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THÈSE POUR L'OBTENTION DU GRADE DE DOCTEUR EN SCIENCES ÉCONOMIQUES

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ÉCOLE DOCTORALE N° 597
Sciences Économiques et Sciences de Gestion
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Par

Nicolas LEFEBVRE

Four Essays in Macroeconomics with Heterogeneous Agents

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"Nunc est bibendum, nunc pede libera pulsanda tellus [C'est maintenant qu'il faut boire, ô mes amis, et frapper la terre d'un pied libre et joyeux]"

Horace, *Odes*

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Contents

- Acknowledgements** **3**
- General introduction** **9**
- 1 How to explain U.S. labor market fluctuations using a simple search and matching model** **11**
 - 1 Introduction 13
 - 2 The model 14
 - 2.1 The setup: aggregate shocks and workers heterogeneity 14
 - 2.2 The matching 15
 - 2.3 The surplus sharing rule 17
 - 2.4 The decision rule of vacancies creation 19
 - 2.5 Turnover 20
 - 2.6 Parameterization 21
 - 2.7 Differences with Robin (2011) 21
 - 3 Data and estimation 21
 - 3.1 Data construction 21
 - 3.2 Data treatment 22
 - 3.3 Asymmetries 22
 - 3.4 The Simulated Method of Moments 23
 - 3.5 Calibration 25
 - 4 Results 25
 - 4.1 Fit of U.S. labor market fluctuations 25
 - 4.2 Estimated parameters 29
 - 5 Concluding remarks 30
- Appendices** **31**
 - 1.A Would an even simpler version work? 31
 - 1.B SMM and statistical inference 34
 - 1.B.1 The procedure 34
 - 1.B.2 The optimal weighting matrix 34

1.B.3	Computation of the variance of moments using GMM	35
1.B.4	Test statistics	35
1.C	Data source	36
2	Structural changes, labor supply heterogeneity and macroeconomic volatility	37
1	Introduction	39
2	The model	41
2.1	Presentation	41
2.2	Households	42
2.3	Firms	45
2.4	Recursive equilibrium	45
3	Calibration	46
4	The Benchmark model	48
5	The impact of U.S. economy's structural changes since the 1960's on the volatility of hours and employment	50
5.1	Numerical experiment's presentation	50
5.2	Analysis of isolated structural changes	51
6	Building a pre-1980's counterfactual	62
6.1	Presenting results	62
6.2	Discussion	63
7	Conclusion	64
	Appendices	65
3	Aging risk, insurance and social security	71
1	Introduction	73
2	Uninsured aging risk	74
2.1	Preferences	74
2.2	Consumption-saving decisions	75
2.3	Substitution effect, wealth effect and the aging risk	77
2.4	Insuring the aging risk	79
3	Welfare improvement through social security	80
3.1	General: Demographics	81
3.2	Equilibrium with PAYG pension system and inelastic labor supply	81
3.3	An extension with variable labor supply	85
4	Conclusion	90
	Appendices	91
3.A	The old agent's program	91
3.B	The youngster's program	92
3.C	Closed form solutions	93

3.C.1	Old agents' closed form solution	93
3.C.2	Young agents' closed form solution	95
3.D	Proof of Proposition 1	96
3.E	Proof of Proposition 2	97
3.F	Equilibrium with inelastic labor supply	98
3.F.1	Aggregation	98
3.F.2	Recursive detrended equilibrium	100
3.F.3	Steady-state equilibrium	100
3.G	Solution when labor supply is elastic	101
3.H	Equilibrium with elastic labor supply	104
3.I	The deterministic case	105
4	Social security and the interest rate: the impact of a pension reform abroad on the french economy	107
1	Introduction	109
2	The model	111
2.1	Aging risk and preferences	111
2.2	Decision rules	113
2.3	Aggregation	118
2.4	Firms	119
2.5	Social security system	120
2.6	Equilibrium	120
3	Simulating a pension reform abroad	123
3.1	Calibration	124
3.2	Numerical experiment	125
3.3	Results	126
4	Concluding remarks	128
	Appendices	129
4.A	Euler equations	129
4.A.1	Senior's program	129
4.A.2	Retired middle-aged program	130
4.A.3	Working middle-aged program	131
4.A.4	Youngsters program	132
4.B	Closed-form solutions	134
4.B.1	Seniors	134
4.B.2	Retired middle-aged	134
4.B.3	Working middle-aged	136
4.B.4	Youngsters	136
4.B.5	Marginal value of wealth and drop in productivity	137

General introduction

In terms of age, wealth or productivity, the heterogeneous nature of economic agents has become a key variable in order to improve our comprehension of various macroeconomic related issues. Among the first macroeconomists that highlighted the important role heterogeneities play was Keynes, who stated in his *General Theory* (1935) [57] " *The amount that the community spends on consumption obviously depends on [...] the principles on which income is divided between the individuals composing it [...] we may have to make an allowance for the possible reactions of aggregate consumption to the change in the distribution of a given real income between entrepreneurs and rentiers resulting from a change in the wage-unit*". The study of aggregate consumption's reaction to changes in the distribution of income and associated marginal propensities to consume is just one example of the many various research fields in macroeconomics where the use of standard representative agent models shows some limitations. Investigating for instance the impact pay-as-you-go (PAYG) pension systems have on welfare or how demographic aging influences the way monetary policy is conducted requires models that take into account the fact that agents do not live forever and differ in term of age: the so-called overlapping generations (OLG) models that were developed following the seminal paper of Diamond (1965) [24]. Studying the cyclical behavior of jobs creation and destruction along the business cycle cannot be done without incorporating heterogeneity in workers' productivity in standard search and matching models, as shown by Mortensen and Pissarides (1994) [66].

The four chapters that constitute this Ph.D. dissertation follow this approach, and making use of heterogeneous agents models, contribute to several fields in macroeconomics. The first chapter focuses on the cyclical behavior of unemployment, job finding and separation rates. As emphasized by Robin (2011) [76], Ferraro (2018) [31], or Lise and Robin (2017) [63], the use of search and matching models with workers that are *ex ante* heterogeneous in productivity and endogenous separations provides a consistent and simple explanation to the high volatility and asymmetrical behavior that characterizes the U.S. unemployment rate. In these models, low skilled workers are associated with a low value of surplus that is potentially negative – i.e. the highest wage firms are ready to offer them given their productivity is below their reservation wage. As the economy goes into a recession, more and more jobs associated with such workers become unproductive and are therefore destroyed, which leads to a fast increase in unemployment. As the economy recovers, search frictions prevent unemployed workers to go back to job immediately, which explains the slow decrease of the unemployment rate. In particular, Lise and Robin's model (2017) [63], which features heterogeneous firms and workers as well as job

openings succeeds almost perfectly at replicating U.S. labor market flows' volatility, but at the cost of a high degree of complexity. This chapter shows how to simplify their model in order to have a "minimal working basis" able to replicate first to third order moments of U.S. unemployment, job finding and separation rates. A model *à la* Lise-Robin that assumes homogeneous firms and that is simulated at a weekly frequency is found to be this minimal representation of U.S. labor market.

The second chapter contributes to the business cycle literature, and studies how structural changes that interact with the distribution of households can reshape the cyclical behavior of hours worked in the U.S.. The labor supply's response of agents to aggregate shocks indeed depend on their individual characteristics, in particular their age, wealth or productivity. The way such heterogeneous agents are distributed in the economy thus influences the response of aggregate hours along the business cycle. For the same reason, some structural changes, because they interact with the shape of this distribution, have an impact on the response of aggregate supply to shocks. This chapter analyzes how three of this type of structural changes that has occurred in the U.S. since the mid-1980's, *i.e.* (i) demographic aging (ii) increasing wage inequalities and (iii) tax/transfer system reforms have influenced the way hours worked react to shocks.

Age is the form of heterogeneity the third chapter focuses on. It contributes to the theoretical welfare economics literature, by putting into evidence the "aging risk" to which savings is subject to in a stochastic aging context. More specifically, in such a framework, an extra euro saved by a youngster today can end up tomorrow in the age-states "young" or "old", which highly differ due to the fact that youngsters only risk to switch to state "old" whereas old agents risk to die. The aging risk arises because the mortality risk is higher in the age-state "old", which makes wealth not having the same value in both states. The chapter then shows that private insurance can take care of such a risk, by redistributing agents' wealth across age-states, *i.e.* from the one where its value is lower to the other one. It also shows that when such markets are missing, and even if taxation distorts labor supply, social security can be welfare improving.

The fourth chapter also uses an OLG model in open economy to study the impact of PAYG pension systems reform undertaken in the rest of the developed world on the french economy, with a particular focus on welfare externalities. Because it increases capital supply at the aggregate level, such a reform is likely to push down the worldwide interest rate, and thus to make the return on capital of french savers go down, which *in fine*, by making them poorer, decreases their welfare.

Chapter 1

How to explain U.S. labor market fluctuations using a simple search and matching model

Abstract

This chapter shows how to extend Robin's (2011) [76] search model which features aggregate uncertainty, heterogeneous workers, endogenous job destruction in order to build a working basis for more complete search and matching models. Adding a simple job openings mechanism with homogeneous firms and simulating the model at a weekly frequency allows it to replicate first to third order moments of unemployment, job finding and destruction rates.

1 Introduction

Despite its exceptional magnitude, the recent spike of the U.S. unemployment rate during the COVID-19 recession is consistent with the two major stylized facts that characterizes U.S. labor market fluctuations: their volatility is high, and they exhibit an asymmetrical pattern¹ – in the case of the unemployment rate, peaks are taller than troughs are deep and increases are steeper than decreases. Following the seminal work of Shimer (2005) [80] who showed that the canonical Diamond-Mortensen-Pissarides (DMP) search and matching model cannot reproduce the first stylized fact – the so-called "Shimer puzzle" –, a large amount of literature was dedicated to the explanation of the volatility of unemployment². More recently, the second stylized fact has caught the eye of the search and matching literature, which provided extensions to the DMP model (Ferraro, 2018 [31] and Adjemian *et al.*, 2018 [1]) to explain the asymmetrical behavior of unemployment.

To my knowledge, the first one to provide a natural candidate that would explain both stylized fact was Robin (2011) [76], who proposed to abandon the use of a representative agent, and to couple workers that are *ex ante* heterogeneous in abilities with endogenous job separations. The mechanism that lets his model generate a satisfying volatility in unemployment is very intuitive. The less skilled workers in the economy are associated with a low value of surplus that is potentially negative – *i.e.* the highest wage firms are ready to offer them given their productivity is below their reservation wage. As the economy goes into a recession, more and more jobs associated with such workers become unproductive and are therefore destroyed, which leads to an increase in unemployment³. As Robin points out, in order to obtain an unemployment rate that varies between 4 and 10%, we just have to associate a 4% exogenous separation rate with 6% of workers whose surpluses are potentially negative, *i.e.* who can be put in unemployment. Moreover, even if his paper did not mention it, this mechanism can explain the asymmetric behavior of unemployment, as shown by Ferraro (2018) [31]. Indeed, the less skilled workers are suddenly laid off at the beginning of the recession, which causes the fast increase of the unemployment rate; once the economy recovers, it takes times for these individuals to get their job back because of search frictions, which explains the slower decrease of unemployment. Even if Robin's model could not capture all the moments of U.S. labor market flows that the standard DMP literature focused on – unemployment rate, job finding and separation rates –, it offered among others both a simple and consistent explanation to the volatility of unemployment, and a solid backbone for more quantitative search and matching models that would aim at going deeper in the analysis of labor market fluctuations,

Such a model was later developed by Lise and Robin (2017) [63]. They added to the 2011 model a richer meeting mechanism by introducing job openings and a two sided heterogeneity, that is to say a labor market where both firms and workers are heterogeneous and where mismatch (or imperfect sorting)

¹This asymmetrical behavior has been well documented, see Neftçi (1984) [69] Sichel (1993) [81], McQueen and Thorley (1993) [65], McKay and Reiss (2008) [64], Ferraro (2018) [31] for instance.

²See, among others, Hall, (2005) [41], Mortensen and Nagypal (2007) [67], Hagedorn and Manovskii (2008) [40], Fujita and Ramey (2012) [33], Gomme and Lkhagvasuren (2015) [38], Petrosky-Nadeau and Zhang (2017) [72].

³Chassambouli (2013) [20] and Ferraro (2018) [31] also used *ex ante* heterogeneity in workers' abilities and endogenous job separations to replicate the cyclical behavior of unemployment. Chassambouli (2013) [20] focused on second order moments, and Ferraro (2018) [31] concentrates on the asymmetrical behavior of employment.

can occur. From the point of view of an economist aiming at explaining the cyclical behavior of the three main DMP literature variables we mentioned above, this extension works substantially better than the 2011 search model, as it succeeds at replicating the main first and second order moments of U.S. labor market fluctuations; these enhanced performances however come with a much higher degree of complexity. The comparison between the two models naturally brings the following question: what crucial ingredients does the original search model need to work as well as Lise and Robin’s model, if one simply aims at replicating the standard U.S. labor market variables – *i.e.* the unemployment, job finding and separation rates – fluctuations? Indeed, using the latter as a working basis might be too costly, whereas using the former is not enough. Between the two might therefore be a minimal representation of U.S. labor market fluctuations, *i.e.* a model as simple as possible but also able to reproduce the latter, that we could use as a working basis for a more complete model.

This paper contributes to the search and matching literature from a methodological point of view, by finding this minimal working basis. It takes the search model of 2011 as a starting point, and adds the simplest layers on it, so that it succeeds at replicating the fluctuations of main U.S. labor market variables. If one focuses on the cyclical behavior of unemployment, job finding and separation rates, it shows that adding a vacancy creation mechanism with homogeneous firms – as in Murin and Robin (2018) [68] – and simulating the model at a weekly frequency leads to performances that are close to Lise and Robin’s (2017) [63] model. It also proposes to go further, and shows that this simple model can also consistently explain the asymmetrical behavior of U.S. labor fluctuations.

The paper is organized as follows. Section 2 presents the models this paper works with, Section 3 presents the data, Section 4 describes the estimation procedure, and Section 5 presents the results of the estimations. Section 6 concludes.

2 The model

2.1 The setup: aggregate shocks and workers heterogeneity

The primary setup is the same as in the one in the model of Robin. Time is discrete, and y_t denotes the state of aggregate productivity at time t , that can take N values among the set $\{y_1, \dots, y_N\}$. At the beginning of each period, a new state of aggregate productivity is drawn according to a Markov chain, where the transition probability is $\mathbb{P}(y_{t+1} = y_j | y_t = y_i) = \pi_{ij}$, with $i, j \in \llbracket 1, N \rrbracket$. The distribution of aggregate shocks is assumed to be symmetrical.

The economy is lived by M types of workers, the amount of workers of a type $m \in \llbracket 1, M \rrbracket$ being denoted l_m . The total population is normalized to unity, *i.e.* $\sum_{m=1}^M l_m = 1$. Workers are either employed or unemployed. Each type m of worker is endowed with an individual productivity x_m , where $x_m < x_{m+1} \forall m \in \llbracket 1, M \rrbracket$. There is a unit measure of homogeneous firms. A worker of type m paired with a firm when the state of aggregate productivity is y_i produce an output $y_i(m) = y_i x_m$.

2.2 The matching

I denote $u_t(m)$ and $e_t(m)$ the unemployment and employment rates of workers of type m at the end of period $t - 1$ (or at the very beginning of period t , before the realization of the productivity shock). The global aggregated unemployment rate is thus given by $u_t = \sum_{m=1}^M u_t(m)l_m$, and the employment rate by $e_t = \sum_{m=1}^M e_t(m)l_m$.

We assume on-the-job search. When a worker m and a firm are matched, the surplus of the match in period t , noted $S_t(m)$ is given by the value of employment minus the value of unemployment for the worker, to which we add the value of a job for the firm (the value of a vacancy being zero in equilibrium, as in any search-and-matching model). The match is destroyed if $S_t(m) < 0$.

The timing of the matching is the following:

1. At the beginning of the period t , a shock occurs, a new state of aggregate productivity level y_t is drawn. If necessary, wage renegotiation takes place (if the shock makes the wage move outside the bargaining set, as it will become clearer later). If it makes the surplus value of m type jobs negative, all these jobs are destroyed, and there will be no matching. Jobs are destroyed exogenously at a rate δ . Let $u_t^*(m)$ and $e_t^*(m)$ be the share of unemployed and employed m type workers on the labor market, that is to say:
 - If the surplus of the match is negative, then $u_t^*(m) = 1$ and $e_t^*(m) = 0$. All matches for type m workers are destroyed, the individuals who have lost their jobs therefore become unemployed and seek for a job.
 - Otherwise, then $u_t^*(m) = u_t(m)$ and $e_t^*(m) = (1 - \delta)e_t$. Note that as in the paper of Robin, exogenously laid-off workers in t are not taken into account in $u_t^*(m)$. They will thus not be matched in the same period, but will be incorporated in $u_{t+1}(m)$.

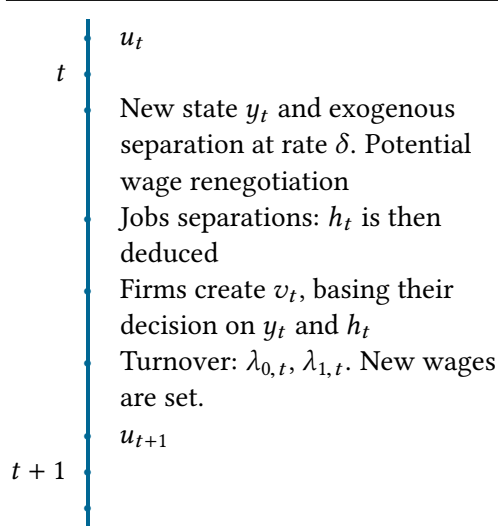
Denoting $u_t^* = \sum_{m=1}^M u_t^*(m)l_m$ and $e_t^* = \sum_{m=1}^M e_t^*(m)l_m$ the number of unemployed and employed workers likely to be matched, the total search effort is given by $h_t = u_t^* + \iota \cdot e_t^*$, where ι is the search effort of employed workers, the one of unemployed workers being normalized to one.

2. Firms observe the search effort h_t , the state of aggregate productivity y_t , and decide of the amount of vacancy v_t to create. I assume random search, *i.e.* firms cannot choose the type of worker they hire.
3. The matching occurs, and wages are set. The number of matches is given by $\mathcal{M}_t = \mathcal{M}(h_t, v_t)$, where $\mathcal{M}(\cdot)$ is the matching function that follows the standard properties. The probability for an unemployed worker to meet a vacancy is $\lambda_{0t} = \frac{\mathcal{M}_t}{h_t}$, and the probability for an employed worker to meet a vacancy is given by $\lambda_{1t} = \iota \lambda_{0t}$. The job filling rate is given by $\phi_t = \frac{\mathcal{M}_t}{v_t}$.
4. The outcomes are u_{t+1} and e_{t+1} , at the end of period t .

The law of motion for unemployment is therefore:

$$u_{t+1}(m) = 1 - [(1 - \delta)(1 - u_t(m)) + \lambda_{0,t}u_t(m)]\mathbb{1}_{\{S_t(m)>0\}}$$

Table 2.1: The timing convention



where $\mathbb{1}_x$ is the indicator function. Thus, we can write

$$u_{t+1}(m) = \begin{cases} 1 & \text{if } S_t(m) \leq 0 \\ u_t(m) + \delta(1 - u_t(m)) - \lambda_{0,t}u_t(m) & \text{if } S_t(m) > 0 \end{cases}$$

Note that at each period, we have $e_t(m) = 1 - u_t(m)$, but $e_t^* \neq 1 - u_t^*$.

I assume in the following a CES matching function (Den Haan, Ramey, and Watson (2000) [23]) that ensures probabilities between 0 and 1, which takes the following form:

$$\mathcal{M}(h_t, v_t) = \frac{h_t \cdot v_t}{(v_t^\psi + h_t^\psi)^{\frac{1}{\psi}}}$$

Where $\psi > 0$ is a shape parameter. The probability for an unemployed workers to meet a firm is therefore

$$\lambda_{0,t} = \frac{v_t}{(v_t^\psi + h_t^\psi)^{\frac{1}{\psi}}} = \frac{\theta_t}{(1 + \theta_t^\psi)^{\frac{1}{\psi}}}$$

the job filling rate being

$$\phi_t = \frac{h_t}{(v_t^\psi + h_t^\psi)^{\frac{1}{\psi}}} = \frac{1}{(1 + \theta_t^\psi)^{\frac{1}{\psi}}}$$

where $\theta_t := v_t/h_t$ is the labor market tightness.

2.3 The surplus sharing rule

2.3.1 The wage setting mechanism

Wages are set *à la* Postel-Vinay and Robin (2002) [74]. The wage is set (or reset) at three occasions: (i) when an unemployed individual is hired, (ii) when an employee matches a poaching firm, and (iii) when it gets out of the bargaining set – I will come back more specifically on this renegotiation that is a little bit technical, but does not change the intuition presented hereafter.

Unemployed workers are assumed to have no bargaining power. For this reason, when they are successfully matched (in the sense that the match is productive), they are offered their reservation wage, and accept it. In other words, when a firm and an unemployed individual are matched, the whole surplus of the match is kept by the firm.

When an employed worker meets another firm, a second price auction is started between the incumbent and the poaching firm, as the former wants to keep its employee and the latter wants her to move. Each firm therefore offers the worker the lowest wage its competitor cannot afford, *i.e.* the wage such that the competitor's value of a job becomes null. As firms are all identical, when there is an on-the-job meeting, both firms make the same offer: they propose the highest wage they can afford, *i.e.* the wage that makes their surpluses null (or equivalently, the wage that makes the worker seize the whole value of the surplus of the match). As the worker is indifferent between staying at his current job or quitting, she is assumed to move to the incumbent firm at an exogenous probability τ .

2.3.2 The surplus of the worker

2.3.2.1 The value of unemployment

The value of unemployment at time t for a m -type unemployed worker is denoted $U_t(m)$. She receives a flow payment $z_i(m)$, which is the opportunity cost of employment (unemployment benefits and valuation of leisure). If at the next period, the unemployed worker finds a job, she will be offered her reservation wage, as she has no bargaining power. This implies that the value of being employed, for a newly hired worker, is equal to the value of being unemployed. The intertemporal value of unemployment is therefore given by the following Bellman equation:

$$U_t(m) = z_i(m) + \beta \sum_{j=1} \pi_{ij} U_j(m) = \alpha y_i(m) + (1 - \alpha) z_0 + \beta \sum_{j=1} \pi_{ij} U_j(m),$$

where z_0 is a constant part of the opportunity cost of employment and $\alpha \in [0, 1]$. where β is the discount factor.

2.3.2.2 The value of employment and the surplus of a worker

The value of employment at period t for a m type worker paid at a wage w is denoted $W_t(w, m)$. The value of the surplus for the worker is therefore $W_t(w, m) - U_t(m)$. Today, her flow value of surplus is given by the difference between her wage and her opportunity cost of employment. Tomorrow, if her job is not destroyed, she can meet another employer with a probability $\lambda_{1,t+1}$ and get all the surplus of the match, or do not meet any outside job opportunity. If it is the case, then depending on the

productivity shock, two outcomes are possible: either the wage is still in the bargaining set (both firms and worker's surpluses are positive) and the wage is not renegotiated, or the wage goes outside the bargaining set and has to be renegotiated. If the wage is too high (the firm's surplus is negative), then it is lowered until the firm's surplus becomes null, as the firm can threat to layoff the worker. If the wage is too low (the worker's surplus is negative) then it is raised so that the worker's surplus becomes null, as the worker can threat to leave. Under the form of Bellman equation, the worker's surplus is written:

$$W_t(w, m) - U_t(m) = w - z_t(m) + \beta(1 - \delta)\mathbb{E}_t \left[\underbrace{\mathbb{1}_{\{S_{t+1}(m) > 0\}}}_{\text{Match has to be viable}} \left[\underbrace{\lambda_{1,t+1} S_{t+1}(m)}_{\text{Get the whole surplus if poached}} + \underbrace{(1 - \lambda_{1,t+1})(W_{t+1}^*(w, m) - U_{t+1}(m))}_{\text{If not, potentially wage renegotiation}} \right] \right],$$

where $W_{t+1}^*(w, m) - U_{t+1}(m) = \min\{\max\{W_{t+1}(w, m) - U_t(m), 0\}, S_{t+1}(m)\}$ is the surplus of the worker in the next period if there was no meeting.

2.3.3 The surplus of the firm

Let us note $J_t(w, m)$ the value for the firm of a job paid at a wage w with a worker of type m , at time t . The current value of the job is given by its output, $y_t(m)$, minus its cost, the wage w . In the next period, if the job is not destroyed, there can be, with probability $\lambda_{1,t+1}$, a meeting with a poaching firm, which will trigger the auction, letting the worker get the whole value of the surplus of the match (*i.e.* the value of the job for the firm becomes null). Otherwise, the same mechanism of potential wage renegotiation takes place: if the wage is still in the bargaining set, then it is still the same; otherwise, it is renegotiated. If the wage is too high, that is to say $W_{t+1}(w, m) - U_{t+1}(m) > S_{t+1}(m)$, it is lowered until the surplus of the worker is $W_{t+1}(w, m) - U_{t+1}(m) = S_{t+1}(m)$ (which amounts to $J_{t+1}(w, m) = 0$). If on the contrary, the wage is too low, so that $J_{t+1}(w, m) > S_{t+1}(m)$, then it is reset such that $J_{t+1}(w, m) = S_{t+1}(m)$. Therefore, the value of the job can be written under the form of the following Bellman equation :

$$\begin{aligned} J_t(w, m) &= y_t(m) - w + \beta(1 - \delta)\mathbb{E}_t \left\{ \underbrace{\mathbb{1}_{\{S_{t+1}(m) > 0\}}}_{\text{Match has to be viable}} \left[\underbrace{\lambda_{1,t+1} \times 0}_{\text{Null surplus if meeting}} + \underbrace{(1 - \lambda_{1,t+1})J_{t+1}^*(w, m)}_{\text{If not, potentially wage renegotiation}} \right] \right\} \\ &= y_t(m) - w + \beta(1 - \delta)\mathbb{E}_t \left\{ \mathbb{1}_{\{S_{t+1}(m) > 0\}} [(1 - \lambda_{1,t+1})J_{t+1}^*(w, m)] \right\}, \end{aligned}$$

where $J_{t+1}^*(w, m) = \min\{\max\{J_{t+1}(w, m), 0\}, S_{t+1}(m)\}$.

2.3.4 The surplus of the match

The surplus of the match is given as the sum of the surplus of the worker and the surplus of the firm, that is to say $S_t(m) = W_t(w, m) - U_t(m) + J_t(w, m)$.

$W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m)$ is given by the following cases:

- The surplus value is negative, *i.e.* there is no wage that makes the job profitable for the firm and that is above the reservation wage of the worker at the same time. The highest wage the firm is able to offer to the employee (the one that makes $J_t(w, m) = 0$) is such as $W_t(w, m) - U_t(m) < 0$;

the lowest wage the employee is ready to work for (the one that makes $W_t(w, m) - U_t(m) = 0$) is such as $J_t(w, m) < 0$. In this case, $W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m) = 0$ as the job is destroyed.

- The surplus value is positive, and the wage is in the bargaining set, *i.e.* $W_{t+1}(w, m) - U_{t+1}(m) \in [0, S_{t+1}(m)[$ and $J_{t+1}(w, m) \in [0, S_{t+1}(m)[$. In this case, there is no renegotiation, and $W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m) = S_{t+1}(m)$
- The surplus value is positive, but not correctly shared, such that $W_{t+1}(w, m) - U_{t+1}(m) < 0$, and $J_{t+1}(w, m) > S_{t+1}(m)$. There is then wage renegotiation, which leads to $W_{t+1}^*(w, m) - U_{t+1}(m) = 0$, and $J_{t+1}^*(w, m) = S_{t+1}(m)$. We therefore get $W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m) = S_{t+1}(m)$
- The surplus value is positive, but not correctly shared, such that $W_{t+1}(w, m) - U_{t+1}(m) > S_{t+1}(m)$, and $J_{t+1}(w, m) < 0$. There is then wage renegotiation, which leads to $W_{t+1}^*(w, m) - U_{t+1}(m) = S_{t+1}(m)$, and $J_{t+1}^*(w, m) = 0$. We thus get $W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m) = S_{t+1}(m)$

Therefore, $W_{t+1}^*(w, m) - U_{t+1}(m) + J_{t+1}^*(w, m) = \max\{0, S_{t+1}(m)\}$. The Bellman equation for the value of the surplus thus takes the following form :

$$S_t(m) = y_t(m) - z_t(m) + \beta(1 - \delta)\mathbb{E}_t \max\{0, S_{t+1}(m)\}$$

The surplus value does only depends on the state of aggregate productivity and the worker's skill, and not on the unemployment rate, nor the wage. Therefore, we can write it as a function of the state of aggregate productivity:

$$S_i(m) = y_i(m) - z + \beta(1 - \delta) \sum_{j=1}^N \pi_{ij} \max\{0, S_j(m)\}$$

The surplus of the match is an increasing function of the worker's type and of the state of aggregate productivity, and can be negative.

2.4 The decision rule of vacancies creation

Firms open vacancies after the realization of the shock; if a match occurs and if it is productive – *i.e.* the bargaining set is non-empty –, the firm and the worker start producing immediately. The cost of opening v_t vacancies is denoted $c(v_t)$. Firms base their decision by observing the current state of productivity, and the number of unemployed and employed workers that can be matched. They post vacancies until the cost of opening a vacancy becomes equal to its expected return. When a firm meets a worker, she can be either (i) an unproductive unemployed worker, which means that the match is not followed by a hiring, *i.e.* the firm's surplus is null, (ii) an unemployed worker, and in this case the firm's surplus is equal to the surplus of the match, or (iii) an employed worker, who will get all the surplus of the match for herself. The probability to meet an unemployed worker of type m being $\frac{u_t^*(m)l_m}{h_t}$, the expected return of a match (before knowing the worker's type and status) is equal to $\sum_{m=1}^M \frac{u_t^*(m)l_m}{h_t} \max\{0, S_t(m)\}$. I simply assume that the cost function is linear, so that it can be written $c(v_t) = c \cdot v_t$. At the equilibrium, as firms are symmetric, they open each the same amount of vacancies.

The free entry condition therefore writes:

$$c'(v_t) = \phi_t \sum_m \frac{u_t^*(m)l_m}{h_t} \max\{0, S_t(m)\} = c$$

The marginal cost of opening a vacancy must be equal to its expected return, which is given by the probability to fill the opened vacancy times the expected value of the job for the firm. With the CES matching function, the optimal amount of vacancies is :

$$v_t = \left[\left(\frac{\kappa \sum_m u_t^*(m)l_m \max\{S_t(m), 0\}}{c} \right)^\psi - h_t^\psi \right]^{\frac{1}{\psi}}$$

Note that the form of the free entry condition makes impossible to write θ_t as a function of the state of aggregate productivity as in the standard DMP model – where it can be denoted θ_i –, since the unemployment rate enters the expression of the free entry condition. When the model is simulated, the amount of vacancies is computed period after period, a method that is feasible with the adopted timing convention (see Table 2.1). Indeed, remember that a period is cut into short sub-periods; a firm decides of the optimal amount of vacancies to open *after* observing the post-shock total search effort.

2.5 Turnover

If the model is simulated at quarterly frequency, the turnover rates are the same as in the original model, except that λ_0 becomes $\lambda_{0,t}$. If the model is simulated at a weekly frequency, the quarterly JDR and JFR are computed as their empirical counterparts (see Section 3.1). We therefore need the number of individuals of type m in unemployment for 5 weeks or more at week w , $u_w^5(m)$, and the total number of individuals experiencing unemployment for 5 weeks or more at week w , u_w^5 , which are given by the following:

$$u_w^5(m) = u_{w-5}(m) \prod_{k=0}^4 (1 - \mathbb{1}_{\{S_{w-5+k}(m) > 0\}} \lambda_{0, w-5+k})$$

$$u_w^5 = \sum_{m=1}^M u_w^5(m)l_m$$

Following the procedure of the survey of the BLS, I build my monthly data for u_m and u_m^5 by picking weeks containing the twelfth day of the month, that is to say weeks $\{2, 7, 11, 15, 19, 24, 28, 32, 37, 41, 46, 50\}$ for each year of simulation. I make the assumption that there are exactly 52 weeks in a year. Monthly job finding rates f_m and separation rates s_m are then computed as their empirical counterparts;

$$f_m = 1 - \frac{u_{m+1}^5}{u_m}$$

$$s_m = \frac{u_{m+1} - u_{m+1}^5}{1 - u_m}$$

Data are then quarterlized as described in Section 3.1. Finally, labor productivity is defined as:

$$\bar{y}_t = \frac{\sum_m (1 - u_t(m)) l_m y_t(m) \mathbb{1}_{\{S_t(m) > 0\}}}{1 - u_t}$$

2.6 Parameterization

We assume l_m to follow a Beta distribution with parameters η and μ . The states of aggregate productivity and the Markovian transition matrix are given by the discretization of the AR(1) process $\log(y_t) = \rho \log(y_{t-1}) + \sigma \sqrt{1 - \rho^2} \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, 1)$, using the Rouwenhorst method⁴ [77] with $N = 150$. The productivity of workers x_m is a grid of M linearly equally spaced points on $[\underline{x}, \underline{x} + 1]$, where $M = 500$.

2.7 Differences with Robin (2011)

To summarize, what did I add to Robin’s model, and why? First, job openings are added, so that fluctuations in JFR are not exclusively driven by the job destruction mechanism. The discrepancy between such a JFR and its empirical counterpart which also includes fluctuations in job openings might indeed be at the origin of an imperfect reproduction of U.S. labor market fluctuations. Second, the model is now written at the weekly frequency, and its simulated data are aggregated at a quarterly frequency. As observed quarterly data are the aggregation of labor market transitions that occur at a higher frequency, write the model at a weekly frequency also contributes at reconciling the model with the data.

3 Data and estimation

Now that the model has been presented, we need to estimate its structural parameters in order to evaluate its capacity to reproduce U.S. labor market fluctuations. This section explains how data the model is supposed to fit are built, then briefly presents the Simulated Method of Moments (SMM) used for the estimation.

3.1 Data construction

The same sample as in Robin (2011) [76] is used at a quarterly frequency (1951Q1-2010Q3), with series of labor productivity, unemployment, JFR and JDR. The size of the sample is restricted to this period for (i) the sake of comparison with Robin’s (2011) results and (ii) as data for output *per capita*, which are needed to estimate the parameters governing aggregate shocks are only available at a quarterly frequency. I follow closely the procedure of Robin to build quarterly series of JDR and JFR. Monthly series are first built using the following:

$$f_m = 1 - \frac{u_{m+1}^5}{u_m}$$

⁴Rouwenhorst’s method is known for performing better in presence of a high persistence, see Galindev and Lkhagvasuren (2010) [36]

$$s_m = \frac{u_{m+1} - u_{m+1}^5}{1 - u_m}$$

Where u_m is the number of unemployed individuals at month m , u_m^5 is the number of individuals unemployed for 5 weeks or more, f_m is the monthly JFR and s_m is the monthly JDR. Quarterly data are then built using the following formulas:

$$\begin{aligned}\bar{s}_m &= s_{m+2} + (1 - s_{m+2} - f_{m+2}) \cdot (s_{m+1} + (1 - s_{m+1} - f_{m+1}) \cdot s_m) \\ \bar{f}_m &= 1 - \bar{s} - (1 - s_m - f_m) \cdot (1 - s_{m+1} - f_{m+1})\end{aligned}$$

Quarterly rates s_q and f_q are retrieved, by taking the average over the quarter. Sources of data are given in appendix in Table 1.C.1.

Finally, as in Robin (2011), we use the BLS quarterly series of seasonally adjusted real output per person in the non-farm business sector to identify the parameters of the aggregate productivity process.

3.2 Data treatment

Raw data are first logged, then HP filtered with a smoothing parameter of 2.5×10^5 . The cycle component is then exponentiated, divided by its mean (so that the mean of the exponentiated cycle component is 1), and multiplied by the mean of the raw series.

More clearly, we note Y_t the raw data, Y_t^T and Y_t^C the trend and cycle components such that $Y_t = Y_t^T \cdot Y_t^C = Y_t^T \times (1 + g_t)$, where g_t is the percentage deviation of the raw data with respect to its trend. The logged series can be written $\log(Y_t) = \log(Y_t^T) + \log(Y_t^C)$. $\log(Y_t^C)$ is then extracted using the H.-P. filter and exponentiated in order to recover $Y_t^C = (1 + g_t)$. We get the series of interest y_t , by centering Y_t^C around the mean of Y_t , such that $y_t = Y_t^C \cdot \frac{1/S \cdot \sum_{s=1}^T Y_s}{1/S \cdot \sum_{s=1}^T Y_s^C}$. Results of the treatment are plotted in Figure 3.1.

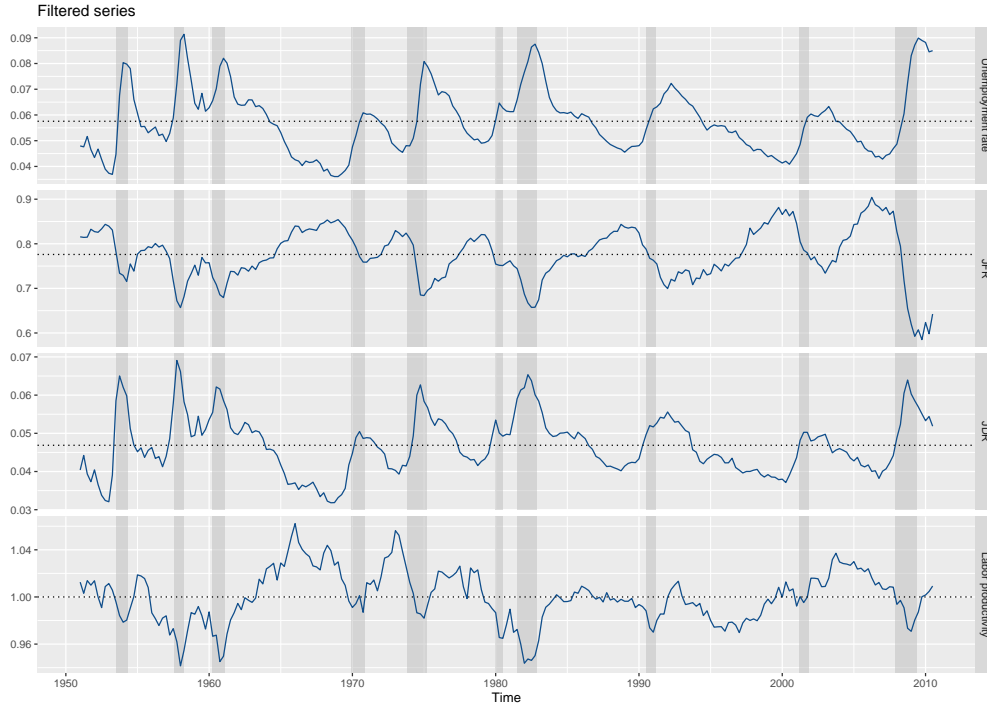
3.3 Asymmetries

It is worth noting that at quarterly frequency, the unemployment rate, JFR and JDR seems to exhibit an asymmetrical behavior, as one can see in Figure 3.2. More specifically, (i) positive deviations from trend unemployment rate and JDR are larger than negative ones (the so-called "deepness asymmetry") and (ii) the unemployment rate and JDR increase faster than they decrease (the so-called "steepness asymmetry"). JFR moves the other way around, as it decreases faster than it increases.

The significancy of the skewnesses of the variables of interest⁵ is tested using the Bai and Ng test [6]. Results are given in Table 3.1. The test fails to reject the null hypothesis of symmetry only for JFR, and confirms the asymmetrical pattern in first difference for all variables.

⁵Details of the test are presented in the appendix.

Figure 3.1: H.P. filtering of JFR, JSR, unemployment rate and productivity



Note: Dark vertical bands stand for NBER recessions. See the section "data treatment" of Section 2.1 for more details.

Table 3.1: Test of significancy of skewness

	u	f	s
Skewness	0.6373*	-0.5208	0.3911*
	[0.0576]	[0.2456]	[0.0802]
Skewness of first difference	1.5421**	-0.7299*	1.7812*
	[0.0294]	[0.0964]	[0.0793]

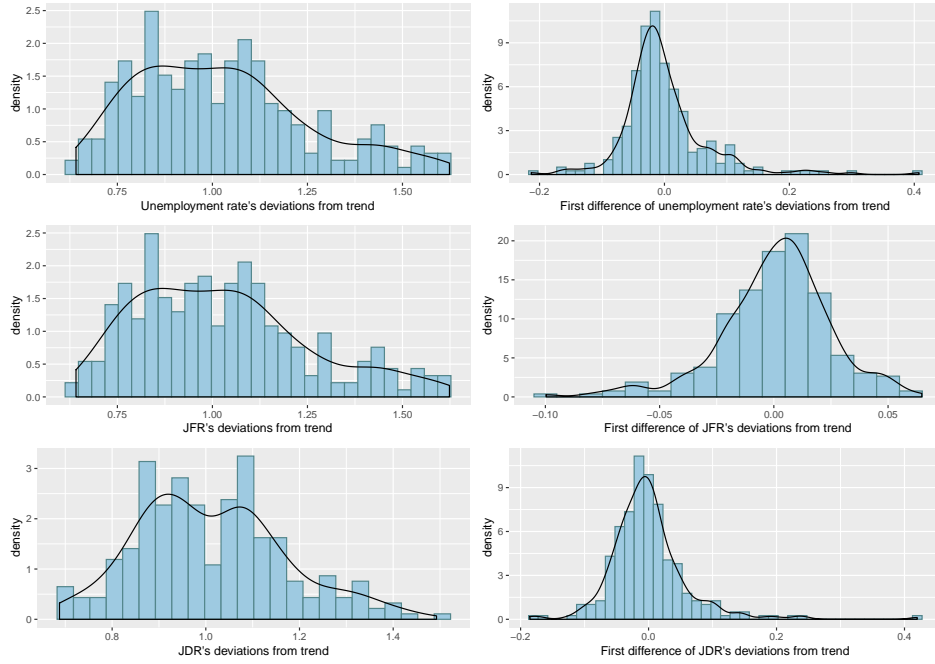
Note: all variables are in deviation from trend. P-values are in brackets.
 * refers to rejection at 10%, ** at 5%, *** at 1%. Variables are not tested jointly. Sample covers 1951Q1-2010Q4

3.4 The Simulated Method of Moments

The vector of parameters to estimate is $\gamma := (\sigma, \rho, \underline{x}, \eta, \mu, \delta, z_0, \omega)$. As the goal of the paper is to find a model that can replicate U.S. labor market fluctuations, and since the information contained in these can be summarized in moments of order up to three, the SMM of Duffie and Singleton (1993) [26] seems to be a natural candidate. To the layman, the essence of SMM can be explained as follows. The objective is to find the set of parameters that let the model generate time series whose moments are as close as possible from their empirical counterparts.

