum_{j=1}^d \nabla_+^j (\psi_r(x)\psi_i(x-e_j) - \psi_i(x)\psi_r(x-e_j)) \left[\partial_{a(x)} \tilde{f}\right] (|\psi(y)|^2, y \in \widetilde{\mathbb{T}}_n^d) \\
&= 2 \sum_{x \in \widetilde{\mathbb{T}}_n^d} \sum_{j=1}^d [\psi_r(x)\psi_i(x-e_j) - \psi_i(x)\psi_r(x-e_j)] \left[(\partial_{a(x)} - \partial_{a(x-e_j)}) \tilde{f}\right] (|\psi(y)|^2, y \in \widetilde{\mathbb{T}}_n^d) \\
&= 2 \sum_{x \in \widetilde{\mathbb{T}}_n^d} \sum_{j=1}^d \sin\Big(\theta_{x-e_j} - \theta_x\Big) |\psi(x)||\psi(x-e_j)| \left[(\partial_{a(x)} - \partial_{a(x-e_j)}) \tilde{f}\right] (|\psi(y)|^2, y \in \widetilde{\mathbb{T}}_n^d),
\end{aligned}
\tag{3.1.16}
$$

where ∇_+^j denotes the discrete gradient in the e_j direction $(\nabla_+^j g)(x) = g(x+e_j) - g(x)$. Since this relation is true pointwise for any $\psi \in S_{m,s}^n$, by choosing a proper ψ (for example one can take θ_y equal to zero, for $y \in \widetilde{\mathbb{T}}_n^d$, except $\theta(x)$, and take $|\psi(y)| = 0$, for all $|y-x|_d = 1$, except $x - e_j$), we have that

$$
(\partial_{a(x)} - \partial_{a(x-e_j)}) \tilde{f}(a(y), y \in \widetilde{\mathbb{T}}_n^d) = 0, \tag{3.1.17}
$$

pointwise for every $x \in \widetilde{\mathbb{T}}_n^d$, and any $1 \le j \le d$.

From (3.1.17), we conclude that $(\partial_{a(x)} - \partial_{a(z)}) \tilde{f}(a(y), y \in \widetilde{\mathbb{T}}_n^d) = 0$ for any $x, z \in \widetilde{\mathbb{T}}_n^d$. This implies

$$
\tilde{f}(a(y), y \in \widetilde{\mathbb{T}}_n^d) = F\left(\sum_{y \in \widetilde{\mathbb{T}}_n^d} a(y)\right) = F(m).
$$

which yields the result. □

Remark 3.1.2. *Similar to Remark 3.B.1, the proof of Theorem 3.1, can be adapted to the case where the non-linearity appearing in the Hamiltonian is given by* $\mathcal{N}(\psi(x)) \sim f(|\psi(x)|^2)$*, with* f *sufficiently smooth, instead of* $\mathcal{N}_n(\psi(x)) \sim |\psi(x)|^4$.

By classical theorems in control theory, given the Hörmander condition, and the existence of a unique invariant measure with full support on $S_{m,s}^n$, it follows the strict positivity of the probability transition (cf. [121], proof of Theorem 2.1) and the following proposition. For any proper signed measure μ, denote its total variation norm by $\|\mu\|_{TV} = \sup_A \mu(A) - \inf_A \mu(A)$ (cf. Sect. 3 of [25]):

Proposition 3.1.3. *Consider the dynamics generated by* (3.1.7)*, denote the law of this process by* $\mu_t^{\beta,n,m}$ *with initial condition* $\mu_0^{\beta,n,m} = \delta_{\psi_0}$*, where* $\psi_0 \in S_{m,s}^n$[3].

[3]Notice that with our definition, $\mu_t^{\beta,n,m}$ actually depends on ψ_0. However, we omit ψ_0 to lighten the notation.

There exists $\gamma_0 = \gamma_0(n, m, \beta, s) > 0$, *and* $C = C(n, m, \beta, s) > 0$ *such that for any* $\psi_0 \in S^n_{m,s}$, *we have:*

$$\|\mu_t^{\beta,n,m} - \mu^n_{\beta,m,s}\|_{TV} \leq Ce^{-\gamma_0 t}. \tag{3.1.18}$$

In particular, we have the weak convergence:

$$\mu_t^{\beta,n,m} \underset{t\to\infty}{\longrightarrow} \mu^n_{\beta,m,s}. \tag{3.1.19}$$

Proof. Since $\mu^n_{\beta,m,s}$ is the unique invariant measure (ergodicity), with full support (for any open set $A \subset S^n_{m,s}$, $\mu^n_{\beta,m}(A) > 0$), given the Hörmander condition we can use the result of [121], (proof of Theorem 2.1 in [121]) and deduce the strict positivity of the probability transition. Furthermore, having the strict positivity of the probability transition, compactness of the phase space, as well as the hypoellipticity of the generator, we can conclude by Theorem 8.9 of [25]. Notice that the fact that C is independent of ψ_0 is deduced from compactness of the phase space. □

The novelty of the stochastic perturbation (3.1.6) can be described as follows: it's a mass-conserving white noise, such that the Gibbs measure is the unique invariant measure for the dynamics, and it provides good ergodic properties as in Proposition 3.1.3. This perturbation is quite "powerful" in the sense that its ergodic properties do not depend on the non-linearity, and we can consider either focusing or defocusing non-linearity. In either of these cases the long time behavior is given by the corresponding Gibbs measure. However, depending on the choice of parameters d, s, h, κ many interesting phenomena can be observed in the large scale limit. In the rest of this note, we focus on one particular case: one-dimensional focusing non-linear Schrödinger Equation on the torus.

3.2 Large Scale Limit and Main Result

3.2.1 Preliminaries about periodic cubic non-linear Schrödinger equation

In this section, we recall rather basic results about the focusing non-linear Schrödinger equation (NLS) with periodic boundary conditions. Consider the following non-linear cubic Schrödinger equation:

$$\begin{aligned} &i\partial_t\psi(x,t) = -\partial_{xx}\psi(x,t) - |\psi(x,t)|^2\psi(x,t),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ &\psi(x,0) = \psi_0(x),\ \psi_0 \in H^1(\mathbb{T}_L), \end{aligned} \tag{3.2.1}$$

where we assume the periodic boundary conditions by the definition of $H^1(\mathbb{T}_L)$ as: $H^1(\mathbb{T}_L) = \{u \in H^1_{loc}(\mathbb{R}, \mathbb{C}) |\ \forall x \in \mathbb{R}, \quad u(x + L) = u(x)\}$, with the following norms and inner product ($\bar{v}$ indicates the complex conjugate):

$$\|u\|_{L^p} = \left(\int_{\mathbb{T}_L} |u|^p dx\right)^{\frac{1}{p}},\ (u, v) = \int_{\mathbb{T}_L} u\bar{v}dx, \quad \|u\|_{H^1} = \left(\int_{\mathbb{T}_L} \left(|\partial_x u|^2 + |u|^2\right) dx\right)^{\frac{1}{2}}. \tag{3.2.2}$$

Global wellposedness of this problem is established in [45], [66]; in particular, $\forall t > 0$, $\psi(x,t) \in H^1(\mathbb{T}_L)$. Note that this equation has two important conserved quantities [4]: the energy or Hamiltonian $\mathcal{H}$, and L^2 norm or mass $\mathcal{M}$, defined by

$$\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbb{T}_L} |\partial_x \psi|^2 dx - \frac{1}{4}\int_{\mathbb{T}_L} |\psi|^4 dx, \qquad \mathcal{M}(\psi) = \int_{\mathbb{T}_L} |\psi|^2 dx. \tag{3.2.3}$$

One of the main features of this equation is the existence of a special class of solutions called the *"standing waves"* or *"periodic waves"*. These are *time periodic* solutions having the following form:

$$\psi(x,t) = e^{i\omega t} u(x). \tag{3.2.4}$$

If $\psi(x,t) = e^{i\omega t}u(x)$ be a solution of (3.2.1), then $u(x)$ should satisfy the following ODE, with periodic boundary condition:

$$u''(x) - \omega u(x) + |u(x)|^2 u(x) = 0. \tag{3.2.5}$$

Notice that the solution of (3.2.5) characterizes the minimum of the energy $\mathcal{H}(u)$, under the constrain $\mathcal{M}(u) = m$, where the frequency ω plays the role of Lagrange multiplier.

In general, we should consider complex valued solutions of (3.2.4). On the other hand, writing this solution as $u(x) = \rho(x)e^{i\theta(x)}$, the corresponding energy is given by

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{T}_L} \left(|\rho'(x)|^2 + \rho(x)^2|\theta'(x)|^2\right) dx - \frac{1}{4}\int_{\mathbb{T}_L} |\rho(x)|^4 dx.$$

This shows that the minimum of the energy $\mathcal{H}(u)$, under the constrain $\mathcal{M}(u) = m$ is attained for $\theta(x) =$ constant. Consequently, this minimum are defined up to a constant phase and we can choose positive real solutions. Also notice that translations $u_y(x) = u(x+y)$ do not change energy and mass.

Here, if we fix the L, and assume u to be real-valued, and positive, and fix the mass of u to be $\mathcal{M}(u) = m$, then under these assumptions, (3.2.5) has a unique (up to a translation) smooth solution, this solution can be written in terms of Jacobi elliptic functions as $u(x) = \alpha \mathfrak{dn}(\lambda x, k)$, where $k \in (0,1), \alpha$, and $\lambda > 0$, $\omega > 0$ are uniquely determined by m, and L (cf. [120], [188], [108], cf. Appendix 1.2.1 for the definition of $\mathfrak{dn}$). We recall the following crucial result from [120], Proposition 3.2, which characterizes this solution as the minimizer of $\mathcal{H}(\psi)$ under the constraint that $\mathcal{M}(\psi) = m$.

Theorem 3.2.1. *Fix $m, L \in \mathbb{R}_+$, and consider the following minimization problem:*

$$E_0(m,L) := \inf\{\mathcal{H}(u) | \mathcal{M}(u) = m,\ u \in H^1(\mathbb{T}_L)\}, \tag{3.2.6}$$

then we have: $-\infty < E_0(m,L) < 0$, and

[4] In fact, since this equation is completely integrable, we have infinite conserved quantities. However, most of the results in this note can be generalized to the sub-critical non-linearities that are not integrable, i.e., we can change the non-linearity term in (3.2.1) into $|\psi|^{p-1}\psi$ with $1 \le p < 5$. Notice that if $p \neq 3$, w do not have the explicit characterization of the Solitions

1. *If* $0 < m \leq \frac{\pi^2}{L}$, *then the constant function* $Q_{m,L}(x) = (\frac{m}{L})^{\frac{1}{2}}$ *is the unique minimizer of* (3.2.6). *This uniqueness is up to a multiplication by a constant phase.*

2. *If* $\frac{\pi^2}{L} < m$, *then* $Q_{m,L}(x) := \alpha \mathfrak{dn}(\lambda x, k)$ *is the unique minimizer of* (3.2.6), *up to a translation and multiplication by a constant phase. Moreover,* $\alpha, \lambda > 0, k \in (0,1)$ *are determined uniquely by* m, L.

Furthermore, we have compactness of the minimizing sequence up to a phase shift and translation in $H^1(\mathbb{T}_L)$, *i.e., for any sequence* u_n *in* $H^1(\mathbb{T}_L)$, *such that* $\mathcal{H}(u_n) \to E_0(m,L)$, *as* $n \to \infty$, *there is a subsequence* u_{n_k}, *and sequences* $\gamma_k \in [0, 2\pi)$, *and* $x_k \in \mathbb{T}_L$, *where* $e^{i\gamma_k} u_{n_k}(. + x_k) \to Q_{m,L}$, *in* $H^1(\mathbb{T}_L)$.

Since each solution of (3.2.5) (and consequently a solution to (3.2.1)) corresponds to the minimization problem (3.2.6), by abusing the terminology, we use the term "*standing wave*" or *Soliton* for $Q_{m,L}$.

Notice that multiplying (3.2.5) by $\bar{u}$ and integrating, we obtain the following relation

$$E_0(m,L) = \frac{1}{4}\int_{\mathbb{T}_L} u^4(x)dx - \frac{\omega m}{2}.$$

that implies $\omega \geq \frac{1}{2m}\int_{\mathbb{T}_L} u^4(x)dx + \frac{m}{2L^2}$.

3.2.2 Stochastic perturbation of discrete focusing NLS

In this section, we are going to perturb the NLS (3.2.1), with the stochastic heat bath, which we defined in Section 3.1, namely (3.1.6). Without loosing generality, in order to simplify notation, we fix the macroscopic length $L = 1$. This means that we fix the following parameters $h = \frac{1}{n}, s = \frac{1}{n}, d = 1, p = 3, \kappa = -1$. Here, we briefly recall the dynamics of Section 3.1 in this particular setup, in order to set the notations.

Fix $n \in \mathbb{N}$, the configuration space is $\chi = \mathbb{C}^n$ and denote a typical element of χ by $\{\psi(x)\}_{x\in\widetilde{\mathbb{T}}_n}$, with $\widetilde{\mathbb{T}}_n = \{1, 2, \ldots, n\}$ is the discrete torus of size n. Equivalently, a function ψ on $\widetilde{\mathbb{T}}_n$ can be seen as discretization of a function u on a unit torus, $u : \mathbb{T} \to \mathbb{C}$, with mesh size $\frac{1}{n}$, i.e., $\psi(x) = u(\frac{x}{n})$, for $x \in \widetilde{\mathbb{T}}_n$. Then the discrete cubic focusing non-linear Schrödinger equation (DNLS) is the following system of ODEs:

$$i\frac{d\psi(x,t)}{dt} = -\Delta\psi(x,t) - |\psi(x,t)|^2\psi(x,t), \tag{3.2.7}$$

where $\Delta\psi(x,t) = n^2(\psi(x+1) - 2\psi(x) + \psi(x-1))$, and we imposed periodic boundary condition $\psi(0) \equiv \psi(n)$. Notice that we define Δ such that formally in the limit $n \to \infty$, this definition coincides with the continuous Laplacian on a unit torus.

Similar to the continuous case, we have the energy or Hamiltonian $\mathcal{H}_n : \mathbb{C}^n \to \mathbb{R}$ as a conserved quantity, that is defined by:

$$\mathcal{H}_n(\psi) = \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n} \frac{n^2}{2}|\psi(x) - \psi(x-1)|^2 - \frac{1}{4n}\sum_{x\in\widetilde{\mathbb{T}}_n} |\psi(x)|^4 = G_n(\psi) - V_n(\psi), \tag{3.2.8}$$

where we have denoted the kinetic energy $G_n(\psi)$ and the potential energy $V_n(\psi)$ as:

$$G_n(\psi) = \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n} \frac{n^2}{2}|\psi(x)-\psi(x-1)|^2, \qquad V_n(\psi) = \frac{1}{4n}\sum_{x\in\widetilde{\mathbb{T}}_n}|\psi(x)|^4. \tag{3.2.9}$$

The other conserved quantity is given by the mass $\mathcal{M}_n : \mathbb{C}^n \to \mathbb{R}$, defined by:

$$\mathcal{M}_n(\psi) = \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|\psi(x)|^2. \tag{3.2.10}$$

Notice that we scaled (3.2.8) and (3.2.10), such that in the limit as $n \to \infty$, we recover $\mathcal{H}$, and $\mathcal{M}$ formally.

The stochastic perturbation we consider will only conserve the mass. Recall $\psi(x) = \psi_r(x) + i\psi_i(x) = |\psi(x)|e^{i\theta(x)}$, the generators of the Hamiltonian and stochastic noise at temperature β^{-1} read

$$A_n = n\sum_{x\in\widetilde{\mathbb{T}}_n}(\partial_{\psi_i(x)}\mathcal{H}_n)\partial_{\psi_r(x)} - (\partial_{\psi_r(x)}\mathcal{H}_n)\partial_{\psi_i(x)}, \tag{3.2.11}$$

$$\mathcal{S}_n = \beta^{-1}\sum_{x\in\widetilde{\mathbb{T}}_n} e^{\beta\mathcal{H}_n}\partial_{\theta(x)}e^{-\beta\mathcal{H}_n}\partial_{\theta(x)}. \tag{3.2.12}$$

Fix $\beta > 0, \gamma > 0$, then the generator of the dynamics and corresponding system of stochastic partial differential equations with values in χ, are as follows:

$$L_n = A_n + \gamma\mathcal{S}_n, \tag{3.2.13}$$

$$\begin{aligned} d\psi(x,t) =& i[\Delta\psi(x,t) + |\psi(x,t)|^2\psi(x,t)]dt - \gamma\psi(x,t)(\beta^{-1} + i\partial_{\theta(x)}\mathcal{H}_n(\psi))dt \\ &- i\sqrt{2\gamma\beta^{-1}}\psi(x,t)dw(x,t), \end{aligned} \tag{3.2.14}$$

where $\{w(x,t), x \in \widetilde{\mathbb{T}}_n\}$ are real independent Wiener processes.

Due to the mass conservation, having an initial condition $\psi(0,t) = \psi_0 \in \mathbb{C}^n$ such that $\mathcal{M}_n(\psi_0) = m$, our dynamic will be confined in the sphere $S_m^n = \{\psi \in \mathbb{C}^n | \mathcal{M}_n(\psi) = m\}$. Denote the uniform probability measure on S_m^n by $d\mu_m^n$, and define the canonical Gibbs measure with inverse temperature β on S_m^n as

$$d\mu_{\beta,m}^n = \frac{1}{Z_n(\beta,m)}e^{-\beta\mathcal{H}_n(\psi)}d\mu_m^n, \tag{3.2.15}$$

Here $Z_n(\beta,m) = \int_{S_m^n} e^{-\beta\mathcal{H}_n(\psi)}d\mu_m^n$. As we observed, $Z_n(\beta,m)$ is finite, and consequently, the existence of $d\mu_{\beta,m}^n$ is evident, since $\mathcal{H}_n(\psi)$ is bounded from below in S_m^n. However, one can find a lower bound for $\mathcal{H}_n(\psi)$, which is uniform in n, using a version of Gagliardo-Nirenberg inequality in the discrete periodic setup. This will be discussed broadly in the Section 3.3 and Appendix 3.C.

Applying the result of Section 3.1, we have the following results: By Theorem 3.1 we know that $d\mu^n_{\beta,m}$ is the unique invariant measure for the dynamics (3.2.13)((3.2.14)). Moreover, let μ_t denotes the law of the process at time $t \geq 0$, generated by (3.2.13), with initial condition $\mu_0 = \delta_{\psi_0}$, where ψ_0 is an arbitrary element of S^n_m[5]. Then, thanks to Proposition (3.1.3), for any $\psi_0 \in S^n_m$, there exists $C = C(n,m) > 0$ and $\gamma_0 = \gamma_0(n,m) > 0$ such that

$$\|\mu_t - \mu^n_{\beta,m}\|_{TV} \leq Ce^{-\gamma_0 t}. \tag{3.2.16}$$

If we run our dynamics for a long time, then take the limit of large n and small temperature β^{-1} properly, we end-up near *Solitons* or *standing waves* ($Q_{1,m}$ from Theorem 3.2.1), with probability one. Notice that here we can take the limit in β and n simultaneously, where we scale β by a factor of $\vartheta(n)$. In order to make these words rigorous, and connect the discrete setup to the continuous one, we need to introduce some notations. For any $\psi_n \in \mathbb{C}^n$, we define its linear interpolation $\bar{\psi}^n : \mathbb{T} \to \mathbb{C}$, on a unit torus by

$$\bar{\psi}_n(y) = \psi_n\Big([ny]\Big)\Big([ny] + 1 - ny\Big) + \psi_n\Big([ny] + 1\Big)\Big(ny - [ny]\Big), \quad \forall y \in \mathbb{T}, \tag{3.2.17}$$

where $[ny]$ denotes the greatest integer less than ny. Denote $H^1(\mathbb{T})$ by $H^1_{per}([0,1]) = H^1_{per}$. For $x \in \mathbb{T}$, let τ_x denotes the translation operator on H^1_{per}, i.e. $(\tau_x f)(y) = f(x+y)$, then, in order to deal with the phase multiplication and translation, define the following seminorm as in [67]:

$$\forall f, g \in H^1_{per}, \quad \|f - g\|_{\tilde{H}^1_{per}} := \inf_{\gamma \in [0,2\pi], x \in \mathbb{T}} \|e^{i\gamma}\tau_x f - g\|_{H^1_{per}}. \tag{3.2.18}$$

In the following we set $Q_{1,m} =: Q_m$. Now we can state the main theorem of this section:

Theorem 3.2.2. *Fix $m > 0$, $\gamma > 0$, and $\beta > 0$, let $\beta_n = \vartheta(n)\beta$, where $\vartheta(n) > 0$ is a scaling parameter, such that*

$$\lim_{n\to\infty} \frac{\vartheta(n)}{n} \to \infty. \tag{3.2.19}$$

Let $\mu_t^{\beta_n,n,m}$ be the law of the process given by its generator (3.2.13)*, with the initial condition $\mu_0^{n,m} = \delta_{\psi_0^{n,m}}$, where $\psi_0^{n,m}$ is a sequence of proper initial conditions, i.e., for all n, $\psi_0^{n,m} \in S^n_m$. Then $\forall\, \epsilon > 0$, we have:*

$$\lim_{n\to\infty} \lim_{t\to\infty} \mu_t^{\beta_n,n,m}\Big(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon\Big) \to 1. \tag{3.2.20}$$

We briefly sketch the proof: we have already proved that $\mu^{\beta_n}_{n,m}$ is the limit in t of $\mu_t^{\beta_n,n,m}$. Consequently, all we have to prove is that

$$\lim_{n\to\infty} \mu^{\beta_n}_{n,m}\Big(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon\Big) \to 1. \tag{3.2.21}$$

[5]Notice that as before, with our definition, μ_t actually depends on ψ_0. However, we omit ψ_0 to lighten the notation.