What moments can summarize the information contained in U.S. labor market fluctuations? The selected set contains 20 moments, as follows:

Figure 3.2: Distribution of deviations from trend of JFR, JDR and unemployment rate



Note: Kernel densities are estimated using standard Gaussian method on R [75]. In both histograms, relative deviation (one plus the deviation in percentage) with respect to the trend are plotted

- Mean of u_t , f_t , s_t and of \bar{y}_t .
- Standard deviation of u_t , f_t , s_t , v_t and of \bar{y}_t .
- AR(1) coefficients of u_t , f_t , s_t and of \bar{y}_t .
- Skewness coefficients of u_t , f_t and s_t .
- Skewness coefficients of Δu_t , Δf_t and Δs_t .
- Correlation between u_t and v_t .

Note that as in Robin (2011) [76], we use moments about labor productivity in the estimation, in order to consistently estimate the parameters governing aggregate shocks. An estimation that does not take into account the information contained in these moments might thus overestimate the size of aggregate shocks, as it would help fitting the high volatility of u , f , s and v . As by definition, aggregate productivity is unobserved, estimating its structural parameters σ and ρ can be done by building – observed – labor productivity series. The underlying aggregate shocks are thus the one which generates a labor productivity whose mean, standard deviation and auto-correlation coefficient are the closest to their empirical counterparts.

Details about the SMM estimation and statistical inference are provided in the appendix.

3.5 Calibration

c – the vacancy posting cost – is calibrated as follows. At the steady state, that we approximate by the median state i , the unemployment rate for each skill group is given by

$$u_t(m) = \begin{cases} 1 & \text{if } S_i(m) \leq 0 \\ \frac{\delta}{\delta + \lambda_0} & \text{if } S_i(m) > 0 \end{cases}$$

The total unemployment rate is thus given by

$$u = \frac{\delta}{\delta + \lambda_0} \sum_{m=1}^M \mathbb{1}_{S_i(m) > 0} l_m + \sum_{m=1}^M \mathbb{1}_{S_i(m) \leq 0} l_m$$

For a given set of parameters γ , the labor market tightness θ and the meeting rate λ_0 are the ones such that we have $u = 0.0576$. Once we get θ , by using the free entry condition we recover c .

Remaining parameters are fixed. α is set to 0.5 and the annual interest rate is 5%; as the model is simulated at a weekly frequency, we have $r = 0.05/52$. Using the results of Faberman *et al.* (2017) [29], we set the relative search effort⁶ ι at 0.12. We then identify the probability for the worker to move to the poaching firm upon meeting τ at 25% using the formula of the quitting rate:

$$q_t = \tau \cdot \iota \cdot \lambda_{0,t} (1 - \delta) \frac{\mathbb{1}_{S_t(m) > 0} (1 - u_t(m))}{1 - u_t}$$

Using data from the JOLTS, we estimate a quarterly quitting rate of 5.5%, which implies an exogenous transition probability of $\tau = 25\%$.

4 Results

Results of the estimation are in Tables 4.1, 4.2 (in appendix) and 4.3. In Table 4.1, the model's simulated weekly data are aggregated at both quarterly and monthly frequencies in order to make comparisons with the results of Robin's model which is simulated at a quarterly frequency, and of Lise and Robin's model, where simulated weekly are aggregated at a monthly one, as consistent as possible. Table 4.2 details goodness-of-the-fit of the model, presenting results of t-tests. Table 4.3 presents estimated parameters.

4.1 Fit of U.S. labor market fluctuations

The results are quite satisfying, especially given the relative simplicity of the model. All standard deviations of unemployment, JFR and JDR are well reproduced, a property that makes the performances of this model regarding its ability to replicate the fluctuations of the main U.S. labor market variables much

⁶They find that 22.4% of employed workers actively searched for a work between 2013 and 2017. The amount of applications they sent and the time they spent searching being approximately half of unemployed workers' ones, a relative search effort of 12% seems consistent. Note that this way to calibrate the search effort parameter leads to the same result as in Robin (2011) [76] or Murtin and Robin (2018) [68].

Table 4.1: Comparison with Robin (2011) and Lise-Robin (2017)

	Observed Monthly	My model Monthly	Lise-Robin (2017)	Observed Quarterly	My model Quarterly	Robin (2011)
Mean u	0.0585	0.0629	0.059	0.0576	0.0619	0.057
Sd($\log(u)$)	0.2174	0.1854	0.203	0.2118	0.1808	0.213
AR(1) of $\log(u)$	0.9889	0.9349	–	0.9582	0.7903	0.94
Skew(u)	0.6277	0.7557	–	0.6373	0.7449	0.69
Skew(Δu)	0.8073	0.9482	–	1.5421	0.2037	–
Mean f	0.4109	0.4469	0.468	0.7759	0.8019	0.753
Sd($\log(f)$)	0.1531	0.1647	0.127	0.0793	0.0808	0.220
AR(1) of f	0.8815	0.8978	–	0.9596	0.7892	0.95
Skew(f)	-0.0649	-0.6350	–	-0.5208	-0.9581	-0.96
Skew(Δf)	0.2081	0.1042	–	-0.7299	-0.0097	–
Mean s	0.0245	0.0292	0.028	0.0469	0.0523	0.0435
Sd($\log(s)$)	0.1120	0.1025	0.095	0.1595	0.1291	0.0573
AR(1) of $\log(s)$	0.8703	0.3185	–	0.9297	0.6402	0.02
Skew(s)	0.3938	2.0129	–	0.3911	0.5784	2.27
Skew(Δs)	0.4307	-0.0043	–	1.7812	0.3411	–
Mean \bar{y}	–	–	–	1	1.000	1
Sd($\log(\bar{y})$)	–	–	–	0.0221	0.0199	0.022
AR(1) of $\log(\bar{y})$	–	–	–	0.9135	0.7341	0.95
Sd($\log(v)$)	0.2182	0.0214	0.105	0.2104	0.0429	–
Correlation between $\log(v)$ and $\log(u)$	-0.8911	-0.5484	-0.9011	-0.975	-0.6746	–
Mean of E2E	0.0103 (0.025)	0.0154	0.025	0.055	0.0509	–
Sd of E2E	0.1467 (0.095)	0.0639	0.112	0.1449	0.0587	–
Sd of v/u	0.423	0.2207	0.306	0.423	0.2121	–

Note: Dataset for my estimation covers 1951-2010, just as Robin (2011), while the one of Lise-Robin (2017) covers 1951-2012. The exception is job-to-job transition rates, which cover 2001-2016. Data between brackets come from Lise-Robin. Moments in blue are the one targeted by the estimation. There are 12 degrees of freedom in my estimation. Data for the output *per capita* being only available at a quarterly frequency, I do not report simulated ones at the monthly one. Note that the difference between the monthly and quarterly unemployment rate only comes from the choice of frequency. E2E stands for "employment to employment" or job-to-job transitions.

closer to the ones of Lise-Robin than the ones of the 2011 search model. As one can see in Table 4.1, in Robin (2011) [76], the standard deviation of JDR was three times higher than in the data, while the one for JFR was 1/3 of its empirical counterpart. However, when this paper's model's simulated weekly data are aggregated at a monthly frequency, the fit of the volatility of u , f and s is close to the one of Lise-Robin. Note that the simulated volatility of vacancies and labor market tightness is too weak, such a result being also obtained by Lise and Robin (2017) [63] who explain it by the difficulty to measure job openings. Results concerning the simulated low volatility of job-to-job transitions must also be considered with caution: the empirical counterpart I chose is the quitting rate coming from the JOLTS data, with a sample covering 2001-2016; since the model is estimated on 1951-2010 data, comparisons are quite difficult to make. Note however that if one uses data from Lise-Robin⁷ (2017) [63] to compute the observed job-to-job transition rate, the simulated volatility becomes much closer to its empirical counterpart. From a somewhat more quantitative point of view, if one focuses on Table 4.2, at a 1% level, the simulated standard deviations of unemployment, JFR and JDR are statistically not different from their empirical counterparts. Note also that this result is obtained together with a simulated volatility

⁷They use data from CPS, covering 1994-2011

Table 4.2: Simulated moments and testing goodness-of-the-fit

	Observed	Simulated		Observed	Simulated
Mean of u	0.0576 (0.0016)	0.0619 (0.0039)	Mean of s	0.0469 (0.0010)	0.0523 (0.0000)
Sd($\log(u)$)	0.2118 (0.0161)	0.1808 (0.0522)	Sd($\log(s)$)	0.1595 (0.0125)	0.1291 (0.0145)
AR(1) of $\log(u)$	0.9582 (0.1475)	0.7903 (0.2543)	AR(1) of $\log(s)$	0.9297 (0.1471)	0.6402 (0.0473)
Skew(u)	0.6373 (0.4261)	0.7449 (0.7894)	Skew(s)	0.3911 (0.3834)	0.5784 (0.5804)
Skew(Δu)	1.542 (0.9763)	0.2037 (0.1703)	Skew(Δs)	1.781 (1.2935)	0.3411 (0.2646)
Mean of f	0.7759 (0.0080)	0.8019 (0.0003)	Mean of \bar{y}	1 (0.0029)	1.00 (0.0065)
Sd($\log(f)$)	0.07928 (0.0087)	0.0808 (0.8576)	Sd($\log(\bar{y})$)	0.0221 (0.0019)	0.0199 (0.2289)
AR(1) of $\log(f)$	0.9596 (0.2152)	0.7892 (0.4286)	AR(1) of $\log(\bar{y})$	0.9135 (0.1625)	0.7341 (0.2694)
Skew(f)	-0.5208 (0.5680)	-0.9581 (0.4167)	Sd($\log(v)$)	0.2104 (0.0136)	0.0429 (0)
Skew(Δf)	-0.7299 (0.5840)	-0.0097 (0.2169)	Corr($\log(u), \log(v)$)	-0.9011 (0.0184)	-0.6746 (0)

Note: In columns "observed", standard deviations are in brackets. In columns "simulated", p-values are in brackets. P-values of rejected moments – at 1% level – are in red. Global p-value is 0.

of labor productivity highly close – even at a 10% level – to the observed one; in other words, the realistic volatility of U.S. labor market fluctuations is obtained using aggregate shocks of a plausible size. Such a property comes from the combination of the "amplification effect" coming from job destruction and the pro-cyclical behavior of vacancies. When we couple aggregate shocks with workers that are heterogeneous in skills, even a small recession shock can increase the unemployment rate by a large amount, as it can suddenly make many jobs unproductive and thus provoke massive layoffs. In addition, during a recession (expansion), firms expect a lower (higher) return of vacancies, and post less (more) of them, which decreases (increases) the unemployment rate. As making firms heterogeneous does not change the fundamental intuition – in Lise and Robin (2017) [63], it is not low-skilled workers who are laid-off *per se*, but mismatched workers – the minimal basis to reproduce the volatility of labor market's fluctuations is indeed this model.

When data are aggregated at the quarterly frequency, deepness asymmetries – third order moments in level – are also well replicated. Skewness coefficients in level are close to their empirical counterparts – especially for unemployment and JDR – while signs of skewness coefficients in first difference are all in the right direction (Note also that statistical tests cannot reject these moments). It is the same job destruction mechanism that explains these asymmetries. Departing from the steady state, decreases of the unemployment rate following economic expansions can only be moderate, as they mainly come from job creations. It can indeed be clearly seen in Figure 4.2 that as only 1% of the population experiences a negative surplus at the median steady-state, an expansion that would make them productive cannot push down the unemployment rate by far. However, recessions can generate spikes in the unemployment rate from its steady state level, as endogenous job destructions can be quite massive. Figure 4.2

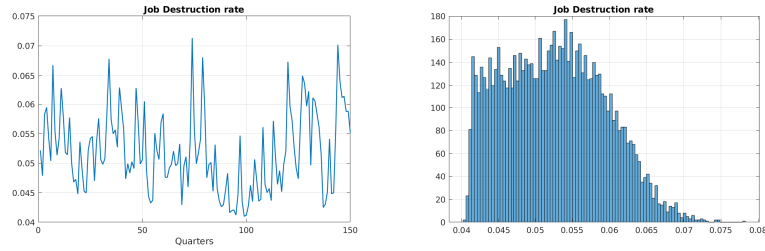
shows that if the economy switches from the median state to the worst one, 13% of the population will be endogenously laid-off. If the model is simulated at a quarterly frequency, such a separation mechanism leads to a JDR that exhibits plateaus on its lower bound: as the job destruction mechanism is triggered only a minority of periods, the JDR is equal to δ most of the time, which is at the origin of the high skewness and null auto-correlation coefficients found by Robin (2011) [76]. The solution is thus to simulate the model at a weekly frequency, in order to aggregate data at a much lower frequency. If one does so, JDR does not exhibit any plateaus anymore, and has therefore a more smoothed shape, as one can see in Figure 4.1. Such a way to generate asymmetries of a realistic size must not be seen as a "dirty trick", as it of course has an economic justification. Aggregate shocks indeed happen at an unknown infra-quarterly frequency: simulating them quarterly thus creates a discrepancy between data and the model. Following Lise and Robin (2017) [63] or Adjemian *et al.* [1] (2018), generating weekly aggregate shocks seems to be the best way to reconcile the model with the data.

Steepness asymmetries are qualitatively well replicated, in the sense that the sign of skewness coefficients are all consistent. Most importantly, the model reproduces the fast increases followed by slower decreases of the unemployment rate. This property comes from the fact that recessions lead to sudden endogenous job destructions, which makes the unemployment rate increase fast. Once the economy recovers, unemployed workers cannot find a job immediately because of search frictions, which makes the unemployment rate go down at a slower pace. In other words, the asymmetry between the speed at which matches are destroyed and created is responsible for the asymmetrical behavior of the unemployment rate in the U.S.

It is finally worth noting that the model is also able to reproduce a Beveridge curve, even if its slope is a bit weaker than in the data. Given the fact that this model is characterized by endogenous separation, the reader who has Fujita and Ramey (2012) [33] in mind might be surprised to see a negative correlation between vacancies and unemployment. In their paper, workers are characterized by an *ex post* idiosyncratic productivity: it is once they are matched that their skill – or the idiosyncratic productivity of the match – is drawn. If the aggregate shock is combined with a "bad" idiosyncratic productivity that makes the match unproductive, the worker becomes unemployed and his skill is reset. Therefore, during a recession, firms face a larger pool of unemployed workers, and can expect to have a productive match with each one of them: as it is now easier to fill a vacancy, firms open more positions, which ends up in a counterfactual Beveridge curve. In our model, the fact that laid-off employees' skill is not reset following a recession changes everything: even if firms face a larger pool of unemployed workers, they know that a match with a "low-skilled" one will be unproductive, which provides them a disincentive to open more vacancies: the Beveridge curve is in the right direction.

Is this model though the minimal representation of U.S. labor market fluctuations? Would it be possible to use an even simpler model, *e.g.* the search model of Robin (2011) [76], but estimated on a different set of moments? Appendix A shows that doing so decreases the performances of the model. Given the low cost of adding a matching function to the model and simulating it a weekly frequency, the model of Robin (2011) [76] augmented with homogeneous firms that open vacancies is thus a strong and consistent working basis to study labor market fluctuations – in the sense of the standard DMP literature, *i.e.* the analysis of the cyclical behavior of u , f and s – in the U.S.

Figure 4.1: Simulated series and distribution of JDR, weekly simulations



Note: Histogram at the right uses 10^6 periods, from which the first periods are discarded.

4.2 Estimated parameters

Table 4.3: Estimation of parameters

Parameter	Value
Beta distribution of abilities η	2.3581 (0.0061)
Beta distribution of abilities μ	11.2163 (0.0264)
Exogenous destruction rate δ	0.0070 (2.6454e-06)
Matching function parameter ω	0.4879 (0.0069)
Persistence of productivity ρ	0.9659 (3.6522e-05)
Unconditional std. of productivity σ	0.0228 (1.9456e-05)
Opportunity cost of employment z_0	0.8469 (0.0015)
Lower bound of productivity \underline{x}	0.8186 (4.4328e-06)
Vacancy opening cost c	0.9503

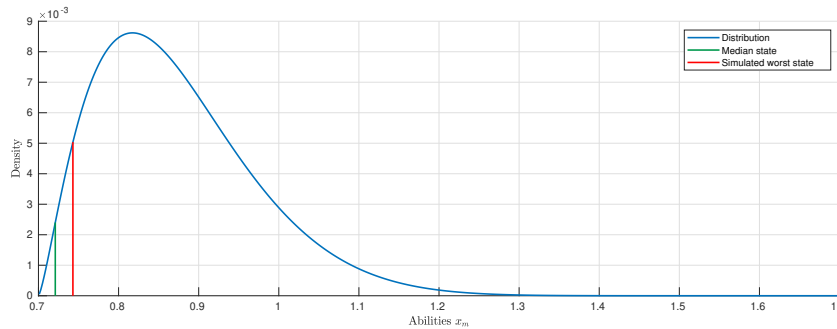
Note: the vacancy opening cost is calibrated. ρ is $0.96^{12} \approx 0.6$ at a quarterly rate. The aggregate productivity is discretized using the Rouwenhorst method (1995)[77] which performs better for high values of ρ (Galindev and Lkhagvasuren, 2010 [36]).

The estimated distribution of abilities is quite skewed as we can see in Figure 4.2, but highly close to the one estimated by Lise and Robin⁸ (2017) [63]. The estimated fixed component of the opportunity cost of employment z_0 is also higher with respect to what can be usually found in the representative agent literature (around 0.7, see for instance Hall and Milgrom, 2008 [42]). However, remember that this is an heterogeneous agents model: as a result, the ratio $\frac{z_i(m)}{y_i(m)}$ differs between individuals. For instance, the 10% less skilled workers – who experience endogenous job destruction – have a ratio of 0.9780 on average, while the ones of top 50% and 10% are respectively 0.7941 and 0.7273. As the individuals considered in the standard DMP literature correspond in this model to these high skill workers, an estimate of z_0 of 0.85 is consistent. Combined with a worker's skill's lower bound $\underline{x} = 0.81$, such an

⁸They find $\eta = 2.15$ and $\mu = 12.00$.

estimation implies that at the median steady-state, 1% of the population is unproductive – *i.e.* their matches surplus would be negative –, whereas at the worst state, it is 14% of the population that would be endogenously unemployed. In other words, an expansion from the median state can, in the best case scenario, lower the unemployment rate by only one percentage point, while a recession can make it increase by 13 percentage points. As we mentioned above, this asymmetry is at the origin of the right skewed distribution of the unemployment rate. Finally, the estimated value of δ is slightly lower than the one found by Lise and Robin (2017) [63].

Figure 4.2: Distribution of skills



Note: Individuals at the left of the green line are laid off during the median state, they represent 1% of the distribution. During the worst recession, individuals at the left of the red line experience a negative surplus, and represent 14% of the population.

5 Concluding remarks

This paper has shown how to reproduce U.S. labor market fluctuations using a relatively simple search and matching model based on Robin (2011) [76]. By just adding a basic job openings mechanism and simulating it at an appropriate frequency, this extension is able to match first to third moments of U.S. unemployment, job finding and destruction rates. If one focuses on the dynamics of these variables along the business cycle, its performances are close to the ones of the highly more complete model of Lise and Robin (2017) [63].

The simplicity of the heterogeneity of agents we considered in this model is at the same time its strong point and one of its limits. On one hand, it indeed let us reproduce in a simple way U.S. labor market fluctuations, and can thus be used a consistent working basis in a more complete model. One could for instance wish to look at the bigger picture and embed this model in a general equilibrium framework, or on the contrary to focus on some labor market flows along the business cycle and thus enrich the model with more details. On the other hand, this heterogeneity of workers is a somewhat abstract concept. For future work, it could therefore be interesting to put some more details in workers' heterogeneity – age related productivity, human capital *etc* – while keeping the model simple to have a better understanding of labor market fluctuations over the business cycle in the U.S.

Appendix

1.A Would an even simpler version work?

Do the imperfect replication of U.S. labor market fluctuations in Robin (2011) come from the chosen set of moments? Could we make it a working basis by simply picking another set of moments?

We follow exactly Robin’s code and estimation procedure, simulating the model at a quarterly frequency. Observed quarterly data are built from monthly ones picking the first month of each quarter⁹, as in Robin (2011) [76]. Simulated JFR and JDR are built¹⁰ as in the following formulas:

$$f_t = \lambda_0 \frac{\sum_{m=1}^M \mathbb{1}_{S_t(m) > 0} u_t(m) l_m}{u_t}$$

$$s_t = \delta + (1 - \delta) \frac{\sum_{m=1}^M \mathbb{1}_{S_t(m) \leq 0} (1 - u_t(m)) l_m}{1 - u_t}$$

Results of the estimations are presented in Table 1.A.1 and 1.A.2.

Table 1.A.2: Estimation of the search model, parameters

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Beta distribution of abilities η	2.048	1.915	1.715	1.824	2.689	1.969	2.214
Beta distribution of abilities μ	5.92	65.11	8.838	5.879	14.62	8.867	4.7848
Exogenous destruction rate δ	0.0425	0.04905	0.0407	0.04757	0.03638	0.02892	0.0445
Exogenous meeting rate λ_0	0.997	0.9926	0.7405	0.9896	0.9998	0.9985	0.9996
Persistence of productivity ρ	0.9288	0.6488	0.6312	0.62	-0.543	-0.1773	0.6096
Unconditional std. of productivity σ	0.0251	0.02295	0.02318	0.02365	0.02657	0.01754	0.0226
Opportunity cost of employment z_0	0.7692	0.9288	0.8854	0.8657	0.8579	0.8291	0.7289
Lower bound of productivity \underline{x}	0.7287	0.9302	0.8715	0.8426	0.8316	0.8125	0.6771

- Column (a) is the same under-identified estimation as in Robin, and delivers quite similar parameters. Just as in Robin (2011), even if it succeeds at replicating the volatility of u , it generates a standard deviation of f (of s) that is three times (the third of) its empirical counterpart.

⁹Some minor differences appear with respect to Robin’s observed moments, they might come from data revision from BLS, a different definition of the skewness estimator or changes in computer’s approximations.

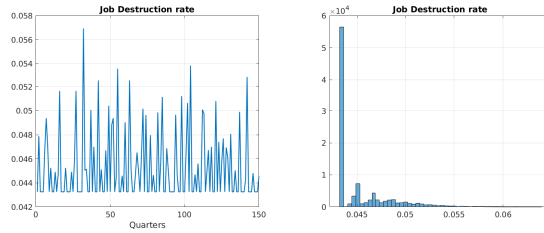
¹⁰Simulated data are not H.-P. filtered, as in Robin’s code \varnothing

Table 1.A.1: Estimation of the search model, simulated moments

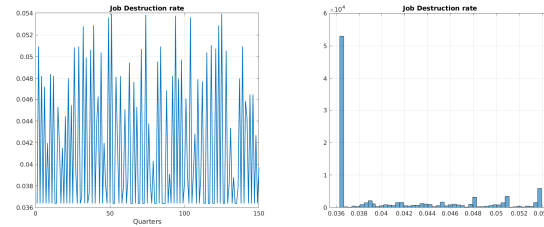
	Observed	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Mean u	0.05748	0.05773	0.0497	0.05886	0.05286	0.04295	0.03379	0.0553
Sd(log(u))	0.2133	0.213	0.166	0.1278	0.1237	0.148	0.1321	0.1004
AR(1) of log(u)	0.9525	0.9253	0.3473	0.6633	0.5793	-0.5089	-0.1728	0.5861
Skew(u)	0.6487	0.9034	6.836	1.166	0.6824	0.007021	0.06338	-0.2040
Skew(Δu)	1.297	-0.04617	-0.3121	0.4756	-0.05732	-0.01441	-0.02062	-0.0462
Kurt(log(u))	2.543	2.56	36.04	2.831	2.079	1.529	1.562	2.3054
Mean f	0.7766	0.7556	0.9828	0.6951	0.9023	0.9194	0.9127	0.7995
Sd(log(f))	0.07992	0.2141	0.09397	0.107	0.1097	0.1017	0.103	0.1027
AR(1) of f	0.9497	0.9317	0.4381	0.6744	0.6971	0.3079	0.3261	0.7151
Skew(f)	-0.5513	-0.1425	-8.173	-1.532	-0.8518	-0.8652	-0.7502	0.2986
Skew(Δf)	-0.6021	0.5684	1.236	0.5025	0.2262	0.1641	0.1343	0.3813
Mean s	0.04683	0.04435	0.0509	0.04302	0.0498	0.04104	0.03171	0.0464
Sd(log(s))	0.1613	0.0644	0.1324	0.09575	0.07001	0.1403	0.1153	0.0594
AR(1) of log(s)	0.9185	0.006541	0.07291	-0.04665	-0.111	-0.5437	-0.4217	-0.1551
Skew(s)	0.3891	2.044	7.899	2.309	1.866	0.9318	1.16	1.6245
Skew(Δs)	1.426	-0.0967	-0.2524	-0.06467	-0.02557	0.001835	0.0001764	0.0289
Mean \bar{y}	0.9999	0.9903	0.9592	1.035	1.081	0.9885	0.9954	1.0001
Sd(log(\bar{y}))	0.02211	0.02313	0.02316	0.0228	0.02313	0.02726	0.01782	0.0219
AR(1) of log(\bar{y})	0.9135	0.9148	0.6531	0.6102	0.5916	-0.5684	-0.217	0.5637

Note: Data in blue are the targeted ones. Only column (g) uses the optimal metric. Differences in observed data with Table 1.A.1 comes from differences in the way quarterly data are built. Here, they are built by picking the first month of each quarter while in other one, quarterly data are the average of monthly ones.

- Column (b) focuses on the standard deviation of all moments. It surprisingly ends up in a heavily skewed distribution of abilities, associated with a low persistence of the aggregate productivity process. Such a result can be explained by the fact that now the model has to lower the standard deviation of f and increase the one of s . Lowering ρ indeed leads to more rare endogenous job destruction, while increasing the weight put onto the low-skilled workers leads to more important endogenous job destruction. In other words, job destruction are rare but massive in this model. Such a property helps to increase the volatility of s , and lowers the one of u and f . Note also that z_0 and \underline{x} are much higher. A high \underline{x} with a high z_0 simply means that matches associated with the less skilled individuals can be destroyed only during the worst recessions. Such a mechanism however leads to large non-linearities (kurtosis and skewness of u are very high), that are therefore important to take into account.
- Column (c) and (d) take partly into account these non-linearities. Kurtosis of u is added in column (c): ρ stays low but the distribution is less skewed. The mechanism explained above (rare layoffs of a large part of the population) is weakened, which therefore ends up in a lowered volatility of u and s . In order to prevent a low unemployment rate (since endogenous layoffs are less important), the estimation identifies a low λ_0 . Column (d) takes into account the skewness of u . It identifies a low ρ for the same reason as before, and reduces even more the right tail of the distribution of ability, in order to limit the non-linearities.
- Column (e) adds skewness of u , f and s , both in level and first difference. Its result is highly surprising: the estimation indeed identifies a negative ρ . If one compares carefully Column (d)



(a) Simulated series and distribution of JDR, column (d) ($\rho > 0$)



(b) Simulated series and distribution of JDR, column (e) ($\rho < 0$)

Figure 1.A.1: Behavior of JDR, depending on the sign of ρ

and (e) of 1.A.1, it is clear that among the added moments, Skew(s) exhibits the largest difference (it is equal to 1.87 in column (d), 0.93 in column (e)). It is first worth noting the fact that the JDR is highly non-linear in this model: JDR hits its lower bound δ most of the time (when null or positive shocks occur), which therefore results in a mass point at the left of its distribution. If ρ becomes negative, aggregate productivity is more likely to have a reversing pattern, *i.e.* be above its average in t , below in $t + 1$, above in $t + 2$... As a result, the JDR is less likely to stay for several periods at its lower bound δ , and *in fine* will have a lower skewness. This difference in behavior can be seen in Figures 1.A.1a and 1.A.1b. When ρ is positive, there are more plateaus than when $\rho < 0$.

- Columns (f) and (g) tries to correct this mis-identification of ρ by including all AR(1) coefficients. When using a calibration identity matrix, ρ is still negative, as the estimation still over-valuates non-linearities. However, in the optimal matrix case, less weight is put onto third order moments, which are less accurately estimated. As a result, the estimation allows for a positive ρ associated with a highly skewed JDR. It is also worth noting that even after the introduction of all moments, the simulated series of JDR are non-persistent, for the same reason that they are too skewed: JDR being equal to its lower bound δ most of the time, the AR(1) coefficient is close to zero. As we have seen in previous sections, this result comes from a discrepancy between the frequency at which data are built/observed and the one at which aggregate shocks occurs. The only way to smooth the JDR is thus to increase the frequency of simulation and to aggregate data.

What have we learn from this battery of estimations? The imperfect replication of U.S. labor market fluctuations of the original search model simulated at a quarterly frequency does not come from a biased estimation of its parameters due to an insufficient set of moments. Even if we take into account all the information about volatility, asymmetries and auto-correlation, the model – among others – cannot generate enough volatility for the unemployment rate and JDR, and fails at replicating the asymmetric

behavior of unemployment. By adding a matching function and simulating at a weekly frequency, which comes with very little cost, the model performs almost as good as the one in Lise and Robin (2017) [63].

1.B SMM and statistical inference

1.B.1 The procedure

The estimation procedure is the following:

1. A vector of moments α , with $\dim(\alpha) = p$ and $\dim(\gamma) = q$ and such that $p > q$ is estimated from the quarterly dataset of length T presented in section 2 and containing u_q, f_q, s_q and the productivity of labor \bar{y}_q . In the estimation E3 I add v_q . This estimation is noted $\widehat{\alpha}_T$.
2. For a set of parameters γ , the model is simulated at a weekly frequency along $S = 52 \times 2000$ periods (*i.e.* 24000 months, which corresponds to a simulation factor of 30). Following Robin, I do not H.-P. filter the simulated data.
3. A vector containing the same moments as in step 1 is then estimated from the simulated dataset, noted $\widehat{\alpha}_S(\gamma)$.
4. The following criterion is computed:

$$J(\gamma) = [\widehat{\alpha}_T - \widehat{\alpha}_S(\gamma)]' W [\widehat{\alpha}_T - \widehat{\alpha}_S(\gamma)]$$

With W a symmetric non-negative weighting matrix.

The simulated method of moments (SMM) estimator $\widehat{\gamma}$ of γ is such that:

$$\widehat{\gamma} := \arg \min_{\gamma} [\widehat{\alpha}_T - \widehat{\alpha}_S(\gamma)]' W [\widehat{\alpha}_T - \widehat{\alpha}_S(\gamma)]$$

Steps 2 to 3 are repeated until the solution $\widehat{\gamma}$ to the minimization problem is found.

1.B.2 The optimal weighting matrix

Let us assume that $\sqrt{T} [\widehat{\alpha}_T - \alpha_0] \xrightarrow{d} \mathcal{N}(0, V)$, and note \widehat{V} the estimation of V . It can be shown that the optimal weighting matrix (*i.e.* the one that guarantees the lowest variance for $\widehat{\gamma}$) is

$$W^* := \left[\left(1 + \frac{T}{S} \right) V \right]^{-1}.$$

The intuition is that the moments that are the most precisely estimated – *i.e.* that have the lowest variance – will be given the largest weights. I use an estimation $\widehat{W}_T^* := \left[\left(1 + \frac{T}{S} \right) \widehat{V}_T \right]^{-1}$ of this weighting matrix when minimizing $J(\gamma)$. In practice, \widehat{V} is computed using the Newey-West estimator.

1.B.3 Computation of the variance of moments using GMM

The observed moments must be such that

$$\mathbb{E}[g(\zeta_t, \alpha_0)] = 0$$

Where $\zeta_t = (u_t, u_{t-1}, f_t, f_{t-1}, s_t, s_{t-1}, v_t, \bar{y}_t, \bar{y}_{t-1})$ is a vector of observations, and $g(\cdot)$ are the moment conditions. For instance, if one simply computes the mean of u_t , then $g(\zeta_t, \alpha_0) = \alpha_0 - u_t$. The moment α then must be such that $\mathbb{E}[g(\zeta_t, \alpha_0)] = 0$, or in practice $\hat{\alpha}_T - \frac{1}{T} \sum_{t=1}^T u_t = 0$, which is the formula of the mean. It can be shown that when T goes to infinity

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T g(\zeta_t, \hat{\alpha}_T) \xrightarrow{d} \mathcal{N}(0, S)$$

Where S is the asymptotic variance. It can then be show that that

$$\sqrt{T}[\hat{\alpha}_T - \alpha_0] \xrightarrow{d} \mathcal{N}(0, V)$$

Where $V = (D'S^{-1}D)^{-1}$, with $D = E \left[\frac{\partial g(\zeta_t, \alpha_0)}{\partial \alpha_0} \right]$ is the jacobian of moments condition with respect to the moments themselves. In practice, S is estimated using the Newey-West estimator:

$$\hat{S}(\zeta, \hat{\alpha}_T) = \Gamma_0 + \sum_{i=1}^I \frac{I+1-i}{I+1} (\hat{\Gamma}_i + \hat{\Gamma}_i')$$

Where $\hat{\Gamma}_i = \frac{1}{T} \sum_{t=i+1}^T g(\zeta_t, \hat{\alpha}_T)g(\zeta_t, \hat{\alpha}_T)'$. The chosen rule of thumb to pick up the maximum lag is the one from Greene (2003) [39], and gives $I = T^{\frac{1}{4}} \approx 4$.

1.B.4 Test statistics

When making use of the optimal weighting matrix W^* , it can be shown that when T goes to infinity,

$$\sqrt{T}[\hat{\gamma}_S - \gamma_0] \xrightarrow{d} \mathcal{N}\left(0, [D(\gamma_0)'W^*D(\gamma_0)]^{-1}\right)$$

where $D(\gamma) := \frac{\partial \hat{\alpha}_S(\gamma)}{\partial \gamma}$.

Since the number of targeted moments is larger than the number of parameters to estimate in both cases, an over-identification test *à la* Hansen (1982) [43] can be conducted. The test statistic is $J - stat := TJ(\hat{\gamma}) \frac{S}{1+S}$ and asymptotically follows a chi-squared distribution with $p - q$ degrees of freedom.

Also, to test the goodness-of-fit of each simulated moment one by one, I make use of the following statistic (see Tauchen, 1998 [84] or Collard *et al.*, 2002 [22])

$$\mu_{T,S} := \left\{ \text{diag} \left[\hat{V}_T - D(\hat{\gamma}) \left(D(\hat{\gamma})' \hat{W}_T^* D(\hat{\gamma}) \right)^{-1} D(\hat{\gamma})' \right] \right\}^{-\frac{1}{2}} \sqrt{T} (\hat{\alpha}_T - \alpha(\hat{\gamma}))$$

which is asymptotically distributed $\mathcal{N}(0, 1)$. Table 4.2 presents the targeted moments of the estimation associated with their standard deviation, as well as simulated moments, associated with the p-value of their t-statistic.

1.C Data source

Table 1.C.1: Description of data

Variable	Source
Unemployment level	BLS serie LNS13000000
Number of civilians unemployed for less than five weeks	BLS serie LNS13008396
Civilian employment level	BLS serie LNS12000000
Nonfarm Business Sector: Real Output Per Person	BLS serie PRS85006163
HWI index	Barnichon (2010)[7]

Chapter 2

Structural changes, labor supply heterogeneity and macroeconomic volatility

This chapter is based on material jointly developed with Laurent Brembilla, post-doctoral researcher at GAINS, Le Mans University.

Abstract

The way heterogeneous households are distributed in an economy plays a key role in the propagation of aggregate shocks, and can soften or amplify the reaction of hours worked to such disturbances. This chapter studies how changes in structural parameters in the economy – the *structural changes* – that influence the propagation mechanism through which aggregate shocks interact with the distribution of agents can impact the volatility of hours worked. It shows that the demographic aging, the increase in income inequalities and the tax/transfer reforms that the U.S. economy has experienced since the 1980's imply an increase of the volatility of hours worked over time. It also shows that these structural changes can explain the increase in the volatility of hours relatively to GDP during the Great Moderation.

1 Introduction

The major changes that has affected the U.S. economy's structure since the end of World War Two have not only modified its potential output or growth, but are also likely to have influenced the way it fluctuates. This conjecture is in particular at the heart of the discussion about the origins of the Great Moderation, *i.e.* the post mid-1980's period characterized – among other – by a drastic slowdown in the volatility of output and worked hours in developed economies – see Table 1.1. From this decades-lasting debate among macroeconomists, three causes to the Great Moderation were identified, as highlighted by Bernanke (2004) [9]: (i) the "good luck hypothesis", which attributes the Great Moderation to less volatile shocks¹, (ii) the "good policy hypothesis", which explains it through better conduct of macroeconomic stabilization policies², and (iii) structural changes that has softened the propagation of shocks in the economy³. There is however no reason to believe that *all* structural changes that has occurred in the U.S. – or developed nations in general – have contributed to the decline of macroeconomic volatility. If we indeed take a step back from the Great Moderation literature to look at the bigger picture, beyond the issue of finding structural changes responsible for a *decrease* in macroeconomic volatility, the third type of theories invites us to think more broadly about the way structural changes through which the U.S. economy has been since the 1960's have affected the way it reacts to shocks. That being said, a question that naturally arises is "what types of structural changes can one focus on?". The list of structural changes that have affected the U.S. economy during the post-war period is indeed quite long, and the channels through which they have influenced – or not – its cyclical behavior are likely to be numerous.

Among all structural changes that are likely to have changed U.S. economy's response to aggregate shocks, some of them share a common particular channel of transmission: those that interact with the distribution of heterogeneous households in the economy. As agents' responses in term of labor supply or consumption to aggregate shocks depend on their individual characteristics (their productivity, wealth, age, *etc*), any structural change that for instance have an impact on the composition of households can be expected to modify the economy's response at the aggregate level, through what we shall call the "distributional channel". Such an intuition making the link between the distribution of agents and aggregate macroeconomic variables is of course not new, and goes back to Keynes (1935) [57], who states that aggregate consumption should react to a variation of the distribution in real income, *i.e.* the distribution of marginal propensity to consume. More recently, the heterogeneous-agents macro literature have taken into account this view, whether by highlighting the differences between traditional New-Keynesian models with a representative agent and their counterparts which allow for heterogeneity (See for instance Kaplan *et al.*, [54]), or by doing the same exercise with real business cycle models, as in Krusell and Smith (1998) [62] or Krueger *et al.* (2016) [60]. In the framework of these papers, a structural change that affects the distribution of wealth also modifies the distribution of marginal

¹See Stock and Watson, 2002 [82]; Ahmed *et al.*, 2004 [3])

²See Clarida *et al.*, 2000 [21]; Blanchard and Simon [12])

³Among the candidates, research suggests better management of inventory (Kahn *et al.*, 2002 [53]) that helps firms to absorb shocks, or financial deregulation and innovation (Dynan *et al.*, 2006[27]).

propensities to consume, and therefore amplifies/softens the response of aggregate consumption along the business cycle. It is a similar intuition that allows us to think that such structural change can also lead to variations in the volatility of labor supply, measured by hours worked, through this same distributional channel. Depending on their age or wealth for instance, households do not take the same decisions in term of labor market participation and amount of worked hours. For instance, households who are eligible for retirement pension and who have accumulated enough wealth are likely to leave the labor market following a drop in real wage; any structural change leading to an increase in their share in the population might thus amplify fluctuations of hours worked.

It is inside this framework that the current chapter contributes to the business cycle literature, by shedding lights on structural changes that have modified fluctuations of hours worked through the distributional channel. More specifically, it studies how demographic aging, tax reforms and changes in income inequalities have affected the cyclical behavior of hours in the U.S., in absolute value and relative to output. These three structural changes that characterize the transition the U.S. economy has been following since the 1960's are indeed natural candidates when it comes to the distributional channel. As given in the example above, because it puts more weight on individuals that are close to retirement age, demographic aging increases the mass of agents potentially subject to labor market exit choice. Labor income tax reforms also interacts with the distribution of households and *in fine* the volatility of hours, by moving the level of wealth above which agents do not wish to work anymore, as put into evidence by Yum (2020) [51]. A tax reform that moves this threshold toward a region of the distribution where the mass of agents is higher (lower) thus increases (decreases) the amount of workers at the frontier between employment/non-employment and hence amplifies (softens) the response of hours worked to an aggregate shock. Finally, a change in wage inequalities, by mechanically reshaping the distribution of wealth, must modify the response of hours worked along the business cycle, as shown by Chang *et al.* (2019) [19].

To study the impact of these structural changes, this chapter uses an incomplete market with overlapping generation, where agents make consumption-saving and labor supply decisions – both at the extensive and intensive margin – calibrated on the post mid 1980's, to build a counterfactual of the mid 1960's- mid 1980's U.S. economy, where the only difference with its "Great Moderation era" counterpart (post mid-1980's) lays in (i) the distribution of income shocks with a lower variance, (ii) a tax-transfer system characterized by both a higher progressivity, lower transfers to seniors and a heavier tax burden, (iii) a younger demographic structure. It first decomposes one by one the impact of these structural changes on the volatility of hours, that all work through the "distributional channel". The lower volatility of income during this period is found to imply a standard deviation of hours that is higher than today: as a less spread distribution of wealth implies more individuals at the neighborhood of the wealth threshold above which households choose to stay non-employed, there are more flows in/out of the labor market and thus a higher volatility of hours. The tax/transfer system that characterizes the pre mid-1980's period differs in many ways from the modern one, as emphasized above: lower transfers to seniors or a higher tax level makes the volatility of hours decrease, while more generous transfers

to low-income households raise the standard deviation of hours. A younger demographic structure, by reducing the share of agents close to retirement age – who are subject to labor market leaving decisions and thus more sensible to the business cycle – is also found to imply a lower volatility of hours worked. It then put together all these structural changes, in order to build a counterfactual of the pre-Great Moderation U.S. economy. We find that they imply a standard deviation of hours that is 15% lower than in the post-1980’s era.

Because it actually predicts a counterfactual volatility of hours that *increases* from the pre-mid 1980’s period to nowadays, such a result highlights the importance of other types of structural changes that have occurred since then and which might explain the observed *slowdown* in volatility – the increasing share of women in labor force for instance, see Albanesi (2019) [4] –, but also the role played by the switch in technological shocks’ volatility in order to explain the Great Moderation. Our results also emphasizes the role played by structural changes that work through the distributional channel to explain another stylized fact that characterizes the Great Moderation, which is the increase in hours volatility relative to output (see Gali, 2009 [35] and Table 1.1): our counterfactual shows that the combination of tax/transfer reforms, demographic aging and higher income inequalities can completely account for this rise.

This chapter is organized as follows: Section 2 presents the model, and Section 3 the calibration of our benchmark economy, that is to say the post-mid 1980’s U.S. economy. Section 4 shows the results of our numerical experiments, and Section 5 concludes.

Table 1.1: The cyclical behavior of hours worked in the U.S.

	Standard deviation of hours (in log)	Ratio of standard deviations of hours over GDP (in log)
1965-1984	0.0168	0.9152
1985-2004	0.0109	1.1094

Note : Data are annual, and H.P. filtered with a smoothing parameter of 6.25. Data come from Bureau of Economic Analysis.

2 The model

2.1 Presentation

Time is discrete, and z_t denotes the state of the aggregate productivity shock at time t , that can take N values among the set $\{z_1, \dots, z_N\}$. At the beginning of each period, a new state of aggregate productivity is drawn according to a Markov chain, with transitions probabilities $\mathbb{P}(z_{t+1} = z_j | z_t = z_i)$.

The economy is lived by J overlapping generations indexed by $j \in \{1, \dots, J\}$, each generation being constituted from a continuum of households. At the beginning of each period, a household from the previous generation j either survives – and thus is part of the generation $j + 1$ – or dies. Death occurs

stochastically, as households face each period an age-dependant survival probability q_j , with $q_J = 0$. Households take utility from the consumption of good and leisure. The intrinsically dynamic nature of life cycle makes households intertemporally optimize their consumption of goods and leisure through saving/disaving decisions – which makes households indexed by their level of wealth a – and labor supply choices. If households hold positive wealth upon their death, the remaining assets are collected by the government.

On top of their age and level of assets a , households are also indexed by their idiosyncratic stochastic state of labor income. As it will become clearer in the following sections, households' labor income depend – among others – on a stochastic component, noted η . These shocks, coupled with the OLG structure of the model, generates heterogeneity, and thus a distribution of wealth, that we note μ : the amount of households aged j , with wealth a and idiosyncratic productivity η is given by $\mu_j(a, \eta)$. As the economy is hit by aggregate shocks, this distribution evolves, following a law of motion $\mu' = \xi(\mu, z)$. To summarize, the individual state-space is given by wealth a , idiosyncratic productivity η and age j , while the aggregate state-space is given by μ and z .

2.2 Households

Overview

Households experience utility from the consumption of goods and leisure. They are indexed by their generation j , their state of idiosyncratic productivity η , and, as they have access to a financial market with an interest rate r , their wealth a . They supply labor at both the extensive and intensive margin, *i.e.* they first chose whether they participate to the labor market or not, and, if they do so, the number of hours they work.

Labor supply and retirement decision

The amount of work households supply is assumed to be a discrete choice: households decide the number of hours l they work among a set \mathbb{D}_j . Households whose age is above J_2 are fully retired, in the sense that they cannot work anymore, *i.e.* $\mathbb{D}_j = \{0\}$ when $j \geq J_2$. Those whose age j is between age J_1 and J_2 (where $J_1 < J_2$) are retired, in the sense that they get a pension, but can keep working. Households whose age is below J_1 do not get pensions.

Labor income process

The labor income of a household is assumed to be made of three components: the wage wl , an age-dependent constant productivity ζ_j and a stochastic component η , such that the total pre-tax labor income is given by $wl\zeta_j\eta$. The natural logarithm of the stochastic process $\log(\eta)$, as usual in the literature, is assumed to follow an auto-regressive process of order 1, with standard deviation σ_η and persistence ρ_η .

Income taxation and transfers

Households get each period streams of market income through interest on their assets and labor income, which is subject to payroll taxes (Medicare and social security). The amount of paid payroll taxes is

aped. Their capital gains are also taxed at rate τ_k .

Households receive transfers of two kinds. The first one is a government guaranteed income floor \underline{y} (Medicaid among others), which is by nature progressive: if a household's disposable income is lower than \underline{y} , the government provides a transfer so that the household gets \underline{y} . On top of this income floor, we assume a second kind of transfers, also progressive, which is a decreasing function of the household's income.

The budget constraint of a household is thus as follows:

$$a' = a + T(a, y) - c(1 + \tau_c)$$

Where $T(a, y)$ is the after-tax income, τ_c being consumption tax rate. Labor income net of payroll taxes is given by

$$y = \begin{cases} w\eta\zeta_j l - T_S(w\eta\zeta_j l) & \text{if } j < J_1 \\ w\eta\zeta_j l - T_S(w\eta\zeta_j l) + B_S & \text{if } J_1 < j < J_2 \\ B_S & \text{if } j \leq J_2 \end{cases}$$

Where B_S is the pension benefit:

$$B_S = \theta \cdot e \text{ where } e \text{ is average labor earnings, } \theta \text{ being a replacement rate.}$$

The payroll taxes is defined as

$$T_S(w\eta\zeta_j l) = \tau_{med} w\eta\zeta_j l + \tau_{SS} \min \{ w\eta\zeta_j l, e_{\max} \}$$

e_{\max} being the maximum level of earning which can be subject to social security payroll taxes.

We approximate the after-tax income function $T(a, y)$, based on Bénabou (2002) [8], Heathcote *et al.* (2017) [47] or Kaymak and Poschke (2016) [56] by the following:

$$T(a, y) = \max \left\{ \left(1 - \tau^k\right) r a + \lambda \left(\frac{y}{e}\right)^{1-\tau_1}, \underline{y} \right\}$$

where e is still the average labor income in the economy. The parameter τ_1 measures the progressivity of the taxes/transfer system: a higher τ_1 means that the whole system is more progressive. The parameter λ controls for the level of taxes, a higher λ reflecting a lower level of taxes; such a tax/transfer function is known to be a good approximation of the U.S. tax/transfer system. If the post-tax income of an individual is lower than the government guaranteed consumption floor \underline{y} , the agents gets \underline{y} . Noting $T_0 = \lambda e^{\tau_1 - 1}$, we can rewrite the tax system under the form:

$$T(a, y) = \max \left\{ \left(1 - \tau^k\right) r a + T_0 (y)^{1-\tau_1}, \underline{y} \right\}$$

Preferences and optimization problem

We assume the instantaneous utility function to be CRRA, the trade-off between consumption of goods and leisure being summarized by a Cobb-Douglas function. Households experience disutility from participating to the labor market, this disutility being assumed to be age-dependent. In order to make the agent chose the amount of labor l he supplies, we finally assume the existence of non-observable random choice-specific taste-shocks $\sigma_\epsilon \epsilon(l)$ as in Iskhakov (2017) [50], where $\epsilon(l)$ follow a type 1 extremum value distribution, are *i.i.d.* and additively separable, σ_ϵ being the scale parameter. The optimal number of hours l is thus the one that solves the following:

$$V_j(a, \eta; \mu, z; \epsilon) = \max \left\{ V_j^l(a, \eta; \mu, z; \epsilon(l)) + \sigma_\epsilon \epsilon(l) \mid l \in \mathbb{D}_j \right\}$$

For an individual of age $j < J_1$ whose taste shock makes him work l hours, the problem is given by the following:

$$\begin{aligned} V_j^l(a, \eta; \mu, z) &= \max_{a', c} \frac{(c^{\alpha_L} (1-l)^{1-\alpha_L})^{1-\sigma}}{1-\sigma} - \mathbb{1}_{l>0} \chi(j) + \beta q_j \mathbb{E}_{\eta, z} V_{j+1}^{\sigma_\epsilon}(a', \eta'; \mu', z') \\ \text{s.t. } a' &= a + T(a, y) - c(1 + \tau_c) \\ y &= w\eta\zeta_j l - T_S(w\eta\zeta_j l) \\ T_S(w\eta\zeta_j l) &= \tau_{med} w\eta\zeta_j l + \tau_{SS} \min \{ w\eta\zeta_j l, e_{\max} \} \\ T(a, y) &= \max \left\{ (1 - \tau^k) r a + T_0(y)^{1-\tau_1}, \underline{y} \right\} \\ \chi(j) &= \chi_0 + \chi_1 j + \chi_2 j^2 \end{aligned}$$

$\chi(j)$ being the participation cost. Because of the assumptions made on the distribution of taste-shocks, we can write the expected value function as follows:

$$V_{j+1}^{\sigma_\epsilon}(a', \eta'; \mu', z') = \sigma_\epsilon \log \left(\sum_{l \in \mathbb{D}_j} \exp \left(\frac{V_{j+1}^l(a', \eta'; \mu', z')}{\sigma_\epsilon} \right) \right)$$

The probability for a given individual to work $l = \ell$ hours is given by the following:

$$\mathbb{P}_j(l = \ell | a, \eta) = \frac{\exp \left(\frac{V_j^\ell(a, \eta; \mu, z)}{\sigma_\epsilon} \right)}{\sum_{i \in \mathbb{D}_j} \exp \left(\frac{V_j^i(a, \eta; \mu, z)}{\sigma_\epsilon} \right)}$$

When the household is aged $J_1 \leq j < J_2$, the program is given by

$$\begin{aligned} V_j^l(a, \eta; \mu, z) &= \max_{a', c} \frac{(c^{\alpha_L} (1-l)^{1-\alpha_L})^{1-\sigma}}{1-\sigma} - \mathbb{1}_{l>0} \chi(j) + \beta q_j \mathbb{E}_{\eta, z} V_{j+1}^{\sigma_\epsilon}(a', \eta'; \mu', z') \\ \text{s.t. } a' &= a + T(a, y) - c(1 + \tau_c) \end{aligned}$$

$$\begin{aligned}
y &= w\eta\zeta_j l - T_S(w\eta\zeta_j l) + B_S \\
T_S(w\eta\zeta_j l) &= \tau_{med} w\eta\zeta_j l + \tau_{SS} \min \{w\eta\zeta_j l, e_{\max}\} \\
T(a, y) &= \max \left\{ (1 - \tau^k) r a + T_0(y)^{1-\tau_1}, \underline{y} \right\} \\
\chi(j) &= \chi_0 + \chi_1 j + \chi_2 j^2
\end{aligned}$$

Finally, when $j \geq J_2$, the program can be written

$$\begin{aligned}
V_j(a, \eta; \mu, z) &= \max_{a', c} \frac{(c^{\alpha_L})^{1-\sigma}}{1-\sigma} + \beta q_j \mathbb{E}_{\eta, z} V_{j+1}^{\sigma_\epsilon}(a', \eta'; \mu', z') \\
\text{s.t. } a' &= a + T(a, y) - c(1 + \tau_c) \\
y &= B_S \\
T(a, y) &= \max \left\{ (1 - \tau^k) r a + T_0(y)^{1-\tau_1}, \underline{y} \right\}
\end{aligned}$$

2.3 Firms

Firms use a standard Cobb-Douglas technology $Y = zK^\alpha L^{1-\alpha}$, where K is the total stock of capital, and L the total labor supply. Profit maximization lead to the usual conditions

$$\begin{aligned}
r &= \alpha \frac{Y}{K} - \delta \\
w &= (1 - \alpha) \frac{Y}{L}
\end{aligned}$$

δ being the depreciation rate.

2.4 Recursive equilibrium

In order to define the equilibrium, we need to describe the budget constraint of the government. We assume that it collects taxes under the form of consumption taxes, bequests, Social Security as well as Medicare payroll taxes and income taxes in order to run a PAYG pension system, provide transfers and make exogenous expenditures G , without any deficits or surpluses.

A recursive competitive dynamic equilibrium is a set of factor prices $r(\mu, z)$, $w(\mu, z)$, aggregate capital $K(\mu, z)$ and $L(\mu, z)$, household decisions rules $a_j^l(a, \eta; \mu, z)$, $c_j^l(a, \eta; \mu, z)$, probabilities $\mathbb{P}_j(l|a, \eta; \mu, z)$, government policy variables $\tau_c, \tau_k, \tau_{SS}, \tau_{med}, \tau_1, T_0, \underline{y}, B_S$, value functions $V_j^l(a, \eta; \mu, z)$, a distribution of households $\mu_j(a, \eta)$, and an aggregate law of motion $\mu' = \xi(\mu, z)$ such that at every period, noting x the individual state (a, η) :

1. Given factor prices $r(\mu, z)$, $w(\mu, z)$ and government policy variables, the decision rules associated to the value functions $V_j^l(a, \eta; \mu, z)$ are given by $a_j^l(a, \eta; \mu, z)$ and $c_j^l(a, \eta; \mu, z)$.
2. Given factor prices, the firm's demand of factor is given by $K(\mu, z)$ and $L(\mu, z)$, following the usual F.O.Cs.

3. Financial and labor markets clear:

$$\int \sum_j \sum_{\ell \in \mathbb{D}_j} a_j^{\ell}(x, \mu, z) \mathbb{P}_j(l = \ell | x; \mu, z) \mu_j(x) dx = K(\xi(\mu, z), z')$$

$$\int \sum_j \sum_{\ell \in \mathbb{D}_j} \mathbb{P}_j(l = \ell | x; \mu, z) \eta \zeta_j \mu_j(x) dx = L(\mu, z)$$

4. Government's budget constraint clears:

$$\underbrace{\int \int \sum_j (1 - q_j) a_j^{\ell}(a, \eta, \mu, z) \mathbb{P}(l = \ell | a, \eta, \mu, z) \mu_j(a, \eta) da d\eta}_{\text{Bequests}} + \underbrace{\tau_c \int \int \sum_j \sum_{\ell \in \mathbb{D}_j} c_j^l(a, \eta) \mathbb{P}_j(l = \ell | a, \eta) \mu_j(a, \eta) da d\eta}_{\text{consumption tax}} +$$

$$\underbrace{\int \int \sum_{j=1}^{J_1-1} \sum_{\ell \in \mathbb{D}_j} [ra + w\eta \zeta_j l - T(a, w\eta \zeta_j l)] \mathbb{P}_j(l = \ell | a, \eta) \mu_j(a, \eta) da d\eta}_{\text{Taxes net of transfers levied on households aged less than } J_1} +$$

$$\underbrace{\int \int \sum_{j=1}^{J_1-1} \sum_{\ell \in \mathbb{D}_j} [ra + w\eta \zeta_j l + B_S - T(a, w\eta \zeta_j l + B_S)] \mathbb{P}_j(l = \ell | a, \eta) \mu_j(a, \eta) da d\eta}_{\text{Taxes net of transfers levied on households whose age is between } J_1 \text{ and } J_2} +$$

$$\underbrace{\int \int \sum_{j=1}^{J_1-1} \sum_{\ell \in \mathbb{D}_j} [ra + B_S - T(a, B_S)] \mathbb{P}_j(l = \ell | a, \eta) \mu_j(a, \eta) da d\eta}_{\text{Taxes net of transfers levied on households whose age is } J_2 \text{ and over}} =$$

$$G + B_S \underbrace{\sum_{j=J_1}^J \int \int \mu_j(a, \eta) da d\eta}_{\text{Pensions expenditures}}$$

3 Calibration

The model is calibrated in order to match some characteristics of the post mid-1980's U.S. economy.

Demographics

We assume that agents are born at age 24, eligible to pensions at $J_1 = 65$ years old, exogenously leave labor market at $J_2 = 80$ years old and can live until 99 years old. Survival rates q_j from Kitao (2015) [58] are used, and we adjust the birth rate at 1% in order to match the ratio of 65+ individuals over 25-64 years old ones.

Earning process

The stochastic idiosyncratic productivity η is assumed to follow an autoregressive process of order 1 in log with a white noise of variance $\sigma_\eta^2 = 0.02$ and autocorrelation $\rho = 0.97$ (Kitao, 2015 [58]) approximated with a five-state, first-order discrete Markov process.

Preferences

In order to match the employment and hours profile by age in the PSID over the period, we calibrate the preference for consumption in the Cobb-Douglas α_L , coefficients in the disutility of labor function χ_0, χ_1, χ_2 , and variance of taste-shocks σ_ϵ . The life-cycle profiles of labor market participation costs and productivity are provided in Figure 3.1. The discount rate β is calibrated at 0.977 in order to get a capital to output ratio of 2.9. The resulting annual interest rate is 5.06%.

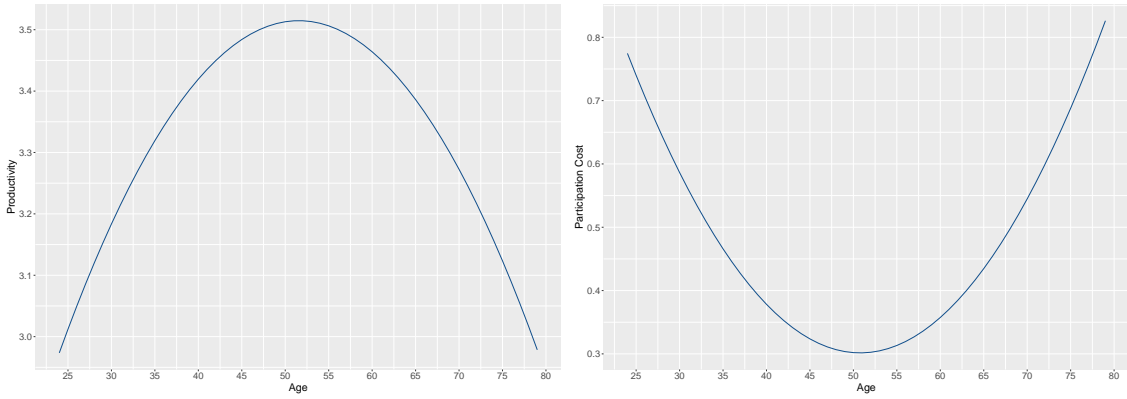


Figure 3.1: Age-related productivity ζ_j and participation cost $\chi(j)$

Note: The left panel shows the productivity profile over the life cycle, while the right one shows the labor market participation cost as a function of age.

Government

The consumption tax rate is $\tau_c = 0.05\%$ (Kitao, 2015 [58]), social security and Medicare tax rates are fixed at $\tau_{SS} = 10.6\%$ and $\tau_{med} = 2.9\%$, while the capital tax rate is $\tau_k = 13.2\%$. Guner et al. (2016) report that corporate taxes amount to 1.7% of GDP on the period 1987-2007. The corporate tax rate consistent with this value is 13.2% given the value of the capital to output ratio in the steady-state. Based on the estimations of Heathcote *et al.* (2017) [47] we set the tax progressivity parameter $\tau_1 = 0.181$. The level of taxes λ is calibrated at $\lambda = 1.33$ (which implies $T_0 = 1.048$) in order to ensure a share of taxes net of transfers and pensions in the GDP of 10%. The replacement rate θ , which defines transfers to seniors is set at 0.5, so that the ratio of pensions and Medicare transfers over GDP is about 7%, as in the mid 2000's. Following the literature (Zhao 2017 [89]), the government guaranteed consumption floor is set at 15% of the average labor earnings.

Production

As usual in the standard macro literature, we set $\alpha = 0.35$ and $\delta = 0.07$. The aggregate technological shocks are assumed to follow an autoregressive process of order 1 in logs with standard deviation of white noise $\sigma_z = 0.0135$ and persistence $\rho_z = 0.815$ (Yum, 2020 [51]) approximated with a three-state, first-order discrete Markov process.

Table 3.1: Parameters for the benchmark economy

Parameter	Description	Value
σ_z	Standard deviation of aggregate shocks's innovation	0.013
ρ_z	Persistence of aggregate shocks	0.8145
σ_η	Standard deviation of income shocks' innovation	0.141
ρ_η	Persistence of income shocks	0.97
α	Share of capital in output	0.35
δ	Depreciation rate	0.07
τ_c	Consumption tax rate	0.05
τ_k	Capital/corporate tax rate	0.132
τ_1	Income tax progressivity	0.181
τ_{SS}	Social security tax rate	0.106
τ_{med}	Medicare tax rate	0.029
β	Discount rate	0.977
λ	Income tax level	1.33
α_L	Share of goods in Cobb-Douglas preferences	0.65
σ_ϵ	Taste shocks	0.127
σ	Inverse of Elasticity of Intertemporal Substitution	2.52
B_S	Pension and Medicare	0.67
\underline{y}	Government guaranteed consumption floor	0.20

4 The Benchmark model

Before moving on to the analysis of the impact of structural changes that has occurred since the 1980's onto the volatility of hours and the employment rate, let us discuss the main properties of our model calibrated on the post mid-1980's U.S. economy, which we will refer to as the "benchmark model".

Table 4.1 presents the first and second moments of employment rates by age, and puts in evidence two stylized facts: labor market participation decreases with age, whereas its volatility increases along the life cycle. Note that this property can be seen in the data – in a somewhat more nuanced way –, as the volatility of 55+ employment rate is higher than its 25-54 counterpart. The decreasing employment rate comes from both the disutility of labor that is an increasing function of age, but also from the wealth effect: as households get older, they get richer and can thus decide to leave the labor market. After age 65 (where household get a pension), the employment rate shrinks, reflecting the decision to retire. The higher volatility of employment rate of seniors has the same origin: as households get wealthier, they get closer to the frontier between employment/non-employment, *i.e.* the level of assets above which agents decide to leave the market. Since there are mechanically more households which are in the neighborhood of this threshold in senior age-classes, aggregate shocks make enter/leave a higher amount of households. This wealth effect can also be seen in Table 4.4, where quartiles by age-classes are presented. As age increases, the median wealth gets higher, and so is the Q_3/Q_2 ratio, which reflects the creation of a wealthy senior upper-class. As one can see in Table 4.2, the employment rate volatility is indeed an increasing function of wealth, which is in turn highly correlated with age: it is not the fact that agents are older *per se* that explains the higher volatility of their labor market fluctuations (apart

from their stronger distaste for work and the fact that they get transfers without working) but the fact that they are wealthier.

Table 4.3 shows both hours worked (per capita and per worker) by age and quartile. Interestingly, the

Table 4.1: Employment rates by age class

Age class	Average (Model)	Volatility (Model)	Average (Data)	Volatility (Data)
25-54	0.884	0.0052	0.800	0.0052
55+	0.388	0.0132	0.302	0.0067
55-64	0.634	0.0083	-	-
65+	0.183	0.0282	-	-
Total (25+)	0.759	0.0062	-	-

Note: data are from BLS, and cover 1985-2004 at a yearly frequency. Data are logged and H.P. filtered. Volatility is the standard deviation in log.

standard deviation of hours worked increases with age, but exhibits a U-shape in the case of wealth, which is closely linked to age. These results come from the combination of two mechanisms. On the one hand, all things being equal, wealthier agents have a less volatile labor supply, as they are better insured against aggregate shocks. On the other hand, all things being equal, retirement age creates a horizon effect as in Gomme (2004): individuals who are close to retirement age are more prone to increase sharply their labor supply in response to a positive wage shock, as they will not be able to work more during the next periods. On the contrary, youngsters can smooth their increase in labor supply on several periods, and are thus less sensitive to wage rises; in other words, volatility in hours increases with age *stricto sensu*. As we have seen, wealth and age are linked, and these two effects thus play against each other. If we consider hours per worker by age, we can see that their volatility increases as agents get older, which indicates that the horizon effect dominates the wealth effect. However, if we look at the volatility of hours per worker by wealth, we see a U-shape curve: indeed, the wealth effect is "boosted" at the left of the distribution, which includes mostly youngsters but also households from other age-classes: the labor supply volatility is thus higher for households with $a < Q_1$ than for those with $Q_1 < a < Q_2$. This positive correlation between standard deviation of hours worked per worker and age can be found in the data⁴, as Table 4.3 shows.

Finally, the model is able to generate a standard deviation of hours relative to output that is quite close to its empirical counterpart: the ratio between standard deviation of hours and output is 1.109 between 1965 and 1984 and 1.002 in our benchmark simulation.

⁴Note that the dataset also includes data for 16-24 years old, whose volatility in worked hours per worker is the highest. This class-age is however not taken into account in our model.

Table 4.2: Employment rates by quartile

Quartile	Average	Volatility
$a < Q_1$	0.903	0.002
$Q_1 < a < Q_2$	0.859	0.004
$Q_2 < a < Q_3$	0.792	0.0077
$Q_3 < a$	0.5487	0.0078

Note: Volatility is the standard deviation in log.

Table 4.3: Volatility of hours worked by age and quartile

Age class	Per worker (Model)	Per worker (Data)	Quartile	Per worker (Model)
25-54	0.0010	0.0026	$a < Q_1$	0.0006
55-64	0.0011	0.0032	$Q_1 < a < Q_2$	0.0005
65+	0.002	-	$Q_2 < a < Q_3$	0.0008
Total	0.001	0.0026	$Q_3 < a$	0.0012

Note: Hours per capita are total worked hours divided by the number of individuals, while "per worker" reflect the intensive margin. Data come from Bick *et al.* (2019) [11], and cover 1985-2004 at yearly frequency.

Table 4.4: Quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}
24-29	1	1	0
30-39	0	4	0.14
40-64	0.17	2.17	0.86
65+	0.28	2.28	1

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the last column, they are expressed in relative value with respect to the median of the class age 65+. In the first line, all quartiles are actually equal to 0.

5 The impact of U.S. economy's structural changes since the 1960's on the volatility of hours and employment

5.1 Numerical experiment's presentation

Our numerical experiments aim at answering two questions. First, how can structural changes in the U.S. economy's fundamentals change the volatility of hours through the "distributional channel"? Second, if the only difference between the pre-mid-1980's U.S. economy and the post-mid-1980's one were these structural changes – as explained later in more details, a different demographic structure, tax/transfer system and variance of wage –, what would the cyclical behavior of hours have looked like fifty years ago? To answer these questions, we build a counterfactual U.S. economy for the mid 1960's-mid 1980's period, and only allow for parameters related to these structural changes to move

with respect to the benchmark post mid-1980’s economy.

The first step is of course to identify the main structural changes that had an influence on the volatility of hours through the distributional channel. We therefore choose to select three of them, based on the literature and empirical observations: (i) demographic aging, (ii) tax reforms and (iii) increasing wage inequalities the U.S. have been experiencing since the 1960s, which are well documented (see Heathcote *et al.*, 2010 [45] and 2010 [46] for instance). More specifically, we study the impact on hours volatility of four types of structural changes in the tax/transfer system. The first one is an increase in transfers to seniors (see Kaymak and Poschke, 2016 [56]), while the second one is a decrease in the progressivity of the U.S. tax system (see Wu, 2021 [86] for instance) . A third characteristic of changes in the tax/transfer system that happened in the U.S. since the 1970s is the decrease in the government guaranteed consumption floor y , as documented by Scholz *et al.* (2006) [78]. Finally, as reported by Wu (2021) [86], the overall tax level was higher in 1978-1980 than fifty years later. We first study one by one the impact of these variations on the labor supply’s volatility – *i.e.* move one parameter keeping all the others constant and simulate the model – before building a counterfactual for the pre-mid-1980s period, where technological shocks have the same magnitude as in the post-mid-1980’s period.

5.2 Analysis of isolated structural changes

5.2.1 Demographic structure

How would our benchmark economy react if we introduce the demographic structure of the pre mid-1980’s period? To adapt the demographic structure, we include the survival rates of 1975⁵. The shorter life expectancy associated with the new demographic structure depresses the saving motive along the life cycle. As a result, the distribution of wealth within age-class 40-64 and 65+ become quite similar. The median wealth of 65+ is also 14% smaller than in the benchmark case, indicating a lower amount of assets at the aggregate level. Lower savings – that is to say a lower supply of capital – *in fine* end up in a higher rate of return of capital – by 0.7 percentage point –, and a wage drop of 3%.

Table 5.1: Demographic aging : quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.2	0.17
40-64	0.2	2.2	1	0.86
65+	0.2	2.2	1	0.86

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

⁵Data come from the Human Mortality Database. The growth rate of the population stays the same, as the generated ratio of 65+ over 25-64 perfectly matches its empirical counterpart of 23%.

Table 5.2: Demographic aging: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.884	0.0050	0.884	0.0052
55+	0.418	0.012	0.388	0.0132
55-64	0.611	0.0070	0.634	0.0083
65+	0.200	0.0277	0.183	0.0282
Total (25+)	0.770	0.0057	0.759	0.0062

Inside age-classes, demographic aging does not change significantly the behavior of households, as employment rate's means and standard deviation are more or less the same. However, this new demographic structure puts less weight on senior households, whose employment rate is the more volatile as we have seen in the previous section. From this simple composition effect, the aggregate employment rate's standard deviation decreases slightly. The volatility of hours worked per worker is almost

Table 5.3: Demographic aging: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0012	0.0011
65+	0.0017	0.0018
Total	0.0009	0.001

insensitive for all class-ages to the change in demographic structure, as one can see in Table 5.3. The same composition effect as in the case of employment rate thus make its volatility at the aggregate level go down by a very small amount. The combination of a lower standard deviation of the employment rate and of a highly slight reduction of hours worked per worker's volatility thus ends up in a drop of total hours worked, which goes from 0.00687 to in the benchmark case to 0.00641, that is to say a reduction of 7%. In other words, a younger demographic structure implies a lower volatility of hours, in absolute value. Relative to output, the standard deviation of hours also slightly decreases as the demographic structure becomes younger, going from 1.002 in the benchmark case to 0.980. The younger demographic structure we use implies a slightly higher productivity of hours, since it put more weight on middle-aged workers – who are at their peak productivity – and reduces the share of 65+ years old workers, who are less productive. As a result, fluctuations in hours that are more productive therefore have more impact on GDP volatility.

5.2.2 Wage inequalities

As documented by the literature (see for instance Heathcote *et al.*, 2010 [45] and 2010 [46]), the post 1960's period is characterized by an increasing trend in wage inequalities. As a variation in the distribution of earning shocks changes mechanically the distribution of wealth, a higher level of wage

inequalities have an impact on the volatility of labor supply, through the distributional channel, as shown by Chang *et al.* (2018) [19]. We thus use estimates from Heathcote *et al.* (2010) [46] of the parameter σ_η , and choose⁶ $\sigma_\eta^2 = 0.092$. As Table 5.4 shows, distribution of wealth is more compact in each

Table 5.4: Lower wage inequalities : quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	2	0.2	0.23
40-64	0.43	2	0.7	0.82
65+	0.4	1.8	1	1.17

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

age-class. Since the earning process is less volatile, there is simply fewer highly wealthy households, which decreases the total stock of capital, leading to a drop in wage of 2% and a higher interest rate by 0.3 percentage points. As Table 5.5 shows, lower wage inequalities lead to significant changes in the

Table 5.5: Lower wage inequalities: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.887	0.0058	0.884	0.0052
55+	0.401	0.014	0.388	0.0132
55-64	0.646	0.0088	0.634	0.0083
65+	0.196	0.0283	0.183	0.0282
Total (25+)	0.767	0.0066	0.759	0.0062

behavior of the employment rate, raising its standard deviation by 6%, a result consistent with Chang *et al.* (2019) [19]. Prime-aged worker (25-54) are the main driver of this increase in the volatility of the overall employment rate, as they are the ones whose employment rate's volatility increases the most. Because it reduces the distribution of wealth's dispersion, a drop in wage inequalities increases the amount of individuals who are in the neighborhood of the participation/non-participation threshold for this age-class, *i.e.* the level of wealth above which 25-54 years old agents choose to leave the labor market. For the sake of comprehension, Figure 5.1 provides graphically the intuition, by presenting two examples of distributions of wealth, one with a high σ_η and another with a lower one. In the case of the high (low) variance, the greater (smaller) dispersion simply translates into a lower (higher) mass of individuals at the threshold between employment and non-employment, and thus into less (more) volatility of the employment rate. As one can see in Table 5.6, the intensive margin does not seem to be particularly hit by a lower level of wage inequalities, as the volatility of hours worked per worker in

⁶We take the average of their estimation of σ_η^2 over the sample 1969-1974

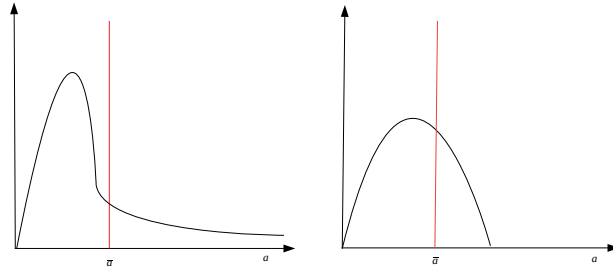


Figure 5.1: An illustration of the impact a lower σ_η has on the distribution of wealth

Note: at the left panel is presented a high σ_η case, at the right one a low σ_η . The red vertical line represents \bar{a} , the level of wealth above which the individual chooses not to work.

Table 5.6: Lower wage inequalities: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0012	0.0010
55-64	0.0014	0.0011
65+	0.0018	0.0018
Total	0.0011	0.001

each age-class is highly close to its counterpart in the benchmark. We therefore conclude that a lower σ_η , summarizing the late 1970's level of wage inequalities, increases the volatility of total worked hours by 9% (0.0075 in this case, and 0.00687 in the benchmark case) through the extensive margin. A lower variance of income shocks also leads hours to become more volatile relatively to output – the standard deviation increases from 1.002 to 1.070, *i.e.* a 7% increase –, in other words, GDP is less sensible to fluctuations in labor supply. This last result simply comes from the fact that the support of the distribution of productivity is less spread out.

5.2.3 Higher consumption floor

Since 1974 where it was at its peak, the consumption floor followed a decreasing trend to the 1990's (Scholz *et al.*, 2006 [78]). Using the estimation of the consumption floor's upper bound from Zhao (2017) [89], we estimate \underline{y} in 1974 to be such that $\underline{y} = 0.2 \cdot e$, 20% of the average labor income in the economy. Implementing a higher consumption floor at first sight does not change the global shape of wealth distribution (see Table 5.7), but increases substantially the volatility of the employment rate by 8%, while keeping the standard deviation of hours worked by worker almost unchanged (see Tables 5.8 and 5.9). A higher government guaranteed consumption floor decreases the level of wealth over which poorest agents choose to no longer participate to the labor market: as the income they can get without working increases, some individuals – the ones who were close to the threshold between employment/non-employment – leave the labor market, which ends up in a lower employment rate. As there are more individuals around the new threshold, aggregate fluctuations increase flows in and

out of the labor market: the volatility increases. Figure 5.2 provides a graphical intuition, by showing how lowering wealth threshold makes more agent subject to participation/non-participation choices ⁷. Note that this increase can be particularly seen in the 25-54 years old age-class: since poorest agents – *i.e.* agents who are eligible to the government provided consumption floor – are concentrated in this age-class, they mainly drive the increase in extensive margin’s volatility. The other component of total hours, *i.e.* hours worked per worker stays quite insensitive to the increase in consumption floor. The combination of these two effects ends up in a significant increase of the volatility of total hours following a higher consumption floor, which goes up from 0.00687 to 0.00734, *i.e.* a 7% increase. The volatility of hours relative to output also increases from 1.002 to 1.043, indicating a lower sensitivity of GDP with respect to hours’ fluctuations. This comes from the fact that a higher consumption floor especially impact low productivity households, whose labor income is modest, and tend to make some of them – who previously made the choice to work – stay non-employed. As a result, there are more of these low-productivity workers at the frontier between employment/non-employment choice. Agents’ flows in/out of employment thus have a soften impact on output.

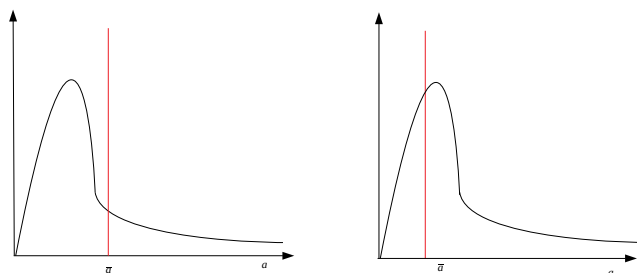


Figure 5.2: The mass of individuals subject to participation choice and transfer rise

Note: at the left panel is presented a low threshold \bar{a} case (higher consumption floor), at the right one a high \bar{a} . The red vertical line represents \bar{a} .

Table 5.7: Higher consumption floor : quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.14	0.14
40-64	0.2	2.6	0.71	0.71
65+	0.29	2.29	1	1

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

⁷See also Yum *et al.* (2020) [51] for an analysis of the role played by higher transfers on business cycles

Table 5.8: Higher consumption floor: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.875	0.0057	0.884	0.0052
55+	0.387	0.0135	0.388	0.0131
55-64	0.630	0.0089	0.634	0.0083
65+	0.185	0.0277	0.183	0.0282
Total (25+)	0.753	0.0067	0.759	0.0062

Table 5.9: Higher consumption floor: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0011	0.0011
65+	0.0018	0.0018
Total	0.0009	0.001

5.2.4 Lower transfers to seniors

As highlighted by Kaymak and Poschke (2016) [56], transfers to seniors (pensions and Medicare) started to rise from the 1970's, a stylized fact we take into account by calibrating B_s to $0.3 \cdot e$ in order to match the Social Security and Medicare expenses over GDP ratio of 1970, which was 3.7%.

Lower pension benefits create an incentive to save for retirement. As a result, households accumulate more wealth along their life cycle. As one can see in Table 5.10, this saving motive generates more inequalities between class ages, in the sense that the gap between 65+ years old and youngsters/prime-aged is wider. At the same time, the distribution of wealth inside given class-ages changes. Notably, there are less households at the lower bound of assets: all quartiles for 24-29 were at the lower bound in the benchmark, while households between Q_2 and Q_3 hold strictly positive wealth. Medians for 24-29, 30-39 and 40-64 stay close to the benchmark case (compare last columns of Table 5.10 and Table 4.4). The wealth of 65+ increases quite substantially, their median wealth being twice higher than its benchmark counterpart. At the macroeconomic level, a higher supply of capital translates into a fall of the natural rate of interest, which is now 4.4%— *i.e.* a 0.6 percentage point fall —, and an increase of wages of 2%, which helps agent at the lower bound to accumulate some assets.

At the aggregate level, lower pensions (*i*) increase employment rate, through an increase of seniors' labor supply, and (*ii*) decrease its volatility, as shown by Table 5.11. The first result comes from a wealth effect: as seniors experience a loss of income, the threshold employment/non-employment increases, *i.e.* seniors need to accumulate more assets by working more years than in the benchmark case to be better-off once they retire. As mechanically fewer of them are able to do so, the employment rate increases. The second one is the conjunction of two mechanisms. On the one hand, the volatility

Table 5.10: Lower transfers to seniors: quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	$+\infty$	0	0
30-39	0	2.5	0.14	0.29
40-64	0.98	2.43	0.5	1
65+	0.43	1.79	1	2

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, both Q_1 and Q_2 are equal to 0, while Q_3 is equal to 0.62. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

Table 5.11: Lower transfers to seniors: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.9107	0.0046	0.884	0.0052
55+	0.53	0.0096	0.388	0.0132
55-64	0.711	0.0064	0.634	0.0083
65+	0.369	0.0142	0.183	0.0282
Total (25+)	0.835	0.0056	0.759	0.0062

of all employment rates decreases, especially the one of seniors. This phenomenon comes from the higher threshold between employment/non-employment lower pensions create: younger individuals have to work more in order to save for their retirement, just as older ones prefer to postpone their retirement. Since there are less individuals at the neighborhood of this threshold in the distribution, there are mechanically less movement in/out of the labor market. As a result, the volatility of the employment decreases. On the other hand, a composition effect pushes the aggregate employment rate's standard deviation up, as the share of high-volatility households (the seniors, despite their lower standard deviation with respect to the benchmark case) in the total labor force increases. As the first effect is stronger, the global employment rate's standard deviation decreases. Since there is almost no adjustments in the intensive margin's volatility, the standard deviation of total hours (the combination of both the extensive and intensive margin) decreases when pensions are lower, from 0.00687 to 0.00639, *i.e.* a drop of 7%. Relatively to output however, lower transfers to seniors do not change the volatility of hours, as it stays highly close to its value in the benchmark case (1.002 in the benchmark, 0.991 in the numerical experiment).

5.2.5 Higher progressivity of tax/transfer system

Empirical evidences seem to suggest that the progressivity of the U.S. tax/transfer system has been declining since the 1970's (see for instance Wu, 2021 [86] or Kaymak and Poschke, 2016 [56]). Using data from the Congressional Budget Office (C.B.O.) as in Heathcote *et al.* (2010) [47] or (2020) [48], we

Table 5.12: Lower transfers to seniors:
volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0011	0.0011
65+	0.0012	0.0018
Total	0.0009	0.001

indeed estimate⁸ a slightly higher progressivity during the 1979-1984 period, with a shape parameter $\tau = 0.2$. As two last columns of Table 5.13 suggests, a change in progressivity of this magnitude is not

Table 5.13: Change in progressivity : quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.14	0.14
40-64	0.16	2.16	0.86	0.86
65+	0.28	2.28	1	1

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

enough to modify significantly the distribution of wealth, and thus the cyclical behavior of employment or hours worked, at both the aggregate and within age class level. As one can see in Tables 5.14 and 5.15, the small increase in the tax system progressivity does not has any significant impact on the volatility at both the intensive and extensive margin. As a result, the standard deviation of total hours stays almost the same (0.00672 in this simulations, vs 0.00687 in the benchmark case). Similarly, relatively to output, the slightly higher progressivity of the tax/transfer system that characterizes the pre mid-1980's U.S. economy does not imply a substantial change in volatility of hours relative to output, as the ratio of standard deviation decreases from 1.002 in the benchmark case to 0.99.

5.2.6 Higher income taxes

As emphasized by Wu (2021) [86], there has been a reduction in tax level since 1978, which translates into a higher value of λ . Calibrating our model to fit a ratio of non-transfers related government expenses over GDP of 15% as in the data, we find a parameter λ for the pre 1980's period of 1.15, which thus implies an increase of the overall level of taxes, our previous calibration leading to value of 1.32. The impact of higher tax level mainly goes through a classical wealth effect, as one can see in Table 5.17, where younger agents seem to be the most sensible. Some of them in the benchmark model choose not

⁸We follow their procedure, an remove medicaid as well as medicare from transfers. Note that estimations on this dataset over the sample 1985-1990 period leads to $\tau = 0.181$, exactly as in Heathcote *et al.* (2010) [47], who mainly use data from PSID. Wu (2021) [86] estimate a parameter of 0.18 for 1978-1980.