We can prove that the measure $\mu_{n,m}^{\beta_n}$ concentrates all its mass on the (discrete) configurations having close to the minimal energy, when we send temperature to zero with a proper speed. It turns out that the proper speed here is to scale β by $\vartheta(n)$, satisfying (3.2.19). Finally, we show that if a configuration has energy close to the minimal, it will be close to Q_m in the sense of (3.2.20). This can be done by adapting certain form of concentration compactness argument to the discrete setup.

Remark 3.2.1. ***About the exchange of limits in*** (3.2.20)***:*** *In the evolution equation* (3.2.14) *the drift term* $\partial_{\theta(x)}\mathcal{H}_n(\psi) = \frac{1}{n}\mathcal{I}m[\psi(x)\Delta\psi^*(x)]$ *would became very singular when* $n \to \infty$ *keeping the temperature positive. But with* $\beta_n \to \infty$ *fast enough the solution should became enough regular in space so that the corresponding limit as* $n \to \infty$ *should be given by the continuous deterministic NLS. This will be investigated in a future work [126]. The later suggests that one could study the joint limit* $n, t \to \infty$*, with* $t_n = n^\alpha t$*. We address the case* $t_n \ll \beta_n$ *in [126]. However, the case* $\beta_n \ll t_n$ *seems more challenging.*

3.3 Discrete "Soliton"

As we already observed in Theorem 3.2.1, the function Q_m (Solitons) can be characterized as the minimizer of a certain variational problem, where we have the compactness of the minimizing sequence. Therefore, one can observe that for a function $u \in H^1_{per}([0,1])$, with $\mathcal{M}(u) = m$, having "close to minimal" energies, means the function itself is close to Q_m in the following sense:

Lemma 3.3.1. *Assume* $u \in H^1_{per}$ *and* $\mathcal{M}(u) = m$*, then,* $\forall \epsilon > 0$, $\exists\, \delta > 0$*, such that if* $\mathcal{H}(u) \leq E_0(m) + \delta$*, then there exists* $\gamma \in [0, 2\pi]$*,* $x \in [0,1]$*, such that* $\|e^{i\gamma}u(.+x) - Q_m\|_{H^1_{per}} < \epsilon$*, equivalently* $\|u(x) - Q_m\|_{\tilde{H}^1_{per}} < \epsilon$*.*

Proof. This is straightforward by the compactness of the minimizing sequence in Theorem 3.2.1. □

In this section, we establish a similar result in the discrete setup (cf. Proposition 3.3.1). There are several works in the literature regarding relation among continuous and "discrete" solitons also known as discrete breathers (cf. [18], [17] and references therein for unbounded domain). Since our proposition is slightly different from these existing works, to be self-contained and for the sake of completeness, we bring a different proof for the above-mentioned proposition. This proof is adapted to our case with bounded domain.

Similar to (3.2.6), fix $n > 1, m > 0$ and define $E_0^n(m)$ as follows:

$$E_0^n(m) := \inf\{\mathcal{H}_n(\psi_n) | \psi_n \in S_m^n\}. \tag{3.3.1}$$

Since $\mathcal{H}_n(\psi_n)$ is a continuous function from the compact set S_m^n to $\mathbb{R}$, the image of this function is compact, hence, $-\infty < E_0^n(m)$, and this infimum is achieved in a compact set, which will be called the set of "discrete Solitons" and denoted by $\emptyset \neq \mathcal{Q}_m^n \subset S_m^n$. By the same argument as in the continuous case, discrete solitons are real-valued and positive up to a constant phase.

For $\psi_n \in S_m^n$, we define $\|\psi_n\|^p_{\ell^p(\widetilde{\mathbb{T}}_n)} = \frac{1}{n}\sum_j |\psi_n(j)|^p$. Then we can write $\mathcal{H}_n(\psi_n) = G_n(\psi_n) - \frac{1}{4}\|\psi_n\|^4_{\ell^4(\widetilde{\mathbb{T}}_n)}$, and by using the discrete Gagliardo-Nirenberg inequality (3.C.11), we have:

$$-\theta(m) \leq E_0^n(m) < 0, \tag{3.3.2}$$

where $\theta(m) = \frac{C^2}{64}m^3 + \frac{C}{4}m^2$. First inequality is a direct consequence of (3.C.11), and the second one can be deduced by considering the constant function $\psi_n(x) = \sqrt{m}$, for all $x \in \widetilde{\mathbb{T}}_n$.

From (3.3.2) we establish a simple but useful lemma:

Lemma 3.3.2. *For every $\epsilon > 0$, there exists $C(m,\epsilon)$, such that for all $n \in \mathbb{N}$ and $\psi_n \in S_n^m$, with $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \epsilon$, we have $G_n(\psi_n) \leq C(m,\epsilon)$.*

Proof. Consider the inequality (3.3.2), and (3.C.11); denote $x = G_n(\psi_n)^{\frac{1}{2}}$ so $x \geq 0$. If $\mathcal{H}_n(\psi_n) \leq E_0^n + \epsilon$ then, thanks to (3.C.11) we have

$$E_0^n(m) + \epsilon \geq x^2 - c'm^{\frac{3}{2}} - c'm^2 \implies x \leq C^{\frac{1}{2}}(m,\epsilon). \tag{3.3.3}$$

where $C^{\frac{1}{2}}(m,\epsilon)$ is given by $C^{\frac{1}{2}}(m,\epsilon) = \frac{c'm^3 + \sqrt{c'm^3 + 4(c'm^2 + E_0^n(m) + \epsilon)}}{2}$, where the expression under the square root is clearly positive, thanks to the expression of $\theta(m)$. □

Lemma 3.3.2 states that if the energy is "small" ($O(1)$), then the configuration should be "smooth" i.e., $G_n \sim O(1)$.

In the rest of this section, we prove that $\bar{\psi}_n$, the linear interpolation of a configuration ψ_n, is arbitrarily close to Q_m in $\tilde{H}^1_{per}$, if we take n sufficiently large, and the energy of ψ_n, $\mathcal{H}_n(\psi_n)$, sufficiently close to $E_0^n(m)$. The proof relies on the fact that the configurations with close to minimal energies are smooth in the sense that their linear interpolation's norm (L^p, H^1 or even the energy) is close to the corresponding discrete norms. This result heavily depends on the inequality of Appendix 3.C. We begin by stating this result:

Proposition 3.3.1. *Fix $m > 0$, for any $\epsilon > 0$, there exists $\eta(\epsilon)$ and $N_0(\epsilon)$, such that for $n > N_0(\epsilon)$, if $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \eta$, then we have: $\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon$.*

We divide the proof of (3.3.1), into a couple of simple lemmas. The advantage of the linear interpolation (3.2.17) is that it conserves the kinetic energy, i.e., $G_n(\psi_n) = \frac{1}{2}\int_0^1 |\partial_x \bar{\psi}_n|^2$. But unfortunately, in general, we have $\|\psi_n\|_{\ell^p(\widetilde{\mathbb{T}}_n)} \geq \|\bar{\psi}_n\|_{L^p}$ for $p \geq 1$, thanks to the Jensen inequality. Consequently, in general we have $\mathcal{H}_n(\psi_n) \leq \mathcal{H}(\bar{\psi}_n)$ and $\mathcal{M}_n(\psi_n) \geq \mathcal{M}(\bar{\psi}_n)$. However, the following lemma helps to establish the fact that these quantities are "close", for configurations with near minimal energies.

Lemma 3.3.3. *For all $n \in \mathbb{N}$, if $\psi_n \in S_m^n$ we have:*

$$\left| \|\bar{\psi}_n\|^p_{L^p(\mathbb{T})} - \|\psi_n\|^p_{\ell^p(\widetilde{\mathbb{T}}_n)} \right| \leq \frac{p(2G_n(\psi_n))^{1/2}\left(m^{1/2} + G_n(\psi_n)^{1/2}\right)^{p-1}}{n},$$

Proof. We have $\psi_n \in S_n^m$, and define $\ell_n = \min\{|\psi(x)| \,\big|\, x \in \widetilde{\mathbb{T}}_n\}$, clearly $\ell_n \leq \sqrt{m}$. Moreover, for any $x \in \widetilde{\mathbb{T}}_n$, we have:

$$|\ell_n - \psi_n(x)| \leq \sum_{j=1}^{n} |\psi_n(j) - \psi_n(j-1)| \leq \sqrt{n} \left(\sum_{j=1}^{n} |\psi_n(j) - \psi_n(j-1)|^2 \right)^{1/2} = \sqrt{2G_n(\psi_n)},$$

where we used a Cauchy-Schwartz inequality. Therefore, we can deduce that

$$\sup_x |\psi(x)| \leq c_1 = m^{\frac{1}{2}} + (2G_n(\psi_n))^{\frac{1}{2}}.$$

Moreover, thanks to the definition of $\bar{\psi}_n(y)$, we have $|\bar{\psi}_n(y)| \leq c_1$, for all $y \in \mathbb{T}$. Then we can simply compute:

$$\begin{aligned}
\left| \|\bar{\psi}_n\|^p_{L^p(\mathbb{T})} - \|\psi_n\|^p_{\ell^p(\widetilde{\mathbb{T}}_n)} \right| \leq & \sum_{x=0}^{n-1} \int_{\frac{x}{n}}^{\frac{x+1}{n}} \Big| |\psi_n(x)|^p - |\bar{\psi}_n(y)|^p \Big| dy \\
\leq & p c_1^{p-1} \sum_{x=1}^{n} \int_{\frac{x-1}{n}}^{\frac{x}{n}} \Big| |\psi_n(x)| - |\bar{\psi}_n(y)| \Big| dy \\
\leq & \frac{p c_1^{p-1}}{n} \sum_{x=1}^{n} |\psi_n(x) - \psi_n(x-1)| \leq \frac{p c_1^{p-1} (2G_n(\psi_n))^{\frac{1}{2}}}{n},
\end{aligned} \tag{3.3.4}$$

where the first inequality comes from the definition, in the second inequality we used the fact that $\psi_n(x)$ and $\bar{\psi}_n(y)$ are bounded uniformly in x and y, and in the third inequality we used the definition of $\bar{\psi}_n(y)$:

$$\Big| |\psi_n(x)| - |\bar{\psi}_n(y)| \Big| \leq |\psi_n(x) - \bar{\psi}_n(y)| \leq |\psi_n(x) - \psi_n(x+1)|.$$

Notice that the last inequality in (3.3.4) is obtained as above. □

As a straightforward consequence of Lemma 3.3.3, we can deduce the following corollaries:

Corollary 3.3.3.1. *For any $c > 0$, there exist $C_1(c, m)$, such that for every $n \in \mathbb{N}$, and $\psi_n \in S_m^n$, such that $G_n(\psi_n) < c$, then $|\mathcal{H}(\bar{\psi}_n) - \mathcal{H}_n(\psi_n)| \leq \frac{C_1(c,m)}{n}$.*

Proof. Thanks to the definition of $\bar{\psi}_n$ (3.2.17), the weak derivative of $\bar{\psi}_n$ is given as follows: for any $y \in [0, 1]$, if $\frac{x}{n} \leq y < \frac{x+1}{n}$ with $x \in \widetilde{\mathbb{T}}_n$, then $\partial_y \bar{\psi}_n(y) = n(\psi_n(x+1) - \psi_n(x))$. Therefore, we have: $\frac{1}{2} \int_0^1 |\partial_y \bar{\psi}_n|^2 = \frac{n}{2} \sum_{x=1}^{n} |\psi(x) - \psi(x-1)|^2 = G_n(\psi_n)$. Hence, we have:

$$\Big| \mathcal{H}_n(\psi_n) - \mathcal{H}(\bar{\psi}_n) \Big| = \frac{1}{4} \Big| \|\bar{\psi}_n\|^4_{L^4(\mathbb{T})} - \|\psi_n\|^4_{\ell^4(\tilde{\widetilde{\mathbb{T}}}_n)} \Big|,$$

and we can conclude thanks to Lemma 3.3.3. □

Corollary 3.3.3.2. *For any $\delta > 0$, there exist $\eta > 0$ and $N_0(\delta)$, such that for $n > N_0(\delta)$ if $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \eta$ then $\mathcal{H}(\bar{\psi}_n) \leq E_0^n(m) + \delta$.*

Proof. It follows immediately from Corollary 3.3.3.1 and Lemma 3.3.2. □

Proposition 3.3.2.

$$\lim_{n\to\infty} E_0^n(m) \to E_0(m). \tag{3.3.5}$$

Proof. Before proceeding, we emphasize the fact that all the constants $c, c1, c2, c', \dots$ are independent of n in this proof.
Recall the definition of Q_m as the minimizer of (3.2.6). Moreover, recall the definition of set of discrete Solitions $\mathcal{Q}_m^n$, as the set of mininizer of (3.3.1). Take $q_n \in \mathcal{Q}_m^n$, notice that thanks to the inequality $|\psi_n(x) - \psi_n(x-1)| \geq ||\psi_n(x)| - |\psi_n(x-1)||$, we can take q_n to be real-valued and positive. Then we have: $\mathcal{H}(Q_m) = E_0(m)$, and for all n, $\mathcal{H}_n(q_n) = E_0^n(m)$.
thanks to Lemma 3.3.2 there exists $c > 0$ uniform in n, such that $G_n(q_n) \leq c$. Therefore, we can use the result of Corollary 3.3.3.1, and deduce that there exists C_1 independent of n, such that:

$$|\mathcal{H}(\bar{q}_n) - \mathcal{H}_n(q_n)| \leq \frac{C_1}{n}. \tag{3.3.6}$$

For any $\psi \in H^1_{per}([0,1])$, define $\lambda_n(\psi)$ as follows:

$$\lambda_n(\psi) = \left(\frac{m}{\mathcal{M}(\psi)}\right)^{\frac{1}{2}}. \tag{3.3.7}$$

In particular, let $\lambda_n = \lambda(\bar{q}_n)$ and observe that for n sufficiently large, $|\lambda_n^2 - 1| \leq \frac{c_0}{n}$, with c_0 independent of n, thanks to Lemma 3.3.3. More precisely, we can take $c_0 = \frac{2\tilde{c}}{m}$, for n sufficiently large, where $\tilde{c}$ is given by Lemma 3.3.3. Now, if we use the definition of $\mathcal{H}$, for n sufficiently large we obtain:

$$\begin{aligned} |\mathcal{H}(\lambda_n\bar{q}_n) - \mathcal{H}(\bar{q}_n)| &\leq |\lambda_n^2 - 1| \int_0^1 \frac{|\partial_y \bar{q}_n(y)|^2}{2} dy + \frac{|\lambda_n^4 - 1|}{4} \int_0^1 |\bar{q}_n(y)|^4 dy \\ &\leq \frac{c_1}{n}, \end{aligned} \tag{3.3.8}$$

where c_1 is independent of n, and we used the estimate $|\lambda_n^2 - 1| \leq \frac{c_0}{n}$; moreover, in order to treat the first term, we take advantage of the fact that $G_n(q_n) = \int_0^1 \frac{|\partial_y \bar{q}_n(y)|^2}{2} dy \leq c$. Lastly, the second term is bounded as follows: we used the bound $\|q_n\|_{\ell^4(\widetilde{\mathbb{T}}_n)} \leq c'$ (thanks to Lemma 3.3.2 and (3.3.2)), then we conclude by using the fact $|\|q_n\|_{\ell^4(\widetilde{\mathbb{T}}_n)} - \|\bar{q}_n\|_{L^4(\mathbb{T})}| \leq \frac{\tilde{c}'}{n}$ which is a direct consequence of Lemma 3.3.3.
Notice that $\mathcal{M}(\lambda_n\bar{q}_n) = m$; therefore, $E_0(m) \leq \mathcal{H}(\lambda_n\bar{q}_n)$. Combining this fact with (3.3.6) and (3.3.8), for n large enough we have:

$$E_0(m) \leq E_0^n(m) + \frac{c''}{n}, \tag{3.3.9}$$

where c'' is a constant independent of n, and we used the fact that $\mathcal{H}_n(q_n) = E_0^n(m)$.

On the other hand, recall that Q_m is smooth, real-valued and non-negative thanks to Theorem 3.2.1. Define $Q_m^n : \widetilde{\mathbb{T}}_n \to \mathbb{C}$ as $Q_m^n(x) = Q_m(\frac{x}{n})$, for $x \in \widetilde{\mathbb{T}}_n$. Let

$$\tilde{\lambda}_n := \left(\frac{m}{\mathcal{M}_n(Q_m^n)}\right)^{\frac{1}{2}}.$$

Thank to the properties of Q_m (in particular the fact that Q_m is smooth with bounded H^1 and L^4 norm), for n large enough we have:

$$|\tilde{\lambda}_n^2 - 1| \leq \frac{c_2}{n}, \tag{3.3.10}$$

where one can take $c_2 = \frac{4\|Q_m\|_{L^\infty}\|Q_m'\|_{L^\infty}}{m}$ (Q' denotes the derivative of Q). Moreover, since Q_m is smooth, $G_n(Q_m^n)$ and $V_n(Q_m^n)$ are bounded uniformly in n by $\frac{\|Q_m'\|_{L^\infty}}{2}$, and $\frac{\|Q_m\|_{L^\infty}^4}{4}$, respectively. Hence, thanks to (3.3.10) for n sufficiently large we have:

$$|\mathcal{H}_n(\tilde{\lambda}_n Q_m^n) - \mathcal{H}_n(Q_m^n)| \leq \frac{c_3}{n}. \tag{3.3.11}$$

Again, since Q_m is at least C^3, by a simple computation we get for n sufficiently large:

$$\begin{aligned}
&|\mathcal{H}_n(Q_m^n) - \mathcal{H}(Q_m)| \leq \frac{1}{2}\sum_{x=1}^{n}\int_{\frac{x-1}{n}}^{\frac{x}{n}}\left|n^2\left|Q_m\Big(\frac{x}{n}\Big) - Q_m\Big(\frac{x-1}{n}\Big)\right|^2 - |\partial_y Q_m(y)|^2\right|dy + \\
&\frac{1}{4}\sum_{x=1}^{n}\int_{\frac{x-1}{n}}^{\frac{x}{n}}\left||Q_m\Big(\frac{x}{n}\Big)|^4 - |Q_m(y)|^4\right|dy \leq \frac{\|Q_m''\|_{L^\infty}\|Q_m'\|_{L^\infty}}{n} + \frac{\|Q_m'\|_{L^\infty}\|Q_m\|_{L^\infty}^3}{n} \\
&\leq \frac{c_4}{n}.
\end{aligned} \tag{3.3.12}$$

Therefore, combining the estimates (3.3.11) and (3.3.12), and recalling the fact that $\mathcal{H}(Q_m) = E_0(m)$, we have for n large enough:

$$|\mathcal{H}_n(\tilde{\lambda} Q_m^n) - \mathcal{H}(Q_m)| \leq \frac{c}{n} \implies E_0^n(m) \leq E_0(m) + \frac{c}{n}, \tag{3.3.13}$$

where we used the fact that $\mathcal{M}(\tilde{\lambda}_n Q_m^n) = m$, hence $E_0^n(m) \leq \mathcal{H}_n(\tilde{\lambda}_n Q_m^n)$. Finally, taking the limit of $n \to \infty$ in (3.3.13) and (3.3.9), properly (lim sup and lim inf, respectively), we deduce the result (3.3.5). □

We finish this section by proving the Proposition 3.3.1:

Proof of Proposition 3.3.1. In consequence of corollary 3.3.3.2 and Proposition 3.3.5 we have that for any $\delta > 0$, there exist $\eta > 0$ and $N_0(\delta)$, such that for $n > N_0(\delta)$ if $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \eta$ then $\mathcal{H}(\bar{\psi}_n) \leq E_0(m) + 2\delta$. Define

$$\lambda_{\bar{\psi}_n} = \Big(\frac{m}{\mathcal{M}(\bar{\psi}_n)}\Big)^{\frac{1}{2}} \geq 1,$$

so that $\mathcal{M}(\lambda_{\bar{\psi}_n}\bar{\psi}_n) = m$. Furthermore by Lemma 3.3.3 $\lambda_{\bar{\psi}_n} \to 1$. We also have that

$$\begin{aligned}\mathcal{H}\left(\lambda_{\bar{\psi}_n}\bar{\psi}_n\right) &= \mathcal{H}(\bar{\psi}_n) + (\lambda^2_{\bar{\psi}_n} - 1)G_n(\psi_n) - (\lambda^4_{\bar{\psi}_n} - 1)V(\bar{\psi}_n) \\ &\leq E_0(m) + 2\delta + (\lambda^2_{\bar{\psi}_n} - 1)C,\end{aligned} \tag{3.3.14}$$

where we bounded G_n thanks to Lemma 3.3.2. By lemma 3.3.1, we have $\|\lambda_{\bar{\psi}_n}\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon/2$, and since

$$\|\lambda_{\bar{\psi}_n}\bar{\psi}_n - \bar{\psi}_n\|_{\tilde{H}^1_{per}} \leq |\lambda^2_{\bar{\psi}_n} - 1|^{1/2}\|\bar{\psi}_n\|_{\tilde{H}^1_{per}} < \epsilon/2$$

for n large enough, we conclude the proof. □

3.4 Large Deviation Estimates

In Proposition 3.3.1, we proved that if the energy $\mathcal{H}_n(\psi_n)$ is sufficiently close to the minimal energy $E^0_n(m)$ for n sufficiently large, then the linear interpolation of a configuration ψ_n is close to Q_m in $\tilde{H}^1$-norm. In this section, we prove that the measure $\mu^n_{\beta_n,m}$ (3.2.15) concentrate on configurations with minimal energy as $n \to \infty$, if we set $\beta_n = \vartheta(n)\beta$, where $\vartheta(n)$ satisfies (3.2.19). As m is fixed in this section, we will drop it from the notations.