Table 5.14: Change in progressivity: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.886	0.0059	0.884	0.0052
55+	0.386	0.0130	0.388	0.0132
55-64	0.634	0.0082	0.634	0.0083
65+	0.176	0.0286	0.183	0.0282
Total (25+)	0.759	0.0060	0.759	0.0062

Table 5.15: Change in progressivity: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0011	0.0011
65+	0.0018	0.002
Total	0.0009	0.001

Table 5.16: Higher income tax level: quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.13	0.14
40-64	0.33	2.17	0.75	0.86
65+	0.2	2	1	1.14

Note: In the three first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

to participate to the labor market, in particular those who have succeeded at accumulating some assets. Such agents therefore make an intertemporal arbitrage, consuming leisure while their productivity (both age-related and stochastic) is low, and expecting to potentially go back on the labor market upon an improvement of the labor income they can pretend to. An increase in labor income taxes distorts this arbitrage: participating to the labor market after an increase in productivity will be less rewarding. Following this decrease in the expected labor income along the life-cycle, some of these non-employed agents thus choose to switch their employment status. In other words, the wealth frontier between employment/non-employment goes up, and establishes in a region of the distribution where less households are in its neighborhood. As we have seen before, such a translation of the participation choice threshold implies a decrease in hours worked volatility. In this case, the total amount of hours worked volatility decreases by almost 20%. Interestingly, the ratio of standards deviations between hours and output also drops from 1.002 in the benchmark case to 0.782: GDP is more sensitive to fluctuations of

hours when the income tax level is higher. Such a drop occurs because higher taxes actually hit young high-productivity workers the most, making more of them entering the labor market. As a result, the level of productivity of agents who are at the frontier between employment-non-employment increases in average, and thus fluctuations around this threshold have a greater impact on output.

Table 5.17: Higher income tax level: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.927	0.0035	0.884	0.0052
55+	0.415	0.0125	0.388	0.0131
55-64	0.665	0.0072	0.634	0.0083
65+	0.205	0.0280	0.183	0.0282
Total (25+)	0.799	0.0048	0.759	0.0062

Table 5.18: Higher labor income tax: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0012	0.0011
65+	0.0018	0.0018
Total	0.0009	0.001

5.2.7 Lower payroll taxes

The last structural change we investigate is an increase in payroll tax rates, as the one the U.S. have been experiencing. Lower payroll taxes that characterize the 1960's-1980's period work symmetrically with respect to the higher income tax case. They translate into a higher disposable income all along the life-cycle for households, leading to a slight increase in the median wealth of seniors with respect to the benchmark case, as Table 5.19 shows. Because payroll tax cuts imply an increase in lifetime

Table 5.19: Lower payroll taxes: quartile of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.13	0.14
40-64	0.17	2.33	0.75	0.86
65+	0.25	2.13	1	1.14

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

labor income, some of the 22-54 agents make an intertemporal labor allocation arbitrage: they choose to delay their (re)-entry in the labor market, expecting to start working later at a higher wage rate and a better productivity. This leads to a small decrease of the employment rate among the 25-54 years old group, but also to a slight rise in its volatility: because the wealth- threshold between employment/non-employment has decreased down to a region where the mass of individuals is higher, business cycle mechanically makes more agents leaver/enter the labor market, as in Section 5.2.3. As in all previous cases, the elasticity of hours worked per worker keeps being low, as both aggregate and age-class standard deviation stay unchanged. The increase of the employment rate's volatility eventually leads to a slight rise in the standard of total hours worked of 4.5%. Relatively to GDP, the standard deviation of hours also rises when payroll taxes are lower, going from 1.002 in the benchmark case to 1.05.

Table 5.20: Lower payroll taxes: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.871	0.0056	0.884	0.0052
55+	0.389	0.0132	0.388	0.0131
55-64	0.631	0.0085	0.634	0.0083
65+	0.187	0.0273	0.183	0.0282
Total (25+)	0.751	0.0065	0.759	0.0062

Table 5.21: Lower payroll taxes: volatility of hours worked by age

Age class	Experiment	Benchmark
25-54	0.0010	0.0010
55-64	0.0011	0.0011
65+	0.0018	0.002
Total	0.0009	0.001

5.2.8 To summarize

Table 5.22 summarizes the previous results.

Table 5.22: Summarizing the impact of structural changes on total hours' volatility

Parameter	Benchmark value	Pre-mid 1980s value	Impact on standard deviation of hours	Impact on ratio of hours and GDP's standard deviations
Younger demographics	–	–	-7%	-2%
Income shocks σ_η	0.141	0.096	+9%	+7%
Consumption floor \underline{y}	$0.15 \times e$	$0.2 \times e$	+7%	+4%
Benefits to seniors B_S	$0.5 \times e$	$0.3 \times e$	-7%	-1%
Progressivity τ_1	0.181	0.2	-2%	-1%
Scale parameter λ	1.33	1.15	-20%	-22%
Payroll taxes τ_{SS} and τ_{med}	0.106 and 0.029	0.08 and 0.01	+4.5%	+5%

Note : a lower scale parameter λ translates into a higher tax level.

6 Building a pre-1980's counterfactual

6.1 Presenting results

If the only differences in the U.S. economy's fundamentals between the pre-1980s period and the decades that followed it were just demographic aging, tax reforms and lower wage inequalities, what would the cyclical behavior of hours worked look like? The interest of building such a counterfactual is twofold, the first one being of course to understand what these differences implied in terms of labor supply volatility back then. The second comes from the comparison of our model predictions and data. As the U.S. economy has been through other major transitions, among which the increasing share of women in total labor supply, or changes in the variance of technological shocks, any significant difference between our models' predictions concerning hours' volatility during this period and data would put in a different perspective these structural changes. This section builds such a counterfactual, and shows that it actually implies a *lower* volatility in hours worked than today, a result completely at odd with data.

In Table 6.1, which reports quartiles of the distribution of wealth, one can see how the combination of lower pensions and inequalities change the shape of the distribution of wealth. Because of the saving motive lower pensions induce, the median level of asset held by seniors is 42% higher than in the benchmark case, while the low σ_η keeps intra age-class inequalities limited. In the case of wages and interest rates, the impact of demographic aging seems to dominate, as the wage rate is 4% below its benchmark counterpart (implying a higher return on capital, by 1 percentage point). Even if lower transfers to household create a saving motive, (i) the fact that seniors – whose wealth is particularly higher than in the benchmark case as we have just seen – represent a smaller share of the population because of demographic aging and (ii) the higher employment rate – because of both an increase in labor supply and a composition effect – pushes down the capitalistic intensity, and thus leads to a higher interest rate. An interesting observation one can make focusing on Table 6.2 is how its result looks like to the ones in Table 5.11, which reports the impact of lower transfers to seniors. The prospect of an

Table 6.1: Counterfactual : quartiles of wealth, by age class

Age class	Q_1/Q_2	Q_3/Q_2	Q_2/Q_2^{65+}	$Q_2/Q_{2,B}^{65}$
24-29	1	1	0	0
30-39	0	4	0.2	0.57
40-64	0.42	1.85	0.7	1
65+	0.5	1.7	1	1.42

Note: In the two first columns, quartiles of a given age-class are expressed in relative value with the median of this same age-class. In the first line, Q_1 , Q_2 and Q_3 are equal to 0. The last column computes the ratio between Q_2 and the median wealth of 65+ in the baseline case.

income loss highly increases participation rates for all classes. Since there are less individuals at the neighborhood of the new threshold between employment/non-employment (these agents being simply wealthier), the amount of flows in/out of employment mechanically decreases, and thus leads to a lower employment rate's standard deviation. This phenomenon is amplified by higher income taxes, which provide an additional incentive to enter the labor market. As in all previous sections, the overall impact

Table 6.2: Counterfactual: employment rates by age class

Age class	Average	Volatility	Average (Benchmark)	Volatility (Benchmark)
25-54	0.936	0.0038	0.884	0.0052
55+	0.540	0.0104	0.388	0.0131
55-64	0.697	0.0070	0.634	0.0083
65+	0.357	0.0173	0.183	0.0282
Total (25+)	0.846	0.0049	0.759	0.0062

on the intensive margin is quite limited, as one can see in Table 6.3.

The last interesting result of our numerical experiment is that the three structural change we investigate can explain the increase of the ratio between standard deviations of hours and GDP observed during the Great Moderation, the main factors being a decrease in the level of income taxes and demographic aging. In particular, a lower level of income taxes hits young high-productivity workers the most, providing them an incentive to leave the labor market. As a result, the level of productivity of agents who are at the frontier between employment-non-employment decreases in average, and thus fluctuations around this threshold have a lower impact on GDP.

6.2 Discussion

To summarize, our counterfactual for the pre-1980's period, which combines all the changes in demographic, fiscal and earnings structure we have tested in the previous sections, predicts a reduction of the volatility of total hours with respect to the benchmark case of 15% and an increase of the volatility of hours relative to output by 22%. In other words, our counterfactual experiment (*i*) shows that all

these changes actually imply a *lower* volatility of hours in the pre-1980's period than in the post-1980s one, while data suggest the inverse and (ii) that these structural changes are good candidates to explain the surge in the ratio between standard deviations of hours and output.

To correctly interpret this result, one has to keep in mind that our numerical experiment aims at evaluating the implications of some of the pre-1980's U.S. economy fundamentals – less wage inequalities, younger demographic structure *etc* – on the volatility of hours, *ceteris paribus*. By showing that they actually implied a *lower* standard deviation, we highlight the role other factors have played in increasing it back then. In particular, it sheds light on the importance a different variance of technological shocks might have played in the reduction of hours' volatility. The prediction of our model indeed implies that if technological shocks in the pre-1980's were to be of the same size as the one the U.S. economy has been experiencing in the post-1980's, the volatility of hours would not have been constant, but actually increasing over time. Said it differently, if the volatility of hours stayed constant during the whole considered period (mid 1960's to mid-2000's), this would have already implied a reduction in aggregate shocks' magnitude, *i.e.* the U.S. economy would have experienced a "hidden" Great Moderation. The observed drop in hours' standard deviation might thus come from a reduction of aggregate shocks' volatility in "two layers": the first layer being the decrease required in order to keep hours' volatility constant, despite structural changes that tend to increase it, and the second layer being the additional lowering that would actually make labor supply's volatility fall.

Table 6.3: Counterfactual: volatility of hours worked per worker by age

Age class	Experiment	Benchmark
25-54	0.0012	0.0010
55-64	0.0013	0.0011
65+	0.0012	0.0018
Total	0.0011	0.001

7 Conclusion

This chapter builds an heterogeneous agents DSGE model to investigate the way structural changes that interact with the distribution of households can modify the cyclical behavior of hours worked. By building a counterfactual of the mid-1960's-mid 1980's U.S. economy where the only differences with respect to the Great Moderation era lay in (i) a younger demographic structure, (ii) less income inequalities and (iii) a different tax/transfer system, we first show that the combination of these structural changes imply a lower volatility of hours, *i.e.* an increase in the standard deviation of hours worked over time. We also show that these structural change can explain the observed rise in the ratio between standard deviations of hours and GDP.

Appendix

We use the well known algorithm of Krusell-Smith (1998) [62] to solve the model with aggregate uncertainty. We use two moments to summarize the aggregate state state: the average stock of capital \mathbf{K} and the wage \mathbf{w} , and use forecasting rules of the following form:

$$\begin{aligned}\log \widehat{\mathbf{K}}' &= H^K(\mathbf{K}, \mathbf{Z}) = a_0 + a_1 \log \mathbf{K} + a_2 \log \mathbf{Z} \\ \log \widehat{\mathbf{w}} &= H^w(\mathbf{k}, \mathbf{Z}) = b_0 + b_1 \mathbf{K} + a_2 \log \mathbf{Z}\end{aligned}$$

To solve the model, the first step is to get steady-state prices, distributions, as well as value functions and policy rules. The procedure we follow is detailed in Algorithm 1. The second step is then to estimate the coefficients of our forecasting rules, by using Algorithm 4. In all our numerical experiments, these forecasting rules lead to highly satisfying R-squared coefficients.

Algorithm 1: STEADY

Input: Guesses of the capital over labor ratio \mathbf{k} and average labor earnings \mathbf{e}

Parameter: Structural parameters: preferences, technology, demographics, fiscal policy and numerical parameters (grids sizes, tolerance levels *etc.*)

Output: Steady state prices, policy rules, value functions and distribution.

1 Build exogenous grids for savings \vec{A}

▸ Below is the main loop, which aims at clearing both labor and financial markets

2 **while** Labor and financial markets are not cleared **do**

3 Call EGM (Algorithm 2) ▸ Endogenous Grid Method from Carroll (2006) [17] computes the decision rules for a given \mathbf{k} and \mathbf{e} . Its inputs are \mathbf{k} and \mathbf{e} . The outputs are value functions $V_j^l(a, \eta)$ – for an age j and labor supply l – and saving policy rules $a_j^l(a, \eta)$

4 Deduce the probabilities to work for a level of assets a and an idiosyncratic productivity η , using the value functions determined just above,

$$\mathbb{P}_j(l = \ell | a, \eta) = \frac{\exp\left(\frac{V_j^\ell(a, \eta)}{\sigma_\epsilon}\right)}{\sum_{l \in \mathbb{D}_j} \exp\left(\frac{V_j^l(a, \eta)}{\sigma_\epsilon}\right)}$$

5 Call YOUNG (Algorithm 3) ▸ Now that we have the decision rules and value function, we compute the stationary distribution, using Young's algorithm (2010) [88] of non-stochastic simulations. Inputs are probabilities $\mathbb{P}_j(l = \ell | a, \eta)$ and savings policy rules. The output is the distribution $\Gamma_j(a, \eta)$

6 Now that we have the distribution, compute the aggregate stock of effective labor and capital:

$$L = \sum_i \sum_k \sum_j \sum_{\ell \in \mathbb{D}_j} \mathbb{P}_j(l = \ell | a_i, \eta_k) \cdot \mu_j(a_i, \eta_k) \cdot \ell \cdot \eta_k \cdot \zeta_j$$
$$K = \sum_i \sum_k \sum_j a_i \mu_j(a_i, \eta_k).$$

7 **if** markets are not cleared *i.e.* $\mathbf{k} \neq K/L$ or $\mathbf{e} \neq wL$ **then**

└ Update guesses \mathbf{k} or \mathbf{e}

8 **return** capitalistic intensity \mathbf{k} , average labor earnings \mathbf{e} , policy rules $a_j^l(a, \eta)$, values functions $V_j^l(a, \eta)$, distribution $\mu_j(a, \eta)$.

Algorithm 2: EGM

Input: Capital over labor \mathbf{k} and average labor earnings \mathbf{e} ▶ We need \mathbf{e} as pensions, consumption floor and tax function are indexed over it.

Parameter: Structural parameters: preferences, technology, demographics, fiscal policy, numerical parameters (grids sizes, tolerance levels) + exogenous grids.

Output: savings policy rules, value functions

- 1 Deduce interest rate $r = \alpha z \mathbf{k}^{1-\alpha}$ and wage $w = (1 - \alpha) z \mathbf{k}^\alpha$
- 2 Solve the following problem using the endogenous grid method (EGM) adapted by Iskhakov (see Iskhakov, 2017 [50]).

$$V_j^l(a, \eta) = \max_{a', c} u_j(c, l) + \beta q_j \mathbb{E} V_{j+1}^{\sigma \epsilon}(a', \eta')$$

$$\text{s.t. } a' \geq 0, c > 0, l \in \mathbb{D}_j$$

$$a' = (1 + r)a + T(a, y) - c(1 + \tau_c)$$

$$y = \begin{cases} w\eta\zeta_j l - T_S(w\eta\zeta_j l) & \text{if } j < J_1 \\ w\eta\zeta_j l - T_S(w\eta\zeta_j l) + B_S & \text{if } J_1 < j < J_2 \\ B_S & \text{if } j \leq J_2 \end{cases}$$

$$\mathbb{D}_j = \begin{cases} \{0; 0.5; 0.85; 1; 1.15; 1.25\} \bar{l} & \text{if } j \leq J_2 \\ \{0\} & \text{otherwise} \end{cases}$$

$$B_S = \theta \cdot \mathbf{e} \text{ where } \mathbf{e} \text{ is average labor earnings, } \mathbf{e} = wL$$

$$T(a, B_S) = \max \left\{ ra + T_0(ra + B_S)^{1-\tau_1}, \underline{y} \right\}, \text{ where } \underline{y} = \omega \cdot \mathbf{e}$$

$$T_S(w\eta\zeta_j l) = \tau_{med} w\eta\zeta_j l + \tau_{SS} \min \{ w\eta\zeta_j l, e_{\max} \}$$

- 3 **return** $a_j^l(a, \eta)$ and $V_j^l(a, \eta)$
-

Algorithm 4: KRUSELL-SMITH MAIN LOOP

Input: Guesses for the forecasting rules of \mathbf{K} and \mathbf{w} , steady state distribution, value functions and decisions rules from Algorithm 1.

Parameter: Structural parameters: preferences, technology, demographics, fiscal policy, numerical parameters (grids sizes, tolerance levels) + exogenous grids \vec{A} .

Output: Decision rules, value functions, forecasting rules

- 1 **while** Forecast errors are beyond tolerance level **do**
 - 2 Using the forecasting rule, deduce endogenous grids $\widehat{\mathbf{K}}' = H^k(\mathbf{K}, \mathbf{Z})$ and $\widehat{\mathbf{w}} = H^w(\mathbf{K}, \mathbf{Z})$, *i.e.* tomorrow's total capital stock and today's wage. Deduce interest rate $\widehat{r} = \mathbf{Z}\alpha(\mathbf{Z}(1 - \alpha)/\widehat{\mathbf{w}})^{\frac{1-\alpha}{\alpha}} - \delta$.
 - 3 Call KRUSELL-SMITH INDIVIDUAL PROBLEM (Algorithm 5) > Deduce decision rules and value function, over the aggregate state space
 - 4 Call SIMULATIONS (Algorithm 6) > Simulate the model over T periods, for a given forecasting rule. Output is the new forecasting rule
 - 5 **if** new forecasting rule \neq guess **then**
 - 6 Update the guess of the forecasting rule.
-

Algorithm 5: KRUSELL-SMITH INDIVIDUAL PROBLEM

Input: Forecasted capital stocks $\widehat{\mathbf{K}}'$, interest rate \widehat{r} and wages $\widehat{\mathbf{w}}$.

- 1 . **Parameter:** Structural parameters: preferences, technology, demographics, fiscal policy, numerical parameters (grids sizes, tolerance levels) + exogenous grids.

Output: savings policy rules, value functions

- 2 Solve the following problem using EGM algorithm of Iskhakov.

$$V_j^l(a, \eta; \mathbf{K}, \mathbf{Z}, H^w(\mathbf{K}, \mathbf{Z})) = \max_{a', c} u_j(c, l) + \beta q_j \mathbb{E} V_{j+1}^{\sigma \epsilon}(a', \eta'; \widehat{\mathbf{K}}', \mathbf{Z}', H^w(\widehat{\mathbf{K}}', \mathbf{Z}'))$$

$$\text{s.t. } a' \geq 0, c > 0, l \in \mathbb{D}_j$$

$$a' = (1 + r)a + T(a, y) - c(1 + \tau_c)$$

$$y = \begin{cases} w\eta\zeta_j l - T_S(w\eta\zeta_j l) & \text{if } j < J_1 \\ w\eta\zeta_j l - T_S(w\eta\zeta_j l) + B_S & \text{if } J_1 < j < J_2 \\ B_S & \text{if } j \leq J_2 \end{cases}$$

$$\mathbb{D}_j = \begin{cases} \{0; 0.5; 0.85; 1; 1.15; 1.25\} \bar{l} & \text{if } j \leq J_2 \\ \{0\} & \text{otherwise} \end{cases}$$

$$B_S = \theta \cdot \bar{e} \text{ where } \bar{e} \text{ is average labor earnings at the steady state}$$

$$T(a, B_S) = \max \left\{ ra + T_0(ra + B_S)^{1-\tau_1}, \underline{y} \right\}, \text{ where } \underline{y} = \omega \cdot \bar{e}$$

$$T_S(w\eta\zeta_j l) = \tau_{med} w\eta\zeta_j l + \tau_{SS} \min \{ w\eta\zeta_j l, e_{\max} \}$$

- 3 **return** $a_j^l(a, \eta; \mathbf{K}, \mathbf{Z}; \mathbf{N})$ and $V_j^l(a, \eta; \mathbf{K}, \mathbf{Z}; \mathbf{w})$.
-

Algorithm 6: SIMULATIONS

Input: Decision rules $a_j^l(a, \eta; \mathbf{K}, \mathbf{Z}; \mathbf{w})$, value functions $V_j^l(a, \eta; \mathbf{K}, \mathbf{Z}, \mathbf{w})$, steady state distribution μ and forecasting rules.

Parameter: Structural parameters: preferences, technology, demographics, fiscal policy, numerical parameters (grids sizes, tolerance levels) + exogenous grids \vec{A} .

Output: OLS coefficients, *i.e.* forecasting rules for simulated series of \mathbf{K}_t , \mathbf{w}_t and \mathbf{Z}_t .

- 1 Initialize simulation's first period as the steady state, *i.e.* use the steady state distribution μ at the beginning.
 - 2 **for** $t \in [1, \dots, T]$ **do**
 - 3 Compute the current total stock of capital \mathbf{K}_t from the distribution deduced in $t - 1$. If $t = 1$, use the steady-state distribution.
 - 4 Forecast the total amount of labor $\widehat{\mathbf{w}}_t$ and next period's $\widehat{\mathbf{K}}'_t$. Deduce from $\widehat{\mathbf{w}}_t$ and \mathbf{K}_t the labor supply $\widehat{\mathbf{L}}_t = \left[\frac{(1-\alpha)\mathbf{Z}_t}{\widehat{\mathbf{w}}_t} \right]^{\frac{1}{\alpha}} \mathbf{K}_t$. ▶ Use the forecasting rule.
 - 5 **while** Labor market is not cleared **do**
 - 6 Interpolate value functions $V_j^l(a, \eta; \mathbf{K}, \mathbf{Z}, \mathbf{w})$ over \mathbf{K}_t and $\widehat{\mathbf{w}}_t$. Deduce working probabilities $\mathbb{P}_j(\ell = l | a, \eta; \mathbf{K}_t, \mathbf{Z}_t; \widehat{\mathbf{w}}_t)$. Compute the total labor supply $L_t = \sum_i \sum_k \sum_j \sum_{\ell \in \mathbb{D}_j} \mathbb{P}_j(\ell = \ell | a_i, \eta_k; \mathbf{K}_t, \mathbf{Z}_t; \widehat{\mathbf{w}}_t) \cdot \mu_{t,j}(a_i, \eta_k) \cdot \ell \cdot \eta_k \cdot \zeta_j$.
 - 7 **if** $L_t \neq \widehat{\mathbf{L}}_t$ **then**
 - 8 Update the guess of labor market, setting $\widehat{\mathbf{L}}_t = L_t$. Update the corresponding forecast of wage, $\widehat{\mathbf{w}}_t = (1 - \alpha)\mathbf{Z}_t \left(\mathbf{K}_t / \widehat{\mathbf{L}}_t \right)^\alpha$
 - 9 Interpolate policy functions $a_j^l(a, \eta; \mathbf{K}, \mathbf{Z}, \mathbf{w})$ over \mathbf{K}_t and \mathbf{w}_t .
 - 10 Call YOUNG (Algorithm 3) ▶ Compute next period's distribution, in order to prepare the computation of next period's total stock of capital
 - 11 Update the forecasting rule, by running OLS of $\log(\mathbf{K}_{t+1})$ over $\log(\mathbf{K}_t)$ and $\log \mathbf{Z}_t$. Same for \mathbf{w}_t .
-

Chapter 3

Aging risk, insurance and social security

Abstract

In a stochastic aging context, an extra euro saved by a youngster today can end up tomorrow in the age-states "young" or "old". These two states are highly different because youngsters can become old, and old people can die. As the mortality risk is higher in the age-state "old", wealth does not have the same value in both states. The paper then studies how insurance markets can help agents to transfer their wealth between age-states – from the one where its value is lower to the other one–. It also shows that when such markets are missing and even if taxation distorts labor supply, social security can be welfare improving. This property however only holds when the interest rate is low enough.

1 Introduction

Aging is subject to uncertainty: individuals do not know how long they are going to live, nor do they know when they will face a higher mortality risk¹. In the life-cycle literature, this uncertainty is modeled using "stochastic aging" as in Blanchard (1985) [13] or Gertler (1999) [37]. In the former – the perpetual youth model –, individuals are assumed to face each period a constant death probability. In the latter, a refinement is offered with a two-stage aging process: newborn immortal youngsters first face a probability to become mortal seniors; it is only once they have switched to this state of nature that they face a constant death probability.

Such an uncertain aging process of course has economic implications on rationale agents' decisions, as it creates risks they want to be insured against. One of these risks this chapter focuses on is the so-called "aging risk". To see it more clearly, let us take the example of a youngster who wants to save an additional unit of wealth. Tomorrow, the agent can take it back as a youngster if he is still one, or as a senior if he switches "age-state". If he switches age-state, he will face a mortality risk; as the agent of course wishes to enjoy his wealth alive, this risk provides an incentive to consume a larger share of his assets. This difference in the consumption behavior between age-states "young" – where the mortality risk is virtually non-existent – and "old" thus creates an aging risk. The value of an extra euro saved by the youngster today is thus subject to uncertainty, as this euro can end up in an age-state where it will be consumed at a higher pace. The two other form of risks associated with stochastic aging, which are (i) the risk to die leaving wealth behind and (ii) the income loss risk upon switching age-state are eliminated in our thought experiment. The first one is gotten rid of by putting in place a perfect annuity market as in Blanchard (1985) [13], and the second one is neutralized by assuming that agents are risk neutral.

This chapter shows that the age-state in which the marginal wealth has the highest value depends on the relative force of a substitution and wealth effect. To do so, in addition to the assumption of risk-neutrality mentioned above, it assumes that agents have a finite elasticity of intertemporal substitution (EIS) by using non-expected recursive preferences *à la* Farmer (1990) [30] as in Gertler (1999) [37]. These so-called Risk Neutrality Constant Elasticity (RINCE) preferences highlight the fact that on the one hand, as the mortality risk makes old agents more impatient and thus makes them consume more, saving becomes more expensive in terms of utility in this age-state. In other words, a substitution effect makes the marginal value of wealth for a senior lower than for a youngster. For this reason, when the substitution effect dominates, an extra euro saved by a youngster will provide him more welfare tomorrow if he stays in the age-state "young" than if he switches to "old". However, on the other hand, as old agents want to consume more, they need more wealth to do so. Because of this wealth effect, the higher mortality risk seniors face makes the marginal value of wealth higher in the age-state "old" than in "young". Said it differently, when the wealth effect is stronger, an additional unit of asset saved

¹Throughout this chapter, the term "aging" does not refer to chronological age, but to states of nature associated with different mortality risks.

by a youngster will provide him more welfare tomorrow if he switches to "old" than if stays young.

A change in the value of marginal wealth upon age-switching, associated to a willingness to optimize intertemporal consumption thus creates a desire for insurance that is independent from risk aversion. Young agents indeed need financial instruments that redirects wealth towards the age-state where its value is the highest. For instance, if the marginal value of wealth is higher in the age-state "old", agents will be willing to buy insurance contracts that would pay them an indemnity if they become seniors. Such a case arises when the wealth effect dominates, which happens when EIS is below unity; as they value saving more in age-state "old" than in "young", youngsters are thus ready to pay an insurance which compensates them if they turn old. When the EIS is over unity, the substitution effect dominates; youngsters are willing to buy a contract that reimburse them if they stay young tomorrow.

What if private insurance was to be missing? This chapter finally shows that if market failures make private insurance absent, social security systems can – imperfectly – take care of aging risk. Even when it takes the form of a tax that distorts labor supply, Pay-as-you-go (PAYG) pension systems can indeed be welfare-increasing, as they redistribute wealth across from state "young" to "old". However, an important condition for this property to hold is an environment where the interest rate, although being high enough to guarantee a dynamically efficient equilibrium, is relatively low; high interest rates indeed make self-insurance more profitable than an imperfect public insurance. Such a conclusion let this chapter thus also contribute to the – long lasting – literature studying the impact of social security on welfare. In particular, it echoes several papers, such as Imrohoroglu *et al.* (1995) [49], Storesletten *et al.* (1999) [83] or Haremborg and Ludwig (2018) [44] which have concluded that despite the presence of idiosyncratic income risk², social security at the stationary equilibrium is inefficient. By considering a different type of idiosyncratic risk – the aging risk –, this paper thus brings a conclusion that is radically different, and justifies the existence of an unfunded pension system³.

2 Uninsured aging risk

2.1 Preferences

In order to put into evidence the "aging risk", we use a stochastic aging model *à la* Gertler (1999) [37], where individuals are born in state "young". Each period, youngsters face a probability $1 - \omega$ to switch to the state "old", and a probability ω to stay in their initial state. On the contrary of youngsters, old individuals are mortal, as they face each period a surviving probability γ . We assume agents to be risk neutral while having a constant elasticity of intertemporal substitution (EIS). These preferences, so-called "RINCE preferences" (Farmer, 1990 [30]), break the standard relationship between risk aversion

²Haremborg and Ludwig (2018) [44] showed that when idiosyncratic income risk is coupled with aggregate uncertainty, social security is welfare improving. They also show that aggregate risk on its own cannot justify the existence of a PAYG pension system, a result similar to the one obtained by Krueger and Kubler (2006) [61].

³Imrohoroglu *et al.* (1995) [49] have a similar conclusion regarding the death risk: if annuity markets are missing, social security can increase welfare. This paper gets rid of this role social security can have by assuming the existence of a perfect annuity market.

and EIS in time additive von Neumann and Morgenstern (VNM) utility function – the coefficient of risk aversion being the inverse of the later–. and can therefore isolate the economic implications of a desire for consumption smoothing across time from risk aversion. It is indeed quite important to make agents risk-neutral in our analysis, in order to make clear the fact that our results do not come from their behavior *vis-à-vis* uncertainty .

RINCE preferences can be represented using recursive non-expected utility functions⁴. They are given by

$$V_t^z = [(c_t^z)^\rho + \beta^z \mathbb{E}_t [V_{t+1}|z]^\rho]^\frac{1}{\rho}$$

where $\rho \in]-\infty, 0[\cup]0, 1[$, $z = \{y, o\}$ indicates the age status of the agent, y denoting young agents, while o denotes old ones, c_t^z is the consumption of the agent,

$$\beta^z = \begin{cases} \beta & \text{if } z = y \\ \beta\gamma & \text{if } z = o \end{cases} \text{ and } \mathbb{E}_t [V_{t+1}|z] = \begin{cases} \omega V_{t+1}^y + (1 - \omega)V_{t+1}^o & \text{if } z = y \\ V_{t+1}^o & \text{if } z = o \end{cases}$$

Thus, preferences of an old agent can be written

$$V_t^o = [(c_t^o)^\rho + \beta\gamma (V_{t+1}^o)^\rho]^\frac{1}{\rho}$$

while the ones of a youngster can be written

$$V_t^y = [(c_t^y)^\rho + \beta (\omega V_{t+1}^y + (1 - \omega)V_{t+1}^o)^\rho]^\frac{1}{\rho}$$

Let us now describe the consumption-saving decisions of both old and young agents.

2.2 Consumption-saving decisions

Old agents

Old agents have access to an annuity market as in Blanchard (1985) [13] and Yaari (1965) [87] that perfectly insures against the uncertainty regarding the timing of death⁵. Competitive mutual funds collect the financial wealth of old individuals, and redistribute the remaining assets of the share γ of agents who did not survive to the share $1 - \gamma$ of agents who are still alive. More specifically, insurance companies' "income" is given by the amount of assets left by agents who did not survive $(1 - \gamma)W$, where W is the total wealth these firms have collected. The redistribution of these assets to the surviving agents takes the form of a premium μ over their wealth, *i.e.* insurance firms have to pay $\mu\gamma W$. As the life insurance market is perfectly competitive, no profits can be made, which implies $\mu = \frac{1-\gamma}{\gamma}$. The gross return of assets of an old agent is thus $(1 + r_t)(1 + \mu) = \frac{1+r_t}{\gamma}$. We finally assume that old individuals get a lump sum income w_t^o .

⁴See Epstein and Zin (1990) [28], Farmer (1990) [30], Kreps and Porteus (1989) [59] and Weil (1990) [85].

⁵I choose to get rid of this uncertainty in order to completely isolate the "aging risk" this chapter focuses on.

The problem of an agent who was born in i , became old in j and is currently in t is given by

$$V_t^{o,i,j} = \max_{c_t^{o,i,j}, a_{t+1}^{o,i,j}} \left[\left(c_t^{o,i,j} \right)^\rho + \beta \gamma \left(V_{t+1}^{o,i,j} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{o,i,j} = \frac{1+r_t}{\gamma} a_t^{o,i,j} + w_t^o - c_t^{o,i,j}$$

The solution of this program takes the form (see Appendix 3.A) of the following Euler equation

$$c_{t+1}^{o,i,j} = (\beta R_{t+1})^\sigma c_t^{o,i,j} \quad (3.1)$$

where $\sigma = \frac{1}{1-\rho}$ is the EIS, and $R_t = 1 + r_t$.

Young agents

Youngsters get a lump sum income w_t^y . The program of a young individual, born in i and currently in t is given by

$$V_t^{y,i} = \max_{c_t^{y,i}, a_{t+1}^{y,i}} \left[\left(c_t^{y,i} \right)^\rho + \beta \left(\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{y,i} = (1+r_t) a_t^{y,i} + w_t^y - c_t^{y,i}$$

His Euler equation is non-standard, as it takes the following form (see Appendix 3.B):

$$\omega c_{t+1}^{y,i} + (1-\omega) \Lambda_{t+1} c_{t+1}^{o,i,t+1} = (\beta R_{t+1} \Omega_{t+1})^\sigma c_t^{y,i} \quad (3.2)$$

Where

$$\Lambda_{t+1} = \left(\frac{\partial V_{t+1}^{o,i,t+1} / \partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i} / \partial a_{t+1}^{y,i}} \right)^\sigma \quad \text{and} \quad \Omega_{t+1} = \omega + (1-\omega) \Lambda_{t+1}^{1-\rho} = \omega + (1-\omega) \frac{\partial V_{t+1}^{o,i,t+1} / \partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i} / \partial a_{t+1}^{y,i}}$$

Note that we have $a_{t+1}^{y,i} = a_{t+1}^{o,i,t+1}$: the wealth of an agent who has just become "old" in $t+1$ is equal to the saving decision $a_{t+1}^{y,i}$ he made in t when he was a youngster.

This Euler equation differs from standard cases, as it features the term $\frac{\partial V_{t+1}^{o,i,t+1} / \partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i} / \partial a_{t+1}^{y,i}}$, the marginal rate of substitution (MRS) between wealth in age "old" and "young". This term is actually the cornerstone of this paper, as it symbolizes the aging risk. If this risk was absent, it would mean that the agent values wealth in both states the same, *i.e.* the MRS would be equal to one. In such a case, we would then have a standard expression,

$$\omega c_{t+1}^{y,i} + (1-\omega) c_{t+1}^{o,i,t+1} = (\beta R_{t+1})^\sigma c_t^{y,i}$$

Yet, under the aging risk, nothing guarantees that the agent is indifferent between increasing his wealth in the state "young" and in the state "old". Because of this difference in the valuation of wealth, an extra euro saved is subject to the "aging risk": if agents value marginal wealth in their old age relatively

more than during their youth – *i.e.* the MRS is higher than one –, then $\Omega > 1$, which increases the expected return of saving. On the contrary, if wealth has relatively more value in the state "young", the MRS is lower than 1, which implies $\Omega < 1$: the agent faces a risk to see the value of his wealth decrease if he becomes old. Following the same logic, consumption in old age does not bring the same utility as its counterpart in young age; the Euler equation thus "translates" old age consumption into young age one, through the term Λ .

2.3 Substitution effect, wealth effect and the aging risk

Now that we have understood that the wealth of a young agent faces an "aging risk", we have to figure out why such a risk exists in the first place. At the optimum, why does the value of wealth depend on age-state? Is the marginal value of wealth higher in the state "old" or young"?

Proposition 1 $\Omega > 1$, *i.e.* the marginal value of wealth is higher in the state "old" than in the state "young" if and only if $\sigma < 1$.

PROOF See Appendix 3.C and 3.D.

The corollary of Proposition 1 is of course that the marginal value of wealth in state "old" is higher in state "young" than in state "old" ($\Omega < 1$) when $\sigma > 1$.

Before explaining the economic reasoning behind this proposition, let us rewrite the closed-form solution of the decision rules. As shown in Appendix 3.C, consumption is linear in financial wealth and human wealth – the discounted sum of lump sum income. At the steady-state, defined here as a situation where variables independent from the life cycle history of agents are supposed constant, the consumption decision of the agent "old" is:

$$c_t^{o,i,j} = \pi^o \left(\frac{R}{Y} a_t^{o,i,j} + h^o \right) \quad \text{with} \quad \pi^o = 1 - \gamma \beta^\sigma R^{\sigma-1}$$

π^o being the marginal propensity to consume (MPC), and where the human wealth h^o is defined as

$$h^o = w^o + \frac{\gamma h^o}{R} \Leftrightarrow h^o = w^o \left(\frac{1+r}{1+r-\gamma} \right)$$

The value function is also linear in wealth:

$$V_t^{o,i,j} = (\pi^o)^{\frac{\sigma}{1-\sigma}} c_t^{o,i,j} = (\pi^o)^{\frac{1}{1-\sigma}} \left(\frac{R}{Y} a_t^{o,i,j} + h^o \right)$$

In the case of the youngster, consumption is also linear in wealth:

$$c_t^{y,i} = \pi^y \left(R a_t^{y,i} + h^y \right)$$

The human wealth being written

$$h^y = w^y + \frac{\omega}{\Omega} \frac{h^y}{R} + \left(1 - \frac{\omega}{\Omega}\right) \frac{h^o}{R} = \frac{\Omega R}{\Omega R - \omega} w^y + \frac{R(\Omega - \omega)}{(R - \gamma)(\Omega R - \omega)} w^o$$

The expression for the MPC being

$$\pi^y = 1 - \beta^\sigma (R\Omega)^{\sigma-1}$$

Where Ω gives the expected return of an extra saved euro, and "corrects" the interest rate from the aging risk:

$$\Omega = \omega + (1 - \omega) \left(\frac{\pi^o}{\pi^y}\right)^{\frac{1}{1-\sigma}}$$

$\left(\frac{\pi^o}{\pi^y}\right)^{\frac{1}{1-\sigma}}$ being the MRS of wealth between states "old" and "young". Indeed, the expression of the value function, linear in wealth, is

$$V_t^{y,i} = (\pi^y)^{\frac{\sigma}{1-\sigma}} c_t^{y,i} = (\pi^y)^{\frac{1}{1-\sigma}} \left(Ra_t^{y,i} + h^y\right)$$

and thus $\frac{\partial V_{t+1}^{o,i,t+1} / \partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i} / \partial a_{t+1}^{y,i}} = \left(\frac{\pi^o}{\pi^y}\right)^{\frac{1}{1-\sigma}}$. These closed form solution make clearly appear the crucial importance of the sign of $\sigma - 1$, which symbolizes the relative strength between substitution and wealth effects. For instance, in the case of an old agent, we indeed have:

$$\frac{\partial \pi^o}{\partial R} \begin{cases} > 0 \text{ if } \sigma < 1 \\ < 0 \text{ if } \sigma > 1 \end{cases}$$

When $\sigma > 1$, the substitution effect dominates. As the return of savings increases, today's consumption becomes more expensive: the agent thus chooses to save more (the MPC decreases) in order to increase his consumption tomorrow. When $\sigma < 1$, the wealth effect is stronger: a higher R makes the agent richer, and thus leads him to consume more, *i.e.* the MPC increases. In both cases, a higher interest rate increases the marginal value of wealth, *i.e.* $\frac{\partial^2 V_{t+1}^{o,i,j}}{\partial a_{t+1}^{o,i,j} \partial R} > 0$.

Now that the role of substitution and wealth effects has been put into evidence using closed form solutions, we can get back to Proposition 1, by studying the impact of a higher mortality risk on the marginal value of wealth in state "mortal". One can first see, using the definition of MPC, that $\frac{\partial \pi^o}{\partial \gamma} < 0$, *i.e.* a lower survival rate increases the MPC. The interpretation is straightforward: a higher mortality risk works just as a lower discount factor: as the agent can be dead tomorrow, he has an incentive to consume more today. The impact of this higher mortality rate on the marginal value of wealth then depends on the sign of $\sigma - 1$. If $\sigma > 1$ - *i.e.* the substitution effect dominates - a lower survival rate decreases the marginal value of wealth: as the agent is more impatient, saving an additional euro becomes more expensive. On the other hand, when $\sigma < 1$, the wealth effect dominates: an agent who

faces a higher mortality risk needs to increase his consumption, and thus values more an additional unit of savings that would help him consume more. In other words, when $\sigma < 1$, $\frac{\partial^2 V_{t+1}^{o,i,j}}{\partial a_{t+1}^{o,i,j} \partial \gamma} < 0$: the marginal value of wealth in state "old" increases as the mortality rate gets higher.

A youngster who wants to save an extra euro today is thus subject to the aging risk, as tomorrow, this euro can end up in the age state "old", where its value changes because of the mortality risk. As the mortal "old" agent is more impatient, his MPC is higher; hence, whether the MRS of wealth between states "young" or "old" $\left(\frac{\pi^o}{\pi^y}\right)^{\frac{1}{1-\sigma}}$ is higher than one only depends on the sign of $\sigma - 1$, following the same economic reasoning as before. If $\sigma > 1$, substitution effect dominates, and saving in the age-state "old" has less value: as the old agent is more impatient, he has less incentive to save than the youngster. In other words, the marginal value of wealth in the state "old" is lower than in the state "young", *i.e.* $\Omega < 1$. On the contrary, an extra euro of saving has a higher value in the age-state "old" than in "young" for a youngster with $\sigma < 1$ because of a wealth effect. Indeed, the state "old" is characterized by a stronger desire of consumption that requires more wealth than the state "young".

Intuitively, this difference of valuation of wealth in a "age-state" or another means that young agents need a financial instrument that would redirect their saving between them. When $\sigma > 1$, they thus would need a contract that redirects their wealth toward the state "young", while if $\sigma < 1$, they would need a financial vehicle that transfers some of their wealth to the state "old", where its marginal value is higher. The following section characterizes formally such financial instruments.

2.4 Insuring the aging risk

As our discussion above about the desire for financial instruments points out, the differential in the marginal value of wealth across age-states is the sign of an inefficiency. If $\sigma < 1$, agents might be willing for an instrument that could redistribute their assets to the state of nature "old", while if $\sigma > 1$, they might need one that does the opposite.

As usual, the market can operate – under some conditions – such a redistribution through private insurance. Let us imagine the following insurance scheme. A young agent in t chooses the amount of goods b_{t+1} he wants to receive in $t + 1$ if he becomes old; in exchange, in $t + 1$, no matter the state of nature, he will have to pay the insurance $p_{t+1}b_{t+1}$, where p_{t+1} is the price of the insurance contract. The problem of a youngster can thus be written

$$V_t^{y,i} = \max_{c_t^{y,i}, a_{t+1}^{y,i}, b_{t+1}^{y,i}} \left[\left(c_t^{y,i} \right)^\rho + \beta \left(\omega V_{t+1}^{y,i} + (1 - \omega) V_{t+1}^{o,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{y,i} = (1 + r_t) a_t^{y,i} + w_t^y - p_t b_t^{y,i} - c_t^{y,i}$$

$p_t b_t^{y,i}$ being the payment of the insurance previously contracted. Let us finally assume insurance market to be perfectly competitive so that free entry condition holds. The resolution of this program thus leads to the following proposition:

Proposition 2 *The young agent is willing to buy an insurance contract that pays him if he becomes "old" (stays "young") if and only if $\sigma < 1$ ($\sigma > 1$)*

PROOF See Appendix 3.E and 3.E.

Proposition 2 is simply the logical consequence of our initial intuition: the economy without insurance is inefficient – despite the fact that agents are risk neutral! – because the value of wealth changes when the agent becomes "old": in this state of nature, as he wants to compensate for the fact that his life is finite by consuming more, he needs more wealth. Thus, when the wealth effect dominates, *i.e.* when $\sigma < 1$, agents are willing to buy contracts that redistribute assets to the state of nature "old" where the marginal value of wealth is higher.

However, if the substitution effect is stronger, agents do not want to buy such contracts anymore. Remember that when $\sigma > 1$, the marginal value of wealth is lower in the state "old" than in "young": as the substitution effect dominates, saving is less costly in states of nature where the agent is more patient – *i.e.* the age "young" –, and thus brings more welfare. For such an agent, the optimal insurance contract works the other way around, and redistributes wealth toward the age-state "young", by paying him an indemnity if stays a youngster.

3 Welfare improvement through social security

What we have learned from the previous section is that despite their risk neutrality, agents who face an uncertain life duration combined with life-insurance contracts can still have – under some assumptions – a desire for insurance to make them better-off. An interesting question that thus arises from this result is the following: what can a government do when market failures prevent these insurance contracts to be subscribed? Can it – to a given extent – replace the missing insurance market? In other words, can social security be welfare improving?

Before we start discussing about applications of the theoretical framework developed in the previous section, one has to make the definitions of "age" or "age switching" somewhat less abstracts. Let us imagine agents who are born as "workers" (as in Gertler's terminology) but who do not know with certainty when they will retire. Workers actually face an aging shock: each period, aging shock might force them to become "retiree"⁶. As their age status changes, retirees leave the labor market and become mortal, which makes thus face each period a surviving probability.

If we assume market failures, such that there is no insurance market, and assuming that $\sigma < 1$ (agents want to redistribute some of their wealth to the state "retiree"), what can a government do in this framework? Taxing workers to redistribute to retirees might be the right answer. The argument behind the welfare improving mechanism of social security is however different from usual: its goal is not to increase the agent's utility by making an income loss risk disappear, but to redirect their wealth to a state where it has a higher value, due to the finiteness of life.

⁶As we already emphasized in introduction, retirement date, subject to a significant degree of uncertainty, see Caliendo *et al.* (2016) [15]

Embedding the previous model in a general equilibrium OLG framework as in Gertler (1999) [37] and using numerical simulations, this section shows that PAYG pension systems can increase welfare if interest rates are low enough. Just as self-insurance, taxation is indeed an imperfect substitute for the private insurance contract presented above, which (i) is priced on a competitive market and (ii) pays an indemnity at the period where the agent switches age-state, and not a lifelong pension. In a framework of incomplete markets, whether the agent prefers self-insurance over taxation thus depends on the return on capital: when it is low enough, it becomes more profitable to rely on a government-ran pension system, even when the equilibrium is dynamically efficient.

An important element we have to take into account is the potentially distorting effect of taxes on labor supply, which are not taken into account in the following subsection as we assume inelastic labor supply. Section 3.3 shows that even when one assumes elastic labor supply, the welfare increasing impact of social security holds for low levels of interest rate.

3.1 General: Demographics

No matter the assumptions about the elasticity of labor supply, the first thing we need is to describe the demographic structure of the population. Individuals are born as "workers" and switch to state "retiree" following the same process as before, the transition rates being the same. Worker's variables are being denoted with a superscript w and retirees with a r . Noting the amount of young individuals in t N_t^w and the population of old individuals N_t^r , let us assume that the amount of newborn individuals in t is given by $(1 - \omega + n)N_t^w$. Each population then has the following law of motions:

$$\begin{aligned} N_{t+1}^w &= \omega N_t^w + (1 - \omega + n)N_t^w = (1 + n)N_t^w \\ N_{t+1}^r &= (1 - \omega)N_t^w + \gamma N_t^r \end{aligned}$$

n being the growth rate of the population at the stationary equilibrium. ψ is defined as the dependence ratio, defined as $\psi := N_t^r / N_t^w = \frac{1-\omega}{1+n-\gamma}$.

3.2 Equilibrium with PAYG pension system and inelastic labor supply

3.2.1 Decision rules

If we assume labor supply to be inelastic, the problem in t of a retiree who was born in i and switched to his current state in j writes:

$$\begin{aligned} V_t^{r,i,j} &= \left[\left(c_t^{r,i,j} \right)^\rho + \beta \gamma \left(V_{t+1}^{r,i,j} \right)^\rho \right]^{\frac{1}{\rho}} \\ \text{s.t. } a_{t+1}^{r,i,j} &= \frac{1+r_t}{\gamma} a_t^{r,i,j} + e_t - c_t^{r,i,j} \end{aligned}$$

Where e_t is the pension benefit. The problem of a worker in t who was born in i is

$$V_t^{w,i} = \left[\left(c_t^{w,i} \right)^\rho + \beta \left(\omega V_{t+1}^{w,i} + (1 - \omega) V_{t+1}^{r,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{w,i} = (1 + r_t)a_t^{w,i} + (1 - \tau_t)w_t - c_t^{w,i,j}$$

Where τ_t is the payroll tax rate and w_t the market wage. Decision rules for the retiree can be rewritten under the following form, noting $R_t = 1 + r_t$:

$$\begin{aligned} c_t^{r,i,j} &= \pi_t^r \left(\frac{R_t}{\gamma} a_t^{r,i,j} + h_t^{r,i,j} \right) \quad \text{with} \quad \pi_t^r = 1 - \gamma \beta^\sigma R_{t+1}^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r} \\ h_t^{r,i,j} &= e_t + \frac{\gamma h_{t+1}^{r,i,j}}{R_{t+1}} \\ V_t^{r,i,j} &= (\pi_t^r)^{\frac{-1}{\rho}} c_t^{r,i,j} \end{aligned}$$

These closed-form solutions – from Gertler (1999) [37] – are derived in Appendix 3.F and are useful in order to aggregate individual decisions. Consumption is linear in financial a_t^r and human wealth h_t^r , the latter being the discounted sum of pensions. π_t^r is thus the marginal propensity to consume (MPC), which does not depend on the agent life-cycle history.

Decision rules for the worker can be written

$$\begin{aligned} c_t^{w,i} &= \pi_t^w \left(R_t a_t^{w,i} + h_t^{w,i} \right) \quad \text{with} \quad \pi_t^w = 1 - \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^w}{\pi_{t+1}^w} \quad \text{and} \quad \Omega_{t+1} = \omega + (1 - \omega) \epsilon_{t+1}^{\frac{1}{1-\sigma}} \\ h_t^{w,i} &= (1 - \tau_t)w_t + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^{w,i}}{1 + r_{t+1}} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) \frac{h_{t+1}^{r,i,t+1}}{1 + r_{t+1}} \\ V_t^{w,i} &= (\pi_t^w)^{\frac{-1}{\rho}} c_t^{w,i} \end{aligned}$$

Where $\epsilon_t = \frac{\pi_t^r}{\pi_t^w}$ is the ratio of MPC across age. Appendix 3.C shows that $\epsilon_t > 1$: retirees have a higher MPC than workers. The interpretation of such a property is straightforward: once an worker becomes a retiree, he increases his consumption because of the finiteness of life.

3.2.2 Aggregation

Decisions rules

Despite the heterogeneity induced by stochastic aging, as consumption is linear in wealth, and since MPC do not depend on the life-cycle history of agents, decision rules can be aggregated like in a representative agent model. The law of motion of wealth is therefore given by:

$$\begin{aligned} A_{t+1}^w &= \omega \left[(1 + r_t)A_t^w + (1 - \tau_t)w_t N_t^w - C_t^w \right] \\ A_{t+1}^r &= (1 + r_t)A_t^r + e_t N_t^r - C_t^r + (1 - \omega) \left[(1 + r_t)A_t^w + (1 - \tau_t)w_t N_t^w - C_t^w \right] \end{aligned}$$

Consumption is given by

$$\begin{aligned} C_t^w &= \pi_t^w (R_t A_t^w + H_t^w) \\ C_t^r &= \pi_t^r (R_t A_t^r + H_t^r) \end{aligned}$$

Note that once aggregated, γ disappears from the consumption's expression, reflecting the life insurance. Aggregated human wealth takes the form

$$H_t^r = E_t + \gamma \frac{H_{t+1}^r}{(1+r_{t+1})(1+n)}$$

$$H_t^w = (1-\tau_t)w_t N_t^w + \frac{\omega}{\Omega_{t+1}} \frac{H_{t+1}^w}{(1+n)(1+r_{t+1})} + \left(1 - \frac{\omega}{\Omega_{t+1}}\right) \frac{H_{t+1}^r}{(1+r_{t+1})(1+n)\psi}$$

Social security

The social security's budget must be balanced, so that

$$e_t \psi = \tau_t w_t$$

Noting the replacement rate $\mu_t = e_t/w_t$, we have

$$\tau_t = \mu_t \psi$$

Firms

Households savings and labor are used by firms to produce, with a standard technology

$$Y_t = K_t^\alpha (N_t^w)^{1-\alpha}$$

Where standard conditions apply

$$w_t = (1-\alpha) \frac{Y_t}{N_t^w} \quad \text{and} \quad r_t = \alpha \frac{Y_t}{K_t} - \delta$$

where δ is the depreciation rate.

3.2.3 Equilibrium

A recursive dynamic equilibrium is defined as a sequence of prices and quantities where (i) households maximize their intertemporal utility, (ii) firms maximize profits, (iii) social security budget is balanced and (iv) financial, labor and good markets clear. At a steady-state equilibrium, all variables that do not depend of the individual life-cycle history – *i.e.* aggregate quantities, prices and MPCs – are constant. For a complete description of the recursive dynamic and steady-state equilibrium, see Appendix 3.F.

3.2.4 Numerical Simulation

Let us now simulate numerically the impact of an increase/decrease in the generosity of public pension (measured by μ) on welfare. Before moving on, we need to define the welfare criterion. In a model where agents are *ex ante* homogeneous but *ex post* heterogeneous, a natural criterion is the *ex ante* intertemporal welfare of a newborn agent. The social security is welfare improving if an agent prefers to be born a world where there is a social security to another one where it is absent.

We use a rather standard calibration. The EIS is set to $\sigma = 0.5$, such a value being standard in the

literature. The share of capital in the total income is $\alpha = 1/3$, and the annual depreciation rate of capital is $\delta = 0.1$, as the frequency is annual. Agents spend 45 years working (newborn at 20) in average – i.e. $\frac{1}{1-\omega} = 45$ –, and expect to spend 15 years in the state retiree, –i.e. $\frac{1}{1-\gamma} = 15$. The replacement rate μ is set at 0.5. In order to rule out the possibility of a dynamically inefficient equilibrium at the initial steady state, we assume no demographic growth, $n = 0$.

For several targets of interest rates (to which a given β that clears financial markets is assigned), the corresponding steady-state equilibria are computed. All the other parameters being kept constant, the replacement rate rate is then lowered to $\mu = 0$, and increased to $\mu = 0.75$.

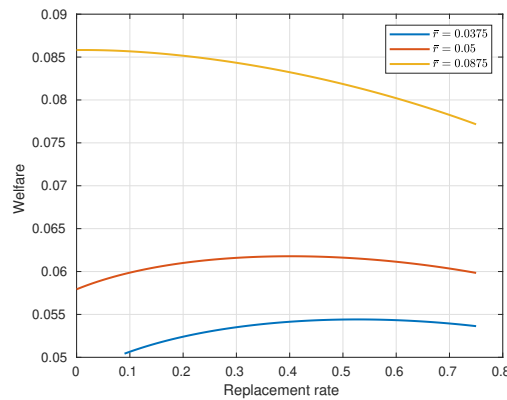


Figure 3.1: Newborn agent welfare

Note: Welfare is computed in absolute value, as a function of the replacement rate. Each curve represents an economy where the interest rate is equal to \bar{r} when $\mu = 0.5$. Blue curve is truncated below $\mu = 0.09$ as interest rate becomes negative.

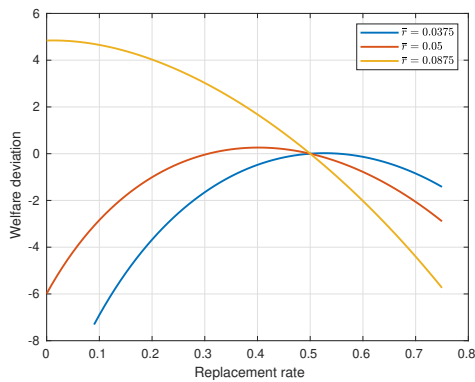


Figure 3.2: Newborn agent welfare's deviation in %

Note: the deviation is expressed in percentage, the initial situation being an equilibrium with $\mu = 0.5$ where the interest rate is equal to \bar{r} , at the intersection of all curves. Blue curve is truncated below $\mu = 0.09$ as interest rate becomes negative.

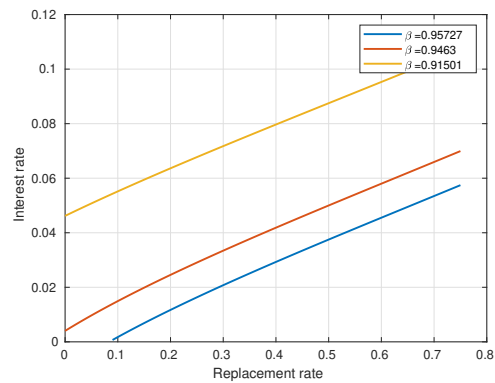


Figure 3.3: Equilibrium interest rate

Note: Each curve gives the interest rate as a function of the replacement rate, for a given β . From the bottom to the top, the targeted interest rate for $\mu = 0.5$ is $\bar{r} = 0.0375$, $\bar{r} = 0.05$, and $\bar{r} = 0.0875$. Blue curve is truncated below $\mu = 0.09$ as interest rate becomes negative.

Figures 3.1 and 3.2 shows the impact of the introduction of a social security system on welfare when

the wealth effect dominates – *i.e.* $\sigma < 1$ –, and confirms the initial intuition. If the interest rate is low enough, PAYG pension systems can make the economy less inefficient, by redistributing wealth to the state "retiree". For instance, when the interest rate associated to the initial steady-state (where $\mu = 0.5$) is 0.05 (red curve), lowering the replacement rate leads to a decrease in welfare. As μ decreases, the saving motive becomes stronger and capital supply increases, which pushes down the interest rate, as one can see in Figure 3.3. Such a low level of r thus makes self insurance less profitable than fiscal redistribution, *i.e.* PAYG pension system is welfare increasing⁷, even if it reduces investment and marginal productivity of labor. However, if the initial steady-state interest rate is equal to 0.087 (yellow curve), social security is welfare decreasing. For such a high return on capital, individuals are better off by imperfectly insuring themselves. Removing pensions would in this case lead to an increase of the welfare of approximately 5%, and lower the interest rate to 4.7%, a value that is high enough to make even a slight introduction of pension system undesirable.

3.3 An extension with variable labor supply

We have assumed so far inelastic labor supply, which thus limits the distortions induced by taxation on labor income. The current section relaxes this assumption, and shows that social security keeps being welfare improving.

3.3.1 Decision rules

Retirees

Let us assume that agents take utility from the consumption of goods and leisure. At the steady-state (where aggregate variables and MPCs are assumed to be constant), preferences for a retiree are given by

$$V_t^{r,i,j} = \left[\left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^\rho + \beta \gamma (V_{t+1}^{r,i,j})^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{r,i,j} = \frac{R}{\gamma} a_t^{r,i,j} + (1 - \tau) w l_t^{r,i,j} + e - c_t^{r,i,j}$$

As in the case with inelastic labor supply, the consumption is linear in financial and human wealth

$$c_t^{r,i,j} = \pi^r \left(\frac{R}{\gamma} a_t^{r,i,j} + h_t^{r,i,j} \right) \quad \text{with } \pi^r = 1 - \gamma \beta^\sigma R^{\sigma-1}$$

Where the human wealth is given by

$$h_t^{r,i,j} = (1 - \tau) w l_t^{r,i,j} + e + \frac{\gamma h_{t+1}^{r,i,j}}{R}$$

⁷Note that social security here does not play the role of a public annuity market, as in Imrohoroğlu and Imrohoroğlu (1995) [49], as we assume the existence of a private perfect one. The welfare improving effect of social security thus only come from the partial insurance of the aging risk.

The labor supply being given by

$$l_t^{r,i,j} = 1 - \frac{\zeta c_t^{r,i,j}}{(1-\tau)w}$$

with $\zeta = \frac{1-\phi}{\phi}$. Upon aggregation, we can express all variables per retiree, and thus write them as constant variables. This let us write labor supply per retiree under the following form:

$$l^r = 1 - \frac{\zeta \pi^r (1+r)}{(1-\tau)w} \left(\frac{(1-\tau)w + e + a^r(1+r-\gamma)}{1+r-\gamma + \zeta(1+r)\pi^r} \right) \quad (3.3)$$

The impact of the mortality risk on the labor supply is ambiguous, and goes through two channels. On the one hand, a lower survival rate works as a higher discount rate⁸ and makes the agent poorer, as he simply expects a lower probability to get income in the future. The retiree thus chooses to consume less leisure, *i.e.* works more. On the other hand, a higher mortality risk increases the MPC which in turn decreases the labor supply; the agent wishes to consume his wealth at a higher pace, this consumption including *both* goods and leisure. Note also that the labor supply decreases with financial assets through a classical wealth effect; as seniors can be expected to hold more assets, their labor supply must be lower than the one of workers.

Workers

In the case of workers, the program writes

$$V_t^{w,i} = \left[\left((c_t^{w,i})^\phi (1 - l_t^{w,i})^{1-\phi} \right)^\rho + \beta [\omega V_{t+1}^{w,i} + (1-\omega)V_{t+1}^{r,i,t+1}]^\rho \right]^{\frac{1}{\rho}}$$

s.t. $a_{t+1}^{w,i} = Ra_t^{w,i} + (1-\tau)wl_t^{w,i} - c_t^{w,i}$

The consumption-leisure choice is given by

$$c_t^{w,i} = \pi^w \left(Ra_t^{w,i} + h_t^{w,i} \right)$$

$$l_t^{w,i} = 1 - \frac{\zeta c_t^{w,i}}{(1-\tau)w}$$

where $(1-\tau)w$ at the denominator symbolizes the substitution effect. The human wealth is given by

$$h_t^{w,i} = (1-\tau)wl_t^{w,i} + \frac{\omega}{\Omega} \frac{h_{t+1}^{w,i}}{R} + \left(1 - \frac{\omega}{\Omega} \right) \frac{h_{t+1}^{r,i,t+1}}{R}$$

The MPC and the aging risk Ω are expressed as follows:

$$\pi^w = 1 - \beta^\sigma (R\Omega)^{\sigma-1}$$

$$\Omega = \omega + (1-\omega) \left(\frac{\pi^r}{\pi^w} \right)^{\frac{1}{1-\sigma}}$$

⁸We have $\frac{\partial l^r}{\partial \gamma} < 0$ when $a^r < \frac{(1-\tau)w+e}{\zeta R \pi^r}$ (always the case when $e = 0$).

Where $\left(\frac{\pi^r}{\pi^w}\right)^{\frac{1}{1-\sigma}}$ is the MRS of wealth between states "retiree" and "worker".

The expressions of the MRS and of MPCs are exactly the same as in the case with inelastic labor. The intuition between the aging risk thus keeps being the same. The mortality risk increases the MPC of retirees: they thus increase the pace at which they consume *both* goods and leisure. When $\sigma > 1$, the substitution effect dominates and $\Omega < 1$, whereas when $\sigma < 1$, the wealth effect is stronger which leads to $\Omega > 1$.

3.3.2 Firms

Aggregate variables are denoted with capital letters. In the case of production, we have

$$Y = K^\alpha L^{1-\alpha}$$

Where $L = L^w + L^r$ and $K = A = A^w + A^r$ at the equilibrium. Standard condition apply:

$$w = (1 - \alpha)\frac{Y}{L} \quad \text{and} \quad r = \alpha\frac{Y}{K} - \delta$$

3.3.3 Social security

Social security's budget must be balanced at every period, so that

$$\tau wL = eN^r$$

Where pensions $e = \mu w$, μ being the replacement rate.

3.3.4 Steady-state equilibrium

Individual decisions are then aggregated the same way as in the previous section. A steady state equilibrium is defined as an equilibrium where prices and aggregate quantities are constant, and where (i) households maximize their intertemporal utility, (ii) firms maximize profits, (iii) social security budget is balanced and (iv) financial, labor and good markets clear. For a complete description of the recursive dynamic and steady-state equilibrium, see Appendix 3.H.

3.3.5 Numerical simulations

We then conduct the same experiment as in the case where labor supply is inelastic. Fixing the initial replacement rate μ at 0.5, several steady-states targeting different levels of interest rate are computed by adjusting β . The initial participation rate of workers l^w is also kept constant at 0.7 by controlling ϕ . Once the calibration done, we allow the replacement rate to move, and compute – among others – the resulting welfare gains/losses. As taxation distorts labor supply, we compare these gains/losses to a case where labor supply would be inelastic, the calibration being kept the same.

Results of the simulations are presented in Figure 3.4, 3.5 and 3.6. Just as in the previous section, despite the distortions it induces on labor supply, social security keeps being welfare-improving when

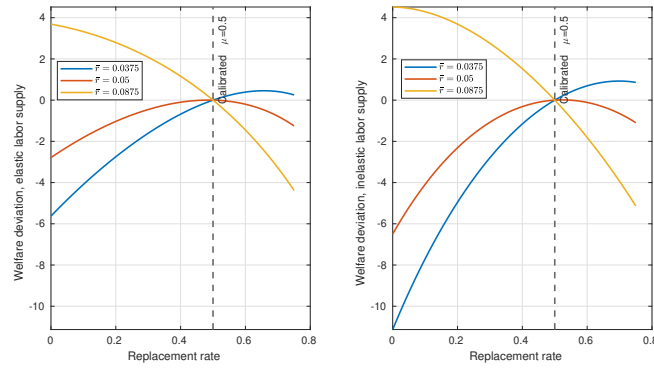


Figure 3.4: Deviation of newborn agent welfare following a variation in the replacement rate, elastic and inelastic labor supply.

Note: In the left panel is computed the deviation of welfare in percentage, the initial situation being an equilibrium with $\mu = 0.5$ (grey dotted line) and elastic labor supply. In the right panel, the initial situation is exactly the same as in the left one; however, as we assume here inelastic labor supply, as μ moves labor supply remains constant.

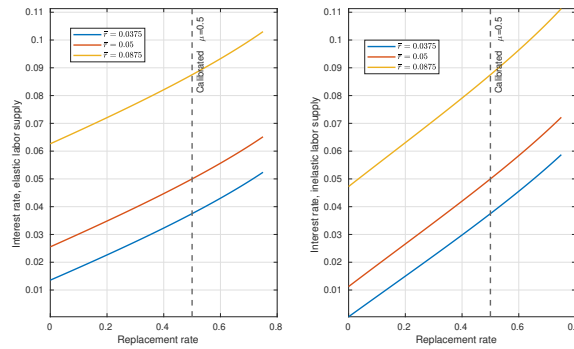


Figure 3.5: The impact of a pension reform on the interest rate

Note: The parameters β and ϕ associated with the blue curve are 0.994 and 0.709, the one with the red curve 0.982 and 0.712, 0.949 and 0.723 for the yellow curve.

interest rates are low enough. Let us remind ourselves that wealth has a higher value in state "retiree" than in "worker" because the mortality risk increases the pace at which goods and leisure are consumed, and therefore creates a need for even more wealth. In order to transfer assets towards the state of nature "retiree", workers have two imperfect tools at their disposal: self insurance and social security. When the initial interest rate \bar{r} is at 3.75% or 5%, social security is more profitable, and pensions cuts decrease welfare as one can see in Figure 3.4 (blue curve and red curves, left panel). Interestingly, removing social security in this context hurts less than when labor supply is inelastic. Indeed, when the initial interest rate is 3.75% (resp. 5%), a full removal of pensions decreases welfare by 5% (resp. 2.7%) when labor supply is elastic, and by 11% (resp. 6.5%) when it is inelastic. The intuition behind this result is straightforward: labor distortions induced by labor taxation limits the welfare increasing impact of social security.

However, if the initial interest rate is above 8.75% (yellow curve), it becomes more interesting to rely on

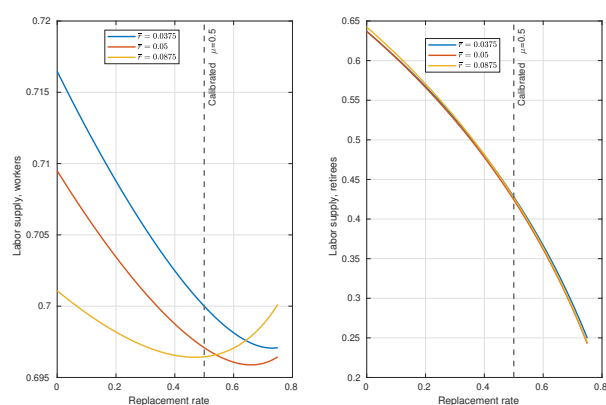


Figure 3.6: The impact of a pension reform on labor supply

Note: The parameters β and ϕ associated with the blue curve are 0.994 and 0.709, the one with the red curve 0.982 and 0.712, 0.949 and 0.723 for the yellow curve.

savings than distorting pensions, *i.e.* social security is welfare decreasing. It is here also worth noting the fact that in this context, removing social security provides more welfare when labor is inelastic than when it is elastic. Indeed, introducing social security in a context where it is dominated by self insurance hurts more agents who cannot adjust their labor supply to dampen the fiscal shock.

It is worth discussing the behavior of both workers and retiree's labor supply that follows an increase of the replacement rate. As one can see in Figure 3.6, as μ increases from 0.5 to 0.75, older workers choose to considerably lower their labor supply, while a pension cut leads to an increase of their participation rate. This is of course mainly due to a wealth effect, that appears at the numerator of the expression of l^r at equation (3.3): pension benefit makes retirees richer, and provides them an incentive to consume more leisure. As we are in a situation where the wealth effect dominates ($\sigma < 1$), social security indeed responds to the retirees' need to transfer part of their wealth from the state "worker" in order to consume more leisure. The impact of pension reforms on workers' labor supply is far less important, but still worth discussing. Cutting pensions from their initial level ($\mu = 0.5$) leads to a higher labor supply, due to a substitution effect: as there are less taxes and a higher wage rate – pension cuts being associated with more investment and thus a higher marginal productivity of labor –, workers have an incentive to work more. Increasing μ however does not necessarily reduce labor supply through this same substitution effect as a wealth effect – taxes indeed makes workers poorer ! – increases slightly the participation rate of workers. Note also that because labor supply increases as μ goes down, the capital over labor ratio does not rise as fast as in the case where labor supply is inelastic. As a result, the interest rate decreases slower following a pension cut, as one can see in Figure 3.5.

4 Conclusion

Aging follows an uncertain process, as individuals do not know how long they will be able to work, or how long they are going to live. Using non standard preferences, where agents are risk neutral while keeping a finite desire for consumption smoothness, this chapter showed that the "aging risk", – the risk to be hit by an aging shock that makes the agent face a survival probability – is the source of a desire for insurance of agents, despite their risk-neutrality. Because they need to consume more once their mortality risk increases, agents value wealth differently between states "young" and "old". This difference creates a need for insurance, which helps transferring wealth towards the age-state where its value is higher.

What if market failures make insurance contracts unavailable? This chapter also showed that in this case, there was room for public intervention to improve welfare, through fiscal policy. In an economy where agents do not know with certainty how long they will be able to stay on the labor market before switching to an "age-state" where they will have to retire and face a surviving probability, if agents value wealth in the age-state "retiree" more than in the state "worker", a public PAYG pension system is welfare improving, even if it creates distortions in agents' labor supply decisions. However, this welfare increasing property of social security only exists when interest rates are sufficiently low – yet in a dynamically efficient equilibrium. As social security is an imperfect substitute for perfect private insurance, in an environment characterized by high interest rates, it is indeed more profitable to be imperfectly self-insured.

Appendix

3.A The old agent's program

The old agent's program is

$$V_t^{o,i,j} = \max_{c_t^{o,i,j}, a_{t+1}^{o,i,j}} \left[\left(c_t^{o,i,j} \right)^\rho + \beta \gamma \left(V_{t+1}^{o,i,j} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{o,i,j} = \frac{1+r_t}{\gamma} a_t^{o,i,j} + w_t^o - c_t^{o,i,j}$$

F.O.Cs are the following:

$$\frac{\partial V_t^{o,i,j}}{\partial c_t^{o,i,j}} = 0 \Leftrightarrow \left(c_t^{o,i,j} \right)^{\rho-1} \left(V_t^{o,i,j} \right)^{1-\rho} = \mu_t^{o,i,j}$$

$$\frac{\partial V_t^{o,i,j}}{\partial a_{t+1}^{o,i,j}} = 0 \Leftrightarrow \gamma \beta \frac{\partial V_{t+1}^{o,i,j}}{\partial a_{t+1}^{o,i,j}} \left(V_{t+1}^{o,i,j} \right)^{\rho-1} \left(V_t^{o,i,j} \right)^{1-\rho} = \mu_t^{o,i,j}$$

Which gives

$$\left(c_t^{o,i,j} \right)^{\rho-1} = \gamma \beta \frac{\partial V_{t+1}^{o,i,j}}{\partial a_{t+1}^{o,i,j}} \left(V_{t+1}^{o,i,j} \right)^{\rho-1}$$

This is a rather standard equation: the agent must be indifferent when given an extra euro between consuming it today or saving it.

The envelope condition gives:

$$\frac{\partial V_t^{o,i,j}}{\partial a_t^{o,i,j}} = \mu_t^{o,i,j} \frac{R_t}{\gamma}$$

an extra euro of asset today means that the agent can consume $(1+r_t)/\gamma$ more, it therefore increase his utility by $\mu_t^{o,i,j} \frac{R_t}{\gamma}$. We therefore get the Euler equation:

$$c_{t+1}^{o,i,j} = (\beta R_{t+1})^\sigma c_t^{o,i,j}$$

Where $\sigma = \frac{1}{1-\rho}$ is the EIS.

3.B The youngster's program

The program of a youngster born in i is given by:

$$V_t^{y,i} = \left[\left(c_t^{y,i} \right)^\rho + \beta \left(\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{y,i} = (1+r_t) a_t^{y,i} + w_t^y - c_t^{y,i}$$

Reminding that $a_{t+1}^{y,i} = a_{t+1}^{o,i,t+1}$, F.O.Cs are the following:

$$\frac{\partial V_t^{y,i}}{\partial c_t^{y,i}} = 0 \Leftrightarrow \left(c_t^{y,i} \right)^{\rho-1} \left(V_t^{y,i} \right)^{1-\rho} = \mu_t^{y,i} \quad (3.4)$$

$$\frac{\partial V_t^{y,i}}{\partial a_{t+1}^{y,i}} = 0 \Leftrightarrow \beta \left[\omega \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} + (1-\omega) \frac{\partial V_{t+1}^{o,i,t+1}}{\partial a_{t+1}^{o,i,t+1}} \right] \left[\omega V_t^{y,i} + (1-\omega) V_t^{o,i,t+1} \right]^{\rho-1} \left(V_t^{y,i} \right)^{1-\rho} = \mu_t^{y,i} \quad (3.5)$$

The envelope condition gives:

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = \mu_t^{y,i} R_t \quad (3.6)$$

Combining (3.4) and (3.6), we get the expression for the marginal value of wealth:

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = R_t \left(\frac{V_t^{y,i}}{c_t^{y,i}} \right)^{1-\rho} \quad (3.7)$$

While combining (3.5) and (3.6) we get a forward equation for the value of wealth.

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = R_t \beta \left[\omega \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} + (1-\omega) \frac{\partial V_{t+1}^{o,i,t+1}}{\partial a_{t+1}^{o,i,t+1}} \right] \left[\frac{\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1}}{V_t^{y,i}} \right]^{\rho-1} \quad (3.8)$$

Substituting the expression for the value of wealth (3.7) in its recursive formulation (3.8), we get

$$\left(\frac{V_t^{y,i}}{c_t^{y,i}} \right)^{1-\rho} = \beta R_{t+1} \left[\omega \left(\frac{V_{t+1}^{y,i}}{c_{t+1}^{y,i}} \right)^{1-\rho} + (1-\omega) \left(\frac{V_{t+1}^{o,i,t+1}}{c_{t+1}^{o,i,t+1}} \right)^{1-\rho} \right] \left[\frac{\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1}}{V_t^{y,i}} \right]^{\rho-1}$$

$$\Rightarrow \left(c_t^{y,i} \right)^{\rho-1} = \beta R_{t+1} \left[\omega + (1-\omega) \left(\frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} \right] \left(\frac{V_{t+1}^{y,i}}{c_{t+1}^{y,i}} \right)^{1-\rho} \left[\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right]^{\rho-1}$$

$$\Rightarrow \left(c_t^{y,i} \right)^{\rho-1} = \beta R_{t+1} \left[\omega + (1-\omega) \left(\frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} \right] \left[\omega c_{t+1}^{y,i} + (1-\omega) \frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} c_{t+1}^{o,i,t+1} \right]^{\rho-1}$$

$$\Rightarrow \omega c_{t+1}^{y,i} + (1-\omega) \frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} c_{t+1}^{o,i,t+1} = \left(\beta R_{t+1} \left[\omega + (1-\omega) \left(\frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} \right] \right)^\sigma c_t^{y,i}$$

Noting $\Lambda_{t+1}^{1-\rho} = \left(\frac{V_{t+1}^{o,i,t+1}/c_{t+1}^{o,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} = \frac{\partial V_{t+1}^{o,i,t+1}/\partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i}/\partial a_{t+1}^{y,i}}$ the MRS between old age and young age wealth⁹, and $\Omega_{t+1} = \omega + (1-\omega)\Lambda_{t+1}^{1-\rho}$, we get a modified Euler equation:

$$\omega c_{t+1}^{y,i} + (1-\omega) c_{t+1}^{o,i,t+1} \Lambda_{t+1} = (\beta R_{t+1} \Omega_{t+1})^\sigma c_t^{y,i}$$

which implies that $\Lambda_{t+1} = \left(\frac{\partial V_{t+1}^{o,i,t+1}/\partial a_{t+1}^{o,i,t+1}}{\partial V_{t+1}^{y,i}/\partial a_{t+1}^{y,i}} \right)^\sigma$.

3.C Closed form solutions

In order to prove Proposition 1 and 2, we first need to rewrite the optimal consumption-saving choice of agents under the form of closed-form solutions.

3.C.1 Old agents' closed form solution

For the closed-form solution of old agents, let us now make the following conjectures: both consumption and the value function are linear in wealth.

$$c_t^{o,i,j} = \pi_t^o \left(\frac{R_t}{\gamma} a_t^{o,i,j} + h_t^o \right) \quad (3.9)$$

$$V_t^{o,i,j} = \Delta_t c_t^{o,i,j} \quad (3.10)$$

π^o being the MPC. The budget constraint remains

$$a_{t+1}^{o,i,j} = \frac{R_t}{\gamma} a_t^{o,i} + w_t^o - c_t^{o,i,j} \quad (3.11)$$

And we define the human wealth as the discounted sum of lump sum incomes

$$h_t^o = w_t^o + \frac{\gamma h_{t+1}^o}{1+r_{t+1}} \quad (3.12)$$

The Euler equations therefore now writes, using the consumption conjecture (3.9):

$$\pi_{t+1}^o \left(\frac{R_{t+1}}{\gamma} a_{t+1}^{o,i,j} + h_{t+1}^o \right) = (\beta R_{t+1})^\sigma \pi_t^o \left(\frac{R_t}{\gamma} a_t^{o,i,j} + h_t^o \right)$$

Which thus can be rewritten

$$a_{t+1}^{o,i,j} + \frac{\gamma h_{t+1}^o}{1+r_{t+1}} = \gamma \beta^\sigma R_{t+1}^{\sigma-1} \frac{\pi_t^o}{\pi_{t+1}^o} \left[\frac{R_t}{\gamma} a_t^{o,i,j} + h_t^o \right]$$

⁹We will see in the next section that Λ does not depend on life-cycle history.

Using the definition of human wealth (3.12), we have $\frac{\gamma h_{t+1}^o}{R_{t+1}} = h_t^o - w_t^o$. Also, the budget constraint states that $a_{t+1}^{o,i,j} = \frac{R_t}{\gamma} a_t^{o,i,j} + w_t^o - c_t^{o,i,j}$. Using both, we can thus write the LHS as $a_{t+1}^{o,i,j} + \frac{\gamma h_{t+1}^o}{1+r_{t+1}} = \frac{R_t}{\gamma} a_t^{o,i,j} + h_t^o - c_t^{o,i,j}$. Using the conjecture about consumption (3.9), we thus get $a_{t+1}^{o,i,j} + \frac{\gamma h_{t+1}^o}{1+r_{t+1}} = \left[\frac{1+r_t}{\gamma} a_t^{o,i,j} + h_t^o \right] (1-\pi_t^o)$. We are in position to write

$$\pi_t^o = 1 - \gamma \beta^\sigma R_{t+1}^{\sigma-1} \frac{\pi_t^o}{\pi_{t+1}^o}$$

Which is the expression for the marginal propensity to consume.

To find the solution for the value function, use the conjecture (3.10), and substitute it in the objective

$$\Delta_t^o c_t^{o,i,j} = \left[\left(c_t^{o,i,j} \right)^{\frac{\sigma-1}{\sigma}} + \beta \gamma \left(\Delta_{t+1}^o c_{t+1}^{o,i,j} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

If we substitute the Euler equation

$$\Delta_t^o c_t^{o,i,j} = \left[\left(c_t^{o,i,j} \right)^{\frac{\sigma-1}{\sigma}} + \beta \gamma \left(\Delta_{t+1}^o (\beta R_{t+1})^\sigma c_{t+1}^{o,i,j} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

Which simplifies to

$$\left(\Delta_t^o \right)^{\frac{\sigma-1}{\sigma}} = 1 + \gamma \left(\Delta_{t+1}^o \right)^{\frac{\sigma-1}{\sigma}} \beta^\sigma R^{\sigma-1}$$

If we rewrite the expression for the MPC

$$\frac{1}{\pi_t^o} = 1 + \gamma \beta^\sigma R_{t+1}^{\sigma-1} \frac{1}{\pi_{t+1}^o}$$

We then identify $\Delta_t^o = (\pi_t^o)^{\frac{\sigma}{1-\sigma}}$.

If we define the steady-state as a situation where variables independent of the life-cycle history of the agent are constant, then we have

$$\begin{aligned} \pi^o &= 1 - \gamma \beta^\sigma R^{\sigma-1} \\ c_t^{o,i,j} &= \pi^o \left(\frac{R_t}{\gamma} a_t^{o,i,j} + h^o \right) \\ h^o &= \frac{1+r}{1+r-\gamma} w^o \\ V_t^{o,i,j} &= (\pi^o)^{\frac{\sigma}{\sigma-1}} c_t^{o,i,j} \\ a_{t+1}^{o,i,j} &= \frac{R_t}{\gamma} a_t^{o,i,j} + w - c_t^{o,i,j} \end{aligned}$$

3.C.2 Young agents' closed form solution

Let us now make the same conjectures as for the old agent, that is to say consumption and the value function are linear in wealth.

$$c_t^{y,i} = \pi_t^y \left(R_t a_t^{y,i} + h_t^y \right) \quad (3.13)$$

$$V_t^{y,i} = \Delta_t^y c_t^{y,i} \quad (3.14)$$

Where h_t^y is the human wealth, that takes the form

$$h_t^y = w_t^y + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^y}{1+r_{t+1}} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) \frac{h_{t+1}^o}{1+r_{t+1}}$$

Note that future streams of income are re-weighted by the term Ω . If the marginal value of wealth is higher in the state "old", then $\Omega > 1$: the weight associated with h_{t+1}^y decreases while the one associated with h_{t+1}^o increases, as flows of income in the state "old" provide more utility.