Theorem 3.4.1. *Recall $\beta_n = \vartheta(n)\beta$, where $\vartheta(n)$ satisfies* (3.2.19)*. For any $\epsilon > 0$, we have:*

$$\lim_{n\to\infty} \mu^n_{\beta_n}(\mathcal{H}_n(\psi_n) \leq E^n_0 + \epsilon) = 1, \tag{3.4.1}$$

The proof of Theorem (3.4.1) depends on two large deviation estimates for the uniform probability measure $d\mu^n_m$ that are proven in appendix 3.A:

1. For any n and any $0 < \mathfrak{g}$ we have:

$$\mu^n_m(G_n(\psi_n) < \mathfrak{g}) \leq \exp(-2n\ln n)\left(\frac{2\mathfrak{g}}{m}\right)^{n-1} 2^n, \tag{3.4.2}$$

   This bound is proven in Lemma 3.A.1, following the same spirit as in [67], Section 10. However, because of our special scaling in G_n, one should follow the dependence of the rate function on n carefully, in contrast to the estimate in [67]. This lemma provides the aforementioned upper bounds. Combining (3.4.2) with Gagliardo-Nirenberg inequality, we can deduce a suitable upper bound for $\mathcal{H}_n$.

2. For any $\epsilon > 0$, there exists $d = d(\epsilon)$, such that for $n > 2$:

$$\mu^n_m(\mathcal{H}_n(\psi_n) < E^n_0 + \epsilon) \geq d^n e^{-2n\ln n}. \tag{3.4.3}$$

   This is proven in Lemma 3.A.2.

We will proceed as follows: first, we state a proof of (3.4.1), when $\beta_n = \beta n \ln n$. This proof is quite simple and illustrates how does the above estimates are involved. Finally, we prove the general case $\beta_n = \vartheta(n)\beta$.

Proof of Theorem 3.4.1 with $\beta_n = \beta n \ln n$. Assume $0 < \epsilon < 1$, in order to prove (3.4.1), it is sufficient to prove that:

$$p_n := \frac{\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n(\psi_n) - E_n^0 \geq \epsilon\}} e^{-\beta_n \mathcal{H}_n(\psi_n)} d\mu_m^n}{\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n(\psi_n) - E_n^0 < \frac{\epsilon}{2}\}} e^{-\beta_n \mathcal{H}_n(\psi_n)} d\mu_m^n} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.4.4}$$

Thanks to the lower bound (3.4.3), there exists $d > 0$ such that:

$$\begin{aligned} &\left(\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n(\psi_n) - E_n^0 < \frac{\epsilon}{2}\}} e^{-\beta_n \mathcal{H}_n(\psi_n)} d\mu_m^n \right)^{-1} \leq e^{\beta_n (E_0^n + \frac{\epsilon}{2})} \left[\mu_m^n \left(\mathcal{H}_n(\psi_n) - E_n^0 \leq \frac{\epsilon}{2} \right) \right]^{-1} \\ &\leq e^{n \ln n \left(\beta (E_0^n + \frac{\epsilon}{2}) + 2 \right) - n \ln d}. \end{aligned} \tag{3.4.5}$$

Let $c > \epsilon + \frac{2}{\beta}$; recall Lemma 3.3.2 and let $c' = C(m, c) > 0$, which is given by this lemma. By using (3.4.2) there exists N_2, such that for $n > N_2$:

$$\begin{aligned} &\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0(m) \geq \epsilon\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n = \\ &\int_{S_m^n} \mathbb{1}_{\{c > \mathcal{H}_n - E_n^0(m) \geq \epsilon\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n + \int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0(m) \geq c\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n \\ &\leq e^{-\beta n \ln n (E_0^n(m) + \epsilon)} \mu_n^m (G_n < c') + e^{-\beta n \ln n (E_0^n(m) + c)} \\ &\leq e^{-n \ln n \left(\beta (E_n^0(m) + \epsilon) + 2 \right) + n \ln(4c'/m)} + e^{-\beta n \ln n (E_0^n(m) + c)}, \end{aligned} \tag{3.4.6}$$

where in the second line we used the fact that $\{c > \mathcal{H}_n - E_0^n(m) \geq \epsilon\} \subset \{G_n \leq c'\}$, thanks to the choice of c', see Lemma 3.3.2. Finally, taking $N > N_2$, and combining (3.4.5) and (3.4.6), gives us the following:

$$0 \leq p_n \leq e^{-n\left(\ln n \frac{\beta\epsilon}{2} - \ln(4c'/dm) \right)} + e^{-n\left(\ln n \left(c\beta - \frac{\beta\epsilon}{2} - 2 \right) - \ln d \right)} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.4.7}$$

thanks to the choice of c. □

Now we prove Theorem 3.4.1 in the general situation with $\beta_n = \beta\vartheta(n)$ satisfying (3.2.19):

Proof of Theorem 3.4.1 with $\beta_n = \vartheta(n)\beta$. Fix $0 < \epsilon < 1$, (the other cases will be straightforward). In order to prove (3.4.1), it is sufficient to prove (3.4.4). As for (3.4.5), there exist $d > 0$ such that :

$$\left(\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0 < \frac{\epsilon}{2}\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n \right)^{-1} \leq e^{\beta\vartheta(n)(E_0^n + \frac{\epsilon}{2})} (d)^{-n} e^{2n \ln n}. \tag{3.4.8}$$

Let us decompose the numerator of (3.4.4) into two parts and denote them by q_n and q'_n:

$$\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0 \geq \epsilon\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n = \underbrace{\int_{S_m^n} \mathbb{1}_{\{\ln n > \mathcal{H}_n - E_n^0 \geq \epsilon\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n}_{q_n} + \underbrace{\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0 \geq \ln n\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n}_{q'_n}. \tag{3.4.9}$$

We simply bound $q'_n \leq e^{-\beta_n(E_n^0 + \ln n)}$ and observe that:

$$\left(\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0(m) < \frac{\epsilon}{2}\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n \right)^{-1} q'_n \leq e^{-\ln n(\beta\vartheta(n) - 2n)} e^{\beta\vartheta(n)\frac{\epsilon}{2} - n\ln d} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.4.10}$$

as $n \to \infty$, where we used the fact that d is a constant independent of n, as well as the condition $\lim_{n\to\infty} \frac{\vartheta(n)}{n} = \infty$.
Now we treat the term corresponding to q_n, thanks to (3.4.2). First, observe that for any $E_n^0(m) < a \leq n$, thanks to the inequality (3.C.11), if we have $\mathcal{H}_n(\psi_n) \leq a$, we can deduce $G_n(\psi_n) \leq c_1(m) + 2a$, where $c_1(m)$ is a constant independent of n. (in fact, $c_1(m) = (\tilde{c}^2 m^3 + 2\tilde{c}m^2)$, with $\tilde{c} = \frac{C}{4}$ and C is the constant in (3.C.11)); consequently, we have for any $E_n^0(m) < a \leq n$:

$$\mu_m^n(\mathcal{H}_n \leq a) \leq \mu_m^n\Big(G_n \leq 2a + c_1(m)\Big). \tag{3.4.11}$$

Recall (3.4.2): for any $0 < \alpha < 2n$ denote $\alpha_o = \frac{2\alpha}{m}$, then

$$\mu_m^n\Big(G_n \leq \alpha\Big) \leq 2^n \alpha_o^{n-1} e^{-2n\ln n} \tag{3.4.12}$$

holds. Therefore, for large n thanks to (3.4.11), (3.4.12), we have for any $E_n^0(m) < a \leq n$:

$$\mu_m^n(\mathcal{H}_n \leq a) \leq 2^n e^{-2n\ln n} \left(\frac{4a + 2c_1(m)}{m} \right)^{n-1}. \tag{3.4.13}$$

Take $h > 0$ independent of n, let $N = \frac{\ln n}{h}$. Then we have for n sufficiently large:

$$\begin{aligned}
&q_n = \sum_{j=0}^{N-1} \int_{S_m^n} \mathbb{1}_{\{E_0^n + \epsilon + jh \leq \mathcal{H}_n < E_0^n + \epsilon + (j+1)h\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n \leq \\
&\sum_{j=0}^{N-1} e^{-\beta_n(E_0^n + \epsilon + jh)} \mu_m^n\Big(E_0^n + \epsilon + jh \leq \mathcal{H}_n < E_0^n + \epsilon + (j+1)h\Big) \leq \\
&2^n e^{-2n\ln n} \sum_{j=0}^{N-1} e^{-\beta_n(E_0^n + \epsilon + jh)} \left(\frac{4}{m}\right)^{n-1} e^{(n-1)\ln\left(E_0^n + \epsilon + (j+1)h + \frac{c_1(m)}{2}\right)} = \\
&2^n \left(\frac{4}{m}\right)^{n-1} e^{-2n\ln n} e^{\beta_n(h + \frac{c_1(m)}{2})} \times \\
&\sum_{j=1}^{N} \exp\left(-\beta_n(E_0^n + \epsilon + jh + \frac{c_1(m)}{2}) + (n-1)\ln\left(E_0^n + \epsilon + jh + \frac{c_1(m)}{2}\right)\right),
\end{aligned} \tag{3.4.14}$$

where we take advantage of the estimate (3.4.13) in the second line. Notice that the term $E_0^n + \frac{c_1(m)}{2} > 0$ thanks to the lower bound (3.3.2). Recall that $\beta_n = \vartheta(n)\beta$, with $\lim_{n\to\infty} \frac{\vartheta(n)}{n} \to \infty$. Therefore, for n sufficiently large $-\beta_n + \frac{n-1}{x} < 0$, for any $x \in [h, 2\ln n]$. However, the later expression is the derivative of $-\beta_n x + (n-1)\ln(x)$, hence, this function is decreasing on the interval $[h + E_0^n + \epsilon + \frac{c_1(m)}{2}, 2\ln n]$ for any n sufficiently large, and $-\beta_n x + (n-1)\ln(x)$ achieves its minimum at $x = h + E_0^n + \epsilon + \frac{c_1(m)}{2}$ in the aforementioned interval. Combining this fact with (3.4.14) we get:

$$\begin{aligned} q_n &\leq 2^n \Big(\frac{4}{m}\Big)^{n-1} e^{-2n\ln n} e^{\beta_n(h+\frac{c_1(m)}{2})} \times \\ &N \exp\Big(-\beta_n(E_0^n + \epsilon + h + \frac{c_1(m)}{2}) + \ln\Big(E_0^n + \epsilon + h + \frac{c_1(m)}{2}\Big)\Big) \\ &= 2^n \Big(\frac{4}{m}\Big)^{n-1} e^{-2n\ln n} N \exp\Big(-\beta_n(E_0^n + \epsilon) + (n-1)\ln\Big(E_0^n + \epsilon + h + \frac{c_1(m)}{2}\Big)\Big). \end{aligned} \tag{3.4.15}$$

Notice that $0 < (E_0^n + \epsilon + h + \frac{c_1(m)}{2}) < (\epsilon + h + \frac{c_1(m)}{2}) =: c'$, and c' is a constant independent of n. Combining the later estimate (3.4.15), with (3.4.8) we get for n sufficiently large:

$$\Big(\int_{S_m^n} \mathbb{1}_{\{\mathcal{H}_n - E_n^0(m) < \frac{\epsilon}{2}\}} e^{-\beta_n \mathcal{H}_n} d\mu_m^n\Big)^{-1} q_n \leq e^{-\beta_n \frac{\epsilon}{2}} \Big(d^{-n} 2^n (\frac{4}{m})^{n-1} (c')^{n-1}\Big) \frac{\ln n}{h} \to 0, \tag{3.4.16}$$

as $n \to \infty$. Notice that (3.4.16) is evident, since the first term $e^{-\beta_n n}$ is super-exponentially small thanks to the assumption $\beta_n = \beta\vartheta(n)$ with $\lim_{n\to\infty} \frac{\vartheta(n)}{n} = \infty$ and the second term is bounded by $e^{n\tilde{c}}$, where $\tilde{c}$ is a constant independent of n. Finally, recalling the decomposition (3.4.9) and combining (3.4.16) with (3.4.10) gives us (3.4.4) and finishes the proof. □

Finally, the proof of Theorem 3.2.2 is a direct consequence of Proposition 3.3.1, and Theorem 3.4.1:

Proof of Theorem 3.2.2. Fix $\epsilon > 0$, thanks to the Proposition (3.1.3), in particular (3.1.19), we have:

$$\lim_{n\to\infty} \lim_{t\to\infty} \mu_t^{\beta_n,n,m}\Big(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon\Big) = \lim_{n\to\infty} \mu^n_{\beta_n,m}\Big(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon\Big).$$

On the other hand let us take $\delta = \delta(\epsilon)$, which is given by Proposition 3.3.1, then for all $n > N_0(\epsilon)$ thank to this proposition we have:

$$1 \geq \mu^n_{\beta_n,m}\Big(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{per}} < \epsilon\Big) \geq \mu^n_{\beta_n,m}\Big(|\mathcal{H}_n(\psi_n) - E_0^n(m)| < \delta\Big). \tag{3.4.17}$$

However, notice that $\lim_{n\to\infty} \mu^n_{\beta_n,m}\Big(|\mathcal{H}_n(\psi_n) - E_0^n(m)| < \delta\Big) = 1$, thanks to Theorem 3.4.1, in particular (3.4.1), and this finishes the proof of Theorem 3.2.2, i.e.,(3.2.20). □

3.A LD estimates for μ_m^n

We collect here some large deviation estimates concerning μ_m^n, the uniform probability on the complex n-dimensional sphere S_m^n, and in particular the estimates (3.4.2) and (3.4.3). Note that in this appendix we slightly change our notations and denote the elements of $\mathbb{C}^n$ by z or $\underline{z}$ instead of ψ.

Lemma 3.A.1. *For any $n \in \mathbb{N}$, let $0 < \mathfrak{g}$. For any $0 < \delta < 1$ we have:*

$$\mu_m^n(G_n(\psi_n) < \mathfrak{g}) \leq \frac{1}{\delta(1-\delta)^{n-1}} \left(\frac{2\mathfrak{g}}{m}\right)^{n-1} \exp(-2n \ln n). \tag{3.A.1}$$

Proof. Let $\{Z_j\}_{j=1}^{\infty}$, be a sequence of i.i.d standard complex normal random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e, for any $n > 0$, the probability density function of $(Z_1, \dots, Z_n)$ is given by:

$$f(\underline{z}) = \prod_{j=1}^{n} \frac{e^{-|z_j|^2}}{\pi}, \qquad \underline{z} := (z_1, \dots, z_n) \in \mathbb{C}^n. \tag{3.A.2}$$

Consequently, the random vector $\{\Psi_n(j) = \frac{\sqrt{mn} Z_j}{\left(\sum_\ell |Z_\ell|^2\right)^{1/2}}, \quad j = 1, \dots, n\}$ is distributed uniformly on S_n^m. For $k \in \widetilde{\mathbb{T}}_n$, let the random variable $\hat{Z}_k$ be defined as the Fourier transform of $Z_1, \dots, Z_n$:

$$\hat{Z}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{-2\pi j \frac{k}{n}} Z_j. \tag{3.A.3}$$

Notice that $(\hat{Z}_1, \dots, \hat{Z}_n)$, has the same distribution as $(Z_1, \dots, Z_n)$. Moreover, we have the following identities thanks to the properties of discrete Fourier transform:

$$\sum_{j=1}^{n} |Z_j|^2 = \sum_{k=1}^{n} |\hat{Z}_k|^2, \tag{3.A.4}$$

$$\sum_{j=1}^{n} |Z_j - Z_{j-1}|^2 = \sum_{k=1}^{n} \omega_k^2 |\hat{Z}_k|^2, \tag{3.A.5}$$

where $\omega_k = 2|\sin\left(\pi \frac{k}{n}\right)|$. Denote

$$\mathfrak{g}_o := \frac{2\mathfrak{g}}{n^2 m}, \tag{3.A.6}$$

and take $0 < \lambda$ such that, $0 < 1 - \mathfrak{g}_o \lambda$. By using Chebyshev's inequality, as well as (3.A.4) and (3.A.5), we have:

$$\begin{aligned}
&\mu_n^m(G_n(\psi_n) \leq \mathfrak{g}) = \mathbb{P}\left(n^2 m \frac{\sum_{j=1}^n |Z_j - Z_{j-1}|^2}{\sum_{j=1}^n |Z_j|^2} \leq 2\mathfrak{g}\right) \\
&= \mathbb{P}\Big(\sum_{k=1}^{n} \omega_k^2 |\hat{Z}_k|^2 \leq \sum_{k=1}^{n} |\hat{Z}_k|^2 \mathfrak{g}_o\Big) = \mathbb{P}\left(\exp\Big(-\sum_{k=1}^{n} \lambda\big(\omega_k^2 - \mathfrak{g}_o\big)|\hat{Z}_k|^2\Big) \geq 1\right) \\
&\leq \mathbb{E}\left(\exp\Big(-\sum_{k=1}^{n} \lambda\big(\omega_k^2 - \mathfrak{g}_o\big)|\hat{Z}_k|^2\Big)\right) = \prod_{k=1}^{n} \mathbb{E}\left(\exp\left(-\lambda(\omega_k^2 - \mathfrak{g}_o)|\hat{Z}_1|^2\right)\right) \\
&= \prod_{k=1}^{n} \frac{1}{\lambda\big(\omega_k^2 - \mathfrak{g}_o\big) + 1}.
\end{aligned} \tag{3.A.7}$$

Notice that in the first line, we used the fact that Ψ_n is uniformly distributed on S_n^m, and in the last line we used the fact that $\hat{Z}_k$ are independent complex Gaussian variable with the same distribution as Z_i, as well as the choice of λ, which permits us to compute the last expectation. We emphasize the fact that the last bound holds for any $0 < \lambda < \mathfrak{g}_o^{-1} = \frac{n^2 m}{2\mathfrak{g}}$, which can depend on n. In fact, our choice of λ depends on n.
Before proceeding, let us recall the following trigonometric identity:

$$\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}, \implies \frac{1}{\prod_{k=1}^{n-1} \omega_k^2} = \frac{1}{n^2}. \tag{3.A.8}$$

For any $0 < \delta < 1$, let us take $\lambda = \frac{(1-\delta)}{\mathfrak{g}_o}$. Notice that we have $1 - \lambda \mathfrak{g}_o = \delta$. Thanks to the choice of λ, by using (3.A.7) and (3.A.8), we obtain

$$\begin{aligned} \mu_n^m(G_n(\psi_n) < \mathfrak{g}) \leq \prod_{k=1}^{n} \frac{1}{\lambda\Big(\omega_k^2 - \mathfrak{g}_o\Big) + 1} \leq \frac{1}{\delta} \prod_{k=1}^{n-1} \frac{1}{\lambda \omega_k^2} \leq \frac{1}{\delta} \frac{1}{(1-\delta)^{n-1}} \mathfrak{g}_o^{n-1} \frac{1}{n^2} \\ = \frac{1}{\delta(1-\delta)^{n-1}} \exp(-2n \ln n) \left(\frac{2\mathfrak{g}}{m}\right)^{n-1}. \end{aligned} \tag{3.A.9}$$

□

Notice that the bound (3.4.2) corresponds to the choice $\delta = 1/2$. Moreover, optimizing (3.A.1) over δ yields: for any $n \in \mathbb{N}$, and $\mathfrak{g} > 0$ we have:

$$\mu_m^n(G_n(\psi_n) < \mathfrak{g}) \leq 4n \left(\frac{2\mathfrak{g}}{m}\right)^{n-1} \exp(-2n \ln n). \tag{3.A.10}$$

We obtain now the lower bound (3.4.3), indicating that set of configurations with close to minimal energy is "large enough".