Using the value function of the old agent, let us also conjecture that $\Delta_t^y = (\pi_t^y)^{\frac{\sigma}{1-\sigma}}$. Let us finally note $\epsilon_t = \pi_t^o / \pi_t^y$ the ratio of MPC between an old and a young agent. These imply

$$\Omega_{t+1} = \omega + (1-\omega)\epsilon_{t+1}^{\frac{1}{1-\sigma}} \quad \text{and} \quad \Lambda_{t+1} = \epsilon_{t+1}^{\frac{\sigma}{1-\sigma}}$$

that is to say the MRS of wealth between age-states "old" and "young" is $\epsilon_{t+1}^{\frac{1}{1-\sigma}}$. Assuming also that $c_t^{o,i,t} = \pi_t^o (R_t a_t^{o,i,t} + h_t^o)$ – agents who has just switched to the state "old" do not get any dividends from the mutual fund –, the Euler equation (3.2) now becomes:

$$\begin{aligned} \omega \pi_{t+1}^y (R_{t+1} a_{t+1}^{y,i} + h_{t+1}^y) + (1-\omega) \epsilon_{t+1}^{\frac{\sigma}{1-\sigma}} \pi_{t+1}^o (R_{t+1} a_{t+1}^{o,i,t+1} + h_{t+1}^o) &= (\beta R_{t+1} \Omega_{t+1})^\sigma \pi_t^y (R_t a_t^{y,i} + h_t^y) \\ \Rightarrow \omega \pi_{t+1}^y (R_{t+1} a_{t+1}^{y,i} + h_{t+1}^y) + (1-\omega) \epsilon_{t+1}^{\frac{1}{1-\sigma}} \pi_{t+1}^y (R_{t+1} a_{t+1}^{o,i,t+1} + h_{t+1}^o) &= (\beta R_{t+1} \Omega_{t+1})^\sigma \pi_t^y (R_t a_t^{y,i} + h_t^y) \end{aligned}$$

Using the fact that $a_{t+1}^{o,i,t+1} = a_{t+1}^{y,i}$ and the definition of Ω_{t+1} , we get

$$\Omega_{t+1} a_{t+1}^{y,i} + \frac{\omega h_{t+1}^y + (1-\omega) \epsilon_{t+1}^{\frac{1}{1-\sigma}} h_{t+1}^o}{R_{t+1}} = \beta^\sigma R_{t+1}^{\sigma-1} \Omega_{t+1}^\sigma \frac{\pi_t^y}{\pi_{t+1}^y} (R_t a_t^{y,i} + h_t^y)$$

Using the definition of budget constraint, the conjecture about the linearity of consumption (3.13) and dividing by Ω_{t+1} :

$$R_t a_t^{y,i} + w_t^y - \pi_t^y (R_t a_t^{y,i} + h_t^y) + \frac{\omega h_{t+1}^y + (1-\omega) \epsilon_{t+1}^{\frac{1}{1-\sigma}} h_{t+1}^o}{R_{t+1} \Omega_{t+1}} = \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} (R_t a_t^{y,i} + h_t^y)$$

Then, using the definition of human wealth:

$$(1 - \pi_t) (R_t a_t^{y,i} + h_t^y) = \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} (R_t a_t^{y,i} + h_t^y)$$

we thus get

$$\pi_t^y = 1 - \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y}$$

To check the conjecture, rewrite the modified Euler equation as

$$\begin{aligned} & \omega \left(\pi_{t+1}^{y,i} \right)^{\frac{\sigma}{\sigma-1}} V_{t+1}^{y,i} + (1-\omega) \left(\epsilon_{t+1} \pi_{t+1}^y \right)^{\frac{\sigma}{\sigma-1}} \epsilon_{t+1}^{\frac{\sigma}{1-\sigma}} V_{t+1}^{o,i,t+1} = (\beta R_{t+1} \Omega_{t+1})^\sigma \left(\pi_t^y \right)^{\frac{\sigma}{\sigma-1}} V_t^{y,i} \\ \Rightarrow & \beta \left[\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right]^{\frac{\sigma-1}{\sigma}} = \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} \left(V_t^{y,i} \right)^{\frac{\sigma-1}{\sigma}} \\ \Rightarrow & V_t^{y,i} = \left[\left(\pi_t^{y,i} \right)^{\frac{\sigma-1}{\sigma}} + \beta \left[\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right]^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

which is consistent.

3.D Proof of Proposition 1

Let us first note that from the closed form solutions, we have

$$\begin{aligned} & \frac{\partial V_{t+1}^{o,i,t+1}}{\partial a_{t+1}^{o,i,j}} = \left(\pi_{t+1}^o \right)^{\frac{1}{1-\sigma}} R_{t+1} \quad \text{and} \quad \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} = \left(\pi_{t+1}^y \right)^{\frac{1}{1-\sigma}} R_{t+1} \\ \Leftrightarrow & \frac{\partial V_{t+1}^{o,i,j}}{\partial a_{t+1}^{o,i,j}} = \left(\epsilon_{t+1} \pi_{t+1}^y \right)^{\frac{1}{1-\sigma}} R_{t+1} \quad \text{and} \quad \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} = \left(\pi_{t+1}^y \right)^{\frac{1}{1-\sigma}} R_{t+1} \end{aligned}$$

Let us first show by contradiction that $\epsilon > 1$, *i.e.* the MPC of old agents is greater than the MPC of youngsters (this proof is given by Gertler (1999) [37]). Let us assume that $\epsilon \leq 1$. Using the steady-state equations of π^y and π^o , we can write $\pi^o - \pi^y = \pi^y(\epsilon - 1) = \beta^\sigma R^{\sigma-1} (\Omega^{\sigma-1} - \gamma)$. Assuming $\epsilon \leq 1$ thus leads to $\gamma \geq \Omega^{\sigma-1}$. Since $\gamma < 1$, we must have $\Omega^{\sigma-1} < 1 \Leftrightarrow \epsilon > 1$. Since there is a contradiction, we must have $\epsilon > 1$.

As $\epsilon_t > 1$, we get

$$\frac{\partial V_t^{o,i,t}}{\partial a_t^{o,i,t}} > \frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} \Leftrightarrow \sigma < 1$$

and

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} > \frac{\partial V_t^{o,i,t}}{\partial a_t^{o,i,t}} \Leftrightarrow \sigma > 1$$

3.E Proof of Proposition 2

When $\sigma < 1$

When the contract pays the agent if he switches to the state "old", the program is thus

$$V_t^{y,i} = \max_{c_t^{y,i}, a_{t+1}^{y,i}, b_{t+1}^{y,i}} \left[\left(c_t^{y,i} \right)^\rho + \beta \left(\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t.} \begin{cases} a_{t+1}^{y,i} = (1+r_t) a_t^{y,i} + w_t^y - p_t b_t^{y,i} - c_t^{y,i} \\ b_{t+1}^{y,i} \geq 0 \end{cases}$$

The pricing condition is given by the following no-profit condition:

$$\underbrace{p_{t+1} B_{t+1}}_{\text{Primes}} = \underbrace{(1-\omega) B_{t+1}}_{\text{Indemnities}} \Leftrightarrow p = (1-\omega)$$

Where B_{t+1} is the total amount of insurance contracts. Noting λ_t the Lagrange multiplier associated with the non-negativity constraint, $\mu_t^{y,i}$ the one associated with the budget constraint of a youngster, $\mu_t^{o,i,j}$ the one associated of an old individual, we can write the first order condition of the insurance contract as

$$\lambda_{t+1} = V_t^{y,i} \beta \left[\omega \left(V_{t+1}^{y,i} \right)^{1-\rho} + (1-\omega) V_{t+1}^{o,i,t+1} \right]^{\rho-1} \left[\omega p_{t+1} \mu_{t+1}^{y,i} - (1-\omega)(1-p_{t+1}) \mu_{t+1}^{o,i,t+1} \right]$$

From the Kuhn and Tucker condition and the pricing condition $p_t = 1 - \omega$, we must have

$$\lambda_{t+1} \geq 0 \Leftrightarrow \mu_{t+1}^{y,i} \geq \mu_{t+1}^{o,i,t+1}$$

If $\mu_{t+1}^{y,i} > \mu_{t+1}^{o,i,t+1}$, we then have a corner solution $b_{t+1} = 0$, and we thus can use closed form solutions derived above. From Proposition 1, we know that the marginal value wealth in state "young" is higher than the one in state "old" if and only if $\sigma > 1$. The corner solution thus arises – *i.e.* young agents refuse an insurance contract that pays back if they switch to state "old" – when $\sigma > 1$. Also, when $\lambda_{t+1} = 0$, we have $\mu_{t+1}^{y,i} = \mu_{t+1}^{o,i,t+1}$, which can never happen in the closed-form solutions – remember that marginal value of wealth is always strictly different between age-states – which thus implies that $b_{t+1} > 0$ when $\sigma < 1$. A young agent is thus willing to buy an insurance contract that pays him if he becomes "old" in the next period if and only if $\sigma < 1$.

When $\sigma > 1$

Now the program writes

$$V_t^{y,i} = \max_{c_t^{y,i}, a_{t+1}^{y,i}, b_{t+1}^{y,i}} \left[\left(c_t^{y,i} \right)^\rho + \beta \left(\omega V_{t+1}^{y,i} + (1-\omega) V_{t+1}^{o,i,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } \begin{cases} a_{t+1}^{y,i} = (1+r_t)a_t^{y,i} + w_t^y - p_t b_t^{y,i} - c_t^{y,i} + b_t^{y,i} \\ b_{t+1}^{y,i} \geq 0 \end{cases}$$

And the pricing condition from insurance companies is $p_t = \omega$. The first order condition is

$$\lambda_{t+1} = V_t^{y,i} \beta \left[\omega \left(V_{t+1}^{y,i} \right)^{1-\rho} + (1-\omega) V_{t+1}^{o,i,t+1} \right]^{\rho-1} \left[(1-\omega) p_{t+1} \mu_{t+1}^{o,i,t+1} - \omega (1-p_{t+1}) \mu_{t+1}^{y,i} \right]$$

From the Kuhn and Tucker condition and the pricing condition $p_t = \omega$, we must have

$$\lambda_{t+1} \geq 0 \Leftrightarrow \mu_{t+1}^{o,i,t+1} \geq \mu_{t+1}^{y,i}$$

If $\mu_{t+1}^{o,i} > \mu_{t+1}^{y,i,t+1}$, we then have a corner solution $b_{t+1} = 0$, and we thus can use closed form solutions derived above. From Proposition 1, we know that the marginal value wealth in state "old" is higher than the one in state "young" if and only if $\sigma < 1$. The corner solution thus arises – *i.e.* young agents refuse an insurance contract that pays back if they stay in state "young" – when $\sigma < 1$. Also, when $\lambda_{t+1} = 0$, we have $\mu_{t+1}^{y,i} = \mu_{t+1}^{o,i,t+1}$, which can never happen in the closed-form solutions – remember that marginal value of wealth is always strictly different between age-states – which thus implies that $b_{t+1} > 0$ when $\sigma > 1$. A young agent is thus willing to buy an insurance contract that pays him if stays "young" in the next period if and only if $\sigma > 1$.

3.F Equilibrium with inelastic labor supply

3.F.1 Aggregation

Let us note $N_t^{r,i,j}$ the number of agents born in t , retired in j who are currently in t , and $N_t^{w,i}$ its counterpart for workers. Total wealth of workers is given by

$$\begin{aligned} A_t^w &= \sum_{i=1}^t a_t^{w,i} N_t^{w,i} \\ &= (1-\omega) \sum_{i=1}^{t-1} a_t^{w,i} N_t^{w,i} + \underbrace{a_t^{w,t} N_t^{w,t}}_{\text{Wealth of newborn, null}} \\ &= (1-\omega) \sum_{i=1}^{t-1} \left[(1+r_{t-1}) a_{t-1}^{w,i} + (1-\tau_{t-1}) w_{t-1} - c_{t-1}^{w,i} \right] N_t^{w,i} \\ &= (1-\omega) \left[(1+r_{t-1}) A_{t-1}^w + (1-\tau_{t-1}) w_{t-1} N_{t-1}^w - C_{t-1}^w \right] \end{aligned}$$

While total wealth for retirees is given by

$$A_t^r = \underbrace{\sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} \frac{a_t^{r,i,j}}{\gamma} N_t^{r,i,j}}_{\text{Wealth coming last period's retirees}} + \underbrace{\sum_{i=1}^{t-1} a_t^{r,i,t} N_t^{r,i,t}}_{\text{Wealth coming last period's workers}}$$

$$\begin{aligned}
&= \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} a_t^{r,i,j} N_{t-1}^{r,i,j} + \sum_{i=1}^{t-1} a_t^{r,i,t} N_t^{r,i,t} \\
&= \sum_{i=1}^{t-3} \sum_{j=i+1}^{t-2} \left\{ \frac{1+r_{t-1}}{\gamma} a_{t-1}^{r,i,j} N_{t-1}^{r,i,j} \right\} + (1+r_{t-1}) \sum_{i=1}^{t-2} \{ a_{t-1}^{r,i,t-1} N_{t-1}^{r,i,t-1} \} + e_{t-1} \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} \{ N_{t-1}^{r,i,j} \} - \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} c_{t-1}^{r,i,j} N_{t-1}^{r,i,j} \\
&+ \sum_{i=1}^{t-1} a_t^{r,i,t} N_t^{r,i,t} \\
&= (1+r_{t-1}) \left[\sum_{i=1}^{t-3} \sum_{j=i+1}^{t-2} \{ a_{t-1}^{r,i,j} N_{t-2}^{r,i,j} \} + \sum_{i=1}^{t-2} \{ a_{t-1}^{r,i,t-1} N_{t-1}^{r,i,t-1} \} \right] + e_{t-1} \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} \{ N_{t-1}^{r,i,j} \} - \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} c_{t-1}^{r,i,j} N_{t-1}^{r,i,j} \\
&+ (1-\omega) \sum_{i=1}^{t-1} \left((1+r_{t-1}) a_{t-1}^{w,i} + (1-\tau_{t-1}) w_{t-1} - c_{t-1}^{w,i} \right) N_{t-1}^{w,i} \\
&= (1+r_{t-1}) A_{t-1}^r + e_{t-1} N_{t-1}^r - C_{t-1}^r + (1-\omega) \left[(1+r_{t-1}) A_{t-1}^w + (1-\tau_{t-1}) w_{t-1} N_{t-1}^w - C_{t-1}^w \right]
\end{aligned}$$

Human wealth can be aggregated as follows for retirees:

$$\begin{aligned}
h_t^r &= e_t + \gamma \frac{h_{t+1}^r}{1+r_{t+1}} \\
\Rightarrow h_t^r N_t^r &= e_t N_t^r + \gamma \frac{h_{t+1}^r N_{t+1}^r}{(1+r_{t+1})(1+n^r)} \\
\Rightarrow H_t^r &= E_t + \gamma \frac{H_{t+1}^r}{(1+r_{t+1})(1+n^r)}
\end{aligned}$$

While for workers, we have

$$\begin{aligned}
h_t^w &= (1-\tau_t) w_t + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^w}{1+r_{t+1}} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) h_{t+1}^r \\
\Rightarrow h_t^w N_t^w &= (1-\tau_t) w_t N_t^w + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^w N_{t+1}^w}{(1+n)(1+r_{t+1})} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) \frac{h_{t+1}^r N_{t+1}^r}{(1+r_{t+1})(1+n)\psi} \\
\Rightarrow H_t^w &= (1-\tau_t) w_t N_t^w + \frac{\omega}{\Omega_{t+1}} \frac{H_{t+1}^w}{(1+n)(1+r_{t+1})} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) \frac{H_{t+1}^r}{(1+r_{t+1})(1+n)\psi}
\end{aligned}$$

Where $\psi = N_t^r/N_t^w$ is the dependency ratio. Consumption can therefore also be aggregated, as

$$\begin{aligned}
C_t^w &= \pi_t \left[(1+r_{t-1}) A_t^w + H_t^w \right] \\
C_t^r &= \epsilon_t \pi_t \left(\frac{R_t}{\gamma} \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} a_t^{r,i,j} N_t^{r,i,j} + R_t \sum_{i=1}^{t-1} a_t^{r,i,t} N_t^{r,i,t} + H_t^r \right) \\
&= \epsilon_t \pi_t \left[R_t A_t^r + H_t^r \right]
\end{aligned}$$

3.F.2 Recursive detrended equilibrium

$\{A_t, A_t^w, H_t^w, H_t^r, C_t^r, C_t^w, C_t, Y_t, K_t\}$ are expressed *per worker* for $\{a_t, a_t^w, c_t, c_t^w, h_t, h_t^w, y_t, k_t\}$ and *per retiree* for $\{c_t^r, a_t^r, h_t^r\}$. The recursive equilibrium is given by

$$\begin{aligned}
y_t &= k_t^\alpha \\
r_t &= \alpha k_t^{\alpha-1} - \delta \\
w_t &= (1 - \alpha)k_t^\alpha \\
a_t &= k_t \\
a_t &= a_t^w + \psi a_t^r \\
a_{t+1}^w &= \omega [R_t a_t^w + (1 - \tau_t)w_t - c_t^w] \\
a_{t+1}^r &= R_t a_t^r + e_t - c_t^r + \frac{1 - \omega}{\psi} [R_t a_t^w + (1 - \tau_t)w_t - c_t^w] \\
c_t^r &= \pi_t^r (R_t a_t^r + h_t^r) \\
c_t^w &= \pi_t^w (R_t a_t^w + h_t^w) \\
c_t &= c_t^w + \psi c_t^r \\
h_t^r &= e_t + \frac{\gamma h_{t+1}^r}{R_{t+1}} \\
h_t^w &= (1 - \tau_t)w_t + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^w}{1 + r_{t+1}} + \left(1 - \frac{\omega}{\Omega_{t+1}}\right) \frac{h_{t+1}^r}{1 + r_{t+1}} \\
\pi_t^r &= 1 - \gamma \beta^\sigma R_{t+1}^{\sigma-1} \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}} \\
\pi_t^w &= 1 - \beta^\sigma (R_{t+1} \Omega_{t+1})^{\sigma-1} \frac{\pi_t^w}{\pi_{t+1}^w} \\
\Omega_{t+1} &= \omega + (1 - \omega) \epsilon_{t+1}^{\frac{1-\sigma}{\sigma}} \\
\epsilon_t &= \frac{\pi_t^r}{\pi_t^w} \\
\tau_t &= \mu_t \psi \\
k_{t+1} &= (1 - \delta)k_t + i_t
\end{aligned}$$

By the Walras Law, we have $y_t = c_t + i_t$, where i_t is the investment.

3.F.3 Steady-state equilibrium

At the steady-state, even if idiosyncratic aging shocks make individual decisions depend on their life-cycle history, at the aggregate level all variables are kept constant. Steady state is given by

$$\begin{aligned}
y &= k^\alpha \\
r &= \alpha k^{\alpha-1} - \delta \\
w &= (1 - \alpha)k^\alpha
\end{aligned}$$

$$\begin{aligned}
a &= k \\
a &= a^w + \psi a^r \\
a^w &= \frac{(1-\tau)w - c^w}{1-\omega R} \\
a^r &= \frac{1}{1-R} \left(e - c^r + \frac{1-\omega}{\psi} [Ra^w + (1-\tau)w - c^w] \right) \\
c^r &= \pi^r (Ra^r + h^r) \\
c^w &= \pi^w (Ra^w + h^w) \\
c &= c^w + \psi c^r \\
h^r &= \frac{R}{R-\gamma} e \\
h^w &= (1-\tau)w + \frac{\omega}{\Omega} \frac{h^w}{1+r} + \left(1 - \frac{\omega}{\Omega}\right) \frac{h^r}{1+r} \\
\pi^r &= 1 - \gamma \beta^\sigma R^{\sigma-1} \\
\pi^w &= 1 - \beta^\sigma (R\Omega)^{\sigma-1} \\
\Omega &= \omega + (1-\omega)\epsilon^{\frac{1}{1-\sigma}} \\
\epsilon &= \frac{\pi^r}{\pi^w} \\
\tau &= \mu\psi \\
k &= \frac{i}{\delta}
\end{aligned}$$

3.G Solution when labor supply is elastic

Retiree

At the steady-state, preferences for a retiree are given by

$$\begin{aligned}
V_t^{r,i,j} &= \left[\left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^\rho + \beta \gamma_t \left(V_{t+1}^{r,i,j} \right)^\rho \right]^{\frac{1}{\rho}} \\
\text{s.t. } a_{t+1}^{r,i,j} &= \frac{R}{\gamma} a_t^{r,i,j} + (1-\tau)w l_t^{r,i,j} + e - c_t^{r,i,j}
\end{aligned}$$

F.O.Cs are given by

$$\begin{aligned}
\phi \left(c_t^{r,i,j} \right)^{\phi\rho-1} (1 - l_t^{r,i,j})^{(1-\phi)\rho} &= \beta \gamma \frac{\partial V_{t+1}^{r,i,j}}{\partial a_{t+1}^{r,i,j}} \left(V_{t+1}^{r,i,j} \right)^{\rho-1} \\
l_t^{r,i,j} &= 1 - \frac{\zeta}{(1-\tau)w} c_t^{r,i,j} \\
\frac{\partial V_t^{r,i,j}}{\partial a_t^{r,i,j}} &= \phi \left(c_t^{r,i,j} \right)^{\phi\rho-1} (1 - l_t^{r,i,j})^{(1-\phi)\rho} \left(V_t^{r,i,j} \right)^{1-\rho} \frac{R}{\gamma}
\end{aligned}$$

By assembling them, one gets:

$$\begin{aligned} (c_t^{r,i,j})^{\rho-1} &= \beta R (c_{t+1}^{r,i,j})^{\rho-1} \\ \Leftrightarrow c_{t+1}^{r,i,j} &= [\beta R]^\sigma c_t^{r,i,j} \end{aligned} \quad (3.15)$$

Where $\sigma = \frac{1}{1-\rho}$ and $\zeta = \frac{1-\phi}{\phi}$. Let us note $h_t^{r,i,j}$ the human capital of an old individual, given by the following:

$$\begin{aligned} h_t^{r,i,j} &= d_t^{r,i,j} + \frac{Y}{R} h_{t+1}^{r,i,j} \\ d_t^{r,i,j} &= (1-\tau)w l_t^{r,i,j} + e \end{aligned}$$

Let us conjecture a solution of the form

$$c_t^{r,i,j} = \pi^r \left(\frac{R}{Y} a_t^{r,i,j} + h_t^{r,i,j} \right) \quad (3.16)$$

Where π^r is the marginal propensity to consume of old individuals. Combining (3.15) with (3.16), one gets:

$$a_{t+1}^{r,i,j} + \frac{Y}{R} h_{t+1}^{r,i,j} = \beta^\sigma R^{\sigma-1} \gamma \left(\frac{R}{Y} a_t^{r,i,j} + h_t^{r,i,j} \right)$$

Combining the budget constraint with the conjecture (3.16) and the definition of human wealth, one gets:

$$(1 - \pi^o) \left(a_t^{r,i,j} + h_t^{r,i,j} \right) = \gamma \beta^\sigma R^{\sigma-1} \left(\frac{R}{Y} a_t^{r,i,j} + h_t^{r,i,j} \right)$$

On thus gets:

$$\pi^o = 1 - \gamma \beta^\sigma R^{\sigma-1} \quad (3.17)$$

To obtain a solution for the value function, first conjecture:

$$V_t^{r,i,j} = \Delta_t^r c_t^{r,i,j} \left(\frac{\zeta}{(1-\tau)w} \right)^{1-\phi} \quad (3.18)$$

Then insert this conjecture into the expression for the value function to solve for Δ_t^r , such that

$$\Delta_t^r c_t^{r,i,j} \left(\frac{\zeta}{(1-\tau)w} \right)^{1-\phi} = \left((c_t^{r,i,j})^\rho + \beta \gamma \left[\Delta_{t+1}^r c_{t+1}^{r,i,j} \left(\frac{\zeta}{(1-\tau)w} \right)^{1-\phi} \right]^\rho \right)^{\frac{1}{\rho}}$$

We then insert the F.O.C. of $c_{t+1}^{r,j}$ and rearrange, which gives:

$$(\Delta_t^r)^\rho = 1 + R_t^{\sigma-1} \gamma (\Delta_{t+1}^r)^\rho \beta^\sigma$$

If we compare this expression to the equation of π^o (3.17), we can see that $\Delta_t^r = (\pi^o)^{-\frac{1}{\rho}}$, which verifies the conjecture.

Worker

In the case of workers, the program writes

$$\begin{aligned} V_t^{w,i} &= \left[\left(c_t^{w,i} \right)^\phi (1 - l_t^{w,i})^{1-\phi} \right]^\rho + \beta \left[\omega V_{t+1}^{w,i} + (1 - \omega) V_{t+1}^{r,i,t+1} \right]^\rho \\ \text{s.t. } a_{t+1}^{w,i} &= R a_t^{w,i} + (1 - \tau) w l_t^{w,i} - c_t^{w,i} \end{aligned}$$

The F.O.C and envelope condition are:

$$\begin{aligned} \phi \left(c_t^{w,i} \right)^{\phi\rho-1} (1 - l_t^{w,i})^{(1-\phi)\rho} &= \beta \left(\omega \frac{\partial V_{t+1}^{w,i}}{\partial a_{t+1}^{w,i}} + (1 - \omega) \frac{\partial V_{t+1}^{r,i,t+1}}{\partial a_{t+1}^{r,i,t+1}} \right) \left(\omega V_{t+1}^{w,i} + (1 - \omega) V_{t+1}^{r,i,t+1} \right)^{\rho-1} \\ l_t^{w,i} &= 1 - \frac{\zeta}{(1 - \tau)w} c_t^{w,i} \\ \frac{\partial V_{t+1}^{w,i}}{\partial a_{t+1}^{w,i}} &= (1 + r) \phi \left(c_{t+1}^{w,i} \right)^{\phi\rho-1} (1 - l_{t+1}^{w,i})^{(1-\phi)\rho} \left(V_{t+1}^{w,i} \right)^{1-\rho} \\ \frac{\partial V_{t+1}^{r,i,t+1}}{\partial a_{t+1}^{r,i,t+1}} &= (1 + r) \phi \left(c_{t+1}^{r,i,t+1} \right)^{\phi\rho-1} (1 - l_{t+1}^{r,i,t+1})^{(1-\phi)\rho} \left(V_{t+1}^{r,i,t+1} \right)^{1-\rho} \end{aligned}$$

Note that the rate of return of assets for a newly retired individual is R , as they do not have access to perfect annuities. Combining, we get

$$\begin{aligned} \left(c_t^{w,i} \right)^{\rho-1} &= \beta R \left[\omega \left(c_{t+1}^{w,i} \right)^{\rho-1} \left(V_t^{w,i} \right)^{1-\rho} + (1 - \omega) \left(c_{t+1}^{r,i,t+1} \right)^{\rho-1} \left(V_t^{r,i,t+1} \right)^{1-\rho} \right] \times \\ &\quad \left[\omega V_{t+1}^{w,i} + (1 - \omega) V_{t+1}^{r,i,t+1} \right] \end{aligned} \quad (3.19)$$

Let us now conjecture that the value function takes the following form:

$$V_t^{w,i} = \Delta_t^w c_t^{w,i} \left(\frac{\zeta}{(1 - \tau)w} \right)^{1-\phi} \quad (3.20)$$

Where $\Delta_t^w = (\pi^w)^{-\frac{1}{\rho}}$. Combining the conjectures (3.18) and (3.20), we can simplify the Euler equation (3.19) as:

$$\left(c_t^{w,i} \right)^{\rho-1} = \beta R \left[\omega + (1 - \omega) \epsilon_{t+1}^{\frac{1-\rho}{\rho}} \right] \left[\omega c_{t+1}^{w,i} + (1 - \omega) c_{t+1}^{r,i,t+1} \cdot \epsilon_{t+1}^{-\frac{1}{\rho}} \right]^{\rho-1}$$

Let us note $\Lambda_{t+1} = \frac{\Delta_{t+1}^r}{\Delta_{t+1}^w} = \epsilon_{t+1}^{-\frac{1}{\rho}} = \epsilon_{t+1}^{\frac{\sigma}{1-\sigma}}$, and $\Omega_{t+1} = \omega + (1 - \omega) \epsilon_{t+1}^{\frac{1-\sigma}{1-\rho}}$. We can rewrite the previous expression under the form:

$$\omega c_{t+1}^{w,i} + (1 - \omega) \Lambda_{t+1} c_{t+1}^{r,i,t+1} = [\beta R \Omega_{t+1}]^\sigma c_t^{w,i} \quad (3.21)$$

Now let us conjecture that the consumption takes the following form:

$$c_t^{w,i} = \pi^w \left(Ra_t^{w,i} + h_t^{w,i} \right) \quad (3.22)$$

Whereas an individual who just has retired has a consumption (no annuity market, as he was a worker the period before):

$$c_t^{r,i,t} = \pi^o \left(Ra_t^{w,i} + h_t^{r,i,t} \right)$$

Where h_t^w is the human capital of a worker, that is to say

$$\begin{aligned} h_t^{w,i} &= d_t^{w,i} + \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^{w,i}}{R} + \left(1 - \frac{\omega}{\Omega_{t+1}} \right) \frac{h_{t+1}^{r,i,t+1}}{R} \\ d_t^{w,i} &= (1 - \tau) \omega l_t^{w,i} \end{aligned}$$

Substituting these conjectures in the Euler equation (3.21), and using the fact that $(1 - \omega)\Lambda_{t+1}\epsilon_{t+1} = \Omega_{t+1} - \omega$ and $h_{t+1}^{r,i,t+1} = \left[h_t^{w,i} - d_t^{w,i} - \frac{\omega}{\Omega_{t+1}} \frac{h_{t+1}^{w,i}}{R} \right] \frac{\Omega_{t+1}}{\Omega_{t+1} - \omega} R$, one gets:

$$\frac{a_{t+1}^{w,i} + h_t^{w,i} - d_t^{w,i}}{Ra_t^{w,i} + h_t^{w,i}} = \beta^\sigma [R\Omega_{t+1}]^{\sigma-1}$$

Using the budget constraint, one can write

$$a_{t+1}^{w,i} - (1 - \tau) \omega l_t^{w,i} + h_t^{w,i} = (1 - \pi^w) \left(Ra_t^{w,i} + h_t^{w,i} \right)$$

Combining with the previous expression, one gets

$$\pi^w = 1 - \beta^\sigma [R\Omega]^\sigma \quad (3.23)$$

This differential equation is independent of both i and j . The conjecture for the value function can be verified by substituting in the Euler equation, as in the retiree's case.

3.H Equilibrium with elastic labor supply

$\{A_t, A_t^r, A_t^w, H_t^r, H_t^w, C_t^r, C_t^w, C_t, Y_t, K_t, L_t^w, L_t^r, L_t\}$ are expressed *per worker* for $\{a_t, a_t^w, c_t, c_t^w, h_t, h_t^w, y_t, k_t, l_t, l_t^w\}$ and *per retiree* for $\{c_t^r, a_t^r, h_t^r, l_t^r\}$. The steady-state equilibrium is given by

$$\begin{aligned} y &= k^\alpha l^{1-\alpha} \\ l &= l^w + \xi l^r \psi \\ r &= \alpha \frac{y}{k} - \delta \\ w &= (1 - \alpha) \frac{y}{l} \end{aligned}$$

$$\begin{aligned}
a &= k \\
a &= a^w + \psi a^r \\
a^w &= \frac{(1 - \tau)wl^w - c^w}{1 - \omega R} \\
a^r &= \frac{1}{1 - R} \left((1 - \tau)wl^r + e - c^r + \frac{1 - \omega}{\psi} [Ra^w + (1 - \tau)wl^r - c^w] \right) \\
c^r &= \pi^r (Ra^r + h^r) \\
c^w &= \pi^w (Ra^w + h^w) \\
c &= c^w + \psi c^r \\
h^r &= \frac{R}{R - \gamma} [(1 - \tau)wl^r + e] \\
h^w &= \left(\frac{\Omega R}{\Omega R - \omega} \right) \left[(1 - \tau)wl^w + \left(1 - \frac{\omega}{\Omega} \frac{h^r}{R} \right) \right] \\
\pi^r &= 1 - \gamma \beta^\sigma R^{\sigma-1} \\
\pi^w &= 1 - \beta^\sigma (R\Omega)^{\sigma-1} \\
\Omega &= \omega + (1 - \omega) \left(\frac{\pi^r}{\pi^w} \right)^{\frac{1}{1-\sigma}} \\
\tau &= \frac{\mu\psi}{l} \\
k &= \frac{i}{\delta}
\end{aligned}$$

3.I The deterministic case

In order to put clearly in evidence the role played by a PAYG pension system in the presence of aging risk, let us make the latter disappear, *i.e.* let the aging process be purely deterministic. In order to be consistent with the calibration used above, where the agent lives 45 years in average in state "worker" and 15 as a "retiree", we assume the length of a period to be 15 years. The agent thus lives three periods in state "worker" and one in state "retiree". The model thus becomes a four period version of the Diamond model (1965) [24] with RINCE preferences.

The intertemporal utility of a newborn is thus given by

$$\begin{cases}
V_t^0 = [(c_t^0)^\rho + \beta (V_{t+1}^1)^\rho] \\
V_t^1 = [(c_t^1)^\rho + \beta (V_{t+1}^2)^\rho] \\
V_t^2 = [(c_t^2)^\rho + \beta (c_{t+1}^3)^\rho]
\end{cases}$$

While the intertemporal budget constraint is given by

$$c_t^0 + \frac{c_{t+1}^1}{R_{t+1}} + \frac{c_{t+2}^2}{R_{t+1}R_{t+2}} + \frac{c_{t+3}^3}{R_{t+1}R_{t+2}R_{t+3}} = (1 - \tau_t)w_t + \frac{(1 - \tau_{t+1})w_{t+1}}{R_{t+1}} + \frac{(1 - \tau_{t+1})w_{t+1}}{R_{t+1}R_{t+2}} + \frac{e_t}{R_{t+1}R_{t+2}R_{t+3}}$$

Parameters are kept constant. β and δ take into account the change of frequency. The welfare of a newborn agent following the introduction of social security (expressed in consumption equivalent variation) is given in Figure 3.I.1. In the absence of dynamical inefficiency, we retrieve the standard result of Diamond, *i.e.* PAYG pension system are inefficient, despite the fact that $\sigma < 1$.

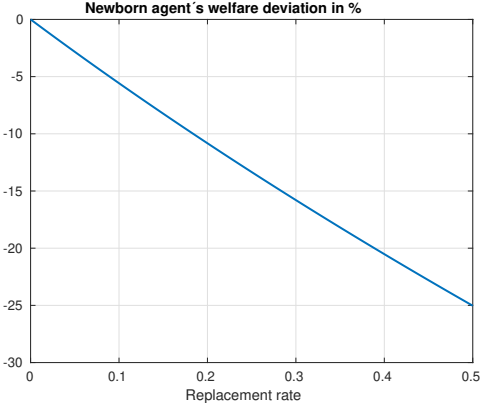


Figure 3.I.1: Newborn agent welfare's deviation in %, $\sigma = 0.5$

Note: the deviation is expressed in percentage, the initial situation being an equilibrium with no social security.

Chapter 4

Social security and the interest rate: the impact of a pension reform abroad on the french economy

Abstract

In a world characterized by a strong financial integration, a structural reform of a pension system is likely, provided that it is implemented in a country that is large enough, to be a source of externalities towards foreign economies. This chapter studies the impact of an unfunded public pension cut in developed countries on the French economy, focusing on its effect it has on welfare. Using a two-countries OLG model, it estimates that if such a reform was to be undertaken abroad, the welfare of french households would decrease in the long run, by pushing down the worldwide interest rate and thus making french households poorer.

1 Introduction

Facing the challenge of demographic aging since several decades, governments of developed countries were left with no other choice but to reform their pay-as-you-go (PAYG) pension systems. Some of them, as France for instance, undertook "parametric reforms" by simply combining an increase in payroll taxes with a longer minimum contribution duration through the so-called Balladur (1993), Fillon (2003), Woerth (2010) or Touraine (2014) reforms. Other countries, as Germany (Riester reforms, 2001), Denmark (2006 and 2011 reforms) or Sweden (1994-1998 reforms) among others, made a more radical choice, by introducing a partially funded pension system. In short, the Old Continent, which used to be characterized by a relative homogeneity among its PAYG pension systems – at least in its western part – is thus now torn between Latin countries with generous pensions fully financed by unfunded systems, and Germanic ones, where pensions partly rely on households' savings.

It is well known that the economic implications of unfunded and funded pension systems are radically different. On the one hand, in the case of an unfunded PAYG pension system, pensions take the form of an instantaneous inter-generational transfer; as no savings are needed to finance them, such a system does not give rise to investment. On the other hand, in a funded pension system, pensions are financed by the capitalized interest of savors. These savings being invested in capital, such a retirement system leads to a higher output in the economy. Which one of these system is socially desirable? Following the seminal paper of Diamond (1965), who showed that an economy characterized by a level of accumulated capital high enough to make the interest rate lower than the growth rate of the economy is "dynamically inefficient", and must put in place instantaneous inter-generational transfers, a large amount of literature offered refinements to his famous two periods model in order to distinguish other conditions under which a PAYG pension system becomes preferable. For instance, Krueger and Kubler (2006) concluded that in the sole presence of aggregate uncertainty, such a pension system is inefficient, just as Imrohorglu *et al.* (1995) [49] who focused on idiosyncratic income risk. Combining both risks however leads to a socially desirable PAYG pension system, as shown by Haremborg and Ludwig (2018) [44]. Using a different kind of idiosyncratic risk, – the "aging risk" – the previous chapter also showed that uncertainty about the retirement age makes unfunded pensions dominate funded one in the sense of Pareto.

The existence of such risks does however not guarantee that current social security systems are necessarily welfare improving with respect to a fully – or partially – funded one, as the level of pensions could indeed be too high, inducing an excessive level of distortions. Several factors could explain such a situation, as myopia or political frictions that make reforms of pensions systems that were Pareto-optimal several decades ago difficult. Yet, as emphasized above, recent history shows that such reforms are politically not impossible to make. Even if population was assumed to remain stationary, if current pensions levels are sub-optimal, one could expect reforms to be undertaken in developed countries, in order to guide them toward a more efficient steady-state.

Given the fact that they would be made in a world characterized by a strong heterogeneity of pen-

sion systems and a high degree of financial integration, the prospect of such reforms should invite us to think about spillover effects. In open economy and perfect mobility of capital, a lower generosity of public pensions in a given country can indeed have an impact abroad: as it provides an incentive for resident households to save more, such a reform pushes down the worldwide interest rate and is thus a source of externalities, as shown by Pemberton (1999) [71]. This interaction between the world interest rate and public pension systems was also studied – among others – by Aglietta *et al.* (2007) [2] or Borsch-Supan *et al.* (2006) [14]. They developed multi-regions large scale OLG models to study the impact of several pension reforms on international capital flows, and show that modeling pensions reforms in closed economy leads to large biased estimations of their macroeconomic impact. In other words, the framework developed by this literature is adapted to answer two main type of questions related with a pension reform. First, does its impact in the country that undertook it differ when one takes into account international capital mobility? Second, what would be its impact on foreign countries?

Its is the second type of question that this chapter will focus on. If we assume France to keep its pension system unchanged, and if the rest of developed countries chooses to cut their pensions, what kind of spillover effects can we expect for the french economy? The scope of this question is multi-dimensional. It is both positive and normative, while aiming at studying both short run and long run impact. Beyond the positive – short run and long run - consideration of purely economic variables such as GDP, Net Foreign Assets (NFA), the interest rate or the labor force, studying pension reforms in the framework of an opened economy brings an additional building block to the normative debate about the link between welfare and social security, a debate in which the previous chapter of this thesis has contributed in the framework of a closed economy. It has shown that in the presence of an "aging risk", that is to say a risk to face a lower survival probability, PAYG pension system might be socially desirable. The current chapter's objective is to go one step further, by making a somewhat more applied work, and study whether if a PAYG pension reform undertaken in the rest of the world will increase welfare in France or not in the long run.

By modelling France and another "region" made up of OECD countries and China¹, this chapter shows that french households are expected to experience welfare losses from such a reform. Because it provides an incentive for foreign households to save more for their old days, the pension cut increases the supply of capital. Some of this capital flows into France and thus increases the marginal productivity of labor and thus wages, which is naturally a source of a higher level of welfare. However, this initial welfare improvement is offset through two main channels. First, since these investments are made by foreign households, they also give rise to a payment of interests which dampens the initial increase of welfare. Second, this "saving glut" – to use the words of Bernanke (2005) [10] – puts downward pressure on the worldwide interest rate, which makes french households poorer.

¹A region "Rest of the world" that includes developed countries (G7 or OECD) and China is standard in the literature of OLG models in open economy, see for instance Ferrero (2010) [32] or Schön and Stähler (2019) [79].

2 The model

There are two asymmetrical countries in the world economy, each of them producing the same good. For the sake of simplicity, we drop the country specific subscript in the presentation of the model.

2.1 Aging risk and preferences

2.1.1 Stochastic aging and demography

As in the previous chapter, we assume the existence of an "aging risk". Aging is assumed to follow a stochastic process. Agents are born in the state "young", and face each period the risk to switch to the state "middle-aged" at a rate $1 - \omega$, ω being the probability to stay "young". When the agent becomes a "middle-aged", two outcomes are possible: either he is, with probability θ , a "retired middle-aged" and gets a pension, or he becomes a "working middle-aged", and does not get any pension with probability $1 - \theta$. "Middle-aged" become "seniors" at a rate η ; "seniors" in turn die at a rate $1 - \gamma$, γ being their survival rate². Noting N_t^y the number of youngsters, N_t^m the total number of middle-aged and N_t^o the number of seniors, we can thus write (dropping country's subscript)

$$\begin{aligned} N_{t+1}^y &= (1 - \omega + n^y)N_t^y + \omega N_t^y = (1 + n^y)N_t^y \\ N_{t+1}^m &= (1 - \omega)N_t^y + \eta N_t^m \\ N_{t+1}^o &= (1 - \eta)N_t^m + \gamma N_t^o \end{aligned}$$

The number of middle-aged eligible to social security's pension is given by $N_t^r = \theta N_t^m$, whereas those who are not are in number $N_t^w = (1 - \theta)N_t^m$.

Stochastic aging in this model reflects the uncertainty agents face along their life-cycle, regarding the age at which they will have to leave the labor market³ and their life duration. Such an uncertainty, combined with some assumptions about preferences of the agent, is responsible for an inefficient competitive equilibrium that can justify the existence of social security, as shown in the previous chapter of this thesis.

2.1.2 RINCE preferences and the "aging risk"

As in Gertler (1999) [37], we assume agents to be risk neutral, while having a finite elasticity of substitution. These preferences, so-called "RINCE preferences" (Farmer, 1990 [30]), break the standard relationship between risk aversion and EIS in time additive von Neumann and Morgenstern (VNM) utility function – the coefficient of risk aversion being the inverse of the later – and can therefore isolate the economic implications of a desire for consumption smoothing across time. We combine such non-standard preferences with stochastic aging for two reasons. First, as emphasized by Carvalho et

²We assume the existence of three age classes instead of two like in the previous chapter in order to better calibrate the model. For instance, employment rate being expressed by cohorts 60-64 years old, 65 years old and more, having only two cohorts would force the model to calibrate the employment rate of 62 years and more (the french retirement age) on 60-64 years old data.

³Retirement age, is for several reasons, subject to a significant degree of uncertainty, see Caliendo *et al.* (2016) [15]

et al. (2014) [18], traditional time separative preferences lead to a counterfactual high level of precautionary savings. Making agents risk neutral while providing them a desire for consumption smoothing helps to circumvent this issue. Second, RINCE preferences combined with stochastic aging let us build an OLG model whose aggregate variables can be derived analytically. Such a property thus let us study more precisely the channels through which pension cuts affects macroeconomic variables. In this model, we assume that agents are endowed with one unit of time per period, that they can use to work or to consume leisure⁴. Preferences are given by:

$$V_t^z = \left[(q_t^z)^\rho + \beta^z \mathbb{E}_t [V_{t+1}|z]^\rho \right]^{\frac{1}{\rho}}$$

where $\rho \in]-\infty, 0[$, $z \in \{y, w, r, o\}$ indicates the age status of the agent, y denoting young agents, w working middle aged ones, r retired middle-aged ones and o seniors.

$$\beta^z = \begin{cases} \beta & \text{if } z \in \{y, w, r\} \\ \beta_Y & \text{if } z = o \end{cases}$$

$$\mathbb{E}_t [V_{t+1}|z] = \begin{cases} \omega V_{t+1}^y + (1 - \omega) [\theta V_{t+1}^r + (1 - \theta) V_{t+1}^w] & \text{if } z = y \\ \eta V_{t+1}^r + (1 - \eta) V_{t+1}^o & \text{if } z = w \\ \eta V_{t+1}^w + (1 - \eta) V_{t+1}^o & \text{if } z = r \\ V_{t+1}^o & \text{if } z = o \end{cases}$$

and

$$q_t^z = (c_t^z)^\phi (1 - l_t^z)^{1-\phi}$$

where c_t^z is consumption of goods and l_t^z is labor supply.

Thus, preference of a senior can be represented under the form of the following recursive utility

$$V_t^o = \left[\left((c_t^o)^\phi (1 - l_t^o)^{1-\phi} \right)^\rho + \beta_Y (V_{t+1}^o)^\rho \right]^{\frac{1}{\rho}}$$

Preferences of retired middle-aged are

$$V_t^r = \left[\left((c_t^r)^\phi (1 - l_t^r)^{1-\phi} \right)^\rho + \beta (\eta V_{t+1}^r + (1 - \eta) V_{t+1}^o)^\rho \right]^{\frac{1}{\rho}}$$

Preferences of a working middle-aged are

$$V_t^w = \left[\left((c_t^w)^\phi (1 - l_t^w)^{1-\phi} \right)^\rho + \beta (\eta V_{t+1}^w + (1 - \eta) V_{t+1}^o)^\rho \right]^{\frac{1}{\rho}}$$

⁴Such a specification is used in Gertler (1999) [37], Kara and von Thadden (2014) [55] or Fujiwara and Teranishi (2008) [34].

and preferences of a youngster are

$$V_t^y = \left[\left((c_t^y)^\phi (1 - l_t^y)^{1-\phi} \right)^\rho + \beta \left[\omega V_{t+1}^y + (1 - \omega) \theta V_{t+1}^r + (1 - \omega)(1 - \theta) V_{t+1}^w \right]^\rho \right]^{\frac{1}{\rho}}$$

2.2 Decision rules

2.2.1 Seniors' decision

Seniors are assumed to be perfectly insured against the risk of death, as they have access to a perfect annuity market as in Blanchard (1985) [13] and Yaari (1965) [87]. Competitive mutual funds collect the financial wealth of old individuals, and redistribute the remaining assets of the share γ of agents who did not survive to the share $1 - \gamma$ of agents who are still alive. More specifically, insurance companies' "income" is given by the amount of assets left by agents who did not survive $(1 - \gamma)W$, where W is the total wealth these firms have collected. The redistribution of these assets to the surviving agents takes the form of a premium μ over their wealth, *i.e.* insurance firms have to pay $\mu\gamma W$. As the life insurance market is perfectly competitive, no profits can be made, which implies $\mu = \frac{1-\gamma}{\gamma}$. The gross return of assets of a retiree is thus $(1 + r_t)(1 + \mu) = \frac{1+r_t}{\gamma}$. They supply elastically labor l_t with a productivity $\xi < 1$, and get in return a wage net of taxes $(1 - \tau_t)w_t l_t \xi$. They also get a pension e_t . The recursive utility of an individual who was born in i , became middle-aged worker in j and a senior in k is given by:

$$V_t^{o,i,j,k} = \max_{c_t^{o,i,j,k}, l_t^{o,i,j,k}} \left[\left((c_t^{o,i,j,k})^\phi (1 - l_t^{o,i,j,k})^{1-\phi} \right)^\rho + \beta \gamma \left(V_{t+1}^{o,i,j,k} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{o,i,j,k} = \frac{1 + r_t}{\gamma} a_t^{o,i,j,k} + (1 - \tau_t) w_t \xi l_t^{o,i,j,k} + e_t - c_t^{o,i,j,k}$$

Noting $\sigma = \frac{1}{1-\rho}$ the elasticity of intertemporal substitution (EIS) and $\zeta = \frac{1-\phi}{\phi}$, it can be shown that first order conditions lead to decision rules that take the following form (see Appendix):

$$l_t^{o,i,j,k} = 1 - \frac{\zeta}{(1 - \tau_t) \xi w_t} c_t^{o,i,j,k} \quad (4.1)$$

$$c_t^{o,i,j,k} = \pi_t^o \left(\frac{R_t}{\gamma} a_t^{o,i,j,k} + h_t^{o,i,j,k} \right) \quad (4.2)$$

$$\pi_t^o = 1 - \gamma \beta^\sigma \left[R_{t+1} \left(\frac{(1 - \tau_t) w_t}{(1 - \tau_{t+1}) w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^o}{\pi_{t+1}^o} \quad (4.3)$$

$$h_t^{o,i,j,k} = e_t + (1 - \tau_t^w) w_t \xi l_t^{o,i,j,k} + \frac{\gamma}{R_{t+1}} h_{t+1}^{o,i,j,k} \quad (4.4)$$

Consumption is linear in financial wealth a^o and human wealth h^o , the latter being the sum of discounted expected labor income. π^o gives the marginal propensity to consume and does not depend on the life-cycle history of the agent. Its value is given by the differential equation (4.3).

At the steady state, the marginal propensity to consume is

$$\pi^o = 1 - \gamma \beta^\sigma R^{\sigma-1}$$

Following an increase of the interest rate, the reaction of the agent depends on the size of the EIS. If $\sigma < 1$, the income effect is stronger than the substitution effect: a higher interest rate makes the agent richer which thus makes him consume more. On the contrary, when $\sigma > 1$, the substitution effect dominates: a higher interest rate provides an incentive to save more, which thus decreases π^o .

The human wealth is given by⁵

$$h^o = \frac{(1+r)[(1-\tau)wl^o\xi + e]}{1+r-\gamma}$$

γ corrects the discounting factor from the death risk: the higher the surviving probability is, the more likely the agent is to perceive income in the future, which thus increases his human wealth. If $\gamma = 1$, it is simply the expression of a perpetual annuity with a coupon given by net labor income and the pension. Combining with the labor supply expression, one gets:

$$h^o = (1+r) \frac{(1-\tau)w\xi + e - \zeta\pi^o(1+r)a^o}{(1+r-\gamma) + \zeta(1+r)\pi^o}$$

Note that if $\phi = 1$, labor supply is inelastic, and the human wealth' expression does not change. As leisure becomes important in the agent's preferences, ζ increases, which reduces labor supply and human wealth. Note that a higher wage increases human wealth, even if can reduce labor supply through the substitution effect. Finally, a higher MPC reduces h^o : as the agent chooses to allocate a larger share of his wealth to consumption, the marginal utility of the latter decreases and thus reduces the incentive to work.

Labor supply can be written

$$l^o = 1 - \frac{\zeta\pi^o(1+r)}{(1-\tau)w\xi} \left(\frac{(1-\tau)w\xi + e + a^o(1+r-\gamma)}{1+r-\gamma + \zeta(1+r)\pi^o} \right)$$

A higher financial wealth decreases labor supply through a classical income effect. An increase of the wage leads to a higher labor supply.

2.2.2 Retired middle-aged decisions

For a retired middle-aged born in i and who switched to his current state in j , the program writes

$$\begin{aligned} V_t^{r,i,j} &= \max_{c_t^{r,i,j}, l_t^{r,i,j}} \left[\left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^\rho + \beta \left(\eta V_{t+1}^{r,i,j} + (1-\eta) V_{t+1}^{o,i,j,t+1} \right)^\rho \right]^{\frac{1}{\rho}} \\ \text{s.t. } a_{t+1}^{r,i,j} &= (1+r_t) a_t^{r,i,j} + (1-\tau_t) w_t l_t^{r,i,j} \xi + e_t - c_t^{r,i,j} \end{aligned}$$

As seniors, retired middle-aged have a productivity $\xi < 1$. Note that if the retired middle-aged becomes a senior in $t+1$, his wealth is $a_{t+1}^{o,i,j,t+1} = a_{t+1}^{r,i,j}$. It can be shown that decision rules take the following

⁵Steady-state variables in this section are expressed *per capita*. Section 2.3 explains how to aggregate all decision rules in order to express them in such a way.

form:

$$\begin{aligned}
l_t^{r,i,j} &= 1 - \frac{\zeta}{(1-\tau_t)w_t\zeta} c_t^{r,i,j} \\
c_t^{r,i,j} &= \pi_t^r \left(R_t a_t^{r,i,j} + h_t^{r,i,j} \right) \\
h_t^{r,i,j} &= (1-\tau_t)w_t l_t^{r,i,j} + e_t + \frac{\eta}{\Omega_{t+1}^r} \frac{h_{t+1}^{r,i,j}}{1+r_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^r} \right) \frac{h_{t+1}^{o,i,j,t+1}}{1+r_{t+1}} \\
\pi_t^r &= 1 - \beta^\sigma \left[R_{t+1} \Omega_{t+1}^r \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)} \right]^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r} \\
\Omega_{t+1}^r &= \eta + (1-\eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

Decision rules are quite similar to the ones derived in the case of the senior. Consumption, as well as labor supply, are linear in financial and human wealth h^r . It can also be shown, as in Gertler (1999) [37], that the MPC of a senior is higher than the one of a retired middle-aged, *i.e.* $\pi^o > \pi^r$: as the senior's life duration is finite, he simply consumes more. Human wealth and π^r follow the same logic as their "senior" counterpart with one exception: they are now adjusted with a term Ω_{t+1}^r , which in turns depend on $\left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}}$. As discussed in the previous chapter and shown in the appendix, $\left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}}$ is actually the marginal rate of substitution between wealth in the state "senior" and "retired middle-aged" $\frac{\partial V_{t+1}^{o,i,j,k} / \partial a_{t+1}^{o,i,j,t+1}}{\partial V_{t+1}^{r,i,j} / \partial a_{t+1}^{r,i,j}}$, and represents the aging risk. Wealth indeed does not have the same value in both states, as savings are consumed at a higher pace in the state "senior" because of the mortality risk. This difference in marginal propensity to consume thus create an "aging risk" wealth is subject to. In which state does wealth have more value? As shown in the previous chapter, this depends on the relative strength of a substitution and an income effect. In a state where the agent is mortal, substitution effect makes the value of marginal wealth decrease, as saving become costlier in terms of utility. However, because he consumes more, the agent also needs more assets to do so *i.e.* a wealth effect makes the marginal value of savings in the state "senior" increase. This difference in the valuation of wealth depending on the "age-state" thus modifies decision rules. In the human wealth's expression, probabilities are re-weighted by Ω_{t+1} : for instance, if the marginal value of wealth is higher in the state "senior", then $\Omega_{t+1} > 1$, which gives less weight to h_{t+1}^r and more to h_{t+1}^o . For the same reason, the interest rate is augmented with the term Ω_{t+1} in the expression of the MPC. The expected return of an additional unit of savings, taking into account the fact that it will not provide the same welfare in the states "senior" and "middle-aged" is $(1+r)\Omega$.

At the steady-state, these decisions can be written

$$\begin{aligned}
l^{r,i,j} &= 1 - \frac{\zeta}{(1-\tau)w\zeta} c^{r,i,j} \\
c^r &= \pi^r (R a^r + h^r) \\
\pi^r &= 1 - \beta^\sigma (R \Omega^r)^{\sigma-1}
\end{aligned}$$

$$\Omega^r = \eta + (1 - \eta) \left(\frac{\pi^o}{\pi^r} \right)^{\frac{1}{1-\sigma}}$$

$$h^r = \frac{R\Omega^r [(1 - \tau)w\xi + e - \zeta\pi^r Ra^r] + (\Omega^r - \eta)h^o}{R\Omega - \eta + \zeta\pi^r R\Omega^r}$$

Note that the steady-state closed form solution of human wealth is still written in recursive form, as it contains h^o , weighted by a factor $\Omega - \eta = (1 - \omega) \left(\frac{\pi^o}{\pi^r} \right)^{\frac{1}{1-\sigma}}$; the human wealth in the state "senior" is thus "converted" in units of middle-aged wealth and weighted by the probability to switch to this state.

2.2.3 Working middle-aged' decision

For a working middle-aged born in i and who switched to his current state in j , the program writes

$$V_t^{w,i,j} = \max_{c_t^{w,i,j}, l_t^{w,i,j}} \left[\left((c_t^{w,i,j})^\phi (1 - l_t^{w,i,j})^{1-\phi} \right)^\rho + \beta \left(\eta V_{t+1}^{w,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

s.t. $a_{t+1}^{w,i,j} = (1 + r_t) a_t^{w,i,j} + (1 - \tau_t) w_t l_t^{w,i,j} - c_t^{w,i,j} - z_t$

Contrary to seniors and retired middle-aged, working middle-aged have a labor productivity equal to one. They also have to pay a lump-sum tax z_t . The resolution of their program is the same as the one of retired middle-aged, and leads to the following decision rules:

$$l_t^{w,i,j} = 1 - \frac{\zeta}{(1 - \tau_t) w_t} c_t^{w,i,j}$$

$$c_t^{w,i,j} = \pi_t^w \left(R_t a_t^{w,i,j} + h_t^{w,i,j} \right)$$

$$h_t^{w,i,j} = (1 - \tau_t) w_t l_t^{w,i,j} - z_t + \frac{\eta}{\Omega_{t+1}^w} \frac{h_{t+1}^{w,i,j}}{1 + r_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^w} \right) \frac{h_{t+1}^{o,i,j,t+1}}{1 + r_{t+1}}$$

$$\pi_t^w = 1 - \beta^\sigma \left[R_{t+1} \Omega_{t+1}^w \left(\frac{(1 - \tau_t) w_t}{(1 - \tau_{t+1}) w_{t+1}} \right)^{(1-\phi)} \right]^{\sigma-1} \frac{\pi_t^w}{\pi_{t+1}^w}$$

$$\Omega_{t+1}^w = \eta + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^w} \right)^{\frac{1}{1-\sigma}} \zeta^{\phi-1}$$

A difference appears with respect to the decisions obtained in the case of retired middle-aged: a term $\zeta^{\phi-1}$, which reflect the possibility of a drop in productivity upon switching state, now appears in the expression of Ω_{t+1}^w . Indeed, the MRS between wealth in the state "senior" and "working middle-aged" is here $\frac{\partial V_{t+1}^{o,i,j,t+1} / \partial a_{t+1}^{o,i,j,t+1}}{\partial V_{t+1}^{w,i,j} / \partial a_{t+1}^{w,i,j}} = \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^w} \right)^{\frac{1}{1-\sigma}} \zeta^{\phi-1}$ and decreases with ζ (see Appendix). Intuitively, the drop in labor productivity increases the relative marginal value of wealth in the state "senior", as it makes the agent rely more on financial wealth to consume goods and leisure. The interpretation thus does not change fundamentally, the age-switching risk now simply includes a risk of productivity drop.

2.2.4 Youngsters' decisions

Youngster have a labor productivity of one, and pay a lump sum tax z_t . The program of a youngster born in i can be written

$$V_t^{y,i} = \max_{c_t^{y,i}, l_t^{y,i}} \left[\left((c_t^{y,i})^\phi (1 - l_t^{y,i})^{1-\phi} \right)^\rho + \beta \left[\omega V_{t+1}^{y,i} + (1 - \omega) \left(\theta V_{t+1}^{r,i,t+1} + (1 - \theta) V_{t+1}^{w,i,t+1} \right) \right]^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{y,i} = (1 + r_t) a_t^{y,i} + (1 - \tau_t) w_t l_t^{y,i} - c_t^{y,i} - z_t$$

It leads to the following consumption-leisure choice:

$$c_t^{y,i} = \pi_t^y \left(R_t a_t^{y,i} + h_t^{y,i} \right)$$

$$l_t^{y,i} = 1 - \zeta \frac{c_t^{y,i}}{(1 - \tau_t) w_t}$$

Where the marginal propensity to consume follows

$$\pi_t^y = 1 - \beta^\sigma \left(R_{t+1} \Omega_{t+1}^y \left(\frac{(1 - \tau_t) w_t}{(1 - \tau_{t+1}) w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y}$$

The term Ω_{t+1}^y here also takes into account the age-switching risk, and is such that

$$\Omega_{t+1}^y = \omega + (1 - \omega) \left[\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1} + (1 - \theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \right]$$

The human wealth is given by

$$h_t^{y,i} = (1 - \tau_t) w_t l_t^{y,i} - z_t + \frac{\omega h_{t+1}^{y,i} + (1 - \omega) \left[\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1} h_{t+1}^{r,i,t+1} + (1 - \theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} h_{t+1}^{w,i,t+1} \right]}{\Omega_{t+1}^y R_{t+1}}$$

The human wealth for a youngster has a somewhat more complicated form than the previous ones, but the interpretation stays the same: the terms $\left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1}$ and $\left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}}$ are the MRS between wealth in state "young" and "middle-aged", and thus "translates" forward flows of income in the same unit.

At the steady-state, these decision rules can be written

$$c^y = \pi^y (R a^y + h^y)$$

$$1 - l^y = \frac{\zeta}{(1 - \tau) w} c^y$$

$$\pi^y = 1 - \beta^\sigma (R \Omega^y)^{\sigma-1}$$

$$h^y = \frac{R \Omega^y [(1 - \tau) w - z - \zeta \pi^y R a^y] + (1 - \omega) \left[\theta \left(\frac{\pi^r}{\pi^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1} h^r + (1 - \theta) \left(\frac{\pi^w}{\pi^y} \right)^{\frac{1}{1-\sigma}} h^w \right]}{(R \Omega^y - \omega) + \zeta \pi^y R \Omega^y}$$

Note that human wealth decreases with financial assets, as a richer agent can indeed work less, his labor income flows are smaller.

2.3 Aggregation

Let us first define the following ratios:

$$\psi_t^{o,y} = \frac{N_t^o}{N_t^y}, \quad \psi_t^{o,m} = \frac{N_t^o}{N_t^m}, \quad \psi_t^{m,y} = \frac{N_t^m}{N_t^y}$$

Along a demographic balanced growth path, we have:

$$\begin{aligned} \psi^{m,y} &= \frac{1 - \omega}{1 + n - \eta} \\ \psi^{o,y} &= \frac{1 - \eta}{1 + n - \gamma} \psi^{m,y} \\ \psi^{o,m} &= \frac{(1 - \eta)\psi^{m,y} + \gamma\psi^{o,y}}{1 - \omega + \eta\psi^{m,y}} \end{aligned}$$

The assumptions about preferences let us aggregate all households decision rules analytically. Denoting aggregate variables with capital letters, we have:

$$L_t^y = N_t^y - \frac{\zeta C_t^y}{(1 - \tau_t)w_t} \quad (4.5)$$

$$L_t^r = N_t^r - \frac{\zeta C_t^r}{(1 - \tau_t)\xi w_t} \quad (4.6)$$

$$L_t^w = N_t^w - \frac{\zeta C_t^w}{(1 - \tau_t)w_t} \quad (4.7)$$

$$L_t^o = N_t^o - \frac{\zeta C_t^o}{(1 - \tau_t)\xi w_t} \quad (4.8)$$

$$L_t = L_t^y + L_t^w + \xi L_t^r + \xi L_t^o \quad (4.9)$$

Aggregate stocks of human capital follow the law of motion:

$$H_t^o = (1 - \tau_t)w_t L_t^o \xi + e_t N_t^o + \frac{\gamma}{(1 + n)(1 + r_{t+1})} H_{t+1}^o \quad (4.10)$$

$$H_t^r = (1 - \tau_t)w_t L_t^r \xi + e_t N_t^r + \frac{\eta}{\Omega_{t+1}^r (1 + n)R_{t+1}} H_{t+1}^r + \left(1 - \frac{\eta}{\Omega_{t+1}^r}\right) \frac{\theta H_{t+1}^o}{(1 + n)R_{t+1}\psi^{o,m}} \quad (4.11)$$

$$H_t^w = (1 - \tau_t)w_t L_t^w - z_t N_t^w + \frac{\eta}{\Omega_{t+1}^w (1 + n)R_{t+1}} H_{t+1}^w + \left(1 - \frac{\eta}{\Omega_{t+1}^w}\right) \frac{(1 - \theta)H_{t+1}^o}{(1 + n)R_{t+1}\psi^{o,m}} \quad (4.12)$$

$$H_t^y = (1 - \tau_t)w_t L_t^y - z_t N_t^y + \frac{\omega H_{t+1}^y}{R_{t+1}\Omega_{t+1}^y (1 + n)} + \frac{(1 - \omega)\theta (\Lambda_{t+1}^{y,r})^{1-\rho} (\xi)^{\phi-1} H_{t+1}^r}{R_{t+1}\Omega_{t+1}^y (1 + n)\theta\psi_t^{m,y}} + \frac{(1 - \omega)(1 - \theta) (\Lambda_{t+1}^{y,w})^{1-\rho} H_{t+1}^w}{R_{t+1}\Omega_{t+1}^y (1 + n)(1 - \theta)\psi^{m,y}} \quad (4.13)$$

Where $\Lambda_{t+1}^{y,r} = (\pi_{t+1}^r / \pi_{t+1}^y)^{\frac{-1}{\rho}}$ and $\Lambda_{t+1}^{y,w} = (\pi_{t+1}^w / \pi_{t+1}^y)^{\frac{-1}{\rho}}$. Note that when aggregate, $(1 + n)$ is added to the discount terms. It ensures that the total sum of future discounted flows of labor income in a given

age-group do not include those coming from agents who are currently not in these groups.

For consumption, we can write

$$C_t^o = \pi_t^o [(1 + r_t)A_t^o + H_t^o] \quad (4.14)$$

$$C_t^r = \pi_t^r [(1 + r_t)A_t^r + H_t^r] \quad (4.15)$$

$$C_t^w = \pi_t^w [(1 + r_t)A_t^w + H_t^w] \quad (4.16)$$

$$C_t^y = \pi_t^y [(1 + r_t)A_t^y + H_t^y] \quad (4.17)$$

Finally, for financial wealth, we can first write the law of motion of aggregate retirees' wealth:

$$A_{t+1}^o = R_t A_t^o + (1 - \tau_t)w_t \xi L_t^o e_t N_t^o - C_t^o + (1 - \eta) [R_t (A_t^r + A_t^w) + (1 - \tau_t)w_t (L_t^w + \xi L_t^r) + e_t N_t^r - z_t N_t^w - (C_t^r + C_t^w)]$$

Whereas for middle aged working and retired individuals, we can write:

$$A_{t+1}^r = \eta [R_t A_t^r + (1 - \tau_t)w_t \xi L_t^r + e_t N_t^r - C_t^r] + (1 - \omega)\theta [R_t A_t^y + (1 - \tau_t)w_t L_t^y - C_t^y - z_t N_t^y]$$

$$A_{t+1}^w = \eta [R_t A_t^w + (1 - \tau_t)w_t L_t^w - C_t^w] + (1 - \omega)(1 - \theta) [R_t A_t^y + (1 - \tau_t)w_t L_t^y - C_t^y - z_t N_t^y]$$

and for youngsters:

$$A_{t+1}^y = \omega [R_t A_t^y + (1 - \tau_t)w_t L_t^y - C_t^y - z_t N_t^y]$$

2.4 Firms

The supply side of the economy is standard. Competitive firms employ labor and capital to produce a homogeneous final good, which is used for both consumption and investment. The production function is Cobb- Douglas. The problem of a representative firm can be written:

$$\max_{L_t, K_t} Y_t - (w_t L_t + r^k K_t)$$

$$\text{s.t. } Y = K_t^\alpha L_t^{1-\alpha}$$

F.O.Cs are the standard ones:

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}$$

$$r_t^k = \alpha \frac{Y_t}{K_t}$$

where $r_t^k - \delta = r_t$, δ being the capital depreciation rate. Note that at the equilibrium, both countries interest rate has to be equalized.

2.5 Social security system

Government operates a PAYG pension system, that we assume to be balanced at each period, so that

$$\tau_t w_t L_t + z_t(N_t^y + N_t^w) = e_t(N_t^r + N_t^o) = \mu_t w_t(N_t^r + N_t^o)$$

Where $\mu_t = \frac{e_t}{w_t}$ is the replacement rate.

2.6 Equilibrium

Since we are in open economy, financial markets is at the equilibrium when the total amount of asset is equal to the total amount of capital, that is to say, using countries subscripts $m = \{1, 2\}$

$$A_{t,1} + A_{t,2} = K_{t,1} + K_{t,2}$$

Defining Net Foreign Asset (NFA) as $F_{t,m} = A_{t,m} - K_{t,m}$, the equilibrium on international financial market writes down

$$F_{t,1} + F_{t,2} = 0$$

Domestic investment is given by

$$I_{t,m} = K_{t+1,m} - (1 - \delta)K_{t,m}$$

The current surplus being defined as the variation of the NFA, *i.e.*

$$CA_{t,m} = F_{t+1,m} - F_{t,m} = S_{t,m} - I_{t,m}$$

Where $S_{t,m}$ is the net saving, that is to say

$$S_{t,m} = Y_{t,m} + r_t F_{t,m} - C_{t,m}$$

The equilibrium of the good market in a given country is given by the following:

$$\begin{aligned} A_{t+1,m} &= F_{t+1,m} + K_{t+1,m} \\ &= (1 + r_t^k - \delta)(F_{t,m} + K_{t,m}) + (1 - \tau_{t,m})w_t L_{t,m} + e_{t,m}(N_{t,m}^r + N_{t,m}^o) - z_{t,m}(N_{t,m}^y + N_{t,m}^w) - C_{t,m} \\ &= (1 + r_t^k - \delta)K_{t,m} + r_t F_{t,m} + F_{t,m} + w_t L_{t,m} - C_{t,m} \\ &= (1 - \delta)K_{t,m} + Y_{t,m} - C_{t,m} + r_t F_{t,m} + F_{t,m} \\ \Leftrightarrow Y_{t,m} &= C_{t,m} + I_{t,m} + \underbrace{CA_{t,m} - r_t F_{t,m}}_{NX_{t,m}} \end{aligned}$$

Where $NX_{t,m}$ is the trade balance of country m .

In each country, $\{A_t, A_t^o, A_t^w, A_t^r, A_t^y, H_t^o, H_t^w, H_t^r, H_t^y, C_t^o, C_t^r, C_t^w, C_t^y, C_t, Y_t, K_t, L_t, L_t^o, L_t^r, L_t^w, L_t^y, F_t, CA_t, NX_t\}$

are then expressed per youngster for $\{a_t, a_t^y, c_t, c_t^y, h_t, h_t^y, y_t, k_t, l_t, l_t^y, f_t, ca_t, nx_t\}$, per working middle-aged for $\{c_t^w, h_t^w, l_t^w\}$, per retired middle-aged for $\{c_t^r, h_t^r, l_t^r\}$, and senior for $\{c_t^o, h_t^o, l_t^o\}$. In each country, we thus have, dropping country's subscript,

$$\pi_t^o = 1 - \gamma\beta^\sigma \left[R_{t+1} \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^o}{\pi_{t+1}^o}$$

$$\pi_t^r = 1 - \beta^\sigma \left[R_{t+1} \Omega_{t+1}^r \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r}$$

$$\pi_t^w = 1 - \beta^\sigma \left[R_{t+1} \Omega_{t+1}^w \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^w}{\pi_{t+1}^w}$$

$$\pi_t^y = 1 - \beta^\sigma \left[R_{t+1} \Omega_{t+1}^y \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y}$$

$$\Omega_{t+1}^r = \eta + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}}$$

$$\Omega_{t+1}^w = \eta + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^w} \right)^{\frac{1}{1-\sigma}}$$

$$\Omega_{t+1}^y = \omega + (1 - \omega)\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} (\xi)^{\phi-1} + (1 - \omega)(1 - \theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}}$$

$$l_t^y = 1 - \frac{\zeta c_t^y}{(1 - \tau_t)w_t}$$

$$l_t^w = 1 - \frac{\zeta c_t^w}{(1 - \tau_t)w_t}$$

$$l_t^r = 1 - \frac{\zeta c_t^r}{(1 - \tau_t)\xi w_t}$$

$$l_t^o = 1 - \frac{\zeta c_t^o}{(1 - \tau_t)\xi w_t}$$

$$l_t = l_t^y + l_t^w(1 - \theta)\psi^{m,y} + \xi l_t^r \theta \psi^{m,y} + \xi \psi^{o,y} l_t^o$$

$$h_t^o = e_t + (1 - \tau_t)w_t \xi l_t^o + \frac{\gamma h_{t+1}^o}{(1 + r_{t+1})}$$

$$h_t^w = (1 - \tau_t)w_t l_t^w - z_t + \frac{\eta}{\Omega_{t+1}^w} \frac{h_{t+1}^w}{R_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^w}\right) \frac{h_{t+1}^o}{R_{t+1}}$$

$$h_t^r = (1 - \tau_t)w_t \xi l_t^r + e_t + \frac{\eta}{\Omega_{t+1}^r} \frac{h_{t+1}^r}{R_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^r}\right) \frac{h_{t+1}^o}{R_{t+1}}$$

$$h_t^y = (1 - \tau_t)w_t l_t^y - z_t + \frac{\omega h_{t+1}^y}{R_{t+1} \Omega_{t+1}^y} + \frac{(1 - \omega)\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y}\right)^{\frac{1}{1-\sigma}} (\xi^r)^{\phi-1} h_{t+1}^r}{R_{t+1} \Omega_{t+1}^y} + \frac{(1 - \omega)(1 - \theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y}\right)^{\frac{1}{1-\sigma}} h_{t+1}^w}{R_{t+1} \Omega_{t+1}^y}$$

$$c_t^o = \pi_t^o [R_t a_t^o + h_t^o]$$

$$c_t^r = \pi_t^r [R_t a_t^r + h_t^r]$$

$$c_t^w = \pi_t^w [R_t a_t^w + h_t^w]$$

$$c_t^y = \pi_t^y [R_t a_t^y + h_t^y]$$

$$c_t = c_t^y + \theta \psi^{m,y} c_t^r + (1 - \theta) \psi^{m,y} c_t^w + \psi_t^{o,y} c_t^o$$

$$(1 + n) a_{t+1}^o = R_t a_t^o + (1 - \tau_t) w_t \xi l_t^o + e_t - c_t^o +$$

$$\frac{1 - \eta}{\psi^{o,m}} (R_t [\theta a_t^r + (1 - \theta) a_t^w] + (1 - \tau_t) w_t [(1 - \theta) l_t^w + \theta l_t^r] + \theta e_t - [\theta c_t^r + (1 - \theta) c_t^w] - (1 - \theta) z_t)$$

$$(1 + n) a_{t+1}^r = \eta [R_t a_t^r + (1 - \tau_t) w_t l_t^r + e_t - c_t^r] + \frac{1 - \omega}{\psi^{m,y}} [R_t a_t^y + (1 - \tau_t) w_t l_t^y - c_t^y - z_t]$$

$$(1 + n) a_{t+1}^w = \eta [R_t a_t^w + (1 - \tau_t) w_t l_t^w - c_t^w] + \frac{1 - \omega_t}{\psi^{m,y}} [R_t a_t^y + (1 - \tau_t) w_t l_t^y - c_t^y - z_t]$$

$$(1 + n) a_{t+1}^y = \omega [R_t a_t^y + (1 - \tau_t) w_t l_t^y - c_t^y - z_t]$$

$$a_t = a_t^y + \theta \psi_t^{m,y} a_t^r + (1 - \theta) \psi_t^{m,y} a_t^w + \psi_t^{o,y} a_t^o$$

$$y_t = k_t^\alpha l_t^{1-\alpha}$$

$$w_t l_t = (1 - \alpha) y_t$$

$$r_t^k k_t = \alpha y_t$$

$$(1 + n) k_{t+1} = (1 - \delta) k_t + i_t$$

$$f_t = a_t - k_t$$

$$y_t = c_t + i_t + n x_t$$

$$n x_t = c a_t - r_t f_t$$

$$c a_t = (1 + n) f_{t+1} - f_t$$

$$\tau_t w_t l_t + z_t [1 + (1 - \theta) \psi_t^{m,y}] = \mu_t w_t (\theta \psi_t^{m,y} + \psi_t^{o,y})$$

The financial market equilibrium writes

$$f_{t,1} + f_{t,2} \Phi_t = 0 \tag{4.18}$$

Where $\Phi_t = \frac{N_{t,2}}{N_{t,1}}$ is an adjustment factor that takes into account the difference in countries' size. Note that at the equilibrium, we have

$$r_{t,1} = r_{t,2}$$

Interest rates are equalized, and give a worldwide interest rate r_t^* .

3 Simulating a pension reform abroad

How can a pension cut abroad affect the french economy? Should we expect a positive externality or a negative one? This section first calibrates the model presented in the previous section to reproduce the main demographic and macroeconomic variables in France and in the rest of the world, the latter being composed with OECD countries and China, in 2015-2018. The model is then used to make the

following numerical experiment. Ensuing a lowering of foreign pension replacement rate from 0.557 to 0.517, we first analyze the short run transition dynamics, before focusing on the long run implications of such a public policy. In particular, in the analysis of the long-run impact of foreign pension cuts, this section emphasizes on the welfare effect it carries for french households.

3.1 Calibration

There are two countries in the world economy: France and the rest of the world (RoW), constituted with OECD countries and China. The relative size of these two economies is given by the ratio of 15-59 years old of the RoW over its french counterpart: RoW is 44,3 times larger than France. Age cohorts in these two regions are given as follows: youngsters are 15-59 years old, middle-aged are 60-64 years old and seniors are 65 years old and more. The demographic growth rate in these two regions is 0.55%, which approximately corresponds to what one could observe in average during the last nine years. In order to reproduce the dependence ratios in France and in the RoW – respectively 32% and 20% –, life expectancy is fixed at 83 and 76.2 years old.

The EIS σ is fixed at 0.25, as in Auerbach and Kotlikoff (1987) [5] in both countries⁶. l_t^y and l_t^o , employment rates of 15-59 years old and 65 years old and more are fixed at 0.7 and 0.03 in France, 0.74 and 0.18 in the rest of the world⁷. Yearly interest rate is fixed at 3%, a value based on the work of Jordà *et al.* (2019) [52]. They indeed find a gap between the return of wealth r and the growth rate of the economy between 3 and 4%, a result in the line with the one of Piketty (2014) [73]. The discount rate β (common to both countries), productivity of retirees ξ and the parameter for consumption's preference ϕ (the two of them being different across countries) are fixed such that the model reproduces l_t^y , l_t^o and r_t ⁸. Note that we find a β slightly over 1, a calibration often met in the OLG literature where the finiteness of life allows for such values. The french replacement rate is fixed at 0.6, while the one for the RoW is 0.567⁹. The payroll tax rate in France is fixed at 31.4%, as estimated by the Conseil d'Orientation des Retraites (2015) [25]. In order to fill the deficit induced by such a rate, we assume that the lump-sum tax z_t adjusts¹⁰. The probability to be retired when switching to the state "middle-aged" θ is used as a reduced form to approximate the retirement age. It is thus fixed at $\frac{3}{5}$ in France – where the retirement age is 62 years old – and $\frac{3.1}{5}$ in the RoW – where the retirement age is 61.9 years old (OECD, 2019 [70]). This calibration implies a french NFA position of -24% of GDP, which is almost equal to its empirical counterpart in 2019. Values of parameters and variables at the steady-state are reported in Table 3.1 and 3.2.

⁶Low values of the EIS parameter are common in this literature, see for instance Carvalho *et al.* (2016) [18] or Ferrero (2010) [32].

⁷Data source: OECD, 2018. Employment rates for the RoW are computed by taking the average between the employment rate of the OECD and china, weighted by their 15-59 population.

⁸We indeed have 5 target variables for 5 parameters.

⁹RoW's replacement rate is the average (weighted by 15-59 population) of the OECD mandatory public replacement rate of 39% and its Chinese counterpart of 71%. Data come from OECD (2019) [70]

¹⁰The french pension system is systematically in deficit since 2007, see for instance the 2018 report of the Conseil d'Orientation des Retraites.

Table 3.1: Parameters

	France	Rest of the World
EIS σ	0.25	0.25
Discount rate β	1.02	1.02
Preference for consumption of goods ϕ	0.69	0.74
Share of capital in income α	1/3	1/3
Capital depreciation	0.1	0.1
Share of middle-aged who get a pension θ	3/5	3.1/5
Payroll tax rate τ	31.4%	21%
Productivity of retirees	0.52	0.48
Replacement rate μ	0.6	0.567
Population growth rate	0.55%	0.55%
Probability to stay "young" ω	$1 - \frac{1}{45}$	$1 - \frac{1}{45}$
Probability to stay "middle-aged" η	$1 - \frac{1}{5}$	$1 - \frac{1}{5}$
Probability to survive γ	$1 - \frac{1}{18}$	$1 - \frac{1}{11.2}$

Table 3.2: Variables

	France	Rest of the World
Interest rate r	4%	4%
Employment rate, 15-59 years old l^y	70%	74%
Employment rate, 60-64 years old $\theta l^r + (1 - \theta)l^w$	37.6%	45.5%
Employment rate, 65 years old and more l^o	3%	18%
Dependence ratio (65 years old and more over 15-64 years old) $\frac{\psi^{oy}}{1 + \psi^{my}}$	32%	20.6%
Consumption over GDP c/y	72.6%	73.2%
Pension over GDP $\frac{e(\theta\psi^{my} + \psi^{oy})}{y}$	23.1%	14.07%
Capital over GDP k/y	2.54	2.54
Current balance, in % of GDP	-0.13%	0.003%
NFA, in % of GDP	-24.1%	0.5%

3.2 Numerical experiment

In order to conclude whether a pension cut in the RoW benefits to french households or not, one must first decompose the impact of such a reform in two phases: the short run transition phase and the new steady-state towards which the economy converges, these phases being computed in perfect foresight. Quite interestingly, the model predicts a very short run effect that goes in the opposite direction of the long run one, as wages are first expected to slightly decrease, and then to go up. Before moving on, three points are worth clarifying. First, why would foreign government cut pensions in an environment where the population is assumed to be stationary? Second, what could be the amplitude of such a reform? Three, can governments lower pensions without any compensations to the current retirees, or those who are close to the age of retirement? The rationale behind a pension cut – among others – in the current framework can be found in the previous chapter, where I have shown that even if "aging risk" combined with an absence of insurance markets makes social security welfare increasing when the EIS is low, government can "overshoot" the optimal replacement rate.. For numerous reasons – e.g. political frictions, myopia – it might be efficient in a long run perspective to lower taxes and thus to

cut pensions. In our calibration, a 5 percentage points reduction of the replacement rate is expected to increase the welfare of youngsters in the final steady-state by 0.35%. Finally, pensions cut without any form of compensation can also be considered as realistic. For instance, putting an end to the indexation of pensions on real wages is equivalent to a decrease in the replacement rate, a policy that has been put in place in several European countries since 2000 (see Carone *et al.*, 2016 [16]). If we associate such a policy with automatic stabilization mechanisms of social security budget as in Sweden, a significant drop in the level of pensions can take place without any compensation.

3.3 Results

The transition phase is plotted in Figure 3.1. The direction in which the interest rate goes following a pension cut depends on the relative strength of the adjustments of labor and savings. During the first periods, senior households face an income loss and therefore immediately increase their labor supply, which thus slightly pushes down the capital intensity. Even if they cut down their consumption, the additional savings it generates (see the saving rate in the RoW in Figure 3.1) is not enough in the first periods to counterbalance the small rise of the marginal productivity of capital that a higher employment rate causes. As the interest rate in the RoW is higher – in the first years –, french savings go abroad, until the capitalist intensity becomes low enough to make the french return of capital equal to the one abroad, preventing any *arbitrage*. This adjustment ends up in a marginally lower french marginal productivity of labor, *i.e.* slightly lower wages. As a result, because of the substitution effect, the french total labor supply decreases. In addition to this substitution effect, expectations of higher labor income (coming from the long run effect we will discuss below) increases human wealth and thus creates a wealth effect that depresses labor supply in the first years. Since the social security budget has to be balanced, the french payroll tax rate increases slightly in order to compensate for the lower employment rate. As the labor force decreases, the GDP also goes down by 0.5%.