Lemma 3.A.2. *For any $\epsilon > 0$, there exists a constant $\mathrm{c} = \mathrm{c}(\epsilon)$, independent of n, such that for $n > 2$ we have:*

$$\mu_m^n(\mathcal{H}_n < E_0^n + \epsilon) \geq \mathrm{c}^n e^{-2n \ln n}. \tag{3.A.11}$$

Proof. Denote by Q a discrete Soliton, we have that $E_0^n = \mathcal{H}_n(Q) = G_n(Q) - V_n(Q)$. we know from the results of Section 3.3 that Q is uniformly bounded in n as well as $G_n(Q)$ and $V_n(Q)$. Observe that

$$\begin{aligned} &\{\psi \in S_m^n : \mathcal{H}_n(\psi) < E_0^n + \epsilon\} \supset \\ &\quad \{\psi \in S_m^n : |G_n(\psi) - G_n(Q)| \leq \epsilon/2, |V_n(\psi) - V_n(Q)| \leq \epsilon/2\}. \end{aligned} \tag{3.A.12}$$

Consequently, we need to construct a neighborhood $\tilde{\mathcal{A}} \subset S_m^n$ of Q that is contained in the set on the RHS of (3.A.12), and such that $\mu_m^n(\tilde{\mathcal{A}}) \geq \mathrm{c}^n e^{-2n \ln n}$ for some constant depending on ϵ.

Let us identify $\mathbb{C}^n \sim \mathbb{R}^{2n}$, and denote the corresponding real components of Q by $(q_1, \dots, q_{2n})$, and the components of a generic $\psi \in S_m^n \sim \mathbb{S}^{2n}_{\sqrt{nm}}$ as $(x_1, \dots, x_{2n})$. We can choose the discrete Soliton Q, such that $q_{2n} \geq q_j \geq 0$.

For any small $\delta > 0$, define the set $\tilde{A}_\delta \subset \mathbb{R}^{2n-1}$ as follows:

$$\tilde{A}_\delta = \left\{ \underline{\xi} \in \Big[-\frac{\delta}{2n}, \frac{\delta}{2n} \Big]^{2n-1} \Big| \quad \Big| \sum_{j=1}^{2n-1} \xi_j q_j \Big| \leq \frac{q_{2n}\delta}{2\sqrt{n}} \right\}. \tag{3.A.13}$$

The volume of this set can be easily estimated by

$$\mathrm{vol}(\tilde{A}_\delta) \geq \frac{1}{3} \left(\frac{\delta}{n} \right)^{2n-1}. \tag{3.A.14}$$

We postpone the proof of (3.A.14) later.

We now define our neighborhood of Q as

$$\tilde{\mathcal{A}}_\delta = \left\{ \underline{x} \in \mathbb{S}^{2n}_{\sqrt{nm}} : x_j = q_j + \xi_j, j = 1 \dots, 2n-1; \ \underline{\xi} \in \tilde{A}_\delta \right\}. \tag{3.A.15}$$

Notice that if $\underline{x} \in \tilde{\mathcal{A}}_\delta$, we have automatically that $x_{2n} = (nm - \sum_{j=1}^{2n-1}(q_j + \xi_j)^2)^{\frac{1}{2}}$. Furthermore, we have that

$$|x_{2n} - q_{2n}| \leq \frac{2\delta}{\sqrt{n}}. \tag{3.A.16}$$

It is easy to check that if $\underline{x} \in \tilde{\mathcal{A}}_\delta$ then $|V_n(\underline{x}) - V_n(Q)| \leq 2c\delta/n \leq 2c\delta$, where c is a constant independent of n. About the gradients term, denoting $\xi_{2n} = x_{2n} - q_{2n}$, we have

$$\begin{aligned} G_n(\underline{\xi}) = &\frac{n}{2} \sum_{j=1}^{n-1} (\xi_{i+1} - \xi_i)^2 + \frac{n}{2}(\xi_n - \xi_1)^2 \\ &+ \frac{n}{2} \sum_{j=1}^{n-2} (\xi_{n+i+1} - \xi_{n+i})^2 + \frac{n}{2}(\xi_{2n} - \xi_{2n-1})^2 + \frac{n}{2}(\xi_{2n} - \xi_{n+1})^2 \leq 6\delta^2, \end{aligned} \tag{3.A.17}$$

and

$$\begin{aligned} |G_n(\underline{x}) - G_n(Q)| &= \left| n \sum_j (q_j - q_{j-1})(\xi_j - \xi_{j-1}) + G_n(\underline{\xi}) \right| \\ &\leq (2G_n(Q))^{1/2} \left(2G_n(\underline{\xi}) \right)^{1/2} + G_n(\underline{\xi}) \\ &\leq C \left(2G_n(\underline{\xi}) \right)^{1/2} + G_n(\underline{\xi}) \leq C'\delta. \end{aligned} \tag{3.A.18}$$

It follows that, choosing $\delta < \min\{\epsilon/2C', \epsilon/2c\}$, the set $\tilde{\mathcal{A}}_\delta$ is contained in the set defined in (3.A.12).

In order to compute $\mu_m^n(\tilde{\mathcal{A}}_\delta)$ we use the following change of variable formula for any measurable $f : \mathbb{S}_r^{2n} \to \mathbb{R}$: (cf. Appendix A of [16])

$$\int_{\mathbb{S}_r^{2n}} f(\underline{x}) d\sigma_r^{2n}(\underline{x}) = \frac{r^2}{} \int_{\mathbb{B}_r^{2n-1}} \frac{f(\underline{y}, \sqrt{r^2 - \|\underline{y}^2\|}) + f(\underline{y}, -\sqrt{r^2 - \|\underline{y}^2\|})}{\sqrt{r^2 - \|\underline{y}\|^2}} dy_1 \dots dy_{2n-1}, \tag{3.A.19}$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^{2n-1}$, and $2r^{2n}nV(\mathbb{B}_1^{2n})$ $V(\mathbb{B}_1^{2n}) = \frac{\pi^n}{n!}$ denotes the volume of the unit ball. Applying the above formula and noticing that $nm \geq \sqrt{nm - \|\underline{y}\|^2}$, we have

$$\begin{aligned}\mu_m^n(\tilde{\mathcal{A}}_\delta) &= \frac{mn!}{2(nm)^n\pi^n}\int_{\mathbb{B}^{2n-1}_{\sqrt{nm}}} \frac{\mathbb{1}_{\tilde{\mathcal{A}}_\delta}(\underline{y},\sqrt{nm-\|\underline{y}\|^2}) + \mathbb{1}_{\tilde{\mathcal{A}}_\delta}(\underline{y},-\sqrt{nm-\|\underline{y}\|^2})}{\sqrt{nm-\|\underline{y}\|^2}} dy_1\dots dy_{2n-1} \\ &\geq \frac{n!}{2n(nm)^n\pi^n}\int_{\mathbb{B}^{2n-1}_{\sqrt{nm}}} \left[\mathbb{1}_{\tilde{\mathcal{A}}_\delta}(\underline{y},\sqrt{nm-\|\underline{y}\|^2}) + \mathbb{1}_{\tilde{\mathcal{A}}_\delta}(\underline{y},-\sqrt{nm-\|\underline{y}\|^2})\right] dy_1\dots dy_{2n-1} \\ &= \frac{n!}{n(nm)^n\pi^n}\int_{-\frac{\delta}{n}}^{\frac{\delta}{n}}\cdots\int_{-\frac{\delta}{n}}^{\frac{\epsilon}{n}} \mathbb{1}_{\tilde{A}_\delta}(\underline{\xi})d\xi_1\dots d\xi_{2n-1} \geq \frac{n!}{n(nm)^n\pi^n}\frac{1}{3}\left(\frac{\delta}{n}\right)^{2n-1}\end{aligned} \tag{3.A.20}$$

and by Stirling approximation we have the desired lower bound.

□

Proof of (3.A.14). Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of i.i.d random variables uniformly distributed on $[-\frac{\delta}{2n},\frac{\delta}{2n}]$. Thanks to Chebyshev's inequality we get:

$$\begin{aligned}&\mathbb{P}\left(\Big|\sum_{j=1}^{2n-1}\xi_j q_j\Big| \leq \frac{q_{2n}\delta}{2\sqrt{n}}\right) = 1-\mathbb{P}\left(\Big|\sum_{j=1}^{2n-1}\xi_j q_j\Big|^2 > \left(\frac{q_{2n}\delta}{2\sqrt{n}}\right)^2\right) \geq \\ &1-\frac{4n}{q_{2n}^2\delta^2}\mathbb{E}\left(\Big|\sum_{j=1}^{2n-1}\xi_j q_j\Big|^2\right) \geq 1-\frac{4n}{\delta^2}\mathbb{E}(\xi_1^2)\sum_{j=1}^{2n-1}\left(\frac{q_j}{q_{2n}}\right)^2 = 1-\frac{1}{3n}\sum_{j=1}^{2n-1}\left(\frac{q_j}{q_{2n}}\right)^2 \geq 1-\frac{2}{3},\end{aligned} \tag{3.A.21}$$

where we used our choice of the discrete Soliton $0 \leq q_j \leq q_{2n}$. □

3.A.1 More Precise LD estimates

We conclude this section mentioning some more precise limits on the large deviations for the uniform measure on the sphere, with a matching lower bound for Large deviation estimate (3.A.1) for $\mathfrak{g} < n^{2^-}$. *These results are not used for proving theorem 3.4.1 and theorem 3.2.2.* However, they have their own interest as it presents a matching lower bound for Large deviation estimate (3.A.1) in certain range. Moreover, these estimates play an important role in our future work [126].
For $0 \leq a < 2$ we have:

$$\limsup_{n\to\infty}\frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n < cn^a)\Big) \leq \ln\Big(\frac{2c}{m}\Big). \tag{3.A.22}$$

$$\liminf_{n\to\infty}\frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n \leq cn^a)\Big) \geq \ln\Big(\frac{2c}{m}\Big). \tag{3.A.23}$$

Before proving above estimates, first we treat the easier case, without correction terms, in Proposition 3.A.1. Then we prove (3.A.22), (3.A.23) in Corollary 3.A.2.1, thanks to the estimates we developed in Proposition 3.A.1.

Proposition 3.A.1. *For $0 \leq a < 2$, and for any $c > 0$ independent of n, we have:*

$$\lim_{n\to\infty} \frac{1}{n \ln n} \ln \mu_m^n(G_n(\psi_n) < cn^a) = -(2-a). \tag{3.A.24}$$

In particular, for $a = 0$

$$\lim_{n\to\infty} \frac{1}{n \ln n} \ln \mu_m^n(G_n(\psi_n) < c) = -2. \tag{3.A.25}$$

Proof. Step1. Upper bound.
First we prove the upper bound:

$$\limsup_{n\to\infty} \frac{1}{n \ln n} \ln \mu_m^n(G_n(\psi_n) < cn^a) \leq -(2-a). \tag{3.A.26}$$

Recall the estimate (3.A.1) from Lemma 3.A.1. Let us choose $\mathfrak{g} = cn^a$, and denotes $c_o := \frac{2c}{m}$, then we have:

$$\mu_n^m(G_n(\psi_n) < cn^a) \leq \frac{1}{\epsilon} \times \frac{1}{(1-\epsilon)^{n-1}} \times \frac{1}{n^a} \times \frac{c_o^{n-1}}{n^{(2-a)n}} \tag{3.A.27}$$

Since $0 < \epsilon < 1$ and c_o are independent of n, for any $\delta > 0$, there exists N_* such that for $n > N_*$ we have

$$\frac{1}{n \ln n} \mu_m^n(G_n(\psi_n) \leq cn^a) \leq -(2-a) + \delta, \tag{3.A.28}$$

This means $\forall \delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n \ln n} \mu_m^n(G_n(\psi_n) \leq cn^a) \leq -(2-a) + \delta.$$

Since the later is true for any $\delta > 0$, taking the limit $\delta \to 0^+$, finishes the proof of (3.A.26).

Step2. Lower Bound
Now we can prove the following lower bound by a proper change of measure:

$$\liminf_{n\to\infty} \frac{1}{n \ln n} \ln \mu_m^n(G_n(\psi_n) \leq cn^a) \geq -(2-a). \tag{3.A.29}$$

Notice that we can adapt the proof of Lemma 3.A.2 as we discussed in Remark 3.A.2 in order to obtain (3.A.29). However, here we bring a slightly longer proof, adapting the idea of [67] to our setup. Although this proof is longer, it illuminates the underlying nature of the problem, since we can obtain the second order correction to (3.A.29) similar to (3.A.22) with this method.
Recall the sequence of i.i.d standard complex normal random variables $\{Z_j\}_{j=1}^{\infty}$. Fix n and recall $\{\hat{Z}_k\}_{k=1}^n$ as the discrete Fourier transform of $(Z_1 \ldots, Z_n)$ (3.A.3), as well as the corresponding identities (3.A.4), (3.A.5). Moreover, let $\lambda = (1-\epsilon)\frac{n^{2-a}}{c_o}$, with $c_o = \frac{2m}{c}$ and $0 < \epsilon < \frac{1}{2}$ (we eventually take the limit $\epsilon \to 0$ after taking the limit n

to infinity). Let us denote $\alpha_k := 1 + \lambda(\omega_k^2 - c_o n^{a-2}) = \epsilon + \frac{1-\epsilon}{c_o} n^{2-a}\omega_k^2$, we proceed similar to (3.A.7), and perform the following change of measure:

$$\mu_m^n(G_n(\psi_n \leq cn^a)) = \mathbb{P}\bigg(\sum_{k=1}^n \Big(|\hat{Z}_k|^2(\omega_k^2 - c_o n^{a-2})\Big) \leq 0\bigg) \tag{3.A.30}$$

Denote the set $A_{c_o,a} := \{\sum_{k=1}^n |\hat{Z}_k|^2(\omega_k^2 - c_o n^{a-2}) \leq 0\}$ and observe:

$$\begin{aligned}\mu_m^n(G_n \leq cn^a)) = \int_{\mathbb{C}^n} \mathbb{1}_{A_{c_o,a}} \prod_{k=1}^n \frac{e^{-|\hat{z}_k|^2}}{\pi} d\hat{z}_k d\hat{z}_k^* = \\ \underbrace{\prod_{k=1}^n \frac{1}{\alpha_k}}_{\tilde{p}_1} \underbrace{\int_{\mathbb{C}^n} \mathbb{1}_{A_{c_o,a}} \prod_{k=1}^n e^{(\alpha_k-1)|\hat{z}_k|^2} \alpha_k \frac{e^{-\alpha_k|\hat{z}_k|^2}}{\pi} d\hat{z}_k \hat{z}_k^*}_{\tilde{p}_2},\end{aligned} \tag{3.A.31}$$

where we denoted $\tilde{p}_1$ and $\tilde{p}_2$ implicitly. Let us first establish a lower bound for $\tilde{p}_2$. Take $c = 4c_o$ and $\delta > 0$, and denote $\tilde{A}_c := \{-c\delta n^{a-1} \leq \sum_{k=1}^n |\hat{Z}_k|^2(\omega_k^2 - c_o n^{a-2}) \leq 0\} = \{-c\delta \leq \frac{1}{n}\sum_{k=1}^n |\hat{Z}_k|^2(n^{2-a}\omega_k^2 - c_o) \leq 0\} \subset \{\sum_{k=1}^n |\hat{Z}_k|^2(\omega_k^2 - c_o n^{a-2}) \leq 0\}$. Since $\tilde{A}_c \subset A_{c_o,a}$ we have:

$$\begin{aligned}\tilde{p}_2 \geq \int_{\mathbb{C}^n} \mathbb{1}_{\tilde{A}_c} e^{\sum_{k=1}^n (\alpha_k-1)|\hat{z}_k|^2} \prod_{k=1}^n \alpha_k \frac{e^{-\alpha_k|\hat{z}_k|^2}}{\pi} d\hat{z}_k \hat{z}_k^* = \int_{\mathbb{C}^n} \mathbb{1}_{\tilde{A}_c} e^{\sum_{k=1}^n \lambda(\omega_k^2 - c_o n^{a-2})|\hat{z}_k|^2} d\gamma_{\underline{\alpha}}^n \\ \geq e^{-c\delta n^{a-1}\lambda} \int_{\mathbb{C}^n} \mathbb{1}_{\tilde{A}_c} d\gamma_{\underline{\alpha}}^n = e^{-n\delta(c\frac{1-\epsilon}{c_o})} \mathbb{P}_{\underline{\alpha}}^n\Big(-c\delta \leq \frac{1}{n}\sum_{k=1}^n |\hat{\mathcal{Z}}_k|^2(n^{2-a}\omega_k^2 - c_o) \leq 0\Big),\end{aligned} \tag{3.A.32}$$

where we denoted $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and $d\gamma_{\underline{\alpha}}^n := \prod_{k=1}^n \alpha_k \frac{e^{-\alpha_k|\hat{z}_k|^2}}{\pi} d\hat{z}_k \hat{z}_k^*$. Notice that $\gamma_{\underline{\alpha}}^n$ is a Complex Gaussian distribution with mean zero and covariance matrix $diag(\alpha_1, \ldots, \alpha_n)$, and correspondingly $\{\mathcal{Z}_k\}_{k=1}^n$ are independent complex Gaussian random variable with mean zero and $\mathbb{E}(|\hat{\mathcal{Z}}_k|^2) = \frac{1}{\alpha_k}$.
In the following we observe that the last term in (3.A.32) can be bounded from below by a term of order $e^{-\delta n}$. Let us split $\mathcal{Z}_n := \frac{1}{n}\sum_{k=1}^n |\hat{\mathcal{Z}}_k|^2(n^{2-a}\omega_k^2 - c_o)$ into two parts:

$$\mathcal{Z}_n^+ := \frac{1}{n}\sum_{k=1}^{n-1} |\hat{\mathcal{Z}}_k|^2(n^{2-a}\omega_k^2 - c_o), \quad \mathcal{Z}_n^o := -\frac{1}{n}|\hat{\mathcal{Z}}_n|^2 c_o,$$

notice the fact that $\omega_n = 0$, and $\mathcal{Z}_n = \mathcal{Z}_n^+ + \mathcal{Z}_n^o$.
We denote $m_n := \mathbb{E}(\mathcal{Z}_n^+)$, $s_n := \mathbb{E}((\mathcal{Z}_n^+ - m_n)^2)$ and observe that $m_n \to \frac{c_o}{1-\epsilon}$ and $s_n \to 0$, as $n \to \infty$: take a' such that $\frac{a}{2} < a' < 1$, define $I_< := ([1, n^{a'}) \cup (n - n^{a'}, n-1]) \cap \mathbb{Z}$, and $I_> := ([1, n-1] \cap \mathbb{Z}) \setminus I_<$ and decompose $\mathcal{Z}_n^+ := \mathcal{Z}_{n,<}^+ + \mathcal{Z}_{n,>}^+$ as follows:

$$\mathcal{Z}_{n,<}^+ := \sum_{k \in I_<} |\hat{\mathcal{Z}}_k|^2(n^{2-a}\omega_k^2 - c_o), \quad \mathcal{Z}_{n,>}^+ := \sum_{k \in I_>} |\hat{\mathcal{Z}}_k|^2(n^{2-a}\omega_k^2 - c_o). \tag{3.A.33}$$

Then by using the fact that $\mathbb{E}(|\hat{\mathcal{Z}}_k|^2) = \frac{1}{\alpha_k}$, with $\alpha_k = (\epsilon + n^{2-a}(\frac{1-\epsilon}{c_o})\omega_k^2)$, we get:

$$|\mathbb{E}(\mathcal{Z}_{n,<}^+)| \leq \frac{2}{n}\sum_{k<n^{a'}} \left|\frac{n^{2-a}\omega_k^2 - c_o}{n^{2-a}\omega_k^2\frac{1-\epsilon}{c_o} + \epsilon}\right| \leq \frac{2}{n}\sum_{k<n^{a'}} \left(\frac{c_o}{1-\epsilon} + \frac{c_o}{\epsilon}\right) \leq \frac{2c_1 n^{a'}}{n} \to 0, \tag{3.A.34}$$

as $n \to 0$, where c_1 is a constant independent of n. Note that for $k \in I_<, \omega_k > 0$. On the other hand, there exists N_1 such that for $n > N_1$, $n^{2a'-2} < \omega^2_{n^{a'}} \leq \omega_k^2$ for all $k \in I_>$. Therefore, we have:

$$0 \leq \frac{1}{n}\sum_{k\in I_>}\frac{1}{\alpha_k} = \frac{1}{n}\sum_{k\in I_>}\frac{1}{n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o})+\epsilon} \leq \frac{c_o}{(1-\epsilon)n}\sum_{k\in I_>}\frac{1}{n^{2a'-a}} \leq \frac{c_o}{(1-\epsilon)n^{2a'-a}} \to 0, \tag{3.A.35}$$

as $n \to \infty$, thanks to the choice of $\frac{a}{2} < a'$. The last expression leads us to the following:

$$(\frac{1-\epsilon}{c_o})\mathbb{E}(\mathcal{Z}^+_{n,>}) = \frac{1}{n}\sum_{k\in I_>}\frac{(\frac{1-\epsilon}{c_o})n^{2-a}\omega_k^2+\epsilon-1}{(\frac{1-\epsilon}{c_o})n^{2-a}\omega_k^2+\epsilon} = \frac{|I_>|}{n} - \frac{1}{n}\sum_{k\in I_>}\frac{1}{\alpha_k} \to 1, \tag{3.A.36}$$

as $n \to \infty$, where we take advantage of (3.A.35) as well as the fact that $n-2n^{a'}-3 \leq |I_>| \leq n-2n^{a'}+1$ and hence, $\lim_{n\to\infty}\frac{|I_>|}{n} \to 1$ by the choice of $a' < 1$. Therefore, by using (3.A.36) and (3.A.34) we have:

$$m_n = \mathbb{E}(\mathcal{Z}^+_n) = \mathbb{E}(cZ^+_{n,<}) + \mathbb{E}(\mathcal{Z}^+_{n,>}) \to \frac{c_o}{1-\epsilon}, \tag{3.A.37}$$

as $n \to \infty$. Recall $s_n = \mathbb{E}((\mathcal{Z}^+_n - m_n)^2) = \mathrm{Var}(\mathcal{Z}^+_n)$, observe that $\mathbb{E}(|\hat{\mathcal{Z}}_k|^4) = \frac{2}{\alpha_k^2}$ and $\mathrm{Var}(|\hat{\mathcal{Z}}_k|^4) = \frac{1}{\alpha_k^2}$. Therefore, thanks to and independence of $\{\hat{\mathcal{Z}}_k\}_{k=1}^n$ we have:

$$0 \leq s_n = \frac{1}{n^2}\sum_{k=1}^{n-1}\left(\frac{n^{2-a}\omega_k^2 - c_o}{n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o})+\epsilon}\right)^2 \leq \frac{2}{n^2}\sum_{k=1}^{n-1}\left(\left(\frac{c_o}{1-\epsilon}\right)^2 + \left(\frac{c_o}{\epsilon}\right)^2\right) \leq \frac{2c_1^2}{n} \to 0, \tag{3.A.38}$$

as $n \to \infty$. Thanks to (3.A.38), by applying the Chebyshev's inequality there exist N_3 such that for $n > N_3$ we have:

$$\mathbb{P}(|\mathcal{Z}^+_n - m_n| < \frac{c\delta}{8}) = 1 - \mathbb{P}(|\mathcal{Z}^+_n - m_n| \geq \frac{c\delta}{8}) \leq 1 - \frac{s_n}{(\frac{c\delta}{8})^2} \geq \frac{1}{2}, \tag{3.A.39}$$

Moreover, thanks to (3.A.37) we can choose N_4 such that for $n > N_4$, $|m_n - \frac{c_o}{1-\epsilon}| \leq \frac{c\delta}{8}$. Therefore, for any $n > \max\{N_3, N_4\}$, we have:

$$\mathbb{P}(|\mathcal{Z}^+_n - \frac{c_o}{1-\epsilon}| \leq \frac{c\delta}{4}) \geq \frac{1}{2}. \tag{3.A.40}$$

Now recall the definition of $\mathcal{Z}^o_n = -\frac{c_o}{n}|\hat{\mathcal{Z}}_n|^2$, where from expression (3.A.32) it is clear that $\hat{\mathcal{Z}}_n$ is a complex Gaussian random variable with probability density function $f(z) = \epsilon\frac{e^{-\epsilon|z|^2}}{\pi}$. Therefore, we can choose N_5 such that for $n > N_5$ we have:

$$\mathbb{P}\Big(-\Big(\frac{c_o}{1-\epsilon}+\frac{3c\delta}{4}\Big) \leq \mathcal{Z}^o_n \leq -\Big(\frac{c_o}{1-\epsilon}+\frac{c\delta}{4}\Big)\Big) \geq \frac{1}{2}\exp\Big(\frac{-n2\epsilon}{1-\epsilon}\Big), \tag{3.A.41}$$

where we performed a Gaussian integration, then we chose N_5 properly. Notice that since $\mathcal{Z}_n = \mathcal{Z}_n^+ + \mathcal{Z}_n^o$ we have

$$\Big\{ -(\frac{c_o}{1-\epsilon} + \frac{3c\delta}{4}) \leq \mathcal{Z}_n^o \leq -(\frac{c_o}{1-\epsilon} + \frac{c\delta}{4}) \Big\} \bigcap \Big\{ \frac{c_o}{1-\epsilon} - \frac{c\delta}{4} \leq \mathcal{Z}_n^+ \leq \frac{c_o}{1-\epsilon} + \frac{c\delta}{4} \Big\} \\ \subset \{ -c\delta \leq \mathcal{Z}_n \leq 0 \}. \tag{3.A.42}$$

Therefore, since $\mathcal{Z}_n^o$ and $\mathcal{Z}_n^+$ are independent, we can factorize the corresponding probabilities. By plugging (3.A.40) and (3.A.41) into (3.A.32), for $n > \max\{N_3, N_4, N_5\}$ we get:

$$\tilde{p}_2 \geq \exp(-n\delta 4(1-\epsilon))\mathbb{P}(-c\delta \leq \mathcal{Z}_n \leq 0) \geq \exp(-n\delta 4(1-\epsilon)) \times \frac{1}{2} \times \frac{1}{2} \exp\Big(-n\epsilon \frac{2}{1-\epsilon}\Big). \\ \tilde{p}_2 \geq \frac{1}{4} \exp\Big(-n\epsilon(1 + \frac{2}{1-\epsilon})\Big), \tag{3.A.43}$$

where we chose $\delta = \frac{\epsilon}{4(1-\epsilon)}$ in the last expression, notice that this choice is legitimate [6] since $0 < \delta < 1$ was arbitrary and $0 < \epsilon < \frac{1}{2}$.
Now we treat the first term in (3.A.31) $\tilde{p}_1$. Recall a' where $\frac{a}{2} < a' < 1$, and the definition of $I_<$ and $I_>$, where we used in (3.A.33). Observe that for $k \in I_<$, $\omega_k^2 \leq 4\pi^2 n^{2a'-a}$. Since $2a' - a > 0$, there exist N_1' such that for $n > N_1'$, we have $\epsilon + n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o}) \leq 2\pi^2 n^{2a'-a}(\frac{1-\epsilon}{c_o})$, and

$$\prod_{k \in I_<} \frac{1}{\alpha_k} = \prod_{k \in I_<} \frac{1}{\epsilon + n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o})} \geq \left(4\pi^2 n^{2a'-a}(\frac{1-\epsilon}{c_o})\right)^{-2n^{a'}} = C_1^{-2n^{a'}} \exp\Big(-c_1 n^{a'} \ln n\Big), \tag{3.A.44}$$

where $C_1 > 0$ and $c_1 > 0$ are independent of n.
On the other hand, for any $k \in I_>$, $n^{2-a}\omega_k^2 \geq n^{2a'-a}$ for n sufficiently large. Hence, $\forall\ \rho > 1$ there exist $N_2(\rho, \epsilon)$, such that for $n > N_2(\rho, \epsilon)$ we have $\rho n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o}) \geq \epsilon + n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o})$, and we can obtain the following bound:

$$\prod_{k \in I_>} \frac{1}{\alpha_k} = \prod_{k \in I_>} \frac{1}{\epsilon + n^{2-a}\omega_k^2(\frac{1-\epsilon}{c_o})} \geq \left(n^{2-a}\right)^{-n+2n^{a'}-1} \left(\frac{\rho(1-\epsilon)}{c_o}\right)^{-n+2n^{a'}-1} \prod_{k \in I_>} \frac{1}{\omega_k^2}. \tag{3.A.45}$$

Observe that for $k \in I_<$ $\omega_k^2 \geq \frac{1}{n^2}$, with n sufficiently large, thanks to (3.A.8) we have:

$$\prod_{k \in I_>} \frac{1}{\omega_k^2} = \prod_{k=1}^{n-1} \frac{1}{\omega_k^2} \prod_{k \in I_<} \omega_k^2 \geq (\frac{1}{n^2})^{2n^{a'}}. \tag{3.A.46}$$

Therefore, thanks to (3.A.45), and (3.A.46) we get:

$$\tilde{p}_1 = \prod_{k=1}^{n} \frac{1}{\alpha_k} = \frac{1}{\epsilon} \prod_{k \in I_<} \frac{1}{\alpha_k} \prod_{k \in I_>} \frac{1}{\alpha_k} \geq n^{-n(2-a)} \left(\frac{\rho(1-\epsilon)}{c_o}\right)^{-n} \mathcal{E}_n, \tag{3.A.47}$$

[6] We only did this choice to make computation easier.

where $\mathcal{E}_n$ is sub-exponential i.e. $\lim_{n\to\infty} \frac{1}{n}\ln(\mathcal{E}_n) = 0$.

Combining (3.A.31), (3.A.43), and (3.A.47), for any $\delta_o > 0$, if we take n sufficiently large, we have:

$$\frac{1}{n\ln n}\ln \mu_m^n(G_n(\psi_n) \le cn^a) \ge -(2-a) - \delta_o. \tag{3.A.48}$$

Taking $\liminf_{n\to\infty}$, and taking $\delta_o \to 0^+$ finishes the proof of (3.A.29). □

Corollary 3.A.2.1. *The expressions* (3.A.26)*, and* (3.A.29) *are basically saying that the probability of the event* $\{G_n \le cn^a\}$ *for* $0 \le a < 2$*, has a super exponential upper and lower bound respectively.*
However, by using the expression (3.A.27)*, we can deduce that in the next order, the upper bound has a exponential correction, which can be expressed explicitly. Formally, we have:*

$$\limsup_{n\to\infty} \frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n < cn^a)\Big) \le \ln\Big(\frac{2c}{m}\Big). \tag{3.A.49}$$

Moreover, the expressions (3.A.47)*,* (3.A.32) *and* (3.A.43) *give us the same correction for the lower bound* (3.A.29)*:*

$$\liminf_{n\to\infty} \frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n \le cn^a)\Big) \ge \ln\Big(\frac{2c}{m}\Big). \tag{3.A.50}$$

Proof. Notice that it is straightforward to deduce (3.A.22) from (3.A.27). Let

$$a_n := \frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n < cn^a)\Big),$$

first we can deduce that for any $\delta > 0$ there exists N_* such that $a_n \le \delta + |\ln(1-\epsilon)| + \ln(c_o)$, which means $\limsup_n a_n \le \delta + |\ln(1-\epsilon)| + \ln(c_o)$. Since the later holds for any $\delta > 0$, taking the limit $\delta \to 0^+$, we have $\limsup_n a_n \le |\ln(1-\epsilon)| + \ln(c_o)$. On the other hand, this inequality is true for any $0 < \epsilon < 1$, taking the limit $\epsilon \to 1^-$ yields the result.
In order to prove (3.A.23), let us denote

$$a'_n := \frac{1}{n}\ln\Big(e^{(2-a)n\ln n}\mu_m^n(G_n \le cn^a)\Big),$$

and recall the expression (3.A.31), as well as the estimates (3.A.43) and (3.A.47), then observe that for n sufficiently large we get:

$$a'_n \ge \ln(c_o) - \ln(\rho(1-\epsilon)) - \epsilon(1 + \frac{2c_o}{1-\epsilon}) + \frac{1}{n}\ln\Big(\frac{\mathcal{E}_n}{4}\Big), \tag{3.A.51}$$

Therefore, we have

$$\liminf_{n\to\infty} a'_n \ge \ln\Big(\frac{2c}{m}\Big) - \ln(\rho(1-\epsilon)) - \epsilon(1 + \frac{2c_o}{1-\epsilon}),$$

The last expression is true for any $\epsilon > 0$ and $\rho > 1$, taking the limit $\epsilon \to 0^+$ and $\rho \to 1^+$ finishes the proof. □

Remark 3.A.2. *Notice that we can adapt the proof of Lemma 3.A.2 in order to obtain the lower bound* (3.A.29)*(in the proof we did not use this lower bound). However, this proof does not provide the "correction" term in* (3.A.23)*.*

3.B Hypoellipticity

In this section, we prove that the generator (3.1.7), is hypoelliptic, and therefore the invariant measure has a smooth density. Notice that we add the subscript n, to emphasize the dependence on n.

Lemma 3.B.1. *Recall the operator $L_n = A_n + S_n$, where $A_n = \mathcal{A}$ and $S_n = \mathcal{S}$ are given by* (3.1.4), (3.1.6), *respectively. Then L_n is hypoelliptic. Consequently, the invariant measure has smooth density with respect to $d\mu_n^m$.*

Proof. Let us the fix the parameters $h = s = \gamma = 1$, the proof for other cases is similar. In order to prove this lemma, it is sufficient to show that L_n satisfies the so-called Hörmander condition. Then the hypoellipticity, and smoothness of the invariant measure follow by the Hörmander's Theorem (hypoellipticity follows from Thorem 22.2.1 of [129], for a general review one can also see [25], and [122]). We prove this condition in the case $d = 1$ in details, the generalization to higher dimensions is a matter of messier algebra (We comment on this at the end of the proof). Let us denote $Y_0 = A_n$ and $Y_x = \partial_{\theta(x)}$ for $x \in \widetilde{\mathbb{T}}_n$. L_n satisfies the Hörmander's condition if the Lie algebra generated by the family

$$\{Y_x\}_{x=1}^n, \quad \{[Y_x, Y_y]\}_{x,y=0}^n, \quad \{[[Y_x, Y_y], Y_z]\}_{x,y,z=0}^n, \dots,$$

has full rank (here $2n-1$) at every point $\psi \in S_n^m$.
Let us define the following notation: for $x, y \in \widetilde{\mathbb{T}}_n$ and symbols i, r, we define $\mathcal{R}_{x^i,y^r}$, $\mathcal{R}_{x^i,y^i}$, $\mathcal{R}_{x^r,y^r}$, and $\mathcal{R}_{x^r,y^i}$ as the following rotations:

$$\begin{aligned} \mathcal{R}_{x^r,y^r} &= \psi_r(x)\partial_{\psi_r(y)} - \psi_r(y)\partial_{\psi_r(x)}, \quad \mathcal{R}_{x^r,y^i} = \psi_r(x)\partial_{\psi_i(y)} - \psi_i(y)\partial_{\psi_r(x)}, \\ \mathcal{R}_{x^i,y^r} &= \psi_i(x)\partial_{\psi_r(y)} - \psi_r(y)\partial_{\psi_i(x)}, \quad \mathcal{R}_{x^i,y^i} = \psi_i(x)\partial_{\psi_i(y)} - \psi_i(y)\partial_{\psi_i(x)}. \end{aligned} \tag{3.B.1}$$

We can rewrite $\partial_{\theta(x)}$, and the Hamiltonian operator A_n in terms of these rotations:

$$\begin{aligned} &\partial_{\theta(x)} = \mathcal{R}_{x^r,x^i}, \\ &A_n = \sum_{x \in \widetilde{\mathbb{T}}_n} \mathcal{R}_{(x+1)^r,x^i} + \mathcal{R}_{x^r,(x+1)^i} - 2\mathcal{R}_{x^r,x^i} + \kappa|\psi(x)|^{p-1}\mathcal{R}_{x^r,x^i}. \end{aligned} \tag{3.B.2}$$

Observe that for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{x^\mu | x \in \widetilde{\mathbb{T}}_n, \mu \in \{r, i\}\}$, (these indices are of the form x^i,x^r), we have (recall $[a, b] = ab - ba$):

$$\begin{aligned} [\mathcal{R}_{\alpha_1,\alpha_2}, \mathcal{R}_{\alpha_3,\alpha_4}] =& \delta_{\alpha_1,\alpha_4}\mathcal{R}_{\alpha_2,\alpha_3} + \delta_{\alpha_2,\alpha_3}\mathcal{R}_{\alpha_1,\alpha_4} - \delta_{\alpha_1,\alpha_3}\mathcal{R}_{\alpha_2,\alpha_4} - \delta_{\alpha_2,\alpha_4}\mathcal{R}_{\alpha_1,\alpha_3} = \\ & \sum_{i,j=1}^{4} \delta_{\{i+j-5\}}\mathcal{R}_{\alpha_i\alpha_j}\delta_{\alpha_k,\alpha_l}(-1)^{i+j+1}, \end{aligned} \tag{3.B.3}$$

where $\{k, l\} := \{1, 2, 3, 4\} \setminus \{i, j\}$.
We rewrite the following commutators in terms of these rotations for every $x \in \widetilde{\mathbb{T}}_n$:

$$\begin{aligned}
&\partial_{\theta(x)} = \mathcal{R}_{x^r,x^i},\\
&\mathcal{A}_x := [A_n, \partial_{\theta(x)}] = \mathcal{R}_{x^r,(x+1)^r} + \mathcal{R}_{x^i,(x+1)^i} - \mathcal{R}_{(x-1)^r,x^r} - \mathcal{R}_{(x-1)^i,x^i},\\
&\mathcal{A}_{x,x+1} := [[A_n, \partial_{\theta(x)}], \partial_{\theta(x+1)}] = \mathcal{R}_{x^r,(x+1)^i} - \mathcal{R}_{x^i,(x+1)^r},\\
&\mathcal{A}_{x,x+1,x} := [[[A_n, \partial_{\theta(x)}], \partial_{\theta(x+1)}], \partial_{\theta(x)}] = \mathcal{R}_{x^r,(x+1)^r} + \mathcal{R}_{x^i,(x+1)^i}.
\end{aligned} \tag{3.B.4}$$

We can compute the following commutators: $\mathcal{A}^{(2)}_{x,x+1} := [\mathcal{A}_{x,x+1}, \mathcal{A}_{x+1,x+2,x+1}]$ and $[\mathcal{A}^{(2)}_{x,x+1}, \partial_{\theta(x+2)}]$ thanks to (3.B.3), and observe that $\mathcal{R}_{x^r,(x+2)^i} - \mathcal{R}_{x^i,(x+2)^r}$ and $\mathcal{R}_{x^r,(x+2)^r} + \mathcal{R}_{x^i,(x+2)^i}$ belong to our Lie algebra. Repeating this process, following an induction, we observe that for $x, y \in \widetilde{\mathbb{T}}_n$ the following terms are in the Lie algebra generated by $\{Y_x\}_{x=1}^n$, $\{[Y_x, Y_y]\}_{x,y=0}^n$, $\{[[Y_x, Y_y], Y_z]\}_{x,y,z=0}^n, \dots$:

$$\mathcal{G}^o_n := \{\mathcal{R}_{x^r,x^i}, \mathcal{R}_{x^r,y^i} - \mathcal{R}_{x^i,y^r}, \mathcal{R}_{x^r,y^r} + \mathcal{R}_{x^i,y^i} \Big| x, y \in \widetilde{\mathbb{T}}_n\}. \tag{3.B.5}$$