In the following years, the increase of saving abroad induced by pension cuts eventually pushes down the RoW interest rate, which thus goes below the french autarky rate. Because of the arbitrage opportunity, foreign households invest in France until both rates equalize. As a result, the global interest rate decreases by 0.5 percentage point, while the french NFA position goes from -23% of GDP to -80%. It is worth noting the fact that the decrease of the interest rate is almost linear, as a pension cut of one percentage point leads to a decrease of the interest rate of 0.1 percentage point. Such a high elasticity of the interest rate comes from the form of the marginal propensity to consume at the steady state, which in the case of seniors writes

$$\pi^o = 1 - \gamma\beta^\sigma R^{\sigma-1}$$

A pension cut starts by reducing the total wealth of foreign seniors at the numerator of the expression

$$c^o = \pi^o R \left(\frac{(1 - \tau)w\xi + e + a^o(R - \gamma)}{R - \gamma} + \zeta R \pi^o \right),$$

Figure 3.1: Transition phase a pension cut in the RoW

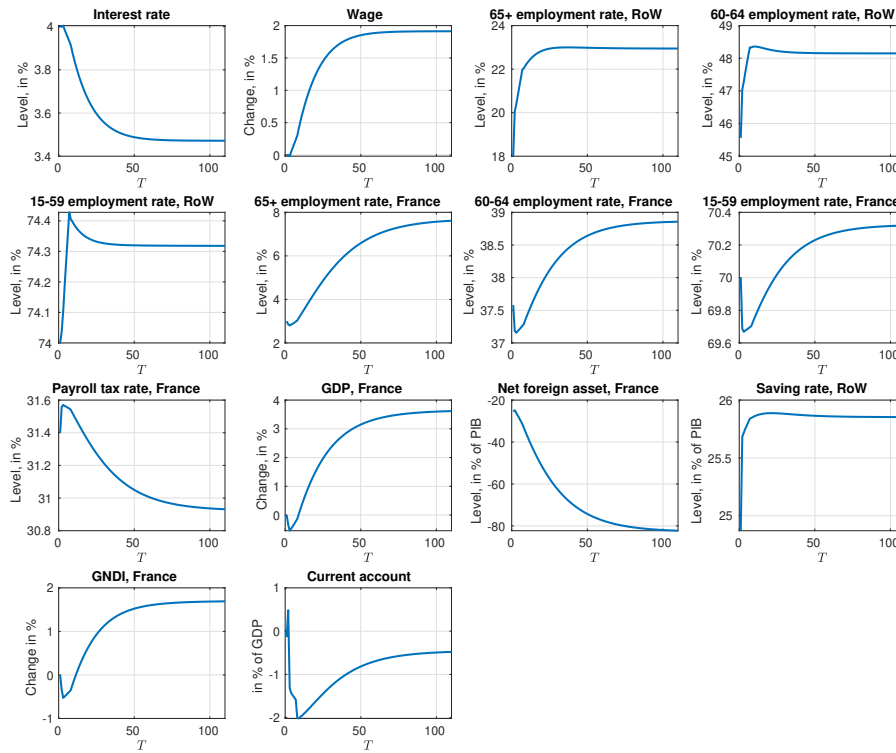
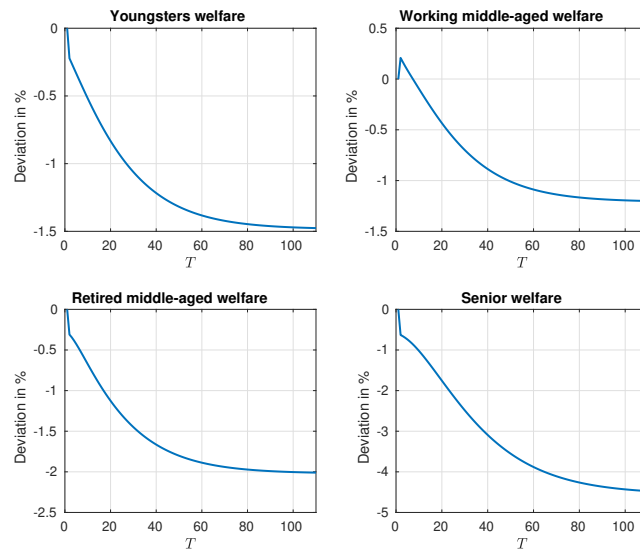


Figure 3.2: Welfare in France following a pension cut in the RoW, several class-ages



they thus reduce their consumption, which pushes down the interest rate. As $\sigma < 1$, a lower r decreases

the marginal propensity to consume, as the income effect is stronger than the substitution effect (a lower interest rates makes the agent poorer, and makes him save more). This additional incentive to save thus put further downward pressure on the interest rate, which therefore decrease almost linearly, despite the decreasing marginal productivity of capital. At the same time, as investments increase the marginal productivity of labor, wages goes up by 2%, which *in fine* leads to a higher participation of all age-classes. Note that seniors' employment rate is particularly sensitive to this pay rise, as it increases by 4.37 percentage points. This dynamic response of seniors' labor supply – and to a lesser extent, of middle-aged – comes from a lower level of financial wealth, due to a lower interest rate: as these households rely more on their financial assets, a lower return of savings reduces their wealth to a greater degree than in the case of the youngsters. To summarize, as the steady-state seniors' consumption of leisure takes the form

$$1 - l^o = \frac{\zeta \pi^o R}{(1 - \tau)w\xi} \left(\frac{(1 - \tau)w\xi + e + a^o(R - \gamma)}{(R - \gamma) + \zeta R\pi^o} \right)$$

a reduction of their financial wealth a^o makes them work more and consume less. On the contrary, youngsters' employment rate stays relatively stable despite the increase in wages, as they rely less on their financial wealth.

Even if they help achieving a higher level of wages, pension cuts have a negative impact on french households welfare. Remember first that the investments responsible for the higher productivity of labor come from abroad, they thus naturally give birth to payments of interests. As they must be deduced from the additional income generated by these investments, the welfare increasing impact of higher wages is soften. In national accounting terms, this is the difference between GDP and Gross National Disposable Income ¹¹ (GNDI): the former is expected to increase by 3.5%, while the latter will rise by 1.7%. Second, pension cuts push the global interest rate downward, which therefore make french households – who are net creditors – poorer. Because of this wealth effect, combined with the mechanism mentioned above, welfare of all age-classes in France decrease, as one can see in Figure 3.2.

4 Concluding remarks

This chapter studies the impact of a public pension cut in the rest of the world on the french economy. Using a two-countries (France and OECD-China) model based on Gertler (1999) [37] which features stochastic aging, RINCE preferences and elastic labor supply, it shows that foreign pension cuts are likely to be a source of negative externalities toward french households, through a decrease of the interest rate. Indeed, pension cuts create an increase of saving abroad, and therefore pushes down the equilibrium interest rate. Even if this translates into investments that improve the marginal productivity of labor and thus wages, it also makes french households poorer and is therefore a source of a decrease in welfare.

¹¹GNDI is given by $GNI_t = Y_t + r_t F_t$. As we are in open economy, we have to make the difference between GDP and GNI. As the french economy is indebted, keep in mind that a share of the GDP goes abroad.

Appendix

4.A Euler equations

4.A.1 Senior's program

The senior's program can be written the following way

$$V_t^{o,i,j,k} = \max_{c_t^{o,i,j,k}, l_t^{o,i,j,k}} \left[\left((c_t^{o,i,j,k})^\phi (1 - l_t^{o,i,j,k})^{1-\phi} \right)^\rho + \beta \gamma \left(V_{t+1}^{o,i,j,k} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$\text{s.t. } a_{t+1}^{o,i,j,k} = \frac{1 + r_t}{\gamma} a_t^{o,i,j,k} + (1 - \tau_t) w_t \xi l_t^{o,i,j,k} + e_t - c_t^{o,i,j,k}$$

First order conditions give

$$\phi (c_t^{o,i,j,k})^{\phi-1} (1 - l_t^{o,i,j,k})^{1-\phi} \left((c_t^{o,i,j,k})^\phi (1 - l_t^{o,i,j,k})^{1-\phi} \right)^{\rho-1} \left(V_t^{o,i,j,k} \right)^{1-\rho} = \mu_t^{o,i,j,k}$$

$$(1 - \phi) (c_t^{o,i,j,k})^\phi (1 - l_t^{o,i,j,k})^{-\phi} \left((c_t^{o,i,j,k})^\phi (1 - l_t^{o,i,j,k})^{1-\phi} \right)^{\rho-1} \left(V_t^{o,i,j,k} \right)^{1-\rho} = (1 - \tau_t) w_t \xi \mu_t^{o,i,j,k}$$

$$\beta \gamma \frac{\partial V_{t+1}^{o,i,j,k}}{\partial a_{t+1}^{o,i,j,k}} \left(V_{t+1}^{o,i,j,k} \right)^{\rho-1} \left(V_t^{o,i,j,k} \right)^{1-\rho} = \mu_t^{o,i,j,k}$$

While the envelope condition is

$$\frac{\partial V_t^{o,i,j,k}}{\partial a_t^{o,i,j,k}} = \frac{R_t}{\gamma} \mu_t^{o,i,j,k}$$

Combining, we get the consumption-leisure trade-off condition:

$$1 - l_t^{o,i,j,k} = \frac{\xi}{(1 - \tau_t) w_t} c_t^{o,i,j,k}$$

Assembling first order conditions and the envelope, one gets the following Euler equation:

$$c_{t+1}^{o,i,j,k} = \left[\left(\frac{(1 - \tau_t) w_t}{(1 - \tau_{t+1}) w_{t+1}} \right)^{(1-\phi)\rho} \beta R_{t+1} \right]^\sigma c_t^{o,i,j,k}$$

4.A.2 Retired middle-aged program

The retired middle-aged program can be written

$$V_t^{r,i,j} = \max_{c_t^{r,i,j}, l_t^{r,i,j}} \left[\left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^\rho + \beta \left(\eta V_{t+1}^{r,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1} \right)^\rho \right]^{\frac{1}{\rho}}$$

s.t. $a_{t+1}^{r,i,j} = (1 + r_t) a_t^{r,i,j} + (1 - \tau_t) w_t l_t^{r,i,j} \zeta + e_t - c_t^{r,i,j}$

First order conditions can be written

$$\phi (c_t^{r,i,j})^{\phi-1} (1 - l_t^{r,i,j})^{1-\phi} \left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^{\rho-1} \left(V_t^{r,i,j} \right)^{1-\rho} = \mu_t^{r,i,j} \quad (4.19)$$

$$(1 - \phi) (c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{-\phi} \left((c_t^{r,i,j})^\phi (1 - l_t^{r,i,j})^{1-\phi} \right)^{\rho-1} \left(V_t^{r,i,j} \right)^{1-\rho} = (1 - \tau_t) w_t \zeta \mu_t^{r,i,j} \quad (4.20)$$

$$\beta \left[\frac{\eta \partial V_{t+1}^{r,i,j}}{\partial a_{t+1}^{r,i,j}} + (1 - \eta) \frac{\partial V_{t+1}^{o,i,j,t+1}}{\partial a_{t+1}^{r,i,j}} \right] \left[\eta V_{t+1}^{r,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1} \right]^{\rho-1} \left(V_t^{r,i,j} \right)^{1-\rho} = \mu_t^{r,i,j} \quad (4.21)$$

The envelope condition gives

$$\frac{\partial V_t^{r,i,j}}{\partial a_t^{r,i,j}} = R_t \mu_t^{r,i,j} \quad (4.22)$$

Combining equations (4.19) and (4.20), we get the consumption-leisure trade-off condition:

$$1 - l_t^{r,i,j} = \frac{\zeta}{(1 - \tau_t) w_t \zeta} c_t^{r,i,j}$$

marginal value of wealth can also be written, combining (4.19), (4.20) and (4.22):

$$\frac{\partial V_t^{r,i,j}}{\partial a_t^{r,i,j}} = \phi R_t \left(\frac{V_t^{r,i,j}}{c_t^{r,i,j}} \right)^{1-\rho} \left(\frac{\zeta}{(1 - \tau_t) w_t \zeta} \right)^{(1-\phi)\rho} \quad (4.23)$$

We can thus rewrite (4.21) as

$$\frac{\partial V_t^{r,i,j}}{\partial a_t^{r,i,j}} = \beta R_t \left[\eta \frac{\partial V_{t+1}^{r,i,j}}{\partial a_{t+1}^{r,i,j}} + (1 - \eta) \frac{\partial V_{t+1}^{o,i,j,t+1}}{\partial a_{t+1}^{r,i,j}} \right] \left[\frac{\eta V_{t+1}^{r,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1}}{V_t^{r,i,j}} \right]^{\rho-1} \quad (4.24)$$

Substituting (4.23), we get

$$\left(c_t^{r,i,j} \right)^{\rho-1} = \beta R_{t+1} \left(\frac{(1 - \tau_t) w_t}{(1 - \tau_{t+1}) w_{t+1}} \right)^{(1-\phi)\rho} \left[\eta \left(\frac{V_{t+1}^{r,i,j}}{c_{t+1}^{r,i,j}} \right)^{1-\rho} + (1 - \eta) \left(\frac{V_{t+1}^{o,i,j,t+1}}{c_{t+1}^{o,i,j,t+1}} \right)^{1-\rho} \right] \left[\eta V_{t+1}^{r,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1} \right]^{\rho-1} \quad (4.25)$$

which gives

$$\begin{aligned} (c_t^{r,i,j})^{\rho-1} = \beta R_{t+1} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \left[\eta + (1-\eta) \left(\frac{V_{t+1}^{o,i,j,t+1}/c_{t+1}^{o,i,j,t+1}}{V_{t+1}^{r,i,j}/c_{t+1}^{r,i,j}} \right)^{1-\rho} \right] \left(\frac{V_{t+1}^{r,i,j}}{c_{t+1}^{r,i,j}} \right)^{1-\rho} \times \\ \left[\eta V_{t+1}^{r,i,j} + (1-\eta)V_{t+1}^{o,i,j,t+1} \right]^{\rho-1} \end{aligned} \quad (4.26)$$

Which ends up with

$$\begin{aligned} (c_t^{r,i,j})^{\rho-1} = \beta R_{t+1} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \left[\eta + (1-\eta) \left(\frac{V_{t+1}^{o,i,j,t+1}/c_{t+1}^{o,i,j,t+1}}{V_{t+1}^{r,i,j}/c_{t+1}^{r,i,j}} \right)^{1-\rho} \right] \times \\ \left[\eta c_{t+1}^{r,i,j} + (1-\eta) \frac{V_{t+1}^{o,i,j,t+1}/c_{t+1}^{o,i,j,t+1}}{V_{t+1}^{r,i,j}/c_{t+1}^{r,i,j}} c_{t+1}^{o,i,j,t+1} \right]^{\rho-1} \end{aligned}$$

Noting

$$(\Lambda_{t+1}^{o,r})^{1-\rho} = \left(\frac{V_{t+1}^{o,i,j,t+1}/c_{t+1}^{o,i,j,t+1}}{V_{t+1}^{r,i,j}/c_{t+1}^{r,i,j}} \right)^{1-\rho} = \frac{\partial V_{t+1}^{o,i,j,t+1}/\partial a_{t+1}^{r,i,j}}{\partial V_{t+1}^{r,i,j}/\partial a_{t+1}^{r,i,j}}$$

the MRS between senior and retired middle-aged wealth, and

$$\Omega_{t+1}^{o,r} = \omega + (1-\omega) (\Lambda_{t+1}^{o,r})^{1-\rho}$$

we get a modified Euler equation:

$$\eta c_{t+1}^{r,i,j} + (1-\eta) \Lambda_{t+1}^{o,r} c_{t+1}^{o,i,j,t+1} = \left[\beta R_{t+1} \Omega_{t+1}^{o,r} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \right]^{\sigma} c_t^{r,i,j}$$

which takes into account the "age-switching" risk.

4.A.3 Working middle-aged program

The retired middle-aged's program can be written

$$\begin{aligned} V_t^{w,i,j} = \max_{c_t^{w,i,j}, l_t^{w,i,j}} \left[\left((c_t^{w,i,j})^\phi (1-l_t^{w,i,j})^{1-\phi} \right)^\rho + \beta \left(\eta V_{t+1}^{w,i,j} + (1-\eta) V_{t+1}^{o,i,j,t+1} \right)^\rho \right]^{\frac{1}{\rho}} \\ \text{s.t. } a_{t+1}^{w,i,j} = (1+r_t) a_t^{w,i,j} + (1-\tau_t) w_t l_t^{w,i,j} - c_t^{w,i,j} \end{aligned}$$

First order conditions can be written

$$\phi (c_t^{w,i,j})^{\phi-1} (1-l_t^{w,i,j})^{1-\phi} \left((c_t^{w,i,j})^\phi (1-l_t^{w,i,j})^{1-\phi} \right)^{\rho-1} \left(V_t^{w,i,j} \right)^{1-\rho} = \mu_t^{w,i,j} \quad (4.27)$$

$$(1-\phi) (c_t^{w,i,j})^{\phi} (1-l_t^{w,i,j})^{-\phi} \left((c_t^{w,i,j})^\phi (1-l_t^{w,i,j})^{1-\phi} \right)^{\rho-1} \left(V_t^{w,i,j} \right)^{1-\rho} = (1-\tau_t) w_t \mu_t^{w,i,j} \quad (4.28)$$

$$\beta \left[\frac{\eta \partial V_{t+1}^{w,i,j}}{\partial a_{t+1}^{w,i,j}} + (1-\eta) \frac{\partial V_{t+1}^{o,i,j,t+1}}{\partial a_{t+1}^{w,i,j}} \right] \left[\eta V_{t+1}^{w,i,j} + (1-\eta) V_{t+1}^{o,i,j,t+1} \right]^{\rho-1} \left(V_t^{w,i,j} \right)^{1-\rho} = \mu_t^{w,i,j} \quad (4.29)$$

The envelope condition gives

$$\frac{\partial V_t^{w,i,j}}{\partial a_t^{w,i,j}} = R_t \mu_t^{w,i,j} \quad (4.30)$$

Using the same methodology as above, we can write a modified Euler equation

$$\eta c_{t+1}^{r,i,j} + (1-\eta) \Lambda_{t+1}^{o,w} \xi^{\phi-1} c_{t+1}^{o,i,j,t+1} = \left[\beta R_{t+1} \Omega_{t+1}^{o,w} \left(\frac{(1-\tau_t) w_t}{(1-\tau_{t+1}) w_{t+1}} \right)^{(1-\phi)\rho} \right]^{\sigma} c_t^{r,i,j}$$

Where

$$\Lambda_{t+1}^{o,w} = \frac{V_{t+1}^{o,i,j,t+1} / c_{t+1}^{o,i,j,t+1}}{V_{t+1}^{r,i,j} / c_{t+1}^{r,i,j}} \xi^{1-\phi}$$

and

$$\Omega_{t+1}^w = \eta + (1-\eta) (\Lambda_{t+1}^{o,w})^{1-\rho} \xi^{\phi-1}$$

The MRS between wealth in states "senior" and "working middle-aged" is here given by $(\Lambda_{t+1}^{o,w})^{1-\rho} \xi^{\phi-1}$.

4.A.4 Youngsters program

The program of a youngster born in i writes

$$V_t^{y,i} = \max_{c_t^{y,i}, l_t^{y,i}} \left[\left((c_t^{y,i})^\phi (1-l_t^{y,i})^{1-\phi} \right)^\rho + \beta \left[\omega V_{t+1}^{y,i} + (1-\omega) \left(\theta V_{t+1}^{r,i,t+1} + (1-\theta) V_{t+1}^{w,i,t+1} \right) \right]^\rho \right]^{\frac{1}{\rho}}$$

s.t. $a_{t+1}^{y,i} = (1+r_t) a_t^{y,i} + (1-\tau_t) w_t l_t^{y,i} - c_t^{y,i} - z_t$

First order conditions can be written

$$\phi (c_t^{y,i})^{\phi-1} (1-l_t^{y,i})^{1-\phi} \left((c_t^{y,i})^\phi (1-l_t^{y,i})^{1-\phi} \right)^{\rho-1} \left(V_t^{y,i} \right)^{1-\rho} = \mu_t^{y,i} \quad (4.31)$$

$$(1-\phi) (c_t^{y,i})^\phi (1-l_t^{y,i})^{-\phi} \left((c_t^{y,i})^\phi (1-l_t^{y,i})^{1-\phi} \right)^{\rho-1} \left(V_t^{y,i} \right)^{1-\rho} = (1-\tau_t) w_t \mu_t^{y,i} \quad (4.32)$$

$$\beta \left[\omega \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} + (1-\omega) \left(\theta \frac{\partial V_{t+1}^{r,i,t+1}}{\partial a_{t+1}^{y,i}} + (1-\theta) \frac{\partial V_{t+1}^{w,i,t+1}}{\partial a_{t+1}^{y,i}} \right) \right] \left[\omega V_{t+1}^{y,i} + (1-\omega) \left(\theta V_{t+1}^{r,i,t+1} + (1-\theta) V_{t+1}^{w,i,t+1} \right) \right]^{\rho-1} \times \left(V_t^{y,i} \right)^{1-\rho} = \mu_t^{y,i} \quad (4.33)$$

The envelope condition gives

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = R_t \mu_t^{y,i} \quad (4.34)$$

We thus get the labor supply equation

$$1 - l_t^{y,i} = \frac{\zeta}{(1 - \tau_t)w_t} c_t^{y,i}$$

the marginal value of wealth can also be written, combining (4.31), (4.32) and (4.34):

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = \phi R_t \left(\frac{V_t^{y,i}}{c_t^{y,i}} \right)^{1-\rho} \left(\frac{\zeta}{(1 - \tau_t)w_t} \right)^{(1-\phi)\rho} \quad (4.35)$$

We can thus rewrite (4.33) as

$$\frac{\partial V_t^{y,i}}{\partial a_t^{y,i}} = \beta \left[\omega \frac{\partial V_{t+1}^{y,i}}{\partial a_{t+1}^{y,i}} + (1 - \omega) \left(\theta \frac{\partial V_{t+1}^{r,i,t+1}}{\partial a_{t+1}^{y,i}} + (1 - \theta) \frac{\partial V_{t+1}^{w,i,t+1}}{\partial a_{t+1}^{y,i}} \right) \right] \left[\frac{\omega V_{t+1}^{y,i} + (1 - \omega) \left(\theta V_{t+1}^{r,i,t+1} + (1 - \theta) V_{t+1}^{w,i,t+1} \right)}{V_t^{w,i,j}} \right]^{\rho-1} \quad (4.36)$$

Substituting (4.23), we get

$$\left(c_t^{y,i} \right)^{\rho-1} = \beta R_{t+1} \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \left[\omega \left(\frac{V_{t+1}^{y,i}}{c_{t+1}^{y,i}} \right)^{1-\rho} + (1 - \omega) \left(\theta \left(\frac{V_{t+1}^{r,i,t+1}}{c_{t+1}^{r,i,t+1}} \right)^{1-\rho} \xi^{(\phi-1)\rho} + (1 - \theta) \left(\frac{V_{t+1}^{w,i,t+1}}{c_{t+1}^{w,i,t+1}} \right)^{1-\rho} \right) \right] \times \left[\omega V_{t+1}^{y,i} + (1 - \omega) \left(\theta V_{t+1}^{r,i,t+1} + (1 - \theta) V_{t+1}^{w,i,t+1} \right) \right]^{\rho-1} \quad (4.37)$$

which gives

$$\left(c_t^{y,i} \right)^{\rho-1} = \left[\omega + (1 - \omega) \left(\theta \left(\frac{V_{t+1}^{r,i,t+1}/c_{t+1}^{r,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} \xi^{(\phi-1)\rho} + (1 - \theta) \left(\frac{V_{t+1}^{w,i,t+1}/c_{t+1}^{w,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \right)^{1-\rho} \right) \right] \times \beta R_{t+1} \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \left[\omega c_{t+1}^{y,i} + (1 - \omega) \left(\theta \frac{V_{t+1}^{r,i,t+1}/c_{t+1}^{r,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} c_{t+1}^{r,i,t+1} + (1 - \theta) \frac{V_{t+1}^{w,i,t+1}/c_{t+1}^{w,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} c_{t+1}^{w,i,t+1} \right) \right]^{\rho-1}$$

Let us note $\Lambda_{t+1}^{r,y} = \frac{V_{t+1}^{r,i,t+1}/c_{t+1}^{r,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}} \xi^{1-\phi}$ and $\Lambda_{t+1}^{w,y} = \frac{V_{t+1}^{w,i,t+1}/c_{t+1}^{w,i,t+1}}{V_{t+1}^{y,i}/c_{t+1}^{y,i}}$. Then we have $(\Lambda_{t+1}^{r,y})^{1-\rho} \xi^{\phi-1} = \frac{\partial V_{t+1}^{r,i,t+1}/\partial a_{t+1}^{r,i,t+1}}{\partial V_{t+1}^{y,i}/\partial a_{t+1}^{y,i}}$, *i.e.* the MRS between wealth in state "retired middle-aged" and wealth in the state "young". Also, we have $(\Lambda_{t+1}^{w,y})^{1-\rho} = \frac{\partial V_{t+1}^{w,i,t+1}/\partial a_{t+1}^{w,i,t+1}}{\partial V_{t+1}^{y,i}/\partial a_{t+1}^{y,i}}$, *i.e.* the MRS between wealth in state "working middle-aged" and wealth in the state "young". Let us finally note

$$\Omega_{t+1}^y = \omega + (1 - \omega) \left[\theta (\Lambda_{t+1}^{r,y})^{1-\rho} \xi^{\phi-1} + (1 - \theta) (\Lambda_{t+1}^{w,y})^{1-\rho} \right]$$

The modified Euler equation is thus

$$\omega c_{t+1}^{y,i} + (1 - \omega) \left[\theta \Lambda_{t+1}^{r,y} \xi^{\phi-1} c_{t+1}^{r,i,t+1} + (1 - \theta) \Lambda_{t+1}^{w,y} c_{t+1}^{w,i,t+1} \right] = \left[\beta R_{t+1} \Omega_{t+1}^y \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{(1-\phi)\rho} \right]^\sigma c_t^{y,i}$$

4.B Closed-form solutions

4.B.1 Seniors

Let us conjecture the following

$$\begin{aligned} c_t^{o,i,j,k} &= \pi_t^o \left(\frac{R_t}{\gamma} a_t^{o,i,i,k} + h_t^{o,i,j,k} \right) \\ h_t^{o,i,j,k} &= (1 - \tau_t) \omega_t \xi l_t^{o,i,j,k} + \frac{\gamma h_{t+1}^{o,i,j,k}}{1 + r_{t+1}} \\ V_t^{o,i,j,k} &= (\pi_t^o)^{\frac{-1}{\rho}} c_t^{o,i,j,k} \left(\frac{\zeta}{(1 - \tau_t) \omega_t \xi} \right)^{1-\phi} \end{aligned}$$

Consumption is linear in financial and human wealth, the latter being the discounted sum of income. The Euler equations therefore now writes:

$$a_{t+1}^{o,i,j,k} + \frac{\gamma h_{t+1}^{o,i,j,k}}{1 + r_{t+1}} = \gamma \beta^\sigma \left[R_{t+1} \left(\frac{(1 - \tau_t) \omega_t}{(1 - \tau_{t+1}) \omega_{t+1}} \right)^{(1-\phi)} \right]^{\sigma-1} \frac{\pi_t^{o,i,j,k}}{\pi_{t+1}^o} \left[(1 + r_t) a_t^{o,i,j,k} + h_t^{o,i,j,k} \right]$$

As $a_{t+1}^{o,i,j,k} + \frac{\gamma h_{t+1}^{o,i,j,k}}{1 + r_{t+1}} = \frac{R_t}{\gamma} a_t^{o,i,j,k} + h_t^{o,i,j,k} - c_t^{o,i,j,k} = \left[\frac{1+r_t}{\gamma} a_t^{o,i,j,k} + h_t^{o,i,j,k} \right] (1 - \pi_t^o)$, we can write

$$\pi_t^o = 1 - \gamma \beta^\sigma \left[R_{t+1} \left(\frac{(1 - \tau_t) \omega_t}{(1 - \tau_{t+1}) \omega_{t+1}} \right)^{(1-\phi)} \right]^{\sigma-1} \frac{\pi_t^o}{\pi_{t+1}^o}$$

Which is the expression for the marginal propensity to consume. To check the conjecture, one simply has to rewrite the Euler equation as

$$\begin{aligned} V_{t+1}^{o,i,j,k} (\pi_{t+1}^o)^{\frac{1}{\rho}} [(1 - \tau_{t+1}) \omega_{t+1}]^{1-\phi} &= (\beta R_{t+1})^\sigma \left(\frac{(1 - \tau_t) \omega_t}{(1 - \tau_{t+1}) \omega_{t+1}} \right)^{(1-\phi)(\sigma-1)} V_t^{o,i,j,k} (\pi_t^o)^{\frac{1}{\rho}} [(1 - \tau_t) \omega_t]^{1-\phi} \\ \Rightarrow V_{t+1}^{o,i,j,k} (\pi_{t+1}^o)^{\frac{1}{\rho}} &= (\beta R_{t+1})^\sigma \left(\frac{(1 - \tau_t) \omega_t}{(1 - \tau_{t+1}) \omega_{t+1}} \right)^{(1-\phi)\sigma} V_t^{o,i,j,k} (\pi_t^o)^{\frac{1}{\rho}} \\ \Rightarrow \gamma \beta \left(V_{t+1}^{o,i,j,k} \right)^\rho &= \gamma \beta^\sigma R_{t+1}^{\sigma-1} \left(\frac{(1 - \tau_t) \omega_t}{(1 - \tau_{t+1}) \omega_{t+1}} \right)^{(1-\phi)(\sigma-1)} \frac{\pi_t^o}{\pi_{t+1}^o} (V_t^o)^\rho \\ \Rightarrow \gamma \beta \left(V_{t+1}^{o,i,j,k} \right)^\rho &= (1 - \pi_t^o) \left(V_t^{o,i,j,k} \right)^\rho \\ \Rightarrow V_t^{o,i,j,k} &= \left[\left(\left(c_t^{o,i,j,k} \right)^\phi \left(1 - l_t^{o,i,j,k} \right)^{1-\phi} \right)^\rho + \beta \gamma \left(V_{t+1}^o \right)^\rho \right]^{\frac{1}{\rho}} \end{aligned}$$

Which is consistent.

4.B.2 Retired middle-aged

Let us now conjecture the following

$$c_t^{r,i,j} = \pi_t^r \left((1 + r_t) a_t^{r,i,j} + h_t^{r,i,j} \right)$$

$$h_t^{r,i,j} = (1 - \tau_t)w_t \xi_t^{r,i,j} + \frac{\eta}{\Omega_{t+1}^r} \frac{h_{t+1}^{r,i,j}}{1 + r_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^r}\right) \frac{h_{t+1}^{o,i,j,t+1}}{1 + r_{t+1}}$$

$$V_t^{r,i,j} = (\pi_t^r)^{\frac{-1}{\rho}} c_t^{r,i,j} \left(\frac{\zeta}{(1 - \tau_t)w_t \xi_t} \right)^{1-\phi}$$

where h_t^r is the human wealth, taking a forward form. As before, π_t^r is the MPC, and does not take into account the life history of the agent. This implies

$$\Omega_{t+1}^r = \eta + (1 - \eta) \left(\frac{\pi_t^o}{\pi_t^r} \right)^{\frac{1}{1-\sigma}} \quad \text{and} \quad \Lambda_{t+1}^r = \left(\frac{\pi_t^o}{\pi_t^r} \right)^{\frac{-1}{\rho}}$$

The Euler equation now becomes:

$$\eta \pi_{t+1}^r (R_{t+1} a_{t+1}^{r,i,j} + h_{t+1}^{r,i,j}) + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}} \pi_{t+1}^r (R_{t+1} a_{t+1}^{o,i,j,t+1} + h_{t+1}^{o,i,j,t+1}) = \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{(1-\phi)(\sigma-1)}$$

$$(\beta R_{t+1} \Omega_{t+1}^r)^\sigma \pi_t^y (R_t a_t^{r,i,j} + h_t^{r,i,j})$$

$$\Rightarrow \Omega_{t+1}^r a_{t+1}^{r,i,j} + \frac{\eta h_{t+1}^{r,i,j} + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}} h_{t+1}^{o,i,j,t+1}}{R_{t+1}} = \beta^\sigma R_{t+1}^{\sigma-1} \Omega_{t+1}^\sigma \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{(1-\phi)(\sigma-1)} \frac{\pi_t^r}{\pi_{t+1}^r} (R_t a_t^{r,i,j} + h_t^{r,i,j})$$

This result coming from the fact that $a_{t+1}^{r,i,j} = a_{t+1}^{o,i,j,t+1}$. Using the definition of the budget constraint and using the conjecture about the linearity of consumption and dividing by Ω_{t+1}^r :

$$R_t a_t^{r,i,j} + (1 - \tau_t)w_t \xi_t^{r,i,j} + e_t - \pi_t^r (a_t^{r,i,j} + h_t^{r,i,j}) + \frac{\eta h_{t+1}^{r,i,j} + (1 - \eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^r} \right)^{\frac{1}{1-\sigma}} h_{t+1}^{o,i,j,t+1}}{R_{t+1} \Omega_{t+1}^r} =$$

$$\beta^\sigma \left(R_{t+1} \Omega_{t+1}^r \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r} (R_t a_t^y + h_t^y)$$

Then, using the definition of human wealth:

$$(1 - \pi_t^r) (R_t a_t^{r,i,j} + h_t^{r,i,j}) = \beta^\sigma \left(R_{t+1} \Omega_{t+1}^r \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r} (R_t a_t^{r,i,j} + h_t^{r,i,j})$$

we thus get

$$\pi_t^r = 1 - \beta^\sigma \left(R_{t+1} \Omega_{t+1}^r \left(\frac{(1 - \tau_t)w_t}{(1 - \tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r}$$

To check for the conjecture, let us substitute the closed form solution of the value function in the Euler equation:

$$\left[\eta V_{t+1}^{r,i,j} + (1 - \eta) V_{t+1}^{o,i,j,t+1} \right] (\pi_{t+1}^r)^{\frac{1}{\rho}} [(1 - \tau_{t+1})w_{t+1}]^{1-\phi} =$$

$$\begin{aligned}
& (\beta R_{t+1} \Omega_{t+1}^r)^\sigma \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)(\sigma-1)} V_t^{r,i,j} (\pi_t^r)^{\frac{1}{\rho}} [(1-\tau_t)w_t]^{1-\phi} \\
\Rightarrow & \left[\eta V_{t+1}^{r,i,j} + (1-\eta)V_{t+1}^{o,i,j,t+1} \right] (\pi_{t+1}^r)^{\frac{1}{\rho}} = (\beta R_{t+1} \Omega_{t+1}^r)^\sigma \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{(1-\phi)\sigma} V_t^{r,i,j} (\pi_t^r)^{\frac{1}{\rho}} \\
\Rightarrow & \beta \left[\eta V_{t+1}^{r,i,j} + (1-\eta)V_{t+1}^{o,i,j,t+1} \right]^\rho = \beta^\sigma \left[R_{t+1} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^r}{\pi_{t+1}^r} \left(V_t^{r,i,j} \right)^\rho \\
\Rightarrow & \beta \left[\eta V_{t+1}^{r,i,j} + (1-\eta)V_{t+1}^{o,i,j,t+1} \right]^\rho = (1-\pi_t^o) \left(V_t^{r,i,j} \right)^\rho \\
\Rightarrow & V_t^{r,i,j} = \left[\left(\left(c_t^{r,i,j} \right)^\phi \left(1-l_t^{r,i,j} \right)^{1-\phi} \right)^\rho + \beta^\sigma \left[\eta V_{t+1}^{r,i,j} + (1-\eta)V_{t+1}^{o,i,j,t+1} \right]^\rho \right]^{\frac{1}{\rho}}
\end{aligned}$$

which is consistent.

4.B.3 Working middle-aged

Let us now conjecture the following

$$\begin{aligned}
c_t^{w,i,j} &= \pi_t^w \left((1+r_t) a_t^{w,i,j} + h_t^{w,i,j} \right) \\
h_t^{w,i,j} &= (1-\tau_t)w_t l_t^{w,i,j} - z_t + \frac{\eta}{\Omega_{t+1}^w} \frac{h_{t+1}^{w,i,j}}{1+r_{t+1}} + \left(1 - \frac{\eta}{\Omega_{t+1}^w} \right) \frac{h_{t+1}^{o,i,j,t+1}}{1+r_{t+1}} \\
V_t^{w,i,j} &= (\pi_t^w)^{\frac{-1}{\rho}} c_t^{w,i,j} \left(\frac{\zeta}{(1-\tau_t)w_t} \right)^{1-\phi}
\end{aligned}$$

Which implies that

$$\Lambda_{t+1}^{o,w} = \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^w} \right)^{\frac{-1}{\rho}} \xi^{1-\phi} \quad \text{and} \quad \Omega_{t+1}^w = \eta + (1-\eta) \left(\frac{\pi_{t+1}^o}{\pi_{t+1}^w} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1}$$

Following the same procedure as before, we end up with

$$\pi_t^w = 1 - \beta^\sigma \left(R_{t+1} \Omega_{t+1}^w \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^w}{\pi_{t+1}^w}$$

4.B.4 Youngsters

Let us finally make the following conjectures

$$\begin{aligned}
c_t^{y,i} &= \pi_t^y \left(R_t a_t^{y,i} + h_t^{y,i} \right) \\
V_t^{y,i} &= (\pi_t^y)^{\frac{-1}{\rho}} c_t^{y,i} \left(\frac{\zeta}{(1-\tau_t)w_t} \right)^{1-\phi} \\
h_t^{y,i} &= (1-\tau_t)w_t l_t^{y,i} - z_t + \frac{h_{t+1}^{y,i}}{R_{t+1} \Omega_{t+1}^y} \\
h_{t+1}^{y,i} &= \omega h_{t+1}^{y,i} + (1-\omega)\theta \left(\Lambda_{t+1}^{r,y} \right)^{1-\rho} (\xi)^{\phi-1} h_{t+1}^{r,i,t+1} + (1-\omega)(1-\theta) \left(\Lambda_{t+1}^{w,y} \right)^{1-\rho} h_{t+1}^{w,i,t+1}
\end{aligned}$$

Where we also have

$$\Lambda_{t+1}^{w,y} = \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{-1}{\rho}}, \quad \Lambda_{t+1}^{r,y} = \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{-1}{\rho}} \quad \text{and} \quad \Omega_{t+1}^y = \omega + (1-\omega) \left[\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1} + (1-\theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \right]$$

The Euler equation can then be rewritten

$$\omega \left(a_{t+1}^{y,i} + \frac{h_{t+1}^{y,i}}{R_{t+1}} \right) + (1-\omega) \left[\theta \left(\frac{\pi_{t+1}^r}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1} \left(a_{t+1}^{r,i,t+1} + \frac{h_{t+1}^{r,i,t+1}}{R_{t+1}} \right) + (1-\theta) \left(\frac{\pi_{t+1}^w}{\pi_{t+1}^y} \right)^{\frac{1}{1-\sigma}} \left(a_{t+1}^{w,i,t+1} + \frac{h_{t+1}^{w,i,t+1}}{R_{t+1}} \right) \right] =$$

$$[\beta \Omega_{t+1}^y]^\sigma \left[R_{t+1} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} \left(R_t a_t^{y,i} + h_t^{y,i} \right)$$

Using the fact that $a_{t+1}^{y,i} = a_{t+1}^{r,i,t+1} = a_{t+1}^{w,i,t+1}$ and the definition of human wealth

$$\Omega_{t+1}^y a_{t+1}^{y,i} + \frac{h_{t+1}^{y,i}}{R_{t+1}} = [\beta \Omega_{t+1}^y]^\sigma \left[R_{t+1} \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} \left(R_t a_t^{y,i} + h_t^{y,i} \right)$$

$$R_t a_t^{y,i} + h_t^{y,i} - c_t^{y,i} = \beta^\sigma \left[R_{t+1} \Omega_{t+1}^y \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right]^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} \left(R_t a_t^{y,i} + h_t^{y,i} \right)$$

Using the conjecture about the form of consumption once again,

$$(1-\pi_t^y) \left(R_t a_t^{y,i} + h_t^{y,i} \right) = \beta^\sigma \left(R_{t+1} \Omega_{t+1}^y \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y} \left(R_t a_t^{y,i} + h_t^{y,i} \right)$$

We end up with the forward equation for the marginal propensity to consume

$$\pi_t^y = 1 - \beta^\sigma \left(R_{t+1} \Omega_{t+1}^y \left(\frac{(1-\tau_t)w_t}{(1-\tau_{t+1})w_{t+1}} \right)^{1-\phi} \right)^{\sigma-1} \frac{\pi_t^y}{\pi_{t+1}^y}$$

The verification of the conjecture follows the same procedure as before.

4.B.5 Marginal value of wealth and drop in productivity

Using the previously derived close-form solution, let us show that the MRS of wealth between (i) states "old" and "working middle-aged" and (ii) states "retired middle-aged" and youngsters increase as the drop in productivity is larger.

From the steady-state solution of $\pi^w = 1 - \beta^\sigma (R\Omega^w)^{\sigma-1}$, and applying the implicit function theorem, we have

$$\frac{\partial \pi^w}{\partial \xi} = \frac{1 + (1-\omega) (\pi^r)^{\frac{1}{1-\sigma}} (\pi^w)^{\frac{1}{1-\sigma}-1} \xi^{\phi-1} \beta^\sigma R^{\sigma-1} (\Omega^w)^{\sigma-2}}{(\phi-1) \xi^{\phi-2} \beta^\sigma R^{\sigma-1} (\sigma-1)}$$

We thus have $\frac{\partial \pi^w}{\partial \xi} > 0$ if $\sigma < 1$ and $\frac{\partial \pi^w}{\partial \xi} < 0$ if $\sigma > 1$. As the MRS of wealth is given by

$$\left(\frac{\pi^o}{\pi^w} \right)^{\frac{1}{1-\sigma}} \xi^{\phi-1}$$

and is an increasing function of ξ , no matter if $\sigma > 1$ or $\sigma < 1$. As the productivity drop is larger (*i.e.* ξ is low), the relative value of marginal wealth in state "senior" increases, as the agent has to rely more on financial assets.

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Titre : Quatre essais en macroéconomie à agents hétérogènes

Mot clés : Macroéconomie, Agents hétérogènes, Cycle, Modèles à générations imbriquées, Systèmes de retraites, Modèles d'appariement, Bien-être

Résumé : Qu'il s'exprime en terme d'âge, de richesse ou encore de niveau de productivité, le caractère hétérogène des agents économiques est une variable dont la prise en compte est devenue primordiale afin d'améliorer notre compréhension de nombreuses problématiques macroéconomiques. À travers quatre chapitres aux thématiques diverses, cette thèse de doctorat s'inscrit dans cette dé-

marche en étudiant la manière dont l'hétérogénéité des agents influence la volatilité de variables macroéconomiques – en particuliers le taux de chômage et les heures travaillées –, et en analysant l'impact qu'ont les systèmes de retraites par répartition sur le bien-être et les principaux agrégats macroéconomiques, tant d'un point de vue normatif que positif.

Title: Four essays in macroeconomics with heterogeneous agents

Keywords: Macroeconomics, Heterogeneous agents, Business cycles, Overlapping generations models, Pension systems, Search and matching models, Welfare

Abstract: In terms of age, wealth or productivity, the heterogeneous nature of economic agents has become a key variable in order to improve our comprehension of various macroeconomic related issues. Throughout four chapters which cover a large spectrum of macroeconomic analysis, this Ph.D. dissertation follows this approach, by study-

ing the way agents' heterogeneity influences the cyclical behavior of macroeconomic variables – in particular unemployment rate and hours worked –, and by analyzing the impact of pay-as-you-go pension systems on welfare and macroeconomic aggregates, from both a normative and positive point of view.