Notice that in the linear case (absence of non-linearity i.e., $p = 2$), the terms appeared in (3.B.5) represent a basis for the Lie algebra. (All the elements are linear combination of these terms).
In the following, we observe that $\mathcal{G}^o_n$ (3.B.5), has rank $2n-1$ for any $\psi \in S^n = \{\psi \in \mathbb{C}^n | \sum_{x=1}^n |\psi(x)|^2 = 1\}$, notice that we consider S^n as a $2n-1$ real sphere $\mathbb{S}^{2n-1}$ (the case where we replace S^n by S^n_m can be treated similarly). Let us prove by an induction. The case $n = 1$ is trivial, since $\partial_{\theta_1} = \psi_r(1)\partial_{\psi_i(1)} - \psi_i(1)\partial_{\psi_r(1)}$ has rank one for any $\psi(1) \in \mathbb{S}^1$ ($|\psi_r(1)|^2 + |\psi_i(1)|^2 = 1$).
Assume $\mathcal{G}^o_n$ has rank $2n-1$ at every point of $\mathbb{S}^{2n-1}$, we prove that $\mathcal{G}^o_{n+1}$ has rank $2n+1$ at every point of $\mathbb{S}^{2n+1}$. We split the proof into two cases:
Case 1.
Take $\psi \in S^{n+1}$, and assume that there exists at least one point x, such that $|\psi(x)| = 0$, we can take $x = n+1$, since we are in the periodic setup. We have $\psi_r(n+1) = \psi_i(n+1) = 0$; therefore, $\hat{\psi} = (\psi_1, \dots, \psi_n) \in S^n$, and $\mathcal{G}^o_n$ has rank $2n-1$ by induction hypothesis. On the other hand, since $\hat{\psi} \in S^n$, there exists $y \in \widetilde{\mathbb{T}}_n$, such that $|\psi(y)| \neq 0$. First, observe that

$$\begin{aligned}
B :=& \{\psi_r(y)\partial_{\psi_i(n+1)} - \psi_i(y)\partial_{\psi_r(n+1)}, \psi_r(y)\partial_{\psi_r(n+1)} + \psi_i(y)\partial_{\psi_i(n+1)}\} =\\
&\{\mathcal{R}_{y^r,(n+1)^i} - \mathcal{R}_{y^i,(n+1)^r}, \mathcal{R}_{y^r,(n+1)^r} - \mathcal{R}_{y^i,(n+1)^i}\} \subset \mathcal{G}^o_{n+1},
\end{aligned} \tag{3.B.6}$$

has rank two (this is straightforward, since $(\psi_r(y), \psi_i(y)) \neq 0$, and one can see a linear combination of elements of B is zero iff $|\psi(y)| = 0$). Then the result follows from the induction hypothesis, as well as the fact that B is orthogonal to $\mathcal{G}^o_n$.
Case2.
Take $\psi \in S^{n+1}$ and assume $|\psi(x)| \neq 0$ for all $x \in \widetilde{\mathbb{T}}_{n+1}$. In this case, we claim the set

$$\mathcal{G}^1_{n+1} := \{\mathcal{R}_{(n+1)^r,(n+1)^i}, \mathcal{R}_{(n+1)^r,x^i} - \mathcal{R}_{(n+1)^i,x^r}, \mathcal{R}_{(n+1)^r,x^r} + \mathcal{R}_{(n+1)^i,x^i} \Big| x \in \widetilde{\mathbb{T}}_n\} \subset \mathcal{G}^o_n, \tag{3.B.7}$$

has rank $2n+1$. In fact, this set has $2n+1$ elements, where we observe that they are linearly independent. Take real [7] coefficients $\{a_x, b_x, c\}_{x=1}^n$ such that

$$c\mathcal{R}_{(n+1)^r,(n+1)^i} + \sum_{x=1}^{n} a_x\Big(\mathcal{R}_{(n+1)^r,x^i} - \mathcal{R}_{(n+1)^i,x^r}\Big) + b_x\Big(\mathcal{R}_{(n+1)^r,x^r} + \mathcal{R}_{(n+1)^i,x^i}\Big) = 0.$$

Computing the coefficients of $\partial_{\psi_r(x)}$ and $\partial_{\psi_i(x)}$, for any $x \in \widetilde{\mathbb{T}}_n$ we get:

$$\begin{aligned}\Big(b_x\psi_r(n+1) - a_x\psi_i(n+1)\Big)\partial_{\psi_r(x)} = 0,\\ \Big(a_x\psi_r(n+1) + b_x\psi_i(n+1)\Big)\partial_{\psi_i(x)} = 0.\end{aligned} \tag{3.B.8}$$

Notice that if $(a_x, b_x) \neq 0$, then $\det\begin{pmatrix} b_x & -a_x \\ a_x & b_x\end{pmatrix} > 0$. However, in order to (3.B.8) holds, the later cannot happen, since $(\psi_r(n+1), \psi_i(n+1)) \neq 0$; therefore $a_x = b_x = 0$ for all $x \in \widetilde{\mathbb{T}}_n$, and we can deduce $c = 0$, which yields the result in the case $d = 1$. In order to prove the result for $d > 1$, for any $x, y \in \widetilde{\mathbb{T}}_n^d$, and any $\mu, \nu \in \{r, i\}$, we define $\mathcal{R}_{x^\mu,y^\nu}$, similar to (3.B.1). Recall $\{e_j\}_{j=1}^d$ as the canonical basis of $\mathbb{R}^d$, then (3.B.2) will be modified as:

$$\begin{aligned}&\partial_{\theta(x)} = \mathcal{R}_{x^r,x^i},\\ &A_n = \sum_{x\in\widetilde{\mathbb{T}}_n}\sum_{j=1}^{d}\Big(\mathcal{R}_{(x+e_j)^r,x^i} + \mathcal{R}_{x^r,(x+e_j)^i} - 2\mathcal{R}_{x^r,x^i}\Big) + \sum_{x\in\widetilde{\mathbb{T}}_n}\kappa|\psi(x)|^{p-1}\mathcal{R}_{x^r,x^i}.\end{aligned} \tag{3.B.9}$$

The identity (3.B.3) remains true by taking $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{x^\mu \big\lfloor x \in \widetilde{\mathbb{T}}_n^d, \mu \in \{r, i\}\}$. This leads to the following modification of (3.B.4), for any $x \in \widetilde{\mathbb{T}}_n^d$, and $1 \leq k \leq d$:

$$\begin{aligned}&\mathcal{A}_x := [A_n, \partial_{\theta(x)}] = \sum_{j=1}^{d}\mathcal{R}_{x^r,(x+e_j)^r} + \mathcal{R}_{x^i,(x+e_j)^i} - \mathcal{R}_{(x-e_j)^r,x^r} - \mathcal{R}_{(x-e_j)^i,x^i},\\ &\mathcal{A}_{x,x+e_k} := [[A_n, \partial_{\theta(x)}], \partial_{\theta(x+e_k)}] = \mathcal{R}_{x^r,(x+e_k)^i} - \mathcal{R}_{x^i,(x+e_k)^r},\\ &\mathcal{A}_{x,x+e_k,x} := [[[A_n, \partial_{\theta(x)}], \partial_{\theta(x+e_k)}], \partial_{\theta(x)}] = \mathcal{R}_{x^r,(x+e_k)^r} + \mathcal{R}_{x^i,(x+e_k)^i}.\end{aligned} \tag{3.B.10}$$

Following the exact same strategy as in the previous case, by an induction we observe that all the terms of the form $\mathcal{R}_{x^r,(x+l_ke_k)^i} - \mathcal{R}_{x^i,(x+l_ke_k)^r}$ and $\mathcal{R}_{x^r,(x+l_ke_k)^r} + \mathcal{R}_{x^i,(x+l_ke_k)^i}$, for any $x \in \widetilde{\mathbb{T}}_n^d$, any $1 \leq k \leq d$, and any $l_k \in \widetilde{\mathbb{T}}_n$, belong to our Lie algebra. Notice that thanks to (3.B.3), we have:

$$\begin{aligned}&[\mathcal{R}_{x^r,(x+l_ke_k)^i} - \mathcal{R}_{x^i,(x+l_ke_k)^r}, \mathcal{R}_{(x+l_ke_k)^r,(x+l_ke_k+l_{k'}e_{k'})^i} - \mathcal{R}_{(x+l_ke_k)^i,(x+l_ke_k+l_{k'}e_{k'})}] =\\ &-\Big(\mathcal{R}_{x^r,(x+l_ke_k+l_{k'}e_{k'})^r} + \mathcal{R}_{x^i,(x+l_ke_k+l_{k'}e_{k'})^i}\Big)\\ &[\mathcal{R}_{x^r,(x+l_ke_k+l_{k'}e_{k'})^r} + \mathcal{R}_{x^i,(x+l_ke_k+l_{k'}e_{k'})^i}, \mathcal{R}_{x^r,x^i}] = \mathcal{R}_{x^i,(x+l_ke_k+l_{k'}e_{k'})^r} - \mathcal{R}_{x^i,(x+l_ke_k+l_{k'}e_{k'})^r}.\end{aligned} \tag{3.B.11}$$

[7] Notice that we are considering the Field $\mathbb{R}$ here, by decomposing ψ into real and imaginary parts.

Repeating the above procedure for $d-1$ times, we can deduce the following set is included in our Lie algebra:

$$\mathcal{G}_n^{o,d} := \{\mathcal{R}_{x^r,x^i}, \mathcal{R}_{x^r,y^i} - \mathcal{R}_{x^i,y^r}, \mathcal{R}_{x^r,y^r} + \mathcal{R}_{x^i,y^i} | x, y \in \widetilde{\mathbb{T}}_n^d\}. \tag{3.B.12}$$

Recall that we observed that the rank of $\mathcal{G}_n^o$ is $2n-1$. However, due to symmetry one can observe that $\mathcal{G}_{n^d}^o$ and $\mathcal{G}_n^{o,d}$ has the same rank and this finishes the proof. □

Remark 3.B.1. *In the above proof, A_n is the Hamiltonian generator corresponding to the Hamiltonian* (3.1.2)*, where the non-linearity is given by $\mathcal{N}_n(\psi(x)) \sim |\psi(x)|^4$. The proof of Lemma 3.B.1 can be adapted to more general cases where the non-linearity is given by $\mathcal{N}(\psi(x)) \sim f(|\psi(x)|^2)$, where f is a sufficiently smooth function.*

3.C Discrete Gagliardo-Nirenberg

We present different versions of the Gagliardo-Nirenberg inequality. This inequality is crucial in the study of the sub-critical non-linear focusing Schrödinger equation, for proving the well-posedness and characterization of the Solitons (cf. [66],[194], [218],[120]). In particular, this inequality has been used in the the proof of Theorem 3.2.1 in [120]. We take advantage of the discrete version of this inequality, so we can establish properties of configurations with minimal or close to minimal energy.
Gagliardo-Nirenberg inequality states that for every $u \in H^1(\mathbb{R}^d)$, and $1 < p < 1 + \frac{4}{d}$, there exists a constant $C(p,d)$, such that (cf. [66],[194],[218]):

$$\|u\|_{L^{p+1}}^{p+1} = \int |u|^{p+1} \leq C(d,p)\Big(\int |\nabla u|^2\Big)^{\frac{d(p-1)}{4}} \Big(\int |u|^2\Big)^{\frac{p+1}{2} - \frac{d(p-1)}{4}}. \tag{3.C.1}$$

While we are focusing on the case where $d=1$, and $p=3<1+\frac{4}{d}=5$ and the domain is periodic, we state the following version from [[120] Section 3.2, [154] Lemma 4.1]. For all $u \in H^1_{per} = H^1(\mathbb{T})$, there exists a constant $C>0$:

$$\begin{aligned} \|u\|_{L^4}^4 &= \int_0^1 |u|^4 \leq C(\|\partial_x u\|_{L^2}\|u\|_{L^2}^3 + \|u\|_{L^2}^4) \\ &= C\left(\Big(\int_0^1 |u|^2\Big)^{\frac{3}{2}}\Big(\int_0^1 |\partial_x u|^2\Big)^{\frac{1}{2}} + \Big(\int_0^1 |u|^2\Big)^2\right). \end{aligned} \tag{3.C.2}$$

We need counterparts of these inequalities in the discrete setting, in order to obtain these inequalities, we generalize results from [[67] Section 17]. First, let us define a handful of notations. Fix $n>0$, and consider a function $f : \widetilde{\mathbb{T}}_n \to \mathbb{C}$, define the discrete $\ell^p(\widetilde{\mathbb{T}}_n)$ norm of f, for $p \geq 1$, as:

$$\|f\|_{\ell^p(\widetilde{\mathbb{T}}_n)} = \Big(\frac{1}{n}\sum_{x \in \widetilde{\mathbb{T}}_n} |f(x)|^p\Big)^{\frac{1}{p}}. \tag{3.C.3}$$

Notice that our definition differs from the conventional one by a factor $n^{-\frac{1}{p}}$. This difference is motivated by the fact that in the limit as $n \to \infty$, we can recover the

continuous L^p norm, formally. Define the $H^1_{per}(\widetilde{\mathbb{T}}_n)$ norm of f as follows:

$$\|f\|_{H^1(\widetilde{\mathbb{T}}_n)} := \Big(\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n} n^2|f(x)-f(x-1)|^2 + \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^2\Big)^{\frac{1}{2}}. \tag{3.C.4}$$

We can also define the space $\ell^p(\mathbb{Z})$, with the following norm: For $f:\mathbb{Z}\to\mathbb{C}$ and $p\geq 1$ define:

$$\|f\|_{\ell^p(\mathbb{Z})} = \Big(\sum_{x\in\mathbb{Z}}|f(x)|^p\Big)^{\frac{1}{p}}. \tag{3.C.5}$$

As usual we have: $\ell^p(\mathbb{Z}) = \{f:\mathbb{Z}\to\mathbb{C} | \|f\|_{\ell^p(\mathbb{Z})}<\infty\}$. We denote the discrete gradient energy of $f:\mathbb{Z}\to\mathbb{C}$ by $G(f)$, and define it as:

$$G(f) := \frac{1}{2}\sum_{x\in\mathbb{Z}}|f(x)-f(x-1)|^2. \tag{3.C.6}$$

Note the difference between $G(f)$ and $G_n(f)$ in (3.2.9), where we scale the second definition by n^2 in order to get the continuous counterpart, formally.

The first version of the discrete Gagliardo-Nirenberg inequality can be recalled from Proposition 17.6 of [67] with a small modification: For every $1<p\leq\infty$, let $\theta=\frac{1}{2}-\frac{1}{p+1}$, obviously $\theta\in(0,1)$, we have: there exists a constant $C(p)>0$, such that $\forall f\in\ell^p(\mathbb{Z})\cap\ell^2(\mathbb{Z})$:

$$\|f\|_{\ell^{p+1}(\mathbb{Z})} \leq C(p)\left(\|f\|_{\ell^2(\mathbb{Z})}\right)^{1-\theta}(G(f))^{\frac{\theta}{2}}. \tag{3.C.7}$$

In particular, for $p=3$, we have $\forall f\in\ell^4(\mathbb{Z})\cap\ell^2(\mathbb{Z})$, there exists a constant C such that:

$$\|f\|^4_{\ell^4(\mathbb{Z})} \leq C(\|f\|_{\ell^2(\mathbb{Z})})^3(G(f))^{\frac{1}{2}}. \tag{3.C.8}$$

We can deduce the following lemma from the later, which is crucial for our purposes.

Lemma 3.C.1. *Recall the definition of G_n (3.2.9), and $\|.\|_{\ell^p(\widetilde{\mathbb{T}}_n)}$ (3.C.3), for every $f:\widetilde{\mathbb{T}}_n\to\mathbb{C}$, there exist a constant C independent of n such that:*

$$\|f\|^4_{\ell^4(\widetilde{\mathbb{T}}_n)} \leq C\Big((\|f\|_{\ell^2(\widetilde{\mathbb{T}}_n)})^3(G_n(f))^{\frac{1}{2}} + \|f\|^4_{\ell^2(\widetilde{\mathbb{T}}_n)}\Big), \tag{3.C.9}$$

we write this inequality in this open form:

$$\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^4 \leq C\left(\Big(\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}\frac{n^2}{2}|f(x)-f(x-1)|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^2\Big)^{\frac{3}{2}} + \Big(\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^2\Big)^2\right). \tag{3.C.10}$$

Usually we have $\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^2 = m$, hence, we have:

$$\frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n}|f(x)|^4 \leq C\left(m^{\frac{3}{2}}G_n(f)^{\frac{1}{2}} + m^2\right). \tag{3.C.11}$$

Before proceeding, let us mention that above inequality in more general form can be found in the literature (cf. [142] (1.6), Theorem 3.2., Theorem 3.3.). However, we find the following demonstration illuminating.

Proof. We prove this lemma by constructing a function $\tilde{f} \in \ell^4(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$, from f as follows: Translate f such that $|f(x)|^2$ achieves its minimum at $x = n$. By this construction, we have $|f(n)|^2 \leq \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n} |f(x)|^2 = m$. Define $\tilde{f}$ on $\mathbb{Z}$ as:

$$\tilde{f}(x) = \begin{cases} f(x), \ \forall x \in \{1,\dots,n\}, \\ f(n)(2-\frac{x}{n}), \ \ \forall x \in \{n+1,\dots,2n\}, \\ f(n)(1+\frac{x}{n}), \ \ \forall x \in \{-n,\dots,-1\}, \\ f(n), \quad \text{if} \quad x = 0, \\ 0 \ \ \text{otherwise.} \end{cases} \tag{3.C.12}$$

By the definition of $\tilde{f}$, for every $p \geq 1$ we have:

$$\|\tilde{f}\|^p_{\ell^p(\mathbb{Z})} = \sum_{x\in\widetilde{\mathbb{T}}_n} |f(x)|^p + |f(n)|^p \sum_{x=0}^{n-1} 2(\frac{x}{n})^p + |f(n)|^p. \tag{3.C.13}$$

By estimating $\frac{1}{n}\sum_{x=1}^{n}(\frac{x}{n})^p$ with its integral value, we have $c_1(p) > 0$, $c_2(p) > 0$ independent of n, such that:

$$\|f\|^p_{\ell^p(\widetilde{\mathbb{T}}_n)} + c_1(p)|f(n)|^p \leq \frac{1}{n}\|\tilde{f}\|^p_{\ell^p(\mathbb{Z})} \leq \|f\|^p_{\ell^p(\widetilde{\mathbb{T}}_n)} + c_2(p)|f(n)|^p. \tag{3.C.14}$$

Moreover, we can compute $G(\tilde{f})$:

$$\begin{aligned} G(\tilde{f}) = &\sum_{x\in\mathbb{Z}} \frac{1}{2}|\tilde{f}(x) - \tilde{f}(x-1)|^2 = \frac{1}{2}\sum_{x=1}^{n} |f(x) - f(x-1)|^2 + \\ &|f(n)|^2\Big(\sum_{x=-n}^{0} \frac{1}{2n^2} + \sum_{x=n+1}^{2n} \frac{1}{2n^2}\Big) = \frac{1}{n}G_n(f) + \frac{1}{n}|f(n)|^2. \end{aligned} \tag{3.C.15}$$

Since we fix n, by (3.C.14) $\tilde{f} \in \ell^2(\mathbb{Z}) \cap \ell^4(\mathbb{Z})$; therefore, we can apply the inequality (3.C.8). By using the fact that $|f(n)|^2 \leq \frac{1}{n}\sum_{x\in\widetilde{\mathbb{T}}_n} |f(x)|^2 = \|f\|^2_{\ell^2(\widetilde{\mathbb{T}}_n)}$, and estimates (3.C.15) and (3.C.14), we get:

$$\begin{aligned} \|f\|^4_{\ell^4(\widetilde{\mathbb{T}}_n)} &\leq \frac{1}{n}\|\tilde{f}\|_{\ell^4(\mathbb{Z})} \leq C\Big(\frac{1}{n}\|\tilde{f}\|^2_{\ell^2(\mathbb{Z})}\Big)^{\frac{3}{2}} \Big(nG(\tilde{f})\Big)^{\frac{1}{2}} \\ &\leq C\Big(\|f\|^2_{\ell^2(\widetilde{\mathbb{T}}_n)} + c_2(2)|f(n)|^2\Big)^{\frac{3}{2}}\Big(G_n(f) + |f(n)|^2\Big)^{\frac{1}{2}} \\ &\leq C'\Big(\|f\|^2_{\ell^2(\widetilde{\mathbb{T}}_n)}\Big)^{\frac{3}{2}}\Big(G_n(f) + |f(n)|^2\Big)^{\frac{1}{2}} \\ &\leq C'\Big(\|f\|^2_{\ell^2(\widetilde{\mathbb{T}}_n)}\Big)^{\frac{3}{2}} G_n(f)^{\frac{1}{2}} + C'\Big(\|f\|^2_{\ell^2(\widetilde{\mathbb{T}}_n)}\Big)^{\frac{3}{2}} \|f\|^4_{\ell^2(\widetilde{\mathbb{T}}_n)}. \end{aligned} \tag{3.C.16}$$

This inequality proves the lemma, since $C' = C(1+c_2(2))^{\frac{3}{2}}$ is a constant independent of n. In the last line, we used the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$. □

Bibliography

[1] Abanin, D. A., Altman, E., Bloch, I., and Serbyn, M. Colloquium: Many-body localization, thermalization, and entanglement. *Rev. Mod. Phys. 91* (May 2019), 021001.

[2] Abdul-Rahman, H., Nachtergaele, B., Sims, R., and Stolz, G. Localization properties of the disordered xy spin chain. *Annalen der Physik 529*, 7 (2017), 1600280, https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201600280.

[3] Ablowitz, M. J., Prinari, B., and Trubatch, A. D. *Discrete and Continuous Nonlinear Schrödinger Systems*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[4] Aizenman, M., and Graf, G. M. Localization bounds for an electron gas. *Journal of Physics A: Mathematical and General 31*, 32 (aug 1998), 6783–6806.

[5] Aizenman, M., and Molchanov, S. Localization at large disorder and at extreme energies: An elementary derivations. *Communications in Mathematical Physics 157*, 2 (Oct 1993), 245–278.

[6] Ajanki, O., and Huveneers, F. Rigorous scaling law for the heat current in disordered harmonic chain. *Communications in Mathematical Physics 301*, 3 (Feb 2011), 841–883.

[7] Alicki, R., and Lendi, K. *Quantum dynamical semigroups and applications*, vol. 717. Springer, 2007.

[8] Amrein, W., and Georgescu, V. On the characterization of bound states and scattering states in quantum mechanics. *Helvetica Physica Acta 46*, 5 (1974), 635–658.

[9] Anderson, P. W. Absence of diffusion in certain random lattices. *Phys. Rev. 109* (Mar 1958), 1492–1505.

[10] Angulo Pava, J. Nonlinear stability of periodic traveling wave solutions to the schrödinger and the modified korteweg-de vries equations. *Journal of Differential Equations 235*, 1 (2007), 1–30.

[11] Aoki, K., Lukkarinen, J., and Spohn, H. Energy transport in weakly anharmonic chains. *Journal of Statistical Physics 124*, 5 (Sep 2006), 1105–1129.

[12] Arnold, L. Random dynamical systems. In *Dynamical systems*. Springer, 1995, pp. 1–43.

[13] Ashcroft, N. W., Mermin, N. D., et al. Solid state physics, 1976.

[14] Attal, S., and Joye, A. The langevin equation for a quantum heat bath. *Journal of Functional Analysis 247*, 2 (2007), 253–288.

[15] Attal, S., Joye, A., et al. *Open Quantum Systems I: The Hamiltonian Approach*. Springer, 2006.

[16] Axler, S., Bourdon, P., and Wade, R. *Harmonic Function Theory*. Graduate Texts in Mathematics. Springer New York, 2013.

[17] Bambusi, D., Faou, E., and Grébert, B. Existence and stability of ground states for fully discrete approximations of the nonlinear schrödinger equation. *Numerische Mathematik 123*, 3 (Mar 2013), 461–492.

[18] Bambusi, D., and Penati, T. Continuous approximation of breathers in one- and two-dimensional DNLS lattices. *Nonlinearity 23*, 1 (dec 2009), 143–157.

[19] Bang, O., Christiansen, P. L., If, F., Rasmussen, K. O., and Gaididei, Y. B. Temperature effects in a nonlinear model of monolayer scheibe aggregates. *Phys. Rev. E 49* (May 1994), 4627–4636.

[20] Barbu, V., Röckner, M., and Zhang, D. Stochastic nonlinear schrödinger equations. *Nonlinear Analysis: Theory, Methods and Applications 136* (2016), 168–194.

[21] baron Fourier, J. B. J. *Théorie analytique de la chaleur*. Chez Firmin Didot, père et fils, 1822.

[22] Basile, G., Bernardin, C., Jara, M., Komorowski, T., and Olla, S. Thermal conductivity in harmonic lattices with random collisions. In *Thermal transport in low dimensions*. Springer, 2016, pp. 215–237.

[23] Basile, G., Bernardin, C., and Olla, S. Momentum conserving model with anomalous thermal conductivity in low dimensional systems. *Phys. Rev. Lett. 96* (May 2006), 204303.

[24] Basko, D. M., Aleiner, I. L., and Altshuler, B. L. Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states. *Annals of physics 321*, 5 (2006), 1126–1205.

[25] Bellet, L. R. *Ergodic Properties of Markov Processes.* Springer Berlin Heidelberg, Berlin, Heidelberg, 2006, pp. 1–39.

[26] Benguria, R., and Kac, M. Quantum langevin equation. *Phys. Rev. Lett. 46* (Jan 1981), 1–4.

[27] Bényi, Á., Oh, T., and Pocovnicu, O. On the probabilistic cauchy theory of the cubic nonlinear schrödinger equation on $\mathbb{R}^d$, $d \geq 3$. *Transactions of the American Mathematical Society, Series B 2*, 1 (2015), 1–50.

[28] BERESTYCKI, H., LIONS, P. L., and PELETIER, L. A. An ode approach to the existence of positive solutions for semilinear problems in r n. *Indiana University Mathematics Journal 30*, 1 (1981), 141–157.

[29] Berlin, T. H., and Kac, M. The spherical model of a ferromagnet. *Phys. Rev. 86* (Jun 1952), 821–835.

[30] Bernard, D., and Jin, T. Open quantum symmetric simple exclusion process. *Phys. Rev. Lett. 123* (Aug 2019), 080601.

[31] Bernard, D., and Jin, T. Solution to the quantum symmetric simple exclusion process: The continuous case. *Communications in Mathematical Physics 384*, 2 (Jun 2021), 1141–1185.

[32] Bernardin, C. Hydrodynamics for a system of harmonic oscillators perturbed by a conservative noise. *Stochastic Processes and their Applications 117*, 4 (2007), 487–513.

[33] Bernardin, C., and Huveneers, F. Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential. *Probability Theory and Related Fields 157*, 1 (2013), 301–331.

[34] Bernardin, C., Huveneers, F., Lebowitz, J. L., Liverani, C., and Olla, S. Green-kubo formula for weakly coupled systems with noise. *Communications in Mathematical Physics 334*, 3 (2015), 1377–1412.

[35] Bernardin, C., Huveneers, F., and Olla, S. Hydrodynamic limit for a disordered harmonic chain. *Communications in Mathematical Physics 365*, 1 (Jan 2019), 215–237.

[36] Bernardin, C., and Olla, S. Fourier's law for a microscopic model of heat conduction. *Journal of Statistical Physics 121*, 3 (Nov 2005), 271–289.

[37] Bernardin, C., and Olla, S. Non-equilibrium macroscopic dynamics of chains of anharmonic oscillators. *preparation, available at www. ceremade. dauphine. fr/ olla/springs* (2010).

[38] Bertini, B., Collura, M., De Nardis, J., and Fagotti, M. Transport in out-of-equilibrium xxz chains: Exact profiles of charges and currents. *Phys. Rev. Lett. 117* (Nov 2016), 207201.

[39] Bludov, Y. V., Konotop, V. V., and Akhmediev, N. Matter rogue waves. *Phys. Rev. A 80* (Sep 2009), 033610.

[40] Boltzmann, L. *Vorlesungen über gastheorie*, vol. 2. Barth, 1912.

[41] Bonetto, F., Lebowitz, J. L., and Rey-Bellet, L. Fourier's law: a challenge to theorists. In *Mathematical physics 2000*. World Scientific, 2000, pp. 128–150.

[42] Bottman, N., Deconinck, B., and Nivala, M. Elliptic solutions of the defocusing NLS equation are stable. *Journal of Physics A: Mathematical and Theoretical 44*, 28 (jun 2011), 285201.

[43] Bourgain, J. Periodic nonlinear schrödinger equation and invariant measures. *Communications in Mathematical Physics 166*, 1 (Dec 1994), 1–26.

[44] Bourgain, J. Invariant measures for the2d-defocusing nonlinear schrödinger equation. *Communications in Mathematical Physics 176*, 2 (Mar 1996), 421–445.

[45] Bourgain, J. *Global Solutions of Nonlinear Schrödinger Equations*. No. v. 46 in American Mathematical Society colloquium publications, Providence, RI. American Mathematical Society Providence RI, 1999.

[46] Bourgain, J. Invariant measures for nls in infinite volume. *Communications in Mathematical Physics 210*, 3 (Apr 2000), 605–620.

[47] Bratteli, O., and Robinson, D. *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras. Symmetry Groups. Decomposition of States*. Operator Algebras and Quantum Statistical Mechanics. Springer, 1987.

[48] Bratteli, O., and Robinson, D. *Operator Algebras and Quantum Statistical Mechanics: Equilibrium States. Models in Quantum Statistical Mechanics*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2003.

[49] Braxmeier-Even, N., and Olla, S. Hydrodynamic limit for a hamiltonian system with boundary conditions and conservative noise. *Archive for Rational Mechanics and Analysis 213*, 2 (Aug 2014), 561–585.

[50] Breuer, H.-P., Petruccione, F., et al. *The theory of open quantum systems*. Oxford University Press on Demand, 2002.

[51] Bricmont, J., and Kupiainen, A. Towards a derivation of fourier's law for coupled anharmonic oscillators. *Communications in mathematical physics 274*, 3 (2007), 555–626.

[52] Brydges, D. C., and Slade, G. Statistical mechanics of the 2-dimensional focusing nonlinear schrödinger equation. *Communications in Mathematical Physics 182*, 2 (Dec 1996), 485–504.

[53] Burlakov, Y. *The phase space of the cubic Schrödinger equation: a numerical study*. PhD thesis, University of California, Berkeley, 1998.

[54] Burq, N., and Tzvetkov, N. Invariant measure for a three dimensional nonlinear wave equation. *International Mathematics Research Notices 2007*, 9 (2007), rnm108–rnm108.

[55] Burq, N., and Tzvetkov, N. Random data cauchy theory for supercritical wave equations i: local theory. *Inventiones mathematicae 173*, 3 (Sep 2008), 449–475.

[56] Burq, N., and Tzvetkov, N. Random data cauchy theory for supercritical wave equations ii: a global existence result. *Inventiones mathematicae 173*, 3 (Sep 2008), 477–496.

[57] Burq, N., and Tzvetkov, N. Probabilistic well-posedness for the cubic wave equation. *Journal of the European Mathematical Society 16*, 1 (2013), 1–30.

[58] Buslaev, V., and Sulem, C. On asymptotic stability of solitary waves for nonlinear schrödinger equations. *Annales de l'I.H.P. Analyse non linéaire 20*, 3 (2003), 419–475.

[59] Carlen, E. A., Fröhlich, J., and Lebowitz, J. Exponential relaxation to equilibrium for a one-dimensional focusing non-linear schrödinger equation with noise. *Communications in mathematical physics 342*, 1 (2016), 303–332.

[60] Carlen, E. A., Fröhlich, J., Lebowitz, J., and Wang, W.-M. Quantitative bounds on the rate of approach to equilibrium for some one-dimensional stochastic nonlinear schrödinger equations. *Nonlinearity 32*, 4 (mar 2019), 1352–1374.

[61] Carmona, R., Klein, A., and Martinelli, F. Anderson localization for bernoulli and other singular potentials. *Communications in Mathematical Physics 108*, 1 (Mar 1987), 41–66.

[62] Carmona, R., and Lacroix, J. *Spectral theory of random Schrödinger operators*. Springer Science & Business Media, 2012.

[63] Casher, A., and Lebowitz, J. L. Heat Flow in Regular and Disordered Harmonic Chains. *Journal of Mathematical Physics 12*, 8 (Aug. 1971), 1701–1711.

[64] Castro-Alvaredo, O. A., Doyon, B., and Yoshimura, T. Emergent hydrodynamics in integrable quantum systems out of equilibrium. *Phys. Rev. X 6* (Dec 2016), 041065.

[65] Cazenave, T., and Lions, P. L. Orbital stability of standing waves for some nonlinear schrödinger equations. *Communications in Mathematical Physics 85*, 4 (Dec 1982), 549–561.

[66] Cazenave, T., Society, A. M., and of Mathematical Sciences, C. I. *Semilinear Schrodinger Equations*. Courant lecture notes in mathematics. American Mathematical Society, Providence, RI, 2003.

[67] Chatterjee, S. Invariant measures and the soliton resolution conjecture. *Communications on Pure and Applied Mathematics 67*, 11 (2014), 1737–1842, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.21501.

[68] Chatterjee, S. A note about the uniform distribution on the intersection of a simplex and a sphere. *Journal of Topology and Analysis 09*, 04 (2017), 717–738, https://doi.org/10.1142/S1793525317500224.

[69] Chatterjee, S., and Kirkpatrick, K. Probabilistic methods for discrete nonlinear schrödinger equations. *Communications on Pure and Applied Mathematics 65*, 5 (2012), 727–757, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.21388.

[70] Chaudhuri, A., Kundu, A., Roy, D., Dhar, A., Lebowitz, J. L., and Spohn, H. Heat transport and phonon localization in mass-disordered harmonic crystals. *Phys. Rev. B 81* (Feb 2010), 064301.

[71] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., and Tao, T. Transfer of energy to high frequencies in the cubic defocusing nonlinear schrödinger equation. *Inventiones mathematicae 181*, 1 (Jul 2010), 39–113.

[72] Colliander, J., and Oh, T. Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. *Duke Mathematical Journal 161*, 3 (2012), 367 – 414.

[73] Cramer, M., and Eisert, J. Correlations, spectral gap and entanglement in harmonic quantum systems on generic lattices. *New Journal of Physics 8*, 5 (may 2006), 71–71.

[74] D'Alessio, L., Kafri, Y., Polkovnikov, A., and Rigol, M. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Advances in Physics 65*, 3 (2016), 239–362, https://doi.org/10.1080/00018732.2016.1198134.

[75] de Bouard, A., and Debussche, A. A stochastic nonlinear schrödinger equation with multiplicative noise. *Communications in Mathematical Physics 205*, 1 (Aug 1999), 161–181.

[76] de Bouard, A., and Debussche, A. The stochastic nonlinear schrödinger equation in H 1. *Stochastic Analysis and Applications 21*, 1 (2003), 97–126, https://doi.org/10.1081/SAP-120017534.

[77] de Bouard, A., Debussche, A., and Di Menza, L. Theoretical and numerical aspects of stochastic nonlinear schrödinger equations. *Journées équations aux dérivées partielles* (2001).

[78] De Bouard, A., Debussche, A., and Fukuizumi, R. Long time behavior of gross–pitaevskii equation at positive temperature. *SIAM Journal on Mathematical Analysis 50*, 6 (2018), 5887–5920, https://doi.org/10.1137/17M1149195.

[79] de Gosson, M. *Symplectic Geometry and Quantum Mechanics*. Operator Theory: Advances and Applications. Birkhäuser Basel, 2006.

[80] Debussche, A., and de Bouard, A. Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise. *The Annals of Probability 33*, 3 (2005), 1078 – 1110.

[81] Debye, P. Zur theorie der spezifischen wärmen. *Annalen der Physik 344*, 14 (1912), 789–839.

[82] del Rio, R., Jitomirskaya, S., Last, Y., and Simon, B. Operators with singular continuous spectrum, iv. hausdorff dimensions, rank one perturbations, and localization. *Journal d'Analyse Mathématique 69*, 1 (Dec 1996), 153–200.

[83] DeMasi, A., and Presutti, E. *Mathematical Methods for Hydrodynamic Limits*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.

[84] Deng, Y., Nahmod, A. R., and Yue, H. Invariant gibbs measures and global strong solutions for nonlinear schrödinger equations in dimension two. *arXiv preprint arXiv:1910.08492* (2019).

[85] Dhar, A. Heat conduction in the disordered harmonic chain revisited. *Phys. Rev. Lett. 86* (Jun 2001), 5882–5885.

[86] Dhar, A. Heat transport in low-dimensional systems. *Advances in Physics 57*, 5 (2008), 457–537.

[87] Dhar, A., and Lebowitz, J. L. Effect of phonon-phonon interactions on localization. *Phys. Rev. Lett. 100* (Apr 2008), 134301.

[88] Dhar, A., and Roy, D. Heat transport in harmonic lattices. *Journal of Statistical Physics 125*, 4 (Nov 2006), 801–820.

[89] Dulong, P. L., and Petit, A.-T. *Recherches sur quelques points importans de la theorie de la chaleur*. 1819.

[90] Dvoretzky, A. On the strong stability of a sequence of events. *Ann. Math. Statist. 20*, 2 (06 1949), 296–299.

[91] Einstein, A. Die plancksche theorie der strahlung und die theorie der spezifischen wärme. *Annalen der Physik 327*, 1 (1907), 180–190.

[92] Enss, V. Asymptotic completeness for quantum mechanical potential scattering. *Communications in Mathematical Physics 61*, 3 (1978), 285–291.

[93] Erdős, L., Schlein, B., and Yau, H.-T. Derivation of the cubic non-linear schrödinger equation from quantum dynamics of many-body systems. *Inventiones mathematicae 167*, 3 (Mar 2007), 515–614.

[94] Erdős, L., Schlein, B., and Yau, H.-T. Rigorous derivation of the gross-pitaevskii equation with a large interaction potential. *Journal of the American Mathematical Society 22*, 4 (2009), 1099–1156. Full publication date: OCTOBER 2009.

[95] Evans, L. C. *Partial differential equations*. American Mathematical Society, Providence, R.I., 2010.

[96] Fermi, E., and Pasta, J. *5. Studies of Nonlinear Problems:*. University of California Press, 2020, pp. 139–154.

[97] Flach, S., Kladko, K., and MacKay, R. S. Energy thresholds for discrete breathers in one-, two-, and three-dimensional lattices. *Phys. Rev. Lett. 78* (Feb 1997), 1207–1210.

[98] Ford, G. W., and Kac, M. On the quantum langevin equation. *Journal of Statistical Physics 46*, 5 (Mar 1987), 803–810.

[99] Ford, G. W., Kac, M., and Mazur, P. Statistical mechanics of assemblies of coupled oscillators. *Journal of Mathematical Physics 6*, 4 (1965), 504–515, https://doi.org/10.1063/1.1704304.

[100] Ford, G. W., Lewis, J. T., and O'Connell, R. F. Quantum langevin equation. *Phys. Rev. A 37* (Jun 1988), 4419–4428.

[101] Fritz, J. *On the Hydrodynamic Limit of a Scalar Ginzburg-Landau Lattice Model: The Resolvent Approach*. Springer US, New York, NY, 1987, pp. 75–97.

[102] Fritz, J., Funaki, T., and Lebowitz, J. L. Stationary states of random hamiltonian systems. *Probability Theory and Related Fields 99*, 2 (Jun 1994), 211–236.

[103] Fröhlich, J., and Spencer, T. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Communications in Mathematical Physics 88*, 2 (1983), 151 – 184.

[104] Furstenberg, H., and Kesten, H. Products of Random Matrices. *The Annals of Mathematical Statistics 31*, 2 (1960), 457 – 469.

[105] Gaididei, Y., Mingaleev, S., Christiansen, P., and Rasmussen, K. Effect of nonlocal dispersion on self-interacting excitations. *Physics Letters A 222*, 3 (1996), 152–156.

[106] Gaididei, Y. B., Mingaleev, S. F., Christiansen, P. L., and Rasmussen, K. O. Effects of nonlocal dispersive interactions on self-trapping excitations. *Phys. Rev. E 55* (May 1997), 6141–6150.

[107] Gallay, T., and Hărăgus, M. Orbital stability of periodic waves for the nonlinear schrödinger equation. *Journal of Dynamics and Differential Equations 19*, 4 (Dec 2007), 825–865.

[108] Gallay, T., and Hărăguş, M. Stability of small periodic waves for the nonlinear schrödinger equation. *Journal of Differential Equations 234*, 2 (2007), 544–581.

[109] Gallay, T., and Pelinovsky, D. Orbital stability in the cubic defocusing nls equation: I. cnoidal periodic waves. *Journal of Differential Equations 258*, 10 (2015), 3607–3638.

[110] Gang, Z., and Sigal, I. M. On soliton dynamics in nonlinear schrödinger equations. *Geometric & Functional Analysis GAFA 16*, 6 (Dec 2006), 1377–1390.

[111] Gidas, B., Ni, W.-M., and Nirenberg, L. Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics 68*, 3 (Oct 1979), 209–243.

[112] Ginibre, J., and Velo, G. On a class of nonlinear schrödinger equations. i. the cauchy problem, general case. *Journal of Functional Analysis 32*, 1 (1979), 1–32.

[113] Glimm, J., and Jaffe, A. *Quantum physics: a functional integral point of view*. Springer Science & Business Media, 2012.

[114] Gradenigo, G., Iubini, S., Livi, R., and Majumdar, S. N. Localization transition in the discrete nonlinear schrödinger equation: ensembles inequivalence and negative temperatures. *Journal of Statistical Mechanics: Theory and Experiment 2021*, 2 (feb 2021), 023201.

[115] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry, i. *Journal of Functional Analysis 74*, 1 (1987), 160–197.

[116] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry, ii. *Journal of Functional Analysis 94*, 2 (1990), 308–348.

[117] Grunewald, N., Otto, F., Villani, C., and Westdickenberg, M. G. A two-scale approach to logarithmic Sobolev inequalities and the hydrodynamic limit. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 45*, 2 (2009), 302 – 351.

[118] Gubinelli, M., and Hofmanová, M. A pde construction of the euclidean Φ^4_3 quantum field theory. *Communications in Mathematical Physics 384*, 1 (May 2021), 1–75.

[119] Guo, M. Z., Papanicolaou, G. C., and Varadhan, S. R. S. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Communications in Mathematical Physics 118*, 1 (Mar 1988), 31–59.

[120] Gustafson, S., Le Coz, S., and Tsai, T.-P. Stability of Periodic Waves of 1D Cubic Nonlinear Schrödinger Equations. *Applied Mathematics Research eXpress 2017*, 2 (06 2017), 431–487, https://academic.oup.com/amrx/article-pdf/2017/2/431/19731346/abx004.pdf.

[121] Hairer, M. A probabilistic argument for the controllability of conservative systems. *arXiv preprint math-ph/0506064* (2005).

[122] Hairer, M. On malliavin's proof of hörmander's theorem. *Bulletin des Sciences Mathématiques 135*, 6 (2011), 650–666. Special issue in memory of Paul Malliavin.

[123] Hall, B. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics. Springer New York, 2013.

[124] Halperin, B. I. Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev. B 25* (Feb 1982), 2185–2190.

[125] Hannani, A. Hydrodynamic limit for a disordered quantum harmonic chain. *Communications in Mathematical Physics 390*, 1 (Feb 2022), 349–423.

[126] Hannani, A., and Olla, S. In preparation.

[127] Hannani, A., and Olla, S. A stochastic thermalization of the discrete nonlinear schrödinger equation, 2021, arXiv:2109.01389 [math-ph].

[128] Higham, N. J. *Functions of Matrices: Theory and Computation*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.

[129] Hörmander, L. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Classics in Mathematics. Springer Berlin Heidelberg, 2007.

[130] Huveneers, F. Drastic fall-off of the thermal conductivity for disordered lattices in the limit of weak anharmonic interactions. *Nonlinearity 26*, 3 (feb 2013), 837–854.

[131] Huveneers, F., and Theil, E. Equivalence of ensembles, condensation and glassy dynamics in the bose–hubbard hamiltonian. *Journal of Statistical Physics 177*, 5 (Dec 2019), 917–935.

[132] Johansson, M., and Rasmussen, K. O. Statistical mechanics of general discrete nonlinear schrödinger models: Localization transition and its relevance for klein-gordon lattices. *Phys. Rev. E 70* (Dec 2004), 066610.

[133] Jordan, R., and Bruce, T. Statistical equilibrium theories for the nonlinear schrodinger equation. *Contemporary Mathematics 283* (2001), 27–40.

[134] Jordan, R., and Josserand, C. Statistical equilibrium states for the nonlinear schrödinger equation. *Mathematics and Computers in Simulation 55*, 4 (2001), 433–447. Nonlinear Waves: Computation and Theory.

[135] Kevrekidis, P. G. *The discrete nonlinear Schrödinger equation: mathematical analysis, numerical computations and physical perspectives*, vol. 232. Springer Science & Business Media, 2009.

[136] Kevrekidis, P. G., Pelinovsky, D. E., and Stefanov, A. Asymptotic stability of small bound states in the discrete nonlinear schrödinger equation. *SIAM journal on mathematical analysis 41*, 5 (2009), 2010–2030.

[137] Kipnis, C., and Landim, C. *Scaling Limits of Interacting Particle Systems*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013.

[138] Kipnis, C., Olla, S., and Varadhan, S. R. S. Hydrodynamics and large deviation for simple exclusion processes. *Communications on Pure and Applied Mathematics 42*, 2 (1989), 115–137, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160420202.

[139] Kirkpatrick, K. Solitons and gibbs measures for nonlinear schrödinger equations. *Mathematical Modelling of Natural Phenomena 7*, 2 (2 2012), 95–112.

[140] Kirkpatrick, K., Schlein, B., and Staffilani, G. Derivation of the two-dimensional nonlinear schrödinger equation from many body quantum dynamics. *American Journal of Mathematics 133*, 1 (2011), 91–130.

[141] Kittel, C. Introduction to solid state physics. fifth edition, Jan 1976.

[142] Kojima, H., Matsuo, T., and Furihata, D. Some discrete inequalities for central-difference type operators. *Mathematics of Computation 86*, 306 (2017), 1719–1739.

[143] Komorowski, T., and Olla, S. On the sector condition and homogenization of diffusions with a gaussian drift. *Journal of Functional Analysis 197*, 1 (2003), 179 – 211.

[144] Komorowski, T., and Olla, S. Ballistic and superdiffusive scales in the macroscopic evolution of a chain of oscillators. *Nonlinearity 29*, 3 (feb 2016), 962–999.

[145] Komorowski, T., and Olla, S. Ballistic and superdiffusive scales in the macroscopic evolution of a chain of oscillators. *Nonlinearity 29*, 3 (feb 2016), 962–999.

[146] Komorowski, T., Olla, S., and Simon, M. Macroscopic evolution of mechanical and thermal energy in a harmonic chain with random flip of velocities. *Kinetic and Related Models 11*, 3 (2018), 615–645.

[147] Kotani, S. Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional schrödinger operators. In *North-Holland Mathematical Library*, vol. 32. Elsevier, 1984, pp. 225–247.

[148] Kowalczyk, M., Martel, Y., and Muñoz, C. Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space. *Journal of the American Mathematical Society 30*, 3 (2017), 769–798.

[149] Kunz, H., and Souillard, B. Sur le spectre des opérateurs aux différences finies aléatoires. *Communications in Mathematical Physics 78*, 2 (Dec 1980), 201–246.

[150] Kwong, M. K. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n. *Archive for Rational Mechanics and Analysis 105*, 3 (Sep 1989), 243–266.

[151] Landau, L. D., and Lifshitz, E. M. *Course of theoretical physics*. Elsevier, 2013.

[152] Lawden, D. F. *Elliptic functions and applications*, vol. 80. Springer Science & Business Media, 2013.

[153] Lebowitz, J. L., Mounaix, P., and Wang, W.-M. Approach to equilibrium for the stochastic nls. *Communications in Mathematical Physics 321*, 1 (Jul 2013), 69–84.

[154] Lebowitz, J. L., Rose, H. A., and Speer, E. R. Statistical mechanics of the nonlinear schrödinger equation. *Journal of Statistical Physics 50*, 3 (Feb 1988), 657–687.

[155] Lepri, S., Livi, R., and Politi, A. Thermal conduction in classical low-dimensional lattices. *Physics Reports 377*, 1 (2003), 1–80.

[156] Letizia, V. *Microscopic models for Fourier's law*. PhD thesis, PSL Research University, 2017.

[157] Letizia, V., and Olla, S. Nonequilibrium isothermal transformations in a temperature gradient from a microscopic dynamics. *The Annals of Probability 45*, 6A (2017), 3987 – 4018.

[158] Lieb, E. H. Exact analysis of an interacting bose gas. ii. the excitation spectrum. *Phys. Rev. 130* (May 1963), 1616–1624.

[159] Lieb, E. H., and Liniger, W. Exact analysis of an interacting bose gas. i. the general solution and the ground state. *Phys. Rev. 130* (May 1963), 1605–1616.

[160] Lieb, E. H., and Mattis, D. C. *Mathematical physics in one dimension: exactly soluble models of interacting particles*. Academic Press, 2013.

[161] Lions, P.-L. The concentration-compactness principle in the calculus of variations. the locally compact case, part 1. *Annales de l'I.H.P. Analyse non linéaire 1*, 2 (1984), 109–145.

[162] Lions, P.-L. The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. *Annales de l'I.H.P. Analyse non linéaire 1*, 4 (1984), 223–283.

[163] Liverani, C., and Olla, S. Toward the fourier law for a weakly interacting anharmonic crystal. *Journal of the American Mathematical Society 25*, 2 (2012), 555–583.

[164] Lukkarinen, J. Multi-state condensation in berlin-kac spherical models. *Communications in Mathematical Physics 373*, 1 (Jan 2020), 389–433.

[165] Lukkarinen, J., and Spohn, H. Anomalous energy transport in the fpu-β chain. *Communications on Pure and Applied Mathematics 61*, 12 (2008), 1753–1786, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.20243.

[166] Lyons, R. Strong laws of large numbers for weakly correlated random variables. *Michigan Math. J. 35*, 3 (1988), 353–359.

[167] Martel, Y., and Merle, F. Asymptotic stability of solitons of the gkdv equations with general nonlinearity. *Mathematische Annalen 341*, 2 (Jun 2008), 391–427.

[168] Martel, Y., Merle, F., and Tsai, T.-P. Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations. *Duke Mathematical Journal 133*, 3 (2006), 405 – 466.

[169] Martinelli, F., and Scoppola, E. Remark on the absence of absolutely continuous spectrum ford-dimensional schrödinger operators with random potential for large disorder or low energy. *Communications in Mathematical Physics 97*, 3 (Sep 1985), 465–471.

[170] Maxwell, J. C. Viii. a dynamical theory of the electromagnetic field. *Philosophical transactions of the Royal Society of London*, 155 (1865), 459–512.

[171] McKean, H. P. Statistical mechanics of nonlinear wave equations (4): Cubic schrödinger. *Communications in Mathematical Physics 168*, 3 (Apr 1995), 479–491.

[172] McKean, H. P., and Vaninsky, K. L. Brownian motion with restoring drift: The petit and micro-canonical ensembles. *Communications in Mathematical Physics 160*, 3 (Mar 1994), 615–630.

[173] McKean, H. P., and Vaninsky, K. L. Action-angle variables for the cubic schrödinger equation. *Communications on Pure and Applied Mathematics 50*, 6 (1997), 489–562, https://onlinelibrary.wiley.com/doi/pdf/10.1002/

[174] McKean, H. P., and Vaninsky, K. L. Cubic schrödinger: The petit canonical ensemble in action-angle variables. *Communications on Pure and Applied Mathematics 50*, 7 (1997), 593–622, https://onlinelibrary.wiley.com/doi/pdf/10.1002/

[175] Minami, N. Local fluctuation of the spectrum of a multidimensional anderson tight binding model. *Communications in Mathematical Physics 177*, 3 (Apr 1996), 709–725.

[176] Mingaleev, S. F., Christiansen, P. L., Gaididei, Y., Johansson, M., and Rasmussen, K. Ø. Models for energy and charge transport and storage in biomolecules. *Journal of Biological Physics 25*, 1 (Mar 1999), 41–63.

[177] Morrey Jr., C. B. On the derivation of the equations of hydrodynamics from statistical mechanics. *Communications on Pure and Applied Mathematics 8*, 2 (1955), 279–326, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160080206.

[178] Nachtergaele, B., Raz, H., Schlein, B., and Sims, R. Lieb-robinson bounds for harmonic and anharmonic lattice systems. *Communications in Mathematical Physics 286*, 3 (Mar 2009), 1073–1098.

[179] Nachtergaele, B., Schlein, B., Sims, R., Starr, S., and Zagrebnov, V. On the existence of the dynamics for anharmonic quantum oscillator systems. *Reviews in Mathematical Physics 22*, 02 (2010), 207–231, https://doi.org/10.1142/S0129055X1000393X.

[180] Nachtergaele, B., Sims, R., and Stolz, G. Quantum harmonic oscillator systems with disorder. *Journal of Statistical Physics 149*, 6 (Dec 2012), 969–1012.

[181] Nachtergaele, B., and Yau, H.-T. Derivation of the euler equations from quantum dynamics. *Communications in Mathematical Physics 243*, 3 (Dec 2003), 485–540.

[182] Nahmod, A. R., Oh, T., Rey-Bellet, L., and Staffilani, G. Invariant weighted wiener measures and almost sure global well-posedness for the periodic derivative nls. *Journal of the European Mathematical Society 14*, 4 (2012), 1275–1330.

[183] Nahmod, A. R., Rey-Bellet, L., Sheffield, S., and Staffilani, G. Absolute continuity of brownian bridges under certain gauge transformations. *arXiv preprint arXiv:1103.4822* (2011).

[184] O'Connor, A. J. A central limit theorem for the disordered harmonic chain. *Communications in Mathematical Physics 45*, 1 (Feb 1975), 63–77.

[185] Oh, T., Quastel, J., and Valko, B. Interpolation of gibbs measures with white noise for hamiltonian pde. *Journal de mathématiques pures et appliquées 97*, 4 (2012), 391–410.

[186] Olla, S., Varadhan, S. R. S., and Yau, H.-T. Hydrodynamical limit for a hamiltonian system with weak noise. *Comm. Math. Phys. 155*, 3 (1993), 523–560.

[187] Oseledets, V. I. A multiplicative ergodic theorem. characteristic ljapunov, exponents of dynamical systems. *Trudy Moskovskogo Matematicheskogo Obshchestva 19* (1968), 179–210.

[188] Pava, J. *Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions*. Mathematical surveys and monographs. American Mathematical Society, Providence, RI, 2009.

[189] Peierls, R. Zur kinetischen theorie der wärmeleitung in kristallen. *Annalen der Physik 395*, 8 (1929), 1055–1101.

[190] Peierls, R. E. *Quantum theory of solids*. Clarendon Press, 1996.

[191] Pelinovsky, D., and Kevrekidis, P. Stability of discrete dark solitons in nonlinear schrödinger lattices. *Journal of Physics A: Mathematical and Theoretical 41*, 18 (2008), 185206.

[192] Pelinovsky, D., and Sakovich, A. Internal modes of discrete solitons near the anti-continuum limit of the dnls equation. *Physica D: Nonlinear Phenomena 240*, 3 (2011), 265–281.

[193] Ponte, P., Papić, Z., Huveneers, F. m. c., and Abanin, D. A. Many-body localization in periodically driven systems. *Phys. Rev. Lett. 114* (Apr 2015), 140401.

[194] Raphaël, P. Stability and blow up for the non linear schrödinger equation. *Lecture notes from the Clay Math 2008 Summer School on Evolution Equations.Clay Mathematics Proceedings, 17. American Mathematical Society, Providence, R.I.; Clay Mathematics Institute, Cambridge, Mass.* (2014), 269–323.

[195] Rasmussen, K., Gaididei, Y., Bang, O., and Christiansen, P. The influence of noise on critical collapse in the nonlinear schrödinger equation. *Physics Letters A 204*, 2 (1995), 121–127.

[196] Rasmussen, K. O., Cretegny, T., Kevrekidis, P. G., and Grønbech-Jensen, N. Statistical mechanics of a discrete nonlinear system. *Phys. Rev. Lett. 84* (Apr 2000), 3740–3743.

[197] Reed, M., and Simon, B. *Methods of Modern Mathematical Physics: Functional analysis*. Methods of Modern Mathematical Physics. Academic Press, 1980.

[198] Rezakhanlou, F. Hydrodynamic limit for attractive particle systems on Z^d. *Communications in Mathematical Physics 140*, 3 (Oct 1991), 417–448.

[199] Rider, B. Fluctuations in the thermodynamic limit of focussing cubic schrödinger. *Journal of Statistical Physics 113*, 3 (Nov 2003), 575–594.

[200] Rider, B. C. On the ∞-volume limit of the focusing cubic schrödinger equation. *Communications on Pure and Applied Mathematics 55*, 10 (2002), 1231–1248, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.10043.

[201] Rieder, Z., Lebowitz, J. L., and Lieb, E. Properties of a harmonic crystal in a stationary nonequilibrium state. *Journal of Mathematical Physics 8*, 5 (1967), 1073–1078, https://doi.org/10.1063/1.1705319.

[202] Rodnianski, I., Schlag, W., and Soffer, A. Dispersive analysis of charge transfer models. *Communications on Pure and Applied Mathematics 58*, 2 (2005), 149–216, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.20066.

[203] Rubin, R. J., and Greer, W. L. Abnormal lattice thermal conductivity of a one dimensional, harmonic, isotopically disordered crystal. *Journal of Mathematical Physics 12*, 8 (Aug 1971), 1686–1701.

[204] Ruelle, D. A remark on bound states in potential-scattering theory. *Il Nuovo Cimento A (1965-1970) 61*, 4 (1969), 655–662.

[205] Rumpf, B. Simple statistical explanation for the localization of energy in nonlinear lattices with two conserved quantities. *Phys. Rev. E 69* (Jan 2004), 016618.

[206] Rumpf, B., and Newell, A. C. Coherent structures and entropy in constrained, modulationally unstable, nonintegrable systems. *Phys. Rev. Lett. 87* (Jul 2001), 054102.

[207] Rémi Carles, E. F. Energy cascades for nls on the torus. *Discrete and Continuous Dynamical Systems 32*, 6 (2012), 2063–2077.

[208] Simon, B. Absence of ballistic motion. *Communications in mathematical physics 134*, 1 (1990), 209–212.

[209] Simon, B., and Wolff, T. Singular continuous spectrum under rank one perturbations and localization for random hamiltonians. *Communications on Pure and Applied Mathematics 39*, 1 (1986), 75–90, https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160390105.

[210] Simon, M. Diffusion coefficient for the disordered harmonic chain perturbed by an energy conserving noise. In *From Particle Systems to Partial Differential Equations II*. Springer, 2015, pp. 355–370.

[211] Soffer, A., and Weinstein, M. I. Resonances, radiation damping and instabilitym in hamiltonian nonlinear wave equations. *Inventiones mathematicae 136*, 1 (Mar 1999), 9–74.

[212] Spohn, H. *Large Scale Dynamics of Interacting Particles*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2012.

[213] Spohn, H. Generalized gibbs ensembles of the classical toda chain. *Journal of Statistical Physics 180*, 1 (Sep 2020), 4–22.

[214] Srednicki, M. Chaos and quantum thermalization. *Phys. Rev. E 50* (Aug 1994), 888–901.

[215] Stefanov, A., and Kevrekidis, P. G. Asymptotic behaviour of small solutions for the discrete nonlinear schrödinger and klein–gordon equations. *Nonlinearity 18*, 4 (may 2005), 1841–1857.

[216] Stolz, G. An introduction to the mathematics of anderson localization. *Entropy and the quantum II. Contemp. Math 552* (2011), 71–108.

[217] Sulem, C., and Sulem, P. *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*. Applied Mathematical Sciences. Springer New York, 1999.

[218] Tao, T., of the Mathematical Sciences, C. B., (U.S.)., N. S. F., and Society, A. M. *Nonlinear Dispersive Equations: Local and Global Analysis*. Conference Board of the Mathematical Sciences. Regional conference series in mathematics. American Mathematical Society, 2006.

[219] Thomann, L., and Tzvetkov, N. Gibbs measure for the periodic derivative nonlinear schrödinger equation. *Nonlinearity 23*, 11 (oct 2010), 2771–2791.

[220] Tzvetkov, N. Invariant measures for the defocusing nonlinear schrödinger equation. *Annales de l'Institut Fourier 58*, 7 (2008), 2543–2604.

[221] Tzvetkov, N. Construction of a gibbs measure associated to the periodic benjamin-ono equation. *Probability theory and related fields 146*, 3-4 (2010), 481.

[222] Verheggen, T. Transmission coefficient and heat conduction of a harmonic chain with random masses: Asymptotic estimates on products of random matrices. *Communications in Mathematical Physics 68*, 1 (Feb 1979), 69–82.

[223] Villani, C. *Hypocoercivity*. American Mathematical Soc.

[224] Weinstein, M. I. Excitation thresholds for nonlinear localized modes on lattices. *Nonlinearity 12*, 3 (jan 1999), 673–691.

[225] Williamson, J. On the algebraic problem concerning the normal forms of linear dynamical systems. *American Journal of Mathematics 58*, 1 (1936), 141–163.

[226] Yau, H.-T. Relative entropy and hydrodynamics of ginzburg-landau models. *Letters in Mathematical Physics 22*, 1 (May 1991), 63–80.

[227] Zakharov, V., and Shabat, A. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Journal of Experimental and Theoretical Physics 34* (1970), 62–69.

[228] Zakharov, V. E. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics 9*, 2 (Mar 1968), 190–194.

RÉSUMÉ

Cette thèse est consacrée à l'étude de la perturbation aléatoire de deux modèles de systèmes de particules en interaction, liés à la mécanique quantique. Le premier est une chaîne désordonnée unidimensionnelle d'oscillateurs harmoniques quantiques, où une limite hydrodynamique dans l'échelle hyperbolique du temps et de l'espace est prouvée ; l'élongation, la quantité de mouvement et l'énergie convergent vers la solution de l'équation d'Euler dans cette échelle.
Dans le deuxième modèle, une perturbation stochastique conservant la masse d'une certaine classe d'équations de Schrödinger non-linéaires discrètes est introduite, modélisant l'action d'un bain de chaleur à une température donnée. La mesure de Gibbs correspondante est la seule mesure invariante de la dynamique, fournissant des propriétés d'ergodicité et de mélange temporel. En guise d'application, on étudie la limite de grand temps, l'approximation du continuum, et la limite de basse température dans le cas cubique unidimensionnelle focalisant.

MOTS CLÉS

Limites hydrodynamiques, Chaînes harmoniques quantiques désordonnées, Équation d'Euler, Conductivité thermique, Localisation d'Anderson, Décroissance de la corrélation, Équation de Schrödinger non-linéaire discrète, Équation de Schrödinger non-linéaire, Processus stochastiques, Diffusion hypoelliptique, Ondes solitaires, Grandes déviations

ABSTRACT

This thesis is devoted to the study of random perturbation of two models of interacting particle systems related to quantum mechanics. First is a one-dimensional disordered chain of quantum harmonic oscillators, where a hydrodynamic limit in the hyperbolic scaling of time and space is proven; elongation, momentum, and energy converge to the solution of the Euler equation in this scaling.
In the second model, a mass conserving stochastic perturbation of a certain class of discrete non-linear Schrödinger equations is introduced, modeling the action of a heat bath at a given temperature. The corresponding Gibbs measure is the unique invariant measure of the dynamics, providing ergodicity and time-mixing properties. As an application, it is proved that in the one-dimensional cubic focusing case, the large time, continuum approximation, and the low-temperature limit of the solution converges to the steady wave of the continuous non-perturbed equation that minimizes the energy for a given mass.

KEYWORDS

Hydrodynamic limits, Disordered quantum harmonic chains, Euler equation, Thermal conductivity, Anderson localization, Decay of correlation, Discrete non-linear Schrödinger equation, Non-linear Schrödinger equation, Stochastic processes, Hypoelliptic diffusion, Solitary waves, Large deviations